Periodic Nonlinear Adaptive Control of Rapidly Time-Varying Linear Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In adaptive control the goal is to deal with systems that have unknown and/or time-varying parameters. Most techniques are proven for the case in which any time-variation is slow, with results for systems with fast time-variations limited to those for which the time-variation is of a known form or for which the plant has stable zero dynamics. Here we propose a new adaptive controller design methodology for which the time-varation can be rapid. While the plant is allowed to have unstable-zero dynamics, it must satisfy several structural conditions which have been proven to be necessary in the literature; we also impose some mild regularity conditions. The proposed controller is nonlinear and periodic, and in each period the parameter values are estimated and an appropriate stabilizing control signal is applied. Under the technical assumptions that the plant is relative degree one and that the plant uncertainty is in terms of a single scalar variable, it is proven that the closed loop system is stable under fast parameter variations with persistent jumps.

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Dedication

To Mom, John, Danielle, and Sandie, for their unending support. To Jesha, for always being there. To Richard, for pushing me to think about the little things.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Background

The primary objective of adaptive control is to handle systems with parameters that are uncertain. A classical example of such an adaptive controller is a linear time-invariant (LTI) controller having adjustable parameters. Typically a tuning mechanism is used to modify the controller in such a way that it is suitable for the uncertain plant. This often results in a nonlinear closed-loop system.

In the 1950's, adaptive control methods were adopted in order to deal with systems for which parameters were both uncertain and time-varying. However, the solution to such a general problem could not be found. Focus shifted to a more modest goal of controlling systems for which parameters were uncertain, but fixed. This simplified scenario was also very difficult and it wasn't until around 1980 that a generalized solution was obtained (e.g. [1], [2], [3]). These controllers typically gave poor transient responses and were not robustly stable in the presence of unmodeled dynamics and bounded disturbances (e.g. [4]). In response, a number of approaches were developed to improve performance. These included the Certainty Equivalence approaches (e.g. [5]), prerouted logic based switching approach (e.g. [6], [7]), and more refined methods such as supervisory and multi-model switching control (e.g. [8], [9], [10], [11], [12]).

The study of time-varying systems has been difficult. With modification, some of the earlier adaptive controllers could handle slow time-variation of plant parameters and/or occasional parameter jumps (e.g. [13], [14], [15], [16], [17], [18]). The study of rapid time-variation has been limited with either the form of the time-variations being known (e.g. [19], [20]), or plants only having stable zero dynamics (the time-varying counterpart of minimum phase) being considered (e.g. [21], [22], [23], [24], [25], [26]). There are a few general results which deal with unstable zero dynamics under moderate time-variations (e.g. [27], [28], [29]). There is also a result which can handle arbitrarily fast, but bounded, time-variation (see [30]). However, the result is limited by a stringent condition in that both the output matrix, C, and the observability matrix of the plant must be independent of the time-varying parameter. The difficulty of unstable zero dynamics comes from the fact that, even if the plant parameters are known up to the present, there are no methods for designing a stabilizing controller.[1]

The approach of gain scheduling developed alongside adaptive control. In the gain scheduling problem, a plant whose parameters depend on a variable (the gain scheduling parameter) is considered. This variable is assumed to be measureable, e.g. a plane whose dynamics depend on the altitude, which is measureable. Despite gain scheduled controller design being a classical, and often ad-hoc, approach, it has re-gained interest since the 1990's (e.g. [31], [32], [33]). There are many different design methods for gain scheduling, however the most common is that of varying the controller coefficients based on the current value of the scheduling variable. An important approach considered is that of converting a nonlinear plant to a linear parameter-varying (LPV) system by either regarding the nonlinearity as the scheduling parameter, or linearizing for a set of operating points regarded as the scheduling parameter. This preserves well-understood linear design tools and allows the utilization of these tools on difficult nonlinear systems. Several controller design approaches have been developed for this situation, typically resulting in a set of LTI compensators where each controller achieves the desired performance specification for a particular instance of the plant (e.g. [33]).

Of significance is the invariant set approach developed in [34], [35], [36], [37], [38], where it is shown that polyhedral Lyapunov functions and associated geometrically intuitive methods can be used for controller synthesis. In particular, in [38] it is shown that under some stringent assumptions, a continuous-time gain-scheduled output feedback controller can be constructed such that the closed-loop system is stable under arbitrarily fast time-variations in the parameter.

1.2 Purpose

The goal of this thesis is to develop a nonlinear adaptive controller that exponentially stabilizes a system with possibly unstable zero dynamics and arbitrarily fast, but bounded, time variation. This will be primarily achieved by extending the work on gain scheduling in [38], but here the scheduling parameter is not assumed to be measureable. The approach is influenced in a large part by [38], [30], and [39].

Here, the case of a plant for which the time-varying parameter is limited to a scalar variable, and for which the parameter is accessible from the plant's first Markov parameter, is considered. The proposed controller utilizes a continuous-time filter in tandem with a discretized version of the gain-scheduled output feedback controller in [38]. However, the time-varying parameter is replaced with an estimate generated by a discrete-time parameter estimator inspired by [39], yielding a nonlinear adaptive controller. It is proven that the closed-loop system is exponentially stable with a bounded gain on the noise under suitable assumptions on the plant[2], even in the presence of persistent parameter discontinuities and arbitrarily fast, but bounded, time variation.

[1] If the time-varying parameter(s) is known in advance, then under very modest conditions we can solve the associated LQR optimal control problem to design the LQR optimal controller.

[2] It is critical that the norm be chosen in just the right way in order to reach this stability result.

1.3 Organization

In Chapter 2, mathematical preliminaries are presented. In Chapter 3, the problem of LPV stability is introduced. In Subsection 3.1, a number of crucial definitions and results regarding the stability of LPV systems are discussed and necessary conditions are also stated. In Subsection 3.2, additional standing assumptions are introduced. In Chapter 4, the proposed controller is expanded upon, and a number of key technical results are proven. Subsection 4.1 provides a brief overview of the controller, and each of Subsections 4.2, 4.3, and 4.4 delve into a specific component of the proposed controller (the filter, the discretized gain-scheduled controller, and the estimator, respectively). In Chapter 5, it is proven that the proposed controller achieves the desired stability objective. In Chapter 6, an illustrative example is provided. Finally, in Chapter 7, a summary and concluding remarks are provided.

Chapter 2

Mathematical Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of non-negative real numbers, and $\mathbb{Z}_+$ denote the set of non-negative integers. We will use both the 1-norm and the ∞-norm for measuring the size of a vector $x \in \mathbb{R}^n$, defined, respectively as

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|, \quad \|x\|_\infty := \max\left\{|x_1|, ..., |x_n|\right\}.$$

Occasionally we will leave the norm of a vector, x, or matrix, A, undecorated when the specific norm used doesn't impact the analysis or results. We will also use the corresponding induced norms of a matrix $A \in \mathbb{R}^{m\times n}$, with $\|A\|$ defined in a usual way:

$$\|A\| := \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}.$$

When handling noise terms we will frequently use a signal norm to measure size, defined as:

$$\|w\|_\infty := \sup_t \|w(t)\|_\infty .$$

For a set $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$, $PC(\mathcal{S})$ denotes the set of all piecewise continuous functions of the form $f : \mathbb{R}_+ \to \mathcal{S}$. A function $f : \mathbb{R}_+ \to \mathcal{S}$ is doubly piecewise smooth on a closed interval $[a, b] \subset \mathbb{R}$ if there exists a finite set $\{t_i\}$ having

$$a = t_1 < t_2 < \cdots < t_k = b$$

so that on each open interval (t_i, t_{i+1}), $i = 1, 2, ..., k-1$, f, $\dot{f}$, and $\ddot{f}$ are continuous, bounded, and have finite limits as $t \to t_i$ and $t \to t_{i+1}$. We say f is doubly piecewise smooth, denoted $f \in PS^1(\mathcal{S})$, if it is doubly piecewise smooth on every finite closed interval in $\mathbb{R}_+$. With $T_0 > 0$ and $\delta_\alpha > 0$, we let $PS^1(\mathcal{S}, T_0, \delta_\alpha)$ denote the set of f for which all discontinuities of $(f, \dot{f}, \ddot{f})$ are at least T_0 seconds apart and satisfy $\operatorname{ess\,sup}_{t\geq 0} \left\| \begin{bmatrix} f \\ \dot{f} \\ \ddot{f} \end{bmatrix} \right\|_\infty \leq \delta_\alpha$.

For a set $\mathcal{F} \subseteq \mathbb{R}$ with the form

$$\mathcal{F} := [\underline{f}_1, \overline{f}_1] \cup [\underline{f}_2, \overline{f}_2] \cup \cdots \cup [\underline{f}_q, \overline{f}_q],$$

having $\underline{f}_1 < \overline{f}_1 < \underline{f}_2 < \overline{f}_2 < \cdots < \underline{f}_q < \overline{f}_q$, we define a projection function $\Pi_{\mathcal{F}} : \mathbb{R} \to \mathcal{F}$ for $a \in \mathbb{R}$ as

$$\Pi_{\mathcal{F}}(a) := \begin{cases} a, & \text{if } a \in \mathcal{F}; \\ \underline{f}_1, & \text{if } a < \underline{f}_1; \\ \overline{f}_j, & \text{if } a \in (\overline{f}_j, \frac{1}{2}(\overline{f}_j + \underline{f}_{j+1})] \text{ and } j = 1, 2, ..., q-1; \\ \underline{f}_{j+1}, & \text{if } a \in (\frac{1}{2}(\overline{f}_j + \underline{f}_{j+1}), \underline{f}_{j+1}) \text{ and } j = 1, 2, ..., q-1; \\ \overline{f}_q, & \text{if } a > \overline{f}_q. \end{cases}$$

We will also take advantage of order notation within the analysis. We say $f : \mathbb{R} \to \mathbb{R}^{n\times m}$ is of order T^j, and write $f = \mathcal{O}(T^j)$, when there exist constants $c > 0$ and $T_1 > 0$ so that

$$\|f(T)\| \leq cT^j,\ T \in (0, T_1).$$

Sometimes we have a function which depends not only on T, but also on a parameter α lying in a set $\mathcal{A} \subset \mathbb{R}$. Then we say $f = \mathcal{O}(T^j)$ if there exists constants $c > 0$ and $T_1 > 0$ so that

$$\|f(T, \alpha)\| \leq cT^j,\ T \in (0, T_1),\ \alpha \in \mathcal{A}.$$

For a set $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ and a function of the form $f : \mathbb{R}_+ \to \mathcal{S}$, with a sampling period T let $f[k] := f(kT)$ for all $k \in \mathbb{Z}_+$.

Chapter 3

Problem Formulation

We consider a time-varying plant of the form

$$\dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t), \quad x(0) = x_0 \tag{3.1a}$$

$$y(t) = C(\alpha(t))x(t), \tag{3.1b}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}$ is the plant input, and $y(t) \in \mathbb{R}$ is the plant output. The plant parameters $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ are assumed to be known functions of α, where the parameter $\alpha(t)$ is unmeasureable[1], though it takes values in a known compact subset $\mathcal{A}$ of an appropriate Euclidean space. Since the case of $n = 1$ corresponds to a minimum phase plant which is well understood (see [21], [22], [23], [24], [25], and [26]), here we will assume that $n \geq 2$. The following assumption is very natural.

Assumption 1: $(A, B)(\alpha)$ is stabilizable for all $\alpha \in \mathcal{A}$, and $(C, A)(\alpha)$ is detectable for all $\alpha \in \mathcal{A}$.

We want to prove a strong exponential form of closed-loop stability. First we ascertain necessary conditions on $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ such that this is achievable. This has been studied in great detail in [38] in the simpler case of gain scheduling in which α is measurable; the conditions which are proven to be necessary there must, clearly, also be necessary here.

3.1 Necessary Conditions

In [38], Blanchini *et al.*, study the control of (3.1) when α is measurable. In Proposition 3.1 and Theorem 3.1 of [38], it is, in essence, argued that a strong exponential form of stability [2] is achievable if, and only if, it is achievable using a so-called LPV controller of the form

$$\dot{\bar{z}}(t) = \bar{F}(\alpha(t))\bar{z}(t) + \bar{G}(\alpha(t))y(t) \tag{3.2a}$$

$$u(t) = \bar{H}(\alpha(t))\bar{z}(t) + \bar{K}(\alpha(t))y(t). \tag{3.2b}$$

[1] The value of $\alpha(t)$ is not available to the control law.

[2] In [38] it is actually argued that an asymptotic form of stability is achievable if, and only if, it is achievable by a controller of the form (3.2). However, it is easy to prove that the controller (3.2) asserted to exist by Theorem 3.1 of [38] actually provides exponential stability.

If we apply this controller to (3.1) then in closed-loop we obtain:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\bar{z}}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A(\alpha(t)) + B(\alpha(t))\bar{K}(\alpha(t))C(\alpha(t)) & B(\alpha(t))\bar{H}(\alpha(t))\\ \bar{G}(\alpha(t))C(\alpha(t)) & \bar{F}(\alpha(t))\end{bmatrix}}_{A_{cl}(\alpha(t))} \underbrace{\begin{bmatrix}x(t)\\ \bar{z}(t)\end{bmatrix}}_{x_{cl}(t)}. \tag{3.3}$$

To proceed we require some definitions in order to formalize the necessary conditions and analysis.

Definition 1 (LPV Exponential Stability): *The system*

$$\dot{\bar{x}}(t) = \bar{A}(\alpha(t))\bar{x}(t), \quad \bar{x}(t_0) = \bar{x}_0 \tag{3.4}$$

(or simply $\bar{A}(\alpha(t))$) is said to be LPV exponentially stable if there exist constants $\gamma \geq 1$ and $\lambda > 0$ such that for every $t_0 \in \mathbb{R}$, $\bar{x}_0 \in \mathbb{R}^n$, and $\alpha \in PC(\mathcal{A})$, the solution of (3.4) *satisfies*

$$\|\bar{x}(t)\| \leq \gamma \mathrm{e}^{-\lambda(t-t_0)}\|\bar{x}(t_0)\|, \quad \text{for } t \geq t_0. \tag{3.5}$$

The controller (3.2) *exponentially stabilizes the plant* (3.1) *if the corresponding closed-loop system* (3.3) *is exponentially stable.*

To present Blanchini's results on control of the closed-loop system we need several additional concepts.

Definition 2 (Class $\mathcal{H}_1$): *A square matrix $H(\alpha)$ is of class $\mathcal{H}_1$ if it is a continuous function of α and if there exists a $\bar{\tau} > 0$ such that $\|I + \tau H(\alpha)\|_1 < 1$ for all $\tau \in (0, \bar{\tau})$ and $\alpha \in \mathcal{A}$.*

Definition 3 (Class $\mathcal{H}_\infty$): *A square matrix $H(\alpha)$ is of class $\mathcal{H}_\infty$ if it is a continuous function of α and if there exists a $\bar{\tau} > 0$ such that $\|I + \tau H(\alpha)\|_\infty < 1$ for all $\tau \in (0, \bar{\tau})$ and $\alpha \in \mathcal{A}$.*

Proposition 1: *(i) For every matrix $H(\alpha) \in \mathcal{H}_1$ there exist $\bar{\lambda} < 0$ and $\bar{T} > 0$ such that for all $\lambda \in (\bar{\lambda}, 0)$ and $T \in (0, \bar{T})$, the following holds:*

$$\|I + TH(\alpha)\|_1 \leq 1 + \lambda T, \quad \alpha \in \mathcal{A}. \tag{3.6}$$

(ii) For every matrix $H(\alpha) \in \mathcal{H}_\infty$ there exist $\bar{\lambda} < 0$ and $\bar{T} > 0$ such that for all $\lambda \in (\bar{\lambda}, 0)$ and $T \in (0, \bar{T})$, the following holds:

$$\|I + TH(\alpha)\|_\infty \leq 1 + \lambda T, \quad \alpha \in \mathcal{A}. \tag{3.7}$$

Proof. The proof of (*i*) is given in Appendix A of [30]. Part (*ii*) follows from part (*i*) on observing that $H(\alpha) \in \mathcal{H}_\infty \iff H(\alpha)^\top \in \mathcal{H}_1$ and $\|I + TH(\alpha)\|_\infty = \left\|I + TH(\alpha)^\top\right\|_1$. □

Proposition 2: *If $H(\alpha) \in \mathcal{H}_1$ or $H(\alpha) \in \mathcal{H}_\infty$, then $H(\alpha)$ is LPV exponentially stable.*

Proof. The proof for the case of $H(\alpha) \in \mathcal{H}_1$ is given in Appendix A of [30]. The case of $H(\alpha) \in \mathcal{H}_\infty$ follows from a slightly modified argument. $\square$

Now we turn to a key result of [38]: the first part of the result is a restatement of part of Theorem 3.1 of [38]; the second part follows from the details of its proof.

Theorem 1: *The system* (3.1) *is LPV exponentially stabilizable via an output feedback controller of the form* (3.2) *if, and only if, there exists a matrix* $P(\alpha) \in \mathcal{H}_1$*, a matrix* $Q(\alpha) \in \mathcal{H}_\infty$*, a full row-rank matrix* X*, a full column-rank matrix* R*, a row vector* $U(\alpha)$*, and a column vector* $L(\alpha)$ *such that the equations*

$$A(\alpha)X + B(\alpha)U(\alpha) = XP(\alpha) \tag{3.8}$$

$$RA(\alpha) + L(\alpha)C(\alpha) = Q(\alpha)R \tag{3.9}$$

are satisfied for all $\alpha \in \mathcal{A}$*; indeed, with* M *any left inverse of* R*, and* Z *chosen so that* $\begin{bmatrix} X \\ Z \end{bmatrix}$ *is square and invertible and* $V(\alpha) := ZP(\alpha)$*, we can choose such a stabilizing controller of the form* (3.2) *in the following way: first define*

$$\begin{bmatrix} K(\alpha) & H(\alpha) \\ G(\alpha) & F(\alpha) \end{bmatrix} := \begin{bmatrix} U(\alpha) \\ V(\alpha) \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1}, \tag{3.10}$$

and from this we obtain the controller

$$\dot{\bar{z}}(t) := \begin{bmatrix} \dot{z}(t) \\ \dot{r}(t) \end{bmatrix} = \begin{bmatrix} F(\alpha) & G(\alpha)M \\ RB(\alpha)H(\alpha) & Q(\alpha) + RB(\alpha)K(\alpha)M \end{bmatrix} \begin{bmatrix} z(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -L(\alpha) \end{bmatrix} y(t), \tag{3.11}$$

$$u(t) = \begin{bmatrix} H(\alpha) & K(\alpha)M \end{bmatrix} \begin{bmatrix} z(t) \\ r(t) \end{bmatrix}; \tag{3.12}$$

it turns out that

$$\begin{bmatrix} A(\alpha) + B(\alpha)K(\alpha) & B(\alpha)H(\alpha) \\ G(\alpha) & F(\alpha) \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} P(\alpha) \begin{bmatrix} X \\ Z \end{bmatrix}^{-1}. \tag{3.13}$$

In light of Theorem 1, we impose the following assumption.

Assumption 2: There exists a matrix $P(\alpha) \in \mathcal{H}_1$, a matrix $Q(\alpha) \in \mathcal{H}_\infty$, a full row-rank matrix X, a full column-rank matrix R, a row vector $U(\alpha)$, and a column vector $L(\alpha)$ such that (3.8)–(3.9) hold for all $\alpha \in \mathcal{A}$.

At this point Z is fixed so that $\begin{bmatrix} X \\ Z \end{bmatrix}$ is non-singular, and we fix a matrix M to be any left inverse of R.

3.2 Additional Assumptions

From the previous section we see that stabilizing a time-varying system is difficult, even when the free parameter is known. The difficulty arises in a very subtle way from the existence of unstable zero dynamics, since it is well known that if the zero dynamics are stable then stabilizing in the face of rapid time-variation is possible, e.g., see [22], [26], [2], [3], [7]. Because of the difficulty of the problem and the lack of general results, in this paper we impose a major structural assumption: we have one degree of freedom in α – it is a scalar; we also require several technical assumptions.

Assumption 3: $\mathcal{A}$ is a compact subset of $\mathbb{R}$, consisting of a finite set of closed intervals.

We will estimate α using ideas from [39]; indeed, we assume that it can be obtained, roughly speaking, from the plant's first Markov parameter. To this end, we define

$$\begin{aligned} f : \mathcal{A} &\to \mathbb{R} \\ \alpha &\mapsto C(\alpha)B(\alpha), \end{aligned}$$

as well as the image of $\mathcal{A}$ under f:

$$\mathcal{F} := f(\mathcal{A}).$$

At this point we impose the second major structural assumption.

Assumption 4: The function $f : \mathcal{A} \to \mathcal{F}$ is one-to-one and its inverse f^{-1} is Lipschitz continuous on $\mathcal{F}$.

Now we turn to more routine regularity assumptions needed to prove that our approach will work. First of all, in this paper we will be constructing a sampled-data controller, which means that $\alpha(t)$ cannot move arbitrarily fast. However we can allow an occasional jump, so we will now impose the following assumption.

Assumption 5: There exist constants $T_0 > 0$ and $\delta_\alpha > 0$ such that $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$.

Next, we assume that A, B, and C are well-behaved as functions of the parameter α.

Assumption 6: $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $\frac{\mathrm{d}C(\alpha)}{\mathrm{d}\alpha}$ are Lipschitz continuous on $\mathcal{A}$.

We impose similar conditions on $P(\alpha)$ and $Q(\alpha)$ of Assumption 2:

Assumption 7: $P(\alpha)$ and $Q(\alpha)$ are Lipschitz continuous on $\mathcal{A}$.

Remark 1: *Assumptions 2, 6, and 7 imply that, if $B(\alpha)$ is full column rank and $C(\alpha)$ is full row rank, then $U(\alpha)$ and $L(\alpha)$ are also Lipschitz continuous on $\mathcal{A}$. It follows from Theorem 1 that the corresponding controller matrices $F(\alpha)$, $G(\alpha)$, $H(\alpha)$, and $K(\alpha)$ are also Lipschitz continuous on $\mathcal{A}$.*

In a realistic situation, the plant is subjected to disturbances from the environment. If we define w_d to be disturbance injected into the plant, and w_n to be measurement noise, then the revised model of the plant is

$$\dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t) + w_d(t), \quad x(0) = x_0, \tag{3.14a}$$
$$y(t) = C(\alpha(t))x(t) + w_n(t). \tag{3.14b}$$

We represent the plant model (3.14) by the triple $(A(\alpha), B(\alpha), C(\alpha))$. From this point on we fix the plant matrices $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ as functions of α. Furthermore, we fix choices of $\mathcal{A}$, T_0, and δ_α. Finally, we require that Assumptions 1 to 7 hold for $(A(\alpha), B(\alpha), C(\alpha))$.

The goal of this paper is to develop a controller that stabilizes $(A(\alpha), B(\alpha), C(\alpha))$ for every $x_0 \in \mathbb{R}^n$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, and $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$ when only the plant output y is measureable.

Chapter 4

The Controller

Here, we present the proposed adaptive controller and prove three key lemmas which are essential to the proof of the main result.

4.1 The Approach

We would like to stabilize the plant (3.14) when α is not measurable. We propose a nonlinear periodic controller to achieve this goal. It operates at a period, T, and consists of several components:

- a continuous-time filter which is used to provide an upper bound on $\|x(t)\|$ and scale a probing signal;
- a discretized version of the gain-scheduled output feedback controller (3.11)–(3.12) with states $z(t)$ and $r(t)$ and α replaced by an estimate $\hat{\alpha}$;
- a sampled-data parameter estimator of $\alpha(t)$ which produces an estimate $\hat{\alpha}[k]$ for use on the control interval $[kT, (k+1)T)$.

We have a base sampling period of $h = \frac{T}{2}$. A block diagram of the closed-loop system is depicted in Figure 4.1.

4.2 The Filter

By Proposition 1, for each $Q(\alpha) \in \mathcal{H}_\infty$ there exists a constant $\lambda < 0$ so that, for sufficiently small T,

$$\|I + TQ(\alpha)\|_\infty \leq 1 + \lambda T, \;\; \alpha \in \mathcal{A}.$$

A method for computing λ is given in the proof of [30, Proposition 1]: letting $q_{ij}(\alpha)$ denote the (i,j)th element of $Q(\alpha)$, define

$$\lambda^* := -\min_i \left| q_{ii}(\alpha) + \sum_{j=1, j\neq i}^{m} |q_{ij}(\alpha)| \right|,$$

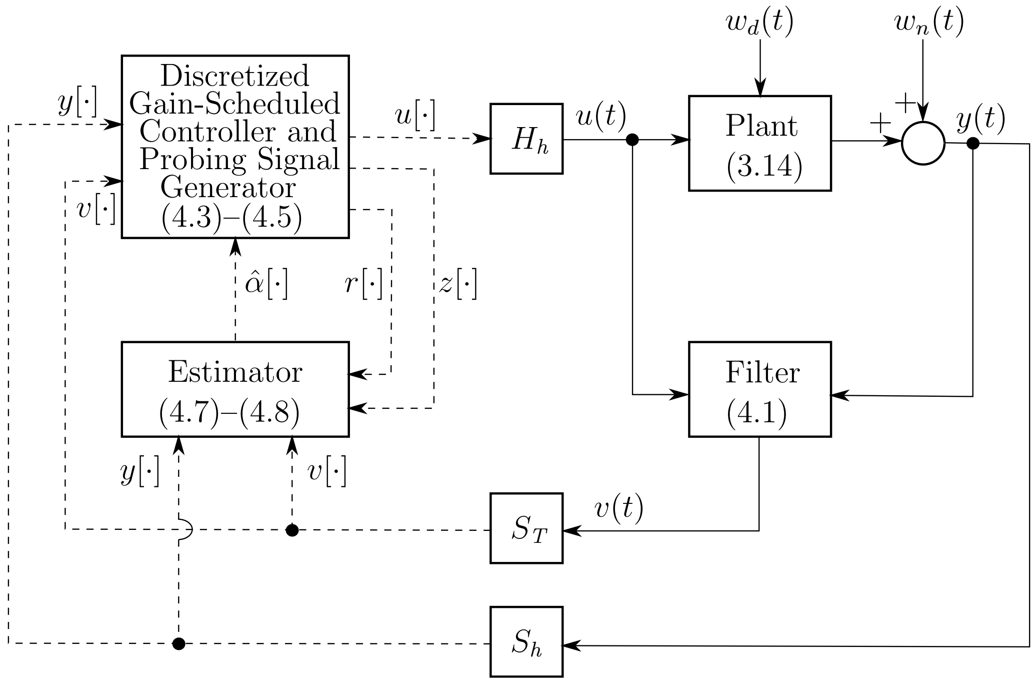

Figure 4.1: Closed-loop system block diagram.

and then choose $\lambda \in (\lambda^*, 0)$. The proposed filter is

$$\dot{v}(t) = \lambda v(t) + \|u(t)\| + \|y(t)\|, \qquad v(0) = 0. \tag{4.1}$$

It turns out that $v(t)$ provides an upper bound on the size of the state.

Lemma 1: *Consider the filter* (4.1) *driven by the input and output of the plant* (3.14)*. There exists a constant $c > 0$ so that for every $u \in PC_\infty$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x_0 \in \mathbb{R}^n$, the plant state satisfies*

$$\|x(t)\| \leq c\mathrm{e}^{\lambda t}\|x(0)\| + cv(t) + c\|w_n\|_\infty + c\|w_d\|_\infty, \quad t \geq 0. \tag{4.2}$$

Proof. The proof of Lemma 1 is in Appendix A.1. □

4.3 The Discretized Gain-Scheduled Controller

With $v(t)$ defined in (4.1) and with $\hat{\alpha}[k]$ denoting an estimate of $\alpha(kT)$ to be defined shortly, we use a suitably modified discretized version of the LPV controller (3.11)–(3.12). The state equation is

$$\begin{bmatrix} z[k+1] \\ r[k+1] \end{bmatrix} = \left(I + T \begin{bmatrix} F(\hat{\alpha}[k]) & G(\hat{\alpha}[k])M \\ RB(\hat{\alpha}[k])H(\hat{\alpha}[k]) & Q(\hat{\alpha}[k]) + RB(\hat{\alpha}[k])K(\hat{\alpha}[k])M \end{bmatrix} \right) \begin{bmatrix} z[k] \\ r[k] \end{bmatrix}$$

$$-T \begin{bmatrix} 0 \\ L(\hat{\alpha}[k]) \end{bmatrix} y[k], \qquad \begin{bmatrix} z(0) \\ r(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

With $\rho \in (0, -\lambda)$, we define a probing signal of the form

$$\delta(t) := \begin{cases} \rho\big(v[k] + \|z[k]\| + \|r[k]\|\big) & t \in [kT, kT+h) \\ -\rho\big(v[k] + \|z[k]\| + \|r[k]\|\big) & t \in [kT+h, kT+2h), \end{cases} \tag{4.4}$$

which we add to the discretized version of the output equation of the LPV controller (3.11)–(3.12) passed through a zero-order hold, yielding

$$u(t) = H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] + \delta(t), \ \ t \in [kT, (k+1)T). \tag{4.5}$$

4.4 The Parameter Estimator

To motivate the choice of parameter estimator, we examine the simplest relative degree one system with $a \neq 0$:

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) + w_d(t) \\ y(t) &= cx(t) + w_n(t). \end{aligned}$$

The goal here is to estimate the first Markov parameter, cb. With $\delta > 0$, if we set

$$u(t) = \begin{cases} \delta & t \in [0, h) \\ -\delta & t \in [h, 2h), \end{cases}$$

then

$$\begin{aligned} x(h) &= e^{ah}x(0) + \frac{b\delta}{a}(e^{ah} - 1) + \int_0^h e^{a(h-\tau)} w_d(\tau) d\tau, \\ x(2h) &= e^{ah}x(h) - \frac{b\delta}{a}(e^{ah} - 1) + \int_h^{2h} e^{a(2h-\tau)} w_d(\tau) d\tau \\ &= e^{2ah}x(0) + \frac{b\delta}{a}(1 - 2e^{ah} + e^{2ah}) + \int_0^{2h} e^{a(2h-\tau)} w_d(\tau) d\tau. \end{aligned}$$

So it follows that

$$\begin{aligned} y(2h) - 2y(h) + y(0) &= c(1 - 2\mathrm{e}^{ah} + \mathrm{e}^{2ah})x(0) + \frac{cb\delta}{a}(3 - 4\mathrm{e}^{ah} + \mathrm{e}^{2ah}) \\ &\quad + c\int_0^{2h} \mathrm{e}^{a(2h-\tau)} w_d(\tau)d\tau \\ &\quad - 2c\int_0^{h} \mathrm{e}^{a(h-\tau)} d\tau + [w_n(2h) - 2w_n(h) + w_n(0)] \end{aligned}$$

Hence,

$$\begin{aligned} \left\| cb - \frac{1}{2h\delta}\Big(- y(2h) + 2y(h) - y(0)\Big)\right\| &= \mathcal{O}(h)\left(\left\| \frac{x(0)}{\delta} \right\| + 1 \right) \\ &\quad + \frac{1}{\delta}\mathcal{O}(1)\left\| w_d \right\|_\infty + \frac{1}{\delta}\mathcal{O}(h^{-1})\left\| w_n \right\|_\infty . \end{aligned} \tag{4.6}$$

So if the last three terms of the RHS are small, then the LHS provides a good estimate of cb. This simple discussion motivates the choice of the estimate in our case where the plant is higher order, the parameter α is time-varying, and the control signal is more complicated. More specifically, with the probing signal defined above in (4.4), we define the estimate of the first Markov parameter by

$$\widehat{CB}[k+1] := \begin{cases} \frac{-y(kT+2h)+2y(kT+h)-y(kT)}{2h\delta(kT)} & \text{if } \delta[k] \neq 0 \\ C(\underline{\alpha})B(\underline{\alpha}) & \text{if } \delta[k] = 0, \end{cases} \tag{4.7}$$

where $\underline{\alpha} := \min\{a : a \in \mathcal{A}\}$, which is well-defined because $\mathcal{A}$ is compact. We then use the estimate of the Markov parameter to form the estimate of $\alpha((k+1)T)$, which we label $\hat{\alpha}[k+1]$:

$$\hat{\alpha}[k+1] := f^{-1}\Big(\Pi_{\mathcal{F}}(\widehat{CB}[k+1])\Big). \tag{4.8}$$

There are two possible stumbling blocks in the estimation procedure:

(i) If the probing signal $\delta[k]$ is small relative to the size of $\|x[k]\|$, then the estimate error term of size $\frac{\|x[k]\|}{|\delta[k]|}$ (see (4.6)) will be large and the estimate may be inaccurate.

(ii) If the probing signal $\delta[k]$ is small relative to $\|w_d\|_\infty$ and to $T^{-1}\|w_n\|_\infty$, then the estimate may also be inaccurate.

In the following result we explain how to avoid these problems.

Lemma 2: *For every $\epsilon > 0$ and $\overline{\delta} > 0$, there exist constants $c > 0$ and $T_1 > 0$ so that, for every $T \in (0, T_1)$, $k \in \mathbb{Z}_+$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x_0 \in \mathbb{R}^n$, when the controller given by (4.1), (4.3)–(4.5), and (4.7)–(4.8) is applied to the plant (3.14), if*

(i) $v[k] + \|z[k]\| + \|r[k]\| > \epsilon \|x[k]\|$,

(ii) $v[k] + \|z[k]\| + \|r[k]\| > c\big(T^{-1} \|w_n\|_\infty + \|w_d\|_\infty\big)$, *and*

(iii) $\alpha(t)$ *is absolutely continuous for* $t \in [kT, (k+1)T]$,

then

$$\|\alpha((k+1)T) - \hat{\alpha}[k+1]\| \le \overline{\delta}. \tag{4.9}$$

Proof. The proof of Lemma 2 is in Appendix A.2. □

Hence if the probing signal is large relative to the plant state and the noise, and if $\alpha(t)$ is absolutely continuous on $[kT, (k+1)T]$, then the estimate $\hat{\alpha}[k+1]$ of $\alpha[k+1]$ will be accurate, in which case the discretized LPV controller (4.3)–(4.5) should perform well over the interval $[(k+1)T, (k+2)T)$ (observe that the effect of the probing signal approximately cancels out over an interval). On the other hand, if any of these conditions fail, then the estimate $\hat{\alpha}[k+1]$ may be inaccurate, so the proposed controller may yield an inappropriate control signal on the following interval; however, condition (iii) fails infrequently, while condition (ii) fails only if the controller state is small relative to the size of the noise, so this case should turn out to be unimportant. The tricky condition (i) will be the problematic one, but it will be carefully handled in the proof of the main result.

Before we get to this, however, we wish to rigorously prove that if the estimate $\hat{\alpha}[k+1]$ is accurate, then the closed-loop system will behave well on the following interval $[(k+1)T, (k+2)T)$. To facilitate analysis, we first transform the plant state x and controller states z and r using the approach adopted in the proof of Theorem 1 of [38]: we define $z(t) := z[k]$ and $r(t) := r[k]$ for $t \in [kT, (k+1)T)$, and we define the transformed states

$$\begin{bmatrix} \bar{x}(t) \\ \bar{z}(t) \end{bmatrix} := \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \qquad s(t) := Rx(t) - r(t). \tag{4.10}$$

At this point we depart from a standard analysis by choosing a norm in just the right way to prove that the closed-loop state is contractive. To this end, it turns out that the system matrix in $\left(\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}, s \right)$-coordinates is upper block triangular with the $(1,1)$ block being in $\mathcal{H}_1$ and the $(2,2)$ block being in $\mathcal{H}_\infty$. So it is natural to use a 1-norm on $\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}$ and an ∞-norm on s. Now we are ready to state a result of the closed-loop behaviour over a period.

Lemma 3: *There exist constants* $T_2 > 0$, $\hat{\lambda} < 0$, $\overline{\delta} > 0$, *and* $c > 0$ *together with an invertible matrix* $N \in \mathbb{R}^{3\times 3}$ *so that with*

$$p(t) := N \begin{bmatrix} v(t) \\ \left\| \begin{bmatrix} \bar{x}(t) \\ \bar{z}(t) \end{bmatrix} \right\|_1 \\ \|s(t)\|_\infty \end{bmatrix},$$

for every $T \in (0, T_2)$, $k \in \mathbb{Z}_+$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, *and* $x_0 \in \mathbb{R}^n$, *when the controller given by* (4.1), (4.3)–(4.5), *and* (4.7)–(4.8) *is applied to the plant* (3.14), *with* $p[k] := p(kT)$ *we have:*

(i) In all cases,

$$\|p(t) - p[k]\|_\infty \le cT \|p[k]\|_\infty + cT \|w_n\|_\infty + cT \|w_d\|_\infty, \quad t \in [kT, (k+1)T). \quad (4.11)$$

(ii) In all cases,

$$\|p[k+1]\|_\infty \le (1 + cT) \|p[k]\|_\infty + cT \|w_n\|_\infty + cT \|w_d\|_\infty. \quad (4.12)$$

(iii) If $\|\hat{\alpha}[k] - \alpha(kT)\| \le \overline{\delta}$ *and if* $\alpha(t)$ *is absolutely continuous for* $t \in [kT, (k+1)T)$, *then*

$$\|p[k+1]\|_\infty \le \mathrm{e}^{\hat{\lambda} T} \|p[k]\|_\infty + cT \|w_n\|_\infty + cT \|w_d\|_\infty. \quad (4.13)$$

Proof. The proof of Lemma 3 is in Appendix A.3. □

Chapter 5

The Main Result

In Lemma 3 we prove that the closed-loop system is well behaved on intervals for which the estimate of $\alpha(t)$ is accurate. Now we will leverage Lemmas 1–3 to prove that we obtain desirable closed-loop behaviour on all of $t \geq 0$. In the following, recall that the initial conditions on the controller states $v(t)$, $z(t)$, and $r(t)$ are zero.

Theorem 2: *There exists contants $T_3 > 0$, $\bar{\lambda} < 0$, and $c > 0$ so that for every $T \in (0, T_3)$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x_0 \in \mathbb{R}^n$, when the controller given by* (4.1)*,* (4.3)*–*(4.5)*, and* (4.7)*–*(4.8) *is applied to the plant* (3.14)*, we have that*

$$\left\| \begin{bmatrix} v(t) \\ x(t) \\ z(t) \\ r(t) \end{bmatrix} \right\| \leq c\mathrm{e}^{\bar{\lambda}t} \left\| \begin{bmatrix} 0 \\ x(0) \\ 0 \\ 0 \end{bmatrix} \right\| + cT^{-1} \left\| w_n \right\|_\infty + c \left\| w_d \right\|_\infty . \tag{5.1}$$

Proof.

Step 1: Bad Estimation

Let $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x_0 \in \mathbb{R}^n$ be arbitrary. Let $c_1 > 0$ be the constant asserted to exist by Lemma 1. Now fix $\epsilon \in (0, \frac{1}{2c_1})$. Let $T_2 > 0$, $\hat{\lambda} < 0$, $\bar{\delta} > 0$, and $c_3 > 0$ be the constants asserted to exist by Lemma 3. Last of all, let $c_2 > 0$ and $T_1 > 0$ be the constants asserted to exist by Lemma 2.

Let $T \in (0, \max\{T_1, T_2\})$ be arbitrary. To proceed, we define two sets of intervals. The first represents times for which the probing signal is too small to ensure a good estimate of $\alpha(t)$, while the second represents times for which the probing signal is of adequate size but is overwhelmed by the noise:

$$\begin{aligned}
B_1(T) \;&:= \Big\{ t \geq 0 : \;\; v(t) + \|z(t)\| + \|r(t)\| \leq \epsilon \|x(t)\| \;\Big\}, \\
B_2(T) \;&:= \Big\{ t \geq 0 : \;\; v(t) + \|z(t)\| + \|r(t)\| > \epsilon \|x(t)\| \,, \\
&\qquad\qquad\quad v(t) + \|z(t)\| + \|r(t)\| \leq c_2 \Big(T^{-1} \|w_n\|_\infty + \|w_d\|_\infty \Big) \Big\},
\end{aligned}$$

and

$$B(T) := B_1(T) \cup B_2(T);$$

notice that $B(T)$ is not empty, since $0 \in B_1(T)$. Although we expect that the estimate of $\alpha(t)$ will be poor on $B(T)$, quite surprisingly, we are still able to obtain a desirable bound on the state. This is because either:

(i) the controller states are small compared to the plant state, so the filter provides an exponential bound ($B_1(T)$), or

(ii) the noise is large compared to the closed-loop state ($B_2(T)$), so it provides a bound on the state.

Claim 1: *There exists a constant $c_4 > 0$, so that*

$$\|p(t)\| \leq c_4 \mathrm{e}^{\lambda t} \|p(0)\| + c_4 T^{-1} \|w_n\|_\infty + c_4 \|w_d\|_\infty \,, \quad t \in B(T). \tag{5.2}$$

Proof of Claim 1.

First we consider the case of $t \in B_1(T)$. By Lemma 1 it follows that

$$\begin{aligned}
\|x(t)\| \;&\leq\; c_1 \mathrm{e}^{\lambda t} \|x(0)\| + c_1 v(t) + c_1 \|w_n\|_\infty + c_1 \|w_d\|_\infty \\
&\leq\; c_1 \mathrm{e}^{\lambda t} \|x(0)\| + c_1 \epsilon \|x(t)\| + c_1 \|w_n\|_\infty + c_1 \|w_d\|_\infty \,;
\end{aligned}$$

since $c_1 \epsilon \in (0, \frac{1}{2})$, this yields

$$\begin{aligned}
\|x(t)\| \;&\leq\; \frac{c_1}{1 - c_1 \epsilon} \mathrm{e}^{\lambda t} \|x(0)\| + \frac{c_1}{1 - c_1 \epsilon} \|w_n\|_\infty + \frac{c_1}{1 - c_1 \epsilon} \|w_d\|_\infty \\
&\leq\; 2 c_1 \mathrm{e}^{\lambda t} \|x(0)\| + 2 c_1 \|w_n\|_\infty + 2 c_1 \|w_d\|_\infty \,.
\end{aligned}$$

Additionally, for $t \in B_1(T)$, we have that

$$\begin{aligned}
v(t) + \|z(t)\| + \|r(t)\| \;&\leq\; \epsilon \|x(t)\| \leq 2 c_1 \epsilon \mathrm{e}^{\lambda t} \|x(0)\| + 2 c_1 \epsilon \|w_n\|_\infty + 2 c_1 \epsilon \|w_d\|_\infty \\
&\leq\; \mathrm{e}^{\lambda t} \|x(0)\| + \|w_n\|_\infty + \|w_d\|_\infty \,.
\end{aligned}$$

So there exists a constant $c_5 > 0$ so that

$$\|p(t)\| \leq c_5 \mathrm{e}^{\lambda t} \|p(0)\| + c_5 \|w_n\|_\infty + c_5 \|w_d\|_\infty \,, \quad t \in B_1(T). \tag{5.3}$$

Now we consider the case of $t \in B_2(T)$. By definition of $B_2(T)$ it is clear that

$$\epsilon \|x(t)\| < v(t) + \|z(t)\| + \|r(t)\| \leq c_2 T^{-1} \|w_n\|_\infty + c_2 \|w_d\|_\infty ,$$

so there exists a constant $c_6 > 0$ such that

$$\begin{aligned} \|p(t)\| &\leq c_6 T^{-1} \|w_n\|_\infty + c_6 \|w_d\|_\infty \\ &\leq c_6 \mathrm{e}^{\lambda t} \|p(0)\| + c_6 T^{-1} \|w_n\|_\infty + c_6 \|w_d\|_\infty , \quad t \in B_2(T). \end{aligned} \tag{5.4}$$

If we combine (5.3) and (5.4), and define $c_4 := \max\{c_5, c_6, c_5 T_2\}$, then the result follows.

□

Step 2: Good Estimation

Now define the remaining set of time as

$$\begin{aligned} G(T) &:= [0,\infty) \setminus B(T) \\ &= \Big\{ t \geq 0 : \quad v(t) + \|z(t)\| + \|r(t)\| > \epsilon \|x(t)\| , \\ &\qquad v(t) + \|z(t)\| + \|r(t)\| > c_2 \Big(T^{-1} \|w_n\|_\infty + \|w_d\|_\infty \Big) \Big\}; \end{aligned}$$

on this set, we expect, roughly speaking, that the estimate of $\alpha(t)$ will be accurate so long as $\alpha(t)$ is smooth.

If $G(T)$ is empty, then Claim 1 provides the desired bound. Now suppose that $G(T)$ is non-empty; then we can write it as a disjoint union of open intervals, possibly an infinite number of them; we will write them as $(t_1, t_2), (t_3, t_4), \ldots$, with $\{t_i\}$ strictly increasing, which we express concisely as $\{(t_i, t_{i+1}) : i \in \mathcal{S}\}$ with $\mathcal{S} := \{n \in \mathbb{N} : n \text{ odd}\}$. $G(T)$ is trickier to handle than $B(T)$, with potential issues being the initial partial periods and intervals containing parameter jumps.

The parameter estimator requires a full period in order to return an accurate estimate of $\alpha(t)$. Furthermore, each interval (t_i, t_{i+1}), $i \in \mathcal{S}$, will always contain a partial interval of the form $[kT, (k+1)T]$ at the beginning and possibly at the end. This leads us to remove an interval from each end of (t_i, t_{i+1}) and define an associated discrete-time index as follows: with i odd, define

$$k_i(T) := \operatorname{int}\Big(\frac{t_i}{T}\Big) + 2$$

and

$$k_{i+1}(T) := \operatorname{int}\Big(\frac{t_{i+1}}{T}\Big).$$

First we obtain a bound on the initial part of the interval (t_i, t_{i+1}), namely $[t_i, k_i(T)T]$.

Claim 2: *There exist constants $T_4 \in (0, \min\{T_1, T_2\})$ and $c_7 > 0$ such that, for all $T \in (0, T_4)$ and $i \in \mathcal{S}$:*

$$\|p(t)\| \leq c_7 \|p(t_i)\| + c_7 T \|w_n\|_\infty + c_7 T \|w_d\|_\infty , \quad t \in [t_i, k_i(T)T]. \tag{5.5}$$

Proof of Claim 2.

Let $T \in (0, \min\{T_1, T_2\})$. From Lemma 3(i)

$$\begin{aligned} \|p(t) - p[k_i(T) - 2]\| &\leq c_3 T \|p[k_i(T) - 2]\| + c_3 T(\|w_n\|_\infty + \|w_d\|_\infty), \\ & \qquad t \in [(k_i(T) - 2)T, (k_i(T) - 1)T], \end{aligned} \tag{5.6}$$

$$\begin{aligned} \|p(t) - p[k_i(T) - 1]\| &\leq c_3 T \|p[k_i(T) - 1]\| + c_3 T(\|w_n\|_\infty + \|w_d\|_\infty), \\ & \qquad t \in [(k_i(T) - 1)T, k_i(T)T]. \end{aligned} \tag{5.7}$$

We know that $t_i \in ((k_i(T) - 2)T, (k_i(T) - 1)T)$; we'd like to use (5.6) to obtain a bound on $p(t)$ in terms of $p(t_i)$. If we evaluate the LHS of (5.6) at $t = t_i$ and rearrange, we see that

$$(1 - c_3 T) \|p[k_i(T) - 2]\| \leq \|p(t_i)\| + c_3 T(\|w_n\|_\infty + \|w_d\|_\infty),$$

so if we define $T_4 := \min\{T_1, T_2, \frac{1}{2c_3}\}$, we see that

$$\|p[k_i(T) - 2]\| \leq 2 \|p(t_i)\| + 2c_3 T \|w_n\|_\infty + 2c_3 T \|w_d\|_\infty .$$

If we now combine this with (5.6) and (5.7), then the result follows easily. □

Now we need a bound on $\|p(t)\|$ for $t \in [k_i(T)T, t_{i+1})$. If this interval is empty, then $t_{i+1} - t_i \leq 2T$ so we can combine Claims 1 and 2 to yield

$$\begin{aligned} \|p(t)\| &\leq c_7 c_4 e^{\lambda t_i} \|p(0)\| + (c_7 c_4 T^{-1} + c_7 T) \|w_n\|_\infty + (c_7 c_4 + c_7 T) \|w_d\|_\infty \\ &\leq c_7 c_4 e^{-2\lambda T} e^{\lambda t} \|p(0)\| + (c_7 c_4 T^{-1} + c_7 T) \|w_n\|_\infty + (c_7 c_4 + c_7 T) \|w_d\|_\infty , t \in [t_i, t_{i+1}], \end{aligned}$$

so there exists a constant $c_8 > 0$ such that

$$\|p(t)\| \leq c_8 e^{\lambda t} \|p(0)\| + c_8 T^{-1} \|w_n\|_\infty + c_8 \|w_d\|_\infty , t \in [t_i, t_{i+1}].$$

Now suppose $[k_i(T), t_{i+1})$ is non-empty; this means that $k_{i+1}(T) \geq k_i(T)$. For every $k \in [k_i(T), k_{i+1}(T) + 1]$ we have

$$v[k-1] + \|z[k-1]\| + \|r[k-1]\| \quad > \quad \epsilon \|x[k-1]\| ,$$

and

$$v[k-1] + \|z[k-1]\| + \|r[k-1]\| \quad > \quad c_2 \Big(T^{-1} \|w_n\|_\infty + \|w_d\|_\infty \Big),$$

so by Lemma 2 and Lemma 3:

a) if $\alpha(t)$ is absolutely continuous for $t \in [(k-1)T, (k+1)T)$, then $|\alpha(kT) - \hat{\alpha}[k]| \leq \overline{\delta}$, so

$$\|p[k+1]\| \leq e^{\hat{\lambda} T} \|p[k]\| + c_3 T \|w_n\|_\infty + c_3 T \|w_d\|_\infty ; \tag{5.8}$$

b) if $\alpha(t)$ is not absolutely continuous for $t \in [(k-1)T, (k+1)T)$, then

$$\|p[k+1]\| \leq (1 + c_3 T) \|p[k]\| + c_3 T \|w_n\|_\infty + c_3 T \|w_d\|_\infty . \tag{5.9}$$

We can now bound the closed-loop state for $t \in [k_i(T)T, (k_{i+1}(T)+1)T)$; notice that $(k_{i+1}(T)+1)T \geq t_{i+1}$, with equality if and only if $t_{i+1} = \infty$.

Claim 3: *For every $\bar{\lambda} \in (\hat{\lambda}, 0)$, there exist constants $T_3 \in (0, T_4)$ and $c_9 > 0$ so that for all $T \in (0, T_3)$ and $i \in \mathcal{S}$, the following holds:*

$$\|p(t)\| \leq c_9 \mathrm{e}^{\bar{\lambda}(t-k_i(T)T)} \|p[k_i(T)]\| + c_9 \|w_n\|_\infty + c_9 \|w_d\|_\infty, \quad t \in [k_i(T)T, (k_{i+1}(T)+1)T). \tag{5.10}$$

Proof of Claim 3.

Fix $\bar{\lambda} \in (\hat{\lambda}, 0)$ and let $T \in (0, T_4)$, $i \in \mathcal{S}$, and $k \in [k_i(T), k_{i+1}(T)]$ be arbitrary. From (5.8) and (5.9) we see that

$$\|p[k+1]\| \leq \begin{cases} \mathrm{e}^{\hat{\lambda}T} \|p[k]\| + c_3 T \|w_n\|_\infty + c_3 T \|w_d\|_\infty & \text{if } \alpha(t) \text{ is a.c. on } [(k-1)T, (k+1)T), \\ (1 + c_3 T) \|p[k]\| + c_3 T \|w_n\|_\infty + c_3 T \|w_d\|_\infty & \text{otherwise.} \end{cases} \tag{5.11}$$

This gives rise to a time-varying gain

$$a[k] = \begin{cases} \mathrm{e}^{\hat{\lambda}T} & \text{if } \alpha(t) \text{ is a.c. on } [(k-1)T, (k+1)T), \\ (1 + c_3 T) & \text{otherwise,} \end{cases}$$

with the corresponding state-transition function labelled Φ. Discontinuities in $\alpha(t)$ are spaced by at least T_0 seconds, so in the time interval $[k_i(T)T, kT]$ there can be at most $\lceil \frac{(k-k_i(T))T}{T_0} \rceil$ parameter jumps; this means there are at most $2\lceil \frac{(k-k_i(T))T}{T_0} \rceil$ values of k for which $a[k] \neq \mathrm{e}^{\hat{\lambda}T}$. Because $\Phi(k, k_i(T)) = \prod_{m=k_i(T)}^{k-1} a[m]$, it follows that

$$\begin{aligned} \|\Phi(k, k_i(T))\| &\leq (1 + c_3 T)^{2\lceil \frac{(k-k_i(T))T}{T_0} \rceil} \mathrm{e}^{\hat{\lambda}T\left(k - k_i(T) - 2\lceil \frac{(k-k_i(T))T}{T_0} \rceil\right)} \\ &= \left[(1 + c_3 T)^2 \mathrm{e}^{-2\hat{\lambda}T}\right]^{\lceil \frac{(k-k_i(T))T}{T_0} \rceil} \mathrm{e}^{\hat{\lambda}T(k-k_i(T))}, \quad k = k_i(T), ..., k_{i+1}(T) + 1. \end{aligned} \tag{5.12}$$

But $\lceil \frac{(k-k_i(T))T}{T_0} \rceil < \frac{(k-k_i(T))T}{T_0} + 1$, so

$$\|\Phi(k, k_i(T))\| \leq \left[(1 + c_3 T)^2 \mathrm{e}^{-2\hat{\lambda}T}\right]^{\frac{(k-k_i(T))T}{T_0} + 1} \mathrm{e}^{\hat{\lambda}T(k-k_i(T))}, \quad k = k_i(T), ..., k_{i+1}(T) + 1. \tag{5.13}$$

We'd like to simplify the first term on the RHS. To do this we first fix $\bar{\lambda} \in (\hat{\lambda}, 0)$; we claim that there exists a constant $c_{10} > 0$ such that

$$\|\Phi(k, k_i(T))\| \leq c_{10} \mathrm{e}^{\bar{\lambda}T(k-k_i(T))}, \quad k = k_i(T), ..., k_{i+1}(T) + 1. \tag{5.14}$$

This will be the case if c_{10} satisfies

$$\left[(1 + c_3 T)^2 \mathrm{e}^{-2\hat{\lambda}T}\right]^{\frac{(k-k_i(T))T}{T_0} + 1} \mathrm{e}^{\hat{\lambda}T(k-k_i(T))} \leq c_{10} \mathrm{e}^{\bar{\lambda}T(k-k_i(T))}, \quad k = k_i(T), ..., k_{i+1}(T) + 1,$$

which will hold if

$$c_{10} \geq \left[(1+c_3T)^2 \mathrm{e}^{-2\hat{\lambda}T}\right] \mathrm{e}^{\left(\hat{\lambda}-\bar{\lambda}+\frac{1}{T_0}\ln\left((1+c_3T)^2\mathrm{e}^{-2\hat{\lambda}T}\right)\right)(k-k_i(T))T}, \quad k = k_i(T), ..., k_{i+1}(T)+1.$$

We have $\hat{\lambda} - \bar{\lambda} < 0$, so it is clear that we can choose $T_3 \in (0, \min\{T_4, 1\})$ sufficiently small so that

$$\hat{\lambda} - \bar{\lambda} + \frac{1}{T_0} \ln\left((1+c_3T)^2 \mathrm{e}^{-2\hat{\lambda}T}\right) \leq 0, \quad T \in (0, T_3),$$

and then set $c_{10} = (1+c_3T_3)^2 \mathrm{e}^{-2\hat{\lambda}T_3}$.

Using the upper bound on the size of Φ given in (5.14) we can now analyze the difference inequality (5.11) and obtain an upper bound on the size of p:

$$\begin{aligned} \|p[k]\| &\leq \|\Phi(k, k_i(T))\| \times \|p[k_i(T)]\| + \Big(\sum_{m=k_i(T)}^{k-1} \|\Phi(k-1, m)\|\Big)\left(c_3T\|w_n\|_\infty + c_3T\|w_d\|_\infty\right) \\ &\leq c_{10}\mathrm{e}^{\bar{\lambda}T(k-k_i(T))}\|p[k_i(T)]\| + c_{10}\Big(\sum_{m=0}^{k-(k_i(T)+1)} \mathrm{e}^{\bar{\lambda}Tm}\Big)\left(c_3T\|w_n\|_\infty + c_3T\|w_d\|_\infty\right), \end{aligned}$$

for $k = k_i(T), ..., k_{i+1}(T)+1$. We can obtain an upper bound on the summation with

$$\sum_{m=0}^{k-(k_i(T)+1)} \mathrm{e}^{\bar{\lambda}Tm} \leq \sum_{m=0}^{\infty} (\mathrm{e}^{\bar{\lambda}T})^m = \frac{1}{1-\mathrm{e}^{\bar{\lambda}T}} = \mathcal{O}(T^{-1}),$$

so there exists a constant $c_{11} > 0$ such that

$$\begin{aligned} \|p[k]\| \;\leq\; & c_{11}\mathrm{e}^{\bar{\lambda}T(k-k_i(T))}\|p[k_i(T)]\| + c_{11}\|w_n\|_\infty + c_{11}\|w_d\|_\infty, \\ & k = k_i(T), ..., k_{i+1}(T)+1. \end{aligned} \tag{5.15}$$

Finally, by Lemma 3(i), we have that

$$\|p(t)\| \leq (1+c_3T)c_{11}\mathrm{e}^{\bar{\lambda}T(k-k_i(T))}\|p[k_i(T)]\| + \left((1+c_3T)c_{11} + c_3T\right)(\|w_n\|_\infty + \|w_d\|_\infty)$$

for $t \in [k_i(T)T, (k_{i+1}(T)+1)T)$, so there exists a constant $c_9 > 0$ such that for all $T \in (0, T_3)$:

$$\|p(t)\| \;\leq\; c_9\mathrm{e}^{\bar{\lambda}(t-k_i(T)T)}\|p[k_i(T)]\| + c_9\|w_n\|_\infty + c_9\|w_d\|_\infty, \quad t \in [k_i(T)T, t_{i+1}).$$

□

Now restrict $T \in (0, T_3)$. We can bound all of $t \in (t_i, t_{i+1})$ as follows. By Claim 1,

$$\|p(t_i)\| \leq c_4\mathrm{e}^{\lambda t_i}\|p(0)\| + c_4T^{-1}\|w_n\|_\infty + c_4\|w_d\|_\infty.$$

So by Claim 2

$$\begin{aligned} \|p(t)\| &\leq c_7\|p(t_i)\| + c_7T\|w_n\|_\infty + c_7T\|w_d\|_\infty \\ &\leq c_7c_4\mathrm{e}^{-2\lambda T}\mathrm{e}^{\lambda t}\|p(0)\| + (c_7c_4T^{-1} + c_7T)\|w_n\|_\infty \\ &\quad + (c_7c_4 + c_7T)\|w_d\|_\infty, \quad t \in [t_i, k_i(T)T]; \end{aligned}$$

if we set $c_{12} := \max\{c_4 c_7 \mathrm{e}^{-2\lambda T_3},\ c_4 c_7 + c_7 T_3^2,\ c_4 c_7 + c_7 T_3\}$, then

$$\|p(t)\| \leq c_{12}\mathrm{e}^{\lambda t}\|p(0)\| + c_{12}T^{-1}\|w_n\|_\infty + c_{12}\|w_d\|_\infty, \quad t \in [t_i, k_i(T)T].$$

By Claim 3, we have

$$\begin{aligned} \|p(t)\| &\leq c_9 \mathrm{e}^{\bar{\lambda}(t-k_i(T)T)}\|p[k_i(T)]\| + c_9\|w_n\|_\infty + c_9\|w_d\|_\infty \\ &\leq c_9 c_{12}\mathrm{e}^{\bar{\lambda}t}\|p(0)\| + (c_9 c_{12}T^{-1} + c_9)\|w_n\|_\infty + (c_9 c_{12} + c_9)\|w_d\|_\infty, \quad t \in [k_i(T)T, t_{i+1}]. \end{aligned}$$

Then, with $c_{13} := \max\{c_9 c_{12} + c_9,\ \ c_{12},\ \ c_9 c_{12} + c_9 T_3\}$,

$$\|p(t)\| \leq c_{13}\mathrm{e}^{\bar{\lambda}t}\|p(0)\| + c_{13}T^{-1}\|w_n\|_\infty + c_{13}\|w_d\|_\infty, \quad t \in (t_i, t_{i+1}). \tag{5.16}$$

Step 3: Final Bound

By Claim 1, for all $t \in B(T)$ we have

$$\|p(t)\| \leq c_4\mathrm{e}^{\lambda t}\|p(0)\| + c_4 T^{-1}\|w_n\|_\infty + c_4\|w_d\|_\infty,$$

and using (5.16), for all $t \in G(T)$ we have

$$\|p(t)\| \leq c_{13}\mathrm{e}^{\bar{\lambda}t}\|p(0)\| + c_{13}T^{-1}\|w_n\|_\infty + c_{13}\|w_d\|_\infty.$$

Defining $c := \max\{c_4,\ c_{13}\}$ gives the desired result. □

Chapter 6

Example and Simulations

Here we consider an illustrative example. In the example we apply the designed controller to a plant that cannot be stabilized by an LTI controller.

6.1 Example

Consider the system

$$\dot{x}(t) = \begin{bmatrix} \alpha(t) & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \alpha(t) \end{bmatrix} u(t), \tag{6.1}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \tag{6.2}$$

The time-varying parameter $\alpha(t)$ takes values in the set $\mathcal{A} = [-9.5, -1.5] \cup [1.5, 9.5]$. The derivative is upper bounded with $|\dot{\alpha}(t)| \leq 100$. There is a minimum time between jumps of $T_0 = 0.5$ seconds. If we freeze α, the plant transfer function is

$$\frac{(1+\alpha)(s+1-\alpha)}{(s-\alpha)(s+1)}.$$

This transfer function has

$$0 < \underbrace{\alpha - 1}_{\text{zero}} < \underbrace{\alpha}_{\text{pole}}$$

for all $\alpha \in [1.5, 9.5]$. Indeed, the frozen plant will have both an unstable pole and a non-minimum phase zero when $\alpha \in [1.5, 9.5]$, with the zero being slower than the pole (a particularly nasty setup). The first Markov parameter is $f(\alpha) = C(\alpha)B(\alpha) = 1 + \alpha$. This plant satisfies Assumptions 1, 3, 4, 5, and 6.

It turns out that this plant is very difficult to stabilize. In fact, we can show that there does not exist an LTI controller that can stabilize the plant for all $\alpha \in \mathcal{A}$. To this end, using Corollary 12 in Section 5.4 of [40], there does not exist an LTI controller stabilizing (6.1)–(6.2) if, and only if, there exists a stable instance of the plant, $P_1(s)$, and another instance of the plant, $P_0(s)$, such that $P_1(s) - P_0(s)$ is not strongly stabilizable. Choosing

$P_1(s)$ as the LTI transfer function of (6.1)–(6.2) when $\alpha = -2$, and $P_0(s)$ as the LTI transfer function of (6.1)–(6.2) when $\alpha = 2$, we get

$$\begin{aligned} P_1(s) - P_0(s) &= -\frac{(s+3)}{(s+1)(s+2)} - \frac{3(s-1)}{(s+1)(s-2)} \\ &\approx -\frac{4(s+2.3)(s-1.3)}{(s+1)(s+2)(s-2)}. \end{aligned}$$

Then $P_1(s) - P_0(s)$ is not strongly stabilizable, and an LTI controller cannot stabilize (6.1)–(6.2) for $\alpha \in \mathcal{A}$.

Following Theorem 1, the matrices

$$\begin{aligned} X &= \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U(\alpha) = \begin{bmatrix} -1.5|\alpha|(\frac{1}{10+\alpha} + \frac{1}{10-\alpha}) & 0 \end{bmatrix}, \\ R &= \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L(\alpha) = \begin{bmatrix} -\frac{10}{33}|\alpha| \\ 0 \end{bmatrix}, \end{aligned}$$

satisfy (3.8)–(3.9). These matrices are used to get the controller described in Section 4.2. Finally, the filter pole is chosen to be $\lambda = -1$ by the method in Section 4.1.

To simulate the closed-loop system, we chose $h = 0.001$ seconds (yielding $T = 0.002$ seconds) and $\rho = 0.25$. The parameter $\alpha(t)$ switches between the trajectories

$$\alpha(t) = 5.5 + 4\sin(25t), \quad \alpha(t) = -5.5 - 4\sin(25t),$$

and will spend 1 second following the former trajectory, and 0.5 seconds following the latter, between jumps. Uniformly distributed noise with $\|w_n\|_\infty = 0.05$ and $\|w_d\|_\infty = 0.05$ is injected in the system for $t \in [5, 15)$; otherwise, the system is noise-free. The plant initial condition is set to $x(0) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$. The simulation results are presented in Figures 6.1, 6.2, and 6.3.

We see that the output of the plant is bounded in response to the initial condition and the measurement noise. The control signal is fairly large, and the plant state x_2 becomes quite large as well, but this is mostly due to the nastiness of the plant rather than a problem of the controller. That being said, the closed-loop system behaviour is relatively well-behaved even in the presence of frequent parameter jumps, and the estimate of the time-varying parameter is quite good when the filter and controller states are large.

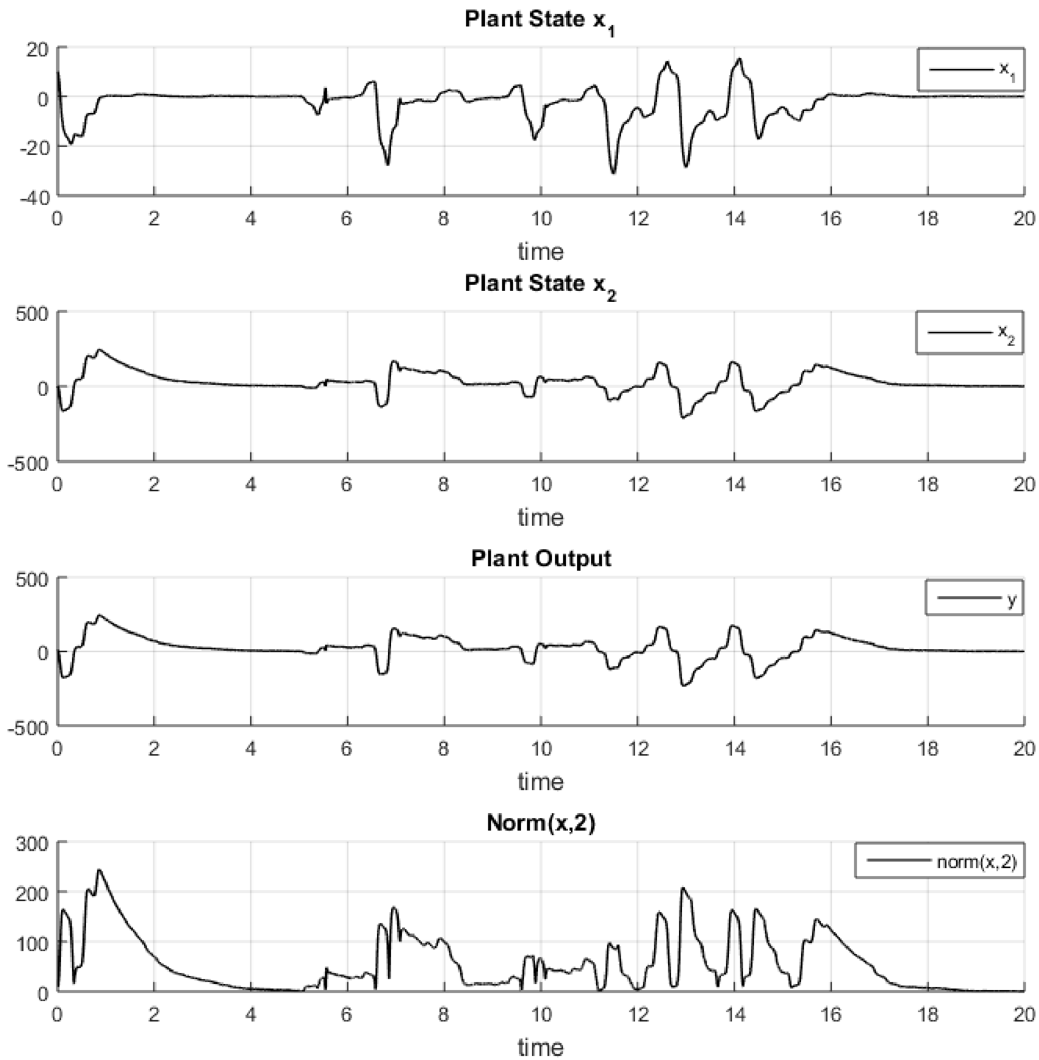

Figure 6.1: The plant states and plant output.

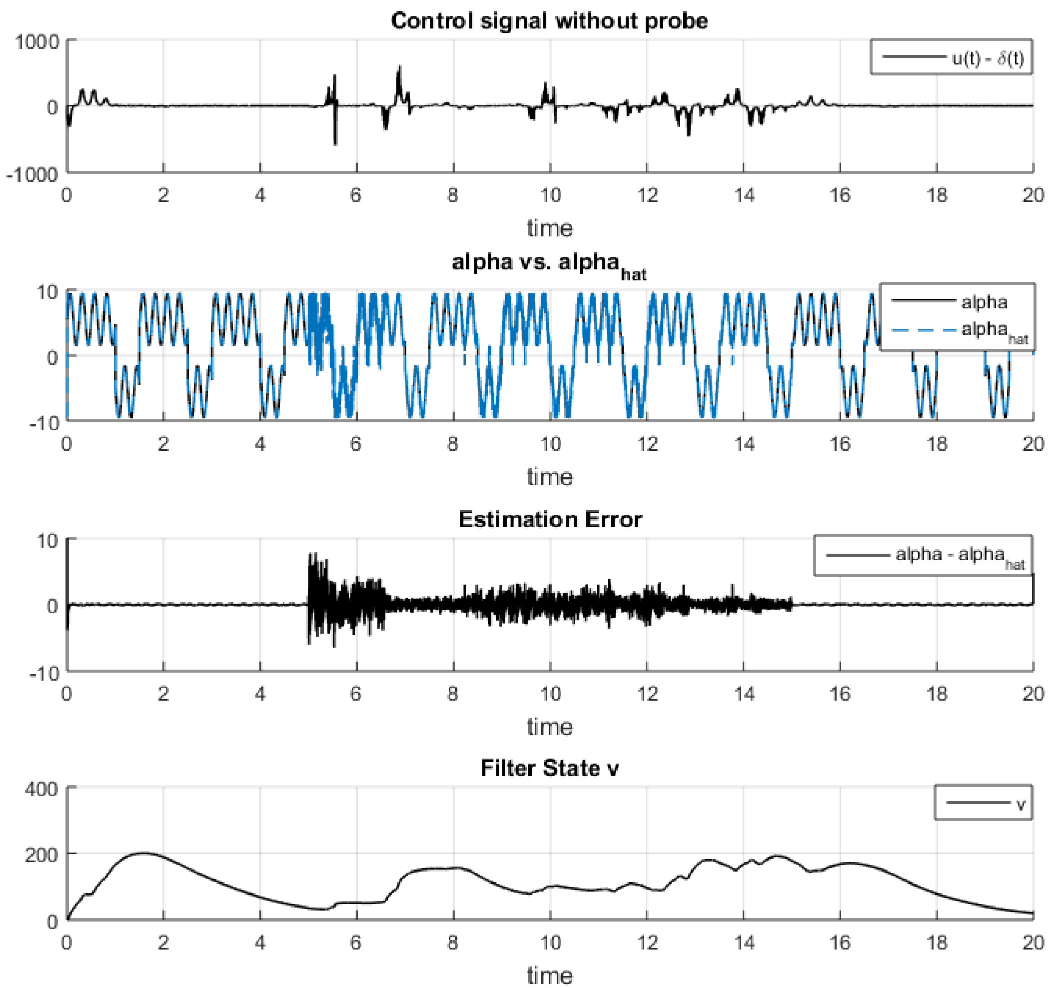

Figure 6.2: Controller output, parameter estimate / estimation error, and filter state.

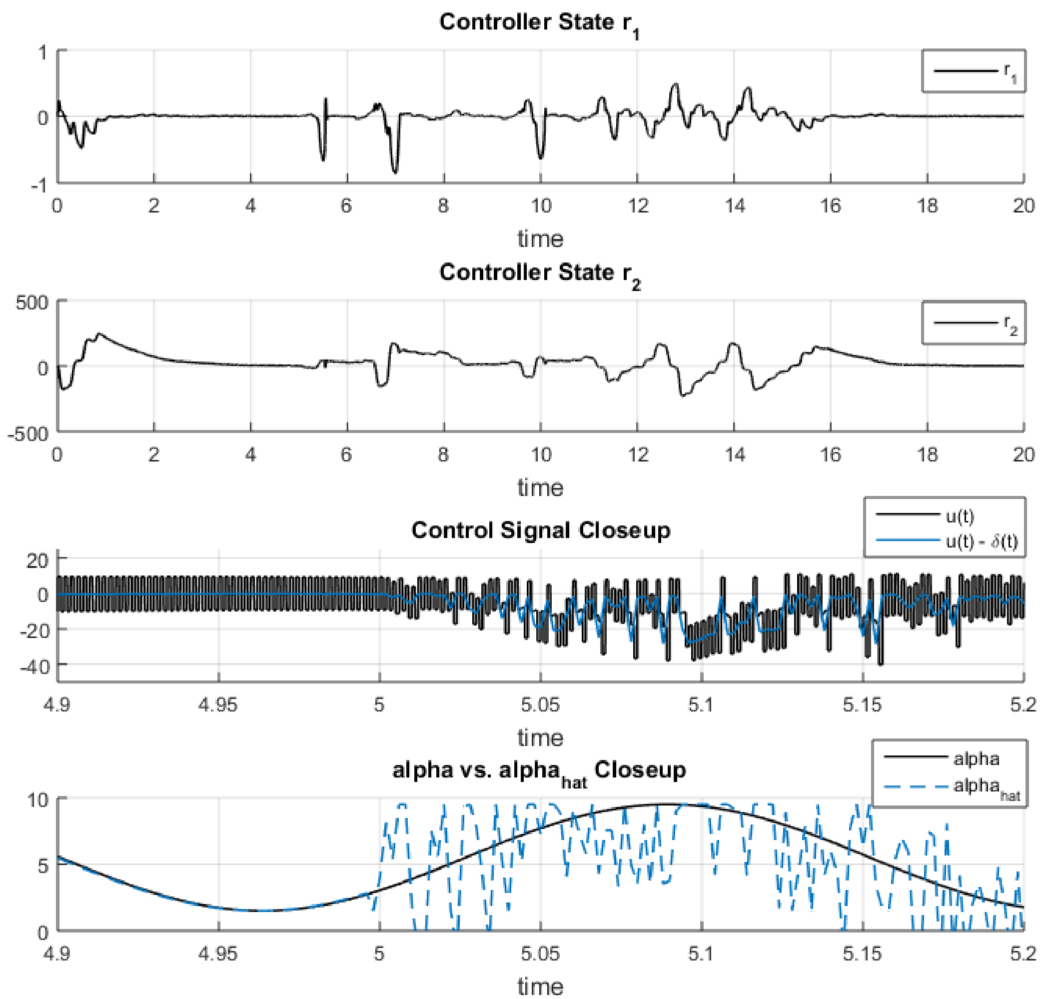

Figure 6.3: Controller states, and controller output and estimates close up.

Chapter 7

Summary and Conclusions

In this thesis we consider the problem of designing a controller to adaptively stabilize an uncertain linear parameter-varying (LPV) plant. This plant could be rapidly time-varying and could have unstable zero dynamics (the time-varying counterpart of non-minimum phase). There are a number of results in the literature for the situation of an uncertain plant with stable zero dynamics. The case of unstable zero dynamics, however, is very challenging. Indeed, results related to plants having unstable zero dynamics suffer very stringent conditions on the plant.

In this thesis a new approach is provided based on results in gain scheduling (particularly [38]). A controller design is presented based on a discretized version of a gain-scheduled output feedback controller, with the scheduling variable replaced by an estimate. The estimate is generated by a discrete-time estimator which uses the state of a filter to appropriately scale a probing signal. This filter provides a surprisingly desirable bound on the closed-loop state when estimation is expected to be inaccurate, i.e., when the probing is small compared to the plant state or when the probing is small compared to the size of noise and disturbance. Under suitable assumptions on the LPV plant, it is proven that if the controller sampling period is small enough, then the closed-loop system is exponentially stable with bounded gain on the noise in the presence of rapid time-variation and persistent parameter jumps. Furthermore, the controller can tolerate noisy measurements and disturbances injected into the state, but the noise gain could be large. Finally, an illustrative example of a plant with unstable zero dynamics is provided. This plant comes from a family of plant models which satisfy the necessary conditions of this paper, which is the subject of a paper that is currently being prepared.

At the moment, the plant is limited to being single-input single-output and having a scalar-valued scheduling variable, however we would like to extend this approach to multiple-input multiple-output plants with vector-valued scheduling variables. Furthermore, due to the difficult nature of handling a plant with a time-varying state-to-output relationship, a major structural assumption that the plant be relative degree one for each value of the scheduling parameter is required. We would like to weaken this assumption so that the applicability of the approach can be extended.

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Appendix A

Proofs

A.1 Proof of Lemma 1

Let $u \in PC_\infty$, $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x_0 \in \mathbb{R}^n$ be arbitrary. The proof is organized into three steps:

1. Starting from equation (3.9), define matrices R_e, $A_e(\alpha)$, and $C_e(\alpha)$, with R_e non-singular, so that $R_e A_e(\alpha) + L(\alpha) C_e(\alpha) = Q(\alpha) R_e$.
2. Use the equation from step 1 to bound a yet to be defined extended state $x_e(t)$.
3. Use the bound on $x_e(t)$ to bound the plant's state $x(t)$.

Step 1

Let v be the number of columns of the constant full rank matrix R from Assumption 2. Let $\bar{R} \in \mathbb{R}^{v\times(v-n)}$ be any matrix so that $R_e := \begin{bmatrix} R & \bar{R} \end{bmatrix}$ is non-singular, and then define

$$\begin{bmatrix} A_{12}(\alpha) \\ A_{22}(\alpha) \end{bmatrix} := R_e^{-1} Q(\alpha) \bar{R},$$

where $A_{12}(\alpha) \in \mathbb{R}^{n\times(v-n)}$, $A_{22}(\alpha) \in \mathbb{R}^{(v-n)\times(v-n)}$. Then, by the definition of R, R_e, A_{12}, and A_{22},

$$\begin{bmatrix} A(\alpha) & A_{12}(\alpha) \\ 0 & A_{22}(\alpha) \end{bmatrix} + R_e^{-1} L(\alpha) \begin{bmatrix} C(\alpha) & 0 \end{bmatrix} = R_e^{-1} Q(\alpha) R_e. \qquad \text{(A.1)}$$

Step 2

Consider the control system

$$\begin{aligned}
\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}A(\alpha(t)) & A_{12}(\alpha(t))\\ 0 & A_{22}(\alpha(t))\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B(\alpha(t))\\ 0\end{bmatrix}u(t) + \begin{bmatrix}w_d(t)\\ 0\end{bmatrix}\\
y_e(t) &= \begin{bmatrix}C(\alpha(t)) & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + w_n(t),
\end{aligned}$$

where $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^{v-n}$. Define $x_e := (x_1, x_2)$; then

$$\begin{aligned}
\dot{x}_e(t) &= \left(\begin{bmatrix}A(\alpha(t)) & A_{12}(\alpha(t))\\ 0 & A_{22}(\alpha(t))\end{bmatrix} + R_e^{-1}L(\alpha(t))\begin{bmatrix}C(\alpha(t)) & 0\end{bmatrix} \right)x_e(t)\\
&\quad -R_e^{-1}L(\alpha(t))\Big(y_e(t) - w_n(t)\Big) + \begin{bmatrix}B(\alpha(t))\\ 0\end{bmatrix}u(t) + \begin{bmatrix}w_d(t)\\ 0\end{bmatrix}.
\end{aligned}$$

Using (A.1), we can write

$$\begin{aligned}
\dot{x}_e(t) &= R_e^{-1}Q(\alpha(t))R_e x_e(t) - R_e^{-1}L(\alpha(t))y_e(t) + R_e^{-1}L(\alpha(t))w_n(t)\\
&\quad + \begin{bmatrix}B(\alpha(t))\\ 0\end{bmatrix}u(t) + \begin{bmatrix}w_d(t)\\ 0\end{bmatrix}.
\end{aligned}$$

The solution to this differential equation, for $t \geq 0$ and any $x_e(0) \in \mathbb{R}^v$, is

$$\begin{aligned}
x_e(t) &= \Phi(t,0)x_e(0) + \int_0^t \Phi(t,\tau)\left(\begin{bmatrix}B(\alpha(\tau))\\ 0\end{bmatrix}u(\tau) - R_e^{-1}L(\alpha(\tau))y_e(\tau) \right)\mathrm{d}\tau\\
&\quad + \int_0^t \Phi(t,\tau)\left(R_e^{-1}L(\alpha(\tau))w_n(\tau) + \begin{bmatrix}w_d(\tau)\\ 0\end{bmatrix} \right)\mathrm{d}\tau,
\end{aligned}$$

where $\Phi(t,0)$ is the state transition function of the unforced system

$$\dot{z}(t) = R_e^{-1}Q(\alpha(t))R_e z(t), \quad t \geq 0.$$

Consider the change of coordinates $p = R_e z$, where R_e is the non-singular matrix from step 1. Then $\dot{p} = Q(\alpha(t))p$ with $Q(\alpha(t)) \in \mathcal{H}_\infty$ so that, by Proposition 2, there exist $\gamma \geq 1$ and $\lambda < 0$ such that

$$\|p(t)\| \leq \gamma \mathrm{e}^{\lambda t}\|p(0)\|, \quad t \geq 0.$$

Again, following the proof of [30, Proposition 1], the constant λ can be taken to be the same as that in the filter (4.1). Then there exists a constant c_1 such that for all $t \geq 0$, $\|z(t)\| \leq c_1\mathrm{e}^{\lambda t}\|z(0)\|$, so we conclude that $\|\Phi(t,0)\| \leq c_1\mathrm{e}^{\lambda t}$ for $t \geq 0$. Using this bound on $\Phi(t,0)$ in the expression for $x_e(t)$ we get

$$\|x_e(t)\| \leq c\mathrm{e}^{\lambda t}\|x_e(0)\| + c\int_0^t \mathrm{e}^{\lambda(t-\tau)}\Big(\|u(\tau)\| + \|y_e(\tau)\| \Big)\mathrm{d}\tau + c\Big(\|w_n\|_\infty + \|w_d\|_\infty \Big), \tag{A.2}$$

with c defined as

$$c := c_1 \max_{\alpha \in \mathcal{A}} \left\{ 1, \left\| R_e^{-1} \right\| \|L(\alpha)\|, \|B(\alpha)\|, -\frac{\|R_e^{-1}\| \|L(\alpha)\|}{\lambda}, -\frac{1}{\lambda} \right\}.$$

Step 3

The subspace $x_2(t) = 0$ is invariant for the extended system. Let $x(t)$ be the solution of the plant's ODE (3.14a) with initial condition $x(0)$. Then the initial condition $x_e(0) = (x(0), 0)$ admits the solution $x_e(t) = (x(t), 0)$ and $y_e(t) = y(t)$. Therefore, using (A.2),

$$\|x(t)\| = \|x_e(t)\| \leq c\mathrm{e}^{\lambda t} \|x(0)\| + c \int_0^t \mathrm{e}^{\lambda(t-\tau)} \Big(\|u(\tau)\| + \|y(\tau)\| \Big) d\tau + c\Big(\|w_n\|_\infty + \|w_d\|_\infty \Big).$$

The solution $v(t)$ of (4.1) equals the integral in the RHS of the above inequality, so we get the desired result.

A.2 Proof of Lemma 2

In the proofs of Lemma 2 and Lemma 3, we utilize a crude bound on the maximum growth of the plant's state over a single period.

Proposition 3: *There exist constants* $\overline{T} > 0$ *and* $c > 0$ *so that for every* $T \in (0, \overline{T})$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, *and* $x[k] \in \mathbb{R}^n$, *when the controller given by* (4.1), (4.3)–(4.5), *and* (4.7)–(4.8) *is applied to the plant* (3.14)*:*

$$\|x(t) - x[k]\| \leq cT \left(v[k] + \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \|s[k]\|_\infty + \|w_d\|_\infty \right), \quad t \in [kT, (k+1)T].$$

Proof of Proposition 3.
Let $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, and $x[k] \in \mathbb{R}^n$ be arbitrary. The solution $x(t)$ to (3.14) with initial condition $x[k]$ satisfies

$$x(t) - x[k] = \int_{kT}^t \Big(A(\alpha(\tau))x[k] + B(\alpha(\tau))u(\tau) + w_d(\tau) \Big) \mathrm{d}\tau + \int_{kT}^t A(\alpha(\tau)) \Big(x(\tau) - x[k] \Big) \mathrm{d}\tau.$$

Taking norms and substituting the expression for $u(t)$ over $[kT, (k+1)T]$, we get the upper bound

$$\|x(t)-x[k]\| \leq T\Big(\max_{\alpha\in\mathcal{A}}\|A(\alpha)\|\,\|x[k]\| + \max_{\alpha\in\mathcal{A}}\|B(\alpha)\|\,(\max_{\alpha\in\mathcal{A}}\|H(\alpha)\|+\rho)\,\|z[k]\|$$
$$+\max_{\alpha\in\mathcal{A}}\|B(\alpha)\|\,(\max_{\alpha\in\mathcal{A}}\|K(\alpha)\|\,\|M\|+\rho)\,\|r[k]\| + \max_{\alpha\in\mathcal{A}}\|B(\alpha)\|\,\rho v[k] + \|w_d\|_\infty\Big)$$
$$+\max_{\alpha\in\mathcal{A}}\|A(\alpha)\|\int_{kT}^{t}\|x(t)-x[k]\|\,\mathrm{d}\tau,\quad t\in[kT,(k+1)T].$$

Using the definition of $\bar{x}$, $\bar{z}$, and s given in (4.10) and invoking the Bellman-Gronwall inequality, there exist constants $c_1>0$ and $c_2>0$ so that

$$\|x(t)-x[k]\| \leq c_1 T\left(v[k] + \left\|\begin{bmatrix}\bar{x}[k]\\ \bar{z}[k]\end{bmatrix}\right\|_1 + \|s[k]\|_\infty\right)\mathrm{e}^{c_2 T} + c_1 T\,\|w_d\|_\infty\,\mathrm{e}^{c_2 T}.$$

Then, for sufficiently small $\overline{T}$, there exists a constant $c>0$ so that, for all $T\in(0,\overline{T})$,

$$\|x(t)-x[k]\| \leq cT\left(v[k] + \left\|\begin{bmatrix}\bar{x}[k]\\ \bar{z}[k]\end{bmatrix}\right\|_1 + \|s[k]\|_\infty + \|w_d\|_\infty\right). \quad \text{(A.3)}$$

□

Proof of Lemma 2.

Let $\epsilon>0$, $\overline{\delta}>0$, $w_n\in PC_\infty$, $w_d\in PC_\infty$, $\alpha\in PS^1(\mathcal{A},T_0,\delta_\alpha)$, $k\in\mathbb{Z}_+$, and $x_0\in\mathbb{R}^n$ be arbitrary. We start with a claim.

Claim 1: *There exist constants $c>0$ and $\overline{T}>0$, so that if $T\in(0,\overline{T})$, $v[k]+\|z[k]\|+\|r[k]\|>\epsilon\|x[k]\|$, and $\alpha(t)$ is absolutely continuous for $t\in[kT,(k+1)T]$, then*

$$\left\|C(\alpha((k+1)T))B(\alpha((k+1)T)) - \widehat{CB}[k+1]\right\| \leq cT + c\frac{\|w_d\|_\infty}{\delta[k]} + cT^{-1}\frac{\|w_n\|_\infty}{\delta[k]}. \quad \text{(A.4)}$$

Proof of Claim 1.
By hypothesis, $v[k]+\|z[k]\|+\|r[k]\|>\epsilon\|x[k]\|$, which implies that $\delta[k]\neq 0$. Then

$$\begin{aligned}-2h\widehat{CB}[k+1]\delta[k] &= y(kT+2h)-2y(kT+h)+y(kT)\\ &= \Big(y_{nf}(kT+2h)-y_{nf}(kT+h)\Big)-\Big(y_{nf}(kT+h)-y_{nf}(kT)\Big)\\ &\quad +w_n(kT+2h)-2w_n(kT+h)+w_n(kT),\end{aligned} \quad \text{(A.5)}$$

where, with some abuse of notation, $y_{nf}(t):=C(t)x(t)$. By this definition of $y_{nf}(t)$, it follows that

$$\dot{y}_{nf}(t) = \Big(\dot{C}(t) + C(t)A(t)\Big)x(t) + C(t)B(t)u(t) + C(t)w_d(t).$$

Then, by the Fundamental Theorem of Calculus and the structure of the control signal (4.5), we have

$$\begin{aligned} y_{nf}(kT+h) - y_{nf}(kT) &= \int_{kT}^{kT+h} \Big(\dot{C}(\tau) + C(\tau)A(\tau)\Big)x(\tau)\mathrm{d}\tau + \int_{kT}^{kT+h} C(\tau)B(\tau)\mathrm{d}\tau\delta[k] \\ &+ \int_{kT}^{kT+h} C(\tau)B(\tau)\mathrm{d}\tau\Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ &+ \int_{kT}^{kT+h} C(\tau)w_d(\tau)\mathrm{d}\tau. \end{aligned}$$

In a similar fashion,

$$\begin{aligned} y_{nf}(kT+2h) - y_{nf}(kT+h) &= \int_{kT+h}^{kT+2h} \Big(\dot{C}(\tau) + C(\tau)A(\tau)\Big)x(\tau)\mathrm{d}\tau \\ &- \int_{kT+h}^{kT+2h} C(\tau)B(\tau)\mathrm{d}\tau\delta[k] \\ &+ \int_{kT+h}^{kT+2h} C(\tau)B(\tau)\mathrm{d}\tau\Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ &+ \int_{kT+h}^{kT+2h} C(\tau)w_d(\tau)\mathrm{d}\tau. \end{aligned}$$

Substituting the previous two expressions into (A.5) yields

$$\begin{aligned} -2h\widehat{CB}[k+1]\delta[k] &= \int_{kT+h}^{kT+2h} \Big(C(\tau)A(\tau) + \dot{C}(\tau)\Big)x(\tau)\mathrm{d}\tau + \int_{kT+h}^{kT+2h} C(\tau)w_d(\tau)\mathrm{d}\tau \\ &+ \int_{kT+h}^{kT+2h} C(\tau)B(\tau)\mathrm{d}\tau\Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ &- \int_{kT}^{kT+h} C(\tau)B(\tau)\mathrm{d}\tau\Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ &- \int_{kT}^{kT+2h} C(\tau)B(\tau)\mathrm{d}\tau\delta[k] - \int_{kT}^{kT+h} \Big(C(\tau)A(\tau) + \dot{C}(\tau)\Big)x(\tau)\mathrm{d}\tau \\ &- \int_{kT}^{kT+h} C(\tau)w_d(\tau)\mathrm{d}\tau + w_n(kT+2h) - 2w_n(kT+h) + w_n(kT) \end{aligned}$$

$$
\begin{aligned}
= \quad & -2hC(kT+2h)B(kT+2h)\delta[k] \\
& + \int_{kT}^{kT+2h} \Big(C(kT+2h)B(kT+2h) - C(\tau)B(\tau) \Big) \mathrm{d}\tau \delta[k] \\
& + \int_{kT+h}^{kT+2h} \Big(\big(C(\tau)A(\tau) + \dot{C}(\tau) \big) x(\tau) - \big(C(\tau-h)A(\tau-h) + \dot{C}(\tau-h) \big) x(\tau-h) \Big) \mathrm{d}\tau \\
& + \int_{kT+h}^{kT+2h} \Big(C(\tau)B(\tau) - C(\tau-h)B(\tau-h) \Big) \mathrm{d}\tau \Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] \Big) \\
& + \int_{kT+h}^{kT+2h} C(\tau)w_d(\tau)\mathrm{d}\tau - \int_{kT}^{kT+h} C(\tau)w_d(\tau)\mathrm{d}\tau \\
& + w_n(kT+2h) - 2w_n(kT+h) + w_n(kT).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\left\| C(kT+2h)B(kT+2h) - \widehat{CB}[k+1] \right\| \leq & \frac{1}{2h} \int_{kT}^{kT+2h} \left\| C(kT+2h)B(kT+2h) - C(\tau)B(\tau) \right\| \mathrm{d}\tau \\
& + \frac{1}{2h\delta[k]} \int_{kT+h}^{kT+2h} \left\| \big(C(\tau)A(\tau) + \dot{C}(\tau) \big) x(\tau) - \big(C(\tau-h)A(\tau-h) + \dot{C}(\tau-h) \big) x(\tau-h) \right\| \mathrm{d}\tau \\
& + \frac{1}{2h\delta[k]} \int_{kT+h}^{kT+2h} \left\| C(\tau)B(\tau) - C(\tau-h)B(\tau-h) \right\| \mathrm{d}\tau \Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] \Big) \\
& + \frac{1}{2h\delta[k]} \left\| \int_{kT+h}^{kT+2h} C(\tau)w_d(\tau)\mathrm{d}\tau - \int_{kT}^{kT+h} C(\tau)w_d(\tau)\mathrm{d}\tau \right\| \\
& + \frac{1}{2h\delta[k]} \left\| w_n(kT+2h) - 2w_n(kT+h) + w_n(kT) \right\|.
\end{aligned}
$$

Utilizing order notation, Proposition 3[1], and applying Assumption 6 to bound the Lipschitz continuous functions, we can write this concisely as

$$
\begin{aligned}
\left\| C(kT+2h)B(kT+2h) - \widehat{CB}[k+1] \right\| \quad = \quad & \mathcal{O}(T) + \mathcal{O}(T) \frac{v[k] + \|x[k]\| + \|z[k]\| + \|r[k]\|}{\delta[k]} \\
& + \mathcal{O}(1) \frac{\|w_d\|_\infty}{\delta[k]} + \mathcal{O}(T^{-1}) \frac{\|w_n\|_\infty}{\delta[k]}.
\end{aligned}
$$

By hypothesis we have

$$
\frac{\|x[k]\| + v[k] + \|z[k]\| + \|r[k]\|}{\delta[k]} \leq \Big(\frac{1}{\epsilon} + 1 \Big) \frac{\big(v[k] + \|z[k]\| + \|r[k]\| \big)}{\rho(v[k] + \|z[k]\| + \|r[k]\|)} = \frac{1}{\rho} \Big(\frac{1}{\epsilon} + 1 \Big),
$$

[1]It is important to note that by using Proposition 3 the estimation error will be upper bounded by the entire closed-loop state, including $v(t)$.

so for all $T \in (0, \overline{T})$, with $\overline{T}$ sufficiently small, we get the desired result,

$$\left\| C((k+1)T)B((k+1)T) - \widehat{CB}[k+1] \right\| = \mathcal{O}(T) + \mathcal{O}(1)\frac{\|w_d\|_\infty}{\delta[k]} + \mathcal{O}(T^{-1})\frac{\|w_n\|_\infty}{\delta[k]}.$$

□

By hypotheses (i) and (iii) and Claim 1, the bound (A.4) holds. By hypothesis (ii) and the definition of $\delta[k]$,

$$\frac{\|w_n\|_\infty}{T\delta[k]} + \frac{\|w_d\|_\infty}{\delta[k]} < \frac{1}{\rho c}. \tag{A.6}$$

With c_1 the constant from Claim 1, substituting (A.6) into (A.4) yields

$$\left\| C((k+1)T)B((k+1)T) - \widehat{CB}[k+1] \right\| < c_1 \left(T + \frac{1}{\rho c} \right). \tag{A.7}$$

We want to bound $\|C(\alpha((k+1)T)B(\alpha((k+1)T) - \Pi_{\mathcal{F}}(\widehat{CB}[k+1])\|$, so we must account for the effect of the projection, $\Pi_{\mathcal{F}}$. To ensure that $\Pi_{\mathcal{F}}$ projects onto the interval of $\mathcal{F}$ containing $C((k+1)T)B((k+1)T)$ it is sufficient that the upper bound in (A.7) be less than half the minimum distance between intervals of $\mathcal{F}$. By Assumptions 3 and 6, the image of $\mathcal{A}$ under f has the form $\mathcal{F} = [\underline{f}_1, \overline{f}_1] \cup \cdots \cup [\underline{f}_q, \overline{f}_q]$, where $\overline{f}_i < \underline{f}_{i+1}$, $i = 1, ..., q-1$, for some finite $q \in \mathbb{Z}_+$.

Let $d_{\min} := \min_j (\underline{f}_{j+1} - \overline{f}_j)$, $j = 1, ..., q-1$, and let $T_1 := \frac{1}{c_1} \min\{\frac{d_{\min}}{4}, \frac{\overline{\delta}}{2\ell}\}$, where ℓ is the Lipschitz constant of f^{-1} on $\mathcal{F}$. Then, for any $c > \frac{c_1}{\rho} \max\{\frac{4}{d_{\min}}, \frac{2\ell}{\overline{\delta}}\}$ and any $T \in (0, T_1)$, we get

$$\left\| C((k+1)T)B((k+1)T) - \widehat{CB}[k+1] \right\| < \frac{c_1 d_{\min}}{4c_1} + \frac{c_1 \rho d_{\min}}{4c_1\rho} = \frac{d_{\min}}{2},$$

so it follows that

$$\left\| C((k+1)T)B((k+1)T) - \Pi_{\mathcal{F}}(\widehat{CB}[k+1]) \right\| \le \left\| C((k+1)T)B((k+1)T) - \widehat{CB}[k+1] \right\|.$$

By Assumption 4 we have

$$\begin{aligned} \|\alpha((k+1)T) - \hat{\alpha}[k+1]\| &\le \ell \left\| C((k+1)T)B((k+1)T) - \Pi_{\mathcal{F}}(\widehat{CB}[k+1]) \right\| \\ &< c_1\ell \left(T + \frac{1}{\rho c} \right) < \frac{c_1\ell\overline{\delta}}{2c_1\ell} + \frac{c_1\ell\rho\overline{\delta}}{2c_1\ell\rho} = \overline{\delta}. \end{aligned}$$

□

A.3 Proof of Lemma 3

Let $w_n \in PC_\infty$, $w_d \in PC_\infty$, $\alpha \in PS^1(\mathcal{A}, T_0, \delta_\alpha)$, $k \in \mathbb{Z}_+$, and $x_0 \in \mathbb{R}^n$ be arbitrary.

Proof of (iii)

In order to accomplish this analysis proceed as follows:

- Analyze all states at the sample points (Steps 1, 2, and 3),
- Upper bound a transformed closed-loop state (Step 4).

Step 1 - Filter Sample Point Analysis

At the sample points, the filter (4.1) satisfies

$$v[k+1] = \mathrm{e}^{\lambda T} v[k] + \int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|u(\tau)\| \, \mathrm{d}\tau + \int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|y(\tau)\| \, \mathrm{d}\tau.$$

By (4.5), the input term satisfies

$$\begin{aligned}\int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|u(\tau)\| \, \mathrm{d}\tau \;&=\; -\frac{1}{\lambda}\Big(1-\mathrm{e}^{\lambda h}\Big)\Bigg[\|H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] + \delta[k]\| \, \mathrm{e}^{\lambda h} \\ &\qquad + \|H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] - \delta[k]\| \Bigg] \\ &\le\; -\frac{1}{\lambda}\Bigg(-\frac{\lambda T}{2}\Bigg)\Bigg(2(\max_{\alpha\in\mathcal{A}} \|H(\alpha)\| + \rho)\,\|z[k]\| + 2(\max_{\alpha\in\mathcal{A}} \|K(\alpha)\|\,\|M\| + \rho)\,\|r[k]\| + 2\rho v[k]\Bigg).\end{aligned}$$

Employing order notation, we can write this compactly as

$$\int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|u(\tau)\| \, \mathrm{d}\tau \;=\; T\rho v[k] + \mathcal{O}(T) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \mathcal{O}(T)\,\|s[k]\|_\infty .$$

The output term satisfies

$$\begin{aligned}\int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|y(\tau)\| \, \mathrm{d}\tau \;&\le\; \int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \mathrm{d}\tau \, \overbrace{\|C(\alpha(kT))x[k]\|}^{=\mathcal{O}(1)\|x[k]\|} \\ &\quad + \int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \Big(\underbrace{\|C(\alpha(\tau)) - C(\alpha(kT))\|\,\|x[k]\|}_{=\mathcal{O}(T)\|x[k]\|} \\ &\quad + \underbrace{\|C(\alpha(\tau))\|\,\|x(\tau)-x[k]\|}_{\text{bound using Proposition 3}} + \|w_n\|_\infty \Big) \mathrm{d}\tau.\end{aligned}$$

Employing order notation, and utilizing Assumption 6 and Proposition 3, we can write this compactly as

$$\int_{kT}^{(k+1)T} \mathrm{e}^{\lambda((k+1)T-\tau)} \|y(\tau)\| \,\mathrm{d}\tau \;=\; \mathcal{O}(T) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \mathcal{O}(T^2) v[k] + \mathcal{O}(T^2) \|s[k]\|_\infty + \mathcal{O}(T) \|w_n\|_\infty + \mathcal{O}(T^2) \|w_d\|_\infty .$$

Combining these upper bounds, for sufficiently small T_2 there exist constants $e_1 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $w_1 > 0$ such that, for all $T \in (0, T_2)$ and all $k \in \mathbb{Z}_+$,

$$\begin{aligned} v[k+1] \;\leq\; & \big(1 + (\lambda+\rho)T + e_1 T^2\big) v[k] + \gamma_1 T \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \gamma_2 T \|s[k]\|_\infty \\ & + w_1 T \|w_n\|_\infty + w_1 T^2 \|w_d\|_\infty . \end{aligned} \tag{A.8}$$

Step 2 - Controller Sample Point Analysis

Starting with $z[k+1]$, define

$$e_{z1}[k] := T\Big(F(\hat{\alpha}[k]) - F(\alpha(kT))\Big) z[k] + T\Big(G(\hat{\alpha}[k]) - G(\alpha(kT))\Big) M r[k],$$

so that we can write

$$z[k+1] = \begin{bmatrix} 0 & I + TF(\alpha(kT)) & TG(\alpha(kT))M \end{bmatrix} \begin{bmatrix} x[k] \\ z[k] \\ r[k] \end{bmatrix} + e_{z1}[k].$$

By Assumptions 6 and 7 (see Remark 1), there exists a constant $\ell_1 > 0$ so that we have

$$\|e_{z1}[k]\| \leq T\ell_1 \|\tilde{\alpha}(kT)\| \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + T\ell_1 \|\tilde{\alpha}(kT)\| \|s[k]\|_\infty ,$$

where $\tilde{\alpha}(kT) = \alpha(kT) - \hat{\alpha}[k]$ denotes the parameter estimation error at time kT. By hypothesis $\|\tilde{\alpha}(kT)\| \leq \overline{\delta}$, so

$$\|e_{z1}[k]\| \leq T\ell_1\overline{\delta} \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + T\ell_1\overline{\delta} \|s[k]\|_\infty . \tag{A.9}$$

We can treat $r[k+1]$ similarly and define

$$\begin{aligned} e_{z2}[k] := & - T\Big(L(\hat{\alpha}[k]) - L(\alpha(kT))\Big) C(\alpha(kT)) x[k] \\ & + TR\Big(B(\hat{\alpha}[k])H(\hat{\alpha}[k]) - B(\alpha(kT))H(\alpha(kT))\Big) z[k] \\ & + T\Big((Q(\hat{\alpha}[k]) - Q(\alpha(kT))) + R(B(\hat{\alpha}[k])K(\hat{\alpha}[k]) - B(\alpha(kT))K(\alpha(kT)))\Big) r[k] \\ & - TL(\hat{\alpha}[k]) w_n[k], \end{aligned}$$

so that

$$\begin{aligned} r[k+1] &= -TL(\alpha(kT))C(\alpha(kT))x[k] + TRB(\alpha(kT))H(\alpha(kT))z[k] \\ &\quad +\Big(I + T\Big(Q(\alpha(kT)) + RB(\alpha(kT))K(\alpha(kT))M\Big)\Big)r[k] \\ &\quad +e_{z2}[k]. \end{aligned}$$

Again, by Assumptions 6 and 7, there exists a constant $\ell_2 > 0$ such that

$$\|e_{z2}[k]\| \le T\ell_2\overline{\delta} \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + T\ell_2\overline{\delta}\, \|s[k]\|_\infty + T\ell_2 \|w_n\|_\infty . \tag{A.10}$$

Finally, we will use

$$e_u[k] := \Big(H(\hat{\alpha}[k]) - H(\alpha(kT))\Big)z[k] + \Big(K(\hat{\alpha}[k]) - K(\alpha(kT))\Big)Mr[k],$$

to write

$$u[k] - \delta(t) = H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k] = \begin{bmatrix} H(\alpha(kT)) & K(\alpha(kT))M \end{bmatrix} \begin{bmatrix} z[k] \\ r[k] \end{bmatrix} + e_u[k].$$

Again, by Assumptions 6 and 7, there exists a constant $\ell_3 > 0$ such that

$$\|e_u[k]\| \le \ell_3\overline{\delta} \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \ell_3\overline{\delta}\, \|s[k]\|_\infty . \tag{A.11}$$

Step 3 - Plant Sample Point Analysis

The value of the plant state at time $t = (k+1)T$ is

$$x[k+1] = x[k] + \int_{kT}^{(k+1)T} A(\alpha(\tau))x(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} B(\alpha(\tau))u(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} w_d(\tau)\mathrm{d}\tau,$$

so that, using the structure of the control signal (4.5),

$$\begin{aligned} x[k+1] &= (I + TA(\alpha(kT)))x[k] + TB(\alpha(kT))\left(\begin{bmatrix} H(\alpha(kT)) & K(\alpha(kT)) \end{bmatrix} \begin{bmatrix} z[k] \\ r[k] \end{bmatrix} + e_u[k] \right) \\ &\quad + \int_{kT}^{(k+1)T} (A(\alpha(\tau)) - A(\alpha(kT)))\mathrm{d}\tau x[k] + \int_{kT}^{(k+1)T} A(\alpha(\tau))(x(\tau) - x[k])\mathrm{d}\tau \\ &\quad + \int_{kT}^{(k+1)T} (B(\alpha(\tau)) - B(\alpha(kT)))\mathrm{d}\tau \Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ &\quad + \int_{kT}^{(k+1)T} B(\alpha(\tau))\delta(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} w_d(\tau)\mathrm{d}\tau. \end{aligned}$$

We define

$$\begin{aligned} e_p[k] \quad := \quad & \int_{kT}^{(k+1)T} (A(\alpha(\tau)) - A(\alpha(kT)))\mathrm{d}\tau x[k] + \int_{kT}^{(k+1)T} A(\alpha(\tau))(x(\tau) - x[k])\mathrm{d}\tau \\ & + \int_{kT}^{(k+1)T} (B(\alpha(\tau)) - B(\alpha(kT)))\mathrm{d}\tau\Big(H(\hat{\alpha}[k])z[k] + K(\hat{\alpha}[k])Mr[k]\Big) \\ & + \int_{kT}^{(k+1)T} B(\alpha(\tau))\delta(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} w_d(\tau)\mathrm{d}\tau, \end{aligned}$$

to be able to compactly write

$$\begin{aligned} x[k+1] = & \begin{bmatrix} I + TA(\alpha(kT)) & TB(\alpha(kT))H(\alpha(kT)) & TB(\alpha(kT))K(\alpha(kT))M \end{bmatrix} \begin{bmatrix} x[k] \\ z[k] \\ r[k] \end{bmatrix} \\ & + TB(\alpha(kT))e_u[k] + e_p[k]. \end{aligned}$$

By Assumptions 6 and 7 (see Remark 1) and Proposition 3, there exists a constant $\ell_4 > 0$ such that

$$\|e_p[k]\| \leq T^2\ell_4 v[k] + T^2\ell_4 \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + T^2\ell_4 \|s[k]\|_\infty + T\ell_4 \|w_d\|_\infty . \tag{A.12}$$

Step 4 - Closed-Loop Difference Inequality

Combining the analysis from the plant and the controller, we get

$$\begin{aligned} \begin{bmatrix} x[k+1] \\ z[k+1] \\ r[k+1] \end{bmatrix} \quad = \quad & \left(I + T \begin{bmatrix} A(\alpha(kT)) & B(\alpha(kT))H(\alpha(kT)) & B(\alpha(kT))K(\alpha(kT))M \\ 0 & F(\alpha(kT)) & G(\alpha(kT))M \\ -L(\alpha(kT))C(\alpha(kT)) & RB(\alpha(kT))H(\alpha(kT)) & Q(\alpha(kT)) + RB(\alpha(kT))K(\alpha(kT))M \end{bmatrix} \right) \begin{bmatrix} x[k] \\ z[k] \\ r[k] \end{bmatrix} \\ & + \begin{bmatrix} TB(\alpha(kT))e_u[k] + e_p[k] \\ e_{z1}[k] \\ e_{z2}[k] \end{bmatrix}. \end{aligned}$$

In $(\bar{x}, \bar{z}, s)$-coordinates, making use of (3.9) and (3.13), we have

$$\begin{aligned} \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \\ s[k+1] \end{bmatrix} \quad = \quad & \left(I + T \begin{bmatrix} P(\alpha(kT)) & -\begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} B(\alpha(kT))K(\alpha(kT))M \\ G(\alpha(kT))M \end{bmatrix} \\ 0 \quad 0 & Q(\alpha(kT)) \end{bmatrix} \right) \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \\ s[k] \end{bmatrix} \\ & + \begin{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} TB(\alpha(kT))e_u[k] + e_p[k] \\ e_{z1}[k] \end{bmatrix} \\ TRB(\alpha(kT))e_u[k] + Re_p[k] - e_{z2}[k] \end{bmatrix}. \end{aligned}$$

To construct the decrescent norm, we'll define a difference inequality and apply two similarity transformations. We start by upper bounding the transformed plant and controller states:

$$\left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \le \|I + TP(\alpha(kT))\|_1 \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + T \left\| \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} B(\alpha(kT))K(\alpha(kT))M \\ G(\alpha(kT))M \end{bmatrix} \right\|_1 \|s[k]\|_1 + \left\| \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} TB(\alpha(kT))e_u[k] + e_p[k] \\ e_{z1}[k] \end{bmatrix} \right\|_1 .$$

Next we take advantage of the $\mathcal{H}_1$ property that $P(\alpha)$ enjoys: by Proposition 1 there exists a constant $\lambda_1 < 0$ so that $\|I + TP(\alpha)\|_1 \le 1 + \lambda_1 T$. Then using (A.9), (A.11), and (A.12), there exists constants $e_2 > 0$, $\gamma_3 > 0$, and $w_2 > 0$ such that

$$\left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \le \left(1 + \lambda_1 T + e_2 T(\overline{\delta} + T)\right) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \left(\gamma_3 T + e_2 T(\overline{\delta} + T)\right) \|s[k]\|_\infty + e_2 T^2 v[k] + w_2 T \|w_d\|_\infty .$$

In a similar fashion, we have an upper bound on $s[k+1]$,

$$\|s[k+1]\|_\infty \le \|I + TQ(\alpha(kT))\|_\infty \|s[k]\|_\infty + T \|RB(\alpha(kT))e_u[k]\|_\infty + \|Re_p[k]\|_\infty + \|e_{z2}[k]\|_\infty ,$$

We can take advantage of the $\mathcal{H}_\infty$ property that $Q(\alpha)$ enjoys: there exists a constant $\lambda_2 < 0$ so that $\|I + TQ(\alpha)\|_\infty \le 1 + \lambda_2 T$. This yields

$$\|s[k+1]\|_\infty \le (1 + \lambda_2 T) \|s[k]\|_\infty + T \|RB(\alpha(kT))\| \|e_u[k]\|_\infty + \|R\| \|e_p[k]\|_\infty + \|e_{z2}[k]\|_\infty ,$$

and using (A.10), (A.11), and (A.12), there exist constants $e_3 > 0$ and $w_3 > 0$ such that

$$\|s[k+1]\|_\infty \le \left(1 + \lambda_2 T + e_3 T(\overline{\delta} + T)\right) \|s[k]\|_\infty + e_3 T(\overline{\delta} + T) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + e_3 T^2 v[k] + w_3 T \|w_n\|_\infty + w_3 T \|w_d\|_\infty .$$

Now we combine the bounds on the states. Choose a positive constant ρ so that $\bar{\lambda} := \lambda + \rho < 0$, and fix $\bar{\lambda} > \lambda_1 > \lambda_2$. Then

$$\begin{bmatrix} v[k+1] \\ \left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \\ \|s[k+1]\|_\infty \end{bmatrix} \le \begin{bmatrix} 1 + \bar{\lambda} T + e_1 T^2 & \gamma_1 T & \gamma_2 T \\ e_2 T^2 & 1 + \lambda_1 T + e_2 T(\overline{\delta} + T) & \gamma_3 T + e_2 T(\overline{\delta} + T) \\ e_3 T^2 & e_3 T(\overline{\delta} + T) & 1 + \lambda_2 T + e_3 T(\overline{\delta} + T) \end{bmatrix} \begin{bmatrix} v[k] \\ \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 \\ \|s[k]\|_\infty \end{bmatrix} + \begin{bmatrix} w_1 T & w_1 T^2 \\ 0 & w_2 T \\ w_3 T & w_3 T \end{bmatrix} \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix} .$$

So we have, with $E \in \mathbb{R}^{3\times 3}$ constant and T_2 sufficiently small, for $T \in (0, T_2)$,

$$\begin{bmatrix} v[k+1] \\ \left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \\ \|s[k+1]\|_\infty \end{bmatrix} \leq \Bigg(\underbrace{\begin{bmatrix} 1+\bar{\lambda}T & \gamma_1 T & \gamma_2 T \\ 0 & 1+\lambda_1 T & \gamma_3 T \\ 0 & 0 & 1+\lambda_2 T \end{bmatrix}}_{=:\Lambda} + T(\overline{\delta}+T)E \Bigg) \begin{bmatrix} v[k] \\ \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 \\ \|s[k]\|_\infty \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} w_1 T & w_1 T^2 \\ 0 & w_2 T \\ w_3 T & w_3 T \end{bmatrix}}_{=:W(T)} \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix}. \tag{A.13}$$

Next we define three states as upper bounds of the above states at periods k and $k+1$. This allows us to get equality, so we can solve and transform a difference equation rather than inequality. Define $\psi := (\psi_1, \psi_2, \psi_3)$ via

$$\psi_1[k] := v[k], \quad \psi_2[k] := \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1, \quad \psi_3[k] := \|s[k]\|_\infty,$$

$$\psi[k+1] := (\Lambda + T(\overline{\delta}+T)E)\psi[k] + W(T) \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix}.$$

It is clear that $v[k+1] \leq \psi_1[k+1]$, $\left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \leq \psi_2[k+1]$, and $\|s[k+1]\|_\infty \leq \psi_3[k+1]$ because

$$\begin{aligned} \begin{bmatrix} v[k+1] \\ \left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \\ \|s[k+1]\|_\infty \end{bmatrix} &\leq (\Lambda + T(\overline{\delta}+T)E) \begin{bmatrix} v[k] \\ \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 \\ \|s[k]\|_\infty \end{bmatrix} + W(T) \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix} \\ &= (\Lambda + T(\overline{\delta}+T)E)\psi[k] + W(T) \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix} \\ &= \psi[k+1]. \end{aligned}$$

Next we perform two similarity transformations with the objective of diagonalizing the matrix Λ in (A.13). The first is a constant transformation of the form

$$V := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & v_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

so that under similiarity transformation we have

$$V(\Lambda + T(\overline{\delta}+T)E)V^{-1} = \begin{bmatrix} 1+\bar{\lambda}T & \gamma_1 T & (\gamma_2 - v_{23}\gamma_1)T \\ 0 & 1+\lambda_1 T & (\gamma_3 + v_{23}(\lambda_2 - \lambda_1))T \\ 0 & 0 & 1+\lambda_2 T \end{bmatrix} + T(\overline{\delta}+T)VEV^{-1}.$$

Choose $v_{23} = \gamma_3/(\lambda_1 - \lambda_2)$ so that we are left with

$$V(\Lambda + T(\overline{\delta} + T)E)V^{-1} = \begin{bmatrix} 1+\bar{\lambda}T & \gamma_1 T & (\gamma_2 - v_{23}\gamma_1)T \\ 0 & 1+\lambda_1 T & 0 \\ 0 & 0 & 1+\lambda_2 T \end{bmatrix} + T(\overline{\delta} + T)VEV^{-1}.$$

To complete the diagonalization of Λ consider a transformation of the form

$$Y := \begin{bmatrix} 1 & \bar{Y} \\ 0 & I \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad \bar{Y}^\top \in \mathbb{R}^2,$$

so that, with $\gamma_4 := \gamma_2 - v_{23}\gamma_1$,

$$\begin{aligned} YV(\Lambda + T(\overline{\delta} + T)E)V^{-1}Y^{-1} \;\; = \;\; & \begin{bmatrix} 1+\bar{\lambda}T & \begin{bmatrix}\gamma_1 & \gamma_4\end{bmatrix} T + \bar{Y}\begin{bmatrix} 1+\lambda_1 T & 0 \\ 0 & 1+\lambda_2 T\end{bmatrix} - (1+\bar{\lambda}T)\bar{Y} \\ \begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix} 1+\lambda_1 T & 0 \\ 0 & 1+\lambda_2 T\end{bmatrix} \end{bmatrix} \\ & + T(\overline{\delta} + T)YVEV^{-1}Y^{-1}. \end{aligned}$$

Choose $\bar{Y} = \begin{bmatrix} \frac{\gamma_1}{\bar{\lambda}-\lambda_1} & \frac{\gamma_4}{\bar{\lambda}-\lambda_2} \end{bmatrix}$ to get

$$YV(\Lambda + T(\overline{\delta} + T)E)V^{-1}Y^{-1} = \begin{bmatrix} 1+\bar{\lambda}T & 0 & 0 \\ 0 & 1+\lambda_1 T & 0 \\ 0 & 0 & 1+\lambda_2 T \end{bmatrix} + T(\overline{\delta} + T)YVEV^{-1}Y^{-1}.$$

Defining $N := YV$, we have

$$N\psi[k+1] = \left(\begin{bmatrix} 1+\bar{\lambda}T & 0 & 0 \\ 0 & 1+\lambda_1 T & 0 \\ 0 & 0 & 1+\lambda_2 T \end{bmatrix} + T(\overline{\delta}+T)NEN^{-1} \right) N\psi[k] + NW(T) \begin{bmatrix} \|w_n\|_\infty \\ \|w_d\|_\infty \end{bmatrix}.$$

All elements of N are non-negative because $0 > \bar{\lambda} > \lambda_1 > \lambda_2$. Using the non-negativity of N,

$$\|N\psi[k+1]\| \geq \left\| N \begin{bmatrix} v[k+1] \\ \left\| \begin{bmatrix} \bar{x}[k+1] \\ \bar{z}[k+1] \end{bmatrix} \right\|_1 \\ \|s[k+1]\|_\infty \end{bmatrix} \right\| = \|p[k+1]\|.$$

Taking the ∞-norm of $p[k+1]$, there exist constants $\gamma > 0$ and $c > 0$ such that

$$\begin{aligned} \|p[k+1]\|_\infty &\leq \|N\psi[k+1]\|_\infty \leq \left(1 + \bar{\lambda}T + \gamma T(\overline{\delta} + T)\right) \|N\psi[k]\|_\infty + cT\|w_n\|_\infty + cT\|w_d\|_\infty \\ &= \left(1 + \bar{\lambda}T + \gamma T(\overline{\delta} + T)\right) \|p[k]\|_\infty + cT\|w_n\|_\infty + cT\|w_d\|_\infty. \end{aligned}$$

Choosing $T_2 < -\frac{\bar{\lambda}}{2\gamma}$ and $\bar{\delta} \in (0, -\frac{\bar{\lambda}}{2\gamma})$, we have that $\hat{\lambda} := \bar{\lambda} + \gamma\bar{\delta} + \gamma T < 0$. Then we have, for all $T \in (0, T_2)$,

$$\begin{aligned}\|p[k+1]\|_\infty \leq & \left(1 + \hat{\lambda} T\right) \|p[k]\|_\infty + cT \|w_n\|_\infty + cT \|w_d\|_\infty \\ \leq & \mathrm{e}^{\hat{\lambda} T} \|p[k]\|_\infty + cT \|w_n\|_\infty + cT \|w_d\|_\infty .\end{aligned}$$

Proof of (i)

By the definition of p,

$$p(t) - p[k] = N \begin{bmatrix} |v(t)| - |v[k]| \\ \left\| \begin{bmatrix} \bar{x}(t) \\ \bar{z}(t) \end{bmatrix} \right\|_1 - \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 \\ \|s(t)\|_\infty - \|s[k]\|_\infty \end{bmatrix}, \quad t \geq 0.$$

Taking the 1-norm and using the reverse triangle inequality, for $t \in [kT, (k+1)T)$, we get

$$\begin{aligned}\|p(t) - p[k]\|_1 \;=\; & \mathcal{O}(1)|v(t) - v[k]| + \mathcal{O}(1) \|x(t) - x[k]\| \\ & + \mathcal{O}(1) \|z(t) - z[k]\| + \mathcal{O}(1) \|r(t) - r[k]\| . \end{aligned} \tag{A.14}$$

The solution to (4.1) with initial condition $v[k]$ is

$$v(t) = v[k] + \int_{kT}^{t} \lambda(v(\tau) - v[k])\mathrm{d}\tau + (t - kT)\lambda v[k] + \int_{kT}^{t} (\|u(\tau)\| + \|y(\tau)\|)\mathrm{d}\tau.$$

Rearranging and taking the absolute value, we have

$$|v(t) - v[k]| \leq |\lambda| \int_{kT}^{t} |v(\tau) - v[k]|\mathrm{d}\tau + T|\lambda|v[k] + \int_{kT}^{t} (\|u(\tau)\| + \|y(\tau)\|)\mathrm{d}\tau,$$

and by applying the Bellman-Gronwall inequality and using Proposition 3, it follows that for sufficiently small T_2, for all $T \in (0, T_2)$ and all $t \in [kT, (k+1)T)$,

$$\begin{aligned}|v(t) - v[k]| \;=\; & \mathcal{O}(T)v[k] + \mathcal{O}(T) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \mathcal{O}(T) \|s[k]\|_\infty \\ & + \mathcal{O}(T) \|w_n\|_\infty + \mathcal{O}(T^2) \|w_d\|_\infty . \end{aligned} \tag{A.15}$$

For $t \in [kT, (k+1)T)$ we also have $\|z(t) - z[k]\| = 0$ and $\|r(t) - r[k]\| = 0$. Using these bounds on (A.14), along with (A.3) and (A.15), we get

$$\begin{aligned}\|p(t) - p[k]\| \;=\; & \mathcal{O}(T) \left\|N^{-1}\right\| \|p[k]\| + \mathcal{O}(T)(\|w_n\|_\infty + \|w_d\|_\infty) \\ =\; & \mathcal{O}(T) \|p[k]\| + \mathcal{O}(T)(\|w_n\|_\infty + \|w_d\|_\infty).\end{aligned}$$

So, for $t \in [kT, (k+1)T)$, there exists a constant $c > 0$ so that

$$\|p(t) - p[k]\| \leq cT \|p[k]\| + cT \|w_n\|_\infty + cT \|w_d\|_\infty .$$

Proof of (ii)

The bound (A.14) derived in the proof of Part (i) remains valid. Additionally, using (4.3) we have

$$z[k+1] - z[k] = TF(\hat{\alpha}[k])z[k] + TG(\hat{\alpha}[k])r[k],$$

so taking the norm and employing order notation,

$$\|z[k+1] - z[k]\| = \mathcal{O}(T) \left\| \begin{bmatrix} \bar{x}[k] \\ \bar{z}[k] \end{bmatrix} \right\|_1 + \mathcal{O}(T) \|s(t)\|_\infty = \mathcal{O}(T) \|p[k]\| .$$

We can also upper bound $\|r[k+1] - r[k]\|$. We have, again from (4.3),

$$\begin{aligned} r[k+1] - r[k] &= TRB(\hat{\alpha}[k])H(\hat{\alpha}[k])z[k] + T\Big(Q(\hat{\alpha}[k]) + RB(\hat{\alpha}[k])K(\hat{\alpha}[k])M\Big)r[k] \\ &\quad -TL(\hat{\alpha}[k])\Big(C(\alpha[k])x[k] + w_n[k]\Big), \end{aligned}$$

so taking the norm and employing order notation,

$$\|r[k+1] - r[k]\| = \mathcal{O}(T) \|p[k]\| + \mathcal{O}(T) \|w_n\|_\infty .$$

Applying these bounds to (A.14), along with Proposition 3 and (A.8), for T_2 sufficiently small and $T \in (0, T_2)$, we get

$$\|p[k+1] - p[k]\| = \mathcal{O}(T) \|p[k]\| + \mathcal{O}(T) \|w_n\|_\infty + \mathcal{O}(T) \|w_d\|_\infty .$$

So there exists a constant $c > 0$ so that, for all $T \in (0, T_2)$,

$$\|p[k+1]\| \leq (1 + cT) \|p[k]\| + cT \|w_n\|_\infty + cT \|w_d\|_\infty .$$