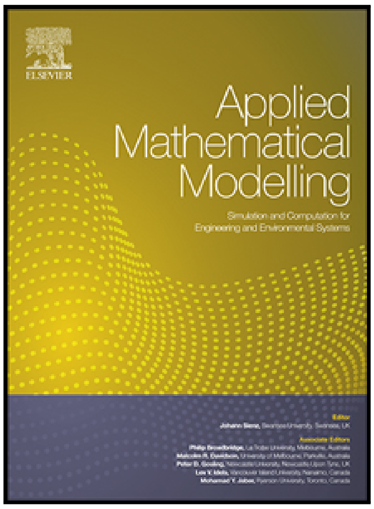

New Exact Series Solutions for Transverse Vibration of Rotationally-Restrained Orthotropic Plates

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Highlights

- Reviewed various analytical methods applicable for plates with general boundary conditions
- Applied finite integral transform to vibration of rotationally-restrained orthotropic plates
- Proposed a new formulation of finite integral transform
- Discussed numerical issues related to application of finite integral transform method

New Exact Series Solutions for Transverse Vibration of Rotationally-Restrained Orthotropic Plates

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Abstract

The exact series solutions of plates with general boundary conditions have been derived by using various methods such as Fourier series expansion, improved Fourier series method, improved superposition method and finite integral transform method. Although the procedures of the methods are different, they are all Fourier-series based analytical methods. In present study, the foregoing analytical methods are reviewed first. Then, an exact series solution of vibration of orthotropic thin plate with rotationally restrained edges is obtained by applying the method of finite integral transform. Although the method of finite integral transform has been applied for vibration analysis of orthotropic plates, the existing formulation requires of solving a highly non-linear equation and the accuracy of the corresponding numerical results can be questionable. For that reason, an alternative formulation is proposed to resolve the issue. The accuracy and convergence of the proposed method are studied by comparing the results with other exact solutions as well as approximate solutions. Discussions are made for the application of the method of finite integral transform for vibration analysis of orthotropic

thin plates.

Keywords: Rectangular orthotropic thin plate, Vibration analysis, Finite integral transform, Rotationally restrained, Rotational fixity factors, Exact series solution

1. Introduction

Over the last few decades, boundary value problems of beams and plates with general boundary conditions have been studied extensively. Exact series solutions have been derived with use of various methods. The first notable method was proposed by Wang and Lin [1] by applying Fourier series to the vibration analysis of beams with general boundary conditions. Subsequently, Wang and Lin [2] extended the use of the Fourier series to obtain exact solutions of several structural mechanics problems with arbitrary boundary conditions by transforming the governing differential equations into integral form with sinusoidal weighting functions. Hurlebaus et al. [3, 4] broadened the use of the method by Wang and Lin [1, 2] to calculate an exact series solution for the free vibration of a completely free orthotropic plate. Other works based on Fourier series were presented in references [5–8] and a short review can be found in [9]. In order to remedy the slow convergence problem of the Fourier series method, Li et al. [10–16] proposed an improved (or modified) Fourier series method in which the displacement functions comprise a Fourier series and an auxiliary function (polynomial function or one-dimensional Fourier series) resulting in remarkable convergence and accuracy.

The method of superposition was thoroughly studied by Gorman [17]. In this method boundary conditions are decomposed into a set of “build blocks” such that analytical solutions can be obtained by means of the generalized Levy method [18]. Recently, Bhaskar et al. [19–21] simplified the method of superposition with use of the so-called untruncated infinite Fourier series instead of conventional Levy-type closed-form expressions to obtain accurate results.

Besides, another remarkable analytical method is the method of finite integral transform. Various types of integral transform were employed to obtain the solutions of a wide variety of boundary value and initial value problems several decades ago [22–31]. Notably, in recent, the double finite integral transforms has been adopted to acquire exact series solution of plates with different complicated boundary conditions with use of various integral kernels, such as fully clamped orthotropic plates by Li et al. [32], free orthotropic rectangular plates in [33–35], and rectangular cantilever thin plates by Tian et al. [36]. Zhang and Xu [37] proposed double finite integral transform for bending of orthotropic plates with edges rotationally restrained. However, dynamic analysis of a plate with rotationally restrained edges has not been explored with use of the method of finite integral transform. Furthermore, the existing formulation [3, 5, 33] for the finite integral transform method in application to vibration analysis of orthotropic plates involves solving a highly non-linear equation, which requires quite laborious computation even for small m and n and consequently numerical results are questionable.

It should be recognised that even though the aforementioned methods are derived from different mathematical principles with various procedures, the methods are all Fourier-series based analytical methods. The inversion formulas of Finite Fourier transforms are exactly Fourier sine/cosine series.

Accordingly, Fourier series expansion and Finite Fourier-integral transform are equivalent but the finite integral transform method is more convenient and automatically involves boundary conditions in the process of conversion. It also can be found that the improved superposition method proposed by Bhaskar et al. [19–21] literally adopted the same concept by using Fourier series expansion to replace conventional Levy-type expressions in the forms of trigonometric and hyperbolic functions. Nevertheless, the superposition process requires skillful decomposition of the original boundary value problems as well as different formulations for each kind of boundary conditions [14]. Furthermore, in the comparison of the Fourier expansion and Finite integral transform method, the improved Fourier series methods developed by Li et al. [10–16] can be quite complicated for some boundary conditions (except classical cases) such as edges elastically restrained against rotations, although the solutions provide accurate results with rapid convergence for arbitrary boundary conditions.

In the present study, with use of the method of finite integral transform, the eigenfrequencies and mode shapes are derived for a rectangular orthotropic thin plate with rotationally restrained edges. An alternative formulation is proposed to obtain the natural frequencies by solving an eigenvalue problem instead of a highly non-linear equation. ~~Moreover, the forced vibration of the plate is investigated by the method of finite integral transform.~~Numerical examples are presented to validate the proposed method by comparing the results with those from different methods. Secondly, several issues arising from numerical calculations will be discussed while applying finite integral transforms for the flexure and vibration of the plates with rotationally restrained edges. In addition, brief comparisons and discussions will be presented for existing exact analytical methods.

2. Vibration of rectangular orthotropic plates with rotationally restrained edges

While a number of studies have been devoted to investigations of vibration of plates with uniform or non-uniform elastic boundary restraints [12, 38–46], most of the studies use approximate methods such as the Rayleigh-Ritz method which is inconvenient comparing to the method of finite integral transform [2]. Moreover, it would also be the first time to examine whether the method of finite integral transform can be applied to plates with different boundary conditions other than completely free conditions reported in [3], whereas its universal application was questioned by Li et al. [14].

2.1. Free vibration

Consider an orthotropic rectangular thin plate with length a, width b and thickness h, as shown in Fig. 1. The plate is assumed to be rigidly supported against transverse displacement around all the edges and the edges are elastically restrained against rotation. The elastic restraints are assumed to be proportional to the rotations, and the restraint stiffness may have any value in the range between simply supported (i.e., perfectly hinged) and fully clamped (i.e., completely fixed) conditions. Although the stiffness of such restraints may vary from point to point, the values are assumed to be uniform along a given boundary for the sake of simplicity.

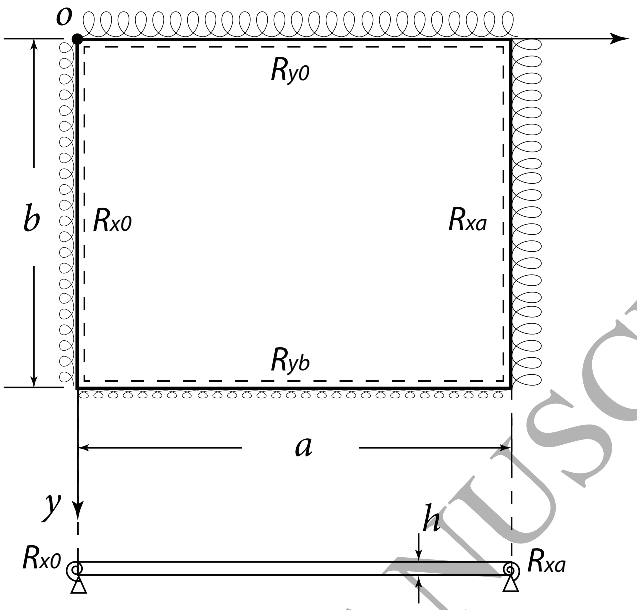


Figure 1: Orthotropic plate with four edges elastically restrained

The governing equation of the free vibration is [47]

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

in which ρ is the density of the plate; D_x and D_y are the flexural rigidity in the x-direction and y-direction, respectively; $D_{xy} = G_{xy}h^3/12$ is torsional rigidity; and $H = D_1 + 2D_{xy}$ is effective torsional rigidity, in which $D_1 = \nu_x D_y = \nu_y D_x$ is defined in terms of the Poisson's ratios ν_x and ν_y of the plate, respectively.

The displacement function $w(x, y, t)$ can be expressed as the product of two functions, one involving only the coordinates x and y, called a mode shape function $W(x, y)$, and the other involving the variable time *T(t)*. An analysis involving separation of variables shows that the function *T(t)* varies sinusoidally with time (either sin or cosine). Denoting the frequency of sinusoidal oscillations by ω, the displacement function can be expressed as

$$w(x,y,t) = W(x,y)e^{i\omega t} \tag{2}$$

Substituting Equation (2) into Eq. (1), it can be obtained

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} - \omega^2 \rho h W = 0 \tag{3}$$

Denoting partial differentiation by a comma, the boundary conditions may be written as

$$w = 0,\ M_x = -D_x\left(w_{,xx} + \nu_y w_{,yy}\right) = -R_{x0} w_{,x} \ \ at\ x = 0 \tag{4a}$$

$$w = 0,\ M_x = -D_x\left(w_{,xx} + \nu_y w_{,yy}\right) = R_{xa} w_{,x} \ \ at\ x = a \tag{4b}$$

$$w = 0,\ M_y = -D_y\left(w_{,yy} + \nu_x w_{,xx}\right) = -R_{y0} w_{,y} \ \ at\ y = 0 \tag{4c}$$

$$w = 0,\ M_y = -D_y\left(w_{,yy} + \nu_x w_{,xx}\right) = R_{yb} w_{,y} \ \ at\ y = b \tag{4d}$$

The pair of the double finite sine transforms is defined as [48]

$$\bar{\bar{W}}(m,n) = \int_0^a \int_0^b W(x,y) \sin \alpha_m x \sin \beta_n y dx dy \tag{5a}$$

$$W(x,y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{\bar{W}}(m,n) \sin \alpha_m x \sin \beta_n y \tag{5b}$$

where

$$\alpha_m = \frac{m\pi}{a},\ \beta_n = \frac{n\pi}{b}\ (m = 1,2,3,...,n = 1,2,3,...) \tag{6}$$

Taking double finite sine transforms on both sides of Eq. (3), it gives

$$\int_0^a \int_0^b \nabla_o^4 W(x,y) \sin \alpha_m x \sin \beta_n y dx dy - \omega^2 \rho h \bar{\bar{W}}(m,n) = 0 \tag{7}$$

where

$$\nabla_o^4 = D_x \frac{\partial^4}{\partial x^4} + 2H \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4} \tag{8}$$

Using integration by parts and considering the boundary conditions of Eqs. (4), the double finite sine transforms of the fourth derivatives in Eq. (7) can be obtained [49]:

$$\int_0^a \int_0^b W_{,xxxx} \sin\alpha_m x \sin\beta_n y dx dy = \alpha_m^4 \bar{\bar{W}}(m,n) - \alpha_m \Big[(-1)^m \bar{W}_{,xx}(a,n) - \bar{W}_{,xx}(0,n) \Big] \tag{9a}$$

$$\int_0^a \int_0^b W_{,xxyy} \sin\alpha_m x \sin\beta_n y dx dy = \alpha_m^2 \beta_n^2 \bar{\bar{W}}(m,n) \tag{9b}$$

$$\int_0^a \int_0^b W_{,yyyy} \sin\alpha_m x \sin\beta_n y dx dy = \beta_n^4 \bar{\bar{W}}(m,n) - \beta_n \Big[(-1)^n \bar{W}_{,yy}(m,b) - \bar{W}_{,yy}(m,0) \Big] \tag{9c}$$

where coefficients $\bar{W}_{,xx}(0,n)$, $\bar{W}_{,xx}(a,n)$, $\bar{W}_{,yy}(m,0)$ and $\bar{W}_{,yy}(m,b)$ are determined from the finite-sine transformed boundary conditions at the four edges by

$$\bar{W}_{,xx}(0,n) = \int_0^b W_{,xx}(0,y)\sin\beta_n y dy \tag{10a}$$

$$\bar{W}_{,xx}(a,n) = \int_0^b W_{,xx}(a,y)\sin\beta_n y dy \tag{10b}$$

$$\bar{W}_{,yy}(m,0) = \int_0^a W_{,xx}(x,0)\sin\beta_n x dx \tag{10c}$$

$$\bar{W}_{,yy}(m,b) = \int_0^a W_{,xx}(x,b)\sin\beta_n x dx \tag{10d}$$

Taking finite sine transform on both sides of Eqs. (4), it yields

$$\bar{W}_{,xx}(0,n) = \frac{R_{x0}}{D_x}\bar{W}_{,x}(0,n) \tag{11a}$$

$$\bar{W}_{,xx}(a,n) = -\frac{R_{xa}}{D_x}\bar{W}_{,x}(a,n) \tag{11b}$$

$$\bar{W}_{,yy}(m,0) = \frac{R_{y0}}{D_y}\bar{W}_{,y}(m,0) \tag{11c}$$

$$\bar{W}_{,yy}(m,b) = -\frac{R_{yb}}{D_y}\bar{W}_{,y}(m,b) \tag{11d}$$

Substituting Eq. (9) into Eq. (7), the following is obtained

$$\bar{\bar{W}}(m,n) = \frac{1}{\Omega_{mn} - \omega^2\rho h}\Big\{\alpha_m D_x\Big[(-1)^m\bar{W}_{,xx}(a,n) - \bar{W}_{,xx}(0,n)\Big] + \beta_n D_y\Big[(-1)^n\bar{W}_{,yy}(m,b) - \bar{W}_{,yy}(m,0)\Big]\Big\} \tag{12}$$

where

$$\Omega_{mn} = D_x\alpha_m^4 + 2H\alpha_m^2\beta_n^2 + D_y\beta_n^4 \tag{13}$$

Taking the inverse finite sine transform of Eq. (12) with respect to the spatial variable x and y, separately, it can be obtained

$$\bar{W}(x,n) = \frac{2}{a}\sum_{m=1}^{\infty}\bar{\bar{W}}(m,n)\sin\alpha_m x \tag{14a}$$

$$\bar{W}(m,y) = \frac{2}{b}\sum_{n=1}^{\infty}\bar{\bar{W}}(m,n)\sin\beta_n y \tag{14b}$$

Using Stokes's transformation and taking the derivative of Eq. (14a) with respect to x and Eq. (14b) to y, respectively, it yields

$$\bar{W}_{,x}(x,n) = \frac{2}{a}\sum_{m=1}^{\infty}\alpha_m\bar{\bar{W}}(m,n)\cos\alpha_m x \tag{15a}$$

$$\bar{W}_{,y}(m,y) = \frac{2}{b}\sum_{n=1}^{\infty}\beta_n\bar{\bar{W}}(m,n)\cos\beta_n y \tag{15b}$$

Applying Eqs. (11) and Eqs. (12), four infinite systems of equations with respect to $\bar{W}_{,xx}(0,n)$, $\bar{W}_{,xx}(a,n)$, $\bar{W}_{,yy}(m,0)$, and $\bar{W}_{,yy}(m,b)$ can be obtained.

$$\bar{W}_{,xx}(0,n) = \frac{2}{a}\frac{R_{x0}}{D_x}\sum_{m=1}^{\infty}\frac{\alpha_m}{\Omega_{mn}-\omega^2\rho h}\left\{\alpha_m D_x\left[(-1)^m\bar{W}_{,xx}(a,n)-\bar{W}_{,xx}(0,n)\right]\right.$$
$$\left.+\beta_n D_y\left[(-1)^n\bar{W}_{,yy}(m,b)-\bar{W}_{,yy}(m,0)\right]\right\} \tag{16a}$$

$$\bar{W}_{,xx}(a,n) = -\frac{2}{a}\frac{R_{xa}}{D_x}\sum_{m=1}^{\infty}\frac{(-1)^m\alpha_m}{\Omega_{mn}-\omega^2\rho h}\left\{\alpha_m D_x\left[(-1)^m\bar{W}_{,xx}(a,n)-\bar{W}_{,xx}(0,n)\right]\right.$$
$$\left.+\beta_n D_y\left[(-1)^n\bar{W}_{,yy}(m,b)-\bar{W}_{,yy}(m,0)\right]\right\} \tag{16b}$$

$$\bar{W}_{,yy}(m,0) = \frac{2}{b}\frac{R_{y0}}{D_y}\sum_{n=1}^{\infty}\frac{\beta_n}{\Omega_{mn}-\omega^2\rho h}\left\{\alpha_m D_x\left[(-1)^m\bar{W}_{,xx}(a,n)-\bar{W}_{,xx}(0,n)\right]\right.$$
$$\left.+\beta_n D_y\left[(-1)^n\bar{W}_{,yy}(m,b)-\bar{W}_{,yy}(m,0)\right]\right\} \tag{16c}$$

$$\bar{W}_{,yy}(m,b) = -\frac{2}{b}\frac{R_{yb}}{D_y}\sum_{n=1}^{\infty}\frac{(-1)^n\beta_n}{\Omega_{mn}-\omega^2\rho h}\left\{\alpha_m D_x\left[(-1)^m\bar{W}_{,xx}(a,n)-\bar{W}_{,xx}(0,n)\right]\right.$$
$$\left.+\beta_n D_y\left[(-1)^n\bar{W}_{,yy}(m,b)-\bar{W}_{,yy}(m,0)\right]\right\} \tag{16d}$$

For each combination of m and n, Eqs. (16) produce $2m+2n$ equations with $2m+2n$ unknown variables. Non-trivial solutions requires the determinant of the coefficient matrix to vanish. Then, the eigenfrequencies of the plate can be calculated as well as the associated vibration modes. This approach was also reported in references [3, 5, 33]. However, such a procedure involves solving a highly non-linear equation, which requires quite laborious computation even for small m and n. This problem cannot be remedied through reducing the $2m+2n$ equations to $m+n$ equations by using the symmetry conditions of modes in the case with symmetric boundary conditions, i.e., $R_{x0} = R_{xa}$ and $R_{y0} = R_{yb}$. For the purpose of illustration, consider the doubly symmetric modes of a clamped plate, from which it can be obtained

$$0 = \sum_{m=1}^{\infty} \frac{\alpha_m}{\Omega_{mn} - \omega^2 \rho h} \left\{ \alpha_m D_x \left[(-1)^m \bar{W},_{xx}(a,n) - \bar{W},_{xx}(0,n) \right] + \beta_n D_y \left[(-1)^n \bar{W},_{yy}(m,b) - \bar{W},_{yy}(m,0) \right] \right\} \tag{17a}$$

$$0 = \sum_{m=1}^{\infty} \frac{(-1)^m \alpha_m}{\Omega_{mn} - \omega^2 \rho h} \left\{ \alpha_m D_x \left[(-1)^m \bar{W},_{xx}(a,n) - \bar{W},_{xx}(0,n) \right] + \beta_n D_y \left[(-1)^n \bar{W},_{yy}(m,b) - \bar{W},_{yy}(m,0) \right] \right\} \tag{17b}$$

$$0 = \sum_{n=1}^{\infty} \frac{\beta_n}{\Omega_{mn} - \omega^2 \rho h} \left\{ \alpha_m D_x \left[(-1)^m \bar{W},_{xx}(a,n) - \bar{W},_{xx}(0,n) \right] + \beta_n D_y \left[(-1)^n \bar{W},_{yy}(m,b) - \bar{W},_{yy}(m,0) \right] \right\} \tag{17c}$$

$$0 = \sum_{n=1}^{\infty} \frac{(-1)^n \beta_n}{\Omega_{mn} - \omega^2 \rho h} \left\{ \alpha_m D_x \left[(-1)^m \bar{W},_{xx}(a,n) - \bar{W},_{xx}(0,n) \right] + \beta_n D_y \left[(-1)^n \bar{W},_{yy}(m,b) - \bar{W},_{yy}(m,0) \right] \right\} \tag{17d}$$

Using the symmetric boundary conditions, it can be found that

$$\begin{aligned} \bar{W}_{,xx}(a,n) &= \bar{W}_{,xx}(0,n) \\ \bar{W}_{,yy}(m,b) &= \bar{W}_{,yy}(m,0) \end{aligned} \tag{18}$$

Thus, terms with even m or n in Eqs. (17) will vanish. After that, Eqs. (17) turn into

$$\sum_{m=1,3,...}^{\infty} \frac{\alpha_m^2 D_x}{\Omega_{mn} - \omega^2 \rho h} \bar{W}_{,xx}(a,n) + \sum_{m=1,3,...}^{\infty} \frac{\alpha_m \beta_n D_y}{\Omega_{mn} - \omega^2 \rho h} \bar{W}_{,yy}(m,b) = 0 \tag{19a}$$

$$\sum_{n=1,3,...}^{\infty} \frac{\alpha_m \beta_n D_x}{\Omega_{mn} - \omega^2 \rho h} \bar{W}_{,xx}(a,n) + \sum_{n=1,3,...}^{\infty} \frac{\beta_n^2 D_y}{\Omega_{mn} - \omega^2 \rho h} \bar{W}_{,yy}(m,b) = 0 \tag{19b}$$

It can be observed that even for the simplified Eqs. (19), it is still required to solve the highly non-linear equation. The infinite series of first term in Eq. (19a) or the second term in Eq. (19b) can be summed without much difficulty in the case of isotropic plate as it will benefit the numerical computation. However, sum of the infinite series will be complex for orthotropic plates. In addition, it can also be recognised that Eqs. (19) are coincidentally identical to Eqs. (16) in reference [21] in which the improved superposition method is applied for isotropic plate. This verifies that the finite integral transform method is essentially the same as the improved superposition method.

Alternatively, instead of solving non-linear equations, Li et al. [10, 14] proposed a simple procedure to obtain the natural frequency. This procedure can also be applied herein for the method of finite integral transform.

Combining Eqs. (11) and Eqs. (15), it yields

$$\bar{W}_{,xx}(0,n) = \frac{2}{a}\frac{R_{x0}}{D_x}\sum_{m=1}^{\infty}\alpha_m\bar{\bar{W}}(m,n) \tag{20a}$$

$$\bar{W}_{,xx}(a,n) = -\frac{2}{a}\frac{R_{xa}}{D_x}\sum_{m=1}^{\infty}(-1)^m\alpha_m\bar{\bar{W}}(m,n) \tag{20b}$$

$$\bar{W}_{,yy}(m,0) = \frac{2}{b}\frac{R_{y0}}{D_y}\sum_{n=1}^{\infty}\beta_n\bar{\bar{W}}(m,n) \tag{20c}$$

$$\bar{W}_{,yy}(m,b) = -\frac{2}{b}\frac{R_{yb}}{D_y}\sum_{n=1}^{\infty}(-1)^n\beta_n\bar{\bar{W}}(m,n) \tag{20d}$$

Substituting Eqs. (20) into Eq. (12) produces

$$\begin{aligned}&\Omega_{mn}\bar{\bar{W}}(m,n) + \frac{2\alpha_m}{a}\sum_{i=1}^{\infty}\left[(-1)^{i+m}R_{xa} + R_{x0}\right]\alpha_i\bar{\bar{W}}(i,n)\\&+\frac{2\beta_n}{b}\sum_{j=1}^{\infty}\left[(-1)^{j+n}R_{yb} + R_{y0}\right]\beta_j\bar{\bar{W}}(m,j) - \omega^2\rho h\bar{\bar{W}}(m,n) = 0\end{aligned} \tag{21}$$

where

$$\alpha_i = \frac{i\pi}{a},\ \beta_j = \frac{j\pi}{b}\ (i = 1,2,3,...,j = 1,2,3,...) \tag{22}$$

In order to reflect the relative stiffness of the plate and the rotational elastic restraints, a rotational fixity factor r was introduced by Zhang and Xu [37] to define elastic restraints along edges and can be expressed as

$$r_{x0} = \frac{1}{1+3\dfrac{D_x}{R_{x0}a}} \tag{23a}$$

$$r_{xa} = \frac{1}{1+3\dfrac{D_x}{R_{xa}a}} \tag{23b}$$

Thus, it can be obtained

$$\frac{R_{x0}a}{D_x} = \frac{3r_{x0}}{1 - r_{x0}} \tag{24a}$$

$$\frac{R_{xa}a}{D_x} = \frac{3r_{xa}}{1 - r_{xa}} \tag{24b}$$

Similarly, it also has

$$\frac{R_{y0}b}{D_y} = \frac{3r_{y0}}{1 - r_{y0}} \tag{25a}$$

$$\frac{R_{yb}b}{D_y} = \frac{3r_{yb}}{1 - r_{yb}} \tag{25b}$$

Substituting Eqs. (24) and Eqs. (25) into Eq. (21), it yields

$$\Omega_{mn}\bar{\bar{W}}(m,n) + \frac{2\alpha_m D_x}{a^2}\sum_{i=1}^{\infty}\left[(-1)^{i+m}\frac{3r_{xa}}{1-r_{xa}} + \frac{3r_{x0}}{1-r_{x0}}\right]\alpha_i\bar{\bar{W}}(i,n)$$
$$+ \frac{2\beta_n D_y}{b^2}\sum_{j=1}^{\infty}\left[(-1)^{j+n}\frac{3r_{yb}}{1-r_{yb}} + \frac{3r_{y0}}{1-r_{y0}}\right]\beta_j\bar{\bar{W}}(m,j) - \omega^2\rho h\bar{\bar{W}}(m,n) = 0 \tag{26}$$

Eq. (26) can be conveniently expressed in the following matrix form:

$$\mathbf{AW} = \omega^2\rho h\mathbf{W} \tag{27}$$

where $\mathbf{W} = [\bar{\bar{W}}(1,1), \bar{\bar{W}}(1,2)...\bar{\bar{W}}(1,N), \bar{\bar{W}}(2,1)...\bar{\bar{W}}(2,N)...\bar{\bar{W}}(M,N)]$ and $\mathbf{A}$ is the corresponding coefficient matrix which can be obtained from Eq. (26). It is assumed that all the series expansions are truncated to finite number M for m and N for n while the upper limit of summation may be theoretically specified as infinity. It can be observed that Eq. (27) is a standard characteristic equation for a matrix and the corresponding eigenfrequencies ω can be conveniently determined. As a result, a complicated highly nonlinear problem of Eqs. (16) is now converted to a simple eigenvalue problem of Eq. (27). For any obtained eigenfrequency, the corresponding eigenvector can be directly determined by substituting the eigenfrequency into Eq. (27). Subsequently, the corresponding mode shape can be derived by substituting the eigenvector of $\bar{\bar{W}}(m,n)$ into Eq. (5b) for each ω.

2.2. Numerical results and comparison

Several representative examples are presented in this section to validate the foregoing proposed analytical procedure. The numerical results are obtained by using built-in *eigs* function in MATLAB® software package. For the sake of convenience, the numbers of double series items are chosen to be same and denoted by N (i.e., $m,\ n = 1, 2, 3, ..., N$) and four edges have the same values of the rotational fixity factors (i.e., $r_{x0} = r_{xa} = r_{y0} = r_{yb} = r$). The results are theoretically exact when $N \to \infty$ while convergent solutions with satisfactory accuracy can be acquired by a finite number of items.

First of all, the convergence of the fundamental frequency is shown in Fig. 2 for the case of an a square isotropic plate with four edges rotationally restrained with $r = 0.999$. Given the fact that flexural solutions of rotational fixity factor $r = 0.999$ are excellently agreed with results of fully clamped plates in [37], the fundamental frequency of plate with $r = 0.999$ is compared with fundamental frequency of a fully clamped plate. The exact value of the fundamental frequency parameter is 35.985 from Li et al. [14] with use of improved Fourier series method. It can be observed from Fig. 2 that the parameter converges to the exact value quite slowly. Since the computation time becomes awfully long when $N > 150$ on a standard PC, the values are examined by truncating the series up to $N = 150$. From the results of convergence study, N is taken to be 100 for all numerical results presented in present study. Figs. 3 illustrate the first six mode shapes of a square isotropic plate with $r = 0.25$. Figs. 4 show the influence of rotational stiffness on the mode shapes of plate. Square isotropic plates with four different rotational fixity factors (0, 0.25, 0.5 and 0.999) are examined. The results indicate that the rotational stiffness may alter the mode shapes (Figs. 4).

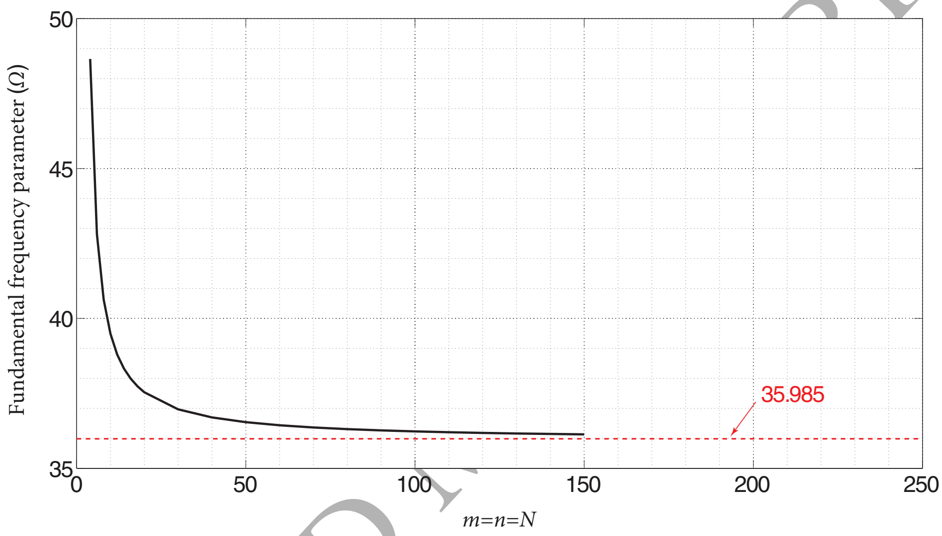
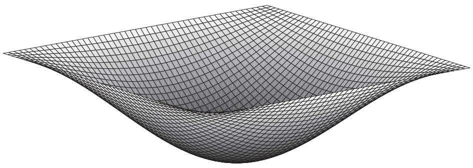
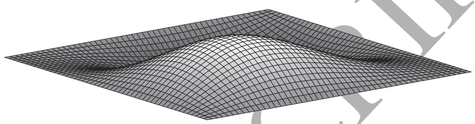


Figure 2: Convergence of the fundamental frequency parameter $\Omega = \omega a^2\sqrt{\rho h/D}$ of a square isotropic plate with $r = 0.999$

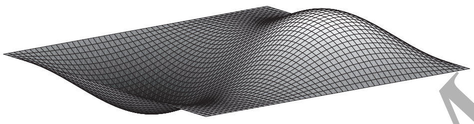
The next example is about a square isotropic plate with four edges rotationally restrained. Various rotational fixity factors from 0.0323 to 0.997 are studied and showed in Table. 1. The present results are compared with those of Mukhopadhyay [43] and Li et al. [14]. The difference of present results and those of Mukhopadhyay [43] are calculated with respect to the exact solutions of Li et al. [14], separately. It can be found that the proposed method provides better predictions than that of Mukhopadhyay [43] and differs from the exact solutions of Li et al. [14] by less than 0.9 percent.



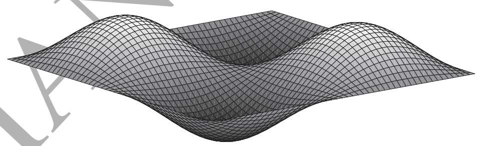
(a) First mode



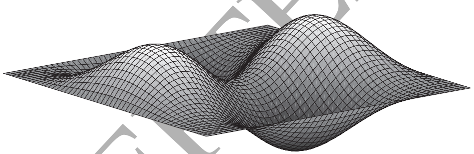
(b) Second mode



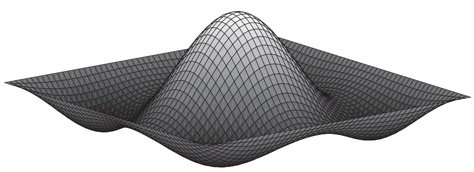
(c) Third mode



(d) Fourth mode

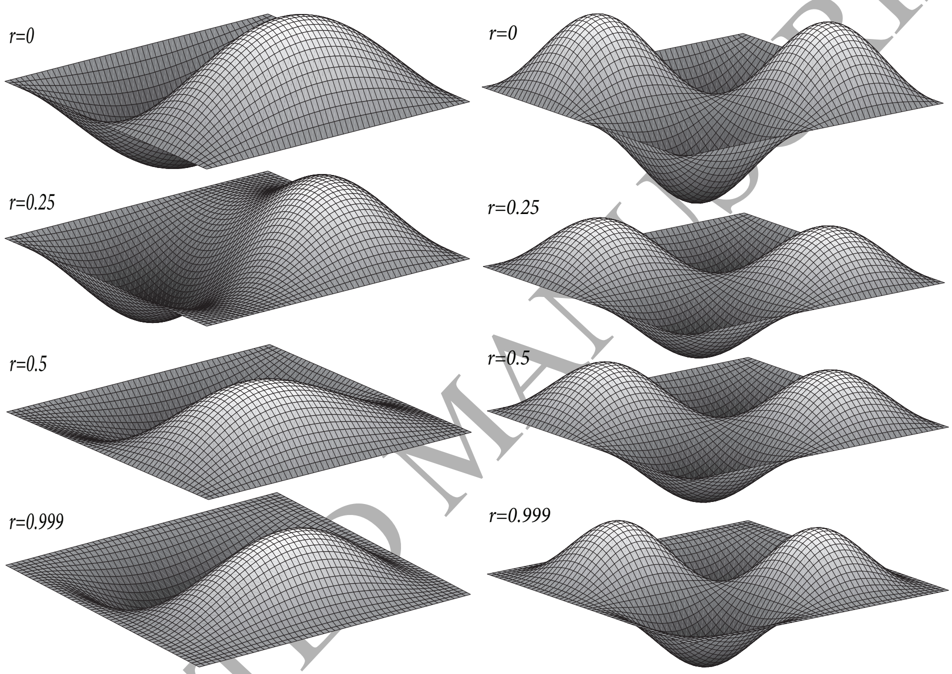


(e) Fifth mode



(f) Sixth mode

Figure 3: First six mode shapes of a square isotropic plate with $r = 0.25$



(a) Third mode (b) Fourth mode

Figure 4: The effect of rotational restraints on the mode shapes of a square isotropic plate with rotational restrained edges

Furthermore, the frequencies obtained from the proposed method are more accurate when the rotational restraint is flexible, say when $r < 0.25$. Thus, it can be concluded that the larger value of rotational fixity factors, the more time consuming to achieve high degree of accuracy.

Then, rectangular orthotropic plates with three edges simply supported ($r_{x0} = r_{xa} = r_{yb} = 0$) and one edge rotationally restrained are considered. The effect of aspect ratios and rotational fixity factors are investigated. The fundamental frequency parameters are tabulated in Table. 2 and compared with results of Laura et al. [41] with the material properties as $D_x/H = D_y/H = 0.5$. Comparisons in Table. 2 indicate well agreements in the results with difference less than 0.8 percent.

At last, Table 3 shows the first five frequency parameters (i.e., $\Omega = \rho h\omega^2 a^2 b^2/(\pi^4 H)$) for a square clamped orthotropic thin plate with elastic constants of $D_x/H = 1.543$ and $D_y/H = 4.810$. The rotational fixity factor $r = 0.9999$ was adopted to simulate the clamped plate by using the present method. The present predictions were compared with those by Dickinson [50] and excellent agreement can be observed.

3. Discussion and remarks

Vibration analysis of rectangular orthotropic plates with rotationally restrained edges has been studied by means of the double finite sine transforms in preceding sections. It can be observed that the method of finite Fourier-integral transform is essentially the same as Fourier series expansion of Wang and Lin [1, 2] and improved superposition method of Bhaskar et al. [19–21]. Comparing with these equivalent methods, the method of finite integral

Table 1: First six frequency parameters Ω for a square isotropic plate with four edges rotationally restrained

r	Ka/D		$\Omega = \omega a^2\sqrt{\rho h/D}$					
			1	2	3	4	5	6
0.0323	0.1	Ref. [43]	19.839	48.894	49.629	79.04	95.678	99.211
		Present	19.936	49.546	49.546	79.155	98.895	98.895
0.25	1	Ref. [14]	21.5	51.187	51.187	80.816	100.58	100.59
		Ref. [43]	20.511	49.116	50.927	79.851	95.777	100.727
		(%)	-4.60^a	-4.05	-0.51	-1.19	-4.78	0.14
		Present	21.505	51.195	51.195	80.831	100.587	100.594
		(%)	0.02^b	0.02	0.02	0.02	0.01	0.00
0.7692	10	Ref. [14]	28.501	60.215	60.215	90.808	111.19	111.41
		Present	28.583	60.337	60.337	90.957	111.352	111.578
		(%)	0.29^b	0.20	0.20	0.16	0.15	0.15
0.8696	20	Ref. [14]	31.08	64.31	64.31	95.85	116.8	117.2
		Ref. [43]	31.111	64.342	64.861	95.85	117.029	118.214
		(%)	0.10^a	0.05	0.86	0.00	0.20	0.87
		Present	31.219	64.535	64.535	96.112	117.181	117.566
		(%)	0.45^b	0.35	0.35	0.27	0.33	0.31
0.9709	100	Ref. [14]	34.671	70.78	70.78	104.45	127.02	127.61
		Ref. [43]	34.753	69.319	70.929	103.377	120.047	127.616
		(%)	0.24^a	-2.06	0.21	-1.03	-5.49	0.00
		Present	34.918	71.259	71.259	105.128	127.845	128.439
		(%)	0.71^b	0.68	0.68	0.65	0.65	0.65
0.997	1000	Ref. [14]	35.842	73.103	73.103	107.79	131.06	131.68
		Present	36.134	73.694	73.694	108.658	132.129	132.756
		(%)	0.81^b	0.81	0.81	0.81	0.82	0.82

note: a–percentage difference of results between [43] and [14]

b–percentage difference of results between the present study and [14]

Ka/D–rotational stiffness coefficient defined in Ref. [14]

Table 2: Fundamental frequency parameter Ω_1 for rectangular orthotropic plates with three edges simply supported ($r_{x0} = r_{xa} = r_{yb} = 0$) and one edge rotationally restrained r_{y0}

k_3	$\frac{R_{y0}b}{D_y}$	r_{y0}	$\Omega_1 = \omega_1 a^2 \sqrt{\rho h/D_x}$								
			b/a=0.5			b/a=1			b/a=1.5		
			Ref. [41]	Present	(%)	Ref. [41]	Present	(%)	Ref. [41]	Present	(%)
0	0	0	56.5685	56.6966	0.23	24.1831	24.1755	-0.03	16.9706	17.0242	0.32
0.5	1	0.25	58.8313	59.1302	0.51	24.4659	24.5448	0.32	17.0963	17.1301	0.20
5	10	0.7692	67.8823	68.0681	0.27	26.1630	26.1534	-0.04	17.7248	17.6557	-0.39
∞	∞	0.9999	75.2362	75.7808	0.72	28.0014	27.9691	-0.12	18.5419	18.4179	-0.67

note: k_3–rotational stiffness coefficient defined in Ref. [41]

Table 3: Frequency parameter $\Omega = \rho h \omega^2 a^2 b^2/(\pi^4 H)$ for a square clamped orthotropic plates with elastic constants of $D_x/H = 1.543$ and $D_y/H = 4.810$

Mode	Frequency parameters		
No.	Ref. [50]	Present ($r = 0.9999$)	difference (%)
1	35.71192	36.29327	1.63
2	96.42569	97.99261	1.63
3	207.0308	210.3881	1.62
4	280.6901	285.3147	1.65
5	290.9267	295.6540	1.62

transform is more convenient and can be routinely applied to more complex boundary value problems by choosing different integral kernels. However, due to the issue of slow convergence, these so-called theoretical-exact series solutions normally produce approximate results for vibration analysis of plates. The larger value of rotational fixity factors, the more time consuming to achieve high degree of accuracy. The improved Fourier series method developed by Li et al. [12, 14] can be applied to improve the convergence and as well as the accuracy.

3.1. Formulation

The method of finite integral transform presented in this study is straightforward in concept and systematic in formulation. First, the governing differential equation is converted into an algebraic equation in terms of the integral form of solution by applying appropriate integral kernel. The initial or boundary conditions will be accounted for automatically in the process of conversion. The resulting algebraic equation can be solved without much difficulty. If the algebraic equation involves some variables which are unknown, the boundary conditions can be applied to determined the variables eventually. Through this procedure, a system of linear algebraic equations will be obtained for unknown variables. Once the integral form of solution is known, the original function can be derived by using the inverse integral transform [51].

As discussed in Section 2.1, two different formulations can be generated. For the case investigated in present research, the first formulation leads to Eq. (12) and then four infinite systems of equations, Eqs. (16), with respect to $\bar{W}_{,xx}(0,n)$, $\bar{W}_{,xx}(a,n)$, $\bar{W}_{,yy}(m,0)$, and $\bar{W}_{,yy}(m,b)$. For each combination of m and n, Eqs. (16) produce $2m+2n$ equations with $2m+2n$ unknown variables. As a results, frequencies can be acquired by solving a highly non-linear equation representing the determinant of the coefficient matrix with dimensions of $(2m+2n)\times(2m+2n)$. This approach was employed by references [3, 33]. However, the number of terms used in the numerical evaluations (size of the matrix) and the numerical method are not reported in [3]. Zhong and Yin [33] computed the eigenfrequencies and corresponding mode shapes by truncating the series up to 13 terms. It would be quite difficult to solve the non-linearly equation resulted from the determinant to obtain the frequency when the large values of m and n are selected. The other operation results in Eq. (21) or Eq. (26) by expressing $\bar{W}_{,xx}(0,n)$, $\bar{W}_{,xx}(a,n)$, $\bar{W}_{,yy}(m,0)$, and $\bar{W}_{,yy}(m,b)$ in terms of $\bar{\bar{W}}(m,n)$ and substituting them into Eq. (12). A systems of linear equation about $\bar{\bar{W}}(m,n)$ with dimensions of $(m\times n)\times(m\times n)$ is derived. Natural frequencies can be easily obtained by determining the eigenvalues of the coefficient matrix. As shown in Section 2.2, the numerical results can be calculated by choosing $m=n=150$ without much difficulty. It can be concluded that the second formulation is more efficient to the first one for the case of free vibration analysis for either one-dimensional elements or two-dimensional elements.

Nevertheless, for the flexural analysis investigated by Zhang and Xu [37], the first formulation leads to the coefficient matrix with dimensions of $(2m+2n)\times(2m+2n)$ but the second one gives that of $(m\times n)\times(m\times n)$. In the view of computational efficiency, the first formulation is more efficient for flexural analysis.

3.2. Convergence

Convergence study has been conducted for free vibration analysis of a square isotropic plate with four edges rotationally restrained with $r = 0.999$ in Section 2.2 by use of MATLAB program carried out on a desktop computer equipped with a 3.40 GHz Intel Core i7-2600 processor and 8 GB of memory. Similarly, the rate of convergence was examined for flexural analysis of plates with four edges rotationally restrained by Zhang and Xu [37]. It was observed that the results were converged slowly. However, for flexural analysis, the numerical results can be easily obtained for the series up to 2000 terms so that the exact solutions can be acquired. On the other hand, overflow problems were occurred shortly when m and n were greater than 200 on the computer programm carried out for the free vibration analysis in this research. Therefore, only approximate values are obtained by applying the method of finite integral transform on the free vibration analysis whereas the exact solutions will be theoretically determined by using infinite series.

However, the convergence of the solutions is extensively accelerated by adopting the improved Fourier series methods developed by Li et al. [10–16] through introducing the supplementary terms to Fourier series. Highly accurate results can be obtained by setting $M = N = 6$ as reported in [12]. Moreover, this improvement seems to be unnecessary for flexural analysis of beams or plates with arbitrary boundary conditions because the issue of the convergence is not significant for the flexural analysis.

3.3. Untruncated and truncated

For numerical calculations, the series solution has to be truncated to a finite number of terms. However, as pointed out in [37], the coefficient matrix will be singular when applying this method for fully clamped plates (i.e., C-C-C-C); therefore, the infinite summations should be first evaluated without truncation. This might be because that the infinite summations are the counterparts of the derivatives of the closed-form Levy-type expressions [21]. Nevertheless, there is no issue of singularity for applying the proposed method on the plates with edges rotationally restrained. Alternatively, for the fully clamped plates, it can be treated as the limiting cases by specifying the rotational fixity factor to be either 0.999 or 0.9999.

3.4. Broad applicability

The broad generality of the method of finite integral transform in solving plate flexural problems was summarized by Li et al. [34]. It is important that the appropriate integral transform kernels should be selected based on the boundary conditions. Accordingly, the accuracy and convergence will be improved. Li [10, 52] proved that the cosine series expansion would converge faster than its sine counterpart for beams with arbitrary elastic restraints but the convergence speed of the sine series solution will be greatly increased when beams is simply supported with only rotational restraints. This might explain why the kernels, $\sin\alpha_m x \sin\beta_n y$, are applied in this research for orthotropic plate with rotationally restrained edges. Similarly, Hurlebaus et al. [3, 4] employed $\cos\alpha_m x \cos\beta_n y$ for free orthotropic plates.

In general, the sinusoidal kernel (i.e., $\sin\alpha_m x$) is taken for edges simply supported, clamped or rotationally restrained (i.e., elastically restrained against rotation). The cosinusoidal kernel (i.e., $\cos\alpha_m x$) is recommended for edges free or translational restrained (i.e., elastically restrained against translation). Alternatively, if a pair of opposite edges, one is fully clamped or simply supported and the other is free, a half-sinusoidal kernel (i.e., $\sin \frac{\alpha_m}{2} x$) can be chosen [34]. The half-sinusoidal kernel is also defined as the modified finite sine transformation as demonstrated by Churchill [53].

4. Conclusion

The method of finite integral transform has been applied to free vibration of a rectangular orthotropic plate with rotationally restrained edges. An alternative formulation is proposed to obtain the natural frequencies by solving an eigenvalue problem instead of a highly non-linear equation. Consequently, the dynamic properties can be determined without much difficulty. Numerical examples validate the present method by comparing the results with different exact solutions and approximate solutions. Several issues in numerical calculations have been noted for applying Finite integral transform. The convergence, accuracy and broad applicability were also discussed. It can be concluded that this unified and systematic method has a general applicability but only provides approximate values for vibration analysis of plates due to slow convergence. In addition, various exact analytical methods for beams and plates with general boundary conditions have been reviewed such as Fourier series expansion, improved Fourier series method, improved superposition method and finite integral transform method. Brief comparisons and discussions are summarized for these exact analytical methods. Although the present research focuses on the investigation of orthotropic plates, conclusions obtained from the research are also applicable for that of isotropic plates.

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Appendix A. Orthogonality properties

Consider two different modes of free vibrations of the plate (satisfying the same boundary conditions), namely $W_{ij}(x,y)$ and $W_{mn}(x,y)$ with the corresponding eigenvalues λ_{ij} and λ_{mn}. The modes satisfy the differential equations

$$\nabla_o^4 W_{ij} - \lambda_{ij}^4 W_{ij} = 0 \tag{A.1a}$$

$$\nabla_o^4 W_{mn} - \lambda_{mn}^4 W_{mn} = 0 \tag{A.1b}$$

in which $\lambda_{ij}^4 = \rho h \omega_{ij}^2$ and $\lambda_{mn}^4 = \rho h \omega_{mn}^2$. By multiplying Eq. (A.1a) with W_{mn} and Eq. (A.1b) with W_{ij}, taking the difference and integrating the result over the area of the plate, it obtains

$$(\lambda_{ij}^4 - \lambda_{mn}^4) \int_0^a \int_0^b W_{ij} W_{mn} dxdy = \int_0^a \int_0^b \left(W_{ij} \nabla_o^4 W_{mn} - W_{mn} \nabla_o^4 W_{ij} \right) dxdy \tag{A.2}$$

The right hand side of Eq. (A.2) can be written as

$$\begin{aligned} &\int_0^a \int_0^b \left(W_{ij} \nabla_o^4 W_{mn} - W_{mn} \nabla_o^4 W_{ij} \right) dxdy \\ &= \int_0^a \int_0^b D_x \left[W_{ij} \frac{\partial^4 W_{mn}}{\partial x^4} - W_{mn} \frac{\partial^4 W_{ij}}{\partial x^4} \right] + H \left[W_{ij} \frac{\partial^4 W_{mn}}{\partial x^2 \partial y^2} - W_{mn} \frac{\partial^4 W_{ij}}{\partial x^2 \partial y^2} \right] dxdy \\ &+ \int_0^a \int_0^b D_y \left[W_{ij} \frac{\partial^4 W_{mn}}{\partial y^4} - W_{mn} \frac{\partial^4 W_{ij}}{\partial y^4} \right] + H \left[W_{ij} \frac{\partial^4 W_{mn}}{\partial x^2 \partial y^2} - W_{mn} \frac{\partial^4 W_{ij}}{\partial x^2 \partial y^2} \right] dxdy \end{aligned} \tag{A.3}$$

Integrate the first and second term of the right hand side of Eq. (A.3) by parts twice with respect to x and y, respectively; and rearrange the terms. Then, it can be obtained

$$
\begin{aligned}
&\int_0^a \int_0^b W_{ij} \nabla_o^4 \left(W_{mn} - W_{mn} \nabla_o^4 W_{ij} \right) dxdy \\
&= \int_0^a \Bigg\{ \left(D_y \frac{\partial^2 W_{ij}}{\partial y^2} + D_1 \frac{\partial^2 W_{ij}}{\partial x^2} \right) \frac{\partial W_{mn}}{\partial y} - \left(D_y \frac{\partial^2 W_{mn}}{\partial y^2} + D_1 \frac{\partial^2 W_{mn}}{\partial x^2} \right) \frac{\partial W_{ij}}{\partial y} \\
&+ \left(D_y \frac{\partial^3 W_{mn}}{\partial y^3} + (D_1 + 4D_{xy}) \frac{\partial^3 W_{mn}}{\partial y \partial x^2} \right) W_{ij} - \left(D_y \frac{\partial^3 W_{ij}}{\partial y^3} + (D_1 + 4D_{xy}) \frac{\partial^3 W_{ij}}{\partial y \partial x^2} \right) W_{mn} \Bigg\} \Bigg|_0^b dx \\
&+ \int_0^b \Bigg\{ \left(D_x \frac{\partial^2 W_{ij}}{\partial x^2} + D_1 \frac{\partial^2 W_{ij}}{\partial y^2} \right) \frac{\partial W_{mn}}{\partial x} - \left(D_x \frac{\partial^2 W_{mn}}{\partial x^2} + D_1 \frac{\partial^2 W_{mn}}{\partial y^2} \right) \frac{\partial W_{ij}}{\partial x} \\
&+ \left(D_x \frac{\partial^3 W_{mn}}{\partial x^3} + (D_1 + 4D_{xy}) \frac{\partial^3 W_{mn}}{\partial x \partial y^2} \right) W_{ij} - \left(D_x \frac{\partial^3 W_{ij}}{\partial x^3} + (D_1 + 4D_{xy}) \frac{\partial^3 W_{ij}}{\partial x \partial y^2} \right) W_{mn} \Bigg\} \Bigg|_0^a dy \\
&- 4D_{xy} \left[W_{ij} \frac{\partial^2 W_{mn}}{\partial x \partial y} - W_{mn} \frac{\partial^2 W_{ij}}{\partial x \partial y} \right] \Bigg|_{0,0}^{a,b}
\end{aligned}
\tag{A.4}
$$

Eq. (A.4) shows that the right hand side of Equation (A.2) will be zero for a plate having any combination of boundary conditions of simply supported, clamped, free, or rotationally restrained. Since $\lambda_{ij} \neq \lambda_{mn}$, Eq. (A.2) is satisfied only if

$$\int_0^a \int_0^b W_{ij} W_{mn} dxdy = 0, \qquad (i \neq m, j \neq n) \tag{A.5}$$

The eigenfunctions of the free vibrations of plates are orthogonal; and their coefficients can also be chosen to satisfy the condition

$$\int_0^a \int_0^b W_{ij}^2 dxdy = 1 \tag{A.6}$$

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