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Highlights

- A matrix method for an inverse conflict analysis is developed.
- Conditions for required preference relations are derived.
- Four solution concepts of human behaviour are considered.
- This method is applicable for both transitive and intransitive preferences.
- Two application approaches are described.

Matrix Representations of the Inverse Problem in the Graph Model for Conflict Resolution

Junjie Wang[a, b], Keith W. Hipel[b, c, d], Liping Fang[e, b, *], and Yaoguo Dang[a]

[a]*College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 211100, China*

[b]*Department of Systems Design Engineering, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*

[c]*Centre for International Governance Innovation, Waterloo, Ontario, N2L 6C2, Canada*

[d]*The Balsillie School of International Affairs, Waterloo, Ontario, N2L 6C2, Canada*

[e]*Department of Mechanical and Industrial Engineering, Ryerson University, 350 Victoria Street, Toronto, Ontario M5B 2K3, Canada*

***Abstract*— Given the final individual stability for each decision maker or an equilibrium of interest, a matrix-based method for an inverse analysis is developed in order to calculate all of the possible preferences for each decision maker creating the stability results based on the Nash, general metarationality, symmetric metarationality, or sequential stability definition of possible human interactions in a conflict. The matrix representations are furnished for the relative preferences, unilateral movements and improvements, as well as joint movements and joint improvements for a conflict having two or more decision makers. Theoretical conditions are derived for specifying required preference relationships in an inverse graph model. Under each of the four solution concepts, a matrix relationship is established to obtain all the available preferences for each decision maker causing the specific state to be an equilibrium. To explain how it can be employed in practice, this new approach to inverse analysis is applied to the Elsipogtog First Nation fracking dispute which took place in the Canadian Province of New Brunswick.**

***Keywords*—Group decision and negotiation, Inverse analysis, Matrix representations, Conflict resolution, Graph model**

1. Introduction

Conflicts occur whenever two or more decision makers (DMs) having differences in value systems, objectives or preferences, interact in the real world. In fact, each DM in a dispute strives to change the course of the conflict and reach a state of interest such as a more preferred state

[*]Corresponding author.

Email addresses: junjie881129@hotmail.com (J. Wang), kwhipel@uwaterloo.ca (K. W. Hipel), lfang@ryerson.ca (L. Fang), iamdangyg@163.com (Y. Dang).

than the status quo. In order to better represent and analyze conflict, many available models to conflict resolution have been proposed within a broad field called game theory. The normal and extensive forms of the game, which are generally considered to be part of classical game theory, were developed by Von Neumann and Morgenstern (1944). Classical game theory is considered to be quantitative in nature because it uses cardinal preferences often expressed as utility values. However, sometimes it is difficult for a DM to determine how much he prefers one state to another. Thus, Howard (1971) designed a fresh approach called metagame analysis which only assumes the availability of relative preference information in which a given DM either prefers one state over another or they are equally preferred. A methodology called conflict analysis put forward by Fraser and Hipel (1979; 1984) was an enhancement and expansion of metagame analysis. The Graph Model for Conflict Resolution (GMCR), which is more comprehensive than existing methodologies, was proposed by Kilgour, Hipel, and Fang (1987) and Fang, Hipel, and Kilgour (1993). The above three methodologies are regarded as qualitative techniques because only relative preference information between any two states is assumed. Because of the foregoing and other reasons, GMCR is widely employed by practitioners and researchers for investigating real world conflict in a highly flexible yet simple way (Madani, 2013).

According to the GMCR procedure, the elements used in this approach can be classified into three main parts which are input, analysis, and output (Fang, Hipel, Kilgour, & Peng, 2003; Kinsara, Petersons, Hipel, & Kilgour, 2015b). The primary items in the input part are the DMs, feasible states in the dispute and DMs' relative preferences over the states. Either an individual or a group, such as a company, can be a DM. A DM can control one or more options, each of which can be selected or not by the DM who controls it. A feasible state is formed as a specific selection of options by the DMs. The analysis part is employed to determine whether a given state is stable for a specified DM or not. The state is said to be stable for a DM if the DM cannot reach a more preferred state in the midst of moves and counter movements by other DMs. An equilibrium of a graph model is a state that is individually stable for all DMs under the same stability definition. A series of stability definitions have been proposed including Nash stability (Nash, 1950, 1951), general metarationality (GMR) (Howard, 1971), symmetric metarationality (SMR) (Howard, 1971), and sequential stability (SEQ) (Fraser & Hipel, 1979, 1984).

The DMs in conflicts may have different purposes when investigating a dispute with different known information. In most situations, one wishes to ascertain the output of an ongoing or a historical dispute by using the analysis engine to calculate various types of individual stability and equilibria after identifying the input part. This is called the forward perspective as portrayed at the top of Fig. 1. In Fig. 1, a check sign (√) means the associated information is known while a question mark (?) indicates an item to determine. Most of the extensions to enrich the theory and applicability of GMCR have been developed under the domain of the forward perspective (Basher, Kilgour, & Hipel, 2012; Bristow, Fang, & Hipel, 2014; He, Kilgour, & Hipel, 2017; Xu, Hipel, & Kilgour, 2009; Xu, Kilgour, Hipel, & Kemkes, 2010). In some cases, the analyst may wish to determine the type of behavior needed to reach a state of interest. This is called the behavioral problem which is depicted as the middle diagram in Fig. 1 (Kinsara et al., 2015b) and for which a mathematical solution was recently provided (Wang, Hipel, Fang, Xu, & Kilgour, 2018).

In some conflict situations, one wishes to know the preferences required by DMs in order to reach an attractive resolution for all parties. In third party intervention, for example, a third party is invited to a negotiation in order to assist the disputants to reach a win/win resolution (see, for instance, Hipel, Sakamoto, and Hagihara (2015)). The third party facilitators may wish to ascertain which preferences are required by the parties in order to reach such an attractive outcome. In order to analyze the resolution of such conflicts in which the preferences for each DM are unknown or partially unknown, the inverse analysis in a graph model is proposed as displayed at the bottom of Fig. 1. As introduced by Kinsara, Kilgour, and Hipel (2015a), the main feature of the inverse analysis is that the preference information must be determined.

In summary, GMCR can be categorized into three perspectives based on the different given information and goals. As can be appreciated, each perspective solves a different kind of conflict problem. The differences among these three perspectives in a graph model are encapsulated as follows:

a) The forward perspective determines the possible equilibria by carrying out the stability analysis based on the preferences of each DM contained in the input.
b) The behavioral perspective ascertains the types of behavior which can produce the outcome of that dispute with the known preferences.
c) The inverse perspective determines the unknown or partially unknown preference relationships for each DM which are required to make a state of interest be an equilibrium under a specific type of behavior.

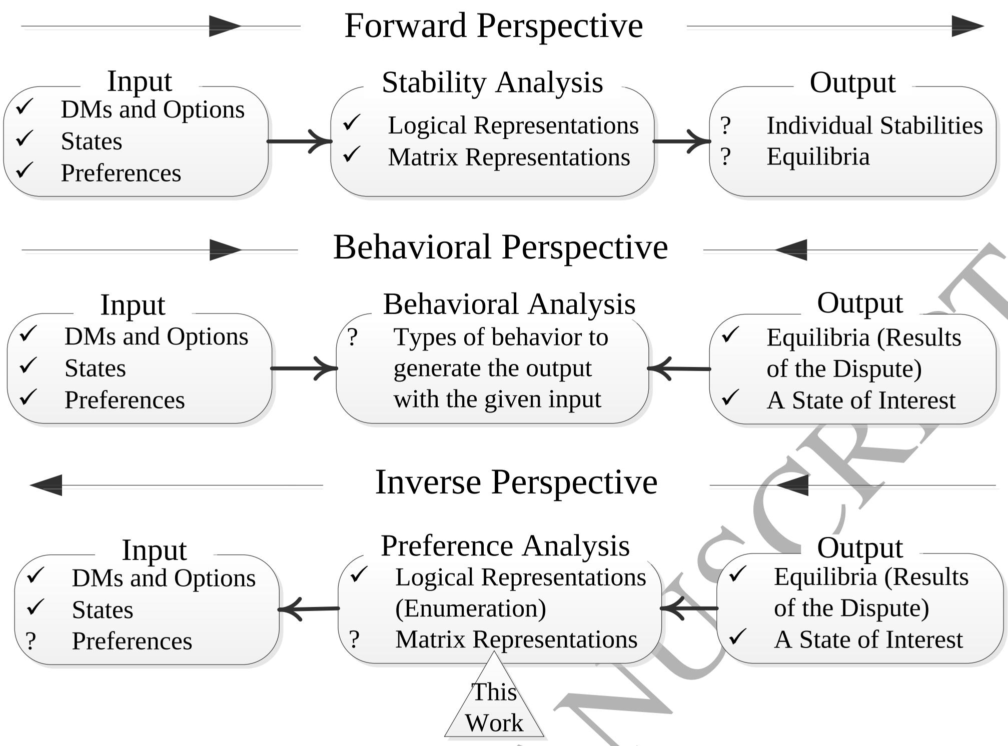

Fig. 1. Three perspectives of carrying out a GMCR study (based on Fig. 1 in the paper by Kinsara et al. (2015b)).

Inverse analysis can provide all of the possible preferences for the DMs to reach a desired resolution. For instance, a third party in a conflict (Hipel et al., 2015), who may be a mediator or analyst, can use the results of the inverse approach to determine how to persuade each DM to select the options resulting in the desired equilibrium according to the needed preferences. In other words, a third-party intervenor can employ inverse analysis to design his mediation strategy based on the required preference relationships to reach a more desired outcome. On the other hand, a particular DM involved in the dispute can take advantage of inverse analysis to change his own preferences and attempt to influence a competitor such that an equilibrium of interest can be reached (Kinsara et al., 2015a). In fact, within engineering and science, inverse analysis is referred to as inverse engineering and constitutes a crucial field of study when addressing physical systems problems (Gladwell, 2005). The topic of this paper is inverse engineering within societal systems in the presence of conflict.

In the field of conflict resolution, techniques for tackling the inverse problem possess some drawbacks. More specifically, the inverse model studied by Kinsara et al. (2015a) assumes the employment of ordinal preferences which mean the preferences are transitive and hence intransitive preferences cannot be handled. Moreover, Sakakibara, Okada, and Nakase (2002) and

Kinsara et al. (2015a) investigated the inverse problem using the logical representation of GMCR and an enumeration method which encounters the issue of computational complexity. In order to address the aforementioned two shortcomings, the preferences of each DM are defined by using pairwise comparisons of every pair of states in conflict as given later in this paper in Definition 1, which means the findings in this paper are valid for both transitive and intransitive preferences. Preference matrices are defined later in Definition 2 to mathematically represent the binary relation between each two states. Preference matrices are employed to derive the mathematical representations of an inverse problem in a graph model having two or more DMs in this research. The preferences for each DM required to make a state of interest be an equilibrium with the given solution concept can be obtained using the inequalities provided in Section 4. As shown in Section 5.4, the computational complexity of the inverse analysis in a graph model can be enormously reduced by using the matrix representations of the inverse GMCR approach proposed in this paper instead of the enumeration method given by Kinsara et al. (2015a).

The remainder of this paper is organized as follows. In Section 2, two potential general applications of this inverse engineering approach to GMCR are described. Within Section 3, matrix representations of preferences, unilateral movements and improvements, and joint movements and improvements are given. Matrix formulations of the inverse analyses for Nash stability, GMR, SMR and SEQ are presented and proven in Section 4. Section 5 consists of a case study of a controversial fracking dispute among the Elsipogtog First Nation, New Brunswick Provincial Government and Southwestern Energy (SWN) Resources in Eastern Canada, which is used to demonstrate how the proposed matrix representations of the inverse problem can be conveniently employed in practice. Finally, some conclusions and ideas for future work are presented in Section 6.

2. Application approaches

In some real life conflicts, the output and part of the input preference information to a graph model are known, but the required preferences of one or more DMs to generate the output are unknown, as portrayed at the bottom of Fig. 1. More specifically, an analyst may wish to ascertain the possible preferences of DMs which satisfy the given final stability results in a historical conflict. For instance, in the past two DMs have reached an equilibrium in a conflict after tough negotiations. However, little information is provided about their preferences. A historian could employ the inverse method provided in this paper to determine the DMs' preferences. Moreover, he could also utilize the inverse methodology to ascertain the preferences needed to attain an even more desirable resolution for both parties, in order to explain for instance why a Pareto superior outcome did not occur. In addition, for an ongoing dispute, the DMs may want to determine what preferences can help them reach a specified preferred equilibrium or an equilibrium of interest. For example, a facilitator or third party (Hipel et al., 2015) may wish to influence the preferences of one or more participants involved in a conflict such that a better or win/win resolution can be achieved. These two potential problems are portrayed in Fig. 2.

As just mentioned, two major ways in which the inverse GMCR approach can be used for addressing conflict are:

a) For a historical conflict, ascertain all the possible preferences of each DM that produced the known equilibrium which is the final result of the dispute;

b) For an ongoing conflict, determine each possible preference relationship of the DMs required to reach a particular desirable resolution or state of interest if it is possible to do so.

These two situations that could occur in an inverse analysis study are depicted as the upper and lower diagrams in Fig. 2, respectively. In Fig. 2, a check sign (√) means the associated information is known while a question mark (?) indicates an item to calculate.

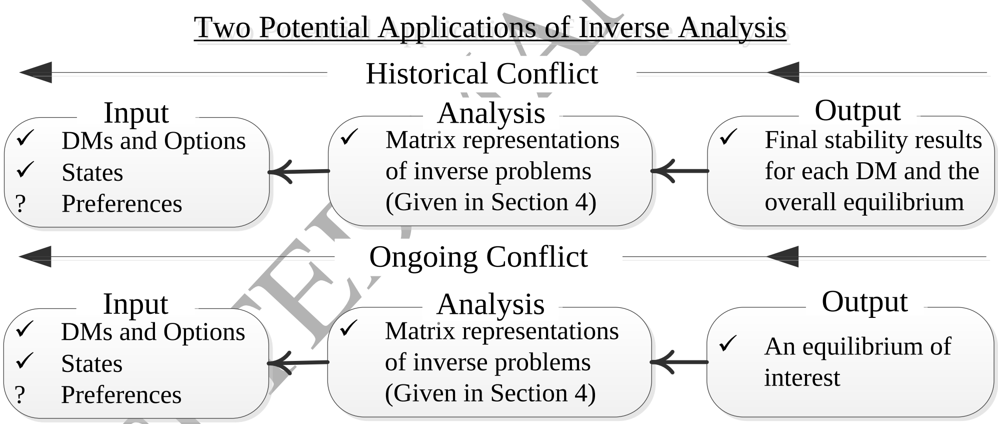

Fig. 2. Two potential applications of inverse problems.

As depicted in Fig. 2, the matrix representations for determining the preferences in an inverse GMCR problem can be used in both historical and ongoing conflicts. In a real conflict, partial preference relations of each DM may be given. With the known part of the preferences, the remaining preferences required to reach the final equilibrium in a historical conflict or produce an equilibrium of interest can be calculated using the matrix representations proposed in Section 4. In some particular situations, it is not possible to reach a state of interest under the known preferences. In addition, this approach can also be used when the preferences of each DM are completely unknown.

3. Modeling of a graph model with matrix representations

GMCR was originally formulated using what is called the “logical” form since solution concepts defining possible human behavior are explained in terms of moves and counter moves. For instance, if all possible unilateral improvements from a given state by a particular DM can be blocked by counter moves by other DMs, the DM is better off not to move and the state under consideration is deemed to be stable. Xu et al. (2009, 2010) provided a clever matrix or algebraic formulation of GMCR including the range of solution concepts describing possible human behavior under conflict. Because the matrix form is especially well suited for performing stability calculations within a decision support system and for theoretically expanding the scope of GMCR, this formulation is employed in this paper for defining inverse GMCR.

To analyze the needed preferences for DMs causing a state of interest to be an equilibrium, the basic concepts for constructing a graph model of the conflict using matrix representations are given in this section. In particular, the matrices used to keep track of preferences, unilateral moves and unilateral improvements for each DM in a conflict model are defined. The definitions of group movements and the joint movement and improvement matrices for the group in a dispute are furnished in this section.

Definition 1 (Fang et al., 1993; Kilgour et al., 1987) (Graph Model): The four main components required to develop a graph model of a conflict with two or more DMs are provided as follows:

1) a finite set of the DMs represented by N where $|N| = n \geq 2$ (“$|\cdot|$” denotes the cardinality of a set);
2) the set of all the feasible states is denoted as $S = \{1, 2, \cdots, m\}$ in which m is the number of feasible states;
3) for each DM $i \in N$, let $A_i \subseteq S \times S$ be a set of arcs containing all the unilateral movements controlled by DM i in one step;
4) for each DM $i \in N$, the simple preference of DM i which contains a pair of relations $\{\succ_i, \sim_i\}$ on S, where $s \succ_i q \, (s \in S, q \in S)$ indicates that DM i strictly prefers state s to q while $s \sim_i q$ implies DM i equally prefers s and q.

Three properties connected to the above preference relations of each DM $i \in N$ are given as follows:

1) $\sim_i$ is reflexive and symmetric (i.e., $\forall s, q \in S, s \sim_i s$, and if $s \sim_i q$, then $q \sim_i s$);
2) $\succ_i$ is asymmetric (i.e., $s \succ_i q$ and $q \succ_i s$ cannot happen simultaneously);

3) $\{\succ_i, \sim_i\}$ is strongly complete which implies for any $s, q \in S$, only one of the following preference relations of DM i can hold: $s \sim_i q$, $s \succ_i q$ or $q \succ_i s$.

Because of the third property, it is assumed any given two states can be compared using strict or equal preference. Roy (1996, Section 7.1.1.2) uses the terminology of indifference by a DM between two states when these two states are equally preferred. In practice, especially when the conflict has existed for a while, preferences are usually transitive whereby the states can be ranked from most to least preferred where ties are allowed. An example of transitivity among three states, $s, q,$ and $p \in S$, for DM i is $s \succ_i q$, $q \succ_i p$, and $s \succ_i p$. However, as discussed in the introduction, the preferences of some DMs may be intransitive in some situations, especially when the conflict is in its early stages. For instance, the preference relations of DM i for the three given states s, q and p are $s \succ_i q$, $q \succ_i p$, and $p \succ_i s$ or $p \sim_i s$. The findings in this research can handle both transitive and intransitive preferences.

Definition 2 (Xu et al., 2009) (Preference): In a graph model, let P_i^+, P_i^-, $P_i^=$ and $P_i^{-,=}$ be the four $m \times m$ preference matrices for DM i whose entry (s, q) for which $s, q \in S$ is defined as follows:

$$P_i^+(s,q) = \begin{cases} 1, & \text{if } q \succ_i s \\ 0, & \text{otherwise,} \end{cases}$$

$$P_i^-(s,q) = \begin{cases} 1, & \text{if } s \succ_i q \\ 0, & \text{otherwise,} \end{cases}$$

$$P_i^=(s,q) = \begin{cases} 1, & \text{if } q \sim_i s \text{ and } q \neq s \\ 0, & \text{otherwise,} \end{cases}$$

$$P_i^{-,=}(s,q) = \begin{cases} 1 - P_i^+(s,q), & \text{if } s \neq q \\ 0, & \text{otherwise.} \end{cases}$$

For the above definition, it is assumed that $P_i^=(s,s) = P_i^{-,=}(s,s) = 0$.

Definition 3 (Xu et al., 2009) (Unilateral Movements and Improvements of a focal DM): In a graph model, let J_i and J_i^+ denote two $m \times m$ 0-1 matrices representing the unilateral moves and unilateral improvements of DM i respectively, as follows:

$$J_i(s,q) = \begin{cases} 1, & \text{if } (s,q) \in A_i \\ 0, & \text{otherwise,} \end{cases}$$

$$J_i^+(s,q) = \begin{cases} 1, & \text{if } J_i(s,q) = 1 \text{ and } q \succ_i s \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the set of unilateral moves of DM *i* from an initial state s, denoted by $R_i(s)$, is defined as

$$R_i(s) = \{q: \text{ if } J_i(s,q) = 1\},$$

and the set of unilateral improvements of DM i from s, represented by $R_i^+(s)$, is given as

$$R_i^+(s) = \{q: \text{ if } J_i^+(s,q) = 1\}.$$

For Definition 3, it is assumed that $J_i(s,s) = 0$.

Assume that $H \subseteq N$ is any subset of the DMs, and let $R_H(s)$ and $R_H^+(s)$ represent the set of all states that can be formed from any sequence of unilateral moves and unilateral improvements, by some or all of the DMs in H, beginning at state s, respectively. In the definitions, one DM can move more than once, but not twice consecutively. If $s_1 \in R_H(s)$, let $\Omega_{Hs}(s_1)$ stand for the set of all last DMs in legal sequences from s to s_1. Similarly, if $s_1 \in R_H^+(s)$, $\Omega_{Hs}^+(s_1)$ is used to denote the set of all last DMs in legal sequences of unilateral improvements from s to s_1.

Definition 4 (Fang et al., 1993) **(**Movements and Improvements by DMs in H **):** Let $s \in S$ and $H \subseteq N$, $H \neq \emptyset$. A unilateral move by H is a member of $R_H(s) \subseteq S$ and a unilateral improvement by H is a member of $R_H^+(s) \subseteq S$, defined inductively by:

Definition for $R_H(s)$:

1) if $j \in H$ and $s_1 \in R_j(s)$, then $s_1 \in R_H(s)$ and $j \in \Omega_{Hs}(s_1)$,

2) if $s_1 \in R_H(s)$, $j \in H$, and $s_2 \in R_j(s_1)$, then

   a) if $|\Omega_{Hs}(s_1)| = 1$ and $j \notin \Omega_{Hs}(s_1)$, then $s_2 \in R_H(s)$ and $j \in \Omega_{Hs}(s_2)$.

   b) if $|\Omega_{Hs}(s_1)| > 1$, then $s_2 \in R_H(s)$ and $j \in \Omega_{Hs}(s_2)$.

Definition for $R_H^+(s)$:

1) if $j \in H$ and $s_1 \in R_j^+(s)$, then $s_1 \in R_H^+(s)$ and $j \in \Omega_{Hs}^+(s_1)$,

2) if $s_1 \in R_H^+(s)$, $j \in H$, and $s_2 \in R_j^+(s_1)$, then

   a) if $|\Omega_{Hs}^+(s_1)| = 1$ and $j \notin \Omega_{Hs}^+(s_1)$, then $s_2 \in R_H^+(s)$ and $j \in \Omega_{Hs}^+(s_2)$,

   b) if $|\Omega_{Hs}^+(s_1)| > 1$, then $s_2 \in R_H^+(s)$ and $j \in \Omega_{Hs}^+(s_2)$.

Definition 5 (Xu et al., 2009) **(**Joint Movement and Joint Improvement Matrices**):** In a graph model, let M_H and M_H^+ be two $m \times m$ matrices called the joint movement matrix and joint improvement matrix whose entry (s,q) is given as follows:

$$M_H(s,q)=\begin{cases}1, & \text{if } q\in R_H(s)\\ 0, & \text{otherwise,}\end{cases}$$

$$M_H^+(s,q)=\begin{cases}1, & \text{if } q\in R_H^+(s)\\ 0, & \text{otherwise.}\end{cases}$$

Note that when $H=\{i\}$, $M_H=J_i$ and $M_H^+=J_i^+$.

Definition 6 (Sign Function): Given an $m\times m$ matrix M. Let $sign(M)$ denote the $m\times m$ matrix with (s,q) entry as follows:

$$sign[M(s,q)]=\begin{cases}1, & M(s,q)>0\\ 0, & M(s,q)=0\\ -1, & M(s,q)<0.\end{cases}$$

4. Inverse analysis using matrix representations

In the inverse problematique of a graph model, one may wish to study the preferences of the DMs given the known behavior of the DMs and a specified equilibrium. The DMs in the conflict may want to ascertain what preferences are needed to cause a state of interest to be stable according to the specified behavior. Four definitions of stabilities consisting of Nash stability (Nash, 1950, 1951), general metarationality (GMR) (Howard, 1971), symmetric metarationality (SMR) (Howard, 1971), and sequential stability (SEQ) (Fraser & Hipel, 1979, 1984) are given in this section. According to the definitions, four theorems and their proofs corresponding to the four types of solution concepts are given to analyze the inverse problems using matrix representations in a graph model of the conflict. In this section, the four equivalent matrix definitions for stability were originally provided by Xu et al. (2009).

As proved by Fang, Hipel, and Kilgour (1989) and Fang et al. (1993), the theoretical relationships among the aforementioned four solution concepts are as portrayed in Fig. 3. The relationships in the Venn diagram are valid for both individual stability and equilibria according to the solution concepts. Notice, for example, in Fig. 3 that if a state is Nash stable then it is also GMR, SMR and SEQ. Additionally, the stability findings for each of Nash, SMR and SEQ are always a subset of GMR results. These special theoretical relationships are used when proving Theorems 3, 6, and 7 in Section 4.

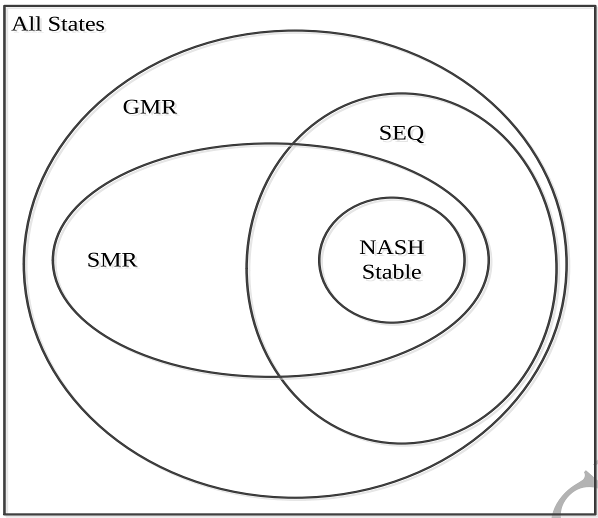

Fig. 3. Relationships among solution concepts for both individual stability and equilibria (based on Fang et al. (1989, 1993)).

In the following definitions and theorems, recall that $|N| = n\,(n \geq 2)$ and m denote the numbers of DMs and feasible states, respectively. Let E be an $m \times m$ matrix and e denote an m-dimensional column vector in both of which each entry is 1. Let e_s be an m-dimensional column vector where the s^{th} entry is 1 and all other entries are 0. Given two $m \times g$ matrices C and Y, define the Hadamard product (Davis, 1962) $Q = C \circ Y$ as the $m \times g$ matrix with (s, q) entry $Q(s, q) = C(s, q) \cdot Y(s, q)$. Let $N - \{i\}$ denote the set of the other DMs except DM i.

A definition for the relationships between matrices and a key theorem are given before the inverse analysis to ascertain what kind of preferences under conflict, referred to as the four solution concepts, are in consonance with known stabilities or states of interest to the DMs.

Definition 7 (Matrix Relations): Let L_1 and L_2 be two $m \times 1$ matrices.

1) Definition for the matrix relation denoted using $\geq$: The relation $L_1 \geq L_2$ means that each entry $L_1(l, 1)\ (l = 1, 2, \cdots, m)$ in matrix L_1 is either larger or equal to the specified entry $L_2(l, 1)\ (l = 1, 2, \cdots, m)$ in matrix L_2.

2) Definition for the matrix relation which uses $\ngeq$: The relation $L_1 \ngeq L_2$ denotes that there is at least one entry $L_1(l, 1)$ in matrix L_1 which is smaller than the specified entry $L_2(l, 1)$ in matrix L_2.

Theorem 1: Let K denote a 0-1 $m \times 1$ matrix and L be an $m \times 1$ matrix whose entry

$L(l,1)$ is a non-negative integer. Define the following function f by

$$f = K^T \cdot (e - sign(L)). \quad (T \text{ denotes matrix and vector transpose.}) \tag{1}$$

Then, $f = 0$ iff $L \geq K$.

Proof: 1) If $f = 0$, $f = \sum_{l=1}^{m} K(l,1) \cdot (1 - sign(L(l,1))) = 0$. Since K is a 0-1 $m \times 1$ matrix, $K(l,1)$ is equal to either 1 or 0.

When $K(l,1) = 1$, $sign(L(l,1))$ must be equal to 1 which means $L(l,1) \geq 1$. Then, $L(l,1) \geq K(l,1)$;

When $K(l,1) = 0$, $sign(L(l,1))$ can be equal to either 0 or 1 which means $L(l,1)$ could be any non-negative integer according to the definiton of L. Then, $L(l,1) \geq K(l,1)$.

Hence, if $f = 0$ then $L \geq K$.

2) If $L \geq K$, $f = \sum_{l=1}^{m} K(l,1) \cdot (1 - sign(L(l,1)))$.

When $K(l,1) = 1$, $L(l,1) \geq K(l,1) = 1$. Then, $sign(L(l,1)) = 1$ which indicates $f = 0$.

When $K(l,1) = 0$, it is obvious that $f = 0$.

Hence, if $L \geq K$, then $f = 0$.□

Definition 8 (Matrix Definition for Nash Stability): State $s \in S$ is Nash stable for DM i iff $e_s^T \cdot J_i^+ \cdot e = 0$.

Theorem 2 (Inverse Analysis for Nash Stability): If state $s \in S$ is Nash stable for DM $i \in N$, the preferences of DM i satisfy the following conditions:

$$e_s^T \cdot J_i \cdot ((P_i^+)^T \cdot e_s) = 0. \tag{2}$$

Proof: Since state $s \in S$ is Nash stable for DM i, then

$$e_s^T \cdot J_i^+ \cdot e = \sum_{s_1=1}^{m} J_i(s, s_1) \cdot P_i^+(s, s_1) = 0 \quad \text{iff} \quad J_i(s, s_1) \cdot P_i^+(s, s_1) = 0, \forall s_1 \in S.$$

Therefore, $\sum_{s_1=1}^{m} J_i(s, s_1) \cdot P_i^+(s, s_1) = e_s^T \cdot J_i \cdot ((P_i^+)^T \cdot e_s) = 0$. Equation (2) indicates that for any $s_1 \in R_i(s)$, the preference relation of DM i between states s_1 and s must be $P_i^+(s, s_1) = 0$.□

Definition 9 (Matrix Definition for GMR): Given the following $m \times m$ matrix M_i^{GMR},

$$M_i^{GMR} = J_i^+ \cdot [E - sign(M_{N-\{i\}} \cdot (P_i^{-,=})^T)]. \tag{3}$$

State $s \in S$ is GMR for DM i iff $M_i^{GMR}(s, s) = 0$.

Theorem 3 (Matrix Inequality for Inverse GMR): State $s \in S$ is GMR for DM $i \in N$ iff the preferences of DM i satisfy the following conditions:

$$M_{N-\{i\}} \cdot ((P_i^{-,=})^T \cdot e_s) \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s). \quad (4)$$

Proof: Based on Definition 2, $P_i^{-,=} = E - I - P_i^+$ where I is the identity matrix. Since state $s \in S$ is GMR for DM i, then according to Definition 9,

$$M_i^{GMR}(s,s) = [(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)]^T \cdot [e - sign(M_{N-\{i\}} \cdot ((P_i^{-,=})^T \cdot e_s))] = 0. \quad (5)$$

In Equation (5), $(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$ is a 0-1 $m \times 1$ matrix and $M_{N-\{i\}} \cdot ((P_i^{-,=})^T \cdot e_s)$ is an $m \times 1$ matrix whose entries are all non-negative integers. According to Theorem 1, $M_i^{GMR}(s,s) = 0$ which implies state s is GMR for DM i iff $M_{N-\{i\}} \cdot ((P_i^{-,=})^T \cdot e_s) \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$.□

Note that Inequality (4) is equivalent to the following inequality,

$$\sum_{s_2=1}^{m} M_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) - J_i(s, s_1) \cdot P_i^+(s, s_1) \geq 0, \ \forall \ s_1 \in S. \quad (6)$$

The above proof used mathematical logic. Actually, Theorem 3 can also be explained by the theoretical relationships between Nash stability and GMR. Hence, if a state is stable according to Nash stability then it is also GMR as displayed in Fig. 4.

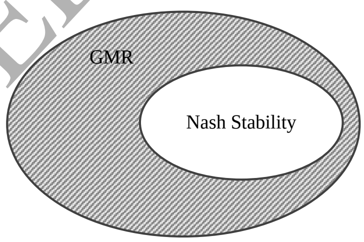

Fig. 4. Theoretical relationships between Nash stability and GMR (based on Fang et al. (1989, 1993)).

In Inequality (6), $J_i(s,s_1) \cdot P_i^+(s,s_1) = J_i^+(s,s_1)$. Then, state s is GMR,

1) If $\forall \ s_1 \in S, J_i^+(s, s_1) = 0$, which means state s is not only GMR stable but also Nash stable for DM i corresponding to the white ellipse in Fig. 4, then $\sum_{s_2=1}^{m} M_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) \geq 0$ according to the definitions of $M_{N-\{i\}}$ and $P_i^{-,=}$.

2) If $\exists s_1 \in S, J_i^+(s, s_1) = 1$, which means state s is not Nash stable but possesses GMR stability for DM i corresponding to the dashed area in Fig. 4, then $\sum_{s_2=1}^{m} M_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) \geq 1$ based on Inequality (6) which implies that for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N-\{1\}}(s_1)$ with $P_i^{-,=}(s, s_2) = 1$.

As indicated in the above situations within 1) and 2), Inequality (6) contains all the conditions of the preference relations for state s to be GMR for DM i.

Definition 10 (Matrix Definition for SMR): Let M_i^{SMR} denote an $m \times m$ matrix as follows:

$$M_i^{SMR} = J_i^+ \cdot [E - sign(M_{N-\{i\}} \cdot W)], \tag{7}$$

where

$$W = (P_i^{-,=})^T \circ [E - sign(J_i \cdot (P_i^+)^T)]. \tag{8}$$

State $s \in S$ is SMR for DM i iff $M_i^{SMR}(s, s) = 0$.

Theorem 4 (Inverse Analysis for SMR): State $s \in S$ is SMR for DM $i \in N$ iff the preferences of DM i satisfy the following conditions:

$$M_{N-\{i\}} \cdot W \cdot e_s \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s). \tag{9}$$

Proof: Since state $s \in S$ is SMR for DM i, then according to Definition 10,

$$M_i^{SMR}(s, s) = [(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)]^T \cdot [e - sign(M_{N-\{i\}} \cdot W \cdot e_s)] = 0. \tag{10}$$

In Equation (10), $(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$ is a 0-1 $m \times 1$ matrix and $M_{N-\{i\}} \cdot W \cdot e_s$ is an $m \times 1$ matrix whose entries are all non-negative integers. According to Theorem 1, $M_i^{SMR}(s, s) = 0$ which implies state s is SMR for DM i iff $M_{N-\{i\}} \cdot W \cdot e_s \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$. □

As noted in Inequality (6), Inequality (4) which is represented by matrix in Theorem 3 can be calculated while the outcome of Inequality (4) should be a set of inequalities whose number is less than m. However, in Theorem 4, a sign function is contained in Inequality (9). Therefore, one can only use the enumeration to solve the inequality with the sign function. A theorem is provided to handle this problem as follows:

Theorem 5: Let K and U denote two 0-1 $m \times 1$ matrices. Let L be an $m \times 1$ matrix whose entry $L(l,1)$ is a non-negative integer. Then,

$$K^T \cdot (U \circ L) = (K^T \circ U^T) \cdot L. \quad (11)$$

Proof: On the left side of Equation (11), $K^T \cdot (U \circ L) = \sum_{l=1}^{m} K(l,1) \cdot (U(l,1) \cdot L(l,1))$ where $K(l,1)$, $U(l,1)$ and $L(l,1)$ are the entries in the three $m \times 1$ matrices, respectively. On the right side of Equation (11), $(K^T \circ U^T) \cdot L = \sum_{l=1}^{m} (K(l,1) \cdot U(l,1)) \cdot L(l,1)$ which is equal to the left side. □

Theorem 6 (Matrix Inequality for Inverse SMR): Inequality (9), that is the matrix condition of preference relations for DM i to generate his individual SMR stability of state s, is equivalent to Formula (12) which can be calculated as:

$$\left\{ J_i(s, s_1) \cdot P_i^+(s, s_1) = 0 \right\} \cup \left\{ J_i \cdot (P_i^+)^T \cdot e_s \not\geq (e_{s_1}^T \cdot M_{N-\{i\}})^T \circ ((P_i^{-,=})^T \cdot e_s) \,\middle|\, J_i(s, s_1) \cdot P_i^+(s, s_1) = 1 \right\},$$
$$\forall s_1 \in S. \quad (12)$$

Proof: From Inequality (9), the following inequality can be obtained,

$$M_{N-\{i\}} \cdot [((P_i^{-,=})^T \cdot e_s) \circ (e - sign(J_i \cdot (P_i^+)^T \cdot e_s))] \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s). \quad (13)$$

Inequality (13) means for each $s_1 \in S$,

$$e_{s_1} \cdot M_{N-\{i\}} \cdot [((P_i^{-,=})^T \cdot e_s) \circ (e - sign(J_i \cdot (P_i^+)^T \cdot e_s))] \geq e_{s_1} \cdot (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s). \quad (14)$$

Since $e_{s_1} \cdot (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s) = J_i(s, s_1) \cdot P_i^+(s, s_1)$, then $e_{s_1} \cdot (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$ must be equal to either 1 or 0.

1) If $J_i(s, s_1) \cdot P_i^+(s, s_1) = 0$, it is obvious that Inequality (13) is true.

2) If $J_i(s, s_1) \cdot P_i^+(s, s_1) = 1$, then

$$e_{s_1} \cdot M_{N-\{i\}} \cdot [((P_i^{-,=})^T \cdot e_s) \circ (e - sign(J_i \cdot (P_i^+)^T \cdot e_s))] \geq 1 \;\; \forall s_1 \in S. \quad (15)$$

Using Theorem 6 provided above, Inequality (15) can be converted to the following inequality:

$$(e_{s_1} \cdot M_{N-\{i\}}) \circ [((P_i^{-,=})^T \cdot e_s)]^T \cdot (e - sign(J_i \cdot (P_i^+)^T \cdot e_s)) \geq 1 \;\; \forall s_1 \in S. \quad (16)$$

The left part of Inequality (16) is equal to a non-negative integer. Therefore, Inequality (16) is equivalent to the following function:

$$(e_{s_1} \cdot M_{N-\{i\}}) \circ [((P_i^{-,=})^T \cdot e_s)]^T \cdot (e - sign(J_i \cdot (P_i^+)^T \cdot e_s)) \neq 0 \;\; \forall s_1 \in S. \quad (17)$$

Consequently, the conditions of preference relations satisfying Inequality (17) can be obtained by removing the outcomes of Equation (18).

$$(e_{s_1} \cdot M_{N-\{i\}}) \circ [\,((P_i^{-,=})^T \cdot e_s)\,]^T \cdot (e - sign(J_i \cdot (P_i^+)^T \cdot e_s)) = 0 \;\; \forall s_1 \in S. \tag{18}$$

Utilizing Theorem 1 put forward in this paper, Equation (18) is equivalent to Inequality (19) shown below:

$$J_i \cdot (P_i^+)^T \cdot e_s \geq (e_{s_1} \cdot M_{N-\{i\}})^T \circ ((P_i^{-,=})^T \cdot e_s) \;\; \forall s_1 \in S. \tag{19}$$

Therefore, Inequality (15) could be presented by Inequality (20):

$$J_i \cdot (P_i^+)^T \cdot e_s \ngeq (e_{s_1} \cdot M_{N-\{i\}})^T \circ ((P_i^{-,=})^T \cdot e_s) \;\; \forall s_1 \in S, \tag{20}$$

Combining the situations 1) and 2), Inequality (12) is equivalent to Inequality (9).□

Note that Inequality (20) means there exists at least one $s_2 \in S$ such that

$$\sum_{s_3=1}^{m} J_i(s_2, s_3) \cdot P_i^+(s, s_3) < M_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) \;\; \forall s_1 \in S, \tag{21}$$

where $J_i(s, s_1) \cdot P_i^+(s, s_1) = 1$.

In Inequality (21), $M_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2))$ must be equal to either 1 or 0. Then, $\sum_{s_3=1}^{m} J_i(s_2, s_3) \times P_i^+(s, s_3) = 0$ if Inequality (21) is true for state $s_2 \in S$.

Similar to the inverse problem for GMR, the above proof used mathematical logic. Actually, Theorem 6 can also be explained by the theoretical relationships between Nash stability and SMR such that if a state is stable according to Nash stability then it is also SMR as portrayed in Fig. 5.

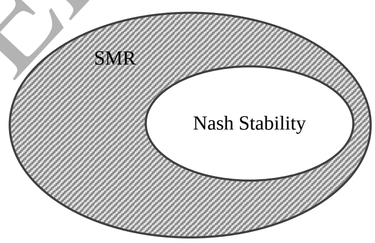

Fig. 5. Theoretical relationships between Nash stability and SMR (based on Fang et al. (1989, 1993)).

In Inequality (12), $J_i(s, s_1) \cdot P_i^+(s, s_1) = J_i^+(s, s_1)$. Hence, state s is SMR,

1) If $\forall s_1 \in S, J_i^+(s, s_1) = 0$, which means that state s is not only SMR stable but also Nash stable for DM i corresponding to the white ellipse in Fig. 5, then $M_{N-\{i\}} \cdot W \cdot e_s \geq 0$ according to the definitions of $M_{N-\{i\}}$ and $P_i^{-,=}$.

2) If $\exists s_1 \in S, J_i^+(s, s_1) = 1$, which means state s is not Nash stable but SMR stable for DM i corresponding to the dashed area in Fig. 5, then as proven in situation 2) of Theorem 6, Inequality (21) is equivalent to the statement that, $\forall s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N-\{i\}}(s_1)$ with $P_i^+(s, s_2) = 0$ and $P_i^+(s, s_3) = 0$ for all $s_3 \in R_i(s_2)$.

As mentioned in the above situations 1) and 2) of Theorem 6, Inequality (12) contains all of the conditions of the preference relations for state s to be SMR for DM i.

Definition 11 (Matrix Definition for SEQ): Given the following $m \times m$ matrix M_i^{SEQ},

$$M_i^{SEQ} = J_i^+ \cdot [E - sign(M_{N-\{i\}}^+ \cdot (P_i^{-,=})^T)]. \tag{22}$$

State $s \in S$ is SEQ for DM i iff $M_i^{SEQ}(s, s) = 0$.

Theorem 7 (Matrix Inequality for Inverse SEQ): State $s \in S$ is SEQ for DM $i \in N$ iff the preferences of DM i satisfy the following conditions:

$$M_{N-\{i\}}^+ \cdot ((P_i^{-,=})^T \cdot e_s) \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s). \tag{23}$$

Proof: Since state $s \in S$ is SEQ for DM i, then according to Definition 11,

$$M_i^{SEQ}(s, s) = [(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)]^T \cdot [e - sign(M_{N-\{i\}}^+ \cdot ((P_i^{-,=})^T \cdot e_s))] = 0 \tag{24}$$

In Equation (24), $(J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$ is a 0-1 $m \times 1$ matrix and $M_{N-\{i\}}^+ \cdot ((P_i^{-,=})^T \cdot e_s)$ is an $m \times 1$ matrix whose entries are all non-negative integers. According to Theorem 1, $M_i^{SEQ}(s, s) = 0$ which implies state s is SEQ for DM i iff $M_{N-\{i\}}^+ \cdot ((P_i^{-,=})^T \cdot e_s) \geq (J_i^T \cdot e_s) \circ ((P_i^+)^T \cdot e_s)$. □

Note that Equation (24) is equivalent to the following inequality,

$$\sum_{s_2=1}^{m} M_{N-\{i\}}^+(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) - J_i(s, s_1) \cdot P_i^+(s, s_1) \geq 0, \forall s_1 \in S. \tag{25}$$

The analysis of preference relationships in an inverse problem for SEQ is a refinement based on the inverse analysis of GMR. Actually, Theorem 7 can also be explained by the theoretical relationships between Nash stability and SEQ since if a state is stable according to Nash stability then it is also SEQ as displayed in Fig. 6.

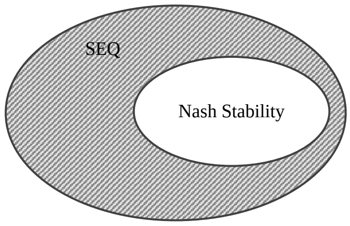

Fig. 6. Theoretical relationships between Nash stability and SEQ (based on Fang et al. (1989, 1993)).

In Inequality (25), $J_i(s, s_1) \cdot P_i^+(s, s_1) = J_i^+(s, s_1)$. Then, state s is SEQ,

1) If $\forall s_1 \in S, J_i^+(s, s_1) = 0$, which means state s is not only SEQ stable but also Nash stable for DM i corresponding to the white ellipse in Fig. 6, then $\sum_{s_2=1}^{m} M^+_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) \geq 0$ according to the definitions for $M^+_{N-\{i\}}$ and $P_i^{-,=}$.

2) If $\exists s_1 \in S, J_i^+(s, s_1) = 1$, which means state s is not Nash stable but SEQ stable for DM i corresponding to the dashed area in Fig. 6, then $\sum_{s_2=1}^{m} M^+_{N-\{i\}}(s_1, s_2) \cdot (P_i^{-,=}(s, s_2)) \geq 1$ based on Inequality (25) which implies for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R^+_{N-\{i\}}(s_1)$ with $P_i^+(s, s_2) = 0$.

As mentioned in the above situations 1) and 2), Inequality (25) can contain all the conditions of the preference relationships for state s to be SEQ for DM i.

According to Theorems 2, 3, 6, and 7, an analyst can easily obtain the set of all possible preference relations for any type of solution concept while the enumeration method given by Kinsara et al. (2015a) needs to test the preference relations one-by-one. It means this novel inverse GMCR approach is much more efficient.

5. Application of inverse analysis to the Elsipogtog First Nation fracking dispute

A New Brunswick fracking dispute in Canada between the provincial Government of Premier Alward and Elsipogtog First Nation is employed to illustrate how the proposed approach for determining the required preferences of each DM for a given state of interest to be an equilibrium works in practice. Because this conflict was previously formulated and analyzed by O'Brien and Hipel (2016), the possible equilibria to this dispute under the preferences provided in their paper have already been calculated using GMCR. Assuming that the preferences of any DM are not known, the new matrix representations for preference analysis presented in Section 4 in this paper are used to ascertain the preferences of each DM to make a state of interest be an equilibrium. Under the assumption that the conflict is current or ongoing, one state of interest for a DM called New Brunswick Provincial Government is selected to illustrate how the algorithms provided in Section 4 for calculating the required preferences of DMs can be used to reach an equilibrium of interest.

In Section 5.1, the background of this dispute is introduced and the conflict is modeled using option form. The matrix form of this model is then presented in Section 5.2 followed by the preference analysis in Section 5.3. The novel approach for solving the inverse problem of GMCR is compared with the technique developed by Kinsara et al. (2015a) in Section 5.4. As shown in Section 5.3, each DM can obtain an inequality containing all of the possible preference relationships to cause the specific state to become an equilibrium under a particular solution concept by using the proposed inverse GMCR approach. This inequality provides a lower computational complexity for determining the preference relationships than the enumeration method as indicated in Table 3.

One should keep in mind that the four solution concepts and connected theorems for establishing the conditions for the inverse problem depend upon pairwise preference information stored in matrices. Because preference comparisons occur only between two states, both transitive and intransitive preference situations can be taken into account and preference cycles cannot arise.

5.1. Background and the option form of the dispute

The Elsipogtog First Nation fracking dispute, which occurred in June of 2013, was first formally modeled and analyzed by O'Brien and Hipel (2016). The Frederick Brook shale deposit where the dispute took place is located in the southern part of the Province of New Brunswick in Canada. The New Brunswick Provincial Government (NBPG) preferred to develop the shale gas resource because of the benefits for its economy. Many residents were worried about potential environmental pollution when allowing shale gas exploration and extraction by fracking. The Elsipogtog First Nation was the most noteworthy party which opposed the shale gas development in New Brunswick. An American company called Southwestern Energy (SWN) Resources was contracted by NBPG to carry out the seismic testing for the shale gas exploration. The New Brunswickers who attempted to prevent the NBPG and SWN Resources from developing the shale gas resource blocked a key road on September 30th, 2013. Assuming that the conflict is current or ongoing, NBPG, who is one of the DMs, has a state or outcome of high interest while it knows nothing about the preferences of each DM. Each DM's preferences required for the state of interest to be an equilibrium is important to NBPG. With the required preferences of DMs, a specific strategy for the negotiation can be formulated to bring about a desirable result. The matrix representations used in preference analysis presented in Section 4 is employed to obtain the required preferences of DMs for the equilibrium of interest to occur. The DMs and options for the New Brunswick fracking dispute are summarized as follows.

1) The **Elsipogtog First Nation** (called **EFN**) can continue to **protest** or not protest by allowing the seismic testing to resume unhindered.
2) The **New Brunswick Provincial Government (NBPG)** cannot allow the shale gas exploration or fracking on Elsipogtog First Nation traditional land, or give a percentage of fracking royalties to the Elsipogtog First Nation.
3) The **Southwestern Energy (SWN) Resources** can **pursue legal action** or not.

As constructed by O'Brien and Hipel (2016), the DMs and their options as well as the twelve feasible states in the conflict are listed in Table 1. As can be seen, each of the three DMs is listed in the left column of this table followed by the option or options it controls. Each column of Ys and Ns in this table represents a specific feasible state. A "Y" opposite an option implies "yes" the option is chosen while an "N" means "no" the option is not taken by the DM. A state number is given below each column for convenience of reference. In Table 1, for example, state s_1 represents the situation in which Elsipogtog First Nation does not protest and allows the seismic testing to resume unhindered (as indicated by the "N" opposite option 1). NBGP selects to neither allow the fracking to take place on Elsipogtog First Nation traditional land nor to give a percentage of fracking royalties. SWN Resources is not taking legal action.

Table 1 Feasible states in the Elsipogtog First Nation fracking dispute.

DMs and Options	Feasible States											
1. Elsipogtog First Nation												
(1) Protest	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y
2. New Brunswick Provincial Government												
(2) No fracking	N	N	Y	Y	N	N	N	N	Y	Y	N	N
(3) Royalties	N	N	N	N	Y	Y	N	N	N	N	Y	Y
3. SWN Resources												
(4) Legal action	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y
State Number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}

A unilateral move by a DM causes the conflict to move from one state to another. Notice, for example, EFN can on its own cause the conflict to go from state s_1 to s_2 by changing its option selection from "N" to "Y". This is a unilateral move because the other two DMs keep the same option selections. The arcs connecting states given as vertices in Fig. 7 graphically display all of the unilateral moves of each DM in this fracking dispute.

In Fig. 7, a two-directional arc means that the movement between two states is reversible. In this conflict, all the movements are reversible. For example, as shown in Fig. 7(a), the move by Elsipogtog First Nation from s_1 to s_2 is reversible, since the arrows or the arc point in opposite directions.

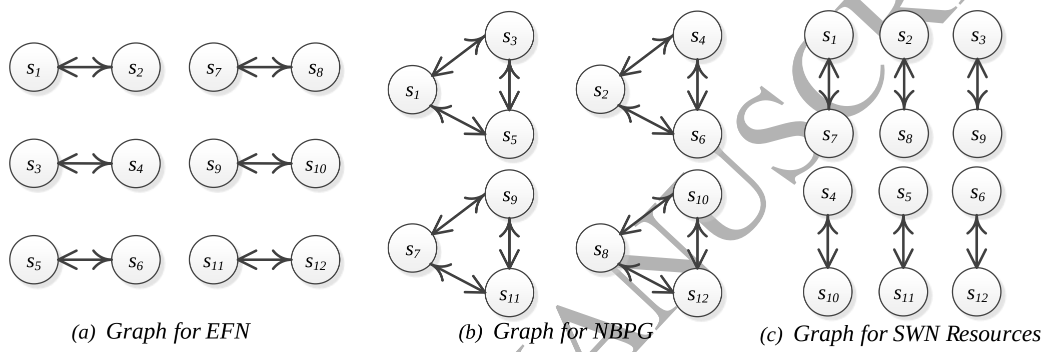

Fig. 7. State transition graphs for the DMs in the Elsipogtog First Nation fracking dispute.

5.2. Matrix form of the graph model

According to the definitions presented in Section 3, the matrix form of the graph model for this conflict is provided for each DM in this subsection. In this dispute, the preferences of each DM are assumed to be unknown. The matrices for unilateral movements and joint movements are listed here. As indicated in Table 1, the DMs Elsipogtog First Nation, New Brunswick Provincial Government and SWN Resources are labeled using the numbers 1 to 3, respectively. The italicized "N" represents the set containing these DMs.

The unilateral movement matrix J_1 and joint movement matrix $M_{N\text{-}\{1\}}$ for Elsipogtog First Nation according to Definitions 3 and 5 and Fig. 7 are:

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad M_{N-\{1\}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Notice, for example, that $J_1(1, 2) = 1$ means that Elsipogtog First Nation can move from s_1 to s_2 in one step (see Fig. 7(a)), as mentioned before.

The unilateral movement matrix J_2 and joint movement matrix $M_{N-\{2\}}$ for NBPG following Definitions 3 and 5 and Fig. 7 are:

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad M_{N-\{2\}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The unilateral movement matrix J_3 and joint movement matrix $M_{N-\{3\}}$ for SWN Resources according to Definitions 3 and 5 and Fig. 7 are:

$$J_3 = \begin{bmatrix} 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \end{bmatrix} \quad M_{N-\{3\}} = \begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1&1&1 \end{bmatrix}$$

5.3. Preference analysis

Under the assumption that the conflict is just starting, NBPG, who is the second DM in this dispute, does not know the preferences of the DMs. State s_1 is of interest to it because this state can provide many benefits. Hence, NBPG may wish to determine the preferences required to make state s_1 be an equilibrium. In order to ascertain how to negotiate with other DMs to reach an equilibrium of interest, the preferences of each DM required for the specified equilibrium are important to NBPG.

State s_1, which is a state of interest to NBPG, is selected for calculating the preferences of each DM required to cause the state to be an equilibrium. An equilibrium may be caused by Nash, GMR, SMR or SEQ behavior. The required preferences of each DM making state s_1 an equilibrium under any of the four behaviors are now determined. In the following analyses, $P_i^+(s,q)$ is used to present the preference relation for DM i between state s and q. For example, $P_1^+(1,2)=0$ means Elsipogtog First Nation less or equally prefers s_2 to s_1.

<u>Nash stability:</u> if state s_1 is Nash stable for each DM, the conditions of the preferences for each DM can be obtained by using Equation (2). The required preferences for each DM are listed as follows, respectively:

a) For DM 1: $P_1^+(1,2)=0$.

b) For DM 2: $P_2^+(1,3)=0$ and $P_2^+(1,5)=0$.

c) For DM 3: $P_3^+(1,7)=0$.

<u>General metarationality:</u> if state s_1 is GMR for each DM, the required preferences for each DM which are listed as given below can be obtained by using Inequality (4).

a) For DM 1: $2P_1^+(1,2) + P_1^+(1,4) + P_1^+(1,6) + P_1^+(1,8) + P_1^+(1,10) + P_1^+(1,12) \le 6$

b) For DM 2:

$2P_2^+(1,3) + P_2^+(1,4) + P_2^+(1,9) + P_2^+(1,10) \le 4$ and $2P_2^+(1,5) + P_2^+(1,6) + P_2^+(1,11) + P_2^+(1,12) \le 4$.

c) For DM 3: $2P_3^+(1,7) + P_3^+(1,8) + P_3^+(1,9) + P_3^+(1,10) + P_3^+(1,11) + P_3^+(1,12) \le 6$.

Symmetric metarationality: if state s_1 is SMR for each DM, the required preferences for each DM which are given as follows can be obtained by using Inequality (12).

a) For DM 1: $P_1^+(1,2) = 0$ or $P_1^+(1,3) + P_1^+(1,4) < P_1^+(1,2)$ or $P_1^+(1,5) + P_1^+(1,6) < P_1^+(1,2)$ or

$P_1^+(1,7) + P_1^+(1,8) < P_1^+(1,2)$ or $P_1^+(1,9) + P_1^+(1,10) < P_1^+(1,2)$ or $P_1^+(1,11) + P_1^+(1,12) < P_1^+(1,2)$.

b) For DM 2:

$P_2^+(1,3) = 0$ or $P_2^+(1,2) + P_2^+(1,4) + P_2^+(1,6) < P_1^+(1,3)$ or $P_2^+(1,7) + P_2^+(1,9) + P_2^+(1,11)$
$< P_1^+(1,3)$ or $P_2^+(1,8) + P_2^+(1,10) + P_2^+(1,12) < P_1^+(1,3)$

and

$P_2^+(1,5) = 0$ or $P_2^+(1,2) + P_2^+(1,4) + P_2^+(1,6) < P_1^+(1,5)$ or $P_2^+(1,7) + P_2^+(1,9) + P_2^+(1,11)$
$< P_1^+(1,5)$ or $P_2^+(1,8) + P_2^+(1,10) + P_2^+(1,12) < P_1^+(1,5)$.

c) For DM 3: $P_3^+(1,7) = 0$ or $P_3^+(1,2) + P_3^+(1,8) < P_3^+(1,7)$ or $P_3^+(1,3) + P_3^+(1,9) < P_3^+(1,7)$ or

$P_3^+(1,4) + P_3^+(1,10) < P_3^+(1,7)$ or $P_3^+(1,5) + P_3^+(1,11) < P_3^+(1,7)$ or $P_3^+(1,6) + P_3^+(1,12) < P_3^+(1,7)$.

Sequential stability: in order to calculate the required preferences for the specific DM when state s_1 is SEQ for it, the other DMs' preferences are needed to obtain its joint improvement matrix. In practice, partial preferences of each DM are often known. However, the complete pairwise comparisons of states to obtain preferences are not known by the DMs or the third party. If any of the preferences for each DM are unknown, the model proposed in this research can be used to obtain the required preferences causing the state under consideration to be a SEQ equilibrium. In this illustration, the known preferences of each DM are given in Table 2.

Table 2 Known preferences for each DM in the Elsipogtog First Nation fracking dispute.

DMs	Known preferences
Elsipogtog First Nation	$s_3 \succ s_4$, $s_{11} \succ s_{12}$
New Brunswick Provincial Government	$s_7 \succ s_9$, $s_2 \succ s_8 \succ s_4 \succ s_{10} \succ s_{12}$
SWN Resources	$s_5 \succ s_{11}$, $s_6 \succ s_{12}$, $s_9 \succ s_{10}$

Then, the remaining preferences for each DM can be obtained by using Inequality (23), as indicated below:

a) For DM 1: $P_2^+(2,6)\times[1-P_1^+(1,6)]+P_3^+(2,8)\times[1-P_1^+(1,8)]\geq P_1^+(1,2)$.

b) For DM 2:

$P_3^+(3,9)\times[1-P_2^+(1,9)]\geq P_2^+(1,3)$ and $P_3^+(3,9)\times P_1^+(9,10)\times[1-P_2^+(1,10)]\geq P_2^+(1,3)$ and $P_3^+(3,9)\times P_1^+(9,10)\times P_3^+(10,4)\times[1-P_2^+(1,4)]\geq P_2^+(1,3)$ and $P_1^+(5,6)\times[1-P_2^+(1,6)]\geq P_2^+(1,5)$.

c) For DM 3: $P_1^+(7,8)\times[1-P_3^+(1,8)]+P_2^+(7,11)\times[1-P_3^+(1,11)]\geq P_3^+(1,7)$.

The required preference relationships for each DM in the fracking dispute are obtained using the proposed mathematical approach in this research to satisfy the four solution concepts consisting of Nash, GMR, SMR and SEQ, respectively. To make the state of interest, s_1, for both NBPG and SWN Resources be an equilibrium (O'Brien & Hipel, 2016), an inequality representing the needed preference relationships for each DM is given according to each of the four solution concepts. For example, if DM 2 wishes to stay at state s_1 and maintain it as a GMR equilibrium, the logical relationships of preferences for each DM based on the inequality for general metarationality can be described as follows:

a) DM 1 prefers state s_1 to at least one of the states s_2, s_4, s_6, s_8, s_{10} and s_{12}.

b) DM 2 cannot prefer state s_3 to s_1, s_4 to s_1, s_9 to s_1, and s_{10} to s_1 together while it cannot prefer s_5 to s_1, s_6 to s_1, s_{11} to s_1, and s_{12} to s_1 at the same time.

c) DM 3 prefers state s_1 to at least one of the states s_7, s_8, s_9, s_{10}, s_{11} and s_{12}.

5.4. Comparison of computational complexity

In this section, the computational complexity of the novel approach proposed in this paper for the inverse GMCR formulation is compared with that of the enumeration method given by Kinsara et al. (2015a). The number of the required preference relationships of DM 1 to make state s_1 be Nash, GMR, SMR or SEQ individually stable in the Elsipogtog First Nation fracking dispute is taken as an example for the comparison. In Section 3, three possible preference relations for each DM between two states are defined in Definition 1. Hence, the number of preference matrices for DM 1 should be $3^{\frac{m\times(m-1)}{2}}$ where m is equal to 12 in the Elsipogtog First Nation fracking dispute. Each possible preference matrix needs to be tested according to the matrix representations of individual stability derived by Xu et al. (2009). However, the number of the required preference relationships is reduced by the constraints obtained in Section 5.3. The results of the comparison are listed in Table 3.

Table 3 Computational complexity of analyzing the preference relationships for DM 1.

Approach	Computational Complexity			
	Nash	GMR	SMR	SEQ
This paper	1	3^6	3^{11}	3^5
Kinsara et al. (2015a)	3^{12}	3^{66}	3^{66}	3^{183}

Actually, the sign function given in Definition 6 for calculating the individual stabilities is an essential component of the matrix representations of GMCR. However, it is really complicated for the inverse problem when the elements in the sign function are uncertain. Therefore, two key theorems - Theorems 1 and 5 - are put forward to convert the equations having sign functions into inequalities without sign functions. The number of needed preference relations is reduced which makes the calculations much easier.

6. Conclusions

In this paper, the matrix representations of the inverse problem for a graph model are formulated to ascertain the required preferences of each DM for reaching a given equilibrium or outcome of interest. Inverse analysis constitutes a powerful extension of the forward and behavioral GMCR methodologies for determining all of the possible preferences causing a state of interest to be an equilibrium as depicted at the bottom of Fig. 1. Four matrix expressions, which are Equation (2), Inequality (4), Inequality (12), and Inequality (23), are given to determine all the available preferences satisfying the four solution concepts consisting of Nash, GMR, SMR and SEQ, respectively. The calculations for executing the inverse analysis are based upon the matrix representation of a conflict and can be readily applied in practice as illustrated by the Elsipogtog First Nation fracking dispute case study in the previous section. In fact, the matrix formulation of the inverse GMCR viewpoint developed in this paper lays the foundations for developing detailed computational implementation algorithms and meaningful expansions of this perspective.

Algebraic expressions for the inverse problem in a graph model are provided in this paper. This unique set of explicit algebraic expressions for ascertaining all possible preference relationships to make a state of interest become an equilibrium can only handle four basic graph model stabilities. It would be worthwhile to extend this inverse GMCR approach to more complex stability definitions such as limited-move stability (Zagare, 1984), non-myopic stability (Brams & Wittman, 1981), and Stackelberg equilibrium concept (Von Stackelberg, 1934).

Additionally, the inverse point view could be expanded for handling conflicts having different preference structures such as unknown (Li, Hipel, Kilgour, & Fang, 2004), fuzzy (Basher et al., 2012), probabilistic (Rego & dos Santos, 2015) and grey (Kuang, Basher, Hipel, & Kilgour, 2015) preferences.

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