Optimal Retirement Planning: Scenario Generation, Preferences, and Objectives

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 1 is to appear in the North American Actuarial Journal under the paper title *Updating Wilkie's Economic Scenario Generator for U.S. Applications*, by Saisai Zhang, Mary R. Hardy, and David Saunders (Zhang et al., 2018b).

Chapter 4 is published by the Canadian Institute of Actuaries under the paper title *Retirement Consumption, Risk Perception and Planning Objectives*, by Saisai Zhang, Mary R. Hardy, and David Saunders (Zhang et al., 2018a).

Abstract

The global trend of shifting from defined benefit (DB) to defined contribution (DC) workplace pension plans is putting growing pressure on individuals to take more ownership in retirement planning and financial decision-making. The essence of the DB is the life-long income guarantee, which requires limited financial planning decisions to be made, either in the accumulation or decumulation phase. The DC on the other hand, is significantly more complex. The lump sum payment at retirement burdens individuals with the task of income generation, in the presence of challenges stemming from an uncertain future lifetime, economic conditions, and evolving consumption needs. The average retiree has limited competency to navigate these challenges, due to low financial literacy, lack of willpower, or deteriorating cognitive abilities with older ages. The high stake of these challenges calls for a normative solution to be proposed – a solution that considers the intricacy of risks, preferences, and normative objective formulations. The objective of this thesis is to explore such a solution.

This thesis comprises three inter-related research directions: long-term economic scenario generators (ESGs), recursive preferences in life-cycle portfolio selection, and retirement objective formulation. A brief description of the subsequent chapters will now follow.

The first chapter conducts a review of Wilkie's ESG, with analysis restricted to series pertinent to retirement planning. Our main findings indicate that there exist challenges in modelling long-term economic series due to the presence of multiple structural shifts in the historical time series. Consequently, certain assumptions of stationarity are violated, and parameters are sensitive to the calibration period. A backtest based on 30-year out-of-sample data indicated that over that period the model had tended to overestimate inflation, underestimate total return on stocks, and performed relatively well for long-term interest rates. Additionally, Wilkie's ESG can be under-representative of the risk in long-term stock investment, particularly in the tails.

The second chapter provides an introductory discussion of Epstein-Zin preferences, which are adopted in the succeeding chapter as a normative preference model. The purpose is to first investigate the implied optimal behaviour and its plausibility. We pay particular attention to whether the output leads to plausible behaviour given the context of retirement planning. Specifically, analytical solutions for a simple consumption problem are derived, isolating the impact of relative risk aversion (RRA), elasticity of intertemporal substitution (EIS), time discounting, and risks stemming from mortality, investment, and inflation. We investigate three Epstein-Zin models employed in the literature, which differ in their treatment of mortality risk, and find that some lead to normatively implausible solutions. Importantly, we find that the EIS is not always monotone in its effect on consumption volatility over time, meaning that its interpretation can be ambiguous when considering an uncertain future lifetime. This has been misinterpreted in the literature to date. We also show that one particular Epstein-Zin specification is not necessarily a generalization of expected utility maximization under constant relative risk aversion, as many works wrongly

claim.

The third chapter investigates the normative validity of the optimal consumption and investment strategies of a discrete-time Epstein-Zin utility maximizing DC retiree who wishes to benefit from stock investment, longevity insurance, and inflation protection. A comparison of three Epstein-Zin specifications is conducted. We use a combination of qualitative and quantitative criteria to evaluate the adequacy of the optimal consumption profile, with special attention paid to the downside risk at extreme old ages. We find that it remains optimal to fully annuitize, but agents with high relative risk aversion hold precautionary savings, the level of which is impacted by the EIS and the preference specification. As discussed in the preceding chapter, the interpretation of EIS on consumption volatility is found to be ambiguous. Investigations of the optimal consumption profile reveal that agents are exposed to relatively high levels of downside risk in the long run. This is partially attributed to a time discounting factor less than 1, which implicitly (and contradictorily) assumes myopia in normative decision-making. An investigation of zero time discounting is conducted, with downside risk found be to significantly reduced in the long run.

The fourth chapter focuses on retirement objective formulation. This chapter is motivated by the unsatisfactory normative solutions found in the preceding chapter under mathematically convenient objective functions. In order to develop more actionable prescriptive solutions, we seek to holistically explore actual retirement decision-making. To this end, we conduct a survey study of 1,000 Canadian (pre-)retirees age 50 to 80, on topics of retirement consumption, wealth, income, risk perception, decision making, and planning objectives. Additionally, we investigate the descriptive validity of the expected lifetime discounted utility maximization framework in predicting optimal planning behaviours. Overall, there is overwhelming evidence of heterogeneity in wealth, income, concerns, and objectives. We find a prevalence of low retirement assets, a severe underestimation of survival probabilities to an extreme old age of 95, and a strong aversion toward life annuities. Pre-retirees appear to have reasonable expectations regarding income and assets in retirement, with the median retiree relying heavily on public pension sources. (Pre-)retirees are primarily concerned with liquidity needs, consumption smoothing, inflation, and longevity in retirement, and are least concerned with bequests. We elicited risk and time preferences, and found an average RRA parameter between 1.74 to 2.48 for pre-retirees and 2.48 to 3.74 for retirees, and a median subjective time discount factor of 0.997. A study of decision-making under risky scenarios reveals dramatic differences between the actual and implied choices under the expected utility maximization framework. Particularly, in the presence of inflation risk, agents lack the understanding of the long-term cumulative impact of inflation on the cost of living. In the presence of investment risk, the upside gain drives decision-making, and the presence of minimum income protection effectively provided by public pension income induces more risk-taking behaviour.

The last chapter concludes the thesis, and proposes general directions for future work in retirement planning research.

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To my parents I dedicate this thesis, and to Jussi.

Table of Contents

List of Figures

List of Tables

Introduction

The global trend of shifting from defined benefit (DB) to defined contribution (DC) workplace pension plans is putting growing pressure on individuals to take more ownership in retirement planning and financial decision-making. The essence of the DB is the life-long income guarantee, which requires limited financial planning decisions to be made, either in the accumulation or decumulation phase. The DC on the other hand, is significantly more complex. The lump sum payment at retirement burdens individuals with the task of income generation, in the presence of challenges stemming from an uncertain future lifetime, economic conditions, and evolving consumption needs (Bodie et al., 1988). The average retiree has limited competency to navigate these challenges, due to low financial literacy (Van Rooij et al., 2012; Lusardi and Mitchell, 2011; Lusardi and Mitchell, 2007a), lack of willpower (Thaler and Benartzi, 2004), or deteriorating cognitive abilities with older ages (Agarwal et al., 2009). The high stake of these challenges calls for a normative solution to be proposed – a solution that considers the intricacy of risks, preferences, and normative objective formulations. The objective of this thesis is to explore such a solution.

First, let us clarify what a normative solution entails. The term 'normative' suggests how one *should* or *ought to* behave, and gives rise to rationality and the *prescription* of actions. It is to be contrasted with 'positive' or 'descriptive' solutions, which attempt to *describe* or *explain* human behaviour, often associated with irrationalities, errors or imperfections (Bell et al., 1988). This is an important distinction, as there are substantial differences in the implications of the two lines of work.

Rational studies of life-cycle investing in the decumulation phase date back to the seminal work of Yaari (1965), where the optimal decisions are obtained using time-additive expected discounted utility maximization. The vast majority of the later work follows this framework, and branches out in two directions. The first, and possibly the most widely known line of work in this field, surrounds the theoretical explanations of *the annuity puzzle*. The annuity puzzle refers to the stark contrast between theoretical predictions in life-cycle theory, where full or high annuitization is shown to significantly improve welfare, and the empirical observation of extremely low demand for annuities around the globe. The most cited conclusion of Yaari (1965) is the normative optimality of full annuitization with no bequest motive (under certain assumptions), which is later extended by Davidoff et al. (2005) with less restrictive conditions. The puzzle has attracted significant attention, following the remark made by the Nobel laureate Franco Modigliani in Modigliani (1986) that,

> It is a well known fact that annuity contracts, other than in the form of group

> insurance through pension systems, are extremely rare. Why this should be so is a subject of considerable current interest. It is still ill-understood.

This phenomenon remains true to date (Bütler and Teppa, 2007; Mitchell et al., 1999) and significant effort has been made to shed light on this puzzle from a descriptive perspective through behavioural economic theory. For a detailed discussion, see Benartzi et al. (2011) and the references therein. Normative theoretical work focuses on the impact of preferences (Iskhakov et al., 2015; Kingston and Thorp, 2005), the objective function (Lockwood, 2012; Milevsky, 1998; Peijnenburg et al., 2016), liquidity concerns (Sinclair and Smetters, 2004; Brugiavini, 1993), the existence of involuntary annuitization through defined benefit pensions (Dushi and Webb, 2004) and social insurance income (Bütler et al., 2017), market imperfections or adverse selection (Brugiavini, 1993; Yagi and Nishigaki, 1993), or some combinations of these factors (Yogo, 2016). No compelling answers have been found.

The second line of work centers around normative retirement consumption and asset allocation, which often times involves annuitization decisions. This line of work does not aim to reconcile theoretical results with empirical observations (either from a rational or behavioural perspective), but rather focuses on the prescription of rational retirement planning decision-making. Such purpose can be either implicit or explicit, depending on the narrative of the work. Both Yaari (1965) and Davidoff et al. (2005) fall under this category. There is a proliferation of work in this area, following two broad frameworks of 1) utility maximization and 2) shortfall risk minimization. For a comprehensive overview of this research, see MacDonald et al. (2013), Horneff et al. (2008a) and the references therein. In more recent years, studies have been primarily conducted in the utility maximization framework, focusing on the investigation of optimal annuitization ratios, including ratios among varying types of annuities (e.g. real or nominally fixed, variable, group self-annuitization) (Boyle et al., 2015; Hanewald et al., 2013; Horneff et al., 2010; Maurer et al., 2013; Milevsky and Young, 2007), and the timing of annuitization (Horneff et al., 2008c; Horneff et al., 2008a; Milevsky and Young, 2007). Others investigate the impact of the preferences or objective formulation on the consumption or asset allocation (Boyle et al., 2015; Butt and Khemka, 2015; Blake et al., 2008). Some, motivated by the empirically observed aversion to annuities, investigate optimal phased withdrawal strategies (Emms, 2010; Horneff et al., 2008a; Bateman and Thorp, 2008; Emms and Haberman, 2008), including the impact of social insurance pension income (Wiafe et al., 2017; Hulley et al., 2013).

This thesis belongs to the second line of work, which is normative retirement planning decision-making. Our primary focus is not on the reconciliation of theoretical and empirical decision-making in retirement (though this is briefly touched on in the fourth chapter), but rather the exploration of the notion of 'optimality' in retirement planning, with the aim of making a practicable contribution to the improvement of retirement financial welfare. This involves the exploration of the inter-relationship among financial risk factors in the long term, normative preference models for intertemporal decision-making, and retirement objective formulations across varying demographics.

This thesis comprises three inter-related research directions: economic scenario generators (Chapter 1), recursive preferences in intertemporal portfolio selection (Chapter 2 and 3), and retirement consumption, risk perception, and objective formulation (Chapter 4). A brief description of the subsequent chapters will now follow.

Economic scenario generators (ESGs) are integrated stochastic models developed for risk aggregation and dependency modelling. They are used to simulate joint observations of financial risk factors and economic variables such as inflation, stock returns, and interest rates. Chapter 1 provides a review of Wilkie's ESG, which is the first such model presented to the actuarial profession. We restrict our analysis to economic series pertinent to retirement planning. The model is updated to 2014 using U.S. data, and used for the subsequent chapters for the projection of economic scenarios.

Chapter 2 provides an introductory discussion of recursive preferences, which are adopted in Chapter 3 as a normative preference model in solving problems in optimal retirement planning. A special case of the recursive specification – the Epstein-Zin utility function is discussed. To set the scene for Chapter 3, this chapter considers a simplified problem, in which analytical expressions of the optimal consumption profile can be derived. This allows one to isolate the impact of the preference, economic, and mortality assumptions, and to appropriately interpret the results presented in Chapter 3.

Chapter 3 investigates the normative validity of the expected discounted lifetime utility (EDLU) maximizing framework, by solving a discrete time optimal consumption-investment strategy of a DC retiree who wishes to benefit from stock investment while seeking longevity insurance and inflation protection. We use a combination of qualitative and quantitative criteria to evaluate the adequacy of the optimal consumption profile, paying particular attention to downside risk at extreme old ages.

Chapter 4 conducts a large-scale survey study on Canadian (pre-)retirees on retirement consumption, risk perception and objective formulation. This work is motivated by an assessment of the normative (and in part, descriptive) validity of traditional applications of mathematical optimization to retirement planning problems. In particular, the study explores three key areas. The first is the difference between expectations and experience among Canadian retirees, as it relates to longevity (life expectancy and survival rates), consumption (annuity prices and expected savings required to maintain a given level of income), and risk and variability of income (expectations of future income requirements and their variability). The second area relates to the current level of wealth in retirement, or savings pre-retirement (in particular in contrast to expectations). The third area addresses preferences and objectives, in particular in assessing whether the revealed preferences are well-represented by commonly employed utility functions and objectives.

Chapter 5 concludes the thesis, and proposes general directions for future work in retirement planning research.

Chapter 1

Wilkie's Economic Scenario Generator

1.1 Introduction

Economic scenario generators (ESGs) are integrated stochastic models developed for simulations of long-run joint distributions of financial risk factors and economic variables. They are powerful tools for risk aggregation and dependency modelling. ESGs have been gaining popularity in recent years as a pricing and risk management tool and have often lent themselves to applications such as asset liability management and the computation of economic capital requirements.

There are many proprietary ESGs being used for actuarial risk management in all areas of practice, but for the independent researcher the field is quite limited. It may be difficult to access the proprietary models, and even with access, if the details of the model are not fully disclosed (or disclosable) for commercial reasons, it may not be possible to interpret model output satisfactorily. For example, if the approach to modelling dependencies is not disclosed then the results of an investigation where tail dependency is critical may be in doubt.

For independent research then, it is highly desirable to utilize a model that is *open access* – that is, the model specification is fully disclosed without restrictions. Even if a proprietary model outperforms an open access model in certain metrics, the open access model may be preferred, as the researcher can assess and adjust for model limitations. The Wilkie model is an open access ESG that was introduced in Wilkie (1986). It was the first comprehensive ESG to be formally presented to the actuarial profession. The model distinguishes itself from other ESGs in that it has been subjected to a high level of public scrutiny, to a degree that no proprietary model has experienced. However, it is thirty years since the introduction of the Wilkie model, and if it is to continue to be of use to researchers, particularly in a US context, it is appropriate to backtest its performance, and explore the potential for future use, particularly to independent researchers, but possibly also to practitioners to benchmark their own models.

The groundwork of the model was laid back in 1980 in a report by the Maturity Guarantees Working Party of the Institute and Faculty of Actuaries (Benjamin et al., 1980). The full model, in its 1986 form, was fitted to UK data and contains retail price index, stock yield, stock dividends, and long-term interest rates ('consols' yield). The model was later updated and extended in Wilkie (1995) with the inclusion of five additional variables: wages index, short-term interest rates, property rentals and yields, index-linked stock yields and currency exchanges rates, and was also calibrated to other countries, including Australia, Canada, Switzerland and the US. Since 2011, a series of updates was introduced: Wilkie et al. (2011), updated parts of the UK model to 2009, discussed the model performance since 1994, and studied the stability of the parameters; Wilkie and Şahin (2016) discussed the experience of the model since 2009, selecting initial conditions, and using "select periods" for shorter-term forecasts; Wilkie and Şahin (2017a), Wilkie and Şahin (2017b) and Wilkie and Şahin (2017c) discussed stochastic interpolation by Brownian and Ornstein-Uhlenbeck bridges; Wilkie and Şahin (2017d) extended the model to include stock earnings.

Numerous reviews and extensions of the Wilkie model have been published, with most focused on Wilkie (1986) and Wilkie (1995). Those relevant to our study are referenced in sections 1.2, 1.3 and 1.4. Following Wilkie, several researchers have developed ESGs for countries other than the UK, including, among many others, Carter (1991) for Australia, Thomson (1996) for South Africa, and Huang et al. (2014) for China. These studies focus on the development of ESGs for these economies using the 'Wilkie' framework, rather than testing Wilkie's ESG for country-specific applications. The contribution of this chapter is to present the model in a US context, with US data, and to analyze the performance of the model historically, and the relevance of the model going forwards, for North American applications. Our main contribution is the backtesting of the model for the past 30 years, and the testing of stationarity assumptions and parameter stability in the US context. We further compare the output of Wilkie's ESG with a generic ESG. The generic ESG is constructed to include features that are, to our best knowledge, commonly found in both public access and proprietary ESGs.

Our main findings indicate that there exist challenges in modelling long-term economic series due to the presence of multiple structural shifts in the historical time series. Consequently, certain assumptions of stationarity are violated; and parameters are sensitive to the calibration period. These challenges can be mitigated by performing a change point analysis; and selecting an appropriate sample period for the parameter estimations. It is critical that these tasks be facilitated by a qualitative understanding of the true data generating process (e.g. historical economic events and economic theories). A backtest based on 30-year out-of-sample data indicated that over that period the model had tended to overestimate inflation (due to a structural shift in an implicit inflation targeting decision by the Federal Reserve in the 1980s), underestimate total return on stocks (due to the unprecedented dot-com bubble in the 1990s), and performed relatively well for long-term interest rates. The comparison to a generic ESG reveals that in the long run, Wilkie's ESG generates a wider range of scenarios for inflation and long-term interest, but a narrower range for stock returns. Wilkie's ESG can be under-representative of the risk in long-term stock investment, particularly in the tails.

This chapter is organized as follows. Section 1.2 describes the model and reports the fitted parameters. Section 1.3 examines model robustness. Section 1.4 compares the performance of Wilkie's ESG with a generic ESG.

1.2 Model Description

The Wilkie model is a $\mathbb{P}$ (real-world) measure model with an annual frequency. It adopts a *cascade* structure, with inflation being the driver of other economic variables. It can be expressed diagrammatically as shown in Figure 1.1. Note that total stock returns are modelled by combining dividend yield and index, hence the dotted lines. We use subscripts to distinguish the series: q for inflation; y for dividend yield; d for dividend index; and c for long-term bond yield.

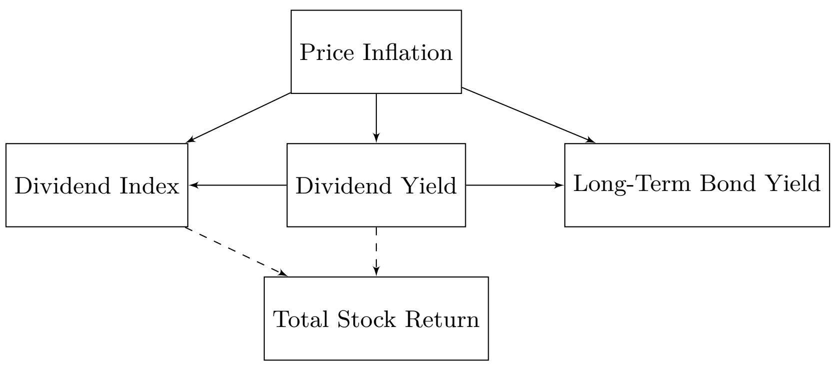


Figure 1.1 Wilkie's ESG – Model Structure

1.2.1 Inflation

Inflation is measured by the continuously compounded rate of change of the Consumer Price Index. The deviation from the long-run mean is modelled by a stationary[1] AR(1) process:

$$\begin{aligned}
&\delta_q(t) = \log \frac{Q(t)}{Q(t-1)}, \\
&\delta_q(t) - \mu_q = a_q \cdot \{\delta_q(t-1) - \mu_q\} + z_q(t), \\
&|a_q| < 1, \quad z_q(t) \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_q^2),
\end{aligned}$$

where $\delta_q(t)$ is the force of inflation in the year $[t-1, t)$, $Q(t)$ is the Consumer Price Index at time t, μ_q is the long-run, unconditional mean, a_q is the parameter governing the strength of autoregression to the long-run average, and σ_q is the standard deviation of the innovation term.

[1]Stationarity refers to 'weak' or 'covariance' stationarity.

The June series of non-seasonally adjusted US City Average All Items Consumer Price Index (CPI) for All Urban Consumers (CPURNSA) from January 1926 to December 2014 (monthly), obtained from Bloomberg, is used. We have 89 observations of CPI, and 88 inflation observations due to log differencing. The fitted parameters are reported in Table 1.1.

	Wilkie (1995) US	Estimated Parameter (SE)			
	1926–1989	1926–1989		1926–2014	
μ_q	0.03	0.0333	(0.0118)	0.0307	(0.0084)
a_q	0.65	0.5843	(0.1009)	0.5731	(0.0862)
σ_q	0.035	0.0389	(0.0035)	0.0337	(0.0026)

Table 1.1 Parameter estimates of the US inflation model: 1926-2014. Standard errors are in parentheses.

Extensions to the UK inflation model include: Clarkson (1991), which investigated the impact of three non-linear components that allow inflation scenarios to exhibit 'quiescent' and 'active' phases; Wilkie (1995), which investigated an autoregressive conditionally heteroscedastic (ARCH) extension to allow for more fat-tailed residuals, and noted several other possible distributions for modelling residuals, including t and gamma distributions; and Chan and Wang (1998), which proposed a revised model that incorporates outliers in inflation. Wilkie (1995) also proposed a vector autoregressive (VAR) approach that jointly models price and wage inflation.

1.2.2 Dividend Yield

The log-transformed dividend yield, taking away its long-run mean, is described as a transfer function model with input inflation and an AR(1) noise.

$$\log y(t) - \log \mu_y = w_y \cdot \delta_q(t) + yn(t),$$
$$yn(t) = a_y \cdot yn(t-1) + z_y(t),$$
$$|a_y| < 1,\ z_y(t) \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_y^2),$$

where $y(t)$ is the dividend yield in the year $[t-1, t)$, μ_y is the long-run mean of the dividend yield, a_y is the autoregressive factor of the noise process, and σ_q is the standard deviation of the innovation term of the noise process.

The June series of price index and total return index of the S&P500 from January 1926 to December 2014 (monthly) are used. Data up to December 2009 are obtained from Morningstar, Inc., with later series obtained from Bloomberg. We have 88 observations for fitting the dividend yield model. The fitted parameters are reported in Table 1.2.

The reduced model in Table 1.2 refers to the model with insignificant parameters (at 5% level) omitted.

	Wilkie (1995)	Estimated Parameter (SE)							
	US	Full				Reduced			
	1926–1989	1926–1989		1926–2014		1926–1989		1926–2014	
w_y	0.5	−0.2683	(0.5219)	−0.4405	(0.4686)	–		–	
μ_y	0.043	0.0449	(0.0048)	0.0313	(0.0101)	0.0444	(0.0047)	0.0309	(0.0097)
a_y	0.7	0.7985	(0.0763)	0.9382	(0.0384)	0.7985	(0.0762)	0.9368	(0.0387)
σ_y	0.21	0.1669	(0.0150)	0.1624	(0.0123)	0.1673	(0.0150)	0.1632	(0.0124)

Table 1.2 Parameter estimates of the US dividend yield model: 1926-2014. Standard errors are in parentheses.

The inclusion of inflation as an input reflects the cascade structure of the model. (Wilkie, 1986) argued that '...dividends, which are measured in money terms, ought, other things being equal, to be related to the general level of money prices elsewhere in the economy'. Hence, a high rate of inflation should theoretically result in a high dividend yield (i.e. $w_y > 0$). However, the estimated w_y for 1926–1989 and 1926–2014 are below zero and far from Wilkie's estimate of 0.5. This is inconsistent with the original model assumption. The high standard error associated with the estimate further suggests that inflation does not drive dividend yields in the way the model specifies. Wilkie (1995) acknowledges that it is not essential to include w_y. It appears that real dividends do not necessarily perform well in times of unstable prices (i.e. high and uncertain inflation). Huber (1997) showed that w_y only captures an extreme relationship between the two and is sensitive to outliers. Hence a reduced model is estimated in Table 1.2. Dropping w_y, the model reduces to an AR(1) for $\log y(t)$.

1.2.3 Dividend Index

The dividend index process is modelled through the exponential rate of dividend growth. It is described as a transfer function model with multiple inputs from inflation, from the random shocks to previous dividend yields, and an independent invertible MA(1) process.

$$\begin{aligned}
&\delta_d(t) = \log \frac{D(t)}{D(t-1)}, \\
&\delta_d(t) - \mu_d = w_d \cdot dm(t) + (1 - w_d) \cdot \delta_q(t) + y_d \cdot z_y(t-1) + b_d \cdot z_d(t-1) + z_d(t), \\
&dm(t) = (1 - d_d) \cdot dm(t-1) + d_d \cdot \delta_q(t), \\
&0 \le d_d \le 1, \quad |b_d| < 1, \quad z_d(t) \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_d^2),
\end{aligned}$$

where $\delta_d(t)$ is the dividend growth in the period $[t-1, t)$, $D(t)$ is the dividend index at time t, μ_d is the long-run mean of dividend growth in excess of inflation, $dm(t)$ is the exponentially weighted average of inflation up to time t, d_d governs the smoothness of

$dm(t)$ (the smaller the value, the more responsive it is to current inflation), y_d measures the effect of the previous shocks to the dividend yield, and σ_d is the standard deviation of the innovation. Consistent with Wilkie (1995), w_d is set to 1 and d_d to 0.38. See Section 1.2.5 for a note on pre-selecting these weights.

The total return index assumes reinvestment of gross dividends. The Dividend Index, $D(t)$, is constructed from the Price Index, $P(t)$, and the Total Return Index $TR(t)$, as $D(t) = \frac{TR(t)}{TR(t-1)} P(t-1) - P(t)$. The implied annual dividend yield is calculated by $y(t) = \frac{D(t)}{P(t)}$. We have 88 observations for dividend index, and 87 dividend growth observations due to log differencing. The fitted parameters are reported in Table 1.3.

	Wilkie (1995)	Estimated Parameter (SE)							
	US	Full				Reduced			
	1926–1989	1926–1989		1926–2014		1926–1989		1926–2014	
d_d	0.38	0.38	–	0.38	–	0.38	–	0.38	–
μ_d	0.0155	0.0095	(0.0080)	0.0128	(0.0073)	0.0096	(0.0078)	0.0129	(0.0069)
y_d	−0.35	0.0539	(0.1008)	0.0880	(0.0901)	–	–	–	–
b_d	0.5	−0.6370	(0.0995)	−0.5753	(0.1083)	−0.6474	(0.0967)	−0.6004	(0.1062)
σ_d	0.09	0.1671	(0.0151)	0.1572	(0.0120)	0.1675	(0.0152)	0.1581	(0.0121)

Table 1.3 Parameter estimates of the US dividend growth model: 1926–2014. Standard errors are in parentheses.

The inflation effect on the dividend growth rate is captured in the first two terms, with the total weight assigned to current inflation being $1 - w_d + w_d \cdot d_d$, and the weight assigned to inflation going back τ years being $w_d \cdot (1 - d_d) \cdot d_d^{\tau}$. It is immediate that past inflation has a diminishing effect on the current rate of dividend growth.

The dividend yield effect is modelled through the inclusion of $y_d \cdot z_y(t-1)$. Note that $z_y(t-1)$ is lagged by 1. This reflects the assumption that 'investors can take into account the unexpected change in dividend yield in the previous period, in order to forecast changes in dividends in the coming year, as dividends are declared early'. This is consistent with the Efficient Market Hypothesis (Huber, 1997). Wilkie (1986) states that 'an unexpected fall in yields results in an upwards change in the dividend index in the following period', which implies $y_d < 0$. Note that the reported parameters are not estimated under such a constraint. If so, the resulting estimate is very close to zero.

The inclusion of $b_d \cdot z_d(t-1)$ assumes that 'companies pay out only part of any additional earnings in dividends in one year, with a further part in the following year' (Wilkie, 1986). This implicitly assumes that the sign of b_d is positive. Mathematically, this means that a shock in dividend growth in the previous year will carry forward the same effect to the current period. However, our estimated b_d is uniformly below zero. This indicates a dividend smoothing effect – previous shocks are carried forward as having opposing effects in the next period.

The dividend yield and index are combined to give the total return on stocks:

$$py(t) = \log\left\{\frac{D(t)/y(t) + D(t)}{D(t-1)/y(t-1)}\right\}$$

where $py(t)$ is the annual total log return on stocks for the period $[t-1, t)$.

1.2.4 Long-Term Interest Rate

The long-term interest rate is modelled as a combination of inflationary and real components through a transfer function model with a single input.

$$\begin{aligned}
&c(t) = w_c \cdot cm(t) + cr(t),\\
&cm(t) = (1-d_c)\cdot cm(t-1) + d_c \cdot \delta_q(t),\\
&\log cr(t) - \log \mu_c = a_c \cdot \{\log cr(t-1) - \log \mu_c\} + y_c \cdot z_y(t) + z_c(t),\\
&0 \le d_c \le 1, \ \ |a_c| < 1, \ \ z_c(t) \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_c^2),
\end{aligned}$$

where $c(t)$ is the long-term interest rate at time t, w_c is a factor moderating the impact of current and past inflation, $cm(t)$ is the exponential weighted moving average of inflation up to time t, $cr(t)$ is the real interest rate component at time t, μ_c is the long-run average of the real interest rate, a_c is the autoregressive parameter, y_c measures the sensitivity of the real interest rate to the current shocks to the dividend yield. Evidently the model is nonlinear.

Huber (1997) pointed out that the parameters w_c and d_c cannot be determined using the method of maximum likelihood or least squares, thus they are set to plausible values and are set such that the inferred real interest rate is above zero. An alternative specification of $cm(t)$ is proposed by Wilkie et al. (2011):

$$cm(t) = \min\left(d_c \cdot \delta_q(t) + (1-d_c)\cdot cm(t-1), c(t) - c_{\min}\right),$$

where $c_{\min} = 0.5\%$. Note that with the changed form, $cr(t)$ is at least $c_{\min}$ so that we can always take the logarithm. To be consistent with Wilkie (1995), we set d_c to be 0.058 and w_c to be 1.0. See Section 1.2.5 for a note on pre-selecting these weights.

The June series of Moody's Seasoned AAA Corporate Bond Yield from January 1927 to December 2014 (monthly), obtained from federalreserve.gov, is used. We have 88 observations. The fitted parameters are reported in Table 1.4.

The Fisher Relation (Fisher, 1930) is explicitly assumed here through $w_c \cdot cm(t)$. Wilkie in his 1986 paper remarked that 'as inflationary expectations have risen, so have interest rates...[we] assume that the market's expectations are influenced by the past history of inflation'.

The dividend yield contribution is captured in the term $y_c \cdot z_y(t)$. It suggests that an unexpected change in dividend yield has an effect on the real long-term interest rate

	Wilkie (1995)	Estimated Parameter (SE)							
	US	Full				Reduced			
	1926–1989	1926–1989		1926–2014		1926–1989		1926–2014	
d_c	0.058	0.058	–	0.058	–	0.058	–	0.058	–
a_c	0.96	0.9598	(0.0372)	0.9176	(0.0439)	0.9953	(0.0077)	0.9175	(0.0439)
μ_c	0.0265	0.0339	(0.0396)	0.0237	(0.0097)	0.0339	(0.0396)	0.0238	(0.0097)
y_c	0.07	0.1060	(0.1265)	0.0237	(0.1398)	–	–	–	–
σ_c	0.21	0.2247	(0.0200)	0.2831	(0.0213)	0.2277	(0.0203)	0.2832	(0.0213)
c_{min}	–	0.005	–	0.005	–	0.005	–	0.005	–

Table 1.4 Parameter estimates of the US long-term interest rate model: 1926-2014. Standard errors are in parentheses.

(although the sign of y_c is not stated). The log-transformed real component can be alternatively seen as an AR(1) process. This implicitly gives rise to the assumption that $cr(t)$ is strictly positive.

Recall that y_c captures the dividend yield effect on the long-term bond yield. We see that the inclusion of y_c has little effect on the other parameters, and the estimate is insignificant at the 5% level. This indicates that in the sample periods studied, we do not tend to observe any significant impact on long-term bond yields from unexpected changes in the dividend yield. Dropping the term, the real component of the long bond yield reduces to an AR(1).

1.2.5 Fitting

Parameters reported are maximum likelihood estimates fitted using R. One can use a least squares approach, which gives a close approximation to the maximum likelihood estimates under normality assumptions on the noise processes. Details on estimation methods are well described in Shumway and Stoffer (2010) and Brockwell and Davis (2013), among others. We note that inconsistent data sources may contribute to the discrepancies between our estimates and Wilkie's. Starting values prior to the commencement of the series are set to the unconditional means.

Certain parameter values were pre-selected (i.e. w_d and d_d in the dividend index model, and w_c and d_c in the long-term interest rate model). This is done to 1) impose certain dependence assumptions, and 2) simplify the estimation process. The values for w_d and w_c are set to 1 to reflect the assumption that expected future inflation (expressed as an exponentially weighted average of past inflation) has a unit impact on the dividend growth and long-term interest rates processes. Pre-selecting d_d and d_c allows the model to be conveniently estimated using pre-existing statistical software packages. In practice, d_d can be empirically determined by maximizing the likelihood function (numerically) within a feasible domain of values (e.g. $[0, 1]$). The parameter d_c on the other hand, cannot be estimated using MLE (Huber, 1997).

	1926–2014		1951–2014	
	PSR Statistic	p-value	PSR Statistic	p-value
Inflation	54.88	0.0000	17.50	0.0015
Log Dividend Yield	6.43	0.2668	6.20	0.1849
Dividend Growth	12.77	0.0257	6.89	0.1419
Log Real Long Bond Yield	5.04	0.4113	5.26	0.2619

Table 1.5 PSR Test for Stationarity

1.2.6 Selected Extensions to Wilkie's ESG

Much work has been conducted extending Wilkie's UK model as a whole. Notably, Chan (1998) presented a continuous-time model constructed using stochastic differential equations driven by Lévy processes; Whitten and Thomas (1999) and Chan et al. (2004) described threshold autoregressive models; and Chan (2002) developed a general vector autoregressive moving average (VARMA) model.

1.3 Model Robustness

This section assesses the robustness of Wilkie's ESG, with particular attention paid to the assumption of stationarity, structural breaks in the empirical processes, and parameter stability. In general, model robustness refers to the insensitivity of the model specifications to slight changes in the input data. Figures 1.2 and 1.3 plot the data.

1.3.1 Stationarity and Structural Breaks

Stationarity is a key assumption in real world economic scenario generators. It implies that the economic series has a constant mean and an auto-covariance function independent of time. This is important when projecting in the long term, since time trends in both the mean and variance may result in explosions, yielding a significant number of implausible scenarios. Recall that in Section 1.2, Wilkie's ESG assumes stationarity for the processes of inflation $\delta_q(t)$, log dividend yield $\log y(t)$, dividend growth $\delta_d(t)$, and log real long bond yield $cr(t)$. We test these assumptions using the Priestley-Subba Rao (PSR) method, a nonparametric test of the overall stationarity of the complete second-order properties of a time series. For details, see Priestley and Rao (1969)[2]. Results are summarized in Table 1.5.

Inflation and Dividend Growth fail the stationarity test at the 5% significance level. The $p-$value for inflation is extremely small – time-varying volatility is suspected to be the primary cause. Inflation appears to fluctuate around a constant mean, but prior to the 1950s – during the world wars and postwar recessions – it is highly volatile. We see

[2]We used the R package `fractal` with default inputs for this analysis.

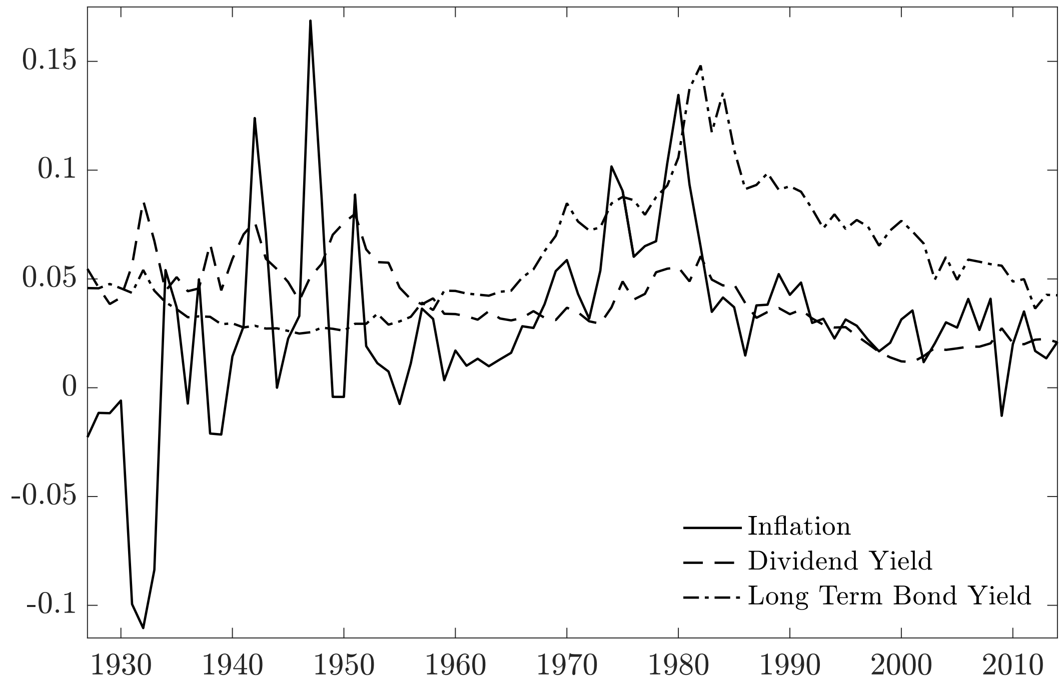


Figure 1.2 Inflation, dividend yield, long-term bond yield (1927-2014)

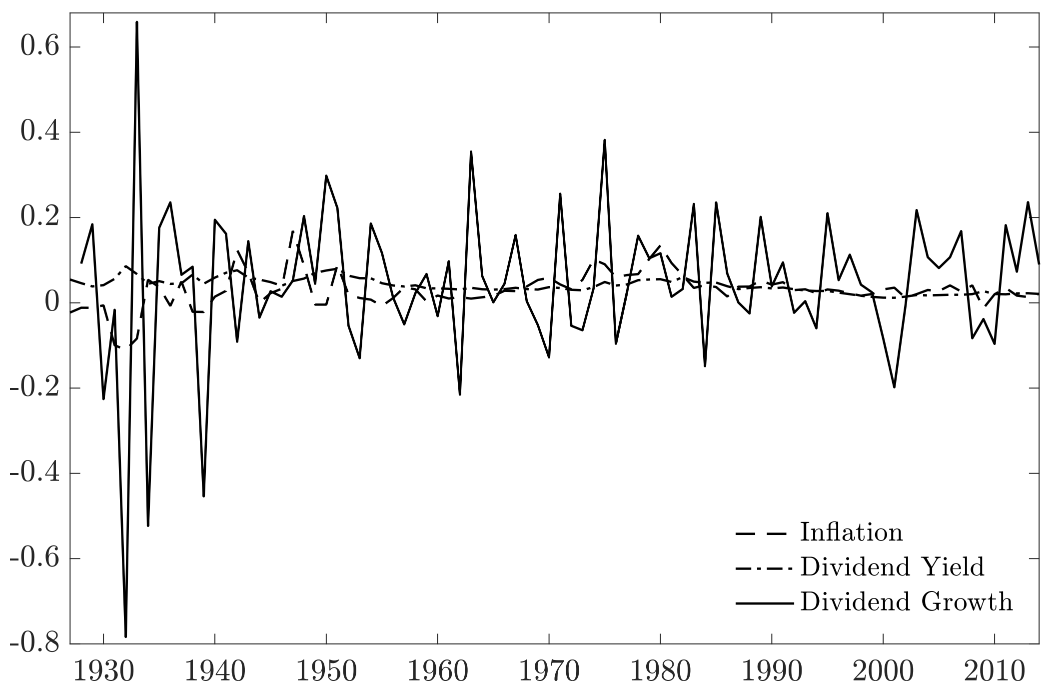


Figure 1.3 Inflation, dividend yield, dividend growth (1927-2014)

similar results for the dividend yield and growth rates. At this stage, one may question the relevance of data prior to the late 1950s, as there appears to be a shift in the dynamics among series around this time. There is little evidence that dividend yields or interest rates are positively driven by inflation prior to the late 1950s. However, shortly after, dividend yields and long-term interest rates begin to follow inflation more closely, in a mildly lagged manner. This raises the question of a structural shift in the underlying data generating process. To assess whether structural shifts were present, we examined the data for statistically significant change points.

Our analysis on structural shifts is relevant in assessing model robustness, since the construction of the model does not assume the presence of change points. In essence, we know that the ESG is an approximation to the true data generating process, but the true process may not be stationary as assumed by Wilkie's ESG, evidently shown in Figures 1.2 and 1.3. The literature on change point analysis is rich, using both parametric and non-parametric methods. For a comprehensive introduction to change point analysis, see for example Basseville and Nikiforov (1993) and Brodsky and Darkhovsky (2013). We have used a univariate multiple change point detection for the individual series, in the absence of an appropriate multivariate method. This will indicate changes in the structure of the individual series, but will not identify changes in the relationships between series.

We use Auto-PARM[3] to detect structural breaks in the inflation, dividend yield, and long-term bond yield models. This method is selected due to its simple execution, and because its parametric assumptions are largely consistent with the Wilkie model. We note that in this study, the procedure can only be applied to series with AR(1) assumptions. Structural breaks in the dividend index model remain to be investigated.

Results are reported in Table 1.6. Change points are estimated to exist for inflation at years 1952 and 1992, and for the log real interest rates at year 2003. This is illustrated in Figure 1.4.

	Segmentation	AR order
Inflation	1926-1951	1
	1952-1991	1
	1992-2014	0
Log Dividend Yield	1926-2014	4
Log Real Interest Rate	1926-2002	1
	2003-2014	0

Table 1.6 Auto-PARM output: 1926-2014

The 1952 inflation breakpoint marks the end of large-scale, post-world-war recessions. The 1992 break in inflation signifies the beginning of the 'modern experience of US inflation'

[3]See Davis et al. (2006) for a detailed description of the method. We would like to thank Professor Davis for providing us with the executable file.

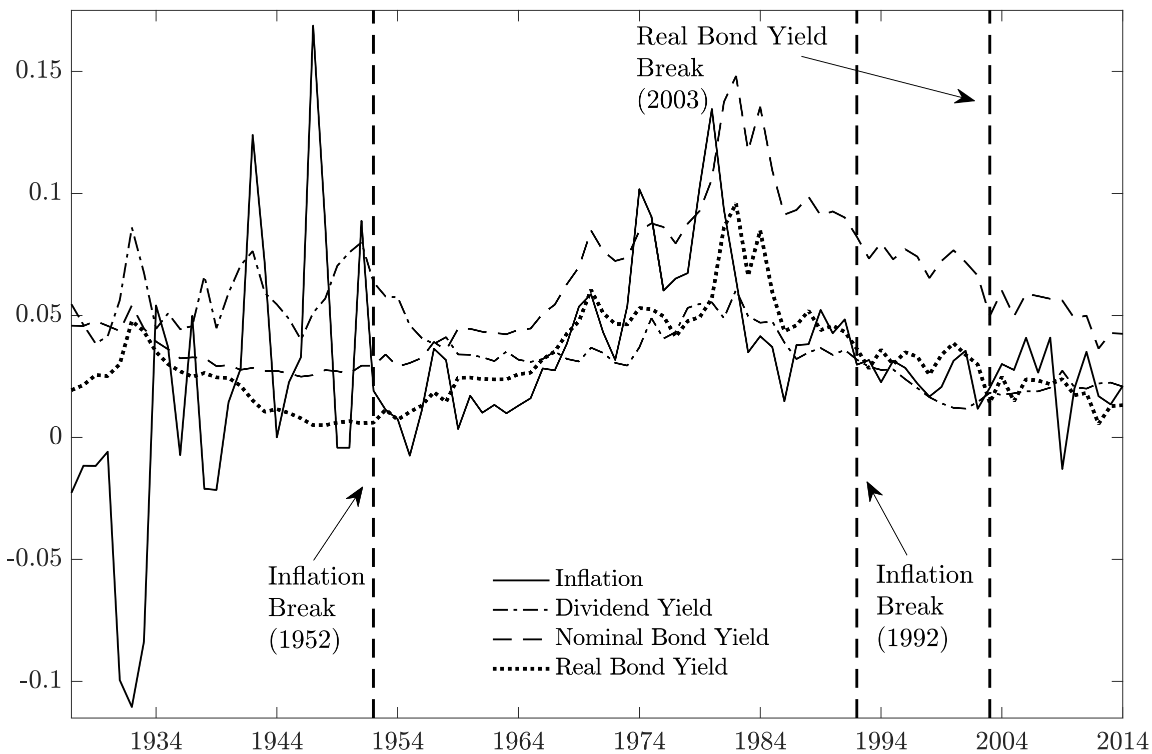


Figure 1.4 Inflation, dividend yield, and long-term bond yield, with estimated structural breaks.

(Reed, 2014). Since 1992, we see very modest volatility in inflation. The deflation in 2009 is caused by the global financial crisis. We also see lesser momentum (that is, no significant serial correlations), which suggests an AR model may have become inappropriate. It is argued that, after the inflation experience in the 1970s and early 1980s, the Federal Reserve system has focused more strongly on maintaining price stability through implicit and explicit (post 2012) inflation-targeting procedures. Price inflation since then has been dramatically more tamed than any other time in the past 90 years (see, e.g., Goodfriend, 2004).

We could reduce the influence of data before the change points by restricting the period of data used to calibrate the model. However, if we are to use the model for long-term projections of economic scenarios, which in actuarial terms means perhaps 10-30 years, we prefer to retain as much data as possible for model calibration. In Table 1.5 a small improvement in the $p-$value for inflation is observed for the data set 1951–2014, though there is still strong evidence to reject the null hypothesis of stationarity. It is suspected that the remaining non-stationarity comes from the high inflation period during the 1970s, largely attributed to the oil crises. The remaining series pass the PSR test.

1.3.2 Parameter Stability

In practice, it would be undesirable to have a model with parameters that are highly sensitive to changes in the input data. This implies high model uncertainty, which leads to unreliable interpretation of the model's output. Wilkie et al. (2011) studied parameter stability for UK data using an iterative approach, considering successive starting dates from 2004 back to 1926, with each data set ending in 2014 (Huber (1997) used a similar approach). We repeated the calculations for US data, and found that results were very similar to the UK experience.

Using an increasing sample window means that the fitted parameters have a tendency to stabilize. We also considered stability using a moving window approach, with fixed 30-year data periods. Note that 30 years of raw data generates 29 observations of inflation, dividend yield, long-term interest rates, and 28 observations of the dividend growth rate. The estimates along with 95% CIs are plotted in Figures 1.5, 1.6, 1.7, and 1.8. The line labeled estimates represents parameters fitted to the intervals $[u, u + 29]$, where $1926 \leq u \leq 1985$.

Inflation

In Figure 1.5, it is observed that the estimates are not smooth over time, due to small numbers of observations. Note that the very high inflation experience during the 1970s causes a considerable rise in a_q, which lasts 30 consecutive periods due to the length of the window. This gives rise to the question of whether these data should be included. A study with observations 1974-1981 removed was conducted with no significant improvement observed.

Dividend Yield

The $[1973, 2002]$ and $[1976, 2005]$ parameters are estimated using conditional least squares, due to the non-convergence of the MLE estimates in the feasible domain of parameters (i.e. $|a_y| < 1$). The $[1974, 2003]$ to $[1976, 2005]$ models have μ_y highly inflated, due to a_y being near its boundary. The parameter w_y rises as late 1970s observations enter the window, indicating a strengthened inflationary effect. This mirrors the conclusion of Huber (1997) which argues that w_y is sensitive to outliers and tends to capture co-movements between inflation and the dividend yield in extreme conditions.

Dividend Growth

The estimates for the dividend growth model change quite sharply compared to the other models. This is due to the MA(1) coefficient, which is more difficult to estimate than the AR(1) coefficient. Overall, we observe considerable uncertainty surrounding the dividend growth model, especially b_d and y_d.

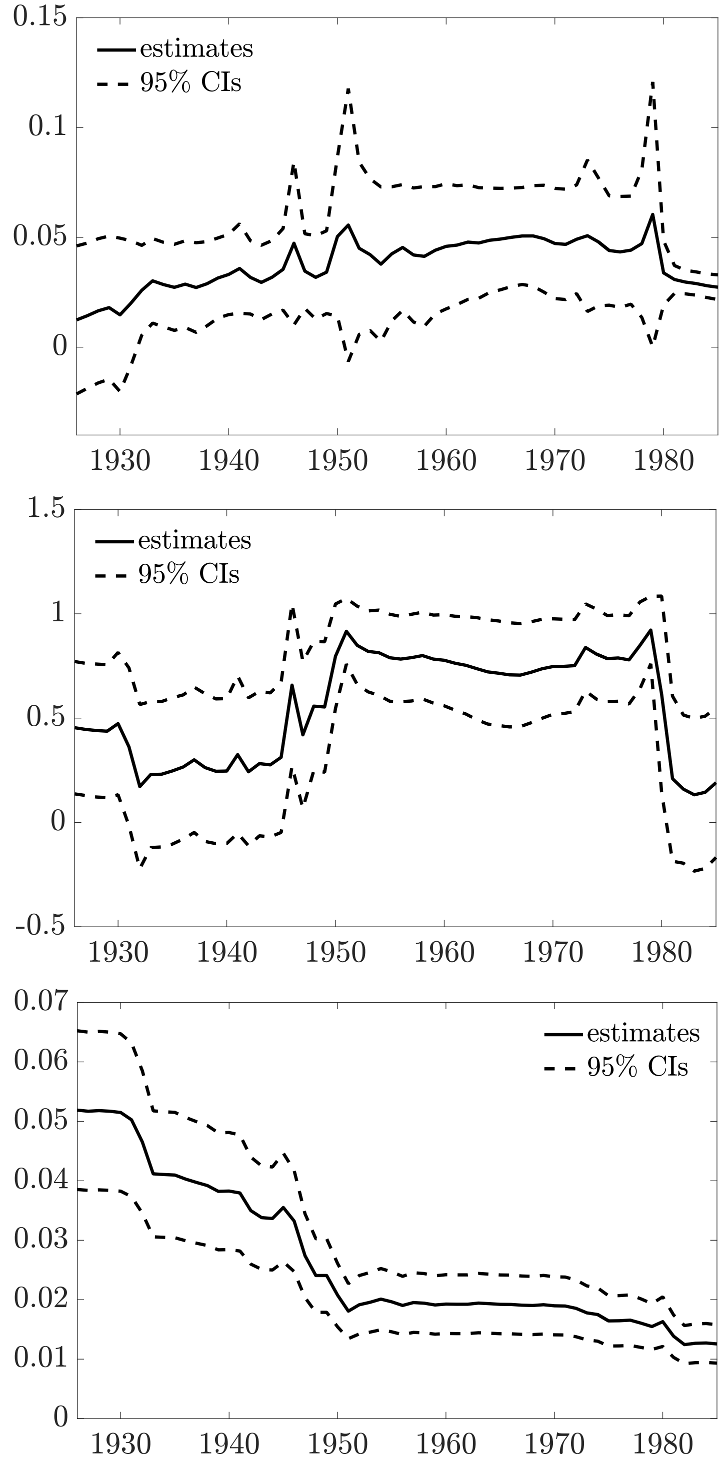


Figure 1.5 Inflation: moving window (30 years) of estimates and 95% CIs. Top to bottom: μ_q, a_q, and σ_q.

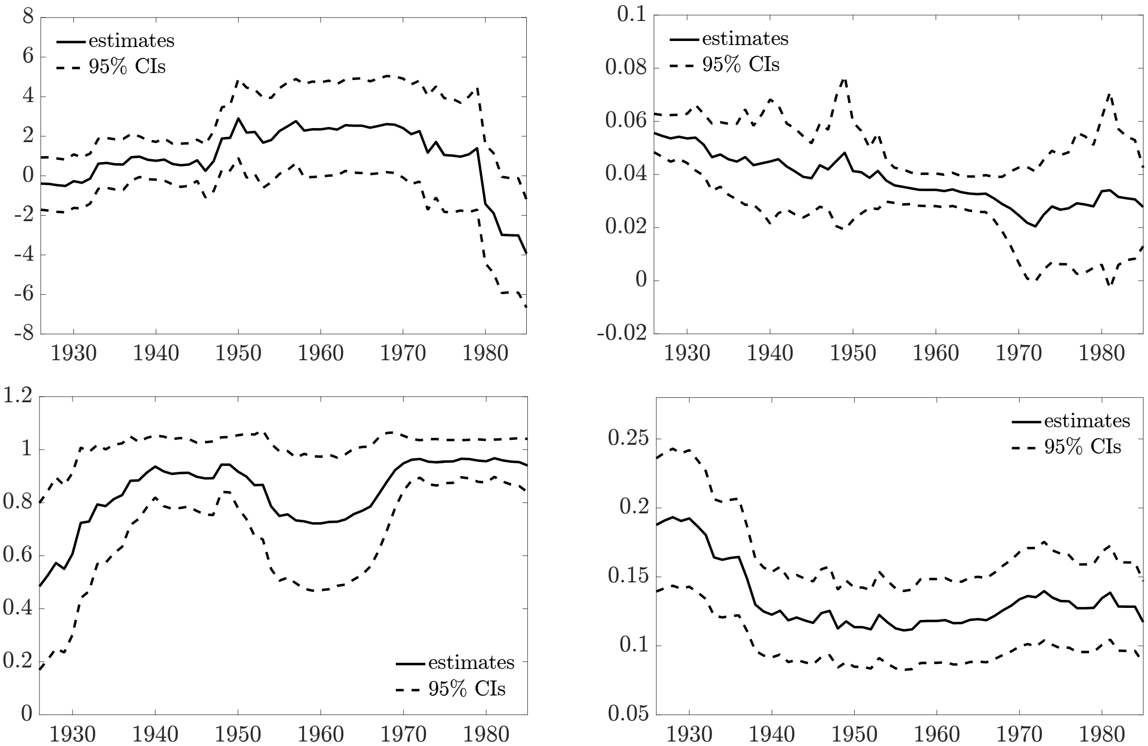


Figure 1.6 Dividend yield: moving window (30 years) of estimates and 95% CIs. Left to right: w_y and μ_y (top), a_y and σ_y (bottom).

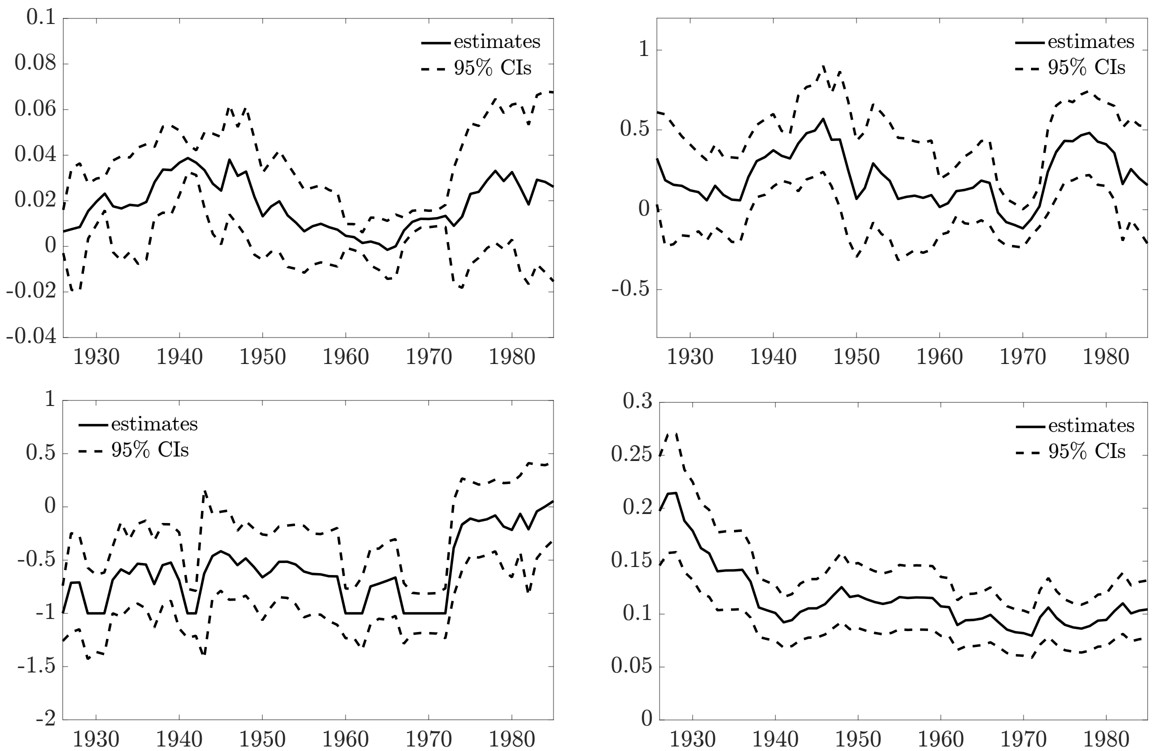


Figure 1.7 Dividend index: moving window (30 years) estimates and 95% CIs. Left to right: μ_d and y_d (top), b_d and σ_d (bottom).

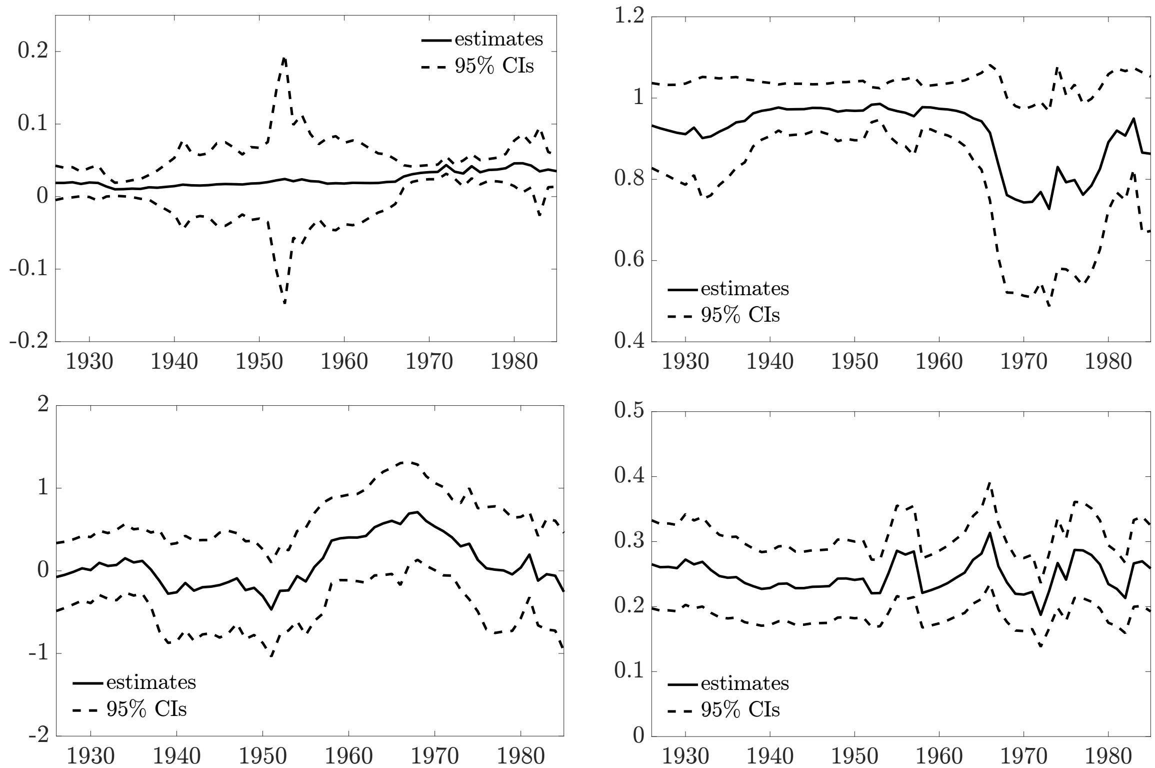


Figure 1.8 Long-term interest rate: moving window (30 years) of estimates and 95% CIs. Left to right: μ_c and a_c (top), y_c and σ_c (bottom).

Long-Term Bond Yield

The moving estimates for long-term bond yields are shown in Figure 1.8. Large uncertainty is seen for the high inflation period, in particular for a_c and σ_c. Parameters are relatively stable for the remaining sample periods.

1.3.3 Backtesting

The performance of the Wilkie model is assessed in this section. To backtest the model for 30 years, we first fit to data from 1951–1984; we use these parameters to project 10,000 scenarios of thirty years, and compare the path of the historical observations since 1985 with the distribution of paths generated by the model. Parameters can be found in Table A.2.

The simulation results are illustrated in Figures 1.9 and 1.10. Figure 1.9 shows the 5%, 25%, 50%, 75% and 95% quantiles of inflation, total log return on stocks, and long-term bond yield, along with three selected (joint) random paths. Scenarios A and C start with similar experiences in inflation, and then differ after around three years. We observe that higher inflation tends to translate to higher volatility in total stock return, with more emphasis on the upside. This is attributed to the positive inflation effect on dividend growth. Dividend yield, which is also positively affected by inflation, is less able to counteract the positive inflation effect on dividend growth. Further note that in the longer term, higher inflation translates to higher long-term interest, whereas shorter term projections (i.e. less than 10 years) appear more random.

Figure 1.10 compares the accumulated simulated quantiles with the out-of-sample data (1985–2014). Let $A(0) = 1$, and let the subscript py denote total return on stocks. The accumulation factors $A(t)$, $t > 0$ are defined as follows:

$$A_q(t) = \exp\left(\sum_{s=1}^{t} \delta_q(s)\right), \quad A_{py}(t) = \exp\left(\sum_{s=1}^{t} py(s)\right), \quad A_c(t) = \prod_{s=1}^{t} (1 + R_c(s))$$

where $A_q(t)$ is the Consumer Price Index (CPI), $A_{py}(t)$ is the total stock returns index, and $A_c(t)$ is the value of a \$1 10-year bond portfolio assuming annual repurchasing[4], respectively at time t.

We observe that the model has tended to overestimate inflation, underestimate total return on stocks, and performed relatively well for long-term interest. It is worth pointing out that the 1990s bull market was historically unprecedented, with persistent returns on stocks between 15% and 20% each year. Inflation also experienced a structural shift, which impacts predictive accuracy. These findings are also apparent in Figure 1.10 (plotted in log scale). Inflation is projected to be much more volatile than observed, and simulated long-term bond yields exhibit behaviour in the right tail that is not reflected in the out-of-sample data. This, together with the study of parameter stability, highlights that regime changes have a discernible impact on the performance of the model and hence the validity and power of its projections. This is a common problem when one uses fixed model structures and parameters for long-term scenario projections.

We illustrate the dependency structure in the model in a scatterplot of pairwise rank pseudo-observations. The pseudo-observations are computed by scaling the rank of each observation by $n + 1$, n being the total number of observations (see Figures 1.11 and 1.12). We additionally report the estimated Kendall's Taus in Table 1.7. We see that total stock returns appear essentially independent of inflation and long-term interest rates, and that bond yields are mildly positively correlated with inflation. So far, we are unable to comment on whether such dependence structure is reasonable − a much larger data set is required. This study is conducted for all $1 \leq t \leq 30$ with similar results obtained.

The purpose of updating the parameters, and assessing the strengths and weaknesses of the model is to allow us to use the model with a better understanding of the underlying assumptions. Although we know that structural changes in the underlying processes may not be well captured, still, the model allows us to investigate potential asset and liability paths over long terms.

[4] $R_c(t) = \frac{(1+c(t-1))^{10}}{(1+c(t))^9} - 1$.

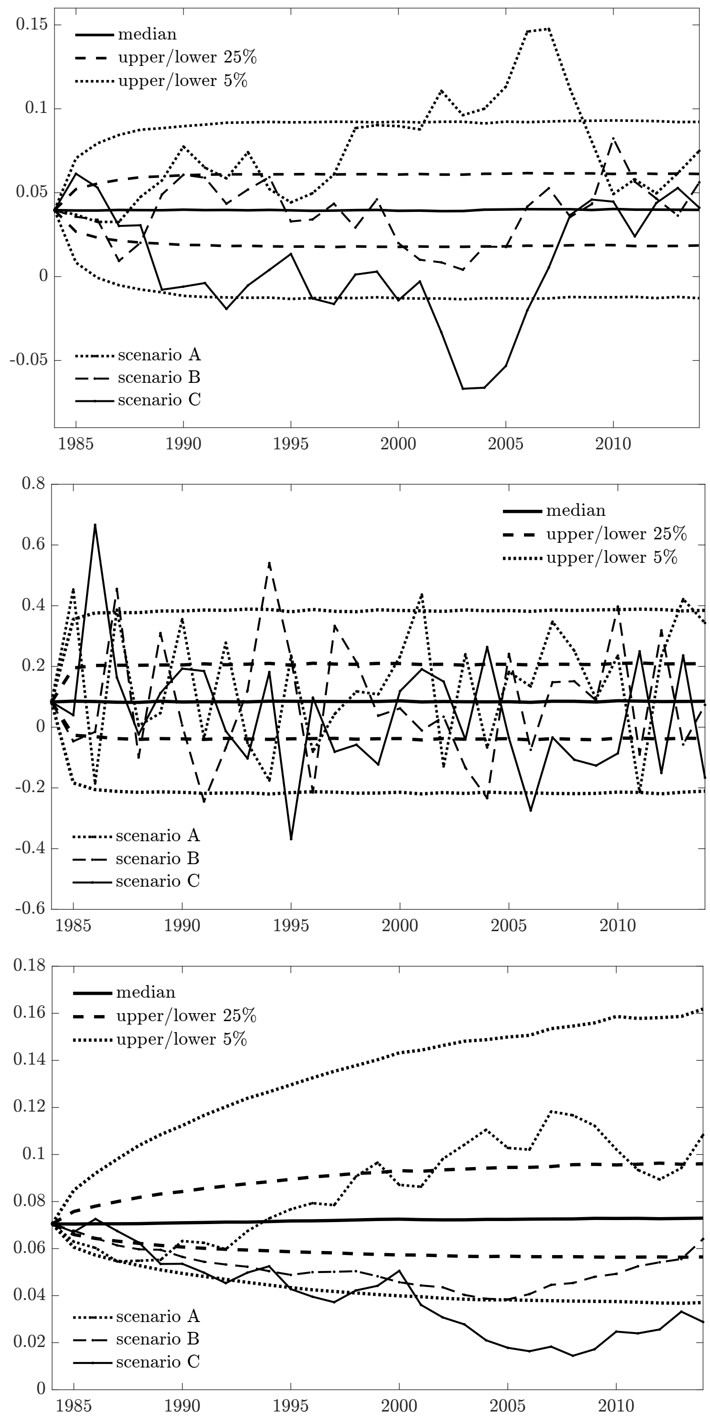


Figure 1.9 Wilkie's ESG (fitted to 1951–1984): Funnel-of-doubt plots of simulated observations with three illustrative joint scenarios. Top to bottom: inflation, total stock returns, long-term bond yields.

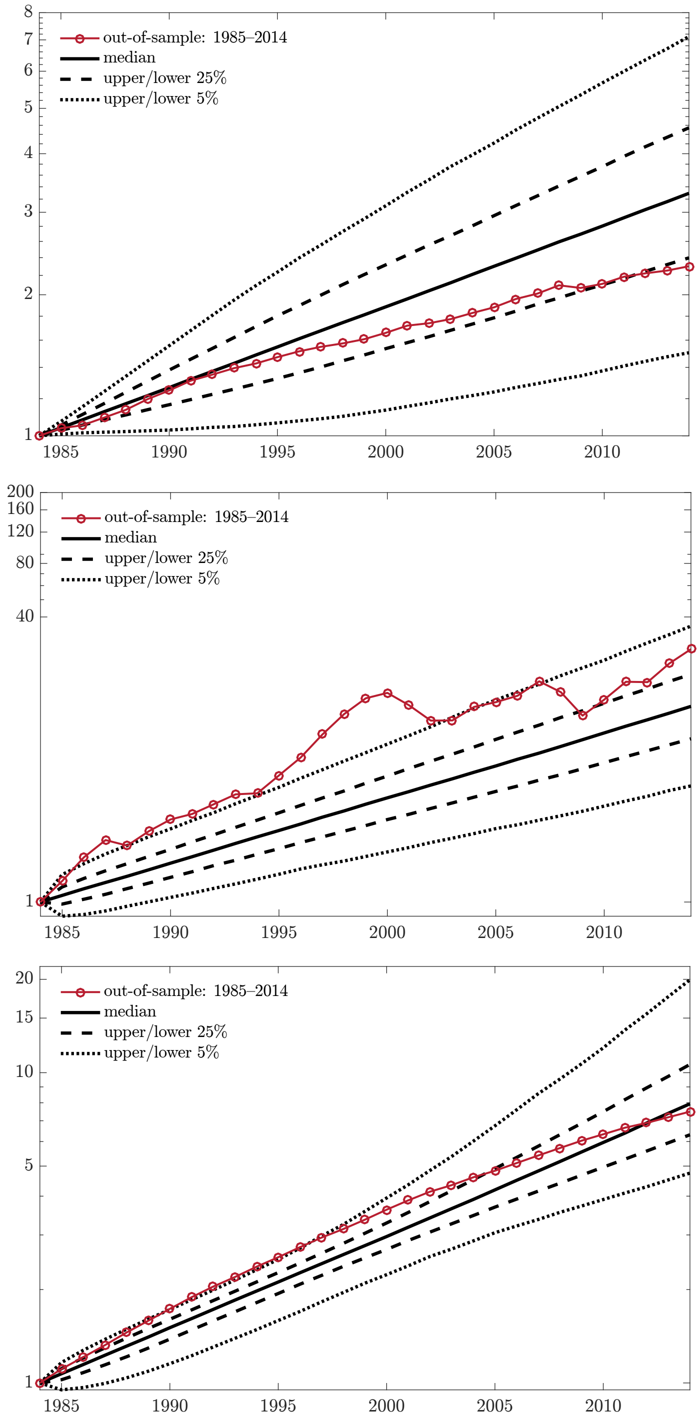


Figure 1.10 Wilkie's ESG (fitted to 1951–1984): Funnel-of-doubt plots (log-scale) of simulated accumulation factors, with out-of-sample (1985-2014) observations. Top to bottom: inflation, total stock returns, long-term bond portfolio returns.

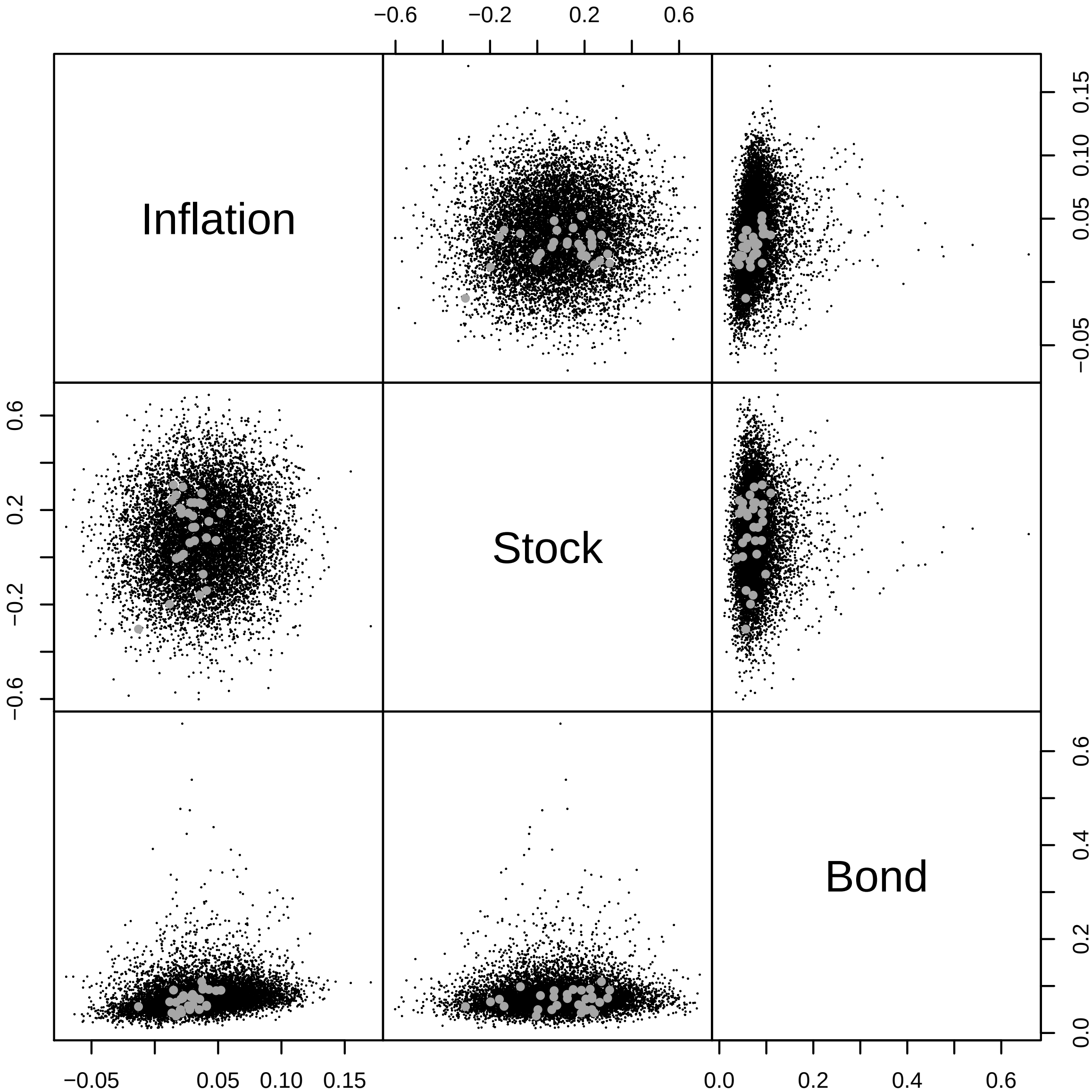


Figure 1.11 Wilkie's ESG (fitted to 1951–1984): Simulated 30-year-forward observations (333 simulations for each year) with out-of-sample observations (1985–2014) overlaid and marked in grey.

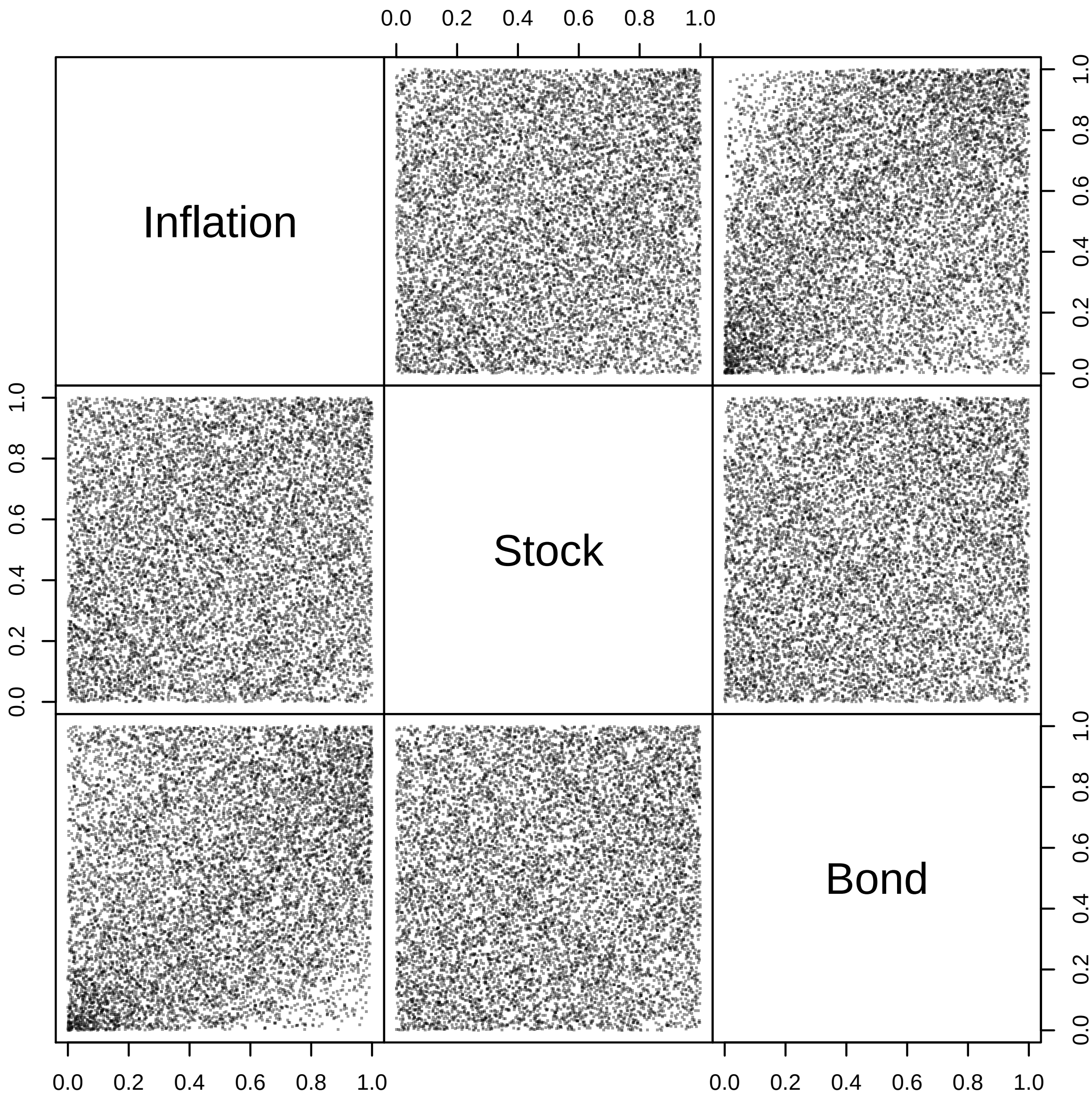


Figure 1.12 Wilkie's ESG (fitted to 1951–1984): 10,000 pseudo observations (component-wise ranks scaled by 1/10001) of simulated observations for year 30.

1.3.4 Other ESGs in the Public Domain

While some research focused on extending Wilkie's ESG by addressing some of its limitations (as referenced in sections 1.2 and 1.3), there exist alternative ways of jointly modelling economic series for long-term use. Here, we briefly comment on how Wilkie's ESG compares with three selected ESGs for which some details are in the public domain: Ahlgrim et al. (2005), Hibbert et al. (2001) and Sherris and Zhang (2009). Our interest in these ESGs primarily stems from their alternative modelling approach.

Hibbert et al. (2001) describe a flexible ESG with four economic series: inflation, real interest rates, the equity risk premium (returns over the risk free short rate) and dividend yields. The frequency of the models varies; the real interest and inflation models are continuous time, two factor models which create a term structure for both real interest and inflation. The equity model is specified as a regime switching Markov model with generic time step, but it is used as a monthly model in the illustrations. The specification as a monthly model does not translate to the same model in other frequencies, however.

The Hibbert et al. (2001) model is influenced by Wilkie (1986), Wilkie (1995), Chan (1998) and Hardy (2001). The parameter estimation method used by Hibbert et al. (2001) is not explicit, which creates a barrier to independent assessment of the model, and to updating and backtesting the model and parameters. They do compare their model with sample scenarios from the Wilkie model, and in particular critique the possibility of low inflation and high interest rates, or low interest rates and high inflation in the Wilkie model. They also opine that the Wilkie model underestimates the probability of significant equity losses over longer terms, as a result of the mean reversion in the Wilkie equity processes.

The Ahlgrim et al. (2005) ESG considers six economic variables: inflation, interest rates, stock returns, dividend yield, property and unemployment rates. It is a $\mathbb{P}$ measure model, calibrated to US data, with different sample periods for each series. It employs a monthly frequency. The model is influenced by Hibbert et al. (2001), with similar mathematical and economic assumptions. The real and nominal interest rate models in the Ahlgrim et al. (2005) ESG are identical to the Hibbert et al. (2001) ESG, except that inflation is described as a one-factor model. It too models the excess return on stocks with a regime-switching process, only it distinguishes between large cap and small cap stocks. Similar to the Hibbert et al. (2001) ESG, the overall dependence is captured by the interplay of the driving forces (i.e. correlated Brownian motions).

The Sherris and Zhang (2009) ESG is a $\mathbb{P}$ measure, multivariate and global Regime-Switching Vector Autoregressive Model (RS-VAR) of a quarterly frequency, calibrated to Australian data from 1979 to 2006. It simultaneously models the log return of eleven economic variables, namely, GDP, CPI, equity, dividend, 90-day treasury notes yields, 2-year treasury bond yields, 10-year treasury bond yields, average weekly earnings, unemployment, property, and US 2-year treasury bond yields. Among these, CPI (i.e. inflation) and the unemployment rates are selected to be regime-switching. The dependencies among the variables are modelled through a regime-switching multivariate normal innovation process.

Similar to the Wilkie model, the Hibbert et al. (2001) and Ahlgrim et al. (2005) ESGs assume the Fisher relation, in that inflation and real interest rates are combined to give nominal interest rates. However, instead of modelling short term and long-term interest rates as the Wilkie model and the Sherris and Zhang (2009) ESG, the Hibbert et al. (2001) and Ahlgrim et al. (2005) ESGs model the entire yield curve. Further, in the Ahlgrim et al. (2005) and Hibbert et al. (2001) ESGs, total returns on equity are directly modelled, as opposed to Wilkie's inferred total returns through dividends. The total stock return in Sherris and Zhang (2009) is obtained from combining price return and dividend yield. The Sherris and Zhang (2009) ESG stands out as being explicitly data driven, whereas the Wilkie model, the Ahlgrim et al. (2005) and Hibbert et al. (2001) ESGs to some extent, combine economic theory and empirical experience.

1.4 Comparison to a generic ESG

To investigate the relative performance of Wilkie's ESG, we construct a generic ESG. The generic ESG is constructed based on the model specifications in Hibbert et al. (2001) and Ahlgrim et al. (2005), which share numerous similarities as described in section 1.3.4, and include features that are, to our best knowledge, commonly found in both public access and certain proprietary ESGs. The generic ESG is simplified yet it captures the essence of these models in that it is a continuous-time model and that dependence is partially captured by correlated Brownian motions. We are interested in the performance of this generic ESG compared to the Wilkie model, particularly in the long-term. This leads to the investigation of the joint distribution of future variables (i.e. the simulated scenarios). Model specification of the generic ESG is given below.

1.4.1 Model Specification of the Generic ESG

i. Inflation: $\delta_q(t)$

$$d\delta_q(t) = \kappa_q \left\{ \mu_q - \delta_q(t) \right\} dt + \sigma_q dB_q(t)$$

ii. Real interest rates: short-term rate $r(t)$, long-term rate $l(t)$

$$\begin{aligned} dr(t) &= \kappa_r \left\{ l(t) - r(t) \right\} dt + \sigma_r dB_r(t) \\ dl(t) &= \kappa_l \left\{ \mu_l - l(t) \right\} dt + \sigma_l dB_l(t) \end{aligned}$$

iii. Total equity returns: $s(t)$, regime switching excess returns $x(t)$

$$\begin{aligned} s(t) &= \delta_q(t) + r(t) + x(t) \\ x(t) \,|\, \rho(t) &\overset{\text{i.i.d.}}{\sim} \mathrm{N}\left(\mu_{\rho(t)}, \sigma^2_{\rho(t)}\right) \end{aligned}$$

where $\rho(t)$ is the regime at time t. The model is governed by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where p_{ij} denotes the probabilities of changing regimes, namely,

$$p_{ij} = Pr\left[\rho(t+1) = j \,|\, \rho(t) = i\right], \qquad i, j = 1, 2.$$

The unconditional probability π_i of being in state i at any point in time is

$$\pi_i = \frac{p_{ji}}{p_{ji} + p_{ij}}.$$

iv. The underlying dependence among the economic series is captured by correlated standard Brownian motions $(B_q(t), B_r(t), B_l(t))^T$ with correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{qr} & \rho_{ql} \\ \rho_{qr} & 1 & \rho_{rl} \\ \rho_{ql} & \rho_{rl} & 1 \end{pmatrix}.$$

v. The generic ESG (excluding stock returns) can be alternatively expressed using independent standard Brownian motions $(W_1(t), W_2(t), W_3(t))^T$, as

$$\begin{aligned} d\delta_q(t) &= \kappa_q\{\mu_q - \delta_q(t)\}dt + \sigma_q dW_1(t) \\ dr(t) &= \kappa_r\{l(t) - r(t)\}dt + \rho_{qr}\sigma_r dW_1(t) + \sigma_r\sqrt{1-\rho_{qr}^2}\,dW_2(t) \\ dl(t) &= \kappa_l\{\mu_l - l(t)\}dt + \rho_{ql}\sigma_l dW_1(t) + \sigma_l\frac{\rho_{rl} - \rho_{ql}\rho_{qr}}{\sqrt{1-\rho_{qr}^2}}dW_2(t) \\ &\qquad + \sigma_l\sqrt{1 - \rho_{ql}^2 - \frac{(\rho_{rl} - \rho_{ql}\rho_{qr})^2}{1-\rho_{qr}^2}}\,dW_3(t) \end{aligned}$$

1.4.2 Data and Fitting of the Generic ESG

The generic ESG is fitted to the same data source as described in section 1.2.5 except on a monthly frequency. The choice of using monthly observations is due to the regime-switching component of the equity model, which requires a large sample size. We additionally use the monthly series of the 3-month US treasury bill, obtained from federalreserve.gov, to fit the short rate model. The reported parameters are maximum likelihood estimates. The R package yuima is used to calibrate the inflation and interest rate models, and MSwM for the stock returns model.

1.4.3 Backtesting the Generic ESG

As with section 1.3.3, we fit the generic ESG to data from 1951–1984 and project 10,000 scenarios 30 years forward. The estimated parameters are reported in Table A.2. Since the ESG is fitted to monthly observations, the projections are simulated on a monthly basis and combined to produce annual observations, such that the output is comparable with Wilkie's ESG. Figure 1.13 gives the funnel-of-doubt plots with three in-sample simulated paths (figure 1.9 is scaled and positioned to the left for the ease of comparison). Figure 1.14 (plotted in log scale) plots the funnel-of-doubt of the simulated accumulation process from the Wilkie and the generic ESG. In the following subsection, we highlight key observations when comparing the output to Wilkie's ESG.

Comparison to Wilkie's ESG

- **Inflation:**

  The generic ESG outputs close medians with Wilkie's ESG, but a much narrower range of scenarios, especially in the lower tail. In the long run, deflations are projected to occur 3.5% of the times under the generic ESG, compared to 10.5% under Wilkie's ESG. The simulated accumulation process for the generic ESG yields lower uncertainties (i.e. a thinner funnel): the out-of-sample accumulation path falls below the 5% quantile after 15 years. The difference is caused by the stronger mean reversion in the generic ESG.

- **Total stock returns:**

  Stock returns scenarios generated by the generic ESG have moderately higher medians. The distribution in each future year is left-skewed, with a thicker left tail than the right. Wilkie's ESG, on the other hand, is almost symmetric. In each projection year, the generic ESG gives a narrower range of scenarios than Wilkie's ESG. However, in the accumulation process, the range of scenarios produced is wider with a thicker left tail (see figure 1.16 for a comparison of estimated densities at year 10). This is due to serial dependence. The out-of-sample path falls consistently above the 25% quantiles.

- **Long-term interest rates:**

  The bond yields scenarios produced by the generic ESG have lower medians than Wilkie's ESG. Overall, a smaller range of scenarios is generated, particularly in the right tail. In year 30, the generic ESG projects a 3.1% chance of yields exceeding 0.1, which is significantly lower than Wilkie's projection of a 22.2% chance. Volatility in a single path is high, resulting in bond yields that change drastically from year to year. The distribution at each year is symmetric, whereas Wilkie's ESG gives significant right-skewness.

- **Pairwise dependence:**

  The generic ESG outputs similar pairwise dependence to Wilkie's ESG. Table 1.7 compares the estimated Kendall's Taus with Wilkie's ESG. Figure 1.12 plots the 30-year-forward observations overlaid with out-of-sample data. We observe positive dependence between inflation and nominal long-term interest rates, and very weak dependence between stock returns and inflation/nominal long-term interest rates. We additionally provide the following comments:

- The generic ESG introduces dependence through addition and correlated driving forces. The former creates dependence based on economic theory; the latter captures leftover dependence that is not explained by the construction. This setup is rigid, with dependence *imposed*. It is also different from the Wilkie model, under which certain parameters governing dependence may be set to zero (or estimated to be statistically insignificant from zero, implying little evidence to support such a relationship).
- The correlation matrix Σ in the generic model could alleviate, if necessary, the degree to which dependence is imposed. For example, a (highly) negative correlation coefficient estimate is obtained for the driving forces underlying inflation and the real short rate, since the real short rate is constructed ex post by subtracting inflation from the nominal short rate.

	Inflation & Stock	Inflation & Bond	Stock & Bond
Wilkie's ESG	0.0557	0.1723	0.0539
Generic ESG	−0.0102	0.2465	0.0613

Table 1.7 Estimated Kendall's Tau at year 30: based on 10,000 simulated observations (models fitted to 1951–1984).

1.4.4 Numerical Challenges

The generic ESG requires a total of 17 parameters, and joint estimation of the full model is computationally challenging due to the large number of parameters involved. The stock returns model can be estimated separately. One may also break down the estimation procedure by first considering the inflation model, then the long rate model, the short rate model, and lastly the correlation parameters. Nonetheless, a large number of parameters (a total of 7) are required to be estimated under the short rate model. In this case, the calibration process is sensitive to the provided initial values for the parameters in the estimation procedure. Wilkie's full model requires 20 parameters, but the model can be more easily calibrated due to the cascade structure. Some parameters are not necessarily required. For example the model fitted to the 1991–2014 data requires 13 parameters.

One additional challenge of fitting the generic ESG lies in the estimation of the variance-covariance matrix of the estimated parameters. This numerically estimated matrix can be non-positive definite. In our analysis, the R function `nearPD` is used to obtain an estimate of a positive-definite variance-covariance matrix. Noise in monthly data (for example, inflation prior to 1966) may also affect the plausibility of the fitted parameters.

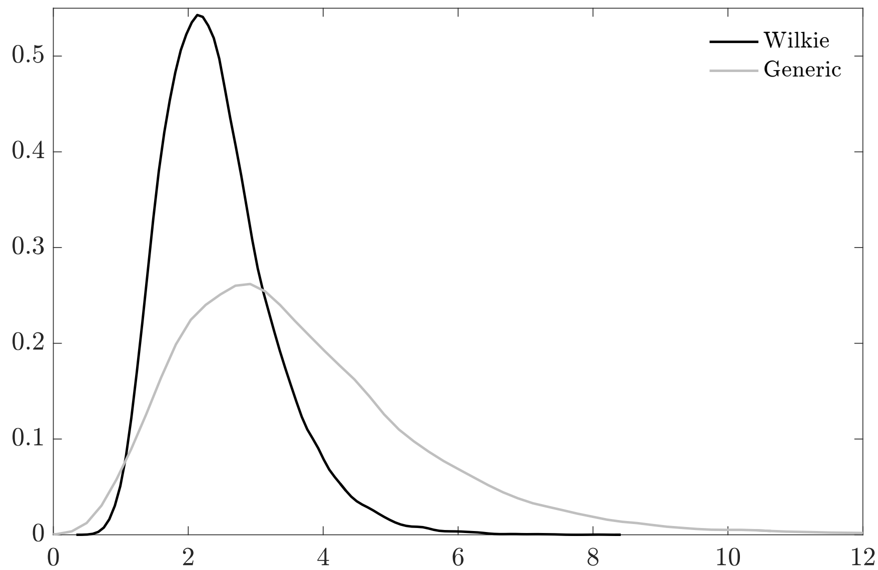


Figure 1.16 Estimated probability density functions for an accumulated unit stock investment at year 10 (models fitted to 1951–1984).

1.5 Conclusion

We have updated Wilkie's ESG to 2014 using US data and assessed its performance since its introduction. We found that due to multiple structural shifts in the historical time series, inflation most strongly violates the assumption of stationarity, and that parameters are sensitive to the calibration period. We studied the dependency among the simulated observations and conclude that the model outputs total stock returns that appear independent of inflation and long-term interest rates, and bond yields that are positively driven by inflation. A backtest based on 30-year out-of-sample data indicated that over that period the model had tended to overestimate inflation (due to a structural shift in inflation targeting policy in the early 1990s), underestimate total return on stocks (due to the unprecedented dot-com bubble in the 1990s), and performed relatively well for long-term interest rates. An analysis that compares Wilkie's ESG with a generically constructed ESG reveals that in the long run, Wilkie's ESG generates a wider range of scenarios for inflation and long-term interest, but a narrower range for stock returns. Wilkie's ESG can be under-representative of the risk in long-term stock investment, particularly in the tails.

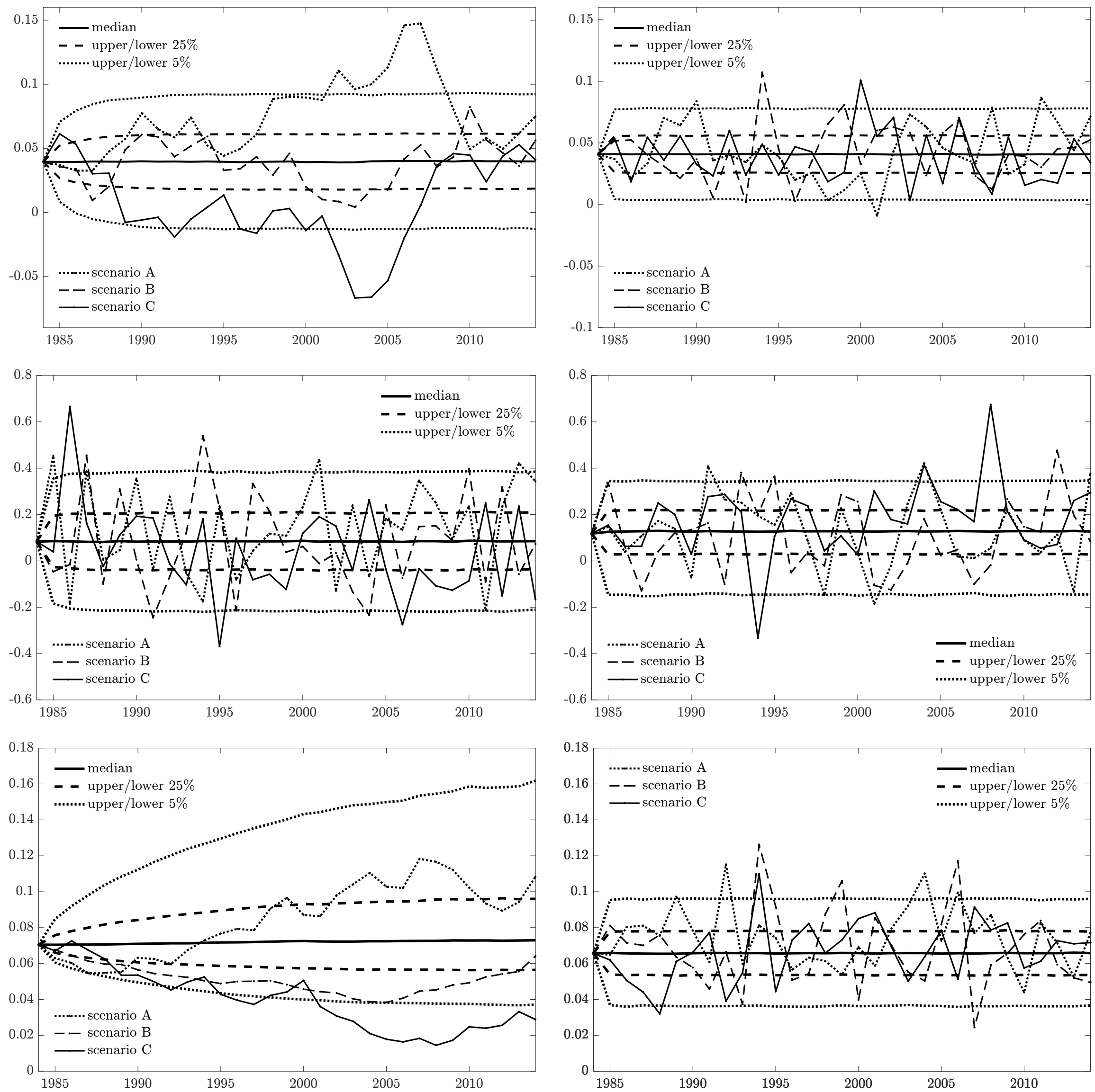


Figure 1.13 Funnel-of-doubt plots of simulated observations with three illustrative joint scenarios. Top to bottom: inflation, total stock returns, long-term bond yields. Left: Wilkie's ESG. Right: Generic ESG. Fitted to 1951–1984.

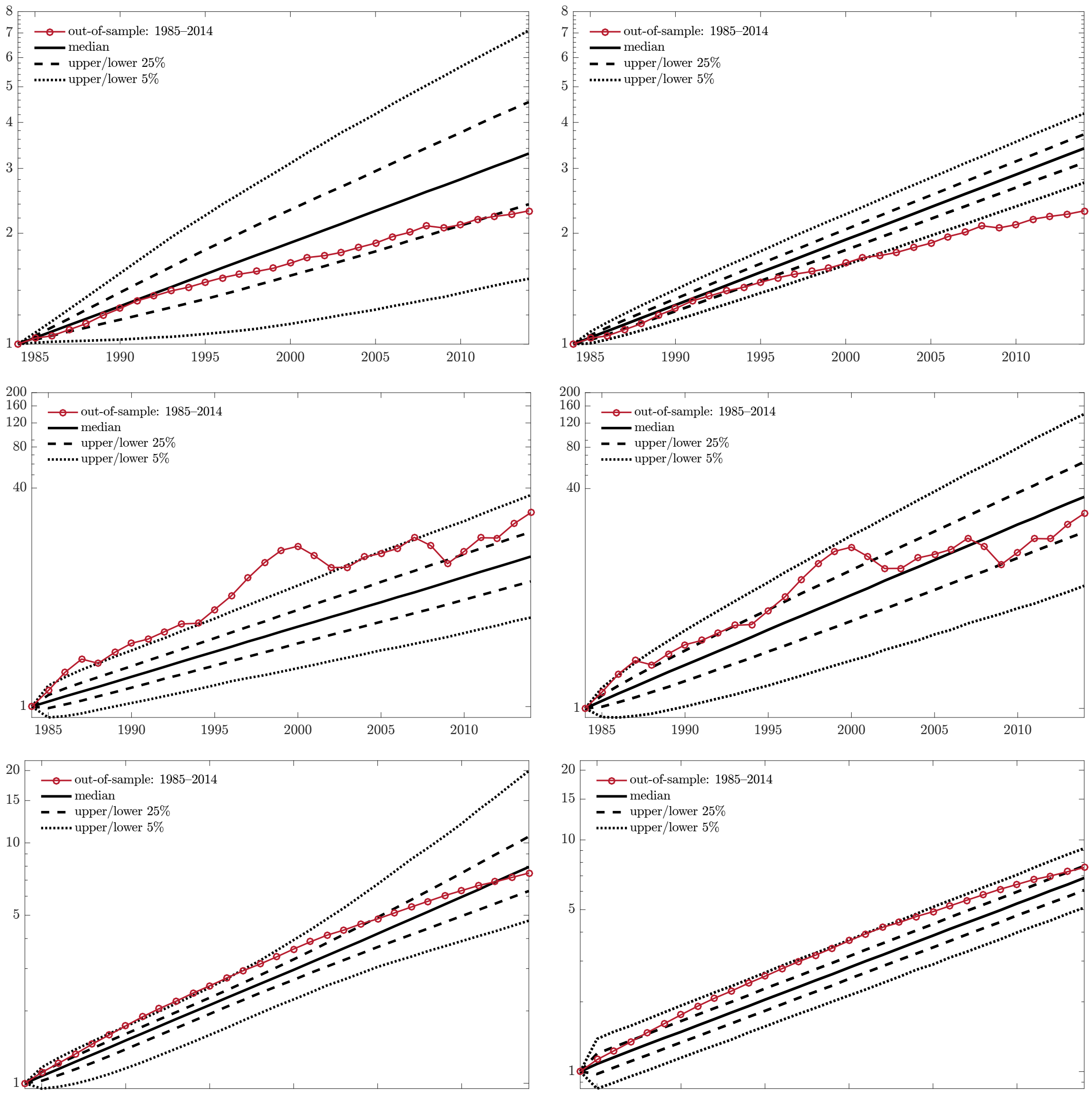


Figure 1.14 Funnel-of-doubt plots (log-scale) of simulated accumulation factors, with out-of-sample (1985–2014) observations. Top to bottom: inflation, total stock returns, long-term bond portfolio returns. Left: Wilkie's ESG. Right: Generic ESG. Fitted to 1951–1984.

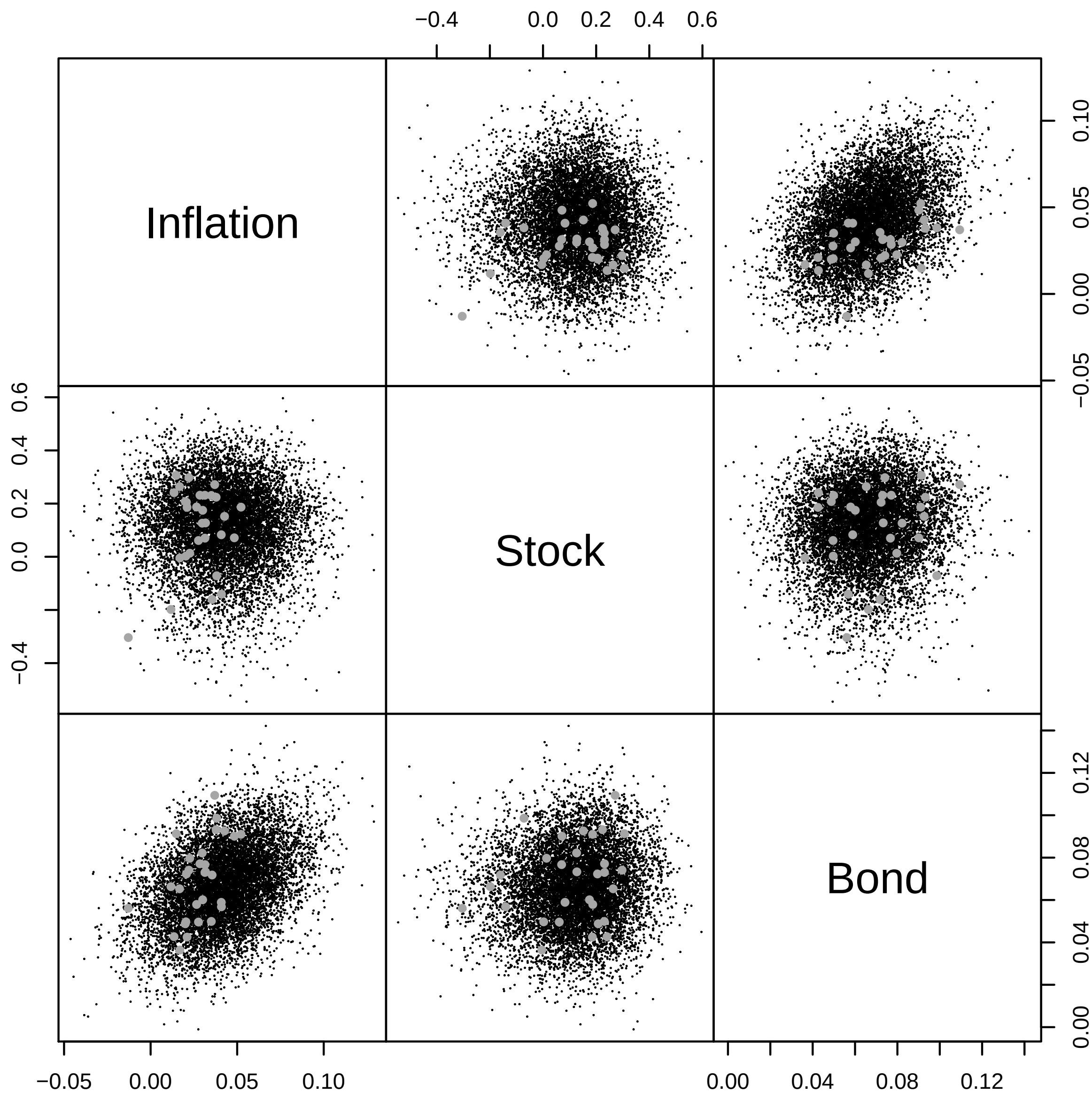


Figure 1.15 Generic ESG (fitted to 1951–1984): Simulated 30-year-forward observations (333 simulations for each year) with out-of-sample observations (1985–2014) overlaid and marked in grey.

For investigating asset-liability management strategies for pension plans, the Wilkie model is quite suitable, because the assessment is long-term and the pension assets are relatively passively invested, so the annual time-step suffices[5]. Additionally, the relationship between the assets and liabilities of a Defined Benefit pension plan is critically connected to future inflation, real rates of interest, and real rates of return on stocks, so the multivariate approach is a good fit to the problem. The uncertainties around the long-term stability of the processes and parameters are relevant, however, there are benefits to a relatively simple, static model, and Wilkie's ESG does offer some insight into risk and volatility of pension plans, taking into consideration the known limitations such as potentially thin left tails for equity returns. Parameter uncertainty may be addressed using sensitivity analysis, and model risk can also be explored, for example, by comparing results using the Wilkie model with more crude independent models for the various processes, or with the Hibbert et al. (2001) model with suitably adjusted parameters.

[5]Where a more frequent time step is required, Wilkie et al. (2003) suggest using a Brownian bridge, or Ornstein-Uhlenbeck (OU) bridge, which create paths constructed to follow a Brownian or OU process, but which are constrained to start and finish at the specified (year end) points generated by the annual model.

Chapter 2

Recursive Preferences

2.1 Introduction

This chapter provides an introductory discussion of recursive preferences, which are adopted in Chapter 3 as a normative preference model in solving problems of optimal retirement planning. Recursive preference models were first introduced by the seminal work of Epstein and Zin (1989), and in essence, characterize the tradeoff between the current-period utility and the utilities to be derived from all future periods, summarized with a single index which is referred to as the certainty equivalent of all future utilities. The most commonly adopted model of this general class of utility functions is the homothetic, constant elasticity of intertemporal substitution (CES) case of Kreps and Porteus (1978) preference with an expected-utility certainty equivalent. We refer to this as the Epstein-Zin utility.

Traditionally, life-cycle and long-term portfolio selection problems consider the decision-making framework of a rational agent who seeks to maximize her time-additive expected discounted lifetime utility (EDLU). The intertemporal aspect of EDLU is the discounted utility (DU) framework, which was introduced by Samuelson (1937). In this work, it was explicitly assumed that future gratification can be discounted and aggregated in a fashion that resembles the present value of future cash flows. This, combined with the expected utility theory formulated by Bernoulli in 1738 and Von Neumann and Morgenstern (1944), gives the time-additive EDLU.

The Epstein-Zin preference characterizes an important extension to this framework in the case of the constant relative risk aversion (CRRA) utility function. The model relaxes the assumption that the parameter governing relative risk aversion (RRA) is set to the reciprocal of the parameter governing the elasticity of intertemporal substitution (EIS), disentangling risk and time preferences. This separation is considered important, since RRA and EIS describe entirely different behaviour, and there is no obvious reason why such a restriction should be imposed. When separately parameterized, the model gains flexibility in describing preferences across time and possibilities.

To date, the Epstein-Zin utility is typically used as a generalization of the traditional approach of time-additive EDLU-CRRA. This extension is straightforward to incorporate numerically in stochastic optimization, since its recursive nature allows multi-period problems to be solved sequentially as multiple single-period problems (i.e. today versus the future), which naturally complements the framework of dynamic programming. In this thesis, we consider Epstein-Zin agents and the implications of this preference model to be used for normative optimal retirement planning.

Before considering a full-blown problem, this chapter first sets the scene by considering a simplified problem, where the agent is age 65 and just about to retire, and has unit wealth to invest (in a single asset) and consume throughout her remaining lifetime. Note that the Epstein-Zin preference is intertemporally homothetic, meaning that behaviour is scale-invariant. The implication is that rich and poor agents have identical preferences, rendering wealth irrelevant. In this case, we are able to derive analytically the optimal consumption decisions, allowing one to isolate the impact of the preference, economic, and mortality assumptions on the optimal solutions. This allows the reader to appropriately interpret the results presented in Chapter 3.

A key contribution of this chapter is on the impact of EIS on the smoothness of the optimal consumption profile. In the literature on life-cycle investing, there exists a misconstrued notion that in the presence of an uncertain future lifetime, the EIS is a parameter that governs consumption smoothing or substitutability over time (see e.g. Horneff et al., 2015; Blake et al., 2008; Horneff et al., 2008b). We show that this does not, in general, hold true. Additionally, we show that the most widely used Epstein-Zin specification (used in: Gomes and Michaelides, 2005; Horneff et al., 2015; Horneff et al., 2009; Horneff et al., 2008b; Horneff et al., 2008c; Blake et al., 2008) is *not* a generalization of the time-additive EDLU-CRRA model: imposing the reciprocal relationship between the RRA and the EIS does not necessarily give rise to the solutions underlying the time-additive EDLU-CRRA models. This implies that it is inaccurate for these works to state that freeing this reciprocal relationship is a relaxation of the time-additive EDLU-CRRA preference. These key findings imply that many existing works using the Epstein-Zin utility model should be revisited, and their conclusions should be reinterpreted with caution.

This chapter is organized as follows. Section 2.2 introduces notation and defines the Epstein-Zin utility function. In section 2.3, a risk-free, single-period problem is solved, and later extended to consider financial risk. In section 2.4, we consider a multi-period problem, and derive analytical expressions for the optimal strategies under 1) risk-free, 2) risky investment, and 3) risky inflation scenarios. In section 2.5, the problem is extended to include a uncertain future lifetime, where three recursive preference specifications that differ in the treatment of mortality risk, are considered, with results of the optimal strategies shown and compared. The key conclusion of this chapter is highlighted in this section. The last section concludes the chapter.

2.2 Definition and Notation

The Epstein-Zin preference model is rigorously defined in Epstein and Zin (1989). Here, we provide a description of the utility function without going into the rigorous mathematical details. Let $t = 0, 1, ..., n$ denote time. Consider an agent aged x at time 0 who dies with probability 1 at time $n+1$. Let C_t and $\tilde{C}_t$ be the nominal and real consumption for the period $[t, t+1)$, respectively. We assume consumption occurs at the beginning of the period. Similarly, let W_t and $\tilde{W}_t$ be the nominal and real wealth before consumption at time t, respectively; $W_0 = \tilde{W}_0 = 1$.

The class of recursive preferences is defined by characterizing the time-t utility index as,

$$V_t = A\left[\tilde{C}_t, \mu_t(V_{t+1})\right],$$

where $A[u, v]$ is the time aggregator function, and $\mu_t(w)$ is the certainty equivalent of a random payoff w at time t. The Epstein-Zin utility function has a CES aggregator and an expected-utility certainty equivalent, namely

$$A(u, v) = \left[u^{1-\frac{1}{\alpha}} + \beta v^{1-\frac{1}{\alpha}}\right]^{1/(1-\frac{1}{\alpha})}, \quad \mu_t(w) = \left[\mathbb{E}_t\left(w^{1-\rho}\right)\right]^{1/(1-\rho)},$$

where $\alpha \in (0, \infty) \setminus \{1\}$ is the parameter governing the EIS[1], $\rho \in (0, \infty) \setminus \{1\}$ the RRA, and $\beta \in (0, 1]$ the subjective impatience factor. The optimal consumption strategy is therefore

$$\{\tilde{C}_0^*, \tilde{C}_1^*, ..., \tilde{C}_n^*\} = \arg\max V_0.$$

Here, we use the superscript * to denote the optimal strategies. It can be easily shown that when $\rho = 1/\alpha$,

$$\max V_0 \equiv \max \mathbb{E}_0\left[\sum_{t=0}^{n} \beta^t \left(\frac{\tilde{C}_t^{1-\rho}}{1-\rho}\right)\right],$$

which yields the maximization of time-additive EDLU-CRRA with exponential discounting factor β.

2.3 Single-Period

2.3.1 Risk-free

Consider a simple single-period problem where $n = 1$. First, suppose there is no risk associated with the future, no inflation, and that the investment return (i.e. interest rate) r_f over the period is deterministic. As such, $\tilde{C}_t = C_t$. Consider an Epstein-Zin utility maximizing agent. Using Bellman's principle of optimality, the optimization problem can be solved using backward induction. At time 1, it is optimal to consume all remaining wealth thus $C_1^* = (1 - C_0)(1 + r_f)$. The utility derived from a given consumption path (C_0, C_1^*) is

$$U(C_0, (1 - C_0)(1 + r_f)) = \{C_0^{1-\frac{1}{\alpha}} + \beta[(1 - C_0)(1 + r_f)]^{1-\frac{1}{\alpha}}\}^{\frac{1}{1-\frac{1}{\alpha}}}.$$

[1]The derivation of EIS is shown in Appendix B.1. Havranek et al. (2015) collected 2735 estimates of the EIS from 169 published studies covering 104 countries, and found majority of these estimates to be less than 1. Hence, we pay particular interest to the parameter range $0 < \alpha < 1$.

For fixed α and r_f, we seek to maximize U with respect to C_0, which yields the optimal consumption at time 0:

$$C_0^* = \arg\max_{C_0} U = \frac{\beta^{-\alpha}(1+r_f)^{1-\alpha}}{1+\beta^{-\alpha}(1+r_f)^{1-\alpha}} = \frac{1}{1+\beta^{\alpha}(1+r_f)^{\alpha-1}}.$$

Note that the optimal savings at time 0 is implied, as $1 - C_0^*$.

In the risk-free case, we focus on investigating the impact of α on the optimal consumption path. In particular, we are interested in the volatility of the optimal consumption over time, since EIS governs intertemporal substitutability. In the single period case, it suffices to look at the slope of the consumption path since it gives a measure equivalent to consumption volatility. Here, the slope can be computed as

$$S^* = C_1^* - C_0^* = 1 + r_f + \frac{2+r_f}{1+\beta^{\alpha}(1+r_f)^{\alpha-1}}.$$

The higher its absolute value, the higher the consumption volatility. Figures 2.1 and 2.2 plot C_0^* as a function of α and r_f for two choices of β. Figures 2.3 and 2.4, similarly, plot $|S^*| = |C_0^* - C_1^*|$. Table 2.1 summarizes (C_0^*, C_1^*) for some key parameter combinations.

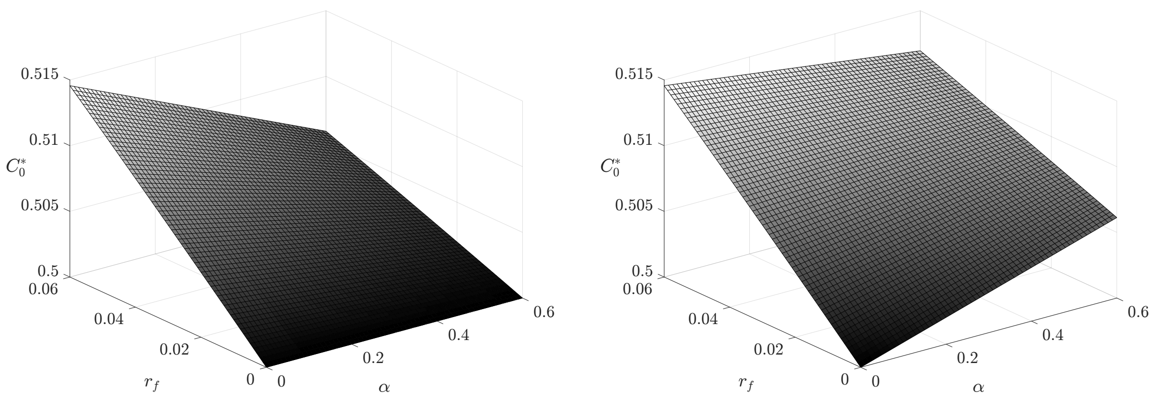


Figure 2.1 Optimal consumption at time 0 as a function of α and r_f. Left: $\beta = 1.0$. Right: $\beta = 0.96$.

	$\boldsymbol{\beta = 1.0}$				$\boldsymbol{\beta = 0.96}$			
	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.2$		$\alpha = 0.5$	
r_f	(C_0^*, C_1^*)	Slope	(C_0^*, C_1^*)	Slope	(C_0^*, C_1^*)	Slope	(C_0^*, C_1^*)	Slope
0.00	$(0.500, 0.500)$	0.000	$(0.500, 0.500)$	0.000	$(0.502, 0.498)$	-0.004	$(0.505, 0.495)$	-0.010
0.02	$(0.504, 0.506)$	0.002	$(0.503, 0.508)$	0.005	$(0.506, 0.504)$	-0.002	$(0.508, 0.502)$	-0.005
0.04	$(0.508, 0.512)$	0.004	$(0.505, 0.515)$	0.010	$(0.510, 0.510)$	-0.000	$(0.510, 0.510)$	-0.000
0.06	$(0.512, 0.518)$	0.006	$(0.507, 0.522)$	0.015	$(0.514, 0.516)$	0.002	$(0.512, 0.517)$	0.005

Table 2.1 Optimal consumption paths: a single-period model

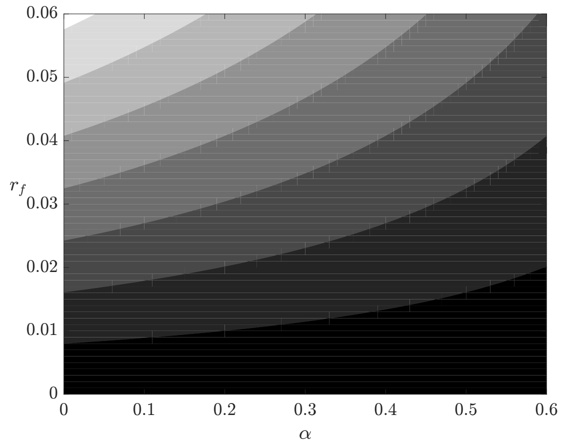


Figure 2.2 Contour plot of optimal consumption at time 0 as a function of α and r_f (lighter shades imply higher C_0^*). Left: $\beta = 1.0$. Right: $\beta = 0.96$.

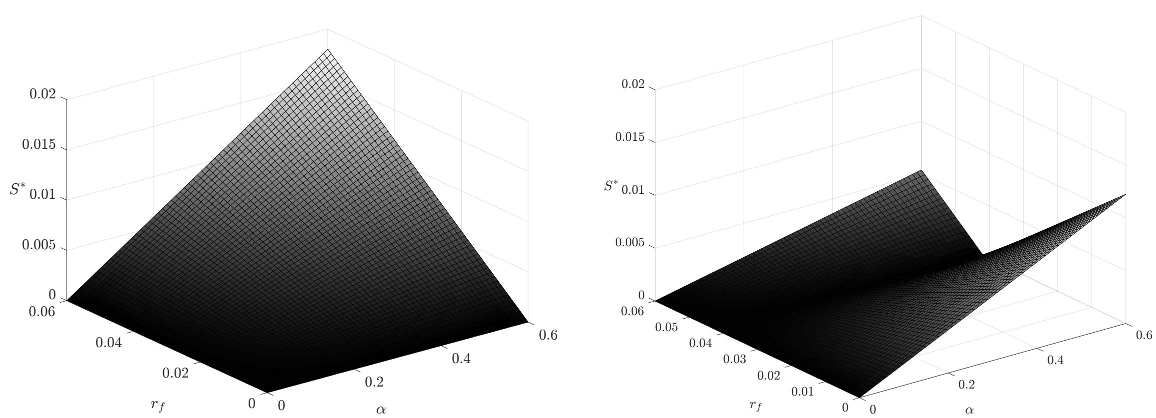


Figure 2.3 Slope of optimal consumption path as a function of α and r_f. Left: $\beta = 1.0$. Right: $\beta = 0.96$.

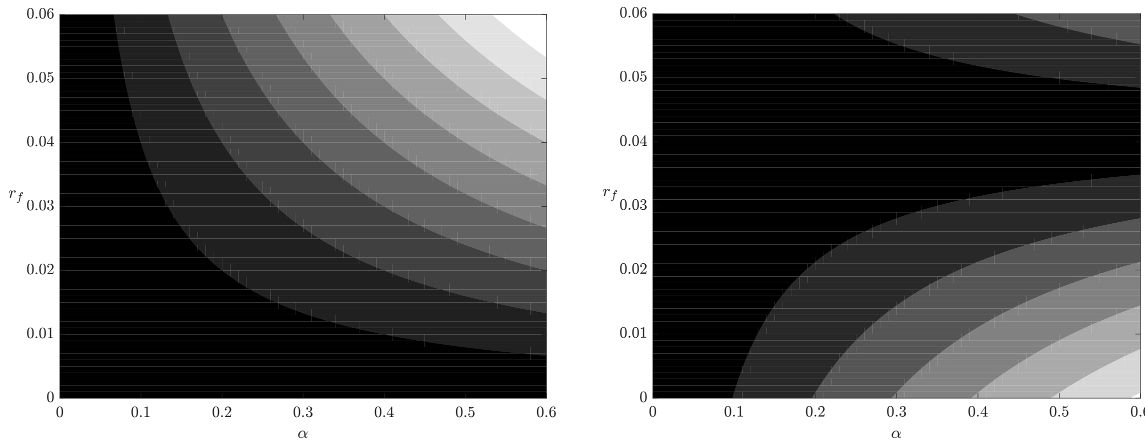


Figure 2.4 Contour plot of slope of optimal consumption path as a function of α and r_f (lighter shades imply higher $|S^*|$). Left: $\beta = 1.0$. Right: $\beta = 0.96$.

We first consider the special case where $\beta = 1.0$ and $r_f = 0$, so that there is no subjective discounting of consumption in the future and that consuming less today will not result in an increase in the aggregate wealth. In this case, the retiree's aggregate consumption over the lifetime is unity regardless of her present consumption choice. We obtain that $C_0^* = C_1^* = 0.5$, implying that the agent chooses the smoothest possible consumption path, regardless of α which governs the strength of the smoothing preference.

We further summarize the following key observations:

- As r_f increases, the slope of the consumption paths rises and the income effect dominates the substitution effect.
- As α increases, the magnitude of the slope rises, implying a more volatile consumption path.
- In absence of subjective discounting, a higher α is associated with higher savings. In the presence of subjective discounting, a higher α implies lower savings when r_f is lower, and higher savings when r_f is higher. The implication is that higher savings behaviour does not automatically translate to smoother consumption behaviour.
- More generally, when $\beta = \frac{1}{1+r_f}$, the optimal consumption is independent of α. This explains the observation that at $r_f = \beta^{-1} - 1$, the contour plots shown in Figure 2.2 become flat. As a result, when the real interest rate is higher than $\beta^{-1} - 1$, a higher α implies higher savings (and lower present consumption); when the real interest rate is lower than $\beta^{-1} - 1$, a higher α implies lower savings (and higher present consumption). Consumption and savings behaviour in the presence of a negative real interest rate follows the same principle.
- In the presence of subjective discounting, agents tend to consume more at time 0 and

save less. This is intuitive as subjective discounting implies that future gratification is considered less important than the present.

2.3.2 Risky investment

The problem can be extended to include a risky investment return. Suppose instead of a risk-free interest rate r_f, the investment return, R, is a random variable. In turn, the optimal consumption at time 0 becomes:

$$C_0^* = \arg\max_{C_0} V_0 = \arg\max_{C_0} \left\{ C_0^{1-\frac{1}{\alpha}} + \beta(1-C_0)^{1-\frac{1}{\alpha}}(1+\mu)^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} = \frac{1}{1+\beta^\alpha \mu^{\alpha-1}}$$

where $\mu = \{\mathbb{E}[(1+R)^{1-\rho}]\}^{\frac{1}{1-\rho}} - 1$. We observe that this is a generalization of the result obtained in the previous example, where μ is the certainty equivalent of the return on investment for an agent with RRA ρ. For a fixed investment opportunity set, a higher RRA translates to a lower certainty equivalent μ, and hence the agent effectively exhibits similar behaviour to that when faced with a lower r_f in the risk-free example. Similarly, for an alternative investment opportunity set, the resulting certainty equivalent investment return can be interpreted in the same vein as with the risk-free example.

2.4 Multiple-Period

2.4.1 Risk-free

Two-period

To further investigate the impact of α on consumption volatility, we extend the single-period case to the n-period case. For illustrative purposes, we first solve the two-period case. Suppose the risk-free rate for the first period and the second period are r_1 and r_2, respectively. Let W_1 denote the wealth before consumption at time 1. The time t utility index, V_t, is therefore

$$\begin{aligned} V_t &= \left\{ C_t^{1-\frac{1}{\alpha}} + \beta V_{t+1}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}}, \qquad \text{for } t = 0, 1 \\ V_2 &= C_2. \end{aligned}$$

Intuitively, we obtain the optimal consumption in the last period based on full consumption of remaining wealth:

$$C_2^* = \arg\max_{C_2} V_2 = (W_1 - C_1)(1 + r_2). \tag{2.1}$$

Hence,

$$\begin{aligned}\max_{C_1} V_1 &= \left\{ C_1^{1-\frac{1}{\alpha}} + \beta \left[\max_{C_2} V_2 \right]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\ &= \left\{ C_1^{1-\frac{1}{\alpha}} + \beta C_2^{*1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} .\end{aligned}$$

Then, the optimal consumption at time 1 is

$$\begin{aligned}C_1^* = \arg\max_{C_1} V_1 &= \arg\max_{C_1} \left\{ C_1{}^{1-\frac{1}{\alpha}} + \beta C_2^{*1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\ &= \arg\max_{C_1} \left\{ C_1{}^{1-\frac{1}{\alpha}} + \beta [(W_1 - C_1)(1+r_2)]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \quad \text{by Equation (2.1)} \\ &= \frac{W_1}{1+\beta^\alpha (1+r_2)^{\alpha-1}}. \qquad (2.2)\end{aligned}$$

Note that this is identical to solving the single-period problem, with starting wealth W_1 instead of 1. We know that $W_1 = (1 - C_0)(1 + r_f)$, meaning that C_1^* and C_2^* can both be expressed as functions of C_0. Thus we have

$$\begin{aligned}C_1^* &= \frac{(1-C_0)(1+r_1)}{1+\beta^\alpha (1+r_2)^{\alpha-1}}, \\ C_2^* &= \beta^\alpha (1+r_2)^\alpha \left(\frac{(1-C_0)(1+r_1)}{1+\beta^\alpha (1+r_2)^{\alpha-1}} \right) \\ &= \beta^\alpha (1+r_2)^\alpha C_1^*.\end{aligned}$$

In turn, the optimal consumption at time 0 can be solved by

$$\begin{aligned}\max_{C_0} V_0 &= \max_{C_0} \left\{ C_0^{1-\frac{1}{\alpha}} + \beta C_1^{*1-\frac{1}{\alpha}} + \beta^2 C_2^{*1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\ &= \max_{C_0} \left\{ C_0^{1-\frac{1}{\alpha}} + \beta \left(\frac{(1-C_0)(1+r_1)}{1+\beta^\alpha (1+r_2)^{\alpha-1}} \right)^{1-\frac{1}{\alpha}} + \beta^2 \left[\beta^\alpha (1+r_2)^\alpha \left(\frac{(1-C_0)(1+r_1)}{1+\beta^\alpha (1+r_2)^{\alpha-1}} \right) \right]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} .\end{aligned}$$

Then, we obtain that

$$C_0^* = \arg\max_{C_0} V_0 = \frac{1}{1+\beta^\alpha (1+r_1)^{\alpha-1} + \beta^{2\alpha} [(1+r_1)(1+r_2)]^{\alpha-1}}.$$

Figure 2.5 shows several pairs of consumption paths for varying β and r_f. We clearly observe that a larger α leads to a steeper slope, therefore larger changes in consumption over time.

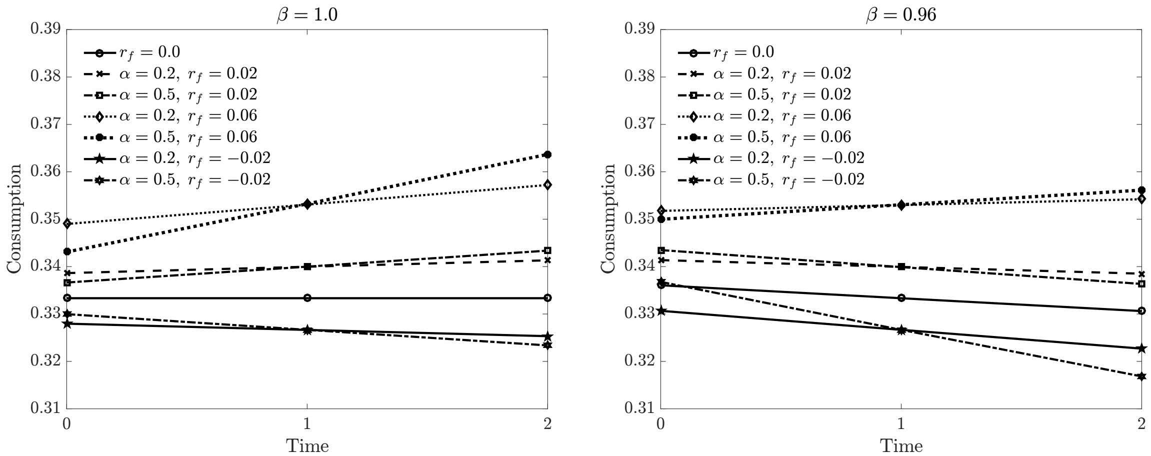


Figure 2.5 Optimal consumption path: two-period example. $r_1 = r_2 = r_f$.

n-period

Comparing the optimal consumption for the single-period and the two-period cases, it is immediate that the optimal solution for an n-period problem can be generalized into an intuitive analytical expression. Consider an n-period problem with consumption $\{C_0, C_1, ..., C_n\}$, wealth before consumption $\{W_0 = 1, W_1, ..., W_n\}$, and risk-free interest rate $\{r_1, r_2, ..., r_n\}$. The time t utility index, V_t, is given by:

$$V_t = \left\{ C_t^{1-\frac{1}{\alpha}} + \beta V_{t+1}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}}, \qquad \text{for } t = 0, ..., n-1$$
$$V_n = C_n.$$

It is immediate that we can obtain the following (by full consumption of wealth at time n and then solving the single-period problem with initial wealth W_{n-1}):

$$\begin{aligned} C_n^* &= W_n = (W_{n-1} - C_{n-1})(1 + r_n), \\ C_{n-1}^* &= \frac{W_{n-1}}{1 + \beta^\alpha (1 + r_n)^{\alpha - 1}}. \end{aligned} \tag{2.3}$$

By substitution, we can obtain the relationship between C_{n-1}^* and C_n^*:

$$\begin{aligned} C_n^* &= (W_{n-1} - C_{n-1}^*)(1 + r_n) \\ &= W_{n-1}(1 + r_n) \left(\frac{\beta^\alpha (1 + r_n)^{\alpha - 1}}{1 + \beta^\alpha (1 + r_n)^{\alpha - 1}} \right) \\ &= \beta^\alpha (1 + r_n)^\alpha C_{n-1}^*. \end{aligned} \tag{2.4}$$

Moving backward to $n-2$, we aim to maximize:

$$
\begin{aligned}
&\max_{C_{n-2}} V_{n-2} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \beta {C_{n-1}^{*}}^{1-\frac{1}{\alpha}} + \beta^2 {C_n^{*}}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \beta {C_{n-1}^{*}}^{1-\frac{1}{\alpha}} + \beta^2 [\beta^\alpha (1+r_n)^\alpha C_{n-1}^{*}]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \quad \text{by Equation (2.4)} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \left(\beta + \beta^2 [\beta^\alpha (1+r_n)^\alpha]^{1-\frac{1}{\alpha}}\right) {C_{n-1}^{*}}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \left(\beta + \beta^2 \left[\beta^{\alpha-1} (1+r_n)^{\alpha-1}\right]\right) \left(\frac{W_{n-1}}{1+\beta^\alpha (1+r_n)^{\alpha-1}}\right)^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \quad \text{by Equation (2.3)} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \left(\beta + \beta^{\alpha+1} (1+r_n)^{\alpha-1}\right) \left(\frac{(W_{n-2}-C_{n-2})(1+r_{n-1})}{1+\beta^\alpha (1+r_n)^{\alpha-1}}\right)^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} \\
&= \max_{C_{n-2}} \left\{ C_{n-2}^{1-\frac{1}{\alpha}} + \left(\frac{\beta + \beta^{\alpha+1} (1+r_n)^{\alpha-1}}{[1+\beta^\alpha (1+r_n)^{\alpha-1}]^{1-\frac{1}{\alpha}}}\right) [(W_{n-2}-C_{n-2})(1+r_{n-1})]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}} .
\end{aligned}
$$

Note that this takes a very similar form as Equation (2.2). We hence obtain that

$$
\begin{aligned}
C_{n-2}^{*} &= \frac{W_{n-2}}{1 + \left(\frac{\beta+\beta^{\alpha+1}(1+r_n)^{\alpha-1}}{[1+\beta^\alpha(1+r_n)^{\alpha-1}]^{1-\frac{1}{\alpha}}}\right)^{\alpha} (1+r_{n-1})^{\alpha-1}} \\
&= \frac{W_{n-2}}{1+\beta^\alpha (1+r_{n-1})^{\alpha-1} + \beta^{2\alpha} [(1+r_n)(1+r_{n-1})]^{\alpha-1}}. \qquad (2.5)
\end{aligned}
$$

Consequently, the relationship between C_{n-2}^{*} and C_{n-1}^{*} can be obtained by combining Equations (2.3) and (2.5):

$$
\begin{aligned}
C_{n-1}^{*} &= \frac{W_{n-1}}{1+\beta^\alpha (1+r_n)^{\alpha-1}} \\
&= \frac{(W_{n-2}-C_{n-2}^{*})(1+r_{n-1})}{1+\beta^\alpha (1+r_n)^{\alpha-1}} \\
&= \frac{\left(W_{n-2} - \frac{W_{n-2}}{1+\beta^\alpha(1+r_{n-1})^{\alpha-1}+\beta^2[(1+r_n)(1+r_{n-1})]^{\alpha-1}}\right)(1+r_{n-1})}{1+\beta^\alpha (1+r_n)^{\alpha-1}} \\
&= \ldots \\
&= \beta^\alpha (1+r_{n-1})^\alpha W_{n-2} \left(1+\beta^\alpha (1+r_{n-1})^{\alpha-1} + \beta^2 [(1+r_n)(1+r_{n-1})]^{\alpha-1}\right)^{-1} \\
&= \beta^\alpha (1+r_{n-1})^\alpha C_{n-2}^{*}. \qquad (2.6)
\end{aligned}
$$

In general, it can be shown that, for all $1 \leq k \leq n$,

$$
\begin{aligned}
C_k^* &= \beta^\alpha (1+r_k)^\alpha C_{k-1}^*, \\
C_{k-1}^* &= \frac{W_{k-1}}{1 + \sum_{i=1}^{n-k+1} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-1+j})^{\alpha-1}}.
\end{aligned}
$$

We prove these relationships by mathematical induction, as follows.

Proof. We have shown that the following hold true:

$$
C_n^* = \beta^\alpha (1+r_n)^\alpha C_{n-1}^*, \tag{2.7}
$$

$$
C_{n-1}^* = \beta^\alpha (1+r_{n-1})^\alpha C_{n-2}^*. \tag{2.8}
$$

And also,

$$
C_n^* = W_n, \tag{2.9}
$$

$$
C_{n-1}^* = \frac{W_{n-1}}{1 + \beta^\alpha (1+r_n)^{\alpha-1}}, \tag{2.10}
$$

$$
C_{n-2}^* = \frac{W_{n-2}}{1 + \beta^\alpha (1+r_{n-1})^{\alpha-1} + \beta^{2\alpha} \left[(1+r_n)(1+r_{n-1})\right]^{\alpha-1}}. \tag{2.11}
$$

Suppose that for some $1 < k < n$, the following hold true:

$$
C_l^* = \beta^\alpha (1+r_l)^\alpha C_{l-1}^*, \quad k \leq l \leq n \tag{2.12}
$$

$$
C_{k-1}^* = \frac{W_{k-1}}{1 + \sum_{i=1}^{n-k+1} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-1+j})^{\alpha-1}} \tag{2.13}
$$

We need to show that,

$$
C_{k-1}^* = \beta^\alpha (1+r_{k-1})^\alpha C_{k-2}^*, \tag{2.14}
$$

$$
C_{k-2}^* = \frac{W_{k-2}}{1 + \sum_{i=1}^{n-k+2} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-2+j})^{\alpha-1}}. \tag{2.15}
$$

We begin by expressing the optimization problem at time $k-2$,

$$
\begin{aligned}
\max_{C_{k-2}} V_{k-2} &= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + \beta \left[\max_{\{C_{k-1},\ldots,C_n\}} V_{k-1} \right]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
&= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + \beta {C_{k-1}^{*}}^{1-\frac{1}{\alpha}} + \beta^2 {C_k^{*}}^{1-\frac{1}{\alpha}} + \ldots + \beta^{n-k+2} {C_n^{*}}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
&= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + \beta {C_{k-1}^{*}}^{1-\frac{1}{\alpha}} + \beta^2 (\beta^{\alpha}(1+r_k)^{\alpha} C_{k-1}^{*})^{1-\frac{1}{\alpha}} + \ldots \right. \\
&\qquad \left. \ldots + \beta^{n-k+2} \left(\beta^{(n-k+1)\alpha} \prod_{j=1}^{n-k+1} (1+r_{k-j})^{\alpha} C_{k-1}^{*} \right)^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
&= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + {C_{k-1}^{*}}^{1-\frac{1}{\alpha}} \beta \left(1 + \sum_{i=1}^{n-k+1} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-j})^{\alpha-1} \right) \right\}^{\frac{1}{1-\frac{1}{\alpha}}}
\end{aligned}
$$

Let $A = 1 + \sum_{i=1}^{n-k+1} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-j})^{\alpha-1}$. Substituting Equation (2.13), we obtain,

$$
\begin{aligned}
\max_{C_{k-2}} V_{k-2} &= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + \beta A \left(\frac{W_{k-1}}{A} \right)^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
&= \max_{C_{k-2}} \left\{ C_{k-2}^{1-\frac{1}{\alpha}} + \beta A^{\frac{1}{\alpha}} \left[(W_{k-2} - C_{k-2})(1+r_{k-1}) \right]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
\Rightarrow C_{k-2}^{*} &= \frac{W_{k-2}}{1 + \beta^{\alpha} A (1+r_{k-1})^{\alpha-1}} \\
&= \frac{W_{k-2}}{1 + \beta^{\alpha}(1+r_{k-1})^{\alpha-1}(1 + \sum_{i=1}^{n-k+1} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-j})^{\alpha-1})} \\
&= \frac{W_{k-2}}{1 + \sum_{i=1}^{n-k+2} \beta^{i\alpha} \prod_{j=1}^{i} (1+r_{k-2+j})^{\alpha-1}}. \qquad (2.16)
\end{aligned}
$$

Hence, we have shown that Equation (2.15) holds. To prove Equation (2.14), we take

$$
\begin{aligned}
C_{k-1}^{*}\beta^{-\alpha}(1+r_{k-1})^{-\alpha} &= \frac{W_{k-1}\beta^{-\alpha}(1+r_{k-1})^{-\alpha}}{1+\sum_{i=1}^{n-k+1}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-1+j})^{\alpha-1}} \\
&= \frac{(W_{k-2}-C_{k-2}^{*})(1+r_{k-1})\beta^{-\alpha}(1+r_{k-1})^{-\alpha}}{1+\sum_{i=1}^{n-k+1}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-1+j})^{\alpha-1}} \\
&= \frac{\left(W_{k-2}-\frac{W_{k-2}}{1+\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}}\right)(1+r_{k-1})\beta^{-\alpha}(1+r_{k-1})^{-\alpha}}{1+\sum_{i=1}^{n-k+1}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-1+j})^{\alpha-1}} \\
&= \frac{W_{k-2}\left(\frac{\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}}{1+\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}}\right)}{\beta^{\alpha}(1+r_{k-1})^{\alpha-1}+\sum_{i=1}^{n-k+1}\beta^{(i+1)\alpha}\prod_{j=1}^{i}(1+r_{k-1+j})^{\alpha-1}(1+r_{k-1})^{\alpha-1}} \\
&= \frac{W_{k-2}\left(\frac{\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}}{1+\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}}\right)}{\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}} \\
&= \frac{W_{k-2}}{1+\sum_{i=1}^{n-k+2}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-2+j})^{\alpha-1}} \\
&= C_{k-2}^{*}
\end{aligned}
$$

Hence, (2.14) is shown. By mathematical induction, it must be true that for all $1 \leq k \leq n$,

$$
\begin{aligned}
C_k^{*} &= \beta^{\alpha}(1+r_k)^{\alpha}C_{k-1}^{*}, \\
C_{k-1}^{*} &= \frac{W_{k-1}}{1+\sum_{i=1}^{n-k+1}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{k-1+j})^{\alpha-1}}.
\end{aligned}
$$

□

The optimal consumption solution for the n-period problem is thus summarized by:

$$C_0^{*} = \frac{1}{1+\sum_{i=1}^{n}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_j)^{\alpha-1}}, \tag{2.17}$$

$$C_t^{*} = \beta^{\alpha}(1+r_t)^{\alpha}C_{t-1}^{*}, \qquad t=1,...,n. \tag{2.18}$$

It can be additionally deduced that the optimal savings can be expressed as follows:

$$W_0 - C_0^{*} = \frac{\sum_{i=1}^{n}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_j)^{\alpha-1}}{1+\sum_{i=1}^{n}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_j)^{\alpha-1}}, \tag{2.19}$$

$$W_t - C_t^{*} = \frac{(1+r_t)\sum_{i=1}^{n-t}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{t+j})^{\alpha-1}}{1+\sum_{i=1}^{n-t}\beta^{i\alpha}\prod_{j=1}^{i}(1+r_{t+j})^{\alpha-1}}\left(W_{t-1}-C_{t-1}^{*}\right). \tag{2.20}$$

The impact of α on the consumption path is shown in Figures 2.6, 2.7 and 2.8, where the volatility in consumption is expressed as a function of α, for short and long period examples. Here, consumption volatility is computed by taking the sample standard deviation, namely $\sqrt{\frac{\sum_{t=0}^{n}(C_t^* - \overline{C^*})^2}{n}}$. It is shown that in general, consumption volatility is positively related to the value of α. Hence, a lower parameter α leads to a stronger preference for consumption smoothing over time.

Figures 2.9, 2.10, and 2.11 plot the optimal consumption paths for $n = 45$ for varying interest rate assumptions. We choose 45 to illustrate the consumption profile for a life aged 65 and living to age 110. It is apparent that in all three cases $\alpha = 0.02$ gives a smoother consumption path, in the sense that consumption is least volatile.

In the case of constant interest rates, we have the following observations:

i. When $\beta(1+r_t) > 1$, consumption rises over time (as per Equation (2.18)). To consume more in the present is perceived[2] as more "costly", in the sense that this effectively reduces the perceived aggregate consumption over the lifetime. In this case, a higher α leads to a lower consumption in the present due to a higher willingness to substitute (i.e. replacing the more expensive good/choice with a less expense alternative).

ii. When $\beta(1 + r_t) < 1$, consumption declines over time (as per Equation (2.18)). To consume less in the present is perceived as more "costly", since perceived returns on savings are negative. In this case, a higher α leads to a higher consumption in the present.

iii. When $\beta(1 + r_t) = 1$, consumption stays constant and maximum smoothing is attained regardless of the value of α. In Figure 2.6 where $\beta = 0.96$, $r_t = 0.04$, and $\beta(1+r_t) \approx 1$, α has a negligible impact on volatility.

In the case of changing interest rates, we simulate a sequence of interest rates from $U(-0.04, 0.06)$ of size 45 (plotted in Figure 2.13). Note that this is treated as a deterministic sequence of returns. Similar observations are made as with constant interest rates, only that the overall direction of the consumption path depends on the combined effect of $\beta(1 + r_t)$, $\forall t$.

Figure 2.12 plots the optimal savings in the case of $r_1 = ... = r_{45} = 0.02$. The main take away is that a lower α does not always imply lower savings when interest rates are positive. Savings again depend on the combined effect of β and r_t.

[2] The term 'perceived' is used due to the presence of subjective discounting β. When $\beta = 1$, the perceived aggregate consumption over lifetime equals the actual aggregate consumption.

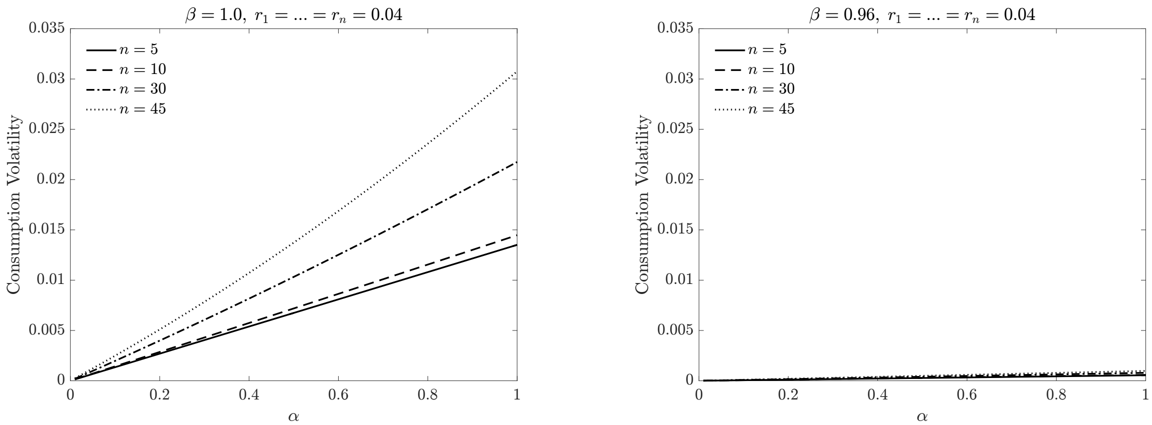


Figure 2.6 Consumption volatility: $r_1 = ... = r_{45} = 0.04$.

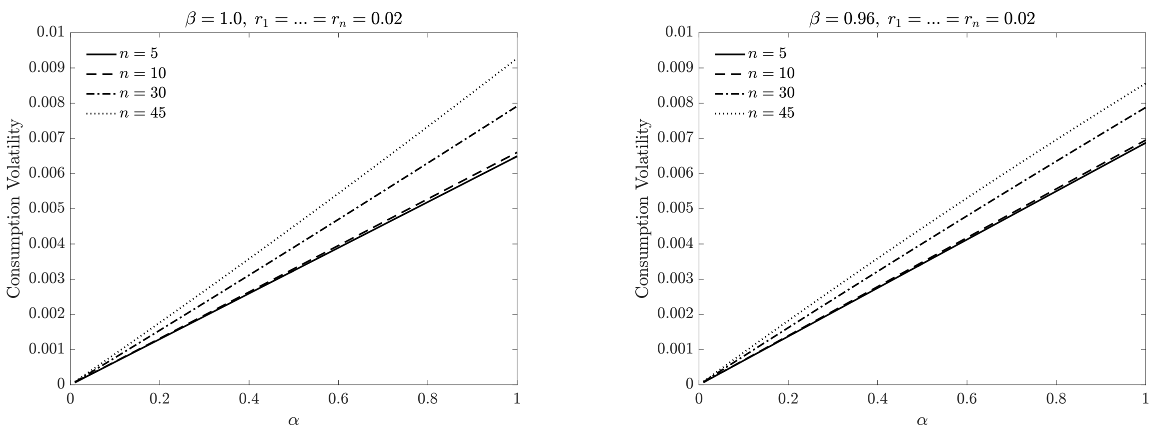


Figure 2.7 Consumption volatility: $r_1 = ... = r_{45} = 0.02$.

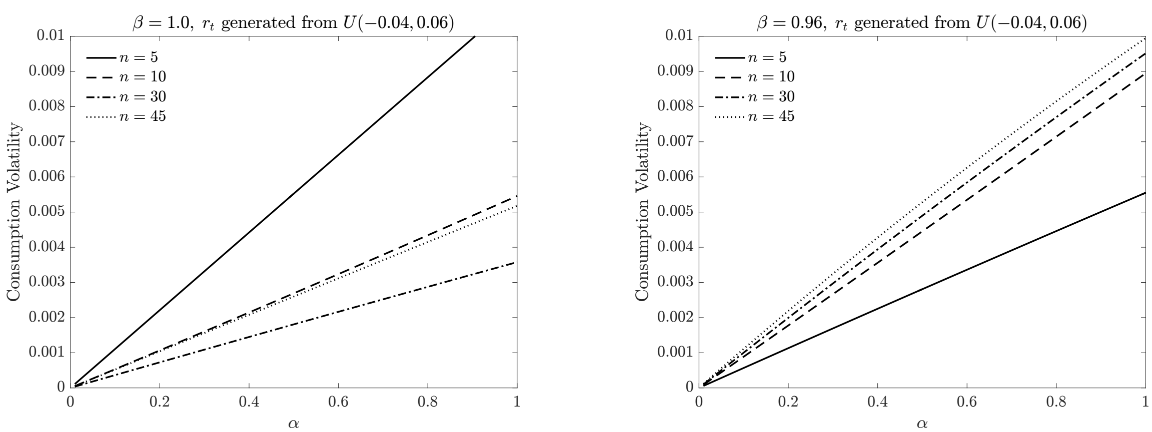


Figure 2.8 Consumption volatility: r_t generated from $\mathrm{U}(-0.04, 0.06)$.

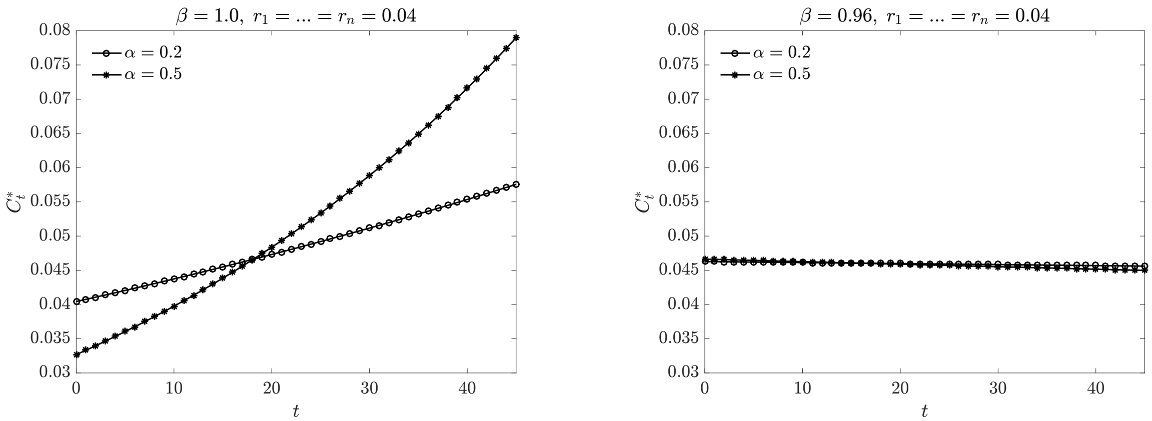


Figure 2.9 Consumption: 45-period example. $r_1 = ... = r_{45} = 0.04$.

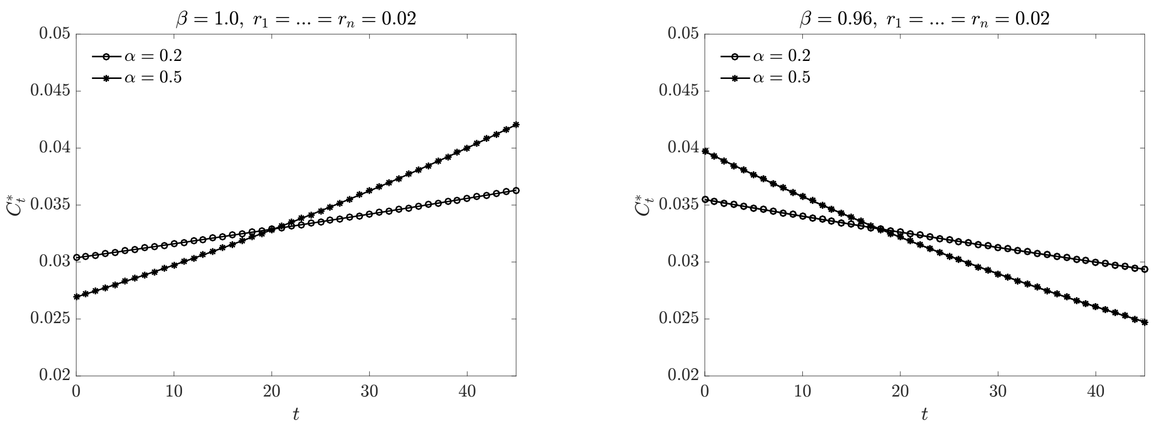


Figure 2.10 Consumption: 45-period example. $r_1 = ... = r_{45} = 0.02$.

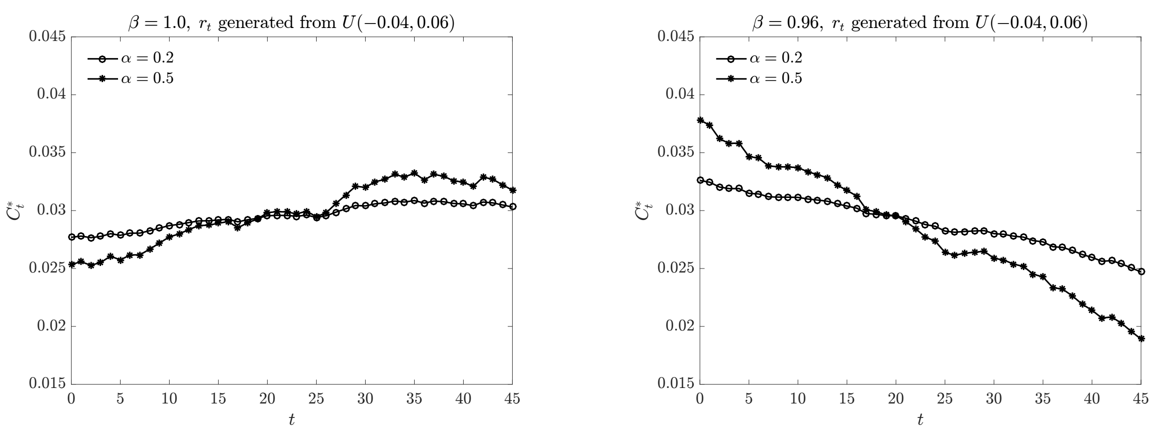


Figure 2.11 Consumption: 45-period example. r_t generated from $\mathrm{U}(-0.04, 0.06)$.

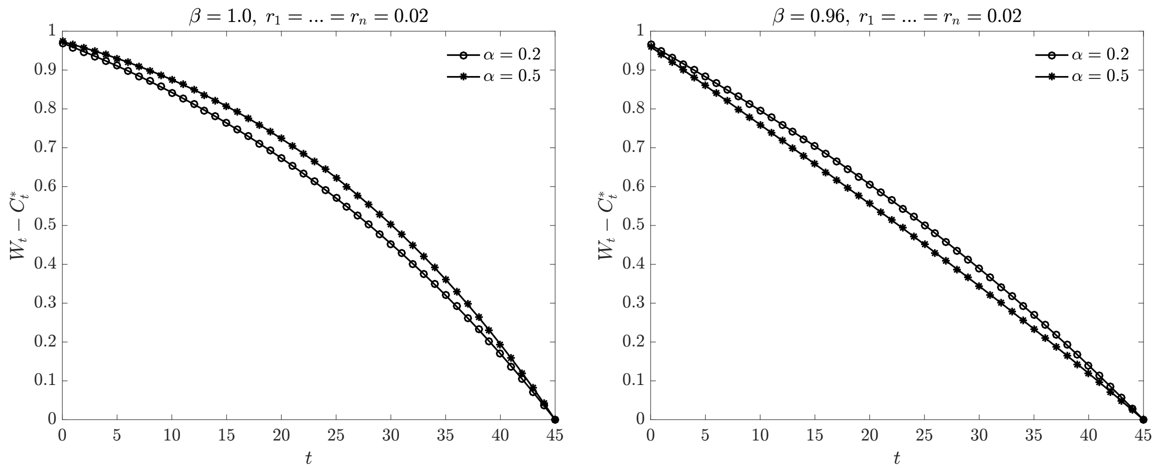


Figure 2.12 Savings: 45-period example. $r_1 = ...r_{45} = 0.02$.

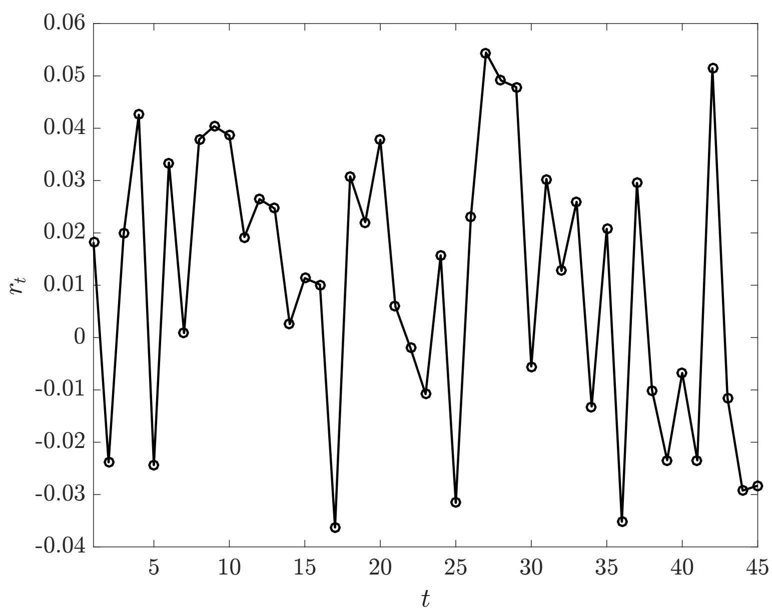


Figure 2.13 Interest rates r_t: one set of scenarios generated from $\mathrm{U}(-0.04, 0.06)$.

2.4.2 Risky investment

The problem can be further generalized by stochastic interest rates or returns on investment. Consider again an n-period problem with stochastic interest rates $\{R_1, R_2, ..., R_n\}$. Note that when interest rates are random variables, at any time t, for all $u > t$, C_u and W_u are random variables (and by extension so is V_u). For simplicity, we assume that interest rates are independent over time. We do not, however, require them to be identically distributed. The time t utility index, V_t, is given by:

$$\begin{aligned} V_t &= \left\{ C_t^{1-\frac{1}{\alpha}} + \beta \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right)^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}}, \qquad \text{for } t = 0, ..., n-1, \\ V_n &= C_n. \end{aligned}$$

The optimal consumption solution takes the following form,

$$\begin{aligned} C_0^* &= \frac{1}{1 + \sum_{i=1}^n \beta^{i\alpha} \prod_{j=1}^i (1+\mu_j)^{\alpha-1}}, \\ C_t^* &= \beta^\alpha (1+\mu_t)^{\alpha-1}(1+R_t)C_{t-1}^*, \qquad\qquad t = 1, ..., n, \\ \mu_t &= \left[\mathbb{E}_t(1+R_t)^{1-\rho}\right]^{\frac{1}{1-\rho}} - 1. \end{aligned}$$

The proof of this result follows analogously from the risk-free case, hence it is omitted. For an agent with RRA ρ, $1 + \mu_t$ is the certainty equivalent of the return $1 + R_t$. The higher the RRA, the lower the certainty equivalent. Thus, solutions involving investment risk can be interpreted similarly to a risk-free problem. This result can be further generalized to a time-evolving RRA, where $\mu_t = \left[\mathbb{E}_t(1+R_t)^{1-\rho_t}\right]^{\frac{1}{1-\rho_t}} - 1$ and ρ_t is the RRA for period $[t-1, t)$. This extension only holds true if the function ρ_t is known to the agent at time 0, which ensures that preferences are time-consistent (hence Bellman's principle of optimality applies).

2.4.3 Risky inflation

We can further incorporate a risky inflation process, since it is more reasonable that utility be derived from real consumption, rather than the nominal level. Consider again an n-period problem with stochastic interest rates $\{R_1, R_2, ..., R_n\}$, and let $\{\delta_1, \delta_2, ..., \delta_n\}$ be the annual inflation rates that are independently but not necessarily identically distributed. Note that R_t and δ_t need not be independent. Let $\{I_0, I_1, ..., I_n\}$ denote the cumulative inflation process such that,

$$I_0 = 1; \quad I_t = \prod_{s=1}^t (1+\delta_t), \quad 1 \le t \le n,$$

Thus, we can define the real consumption and wealth before consumption to be, respectively,

$$\tilde{C}_t = \frac{C_t}{I_t}, \quad \tilde{W}_t = \frac{W_t}{I_t}, \quad 0 \le t \le n.$$

The time t utility index based on real consumption, V_t, is given by:

$$\begin{aligned} V_t &= \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right)^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}}, \qquad \text{for } \ t = 0, ..., n-1, \\ V_n &= \tilde{C}_n. \end{aligned}$$

Note that when $\delta_1 = \delta_2 = ... = \delta_n = 0$, this reduces to the zero inflation problem discussed in the previous section. The optimal consumption solution is thus,

$$\begin{aligned} \tilde{C}_0^* &= \frac{1}{1 + \sum_{i=1}^n \beta^{i\alpha} \prod_{j=1}^i (1 + \tilde{\mu}_j)^{\alpha-1}}, \\ \tilde{C}_t^* &= \beta^\alpha (1 + \tilde{\mu}_t)^{\alpha-1} \left(\frac{1 + R_t}{1 + \delta_t} \right) \tilde{C}_{t-1}^*, \qquad t = 1, ..., n, \\ \tilde{\mu}_t &= \left[\mathbb{E}_t \left(\frac{1 + R_t}{1 + \delta_t} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}} - 1. \end{aligned}$$

It is immediate that $\frac{1+R_t}{1+\delta_t}$ yields the real rate of return on investment for the period $[t-1, t)$. It may be convenient to simply define a real return process. However, under such an assumption all values are expressed in real terms. This would restrict the investment opportunities or financial products that are made available to the agent. For example, a nominally fixed life annuity is not properly defined without an inflation process.

2.5 Uncertain lifetime

So far, we have worked under the assumption of a certain lifetime. For life-cycle modelling purposes, such a stringent assumption is highly undesirable. It is natural that the n-period deterministic case be extended to include an uncertain future lifetime.

With all else being equal, suppose that instead of being n, the remaining lifetime of an agent aged x at time t is a continuous random variable, denoted by $T_{x,t}$. The survival probability of this agent at time t, for at least u years, is denoted by

$${}_u p_{x,t} = \prod_{s=1}^{u} (1 - q_{x+s-1,t+s-1}).$$

where $q_{x+s-1,t+s-1}$ is the mortality rate of a life aged $x+s-1$ at time $t+s-1$. Note that $q_{\omega-1} = 1$ where $\omega \in Z_+$ is the terminal age, and that there is no uncertainty associated with the mortality rates.

The original work of Epstein and Zin (1989) defines a class of recursive preference models for an infinitely lived agent. To date, there is no general agreement on how an uncertain lifetime can be incorporated into these specifications. In this section, we examine three preference specifications that incorporate uncertain future lifetimes. In subsection 2.5.1, we first work in the absence of consumption risks (contingent on survival) to isolate the effect of mortality risk. We refer to this case as "risk-free". We then present the general solutions, in subsection 2.5.2, where consumption risks are included.

2.5.1 Risk-free

Córdoba and Ripoll (2016)

Suppose the agent has age x at time 0, and recall that ρ denotes relative risk aversion. The following formulation was proposed by Córdoba and Ripoll (2016),

$$V_t = \left\{ C_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} V_{t+1}^{1-\rho} + (1-p_{x+t,t}) \underline{V}^{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}}, \qquad t = 0, ..., \omega - x - 1, \tag{2.21}$$

where $\underline{V} \geq 0$ is the perceived utility in the death state. Mechanically fixing $\underline{V}^{1-\rho} = 0$, it ultimately yields that,

$$V_t = \left\{ C_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} V_{t+1}^{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}}, \tag{2.22}$$

$$= \left\{ C_t^{1-\frac{1}{\alpha}} + \beta (p_{x+t,t})^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \left[V_{t+1} \right]^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}}. \tag{2.23}$$

The optimal consumption, **contingent on survival**, is therefore (steps omitted due to simplicity),

$$C_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_ip_{x,0})^{\frac{\alpha-1}{1-\rho}} \prod_{j=1}^{i} (1+r_j)^{\alpha-1}}, \tag{2.24}$$

$$C_t^* = \left[\beta \left(p_{x+t-1,t-1} \right)^{\frac{1-1/\alpha}{1-\rho}} \right]^{\alpha} (1+r_t)^{\alpha} C_{t-1}^*, \qquad t = 1, ..., \omega - x - 1, \tag{2.25}$$

The impact is that $\beta \left(p_{x+t-1,t-1} \right)^{\frac{1-1/\alpha}{1-\rho}}$ constitutes the new impatience factor for the agent, which we denote as $\beta^\star$. Notice that $0 \leq \beta^\star \leq 1$ only if $1 - \frac{1}{\alpha}$ and $1 - \rho$ are of the same sign. This means if one is not careful, certain parameter combinations can result in $\beta^\star \gg 1$, especially when mortality rates are high. In the case of CRRA where $\alpha = \frac{1}{\rho}$, this cannot happen. As such, this preference formulation can potentially produce strange consumption paths, where mortality risk reduces impatience, rather than adding to it as it intuitively should. A simple example that illustrates this counterintuitive behaviour is that when $\alpha = 0.2$, $\rho = 0.5$ and $p_{x,t} = 0.01$, meaning that survival to the next period is extremely unlikely, the impact of mortality on discounting is 1×10^{16}, which effectively yields an extremely small consumption in the present, whereas intuition dictates that the agent should consume more today since no utility is derived upon death, which is an extremely likely event in the immediate future. In an extreme case when $p_{x,t} = 0$, $\beta^\star$ can take the value ∞ which implies that $V_t = 0$ for all C_t, which contradicts the logical unique solution that $C_t^* = W_t$. These observations suggest that such a formulation may not be ideal for an application in retirement planning, which involves modelling the behaviour of agents with rising mortality rates. If used, one should be careful in choosing the parameters governing RRA and EIS such that mortality does not reduce impatience. For commonly adopted parameters, this translates to cases where $\rho > 1$, $0 < \alpha < 1$, and $\rho < \frac{1}{\alpha}$.

Bommier et al. (2017) provides a more in-depth critique of this preference model (including the above observations), which is beyond the scope of this thesis. Instead, we proceed to investigate numerically the impact of α on the optimal consumption path, within the set of parameters that are relevant to our studies. We are particularly interested in the cases where $\rho > 1$ and $0 < \alpha < 1$, so that $\beta^\star$ cannot be greater than 1 (resulting in peculiar consumption profiles). A full-blown investigation is of interest, but beyond the scope of this thesis. It suffices to point out the main finding that under this preference formulation, **a smaller α does not necessarily lead to higher smoothness in consumption contingent on survival**. This is due to the distortion produced by the mortality component of $\beta^\star$, which acts against consumption substitution: for $0 < \alpha < 1$, a smaller α leads to higher impatience (i.e. a smaller $\beta^\star$), which leads to higher consumption in the present; however, a smaller α also means the agent is less willing to give up consumption today for consumption in the future. The net effect of the two opposing forces yields the overall consumption smoothness, which is no longer monotone in α. This gives rise to an important message: under this preference formulation, it may be difficult to evaluate the impact of the parameter α on the optimal solutions, since its net effect on intertemporal substitutability and smoothing is ambiguous. While it is meaningful to separate the parameterization of RRA and EIS, users of this preference model should be aware of the non-monotone net effect of EIS on consumption substitution in the presence of mortality, in order to accurately interpret the optimal solutions.

Below, we illustrate this finding numerically and graphically (Figure 2.14), assuming mortality[3] follows the Cairns et al. (2006) model described in Appendix C.1.

[3] Mortality rates are deterministic over time, computed based on the median best estimates.

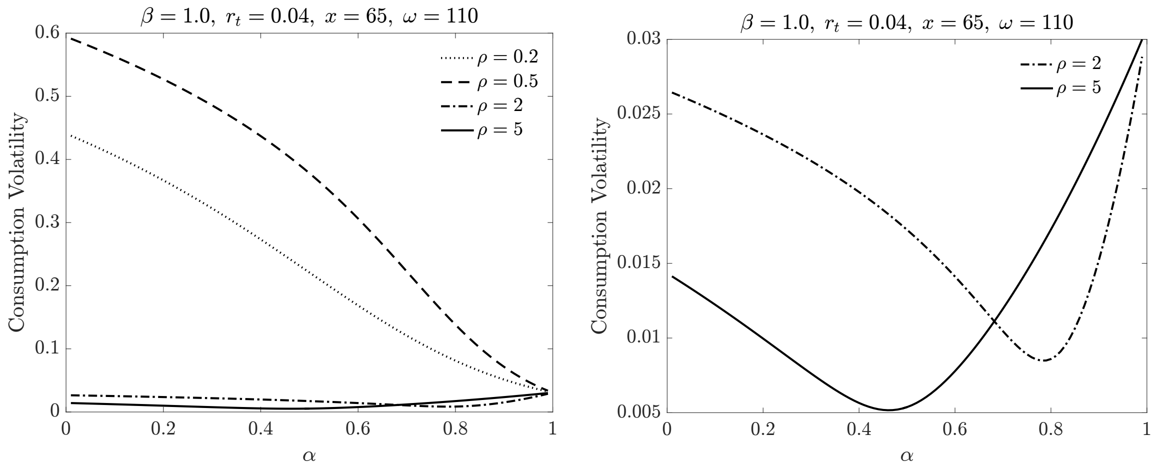


Figure 2.14 Consumption volatility: Córdoba and Ripoll (2016).

Gomes and Michaelides (2005)

A commonly adopted recursive preference model in studies of life-cycle investment takes the following form, which was to our best knowledge first described in Gomes and Michaelides (2005). This preference model is adopted in many other studies in similar contexts, including Horneff et al. (2015), Horneff et al. (2009), Horneff et al. (2008b), Horneff et al. (2008c) and Blake et al. (2008). For $t = 0, ..., \omega - x - 1$, we have

$$V_t = \left\{ (1 - \beta p_{x+t,t}) C_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} V_{t+1}^{1-\rho} + (1 - p_{x+t,t}) b \frac{(W_{t+1}/b)^{1-\rho}}{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}}, \tag{2.26}$$

where b governs the strength of the bequest motive. Note that this is a very similar formulation to Córdoba and Ripoll (2016), except that utility in the death state is defined as the utility derived from bequesting the remaining wealth, and that current consumption is being scaled by a time-dependent factor between 0 and 1. Considering a zero bequest motive and the transformation $V_t^\star = \frac{V_t}{(1-\beta p_{x+t,t})^{\frac{1}{1-1/\alpha}}}$, the form is equivalent to (due to intertemporal homotheticity),

$$V_t^\star = \left\{ C_t^{1-\frac{1}{\alpha}} + \beta \left(\frac{1 - \beta p_{x+t+1,t+1}}{1 - \beta p_{x+t,t}} \right) (p_{x+t,t})^{\frac{1-\frac{1}{\alpha}}{1-\rho}} {V_{t+1}^\star}^{1-\frac{1}{\alpha}} \right\}^{\frac{1}{1-1/\alpha}}. \tag{2.27}$$

The optimal consumption, contingent on survival, is therefore (steps omitted due to simplicity),

$$C_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_ip_{x,0})^{\frac{\alpha-1}{1-\rho}} \prod_{j=1}^{i} \left(\frac{1-\beta p_{x+j,j}}{1-\beta p_{x+j-1,j-1}}\right)^{\alpha} (1+r_j)^{\alpha-1}}, \tag{2.28}$$

$$C_t^* = \left[\beta \left(\frac{1-\beta p_{x+t,t}}{1-\beta p_{x+t-1,t-1}}\right) (p_{x+t-1,t-1})^{\frac{1-1/\alpha}{1-\rho}}\right]^{\alpha} (1+r_t)^{\alpha} C_{t-1}^*, \; t = 1, ..., \omega - x - 1. \tag{2.29}$$

Unlike Córdoba and Ripoll (2016), the net impatience factor $\beta^\star = \beta \left(\frac{1-\beta p_{x+t+1,t+1}}{1-\beta p_{x+t,t}}\right) (p_{x+t,t})^{\frac{1-\frac{1}{\alpha}}{1-\rho}}$ can be greater than 1 even if $1 - \frac{1}{\alpha}$ and $1 - \rho$ are of the same sign, since $p_{x+t+1,t+1} < p_{x+t,t}$. This also implies that mortality can result in lower[4] impatience, which as discussed above is counterintuitive.

Moreover, when $\rho = \frac{1}{\alpha}$, maximizing the time 0 utility index is not necessarily equivalent to maximizing the time-additive EDLU-CRRA with exponential discounting. Setting $\alpha = \frac{1}{\rho}$, to maximize $V_0^\star$ from Equation (2.27) is equivalent to:

$$\max_{\{C_0,...,C_{\omega-x-1}\}} \left[\sum_{t=0}^{\omega-x-1} \left(\frac{1-\beta p_{x+t,t}}{1-\beta p_{x,0}}\right) \beta^t {}_tp_{x,0} \left(\frac{C_t^{1-\rho}}{1-\rho}\right)\right],$$

which is not necessarily[5] equivalent to $\max_{\{C_0,...,C_{\omega-x-1}\}} \left[\sum_{t=0}^{\omega-x-1} \beta^t {}_tp_{x,0} \left(\frac{C_t^{1-\rho}}{1-\rho}\right)\right]$ due to the extra factor $\left(\frac{1-\beta p_{x+t,t}}{1-\beta p_{x,0}}\right)$. Hence, it is inaccurate to suggest that freeing the parameter α is to move away from the framework of EDLU-CRRA, since maximizing the two expressions does not necessarily give the same optimal solutions.

Figure 2.15 plots the consumption volatility as a function of α, which results in a similar behaviour as the specification described in Equation (2.21), since the distortion of the survival rate persists. Comparing the optimal consumption paths (Figure 2.16), the two preferences lead to rather different consumption behaviour. Regardless, both specifications lose the interpretation of α as a parameter governing consumption smoothing.

Li and Smetters (2010)

The following preference model is considered in Li and Smetters (2010) and Blake et al. (2014), in which mortality acts as a source of impatience:

$$V_t = \left\{C_t^{1-\frac{1}{\alpha}} + \beta p_{x+t,t} V_{t+1}^{1-\frac{1}{\alpha}}\right\}^{\frac{1}{1-1/\alpha}}, \;\; t = 0, ..., \omega - x - 1. \tag{2.30}$$

[4]For example, consider $\rho = 5$, $\alpha = 0.6$, $\beta = 0.96$, $p_{65,0} = 0.9862$, and $p_{66,1} = 0.9851$. This yields $\beta^\star = 0.9769 > 0.96$.

[5]The equivalence is established when mortality is described by a constant force model, which is rarely used as a realistic mortality model due to its strong assumptions.

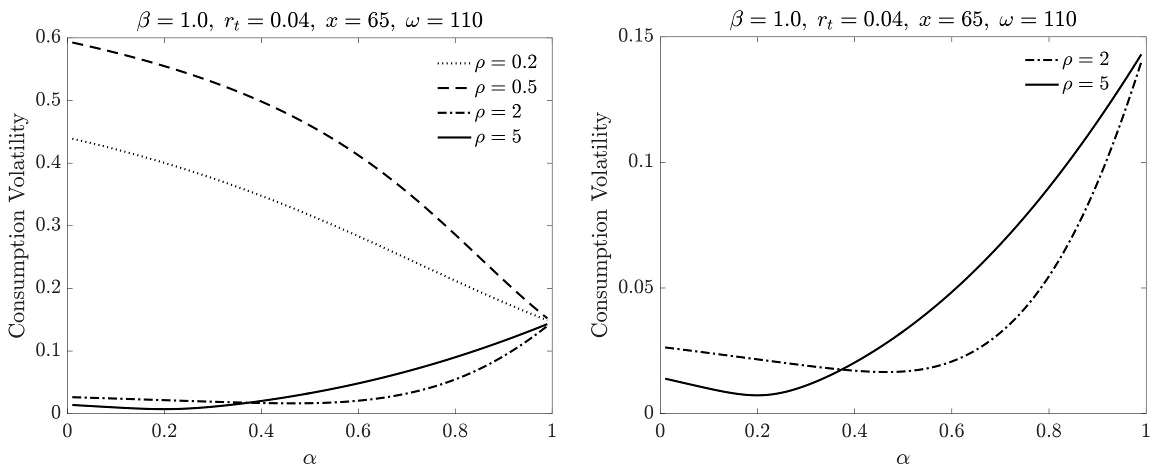


Figure 2.15 Consumption volatility: Gomes and Michaelides (2005)

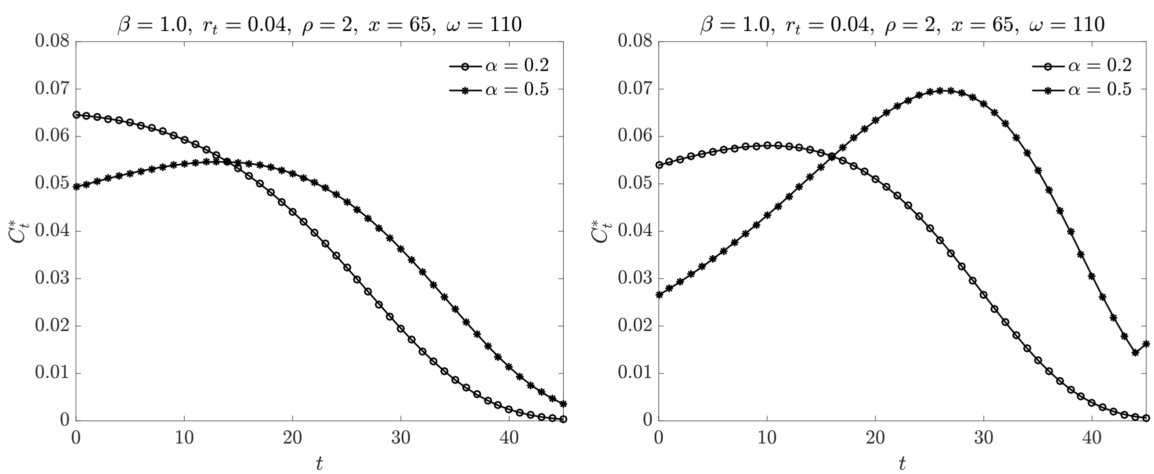


Figure 2.16 Consumption: Left: Córdoba and Ripoll (2016); Right: Gomes and Michaelides (2005).

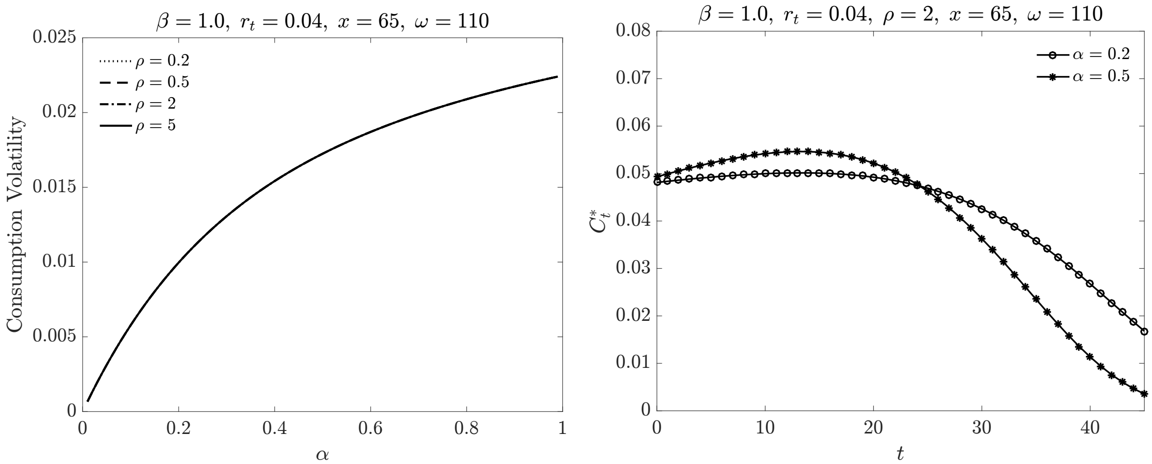


Figure 2.17 Consumption volatility and consumption: Li and Smetters (2010).

The optimal consumption solution is therefore,

$$C_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_ip_{x,0})^\alpha \prod_{j=1}^{i} (1+r_j)^{\alpha-1}},$$
$$C_t^* = \beta^\alpha \left(p_{x+t-1,t-1}\right)^\alpha (1+r_t)^\alpha C_{t-1}^*, \qquad t = 1, ...,, \omega - x - 1.$$

This circumvents the need to define a utility index in the state of death, which is beyond the scope of this thesis. This specification assumes that mortality expectation, combined with β, gives the overall time impatience. It has several desirable properties:

i. When survival to the next period has probability zero, a unique consumption solution is given at $C_t^* = W_t$, which is logically sound.

ii. The new impatience factor $\beta p_{x+t,t}$ is in $[0, 1]$, and will not be inflated to produce unrealistic consumption paths due to the unconstrained relationship between α and ρ.

iii. The interpretation of the parameter α is made clear, which is the preference on consumption volatility over time. Hence, its sole impact on the optimal consumption profile is straightforward to interpret.

An implication of this specification is that mortality increases impatience, and this impact is not related to the degree of risk aversion. Specifically, in a world without financial risks and fixing EIS, all agents behave identically. This effectively means that agents are risk neutral towards mortality risks and risk averse towards financial risks, which implies that mortality and consumption risk aversions are parameterized differently. This is a potential flaw, if one wishes to study the varying mortality risk aversion, or if capturing mortality risk aversion is of particular importance to the investigation.

2.5.2 General solutions

Below, we present the general solution to an optimal intertemporal consumption problem with unit wealth at time 0, assuming that the agent is age x at time 0, and faces investment risk, inflation risk, and mortality risk. Assuming no utility is derived from the death state, the following Epstein-Zin specifications are considered. For $t = 0, ..., \omega - x - 1$, we have

- S1: Córdoba and Ripoll (2016)

$$V_t = \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{2.31}$$

- S2: Gomes and Michaelides (2005)

$$V_t = \left\{ (1 - \beta p_{x+t,t}) \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{2.32}$$

- S3: Li and Smetters (2010)

$$V_t = \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta p_{x+t,t} \left[\mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{2.33}$$

Note that the $\mathbb{E}_t[\cdot]$ is the conditional expectation at time t, taken on consumption risks upon survival, since mortality risk is treated separately. To obtain the optimal consumption profile, we seek to maximize V_0 with respect to $\{\tilde{C}_0, \tilde{C}_1, ..., \tilde{C}_{\omega-x-1}\}$, which yields the following,

- S1: Córdoba and Ripoll (2016)

$$\tilde{C}_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_i p_{x,0})^{\frac{\alpha-1}{1-\rho}} \prod_{j=1}^{i} (1 + \tilde{\mu}_j)^{\alpha-1}}, \tag{2.34}$$

$$\tilde{C}_t^* = \left[\beta \left(p_{x+t-1,t-1} \right)^{\frac{1-1/\alpha}{1-\rho}} \right]^{\alpha} (1 + \tilde{\mu}_t)^{\alpha-1} \left(\frac{1 + R_t}{1 + \delta_t} \right) \tilde{C}_{t-1}^*, \tag{2.35}$$

$$\tilde{\mu}_t = \left[\mathbb{E}_t \left(\frac{1 + R_t}{1 + \delta_t} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}} - 1. \tag{2.36}$$

- S2: Gomes and Michaelides (2005)

$$\tilde{C}_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_ip_{x,0})^{\frac{\alpha-1}{1-\rho}} \prod_{j=1}^{i} \left(\frac{1-\beta p_{x+j,j}}{1-\beta p_{x+j-1,j-1}}\right)^{\alpha} (1+\tilde{\mu}_j)^{\alpha-1}}, \tag{2.37}$$

$$\tilde{C}_t^* = \left[\beta \left(\frac{1-\beta p_{x+t,t}}{1-\beta p_{x+t-1,t-1}}\right) (p_{x+t-1,t-1})^{\frac{1-1/\alpha}{1-\rho}}\right]^{\alpha} (1+\tilde{\mu}_t)^{\alpha-1} \left(\frac{1+R_t}{1+\delta_t}\right) \tilde{C}_{t-1}^*, \tag{2.38}$$

$$\tilde{\mu}_t = \left[\mathbb{E}_t \left(\frac{1+R_t}{1+\delta_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} - 1. \tag{2.39}$$

- S3: Li and Smetters (2010)

$$\tilde{C}_0^* = \frac{1}{1 + \sum_{i=1}^{\omega-x-1} \beta^{i\alpha} ({}_ip_{x,0})^{\alpha} \prod_{j=1}^{i} (1+\tilde{\mu}_j)^{\alpha-1}},$$

$$\tilde{C}_t^* = \beta^{\alpha} (p_{x+t-1,t-1})^{\alpha} (1+\tilde{\mu}_t)^{\alpha-1} \left(\frac{1+R_t}{1+\delta_t}\right) \tilde{C}_{t-1}^*,$$

$$\tilde{\mu}_t = \left[\mathbb{E}_t \left(\frac{1+R_t}{1+\delta_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} - 1,$$

for $t = 1, ..., , \omega - x - 1$. As discussed, this can be further generalized to consider a time-changing RRA, where ρ is replaced with $\rho_t > 0$. The proof follows analogously from the risk-free case described in Section 2.4.1, and is hence omitted.

2.6 Conclusion

In this chapter, we provide an introductory discussion of Epstein-Zin preferences, which are cases of the recursive specification of Epstein and Zin (1989), under the CES aggregator and expected-utility certainty equivalent. The Epstein-Zin preference model is more flexible than the time-additive EDLU models in that it allows for the separation of risk and time preferences, which is appealing given the intricacy and the long-term nature of optimal retirement planning.

The purpose of this chapter is to first investigate the implied behaviour for the representative agent and its plausibility. This is especially important since the study is of a normative nature, that is, to *prescribe* rational retirement planning strategies. In this investigation, we pay particular attention to whether the output can be interpreted, that is, whether it leads to plausible behaviour given the context and expert knowledge. This allows sanity checks to be performed in order to verify the validity and convergence of a more involved numerical procedure. Specifically, we derive analytical solutions for a simple optimal consumption problem for Epstein-Zin agents. We isolate the impact of relative risk aversion, elasticity of intertemporal substitution, time discounting, mortality risk, investment risk, and inflation risk on the optimal consumption profile. We find that in the absence of mortality risk,

i. RRA impacts the certainty equivalent of risky returns on investment, such that a higher RRA gives a lower certainty equivalent. The implication is that risky problems (including risky investment and inflation) can be interpreted similarly to risk-free problems;

ii. EIS governs consumption smoothing over time such that a lower EIS leads to a higher smoothing preference. The effect of EIS depends on the relationship between time discounting and the certainty equivalent of return on wealth, such that in cases when the perceived certainty equivalent of real returns is close to 1, EIS has an insignificant effect on smoothing;

For studying uncertain lifetime, we investigate three Epstein-Zin specifications that are used in the literature on life-cycle investment. These models differ in their treatment of mortality risks. We find that the preference models lead to different consumption profiles, and that models S1 and S2 can be normatively implausible. Particularly the most commonly adopted model S2, first described in Gomes and Michaelides (2005) and later used in Horneff et al. (2015), Horneff et al. (2009), Horneff et al. (2008b), Horneff et al. (2008c) and Blake et al. (2008), is found to contain highly undesirable normative properties due to high time discounting (greater than 1 in some instances). This results in extremely high savings at the expense of low consumptions at younger ages. We further clarify that S2 is not necessarily a generalization of expected discounted lifetime utility maximization in the case of constant relative risk aversion. This has been wrongly understood by the users of this model. Additionally, we find that under models S1 and S2, the EIS no longer governs consumption smoothing, implying that the parameter is not monotone in its effect on consumption volatility over time. This suggests that the effects of EIS are ambiguous and have been misinterpreted by the users of these preference specifications. The model S3, in contrast, does not exhibit these behaviours and is hence more preferable in retirement planning applications.

These results set the scene for the interpretation of results in Chapter 3, where a more complex optimal retirement planning problem is considered. In treatment of mortality risk, all three Epstein-Zin preference models are considered with results compared.

Chapter 3

Optimal Retirement Planning under Recursive Preferences

3.1 Introduction

Much effort has been made in modelling life-cycle investment and consumption problems, with the first work of such dating back to Yaari (1965). More recent studies include Boyle et al. (2015), Hanewald et al. (2013), Maurer et al. (2013), Horneff et al. (2010), Horneff et al. (2008c), Blake et al. (2008) and Horneff et al. (2008b), etc. These studies differ in the bundles of investment opportunities, annuitization products, restrictions on annuity purchases, treatment of systematic or idiosyncratic longevity risks, preferences, etc. Regardless, there is significant ambiguity surrounding the interpretation of the optimal solutions. Specifically as to whether the resulting optimal solutions ought to be adopted as practicable financial planning strategies, given the limitations of the assumptions (with exception of Boyle et al. (2015)). Fundamentally, this ambiguity stems from the obscurity of whether these researchers subscribe to the normative validity of the adopted preference models.

This chapter takes a step in the direction of addressing the above concern. We investigate the *normative validity* of the optimal consumption and investment strategies of a discrete-time utility maximizing DC retiree who wishes to benefit from stock investment, longevity insurance and inflation protection. We use a combination of qualitative and quantitative criteria to evaluate the adequacy of the optimal consumption profile. Particular attention is paid to the downside risk at extreme old ages.

This chapter considers the following investment vehicles: stocks, bonds, traditional annuities products such as fixed, escalating nominal and inflation-indexed annuities, and a Variable Payout Annuity (VPA), which is a group self-annuitization scheme offered as a retirement income option by some DC pension plans. Additionally, this chapter extends previous work by

i. investigating a broader set of risks,

This chapter was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada.

ii. adopting an economic scenario generator model that better reflects the interdependence of financial risk factors,

iii. using Epstein-Zin preferences that are more flexible for modelling intertemporal decision-making.

Specifically, we extend Boyle et al. (2015) by considering inflation risk, a stochastic investment opportunity set and Epstein-Zin preferences. Similar to Horneff et al. (2010), this chapter pays particular attention to the value of the VPA. However, systematic longevity and inflation risks are also incorporated. This work also extends Hanewald et al. (2013) such that instead of optimizing among a set of pre-specified options, it investigates all strategies within the given assumptions and the investment opportunity set. Our work further compares three Epstein-Zin utility functions used in the literature, which we refer to as S1 (see Córdoba and Ripoll, 2016), S2 (see Gomes and Michaelides, 2005; Blake et al., 2008; Horneff et al., 2008c; Horneff et al., 2008b), and S3 (see Li and Smetters, 2010; Blake et al., 2014).

The key contributions of this chapter are threefold: 1) the comparison of three Epstein-Zin models in the context of retirement planning involving annuitization decisions, 2) proposing a quantitative criterion for assessing the normative adequacy of the resulting optimal consumption profiles, 3) the evaluation of the impact of pure time discounting on old-age financial security. Specifically to the third point, we bridge the gap between the literature on normative life-cycle investment and the economics literature on intertemporal decision-making, by highlighting the paradox of assuming myopia in prescriptive long-term planning. This contribution is substantial, since it raises serious doubt over the conclusions of almost all existing works on normative life-cycle investing that involve pure time discounting.

This chapter is organized as follows. Section 3.1.1 describes the background of VPAs, including recent work on this topic. Section 3.2 lays out the investment, mortality and economic assumptions. Section 3.3 describes the criteria used to assess the normative adequacy of the optimal consumption strategies. Section 3.4 extends the work of Boyle et al. (2015) by introducing inflation risk and considering Wilkie's economic scenario generator (ESG) under time-additive expected discounted lifetime utility maximization framework. Section 3.5 describes the full problem under Epstein-Zin preferences, with numerical results presented in Section 3.6. Section 3.7 examines the optimal consumption profile, paying particular attention to comparison of Epstein-Zin specifications (refer to the discussions in Chapter 2), the normative validity of the results, and model uncertainty of the ESG. Section 3.8 discusses the pure rate of time discounting, and assesses its critical impact on old-age financial security. Section 3.9 discusses numerical challenges. The last section concludes the chapter and discusses future directions.

3.1.1 Variable Payout Annuities

VPAs are also known as 'Variable Income Annuities' and 'Variable Participating Life Annuities', and resemble a group self-annuitization (GSA) scheme. It is a life annuity with benefit payments that vary subject to the mortality experience of the pool, and the investment performance of the fund in which the underlying reserve assets are invested, relative to a pre-determined interest rate. Depending on the treatment of longevity risk, the underlying mortality assumptions can be either time-varying or pre-specified. VPAs have flexible designs that vary depending on the presence of guaranteed minimum benefits and annuitants' exposure to investment and longevity risks. For a summary of common VPA features, see Vittas (2011).

VPAs can act as an annuity option offered to members of a defined contribution pension plan, or as a retirement income product that can be purchased directly from a private annuity provider[1]. An open VPA has the advantage of allowing for participation in the higher but more volatile returns of stocks while providing protection against diversifiable longevity risk, as well as avoiding annuitization risk and being offered at cheaper prices (Vittas, 2011). On the other hand, in the absence of guaranteed minimum benefits, surviving annuitants are to share the nondiversifiable (or systematic) longevity risk, that is to say, the impact coming from the unexpected increase in the average longevity is not protected.

VPAs are attracting attention in recent years. For a comprehensive treatment, see Dellinger (2006) and Vittas (2011). Horneff et al. (2010) studies the value of the VPA to a retiree who is given the option of dynamically investing in stocks and bonds, and gradually purchasing a VPA. Their work, however, does not take into account systematic longevity risk. Boyle et al. (2015) conducts a similar study, allowing for systematic longevity risk and access to the fixed annuity market. Both works consider a constant investment opportunity set and a zero inflation environment. Hanewald et al. (2013), on the other hand, studies the management of longevity risk (both idiosyncratic and systematic) through optimally selecting among ten pre-specified portfolios comprised of fixed immediate/deferred life annuities, inflation-indexed annuities, self-annuitization with phased withdrawals and a GSA. All of these works, at least in part, consider an EDLU maximizing framework with a risk averse CRRA agent.

3.2 Assumptions

Suppose at the time of retirement, assumed to be fixed at age $x^\star = 65$, the individual has wealth $W^\star$ and needs to determine the optimal wealth allocation among,

i. investing in the stock market,

[1] For example, the University of British Columbia Faculty Pension Plan offers members the option to convert all or a part of their DC account balance to a Variable Payment Life annuity. Sun Life Assurance Company of Canada offers a VPA policy under the name "Sunflex Retirement Income".

ii. investing in long term bonds,

iii. annuitizing with an open-fund[2] VPA with α_V invested in stocks and the rest in long term bonds, and

iv. annuitizing with a $k \times 100\%$[3] inflation-indexed annuity (IIA) with the fee load λ,

subject to some objective function and preference model. At the beginning of each year during retirement, say time t, the individual needs to decide on the optimal level of consumption, and a portfolio strategy between stocks and bonds. In addition to past events, she has the knowledge of

i. her wealth at t,

ii. the probability of her survival to the next period, $p_{x+t,t}$,

iii. the true conditional distribution of the returns on stocks and bonds over the period $[t, t+1)$, and

iv. the true conditional distribution of inflation over the period $[t, t+1)$.

The two-factor Cairns, Blake and Dowd (CBD) mortality model, introduced by Cairns et al. (2006), is used to allow for systematic longevity risk. Parameter choice is consistent with Boyle et al. (2015) and Maurer et al. (2013). No idiosyncratic mortality is considered, assuming that the VPA fund is open and is large enough to fully diversify away this risk. We further assume that the annuity factor used to determine the VPA benefit is adjusted on an annual basis to reflect the most up to date mortality experience. More precisely, we define the annuity factor for a life currently aged x at time t to be,

$$\ddot{a}_{x,t} = \sum_{u=0}^{\omega-x-1} v^u {}_u\hat{p}_{x,t}|\mathcal{F}(t) = 1 + \sum_{u=1}^{\omega-x-1} (1+\mathrm{AIR})^{-u} \left\{ \prod_{s=1}^{u} (1 - \hat{q}_{x+s-1,t+s-1}|\mathcal{F}(t)) \right\}$$

where $\mathcal{F}(t)$ is the filtration process generated by all information up to time t, ω is the terminal age, and AIR stands for the assumed (or annuity) interest rate. The symbol *hat* is used to denote the best estimate of the random variable, which is defined as the median. As such, ${}_u\hat{p}_{x,t}|\mathcal{F}(t)$ is the time t best estimate of the survival probability of a life aged x at time t for at least u years, and $\hat{q}_{x+s-1,t+s-1}|\mathcal{F}(t)$ denotes the best estimate, at time t, of the mortality rate of a life aged $x+s-1$ at time $t+s-1$. For more information on the computation of the best estimates, see Appendix C.1. In essence, the annuity factors are computed annually, based on the most up-to-date best estimate of future mortality rates (i.e. the median of the projected mortality rates given the most up-to-date information).

[2] At $t = 0$, the VPA has a large number of survivors all aged 65, with an equal number of new entrants allowed at the beginning of each year starting from $t = 1$. For details, including the derivation of benefit adjustment factors at time t, denoted by j_t, see Boyle et al. (2015).

[3] The indexing factor k governs the level of inflationary effects on benefits from an IIA. For example, $k = 0.6$ implies that benefits are indexed to 60% of inflation in each year. Note that when $k = 0$, the IIA is equivalent to a nominally fixed annuity (FA).

We use the superscripts V and I to distinguish the annuity factor associated with the VPA (i.e. $\ddot{a}^V_{x^\star,t}$) and the inflation-indexed annuity (i.e. $\ddot{a}^I_{x^\star,t}$).

3.3 Criteria for Normative Adequacy in Consumption

Clearly, retirement consumption adequacy is highly subjective to individual circumstances. One feasible approach is to set a minimum consumption threshold that represents the very basic bundle of goods and services, including food, shelter, health expenses, and leisure. Such a measure is however not suitable in our study, due to the scale-invariance property of the preference model. A comparison to a minimum consumption threshold would be ad hoc. Certainly, an investigation of a varying wealth level can be conducted, as such, the focus shifts to savings adequacy, which is not the focus of this chapter.

As a benchmark, we use the consumption profile implied by a fully-indexed IIA payment purchased with full wealth at retirement with fee load λ. A fully-indexed IIA provides a consumption path fully protected under longevity, investment, and inflation risks, and implies maximum consumption smoothness. For each time t, we compute the following quantity as a relative measure for downside risk in consumption,

$$\mathrm{VaR}_a(\mathrm{RC}_t) = \mathrm{VaR}_a\left(\frac{\tilde{C}^*_t}{\frac{W^\star}{\ddot{a}^I_{x^\star,0}(1+\lambda)}}\right) = \frac{\mathrm{VaR}_a(\tilde{C}^*_t)}{\frac{W^\star}{\ddot{a}^I_{x^\star,0}(1+\lambda)}} \tag{3.1}$$

where RC_t stands for the real consumption at time t relative to the benchmark consumption, and $\mathrm{VaR}_a(\cdot)$ denotes the Value-at-Risk operator at a. We compute this quantity for $a = 0.01, 0.05$.

The implication of this benchmark is a flat consumption profile in retirement, which may not be supported by empirical findings. Empirical studies on consumption do not generally focus on adequacy, but rather patterns in retirement. The general finding is that consumption as a function of age is hump shaped, tracking the age profile of income (Attanasio and Weber, 2010); and that consumption declines in retirement (jointly with income) in many countries, including the US (Bernheim et al., 2001), the UK (Banks et al., 1998), Japan (Wakabayashi, 2008) and Germany (Lührmann, 2010). Hurd and Rohwedder (2013) summarizes several explanations for this phenomenon, ranging from voluntary to involuntary reduction in consumption. Voluntary reduction includes 1) the cessation of work-related expenses and 2) the substitution of home-produced goods for purchased goods. Involuntary reduction includes 1) less than anticipated retirement financial resources, hence the need to reduce consumption, 2) under-saving due to myopia, or involuntary retirement.

Nonetheless, there appears to be significant heterogeneity in consumption change at retirement (Hurd and Rohwedder, 2013). Hence, the benchmark is not meant to represent an 'ideal' consumption path, but rather as a maximally attainable consumption path with zero risk. Further work on normative consumption adequacy is warranted, but beyond the scope of this thesis.

3.4 EDLU-CRRA and Single-Variable Grid Search

First, consider a representative agent with CRRA preference with relative risk aversion parameter $\rho = 2$. Let $\boldsymbol{C} = (C_0, C_1, ..., C_T)$ be the vector of nominal consumptions, $\boldsymbol{\omega} = (\omega_V, \omega_F, \omega_0, \omega_1, ..., \omega_T)$ be the vector of asset allocation decisions, where ω_V and ω_F denote the proportion of wealth allocated in the VPA and FA at time 0, and ω_t, $0 \leq t \leq T$, denotes the proportion of equity allocation in the liquid wealth being held at time t. At each time t, the individual seeks to maximize her expected discounted lifetime utility

$$\max_{\boldsymbol{\omega},\boldsymbol{C}} \mathbb{E}\left[U(\boldsymbol{C})\right] = \max_{\boldsymbol{\omega},\boldsymbol{C}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(C_t)\right] = \max_{\boldsymbol{\omega},\boldsymbol{C}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\rho}}{1-\rho}\right)\right],$$

where $0 < \beta \leq 1$ is the impatience or time discounting factor, suggesting that the individual prefers present to future pleasure. In this section, we take $\beta = 0.96$, which is a standard parameter choice in the existing literature (see, e.g., Yogo, 2016; Horneff et al., 2015; Blake et al., 2014; Horneff et al., 2010; Horneff et al., 2009). We later re-examine this assumption in Section 3.8.

Boyle et al. (2015) shows that under a certain set of assumptions, the optimal portfolio strategy involves full annuitization and full consumption of annuity benefits. In this subsection, we follow their footsteps and work under the assumption that *the retiree optimizes within this subset of strategies.* The optimization problem is solved using the Monte Carlo single-variable grid search method. We use 10,000 simulations to approximate the expectation of total discounted lifetime utility. Since the optimal controls are reduced to the annuitization choice at time 0 (i.e. $\omega_V + \omega_F = 1$), a single-variable grid search method should suffice.

3.4.1 Optimizing Under Normal Stock Returns

To make our results comparable with Boyle et al. (2015), Maurer et al. (2013), Blake et al. (2008), and Horneff et al. (2010), we assume that the continuously compounded stock return process follows an i.i.d. normal distribution with $\mu = 4.078\%$ and $\sigma = 18.703\%$, and a fixed 2% risk-free interest rate. Ignoring the impact of inflation, where the IIA reduces to a fixed annuity (FA, denoted by superscript F), we are able to reproduce the results of Boyle et al. (2015).

3.4.2 Optimizing Under Wilkie's ESG

We repeat the grid-search procedure with Wilkie's ESG with 1991-2014 parameters (see Table A.2 (L)) and 2014 starting values. The parameter σ_c is adjusted from 0.3893 to 0.1 to obtain a smoother bond yield process. The equity log return process under the Wilkie model has a similar mean return of 4.5% and standard deviation of 17.7% compared to the $\mathrm{LN}(4\%, 18\%^2)$. The bond yield process, on average, fluctuates around 4.4% with standard deviation 0.86%. For justifications of parameter choice see Chapter 1. We first consider the optimal annuitization choice under the VPA and FA, without taking into account purchasing power. For $\mathrm{AIR}_V = \mathrm{AIR}_F = 3\%$, it is found that there exists a range of α_V, $[33\%, 62\%]$, under which it is always optimal for the retiree to invest 100% in the VPA. This range widens as the fee load λ rises, for details see Table 3.1.

λ	0.000	0.025	0.05	0.075	0.100
α_V	$[33\%, 62\%]$	$[19\%, 76\%]$	$[9\%, 84\%]$	$[4\%, 91\%]$	$[0\%, 96\%]$

Table 3.1 Ranges of α_V that result in optimal allocation being 100% in the VPA, based on 10,000 simulations

The VPA becomes significantly more attractive, being a direct result of Wilkie's more optimistic projections of fund returns. A higher AIR_F tilts the balance, since the FA guarantees a higher annual benefit. Table 3.2 summarizes the optimal allocation under an AIR of 4%, and $\alpha_V = 60\%$. A 60-40 asset mix is common in VPAs offered in the market, hence of particular interest.

λ	0.000	0.025	0.050	0.075	0.100
ω_V	0.29	0.50	0.70	0.88	1.00
ω_F	0.71	0.50	0.30	0.12	0.00
B_0(\$000)	76.1	75.2	75.0	75.5	76.1
B_0^V(\$000)	22.1	38.0	53.3	66.9	76.1
B_0^F(\$000)	54.0	37.2	21.7	8.6	0.0

Table 3.2 Optimal allocations between the VPA and the FA under Wilkie's ESG, $W^\star =$ \$1,000,000, $\alpha_V = 0.6$, $\mathrm{AIR}_V = \mathrm{AIR}_F = 0.04$, 10,000 simulations.

3.4.3 Seeking Protection Against Inflation

The question remains what should the retiree do to maintain a decent lifestyle (i.e. a decent level of purchasing power) throughout retirement. To achieve this, a retiree would seek protection against longevity risk and inflation risk. In this section, we take into account the impact of inflation and study a retiree's optimal consumption-investment decision under the objective of maximizing expected total lifetime utility derived from real consumption, $\tilde{C}_t$. Recall in Chapter 2 that $\tilde{C}_t$ is defined as the nominal consumption C_t divided by the cumulative inflation, where δ_t is the annual inflation of year $[t-1,t)$, and $\{I_t\}_{t=0}^{t=T}$ is the cumulative inflation process, such that

$$I_0 = 1; \quad I_t = \prod_{s=1}^{t} (1+\delta_t), \quad 1 \le t \le T,$$

and that real consumption for year t, conditional on survival, is,

$$\tilde{C}_t = \frac{C_t}{I_t}, \;\; 0 \le t \le T.$$

Table 3.3 shows that when real consumptions are considered, the VPA remains attractive, but to a lesser degree. This marginal difference is surprising as the FA does not provide an inflation protection. However, this can be explained by the fact that, inflation on average lowers consumption levels by roughly 2.4% per annum, which implies for each additional unit of risk, the individual is gaining less, resulting in her moving away from the previous optimal decision and adjusting the risk exposure.

λ	0.000	0.025	0.050	0.075	0.100
ω_V	0.28	0.47	0.65	0.81	0.97
ω_F	0.72	0.53	0.35	0.09	0.03
B_0($000)	76.1	75.1	74.8	75.1	75.9
B_0^V($000)	21.3	35.8	49.5	61.6	73.8
B_0^F($000)	54.8	39.3	25.3	13.5	2.1

Table 3.3 Optimal allocations between the VPA and the FA under Wilkie's ESG, $W^\star =$ $1,000,000, $\alpha_V = 0.6$, $\text{AIR}_V = \text{AIR}_F = 0.04$, inflation adjusted, 10,000 simulations.

Consider two alternatives to a FA that provide some degree of inflation protection, an escalating nominal annuity (ENA) and an fully- indexed IIA distinguished by the subscripts E and I. In an ENA, the benefits increase at a pre-determined rate and are fully known at the time of purchase. A fully-indexed IIA, on the other hand, provides benefits that vary at the rate of inflation, which means benefits are effectively fixed in real terms.

For this section, we assume the current market AIR is at the long term government bond yield of 2.6%. The first benefit payment is computed by $\frac{W^\star}{\ddot{a}_{x^\star,0}(1+\lambda)}$. We note that retirees are exposed to some level of annuitization risk, as the current long term interest rate is very low compared to the observations in the past decade. Wilkie's ESG projects the long term nominal interest rate to be 4.4%, and a long run average inflation expectation of 2.4%, hence the chosen AIR is close to the long term real rate of interest, which Vittas (2011) indicates to be an appropriate discount rate.

Escalating Nominal Annuities

Escalating Nominal Annuities provide partial protection against inflation, depending on the pre-determined guaranteed rate. We first consider an ENA with benefits growing annually at 2%. Suppose the retiree only has access to an open-fund VPA and the ENA. We are interested in the optimal annuitization decision that maximizes the retiree's EDLU when previous wealth and utility assumptions are retained. Let ω_E denote the optimal proportion of initial wealth allocated in the ENA, and B_t^E denote the ENA benefit received at time t. Table 3.4 summarizes the optimal annuitization decisions.

λ	0.00	0.05	0.10	0.15	0.20
ω_V	0.00	0.18	0.54	0.84	1.00
ω_E	1.00	0.82	0.46	0.16	0.00
B_0($000)	66.9	64.3	64.1	65.5	66.9
B_0^V($000)	0.0	12.0	36.1	56.2	66.9
B_0^E($000)	66.9	52.3	28.0	9.3	0.0

Table 3.4 Optimal allocations between the VPA and the ENA at 2%, under Wilkie's ESG, $W^\star = \$1{,}000{,}000$, $\alpha_V = 0.6$, $\text{AIR}_V = \text{AIR}_E = 0.026$, inflation adjusted, 10,000 simulations.

It is within expectation that despite the lower AIR, the ENA is more attractive to the retiree than a FA as it can be sold at a much higher fee load. Consider the case when $\lambda = 0.10$. The annuitant would allocate half of her initial wealth in the VPA and half in the ENA. Figure 3.1 shows the estimated mean, median, the 5% and 95% quantiles of the projected real consumption paths as a proportion of the initial consumption at time 0, based on 10,000 simulations. 50 simulated scenarios are also shown (see coloured paths), which represent the same 50 joint mortality and economic scenarios. It can be observed that on average, the individual is able to maintain roughly 80-85% of her purchasing power throughout retirement. However, her downside risk is unprotected. The chance of declining purchasing power is rapidly rising, whereas the upside potential is relatively flat, at 120%. Despite being the optimal strategy, its downside risk, which stems from the high equity investment in the VPA fund, may be seen to be too high while the upside is relatively limited. The financial integrity of the retiree at older ages is not upheld with acceptable confidence.

Inflation-Indexed Annuities

Fully-indexed inflation-indexed annuities provide full protection against inflation risk as benefits are effectively fixed in real terms. Table 3.5 summarizes the optimal annuitization decisions.

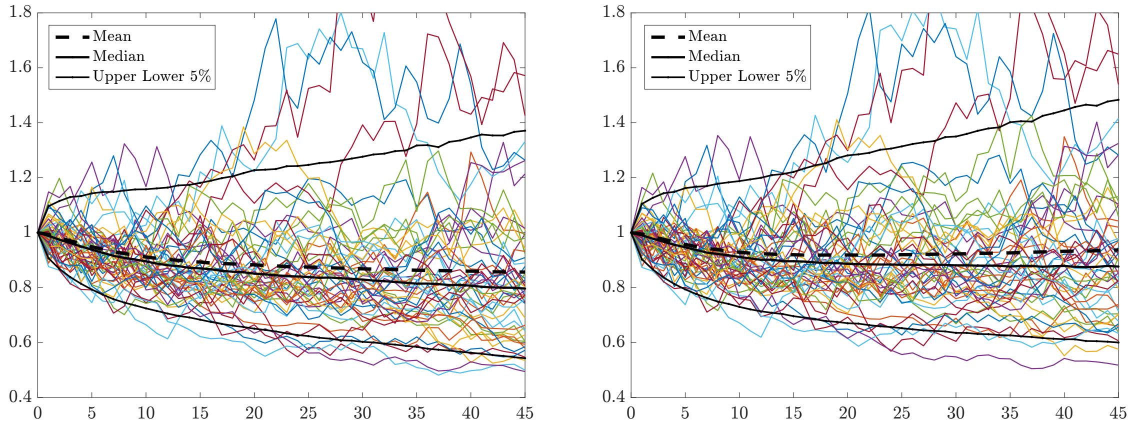


Figure 3.1 Simulated real consumption paths following optimal strategy as a proportion of initial benefit under Wilkie's ESG, $\lambda = 0.10$(L)/0.15 (R), with 54-46 (L)/ 55-45 (R) allocation in the VPA and the ENA (L)/IIA (R), $\alpha_V = 0.6$, $\text{AIR}_V = \text{AIR}_E = 0.026$, 10,000 simulations.

λ	0.05	0.10	0.15	0.20	0.25
ω_V	0.00	0.23	0.55	0.82	1.00
ω_I	1.00	0.77	0.45	0.18	0.00
B_0(\$000)	63.7	62.2	63.0	64.9	66.9
B_0^V(\$000)	0.0	15.4	36.8	54.9	66.9
B_0^I(\$000)	63.7	46.8	26.2	10.0	0.0

Table 3.5 Optimal allocations between the VPA and the IIA under Wilkie's ESG, $W^\star =$ \$1,000,000, $\alpha_V = 0.6$, $\text{AIR}_V = \text{AIR}_I = 0.026$, inflation adjusted, 10,000 simulations.

The IIA is seen to be more attractive than the ENA considered, as it remains appealing at higher fee loads up to $\lambda = 0.25$. Consider the optimal strategy when $\lambda = 0.15$, the retiree allocates roughly half of her initial wealth in the VPA and the IIA – a similar strategy to the one considered in the ENA case with $\lambda = 0.10$. The individual's projected real retirement income is shown in Figure 3.1, which indicates similar consumption paths. The IIA provides a marginally better inflation protection and an upside potential, at the cost of a lower initial benefit due to a higher fee charge. In both cases, the retiree's purchasing power remains above 85% of her initial wealth with more than 50% probability throughout retirement. However, she still faces relatively sustained and increasing probability of purchasing power falling below 70% starting from year 10 to 15.

3.5 Epstein-Zin Preferences and Dynamic Programming

In this section, we solve the optimal retirement planning problem using the following three recursive/Epstein-Zin specifications.

- S1: Córdoba and Ripoll (2016)

$$V_t = \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{3.2}$$

- S2: Gomes and Michaelides (2005)

$$V_t = \left\{ (1-\beta p_{x+t,t}) \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta \left[p_{x+t,t} \mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{3.3}$$

- S3: Li and Smetters (2010)

$$V_t = \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta p_{x+t,t} \left[\mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-1/\alpha}} \tag{3.4}$$

Note that the $\mathbb{E}_t[\cdot]$ is the short hand for the conditional expectation at time t, given past observations up until time t, and is taken on risks that impact consumption upon survival, including financial risks and systematic mortality risks that impact VPA payoffs. Mortality of the agent is treated separately, as discussed in Chapter 2. In subsections 3.5.1, 3.5.2, and 3.5.3, the problem is formulated using preference model S3. Formulations under S1 and S2 are analogous, hence omitted.

3.5.1 The Optimization Problem

Under S3, the individual has an Epstein-Zin preference that takes the following form,

$$V_t = \left\{ \tilde{C}_t^{1-\frac{1}{\alpha}} + \beta p_{x+t,t} \left[\mathbb{E}_t \left(V_{t+1}^{1-\rho} \right) \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{1/(1-\frac{1}{\alpha})}, \tag{3.5}$$

where $\rho \in (0,\infty) \setminus \{1\}$ captures the level of relative risk aversion, and $\alpha \in (0,\infty) \setminus \{1\}$ the elasticity of intertemporal substitution. The expected discounted lifetime utility assuming CRRA can be obtained by fixing $\rho = \frac{1}{\alpha}$ and computing V_0. The conditional expectation $\mathbb{E}_t$ is taken with respect to the joint distribution of stock returns, long term bond returns, inflation and mortality rates. The probability of survival $p_{x+t,t}$ is known at time t. Recall $\tilde{C}_t$ is the real consumption at time t. The individual is additionally assumed to know her real wealth at time t (i.e. inflation). We consider the problem under the settings of normal stock returns, normally distributed force of inflation, and constant nominal risk-free interest rate. We further generalize the retiree's annuitization choice by allowing k, the inflation-indexing factor, to vary between 0 and 1. This is determined at the time of purchase and is fixed throughout the contract term.

Let W_t be the nominal liquid wealth process of the retiree at time t, after receiving annuity benefits. Let B_t^V, B_t^I be the nominal annuity benefit received from the VPA and the IIA at time t, respectively, and C_t be the nominal consumption of year $t+1$, assumed to occur at the beginning of the year, immediately after annuity benefits are paid out. As usual, we use the symbol $\sim$ on top to denote the corresponding real process. More precisely,

$$\tilde{W}_t = \frac{W_t}{I_t},\ \tilde{B}_t^V = \frac{B_t^V}{I_t},\ \tilde{B}_t^I = \frac{B_t^I}{I_t}.$$

Note that when $k=1$, the IIA is effectively fixed in real terms, hence $\{\tilde{B}_t\}$ becomes a deterministic process as $\tilde{B}_t^I = B_0^I$, $\forall t$. Further, let $\boldsymbol{\omega} = (\omega_V, \omega_I, \omega_0, \omega_1, ..., \omega_T)$ be the vector of asset allocation decisions, where ω_V and ω_I denote the proportion of wealth allocated in the VPA and IIAs at time 0, and ω_t, $0 \le t \le T$, denotes the proportion of equity allocation in the liquid wealth being held at time t. Let $\tilde{\boldsymbol{C}} = (\tilde{C}_0, \tilde{C}_1, ..., \tilde{C}_T)$ be the vector of real consumptions. Suppose the individual has initial wealth $W^\star$ and age $x^\star$. The optimization problem becomes,

$$\max_{\boldsymbol{\omega}, \tilde{\boldsymbol{C}}} V_0,$$

with wealth process,

$$\begin{aligned}
\tilde{W}_0 &= W_0 = (1-\omega_V-\omega_I)W^\star + B_0^V + B_0^I, \\
\tilde{W}_t &= (\tilde{W}_{t-1} - \tilde{C}_{t-1})\left(\frac{1+R_t^W}{1+\delta_t}\right) + \tilde{B}_t^V + \tilde{B}_t^I, \quad 0 \le t \le T
\end{aligned}$$

where,

$$R_t^W = \omega_{t-1}[1+R_t^{Eq}] + (1-\omega_{t-1})[1+R_t^{LB}];$$

and benefits $\tilde{B}_t^V$, $\tilde{B}_t^I$ that follow,

$$\begin{aligned}
\tilde{B}_0^V &= B_0^V = \frac{\omega_V W^\star}{\ddot{a}_{x^\star,0}^V}, &\quad \tilde{B}_t^V &= \tilde{B}_{t-1}^V\left(\frac{1+j_t}{1+\delta_t}\right), \\
\tilde{B}_0^I &= B_0^I = \frac{\omega_I W^\star}{\ddot{a}_{x^\star,0}^I(1+\lambda)}, &\quad \tilde{B}_t^I &= \tilde{B}_{t-1}^I\left(\frac{1+k\delta_t}{1+\delta_t}\right),
\end{aligned}$$

subject to the following constraints that for all t,

$$0 \le \tilde{C}_t \le \tilde{W}_t, \qquad 0 \le \tilde{W}_t, \tilde{B}_t^V, \tilde{B}_t^I;$$
$$0 \le \omega_V, \omega_I, \omega_t \le 1, \quad \omega_V + \omega_I \le 1.$$

Note that $j_t = \frac{B_t^V}{B_{t-1}^V}$ denotes the adjustment factor at time t, which is used to compute the VPA benefits based on the investment performance and the mortality experience of the VPA fund. The derivations of the adjustment factors are described in Appendix C.2.

3.5.2 The Normalized Optimization Problem

The complexity associated with dimensionality can be alleviated by solving the equivalent normalized optimization problem. Define $\tilde{c}_t = \frac{\tilde{C}_t}{\tilde{B}_t}$, $\tilde{w}_t = \frac{\tilde{W}_t}{\tilde{B}_t}$, $\tilde{b}_t = \frac{\tilde{B}_t^V}{\tilde{B}_t}$ and $\tilde{\boldsymbol{c}} = (\tilde{c}_0, \tilde{c}_1, ..., \tilde{c}_T)$. It is equivalent to solve,

$$\max_{\omega_V, \omega_I} \left[\left(\frac{\omega_V}{\ddot{a}_{x^\star,0}^V} + \frac{\omega_I}{\ddot{a}_{x^\star,0}^I (1+\lambda)} \right) \max_{\boldsymbol{\omega}, \tilde{\boldsymbol{c}}} v_0 \right], \tag{3.6}$$

where,

$$v_t = \left\{ \tilde{c}_t^{1-\frac{1}{\alpha}} + \beta p_{x+t,t} \left\{ \mathbb{E}_t \left[\left(\frac{1 + \tilde{b}_t j_{t+1} + k\delta_{t+1}(1-\tilde{b}_t)}{1+\delta_{t+1}} v_{t+1} \right)^{1-\rho} \right] \right\}^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}}, \tag{3.7}$$

with normalized wealth process,

$$\tilde{w}_0 = w_0 = 1 + \frac{(1-\omega_V-\omega_I)}{\frac{\omega_V}{\ddot{a}_{x^\star,0}^V} + \frac{\omega_I}{\ddot{a}_{x^\star,0}^I(1+\lambda)}},$$
$$\tilde{w}_t = 1 + \frac{(\tilde{w}_{t-1} - \tilde{c}_{t-1})(1+R_t^W)}{(1+k\delta_t) + \tilde{b}_{t-1}(j_t - k\delta_t)}, \quad 0 \le t \le T,$$

where,

$$R_t^W = \omega_{t-1}[1+R_t^{Eq}] + (1-\omega_{t-1})[1+R_t^{LB}] - 1;$$

and normalized benefits that follow,

$$\tilde{b}_0 = b_0 = \frac{\omega_V / \ddot{a}_{x^\star,0}^V}{\frac{\omega_V}{\ddot{a}_{x^\star,0}^V} + \frac{\omega_I}{\ddot{a}_{x^\star,0}^I(1+\lambda)}}, \quad \tilde{b}_t = \left(\frac{\tilde{b}_{t-1}(1+j_t)}{(1+k\delta_t) + \tilde{b}_{t-1}(j_t - k\delta_t)} \right),$$

subject to the constraints,

$$0 \le \tilde{c}_t \le \tilde{w}_t, \quad 0 \le \omega_V, \omega_I, \omega_t \le 1, \quad \omega_V + \omega_I \le 1.$$

3.5.3 Value Function Iteration

The value function iteration method is used to numerically approximate the optimal solution. Due to the fact that the terminal optimal decisions are known, and that recursive preferences allow the separation of a multi-period problem into numerous single-period problems, one can apply backward induction and Bellman's principle of optimality to iteratively arrive at the optimal decision to time 0.

Because there is no bequest motive, at time T, it is optimal to consume all remaining wealth, as the individual's death is certain at time $T+1$,

$$\max_{\omega_T,\tilde{c}_T} v_T = \max_{\omega_T,\tilde{c}_T} (\tilde{c}_T^{1-\frac{1}{\alpha}})^{1/(1-\frac{1}{\alpha})} = \max_{\omega_T,\tilde{c}_T} \tilde{c}_T = \tilde{w}_T.$$

We obtain that the optimal controls at time T, which we denote using $\tilde{c}_T^*$ and ω_T^*, are

$$\tilde{c}_T^* = \tilde{w}_T, \qquad 0 \leq \omega_T^{*\,4} \leq 1.$$

Moving back one period to $T-1$,

$$\begin{aligned}
&\max_{\omega_{T-1},\tilde{c}_{T-1}} v_{T-1} \\
&= \max_{\omega_{T-1},\tilde{c}_{T-1}} \left\{ \tilde{c}_{T-1}^{1-\frac{1}{\alpha}} + \beta p_{x+T-1,T-1} \left[\mathbb{E}_{T-1} \left(\frac{1+\tilde{b}_{T-1} j_T + k\delta_T(1-\tilde{b}_{T-1})}{1+\delta_T} v_T \right)^{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}} \\
&= \max_{\omega_{T-1},\tilde{c}_{T-1}} \left\{ \tilde{c}_{T-1}^{1-\frac{1}{\alpha}} + \beta p_{x+T-1,T-1} \left[\mathbb{E}_{T-1} \left(\frac{1+\tilde{b}_{T-1} j_T + k\delta_T(1-\tilde{b}_{T-1})}{1+\delta_T} \max_{\omega_T,\tilde{c}_T} v_T \right)^{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}}.
\end{aligned}$$

Note that the one-period conditional survival probability $p_{x,t}$ is governed by the two-dimensional stochastic process $\boldsymbol{A}_t$ from the CBD mortality model. We thereby denote $p_{x,t}$ as $p_x(\boldsymbol{A}_t)$, and after some rearranging, $\max_{\omega_{T-1},\tilde{c}_{T-1}} v_{T-1}$ can be rewritten as,

$$\max_{\omega_{T-1},\tilde{c}_{T-1}} \left\{ \tilde{c}_{T-1}^{1-\frac{1}{\alpha}} + \beta p_{x+T-1}(\boldsymbol{A}_{T-1}) \left[\mathbb{E}_{T-1} \left(\frac{1+\tilde{b}_{T-1} j_T + k\delta_T(1-\tilde{b}_{T-1})}{1+\delta_T} \tilde{w}_T \right)^{1-\rho} \right]^{\frac{1-\frac{1}{\alpha}}{1-\rho}} \right\}^{\frac{1}{1-\frac{1}{\alpha}}}.$$

We need to evaluate a conditional expectation $\mathbb{E}_{T-1}[\cdot]$, which cannot be obtained analytically but is relatively simple to approximate numerically (using Monte Carlo methods

[4] Note that given $\tilde{c}_T^*$, the objective function is maximized for all possible ω_T, due to liquid wealth being fully consumed at time T immediately after annuity payments.

or numerical integration) under the current setting of normally distributed stock returns, inflation and fixed risk-free rate, due to the i.i.d. assumption. The expectation is a function of $T-1$, $\tilde{w}_{T-1}$, $\tilde{b}_{T-1}$, $\boldsymbol{A}_{T-1}$ and the controls $\tilde{c}_{T-1}$, ω_{T-1}. Consequently, we define the normalized value function $\mathcal{J}(\cdot)$ to be

$$\mathcal{J}(t,\tilde{w}_t,\tilde{b}_t,\boldsymbol{A}_t)=\max_{\omega_t,\tilde{c}_t} v_t,\quad \forall 0\le t\le T.$$

It is immediate that we have the following relationship,

$$\max_{\tilde{\boldsymbol{C}},\boldsymbol{\omega}} V_0 = W^\star \max_{\omega_V,\omega_I}\left\{\left(\frac{\omega_V}{\ddot{a}^V_{x^\star}}+\frac{\omega_I}{\ddot{a}^I_{x^\star}(1+\lambda)}\right)\mathcal{J}\left(0,1+\frac{(1-\omega_V-\omega_I)}{\frac{\omega_V}{\ddot{a}^V_{x^\star,0}}+\frac{\omega_I}{\ddot{a}^I_{x^\star,0}(1+\lambda)}},\frac{\omega_V/\ddot{a}^V_{x^\star,0}}{\frac{\omega_V}{\ddot{a}^V_{x^\star,0}}+\frac{\omega_I}{\ddot{a}^I_{x^\star,0}(1+\lambda)}},A_0\right)\right\}.$$

The two-dimensional state vector $\boldsymbol{A}_t$ follows, $\boldsymbol{A}_t=\tau+\boldsymbol{A}_{t-1}+V\boldsymbol{Z}_t$. The optimal controls are obtained from value function iteration using backward induction as in Carroll (2011). We begin with discretizing the state variables $\tilde{w}_t,\tilde{b}_t,A_{0,t},A_{1,t}$,

$$[1,1000]\times[0,1]\times[-12,-10]\times[0.09,0.11]$$

by constructing a four-dimensional non-equidistant grid of size $13\times101\times4\times4$. The first three grid points for $\tilde{w}_t$ are concentrated near 1 (i.e. $[1,1.5,2]$) to obtain a closer approximation near full annuitization. The remaining are grids are equidistant (i.e. $[100,200,...,1000]\times[0,0.01,...1]\times[-12,-11.3333,-10.6667,-10]\times[0.09,0.0967,0.1033,0.11]$). The value function in between and outside the grid points is evaluated using multilinear interpolation and extrapolation.

3.6 Numerical Results under Epstein-Zin Preferences

	Assumption 1	Assumption 2
Total Stock Returns	$LN(0.04078, 0.18703^2)$	$LN(0.06078, 0.18703^2)$
Force of Inflation	Nil	$N(0.02, 0.01^2)$
Risk-Free Interest Rate	0.02	0.03
AIR_V	0.03	0.03
AIR_F	0.03	0.03
AIR_I $(k=1)$	Nil	0.0096
AIR_I $(k=0.5)$	Nil	0.0198

Table 3.6 Economic scenario assumptions (two sets) used for numerical optimization.

The assumptions of normal stock returns and fixed risk-free interest rate are retained. This greatly simplifies the optimization numerically. We consider an additional set of assumptions (Assumption 2) under which inflation varies stochastically over time according to $N(0.02, 0.01^2)$. The parameters are selected based on historical observations[5] in the U.S. from 1991 to 2014. Note that although this implies a low inflation environment, the long term accumulated effect of inflation on purchasing power is not trivial. In a 10-year horizon, inflation will on average wear off 18% of an individual's purchasing power; this rises to 45% in a 30-year horizon.

We consider four types of retirees[6] in Table 3.7.

	Lower RRA $\rho = 2$	Higher RRA $\rho = 5$
Lower EIS $\alpha = 0.2$	Type 1: Less risk averse Stronger smoothing preference	Type 3: More risk averse Stronger smoothing preference
Higher EIS $\alpha = 0.5$	Type 2: Less risk averse Weaker smoothing preference	Type 4: More risk averse Weaker smoothing preference

Table 3.7 Types of retirees (description in relative terms)

As shown in Chapter 2, the elasticity of intertemporal consumption governs the individual's preference towards different types of consumption paths over time, or 'consumption smoothing', contingent on survival. This effect can be directly observed from the marginal rate of intertemporal substitution (see Appendix B.1), which governs the rate of change of the indifference curve between present and future consumption. A lower α results in more curvature in the indifference curve, therefore suggesting that the same drop in current consumption is to be compensated by a higher level of future consumption to remain indifferent. This is of course all given the fact that consumption is not subject to mortality risk, or when preference model S3 is considered. As shown, under an uncertain lifetime, the impact of EIS on consumption smoothing can be ambiguous under preferences S1 and S2.

In the following subsection, we proceed by solving the optimal retirement planning problem. We first investigate a reduced problem, aiming to reproduce the results of Boyle et al. (2015) to check the validity of the numerical procedure. We then gradually relax the assumptions of preferences (from time-additive EDLU-CRRA to Epstein-Zin), systematic longevity development (from zero to stochastic), and inflation (from zero to stochastic). The goal is to investigate the impact of these assumptions on the optimal decision variables. We pay particular attention to the normative adequacy of the resulting consumption paths.

[5]For inflation, we use non-seasonally adjusted U.S. City Average All Items Consumer Price Index (CPI) for All Urban Consumers (CPURNSA), obtained from Bloomberg. For total stock returns, we use the total return index of the S&P500, obtained from Morningstar, Inc. (prior to 2010) and Bloomberg (2010–2014).

[6]The representative agents are consistent with Blake et al. (2008). The magnitude and sign of EIS are a source of debate. See for example Hall (1988) and Havranek et al. (2015).

3.6.1 Working in Absence of Inflation – Assumption One

Working in Absence of Stochastic Systematic Longevity Development

To be consistent with Boyle et al. (2015), we first work in absence of stochastic systematic longevity development to reproduce their results for time-additive EDLU-CRRA retirees. In this case, the optimization problem becomes a simple one with two state variables $\tilde{w}_t$ and $\tilde{b}_t$. The IIA reduces to a conventional fixed annuity. Figure 3.2 plots the optimal annuitization level in the VPA against the fee load on the FA, for all four representative agents.

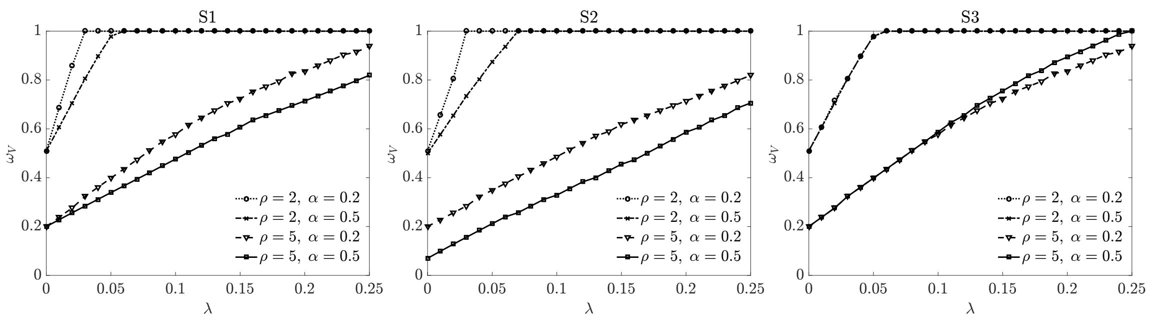


Figure 3.2 Optimal annuitization in a 40-60 VPA under assumption 1, without systematic longevity development.

The following key results are obtained

- under S1 and S3:
  - i. full annuitization in the VPA and the FA;
  - ii. full consumption of annuity benefit throughout retirement, which implies zero precautionary savings;
  - iii. the retiree's liquid wealth at t, prior to consumption, is equal to the sum of the VPA and FA benefits;
- under S2:
  - i. full annuitization in the VPA and the FA;
  - ii. partial consumption of annuity benefit throughout retirement, which implies positive precautionary savings;
  - iii. the retiree's liquid wealth at t, prior to consumption, is equal to the sum of annuity benefits and return on savings in $[t-1, t)$.

Boyle et al. (2015)'s findings for time-additive EDLU-CRRA agents correspond to cases where $\rho = 1/\alpha$ under S1 and S3, for which we obtained consistent results. As discussed in

Chapter 2, setting $\rho = 1/\alpha$ under S2 does not necessarily yield solutions consistent with time-additive EDLU-CRRA, which is clearly shown in Figure 3.2.

We further find that the optimal annuitization decision is highly sensitive to RRA, where a high RRA significantly reduces VPA purchase. Under S1 and S2, a lower EIS leads to a higher incentive to purchase the VPA; whereas under S3, EIS has an almost negligible impact on the annuitization decisions, except when the fee load is high, then a lower EIS gives a slightly lower VPA incentive. In a way, the impact of EIS under S3 contradicts that under S1 and S2. This is due to the differences in the treatment of mortality risk. In Chapter 2, we have extensively discussed the impact of EIS on consumption smoothing under all three preference specifications. We re-iterate that under S1 and S2 (but not S3), α is not a parameter that governs preferences over consumption smoothing. This means that a lower α can lead to higher consumption volatility over time, as is often the case under a higher annuitization in VPA at retirement. We expand the discussion on consumption smoothing in later sections.

Working Under Stochastic Systematic Longevity Development

We extend the problem to allow for stochastic systematic longevity development, under which there are a random number of survivors in the VPA pool, and a stochastic annuity factor that is adjusted annually to reflect the most up-to-date mortality experience.

Boyle et al. (2015) derives the expression for the adjustment factor for an open-fund VPA in the presence of systematic longevity development. The expression, shown in Equation (3.8), remains unchanged under our best estimate approach. The mortality component of the adjustment factor becomes random, since at time $t-1$, the value of $\ddot{a}^V_{x_k+t,t}$ is unknown. The computation of the adjustment factor in this case requires keeping track of the benefits paid to each surviving member in the previous period. This is computationally expensive, even under the assumption of homogeneous new entrants, since the adjustment factor is frequently computed in the optimization process. Thus, we simplify its computation by an approximation, which assumes homogeneous new entrants (i.e. enter at age 65 with equal wealth) and all survivor benefits at time t are equal. This gives an acceptable tradeoff between accuracy and efficiency, since the variation of the mortality component has an insignificant impact on the adjustment factor compared to the dominating impact of investment returns. Mathematically, this yields

$$\begin{aligned} 1 + j_t &= \left(\frac{\sum_{k \in S^+_{t-1}} B^V_k(t-1) p_{x_k+t-1,t-1}\, \ddot{a}^V_{x_k+t,t-1}}{\sum_{k \in S_t} B^V_k(t-1) \ddot{a}^V_{x_k+t,t}} \right) \left(\frac{1+R^V_t}{1+\text{AIR}} \right) \\ &\approx \left(\frac{\sum_{c=0}^{t-1} \ddot{a}^V_{x+t-c,t-1}}{\sum_{c=0}^{t-1} \ddot{a}^V_{x+t-c,t}} \right) \left(\frac{1+R^V_t}{1+\text{AIR}} \right). \end{aligned} \tag{3.8}$$

The mortality component is approximated by the expected over actual ratio of the sum of the annuities factors for all cohorts. The approximation gradually deteriorates as $t \to T$.

This is caused by survivor benefits at older ages having a higher variation which more strongly violates the assumption of equal benefits among cohorts. Effectively, due to their small magnitudes, combined with time discounting and higher mortality rates, these deviations have a negligible impact on the optimal decisions at time 0. Figure 3.3 shows the mean error for the approximation for the mortality component. In fact, we are able to reproduce the result of Boyle et al. (2015) for a retiree with $\rho = 1/\alpha = 2$, which suggests the approximation is reasonably accurate.

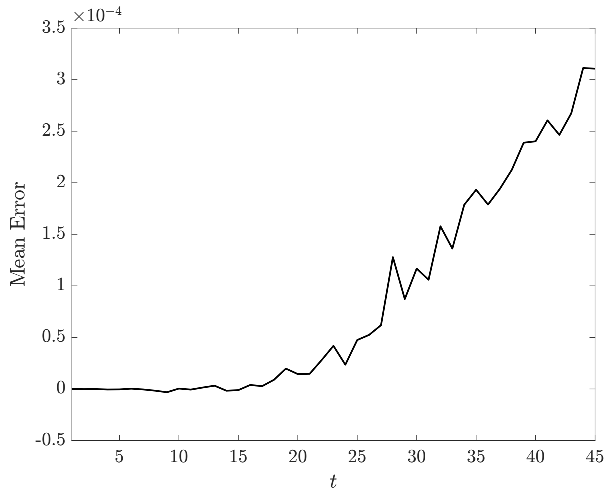


Figure 3.3 Mean error: approximation of the mortality component of j_t.

Figure 3.4 shows the optimal annuitization decisions. In the presence of stochastic systematic longevity where annuity factors are adjusted annually, retirees optimal decisions generally remain unchanged except opting for lower annuitization in the VPA. This is because systematic longevity risk cannot be diversified away. Consequently, annuitants are fully exposed to the risk of the members in the pool being longer-lived over time, resulting in the VPA being a riskier choice. The VPA and the FA both yield lower initial benefits due to the positive trend in longevity development. The annuity factor at time 0 is 14.7673 (instead of 14.3898 in the case of a zero trend).

3.6.2 Working under Risky Inflation – Assumption Two

In the previous two sections, we solved two simplified problems in order to gradually build upon previous work, and to break down the effect of EIS and systematic longevity risk. Here, we solve the full problem introduced in Section 3.5. We assume a stochastic systematic mortality process and a risky inflation process as described in assumption set 2. We adopt a different set of economic assumptions from previous work (i.e. assumption set 1) due to our interest in investigating the impact of risky inflation and the value of

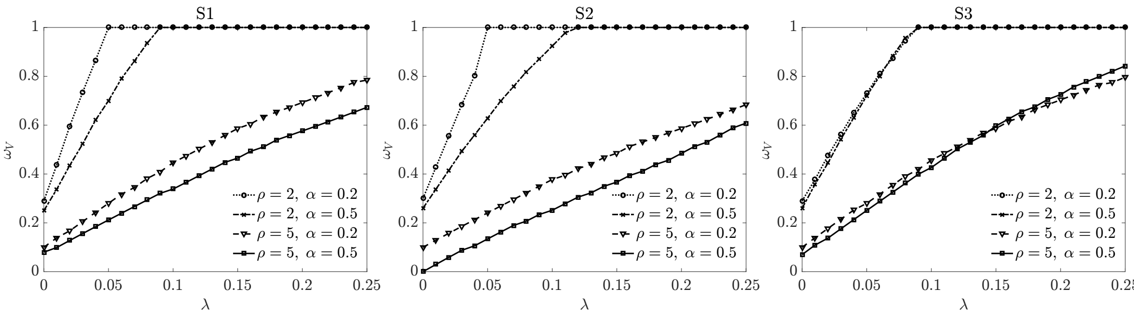


Figure 3.4 Optimal annuitization in a 40-60 VPA under Assumption 1, with stochastic systematic longevity development. x-axis: λ, y-axis: ω_V.

inflation protection instruments. We consider three values of inflation-indexing factor $k = 0,\ 0.5$ and 1.0. Figures 3.5, 3.6, and 3.7 summarize the optimal annuitization decisions for the representative agents under preference model S1, S2, and S3.

We highlight the following key observations.

- In general:
  - i. it is optimal to fully annuitize at retirement; for $\rho = 2$, optimal annuitization in the VPA is 100%;
  - ii. for $\rho = 5$,
    - a) it is optimal to hold some precautionary savings, the level of which depends on α;
    - b) the need to hold precautionary savings is lower when inflation-indexed products are available (i.e. $k > 0$).
    - c) the parameter α has a small effect on the annuitization ratios; the effect increases as k and λ increase.
- under S3:
  - i. for $\rho = 2$ and $\alpha = 0.2$, when $k = 0$, the agent holds a very small amount of savings to cushion consumption volatility over time. This behaviour is not observed for other $\rho = 2$ agents. Nonetheless, this has a negligible impact on the optimal consumption profile.

Retirees with RRA $\rho = 5$

For the rest of this section, we discuss the solutions for retirees with RRA parameter $\rho = 5$. We focus on the impact of α, k, and the preference specification, on the optimal consumption, savings, and investment behaviour. We pay particular attention to the smoothness of the

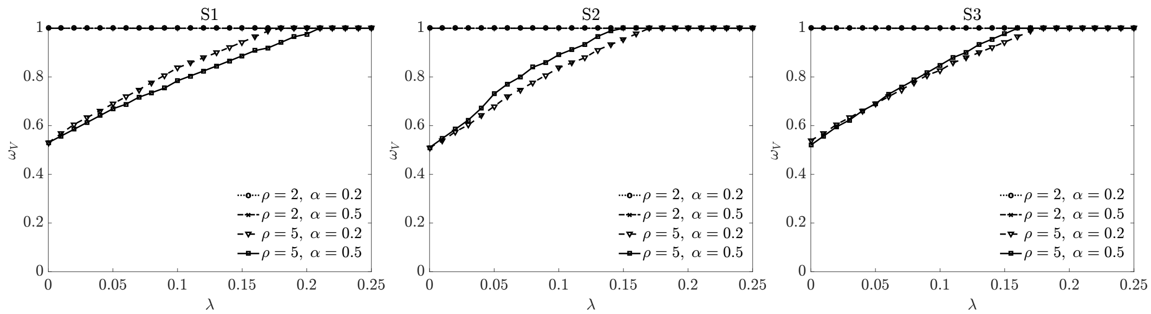


Figure 3.5 Optimal annuitization in a 40-60 VPA under assumption 2. $k = 0.0$.

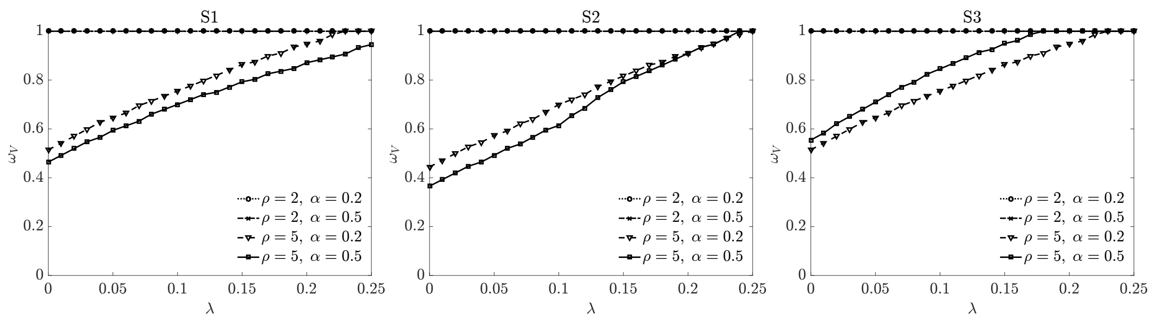


Figure 3.6 Optimal annuitization in a 40-60 VPA under assumption 2. $k = 0.5$

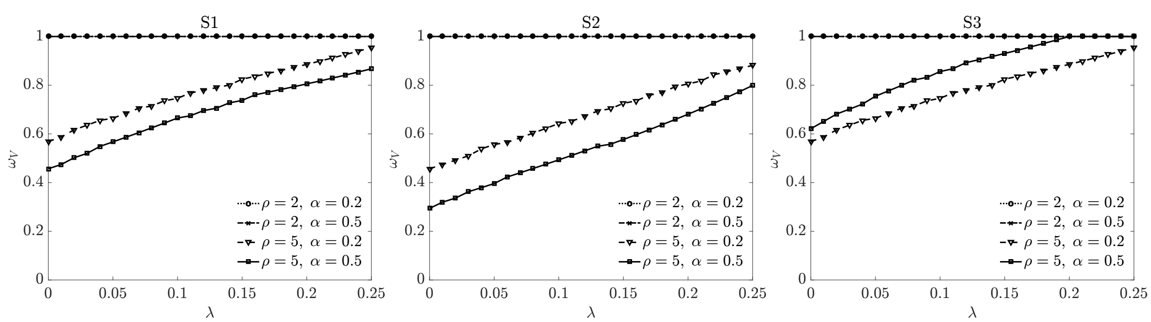


Figure 3.7 Optimal annuitization in a 40-60 VPA under assumption 2. $k = 1.0$

consumption path over time. The simulated results are based on 10,000 scenarios and a fee load assumption of $\lambda = 0.1$. IIAs tend to be more costly but quotes are difficult to obtain.

We first comment on the optimal annuitization decisions. There are several factors at play and together the net effect gives the overall annuitization ratio. The first, and the most intuitive, is RRA ρ, which drives the agent toward the IIA since it is less risky than the VPA, and when $k > 0$ creates protection against inflation risk. The second is the fee load λ. The higher the fee, the more expensive the IIA is and this in turn creates a higher incentive for the VPA. The third is k, which determines the level of initial IIA benefits. The higher k is, the lower the initial IIA benefits are compared to the VPA, which creates an incentive for a higher VPA allocation since lifetime is uncertain (i.e. the agent faces the risk of a low income stream in the case of a short lifespan). The fourth factor, and the most complex, is α, since α impacts both the agent's preference over intertemporal substitutability and impatience (this is discussed at length in Chapter 2). In S1 and S2, the impact of α on consumption smoothing in the presence of mortality risk is not necessarily monotonic, meaning that α cannot be interpreted as a parameter that governs consumption smoothing. This is due to the impact of α on impatience, which can create an opposite force to smoothness. In S3, α has no impact on impatience, hence its interpretation of being a consumption smoothing parameter is retained. This is an important point when interpreting the numerical results in this section.

Precautionary savings

We first investigate savings. Figures 3.8, 3.9, and 3.10 plot the median optimal savings under S1, S2, and S3 for $\alpha = 0.1$ to 0.6 and $\lambda = 0.1$. We observe that under all three preference specifications, retirees gradually build up savings which peak after 10–20 years, and deplete them towards the end of their lifetime. This is a direct result of declining survival probabilities with age, since zero utility is derived from wealth upon death and the benefit of savings is less likely to be realized with age. In general, the overall savings are significantly higher under S2 compared to S1 and S3. This is interesting since the annuitization decisions are fairly close. In S2, the kink in savings at age 109 is due to the sudden drop in survival probability at age 110. The impact of α on savings level is also dramatically different in S2, in that a lower α gives higher savings. This is opposite to the solutions under S1 and S3.

Note that, regardless of the relative sizes of $0 < \alpha < 1$, a consumption smoothing response is triggered and agents with strong enough net smoothing preference would build up savings to cushion consumption volatility over time (as we observe that some agents acting under S1 and S3 do not prefer to save, indicating insufficiently strong smoothing preference to generate savings behaviour). Savings significantly drop for higher k since consumption volatility is partially mitigated through receiving IIA benefits.

In Chapter 2, we have shown that higher savings do not necessarily lead to smoother consumption, depending on the perceived expected rate of return on savings. We are ultimately interested in the consumption behaviour, which we investigate in the next subsection.

Consumption volatility
Figures 3.11, 3.12 and 3.13 show the median optimal consumption profile under S1, S2, and S3 for $\alpha = 0.1$ to 0.6 and $\lambda = 0.1$. We highlight the following key observations regarding median optimal consumption:

i. S1 and S3 yield very similar consumption paths, especially for high α's. It is also evident that when $\alpha = 0.2$, the two median consumption paths are identical since the two preference models reduce to the same time-additive EDLU-CRRA case.

ii. When α is lower, all three preference models appear to yield relatively smoother consumption paths prior to reaching age 85–90, which is roughly around the agent's expected age at death (85.4). The difference in average consumption smoothness reduces when agents can access higher inflation protection through a higher k.

iii. Under S2, the agent with higher α holds significant savings to ensure higher consumption later in life. This behaviour is counterintuitive but mirrors the analytical solution presented in Chapter 2. This behaviour is caused by the fact that the impatience factor can be greater than 1 at older ages when the mortality rate rises quickly[7]. This means the agent would more strongly value consumption later in life at the expense of consumption in the present. This explains the significantly higher savings under S2 compared to S1 and S3, and the resulting higher consumption levels at older ages. This observation provides further evidence that the preference specification S2 exhibits undesirable properties and is inappropriate for retirement planning applications.

iv. We further note that the sudden rise in consumption at age 110 under S2 is due to the sudden drop in survival probability to the terminal age.

Figures 3.14, 3.15, and 3.16 show the boxplots of consumption volatility which is defined as the sample standard deviation taken over each simulated consumption path. In Chapter 2, we showed that the interpretation of α is ambiguous under S1 and S2 due to its impact on impatience, which acts against its effect on preferences over consumption smoothing; whereas under S3, α governs consumption smoothing over time. Here, consistent results are obtained. We highlight key observations below:

i. In general, the impact of α on consumption volatility is more pronounced for a higher k. This is due to the higher sensitivity of income level to the decision variables, since a higher k results in lower IIA benefits compared to VPA benefits.

ii. Under S1, a lower α leads to higher consumption volatility. Nonetheless, the impact is very small and only visible for higher k.

iii. Under S2, the impact of α on consumption volatility is V shaped, which mirrors the findings in the analytical case in Chapter 2.

[7] The impatience factor under S1 and S3 cannot be greater than 1 when $1 - \frac{1}{\alpha}$ and $1 - \rho$ are of the same sign. For details, see Chapter 2.

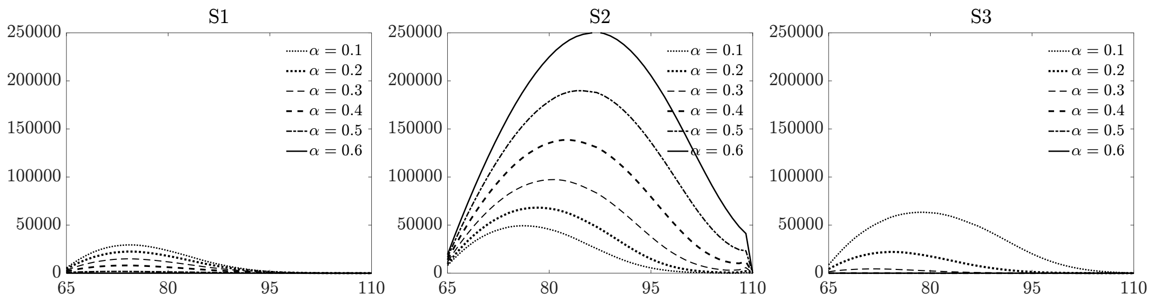


Figure 3.8 Median optimal real savings: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{W}_t - \tilde{C}_t$.

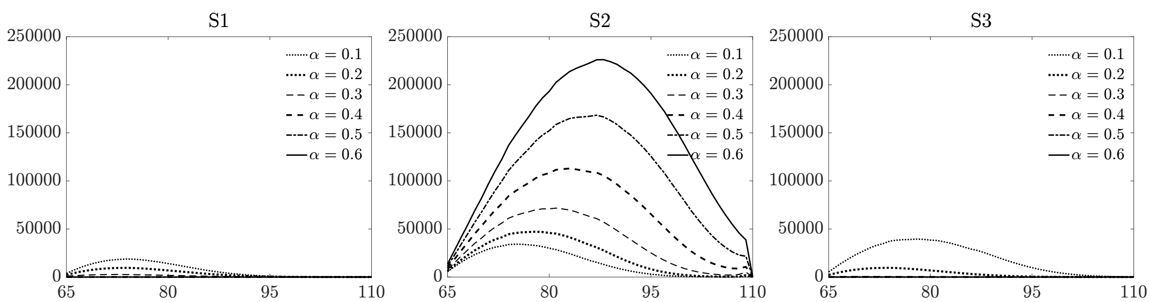


Figure 3.9 Median optimal real savings: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.5$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{W}_t - \tilde{C}_t$.

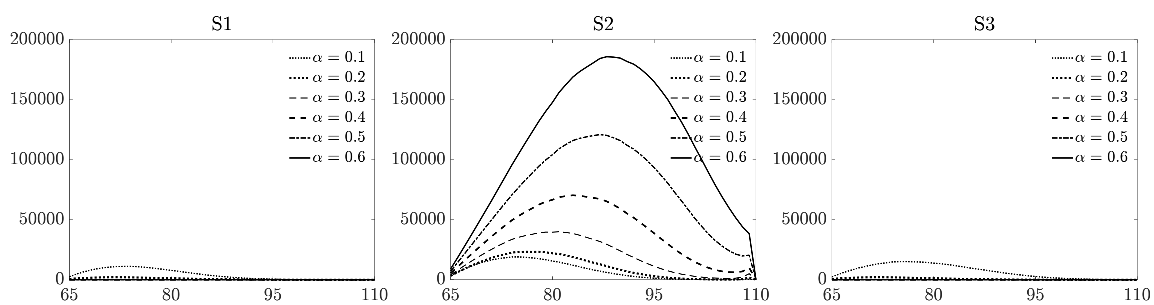


Figure 3.10 Median optimal real savings: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{W}_t - \tilde{C}_t$.

iv. Under S3, a lower α yields lower consumption volatility over time. This is as expected since α governs intertemporal substitutability and does not impact impatience.

Stock holdings
The median optimal investment strategies are shown in Figures 3.17, 3.18, and 3.19. The plots show the smoothed allocation in stocks (i.e. 10-year moving average of ω_t). We plot the smoothed process due to the high volatility in the median of ω_t in the first 25 years. This is a result of 1) error accumulation in dynamic programming, which is an inherent issue and is difficult to mitigate; and 2) high absolute tolerance level set for ω_t in the iterations in order to improve computational efficiency. Regardless, this is not a major concern since small variations in ω_t do not have a significant impact on the optimal consumption and annuitization choice, which we are primarily interested in. The key takeaway is that in general, stock allocation is sensitive to the precautionary savings, such that higher savings (compared to consumption) are associated with lower stock allocation.

3.7 Optimal Consumption

We show and discuss the long-term real consumption scenarios ($\tilde{C}_t$) under varying assumptions of

i. RRA (ρ),

ii. EIS (α),

iii. inflation protection (k), and

iv. economic scenarios.

We examine results obtained from assumption set 2. All estimated quantiles and moments are based on 10,000 simulations. We show scenarios for ages 65 to 95 only, due to high uncertainties in estimated mortality rates at advanced ages. We show results for $\lambda = 0.1$.

We further comment on the normative adequacy of the optimal consumption profile, using the measure introduced in Section 3.3. For each time t, we compute the following quantity as a relative measure for downside risk in consumption,

$$\mathrm{VaR}_a(\mathrm{RC}_t) = \frac{\mathrm{VaR}_a(\tilde{C}_t^*)}{\frac{W^\star}{\ddot{a}^I_{x^\star,0}(1+\lambda)}} \tag{3.9}$$

where RC_t stands for the real consumption at time t relative to the benchmark consumption, and $\mathrm{VaR}_a(\cdot)$ denotes the Value-at-Risk operator at a. We compute this quantity for $a = 0.01, 0.05$, which represents the threshold that relative consumption falls below with probability 0.01 and 0.05 at each point in time. The benchmark consumption profile is defined as the benefit of a fully-indexed IIA implied under full annuitization, which is, in real terms, constant over one's lifetime.

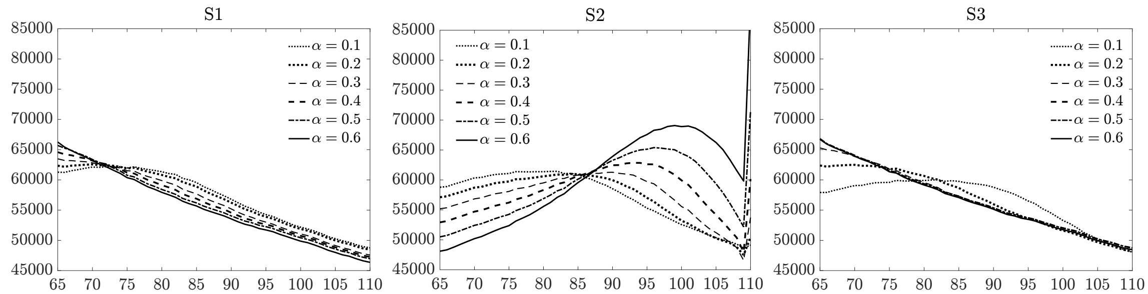


Figure 3.11 Median optimal real consumption: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$.

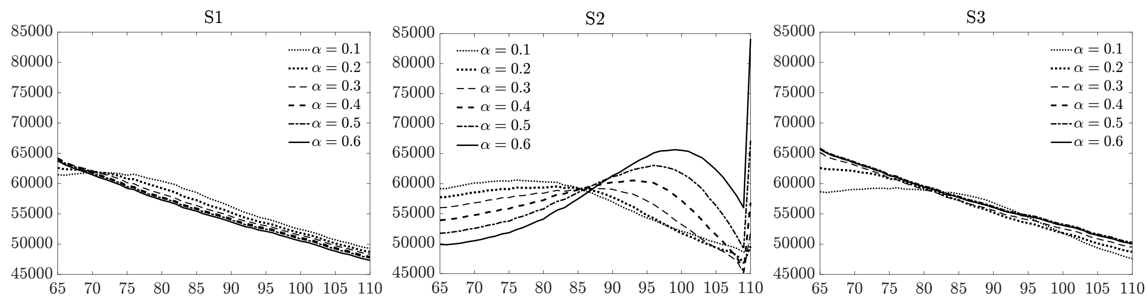


Figure 3.12 Median optimal real consumption: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.5$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$.

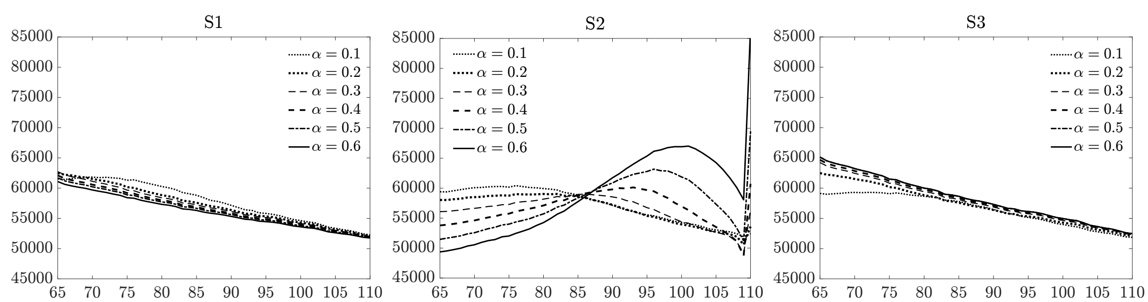


Figure 3.13 Median optimal real consumption: assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$.

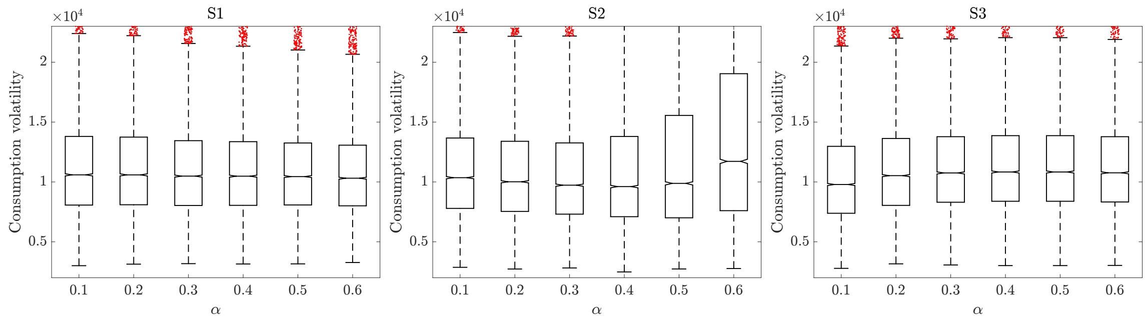


Figure 3.14 Boxplot: consumption volatility, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1$million.

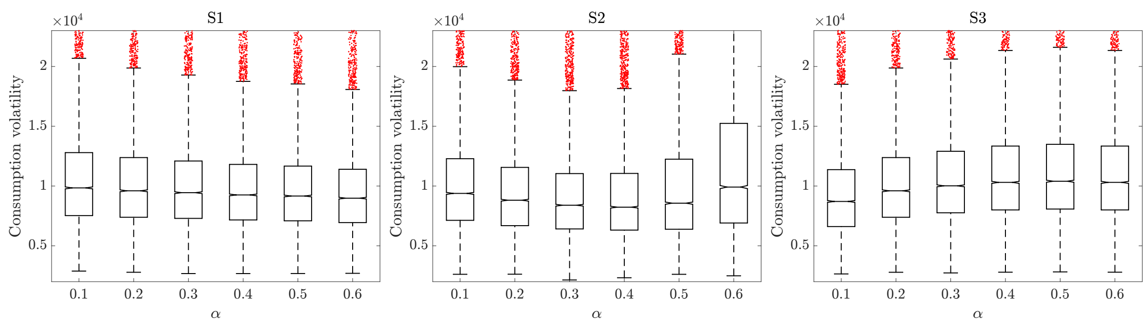


Figure 3.15 Boxplot: consumption volatility, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.5$, $W^\star = \$1$million.

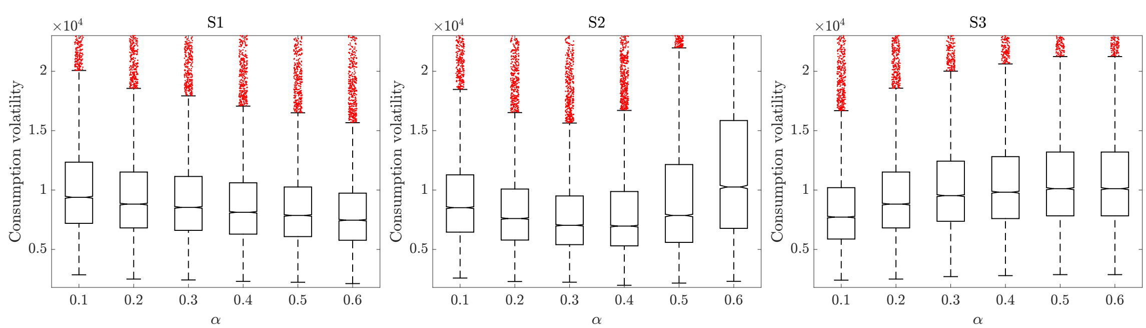


Figure 3.16 Boxplot: consumption volatility, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million.

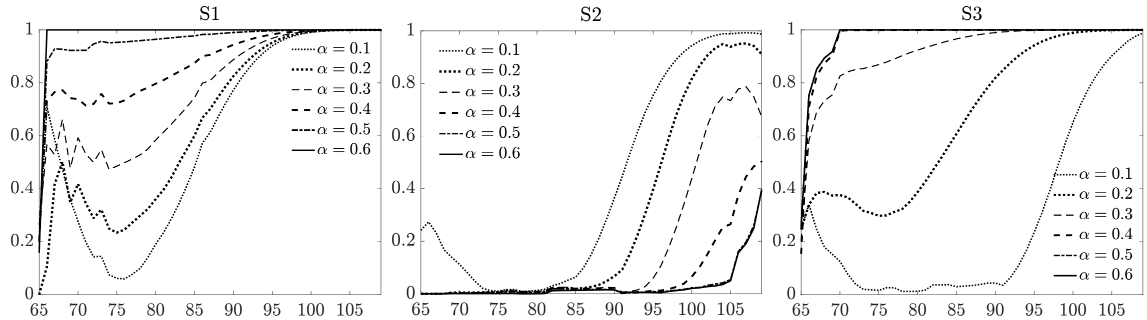


Figure 3.17 Median optimal stock allocation: Assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1$million. x-axis: age, y-axis: ω_t. 10-year Moving average.

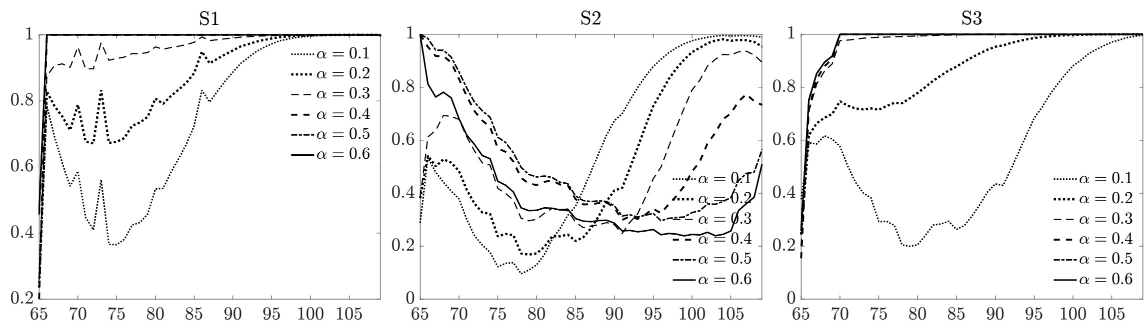


Figure 3.18 Median optimal stock allocation: Assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.5$, $W^\star = \$1$million. x-axis: age, y-axis: ω_t. 10-year Moving average.

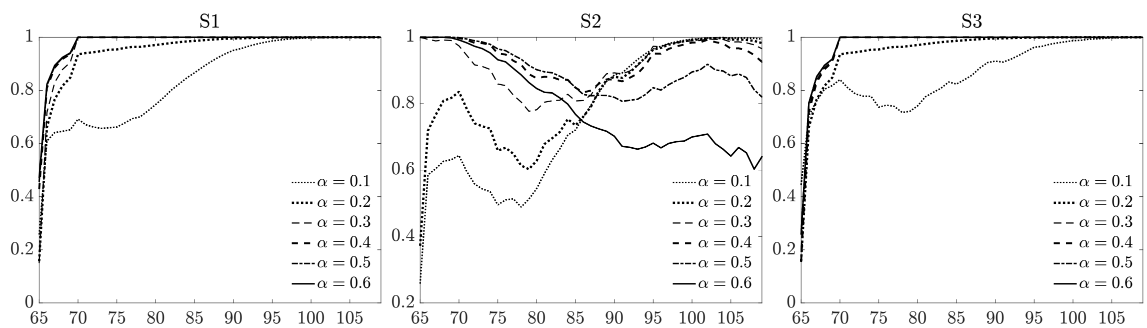


Figure 3.19 Median optimal stock allocation: Assumption 2, $\rho = 5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. x-axis: age, y-axis: ω_t. 10-year Moving average.

3.7.1 Under Assumption 2

Figures 3.20 to 3.23 plot the mean, median, the 5% and 95% estimated quantiles of the real consumption paths, under Assumption 2. 50 simulated consumption scenarios are also shown (see coloured paths), which in each plot represent the same 50 joint mortality and economic scenarios.

RRA (ρ)

Overall, the real average consumption appears constant over time, suggesting that the expected rate of growth of VPA benefits closely matches the rate of inflation. Comparing Figures 3.20 and 3.21, a stronger RRA leads to lower consumption volatility, at the expense of lower average consumption across all ages. An agent with $\rho = 2$ faces higher downside risk, which increases with age, but such prospects are being compensated by the upside scenarios with low frequencies but high magnitudes.

The left and right of Figure 3.20 appear almost identical. As mentioned in the previous section, under S3, the agent holds a very small level of savings to substitute consumption in the first 10–15 years. This effect is very small but it is visible that the agent trades off early consumption for lower volatility in the earlier years of retirement.

In Figure 3.21, S1 and S3 yield identical results since the model reduces to the time-additive EDLU-CRRA. Under S2, the agent consumes less in the earlier years in order to have higher consumption in the future. This, as discussed, is due to the effect of α on reducing impatience under the preference specification.

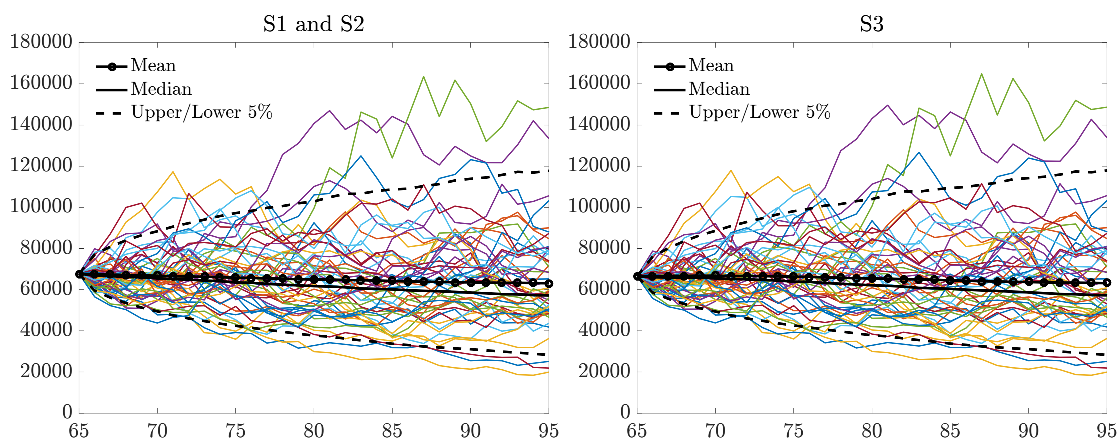


Figure 3.20 Simulated real consumption paths following optimal strategy: $\rho = 2$, $\alpha = 0.2$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1\text{million}$. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2.

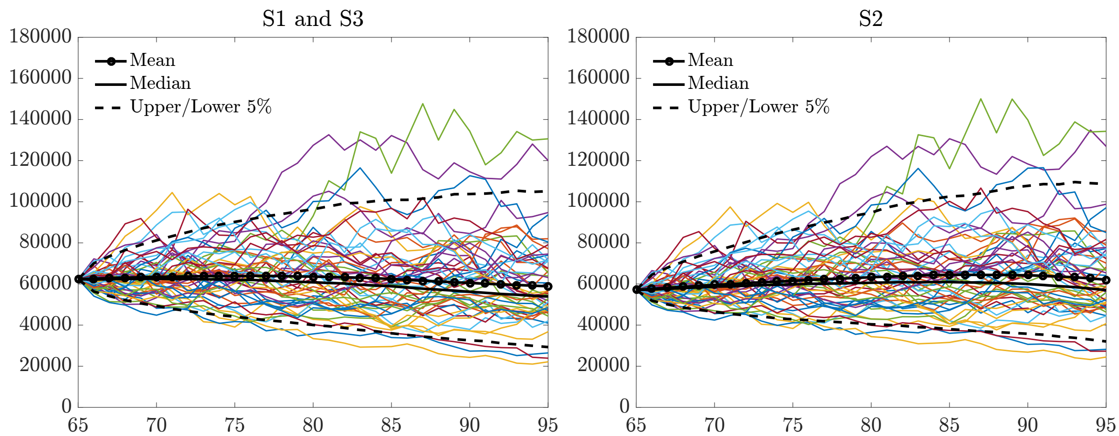


Figure 3.21 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.2$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^{\star} = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2.

EIS (α)

Comparing Figures 3.21 and Figure 3.22, we observe that the impact of α on the optimal consumption paths is more pronounced under S2, where a higher α leads to a drop in earlier consumption in exchange for higher consumption in the future (this has been discussed at length earlier). Under S1 and S3, the initial consumption levels are higher; the distributions of future consumption are highly similar. This reflects the highly comparable annuitization and savings decision when $k = 0$.

Inflation protection (k)

Comparing Figures 3.23 and 3.22, we see that a fully indexed IIA reduces consumption volatility in general. The average consumption is more smooth for $k = 1$. This difference is more visible under S1 and S2. Regardless, the overall differences are small since in the case of $k = 0$ the agent can purchase more VPA to achieve higher average consumption growth that matches the inflation rates. The results shown assume that the IIA and the FA are subject to the same fee load. In practice, the IIA may be sold at a much higher price to compensate for the extra inflation risk born by the provider.

Consumption downside risk

Figures 3.24 and 3.25 plot the estimated 5% and 1% Value-at-Risk (quantiles) of real consumption relative to the benchmark consumption profile. It is found that in all three preference models, downside risk in consumption is high and rises with age. The exposure of downside risk is higher for lower k, due to the cumulative impact of inflation on purchasing power. Comparing across preference models, downside risk in consumption is the lowest

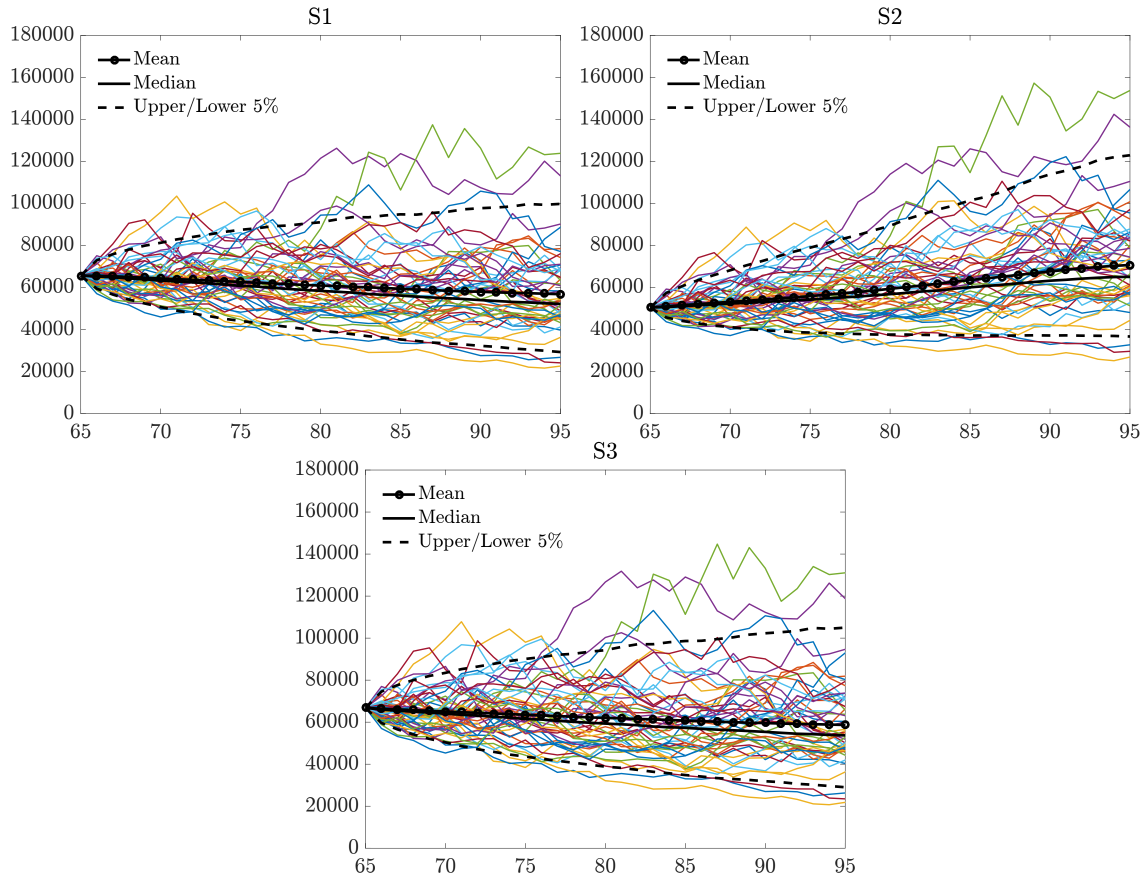


Figure 3.22 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2.

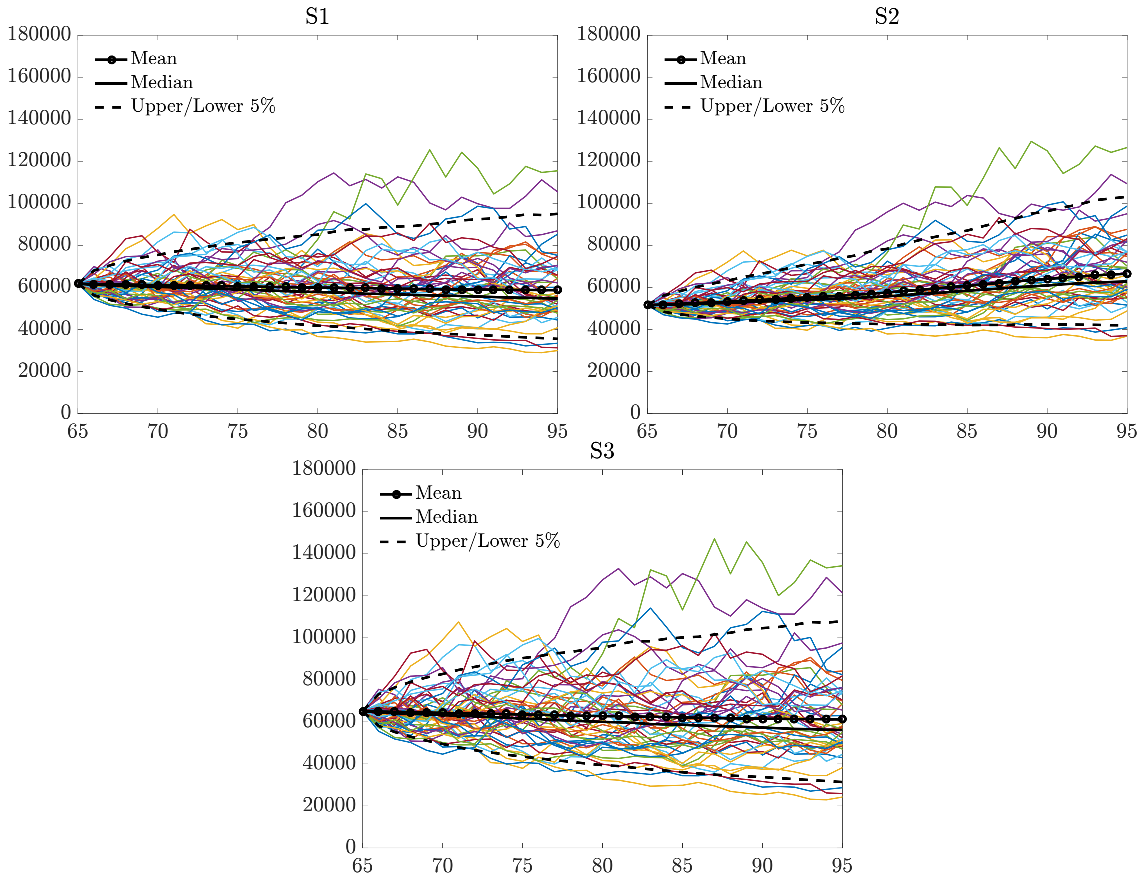


Figure 3.23 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2.

under S2, since the model yields lower impatience, hence higher savings. Under S1 and S3, downside risk in consumption is similar across RRA and EIS parameters. Figures 3.33 and 3.34 repeat the analysis under Wilkie's ESG, with similar observations.

3.7.2 Under Wilkie's ESG

Here, we investigate the optimal consumption behaviour in the presence of model uncertainty. Suppose that the optimal strategies are unchanged, and are executed by the agent as implied under the economic conditions specified in Assumption 2 (see Table 3.6). We are interested in the impact of model uncertainty on the resulting consumption profile.

Suppose that the 'actual' economy follows Wilkie's ESG fitted to historical data from 1991 to 2014, with parameters specified in Table A.1. We adjust the parameter $\sigma_c = 0.1$ to reduce the volatility in the long-term bond yields. The distribution of the force of inflation, total log-returns on stocks, and annual returns on long-term bonds are shown in Figures 3.26, 3.27, and 3.28, respectively. The agent behaves according to optimal strategies solved under Assumption 2.

Before examining consumption, we note the following key points regarding the scenarios of Wilkie's ESG:

i. the economic scenarios are generated based on biased (rather than 'neutral') starting points: the latest observations in 2014 are used as initial values,
ii. the force of inflation scenarios are effectively i.i.d. $\mathrm{N}(0.024, 0.01^2)$,
iii. the total stock log-returns are approximately i.i.d. $\mathrm{N}(0.048, 0.18^2)$,
iv. the long-run mean for annual nominal returns on long-term bonds is 0.039; the standard deviation is 0.044.

First consider an agent with RRA $\rho = 2$, Figure 3.29 plots the optimal consumption behaviour under an economy described by Wilkie's ESG. Compared to Figure 3.20, there is an evident rapid decline in average future consumption. This decline is a combined effect of higher average inflation and a lower average return on stocks. The differences are small when considered separately, but combined in the long term, they have a material impact on the agent's real consumption due to the high annuitization ratio (100%) in the VPA. Some of this impact is offset by higher returns on long-term bonds. Additionally, there is a considerable reduction in consumption volatility in each year (note the difference in axis scale). This reduction stems from the mild positive dependence between inflation and stock returns (through a negative w_y), and the negative dependence between stock and bond returns (through inflation). Recall that the VPA invests in a combination of stocks and bonds, implying that the volatility of VPA fund returns is lower.

Similar observations are made for $\rho = 5$, as agents still have relatively high exposures in the VPA and are consequently influenced by the higher erosion in purchasing power and the lower returns from stocks. The impact is however smaller compared to RRA $\rho = 2$, since the agents carry out less risk-taking strategies.

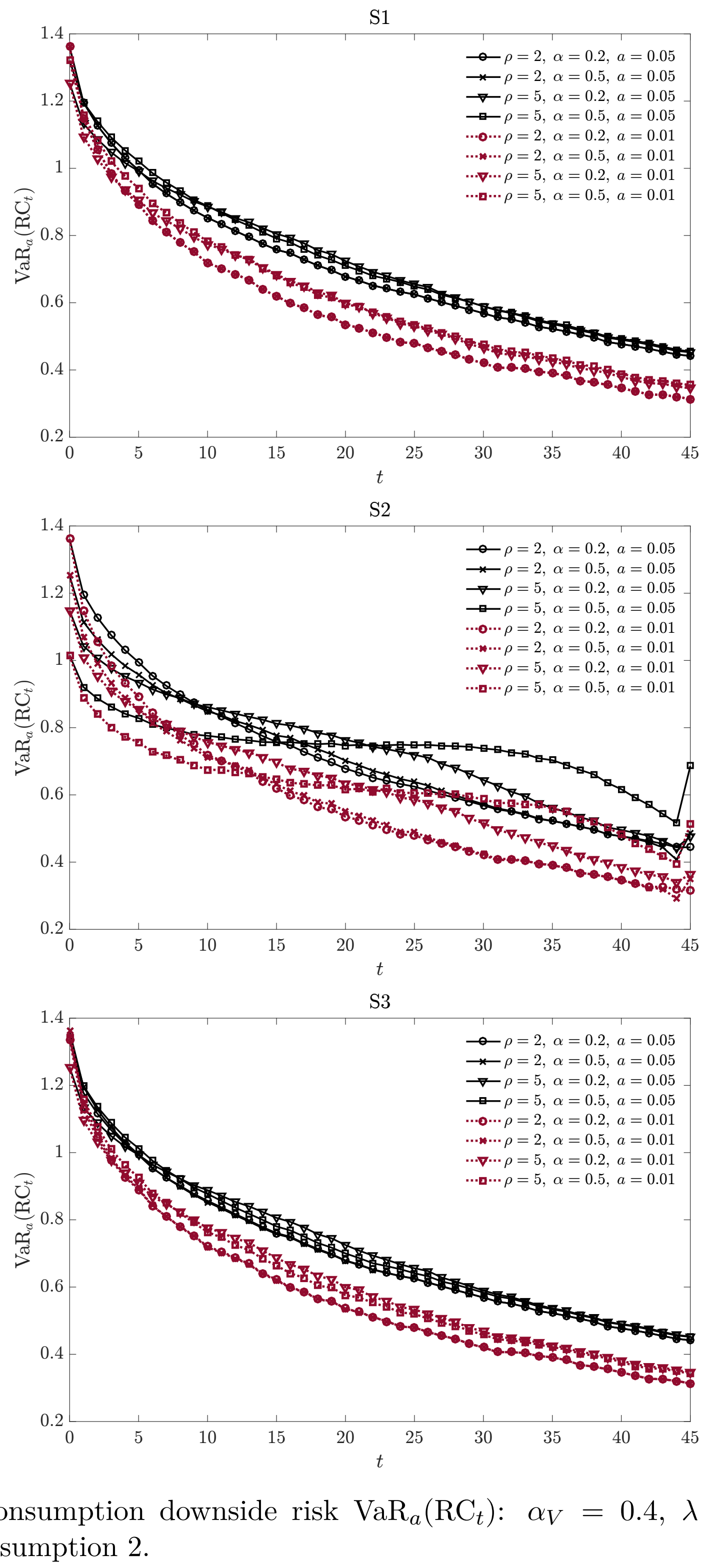


Figure 3.24 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $B_0^I = 49,746$. Assumption 2.

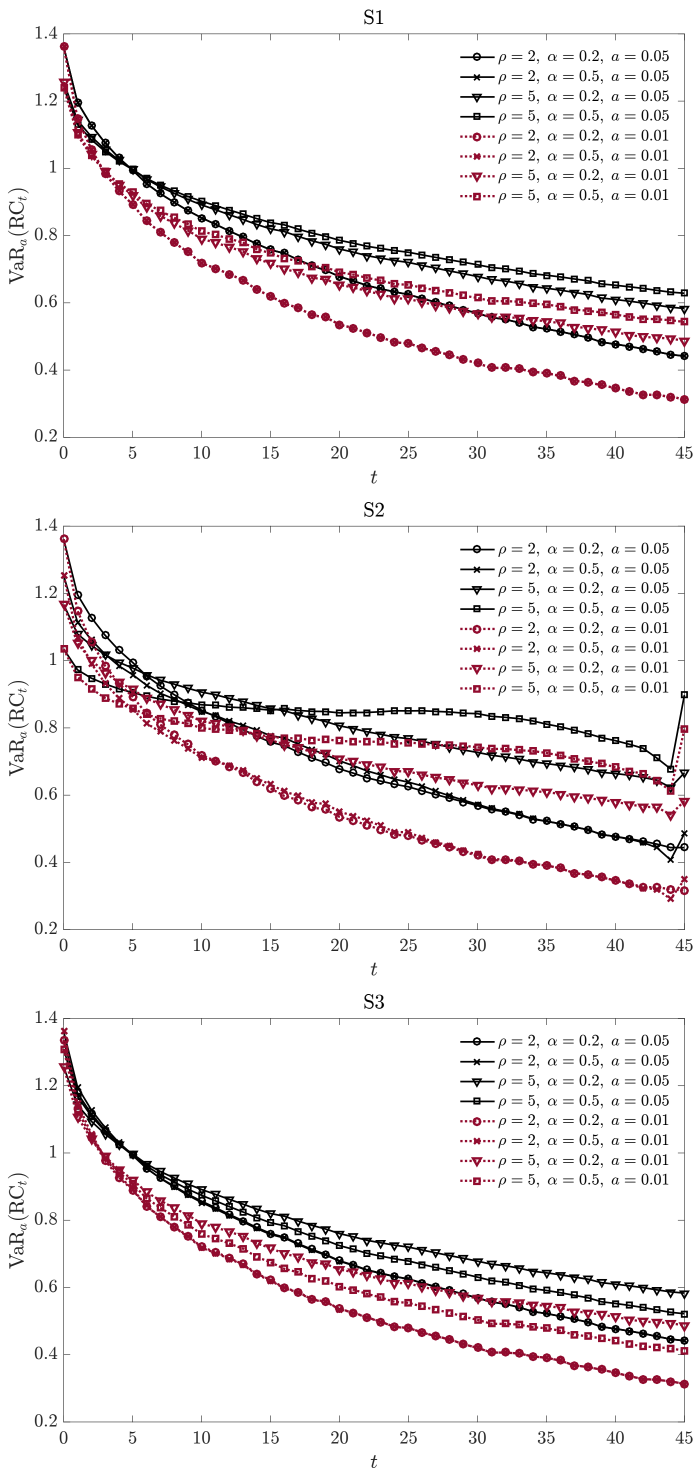


Figure 3.25 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $B_0^I = 49,746$. Assumption 2.

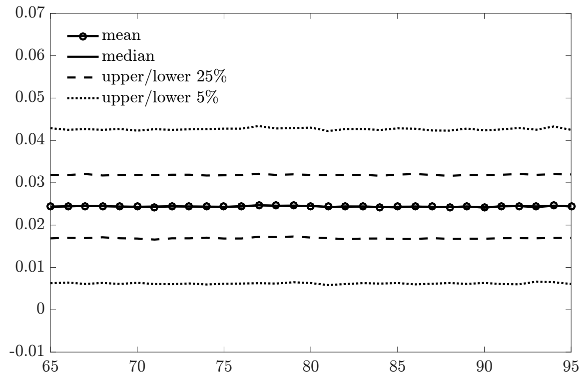


Figure 3.26 Force of inflation: Wilkie's ESG 1991–2014 (2014 starting point).

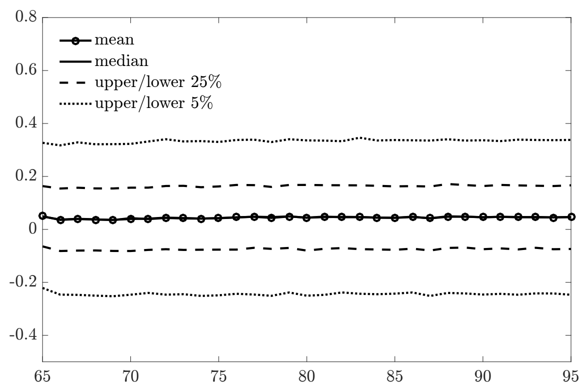


Figure 3.27 Total log-returns on stocks: Wilkie's ESG 1991–2014 (2014 starting point).

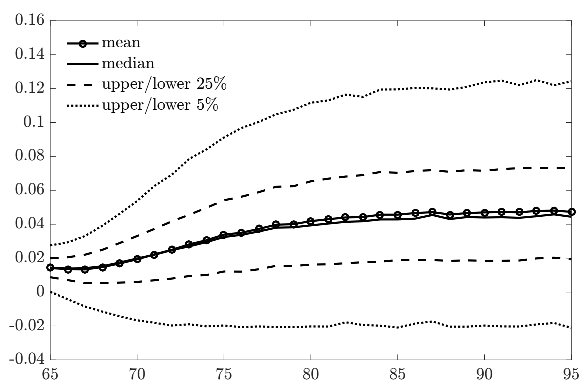


Figure 3.28 Annual returns on long bonds: Wilkie's ESG 1991–2014 (2014 starting point).

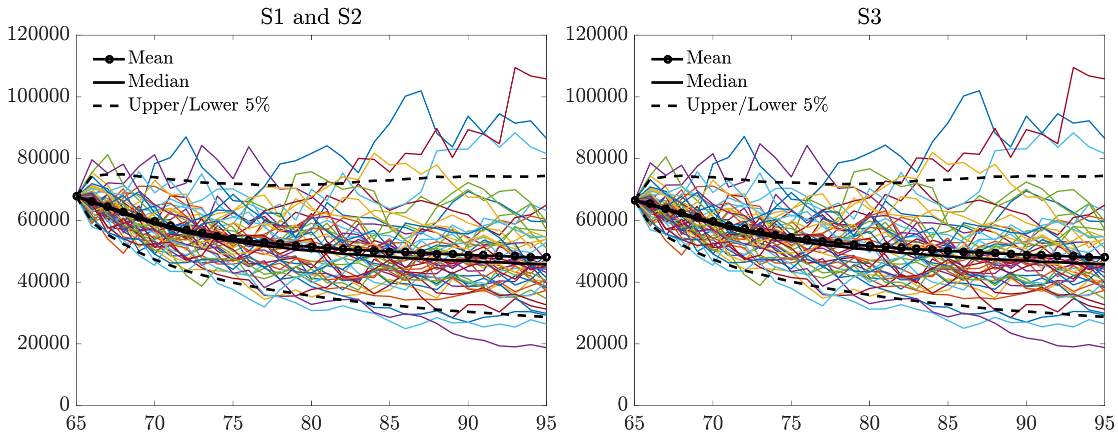


Figure 3.29 Simulated real consumption paths following optimal strategy: $\rho = 2$, $\alpha = 0.2$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1\text{million}$. x-axis: age, y-axis: $\tilde{C}_t$. Wilkie's ESG.

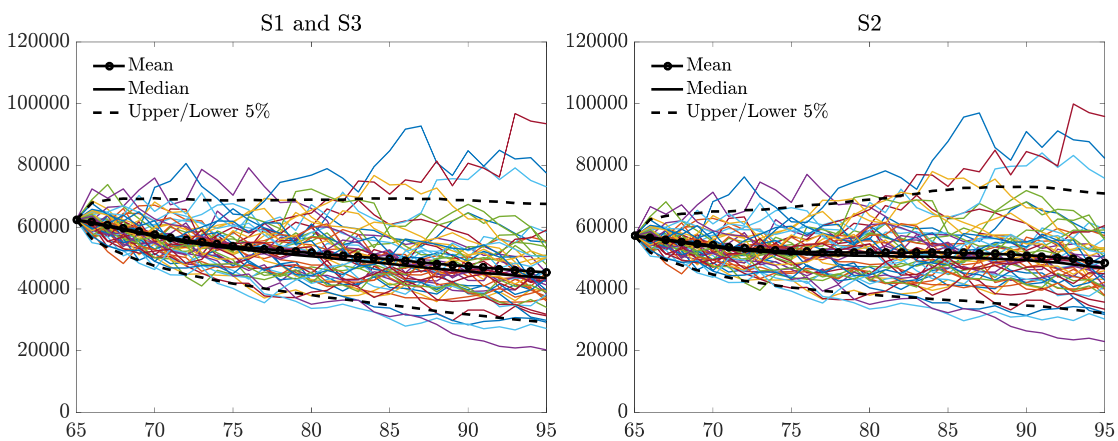


Figure 3.30 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.2$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^\star = \$1\text{million}$. x-axis: age, y-axis: $\tilde{C}_t$. Wilkie's ESG.

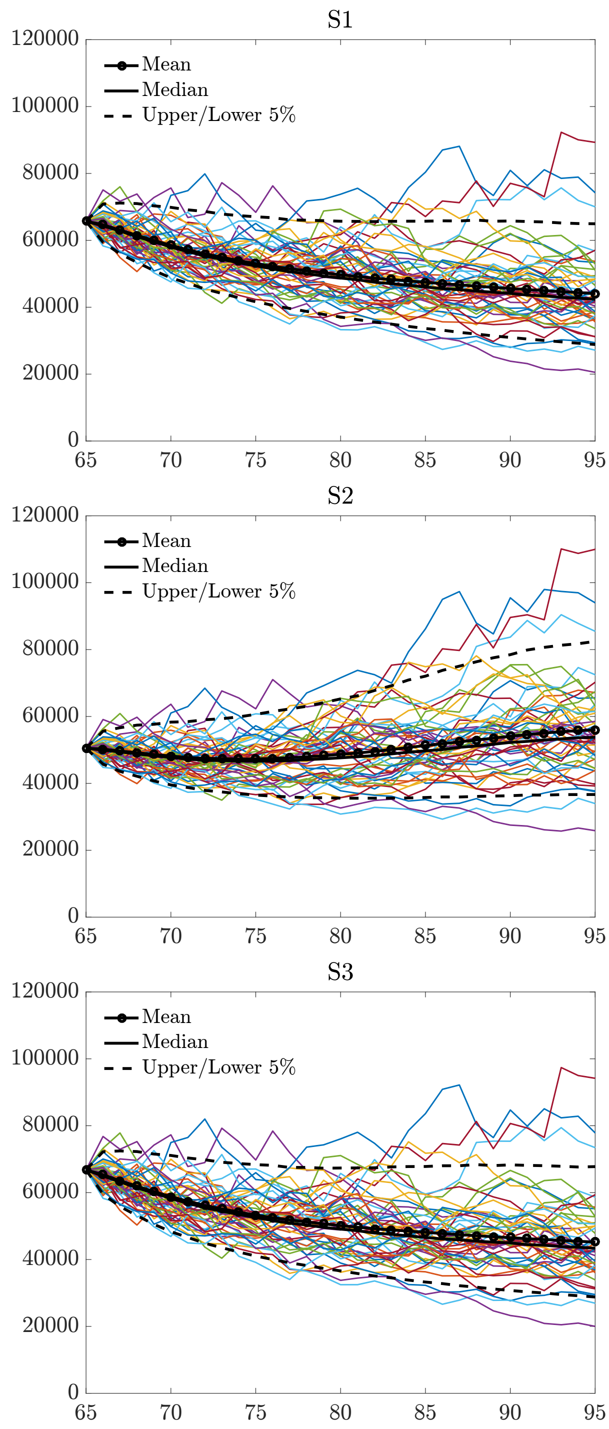


Figure 3.31 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $W^{\star} = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$. Wilkie's ESG.

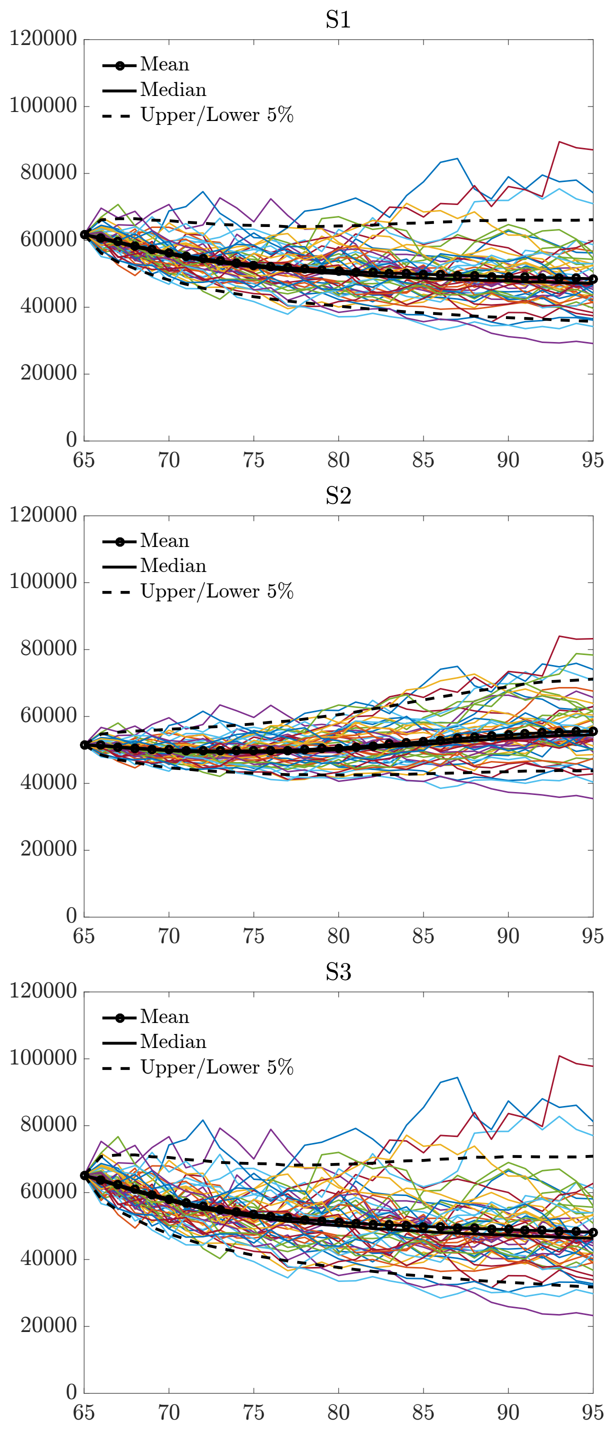


Figure 3.32 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. x-axis: age, y-axis: $\tilde{C}_t$. Wilkie's ESG.

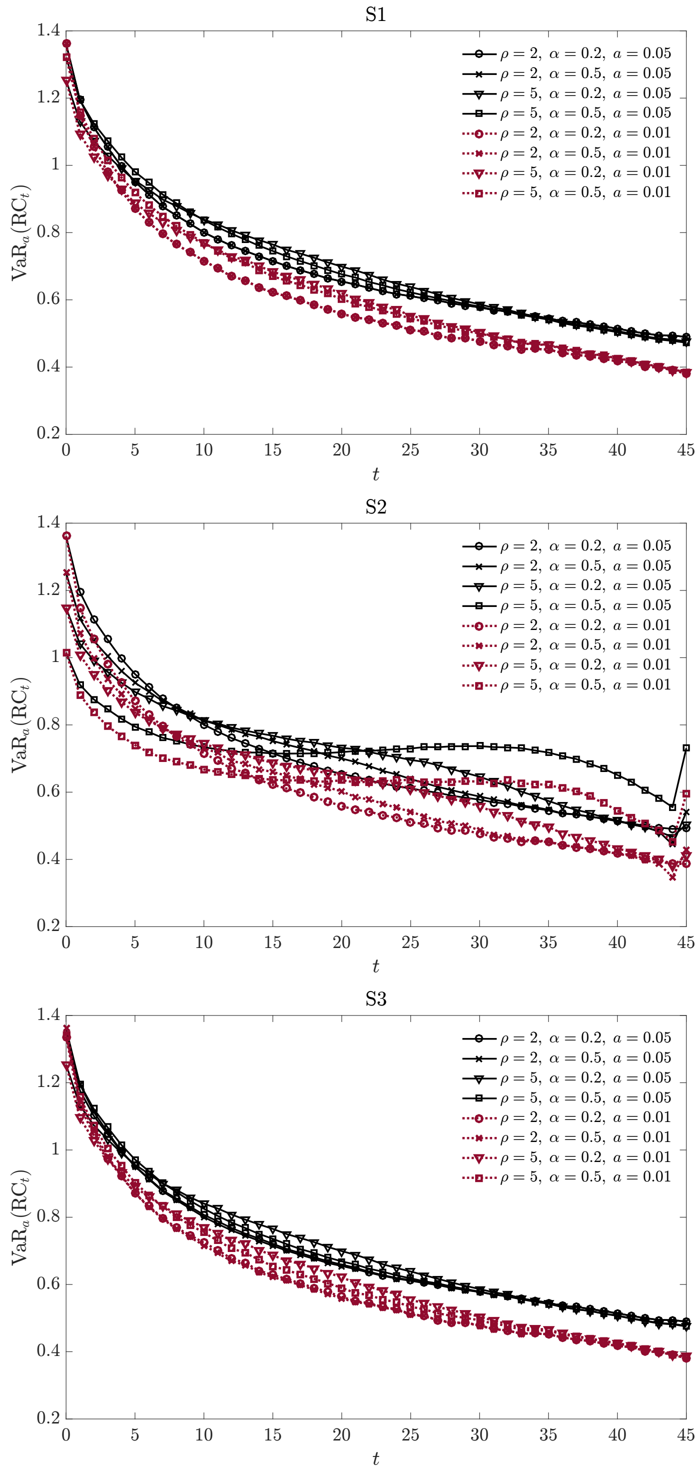


Figure 3.33 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 0.0$, $B_0^I = 49,746$. Wilkie's ESG.

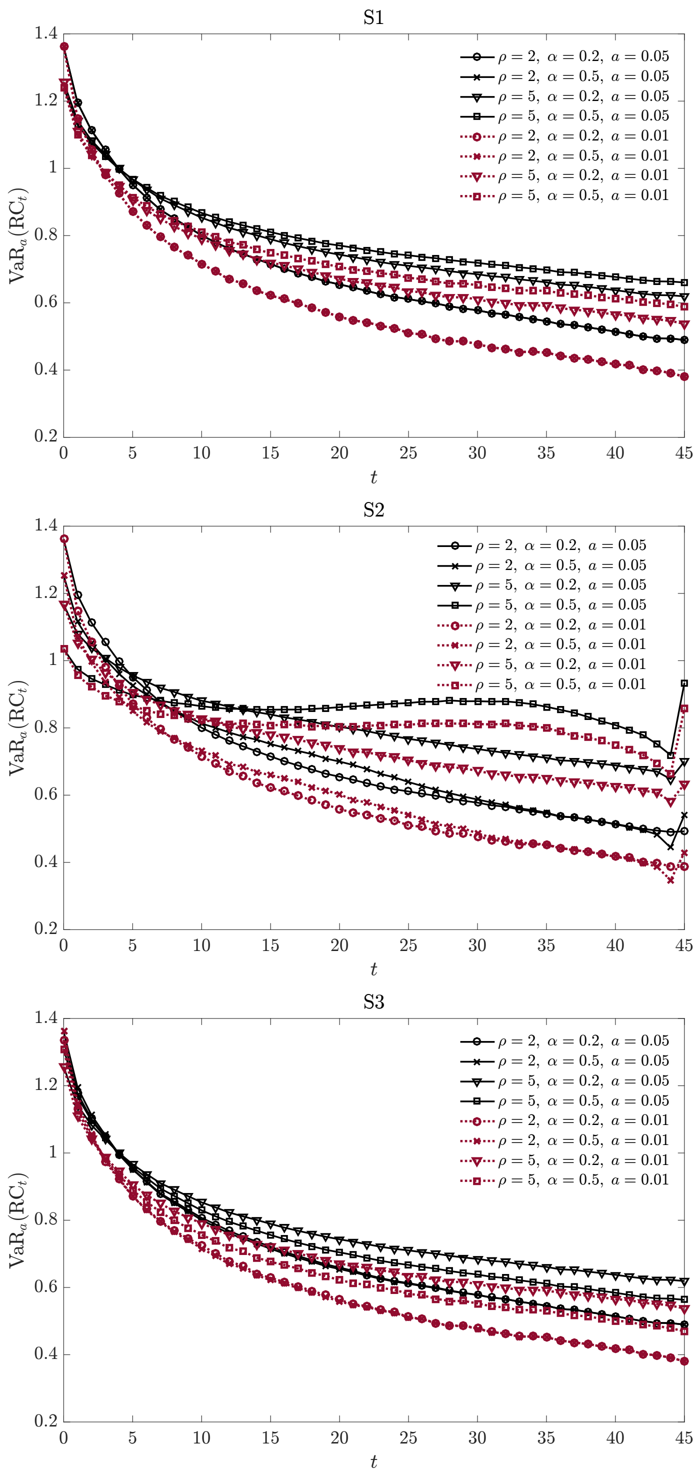


Figure 3.34 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\alpha_V = 0.4,\ \lambda = 0.1,\ k = 1.0,$ $B_0^I = 49,746$. Wilkie's ESG.

3.8 Time Discounting

One significant source of complexity in life-cycle problems stems from the modelling of long-term economic scenarios, as discussed in Chapter 1 and the previous sections. Another involves choosing an appropriate intertemporal preference model. This is a particularly challenging task, as the future inherently involves risks, uncertainties, and evolving preferences. The exploration of normative and descriptive models for choice over time has been an ongoing effort for economists.

The two pillars of the expected discounted lifetime utility (EDLU) used in intertemporal decision-making under uncertainty are the EU (expected utility) and the DU (discounted utility). The EU was first formulated by Bernoulli in 1738, and axiomatized in Von Neumann and Morgenstern (1944). To date, it has been accepted as the right normative model for decision-making under uncertainty, supported by numerous works including Broome (2017), Edwards (2013), Baron (1996), and many others. The DU, on the other hand, was first mathematically formulated by Samuelson (1937), which explicitly assumes that future gratification (utility) can be discounted and aggregated in a fashion that resembles the present value of future cash flows.

Both the EU and the DU have been criticized on normative grounds. Critics of the EU include Allais (1953), Ellsberg (1961), Tversky (1975), and Schmeidler (1989), among others. The focus of this section, however, is on DU. The original intent of DU is to propose a generalized intertemporal choice model that can be used in many applications. Although the author demonstrated serious cautiousness regarding the normative and descriptive validity of this formulation, stressing the arbitrariness of these underlying assumptions, the DU was quickly accepted as an adequate normative or descriptive model for applications ranging from savings behaviour, labour supply, security valuation, education decisions, and crime (Loewenstein and Prelec, 1992).

The DU in its most restrictive form includes the exponential discounting of future utility governed by the parameter β, which often reflects the "pure rate of time preference". Loewenstein and Elster (1992) provides a thorough historical overview of the psychological explanations of the fundamental question of "why do people discount the future?". Explanations include, among others, *the brevity and uncertainty of human life*, *the pain of abstinence*, and *imperfections in the translation of future events into the present*. These propositions evolve over time, at times reflecting conflicting philosophical beliefs about what constitutes human rationality. Among these scholarly discussions, some viewed equal treatment of present and future as a norm of behaviour, and strived to understand why human behaviour deviated from it (Loewenstein and Elster, 1992). For instance, the 19th century economist and logician, William Stanley Jevons, makes the following remark in Jevons (1871):

> To secure a maximum of benefit in life, all future events, all future pleasures or pains, should act upon us with the same force as if they were present, allowance

> being made for their uncertainty. The factor expressing the effect of remoteness should, in short, always be unity, so that time should have no influence. But no human mind is constituted in this perfect way: a future feeling is always less influential than a present one.

This remark points to the paradox of the inclusion of time discounting in a normative setting, where it fundamentally is a manifestation of irrationality (i.e. arguably a deviation from "optimality"). This raises serious doubt over the normative validity of the classical approach of EDLU maximization involving a pure time discounting rate smaller than 1.

In the current literature, the parameter governing the pure rate of time preference, β, is often assumed to be in the range of 0.95 to 0.98 (see e.g. Horneff et al., 2015; Hanewald et al., 2013; Horneff et al., 2010; Blake et al., 2008; Li and Smetters, 2010). These parameter values are rarely justified, nor is the implication on the preference of the representative agent. Considering a simple time-additive EDLU model, a β value of 0.96 suggests that gratification or utility in t years' time is 0.96^t times as important as the utility at the present. Specifically, this means the agent values gratification in 10 years as 34% less important than happiness today, and in 30 years, 71% less important. This assumption leads to a significant devaluation of consumptions at extreme-old ages. This is on top of the devaluation due to mortality risk, which in itself was historically argued as an explanation for the discounting of future events by the 18th century economist John Rae (i.e. the above-mentioned "the brevity and uncertainty of human life"). This devaluation of the future is highly contradictory to any rational description of long-term planning, due to the implicit myopia in the model assumptions.

This section proceeds to quantify the impact of pure time discounting on the optimal consumption profile. In previous sections, a $\beta = 0.96$ is assumed so that our results are comparable with the existing literature. Here, we set $\beta = 1.0$, assuming that the pure rate of time preference is unity, such that the agent equally values the happiness throughout her remaining lifetime. This reflects the philosophical belief cited above in Jevons (1871).

Setting $\beta = 1.0$

Setting $\beta = 1.0$ yields zero discounting stemming from pure time preferences. Here, we repeat the analysis in Section 3.5 and compare the optimal consumption path for $\beta = 0.96$ and $\beta = 1.0$. We show the results for S1 and S3 only, since the impact of mortality on time discounting can be highly inflated under S2; and for $\rho = 5.0$ and $\alpha = 0.5$. For simplicity, we consider the case where $k = 1.0$. The impact of β on other agents and the parameter k is similar, hence these cases are omitted.

Figure 3.35 is to be compared with Figure 3.7, and it shows the optimal annuitization decisions for the representative agents with zero pure time discounting. While full annuitization remains optimal, equal or less wealth is allocated in the VPA. Intuitively, a lower ω_V gives lower downside risk in the long run, mirroring the fact that higher weight is placed on consumption at older ages.

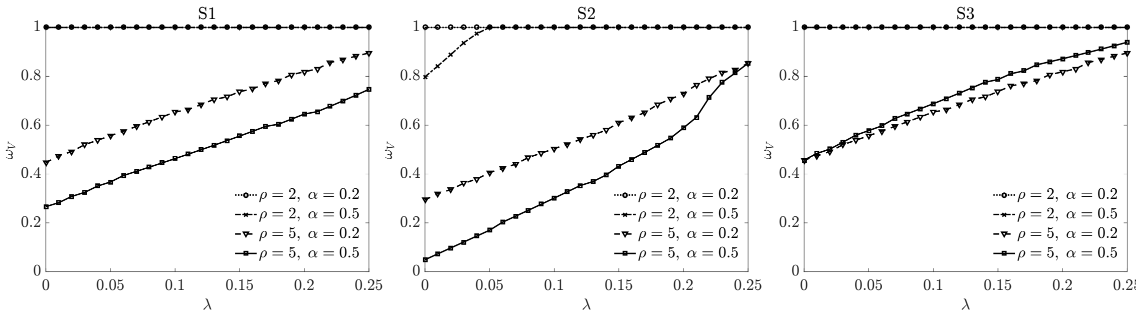


Figure 3.35 Optimal annuitization in a 40-60 VPA under Assumption 2, with stochastic systematic longevity development. $\beta = 1.0$. $k = 1.0$.

Comparing Figure 3.36 with Figure 3.23, we observe less volatility in the consumption due to the lower exposure in the VPA, and hence lower downside risk at older ages. This is more clearly shown in Figure 3.37 where downside risk in consumption is quantified. We repeat the optimization study for a more aggressively invested VPA with $\alpha_V = 0.6$ and $\text{AIR}_V = 0.06$, with the resulting consumption profile shown in Figures 3.38 and 3.39. In this case, the impact of β is more pronounced. Under zero time discounting, agents have a flatter consumption profile on average since future gratification is treated equally as present gratification (in absence of mortality), and lower downside risk in consumption. For instance, in Figure 3.40 under S1, one can clearly observe that for $\beta = 1.0$, after $t = 17$, the downside risk measured at $a = 0.01$ is even lower than the downside risk measured at $a = 0.05$ for $\beta = 0.96$, indicating a significant reduction in the risk of old-age poverty. This clearly motivates that, in a normative setting, pure time discounting should be set to unity, especially when a source of time discounting, mortality, is already accounted for in the model specifications.

3.9 Numerical Challenges

Numerical challenges centre around the computational speed, which is found to be linear in the number of optimizations being solved, hence linear in the number of grid points. Solutions are found to be more sensitive to the fineness of the $\tilde{b}_t$ grid than the $\tilde{w}_t$ grid, hence a coarser $\tilde{w}_t$ grid is used, especially since it becomes readily apparent that the optimal solution is concentrated around $\tilde{w}_t = 1$. The adopted optimization algorithm is Constrained Optimization by Linear Approximations (COBYLA), described in Powell (1994) and implemented by Johnson (2014) through `NLopt`.

The total number of optimization problems is approximately the number of grid points $\times$ $(T - 1)$ which is, for example, $13 \times 101 \times 4 \times 4 \times 45 = 945{,}360$. Efficiency is highly sensitive to the initial values of the controls. We find that dynamically setting the initial values using

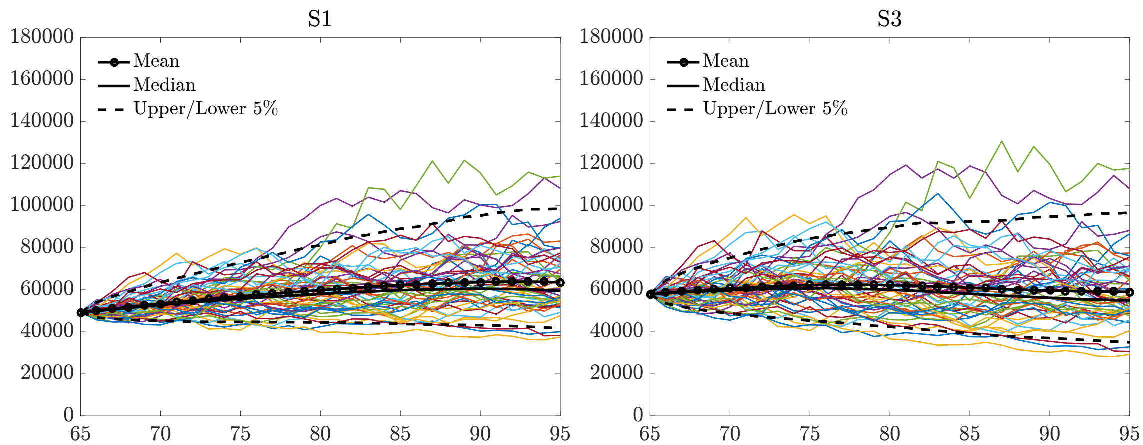


Figure 3.36 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1$million. $\beta = 1.0$. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2.

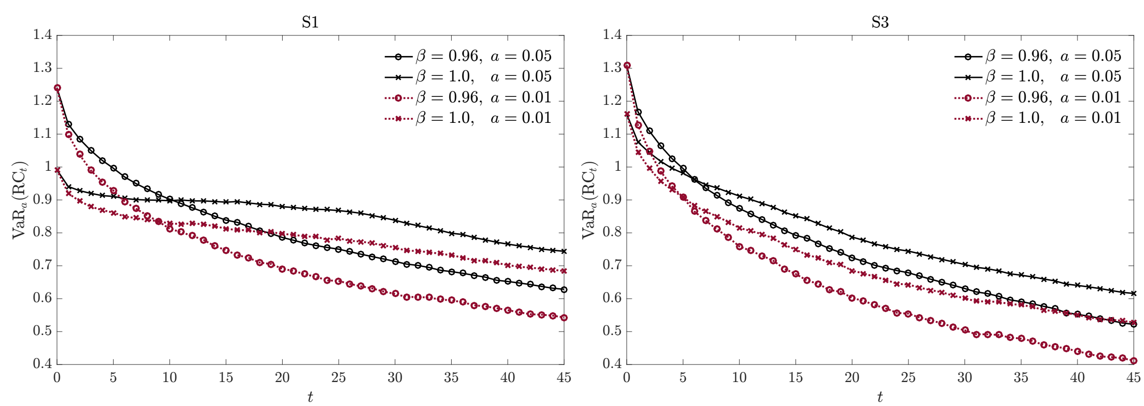


Figure 3.37 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.4$, $\mathrm{AIR}_V = 0.03$, $\lambda = 0.1$, $k = 1.0$, $B_0^I = 49,746$. Assumption 2.

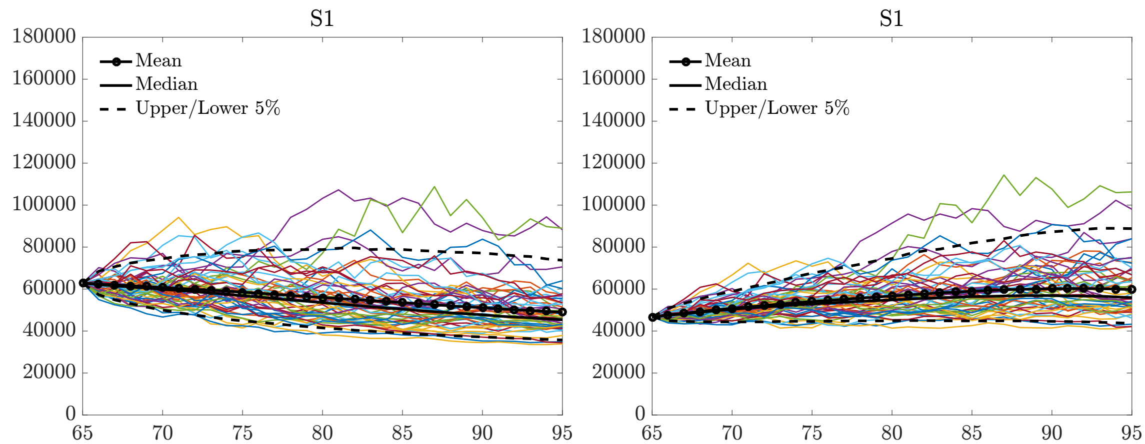


Figure 3.38 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.6$, $\text{AIR}_V = 0.06$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1\text{million}$. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2. Left: $\beta = 0.96$. Right: $\beta = 1.0$.

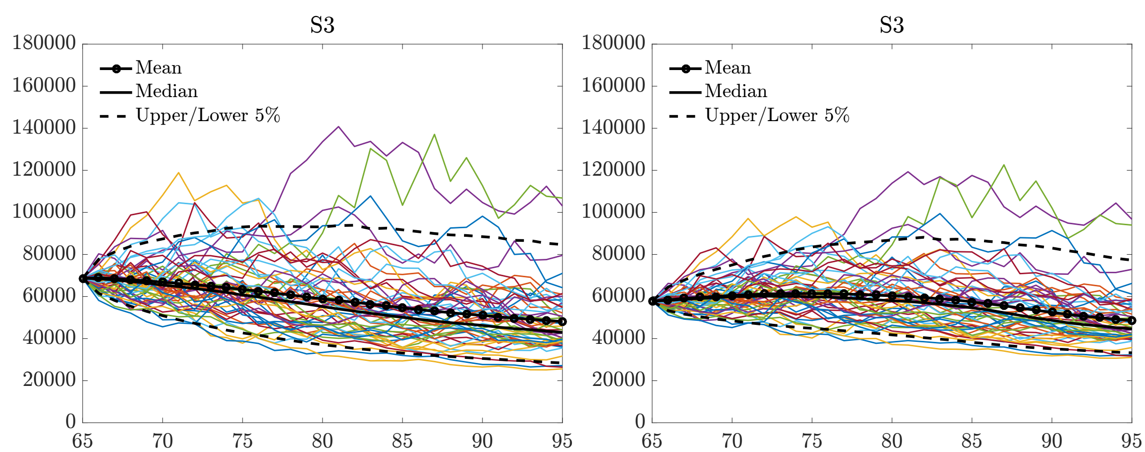


Figure 3.39 Simulated real consumption paths following optimal strategy: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.6$, $\text{AIR}_V = 0.06$, $\lambda = 0.1$, $k = 1.0$, $W^\star = \$1\text{million}$. $\beta = 1.0$. x-axis: age, y-axis: $\tilde{C}_t$. Assumption 2. Left: $\beta = 0.96$. Right: $\beta = 1.0$.

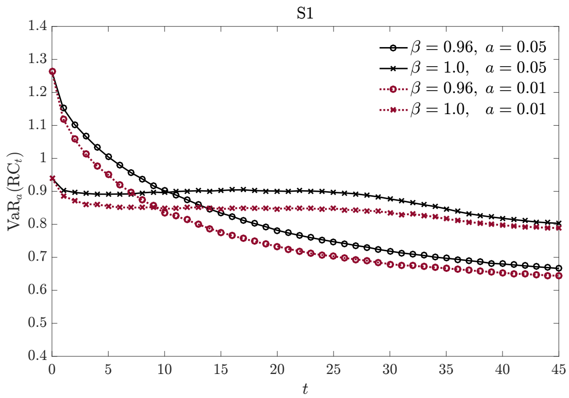


Figure 3.40 Consumption downside risk $\mathrm{VaR}_a(\mathrm{RC}_t)$: $\rho = 5$, $\alpha = 0.5$, $\alpha_V = 0.6$, $\mathrm{AIR}_V = 0.06$, $\lambda = 0.1$, $k = 1.0$, $B_0^I = 49,746$. Assumption 2.

optimized controls in the previous period dramatically improves computational efficiency. Runtime is linearly proportional to the number of simulations used to approximate the conditional expectation of the continuation values. It is further found that when RRA is high, finer grids are required to obtain more stable solutions. We report that on a 4-core AMD Opteron @2.2GHz, jobs with 4-dimensional state space take on average 40–60 hours. The program is written in the `C++` language.

The problem of efficiency can be improved through the quadrature method in evaluating the conditional expectations, and through parallel computing. At each time step of the backward recursion, the grids can be separated and solved simultaneously, subject to the condition that it does not access unknown values at the previous step. This is in strict sense not parallel computing, due to the reliance on recursions. At each time step the program cannot proceed if it tries to access an unsolved grid point in the previous period. This can be improved by either dynamically assigning optimizations to different positions in the grid, or considering alternative sequential arrangements. The computation speed will theoretically be improved by a factor equal to the number of threads, plus some overhead arising from the communication of information among processing elements.

3.10 Conclusion

We solved an optimal consumption-investment problem for a retired individual facing longevity and inflation risks in the presence of VPAs. We found that, assuming full annuitization and consumption of wealth, and under stochastic economic conditions projected by Wilkie's ESG, a time-additive EDLU-CRRA maximizing agent with zero bequest motive more highly values conventional annuity products (i.e. fixed, escalating nominal, and inflation-indexed annuities) when inflation is introduced.

We further considered four types of Epstein-Zin agents under three preference specifications, S1, S2 and S3, discussed in Chapter 2. The preference models differ in the treatment of mortality risk. We found that the optimality of full annuitization remains unchanged. However, agents with higher RRA ($\rho = 5$) would hold precautionary savings, the level of which differs dramatically for the three preference specifications.

Overall, the annuitization decision is more strongly affected by RRA than the EIS, such that higher RRA leads to lower annuitization in the VPA. The effect of EIS is difficult to interpret under S1 and S2, since EIS no longer governs preferences over consumption smoothing over time. Under S3, a lower EIS reduces annuitization in the VPA when RRA is high. Overall, the effect of EIS is stronger when the indexing factor k is larger. Incorporating stochastic systematic longevity development in the VPA mortality assumptions is found to lower the attractiveness of the VPA, as the systematic longevity risk is nondiversifiable and the retiree's risk exposure increases subsequently.

We projected retirees' real consumption paths under the optimal strategy. Comparing the consumption behaviour under S1, S2, and S3, we conclude the following:

i. under S1 and S3, altering the EIS had a minor impact on the optimal consumptions. In particular,
   - under S1, a lower EIS leads to higher consumption volatility;
   - under S3, a lower EIS leads to lower consumption volatility;

ii. under S2, altering the EIS had a more significant impact. However, certain agents would save in order to have significantly higher consumption later in life. This is due to the impact of EIS on impatience, causing time discounting to be greater than 1 when mortality is high. This is an undesirable property of the preference model, especially when used in the context of retirement planning.

Overall, we note relatively high levels of downside risk in the long run, despite the extension to the Epstein-Zin specification which allowed for a more flexible preference specification for long-term intertemporal problems. This is partially attributed to the implicit assumption of myopia by allowing the pure rate of time discounting to be smaller than one. Drawing on the literature on choice over time, we argue that in a normative setting, the pure rate of time discounting should be set to unity to reflect the equal treatment of gratification across one's lifetime. We repeat the optimization with zero time discounting and find a significant reduction in consumption downside risk in the long run, highlighting the impact of this assumption on the risk of old-age poverty. In the next chapter, we present the empirical finding that individuals are significantly less myopic than most prescriptive models assume, further highlighting the paradoxical nature of the parameter β in either prescriptive or descriptive modeling of retirement planning.

Chapter 4

Retirement Consumption, Risk Perception, and Planning Objectives

4.1 Introduction and Motivation

This chapter focuses on retirement objective formulation, which requires a comprehensive understanding of actual retirement experience, as it relates to longevity, consumption, wealth, risk perception, and variability of income, etc. To this end, a survey study is suitable. In this chapter, we report the findings of a large-scale web-based survey study of 1,000 Canadian (pre-)retirees on retirement planning objectives. This investigation is conducted in a country-specific context, since social insurance programs, including public pensions and health care, vary dramatically across countries and potentially play an important role in retirement planning concerns. In addition to normative objective formulation, we are interested in the descriptive validity of the expected lifetime discounted utility maximization framework in predicting optimal planning behaviours. This is grounded on the work of Beshears et al. (2014), which designed a set of comprehensible quantitative survey questions on retirement income preferences and is used as a basis for our study.

The key contribution of this chapter is to shed light on actual retirement decision-making, and compare and contrast our findings with theoretical retirement objective formulation. This has both normative and descriptive implications. Motivated by the concern over the current state of research on retirement planning, which has tended to be driven by mathematical elegance rather than practicability or efficacy, we bring forward empirical evidence of retirement planning, as it relates to consumption needs, risks, and preferences over annuities and pension incomes. This contributes to the literature by 1) identifying actual challenges in individual retirement planning, and their causes, and 2) driving future research and public policy changes that address these challenges. Through this study, we seek to stimulate new discussions on retirement decision-making and inspire innovation. It may appear to the reader that this work raises more questions than it answers.

This work was partially funded by the Canadian Institute of Actuaries, for which a technical report was prepared, peer-reviewed, and published. This chapter attaches this report with minimal changes.

4.2 Executive Summary

4.2.1 Overview

The Ontario Retirement Survey (ORS), conducted by Saisai Zhang, Mary Hardy, and David Saunders at the Department of Statistics and Actuarial Science, University of Waterloo, explores retirement consumption, risk perception and alternative objective functions and decision-making models in the retirement planning phase of Canadians' lives. The study carried out an online survey of 1,000 randomly selected Ontario pre-retirees and retirees aged 50 to 80, with the purpose of determining key elements of the retirement planning and experience of Canadians, including retirement age, subjective survival beliefs, private pension wealth and income, preferences among different consumption patterns, key risks concerning retirees and pre-retirees, and subjective time discounting factors.

This study is motivated by concerns regarding the adequacy of traditional applications of mathematical optimization to retirement planning problems. These applications typically focus on either the mean-variance tradeoff, or the maximization of expected discounted lifetime utility derived from consumption and wealth. While the resulting optimization models have nice mathematical properties, it is not clear that the optimal portfolio strategies genuinely reflect the retirement planning concerns of individual Canadians. The ORS aims to explore the adequacy of the utility maximizing framework under risk-averse agents, by gaining a more complete picture of retirement planning concerns and behaviours of Canadians, and by comparing and contrasting observed preferences with preferences implied under such a theoretical framework.

The study explores three key areas. The first is the difference between expectations and experience among Canadian retirees, in particular, as it relates to longevity (life expectancy and survival rates), consumption (annuity prices and expected savings required to maintain a given level of income), and risk and variability of income (expectations of future income requirements and their variability). The second area relates to the current level of wealth in retirement, or savings pre-retirement (in particular in contrast to expectations). The third area addresses preferences and objectives. In this area, following in the vein of some of the literature on the elicitation of utility functions, survey participants are presented with a sequence of choices aimed at determining their risk and time preferences, and in particular in assessing whether the emergent preferences are well-represented by commonly employed utility functions and objectives

The survey was launched on September 20, 2016 and closed on October 13, 2016. The survey had a response rate of 7.7% and an average survey length of 27 minutes. The sample is reasonably well-representative of the population in terms of key demographic factors such as age, gender, marital status, household income, and retirement assets.

4.2.2 Main Findings

The main findings of this study are highlighted below.

- **Expected retirement age:** on average, pre-retirees expect to retire at a later age than retirees have experienced: the median and mean retirement age for retirees is 60 and 58.6, respectively; pre-retirees' expectation for the age of retirement has a median of 65 and a mean of 67.1.
- **The prevalence of low retirement wealth:** the study finds that a majority (61%) of the respondents have/expect relatively "low[1]" liquid retirement assets, with the phenomenon more pronounced for female respondents. Alarmingly, 10% have/expect less than $25,000 of liquid retirement assets and do not own their home or other properties, implying low living standards in retirement. Overall, 21% expect/have liquid retirement assets under $25,000, 40% under $100,000.
- **Severe underestimation of survival probability to an extreme old age:** while respondents have reasonable beliefs for their own life expectancies, there is strong evidence that they highly underestimate the probability of survival to at least age 95 or more. 92% of respondents report a subjective survival rate below the objective rate implied by the most recent life table, 27% believe that they have zero chance of surviving to age 95.
- **Retirement income expectations:** overall, when compared to retirees' experience, pre-retirees have reasonable expectations regarding income in retirement from a variety of sources. Fewer pre-retirees (than retirees) have defined benefit (DB) workplace pensions, mirroring a shift away from DB arrangements. Pre-retirees expectations for social insurance pensions are quite strongly related to the highest level of education attained (which is positively correlated with financial literacy). Certain pre-retirees may underestimate public pension benefits due to a lack of knowledge of the Canadian retirement income system.
- **Retirement planning and spending concerns:** the four most important concerns when making planning and spending decisions are: liquidity, consumption/income smoothing, inflation and longevity. Bequest and investment risk-taking are considered the least important. Concerns vary according to retirement wealth levels: high wealth respondents are more concerned with meeting home care or nursing home expenses than longevity risk; extremely low wealth respondents are more concerned with consumption/income smoothing. Female respondents, in general, are found to be more concerned than male.
- **Bequest motives:** bequest is generally viewed as fairly unimportant. Nonetheless, respondents with more dependents tend to have higher bequest motives. The strength of bequest motives is also found to be affected by when death occurs: respondents have mildly higher bequest motives if death were to occur earlier in life.

[1] Below $200,000 for single respondents; below $300,000 for married/common-law respondents.

- **Seeking professional financial advice in retirement planning:** the overall attitude towards seeking professional financial advice is positive. Behaviour, however, is found to be strongly related to liquid retirement assets: respondents with low liquid assets show little interest in seeking advice, mainly due to affordability. In general, respondents show high concerns over potential issues such as accessing quality service, conflicts of interest, and frauds.
- **Attitudes towards life annuities:** the study found that respondents profoundly undervalue life annuities. 84% of respondents reported a subjective price that is below half of the average market price in 2015. Respondents additionally showed extremely low interest in purchasing annuities at any price, with top concerns being credit risk (i.e. the fear of provider default), a loss of flexibility and control, and a loss of financial security.
- **Risk preferences elicitation:** qualitatively, the average level of risk aversion among respondents is "moderate". The average elicited constant relative risk aversion (CRRA) parameter is between 1.74 and 2.48 for pre-retirees, and between 2.48 and 3.74 for retirees, implying that retirees are more risk averse. The elicited CRRA parameter is found to also depend on age, gender, education and wealth level, such that those who are female, with fewer years of formal education, or poorer tend to be more risk averse.
- **Time preferences elicitation:** the median subjective time discount factor elicited is 0.997, and the mean is 0.965. 85% of the elicited factors are greater than 0.96, 71% are greater than 0.98. The two discount factors, 0.96 and 0.98, are commonly used in the retirement planning literature, and imply much stronger myopia in long-term problems than empirically observed in this study.
- **Decision-making under risky scenarios:** this quantitative section studies the descriptive validity of the traditional life-cycle investment approach to retirement planning, which is to maximize welfare measured by the expected discounted lifetime utility derived from real consumption. The study finds dramatic differences between the actual and implied choices under the welfare maximization framework.
  - *Inflation-indexed pension versus steady pension:* the study of decision-making under risky inflation scenarios found that respondents, in general, lack the understanding of the long-term cumulative impact of inflation on the cost of living. This is manifested in the dramatic change in preferences when cumulative inflation impact is depicted. Respondents more concerned over the impact of inflation on the costs of living, and those with more optimistic longevity beliefs, are willing to "pay" more in exchange for inflation protection.
  - *Equity-linked pension versus steady pension:* the study of decision-making under risky equity returns scenarios reveals that when choosing between an equity-linked pension and a steady pension, the potential for upside gain (in the equity scenarios) drives decision-making. It is also found that having a higher minimum income protection,

which in most cases, is provided by a social insurance pension income, induces more risk-taking behaviour.

4.3 Introduction

The Ontario Retirement Survey explores retirement consumption, risk perception and alternative objective functions and decision-making models in the retirement planning phase of Canadians. We carry out a survey of 1,000 randomly selected Ontario pre-retirees and retirees. The objective of the survey is to obtain a more complete picture of the retirement objectives of Canadians, and to compare and contrast them with the objectives that are commonly assumed in models of lifetime portfolio selection. We are interested in three key areas. The first is the difference between expectations and experience among Canadian retirees. In particular, as it relates to longevity (life expectancy and survival rates), consumption (annuity prices and expected savings required to maintain a given level of income), and risk and variability of income (expectations of future income requirements and their variability). The second area relates to the current level of wealth in retirement, or savings pre-retirement (in particular in contrast to expectations). The third area addresses preferences and objectives. In this area, following in the vein of some of the literature on the elicitation of utility functions, survey participants will be presented with a sequence of choices aimed at determining their risk and time preferences, and in particular at assessing whether the emergent preferences are well-represented by commonly employed utility functions and objectives.

The purpose of this study is to determine key elements of the retirement planning and experience of Canadians, including preferences among different consumption patterns, key risks concerning retirees and pre-retirees, and subjective discount factors and mortality expectations. We further elicit quantitative risk preferences that will enable us to develop more realistic models for retirees' preferences and needs.

The report is organized as follows. Section 4.4 describes survey methodology. Section 4.5 discusses the design of the survey, which includes descriptions of each major section of the survey, and the assumptions and calculations made for the quantitative survey questions. Section 4.6 describes the data and assumptions used in the analysis. Section 4.7 assesses the representativeness of the sample, in terms of key demographic information, by comparing with the most recent census and national survey data. Section 4.8 provides an analysis of the survey data and detailed discussions of their implications. Section 4.9 concludes the chapter. The survey questions are included in Appendix D.1. Appendix D.2 reports the economic and pension scenarios used in the survey.

The survey is conducted in conjunction with the University of Waterloo's Survey Research Centre. The survey has received ethics clearing from the Office of Research Ethics at the University of Waterloo. We acknowledge the support of the Canadian Institute of Actuaries, the University of Waterloo, and the Natural Sciences and Engineering Research Council of

Canada (NSERC).

4.4 Methodology

The Ontario Retirement Survey (the survey, or ORS) was administered by the Survey Research Centre, officially launched on September 20, 2016 and closed on October 13, 2016. Respondents were recruited using the online panel firm, Leger. A total of 1,001 completed and 10 partially-completed survey responses were collected. The average length of a completed survey was 27 minutes.

The target group of participants was 1,000 Ontario residents aged 50 to 80, with 500 self-identified as retired and 500 as pre-retired. Other demographic data such as gender, marital status, age group and urban or rural residence were monitored. An email was first sent by the panel firm to collect information regarding age, region of residence, and retirement status. A second email was sent, inviting the member to participate, if they satisfied the condition for completing the survey (i.e. belonged to the target group). During the launching period, invited members would receive up to two email reminders if they had not yet completed the survey. In total, 13,275 unique invitations and 6,100 reminders were sent. The detailed invitation timeline is included in Table 4.1.

Date	Number of invites	New/Reminder
9/20/2016	200	New (Soft Launch)
9/22/2016	2000	New
9/23/2016	2250	New
9/26/2016	3000	New
9/27/2016	500	New
9/28/2016	400	New
9/28/2016	900	Reminders
10/3/2016	100	New
10/3/2016	1400	Reminders
10/4/2016	1000	Reminders
10/6/2016	1500	New
10/7/2016	1250	New
10/11/2016	1075	New
10/11/2016	2800	Reminders
10/12/2016	1000	New

Table 4.1 Survey contact timeline (source: Survey Research Centre, University of Waterloo)

Respondents[2] who accepted the invitation to participate (i.e. who clicked on the link to the web survey) were categorized as two group, Completed or Drop-outs. Completed respondents were those who reached the very end of the survey and exited from the “Thank

[2]Used interchangeably with participants and subjects.

you for participating...” page (henceforth the “Thank you” page). Note that a Completed respondent needed not complete the survey by answering every question, since questions were allowed to be skipped. Drop-out respondents were those who last exited the survey before reaching the “Thank you” page. Note that respondents were permitted to exit and return to the survey by re-clicking on the invitation link. A Completed respondent was marked as a Speeder, if their total survey time was less than 1/3 of the median survey time of the up to date sample. Speeders’ responses were considered unreliable and were removed from the sample of Completed respondents. A Drop-out respondent was marked as Partially-completed, if a significant proportion of the survey was completed. Partially-completed respondents’ responses were considered to contain sufficient information and were collected in addition to the Completed responses.

The survey had a total number of 306 Drop-outs, 16 Speeders, and 10 Partial-completes. Out of the 306 Drop-outs, 125 occurred on the instruction page which is a typical observation for web surveys. The survey has a response rate of 7.7%, which is lower than the typical response rates seen for web studies (10% to 15%). This is likely due to the length of the survey and the complexity of the questions being asked. A summary of survey completion breakdown is given in Table 4.2.

Category	Number
Completed	1,001
Partially-Completed	10
Drop-outs	306
Speeders (removed)	16

Table 4.2 Survey completion breakdown (source: Survey Research Centre, University of Waterloo)

4.5 Survey Design

The objective of the study is to develop a better understanding of the concerns and risk preferences of individuals who are either close to retirement, or who are already retired. It is designed to include four major sections[3]: Preliminary, Expectations and Experience, Preferences, and Retirement Planning Objectives. A short list of demographic information is collected at the end of the survey. The survey is designed such that the questions presented to participants are adaptive of answers provided in the earlier section of the survey, in particular, on gender (female, male), marital status (married, single), retirement status (pre-retired, retired) and wealth status (low, medium, high). As a result, respondents are gradually divided into 24 categories. This process allows the survey to target the questions to the respondents. For example, when asked about annuity purchases, a female-identifying

[3]Section information is not shown to the respondents.

respondent would be presented with an annuity price that is computed based on female mortality assumptions.

4.5.1 Preliminary & Expectations and Experience

The section ‘Preliminary’ is designed to identify relevant information such that the remaining part of the survey can be properly presented. The section ‘Expectations and Experience’ collects information on expected and actual retirement age; subjective life expectancy; expected and actual asset level at retirement (liquid and real estate); expected and actual retirement consumptions with income sources including private and public pension, annuities, investment and employment; subjective annuity prices; and attitude towards seeking professional financial advice.

4.5.2 Preferences

The section ‘Preferences’ elicits preferences towards risk and time. Risk preferences are elicited both qualitatively and quantitatively. Qualitative elicitation questions, for example, include describing risk taking behaviour in managing investable assets before and after retirement. Quantitative elicitation questions are more involved and require participants to make hypothetical “lottery” choices based on smaller and larger stakes. There exists a variety of experimental methods on eliciting risk preferences. For a review of commonly used methods, see Charness et al. (2013). In this study, we use the Eckel and Grossman method developed by Eckel and Grossman (2002). Specifically, participants are presented with two lists of payoff options. Each list consists of 10 options that have a 50-50 chance of giving a higher or a lower payoff. Participants are asked to choose ONE option they prefer the most. The list of payoffs based on smaller stakes is presented in Table 4.3[4], along with the expected returns, standard deviations and the corresponding implied range of CRRA parameters. This implied CRRA range assumes zero background consumption, and that the payoffs are immediately consumed upon being received.

In particular, let p_{il} and p_{ih} denote the low and high payoff of option i. Then, a participant with CRRA preference prefers option i to j if

$$\frac{u(\omega + p_{il}) + u(\omega + p_{ih})}{2} > \frac{u(\omega + p_{jl}) + u(\omega + p_{jh})}{2} \tag{4.1}$$

where $u(\omega + c) = \dfrac{(\omega + c)^{1-\rho}}{1-\rho}$, ω represents background consumption and $\rho \neq 1$ is the parameter that governs the degree of risk aversion. The implied CRRA range can be obtained by solving for ρ such that chosen option yields the highest expected utility among all ten options.

[4]The last three columns of Table 4.3 are not shown to the respondents.

Option	Low Payoff	High Payoff	Expected Return	Standard Deviation	Implied CRRA range
1	\$10.00	\$10.00	\$10.00	\$0.00	$9.73 < \rho < \infty$
2	\$9.50	\$11.00	\$10.25	\$0.75	$5.87 < \rho < 9.73$
3	\$9.20	\$12.00	\$10.60	\$1.40	$3.74 < \rho < 5.87$
4	\$8.90	\$13.00	\$10.95	\$2.05	$2.48 < \rho < 3.74$
5	\$8.30	\$15.00	\$11.65	\$3.35	$1.74 < \rho < 2.48$
6	\$7.70	\$17.00	\$12.35	\$4.65	$1.36 < \rho < 1.74$
7	\$7.10	\$19.00	\$13.05	\$5.95	$0.88 < \rho < 1.36$
8	\$6.00	\$22.00	\$14.00	\$8.00	$0.26 < \rho < 0.88$
9	\$4.00	\$25.00	\$14.50	\$10.50	$0 < \rho < 0.26$
10	\$2.00	\$27.00	\$14.50	\$12.50	$-\infty < \rho < 0$

Table 4.3 An example of a quantitative risk preference elicitation question with low stake options.

Time preferences are elicited quantitatively using the multiple-horizon treatment based on the experimental method introduced by Coller and Williams (1999) and extended by Harrison et al. (2002). Specifically, participants are asked to consider seven hypothetical *binary* payoffs that are made with certainty (i.e. no risk associated with the payoffs). The options are presented such that a lower payoff is made 1 month from today, and a higher payoff is made 13 months from today. Both options are designed to be future payoff options to avoid the potential problem of subjects facing extra risk or transaction costs with the future income option, as compared to the "instant" income option (for a more detailed discussion of the experimental design, see Andersen et al. (2008)). A total of 7 choices are made. The question is presented in Table 4.4.

Scenario	Payoff Option A (pays 1 month from today)	Payoff Option B (pays 13 months from today)	I prefer: **Option A**	I prefer: **Option B**
1	\$1,000	\$1,020	☐	☐
2	\$1,000	\$1,040	☐	☐
3	\$1,000	\$1,080	☐	☐
4	\$1,000	\$1,140	☐	☐
5	\$1,000	\$1,240	☐	☐
6	\$1,000	\$1,340	☐	☐
7	\$1,000	\$1,440	☐	☐

Table 4.4 Time preference elicitation.

A rational respondent should fall into one of the three cases: 1) always prefers option A, 2) always prefers option B, or 3) starts with option A then switches to option B (i.e. a unique switching point). The first case implies high impatience; the second low impatience; the third in-between. Responses that involve multiple switching points, or from B to A are difficult to interpret from a rational standpoint, and hence should be ignored in the analysis. Let p_{kA} and p_{kB} denote the payoff from Option A and B of scenario k. Assuming the switching point is scenario i, the range of discount rates is obtained by solving the following two inequalities,

$$u(\omega + p_{(i-1)A}) + \beta u(\omega) > u(\omega) + \beta u(\omega + p_{(i-1)B}) \tag{4.2}$$

$$u(\omega + p_{iA}) + \beta u(\omega) < u(\omega) + \beta u(\omega + p_{iB}) \tag{4.3}$$

which yields,

$$\frac{(\omega + p_{iA})^{1-\rho} - \omega^{1-\rho}}{(\omega + p_{iB})^{1-\rho} - \omega^{1-\rho}} < \beta < \frac{\left(\omega + p_{(i-1)A}\right)^{1-\rho} - \omega^{1-\rho}}{\left(\omega + p_{(i-1)B}\right)^{1-\rho} - \omega^{1-\rho}} \tag{4.4}$$

When there is no switching (i.e. consistent choice of either Option A or B), then,

$$0 < \beta < \frac{(\omega + p_{7A})^{1-\rho} - \omega^{1-\rho}}{(\omega + p_{7B})^{1-\rho} - \omega^{1-\rho}}, \qquad \text{if Option A} \tag{4.5}$$

$$\frac{(\omega + p_{1A})^{1-\rho} - \omega^{1-\rho}}{(\omega + p_{1B})^{1-\rho} - \omega^{1-\rho}} < \beta < 1, \qquad \text{if Option B} \tag{4.6}$$

4.5.3 Retirement Planning Objectives

The section Retirement Planning Objectives presents participants with hypothetical retirement scenarios on annuitization and retirement income decisions under inflation and investment risk exposures. This section is an extension of the study conducted by Beshears et al. (2014), where participants are asked to make hypothetical annuitization choices. One main focus of Beshears et al. (2014) is the effect of framing in annuitization decision-making (for more details, see the cited paper). Our study focuses on annuitization and retirement income decision-making under risky scenarios (with minimal framing to avoid framing biases). The remaining section describes the survey questions and the way they are presented to the respondents.

Match-Inflation Pension

This section examines decision-making under risky inflation and comprises of two parts. In the first part, participants are given no extra information on future inflation outlook (see Q22). In the second part, participants are provided with 10 possible scenarios of future costs of living (see Q23). The purpose is to study the impact of subjective inflation beliefs.

i. Subjective Inflation Beliefs – Q22

Participants are first asked to consider the following hypothetical scenario:

> Suppose that you are 65 years old and you are JUST about to retire. Your employer will pay you monthly pension income payments for the rest of your life. The pension will stop when you die. Your employer presents you with the following pension options:
>
> 1. **Match-Inflation Pension Option**: This option pays a monthly pension that increases at a rate that exactly matches inflation (i.e. the increase in the cost of living).
> 2. **Steady Pension Option**: This option pays a constant monthly pension that does not change.

Participants are then presented with six scenarios, with varying first monthly income payment under the Match-Inflation pension, as shown in Table 4.5. The dollar values are determined by the individual wealth level and gender, collected in the Preliminary section. Participants are prompted to choose either the Match-Inflation or Steady pension, under each of the six scenarios, meaning that six choices are made. At this stage, the choices are entirely based on each participant's subjective belief on future macroeconomic conditions.

First Monthly Income Payment		**I would choose...**	
Match-Inflation Option This option provides income payments that increase at the rate of inflation	Steady Option This option provides income payments that do not change	**Match-Inflation Option**	**Steady Option**
$1,551	$2,358	☐	☐
$1,650	$2,358	☐	☐
$1,752	$2,358	☐	☐
$1,857	$2,358	☐	☐
$1,965	$2,358	☐	☐
$2,358	$2,358	☐	☐

Table 4.5 An example of Match-Inflation pension option versus steady pension option under subjective inflation beliefs (high wealth & female).

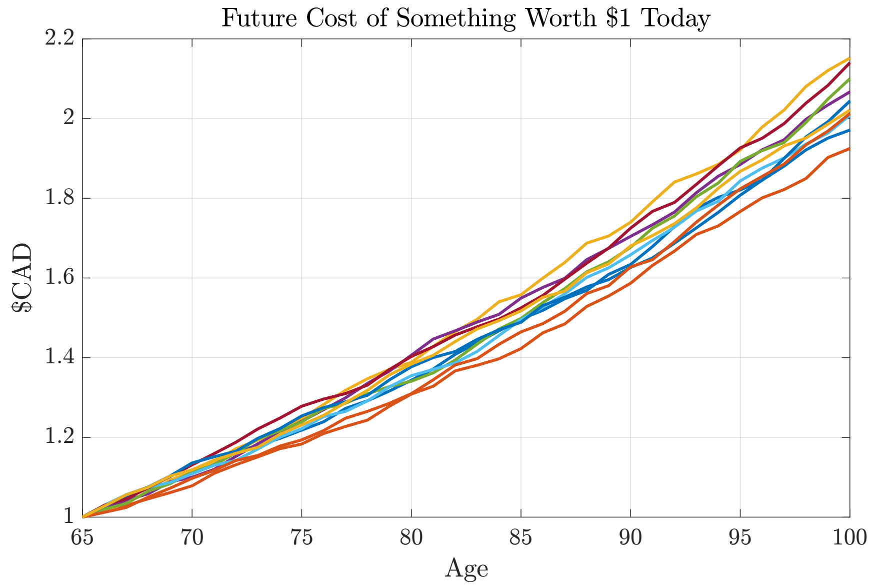
ii. Objective Inflation Information – Q23

The next question repeats the previous hypothetical scenario but provides participants with *additional* information on inflation outlook. Participants are informed that inflation is targeted, but the actual rates can deviate from the target from year to year.

> Consider the same hypothetical situation where you are 65 years old and JUST about to retire. This time, you will be provided with more information on what inflation will look like in the future.
>
> Suppose the Canadian government targets inflation to be 2% per year, but the actual inflation in each year can be anywhere between 1% to 3%. You can

expect inflation to be 2% per year on average for the rest of your life, but you do not know for certain what the actual future inflation rates will be.

To help you make a decision, the chart below presents 10 possible paths of how much things that cost $1 today could cost you in the next 35 years. Keep in mind there is an equal chance for any one of these paths to happen.



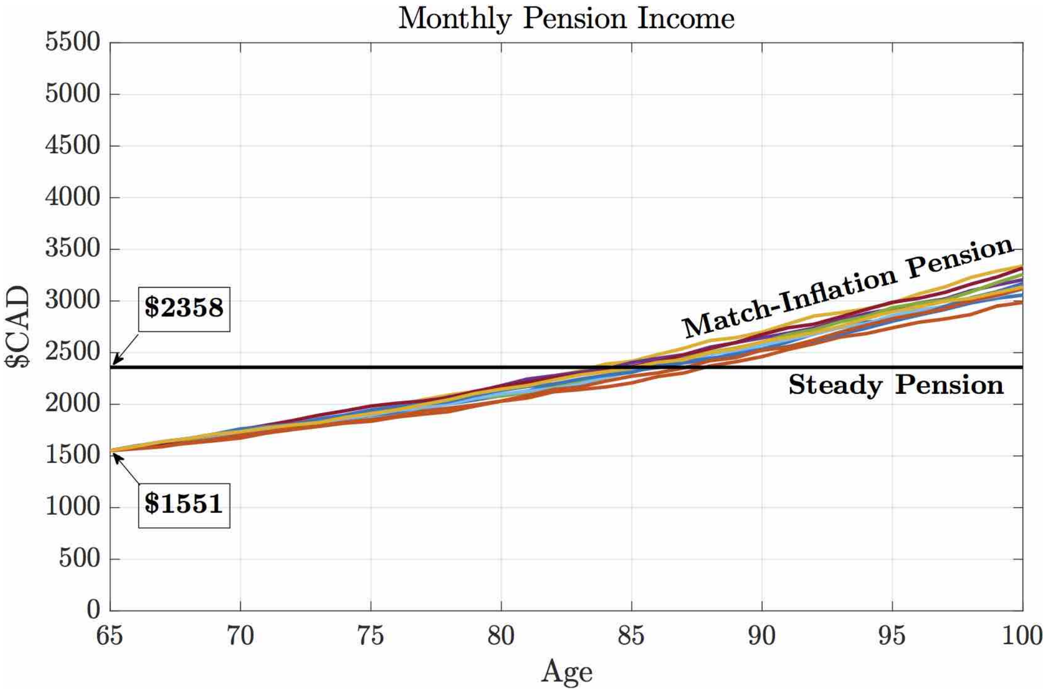
Now, consider again the two pension options your employer presents you...

Respondents are asked to make the same choices as described in Table 4.5, with each choice supported with a chart of possible future pension payments. When prompted to make a choice, the respondent is reminded that future inflation is random, and that the Match-inflation pension has an equal chance of following any one of the ten coloured paths shown in the chart. An example of case 1 is given below:

Consider Case 1.

- If you choose the **Match-Inflation Pension** Option,
  - Your first monthly income payment will be ***$1,551***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension** Option,
  - Your first monthly income payment will be ***$2,358***.

- Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

☐ Steady Pension Option

Respondents are shown the next case if they choose the steady option, with each succeeding case offering a higher starting value for the Match-inflation pension (as described in Table 4.5). The last case (i.e. case 6) is shown in Figure 4.1, under which the Match-Inflation pension gives strictly higher payments than the steady option. To minimize survey fatigue, at any point the respondent switches to the Match-Inflation option, the question terminates, skipping the remaining cases. The underlying assumption[5] is that the respondent would prefer the Match-Inflation option for the remaining cases since they give strictly higher payments.

[5]We acknowledge that, in reality, certain respondents may make inconsistent choices.

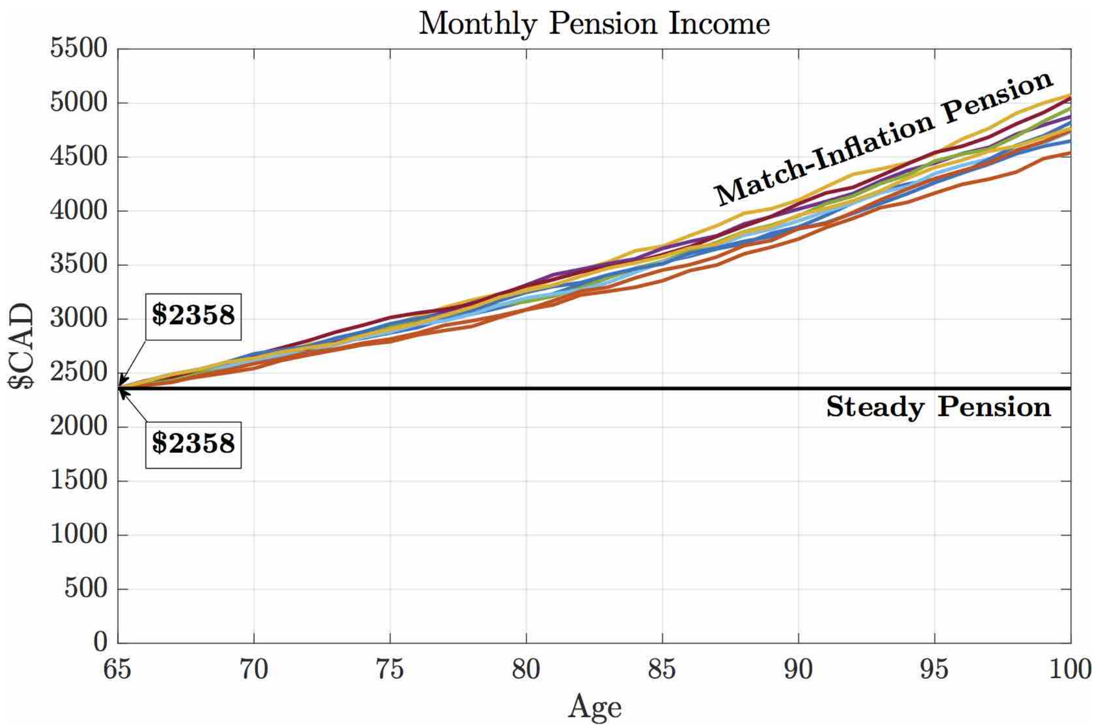


Figure 4.1 An example of the chart shown to respondents: Match-Inflation pension option versus steady pension option with objective inflation information given (high wealth & female, case 6).

We are interested in the point at which the respondent switches to the Match-Inflation option. Certainly, this is related to the subject's beliefs about longevity, inflation (single period and long term), and to a large degree, financial literacy. A follow-up question asks participants to rate the importance of four concerns/statements in making the choices regarding match-inflation pensions (see Q23A). Following this, an additional question is asked to assess participants' understanding (see Q23B). The first pension payments are summarized in Table D.3 in the Appendix.

Equity-Linked Pension

The third section alters the pension scenario and introduces an Equity-Linked pension option. Participants are asked to consider the following hypothetical pension scenario:

> Suppose that you are 65 years old and you are JUST about to retire. Your employer will pay you monthly pension income payments for the rest of your life. The pension will stop when you die. Your employer presents you with the following pension options:

1. **Equity-Linked Pension Option**

   This option pays a monthly pension with payments linked to stock market performance. Every year your pension will be re-evaluated based on the performance of the stock market. *This means that your pension income payments are uncertain and will experience some ups and downs.*

2. **Steady Pension Option**

   This option pays a constant monthly pension that *does not change.*

To help you make a decision, for each case, we will present you *10 possible future paths of what your future monthly pension will look like under the Equity-Linked Pension Option.*

Keep in mind there is an *equal chance* for any one of the paths to happen.

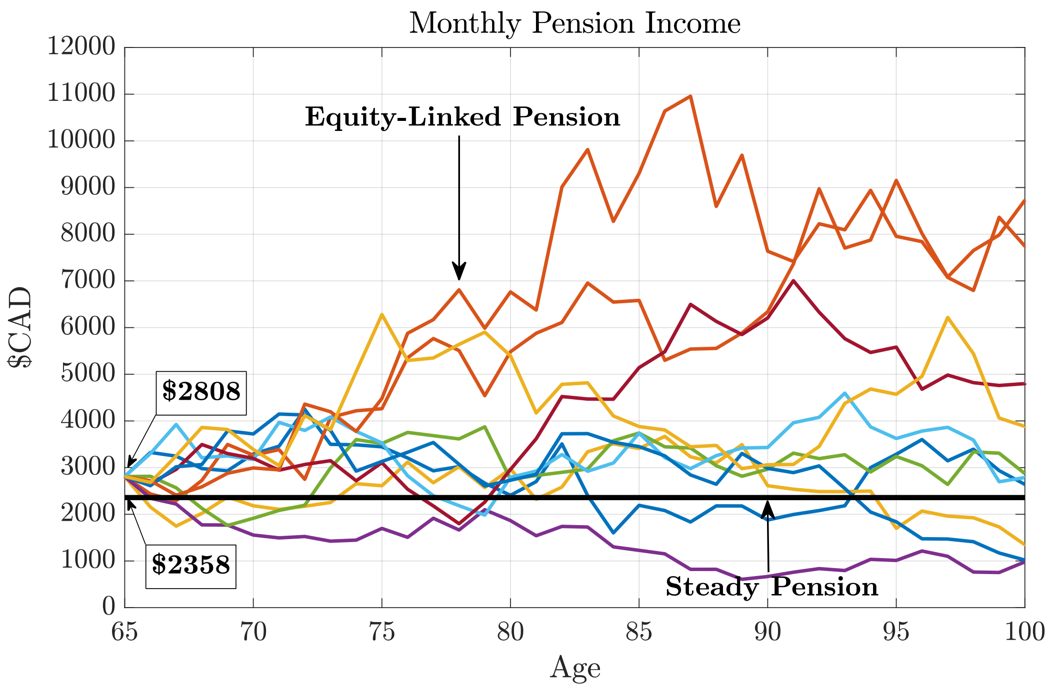
Participants are then presented with six cases (i.e. 1A, 1B, 1C, 2A, 2B, 2C). The cases differ in the annuity interest rate assumptions (AIRs) and risk exposures (i.e. equity allocation). For details, see Section 4.5.4. In essence, the equity-linked pension becomes increasingly risky from cases A to C (i.e. 1A to 1C, 2A to 2C). Compared to the steady pension, the upside gains and downside losses of the equity-linked option become more pronounced. The difference between the two sets of scenarios is in the average trend of the pension income. For 1A to 1C, the trend is *flat*, and for 2A to 2C, the trend tends *downwards.* We are interested in the degree to which the trend impacts decision-making.

An example of case 1C is described below:

Consider Case 1C.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$2,808***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

☐ Steady Pension Option

After making the six choices, a follow-up question asks participants to rate the importance of seven concerns/statements in making the choices regarding equity-linked pensions (see Q24A). Following this, an additional question is presented to assess participants' understanding (see Q24B). The first pension payments are summarized in Table D.4 in the Appendix.

4.5.4 Scenario Calculations, Economic and Mortality Assumptions

Pension Scenario Calculations

In this section, we describe the calculations of the hypothetical pension scenarios in the second last section of the survey. We start by assigning a hypothetical wealth level W, which can be thought of as the accumulated individual pension account balance at retirement. W is dependent on the participant's wealth status (i.e. low, medium and high) and is assumed to take the following values:

	Low wealth	Medium wealth	High wealth
W	150,000	300,000	500,000

Table 4.6 Hypothetical pension account balance at retirement (not disclosed to participants).

Let $t = 0, 1, 2, 3...$ denote time in years. Let $\boldsymbol{P_t} = \left(P_t^S, P_t^M, P_t^E\right)^T$ be the yearly pension income for the Steady, Match-inflation and Equity-linked pension options, respectively. Then the monthly pension income, $\boldsymbol{p_t} = \left(p_t^S, p_t^M, p_t^E\right)^T$, is computed as,

$$p_t^S = \frac{P_t^S}{12}, \quad p_t^M = \frac{P_t^M}{12}, \quad p_t^E = \frac{P_t^E}{12}. \tag{4.7}$$

Let $\boldsymbol{i} = \left(i^S, i^M, i^E\right)^T$ denote the AIRs. The pension income in the first year is,

$$P_0^S = \frac{W}{\ddot{a}_{65:i^S}^{(12)}}, \quad P_0^M = \frac{W}{\ddot{a}_{65:i^M}^{(12)}}, \quad P_0^E = \frac{W}{\ddot{a}_{65:i^E}^{(12)}}. \tag{4.8}$$

The annuity factor is computed using the approximation $\ddot{a}_x^{(12)} = \ddot{a}_x - \frac{11}{24}$, and is calculated based on gender-specific mortality assumptions described in Section 4.5.4.

Recall that each risky pension choice involves 10 economic scenarios (i.e. ten paths). The remaining section describes the calculation for *each economic scenario*.

i. Steady Pension

The Steady pension option pays a constant income stream, therefore for all $t > 0$,

$$P_t^S = P_0^S. \tag{4.9}$$

ii. Match-Inflation Pension

Let δ_t denote the annual inflation in year t, then,

$$P_t^M = P_{t-1}^M \left(1 + \delta_t\right), \qquad \forall t > 0. \tag{4.10}$$

iii. Equity-Linked Pension

Let r_t denote the annual total return on equity in year t, r_f denote the risk-free interest rate, and α be the percentage of equity allocation in the underlying fund invested, then,

$$P_t^E = P_{t-1}^E \frac{1 + r_f + \alpha(r_t - r_f)}{1 + i^E}, \qquad \forall t > 0. \tag{4.11}$$

Economic Assumptions

The used inflation scenarios are shown in Table D.2 in the appendix. The scenarios are randomly generated from $U(0.01, 0.03)$, which reflects the most recent inflation-control target adopted by the Bank and the Government of Canada in 2016 that aims to keep inflation at 2% (and within a target range of 1% to 3%). The total log returns scenarios (i.e. $\log(1 + r_t)$) used are described in Table D.1 in the appendix. The scenarios are randomly generated from $N(0.04078, 0.18703^2)$, which is a common assumption used in the life-cycle retirement planning literature (see, e.g., Maurer et al., 2013; Blake et al., 2008; Horneff et al., 2010). The risk-free interest rate r_f is assumed to be 2%, which is set based on the Canadian long-term government bond yields in 2016.

The equity scenarios are handpicked to represent a reasonably wide-spread range of scenarios. One may argue that the scenarios are “too good to be true”, or “not reasonably representative of reality”, since the changes for the upside largely outweigh the downside. This is by design, as we are interested in whether the unlikely worst case scenario would have a significant impact on decision-making. We acknowledge that in reality, the downside frequency may be low but the severity may be highly significant, especially in the event of unfavourable systematic mortality shifts and stock market crashes.

The annuity interest rates (AIRs) used to compute the incomes in Q14, Q22, Q23 and Q24 are based on the average of the three most competitive hypothetical quotes at December 31, 2015, obtained in Committee on Pension Plan Financial Reporting (2016). The AIRs used for Q14 and the Steady pension option in Q22, Q23 and Q24 are 2.85%, based on the non-indexed annuity proxy. The AIRs used in Q22 and Q23 for the Match-Inflation pension options are:

Scenario	1	2*	3	4	5	6
AIR (i^M)	−0.51%	−0.06%	0.39%	0.84%	1.74%	2.85%

Table 4.7 Annuity Interest Rates used in the Match-Inflation Pension Options.

Among these, the asterisked scenario AIR = −0.06% is based on the CPI-Indexed annuity proxy, representing a realistic market quote in early 2016 (when the survey was designed). For Q24, the AIRs used in computing the Equity-Linked pension incomes are:

Case	1A	1B	1C	2A	2B	2C
AIR (i^E)	3.45%	4.04%	4.53%	4.83%	5.66%	6.34%
Equity allocation (α)	40%	60%	80%	40%	60%	80%

Table 4.8 Annuity Interest Rates and equity allocation assumptions used in the Equity-Linked Pension Options.

Mortality Assumptions

The mortality assumptions used are Canadian Pensioners' Mortality Table 2014 (CPM2014) combined with mortality improvement scale CPM Improvement Scale B (CPM-B) with no adjustments for sub- or super-standard mortality (CPM2014proj), for both female and male[6]. Details on CPM2014 can be found in Pension Experience Subcommittee (2014).

Four questions in this survey require mortality assumptions: Q14, Q22, Q23 and Q24. In Q14, mortality rates are used to compute the default (i.e. when participants skip the previous question) maximum price for the monthly income of $100. In Q22, Q23 and Q24, mortality rates are used to compute the monthly pension incomes for the Match-Inflation, Steady and Equity-Linked pension options.

4.6 Data and Assumptions

Historical inflation rates are computed based on monthly Consumer Price Index (all items) in Ontario from January 1979 to October 2016, obtained from Statistics Canada (table 326-0020). At times, future inflation assumptions need to be made to convert expected future monetary values to the present. In these cases, inflation is assumed to be 2% per annum.

Some responses regarding wealth and income are collected over dollar intervals. In our analysis, responses are assumed to be the midpoints of the intervals. For example, if one selects $10,000 to $20,000, we assume a response of $15,000. This may bias the results, especially in cases where responses are concentrated in the lower end. This can be improved by always giving options for choosing $0. Nonetheless, obtaining an accurate estimate of actual values is difficult without compromising the participation/completion rate, since respondents are far more likely to choose an interval than to provide a point estimate.

4.7 Assessing Sample Representativeness

To ensure that the collected data is representative of residents in Ontario, a comparison to the most recent census and national survey data is conducted. Key aspects of interest are age, marital status, household income, retirement liquid assets and real estate/home ownership. The following data provided by Statistics Canada are used,

- 2011 National household survey (sample size 4.5 million),
- 2012 Survey of Financial Security (sample size 20,000),
- CANSIM Database (updated daily).

[6] Recall that a participant's gender is identified in this study. In the event that the question on gender identification is skipped, male mortality assumptions are used.

4.7.1 Age

	ORS 2016	NHS 2011
Age Group	Percentage	Percentage
50 – 54	17%	25%
55 – 59	20%	21%
60 – 64	22%	19%
65 – 69	21%	14%
70 – 74	14%	11%
75 – 79	6%	9%
Sum	100%	100%

Table 4.9 Age Distribution of ORS 2016 and NHS 2011.

Comparison to the National Household Survey 2011 shows that the sample has a higher concentration in age group 60 – 69 and a lower concentration in age group 50 – 54. Considering that the Canadian population is on average aging, we conclude that the sample represents the population reasonably well.

4.7.2 Marital Status

	ORS 2016		CANSIM 2016	
Age Group	Married	Single	Married	Single
50 – 80	71.2%	28.8%	70.3%	29.7%

Table 4.10 Marital Status of ORS 2016 and CANSIM 2016.

Comparison to CANSIM data in 2016 shows that the sample well-represents the population in terms of marital status. Here, the married status described in CANSIM 2016 includes those in common-law relationships and excludes separated, widowed or divorced married couples not living common law.

4.7.3 Household Income

	ORS 2016			NHS 2011	
	Frequency	Percentage	Cumulative	Percentage	Cumulative
Below $25,000	101	10%	10%	16%	16%
$25,000 to $49,999	185	19%	29%	22%	37%
$50,000 to $74,999	186	19%	48%	19%	56%
$75,000 to $99,999	164	17%	65%	15%	71%
$100,000 to $124,999	110	11%	76%	10%	81%
$125,000 to $149,999	82	8%	84%	7%	87%
$150,000 to $174,999	50	5%	89%	|	|
$175,000 to $199,999	37	4%	93%	|	|
$200,000 to $224,999	39	4%	97%	13%	100%
$225,000 to $250,000	7	1%	98%	|	|
Above $250,000	21	2%	100.0%	|	|
Skipped	19				
Sum	1001	100%		100%	
Median		$75,000 to $99,999		$66,358	
Average		$75,000 to $99,999		$85,772	

Table 4.11 Income Distribution of ORS 2016 (adjusted to 2011 dollars) and NHS 2011 (Total Household Income, Ontario).

To ensure that our results are comparable to NHS 2011, inflation adjustments to monetary values need to be made. The survey collects information on total household income for the year 2015 (for pre-retirees) and the year prior to retirement (for retirees). As information on retirement age is also collected (in Q5), combining with information on birth year, we can deduce the year of retirement and convert the income to 2011 dollars using historical inflation rates. For pre-retirees, no extra information is required since incomes are all expressed in the monetary value in 2015. Inflation adjustment is done by adjusting the midpoint of selection by historical inflation and then reassigning the selection.

The total household income reported in NHS 2011 refers to income from all sources, including employment income, income from government programs, pension income, investment income and any other money income. This is a reasonable benchmark because retirement status is self-identified, hence there is no restriction on the income source pre or post retirement (e.g. a self-identified pre-retiree may be receiving government pension income in 2015).

4.7.4 Retirement Liquid Assets

	ORS 2016		SFS 2012					
	Retirement Liquid Assets		Private Pension Assets		TFSA		Deposits	
Age Group	Below $25,000	Median (Excl. Below $25,000)	None	Median (Excl. None)	None	Median (Excl. None)	None	Median (Excl. None)
55 – 64	21.4%	$200,000 to $299,999	23.70%	$285,000	60.5%	$11,000	6.7%	5,000
65 +	18.7%	$200,000 to $299,999	25.80%	$266,000	55.1%	$18,000	2.6%	10,000

Table 4.12 Liquid Asset Distribution of ORS 2016 and SFS 2012 by age group.

The amount of liquid assets (actual and expected) at the time of retirement is compared with the 2012 Survey of Financial Security. In particular, comparisons are made to the level of private pension assets, tax-free savings account (TFSA) and any deposits held in financial institutions, for all family units in Ontario. The age group reflects the age of the major income earner of the family unit. SFS 2012 reports other liquid assets sources, such as bonds and stocks, but the percentage owned and median values are very small hence excluded. The ORS finds that roughly 20% had/expected liquid assets to be below $25,000 (including none). This mirrors the findings of SFS that nearly a quarter of the population had no private pension assets. It appears that our sample may be slightly wealthier than the population on average. This is reasonable since we asked the respondents to either indicate their expected assets or actual assets at retirement, at which asset level is typically high (i.e. the end of the retirement asset accumulation stage), whereas the SFS looks at actual asset level at the time. Other factors including pre-retirees' salary inflation, interest rate expectations, as well as past inflation also may impact the value of the assets. Overall, we conclude that our sample represents the population to a reasonable degree.

4.7.5 Real Estate/Home Ownership

	ORS 2016		SFS 2012					
	Real Estate (net Mortgage)		Principal Residence		Other Real Estate		Mortgages	
Age Group	Do not own	Median (Excl. Do not own)	None	Median (Excl. None)	None	Median (Excl. None)	None	Median (Excl. None)
55 – 64	20.3%	$200,000 to $399,999	27.9%	$300,000	79.9%	$230,000	65.7%	$135,000
65 +	16.7%	$200,000 to $399,999	25.3%	$300,000	20.5%	$200,000	86.9%	$60,000

Table 4.13 Real estate distribution of ORS 2016 and SFS 2012 by age group.

Comparison to SFS 2012 suggests that the sample contains a slightly higher proportion of households that own real estate (including principal residence and other properties). Among those households that own real estate, the median of value net mortgage is reasonably close to the value reported in SFS. In terms of home ownership (principal residence), the sample has 20.5% (age 55 – 64) and 19.1% (65 – 80) who do not own a home.

4.7.6 Overall

Overall, the sample is reasonably representative of the population in terms of age, marital status, household income, retirement liquid assets and real estate. There is some evidence that the sample is slightly biased towards wealthier households. This is a reasonable observation given that selection bias is present in web based surveys. For example, those who have less access to the internet may be under-represented (see, e.g., Fowler Jr, 2013; Couper, 2000).

4.8 Data Analysis

4.8.1 Introduction

This section provides an analysis of the survey data and detailed discussions of their implications. Specifically, the relationships between any two variables are shown graphically using 'frequency plots', and are quantified by Kendall's rank correlations (i.e. Kendall's tau).

A frequency plot resembles a heat-map, which shows the proportions of respondents that fall under each pair of choices. It can also be viewed as a graphically represented 2-dimensional matrix, with the colour of each square representing the relative size of each element. The first frequency plot can be found in Section 4.8.2 and is described with more details.

Kendall's rank correlation, or Kendall's tau, measures the concordance, or pairwise dependence, for two random variables (X, Y), and is formally defined as

$$\tau_k(X, Y) = \mathbb{E}\left(\text{sign}((X - \tilde{X})(Y - \tilde{Y}))\right) \tag{4.12}$$

where $(\tilde{X}, \tilde{Y})$ is an independent copy of (X, Y). Kendall's tau takes values in $[-1, 1]$, where 1 implies comonotonicity, and -1 implies countermonotonicity. The intuition is that if larger values of X (relative to its range) are associated with larger relative values of Y, and smaller values of X with smaller values of Y, then X and Y are concordant. On the other hand, if larger relative values of X are associated with smaller relative values of Y (and vice versa), then X and Y are discordant. The estimator is computed by,

$$\widehat{\tau}_k = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{sign}((X_i - X_j)(Y_i - Y_j)) \tag{4.13}$$

where n is the total number of pairs observed. We further conduct the Tau test to establish whether the pair of variables can be considered as statistically dependent. Under the null hypothesis of independence, the test statistic $\widehat{\tau}_k$ is approximately Normally distributed with mean 0 and variance $\frac{2(2n+5)}{9n(n-1)}$. We use * to denote a p-value ≤ 0.05, and ** for a p-value ≤ 0.01.

4.8.2 Retirement wealth

We find that the majority of the respondents fall under "low" under expected and actual liquid assets at retirement, with the phenomenon more pronounced for female respondents. The breakdown for all respondents is shown in Figure 4.2.

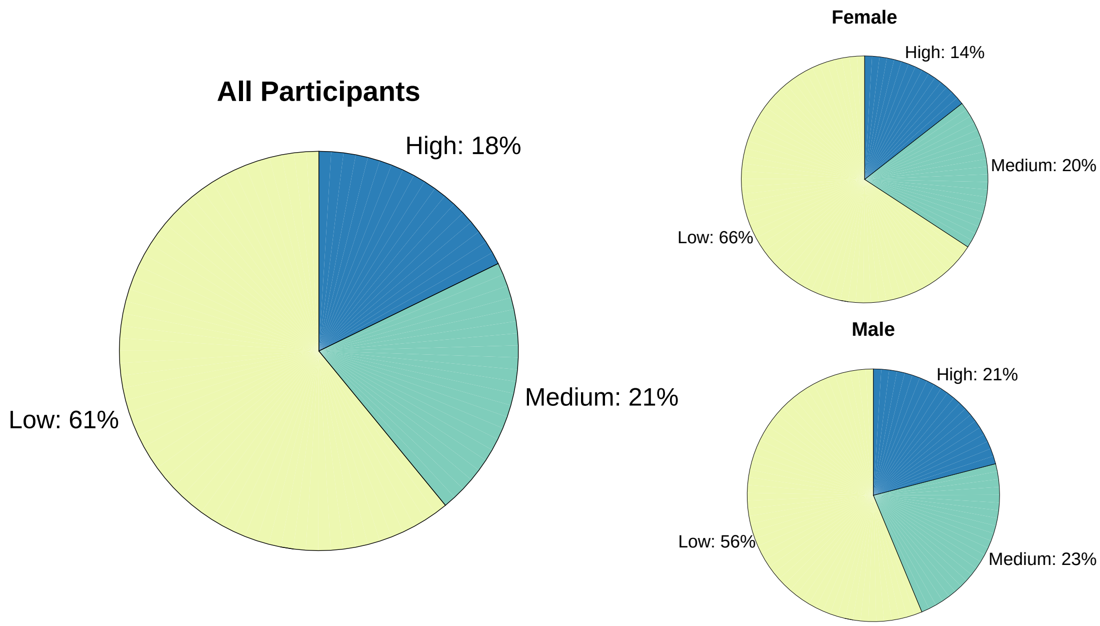


Figure 4.2 All participants: liquid assets at retirement.

	Single	Married
Low	Below \$200,000	Below \$300,000
Medium	\$200,000 to \$399,999	\$300,000 to \$699,999
High	Above \$400,000	Above \$700,000

Table 4.14 Definition of liquid wealth levels.

It is important to note that the wealth categories are defined differently for single and married participants, as shown in Table 4.14. We highlight that the result does not necessarily suggest that those having "low" retirement liquid assets have inadequate retirement savings, since the amount of liquid assets required to provide for an adequate retirement living is highly subjective. Contributing factors may include lifestyle, life expectancy, household size, health, pension income (e.g. workplace pension or public pension), other income (e.g. employment and investment) and fixed assets.

Comparing liquid and fixed assets reveals respondents' overall asset level at retirement. Fixed assets include home (i.e. principal residence) and other property investment, and are highly relevant to the cost of retirement and income in retirement. Specifically, home ownership directly impacts the cost of living in retirement (i.e. renting costs); it is also a source of liquidity through downsizing or a reverse mortgage. Investment properties bring in rental income, which is a regular income that hedges price inflation, and produces capital gains if sold.

Figure 4.3 is a frequency plot, which we use extensively to illustrate the relationship between two variables. In this case, the figure shows the proportion of respondents who fall under each pair of liquid and fixed asset levels. For example, the dark square in the bottom left indicates that 10% of respondents do not own real estate *and* expect/have had less than $25,000 of liquid assets at retirement. This proportion is alarming, along with the evident positive relationship between liquid and fixed assets (Kendall's tau: 0.4323^{**}; correlation: 0.4791^{**}), suggesting that those with low retirement savings tend to have low fixed assets, which implies higher costs of living, lower liquidity, and lower income from property investment.

Overall, 21% expect/have had liquid assets under $25,000, 40% under $100,000; 19% do not own property. This issue is discussed in more detail in Section 4.8.4.

Respondents with extremely low assets

One may be interested in investigating those with extremely low private pension assets, in particular relating to income at retirement. For this part, we examine the 101 respondents who do not own property and expect/have had less than $25,000 at retirement (i.e. the bottom left square). We find the following:

- 71 do not have defined benefit pensions, those who do receive, on average, $14,667
- 85 do not have life annuities, those who do receive, on average, $12,333
- 97 expect/have made very little withdrawal from liquid assets
- respondents expect/have public pension income of $10,860 on average
- respondents on average expect/have income from other sources that averages around $9,167

Overall, the 101 respondents expect/had an income of $34,083 in the first year of retirement, of which 32% comes from public pension benefits, and 27% from other income sources. When asked to specify these income sources, 44 respondents provided written responses,

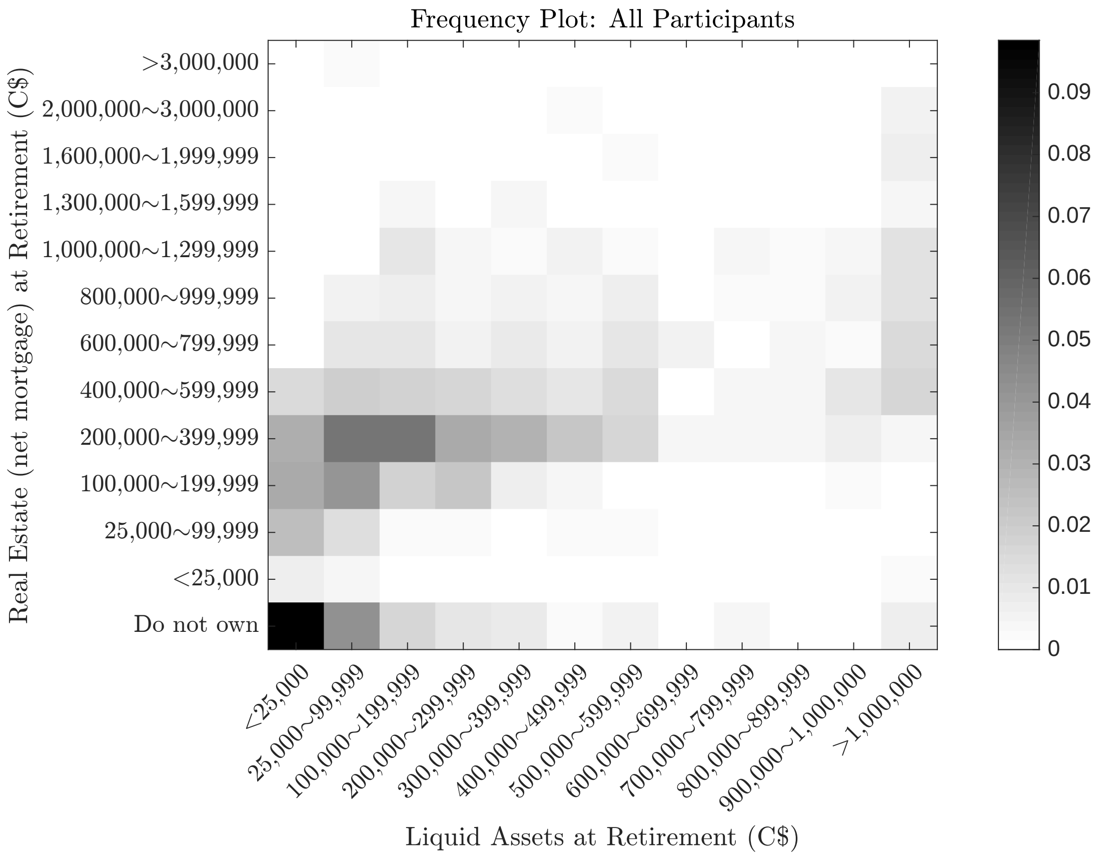


Figure 4.3 All participants: liquid assets and real estate at retirement.

among which 15 indicated there are simply no other income sources, 14 indicated some form of work or employment, 7 mentioned income associated with disabilities, others mentioned spouse's salary, inheritance etc. These findings highlight that retirees with extremely low assets are highly dependent on public pensions, and potentially expect to re-enter the workforce to supplement their retirement income.

4.8.3 Longevity and survival beliefs

The survey collects three responses that measure respondents' subjective beliefs on survival rates and life expectancy (see Q6, Q7 and Q8). We are interested in respondents' beliefs about their 1) life expectancies, 2) survival probabilities for at least 10 more years, and 3) survival probabilities to an extreme old age of 95. We explore the degree to which these subjective beliefs differ from actual mortality experience, and their corresponding influence on retirement decision-making.

The mortality data[7] for Ontario male and female residents are obtained from Statistics Canada. As of August 2017 (the time of writing), the most up-to-date mortality information from Statistics Canada is for 2011-2013. The mortality experience from 2014 to 2016 is not reflected, thus comparisons should be interpreted with caution. Results are illustrated in Figures 4.4, 4.5, and 4.6. We highlight the following findings:

i. **Severe underestimation of survival probabilities to an extreme old age:** we find strong evidence that respondents highly underestimate their probability of survival to at least age 95 or more. As shown in Figures 4.5 and 4.6, 92% report a survival rate below the objective rate, and 27% believe that they have zero chance of surviving to age 95. This evidence points to a severe underestimation of longevity risk. This raises questions as to whether such underestimation plays a key role in retirement decision-making.

ii. **Mild underestimation of life expectancy:** the objective expected age at death for Ontario individuals aged 50 to 80 is 85.6 to 90.6 for females and 82.1 to 88.9 for males. The left plot of Figure 4.4 shows the distribution of responses, which ranges from 60 to 200, centred around 85. On the right hand side, we take each of the subjective responses and divide them with the expected age at death computed based on population mortality experience (i.e. objective mortality rates) and plot the distribution of this ratio. We find that 54% report an expected age at death lower than the objective estimate, and that the median ratio is 0.98. This suggests that on average, respondents mildly underestimate their life expectancies. While 54% and 0.98 are not strong evidence, the degree of underestimation may be more pronounced if mortality improvement[8] from year 2014 to 2016 is taken into account.

[7] Data source: Statistics Canada. 2017. Life tables, Canada, provinces and territories, catalogue no. 84-537-X.

[8] The expected age at death for Ontario females aged 50 to 80 is 85.4 to 90.5 in 2010-2012, and 85.2 to 90.4

iii. **Close estimation of survival probabilities for 10 or more years:** on average, the subjective estimates for survival probabilities for 10 or more years are relatively close to the objective estimates. 47% reported a probability estimate below the objective rate. The median ratio (i.e. subjective divided by objective rates) is 1.02. We conclude that these are close estimates, taking into account mortality improvement in the past 3 years.

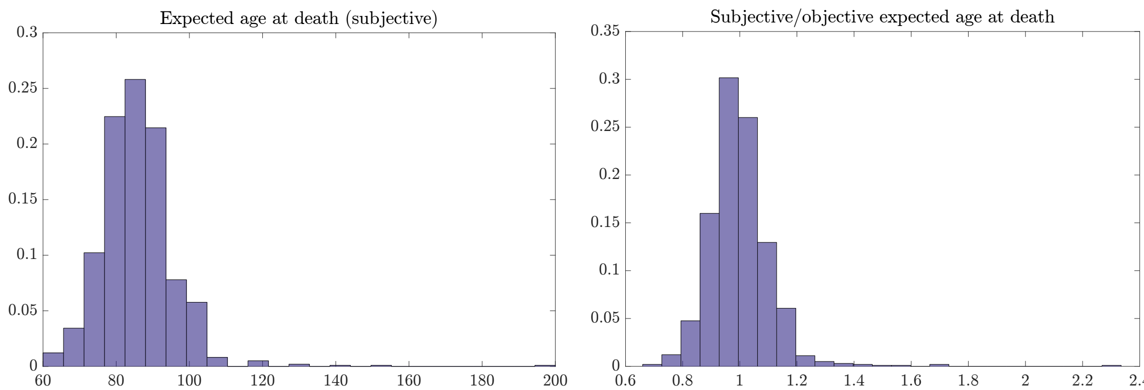


Figure 4.4 Expected age at death: subjective estimates (reported) and subjective/objective estimates.

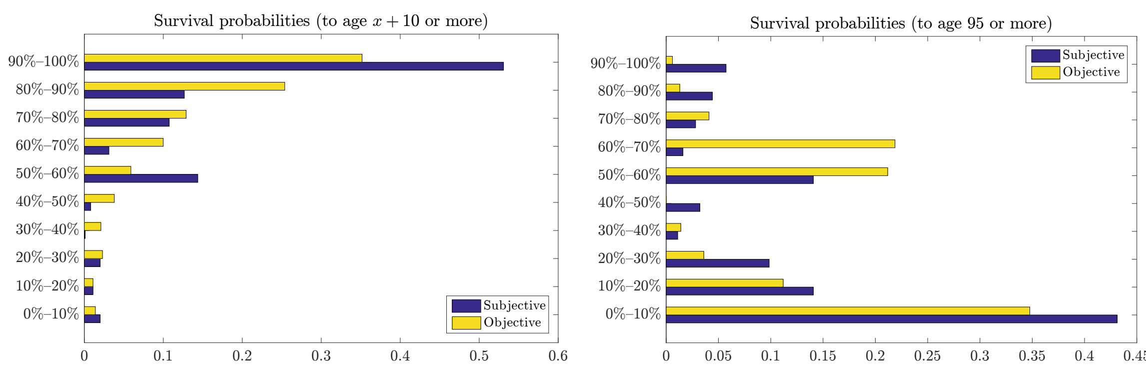


Figure 4.5 Survival probabilities: subjective and objective estimates.

4.8.4 Retirement expectation and experience

In this section, we separate responses from pre-retirees and retirees regarding retirement expectation and experience. For responses involving dollar values, such as assets and income,

in 2009-2011.

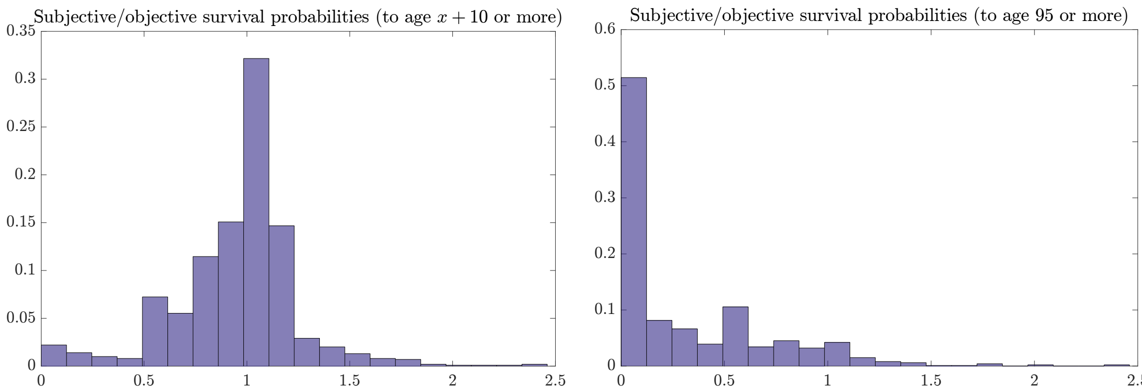


Figure 4.6 Survival probabilities: subjective/objective estimates.

we report findings without adjusting for inflation. This is because adjusting for inflation requires ad hoc assumptions to be made about future inflation rates. Inflation adjustment also does not make a large impact when responses are given within a range. For example, a response within the range of \$10,000 to \$20,000 will most likely be within the same range after adjusting for inflation for 5 to 10 years.

We highlight that this is *not* an expected versus observed comparison, as many objective factors may contribute to the differences. In other words, if pre-retirees' expectation is significantly different from retirees' experience, it is not necessarily the *wrong* expectation. Nonetheless, the direction and magnitude of such differences, combined with other evidence collected in the survey, may reveal important insights on the retirement planning behaviours of Canadian individuals.

We find that on average, pre-retirees' expectation of asset values at retirement quite reasonably matches retirees' experience, taking a number of factors into account. The following section discusses retirement age, assets, income and planning concerns in detail.

Retirement age

Compared to retirees' experience, pre-retirees on average expect a higher retirement age. Figure 4.7 plots the distribution of expected and experienced retirement age. Table 4.15 summarizes the average retirement age. A myriad of factors may contribute to the difference in expectation and experience, including life expectancy, age eligibility of public pension schemes, involuntary retirement, retirement readiness (e.g. asset and retirement income) and survey bias. We provide an explanation for each.

i. **Life expectancy:** on average, pre-retirees estimate a lower expected age at death than retirees (84.99 and 86.05). It therefore cannot be concluded that pre-retirees expect to retire later because they expect to live longer.

ii. **Age eligibility:** the normal retirement age for receiving CPP and OAS benefits has stayed at 65 since 1970 (CPP benefits can be taken at a reduced rate as early as age 60 since 1987). This means that pre-retirees are not compelled to retire later to be eligible to receive public benefits. Nonetheless, it is still possible for pre-retirees to anticipate a future increase in the eligibility age, since such increases are taking place in other aging countries such as Australia, Sweden, and the United Kingdom.

iii. **Involuntary retirement:** pre-retirees may overestimate their remaining working life, by underestimating the probability of declining health, direct or indirect involuntary termination, or other relevant personal circumstances. The survey does not collect information on the reasons behind retirement for retirees; doing so may reveal more insight.

iv. **Retirement readiness:** it is plausible that pre-retirees expect to retire later due to lower retirement readiness. Particularly for those in the private sector, the shift from having generous defined benefit pension arrangements to defined contribution or no arrangements means that pre-retirees need to accumulate more assets to meet retirement needs, which translates to a longer working life and shorter retirement length (see section 4.8.4 on defined benefit pension income).

v. **Survey bias:** the survey targets an audience between ages 50 to 80, which means pre-retirees cannot report a retirement age less than 50, whereas retirees can. This may create a biased sample and contribute to a larger average expected retirement age, which we observe.

	Mean	Median
Expectation of pre-retirees	67.11	65.00
Experience of retirees	58.59	60.00

Table 4.15 Average age at retirement.

Assets at retirement

We compare pre-retirees' expectation of asset values at retirement with retirees' experience, and find that, on average the expectations are very close to retirees' experience. Table 4.16 summarizes the mean and median of reported asset values at the year of retirement. Figure 4.8 shows the proportion of each response for pre-retirees and retirees.

We see that for pre-retirees, the means are higher for both liquid and fixed assets while the medians are the same. This shows that the larger means are driven by higher expectations in the right tails (see Figure 4.8), meaning that a small proportion of pre-retirees expect much higher assets at retirement, while the majority expect similar asset levels with retirees' experience. For liquid assets, the largest difference in expectation and experience comes from the response "> 1,000,000". This may be a manifestation of a tendency to overweigh

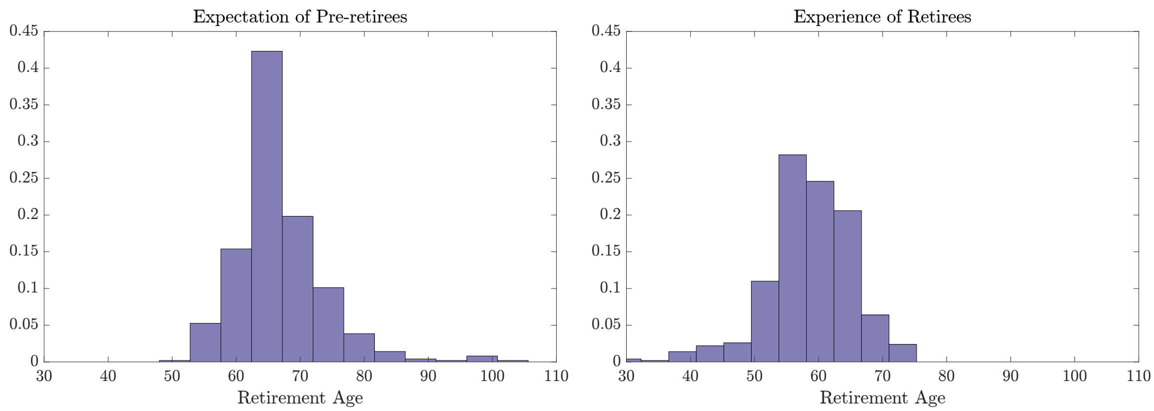


Figure 4.7 Distribution of retirement age.

small probability events, as behavioural economic theory suggests (see Kahneman and Tversky, 1979), or overconfidence in one's investment skills. For fixed assets, pre-retirees with medium to high property holdings consistently expect higher property values at retirement. This is reasonable, since historical property inflation has been high in recent years. Pre-retirees may also have high confidence in the Ontario housing market based on these favourable observations. Additionally, historical inflation and pre-retirees' beliefs of future price inflation contribute to the difference.

Further, pre-retirees on average expect to retire later than retirees have, which provides additional explanation for much higher fixed asset values at retirement (the accumulation of property inflation over a longer period). For liquid assets, the difference is small, which provides further evidence that lower retirement readiness, in terms of liquid retirement assets and income on average contributes to the large difference in expected retirement age for pre-retirees. A shorter retirement life, combined with a similar level of assets, sheds light on pre-retirees' lower retirement income expectations. We investigate retirement income in the next section.

	Liquid assets		Fixed assets	
	Mean	Median	Mean	Median
Expectation of pre-retirees	\$320,230	\$150,000	\$462,960	300,000
Experience of retirees	\$306,210	\$150,000	\$357,640	300,000

Table 4.16 Liquid and fixed assets at retirement.

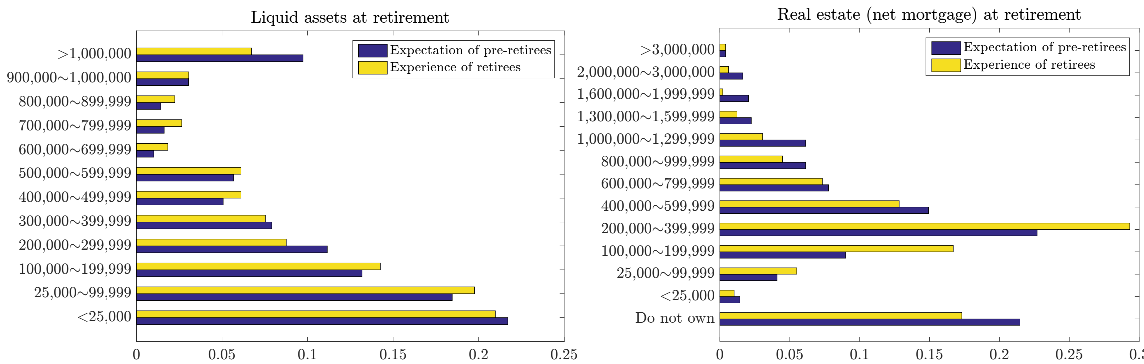


Figure 4.8 Liquid and fixed assets at retirement.

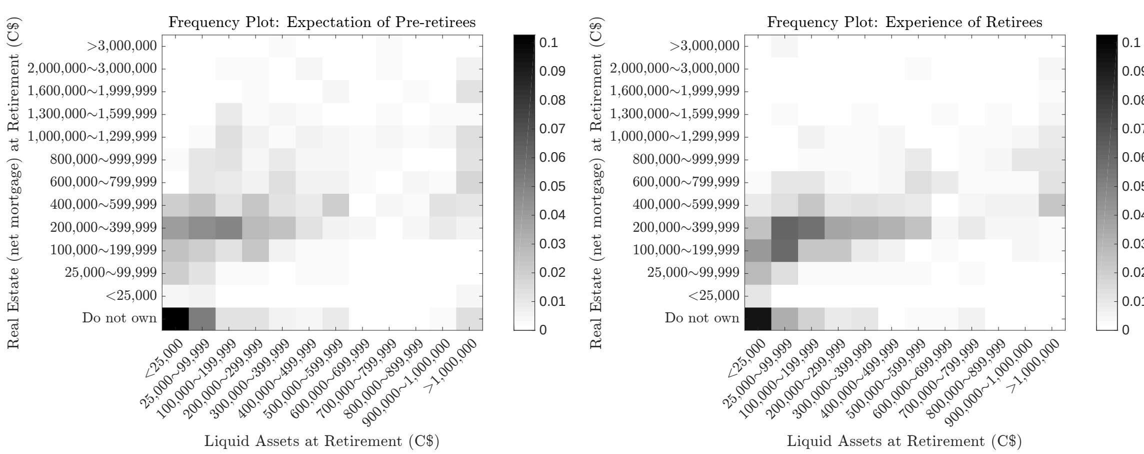


Figure 4.9 Liquid and fixed assets at retirement.

Retirement income

We compare pre-retirees expectations of retirement income from five sources with retirees' experience. Table 4.17 summarizes the average income expected and experienced in the first year of retirement. Figures 4.10 to 4.14 show the distribution of expected and experienced income from various sources in the first year of retirement, separated by marital status[9]. We highlight the following main findings:

i. **Reasonable expectations by pre-retirees:** taking inflation into account, the mean of the expected total income is close to the experience (i.e. 2,101 higher), but the median is considerably *lower* (i.e. 8,000 lower).

ii. **Defined benefit pension:** the most notable difference lies in income from defined benefit pension plans, where pre-retirees' expectation is significantly *lower* (i.e. 4,707 lower in the mean, 10,000 lower in the median). Overall, 48% of pre-retirees and 36% of retirees do not have a defined benefit pension. The shift away from the defined benefit arrangement is evident.

iii. **Social insurance pension:** the pension income from social insurance programs[10] (i.e. public pensions), from which pre-retirees expect significantly *higher* amounts (i.e. 3,887 higher in the mean, 6,000 higher in the median). We examine public pensions more closely in the next paragraph.

iv. **Withdrawals:** comparing the two means, pre-retirees expect to make moderately higher withdrawals from liquid assets (i.e. 3,295 higher), which is consistent with their moderately higher expectation of liquid assets at retirement.

v. **Life annuities:** few respondents have or expect to purchase private life annuities (i.e. 21% of pre-retirees and 12% of retirees).

vi. **Other income sources:** expectation of income from other sources, such as investment income or employment, is moderately lower than retirees' experience. This may reflect a lower expectation for the need for employment in retirement.

As mentioned above, the seemingly high expectation of public pension benefits among pre-retirees compared to retirees' experience needs to be more closely examined. At first glance, this may seem to indicate overconfidence in the Canadian public pensions program among pre-retirees. However, this does not account for the age eligibility requirement to receive these pension benefits. The normal retirement age for the OAS, GIS and CPP is 65 (though CPP benefits can be taken at a reduced rate starting from age 60), meaning that those who retire prior to age 65 (or 60, in the case of CPP only) are not eligible, in the first year of retirement, for receiving public pension benefits.

Table 4.18 summarizes the percentages of respondents with expected or actual retirement

[9] Survey options presented differ for married and single respondents.

[10] Public pension programs in Canada include the Canada Pension Plan, Old-age Security and Guaranteed Income Supplement.

	Expectation of pre-retirees		Experience of retirees	
	Mean	Median	Mean	Median
Defined benefit	$23,478	$5,000	$28,185	$15,000
Life annuities	$6,298	$0	$2,833	$0
Withdrawals	$21,793	$10,000	$18,498	$10,000
Public pension	$15,505	$15,000	$11,618	$9,000
Other income	$17,782	$10,000	$21,759	$10,000
Total income	$85,194	$61,000	$83,093	$ 69,000

Table 4.17 Average income in the first year of retirement.

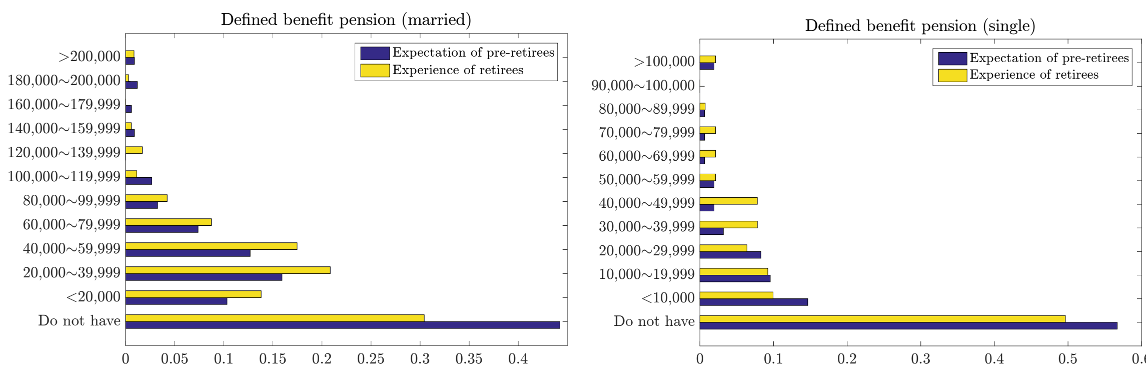


Figure 4.10 Defined benefit pension in the first year of retirement.

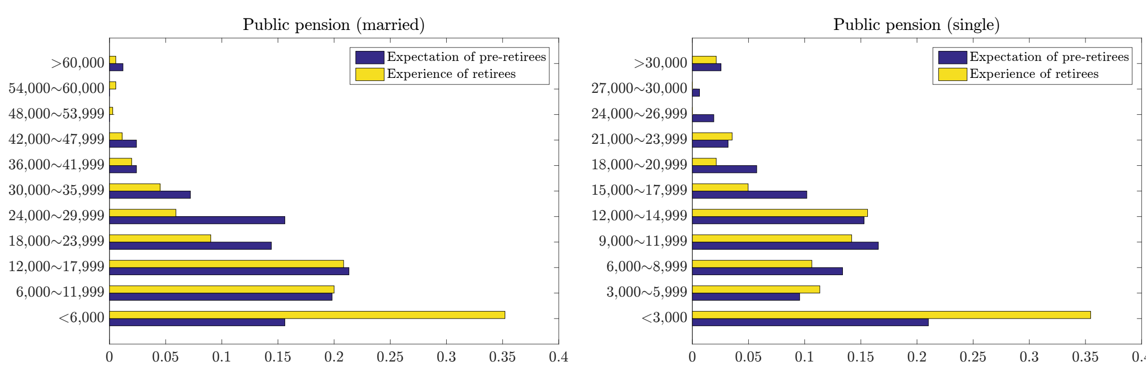


Figure 4.11 Public pension in the first year of retirement.

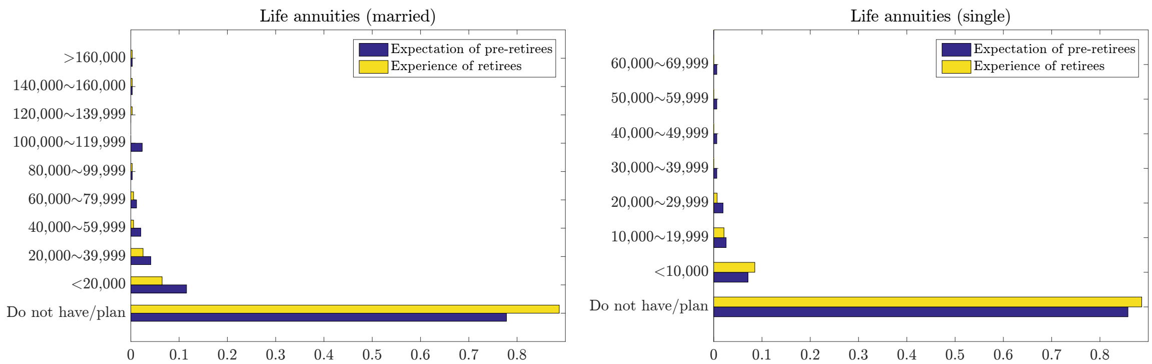


Figure 4.12 Life annuities in the first year of retirement.

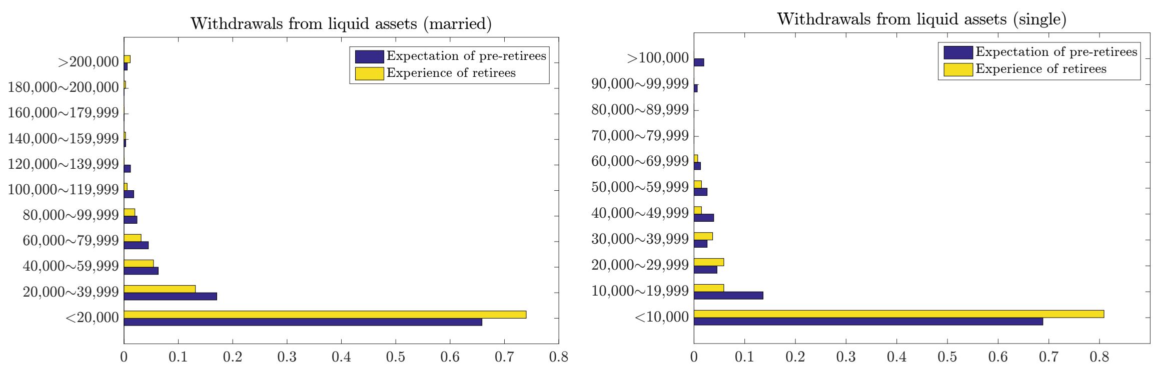


Figure 4.13 Withdrawals from liquid assets in the first year of retirement.

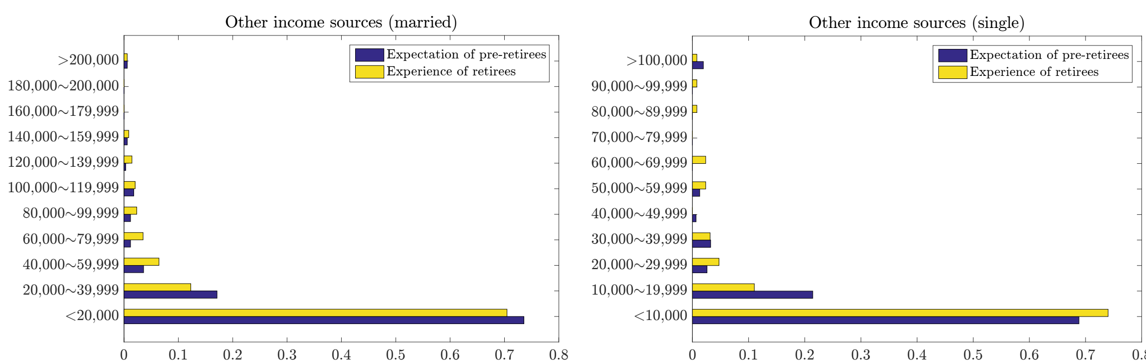


Figure 4.14 Income from other sources in the first year of retirement.

	60 and above			65 and above		
	All	Married	Single	All	Married	Single
All	72%	70%	75%	49%	45%	59%
Pre-retirees	93%	92%	96%	76%	72%	84%
Retirees	50%	50%	52%	22%	18%	31%

Table 4.18 Percentages of respondents with retirement age at least 60 or 65.

age at least 60 and 65. It can be observed that half of the retirees were not eligible for OAS and GIS benefits in the first year of retirement, which explains the high proportions of "Below $6,000" and "Below $3,000" responses from retirees. It would be sensible to repeat the analysis excluding those who expect to, or have retired before the age 60 or 65, which is summarized by Table 4.19, and Figures 4.15 and 4.16.

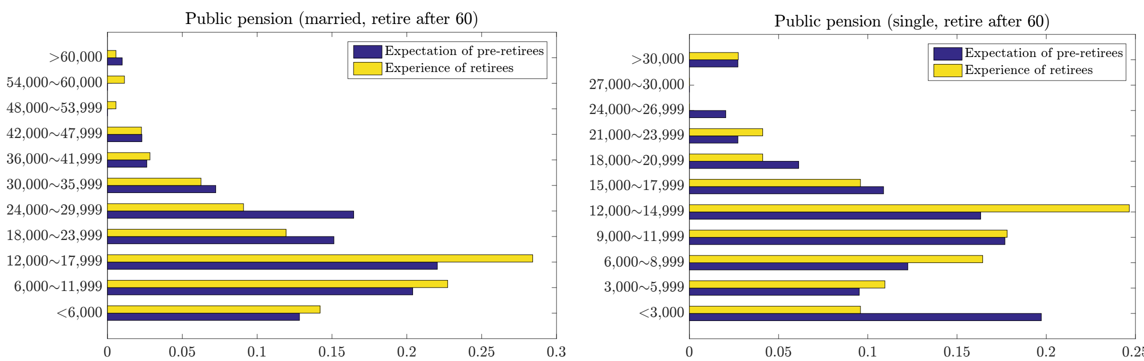


Figure 4.15 Public pension in the first year of retirement (retirement age 60 and over).

	Expectation of pre-retirees		Experience of retirees	
	Mean	Median	Mean	Median
60 and above	$15,762	$15,000	$15,440	$15,000
65 and above	$16,111	$15,000	$18,179	$15,000

Table 4.19 Average public pension income in the first year of retirement (excluding ineligible respondents).

We observe that when ineligible respondents are excluded, the difference between expectation and experience diminishes. We attribute this to the rise in the average of retirees' public pension income, since ineligible respondents are excluded. For the 65 and above group, pre-retirees' lower[11] expectation is largely driven by choices at the lower end, which

[11] The Mann-Whitney U-test is performed with $p = 0.0817$. The null hypothesis of equal distributions is not

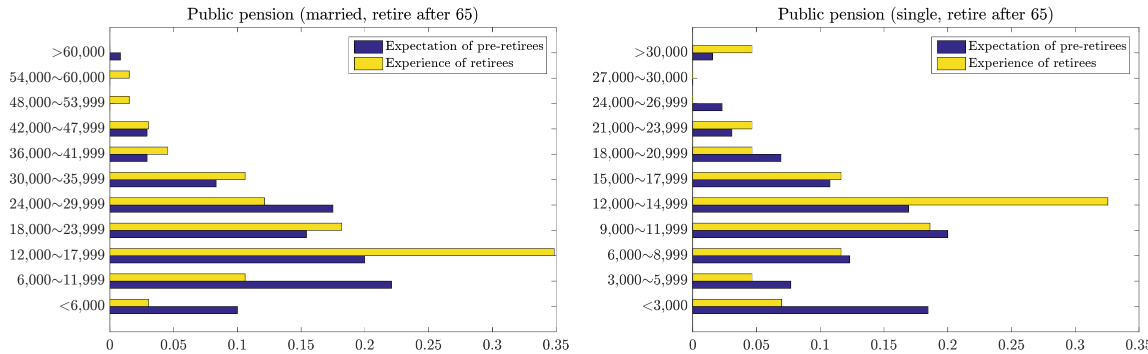


Figure 4.16 Public pension in the first year of retirement (retirement age 65 and over).

we discuss below. It should be noted that inflation and inflation expectations play a role, as benefits are indexed[12] to price inflation.

i. **Complexity/Lack of knowledge:** The Canadian pubic pension system is complex, and includes two pillars. The first pillar includes OAS and GIS which are means-tested. The second pillar, the CPP,[13] has earnings-related contributions and benefits. As such, estimating future public pension benefits is difficult, especially for those with low financial literacy who have not sought retirement planning advice. This is particularly the case for the CPP, where the calculation of the benefit is tied to past contributions. The CPP benefits, as well as income from other sources, also interact with the first pillar through "clawbacks", that is, higher CPP benefits and other income may reduce benefits from the OAS and GIS.

ii. **Age of application:** We solicit responses based on the first year of retirement, which means even if the retirement age is above 60 or 65, respondents may choose to postpone taking benefits to a later age, either irrationally (e.g. lack of knowledge, inaction, etc.), or rationally (e.g. CPP benefits can be taken as early as 60, at a reduced rate, or postponed to as late as age 70, with an incentive of increased benefits).

iii. **Changed incentives on age of application:** from 2012, the CPP introduced changes to further incentivize late retirement and disincentivize early retirement. Gradually by 2016, the benefit reduction rate for each year of early retirement increased from 6% to 7.2%; the adjustment rate for each year of late retirement increased from 6% to 8.4%. Consequently, pre-retirees in this study have a *higher* incentive to postpone

rejected at the 5% level.

[12]The OAS and GIS benefits are indexed to quarterly changes in the Consumer Price Index, whereas the CPP benefits are adjusted annually.

[13]Or the Quebec pension plan (QPP) for residents in Quebec, which we do not consider since respondents are residents of Ontario. Some (very few) respondents report income from QPP in the "other income sources" category.

CPP benefits than retirees, meaning that they may report *lower* expected public pension income in the first year of retirement.

iv. **Increase in future benefits:** Pre-retirees may expect public pension benefits to change due to future changes in the public retirement income system. This is a reasonable expectation since historically the Canadian public pension system has evolved significantly. Specifically, GIS benefits for single retirees increased in 2016; the CPP was also set to increase its benefits starting in 2019. Pre-retirees may or may not be aware of these introduced changes, and may or may not accurately translate legislative changes to monetary changes in their retirement income expectations. The effect on expectations due to future changes in public pensions is inconclusive since the survey does not specifically solicit directly relevant responses.

v. **Increase in future eligibility age:** In 2012, the Canada government announced plans to increase the age of eligibility for OAS and GIS from 65 to 67, taking effect between 2023 and 2029. The proposed change was reversed in March 2016, prior to the time this survey took place. Assuming that the survey respondents were informed of both events, it is reasonable that certain pre-retirees may hold the belief that the age of eligibility may increase in the future (since it has been proposed in the past). If these respondents expect to retire before the age of eligibility (say 67), then their expected social insurance pension should be low in the first year of retirement.

vi. **Immigration:** one factor that may contribute to the high concentration of low expected public pension income among pre-retirees is immigration. Since the early 1990s, the inflow of immigrants to Canada has been on average roughly 235,000 per year, which is nearly 50% higher than the average number from the 60s to 80s (source: Statistics Canada Catalogue no. 11-630-X). Pre-retirees who migrated to Canada at older ages would receive lower CPP benefits, since benefits are linked to historical contributions throughout one's working life. We do not collect immigration information in the survey, however, 21% of Canada's total population are foreign-born (source: NHS 2011) it is likely that we have a similar composition among the survey respondents. Nonetheless, it remains inconclusive as to what extent immigration contributes to the difference in expectation and experience.

Looking at those who retire at age 65 and above, the difference between expectation and experience is most significant for the first choice "below 6,000" and "below 3,000". It may be interesting to single out these respondents, and compare certain demographic information. For this we make the following comments.

We find that these pre-retirees on average have fewer years of formal education, hence are less likely to be financially literate (see, for example, Lusardi and Mitchell (2007a)): 62% report the highest level of education completed being "trades or college certificate or diploma" or below, whereas only 40% of the retirees in the same choice category report a similar education level. This is reasonable, since income is typically positively correlated with education received[14], and higher income earners receive less from means-tested public pension schemes. Further, 59% of the pre-retirees have not sought professional financial planning advice in the past, which, combined with likely lower financial literacy, raises questions as to whether they can accurately estimate future benefits. This percentage is higher than the 51% among the remaining pre-retirees. The above evidence suggests that it is quite likely that these pre-retirees underestimate public pension benefits due to lack of knowledge, which results in the mean being lower than retirees' experience even when future benefits are to increase due to recent legislative changes.

	Expectation of pre-retirees		Experience of retirees	
	Mean	Median	Mean	Median
Defined benefit	$18,562	$0	$16,822	$0
Life annuities	$5,538	$0	$3,148	$0
Withdrawals	$20,082	$10,000	$23,619	$10,000
Public pension	$16,111	$15,000	$18,179	$15,000
Other income	$18,003	$10,000	$19,688	$10,000
Total income	$78,296	$35,000	$81,456	$ 35,000

Table 4.20 Average income in the first year of retirement (retirement age 65 and above).

For pre-retirees who expect to retire at 65 or above (i.e. eligible for public pension benefits), we observe a positive relationship between public pension income and liquid assets expectations (the estimated Kendall's tau is 0.08*). In Table 4.20, we observe that public pension income is a primary source of income for an average (median) individual. This implies that those who expect more public pensions also tend to have higher liquid assets at retirement. This appears counterintuitive since one would imagine that with such low expectations, these individuals would save more. This may be due to the fact that the major source of public pension income, the CPP, is an earnings-related program.

4.8.5 Retirement planning and spending

Concerns

The results, summarized in Figure 4.17, show that for **all respondents**, the top four important concerns when making retirement planning and spending decisions are:

- **Liquidity** (setting money aside to access quickly when unforeseen expenses arise)
- **Consumption/Income smoothing** (avoiding ups and downs in my income / having a smooth income stream during retirement)

[14] Our results show a positive estimated Kendall's tau of 0.2503** and a correlation coefficient of 0.3280** for income and the highest education received among all respondents. This is consistent with other research findings such as Lusardi and Mitchell (2007b).

- **Inflation** (the impact of inflation on my standard of living)
- **Longevity** (the possibility of living longer than expected)

The least important concerns are:

- **Bequest** (leaving a bequest)
- **Investment risk** (taking some investment risk with my savings during retirement)

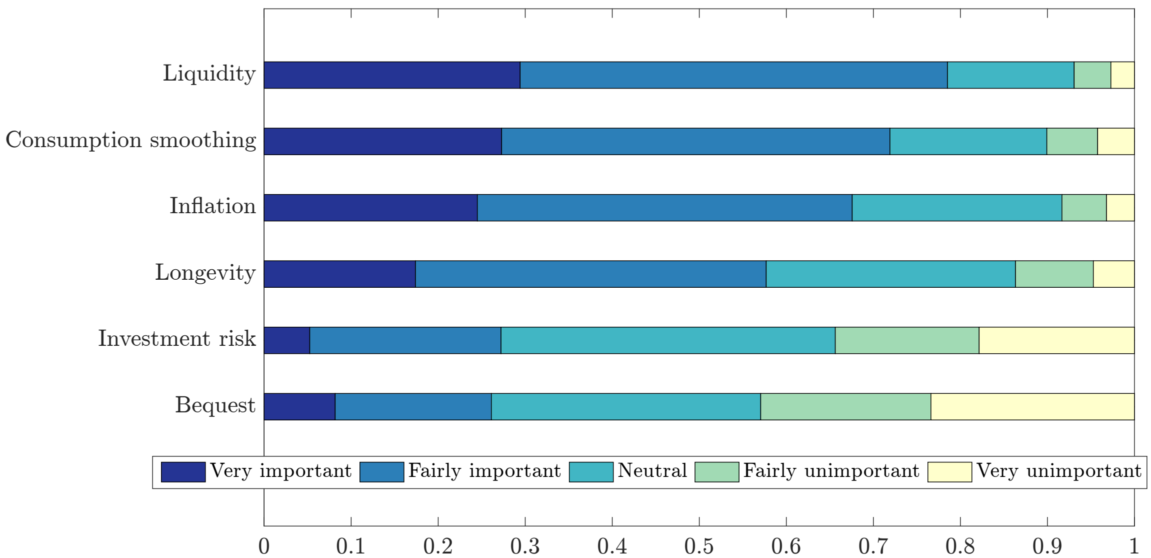


Figure 4.17 All participants: retirement planning and spending concerns (top to bottom: most important to least important).

Concerns change when separately considering the wealthy (i.e. “high” wealth level) and the extremely poor (i.e. liquid assets below $25,000) (see Figure 4.18). High wealth respondents are more concerned with home care or nursing home expenses than with longevity. Extremely low assets respondents are most concerned with consumption/income smoothing. These observations are reasonable, since wealthy respondents can potentially afford home care or nursing home services (while others cannot, hence are not too concerned); and respondents with extremely low liquid assets cannot tolerate income or consumption volatility due to having extremely limited capacity to make up for the differences.

We further quantify the relationship between the strength of the concerns and basic demographic information of the respondents. Table 4.21 summarizes the estimated Kendall’s taus of each concern and demographic item. The strength of the concerns is measured on a scale of 1 to 5, with 1 being very unimportant and 5 being very important. Retirement, gender and marital status are indicator variables, with 1 being pre-retiree, female, or married, and 2 being retiree, male or single (as shown in the tables). The concerns are listed in the order of high to low average importance among all respondents. We find the

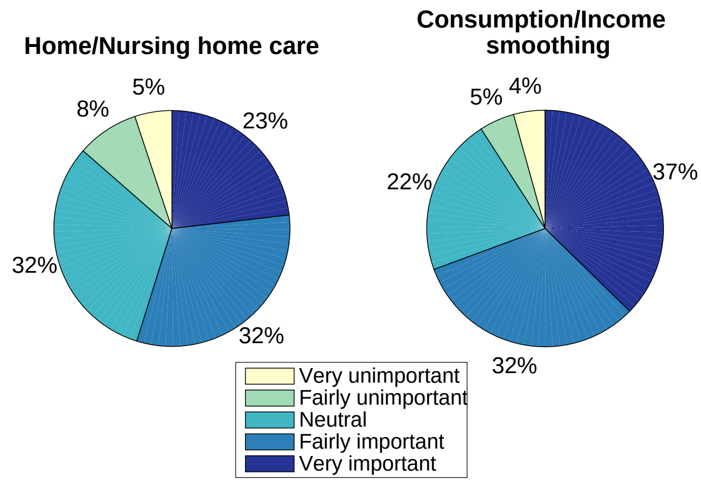


Figure 4.18 Retirement planning and spending concerns. Left: participants with "high" wealth. Right: participants with liquid assets below $25,000.

following evidence:

i. **Gender:** gender has a significant impact on the level of concern respondents express. For the top 8 concerns, the estimated Kendall's tau with gender are all negative and statistically significant, indicating that female respondents tend to rate the concerns as more important than male respondents. The relationship is stronger for top concerns. Male respondents, on the other hand, tend to rate taking some investment risk with their retirement savings as more important than females do.

ii. **Assets:** as discussed previously, those with high liquid assets tend to be more concerned over nursing care/home care expenses, and less concerned over longevity risk. Also, respondents with higher assets (liquid and fixed) tend to consider taking some investment risk as more important, suggesting that risk preferences may be dependent on wealth. More discussion on risk preferences and decision-making under risk can be found in sections 4.8.6 and 4.8.7.

	Age	Retirement	Gender	Marital	Liquid Assets	Fixed Assets
(1: unimp 5: imp)		(1: pre 2: ret)	(1: F 2: M)	(1. mar 2: sin)		
Unforeseen exps	−0.02	−0.01	−0.20**	0.00	0.06**	0.02
Smooth income	0.02	−0.02	−0.17**	0.03	−0.05*	−0.11**
Inflation	−0.07**	−0.08**	−0.15**	0.06*	−0.02	−0.06*
Living longer	−0.02	−0.04	−0.15**	0.02	0.01	0.00
Less spending	−0.09**	−0.11**	−0.11**	−0.07*	−0.04	−0.03
Nursing Care	0.01	−0.04	−0.10**	−0.02	0.08**	0.05
More spending	0.00	−0.03	−0.09**	0.07*	−0.05*	−0.06**
Dying early	−0.08**	−0.02	−0.07**	−0.07**	−0.08**	−0.03
Taking inv risk	0.00	0.01	0.07**	−0.04	0.17**	0.10**
Bequest	0.00	−0.02	−0.05	−0.05	0.03	0.03

Table 4.21 Kendall's taus: relationship between concerns and demographic information.

In addition to the specific planning and spending concerns, respondents are asked to rate how concerned they are, currently, regarding the impact of aging on their and/or their spouses' ability to manage their own finances. The average response shows a lack of concern (see Figure 4.19, the average is "fairly unconcerned"), especially among retired respondents (see Table 4.22). Mild relationship between the respondent's current health status, such that those with poorer health tend to be more concerned, is found. Single respondents show higher relative concern over the issue than married ones.

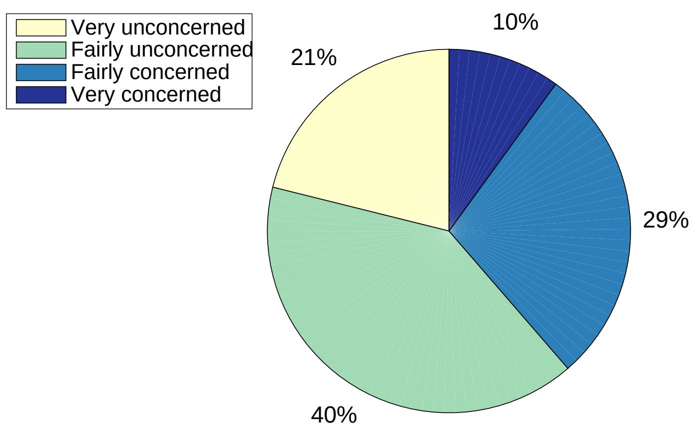


Figure 4.19 All participants: current concerns over the ability to manage own finances as one ages.

	Age	Retirement	Gender	Marital	Health	Liquid	Education
(1: unc 4: con)		(1: pre 2: ret)	(1: F 2: M)	(1. mar 2: sin)	(1: Exc 5: Poor)		
Aging	-0.06^{*}	-0.22^{**}	-0.06	0.07^{*}	0.08^{**}	-0.06^{*}	-0.04

Table 4.22 Kendall's taus: relationship between current concerns over the impact of aging on the capability of managing finances and demographic information.

Attitude towards professional advice

Figure 4.20 summarizes responses on seeking professional financial advice (henceforth advice). Overall, attitude towards seeking advice is positive: 60% indicate that it plays a role in future planning, a majority of which have sought advice, implying a comparably positive experience. A majority of those who will not be seeking advice have not had the experience of doing so in the past (i.e. "No/No").

Attitude is found to be strongly related the level of actual/expected liquid assets at retirement. Those with very low liquid assets (i.e. below $25,000, a total of 210 respondents) have dramatically different attitudes from the rest: 80% of these respondents report no interest in seeking advice in the future (34% for the rest). This finding is reasonable, as these respondents have little assets to manage. This also mirrors the findings of Iannicola and Parker (2010), which discusses barriers to financial advice for non-affluent individuals in the US. The following discussion excludes the 210 respondents with liquid retirement assets below $25,000.

Figure 4.21 summarizes the results: concerns are shown from top left to bottom right in the order of high to low average importance. Table 4.24 outlines the relationship between concerns and demographic information. We highlight the following findings:

i. **The general concern is high:** respondents on average show high concerns over all issues raised, with the top two being 'having access to quality service' and 'conflicts of interest'.

ii. **Age:** a mild relationship is found between age and concerns over quality assessment and conflict of interest, such that younger subjects are more concerned. Pre-retirees (who are on average younger) are more concerned about conflicts of interest.

iii. **Written comments:** A total of 84 respondents provided written comments on additional concerns (on top of those shown in Figure 4.21) over seeking advice or financial advisors in general, among which only 52 are relevant to the question. Top comments/concerns are recorded in Table 4.23. Interestingly, the most prevalent response is expressing trust in the current advisors (due to long-term relationships). Other concerns center around the fiduciary duties of financial advisors (i.e. conflicts of interest, fees transparency, and government oversight).

Top comments/concerns	Frequency	Percentage
“I trust my advisor”	11	21%
Conflict of Interest	7	13%
Affordability/cost	7	13%
Negative past experience	5	10%
Government oversight	5	10%
Fees transparency	4	7%

Table 4.23 Top comments/concerns towards seeking professional financial advice (collected from written responses).

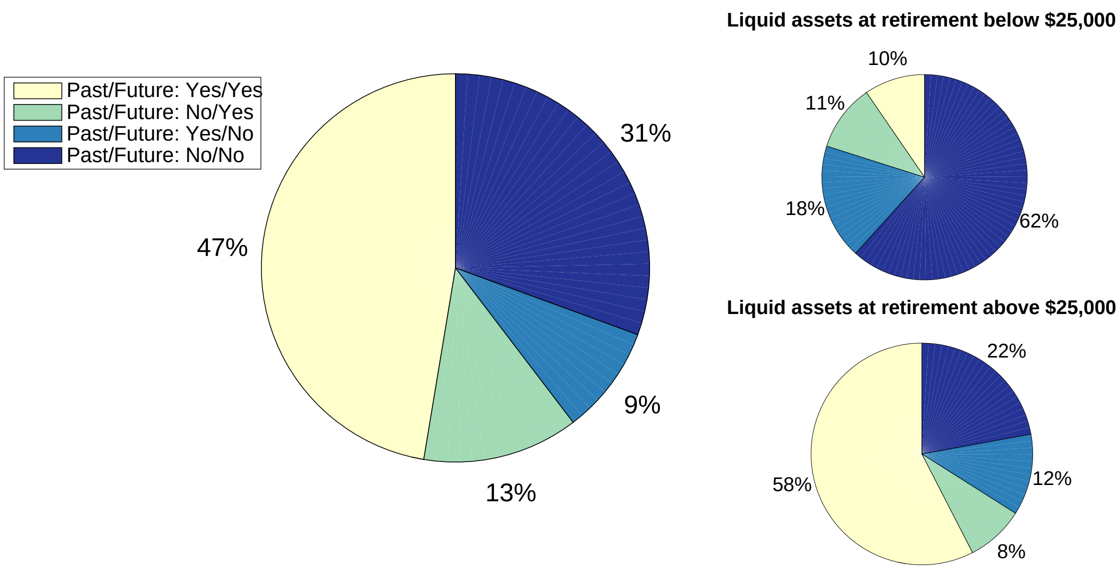


Figure 4.20 All participants: attitude towards seeking professional financial advice.

	Age	Retirement	Gender	Marital	Liquid	Fixed
(1: uncon 5: con)		(1: pre 2: ret)	(1: F 2: M)	(1. mar 2: sin)		
Access quality service	0.04	0.03	−0.06	−0.02	0.04	0.03
Victim of fraud	−0.05	−0.03	−0.02	−0.02	−0.04	0.00
Unable to assess quality	−0.06*	−0.06	−0.05	−0.03	−0.04	−0.01
Incapable advisors	−0.02	−0.02	−0.03	0.01	−0.01	−0.02
Self-serving advisors	−0.06*	−0.08*	−0.04	0.00	−0.04	0.00

Table 4.24 Kendall’s taus: seeking professional financial advice (excluding participants with liquid assets below $25,000).

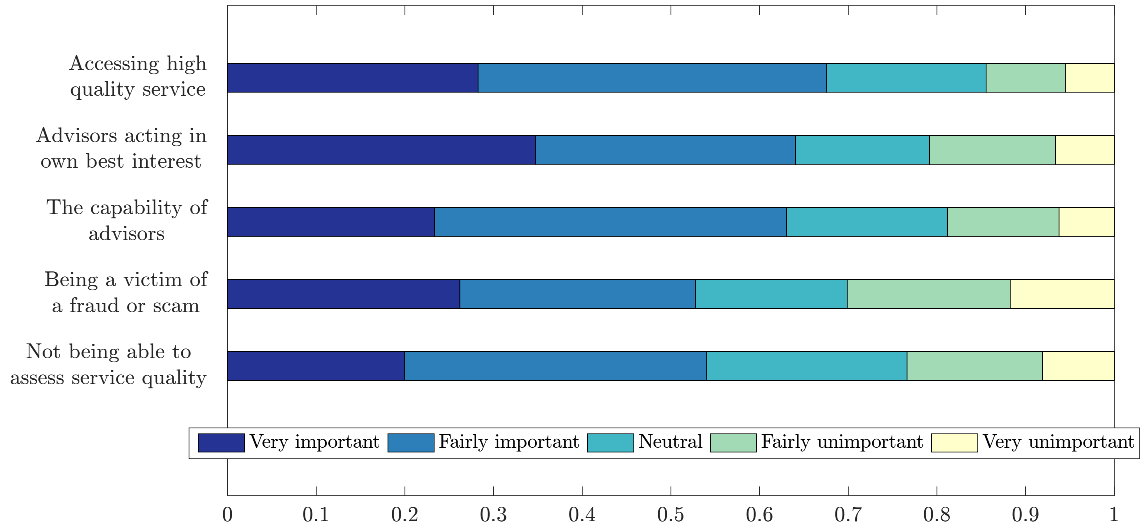


Figure 4.21 Concerns regarding professional financial advisors (excluding participants with liquid assets below $25,000). From top to bottom: high to low average importance.

Bequest motives

In the beginning of section 4.8.7, bequest motives are found to play the least important role in dictating retirement planning and spending decisions. However, bequest motives are common assumptions in traditional life-cycle models, and are often quantified by a utility function multiplied by a strength parameter, which governs the willingness to bequest. For example, an agent who more strongly values bequest would have a higher strength parameter. Consequently, the utility derived from bequest is more amplified, resulting in bequest playing a significant role in retirement decision-making in the academic literature. To date, the strength parameters used in these models are constant throughout an agent's remaining lifetime. We postulate that bequest motives vary, at the very least with age which broadly measures personal circumstances. For example, younger individuals are more likely to have dependents, hence should have stronger bequest motives if they were to die within the next couple of years than older individuals with grown children (i.e. fewer or no dependents). Younger individuals may also have younger dependents, who need more sizeable financial support, hence influencing the size of the bequest.

The survey collects responses on how important bequest is if respondents were to die within the next 10 years, 10 to 20 years and over 20 years from now. We are interested in exploring whether the strength of bequest changes with age, or other personal circumstances. Results are summarized in Figure 4.22 and Table 4.25. We report the following findings:

i. **Overall:** the average importance of bequest is between "fairly unimportant" and "neutral". This is consistent with our previous findings. The average level of importance *drops* when death is more distant: some of this difference may be due to myopia. Additionally, little evidence suggests that the strength of bequest motives is related to current health, or the amount of liquid assets.

ii. **Dependents[15]:** the number of dependents is found to have the strongest relationship with bequest motives, in that those with more dependents tend to value bequests more strongly. This finding implies that bequest motives are strongly driven by financial needs.

iii. **Marital status:** married respondents tend to have stronger bequest motives than single respondents, regardless of when death might occur.

iv. **Gender:** women tend to have stronger bequest motives than men, regardless of when death might occur. This is consistent with the previous findings that women tend to be more concerned over retirement planning and spending issues than men.

v. **Fixed assets:** respondents with higher fixed assets (i.e. properties) tend to consider bequest as more important. Interestingly no such relationship is observed for liquid assets. This implies that, potentially, respondents view properties as their major source of bequest.

vi. **Retirement and age:** since retirement and age are both negatively related[16] to the number of dependents (i.e. younger or pre-retirees tend to have more dependents), we find a mild relationship between each of the two factors and bequest.

(1: unimp 5: imp)	Age	Retirement (1: pre 2: ret)	Gender (1: F 2: M)	Marital (1. mar 2: sin)	Dependents (1: 1 5: ≥ 5)	Liquid	Fixed
< 10	-0.05^{*}	-0.10^{**}	-0.11^{**}	-0.13^{**}	0.21^{**}	0.02	0.07^{**}
$10-20$	-0.05^{*}	-0.06^{*}	-0.09^{**}	-0.12^{**}	0.20^{**}	0.02	0.07^{**}
> 20	-0.06^{*}	-0.06	-0.09^{**}	-0.11^{**}	0.20^{**}	0.02	0.06^{*}

Table 4.25 Kendall's taus: relationship between bequest motives and demographic information.

[15] Q10 collects responses on the number of people the household income supports in 2015 (for pre-retirees) and in the first year of retirement (for retirees). The average is 2.19 for pre-retirees and 1.98 for retirees (the Mann-Whitney U-test is performed with $p = 0.0116$. The null hypothesis of equal distribution is rejected at the 5% level.). Since no data is collected on the number of dependents retirees currently have, we make the assumption that the number of current dependents is equal to that at retirement.

[16] The estimated Kendall's tau for age and retirement, against the number of dependents are -0.14^{**} and -0.07^{*}.

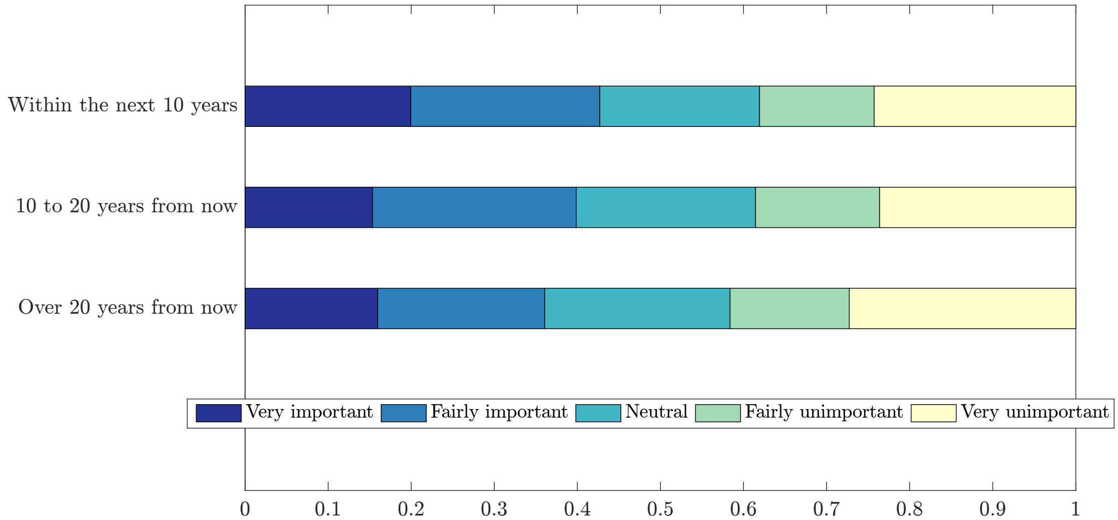


Figure 4.22 All participants: the importance of bequest if respondents were to die within different time periods.

4.8.6 Risk and time preferences

Risk preferences

We elicit risk preferences both qualitatively and quantitatively. Qualitative results are summarized in Figures 4.23 and 4.24, quantitative results in Figures 4.25 and 4.26, and Table 4.26.

i. **Average risk aversion is "moderate":** For pre-retirees, the median description for current risk attitude with respect to managing assets is that "I am willing to accept moderate risk". The same median is observed for retirees when asked to state their risk attitude when first retired. The distributions of risk attitude for pre-retirees and retirees show strong similarities (see Figure 4.23).

ii. **Risk aversion over time:** When asked to compare risk attitude across time (see Figure 4.24), almost all respondents expect/express that their risk tolerance level will be/is no greater than when they were younger. In other words, there is strong evidence that risk aversion is perceived to either stay the same or increase with age. This perception is consistent with empirical research findings such as Eckel and Grossman (2008) and Halek and Eisenhauer (2001). An interesting observation is that pre-retirees have a very reasonable expectation as to how their risk attitude would change entering retirement.

iii. **CRRA parameter:** The median implied CRRA range for pre-retirees is $1.74 < \rho < 2.48$ under *lower stakes* and $2.48 < \rho < 3.74$ under *higher stakes*. For retirees, the range is $2.48 < \rho < 3.74$ under both cases. Close to 90% of respondents have an implied CRRA parameter greater than zero (i.e. risk averse). There is strong dependency between the implied CRRA range under the lower and higher stake options (see Figure 4.26). We highlight the following implications and findings:

- For pre-retirees, the CRRA parameter is dependent on the level of consumption, such that the higher the stakes, the higher the aversion to risk. This is consistent with findings by Harrison et al. (2007), which offered evidence of lower estimates of relative risk aversion when the stakes are reduced significantly. These empirical observations question the validity of the restrictive representation of the CRRA utility function, and potentially point to a decreasing relative risk aversion utility form. Nonetheless, we find no such evidence for retirees. It could be that the difference exists but is too small to be detected. We postulate that this difference may reduce with age.
- Relative risk aversion is related to age, retirement status, gender, education, and wealth (see Table 4.26), such that the *retired/female/those with fewer years of formal education/poorer* are more risk averse. Risk aversion has the strongest relationship with wealth and retirement.

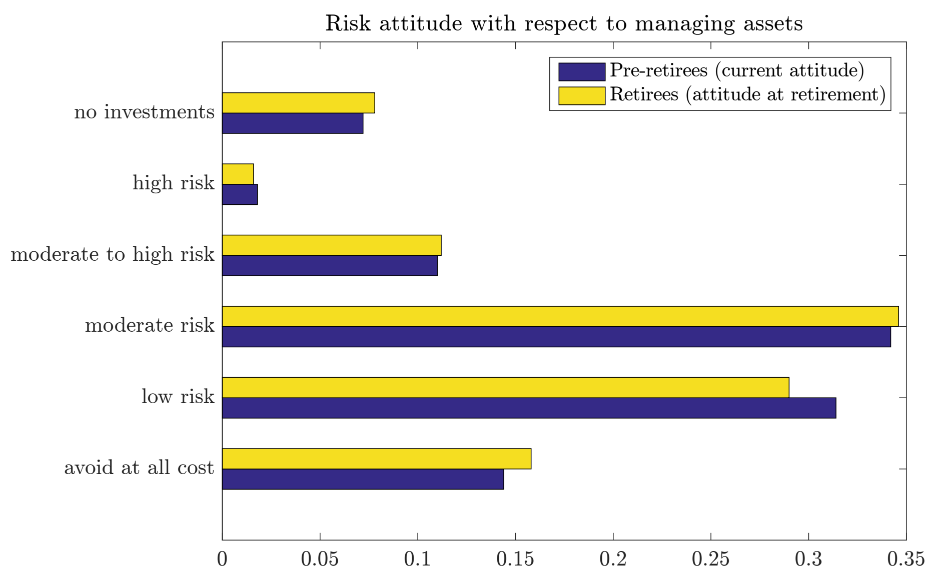


Figure 4.23 All participants: risk attitude (the amount of risks willing to accept).

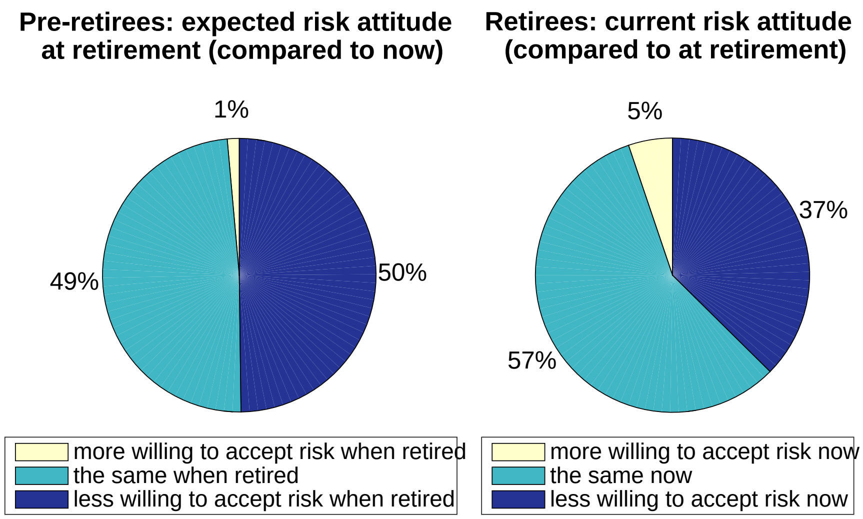


Figure 4.24 All participants: risk attitude (compared to current (pre-retirees) and at retirement (retirees)).

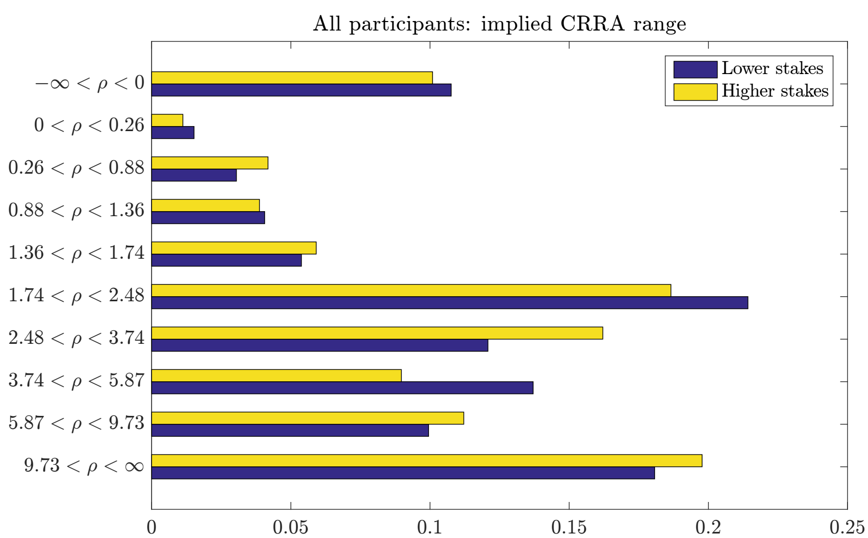


Figure 4.25 All participants: implied CRRA range for choices under lower and higher stakes.

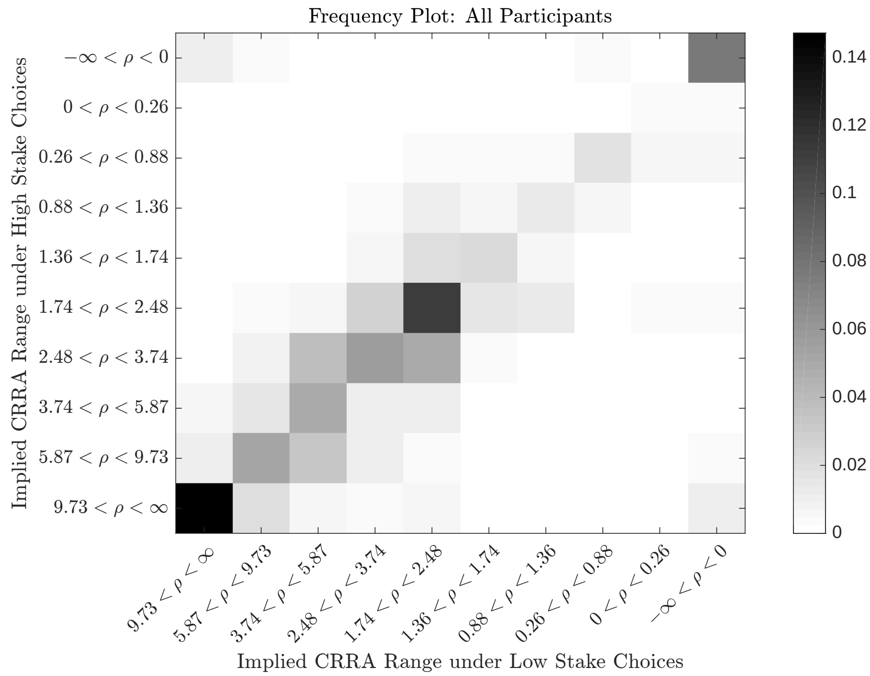


Figure 4.26 All participants: the relationship between the implied CRRA range under lower and higher stakes. Kendall's tau: 0.69**.

	Age	Retirement	Gender	Marital	Education	Liquid	Fixed
(1: low 10: high)		(1: pre 2: ret)	(1: F 2: M)	(1. mar 2: sin)	(1: low 6: high)		
Lower stakes	0.06*	0.09**	−0.06*	0.05	−0.06*	−0.08**	−0.10**
Higher stakes	0.05*	0.10**	−0.05	0.06*	−0.06*	−0.07**	−0.07**

Table 4.26 Kendall's taus: relationship between risk aversion and demographic information.

Time preferences

Similar to risk, time preferences are elicited through hypothetical monetary choices. Participants are asked to make 7 choices with binary payoff options, with a lower more immediate payoff and a higher payoff made in the distant future. The goal is to elicit the annual time discount factor β.

As described in section 4.5.2, β can only be obtained from consistent choices. Consistent choices include three cases: 1) consistently choose Option A, 2) consistently choose Option B, and 3) switch from A to B, once only. Among all participants, 715 made consistent choices. We obtain 711 time discount factors due their entanglement with the parameter of risk aversion ρ (see Equation (4.4) and (4.5)). Here, we use the ρ obtained from the higher stake options, and assume a background consumption[17] of \$172 which is the average daily consumption of Ontario residents in 2015. A histogram of elicited[18] discount factors is shown in Figure 4.27.

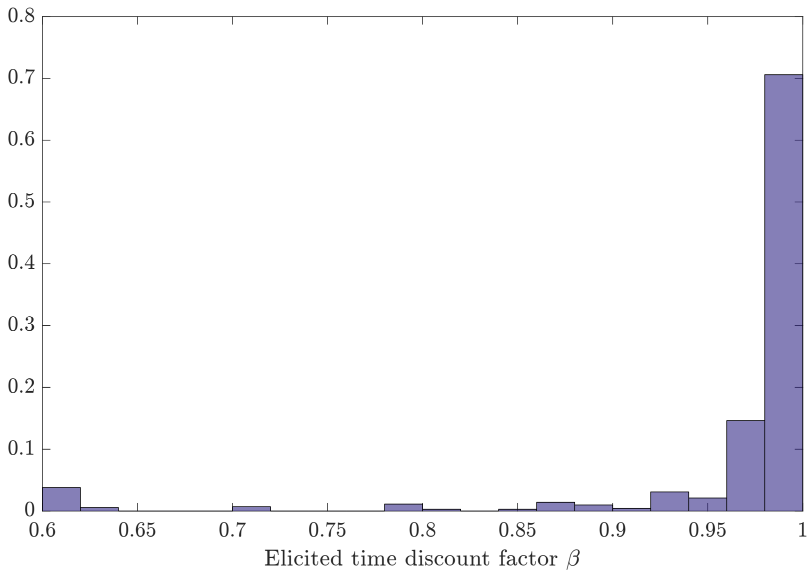


Figure 4.27 All participants: elicited annual time discount factor β.

	Age	Retirement (1: pre 2: ret)	Gender (1: F 2: M)	Marital (1. mar 2: sin)	Education (1: low 6: high)	Liquid	Fixed	Risk Aversion
β	0.02	0.09^{**}	-0.04	0.03	-0.01	-0.05	-0.04	0.86^{**}

Table 4.27 Kendall's taus: relationship between elicited time discount factor and demographic information.

We find the following:

i. **Average discount factor above 0.96:** the median elicited β is 0.997, and the mean is 0.965. 85% of the elicited β are greater than 0.96, 71% are greater than 0.98. The

[17]The assumption of background consumption ω impacts the elicited time discounting parameter β. A lower ω gives rise to a higher β. However, this impact is small within reasonable bounds. We repeated the calculations assuming average daily background consumptions of \$55 and \$300 (i.e. an average yearly consumption of \$20,075 and \$109,500) and found negligible differences. The average household consumption for Ontario residents is \$62,719 in 2015 (Source: Statistics Canada, CANSIM, table 203-0021 and Catalogue no. 62F0026M).

[18]We obtain the discount factors by taking the midpoint of the elicited range. For those who provide boundary responses (i.e. consistent choice of A or B), the elicited factor is taken as the upper or lower bound. This avoids having discount factors that imply extreme myopia.

two discount factors, 0.96 and 0.98, are commonly used in the retirement planning literature, and imply much stronger myopia in long-term problems than our empirical findings.

ii. **Retirement:** the elicited discount factor is related to the retirement status (see Table 4.16) in that retirees tend to have higher discount factors than pre-retirees. This implies that retirees may be more forward thinking in terms of financial planning and spending, due to lower income compared to pre-retirees.

iii. **Other demographic factors:** there is no evidence that the discount factor is dependent on age, gender, marital status, education, health, or asset levels. There is a strong positive relationship between β and ρ since the boundaries of β are increasing functions of ρ (see Equations (4.4) and (4.5)).

The elicited discount rates are based on short-term choices (i.e. 1 year) and are later applied to longer term problems (i.e. 30+ years) in Section 4.8.7. The assumption is that intertemporal choice is time-consistent, meaning that preferences do not change over time. This type of time discounting is called time-consistent discounting, or exponential discounting. Though widely used in the economic and retirement planning literature, there exists evidence that exponential time discounting does not reflect the observed behaviour of individuals. Loewenstein and Elster (1992), among others, provides an interesting and comprehensive discussion on intertemporal preferences. An alternative assumption is hyperbolic discounting, under which the discounting effect declines for longer periods and choice becomes time-inconsistent. Our study restricts the investigation to time-consistent models, which have been commonly adopted in the actuarial science literature to date. Investigations of alternative decision-making models are reserved for future studies.

4.8.7 Retirement planning under risky scenarios

The annuity puzzle

In the retirement planning literature, life annuities are shown to significantly improve the welfare of an individual in retirement (see Yaari, 1965; Davidoff et al., 2005; Peijnenburg et al., 2016). Despite these theoretical findings, there is ample evidence that, in practice, annuity sales are extremely low (with the exception of a few countries, such as the United Kingdom which until recently mandated partial annuitization of pension funds). This contrast is referred to as the *Annuity Puzzle*. To explore this puzzle, we construct a hypothetical annuity[19] product of \$100 per month and ask respondents to 1) value its worth, 2) make purchasing decisions, and 3) explain the reasons behind these decisions. We first ask respondents to specify the maximum amount that they are willing to pay for this

[19]The annuity product is presented under the name "guaranteed income for life" to avoid any pre-existent negative sentiment towards the word annuity.

product, followed by a question that offers the annuity at the selected price[20] and solicits responses on the number of units they are willing to purchase. Respondents are then asked to rate how important certain considerations are in their choices. Results are summarized in Figures 4.28, 4.29, and 4.30, and Tables 4.28 and 4.29. We highlight the following findings:

i. **Extreme undervaluation of annuities:** Overall, 66% of respondents report that the maximum price is below 3,000, 79% report below 6,000 and 84% report below 9,000 which is approximately half of the average market price in 2015. The 2015 average market quote for a life annuity of $100 per month for a life aged 65 is $19,776 for male and $21,202 for female. This suggests that annuity products have extremely low value to respondents. Figure 4.28 plots the relationship between liquid assets and the reported maximum prices. We observe that the majority of the choices are concentrated on "Below 3,000", regardless of the level of liquid funds. This is an interesting observation, since for a price of $3,000 the break-even point for purchasing the annuity is roughly 3 years. A financially literate respondent should be willing to purchase at this price if they believe that survival for another 3 years is highly likely.

ii. **Extremely low interest in purchasing annuities at any price:** Since respondents reported extremely low prices, the annuities are correspondingly offered at extremely low prices (e.g. a price of $1,500 is offered for those who chose "Below 3,000" in the valuation question). Surprisingly, 53% still choose not to purchase. The average offering price for these respondents is extremely low at $3,530, which is less than one fifth of the market price.

iii. **Liquid assets, gender and education:** We find that the maximum price is more strongly impacted by the amount of liquid assets, gender and the highest level of education, all of which have been shown to be strongly related to financial literacy (see, e.g., Lusardi and Mitchell, 2007b; Lusardi and Mitchell, 2007a). Male, more educated and those with higher liquid assets tend to value annuities more highly. The impact of gender is particularly interesting since females are more exposed to longevity risk than males, and should theoretically value annuities more highly.

iv. **Retirement status:** We find that pre-retirees tend to purchase more units than retirees. This mirrors the fact that pre-retirees have on average lower retirement income than retirees due to decreasing defined benefit workplace pensions.

[20] If the previous question is skipped, the 2015 market price is shown.

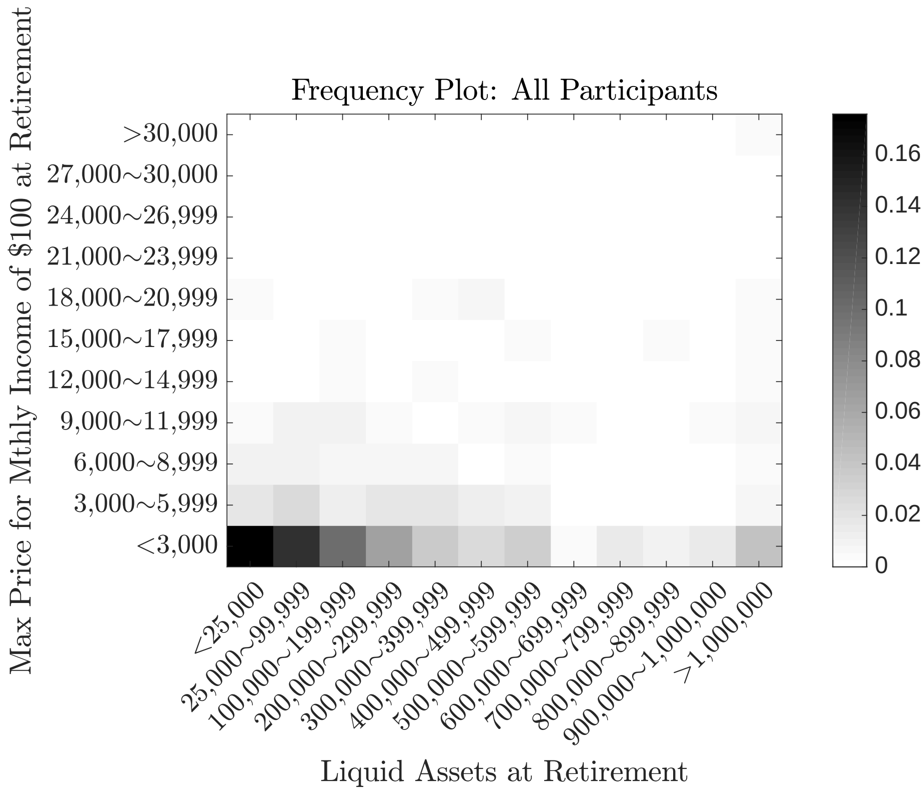


Figure 4.28 All participants: liquid assets and the maximum prices willing to pay for a monthly (annuity) income of $100.

Further evidence (summarized in Figure 4.30) suggests that the **top three concerns** over annuity purchases are:

- Credit risk (default of the annuity provider)
- Loss of flexibility and control of personal finances
- Loss of financial security

The above findings shed important light on the annuity puzzle. In the current literature, full annuitization is considered optimal when retirement planning (or welfare maximization) is based solely on maximizing the CRRA utility derived from consumption and bequest. We are not aware of any quantitative model that considers utility derived from *holding* wealth, which appears to have high psychological importance to individuals (i.e. a sense of control, liquidity and security). The fear of provider default, which can be argued to be rational or irrational, is also rarely considered in modelling descriptive decision-making. In fact, Canadian annuitants face very limited credit default risk due to the protection provided by Assuris – a Canadian consumer protection agency that insures full or partial annuity payments in the event of provider default. The strong concern demonstrated by respondents regarding provider default points to a lack of understanding of this protection. It should be highlighted that there is some framing effect at play, since the option of provider default is presented as a possible concern.

In making the annuity purchase decision, respondents show low concerns over longevity risk, possibly due to existing access to public/workplace pension income which provides some longevity protection. Respondents are least concerned over the possibility of family facing hardship if they were to die early, which gives rise to three possible explanations: 1) they do not consider dying early as a plausible scenario; 2) there are no significant financial consequences; 3) there might be significant financial consequences, but the respondent fails to foresee, or chooses to ignore them in the present (i.e. myopia). This, combined with the findings of low bequest motives in the section 4.8.5, suggests that the financial consequences in the event of death play an unimportant role in respondents' retirement planning and spending decisions.

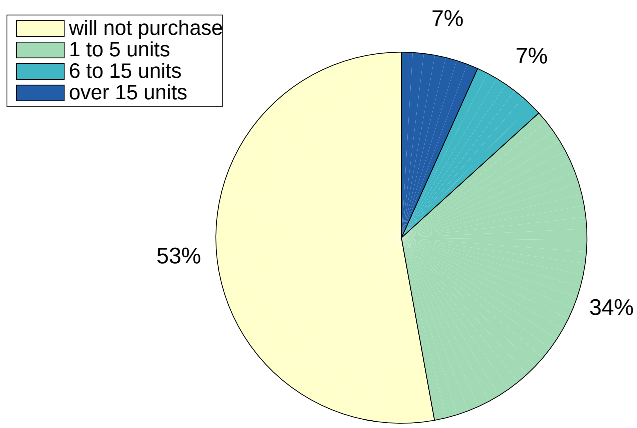


Figure 4.29 All participants: units of purchase when annuities are offered at agreeable prices.

	Maximum price	Units of purchase
Credit risk	0.01	0.04
Loss of flexibility and control	0.05	0.01
Loss of financial security	−0.07**	−0.10**
Longevity risk	0.07**	0.07**
Family hardship	0.00	0.03

Table 4.28 Kendall's taus: the relationship between maximum price/units of purchase and concerns.

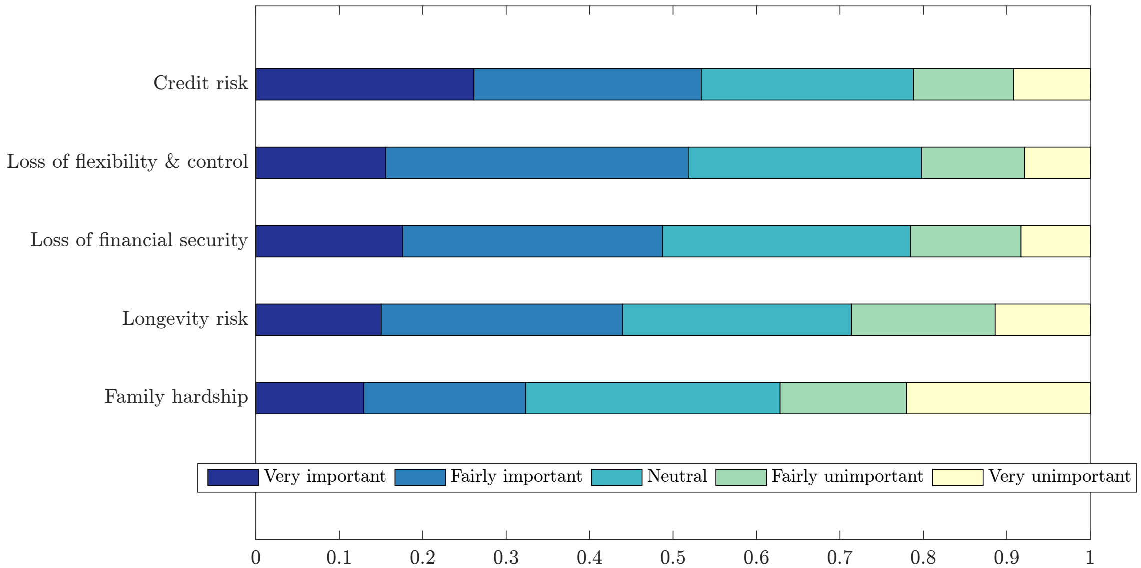


Figure 4.30 Concerns about purchasing annuities. From top to bottom: high to low average importance.

	Retirement	Gender	Marital	Health	Education	Liquid
	(1: pre 2: ret)	(1: F 2: M)	(1. mar 2: sin)	(1: exc 5: poor)	(1: low 6: high)	
Price	0.05	0.17^{**}	-0.10^{**}	-0.07^{*}	0.17^{**}	0.23^{**}
Units	-0.12^{**}	0.05	−0.01	−0.03	0.08^{**}	0.09^{**}
Credit risk	-0.10^{**}	-0.10^{**}	−0.04	0.03	−0.03	−0.01
Loss of flexibility	-0.09^{**}	-0.08^{**}	−0.01	0.05	−0.02	0.02
Loss of security	-0.09^{**}	-0.16^{**}	−0.03	0.07^{**}	-0.06^{**}	-0.09^{**}
Longevity	-0.17^{**}	-0.17^{**}	0.00	0.06^{*}	-0.06^{**}	-0.08^{**}
Family hardship	-0.08^{**}	−0.02	-0.18^{**}	0.08^{**}	-0.11^{**}	-0.08^{**}

Table 4.29 Kendall's taus: relationship between annuity purchases, concerns and demographic information.

Decision-making under risky pension scenarios

The purpose of this section is to assess the descriptive validity of the traditional life-cycle investment approach to retirement planning, which is to maximize welfare measured by the expected discounted lifetime utility (EDLU) derived from real consumption. In this section, we compare the pension choices implied under the EDLU approach, and the actual choices made by survey subjects. The maximized EDLU is expressed mathematically as

$$\max_{p\in\mathcal{P}} \mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t u\left(c_t(p)\right)\right\} \tag{4.14}$$

where $\mathcal{P} = \{$"pension option 1", "pension option 2"$\}$, and $\{c_t(p)\}_{t=0}^{\infty}$, real consumption, is a function of p. The implied decision under EDLU is that the subject should choose "pension option i" if

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t u\left(c_t(i)\right)\right\} > \mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t u\left(c_t(j)\right)\right\}, \qquad i \neq j. \tag{4.15}$$

Under the CRRA assumption, the utility function is $u(c) = \dfrac{(\omega + c)^{1-\rho}}{1-\rho}$. In our analysis, we compute the pension choices implied under the EDLU maximization method using elicited time and risk preference parameters $\widehat{\beta}$ and $\widehat{\rho}$. We additionally assume zero background consumption and that the monthly pension income is fully consumed. In reality, individuals may choose to save, invest or borrow, but data on such specific behaviour is not collected (and would be extremely difficult to do so, especially in a cross-sectional survey). The assumption simplifies the problem and the results should be interpreted accordingly.

The elicited time discount rates are based on short-term choices (i.e. 1 year) and are being applied to longer term problems (i.e. 30+ years). The assumption is that intertemporal choice is time-consistent, meaning that preferences do not change over time. This is referred to as time-consistent discounting, or exponential discounting. Though widely used in the economic and retirement planning literature, there exists evidence that exponential time discounting does not reflect the observed behaviour of individuals (Loewenstein and Prelec, 1992). A prominent alternative to exponential discounting is hyperbolic discounting, under which the discounting effect declines for longer periods (i.e. β in more distant periods becomes closer to 1) and choice becomes time-inconsistent (see Laibson, 1997). Our current study restricts the investigation to the time-consistent model described in Equation (4.14), which is commonly adopted in the retirement planning literature to date. Investigations of alternative decision-making models are reserved for future studies.

One may be interested in the implication of hyperbolic discounting on long-term decision-making investigated in this study. For this we provide the following comments. If the observed preferences are well-described by hyperbolic discounting, then future gratifications would be discounted less heavily (compared to exponential discounting), which would potentially impact decision-making at the present. This effect would be more prominent for those who exhibit stronger hyperbolic preferences, potentially agents with a lower short-term discount factor β. Overall, most of the elicited β's are very close to unity, meaning that very little short-term discounting is observed. This implies that for most surveyed individuals, even under the assumption of exponential discounting (where the future is more heavily discounted than in the hyperbolic discounting), time discounting does not have a dominant impact on decision-making. The implication is that if preferences follow hyperbolic discounting, most of the differences in model-implied choice would be quite small.

i. Inflation-indexed pension (with subjective beliefs)

Respondents are first asked to choose between Match-inflation pension (MIP,"pension option 1") and Steady pension (SP, "pension option 2") without any given information on future inflation outlook. The breakdown of choices for all six cases is summarized in Figure 4.31. The relationships between pension choices and certain decision-making factors are presented in Table 4.30. We highlight the following findings:

i. **Consistency:** respondents on average make consistent choices. The proportion of MIP choices increases for succeeding choices, which is reasonable since the next MIP choice gives strictly higher income. This is also observed in the next section when objective inflation information is provided.

ii. **Low expectation of cumulative inflation impact in the long term:** when provided with no information on inflation outlook, respondents place lower value on the MIP. Comparing Figure 4.31 to Figures 4.32 and 4.33, the percentage of those who prefer the MIP increases dramatically, implying that the subjects either 1) believe that inflation has a lower impact on the cost of living than described (i.e. on average 2% per annum with some variations), or 2) lack the understanding of the impact of cumulative inflation on the cost of living. A combination of both factors is also likely.

iii. **Financial literacy:** certain choices are quite strongly clouded by low financial literacy. Certain respondents with low financial literacy may not grasp the concept of inflation or be able to quantify its cumulative impact in the long term. This is clearly observed in Figure 4.31, where 31% choose SP over MIP when MIP gives strictly higher income in all scenarios. This is strongly related to subjects' understanding of the MIP option: Table 4.30 shows an increasing estimated Kendall's tau for "MIP is difficult to understand", suggesting that those who find the MIP difficult to understand are far more likely to choose the steady option regardless of the case.

iv. **Other relationships:** respondents concerned with inflation impact on living standard tend to identify, correctly, that the MIP provides inflation protection, and are willing to "trade" such protection with lower income in the earlier years. Those not concerned with inflation risk are more likely to choose the SP in all cases. However, both relationships are rather mild. An interesting observation is that "maintaining a stable living standard" is not associated with either pension choice, as one would expect. The implication is that respondents may fail to recognize that the MIP is the option that provides "steady" purchasing power.

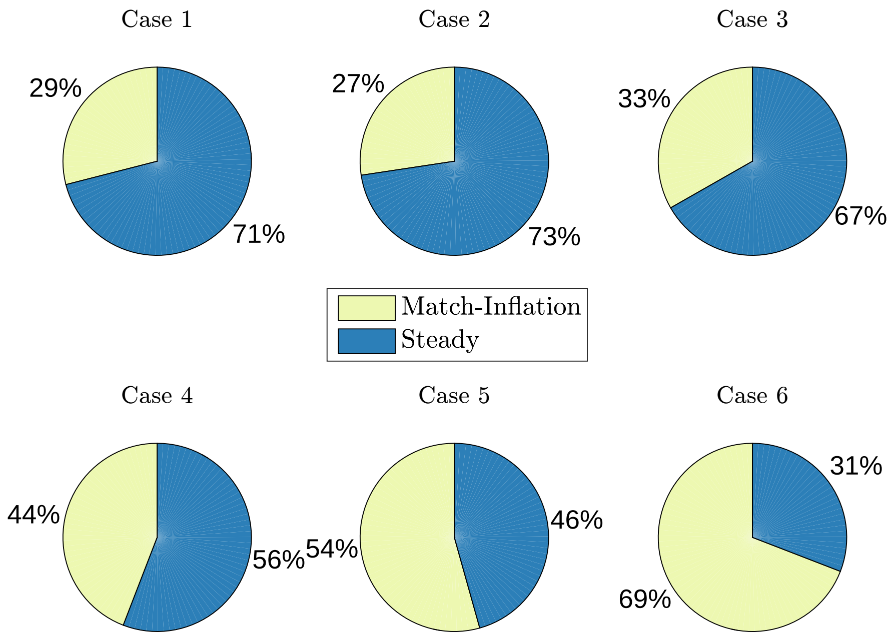


Figure 4.31 All participants: choices under match-inflation pension (subjective beliefs on future inflation).

(1: unimp 5: imp) \ (1: MI 2: S)	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Stable living standard	0.05	0.04	0.04	0.05	0.06	0.02
Inflation impact	-0.11^{**}	-0.10^{**}	-0.10^{**}	-0.08^{*}	-0.03	-0.05
Inflation risk is low	0.04	0.02	0.07^{*}	0.09^{**}	0.08^{**}	0.08^{**}
MIP difficult to understand	0.03	0.05	0.10^{**}	0.17^{**}	0.22^{**}	0.20^{**}

Table 4.30 Kendall's taus: relationship between match-inflation pension choices and retirement planning concerns (subjective beliefs on future inflation).

ii. Inflation-indexed pension (with objective information)

In the second part of this section, participants are asked to remake the six choices between Match-inflation pension (MIP,"pension option 1") and Steady pension (SP, "pension option 2"), but this time with objective information on inflation provided. Participants are told that inflation is on average 2% per year, but can vary between 1% and 3% from year to year (i.e. is risky). A graph on the cost of living for the next 35 years is provided, along with a chart that depicts the income paths for options 1 and 2. For a detailed description of the survey questions, see Section 4.5.3.

Figures 4.32 and 4.33 compare the actual choices made by subjects and the implied (or "predicted") choices under the EDLU maximization approach. The relationships between pension choices and selected demographic factors, retirement income and planning concerns are summarized in Tables 4.31, 4.33, and 4.34, respectively. The importance of concerns are summarized in Figure 4.34. We additionally analyze the point at which subjects switch from SP to MIP, as it reveals important insight on the perceived "cost" of inflation protection. The switching point is defined as the pension scenario where the respondent selects the MIP for the first time. Those who select SP for all six cases are ignored (83 individuals). The relationships between the switching points and subjective survival beliefs are summarized in Table 4.32.

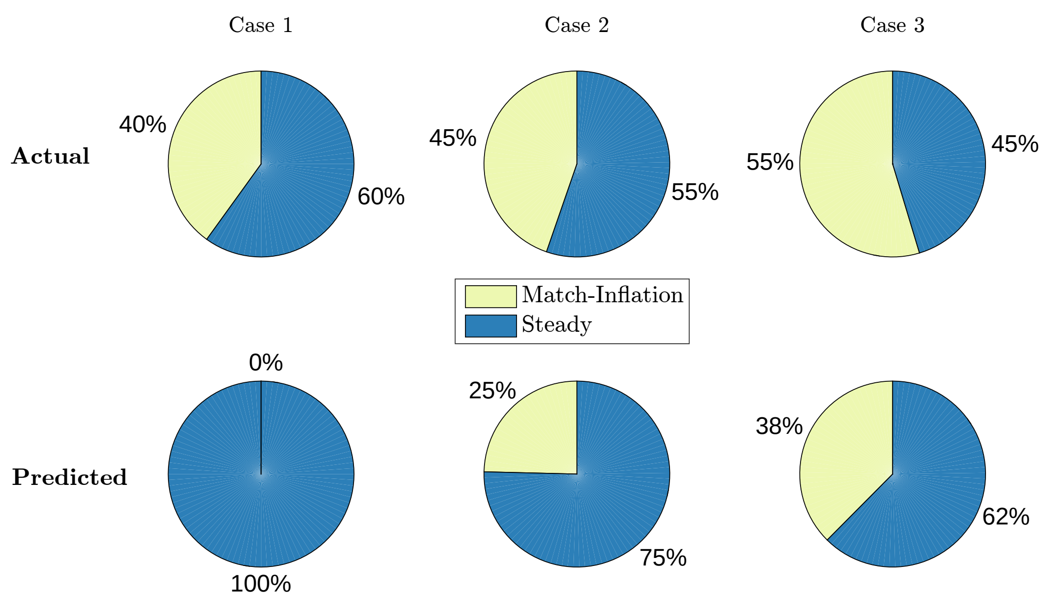


Figure 4.32 All participants: choices under match-inflation pension scenarios 1 to 3 (inflation information provided).

We highlight the following findings:

i. **Dramatic differences between actual and implied choices when the MIP**

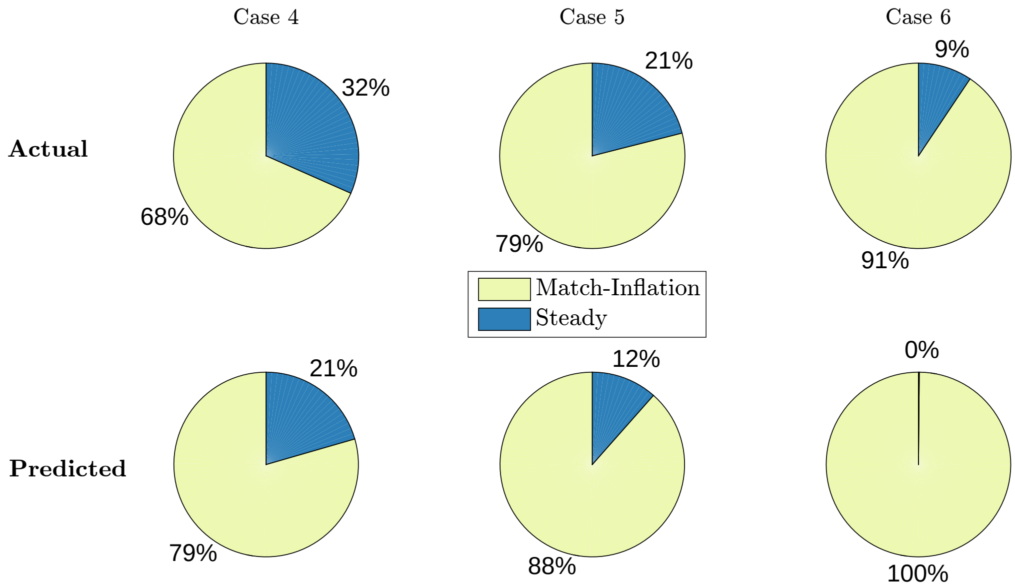


Figure 4.33 All participants: choices under match-inflation pension scenarios 4 to 6 (inflation information provided).

is costly: when the MIP is more "costly" (cases 1, 2, and 3), we observe stronger preferences towards the MIP than those implied under the EDLU maximization approach. This suggests that inflation protection in retirement income is valued much more strongly than the model predicts. Note that in case 1, where the MIP is offered at a higher price than the average 2016 market price for inflation-indexed annuities, 40% still prefer the MIP to the SP. The differences are smaller for higher MIP benefits (cases 4, 5, and 6), as the appeal of MIP become more apparent to the participants. Nonetheless, 9% of participants prefer the SP when the MIP gives less risky and strictly higher real payments. This may be a result of extreme risk-seeking behaviour, irrationality, mindless decision-making or confusion/misinterpretation.

ii. **Average switching points:** the mean and median switching point to the MIP are 2.84 and 3, respectively. Both suggest a higher starting value than the average 2016 market starting value for inflation-indexed annuities, implying that the average price is moderately higher than what an average individual would accept (in the presence of a fixed annuity).

iii. **The cost of inflation protection related to subjective longevity beliefs:** the switching point is found to be related to age, subjective expected age at death, and subjective beliefs of survival rate to an extreme old age of 95. The impact of age in decision-making is evident, since older participants have lower uncertainties regarding future longevity (since age 65, as the hypothetical scenario describes). Those who believe that they will live longer (measured by subjective expected age at death and survival rate to 95), are more likely to choose the MIP due to the stronger long term impact of inflation. The switching point is also found to be unrelated to retirement status, gender, marital status, education, assets or risk aversion.

iv. **The impact of retirement savings:** subjects with lower retirement liquid assets tend to prefer the MIP when the MIP is more expensive. This is a reasonable observation since lower liquid assets are less likely to provide a long-lasting income stream that protects against declining purchasing power. As the MIP becomes less costly, the relationship with savings changes. This is due to the fact that the group of subjects who consistently choose the SP (even when the MIP is quite attractive) are increasingly dominated by subjects with lower financial literacy. These individuals tend to have lower assets and fewer years of formal education (see column "Education" in Table 4.31).

v. **Retirement income has little impact:** we observe little significant relationship between pension choices and retirement income. It is unclear whether respondents, while making the hypothetical pension choices, take existing retirement income into consideration.

	Age	Retirement	Gender	Marital	Education	Liquid	Fixed	Risk aversion
(1: MI 2: S)		(1: pre 2: ret)	(1: F 2: M)	(1: mar 2: sin)	(1: low 6: high)			
Case 1	−0.04	−0.02	0.03	0.04	−0.01	0.07**	0.05	−0.02
Case 2	−0.06*	−0.03	0.06*	0.02	−0.01	0.09**	0.05	−0.01
Case 3	−0.08**	−0.05	0.07*	0.01	−0.02	0.02	0.01	0.01
Case 4	−0.11**	−0.10**	−0.01	0.01	−0.08**	−0.08**	−0.06*	0.03
Case 5	−0.09**	−0.04	−0.04	0.00	−0.12**	−0.12**	−0.09**	0.07*
Case 6	−0.06*	0.00	−0.01	−0.03	−0.07*	−0.07**	−0.07*	0.06*

Table 4.31 Kendall's taus: relationship between match-inflation pension choices and demographic information (inflation information provided).

	Age	Expected age at death	Subjective survival rate to age 95	Time discounting
Switch	−0.06*	−0.11**	−0.12**	−0.06

Table 4.32 Kendall's taus: relationship between match-inflation pension choices and age, subjective survival and time discounting (inflation information provided).

iii. Equity-linked pension

In this section of the survey, participants consider a slightly different hypothetical pension scenario. The format is similar to the previous section on inflation-indexed pension with

(1: MI 2: S)	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Defined Benefit	−0.05	−0.04	−0.05	−0.06*	−0.01	0.01
Public pension	−0.07*	−0.08**	−0.04	0.00	0.01	−0.02
Life annuities	−0.07*	−0.07*	−0.05	0.01	0.03	0.04
Withdrawals	0.02	0.03	0.01	−0.02	−0.05	0.00
Other income	−0.01	0.00	0.00	−0.01	−0.02	−0.01

Table 4.33 Kendall's taus: relationship between match-inflation pension choices and retirement income (inflation information provided).

(1: unimp 5: imp) \ (1: MI 2: S)	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Stable living standard	−0.03	−0.04	−0.05	−0.01	0.00	−0.04
Inflation impact	−0.20**	−0.21**	−0.20**	−0.15**	−0.11**	−0.05
Inflation risk is low	−0.01	−0.02	0.02	0.05	0.07*	0.08**
MIP difficult to understand	−0.06*	−0.07*	−0.03	0.05	0.10**	0.16**

Table 4.34 Kendall's taus: relationship between match-inflation pension choices and retirement planning concerns (inflation information provided).

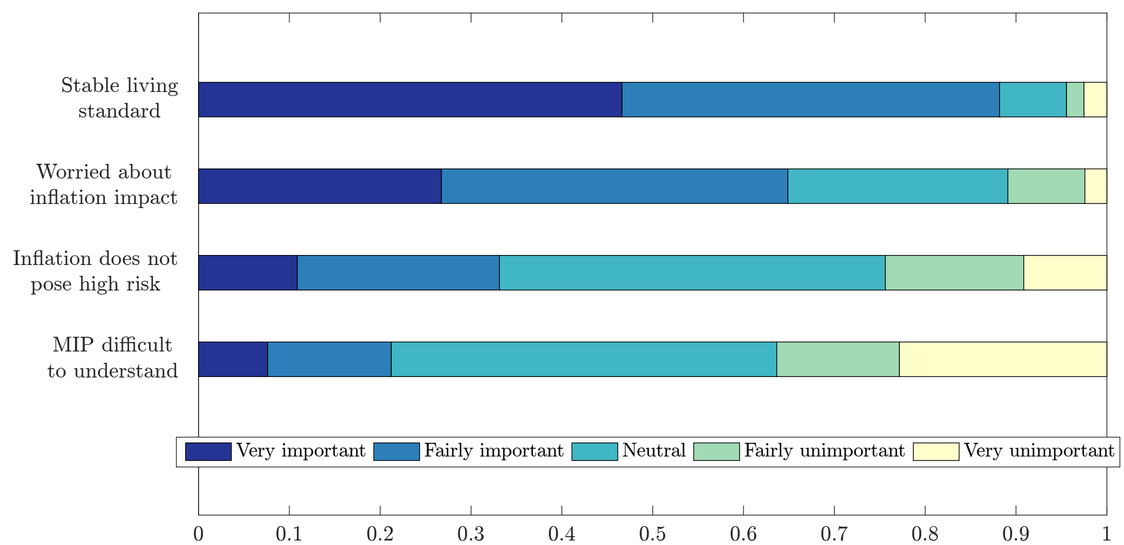


Figure 4.34 Importance of concerns when making choices between MIP and SP (top to bottom: high to low average importance).

inflation information provided. Here, participants are provided with six cases. In each, participants are asked to choose between an Equity-Linked pension (ELP, "pension option 1") and the Steady pension (SP, "pension option 2"). The six scenarios vary in the starting pension incomes and the level of risk exposed in the ELP. For a detailed description of the survey question, see Section 4.5.3. The goal is to study the perception of the risk and return tradeoff in retirement income. In Section 4.8.5, we observe that subjects report low interest in taking investment risk in retirement, and a high preference for a smooth consumption/income stream. We are interested in the consistency of these responses. Participants' choices, along with implied (predicted) choices under the EDLU maximization approach are summarized in Figures 4.35 and 4.36. The relationships between the pension choices and demographic information, retirement income, and concerns are included in Tables 4.35, 4.36 and 4.37.

We highlight the following key findings:

i. **Dramatic differences between actual and predicted choices:** similar to MIP, we observe dramatic differences between the actual choices made by subjects and the predicted choices implied by the EDLU maximization approach. In particular, we note the following:

   - **The upside gain drives decision-making:** as the ELP becomes riskier, the percentage of ELP choices increases. This has an important implication, that the main driving force behind decision-making in risky scenarios is the *upside gain* (in the context of our framing). The predicted choices imply the opposite, that fewer subjects should prefer the ELP as risk grows higher. Note that the worst case scenario (i.e. the purple path) becomes increasingly worsened. Regardless, on average, this does not seem to discourage participants from choosing the ELP.
   - **Downside protection induces more risk-taking behaviour:** a close examination of the relationship with existing pension income shows that those with higher social insurance pensions exhibit higher risk-taking behaviour. A social insurance pension acts as a minimum income guarantee: the higher the guarantee, the lower the impact of downside scenarios on living standard, hence, the lower the downside consequences for choosing the riskier workplace pension (with the upside gain unaffected). This observation has important implications on the design of retirement income products, especially those with some form of guarantee structures, such as variable payout annuities, hybrid pension plans and segregated funds products.
   - **Stronger implied preferences towards the ELP by EDLU maximization:** the EDLU maximization approach suggests stronger preferences for the ELP than observed (see Figures 4.35 and 4.36), except for the case 2C where the risk is at its highest. We postulate that EDLU over-predicts risk-taking behaviour when the risk/return is lower, and under-predicts when risk/return is higher. This may imply some form of S-shaped preference model, such as those described in Cumulative

Prospect Theory (see Tversky and Kahneman, 1992).

ii. **Consistency with risk aversion:** choices are consistent with the elicited relative risk aversion, such that those with higher CRRA parameters tend to prefer the SP. Particularly, female or lower wealth subjects tend to be less risk-taking and show stronger preferences towards the SP (see Table 4.35).

iii. **Perception of risk and return:** Respondents report that the two most important deciding factors are:

- Old-age financial security[21]
- Persistently poor stock performance[22].

The two least important concerns are:

- Incomprehensible pension choices[23]
- Risk-taking for long term gains[24]

Additionally, Table 4.37 shows that the top two important concerns are associated with a preference for the SP, along with the fear of stock market crashes and a preference for choices that are comprehensible (i.e. the implication is that the SP is easier to understand than the ELP, hence more preferable). In particular for the most important concern, the association shows *farsightedness*, as respondents tend to recognize the long term risk in the extremely low income paths. This is consistent with the high average elicited time discount factor.

Although risk-taking for long term gains is rated as the least important on average, those who believe in its importance show a very strong preference for the ELP. A contradiction is observed for "changing risk attitude"[25], as those who consider this an important reason tend to choose the ELP. Recall that in Section 4.8.6, almost half of subjects report that aversion to risk will increase as they age. This implies that these respondents should, theoretically, prefer the SP as the ELP has higher volatility and downside consequences as they age. This brings about important questions about how respondents perceive risk and return. We postulate that respondents may perceive the SP as "risky" as it would provide "insufficient" income when compared to the highly attractive upside of the ELP, hence putting their financial security at risk.

[21] I want to make sure I have enough income later in life.
[22] I am worried about persistent poor stock market performance and deteriorating income stream.
[23] The concept of the Equity-Linked pension option is difficult to understand.
[24] I expected to take some risk to gain exposure of possible higher income in the long term.
[25] I expect my risk attitude to change as I grow older.

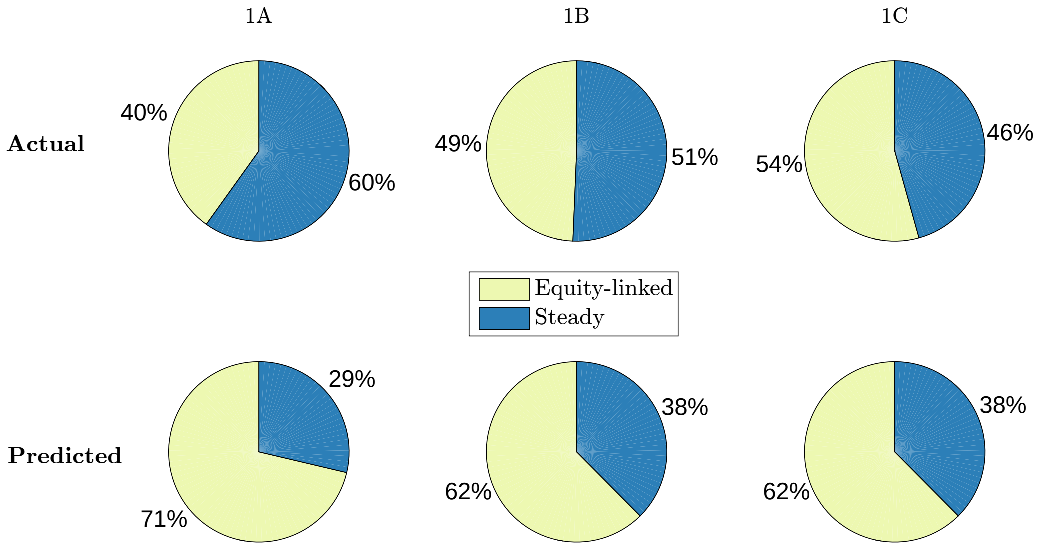


Figure 4.35 All participants: choices under risk pension scenarios 1A to 1C.

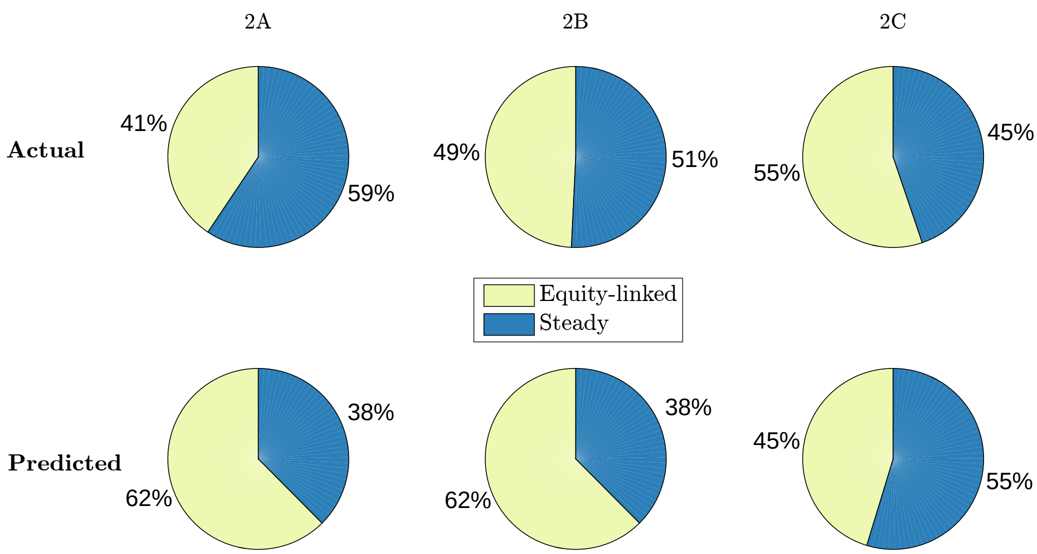


Figure 4.36 All participants: choices under risk pension scenarios 2A to 2C.

(1: EL 2: S)	Age	Retirement (1: pre 2: ret)	Gender (1: F 2: M)	Marital (1: mar 2: sin)	Education (1: low 6: high)	Liquid	Fixed	Risk aversion
1A	-0.06^{*}	-0.03	-0.18^{**}	0.03	-0.08^{**}	-0.10^{**}	-0.08^{**}	0.14^{**}
1B	-0.05	0.01	-0.14^{**}	0.03	-0.07^{*}	-0.08^{**}	-0.07^{*}	0.18^{**}
1C	-0.03	0.03	-0.14^{**}	0.04	-0.07^{*}	-0.07^{**}	-0.07^{*}	0.16^{**}
2A	-0.01	0.03	-0.11^{**}	0.03	-0.03	-0.09^{**}	-0.06^{*}	0.12^{**}
2B	-0.05	-0.02	-0.10^{**}	0.02	-0.06^{*}	-0.08^{**}	-0.06^{*}	0.16^{**}
2C	-0.05	-0.01	-0.14^{**}	0.03	-0.05	-0.08^{**}	-0.06^{*}	0.19^{**}

Table 4.35 Kendall's taus: relationship between equity-linked pension choices and demographic information.

(1: EL 2: S)	1A	1B	1C	2A	2B	2C
Defined Benefit	0.05	0.02	0.00	0.04	0.00	0.00
Public pension	-0.06^{*}	-0.10^{**}	-0.06^{*}	-0.06^{*}	-0.06^{*}	-0.04
Life annuities	-0.02	-0.05	-0.05	-0.03	-0.05	-0.05
Withdrawals	-0.06^{*}	-0.06^{*}	-0.04	-0.07^{*}	0.04	-0.03
Other income	-0.04	-0.06	-0.02	0.04	0.01	0.01

Table 4.36 Kendall's taus: relationship between equity-linked pension choices and retirement income.

(1: unimp 5: imp) \ (1: EL 2: S)	1A	1B	1C	2A	2B	2C
Take risk for return in long term	-0.33^{**}	-0.36^{**}	-0.36^{**}	-0.31^{**}	-0.37^{**}	-0.39^{**}
Stock market crashes	0.30^{**}	0.31^{**}	0.28^{**}	0.20^{**}	0.24^{**}	0.25^{**}
Persistently poor stock performance	0.29^{**}	0.29^{**}	0.26^{**}	0.21^{**}	0.23^{**}	0.24^{**}
Prefer understandable options	0.23^{**}	0.22^{**}	0.24^{**}	0.13^{**}	0.18^{**}	0.18^{**}
ELP difficult to understand	0.13^{**}	0.13^{**}	0.11^{**}	0.07^{**}	0.11^{**}	0.10^{**}
Changing risk attitude	-0.10^{**}	-0.10^{**}	-0.13^{**}	-0.09^{**}	-0.13^{**}	-0.12^{**}
Enough income later in life	0.10^{**}	0.11^{**}	0.10^{**}	0.09^{**}	0.07^{*}	0.09^{**}

Table 4.37 Kendall's taus: relationship between equity-linked pension choices and retirement planning concerns.

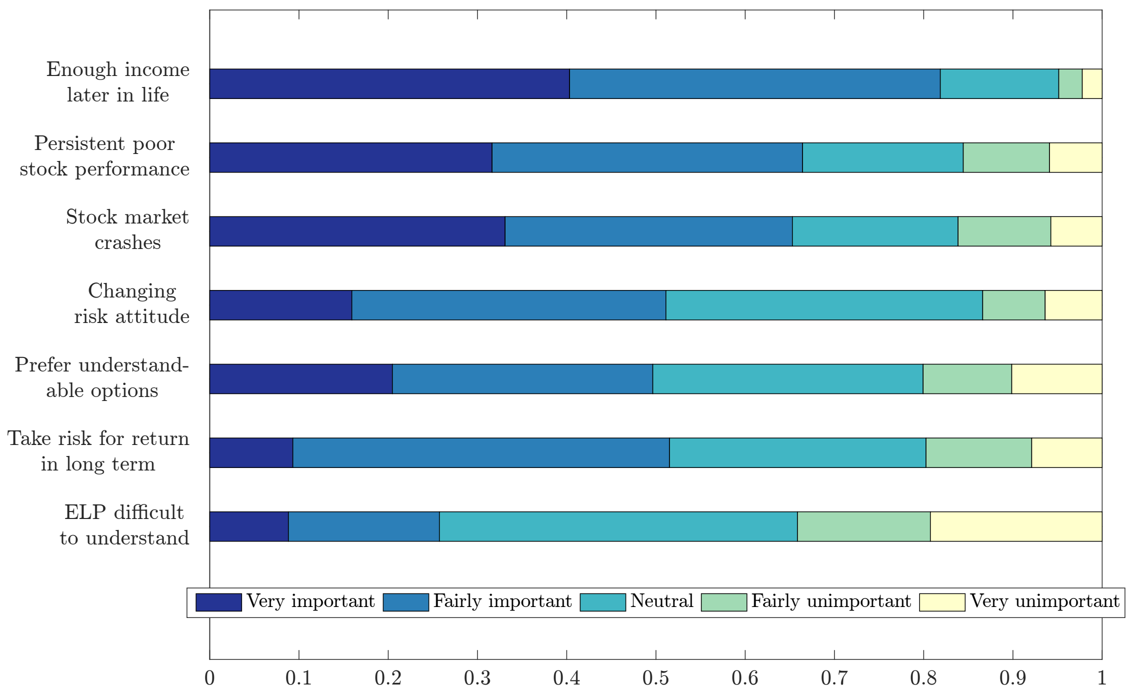


Figure 4.37 Importance of concerns when making choices between ELP and SP (top to bottom: high to low average importance).

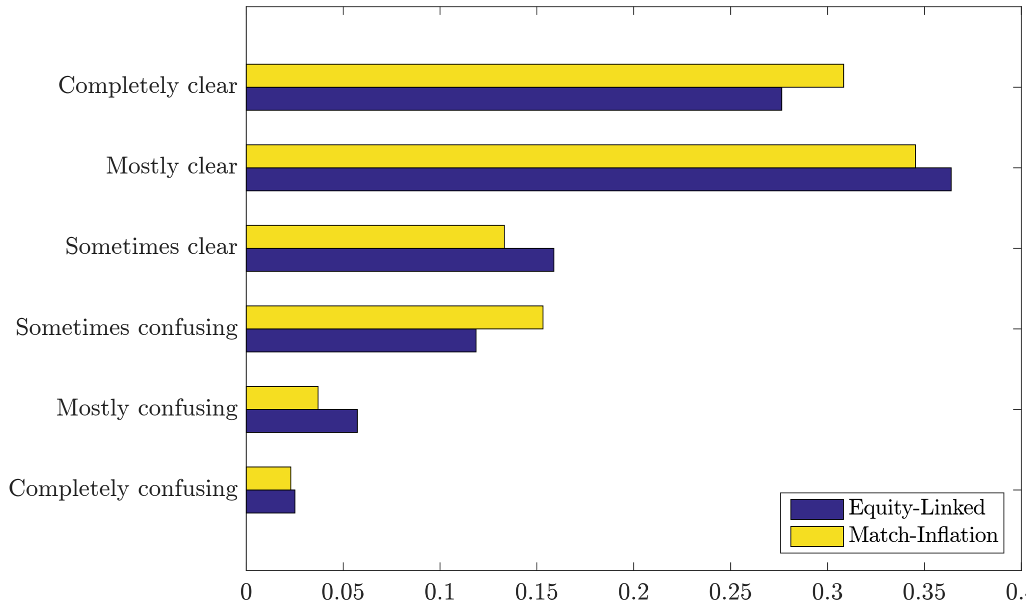


Figure 4.38 Reported clarity of the questions involving the Match-Inflation and Equity-Linked pension options.

4.9 Conclusion and Implications

This report presents the findings of the Ontario Retirement Survey, which studies retirement consumption, wealth, income, risk perception, decision-making, and planning objectives of Canadian pre-retirees and retirees. Specifically, following the footsteps of Beshears et al. (2014), we study the decision-making process of participants facing longevity, inflation and investment risks. It should be noted that these choices are hypothetical in nature, and the results should be interpreted with caution.

Several findings of this study have important implications on the design of retirement income and pension solutions, and the quantitative literature on retirement planning, especially those with a focus on the decumulation phase.

The study finds that while (pre-)retirees have reasonable beliefs for their own life expectancies, the meaning of life expectancies may be wrongly interpreted. It is possible that individuals believe that expected age at death is the maximum age achievable. For example, the official Canadian Retirement Income Calculator on Canada.ca (see *Canadian Retirement Income Calculator*) uses an individual's expected age at death as the default age at death for calculating retirement income, meaning that the status quo (Kahneman et al., 1991) would lead to a withdrawal plan that ceases at the default age at death, exposing the individual to the risk of longevity or a sharp drop in living standard at an advanced age. We further find that respondents severely underestimate their survival probabilities to an extreme old age of 95, where almost a third believe that the chance is zero. Such beliefs have potentially large impact on retirement planning behaviours and may result in severe financial consequences at extreme old ages.

There has been a strong emphasis on bequest motives in the literature of utility maximization (see thesis introduction). Modelling bequests means that some utility can be derived from the death state, creating a tradeoff between consumption and savings, thus leading to more flexible optimal consumption profiles. Our study shows that leaving a bequest is, in general, unimportant. This questions the necessity of modelling utilities in the death state, especially when it adds to the computational complexity of dynamic programming. More important concerns revealed by our study, such as liquidity, consumption smoothing, home care or nursing home care expenses, were given less attention in quantitative works. More effort should be made to integrate these concerns into new or existing modelling frameworks in future work.

The study again confirms the widespread aversion for life annuities, which results in a profound subjective undervaluation of their monetary worth, and extreme reluctance in purchasing annuities at any price. Such aversion is mainly grounded on a general fear for provider default (irrational in the Ontario context), which is also found by Beshears et al. (2014). Although theoretically, a life annuity provide longevity protection, it tends to be perceived as a 'risky gamble' of whether a retiree can 'win' by outliving the pool of annuitants. As shown, respondents profoundly undervalue the risk of living longer than expected, which further implies an over-estimation of the risk of 'losing' such a gamble. The lump sum purchase, on the other hand, results in the perception of losing financial flexibility, control and security; none of which, to our knowledge, is currently incorporated in descriptive retirement decision-making models that give rise to the annuity puzzle. In essence, we highlight that the annuity puzzle is *no longer* a puzzle, when one chooses to recognize the psychological deterrents associated with annuity purchases (see MacDonald et al., 2013). Existing effort made on the reconciliation of theoretical explanations and empirical observations of life annuities largely ignores these important psychological dimensions.

We find that a moderate level of risk aversion is measured by a constant relative risk aversion parameter (ρ) between 1.74 to 3.74. This finding sets a benchmark for future parameter choice in similar contexts. We point out that this is significantly lower than what is commonly assumed in the literature (i.e. $\rho = 5$), raising doubt over the practical relevance of the results arise from these studies. We also find that (pre-)retirees are less myopic than typically assumed. This is manifested in the high time discount factors elicited, which are very close to 1, suggesting close to zero discounting for future gratifications. This further underscores the paradoxical nature of subjective time discounting in descriptive and prescriptive modelling of retiree behaviour, especially given its negative cumulative impact on old-age financial security. Existing works that assume stronger subjective time discounting should be revisited.

Our study on decision-making under risk pension scenarios invalidates the CRRA utility maximization approach as a descriptive model in the context of long-term retirement planning. We find dramatic differences between the actual and implied choices under this approach. Our investigation reveals that in the context of our framing, when facing investment risks, the potential for the upside gain drives decision-making, and that having a higher minimum income protection (often provided by public pension income) induces more risk-taking behaviour. Our work further reveals that when facing risky inflation, participants in general lack the understanding of its long-term cumulative impact on the cost of living. These findings highlight the complexity of the interplay among various retirement income sources and decision-making. They also have important implications on the design of retirement income products involving risk taking and sharing, especially those with embedded choices, and guarantee structures, such as hybrid pension plans and segregated funds products.

As with most survey studies, the ORS has some limitations. But the study was conducted following best practices in statistical sampling. General limitations and errors present in survey studies are well-described in Fowler Jr (2013). Specific limitations in the survey design and the analysis of results are discussed as they emerge in the main text. The conclusions should be interpreted with these limitations in mind.

Chapter 5

Future Work

This thesis explores three important topics in retirement decision-making, namely long-term economic scenario modelling and generation, optimal life-cycle investment, and preferences and objective formulation. This chapter discusses future work in each of these topics. General directions that require more attention are also highlighted.

Chapter 1 investigates Wilkie's ESG and translates the model for uses in the US context. Future work may consider the development of a long-term ESG more appropriate for US or Canadian use, taking into considerations the limitations of Wilkie's ESG discussed in this chapter. Similarly, model risk can be explored by comparing the model with open-access ESGs. Importantly, there still lacks a rigorous definition of what constitutes an appropriate ESG, for practitioners or independent researchers, in general or for specific applications. The literature would benefit from such a discussion. Additional practical aspects of ESGs should be regarded in the development and critique of these models, these include user friendliness, flexibility, calibration, scalability, and the communication of model assumptions and outputs.

Chapters 2 and 3 consider the extension to Epstein-Zin preferences in modelling intertemporal portfolio choice. This extension retains a major flaw of EDLU maximization such that the certainty equivalent of future utility is computed as an expected value. This measure of certainty equivalent effectively trades off extremely high and low consumption scenarios. Extensions to non-expected certainty equivalent, such as the Chew-Dekel class (see Epstein and Zin, 1989) should be considered for future work. Along with this, more efficient numerical procedures should be explored (see Brandt et al., 2005), given the sensitivity of speed to the complexity of the preference model. To date, the most common approach remains to be value function iteration, using either a serial or parallel method.

Studies of life-cycle investment continue to dominate the retirement planning literature, with the majority putting forward normative solutions. It should be highlighted that these solutions are based on greatly simplified objective functions. Consequently, the resulting optimal strategies are most often not adequate to be given as plausible financial planning advice. As shown in Chapter 4, individuals face a significantly more intricate and evolving world, ranging from economic, financial and health risks, to pensions, evolving preferences and spending concerns, etc. The prescription of retirement planning strategies should *not* be made in absence of a thorough understanding of actual decision-making in retirement (rationality aside). Future investigations should be driven by practical usefulness, rather than mathematical complexity or descriptive validity of the preference models. At the very least, researchers in the prescriptive line of work ought to ask the question whether the resulting optimal solutions appear “reasonable”, or can be adopted as practicable retirement planning strategies in real life. Additionally, on top of individual optimality, researchers should consider the broader impact of the proposed normative solutions. After all, the prescription of microeconomic decision-making should not be made independently of a nation’s macroeconomic security and sustainability.

Those who wish to extend the survey study in Chapter 4 can consider the following improvements to the survey design. First, participants indicate that they would like to have the option of selecting \$0, or “do not have” when asked about wealth and income. Note that the option of \$0 is included in the option “Below \$X”, but respondents may interpret “Below \$X” as a strictly positive response. These respondents believe that they are not given a selectable option, hence, may randomly select or skip the question altogether. Second, the hypothetical annuity purchase question can be improved. In the follow-up question on annuity purchases, instead of offering the annuity at the midpoint of the selection, it may be more realistic to offer them at the lower bound of the selection. This way, the offer prices will be strictly below or at the maximum specified price. Third, a question on the immigration status of the respondents may be included. Immigrants who arrived in Canada after adulthood face challenges when it comes to retirement planning and savings in three ways: 1) disruptive careers and differences in the nature and characteristics of employment, hence reduced capacity and incentives to save privately for retirement, 2) OAS eligibility, 3) CPP contributions. Fourth, future work may include survey questions that explore Canadians’ attitude towards purchasing reverse mortgage, term and whole life insurance products. Finally, this survey study can be replicated for similar investigations in countries other than Canada, with a few questions on social insurance programs altered to reflect the country-specific programs.

In Chapter 4, we find significant heterogeneity in retirement savings, income, consumption, and planning concerns. This questions the existence of a truly “representative” agent, since retirement concerns the entire working-age population. Significant effort should be made to incorporate this heterogeneity in the mathematical formulations of objectives, and this should be done well beyond simply specifying varying levels of risk aversion and time preferences. Future work may explore techniques from predictive analytics. Moreover, this heterogeneity creates challenges for researchers and policymakers to assess the normative adequacy of theoretical and prescriptive results. Future work should consider the construction of a set of qualitative and quantitative criteria, taking into account micro-level characteristics of the retirees. Some effort along this line was made in this thesis, but significant improvement and extensions should be explored. Future work may consider the framework of the Living Standards Replacement Rate described in MacDonald et al. (2016).

Lastly, considerably more effort should be made on the interpretation, and consequently translation, of theoretical results to practice.

Appendix A – Chapter 1

A.1 Updating Wilkie's ESG to 2014

Wilkie's ESG (US) updated to 2014 remains an integrated stochastic economic model. The relationship between total stock return and long-term bond yield sustains through the inflationary effect. Parameters appropriate for long-term economic scenario generation are reported in Table A.1. The updated model structure is shown in Figure A.1.

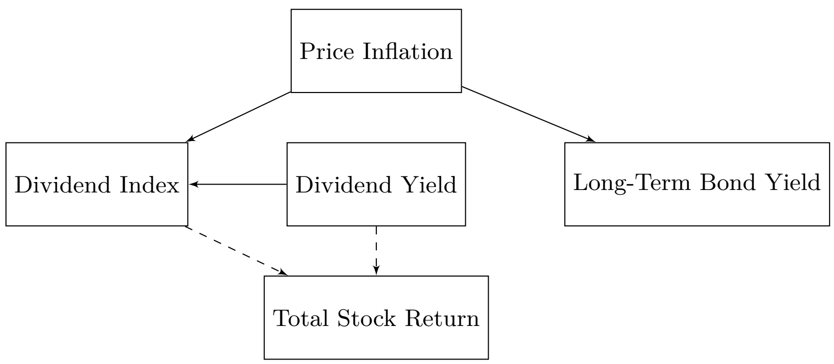


Figure A.1 Wilkie's ESG: model structure (fitted to 1991–2014).

	Estimates	(SE)	Simulation
μ_q	0.0338	(0.0085)	0.0338
a_q	0.7575	(0.0793)	0.7575
σ_q	0.0171	(0.0015)	0.0171
w_y	0.0504	(0.9249)	0
μ_y	0.0331	(0.0108)	0.0331
a_y	0.9582	(0.0341)	0.9582
σ_y	0.1310	(0.0116)	0.1310
w_d	1	–	1
d_d	0.3800	–	0.3800
μ_d	0.0202	(0.0098)	0.0202
y_d	0.2138	(0.1055)	0.2138
b_d	−0.3468	(0.1563)	−0.3468
σ_d	0.1146	(0.0103)	0.1146
w_c	1	–	1
d_c	0.0580	–	0.0580
μ_c	0.0213	(0.0103)	0.0213
a_c	0.9109	(0.0576)	0.9109
y_c	−0.1958	(0.2337)	0
σ_c	0.3157	(0.0280)	0.3157
c_{min}	0.0050	–	0.0050

	Estimates	(SE)	Simulation
μ_q	0.0244	(0.0021)	0.0244
a_q	-0.1011	(0.2043)	0
σ_q	0.0111	(0.0016)	0.0111
w_y	−4.5762	(1.5249)	−4.5762
μ_y	0.0252	(0.0055)	0.0252
a_y	0.9112	(0.0759)	0.9112
σ_y	0.1159	(0.0167)	0.1159
w_d	1	–	1
d_d	0.3800	–	0.3800
μ_d	0.0306	(0.0266)	0
y_d	0.1567	(0.2187)	0
b_d	0.1215	(0.2013)	0
σ_d	0.1097	(0.0165)	0.1097
w_c	1	–	1
d_c	0.0580	–	0.0580
μ_c	0.0338	(0.0081)	0.0338
a_c	0.8020	(0.1322)	0.8020
y_c	−0.2785	(0.4551)	0
σ_c	0.2538	(0.0372)	0.2538
c_{min}	0.0050	–	0.0050

Table A.1 Wilkie's ESG: fitted parameters. Left: 1951–2014. Right: 1991–2014.

A.2 Backtesting Parameters

	Estimates	(SE)	Simulation
μ_q	0.0396	(0.0153)	0.0396
a_q	0.8067	(0.0935)	0.8067
σ_q	0.0189	(0.0024)	0.0189
w_y	1.9414	(1.0897)	1.9414
μ_y	0.0412	(0.0066)	0.0412
a_y	0.8787	(0.0853)	0.8787
σ_y	0.1186	(0.0142)	0.1186
w_d	1	–	1
d_d	0.3800	–	0.3800
μ_d	0.0152	(0.0119)	0
y_d	0.2080	(0.2130)	0
b_d	−0.4954	(0.1967)	−0.4954
σ_d	0.1157	(0.0147)	0.1157
w_c	1	–	1
d_c	0.0580	–	0.0580
μ_c	0.0309	(0.0302)	0.0309
a_c	0.9633	(0.0414)	0.9633
y_c	0.0139	(0.2477)	0
σ_c	0.2307	(0.0279)	0.2307
c_{min}	0.0050	–	0.0050

	Estimates	(SE)	Simulation
μ_q	0.0034	(0.0003)	0.0034
κ_q	0.4451	(0.0404)	0.4451
σ_q	0.0031	(0.0000)	0.0031
κ_r	1.1643	(0.0570)	1.1643
σ_r	0.0035	(0.0000)	0.0035
μ_l	0.0019	(0.0003)	0.0019
κ_l	0.5619	(0.0436)	0.5619
σ_l	0.0030	(0.0000)	0.0030
μ_1	0.0079	(0.0025)	0.0079
σ_1	0.0339	(0.0027)	0.0339
μ_2	−0.0078	(0.0105)	−0.0078
σ_2	0.0541	(0.0076)	0.0541
p_{12}	0.0283	(0.0215)	0.0283
p_{21}	0.1027	(0.0852)	0.1027
ρ_{qr}	−0.8638	(0.0085)	−0.8638
ρ_{rl}	0.8284	(0.0188)	0.8284
ρ_{ql}	−0.9121	(0.0062)	-0.9121

Table A.2 Backtesting parameters (fitted to 1951–1984). Left: Wilkie's ESG. Right: Generic ESG (monthly).

A.3 Long Term Bonds Annual Reinvestment Calculations

Assuming a portfolio value of \$1 at time 0, below we set out how the portfolio value evolves over time, assuming an annual reinvestment of proceeds in long term bonds (n-year, $n >> 1$).

Let $A_c(t)$ denote the accumulation factor, or, the unit portfolio value at time t. $A_c(0) = 1$. Let BP_t denote the price of an n-year zero-coupon bond at time t with a face value of \$1, maturing in 10 years. Let BP_t^+ denote the re-sale price of the bond, purchased at time t, evaluated 1 year later.

$$BP_t = (1 + c(t))^{-n}, \quad BP_t^+ = (1 + c(t+1))^{-(n-1)}$$

Let N_t denote the number of bond contracts being held at time t, after reinvestment of proceeds. It is immediate that

$$N_t = \frac{A_c(t)}{BP_t}.$$

Consider the simple case when $t = 0$.

$$N_0 = \frac{1}{BP_0} = (1 + c(0))^n$$

At $t = 1$, the portfolio is worth $A_c(1) = \frac{BP_0^+}{BP_0} = \frac{(1+c(0))^n}{(1+c(1))^{n-1}}$. The n-year bond is currently worth $BP_1 = (1 + c(1))^{-n}$. Current holdings are converted to cash in the secondary market and reinvested in n-year bonds. The number of bonds being held at $t = 1$, after reinvestment is

$$N_1 = A(1)/BP_1 = (1 + c(0))^n(1 + c(1)).$$

At $t = 2$, the portfolio grows to $A_c(2) = N(1) \times BP_1^+ = (1+c(0))^n(1+c(1))(1+c(2))^{-(n-1)}$. After reinvestment, the number of bonds being held is:

$$N_2 = A_c(2)/BP_2 = (1 + c(0))^n(1 + c(1))(1 + c(2))$$

Thus, it can be iteratively shown that the following equations hold, $\forall t > 0$,

$$N_t = (1 + c(0))^n \prod_{i=1}^{t}(1 + c(i)), \quad A_c(t) = \frac{(1 + c(0))^n}{(1 + c(t))^n} \prod_{i=1}^{t}(1 + c(i))$$

Consequently, the effective annual return on reinvesting an n-year long term bond, $R_c(t)$, is,

$$R_c(t) = \frac{A_c(t)}{A_c(t-1)} - 1 = \frac{(1 + c(t-1))^n}{(1 + c(t))^{n-1}} - 1$$

Appendix B – Chapter 2

B.1 Elasticity of Intertemporal Substitution

At any time t, the EIS is given by

$$\text{EIS} = -\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log(\text{MRS}_{t,t+1})}$$

where $\text{MRS}_{t,t+1}$ is the marginal rate of substitution between C_t and C_{t+1}, alternatively referred to as the marginal rate of intertemporal substitution. In economics, the marginal rate of substitution refers to the rate at which an agent is ready to give up one good in exchange for another good while maintaining the same level of utility. Consequently, the marginal rate of (intertemporal) substitution is the rate at which an individual is willing to delay current consumption in exchange for future consumption, while keeping the same level of utility V_t. Hence, it is given by

$$\text{MRS}_{t,t+1} = \frac{\partial V_t/\partial C_{t+1}}{\partial V_t/\partial C_t} = \frac{A_2(C_t, \mu_t(V_{t+1}))A_1(C_{t+1}, \mu_{t+1}(V_{t+2}))}{A_1(C_t, \mu_t(V_{t+1}))}$$

where A_1 and A_2 denote the corresponding partial derivatives with respect to the first and second argument. From this point on we suppress the argument for μ_t. It can be readily obtained that

$$\begin{aligned}
\frac{\partial V_t}{\partial \mu_t} &= \frac{1}{1-\frac{1}{\alpha}} V_t^{\frac{1}{\alpha}} \beta(1-\frac{1}{\alpha})\mu_t^{-1/\alpha} = \beta\left(\frac{V_t}{\mu_t}\right)^{1/\alpha}, \\
\frac{\partial V_t}{\partial C_t} &= \frac{1}{1-\frac{1}{\alpha}} V_t^{\frac{1}{\alpha}} (1-\frac{1}{\alpha})C_t^{-1/\alpha} = \left(\frac{V_t}{C_t}\right)^{1/\alpha}.
\end{aligned}$$

Consequently, we have

$$\text{MRS}_{t,t+1} = \beta\left(\frac{C_t}{C_{t+1}}\right)^{1/\alpha}\left(\frac{V_{t+1}}{\mu_t}\right)^{1/\alpha},$$

and that the EIS is given by

$$\text{EIS} = -\frac{d\log\left(\frac{C_{t+1}}{C_t}\right)}{d\log(MRS_{t,t+1})} = -\frac{\frac{C_t}{C_{t+1}}d(\frac{C_{t+1}}{C_t})}{-\frac{1}{\alpha}\frac{C_t}{C_{t+1}}d\left(\frac{C_{t+1}}{C_t}\right)} = \alpha.$$

Appendix C – Chapter 3

C.1 The CBD Mortality Model

Under the CBD model described in Cairns et al. (2006), the logit of the conditional mortality rate $q_{x,t} = 1 - {}_1p_{x,t}$ is

$$\operatorname{logit} q_{x,t} = \log \frac{q_{x,t}}{1 - q_{x,t}} = A_{0,t} + A_{1,t}x,$$

and the two dimensional process $A_t = (A_{0,t}, A_{1,t})^T$ is given by

$$A_{t+1} = \tau + A_t + VZ_{t+1},$$

where $V^TV = \Sigma$ is the covariance matrix and Z_{t+1} is a standard normal random vector. Assuming a terminal age of 110, we use the following parameters obtained from Maurer et al. (2013):

$$A_0 = \begin{bmatrix} -10.1502416 \\ 0.0904819 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.0019766 & -0.0000291 \\ -0.0000291 & 0.0000006 \end{bmatrix}$$
$$\tau = \begin{bmatrix} -0.0337497 \\ 0.0003242 \end{bmatrix}, \quad V = \begin{bmatrix} 0.0444590 & -0.0006545 \\ 0 & 0.0004142 \end{bmatrix}$$

In Chapter 3, the best estimates of future mortality rates, based on information up to time u, are used for computing the annuity factors. Here, we clarify how the best estimates are computed. Consider the mortality rate for a life aged x at time t (i.e. $q_{x,t}$). The best estimate of this rate computed at time u, where $u \leq t$, is:

$$\begin{aligned} \hat{q}_{x,t}|\mathcal{F}(u) &= \frac{\exp\left\{\hat{A}_{0,t}|\mathcal{F}(u) + x\hat{A}_{1,t}|\mathcal{F}(u)\right\}}{1 + \exp\left\{\hat{A}_{0,t}|\mathcal{F}(u) + x\hat{A}_{1,t}|\mathcal{F}(u)\right\}} \\ &= \frac{\exp\{A_{0,u} + (t-u)\tau_{11} + x(A_{1,u} + (t-u)\tau_{21})\}}{1 + \exp\{A_{0,u} + (t-u)\tau_{11} + x(A_{1,u} + (t-u)\tau_{21})\}}. \end{aligned}$$

C.2 Adjustment Factors Calculation

Here, we demonstrate the calculation of the adjustment factors for an open-fund VPA where annuity factors are computed annually based on the current best estimate to reflect the most up-to-date mortality experience. Using notation consistent with Boyle et al. (2015), at any time $t-1$, prior to benefit payment, the total fund balance $F(t-1)$ is calculated by aggregating the expected present values of the benefits for all surviving members of the fund, namely

$$F(t-1) = \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)\ddot{a}_{x_k+t-1,t-1}.$$

Then, the balance after benefit payment, $F(t-1)^+$, is

$$\begin{aligned} F(t-1)^+ &= F(t-1) - \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) \\ &= \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)a_{x_k+t-1,t-1}. \end{aligned}$$

The fund balance before benefit payment at time t is therefore

$$\begin{aligned} F(t) &= F(t-1)^+(1+R_t^V) \\ &= (1+R_t) \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)a_{x_k+t-1,t-1}, \end{aligned}$$

where R_t^V is the one-year return on the VPA fund. Since $F(t)$ can be alternatively expressed as

$$\begin{aligned} F(t) &= \sum_{k \in \mathcal{A}_t} B_k(t)\ddot{a}_{x_k+t,t} \\ &= \sum_{k \in \mathcal{A}_t} B_k(t-1)(1+j_t)\ddot{a}_{x_k+t,t}. \end{aligned}$$

We have

$$\begin{aligned} \sum_{k \in \mathcal{A}_t} B_k(t-1)(1+j_t)\ddot{a}_{x_k+t,t} &= (1+R_t^V) \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)a_{x_k+t-1,t-1} \\ 1+j_t &= \frac{(1+R_t^V)\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)a_{x_k+t-1,t-1}}{\sum_{k \in \mathcal{A}_t} B_k(t-1)\ddot{a}_{x_k+t,t}}. \end{aligned}$$

Since,

$$a_{x_k+t-1,t-1} = p_{x_k+t-1,t-1}(1+\text{AIR})^{-1}\ddot{a}_{x_k+t,t-1}.$$

Then,

$$1+j_t = \left(\frac{1+R_t^V}{1+\text{AIR}}\right) \frac{\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)p_{x_k+t-1,t-1}\ddot{a}_{x_k+t,t-1}}{\sum_{k \in \mathcal{A}_t} B_k(t-1)\ddot{a}_{x_k+t,t}}$$

It is important to note that at time $t-1$, the randomness of j_t stems from the following:

i. the random investment return on the VPA fund R_t, and

ii. Z_t, which drives the random vector A_t, and hence the random variable $\ddot{a}_{x_k+t,t}$ since it is defined as the best estimate given the information up to time t.

Since no idiosyncratic risk is allowed, the number of surviving members to time t is known at $t-1$. As such, $p_{x_k+t-1,t-1}$ and $\ddot{a}_{x_k+t,t-1}$ are also known at $t-1$.

C.3 Backward Induction

The optimal decision at time T is known, since death is certain at $T+1$ and therefore it is optimal to consume all remaining wealth. The optimization procedure follows below,

i. Initialize four vectors storing the discretized state variables.

ii. Construct a five-dimensional array that stores the value function evaluated at discretized state variables and time.

iii. Initialize two five-dimensional arrays that store the corresponding optimal normalized consumption and investment decisions ω_t.

iv. Start with T. The value function and optimal consumption depend only on one state variable w_T, Compute these values based on w_T and fill the grid.

v. Moving back to $T-1$, first evaluate the conditional expectation. Note that randomness stems from the risky asset returns (i.i.d.) and the adjustment factor which depends only on the state variable $\boldsymbol{A}_{T-1}$. Simulate N scenarios conditional on each state of $A_{0,T-1}$ and $A_{1,T-1}$.

vi. Solve for the optimal controls for each combination of the state variables, record them and record the corresponding value functions.

vii. Repeat steps 6 and 7 for $T-2, ..., 0$. At time 0, obtain the optimal controls $\boldsymbol{c}$, $\boldsymbol{\omega}$ for a range of initial states (w_0, b_0).

viii. Solve the final optimization problem with control variables ω_V and ω_I.

Note that the adjustment factor is evaluated differently depending on if systematic mortality development is considered. If so, then,

$$1 + j_t = \left(\frac{\sum_{k \in S_{t-1}^+} {}^k B_{t-1}^V p_{x_k+t-1,t-1} \, \ddot{a}_{x_k+t,t-1}}{\sum_{k \in S_t} {}^k B_{t-1}^V \ddot{a}_{x_k+t,t}} \right) \left(\frac{1 + R_t^V}{1 + \text{AIR}} \right).$$

It not, the adjustment factor depends only on the VPA fund returns. The optimization problem can be solved using a two-dimensional grid as $\boldsymbol{A}_t$ is no longer relevant.

C.4 Multilinear Interpolation and Extrapolation

Here, we describe the multilinear interpolation and extrapolation method used to approximate the value function. The method searches for an estimate of a function $f(x_1, x_2, x_3, ..., x_z)$ from a z-dimensional grid of tabulated values f and z one-dimensional vectors giving the computed values of each of the independent variables $x_1, ..., x_z$.

Consider when $n = 4$. Suppose the value of f is known on $(x_{1,1}, ..., x_{1,m})^T$, $(x_{2,1}, ..., x_{2,n})^T$, $(x_{3,1}, ..., x_{3,p})^T$, and $(x_{4,1}, ..., x_{4,q})^T$. We seek to approximate $f(x_1^*, x_2^*, x_3^*, x_4^*)$. First, we locate the 4-dimensional grid space in which the point $(x_1^*, x_2^*, x_3^*, x_4^*)$ falls, that is, to find the eight tabulated points that surround the desired interior point. Suppose,

$$x_{1,al} \leq x_1^* \leq x_{1,au}, \quad x_{2,bl} \leq x_2^* \leq x_{2,bu}, \quad x_{3,cl} \leq x_3^* \leq x_{3,cu}, \quad x_{4,dl} \leq x_4^* \leq x_{4,du}.$$

Then, let

$$u = \frac{x_1^* - x_{1,al}}{x_{1,au} - x_{1,al}}, \quad v = \frac{x_2^* - x_{2,bl}}{x_{2,bu} - x_{2,bl}}, \quad k = \frac{x_3^* - x_{3,cl}}{x_{3,cu} - x_{3,cl}}, \quad l = \frac{x_4^* - x_{4,dl}}{x_{4,du} - x_{4,dl}}.$$

We obtain that

$$\begin{aligned} f(x_1^*, x_2^*, x_3^*, x_4^*) \approx & (1-u)(1-v)(1-k)(1-l)f(x_{1,al}, x_{2,bl}, x_{3,cl}, x_{4,dl}) \\ & + u(1-v)(1-k)(1-l)f(x_{1,au}, x_{2,bl}, x_{3,cl}, x_{4,dl}) \\ & + uv(1-k)(1-l)f(x_{1,au}, x_{2,bu}, x_{3,cl}, x_{4,dl}) \\ & + (1-u)v(1-k)(1-l)f(x_{1,al}, x_{2,bu}, x_{3,cl}, x_{4,dl}) \\ & + \cdots \\ & + uvklf(x_{1,au}, x_{2,bu}, x_{3,cu}, x_{4,du}). \end{aligned}$$

Note that it is also possible for the point to lie outside of the grid range. In such cases extrapolation is used. Extrapolation follows the same concept by creating a tangent line outside the range, except that the point of interest does not fall in any grid space. More precisely, for each dimension, we need to locate the two closest points such that,

$$\begin{aligned} x_{1,al} \leq x_{1,au} \leq x_1^*, \quad x_{2,bl} \leq x_{2,bu} \leq x_2^*, \\ x_{3,cl} \leq x_{3,cu} \leq x_3^*, \quad x_{4,dl} \leq x_{4,du} \leq x_4^*. \end{aligned}$$

The function approximation then follows similarly as above.

Appendix D – Chapter 4

D.1 Survey

This section presents the survey used to conduct the Ontario Retirement Survey study referenced in Chapter 4. This is a web-based survey, where questions are dynamic, such that certain text, numbers and graphics are presented based on the previous responses provided by the participant. Details are given in the Section "Notes", and are included as [blue text] in the survey.

Introduction

You are invited to participate in a research study conducted by Mary Hardy, David Saunders and Saisai Zhang at the Department of Statistics and Actuarial Science, University of Waterloo.

The objective of this study is to develop a better understanding of the concerns and risk preferences of individuals who are either close to retirement, or who are already retired. This will allow us to assess whether the retirement income plans offered by insurance companies, or provided through occupational pension plans, are adequately meeting the needs of retirees.

The on-line survey should take about 20 to 30 minutes to complete. We are not asking for money or selling anything. The survey includes questions about your assets, income, and some demographic information, as these are important considerations in assessing appropriate retirement income plans.

The on-line survey data is being collected by the Survey Research Centre at the University of Waterloo, on behalf of the Investigators. All survey data will be anonymous to the Investigators as they are not provided with any names or identifying information of respondents. No individual information, including participation status, will be shared with the researchers or with any other institution. When analyzed, all of the data will be summarized and no individual could be identified from these summarized results. Furthermore, the web site is programmed to collect responses alone and will not collect any information that could potentially identify you (such as machine identifiers). An analysis of the results will be published in Saisai Zhang's PhD thesis; the Investigators also intend to summarize the results in a report that will be made available to participants on request, and may include some analysis in scientific journal articles.

Participation in this study is voluntary. You can decline to respond to any question by leaving it blank, with the exception of the first two questions that determine whether you qualify to complete the survey. The first two questions are about your age and retirement status, as the Investigators are interested in the changing attitude to financial security as people transition from work to retirement. You may withdraw from participating in the survey by simply closing the browser without submitting your responses. However, once you have submitted your responses it is not possible to withdraw your consent to participate, as we have no way of knowing which responses are yours. By indicating your consent you are not waiving your legal rights or releasing the investigators or involved institutions from their legal and professional responsibilities.

The Survey Research Centre has complete control over data security and participant anonymity. The data collected from this study, with no personal identifiers, will be maintained on a password-protected computer database in a restricted access area of the university. The data will be electronically archived after completion of the study and retained for a minimum of 7 years.

There are no known or anticipated risks from participating in this study.

The survey has been partially funded by a research grant from the Canadian Institute of Actuaries. The Canadian Institute of Actuaries is the national organization of the actuarial profession, with a mission to serve the public through actuarial education and research, and is independent of the commercial interest of any financial institutions. The funders of this study have no influence over the analysis of the survey responses or the dissemination of the survey results.

We would like to assure you that this study has been reviewed and received ethics clearance through a University of Waterloo Research Ethics Committee. However, the final decision about participation is yours. If you have any comments or concerns resulting from your participation in this study, please contact the Chief Ethics Officer, Office of Research Ethics, at 519-888-4567, Ext. 36005 or ore-ceo@uwaterloo.ca.

Should you have any questions about the study, please contact Mary Hardy at mrhardy@uwaterloo.ca. If you would like to receive a copy of the results of this study, please contact Saisai Zhang at s288zhan@uwaterloo.ca. If you are experiencing technical issues when completing the survey, please contact srccinb@uwaterloo.ca.

Thank you for considering participating in this study.

[Button] I CONSENT TO PARTICIPATE

Notes

- Texts in square brackets [marked in blue] are notes for the programmer and are not shown to the respondents. Further, section/subsection titles are not shown to the respondents.
- Respondents are gradually divided into 24 categories based on,

  i. Gender (Female, Male), determined in Q3.

  ii. Marital status (Married, Single), determined in Q4.

  iii. Retirement status (Pre-Retiree, Retiree), determined in Q2.

  iv. Wealth status (Low, Medium, High), determined in Q11.

- Questions are presented to respondents based on their specified category. For example, a question labeled [Retiree, Single] is shown to a respondent identified as single and retired, regardless of gender and wealth status.
- In particular in Section D.1.4, respondents will see charts and monetary values based on their Gender and Wealth category (e.g. Female and High Wealth Status). These adaptive figures are highlighted in magenta.
- Adaptive income figures for the last section are included in Table D.3 and Table D.4.

D.1.1 Preliminary

Q1 In what year were you born?

____

[Determine Age = 2016−Entry]

[IF Age < 50 OR IF Age > 80, THEN SHOW "Thank you but to qualify to complete this survey, we require a different age group. Thank you for your time."]

[IF SKIPPED, THEN SHOW "In order to present you with the remaining questions, please provide an answer to this question. Please click the PREVIOUS button to go back to provide an answer to continue, or the NEXT button to exit the survey."]

Q2 Which one of the statements best describes your current work/retirement status?

- ☐ I work full-time and I do NOT consider myself retired [→ Pre-retiree]
- ☐ I work full-time and I consider myself retired [→ Retiree]
- ☐ I work part-time and I do NOT consider myself retired [→ Pre-retiree]
- ☐ I work part-time and I consider myself retired [→ Retiree]
- ☐ I do NOT work (outside the home) and I do NOT yet consider myself retired [→ Pre-retiree]
- ☐ I do NOT work (outside the home) and I consider myself retired [→ Retiree]

[Determine Retirement Status]

[IF SKIPPED, THEN SHOW "In order to present you with the remaining questions, please provide an answer to this question. Please click the PREVIOUS button to go back to provide an answer to continue, or the NEXT button to exit the survey."]

Q3 How do you identify your gender?

- ☐ Female [→ Female]
- ☐ Male [→ Male]
- ☐ Other [→ Male]
- ☐ Prefer not to say [→ Male]

[Determine Gender]

[IF SKIPPED, THEN → Male]

Q4 Do you currently have a spouse or common-law partner?

- ☐ Yes [→ Married, SHOW Q4a]
- ☐ No [→ Single, SKIP Q4a]

[Determine Marital Status]
[IF SKIPPED, THEN → Single]

Q4a Is your spouse/partner...

- ☐ Working full-time
- ☐ Working part-time
- ☐ Not Working (outside the home)

D.1.2 Expectations and Experience

Retirement Age

Q5 [Pre-Retirees] At what age do you expect to retire?

________________________ [RESTRICTION: ENTRY $\geq$ Age – 1]

[Retirees] At what age did you retire?

________________________ [RESTRICTION: ENTRY $\leq$ Age]

Life Expectancy

Q6 Approximately what age do you expect to live to?

________________________ [RESTRICTION: ENTRY $\geq$ Age]

Q7 On a scale of 0% to 100%, where 0% means absolutely no chance and 100% means absolutely certain, **what do you think is the chance that you will live to be** [SHOW:Age + 10] **or more**?

________________________% [RESTRICTION: 0 $\leq$ ENTRY $\leq$ 100]

Q8 On a scale of 0% to 100%, where 0% means absolutely no chance and 100% means absolutely certain, **what do you think is the chance that you will live to be** 95 **or more**?

________________________% [RESTRICTION: 0 $\leq$ ENTRY $\leq$ 100]

Wealth

Q9 [Pre-Retirees, Married] What was your and your spouse/partner's total income in 2015?
(Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Pre-Retirees, Single] What was your income in 2015? (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Retirees, Married] What was your and your spouse/partner's total income, **in the year prior to your retirement**? (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Retirees, Single] What was your income, **in the year prior to your retirement**? (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

☐ Below $25,000

☐ $25,000 to $49,999

☐ $50,000 to $74,999

☐ $75,000 to $99,999

☐ $100,000 to $124,999

☐ $125,000 to $149,999

☐ $150,000 to $174,999

☐ $175,000 to $199,999

☐ $200,000 to $224,999

☐ $225,000 to $250,000

☐ Above $250,000

Q10 [Pre-Retirees] How many people did this income support?

[Retirees] How many people was this income supporting at the time?

☐ 1

☐ 2

☐ 3

☐ 4

☐ 5 or more

Q11 [Pre-Retirees, Married] **When you retire**, approximately how much money do you and your spouse/partner ***expect*** to have in the following savings and investment categories: (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Pre-Retirees, Single] **When you retire**, approximately how much money do you

expect to have in the following savings and investment categories: (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Retirees, Married] **Part a**: **When you first retired**, approximately how much money did you and your spouse/partner have in the following savings and investment categories: (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Retirees, Single] **Part a**: **When you first retired**, approximately how much money did you have in the following savings and investment categories:(Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

a. **Liquid assets**, including Registered Retirement Savings Plan (RRSP), Registered Retirement Income Fund (RRIF), Defined Contribution Pension Fund, Tax-Free Savings Account (TFSA), Guaranteed Income Certificates (GICs), chequing and savings accounts, and any other savings in which you can decide how the money is invested (*please do not include the value of your home or other investment properties*).

☐ Below $25,000 [IF Married, Wealth Status→ Low; IF Single, Wealth Status→ Low]

☐ $25,000 to $99,999 [IF Married, Wealth Status→ Low; IF Single, Wealth Status→ Low]

☐ $100,000 to $199,999 [IF Married, Wealth Status→ Low; IF Single, Wealth Status→ Low]

☐ $200,000 to $299,999 [IF Married, Wealth Status→ Low; IF Single, Wealth Status→ Medium]

☐ $300,000 to $399,999 [IF Married, Wealth Status→ Medium; IF Single, Wealth Status→ Medium]

☐ $400,000 to $499,999 [IF Married, Wealth Status→ Medium; IF Single, Wealth Status→ High]

☐ $500,000 to $599,999 [IF Married, Wealth Status→ Medium; IF Single, Wealth Status→ High]

☐ $600,000 to $699,999 [IF Married, Wealth Status→ Medium; IF Single, Wealth Status→ High]

☐ $700,000 to $799,999 [IF Married, Wealth Status→ High; IF Single, Wealth Status→ High]

☐ 800,000 to 899,000 [IF Married, Wealth Status→ High; IF Single, Wealth Status→ High]

☐ 900,000 to 1,000,000 [IF Married, Wealth Status→ High; IF Single, Wealth Status→ High]

☐ Above 1,000,000 [IF Married, Wealth Status→ High; IF Single, Wealth Status→ High]

[Determine Wealth Status]

[IF SKIPPED, THEN → Medium]

b. **Real Estate**, including the value of your primary home and other investment properties, minus any outstanding mortgage amounts.

☐ Do not [IF Retiree, Did not] own any property

☐ Below $25,000

☐ $25,000 to $99,999

☐ $100,000 to $199,999

☐ $200,000 to $399,999

☐ $400,000 to $599,999

☐ $600,000 to $799,999

☐ $800,000 to $999,999

☐ $1,000,000 to $1,299,999

☐ $1,300,000 to $1,599,999

☐ $1,600,000 to $1,999,999

☐ $2,000,000 to $3,000,000

☐ Above $3,000,000

[IF GO BACK, THEN SHOW "Sorry, your answers up to this point have been saved. They are being used to determine the remaining survey questions and therefore, cannot be changed. Please click"Next" to continue the survey."]

[Retirees, Married] **Part b**: **CURRENTLY**,approximately how much money do you and your spouse/partner have in the following savings and investment categories: (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

[Retirees, Single] **Part b**: **CURRENTLY**, approximately how much money do you have in the following savings and investment categories: (Please note that in this survey, all monetary values are expressed in Canadian Dollars $)

a. **Liquid Assets**, including Registered Retirement Savings Plan (RRSP), Registered Retirement Income Fund (RRIF), Defined Contribution Pension Fund, Tax-Free Savings Account (TFSA), Guaranteed Income Certificates (GICs), chequing and savings accounts, and any other savings in which you can decide how the money is invested (*please do not include the value of your home or other investment properties*).

☐ Below $25,000

☐ $25,000 to $99,999

☐ $100,000 to $199,999

☐ $200,000 to $299,999

☐ $300,000 to $399,999

☐ $400,000 to $499,999

☐ $500,000 to $599,999

☐ $600,000 to $699,999

☐ $700,000 to $799,999

☐ $800,000 to $899,000

☐ $900,000 to $1,000,000

☐ Above $1,000,000

b. **Real Estate**, including the value of your primary home and other investment properties, minus any outstanding mortgage amounts.

☐ Do not own any property

☐ Below $25,000

☐ $25,000 to $99,999

☐ $100,000 to $199,999

☐ $200,000 to $399,999

☐ $400,000 to $599,999

☐ $600,000 to $799,999

☐ $800,000 to $999,999

☐ $1,000,000 to $1,299,999

☐ $1,300,000 to $1,599,999

☐ $1,600,000 to $1,999,999

☐ $2,000,000 to $3,000,000

☐ Above $3,000,000

Income and Consumption

Q12 [Pre-Retirees, Married] Please indicate the approximate amount you and your spouse/-partner ***expect*** to receive, **in the first year of your retirement**, from the following sources:

[Pre-Retirees, Single] Please indicate the approximate amount you ***expect*** to receive, **in the first year of your retirement**, from the following sources:

[Retirees, Married] **Part a**: Please indicate the approximate amount you and your spouse/partner received from the following sources **in the first year of your retirement**:

[Retirees, Single] **Part a**: Please indicate the approximate amount you received from the following sources **in the first year of your retirement**:

a. Defined Benefit Pension Plan
(A defined benefit pension plan provides members with a defined pension income when they retire. The formula used to determine a member's benefit usually involves factors such as years of membership in the pension plan and the member's salary, and is not dependent on the investment returns of the plan fund.)

☐ Do not have Defined Benefit Pension Plans

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

b. Government Pension Provisions, including Old Age Security (OAS) Pension, Guaranteed Income Supplement (GIS), and Canada Pension Plan (CPP)

☐ Below $3,000 [IF Married, Below $6,000]

☐ $3,000 to $5,999 [IF Married, $6,000 to $11,999]

☐ $6,000 to $8,999 [IF Married, $12,000 to $17,999]

☐ $9,000 to $11,999 [IF Married, $18,000 to $23,999]

☐ $12,000 to $14,999 [IF Married, $24,000 to $29,999]

☐ $15,000 to $17,999 [IF Married, $30,000 to $35,999]

☐ $18,000 to $20,999 [IF Married, $36,000 to $41,999]

☐ $21,000 to $23,999 [IF Married, $42,000 to $47,999]

☐ $24,000 to $26,999 [IF Married, $48,000 to $53,999]

☐ $27,000 to $30,000 [IF Married, $54,000 to $60,000]

☐ Above $30,000 [IF Married, Above $60,000]

c. Life Annuities, privately purchased from an annuity provider (e.g. a life insurance company)
(Life annuities guarantee you a predetermined income for as long as you live, in exchange for a lump-sum payment upfront.)

☐ [Pre-Retirees: Do not plan to not purchase life annuities when I retire. Retirees: Do not have life annuities.]

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $80,000 [IF Married, $140,000 to $160,000]

☐ Above $80,000 [IF Married, Above $160,000]

d. Withdrawals from Liquid Assets, including for example, Registered Retirement Savings Plan (RRSP), Registered Retirement Income Fund (RRIF), Defined Contribution Pension Plan, Tax-Free Savings Account (TFSA), or other savings and chequing accounts.

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

e. Other Income Sources, including for example, full-time or part-time employment [IF Married, of you and your spouse/partner], property investment, etc.

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

Please specify these income sources ______________________________

[Retirees, Married] **Part b**: Please indicate the approximate amount you and your spouse/partner received from the following sources **in the year 2015**:

[Retirees, Single] **Part b**: Please indicate the approximate amount you received from the following sources **in the year 2015**:

a. Defined Benefit Pension Plan

(A defined benefit pension plan provides members with a defined pension income when they retire. The formula used to determine a member's benefit usually involves factors such as years of membership in the pension plan and the member's salary, and is not dependent on the investment returns of the plan fund.)

☐ Do not have Defined Benefit Pension Plans

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

b. Government Pension Provisions, including Old Age Security (OAS) Pension, Guaranteed Income Supplement (GIS), and Canada Pension Plan (CPP)

☐ Below $3,000 [IF Married, Below $6,000]

☐ $3,000 to $5,999 [IF Married, $6,000 to $11,999]

☐ $6,000 to $8,999 [IF Married, $12,000 to $17,999]

☐ $9,000 to $11,999 [IF Married, $18,000 to $23,999]

☐ $12,000 to $14,999 [IF Married, $24,000 to $29,999]

☐ $15,000 to $17,999 [IF Married, $30,000 to $35,999]

☐ $18,000 to $20,999 [IF Married, $36,000 to $41,999]

☐ $21,000 to $23,999 [IF Married, $42,000 to $47,999]

☐ $24,000 to $26,999 [IF Married, $48,000 to $53,999]

☐ $27,000 to $30,000 [IF Married, $54,000 to $60,000]

☐ Above $30,000 [IF Married, Above $60,000]

c. Life Annuities, privately purchased from an annuity provider (e.g. a life insurance company)

(Life annuities guarantees you a predetermined income for as long as you live, in exchange for a lump-sum payment upfront.)

☐ Do not have life annuities.

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $80,000 [IF Married, $140,000 to $160,000]

☐ Above $80,000 [IF Married, Above $160,000]

d. Withdrawals from Liquid Assets, including for example, Registered Retirement Savings Plan (RRSP), Registered Retirement Income Fund (RRIF), Defined Contribution Pension Plan, Tax-Free Savings Account (TFSA), or other savings and chequing accounts.

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

e. Other Income Sources, including for example, full-time or part-time employment [IF Married, of you and your spouse/partner], property investment, etc.

☐ Below $10,000 [IF Married, Below $20,000]

☐ $10,000 to $19,999 [IF Married, $20,000 to $39,999]

☐ $20,000 to $29,999 [IF Married, $40,000 to $59,999]

☐ $30,000 to $39,999 [IF Married, $60,000 to $79,999]

☐ $40,000 to $49,999 [IF Married, $80,000 to $99,999]

☐ $50,000 to $59,999 [IF Married, $100,000 to $119,999]

☐ $60,000 to $69,999 [IF Married, $120,000 to $139,999]

☐ $70,000 to $79,999 [IF Married, $140,000 to $159,999]

☐ $80,000 to $89,999 [IF Married, $160,000 to $179,999]

☐ $90,000 to $100,000 [IF Married, $180,000 to $200,000]

☐ Above $100,000 [IF Married, Above $200,000]

Please specify these income sources ____

Q13 Please indicate how important the following considerations are in making retirement savings and spending decisions. Please select one importance rating that best represents your opinion.

	Very Unim-portant	**Fairly Unim-portant**	**Neu-tral**	**Fairly Impor-tant**	**Very Impor-tant**
Setting money aside for nursing home or home care expenses	☐	☐	☐	☐	☐
Leaving a bequest	☐	☐	☐	☐	☐
Setting money aside to access quickly when unforeseen expenses arise	☐	☐	☐	☐	☐
The impact of inflation on my standard of living	☐	☐	☐	☐	☐
The possibility of living longer than expected	☐	☐	☐	☐	☐
The possibility of dying early	☐	☐	☐	☐	☐
Taking some investment risk with my savings during retirement	☐	☐	☐	☐	☐
Avoiding ups and downs in my income; having a smooth income stream during retirement	☐	☐	☐	☐	☐
Expecting my spending needs in the future to be less than at retirement	☐	☐	☐	☐	☐
Expecting my spending needs in the future to be more than at retirement	☐	☐	☐	☐	☐
Please specify any other important considerations in your retirement savings and spending decisions					

Q14 [Pre-Retirees] Consider the following hypothetical situation. **At the time of your retirement**, you are given an option by your financial service provider (e.g. bank, financial advisor, or insurance company) to purchase a product that guarantees you a **monthly** income of $100 for the rest of your life. This income will stop when you die.

Please indicate the ***maximum*** amount of money you are willing to pay for this product (Note that this is a one-time purchase).

- ☐ Below $3,000 [$P \to \$1,500$]
- ☐ $3,000 to $5,999 [$P \to \$4,500$]
- ☐ $6,000 to $8,999 [$P \to \$7,500$]
- ☐ $9,000 to $11,999 [$P \to \$10,500$]
- ☐ $12,000 to $14,999 [$P \to \$13,500$]
- ☐ $15,000 to $17,999 [$P \to \$16,500$]
- ☐ $18,000 to $20,999 [$P \to \$19,500$]
- ☐ $21,000 to $23,999 [$P \to \$22,500$]

☐ $24,000 to $26,999 [$P \rightarrow \$25,500$]

☐ $27,000 to $30,000 [$P \rightarrow \$28,500$]

☐ Above $30,000 [$P \rightarrow \$31,500$]

[Determine: P = Midpoint value of selection]

[IF SKIPPED AND MALE, THEN $P = \$19,776$. IF SKIPPED AND FEMALE, THEN $P = \$21,202$]

[Pre-Retirees] Now assume that **at the time of your retirement**, you can purchase multiple units of the product.

For example, if you purchase 2 units, you will receive a guaranteed income of $200 per month for the rest of your life, and pay twice the price.

If the price for each $100 per month is [SHOW P], how many units of the product would you purchase?

I will purchase ______________________ unit(s). (Indicate 0 if you are unwilling to purchase the product.)

[Retirees] **Part a**: Consider the following hypothetical situation. **When you retired**, you were given an option by your financial service provider (e.g. bank, financial advisor, or insurance company) to purchase a product that guaranteed you a **monthly** income of $100 for the rest of your life. This income will stop when you die.

Please indicate the ***maximum*** amount of money you would have been willing to pay for this product. (Note that this is a one-time purchase)

☐ Below $3,000 [$P_a \rightarrow \$1,500$]

☐ $3,000 to $5,999 [$P_a \rightarrow \$4,500$]

☐ $6,000 to $8,999 [$P_a \rightarrow \$7,500$]

☐ $9,000 to $11,999 [$P_a \rightarrow \$10,500$]

☐ $12,000 to $14,999 [$P_a \rightarrow \$13,500$]

☐ $15,000 to $17,999 [$P_a \rightarrow \$16,500$]

☐ $18,000 to $20,999 [$P_a \rightarrow \$19,500$]

☐ $21,000 to $23,999 [$P_a \rightarrow \$22,500$]

☐ $24,000 to $26,999 [$P_a \rightarrow \$25,500$]

☐ \$27,000 to \$30,000 [$P_a \rightarrow \$28,500$]

☐ Above \$30,000 [$P_a \rightarrow \$31,500$]

[Determine: P_a = Midpoint value of selection]

[IF SKIPPED AND MALE, THEN $P_a = \$19,776$. IF SKIPPED AND FEMALE, THEN $P_a = \$21,202$]

Now assume that **at the time of your retirement**, you could purchase multiple units of the product.

For example, if you purchase 2 units, you will receive a guaranteed income of \$200 per month for the rest of your life, and pay twice the price.

If the price for each \$100 per month is [SHOW P_a], how many units of the product would you purchase?

I would purchase ________________________ unit(s). (Indicate 0 if you would be unwilling to purchase the product.)

[Retirees] **Part b**: Now, imagine that you are **CURRENTLY** given an option by your financial service provider (e.g. bank, financial advisor, or insurance company) to purchase a product that guarantees you a **monthly** income of \$100 for the rest of your life. This income will stop when you die.

Please indicate the ***maximum*** amount of money you are willing to pay for this product. (Note that this is a one-time purchase)

☐ Below \$3,000 [$P_b \rightarrow \$1,500$]

☐ \$3,000 to \$5,999 [$P_b \rightarrow \$4,500$]

☐ \$6,000 to \$8,999 [$P_b \rightarrow \$7,500$]

☐ \$9,000 to \$11,999 [$P_b \rightarrow \$10,500$]

☐ \$12,000 to \$14,999 [$P_b \rightarrow \$13,500$]

☐ \$15,000 to \$17,999 [$P_b \rightarrow \$16,500$]

☐ \$18,000 to \$20,999 [$P_b \rightarrow \$19,500$]

☐ \$21,000 to \$23,999 [$P_b \rightarrow \$22,500$]

☐ \$24,000 to \$26,999 [$P_b \rightarrow \$25,500$]

☐ \$27,000 to \$30,000 [$P_b \rightarrow \$28,500$]

☐ Above $30,000 [$P_b \rightarrow \$31,500$]

[Determine: P_b = Midpoint value of selection]

[IF SKIPPED AND MALE, THEN $P_b = \$19,776$. IF SKIPPED AND FEMALE, THEN $P_b = \$21,202$]

Now assume that **CURRENTLY**, you can purchase multiple units of the product.

For example, if you purchase 2 units, you will receive a guaranteed income of $200 per month for the rest of your life, and pay twice the price.

If the price for each $100 per month is [SHOW P_b], how many units of the product would you purchase?

I will purchase ____________________________ unit(s). (Indicate 0 if you are unwilling to purchase the product.)

Q14a Please indicate how important the following considerations were in answering the previous question.

	Very Unimportant	**Unimportant**	**Neutral**	**Important**	**Very Important**
I am concerned that I might outlive my assets	☐	☐	☐	☐	☐
I am concerned that I would lose flexibility and control in some of my own finances	☐	☐	☐	☐	☐
I am concerned that the financial service provider might not honour this guarantee	☐	☐	☐	☐	☐
I am concerned that spending on this type of product would lead to a loss of financial security	☐	☐	☐	☐	☐
I am concerned that my family would face hardship if I were to die early	☐	☐	☐	☐	☐
Please specify any other important considerations					

Financial Management and Advice

Q15 [Pre-Retirees] As you age, there might come a time when it becomes difficult for you and/or your spouse to manage your own finances. Please indicate how concerned you are about this.

- ☐ Very Unconcerned
- ☐ Fairly Unconcerned
- ☐ Fairly Concerned
- ☐ Very Concerned

[Retirees] **Part a**: As you age, there might come a time when it becomes difficult for you and/or your spouse to manage your own finances. Please indicate how concerned you were about this when you **first retired**.

- ☐ Very Unconcerned
- ☐ Fairly Unconcerned
- ☐ Fairly Concerned
- ☐ Very Concerned

[Retirees] **Part b**: How concerned are you about this, **CURRENTLY**?

- ☐ Very Unconcerned
- ☐ Fairly Unconcerned
- ☐ Fairly Concerned
- ☐ Very Concerned

Q16 Some people seek professional financial advice to assist in their retirement planning.Which of the following best represents your situation?

- ☐ I have consulted with a financial advisor in the past and I will continue to do so in the future.
- ☐ I have consulted with a financial advisor in the past but I do NOT plan to do so in the future.
- ☐ I have NOT consulted with a financial advisor in the past but I intend to in the future.
- ☐ I have NOT consulted with a financial advisor in the past and I do NOT intend to in the future.

Q16a Please indicate how concerned you are about each of the following regarding professional financial advisors. Please select the one rating that best represents your opinion.

	Very Unconcerned	Fairly Unconcerned	Neutral	Fairly Concerned	Very Concerned
Accessing high quality services	☐	☐	☐	☐	☐
Being a victim of a fraud or scam	☐	☐	☐	☐	☐
Not being able to assess the quality of the service	☐	☐	☐	☐	☐
The capability of financial advisors to address my concerns or improve my financial welfare	☐	☐	☐	☐	☐
Professional financial advisors acting in their own best interest rather than mine	☐	☐	☐	☐	☐
Please specify any other important considerations					

Q17 Please indicate how important it is to leave money for your heirs (select one importance rating for each option)

If you were to die...	**Very Unimportant**	**Fairly Unimportant**	**Neutral**	**Fairly Important**	**Very Important**
Within the next 10 years from now	☐	☐	☐	☐	☐
10 to 20 years from now	☐	☐	☐	☐	☐
Over 20 years from now	☐	☐	☐	☐	☐

D.1.3 Preferences

General

The following questions are related to your **risk attitude**. Your risk attitude describes your willingness to accept risk of gaining or losing money when managing your finances and investments.

Q18 [Pre-Retirees] **CURRENTLY**, how would you describe your **risk attitude** with respect to managing your investable assets?

☐ I avoid risk at all cost

☐ I am only willing to accept low risk

☐ I am willing to accept moderate risk

- ☐ I can live with moderate to high risk
- ☐ I feel comfortable with high risk
- ☐ I do not have any investments

[Retirees] **When you first retired**, how would you describe your **risk attitude** with respect to managing your investable assets?

- ☐ I avoided risk at all cost
- ☐ I was only willing to accept low risk
- ☐ I was willing to accept moderate risk
- ☐ I could live with moderate to high risk
- ☐ I felt comfortable with high risk
- ☐ I did not have any investments

Q19 [Pre-Retirees] **When you retire**, how do you expect your **risk attitude** to change?

- ☐ I expect to be more willing to accept risk when I retire.
- ☐ I expect my risk attitude to stay the same when I retire.
- ☐ I expect to be less willing to accept risk when I retire.

[Retirees] Which one of the following statements best describes your **CURRENT risk attitude** compared to when you first retired?

- ☐ I am more willing to accept risk now than when I first retired.
- ☐ My risk attitude is the same compared to when I first retired.
- ☐ I am less willing to accept risk now than when I first retired.

Elicitation

Q20 In the following task, you will be presented with a list of hypothetical options. Each option has a **50/50 chance** of giving either a high payoff or a low payoff. We are interested in which option you prefer the most.

First, take a look at the following example. You do not need to make a choice in this example.

Option	Low Payoff	High Payoff	[Expected Return]	[Standard Deviation]	[Implied CRRA range]
1	\$10.00	\$10.00	\$10.00	\$0.00	$9.72 < \rho < \infty$
2	\$9.50	\$11.00	\$10.25	\$0.75	$-\infty < \rho < 9.72$

[NOTE TO PROGRAMMER: The last three columns are not shown to the respondents.]

- If you choose Option 1, there is a **50% chance** that you will receive $10.00, and a **50% chance** that you will receive $10.00 (in this special case, you will receive $10.00 for certain).
- If you choose Option 2, there is a **50% chance** that you will receive $9.50, and a **50% chance** that you will receive $11.00.
- Choosing any one of the options will not cost you any money.
- Your task is to choose the ONE option you prefer the most.

Now it is time to choose. Consider the 10 options listed below. Remember that the chances are 50/50.

Option	Low Payoff	High Payoff	[Expected Return]	[Standard Deviation]	[Implied CRRA range]
1	$10.00	$10.00	$10.00	$0.00	$9.73 < \rho < \infty$
2	$9.50	$11.00	$10.25	$0.75	$5.87 < \rho < 9.73$
3	$9.20	$12.00	$10.60	$1.40	$3.74 < \rho < 5.87$
4	$8.90	$13.00	$10.95	$2.05	$2.48 < \rho < 3.74$
5	$8.30	$15.00	$11.65	$3.35	$1.74 < \rho < 2.48$
6	$7.70	$17.00	$12.35	$4.65	$1.36 < \rho < 1.74$
7	$7.10	$19.00	$13.05	$5.95	$0.88 < \rho < 1.36$
8	$6.00	$22.00	$14.00	$8.00	$0.26 < \rho < 0.88$
9	$4.00	$25.00	$14.50	$10.50	$0 < \rho < 0.26$
10	$2.00	$27.00	$14.50	$12.50	$-\infty < \rho < 0$

[NOTE TO PROGRAMMER: The last three columns are not shown to the respondents.]

Among options 1 to 10, which do you prefer the most? Please indicate ONE option using the number 1 to 10.

_________________________[RESTRICTION: Number 1 to 10]

Now, consider an alternative set of 10 options listed below. Remember that the chances are 50/50.

Option	Low Payoff	High Payoff	[Expected Return]	[Standard Deviation]	[Implied CRRA range]
11	$5,000	$5,000	$5,000	$0	$9.73 < \rho < \infty$
12	$4,750	$5,500	$5,125	$375	$5.87 < \rho < 9.73$
13	$4,600	$6,000	$5,300	$700	$3.74 < \rho < 5.87$
14	$4,450	$6,500	$5,475	$1,025	$2.48 < \rho < 3.74$
15	$4,150	$7,500	$5,825	$1,675	$1.74 < \rho < 2.48$
16	$3,850	$8,500	$6,175	$2,325	$1.36 < \rho < 1.74$
17	$3,550	$9,500	$6,525	$2,975	$0.88 < \rho < 1.36$
18	$3,000	$11,000	$7,000	$4,000	$0.26 < \rho < 0.88$
19	$2,000	$12,500	$7,250	$5,250	$0 < \rho < 0.26$
20	$1,000	$13,500	$7,250	$6,250	$-\infty < \rho < 0$

[NOTE TO PROGRAMMER: The last three columns are not shown to the respondents.]

Among options 11 to 20, which do you prefer the most? Please indicate ONE option using the number 11 to 20.

__________________________[RESTRICTION: Number 11 to 20]

Q21 In the following task, instead of payoffs with chances, please consider these hypothetical payoffs that are made with certainty. **This means that there is NO risk associated with the payoffs.**

This task has 7 scenarios. For each scenario, you are given the option to choose between

- Payoff Option A: a lower payoff to be paid **1 month from today**, or
- Payoff Option B: a higher payoff to be paid **13 months from today**.

First, consider the following example. You do not need to make a choice in this example.

Scenario	Payoff Option A (pays 1 month from today)	Payoff Option B (pays 13 months from today)	I prefer: **Option A**	I prefer: **Option B**
1	$1,000	$1,020	☐	☐

- If you choose Option A, you will receive a payoff of $1,000 **1 month from today**.
- If you choose Option B, you will receive a payoff of $1,020 **13 months from today**.
- Choosing any one of the options will not cost you any money.
- Again, there is NO risk associated with any of these payoffs.

Now it is time to choose. For each scenario below, **please choose either Option A or B**.

Scenario	Payoff Option A (pays 1 month from today)	Payoff Option B (pays 13 months from today)	I prefer: **Option A**	I prefer: **Option B**
1	$1,000	$1,020	☐	☐
2	$1,000	$1,040	☐	☐
3	$1,000	$1,080	☐	☐
4	$1,000	$1,140	☐	☐
5	$1,000	$1,240	☐	☐
6	$1,000	$1,340	☐	☐
7	$1,000	$1,440	☐	☐

D.1.4 Retirement Planning Objectives

Inflation Risk

i. Subjective Inflation Beliefs

Q22 For this section, please consider the following hypothetical situation: Suppose that you are 65 years old and you are JUST about to retire. Your employer will pay you monthly pension income payments for the rest of your life. The pension will stop when you die. Your employer presents you with the following pension options:

1. **Match-Inflation Pension Option**

   This option pays a monthly pension that *increases at a rate that exactly matches inflation (i.e. the increase in the cost of living).*

2. **Steady Pension Option**

   This option pays a constant monthly pension that *does not change.*

In each of the following six scenarios, which option would you choose, assuming that you are 65 and JUST about to retire?

[EXAMPLE: HIGH, FEMALE. SEE APPENDIX FOR OTHER CATEGORIES.]

First Monthly Income Payment		**I would choose...**	
Match-Inflation Option This option provides income payments that increase at the rate of inflation	Steady Option This option provides income payments that do not change	**Match-Inflation Option**	**Steady Option**
$1,551	$2,358	☐	☐
$1,650	$2,358	☐	☐
$1,752	$2,358	☐	☐
$1,857	$2,358	☐	☐
$1,965	$2,358	☐	☐
$2,358	$2,358	☐	☐

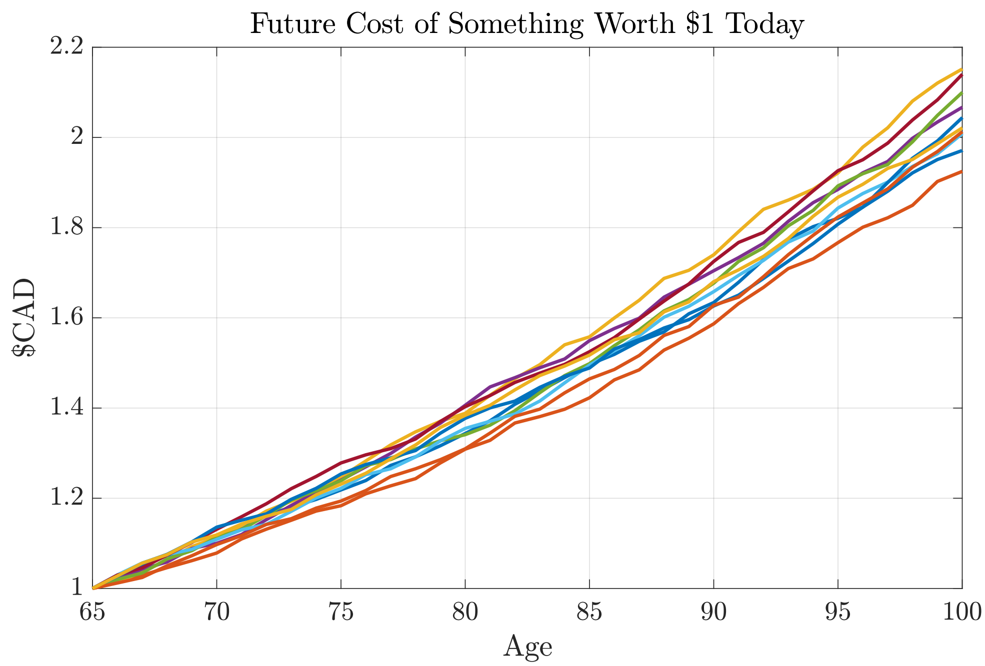
ii. Objective Inflation Information

Q23 Consider the same hypothetical situation where you are 65 years old and JUST about to retire. This time, you will be provided with more information on what inflation will look like in the future.

Suppose the Canadian government targets inflation to be 2% per year, but the actual inflation in each year can be anywhere between 1% to 3%. You can expect inflation to be 2% per year on average for the rest of your life, but you do not know for certain what the actual future inflation rates will be.

To help you make a decision, the chart below presents 10 possible paths of how much things that cost $1 today could cost you in the next 35 years. Keep in mind there is an equal chance for any one of these paths to happen.

[INSERT GRAPH COLA]



Now, consider again the two pension options your employer presents you:

1. **Match-Inflation Pension Option**

   This option pays a monthly pension that *increases at a rate that exactly matches inflation (i.e. the increase in the cost of living)*.

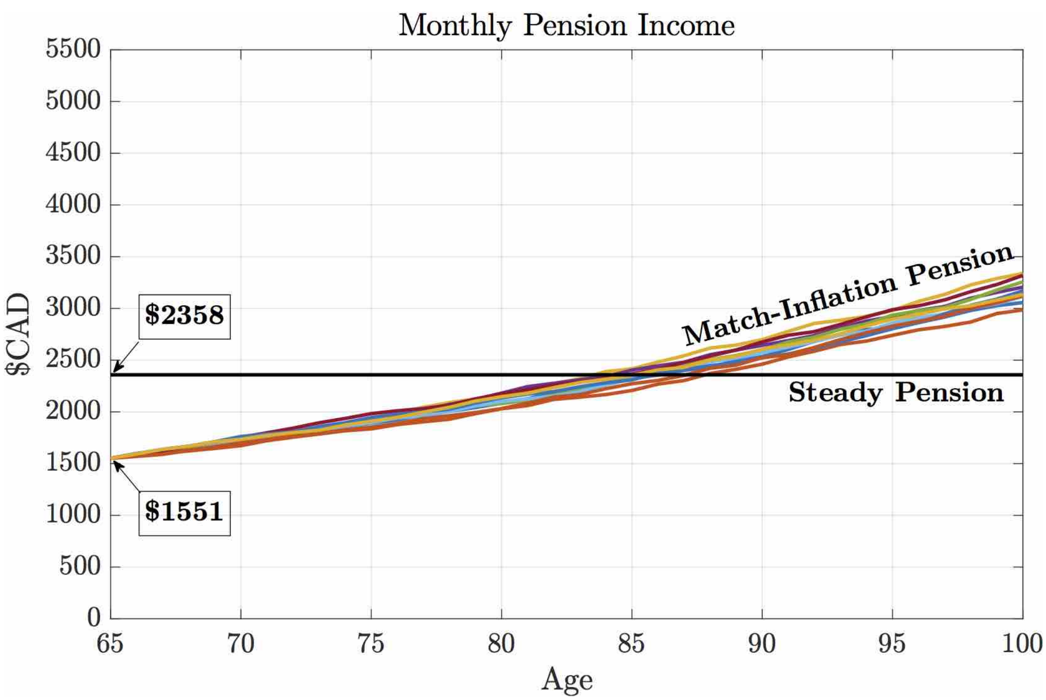
2. **Steady Pension Option**

   This option pays a constant monthly pension that *does not change*.

Consider Case 1.

- If you choose the **Match-Inflation Pension** Option,
  - Your first monthly income payment will be ***$1,551***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension** Option,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI1] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

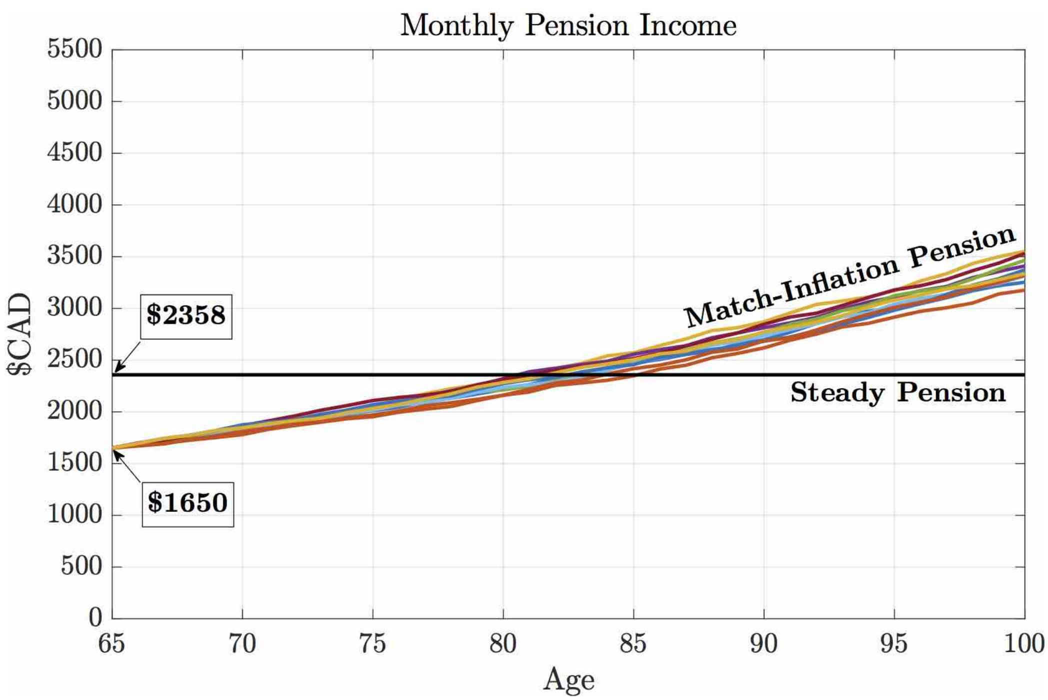
☐ Steady Pension Option

[IF CHOICE = "Match-Inflation Pension Option", SKIP TO Q23A]
[IF CHOICE = "Steady Pension Option", SHOW NEXT CASE]

Consider Case 2.

- If you choose the **Match-Inflation Pension Option**,
  - Your first monthly income payment will be ***$1,650***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI2] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

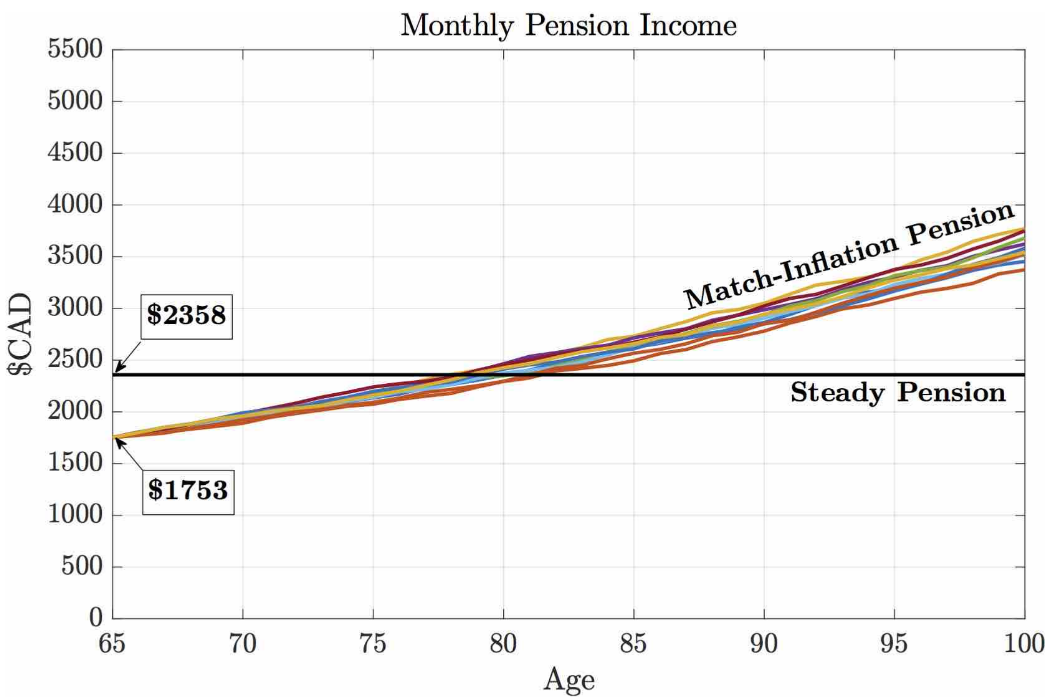
☐ Steady Pension Option

[IF CHOICE = "Match-Inflation Pension Option", SKIP TO Q23A]
[IF CHOICE = "Steady Pension Option", SHOW NEXT CASE]

Consider Case 3.

- If you choose the **Match-Inflation Pension Option**,
  - Your first monthly income payment will be ***$1,753***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI3] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

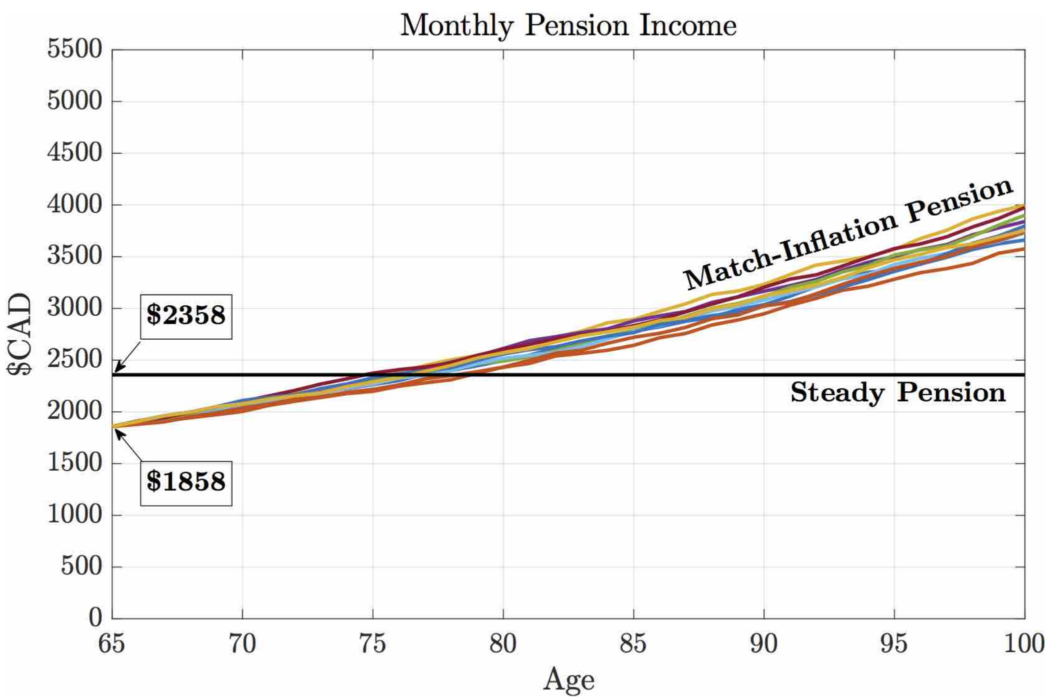
☐ Steady Pension Option

[IF CHOICE = "Match-Inflation Pension Option", SKIP TO Q23A]
[IF CHOICE = "Steady Pension Option", SHOW NEXT CASE]

Consider Case 4.

- If you choose the **Match-Inflation Pension Option**,
  - Your first monthly income payment will be ***$1,858***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI4] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

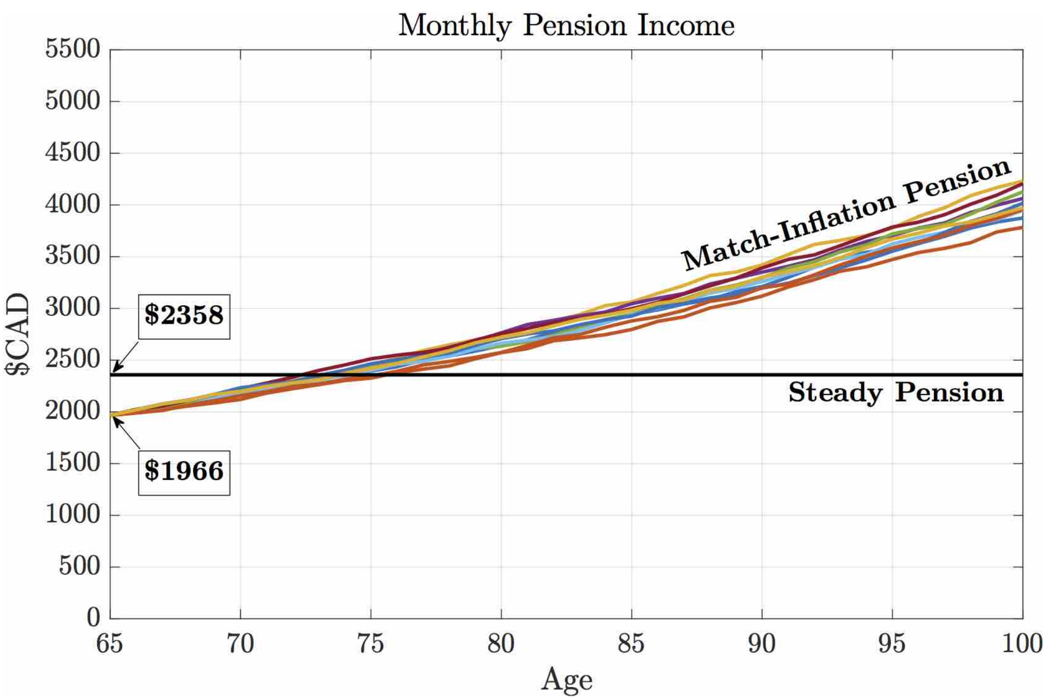
☐ Steady Pension Option

[IF CHOICE = "Match-Inflation Pension Option", SKIP TO Q23A]
[IF CHOICE = "Steady Pension Option", SHOW NEXT CASE]

Consider Case 5.

- If you choose the **Match-Inflation Pension Option**,
  - Your first monthly income payment will be ***$1,966***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI5] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

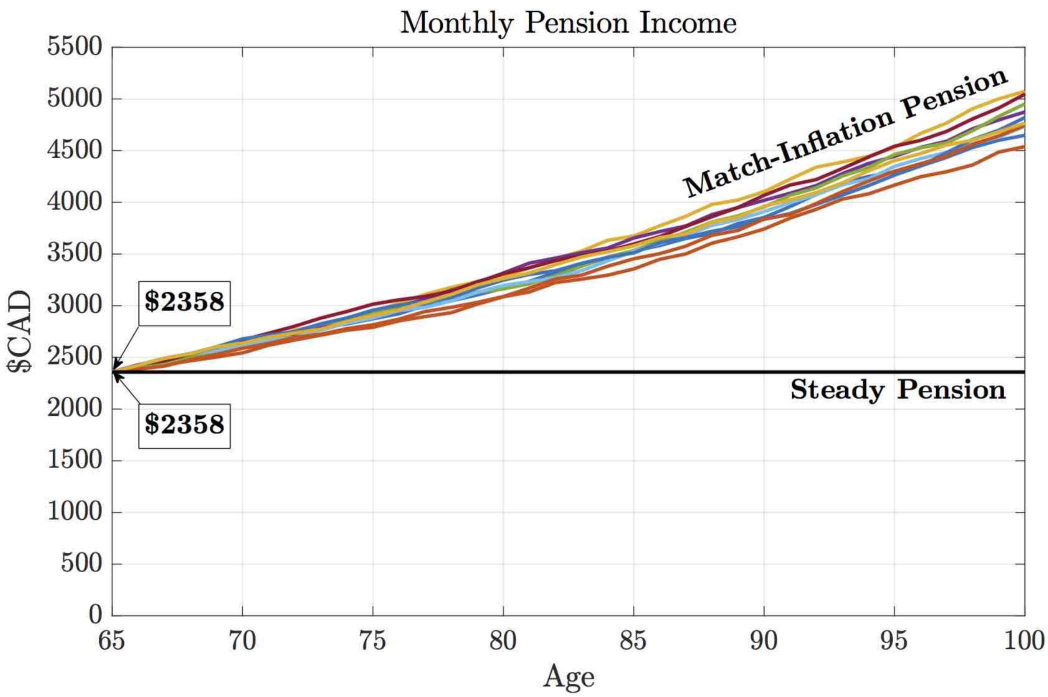
☐ Steady Pension Option

[IF CHOICE = "Match-Inflation Pension Option", SKIP TO Q23A]
[IF CHOICE = "Steady Pension Option", SHOW NEXT CASE]

Consider Case 6.

- If you choose the **Match-Inflation Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

[INSERT PLOT MI6] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Match-Inflation Pension Option

☐ Steady Pension Option

Q23A Please rate the importance of the following reasons behind your choices between **match-inflation** and **steady** pension options.

For each reason, please select the one importance rating that best represents your opinion.

	Very Unimportant	**Fairly Unimportant**	**Neutral**	**Fairly Important**	**Very Important**
I want to maintain a stable and adequate living standard	☐	☐	☐	☐	☐
I am worried that inflation will have a severe impact on living standards	☐	☐	☐	☐	☐
I do not believe inflation poses high risk to my retirement financial security	☐	☐	☐	☐	☐
The concept of the match-inflation income option is difficult to understand	☐	☐	☐	☐	☐
Please specify any other important considerations					

Q23B Please indicate how you feel about the clarity of questions involving the **match-inflation pension**.

☐ Completely clear

☐ Mostly clear

☐ Sometimes clear

☐ Sometimes confusing

☐ Mostly confusing

☐ Completely confusing

Investment Risk

Q24 For this section, please consider a similar hypothetical situation.

Suppose that you are 65 years old and you are JUST about to retire. Your employer will pay you monthly pension income payments for the rest of your life. The pension will stop when you die. Your employer presents you with the following pension options:

1. **Equity-Linked Pension Option**

   This option pays a monthly pension with payments linked to stock market performance. Every year your pension will be re-evaluated based on the performance of the stock market. *This means that your pension income payments are uncertain and will experience some ups and downs.*

2. **Steady Pension Option**

   This option pays a constant monthly pension that *does not change*.

To help you make a decision, for each case, we will present you *10 possible future paths of what your future monthly pension will look like under the Equity-Linked Pension Option.*

Keep in mind there is an *equal chance* for any one of the paths to happen.

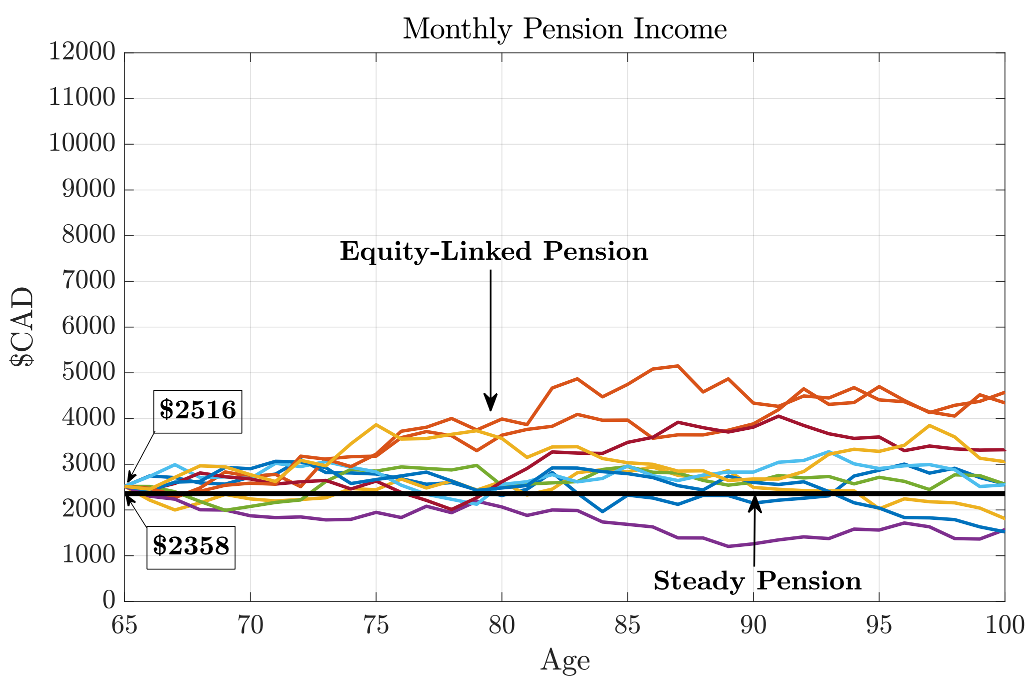
i. Fair AIRs

Consider Case 1A.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$2,516***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL1A] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

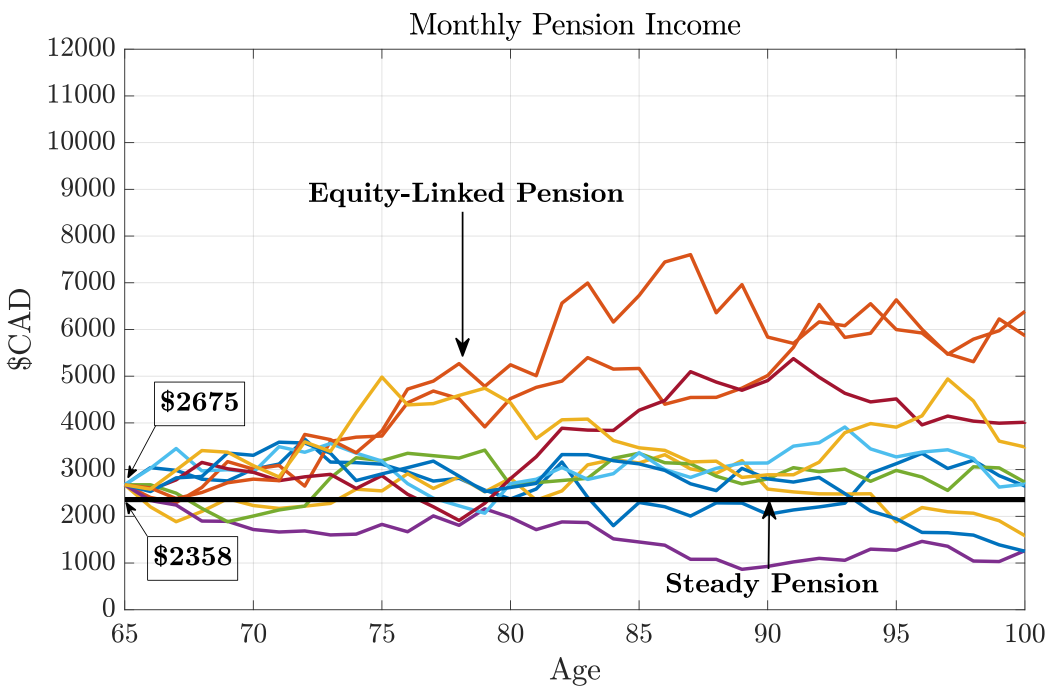
☐ Steady Pension Option

Consider Case 1B.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$2,675***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL1B] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

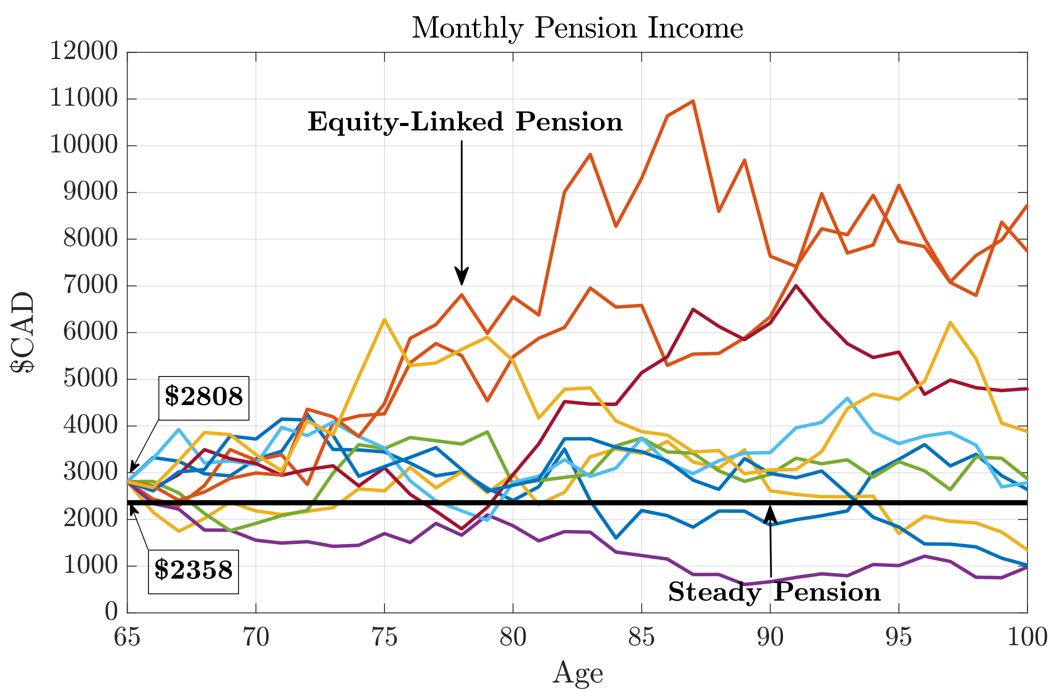
☐ Steady Pension Option

Consider Case 1C.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$2,808***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL1C] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

☐ Steady Pension Option

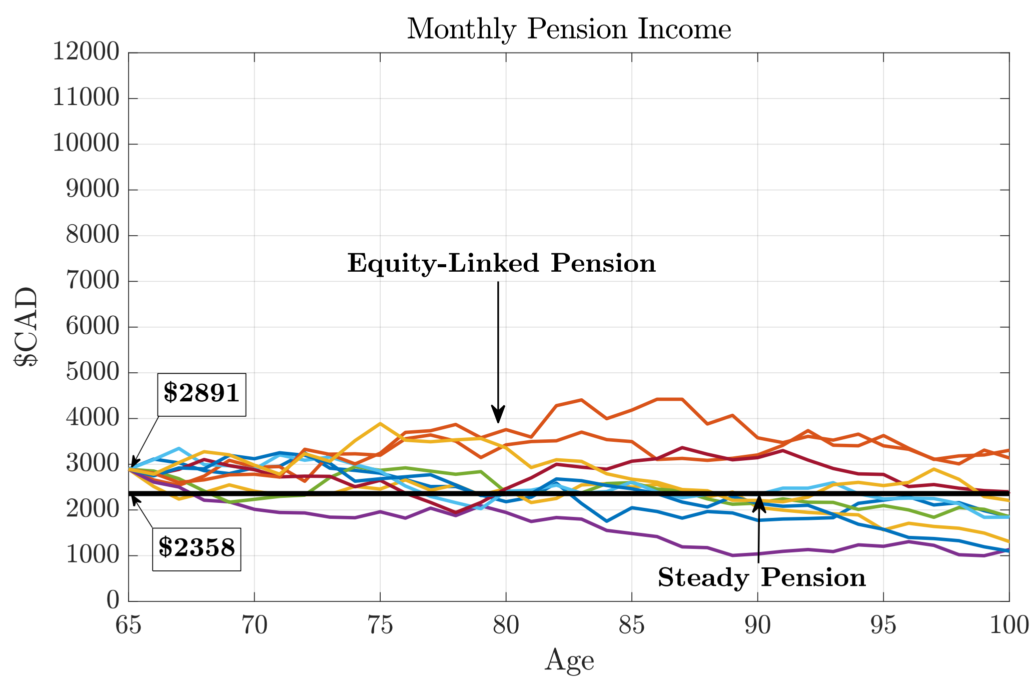
ii. High AIRs

Consider Case 2A.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$2,891***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL2A] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

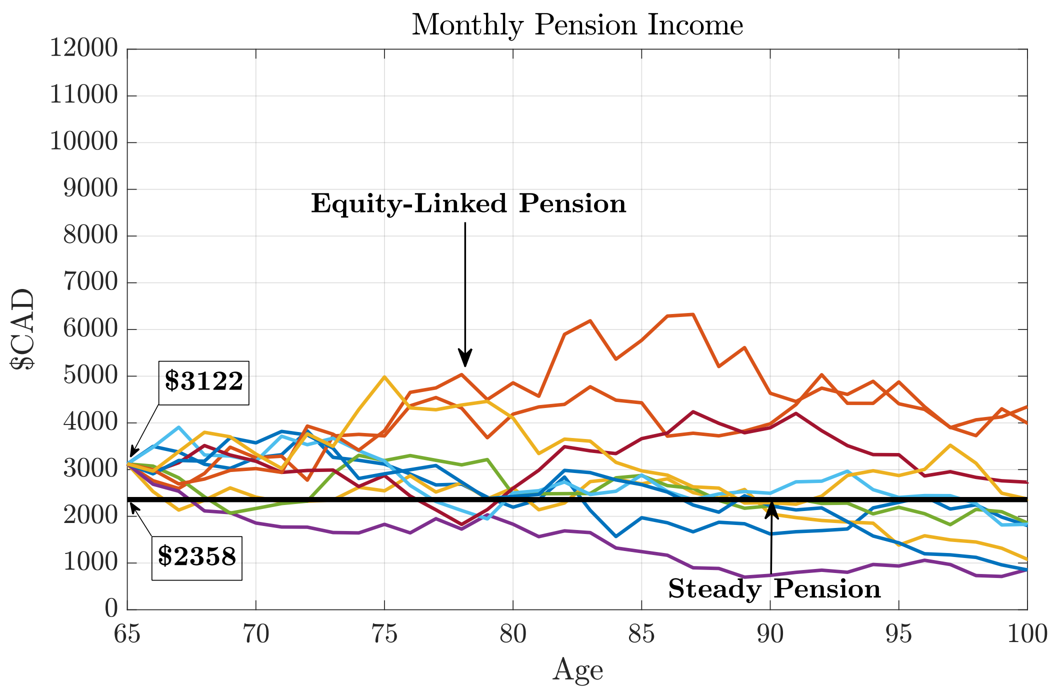
☐ Steady Pension Option

Consider Case 2B.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$3,122***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL2B] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

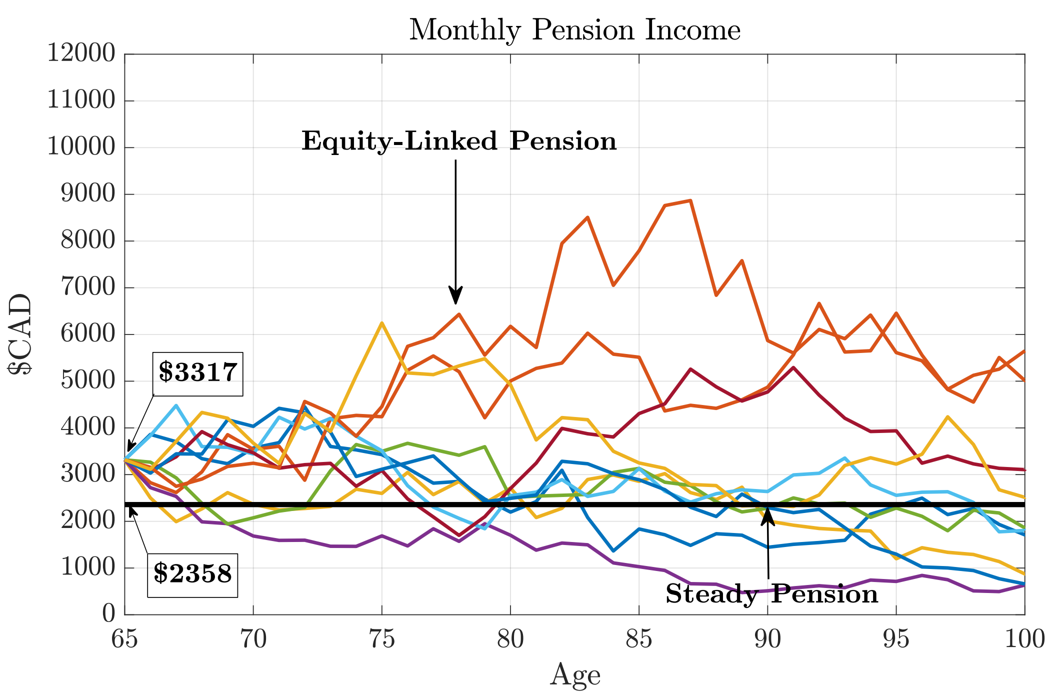
☐ Steady Pension Option

Consider Case 2C.

- If you choose the **Equity-Linked Pension Option**,
  - Your first monthly income payment will be ***$3,317***.
  - Your future monthly pension has an equal chance of being any one of the 10 coloured pension paths as shown in the chart below.
- If you choose the **Steady Pension Option**,
  - Your first monthly income payment will be ***$2,358***.
  - Your future monthly pension will stay the same for the rest of your life, as shown by the black line in the chart below.

**Note the difference in the first monthly income payments.*

[INSERT PLOT EL2C] [EXAMPLE: HIGH, FEMALE, SEE GRAPH BELOW]



Which option would you choose, assuming that you are 65 and JUST about to retire?

☐ Equity-Linked Pension Option

☐ Steady Pension Option

Q24A Please rate the importance of the following reasons behind your choices between the **equity-linked** and **steady** pension options.

For each reason, please select the one importance rating that best represents your opinion.

	Very Unimportant	**Fairly Unimportant**	**Neutral**	**Fairly Important**	**Very Important**
I am worried about stock market crashes	☐	☐	☐	☐	☐
I am worried about persistent poor stock market performance and deteriorating income stream	☐	☐	☐	☐	☐
The concept of the Equity-Linked pension option is difficult to understand	☐	☐	☐	☐	☐
I prefer income payment options that are easier for me to understand	☐	☐	☐	☐	☐
I expect to take some risk to gain exposure of possible higher income in the long term	☐	☐	☐	☐	☐
I want to make sure I have enough income later in life	☐	☐	☐	☐	☐
I expect my risk attitude would change as I grow older	☐	☐	☐	☐	☐
Please specify any other important considerations					

Q24B Please indicate how you feel about the clarity of the questions involving the **equity-linked pension**.

- ☐ Completely clear
- ☐ Mostly clear
- ☐ Sometimes clear
- ☐ Sometimes confusing
- ☐ Mostly confusing
- ☐ Completely confusing

D.1.5 Demographic Information

Q25 What is the highest level of education you completed?

- ☐ Some high school or less
- ☐ High school graduate
- ☐ Trades or college certificate or diploma
- ☐ University certificate or diploma below the bachelor level
- ☐ Bachelor's Degree
- ☐ Graduate or Postgraduate Degree
- ☐ Other professional degree

Q26 In general, would you say your health is ...?

- ☐ Excellent
- ☐ Good
- ☐ Fair
- ☐ Poor
- ☐ Very Poor

Q27 Do you own a home that you currently reside in?

- ☐ Yes
- ☐ No

Q28 Where is your permanent residence?

- ☐ In an area with a population of 1,000 or over
- ☐ In an area with a population under 1,000

Q29 Please explain what, if anything, you found to be unclear or confusing about the questions in this survey.

Thank you for participating in the survey! Your feedback is extremely valuable.

If you would like to receive a copy of the results of this study, please contact Saisai Zhang at s288zhan@uwaterloo.ca.

We would like to assure you that this project has been reviewed and received ethics clearance through a University of Waterloo Research Ethics Committee. Should you have any comments or concerns resulting from your participation in this study, please contact the Chief Ethics Officer, Office of Research Ethics, at 1-519-888-4567 ext. 36005 or ore-ceo@uwaterloo.ca.

If you have any general comments or questions related to this study, please contact Mary Hardy at mrhardy@uwaterloo.ca.

[**Exit**]

D.2 Economic and Pension Scenarios

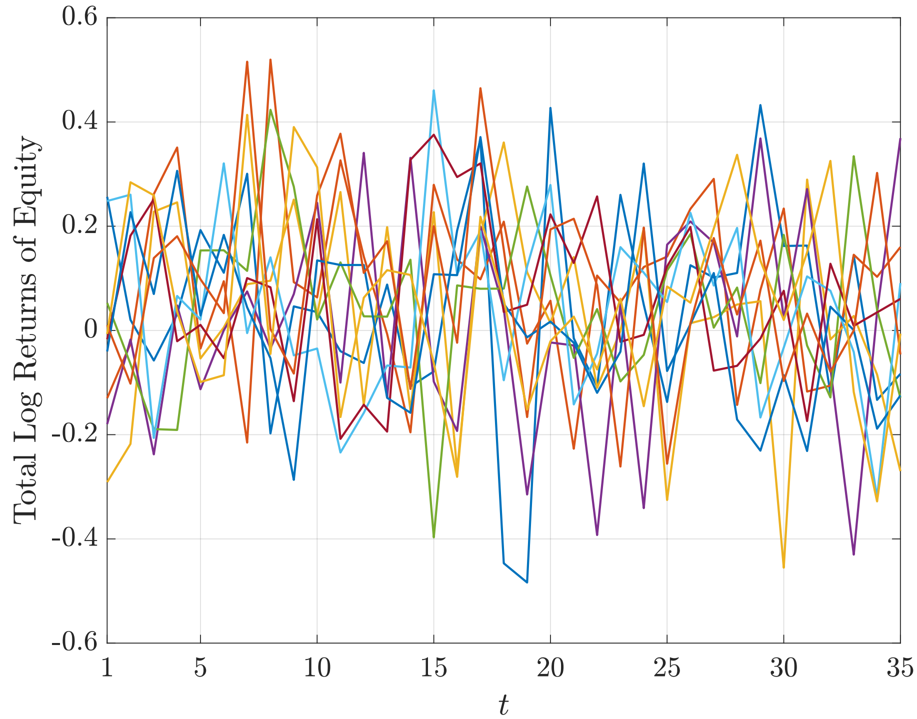
This section reports the economic and pension scenarios used in projecting future Match-Inflation and Equity-Linked pension options.

Scenario	1	2	3	4	5	6	7	8	9	10
$t = 1$	0.2564	−0.1299	−0.2911	−0.1796	0.0528	0.2483	−0.0169	−0.0403	0.0100	−0.0068
$t = 2$	0.0203	−0.0254	−0.2177	−0.0174	−0.0645	0.2605	0.1819	0.2270	−0.1021	0.2843
$t = 3$	−0.0577	0.2615	0.2279	−0.2377	−0.1893	−0.2063	0.2518	0.0699	0.1389	0.2594
$t = 4$	0.0310	0.3510	0.2458	0.0479	−0.1909	0.0662	−0.0207	0.3063	0.1808	0.0358
$t = 5$	0.1925	−0.0347	−0.0537	−0.1128	0.1535	0.0210	0.0107	0.0287	0.0982	−0.0989
$t = 6$	0.1107	0.0945	0.0058	0.0004	0.1539	0.3206	−0.0532	0.1834	0.0332	−0.0860
$t = 7$	0.3007	−0.2151	0.0888	0.0749	0.1143	−0.0051	0.1005	0.0444	0.5157	0.4137
$t = 8$	−0.1975	0.5197	0.0954	−0.0344	0.4234	0.1403	0.0828	−0.0534	0.0028	−0.0451
$t = 9$	0.0462	0.0927	0.2510	0.0697	0.2759	−0.0478	−0.1358	−0.2867	−0.0828	0.3903
$t = 10$	0.0352	0.0634	0.0294	0.2451	0.0210	−0.0347	0.2135	0.1344	0.2590	0.3130
$t = 11$	−0.0398	0.3265	0.2656	−0.1001	0.1306	−0.2342	−0.2080	0.1256	0.3775	−0.1662
$t = 12$	−0.0624	0.1417	−0.1429	0.3407	0.0271	−0.1582	−0.1425	0.1256	0.1104	0.0629
$t = 13$	0.0880	−0.0068	0.1982	−0.1279	0.0265	−0.0672	−0.1941	−0.1293	0.1712	0.1157
$t = 14$	−0.1075	−0.1955	−0.1529	0.3313	0.1357	−0.0712	0.3275	−0.1580	−0.1123	0.1065
$t = 15$	−0.0787	0.2795	0.2270	−0.0982	−0.3970	0.4607	0.3756	0.1079	0.2000	−0.0624
$t = 16$	0.1916	0.1362	−0.2682	−0.1929	0.0864	0.1082	0.2944	0.1061	−0.0236	−0.2810
$t = 17$	0.3660	0.0981	0.1824	0.1994	0.0800	0.1904	0.3206	0.3711	0.4648	0.2182
$t = 18$	−0.4468	0.2088	0.3607	0.0410	0.0803	−0.0956	0.0355	0.0513	0.1550	0.0586
$t = 19$	−0.4837	−0.0253	0.1118	−0.3147	0.2762	0.1226	0.0493	−0.0131	−0.1662	−0.1519
$t = 20$	0.4270	0.0570	0.0141	−0.0231	0.1059	0.2791	0.2227	0.0170	0.1939	−0.0207
$t = 21$	−0.0151	−0.2269	0.1408	−0.0287	−0.0527	−0.1419	0.1295	−0.0237	0.2143	0.0266
$t = 22$	−0.1083	0.1053	−0.1102	−0.3925	0.0410	−0.0446	0.2572	−0.1197	0.0868	−0.0749
$t = 23$	0.2602	0.0535	−0.0010	0.0511	−0.0978	0.1599	−0.0219	−0.0407	−0.2611	0.0610
$t = 24$	0.0494	0.1217	0.1959	−0.3409	−0.0469	0.1109	−0.0085	0.3203	0.1974	−0.1454
$t = 25$	−0.1372	0.1417	−0.3253	0.1646	0.1150	0.0546	0.1229	−0.0778	−0.2558	0.0843
$t = 26$	0.1251	0.2333	0.0139	0.2091	0.1850	0.2257	0.1987	0.0120	0.0139	0.0532
$t = 27$	0.1014	0.2908	0.0254	0.1685	0.0053	0.0861	−0.0771	0.1102	0.1773	0.1953
$t = 28$	0.1103	−0.1430	0.0498	−0.0115	0.0823	0.1968	−0.0679	−0.1709	0.0306	0.3370
$t = 29$	0.4325	0.0775	0.0561	0.3684	−0.1011	−0.1670	−0.0155	−0.2307	0.1724	0.1339
$t = 30$	0.1621	0.2339	−0.4552	0.0226	0.1834	−0.0329	0.0761	−0.0857	−0.0969	0.0201
$t = 31$	0.1628	−0.1172	0.2895	0.2713	−0.0290	0.1037	−0.1739	−0.2314	0.0323	0.1495
$t = 32$	−0.1207	−0.1058	−0.0173	−0.0725	−0.1285	0.0763	0.1280	0.0456	−0.0791	0.3253
$t = 33$	0.1440	0.1453	0.0278	−0.4301	0.3343	−0.0413	0.0088	0.0006	0.0000	−0.1181
$t = 34$	−0.1332	0.1029	−0.0846	0.0350	0.0399	−0.3200	0.0346	−0.1885	0.3023	−0.3280
$t = 35$	−0.0833	0.1600	−0.2699	0.3690	−0.1293	0.0909	0.0606	−0.1247	−0.0460	−0.0061

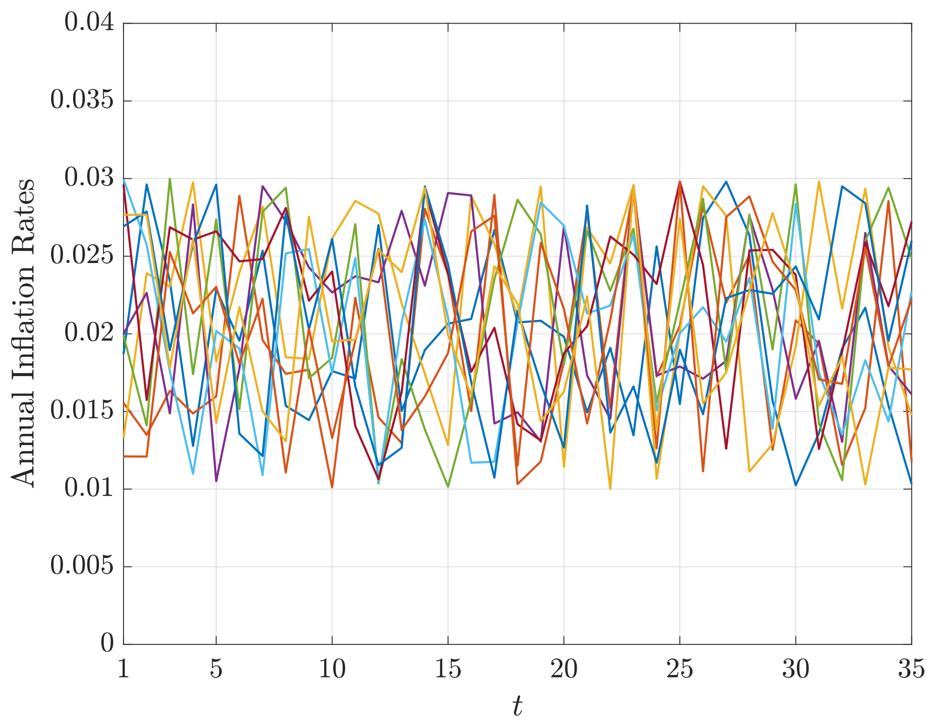
Table D.1 Total Log Returns of Equity used in Equity-Linked Pension Options.

Scenario	1	2	3	4	5	6	7	8	9	10
$t = 1$	0.0187	0.0156	0.0133	0.0200	0.0199	0.0299	0.0296	0.0269	0.0121	0.0277
$t = 2$	0.0296	0.0135	0.0239	0.0226	0.0141	0.0257	0.0157	0.0279	0.0121	0.0277
$t = 3$	0.0234	0.0163	0.0230	0.0149	0.0300	0.0176	0.0269	0.0190	0.0253	0.0178
$t = 4$	0.0128	0.0149	0.0298	0.0283	0.0174	0.0110	0.0261	0.0258	0.0213	0.0261
$t = 5$	0.0229	0.0160	0.0182	0.0105	0.0274	0.0202	0.0266	0.0296	0.0230	0.0143
$t = 6$	0.0196	0.0289	0.0240	0.0178	0.0152	0.0191	0.0247	0.0136	0.0181	0.0217
$t = 7$	0.0254	0.0196	0.0282	0.0295	0.0279	0.0109	0.0248	0.0121	0.0223	0.0150
$t = 8$	0.0154	0.0174	0.0185	0.0273	0.0294	0.0252	0.0281	0.0277	0.0111	0.0131
$t = 9$	0.0144	0.0177	0.0184	0.0243	0.0171	0.0255	0.0221	0.0203	0.0203	0.0275
$t = 10$	0.0176	0.0101	0.0261	0.0227	0.0185	0.0175	0.0240	0.0261	0.0133	0.0195
$t = 11$	0.0171	0.0223	0.0286	0.0237	0.0271	0.0249	0.0141	0.0166	0.0195	0.0196
$t = 12$	0.0270	0.0146	0.0277	0.0233	0.0104	0.0105	0.0106	0.0116	0.0255	0.0254
$t = 13$	0.0150	0.0130	0.0220	0.0279	0.0184	0.0207	0.0162	0.0127	0.0138	0.0240
$t = 14$	0.0190	0.0281	0.0174	0.0231	0.0138	0.0274	0.0293	0.0295	0.0160	0.0294
$t = 15$	0.0207	0.0238	0.0128	0.0291	0.0102	0.0210	0.0239	0.0245	0.0188	0.0199
$t = 16$	0.0210	0.0150	0.0289	0.0289	0.0158	0.0117	0.0176	0.0169	0.0266	0.0160
$t = 17$	0.0267	0.0290	0.0258	0.0142	0.0235	0.0118	0.0204	0.0107	0.0276	0.0244
$t = 18$	0.0207	0.0103	0.0211	0.0150	0.0286	0.0207	0.0142	0.0215	0.0115	0.0219
$t = 19$	0.0208	0.0118	0.0295	0.0131	0.0264	0.0285	0.0131	0.0168	0.0259	0.0144
$t = 20$	0.0198	0.0182	0.0114	0.0269	0.0180	0.0270	0.0187	0.0127	0.0216	0.0163
$t = 21$	0.0149	0.0279	0.0268	0.0173	0.0266	0.0213	0.0205	0.0283	0.0142	0.0224
$t = 22$	0.0191	0.0152	0.0245	0.0145	0.0228	0.0218	0.0263	0.0136	0.0209	0.0100
$t = 23$	0.0135	0.0295	0.0296	0.0292	0.0268	0.0264	0.0251	0.0166	0.0293	0.0295
$t = 24$	0.0256	0.0173	0.0107	0.0173	0.0156	0.0151	0.0232	0.0117	0.0126	0.0138
$t = 25$	0.0155	0.0205	0.0202	0.0179	0.0221	0.0200	0.0298	0.0190	0.0298	0.0274
$t = 26$	0.0275	0.0282	0.0295	0.0171	0.0287	0.0217	0.0244	0.0148	0.0111	0.0155
$t = 27$	0.0298	0.0220	0.0276	0.0183	0.0176	0.0195	0.0126	0.0223	0.0275	0.0175
$t = 28$	0.0263	0.0251	0.0111	0.0276	0.0277	0.0236	0.0254	0.0228	0.0289	0.0228
$t = 29$	0.0157	0.0125	0.0129	0.0229	0.0190	0.0139	0.0254	0.0226	0.0246	0.0278
$t = 30$	0.0102	0.0209	0.0192	0.0158	0.0296	0.0284	0.0238	0.0243	0.0228	0.0232
$t = 31$	0.0136	0.0194	0.0298	0.0196	0.0142	0.0175	0.0126	0.0209	0.0171	0.0154
$t = 32$	0.0191	0.0115	0.0216	0.0130	0.0106	0.0135	0.0188	0.0295	0.0168	0.0185
$t = 33$	0.0217	0.0152	0.0294	0.0265	0.0260	0.0183	0.0259	0.0284	0.0256	0.0103
$t = 34$	0.0154	0.0286	0.0191	0.0179	0.0294	0.0144	0.0218	0.0195	0.0179	0.0178
$t = 35$	0.0103	0.0118	0.0148	0.0161	0.0248	0.0228	0.0272	0.0260	0.0223	0.0177

Table D.2 Annual Inflation Rates used in Match-Inflation Pension Options.



Total Log Returns of Equity
t
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
1
5
10
15
20
25
30
35



Annual Inflation Rates
t
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
1
5
10
15
20
25
30
35

	Low & Female		Low & Male	
Cases	Match-Inflation	Steady	Match-Inflation	Steady
1	465	707	515	759
2	495	707	545	759
3	526	707	576	759
4	557	707	608	759
5	590	707	641	759
6	707	707	759	759
	Med & Female		Med & Male	
Cases	Match-Inflation	Steady	Match-Inflation	Steady
1	931	1415	1029	1517
2	990	1415	1090	1517
3	1052	1415	1152	1517
4	1115	1415	1216	1517
5	1179	1415	1281	1517
6	1415	1415	1517	1517
	High & Female		High & Male	
Cases	Match-Inflation	Steady	Match-Inflation	Steady
1	1551	2358	1715	2528
2	1650	2358	1817	2528
3	1753	2358	1920	2528
4	1858	2358	2026	2528
5	1966	2358	2135	2528
6	2358	2358	2528	2528

Table D.3 First monthly pension payments for choices between Match-Inflation and Steady pension options.

	Low & Female		Low & Male	
Cases	Equity-Linked	Steady	Equity-Linked	Steady
1A	755	707	806	759
1B	802	707	853	759
1C	842	707	893	759
2A	867	707	918	759
2B	937	707	986	759
2C	995	707	1044	759
	Med & Female		**Med & Male**	
Cases	Equity-Linked	Steady	Equity-Linked	Steady
1A	1510	1415	1611	1517
1B	1605	1415	1706	1517
1C	1685	1415	1786	1517
2A	1735	1415	1835	1517
2B	1873	1415	1973	1517
2C	1990	1415	2088	1517
	High & Female		**High & Male**	
Cases	Equity-Linked	Steady	Equity-Linked	Steady
1A	2516	2358	2686	2528
1B	2675	2358	2844	2528
1C	2808	2358	2976	2528
2A	2891	2358	3058	2528
2B	3122	2358	3288	2528
2C	3317	2358	3480	2528

Table D.4 First monthly pension payments for choices between Equity-linked and Steady pension options.

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