Pricing and Hedging of Emerging Products in Finance and Insurance

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis addresses the pricing and hedging issues on the newly-developed financial and insurance products, including simplified hedges for path-dependent options, variable annuities tied with state-dependent fees, and defaultable reverse mortgage contracts.

In Chapter 1, we present a method to construct a simplified alternative derivative that resembles a given highly path-dependent derivative. Path-dependent derivatives are typically difficult to hedge. Traditional dynamic delta hedging does not perform well because of the difficulty to evaluate the Greeks and the high cost of constantly rebalancing. We propose to price and hedge path-dependent derivatives by constructing simplified alternatives that preserve certain distributional properties of their terminal payoffs, and that can be hedged by semi-static replication. The method is illustrated by a geometric Asian option and by a lookback option in the Black-Scholes setting, for which explicit forms of the simplified alternatives exist. An extension to a Heston stochastic volatility model is discussed as well.

In Chapter 2, we model and study the benefits of charging state-dependent fees in variable annuities tied to the market volatility. Variable annuities (VAs) and other long-term equity-linked insurance products are typically difficult to hedge in incomplete markets. A state-dependent fee structure tied to market volatility is proposed in these products to contribute to the risk sharing mechanism between policyholders and insurers and also to reduce the hedging difficulty. We provide criteria for the fair-fee determination in the context of reducing the risk related to writing the VA contract. A method of optimal static hedging as a benchmark compared to other strategies is proposed in the stochastic volatility setting. We formulate our problem with guaranteed minimum accumulation benefits (GMABs), but it is also applicable to other equity-linked insurance contracts.

In Chapter 3, we propose a pricing scheme based on default risk models for Home Equity Conversion Mortgages (HECM). HECM Reverse mortgages are designed to allow elder homeowners aged 62 or over to convert the equity in their homes to regular revenues or a line of credit and to retain full ownership of their property for the whole life of the loan. Unlike a traditional mortgage, reverse mortgage loans do not need to be paid off as long as the borrowers remain in their home and pay due obligations such as home insurance and property taxes. HECM are non-recourse reverse mortgage loans insured by the Federal Housing Administration (FHA). HECM reverse mortgages confront a rising default risk in the wake of 2008, jeopardising the financial soundness of FHA's Mutual Mortgage Insurance Fund. The fairness of the HECM insurance premium has therefore been challenged. In this chapter, we initiate to price the reverse mortgage contract according to borrowers'

individual credit and default risk. The proposed method achieves a closed-form valuation with mortality risk, interest rate risk, housing price risk, and default risk. The impact on fair HECM insurance premiums of these risks is then investigated. Our work demonstrates that the proposed pricing solution and the corresponding newly-designed rating system will provide HECM lenders a better payment arrangement for the risk management and also support the effectiveness of recent policy changes in the HECM program.

The products described as above are designed in incomplete markets, which renders perfect hedging of these contracts impossible. The goal of Chapter 4 is to develop optimal static hedging in the context of minimizing the shortfall risk either for path-dependent options, hedging liabilities with insufficient budget, or hedging liabilities under the stochastic volatility environment. The shortfall risk is defined as the expectation of the potential loss from the imperfect hedging strategy, weighted by some loss function reflecting the hedger's risk preferences. In Chapter 4, we take examples on the Asian option and the GMAB contract in Chapter 2 and further develop the optimal static hedging for our products under the Heston-type stochastic volatility environment.

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I am truly indebted to my parents, my cousins, my aunts and uncles, for their encouragement and faith in me throughout my life. I dedicate my Ph.D. thesis to their understanding, love, and unconditional support.

Dedication

To my parents, Dunlong Tang and Huijun Jin

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Chapter 1

Simplified Hedge for Path-dependent Derivatives

This chapter is based on the published manuscript in the International Journal of Theoretical and Applied Finance by Bernard and Tang (2016).

Path-dependent derivatives have payoffs that depend on past and current values of the underlying variables. Pricing highly path-dependent options usually requires numerical methods and intensive computations. For instance, Forsyth, Vetzal, and Zvan (1999) use PDE techniques to price lookback options, Petrella and Kou (2004) use Laplace transforms to price barrier and lookback options, Fang and Oosterlee (2011) and Kwok, Leung, and Wong (2012) use Fourier transforms to price exotic derivatives. Approximation of discretely monitored path-dependent options by continuously monitored path-dependent options can lead to accurate approximations (Broadie, Glasserman, and Kou (1997), Fuh, Luo, and Yen (2013)). Monte Carlo methods are always a possible alternative: they have the advantage to incorporate flexible and general modeling assumptions but they are typically slow. Under the assumption of a Black-Scholes market, prices of certain path-dependent derivatives are known explicitly (e.g., geometric Asian options by Kemna and Vorst (1990), lookback options by Conze and Viswanathan (1991)).

In this chapter, we present a method to construct a simplified alternative derivative that resembles a given highly path-dependent derivative. This construction preserves certain distributional properties of the original path-dependent payoff but it is simpler in that it only depends on the underlying asset at a finite number of dates. As a consequence, it can then be easily hedged by semi-static replication. Moreover, since the cost of the hedging program typically increases with the number of transactions, hedging the simplified

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alternative instead of the original path-dependent derivative will lower the hedging cost because it requires only a limited number of rebalancing dates.

Our work makes use of the concept of cost-efficiency of Dybvig (1988a,b), further developed by Bernard, Boyle, and Vanduffel (2014). Cost-efficiency consists of minimizing the cost of achieving a desired distribution at some given maturity time. Bernard, Moraux, Rüschendorf, and Vanduffel (2015) extend this study to include state-dependent constraints. One application of this work is to "simplify" highly path-dependent payoffs by preserving distributional properties. For example, a geometric Asian option with maturity T can be correlated at more than 97 % with a payoff involving only the underlying value at time $T/2$ and at time T. This payoff has the same distribution at time T as the geometric Asian option and the same joint distribution with $S_{T/2}$ as the geometric Asian option. This chapter builds on this idea and shows how it is possible to add additional constraints to construct path-dependent derivatives that are even closer to the path-dependent option under study. For example, we show how to construct a derivative depending on the underlying process at three dates such that it has the same distribution at maturity as the geometric Asian option, and also the same joint distribution with the underlying price at $T/3$ and $2T/3$ as the geometric Asian option. It is clear that there is a trade-off between getting the alternative payoff as close as possible to the original path-dependent payoff and keeping it as simple as possible. We then propose to hedge path-dependent derivatives by hedging their simplified alternatives with semi-static hedging techniques.

The semi-static approach offers several advantages over dynamic hedging. Traditional dynamic delta hedging does not perform well in particular because of the difficulty to evaluate Greeks and the high cost of constantly rebalancing. When volatility is high, dynamic hedging is expensive. If there are random jumps in the underlying price, then dynamic hedges may result in large errors. By contrast, semi-static hedging is more robust to model risk: it requires fewer portfolio adjustments and random jumps in the underlying price do not significantly affect its performance. A significant literature on static hedging focuses on barrier options. Derman, Ergener, and Kani (1995) introduce a static replicating approach to hedge barrier options in a binomial tree model using standard options with varying maturities. Carr and Bowie (1994), Carr and Chou (1997), and Carr, Ellis, and Gupta (1998) develop static hedges for barrier options using standard call and put options.

The remainder of this chapter is organized as follows. Section 1.1 describes the framework and the main assumptions of the model for the underlying stock price. Section 1.2 illustrates why only hedging the distribution of the terminal payoff of a path-dependent option may result in significant hedging errors. In Section 1.3, we formulate the theoretical results that we then use to improve the approximation by using additional information. Section 1.4 illustrates the method with geometric Asian options and lookback options in

the Black-Scholes model. We show how to apply several criteria, based on correlation, variance and Value-at-Risk to select the best simplified alternative. We compare the hedging performance of the selected alternative with the cheapest payoff with the same distribution. The results in Sections 1.2-1.4 are under the Black-Scholes framework. Section 1.5 proposes two extensions. First, we generalize the construction respectively to obtain explicit forms and approximations of multivariate derivatives in the cases of geometric Asian options and any other path-dependent derivatives that are closer to the original path-dependent payoff than the alternative bivariate derivatives derived in Section 1.4. Second, we discuss how to deal with the example of the Heston stochastic volatility model. Conclusions are given in Section 1.6.

1.1 Market Model

1.1.1 Black-Scholes Model

For ease of exposition, we mostly use the Black-Scholes model to illustrate the methodology. A class of Lévy markets and the Heston stochastic volatility model will be presented in Section 1.1.4. In the Black-Scholes market, the price process S_t follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian motion under the physical measure $\mathbb{P}$, μ is the drift and σ is the volatility. The solution for the stock price is $S_T = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right)$, with cdf F_{S_T} at time T given by

$$F_{S_T}(x) = \mathbb{P}(S_T \leq x) = \Phi\left(\frac{\ln\left(\frac{x}{S_0}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \tag{1.1}$$

where Φ is the cdf of a standard normal random variable. Let r denote the continuously compounded risk-free interest rate. The Black-Scholes market is arbitrage-free and complete and the (unique) state-price density at time t (ensuring that $(\xi_t S_t)_t$ is a martingale) can be computed explicitly as $\xi_t = e^{-rt}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t}e^{-\left(\frac{\mu-r}{\sigma}\right)W_t}$. Consequently, ξ_T can be writ-

ten as an explicit function of the stock price S_T as follows

$$\xi_T = \alpha \left(\frac{S_T}{S_0} \right)^{-\frac{\theta}{\sigma}}, \tag{1.2}$$

where $\alpha = \exp\left(\frac{\theta}{\sigma} \left(\mu - \frac{\sigma^2}{2} \right) T - \left(r + \frac{\theta^2}{2} \right) T \right)$ and $\theta = \frac{\mu - r}{\sigma}$.

Definition 1.1.1 recalls the expression of the price of a European derivative with maturity T using the physical and risk-neutral measures respectively.

Definition 1.1.1. The prices at time 0 and at time t of a payoff X_T paid at T are computed as

$$c_0(X_T) := \mathbb{E}_\mathbb{P}[\xi_T X_T] = \mathbb{E}_\mathbb{Q}[e^{-rT} X_T]$$

and

$$c_t(X_T) := \mathbb{E}_\mathbb{P}\left[\frac{\xi_T}{\xi_t} X_T | \mathcal{F}_t \right] = \mathbb{E}_\mathbb{Q}[e^{-r(T-t)} X_T | \mathcal{F}_t]$$

respectively, where $\mathbb{Q}$ denotes the risk-neutral probability, and $\xi_t = e^{-rt} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)_t$.

1.1.2 Cost-efficiency Approach

We first construct simplified alternatives to a given highly path-dependent payoff using the concept of cost-efficiency. Formally, the "cost-efficient" payoff is the cheapest strategy with distribution F: it is the solution to the following optimization problem

$$\min_{\{X_T | X_T \sim F\}} c_0(X_T) \tag{1.3}$$

where $\{X_T | X_T \sim F\}$ denotes the set of payoffs X_T that have the distribution F and $c_0(X_T)$ is the initial budget needed at 0 to pay X_T at T. Bernard, Boyle, and Vanduffel (2014) prove that the optimization (1.3) has a unique (a.s.) solution $F^{-1}(1 - F_{\xi_T}(\xi_T))$, when the initial price is computed as $c_0(X_T) = \mathbb{E}_\mathbb{P}[\xi_T X_T]$ and the cdf of ξ_T, F_{ξ_T}, is distributed. Intuitively, the outcomes of the payoff with the distribution F are rearranged in reverse order of the values taken by ξ_T (which was first observed by Dybvig (1988a,b)). The cost-efficiency problem (1.3) is then extended by Bernard, Moraux, Rüschendorf, and Vanduffel (2015) to solve the cheapest strategy that has a given joint distribution with an additional market variable A_T:

$$\min_{(X_T, A_T) \sim G} c_0(X_T).$$

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We use the above idea to construct simplified hedges for path-dependent derivatives. Specifically, to hedge a highly path-dependent option with payoff X_T paid at maturity T (e.g., that depends on the entire path of the underlying), we consider an alternative derivative that depends for example on the underlying asset prices at a finite number of dates $t_0 = 0 < t_1 < t_2 < .. < t_n = T$ and that is "close" to the original payoff. Let $V(t_1, t_2, ..., t_n)$ denote the payoff of the alternative derivative such that

$$X_T \approx V(S_{t_1}, S_{t_2}, ..., S_{t_n}) \tag{1.4}$$

where the approximation is done through using criteria based on distributional properties of X_T. Typically, we choose a benchmark A_T (which can be a multidimensional vector) and (1.4) means that $(V(S_{t_1}, S_{t_2}, ..., S_{t_n}), A_T)$ has the same joint distribution as (X_T, A_T). Loosely speaking, we write that $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$ is an approximation for the path-dependent payoff X_T, also called a "simplified hedge".

1.1.3   Semi-static Hedge

To hedge a path-dependent payoff X_T, we derive approximations as given in (1.4) and hedge $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$ instead. To do so, we use semi-static hedging techniques by rebalancing n times the portfolio between 0 (first rebalancing date) and the maturity T. The semi-static replication to hedge the simplified derivative $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$ consists of replicating using European path-independent claims. This method builds on the static hedging strategy introduced by Breeden and Litzenberger (1978) and extended by Carr and Wu (2013) who show how to hedge a given payoff $h(S_T)$ where h is twice differentiable with an explicit portfolio consisting of bonds, stocks, and vanilla call and put options. At time $u > 0$, the payoff $h(S_T)$ is replicated by a position $h(S_u) - h'(S_u)S_u$ in the bank account, $h'(S_u)$ shares and $h''(K)dK$ out-of-the-money put and call options with strike K, for all $K \geq 0$, with maturity $T - u$. Specifically, at time T, the payoff $h(S_T)$ is decomposed as follows

$$
\begin{aligned}
h(S_T) &= [h(S_u) - h'(S_u)S_u] + h'(S_u)S_T \\
&+ \int_0^{S_u} h''(K)(K - S_T)^+ dK + \int_{S_u}^{\infty} h''(K)(S_T - K)^+ dK.
\end{aligned} \tag{1.5}
$$

The value at time u of this payoff can then be computed as

$$
\begin{aligned}
C_u[h(S_T)] &= [h(S_u) - h'(S_u)S_u]e^{-r(T-u)} + h'(S_u)S_u \\
&+ \int_0^{S_u} h''(K)\mathcal{P}_u(K)dK + \int_{S_u}^{\infty} h''(K)\mathcal{C}_u(K)dK.
\end{aligned} \tag{1.6}
$$

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where $\mathcal{P}_u(K)$ and $\mathcal{C}_u(K)$ denote respectively the prices at time u of a standard European put and of a call.

The semi-static hedging portfolio for a multivariate derivative $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$ can be constructed as follows. For ease of exposition, we first illustrate this construction with a bivariate derivative with payoff $X_T = f(S_t, S_T)$ for $0 < t < T$. The first step is to find its value at time t, i.e. $c_t(X_T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}f(S_t, S_T)|\mathcal{F}_t\right]$. It can be expressed as a function $g(S_t)$ of S_t. If g is twice differentiable, we can apply the above method using puts and calls with maturity t. Then, formula (1.5) can be applied directly with $h = g$ and $u = 0$, $T = t$. At time t, the static hedge matches perfectly $g(S_t)$, which is the exact budget needed at time t to buy the portfolio of bonds, stocks and vanilla calls and puts that will hedge the final payoff $X_T = f(S_t, S_T)$ in which S_t is now known. For this second hedge, we use options with maturity $T - t$ bought at time t at all strikes and apply a second time the static hedge recalled in (1.5) and $h(\cdot) = f(S_t, \cdot)$, S_t is known, and $u = t$. In practice, only a limited number of strikes are available in the market and thus some hedging errors remain. However, we can expect that transaction costs will be lower than in a delta hedging strategy as the portfolio is only rebalanced once (at time t) before maturity, and that this hedge is more robust to model risk than a delta hedge.

It is then straightforward to extend the semi-static hedging strategy to hedge multivariate derivatives. For example, to hedge a payoff of the form $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$, we choose a portfolio at time 0 and then rebalance it $n-1$ times at time t_i for $i = 1, 2, ..., n-1$. At time t_i, we compute $\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t_{i+1})}V(S_{t_1}, S_{t_2}, ..., S_{t_n})|\mathcal{F}_{t_{i+1}}]$, which is a function of $S_{t_1}, S_{t_2}, ..., S_{t_{i+1}}$ in which all values are known up to S_{t_i} are known at time t_i. It is thus a payoff of one random component $S_{t_{i+1}}$ and the expression (1.5) can then be applied at time $u = t_i$, for a maturity $T = t_{i+1}$. At each date, the proposed hedge matches exactly the cost needed to pursue the strategy, so that in theory it finally hedges the final maturity payoff $V(S_{t_1}, S_{t_2}, ..., S_{t_n})$ perfectly. Remaining hedging errors come from the fact that only a limited number of strikes is available in practice.

1.1.4   More General Markets

Most of the derivations in this chapter are given in the Black-Scholes setting. However, it is possible to apply the methodology to more realistic market models.

One extension is to construct and hedge simplified alternatives in exponential Lévy markets by applying the methodology of Von Hammerstein, Lütkebohmert, Rüschendorf, and Wolf (2014) and Rüschendorf and Wolf (2015). Specifically, they provide various theoretical and empirical grounds to build the state-price density in general Lévy markets using

the Esscher martingale measure, in the case of price processes driven by multivariate NIG- and VG-processes. Rüschendorf and Wolf (2015) provide an expression for ξ_T extending (1.2) and a powerful estimation technique.

We now explain how to extend the Black-Scholes model to use the Heston stochastic volatility model for which an explicit expression of ξ_T is also available. In the Heston model, under the real-world probability measure, the price process S_t follows the system of SDEs

$$
\begin{cases}
dS_t = (r + \mu^S v_t)S_t dt + \sqrt{v_t} S_t dB_t^S, & S_0 > 0, \\
dv_t = \kappa \left(\theta - v_t\right) dt + \sigma \sqrt{v_t} dB_t^v, & v_0 > 0,
\end{cases}
\tag{1.7}
$$

where μ^S governs the equity premium, B_t^S and B_t^v are two correlated standard Brownian motions with correlation ρ such that $\mathbb{E}\left[dB_t^S dB_t^v\right] = \rho dt$, v_t is the variance or squared volatility, $\kappa > 0$ determines the speed of mean reversion for the variance process, θ is the long-run variance level, and σ is the volatility of volatility. Then, assuming that the state-price density takes a specific exponential form, Christoffersen, Heston, and Jacobs (2013) show that the unique state-price density ξ_T reconciling the physical and risk neutral measures is given by

$$
\xi_T = \beta(T) \left(\frac{S_T}{S_0}\right)^\psi,
\tag{1.8}
$$

where

$$
\begin{aligned}
\beta(T) &= \exp\left(\delta T + \eta \int_0^T v_s ds + \zeta \left(v_T - v_0\right)\right), \\
\delta &= -\left(1 + \psi\right) r - \zeta \kappa \theta, \\
\eta &= -\psi \mu^S + \frac{1}{2}\psi + \zeta \kappa - \frac{1}{2}\left(\psi^2 + 2\psi\zeta\sigma\rho + \psi^2\sigma^2\right), \\
\psi &= \frac{-\mu^S + \frac{\lambda\rho}{\sigma}}{1 - \rho^2}, \\
\zeta &= \frac{\mu^S \sigma\rho - \lambda}{\sigma^2(1 - \rho^2)},
\end{aligned}
$$

and λ represents the price of volatility risk, which might be determined by the dynamics of a volatility-dependent asset. These parameters are found by imposing the condition that the product of the price of any traded asset and the state-price density is a martingale under the physical probability measure.

Note that the expression of the state-price density ξ_T in (1.8) extends naturally the one

in Black-Scholes model in (1.2). However, in the Heston model, ξ_T does not only depend on the price process S_T but depends also on the variance process $(v_t)_t$ for $t \in [0, T]$. We will slightly adapt the results of Bernard, Moraux, Rüschendorf, and Vanduffel (2015) to account for this generalization.[1]

1.2 Cheapest Hedge of a Path-independent Derivative using Its Distribution Only

In this section, we propose to hedge a path-dependent payoff by the cheapest payoff that has the same terminal distribution under the physical probability distribution $\mathbb{P}$. We illustrate this idea with the study of a geometric Asian option. This example is useful to highlight the limitations of hedging using the distribution of the payoff only. However, it is important to note that this observation related to the poor performance of the hedge is also valid for most path-dependent derivatives. In the remainder of the chapter, we suggest to use additional constraints to ensure that the hedge is effective.

1.2.1 Geometric Asian Call Option and Cheapest Alternative

The payoff of a continuously monitored geometric average G_T and a geometric Asian call option Y_T are given by

$$G_T = e^{\frac{1}{T} \int_0^T \ln(S_t) dt}, \qquad Y_T = (G_T - K)^+,$$

where K denotes the strike price. As recalled in Section 1.1.2 the cheapest payoff with the same distribution F as G_T is given explicitly as $F^{-1}(1 - F_{\xi_T}(\xi_T))$ where F_{ξ_T} denotes the cdf of the state price density ξ_T. In the Black-Scholes setting, when $\mu > r$, this expression can be simplified to $F^{-1}(F_{S_T}(S_T))$, which can be explicitly computed for a geometric Asian option. In this case, it is a power call option,

$$Y_T^* = d \left(S_T^{1/\sqrt{3}} - \frac{K}{d} \right)^+, \tag{1.9}$$

---

[1]They assume that ξ_t is a decreasing function of S_t for the ease of exposition but their results hold more generally.

where $d = S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$ (proof in the Appendix of Bernard, Boyle, and Vanduffel (2014)). There is a simple analytical solution for the initial price of the payoff $Y_T^\star$

$$C_0 = S_0 e^{\left(\frac{1}{\sqrt{3}}-1\right)rT+\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\mu T-\frac{\sigma^2 T}{12}}\Phi(h_1) - Ke^{-rT}\Phi(h_2), \tag{1.10}$$

where $h_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\mu T+\frac{r}{\sqrt{3}}T+\frac{1}{12}\sigma^2 T}{\sigma\sqrt{\frac{T}{3}}}$ and $h_2 = h_1 - \sigma\sqrt{\frac{T}{3}}$. Note that letting K tend to zero provides the cost-efficient payoff Y_T^* that has the same distribution as the geometric average G_T. We denote it by

$$G_T^* = dS_T^{1/\sqrt{3}}. \tag{1.11}$$

1.2.2 Hedging Performance

Assume that to hedge the original payoff G_T, respectively the Asian call payoff Y_T, we replicate G_T^* (respectively Y_T^*) instead. The hedging error is then related to the distance between the original payoff for the geometric average G_T and its cost-efficient counterpart $G_T^\star$, respectively between the Asian call $Y_T = (G_T - K)^+$ and $Y_T^\star$. To do so, we use Monte Carlo simulations, and simulate 50,000 times the geometric average G_T (as a log-normal variable) with the following parameters $K=S_0=100$, $r=0.05$, $\mu=0.1$, $\sigma=0.2$ and $T=1$.

By construction, the cdf of G_T (respectively of Y_T) is equal to the cdf of $G_T^\star$ (respectively of $Y_T^\star$), however, these two payoffs are not equal as they do not have the same cashflows. Let us denote by H_G and H_Y the payoffs of the respective hedging strategies when you sell G_T (resp. Y_T) and you replicate it by $G_T^\star$ (resp. $Y_T^\star$). To investigate the performance of this hedge, we thus study the difference between the desired payoff G_T (or Y_T) and the cost-efficient one. Given that the prices $c_G^\star$ and $c_Y^\star$ of $G_T^\star$ and of $Y_T^\star$ are strictly smaller than the respective prices c_G and c_Y of G_T and Y_T, we need to account for this difference. We assume that the extra premium received at time 0 is invested in a bank account so that the hedge portfolios are respectively equal to $G_T^\star + (c_G - c_G^\star)e^{rT}$ for G_T and $Y_T^\star + (c_Y - c_Y^\star)e^{rT}$ for Y_T. The hedging errors can then be defined by

$$H_G(T) := G_T^* - G_T + (c_G - c_G^*)e^{rT}, \qquad H_Y(T) := Y_T^* - Y_T + (c_Y - c_Y^*)e^{rT}. \tag{1.12}$$

We display $H_G(T)$ in the left panel of Figure 1.1 and $H_Y(T)$ in the right panel.

From the scatterplots in Figure 1.1, it is clear that the hedging performance is rather poor and that hedging the distribution is not enough. Positive values correspond to gains
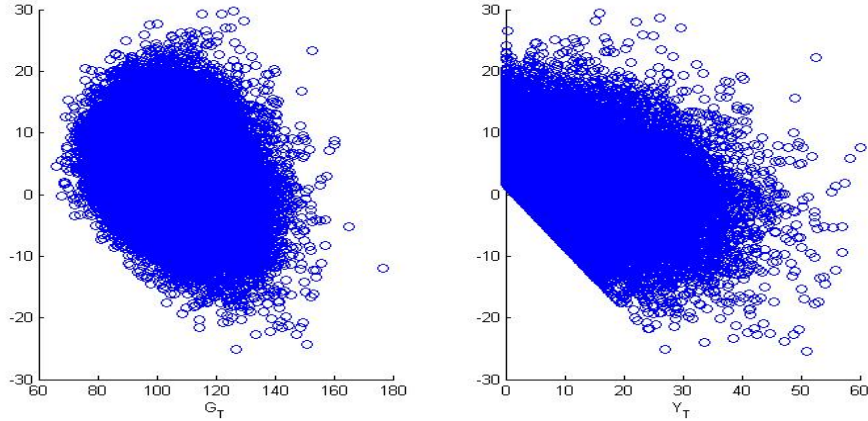
Figure 1.1: Scatterplots of $H_G(T)$ and $H_Y(T)$ as a function of G_T with $K = S_0 = 100, r = 0.05, \mu = 0.1, \sigma = 0.2$ and $T = 1$.

and negative values to losses. The hedging errors of the payoff G_T, denoted by $H_G(T)$, range roughly from -20 to 20 when G_T takes values between 80 and 150 (the corresponding relative errors from -16.7% to 18.2%). Similarly, $H_Y(T)$, the hedging errors of the payoff Y_T, range roughly from -20 to 20 when Y_T takes values between 0 and 50 (the corresponding relative errors from -80% to $+\infty$). These errors are very significant and are due to the fact that cashflows do not match in general, only the distributions of the payoffs between the original payoffs (G_T or Y_T) and their cost-efficient counterparts (G_T^* or Y_T^*) are similar. In particular, the correlation between the target payoff to hedge and the cost-efficient counterpart is not enough. In the next section, we add an additional constraint to improve the matching between the original payoff and the simplified alternative. To do so, we impose for example that the hedge should behave similarly as the original payoff with respect to some other benchmarks in the market.

1.3 Improved Hedge using Additional Distributional Information

We propose to improve the hedge discussed in the previous section by including more information apart from the distribution of the final payoff. Specifically, we assume that we construct an alternative derivative that has the same *joint* distribution with a well-chosen

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benchmark in the market as the target path-dependent payoff to hedge. As we will see next, in the subsequent sections, this technique allows to improve the hedging performance significantly.

The improved hedge can be constructed by taking the state-price density ξ_T as benchmark as stated hereafter in Theorem 2 and Theorem 4. Equivalently, the benchmark can also be the terminal underlying price S_T when the state-price density ξ_T is decreasing in S_T. The state-price density plays an important role as it reflects the market: it can be interpreted as the inverse of the growth optimal portfolio (Platen (2006)). In what follows, we first choose a benchmark that is either the state-price density ξ_T at maturity T or the stock price at maturity S_T. We then explain how to extend it to the case of a general benchmark A_T that has a joint density with the state-price density ξ_T, and could be multidimensional (a reference vector A_T of market variables).

1.3.1 Using Information on the Conditional Distribution of S_T or ξ_T

To construct the hedge, we apply the following theorem from Bernard, Moraux, Rüschendorf, and Vanduffel (2015).

Theorem 1 (Payoffs with a given joint distribution with S_T and price c)**.** Under the assumptions that (S_t, S_T) has a joint density with respect to the Lebesgue measure, and $\xi_T = g(S_T)$ where g is a decreasing deterministic function, consider a payoff X_T with price c having joint distribution G with S_T. Then, for any $0 < t < T$,

$$f(S_t, S_T) := F_{X_T|S_T}^{-1}(F_{S_t|S_T}(S_t)). \tag{1.13}$$

has the same price c and the same joint distribution G with S_T.

Given that the cost of a strategy is given by the expected value of the product of the payoff with the state-price density ξ_T, that the joint cdf between S_T and X_T is the same as the joint cdf between S_T and $f(S_t, S_T)$, and that $\xi_T = g(S_T)$, it follows that

$$c_0(X_T) = \mathbb{E}\left[g(S_T)X_T\right] = \mathbb{E}\left[g(S_T)f(S_t, S_T)\right] = c_0(f(S_t, S_T)). \tag{1.14}$$

The assumptions of Theorem 1 are verified for instance in the Black-Scholes market model when $\mu > r$. We then propose to extend it to more general state-price densities to be able to deal, for example, with the Heston stochastic volatility model. In the case when

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the state price density is not decreasing in the underlying stock price, we can modify the above theorem as follows.

Theorem 2 (Payoffs with a given joint distribution with ξ_T and price c)**.** Let X_T be a payoff with price c having joint distribution G with ξ_T in (1.8). Then, for any $0 < t < T$,

$$f(\xi_t, \xi_T) := F_{X_T|\xi_T}^{-1}(F_{\xi_t|\xi_T}(\xi_t)). \tag{1.15}$$

has the same price c and the same joint distribution G with ξ_T.

Two payoffs that have the same joint distribution with ξ_T have the same price given that $\mathbb{E}[\xi_T X_T]$ for a payoff X_T only depends on the joint distribution of ξ_T and X_T: when $(\xi_T, f(\xi_t, \xi_T)) \sim (\xi_T, X_T) \sim G$, $c = c_0(f(\xi_t, \xi_T)) = \mathbb{E}[\xi_T f(\xi_t, \xi_T)] = \mathbb{E}[\xi_T X_T] = c_0(X_T)$. The proof of Theorem 2 is similar to the proof of Theorem 1 (and is thus omitted: it can be found in Bernard, Moraux, Rüschendorf, and Vanduffel (2015)).

1.3.2 Using Information on the Conditional Distribution with a General Benchmark Vector A_T

We extend the result to construct the cheapest bivariate derivative with the right joint distribution with a vector A_T of market indicators instead of a single market variable.

Theorem 3. Under the same assumptions for the state-price density as in Theorem 1, let A_T be a d-multidimensional vector and assume that (S_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a $(d+1)$-variate cumulative distribution function. The following optimization problem

$$\min_{(X_T, A_T) \sim G} c_0(X_T) \tag{1.16}$$

has an almost surely unique solution X_T^* which is of the form $f(S_T, A_T)$, almost surely increasing in S_T conditionally on A_T, and given by

$$X_T^* := F_{X_T|A_T}^{-1}(F_{S_T|A_T}(S_T)). \tag{1.17}$$

Theorem 3 is proved in Bernard, Moraux, Rüschendorf, and Vanduffel (2015) in the case when A_T is a one-dimensional benchmark but the proof is identical when A_T is a vector and thus omitted. In the case of the Heston model for instance, the state-price density is not decreasing in S_T and thus the theorem needs to be modified to account for this generalization. In this case, the above theorem can be formulated using the state-price density as follows.

Theorem 4 (Cost-efficiency of the simplified alternatives)**.** Let A_T be a d-dimensional vector such that (ξ_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a $d + 1$ variate distribution function. The following optimization problem

$$\min_{(X_T, A_T) \sim G} c_0\left(X_T\right) \tag{1.18}$$

has an almost surely unique solution X_T^* which is of the form $f(\xi_T, A_T)$ almost surely decreasing in ξ_T conditionally on the benchmark A_T and given by

$$X_T^* := F_{X_T|A_T}^{-1}(1 - F_{\xi_T|A_T}(\xi_T)). \tag{1.19}$$

The proof follows the characterization of cost-efficiency by Bernard, Boyle, and Vanduffel (2014). X_T^* solves (1.18), i.e. it is the cheapest alternative that satisfies the constraint that the joint distribution with the benchmark $(X_T, A_T) \sim G$, because X_T^* and ξ_T are antimonotonic conditionally on A_T.

Section 1.3 has given the theoretical results that are needed to improve the hedging of path-dependent derivatives. The next section provides examples to illustrate their use in the Black Scholes setting. We provide several extensions in Section 1.5 including an example in the Heston stochastic volatility model.

1.4 Examples for the Construction of a Simplified Hedge as a Bivariate Derivative in the Black-Scholes Setting

In this section, we develop two examples in the Black-Scholes setting. We first illustrate the approach using a geometric Asian option, which we study in full detail. We then develop an additional example using the lookback option. In Section 1.5.1, we will explain how to deal with any path-dependent derivative in the Black-Scholes market model.

1.4.1 Example of Hedging the Geometric Asian Option with a Bivariate Derivative that Has the Right Joint Distribution with S_T

In the Black Scholes setting, ξ_T is a decreasing function of S_T. We can thus apply Theorem 1. By applying Theorem 1 to the payoff G_T in the Black-Scholes model, we construct $R_T(t) = f(S_t, S_T)$ such that

$$(R_T(t), S_T) \sim (G_T, S_T), \tag{1.20}$$

and the initial cost of G_T is equal to the bivariate derivative $R_T(t)$. We can compute $R_T(t)$ by applying (1.13) in Theorem 1 and we find that

$$R_T(t) = S_0^{\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}, \tag{1.21}$$

where t is freely chosen in $(0, T)$. Details on how (1.13) becomes (1.21) can be found in Bernard, Moraux, Rüschendorf, and Vanduffel (2015). Note that there is no uniqueness. For example, $1 - F_{S_t|S_T}(S_t)$ is also independent of S_T, we can thus also consider $H_T(t) := F_{X_T|S_T}^{-1}(1 - F_{S_t|S_T}(S_t))$ as a suitable bivariate derivative $(0 < t < T)$ satisfying the constraint about the joint distribution as in (1.20). In this case, one obtains $H_T(t) = S_0^{\frac{1}{2} + \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{-\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2} + \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}$.

By definition, this bivariate derivative preserves the dependence between G_T and S_T. Compared to the original contract, however, it is weakly path-dependent as it only depends on S_t and S_T and not on the entire path of the underlying S. Consequently, both the call option written on $R_T(t)$ and the call option written on G_T have the same joint distribution with S_T. More formally, one has

$$\left((R_T(t) - K)^+, S_T\right) \sim \left((G_T - K)^+, S_T\right). \tag{1.22}$$

$(R_T(t) - K)^+$ is therefore a bivariate alternative equivalent to the fixed strike (continuously monitored) geometric Asian call.

To implement the semi-static hedge, we will need to know the price of this call option with payoff $(R_T(t) - K)^+$ at any time. It can be priced explicitly at time 0. Indeed, the equality of joint distributions exposed in (1.22) implies that the call option written on $R_T(t)$ has the same price as the original fixed strike (continuously monitored) geometric

Asian call Y_T. The price at time 0 is therefore

$$c_0((G_T - K)^+) = c_0((R_T(t) - K)^+) = S_0 e^{-\frac{1}{2}\left(r+\frac{\sigma^2}{6}\right)T}\Phi(\tilde{d}_1) - Ke^{-rT}\Phi(\tilde{d}_2), \qquad (1.23)$$

where $\tilde{d}_1 = \frac{\ln(S_0/K)+\frac{1}{2}\left(r+\frac{\sigma^2}{6}\right)T}{\sigma\sqrt{T/3}}$ and $\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T/3}$ (see Kemna and Vorst (1990)). Again, letting K tend to zero provides the price of the alternative payoff corresponding to the geometric average G_T.

The pricing formula in (1.23) works for all choices of t in the payoff $R_T(t)$ to be paid at time T. In addition, given the log-normal property of $R_T(t)$, it is possible to compute explicitly the no-arbitrage price of $R_T(t)$ at any time u for a given time t. This price will be useful to implement the semi-static hedge.

Proposition 1. The price of $(R_T(t) - K)^+$ conditional on $\mathcal{F}_u$ at any time $u, t \in [0, T]$, can be computed explicitly

$$c_u\left((R_T(t)-K)^+\right) = \begin{cases} e^{-r(T-u)}\left[e^{\left(m_{R_1}+\frac{v_{R_1}}{2}\right)}\Phi\left(\frac{m_{R_1}+v_{R_1}-\ln K}{\sqrt{v_{R_1}}}\right) - K\Phi\left(\frac{m_{R_1}-\ln K}{\sqrt{v_{R_1}}}\right)\right], & u \le t, \\[4mm] e^{-r(T-u)}\left[e^{\left(m_{R_2}+\frac{v_{R_2}}{2}\right)}\Phi\left(\frac{m_{R_2}+v_{R_2}-\ln K}{\sqrt{v_{R_2}}}\right) - K\Phi\left(\frac{m_{R_2}-\ln K}{\sqrt{v_{R_2}}}\right)\right], & u \ge t, \end{cases}$$

$$(1.24)$$

where

$$m_{R_1} = \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}\right)\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}\right)\left[\ln S_u - \left(r - \frac{\sigma^2}{2}\right)u\right],$$

$$v_{R_1} = \sigma^2\left[\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}\right)^2(t-u) + \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\right)^2(T-t)\right],$$

$$m_{R_2} = \frac{\ln S_0}{2} - \frac{\ln S_0}{2\sqrt{3}}\sqrt{\frac{T-t}{t}} + \frac{1}{2\sqrt{3}}\frac{T}{t}\sqrt{\frac{t}{T-t}}\ln S_t$$
$$+ \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\right)\left[\ln S_u + \left(r - \frac{\sigma^2}{2}\right)(T-u)\right],$$

$$v_{R_2} = \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\right)^2\sigma^2(T-u).$$

Taking $u = 0$, the price in Proposition 1 is exactly equal to the initial price in (1.23) which is consistent with the property (1.14). To prove Proposition 1, apply Lemma 6 (given in Appendix A) with $a = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}, b = \frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$ and $c = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$.

Choosing among $R_T(t)$ by correlation. The construction in Theorem 1 depends on t. Thus there is an infinite number of bivariate derivatives satisfying the constraint that the joint distribution with S_T is preserved. Maximizing the correlation or equivalently minimizing the variance[2] of the difference between $\ln(R_T(t))$ and $\ln(G_T)$ is nevertheless a possible way to select the intermediary time t. The covariance between $\ln(R_T(t))$ and $\ln(G_T)$ is given by

$$\text{cov}\left(\ln\left(R_T\left(t\right)\right),\ln\left(G_T\right)\right) = \frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right).$$

and, by construction of $R_T(t)$, standard deviations of $\ln(R_T(t))$ and $\ln(G_T)$ are both equal to $\sigma\sqrt{\frac{T}{3}}$. So maximizing the correlation coefficient is equivalent to maximizing the

---

[2] The expansion of $\text{var}(\ln G_T - \ln R_T(t)) = \sigma^2_{\ln G_T(t)} + \sigma^2_{\ln R_T(t)} - 2\rho\sigma_{\ln G_T(t)}\sigma_{\ln R_T(t)}$. We can see that the following equivalence holds: $\max_t \rho(\ln R_T(t), \ln G_T) \Leftrightarrow \min_t \text{var}(\ln G_T - \ln R_T(t))$.

covariance and thus maximizing $f(t) = (T-t)\,t$. This is obtained at $t^* = \frac{T}{2}$ and the maximal correlation $\rho_{\max}$ between $\ln(R_T(t))$ and $\ln(G_T)$ is

$$\rho_{\max} = \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t^*)\,t^*}}{4T} = \frac{3}{4} + \frac{\sqrt{3}}{8} \approx 0.9665, \qquad (1.25)$$

reflecting that the newly designed bivariate derivative is highly correlated to the geometric average, while being considerably simpler.

Note that both the maximum correlation and the optimum $R_T(\frac{T}{2})$ are robust to changes in market parameters, given that the derivations do not involve the volatility parameter σ, the interest rate r or the instantaneous expected return μ.

Remark. Minimizing the value-at-risk (VaR) under $\mathbb{P}$ is another possible way to select the value for t. Given a confidence level $\alpha \in (0,1)$ we define the loss function as $L := \ln G_T - \ln R_T(t)$, then $VaR_\alpha(L)$ is the level α-quantile, i.e. $VaR_\alpha(L) = \inf\{l \in \mathbb{R} : \mathbb{P}(L > l) \le 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \ge \alpha\}$. We evaluate $VaR_\alpha(L)$ at different chosen times t and find out the optimal $R_T(t)$ that minimizes the VaR of the loss function L:

$$\min_t VaR_\alpha\left[\ln G_T - \ln R_T(t)\right]. \qquad (1.26)$$

Note that $(\ln G_T, \ln R_T(t))$ follows a bivariate normal so that $L := \ln G_T - \ln R_T(t)$ also follows a normal distribution with $\mu_L = 0$ and variance σ_L^2. The expression of $VaR_\alpha(L)$ can be simplified as

$$VaR_\alpha(L) = \mu_L + \sigma_L \Phi^{-1}(\alpha) = \sigma_L \Phi^{-1}(\alpha). \qquad (1.27)$$

From equation (1.27), minimizing $VaR_\alpha(L)$ is equivalent to minimizing σ_L^2, which is also equivalent to maximizing the correlation coefficient ρ (see Footnote 2). From (1.25), we take $\rho_{\max}^* = 0.9665$ when $t = 0.5T$ to compute $\min \sigma_L = 0.02989$ and thus we obtain $\min VaR_\alpha(L) = 0.0695$ when $\alpha = 99\%$. In more general market settings, or for more general path-dependent derivatives, minimizing VaR will generally not be the same as maximizing correlation.

We now hedge the geometric average of the stock price G_T and a call on this average Y_T using the above constructions $R_T(t^*)$ and $(R_T(t^*) - K)^+$. We compute the hedging errors as H_G and H_Y in (1.12):

$$L_G(T) := R_T^g - G_T, \qquad L_Y(T) := R_T^y - Y_T, \qquad (1.28)$$

where $R_T^g := R_T(\frac{T}{2})$ and $R_T^y := (R_T(\frac{T}{2}) - K)^+$ (where the time t^* has been set to the optimal value $\frac{T}{2}$). Note that the initial costs do not matter in the formulas as the alternative payoff

has the same cost as the original path-dependent derivative. The payoffs of the alternative derivatives R_T^g and R_T^y are now "closer" to the original payoffs G_T and Y_T, and thus the performance of the hedge is improved.
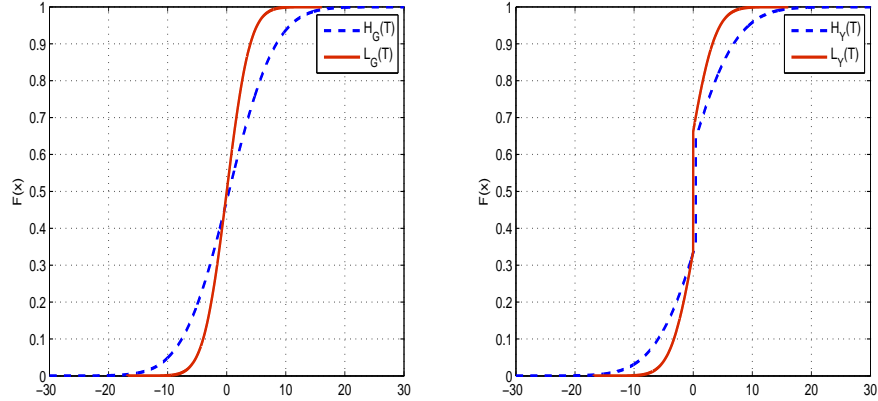
Figure 1.2: Empirical cdfs of $L_G(T)$ (on the left panel) and of $L_Y(T)$ (on the right panel) on top of the cdfs of $H_G(T)$ and $H_Y(T)$.

Figure 1.2 illustrates the fact that the hedging performance when hedging with the bivariate derivative R_T^g (resp. R_T^y) is greatly improved compared to the use of the cost-efficient payoff G_T^* (resp. Y_T^*). The empirical cdfs in Figure 1.2 are now steeper and the chance of the difference approaching zero becomes larger in both cases. Note that a perfect hedge would consist of a payoff for the difference equal to 0 almost surely and thus a cdf equal to $F(x) = 0$ for all $x < 0$ and 1 for all $x \geq 0$.

1.4.2 Example of hedging the geometric Asian option with a bivariate derivative that has the right joint distribution with S_t for $t \in (0, T)$

Given the benchmark $A_T = S_t$, we apply Theorem 3 to the payoff $G_T = e^{\frac{1}{T} \int_0^T \ln(S_t) dt}$ in the Black-Scholes setting and find a payoff $M_T(t) := f(S_T, S_t)$ such that

$$(M_T(t), S_t) \sim (G_T, S_t). \tag{1.29}$$

We find the following explicit expression of $M_T(t)$ by applying Theorem 3,

$$M_T(t) = a \cdot S_t^b \cdot S_T^p, \tag{1.30}$$

where

$$a = S_0^{\frac{t}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\left(\frac{(T-t)^2}{2T} - p(T-t)\right)}, b = 1 - p - \frac{t}{2T}, p = \sqrt{\frac{4T^3 - 12tT^2 - 3t^3 + 12t^2T}{12T^2(T-t)}}, \tag{1.31}$$

and t is freely chosen in $(0, T)$. Details on how (1.17) becomes (1.30) are given in Appendix 1.6.

Using (1.29), both the call option written on $M_T(t)$ and the call option written on G_T have thus the same joint distribution with S_t. More formally,

$$\left((M_T(t) - K)^+, S_t\right) \sim \left((G_T - K)^+, S_t\right). \tag{1.32}$$

Similar to the $R_T(t)$ case, we can derive the explicit form of the call price on $M_T(t)$ by log-normality. The pricing formula at time t is given in the following proposition.

Proposition 2. For $t \in [0, T]$, the price of $(M_T(t) - K)^+$ at time u can be computed using Definition 1.1.1.

$$c_u\left((M_T(t) - K)^+\right) = \begin{cases} e^{-r(T-u)}\left[e^{\left(m_{M_1} + \frac{v_{M_1}}{2}\right)}\Phi\left(\frac{m_{M_1} + v_{M_1} - \ln K}{\sqrt{v_{M_1}}}\right) - K\Phi\left(\frac{m_{M_1} - \ln K}{\sqrt{v_{M_1}}}\right)\right], & u \leq t, \\ e^{-r(T-u)}\left[e^{\left(m_{M_2} + \frac{v_{M_2}}{2}\right)}\Phi\left(\frac{m_{M_2} + v_{M_2} - \ln K}{\sqrt{v_{M_2}}}\right) - K\Phi\left(\frac{m_{M_2} - \ln K}{\sqrt{v_{M_2}}}\right)\right], & u \geq t, \end{cases}$$
$$\tag{1.33}$$

where

$$\begin{cases} m_{M_1} = \ln a + \left(1 - \frac{t}{2T}\right)\ln S_u + \left(r - \frac{\sigma^2}{2}\right)\left[\frac{T}{2} + p(T-t) - \frac{(T-t)^2}{2T} - \left(1 - \frac{t}{2T}\right)u\right], \\ v_{M_1} = \sigma^2\left[\frac{T}{3} - (b^2 + p^2 + 2bp)u\right], \\ m_{M_2} = \ln a + b\ln S_t + p\left[\ln S_u + \left(r - \frac{\sigma^2}{2}\right)(T-u)\right], \\ v_{M_2} = p^2\sigma^2(T-u), \end{cases}$$

and a, b and p are given by (1.31).

The proof of Proposition 2 follows immediately from Lemma 6 given in the Appendix. $\square$

We select the best alternative by maximizing the correlation of log-returns between the alternative and the geometric average. The correlation coefficient for $M_T(t)$ is derived in

the Appendix and is equal to

$$\rho = \left(1 - p - \frac{t}{2T}\right)\frac{3t}{T}\left(1 - \frac{t}{2T}\right) + \frac{3p}{2}. \tag{1.34}$$

Due to the complexity of this expression, Figure 1.3 illustrates the pattern of the correlation coefficient ρ with respect to the chosen time t on (0,1). We observe that the maximum correlation $(\rho^*_{\max} = 0.9650)$ is achieved while taking t equal to $t^* = 0.48$. Note that it is also robust to market changes as it does not depend on μ, r and σ.
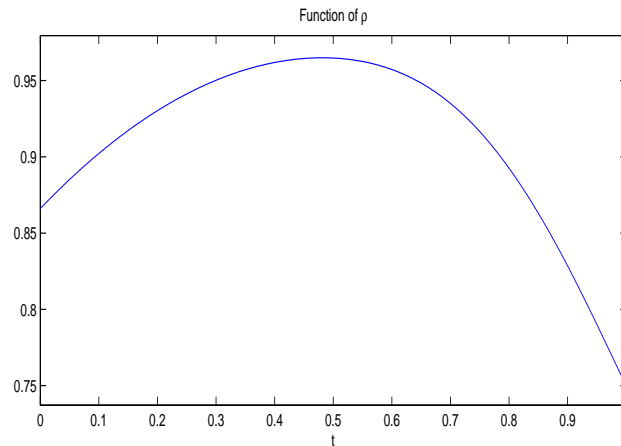
Figure 1.3: Correlation coefficient $\rho\left(\ln M_T(t), \ln G_T\right)$ with respect to the chosen time t by benchmark S_t.

We then note that as $\ln(M_T(t))$ is normally distributed, the minimum VaR is achieved when correlation is maximum between $\ln(M_T(t))$ and $\ln(G_T)$. We report our results in Table 1.1 hereafter.

	t^*	$\max \rho$	$\min$ std.dev.	$\min VaR_{99\%}$
$R_T(t)$	0.5	0.9665	0.0299	0.070
$M_T(t)$	0.48	0.9650	0.0306	0.071

Table 1.1: Comparison between $R_T(t)$ and $M_T(t)$ in terms of correlation, standard deviation and Value-at-Risk of the loss function.

Similarly to (1.12), we define the hedging errors as follows

$$
\begin{aligned}
D_G(T) &:= M_T^g(t^*) - G_T + (c_0(G_T) - c_0(M_T^g(t^*)))\, e^{rT}, \qquad (1.35) \\
D_Y(T) &:= M_T^y(t^*) - Y_T + (c_0(Y_T) - c_0(M_T^y(t^*)))\, e^{rT},
\end{aligned}
$$

where $M_T^g(t^*)$ and $M_T^y(t^*)$ are the optimal simplified bivariate derivatives to the geometric Asian average G_T and the geometric Asian call Y_T respectively. Note that the initial price of $(M_T(t) - K)^+$ at $u = 0$ in (1.33) is not the same as the initial price of $(G_T - K)^+$ in (1.23). Finally, we find that the cdfs of $D_G(T)$ and $D_Y(T)$ are very close to that of $L_G(T)$ and $L_Y(T)$ in Figure 1.2. This example illustrates that the choice of the benchmark may not always be significant. In this case the benchmark can be chosen to be S_{t^*} or S_T and the hedging performance is similar.

The above example deals with the geometric Asian option but the methodology is not specific to that option and other derivatives can be considered as well. Next, we briefly illustrate the methodology with the Lookback option. We then explain how to deal with a general path-dependent payoff in Section 1.5.1.

1.4.3 Approximating a Lookback Option with a Bivariate Derivative

In this section, we develop a second example. Specifically, in the Black-Scholes setting, we consider $A_T = S_t$ as benchmark and derive the corresponding simplified alternative for a lookback call option with strike K and payoff

$$
L_T = \left(\max_{u \in [0,T]} \{S_u\} - K \right)^+ .
$$

that has the same joint distribution with the chosen benchmark S_t. To do so, we need the conditional distribution of L_T given S_t. We make use of the properties of the Brownian bridge that can be found for instance in Beghin and Orsingher (1999).

Specifically, assume that $0 < t < T$, then the conditional distribution of L_T given $A_T = S_t$ depends on the value of S_0 and S_t. Since L_T given $A_T = S_t$ has a non-negative payoff, $F_{L_T|S_t}(\ell) = 0$ when $\ell < 0$. When $\ell \geq 0$, it depends on S_0, S_t and K:

In the case when $S_0 \geq K$ or $S_t \geq K$, the Lookback option always terminates in-the-money as $L_T > 0$ almost surely. Note that $F_{L_T|S_t}(\ell) = 0$ if $0 \leq \ell \leq \max(S_0, S_t) - K$.

When $\ell > \max(S_0, S_t) - K$, we have

$$
\begin{aligned}
F_{L_T|S_t}(\ell) & = \mathbb{P}\left(L_T \leq \ell \big| S_t = s\right) = \mathbb{P}\left(\max_{u \in [0,T]}\{S_u\} \leq K + \ell \big| S_t = s\right) \\
& = \mathbb{P}\left(\max_{u \in [0,T]}\left\{\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)u + W_u\right\} \leq x \big| \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t + W_t = \eta\right),
\end{aligned}
$$

where $x = \frac{1}{\sigma}\ln\left(\frac{K+\ell}{S_0}\right)$ and $\eta = \frac{1}{\sigma}\ln\left(\frac{s}{S_0}\right)$. By Beghin and Orsingher (1999) for any $x > 0$, when $\eta < x$, i.e. when $\ell > s - K$,

$$
F_{L_T|S_t}(\ell) = 1 - e^{-2x(x-\eta)/t} - \left(1 - e^{-2x(x-\eta)/t}\right)\left(1 - \Phi(d_1) + e^{2\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)(x-\eta)}\left(1 - \Phi(d_2)\right)\right),
$$

$$(1.36)$$

where Φ is the standard normal cumulative distribution function, $d_1 = \frac{x-\eta}{\sqrt{T-t}} - \frac{\sqrt{T-t}}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)$ and $d_2 = \frac{x-\eta}{\sqrt{T-t}} + \frac{\sqrt{T-t}}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)$. When $\eta \geq x$, i.e. when $\ell \leq s - K$, $F_{L_T|S_t}(\ell)$ is equal to 0. In the case exposed above, $F_{L_T|S_t}(\ell)$ is invertible for any $y \in (0,1)$ since it is continuous and strictly increasing from $\mathbb{R}^+$ to $(0,1)$.

In the case when $S_0 < K$ and $S_t < K$, it is easy to check that $x > 0$ and $\eta < x$. Thus formula (1.36) holds as well. In the case when $S_0 < K$ and $S_t \geq K$, we first note that $x > 0$. Then, for $0 \leq \ell \leq S_t - K$, i.e., for $\eta \geq x$, $F_{L_T|S_t}(\ell) = 0$. When $\ell > S_t - K$, i.e. when $\eta < x$, formula (1.36) holds for positive $F_{L_T|S_t}(\ell)$.

To construct an approximate bivariate derivative for L_T, we need $F_{L_T|S_t}$ as computed above and we need the distribution of S_T conditional on S_t as well, which we recall here

$$
y_S := F_{S_T|S_t}(S_T) = \Phi\left(\frac{\ln\frac{S_T}{S_t} - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right),
$$

and by Theorem 3, the simplified payoff for L_T is then written as $L_T^* = F_{L_T|S_t}^{-1}(y_S)$.

The original payoff L_T and the corresponding simplified alternative L_T^* do not have the same initial costs. Thus we define the hedging error of the Lookback call option L_T by the alternative payoff L_T^* as $K_Y(T) := L_T^* - L_T + (c_0(L_T) - c_0(L_T^*))e^{rT}$. We then represent the cdf of the hedging error in Figure 1.4 for $t = T/2$.

We observe from Figure 1.4 that the distribution of hedging errors for $K_Y(T)$ has some spread and may incur losses of size up to 40. It means that hedging a lookback option with

one intermediary date is not enough and that we need to impose additional constraints on the simplified derivative to achieve a better hedge.
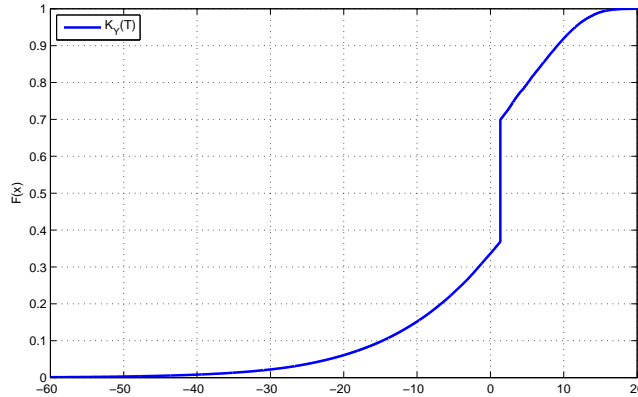
Figure 1.4: cdfs of $K_Y(T)$ with the parameters: $S_0 = 100$, $K = 110$, $\mu = 0.1$, $\sigma = 0.2$, $T = 1$ and chosen time $t = 0.5$.

1.5 Extensions

Here, we extend the previous results in several directions. First, we explain how to deal with the hedge of a general path-dependent derivative with several intermediary distributional constraints. We then illustrate on an example the case when the benchmark is a multivariate vector. In this case, the simplified alternative depends on a finite number of dates (and not only on one intermediary date). Finally, we discuss the more general market model of Heston to incorporate stochastic volatility.

1.5.1 General Path-dependent Derivative Approximated by a Multivariate Derivative

In the previous section, we have approximated path-dependent payoffs with bivariate derivatives. However, even though the hedging errors are significantly reduced compared to the case of hedging the terminal distribution only, they can remain significant as it is the case for the Lookback option. We now show how to extend the construction method to

multivariate derivatives to improve the hedge considered in the previous section. Specifically, we choose a benchmark vector $A_T = (A_1, ... A_n)$ and then construct a simple payoff $V_T^{(n)}$ such that $(V_T, A_T) \sim (X_T, A_T)$, where X_T is the target payoff to hedge. Applying Theorem 4 in Section 1.3

$$V_T^{(n)} := F_{X_T|A_T}^{-1}(1 - F_{\xi_T|A_T}(\xi_T)) \tag{1.37}$$

where both $F_{X_T|A_T}$ and $F_{\xi_T|A_T}$ need to be estimated and where we choose $A_T = (S_{t_1}, ..., S_{t_n})$. In the Black-Scholes model (and in more general market model), it is always possible to proceed numerically and develop an approximation of the path-dependent derivative by a simplified multivariate alternative payoff using numerical methods when explicit closed-form expressions for the conditional distributions appearing in the approximation (1.37) are not available.

There exists an extensive literature on nonparametric estimation of conditional distributions such as Hall, Wolff, and Yao (1999), Cai (2002), Hansen (2004) and Hall and Yao (2005). Hall and Yao (2005) show that conventional nonparametric estimators can suffer poor accuracy and slow convergence rates and suggest approximating the distribution of a random variable Y given a dependent random n-variate vector X, $Y|X$, by that of $Y|\theta^T X$ instead, where the unit vector θ is selected so that the approximation is optimal under a least-squares criterion. We use this technique to approximate by the conditional distributions to the n-variate benchmark vector $A_T = (S_{t_1}, S_{t_2}, \cdots, S_{t_n})$ ($\hat{F}_{X_T|A_T}$ and $\hat{F}_{S_T|A_T}$).

1.5.2 Example of the Hedge of an Asian Option with a Trivariate Derivative in the Black-Scholes Model

We now illustrate the idea of the previous paragraph in the Black-Scholes market by the explicit construction of a trivariate derivative V_T that approximates the Asian option $Y_T = (G_T - K)^+$ in the sense that $(S_{t_1}, S_{t_2}, V_T) \sim (S_{t_1}, S_{t_2}, Y_T)$. To do so, we first find

$$(V_T(t_1, t_2), S_{t_1}, S_{t_2}) \sim (G_T, S_{t_1}, S_{t_2})$$

for $t_1 < t_2$. To do so, we apply Theorem 3 to the payoff G_T and find a trivariate payoff $V_T(t_1, t_2) = f(S_T, S_{t_1}, S_{t_2})$ such that

$$(S_{t_1}, S_{t_2}, V_T(t_1, t_2)) \sim (S_{t_1}, S_{t_2}, G_T). \tag{1.38}$$

The explicit form of $V_T(t_1, t_2)$ by applying Theorem 3 can be expressed as

$$V_T(t_1, t_2) = a S_{t_1}^b S_{t_2}^c S_T^d, \qquad (1.39)$$

where

$$a = S_0^{\frac{t_1}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\left[\frac{(T-t_2)^2}{2T} - \sqrt{(T-t_2)h}\right]}, \quad b = \frac{t_2}{2T}, \quad c = 1 - \frac{t_1 + t_2}{2T} - \sqrt{\frac{h}{T - t_2}},$$

$$d = \sqrt{\frac{h}{T - t_2}}, \quad h = \frac{T}{3} - t_2 + \frac{t_2^2}{T} - \frac{t_1 t_2 (t_2 - t_1) + t_2^3}{4T^2}. \quad (1.40)$$

Details on how (1.17) becomes (1.39) are shown in Appendix 1.6.

We are able to compute the price of $(V_T(t_1, t_2) - K)^+$. The trivariate derivative $V_T(t_1, t_2)$ is more path-dependent than $R_T(t)$ or $M_T(t)$ (see Section 1.4) but it is still less path-dependent than the original Asian option. The call option on $V_T(t_1, t_2)$ and the call option on G_T have the same joint distribution with S_{t_1} and S_{t_2}, i.e,

$$\left((V_T(t_1, t_2) - K)^+, S_{t_1}, S_{t_2}\right) \sim \left((G_T - K)^+, S_{t_1}, S_{t_2},\right). \qquad (1.41)$$

$(V_T(t_1, t_2) - K)^+$ is therefore a trivariate equivalent to the fixed strike (continuously monitored) geometric Asian call. Applying Lemma 6, Proposition 3 provides the pricing formula for the call on $V_T(t_1, t_2)$.

Proposition 3. Let $t_1, t_2 \in [0, T]$. The price of $(V_T(t_1, t_2) - K)^+$ at any time u is given by

$$c_u\left((V_T(t_1, t_2) - K)^+\right) = \begin{cases} e^{-r(T-u)} \left[e^{\left(m_{V_1} + \frac{v_{V_1}}{2}\right)} \Phi\left(\frac{m_{V_1} + v_{V_1} - \ln K}{\sqrt{v_{V_1}}}\right) - K\Phi\left(\frac{m_{V_1} - \ln K}{\sqrt{v_{V_1}}}\right) \right], & u < t_1, \\[3mm] e^{-r(T-u)} \left[e^{\left(m_{V_2} + \frac{v_{V_2}}{2}\right)} \Phi\left(\frac{m_{V_2} + v_{V_2} - \ln K}{\sqrt{v_{V_2}}}\right) - K\Phi\left(\frac{m_{V_2} - \ln K}{\sqrt{v_{V_2}}}\right) \right], & t_1 \le u < t_2, \\[3mm] e^{-r(T-u)} \left[e^{\left(m_{V_3} + \frac{v_{V_3}}{2}\right)} \Phi\left(\frac{m_{V_3} + v_{V_3} - \ln K}{\sqrt{v_{V_3}}}\right) - K\Phi\left(\frac{m_{V_3} - \ln K}{\sqrt{v_{V_3}}}\right) \right], & u \ge t_2, \end{cases}$$

$$(1.42)$$

where

$$
\begin{cases}
m_{V_1} = \ln a + \left(1 - \frac{t_1}{2T}\right) \ln S_u + (r - \sigma^2) \left[\frac{T}{2} + \sqrt{(T - t_2)\, h} - \frac{(T - t_2)^2}{2T} - \left(1 - \frac{t_1}{2T}\right) u\right], \\
v_{V_1} = \sigma^2 \left[\frac{T}{3} - \left(1 - \frac{t_1}{2T}\right)^2 u\right], \\
m_{V_2} = \ln a + b \ln S_{t_1} + (c + d) \ln S_u + \left(r - \frac{\sigma^2}{2}\right) [c t_2 + dT - (c + d)\, u], \\
v_{V_2} = \sigma^2 \left[c^2 (t_2 - u) + d^2 (T - u) + 2cd\, (t_2 - u)\right], \\
m_{V_3} = \ln a + b \ln S_{t_1} + c \ln S_{t_2} + d \ln S_u + \left(r - \frac{\sigma^2}{2}\right) d\, (T - u), \\
v_{V_3} = d^2 \sigma^2 \, (T - u),
\end{cases}
$$

and a, b, c, d and h are defined in (1.40).

The initial price of $V_T(t_1, t_2)$ at $u = 0$ in (3) is not the same as the initial price of G_T.

The previous results show that the criteria of correlation, variance and VaR between $\ln V_T(t_1, t_2)$ and $\ln G_T$ are equivalent under the assumption of log-normality. We compute the correlation coefficient for $\ln V_T(t_1, t_2)$ and $\ln G_T$, which is given by

$$
\rho = \frac{3 t_1 t_2}{2 T^3} \left(T - \frac{t_1}{2}\right) + \frac{3 t_2}{T^2} \left(T - \frac{t_2}{2}\right) \left(1 - \frac{t_1 + t_2}{2T} - \sqrt{\frac{h}{T - t_2}}\right) + \frac{3}{2} \sqrt{\frac{h}{T - t_2}}, \quad (1.43)
$$

where h can be found in (1.40). Detailed derivations are in Appendix 1.6. Maximizing the above correlation is a non-linear optimization problem with respect to t_1 and t_2 under the constraint such that $0 < t_1 < t_2 < T$. The generalized reduced gradient algorithm solves that $\rho_{\max}$ is equal to 0.9839 while $t_1^* = 0.32$ and $t_2^* = 0.64$, which is about 2% higher correlation compared to those of $M_T(t)$ and $R_T(t)$.

	t_1^*	t_2^*	$\max \rho$	$\min$ std.dev.	$\min$ VaR$_{99\%}$
$V_T(t_1, t_2)$	0.32	0.64	0.9839	0.0207	0.048

Table 1.2: Correlation, standard deviation and Value-at-Risk of the loss function for $V_T(t_1, t_2)$ and G_T under the optimal chosen time.

We define the error corresponding to the approximation by hedging strategy $V_T(t_1, t_2)$ of the geometric average.

$$
\begin{aligned}
J_G(T) : &= V_T^g(t_1^*, t_2^*) - G_T + (c_0(G_T) - c_0(V_T^g(t_1^*, t_2^*)))\, e^{rT}, \\
J_Y(T) : &= V_T^y(t_1^*, t_2^*) - Y_T + (c_0(Y_T) - c_0(V_T^y(t_1^*, t_2^*)))\, e^{rT}. \quad (1.44)
\end{aligned}
$$

where $V_T^g(t_1^*, t_2^*)$ and $V_T^y(t_1^*, t_2^*)$ are the optimal simplified trivariate derivatives to the geometric Asian average G_T and the geometric Asian call Y_T respectively.
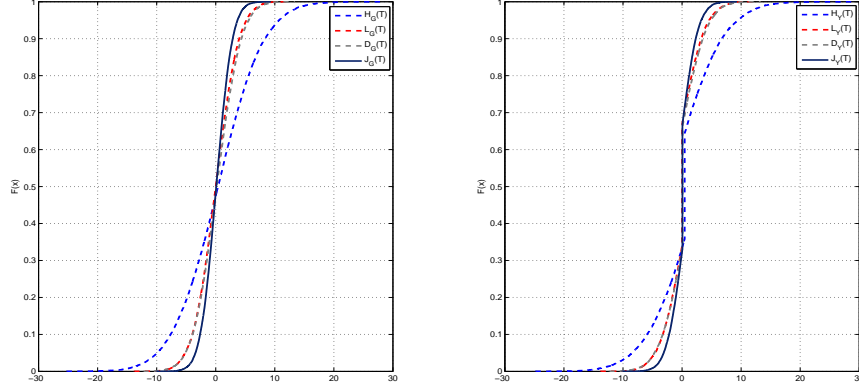
Figure 1.5: cdfs of $J_G(T)$ (on the left panel) and of $J_Y(T)$ (on the right panel) on top of the cdfs of $H_G(T)$, $H_Y(T)$, $L_G(T)$, $L_Y(T)$, $D_G(T)$ and $D_Y(T)$.

From Figure 1.5, we see that the theoretical deviation between the geometric average or the Asian call and the corresponding simplified derivatives is very small and thus it is satisfactory to replace the hedging of the path-dependent derivative by the hedging of the simplified one.

1.5.3 Simplified Alternative in the Heston Volatility Model

We now illustrate the construction method of the simplified alternative for the geometric Asian call option in the Heston stochastic volatility model using Theorem 4. To do so, we numerically evaluate the corresponding simplified payoff with the (quadruplet) benchmark $A_T = \left(\xi_t, S_t, v_t, \int_0^t v_s ds \right)$. There is no guarantee that the constructed payoff has the same initial cost. We thus define, similarly as before, the hedging error at maturity time T under the Heston stochastic volatility model as

$$I_G(T) := X_T^{*,g} - G_T + (c_G - c_{X^*,g}) e^{rT}, \qquad I_Y(T) := X_T^{*,y} - Y_T + (c_Y - c_{X^*,y}) e^{rT}, \quad (1.45)$$

where $X_T^{*,g}$ is the simplified alternative for G_T in the Heston stochastic volatility model, $c_{X^*,g}$ is its cost at time 0 and $X_T^{*,y}$ is the simplified alternative for Y_T in the Heston stochastic

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volatility model, $c_{X^{*,y}}$ is its cost at time 0. Note that $c_{X^{*,y}} = c_0\left((X_T^{*,g} - K)^+\right)$.

From (1.8), we rewrite the Heston state-price density as

$$\xi_T = \xi_t \left(\frac{S_T}{S_t}\right)^\psi \times \exp\left(\delta(T - t) + \eta \int_t^T v_s ds + \zeta(v_T - v_t)\right), \qquad (1.46)$$

and it depends on the quadruplet benchmark A_T. By Theorem 4, we construct the bivariate alternative $X_T^{*,g}$ conditional on the quadruplet $A_T = \left(\xi_t, S_t, v_t, \int_0^t v_s ds\right)$ with the following procedure:

1. Use the Euler scheme to simulate the dynamics in (1.7) and the state-price density ξ_t in (1.8) simultaneously to simulate the quadruplet $A_T = \left(\xi_t, S_t, v_t, \int_0^t v_s ds\right)$.

2. Given the intermediate state vector A_T, by (1.46), we use forward simulation to generate the future paths of ξ_t over the period $[t, T]$ and obtain the random state prices ξ_T at maturity. We thus find the distribution of ξ_T on A_T.

3. Compute the quantile $q = 1 - \hat{F}_{\xi_T|A_T}(\xi_T) \sim \text{Uniform}(0, 1)$

4. To hedge the original path-dependent payoff X_T with conditional cdf $\hat{F}_{X_T|A_T}$, we use Theorem 4 to construct the cheapest alternative $X_T^{*,g}$ with the choice of benchmark $A_T = \left(\xi_t, S_t, v_t, \int_0^t v_s ds\right)$. This is given by $X_T^{*,g} = \hat{F}_{X_T|A_T}^{-1}(q)$

5. The costs of the alternative payoff $X_T^{*,g}$ and its corresponding call $(X_T^{*,g} - K)^+$ can be respectively evaluated by

$$c_{X^*}^g := \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\xi_T X_T^{*,g}\big|A_T\right]\right], \qquad c_{X^*}^Y := \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\xi_T (X_T^{*,g} - K)^+\big|A_T\right]\right].$$

Figure 1.6 illustrates the hedge between the simplified bivariate derivative and the geometric average under the Heston stochastic volatility model. We find that the correlation between $\ln(I_G(t))$ and $\ln(G_T)$ is approximately equal to 97%, while the corresponding correlation between $\ln(I_Y(t))$ and $\ln(Y_T)$ is about 91%. Figure 1.6 illustrates that the theoretical deviation between the goemetric average or the Asian call and the corresponding bivariate derivatives in the Heston market is very small and thus satisfactory to replace the original payoffs by the simplified ones.
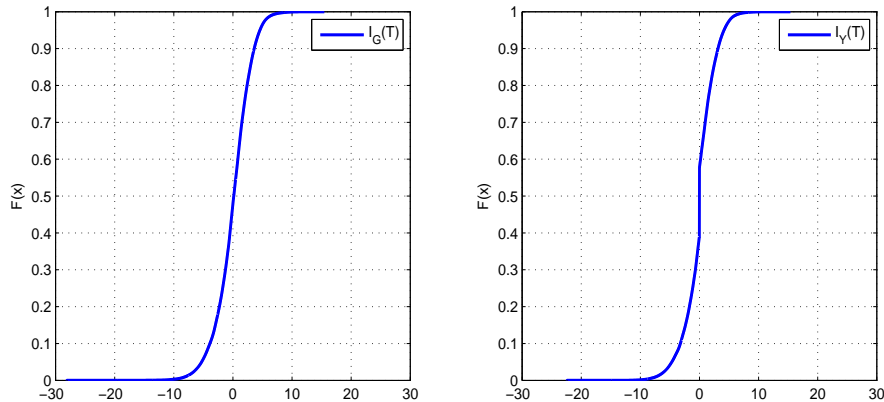
Figure 1.6: Empirical cdfs of $I_G(T)$ (on the left panel) and of $I_Y(T)$ (on the right panel) with the Heston parameters: $\rho = -0.5$, $\kappa = 0.3$, $\sigma = 0.1$, $\theta = v_0 = 0.015$, $\lambda = -0.02$, $K = S_0 = 100$, $r = 0.02$, $\mu^S = 0.2$, $T = 1$ and chosen time $t = 0.5$.

1.6 Concluding Remarks

We develop a method to hedge a highly path-dependent payoff using simplified derivatives that preserve certain distributional properties of the original payoff. Our work builds on the study of Bernard, Moraux, Rüschendorf, and Vanduffel (2015) and shows how to apply it in practice in the context of hedging. In particular, we are able to extend it to the case when one has a finite number of constraints (which corresponds to a n-dimensional benchmark vector A_T). We illustrate the study by a geometric Asian option and a lookback option but our method can be extended to hedge other path-dependent options by numerically approximating their distributions and conditional distributions when they are not available explicitly. It is also possible to simplify path-dependent payoffs under more general models than the Black-Scholes setting, such as the Lévy and Heston models that are mentioned in the chapter. More generally, affine jump-diffusion models can be considered. In another context, Bernard, Chen, and Vanduffel (2015) use the cost-efficiency theory to find the utility function of an investor with law-invariant preferences. The use of cost-efficiency conditional to a benchmark can be used to extend to state-dependent preferences. Another research direction is to extend the work of Amin and Kat (2003), who use the cost-efficiency theory to replicate the distribution of hedge fund returns at lowest cost. However, the hedge fund performance is not exactly "replicated," only its probability distribution is preserved. Our work can also be used to provide a closer replication in this context.

APPENDIX

Proofs

Lemma 5 (Conditional Multivariate Gaussian). Assuming that Σ is positive definite, the conditional distributions of X_2 given X_1 and of X_1 given X_2 are multivariate normal. For example, $X_2|X_1 = x_1$ is a multivariate normal with dimension $d - k$ with mean $\hat{\mu} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$ and variance $\hat{\Sigma} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, where $\mu_1 = \mathbb{E}[X_1]$, $\mu_2 = \mathbb{E}[X_2]$ and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$.

We use the above method to generate $(\ln S_t, \ln S_T, \ln G_T)$ exactly (without any discretization for the geometric average). The covariance matrix we derived for this case is

$$\Sigma = \begin{bmatrix} \sigma^2 t & \sigma^2 t & \frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) \\ \sigma^2 t & \sigma^2 T & \frac{\sigma^2 T}{2} \\ \frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) & \frac{\sigma^2 T}{2} & \frac{\sigma^2 T}{3} \end{bmatrix}.$$

Lemma 6 (Pricing formula). In the Black-Scholes model, the price at time u of the payoff $X = aS_t^b S_T^c$ (to be paid at time T) is equal to

$$c_u(X) = \begin{cases} e^{-r(T-t)}\left[\exp\left(m_1 + \frac{v_1}{2}\right)\Phi\left(\frac{m_1 + v_1 - \ln K}{\sqrt{v_1}}\right) - K\Phi\left(\frac{m_1 - \ln K}{\sqrt{v_1}}\right)\right], & \text{if } u < t, \\ e^{-r(T-t)}\left[\exp\left(m_2 + \frac{v_2}{2}\right)\Phi\left(\frac{m_2 + v_2 - \ln K}{\sqrt{v_2}}\right) - K\Phi\left(\frac{m_2 - \ln K}{\sqrt{v_2}}\right)\right], & \text{if } u \geq t, \end{cases}$$

where m_1, m_2, v_1 and v_2 are explicitly known and given below.

To prove Lemma 6, note that $\ln(X)$ is normally distributed with conditional mean m and conditional variance v conditionally on $\mathcal{F}_u$. i.e. $\ln(X)|\mathcal{F}_u \sim N(m, v)$, then the price

of $(X - K)^+$ at time u is

$$
\begin{aligned}
c_u\left((X-K)^+\right) &= e^{-r(T-u)}\mathbb{E}_u\left[(X-K)^+\right] \\
&= e^{-r(T-u)}\left[\exp\left(m+\frac{v}{2}\right)\Phi\left(\frac{m+v-\ln K}{\sqrt{v}}\right) - K\Phi\left(\frac{m-\ln K}{\sqrt{v}}\right)\right].
\end{aligned}
$$

Recall that $\ln X = \ln a + b \cdot \ln S_t + c \cdot \ln S_T$. Then for $0 \le u < t$, both S_t and S_T are random and

$$
\ln X = \ln a + b\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(t-u)+\sigma W_{u,t}\right] + c\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(T-u)+\sigma W_{u,T}\right],
$$

where $W_{u,t} = W_t - W_u$.

$$
m_1 = \mathbb{E}_u\left[\ln X\right] = \ln a + b\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(t-u)\right] + c\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(T-u)\right],
$$

$$
v_1 = \mathrm{var}_u\left[\ln X\right] = \mathrm{var}\left(b\sigma W_{u,t}+c\sigma W_{u,T}\right) = b^2\sigma^2\,(t-u)+c^2\sigma^2\,(T-u)+2bc\sigma^2\,(t-u).
$$

For $t \le u < T$, $\ln X = \ln a + b\ln S_t + c\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(T-u)\right]+\sigma W_{u,T}$,

$$
\begin{aligned}
m_2 &= \mathbb{E}_u\left[\ln X\right] = \ln a + b\ln S_t + c\left[\ln S_u + \left(r-\frac{\sigma^2}{2}\right)(T-u)\right], \\
v_2 &= \mathrm{var}_u\left[\ln X\right] = \mathrm{var}\left(c\sigma W_{u,T}\right) = c^2\sigma^2\,(T-u).
\end{aligned}
$$

Derivation of Equation (1.30)

Given the benchmark $A_T = S_t$, we take $X|Y = \ln(S_T/S_0)\,|\,\ln(S_t/S_0)$. We thus obtain that $\mathbb{E}(X) = \mathbb{E}[\ln(S_T/S_0)] = \left(\mu-\frac{1}{2}\sigma^2\right)T$, $\mathbb{E}(Y) = \mathbb{E}[\ln(S_t/S_0)] = \left(\mu-\frac{1}{2}\sigma^2\right)t$, $\mathrm{var}(X) = \mathrm{var}[\ln(S_T/S_0)] = \sigma^2 T$, $\mathrm{var}(Y) = \mathrm{var}[\ln(S_t/S_0)] = \sigma^2 t$, $\mathrm{cov}(X,Y) = \sigma^2\mathrm{var}(W_t) = \sigma^2 t$, and $\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{\sigma^2 t}{\sigma\sqrt{T}\sigma\sqrt{t}} = \sqrt{\frac{t}{T}}$.

The conditional expectation and conditional variance of $X|Y$ can be derived as

$$\mathbb{E}(X|Y) = \mathbb{E}\left[\ln\left(S_T/S_0\right)\,|\,\ln\left(S_t/S_0\right)\right] = \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \ln\left(S_t/S_0\right),$$

$$\operatorname{var}(X|Y) = \left(1 - \rho_{X,Y}^2\right)\operatorname{var}(X) = \left(1 - \rho_{X,Y}^2\right)\operatorname{var}\left(\ln\left(S_T/S_0\right)\right) = \left(1 - \frac{t}{T}\right)\sigma^2 T = \sigma^2(T-t).$$

Thus $X|Y$ follows normal distribution such that

$$\ln\left(S_T/S_0\right)\,|\,\ln\left(S_t/S_0\right) \sim N\left(\ln\left(S_t/S_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t),\,\sigma^2(T-t)\right).$$

The distribution of S_T conditional on S_t is given by

$$y = F_{S_T|S_t}(S_T) = \Phi\left(\frac{\ln\left(S_T/S_t\right) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

The couple $\left(\ln\left(G_T\right),\ln\left(S_t\right)\right)$ is bivariate normally distributed with mean and variance given by $\mathbb{E}\left[\ln G_T\right] = \ln\left(S_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{T}{2}$, $\operatorname{var}\left[\ln G_T\right] = \frac{\sigma^2 T}{3}$, and $\mathbb{E}\left[\ln S_t\right] = \ln\left(S_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)t$, $\operatorname{var}\left[\ln S_t\right] = \sigma^2 t$.

$$\begin{aligned}
\operatorname{cov}\left(\ln\left(S_t\right), \frac{1}{T}\int_0^T \ln\left(S_s\right)ds\right) &= \frac{\sigma^2}{T}\int_0^T \operatorname{cov}\left(\ln\left(S_t\right), \ln\left(S_s\right)\right)ds \\
&= \frac{\sigma^2}{T}\int_0^T (s \wedge t)\,ds = \frac{\sigma^2}{T}\left[\int_0^t s\,ds + \int_t^T t\,ds\right] \\
&= \frac{\sigma^2}{T}\left[\frac{1}{2}t^2 + t(T-t)\right] = \frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right).
\end{aligned}$$

The correlation coefficient between $\ln(S_t)$ and $\ln(G_T)$

$$\rho\left(\ln\left(S_t\right), \ln\left(G_T\right)\right) = \frac{\operatorname{cov}\left(\ln\left(S_t\right), \ln\left(G_T\right)\right)}{\sqrt{\operatorname{var}\left(\ln\left(S_t\right)\right)}\sqrt{\operatorname{var}\left(\ln\left(G_T\right)\right)}} = \frac{\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right)}{\sigma\sqrt{t}\sigma\sqrt{\frac{T}{3}}} = \sqrt{3}\sqrt{\frac{t}{T}} - \frac{\sqrt{3}}{2}\frac{t\sqrt{t}}{T\sqrt{T}}.$$

The conditional expectation and conditional variance of $\ln(G_T)|\ln(S_t)$

$$\mathbb{E}\left[\ln\left(G_T\right)|\ln\left(S_t\right)\right] = \ln\left(S_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{T}{2} + \frac{\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right)}{\sigma^2 t}\left[\ln\left(S_t\right) - \left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\right)t\right]$$

$$= \ln\left(S_0^{\frac{t}{2T}}S_t^{1-\frac{t}{2T}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2T}.$$

$$\text{var}\left[\ln\left(G_T\right)|\ln\left(S_t\right)\right] = \left(1 - \rho^2\right)\text{var}\left(\ln\left(G_T\right)\right)$$

$$= \left[1 - \left(\sqrt{3}\sqrt{\frac{t}{T}} - \frac{\sqrt{3}}{2}\frac{t\sqrt{t}}{T\sqrt{T}}\right)^2\right]\frac{1}{3}\sigma^2 T = \left(\frac{4T^3 - 12tT^2 - 3t^3 + 12t^2 T}{12T^2}\right)\sigma^2.$$

The conditional distribution of $G_T|S_t$ can be expressed as

$$F_{G_T|S_t}\left(x\right) = \Phi\left(\frac{\ln\left(x\right) - \ln\left(S_0^{\frac{t}{2T}}S_t^{1-\frac{t}{2T}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2T}}{\sigma\sqrt{\frac{4T^3 - 12tT^2 - 3t^3 + 12t^2 T}{12T^2}}}\right).$$

We thus get the form of bivariate derivative $M_T^g(t)$ by

$$M_T^g(t) = F_{G_T|S_t}^{-1}\left(y\right)$$

$$= \exp\left(\ln\left(S_0^{\frac{t}{2T}}S_t^{1-\frac{t}{2T}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2T} + \sigma\sqrt{\frac{4T^3 - 12tT^2 - 3t^3 + 12t^2 T}{12T^2}}\Phi^{-1}\left(y\right)\right)$$

$$= S_0^{\frac{t}{2T}}S_t^{1-\frac{t}{2T}}\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2T} + p\left(\ln\left(S_T/S_t\right) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)\right],$$

where $y = F_{S_T|S_t}\left(S_T\right)$ and $p = \sqrt{\frac{4T^3 - 12tT^2 - 3t^3 + 12t^2 T}{12T^2(T-t)}}$. Thus,

$$M_T^g(t) = S_0^{\frac{t}{2T}}S_t^{1-\frac{t}{2T}}\left(\frac{S_T}{S_t}\right)^p \underbrace{\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\left(\frac{(T-t)^2}{2T} - p(T-t)\right)\right]}_{q} = S_0^{\frac{t}{2T}}S_t^{1-p-\frac{t}{2T}}S_T^p q.$$

Derivation of Equation (1.34)

Letting $\ln M_T^g(t) = \frac{t}{2T}\ln S_0 + \left(1 - p - \frac{t}{2T}\right)\ln S_t + p\ln S_T + \ln q = a\ln S_0 + b\ln S_t + p\ln S_T + c$, we have

$\operatorname{cov}\left(\ln M_T^g(t), \ln G_T\right)$

$$
\begin{aligned}
&= b \times \operatorname{cov}\left(\ln(S_t), \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) + p \times \operatorname{cov}\left(\ln(S_T), \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) \\
&= b\frac{1}{T}\int_0^T \operatorname{cov}\left(\ln S_t, \ln S_s\right)ds + p\frac{\sigma^2 T}{2} \\
&= b\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) + p\frac{\sigma^2 T}{2} = \left(1 - p - \frac{t}{2T}\right)\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) + p\frac{\sigma^2 T}{2}.
\end{aligned}
$$

Let $\sigma_{\ln M_T^g(t)}$ and $\sigma_{\ln G_T}$ be the standard deviations.

$$
\begin{aligned}
\rho\left(\ln\left(M_T^g(t)\right), \ln(G_T)\right) &= \frac{\operatorname{cov}\left(\ln\left(M_T^g(t)\right), \ln(G_T)\right)}{\sqrt{\operatorname{var}\left(\ln\left(M_T^g(t)\right)\right)}\sqrt{\operatorname{var}\left(\ln(G_T)\right)}} = \frac{\left(1 - p - \frac{t}{2T}\right)\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) + p\frac{\sigma^2 T}{2}}{\frac{\sigma^2 T}{3}} \\
&= \left(1 - p - \frac{t}{2T}\right)\frac{3t}{T}\left(1 - \frac{t}{2T}\right) + \frac{3p}{2}.
\end{aligned}
$$

Derivation of Equation (1.39)

Since $t_1 < t_2$ and by the Markov property, $F_{S_T|\left(S_{t_1}, S_{t_2}\right)}(S_T) = F_{S_T|S_{t_2}}(S_T)$. Note that $\ln(S_T/S_0)\,|\,\ln(S_{t_2}/S_0) \sim N\left(\ln(S_T/S_{t_2}) + \left(\mu - \frac{1}{2}\sigma^2\right)(T - t_2), \sigma^2(T - t_2)\right)$, the distribution of S_T conditional on S_{t_2} is given by

$$
z = F_{S_T|S_{t_2}}(S_T) = \Phi\left(\frac{\ln(S_T/S_{t_2}) - \left(\mu - \frac{1}{2}\sigma^2\right)(T - t_2)}{\sigma\sqrt{T - t_2}}\right),
$$

We then construct the random vector $\left(\ln S_{t_1}, \ln S_{t_2}, \ln G_T\right)$ using Lemma 5. The covariance matrix is $\Sigma = \begin{bmatrix} \sigma^2 t_1 & \sigma^2 t_1 & \frac{\sigma^2 t_1}{T}\left(T - \frac{t_1}{2}\right) \\ \sigma^2 t_1 & \sigma^2 t_2 & \frac{\sigma^2 t_2}{T}\left(T - \frac{t_2}{2}\right) \\ \frac{\sigma^2 t_1}{T}\left(T - \frac{t_1}{2}\right) & \frac{\sigma^2 t_2}{T}\left(T - \frac{t_2}{2}\right) & \frac{\sigma^2 T}{3} \end{bmatrix}$.

Then we decompose the covariance matrix and define the sub-matrices

$$\Sigma_{11} = \begin{bmatrix} \sigma^2 t_1 & \sigma^2 t_1 \\ \sigma^2 t_1 & \sigma^2 t_2 \end{bmatrix}, \Sigma_{12} = \begin{bmatrix} \frac{\sigma^2 t_1}{T}\left(T - \frac{t_1}{2}\right) \\ \frac{\sigma^2 t_2}{T}\left(T - \frac{t_2}{2}\right) \end{bmatrix} = \Sigma_{21}^T \text{ and } \Sigma_{22} = \begin{bmatrix} \frac{\sigma^2 T}{3} \end{bmatrix}.$$

Note that $\ln G_T \sim N(\mu_2, \Sigma_{22})$, by Lemma 5 letting $x_1 = \begin{bmatrix} \ln S_{t_1} & \ln S_{t_2} \end{bmatrix}^T$ and $\mu_1 = \begin{bmatrix} \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t_1 & \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t_2 \end{bmatrix}^T$, then after some tedious calculations, we find that $\ln G_T | (\ln S_{t_1}, \ln S_{t_2}) \sim N(\tilde{m}, \tilde{v})$, where $\tilde{m} = \ln\left(S_0^{\frac{t_1}{2T}} S_{t_1}^{\frac{t_2}{2T}} S_{t_2}^{1 - \frac{t_1+t_2}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\frac{(T-t_2)^2}{2T}}\right)$ and $\tilde{v} = \sigma^2 h$ with $h = \frac{T}{3} - t_2 + \frac{t_2^2}{T} - \frac{t_1 t_2(t_2 - t_1) + t_2^3}{4T^2}$,

$$F_{G_T|S_{t_1}, S_{t_2}}(x) = \Phi\left(\frac{\ln(x) - \ln\left(S_0^{\frac{t_1}{2T}} S_{t_1}^{\frac{t_2}{2T}} S_{t_2}^{1 - \frac{t_1+t_2}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\frac{(T-t_2)^2}{2T}}\right)}{\sigma\sqrt{h}}\right),$$

$$\begin{aligned}
V_T(t_1, t_2) &= F_{G_T|S_{t_1}, S_{t_2}}^{-1}(z) \\
&= \exp\left[\ln\left(S_0^{\frac{t_1}{2T}} S_{t_1}^{\frac{t_2}{2T}} S_{t_2}^{1 - \frac{t_1+t_2}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\frac{(T-t_2)^2}{2T}}\right) + \sigma\sqrt{h}\Phi^{-1}(z)\right] \\
&= S_0^{\frac{t_1}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\left[\frac{(T-t_2)^2}{2T} - \sqrt{(T-t_2)h}\right]} S_{t_1}^{\frac{t_2}{2T}} S_{t_2}^{1 - \frac{t_1+t_2}{2T} - \sqrt{\frac{h}{T-t_2}}} S_T^{\sqrt{\frac{h}{T-t_2}}} = a S_{t_1}^b S_{t_2}^c S_T^d,
\end{aligned}$$

where $z = F_{S_T|S_{t_2}}(S_T)$ and $a = s_0^{\frac{t_1}{2T}} e^{\left(\mu - \frac{\sigma^2}{2}\right)\left[\frac{(T-t_2)^2}{2T} - \sqrt{(T-t_2)h}\right]}$, $b = \frac{t_2}{2T}$, $c = 1 - \frac{t_1+t_2}{2T} - \sqrt{\frac{h}{T-t_2}}$ and $d = \sqrt{\frac{h}{T-t_2}}$.

Derivation of Equation (1.43)

For $\ln V_T(t_1, t_2) = \ln a + b\ln S_{t_1} + c\ln S_{t_2} + d\ln S_T$,

$$\begin{aligned}
\text{cov}(\ln V_T(t_1, t_2), \ln G_T) &= b \times \text{cov}\left(\ln S_{t_1}, \frac{1}{T}\int_0^T \ln(S_s)ds\right) + c \times \text{cov}\left(\ln S_{t_2}, \frac{1}{T}\int_0^T \ln(S_s)ds\right) \\
&\quad + d \times \text{cov}\left(\ln S_T, \frac{1}{T}\int_0^T \ln(S_s)ds\right) \\
&= b\frac{\sigma^2 t_1}{T}\left(T - \frac{t_1}{2}\right) + c\frac{\sigma^2 t_2}{T}\left(T - \frac{t_2}{2}\right) + d\frac{\sigma^2 T}{2}
\end{aligned}$$

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$$
\begin{aligned}
\rho\left(\ln V_T(t_1, t_2), \ln G_T\right) &= \frac{\operatorname{cov}\left(\ln V_T(t_1, t_2), \ln G_T\right)}{\sqrt{\operatorname{var}\left(\ln V_T(t_1, t_2)\right)}\sqrt{\operatorname{var}\left(\ln G_T\right)}} \\
&= \frac{3t_1 t_2}{2T^3}\left(T - \frac{t_1}{2}\right) + \frac{3t_2}{T^2}\left(T - \frac{t_2}{2}\right)\left(1 - \frac{t_1 + t_2}{2T} - \sqrt{\frac{h}{T - t_2}}\right) \\
&\quad + \frac{3}{2}\sqrt{\frac{h}{T - t_2}},
\end{aligned}
$$

where $0 < t_1 < t_2 < T$.

Chapter 2

Variable Annuity with State-dependent Fee Linked to Market Volatility

2.1 Introduction

A variable annuity (VA) is a tax-deferred and unit-linked insurance product that provides various forms of guarantee riders to investors via equity participation in a collective investment selected by the policyholder. With a variable annuity, investors make payments until their retirement and then begin receiving regular retirement income from the insurance company. VA guarantees can be classified into two broad types: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs). The GMLBs include guaranteed minimum accumulation benefits (GMABs), guaranteed minimum income benefits (GMIBs) and guaranteed minimum withdrawal benefits (GMWBs).

In this research, we will work on pricing and hedging issues of variable annuities with a state-dependent fee structure. Such state-dependent fees can depend on several uncertainty sources, such as fund value, index value, and market volatility. For example, when the market is volatile, the hedging program of guarantees in VAs is typically very expensive. Traditional VAs charge a constant fee rate that does not depend on market conditions. Our motivation comes from the variable annuity contracts offered by American General Life in 2014 (see the prospectus dated May 1, 2014 of the "Polaris Variable Annuities Choice IV"). In this contract, the fee rate adjustment is tied to the change in the market Volatility Index (VIX) reported by the Chicago Board Options Exchange. We propose to model and

study the benefits of charging state-dependent fees in VAs that depend on the volatility level. We expect that the fees will be lower on average for the policyholder and that they will provide better matching for the insurer between the actual value of the guarantee and the premium collected by the insurer. We thus expect to improve the hedging program of VAs using state-dependent fees. Such an approach challenges the existing VA modeling: see Coleman, Li, and Patron (2006), Milevsky and Salisbury (2006), Bauer, Kling, and Russ (2008), Chen, Vetzal, and Forsyth (2008), Dai, Kuen Kwok, and Zong (2008), Lin et al. (2009), Hürlimann (2010), and references therein. Almost all the research to date has focused on a fixed fee rate, i.e. the policyholder has to pay a constant percentage of the fund. The pricing and hedging of VAs with state-dependent fees linked to volatility, however, are realistic and attractive in the insurance industry.

This chapter aims to solve some practical problems arising from the design of variable annuities with a state-dependent fee. In particular, we investigate the following questions:

(1) How to design a model that takes into account the random evolution of the VA fund value, the volatility of the fund and the state-dependent fee in response to market conditions such as the volatility level?

(2) How to find fair values of derivative claims embedded in VAs with a state-dependent fee structure, and correspondingly how to determine the state-dependent fee rate?

(3) In the incomplete market, the long-term liabilities such as most VAs are typically difficult to hedge. What will be a benchmark hedging strategy compared to other strategies when aligned with the state-dependent fees?

To tackle the first issue, a three-factor affine model can be generally used to model the dynamics of the price of the VA fund, the variance of the fund price and the state-dependent fee rate. When the fee rate structure has particular properties, such as linearity with respect to the variance process, we can lower the dimension in the affine framework and thus reduce the three-factor model to the simpler two-factor one. To solve the second problem, we develop approaches based on characteristic function. We use the Black-Scholes Fourier method to price contingent claims on these variables. To determine the state-dependent fee, it is clear that a fair fee rate for the policyholder will match the expected liability associated with the guarantee of the VA fund at issue. Depending on the assumed fee rate structure, we may consider particular constraints, such as minimizing the mean squared error between the cost of protection and the expected fee collected during the life of the contract. Alternatively, we can use constraints based on risk measures that reflect certain fee arrangements between the insurer and the policyholder. For the third problem, we develop an efficient static hedging strategy in the spirit of Kolkiewicz (2016) for VA liabilities that can be considered as a benchmark to compare other strategies, such

as traditional dynamic hedging and short-dated static hedging. This hedging option can also be used for hedging long-term and path-dependent liabilities such as most VAs that require working with more general forms of dependency among risk factors.

The continuous-time affine models have played a prominent role in both the term structure literature and the stochastic volatility literature due to their analytic tractability. The early literature focused on specific models; for example, Vasicek (1977) and Cox, Ingersoll, and Ross (1985) deal with single-factor term structure models, Chen (1996), Balduzzi, Das, Foresi, and Sundaram (1996) deal with multiple-factor term structure models, and Hull and White (1987) and Heston (1993) with asset price models with stochastic volatility. A more recent strand of the literature focuses on broader classes of models rather than specific cases. For the case of affine term structure models, we refer to Duffie and Kan (1996) and Duffie, Filipović, and Schachermayer (2003), for systematic treatments, Dai and Singleton (2000), for an empirical investigation and classification scheme, Duffee (2002) and Cheridito, Filipović, and Kimmel (2007), for extended market price of risk specifications, Collin-Dufresne, Goldstein, and Jones (2008) and Joslin (2006), for alternate classification schemes, and Duffie, Pan, and Singleton (2000), for additional applications of affine processes. Furthermore, the square-root diffusion, which is a particular example of affine models, became the central component of many important financial models, including the CIR interest rate model (Cox, Ingersoll, and Ross (1985)) and the above mentioned models. The attractiveness of the square-root diffusion is motivated by several essential properties, including positivity, mean-reversion, and closed-form solution for the transition density function. In particular, the European calls and puts under affine models have the closed-form solution that makes the calibration to market prices quick and efficient. Combined with the ability to capture volatility smiles and skews, all these make affine models a viable tool in many pricing applications, including modeling of equity, foreign exchange and credit products.

It is often possible to find the characteristic function of the joint distribution of the involved factors under affine models. Therefore, we can price options using some of the existing techniques based on characteristic functions. A simplest, but not the most efficient approach, is to take an integral of the payoff function over the probability distribution obtained by inverting the corresponding Fourier transform. There is a growing interest in applying such methods to pricing, which is due to the fact that more realistic models are often more conveniently represented through characteristic functions than through their probability distributions. Inside the field of Finance, Stein and Stein (1991) first used the Fourier inversion method to find the distribution of the underlying asset with the stochastic volatility model. Heston (1993) demonstrated the importance of the characteristic function in finding a closed-form solution for options with stochastic volatility. Bakshi, Kapadia, and

Madan (2003) provided an economic foundation for characteristic functions, and advanced the approach of Heston (1993) and Stein and Stein (1991) in many significant ways. In particular, the authors developed valuation formulas for a wide variety of contingent claims. Duffie, Pan, and Singleton (2000) extended the Fourier methods to a broader range of stochastic processes, the class of exponential affine jump diffusions. A numerically very efficient methodology is introduced in Carr and Madan (1999), who employed the use of fast Fourier transform (FFT) algorithms by mapping the Fourier transform directly to call option prices with arbitrary strikes via the characteristic function at one time. Lee (2004) generalized the use of FFT to price with multidimensional state variables in the framework of Duffie, Pan, and Singleton (2000) and Bakshi, Kapadia, and Madan (2003). In Carr and Wu (2004), the authors extended the Carr and Madan (1999) methodology to general claims and applied it to time changed Lévy processes, the class of generalized affine models (Filipović (2001)) and quadratic activity rate models (Leippold and Wu (2002)).

There is a quite limited number of studies on pricing variable annuity funds based on affine models. Benhamou and Gauthier (2009) priced variable annuity contracts with stochastic interest rate and volatility models. Guan and Liang (2014) used a three-factor affine model to search optimal management of DC pension plans. In both papers, they specified the models with either CIR or Ornstein-Uhlenbeck process for dynamics that will produce closed formulas for pricing with characteristic functions by time-invariant Riccati equations. Cui, Feng, and MacKay (2017) price a guaranteed minimum maturity benefit with VIX-linked fees in a Heston-type stochastic volatility setting. Their numerical examples show that the VIX-linked fee reduces the sensitivity of the insurer's liability to market volatility when compared to a VA with the traditional fixed fee rate. Kouritzin and MacKay (2018) further assess the effectiveness of the VIX-linked fee structure in decreasing the sensitivity of the insurer's liability to volatility risk for a GMWB contract. Despite the similar framework of modeling VA contracts with the VIX-linked fee structure, our contributions are two-fold: (1) We address the fee-rate determination of the VIX-linked fee structure in the context of minimizing the risk related to writing a GMAB contract with state-dependent fees. Specifically, we want to minimize the expected shortfall between the GMAB liability and the fees to be collected. (2) We formulate the optimal hedging strategy for a GMAB with state-dependent fees under the Heston-type stochastic volatility model. The proposed hedging strategy is further discussed in Chapter 4.

Market consistent valuation. VAs issuers face new challenges as Solvency II regulations request a market consistent valuation of liabilities. Reserve requirements for VA products are then highly "scenario dependent" and are not known at time of issue. Technical provisions and solvency capital requirements change over time depending on market

conditions (e.g., interest rates, volatility, equity prices). A widely used interpretation of the market-consistent approach is that the values of liabilities should be equal to the amount that the insurer would pay to transfer its remaining contractual obligations immediately to another entity. For certain types of insurance liabilities, this approach is relatively easy to adopt. In many other cases, however, the correct approach is less clear. In particular, insurance liabilities containing options and guarantees are typically valued using stochastic simulation precisely because a market price for the liability cannot be observed in the market. In such cases, a suitably calibrated economic scenario generator (ESG), which can reproduce the market prices of instruments that reflect the nature and term of the liabilities being valued, is likely to be used.

The issuer's income on most VAs is typically computed as a fixed percentage fee of the fund under management and fluctuates as a function of the underlying value. However, VAs are long term contracts and market conditions will change throughout the term of the contract. For example, the market value of guarantees goes up when equity goes down and volatility goes up. Moreover, when equity goes down, the volatility typically increases. Our analysis highlights the need to use models with sufficient degrees of freedom to capture all relevant market data in the valuation of VA products. To illustrate this point, we then look at a more realistic extension of the simple Black-Scholes model, such as the Heston volatility model. The Heston volatility model incorporates both stochastic processes for equity values and equity volatility. In the context of market consistency, the pricing formula can be a suitable ESG by associating the calibration of the market prices of guarantees with the state-dependent fees tied to the Heston market volatility.

2.2   Polaris Choice IV Contract

Our proposed design of state-dependent fees is inspired by the design of a recent VA contract called Polaris Choice IV issued in the U.S. insurance market.

The Polaris Choice IV is a variable annuity launched by American General Life in May 2014. In the Accumulation Phase, it builds assets on a tax-deferred basis. In the Income Phase, it provides the policyholder with guaranteed income through annuity income payments. The Polaris Choice IV allows the policyholder to invest in Variable Portfolios which, like mutual funds, have different investment objectives and performance. Policyholders can gain or lose money if they invest in these Variable Portfolios. The amount of money that accumulates in the contract then depends on the performance of the selected Variable Portfolios.

To grow and secure income, including rising lifetime income, the policyholder may elect one of the optional Living Benefits, all of which are guaranteed minimum withdrawal benefits, for an additional fee. These Living Benefits may offer protection in the event that the contract value declines due to unfavorable investment performance, certain withdrawal activity or changing longevity risk. The two typical living benefits from the Polaris Choice IV are called Polaris Income Plus and Polaris Income Builder. They impose an investment requirement on the policyholder so that he needs to invest in high-risk variable portfolios that are predetermined in the contract. In these two benefits, the policyholder may add a 6% Income Credit to the Income Base, the amount on which guaranteed withdrawals are based, each year for the first 12 Benefit Years. In the Polaris Income Plus, the 6% Income Credit is reduced but not eliminated in any Benefit Year in which cumulative withdrawals are less than 6% of the Income Base and not greater than the Maximum Annual Withdrawal Amount applicable to the selected income option, thereby providing a guarantee that income can increase during the first 12 years even after starting withdrawals. In the Polaris Income Builder, the 6% Income Credit is only available in years when no withdrawals are taken. If the investors do not withdraw during the first 12 years, the investors will be eligible for the Minimum Income Base that is equal to 200% of the first Benefit Year's Eligible Purchase Payments. Unlike many other single-premium contracts in the market, Polaris Choice IV allows subsequent purchase payments of as little as $100 for the policyholder aged 86 or younger. Such a flexible premium allows the investor to make additional payments to the annuity after it is issued, but also leads to an increase of the annual fee, which is calculated as a percentage of the Income Base.

The living benefits of Polaris Income Plus and Polaris Income Builder aim to increase the Income Base for a larger withdrawal amount but also lead to increase an annual fee which is calculated as a percentage of the Income Base. The fee for Polaris Income Plus and Polaris Income Builder is assessed against the Income Base and deducted from the contract value at the end of each benefit quarter. The Initial Annual Fee Rate is guaranteed not to change for the first benefit year. Subsequently, the fee rate may change quarterly subject to the parameters identified in the contract. Any fee adjustment is based on a non-discretionary formula tied to the change in the Volatility Index (VIX), an index of market volatility reported by the Chicago Board Options Exchange. In general, as the average value of the VIX decreases or increases, the fee rate will decrease or increase accordingly, subject to the maximums and the minimums described in the contract. The non-discretionary formula used in the calculation of the Annual Fee Rate applicable after the first Benefit Year is of the form

$$\text{Initial Annual Fee Rate} + [0.05\% \times (\text{Average Value of the VIX} - 20)], \qquad (2.1)$$

where the initial annual fee rate is set to 1.1% for a policyholder aged from 65 to 85 at the inception of the contract.

In the following, we simplify the above fee-rate formula by assuming a linearity between the fee rate and the volatility factor to analyze the state-dependent fees in the stochastic affine model.

2.3   Affine Model

Given a VA fund that is fully invested in a market equity S, we need a model for the dynamics of the market index. In this chapter, we formulate the VA pricing problem under the framework of affine models. We use this set-up to analyze VA products with fees that depend on the market volatility. The advantage of affine models is that they lead to closed-form representations of characteristic functions. We give a definition of affine models in Section 2.3.1. In Section 2.3.2, we start with the classic stochastic volatility model of Heston (1993).

2.3.1   Definition

In this section we introduce affine models, which rank now among the most popular models in theory and practice. For instance, the very first term structure model, the Vasicek model, is an affine model. Other popular models such as Cox-Ingersoll-Ross (CIR), Hull-White (HW) or Longstaff and Schwartz are also of this type. Their popularity is due to the fact that they often produce explicit formulas for option prices. An affine model is defined as follows:

Definition 2.3.1. Suppose that a state vector Y_t follows

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW(t). \tag{2.2}$$

The system (2.2) is said to be of affine form if

$$\mu(Y_t) = a_0 + a_1 Y_t, \qquad \text{for} \quad (a_0, a_1) \in \mathcal{R}^n \times \mathcal{R}^{n \times n}, \tag{2.3}$$

$$\left(\sigma(Y_t)\sigma(Y_t)^T \right)_{ij} = (c_0)_{ij} + (c_1)_{ij}^T Y_t, \qquad \text{with} \quad (c_0, c_1) \in \mathcal{R}^{n \times n} \times \mathcal{R}^{n \times n \times n}, \tag{2.4}$$

for $i, j = 1, \ldots, n$.

In the class of affine diffusion processes, a closed-form solution of the characteristic function exists. For any $u \in \mathcal{R}^n$, it is known that the discounted joint characteristic function of Y is given by

$$\phi(u, Y_t, t, T) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-r\tau + iu^T Y_T\right) \big| \mathcal{F}_t \right] = e^{A(u,\tau) + B^T(u,\tau)Y_t}, \qquad (2.5)$$

where $\tau := T - t$ and the interest rate r is constant. The coefficients $A(u, \tau)$ and $B^T(u, \tau)$ have to satisfy the following complex-valued Riccati equation

$$\begin{cases} \frac{d}{d\tau} B(u, \tau) = a_1^T B(u, \tau) + \frac{1}{2} B(u, \tau)^T c_1 B(u, \tau), \\ \frac{d}{d\tau} A(u, \tau) = -r + B^T(u, \tau) a_0 + \frac{1}{2} B^T(u, \tau) c_0 B(u, \tau), \end{cases}$$

subject to the terminal condition $A(u, 0) = 0$ and $B(u, 0) = iu$. In general the solutions to the above system of ordinary differential equations have to be computed numerically. However, there are some models for which A and B can be represented in analytical forms.

In the context of affine models with stochastic interest rate, stochastic volatility or hybrid models with stochastic interest rate and volatility, expression (2.5) provides an explicit form for the characteristic function of the logarithm of the asset price. Closed-form prices of European options can be obtained by inverting this characteristic function (see Duffie, Pan, and Singleton (2000)). Below we derive characteristic functions for models that we use to analyze VAs with a state-dependent fee structure.

2.3.2   The Affine Heston Model

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t, t \geq 0\}$ for the stochastic VA fund jointly with its stochastic volatility, where $\mathbb{P}$ is the physical probability measure. In the Heston framework of Christoffersen, Heston, and Jacobs (2013), the VA fund value F_t has the following dynamics

$$\begin{cases} \dfrac{dF_t}{F_t} = \mu dt + \sqrt{v_t} d\tilde{W}_x(t), & F_0 > 0, \\ dv_t = \kappa^* \left(\bar{v}^* - v_t\right) dt + \gamma \sqrt{v_t} d\tilde{W}_v(t), & v_0 > 0, \end{cases} \qquad (2.6)$$

where μ is the physical return. $\tilde{W}_x(t)$ and $\tilde{W}_v(t)$ are two correlated standard Brownian motions under $\mathbb{P}$ with correlation ρ. $\kappa^* > 0$, determines the speed of adjustment of the volatility towards its long-run mean $\bar{v}^* > 0$, and $\gamma > 0$ is the volatility of the volatility. The variance process v_t is strictly positive when the Feller condition, $2\kappa^* \bar{v}^* > \gamma^2$, is satisfied.

In the Heston model, the risk-neutral measure used for pricing purposes is obtained by specifying a so-called "market price of volatility risk" $\Lambda(F, v, t); = \lambda\sqrt{v}$ for some constant λ, which is assumed to be proportional to the volatility. Then under the resulting risk-neutral measure $\mathbb{Q}$, the fund value follows the dynamics

$$
\begin{cases}
\dfrac{dF_t}{F_t} = rdt + \sqrt{v_t}dW_x(t), & F_0 > 0, \\
dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_v(t), & v_0 > 0,
\end{cases}
\tag{2.7}
$$

where $r > 0$ is a constant risk-free rate, $\kappa = \kappa^* + \lambda$, $\bar{v} = \kappa^*\bar{v}^*/(\kappa^* + \lambda)$, and λ is the volatility risk premium. $W_x(t)$ and $W_v(t)$ are $\mathbb{Q}$-Brownian motions with $dW_x(t)dW_v(t) = \rho_{x,v}dt$ and $|\rho_{x,v}| < 1$.

The above model is not in the class of affine processes, whereas under the log-transform for the underlying asset, $x_t = \log F_t$, the process becomes

$$
\begin{cases}
dx_t = \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_x(t) \\
dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_v(t), & v_0 > 0,
\end{cases}
\tag{2.8}
$$

and hence it is affine.

For the option payoff $U(x, v, t)$ on F, the Heston PDE in terms of the log price x_t is given by

$$
\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + \rho\gamma v\frac{\partial^2 U}{\partial v\partial x} + \frac{1}{2}\gamma^2 v\frac{\partial^2 U}{\partial v^2} - rU + [\kappa(\bar{v} - v) - \lambda v]\frac{\partial U}{\partial v} = 0. \tag{2.9}
$$

Duffie, Pan, and Singleton (2000) characterized the discounted characteristic function of the log-price for underlying asset in the Heston model. It has the form of

$$
\phi_H(u, x_t, v_t, \tau) := \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-r\tau + iux_T\right)|\mathcal{F}_t\right) = \exp\left(A(u, \tau) + B(u, \tau)x_t + C(u, \tau)v_t\right), \tag{2.10}
$$

where the functions $A(u, \tau)$, $B(u, \tau)$ and $C(u, \tau)$ are known in closed form. Because of the form of the characteristic function, we cannot get its inverse analytically, and a numerical method for integration, such as Fourier methods, has to be used.

According to Heston (1993), an analytical representation of a European call option on the fund F with no dividend is given by

$$
C(F_t, v_t, t, T) = F_t P_1 - Ke^{-r(T-t)}P_2, \tag{2.11}
$$

where

$$P_j(x_t, v_t, \tau, K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{iu \ln(K) \phi_j(u, x_t, v_t, \tau)}}{iu} \right) du,$$

$$\phi_j(u, x_t, v_t, \tau) = \exp \left[E(u, \tau) + F(u, \tau) v_t + iu x_t \right],$$

$$E(u, \tau) = r u i \tau + \frac{a}{\sigma^2} \left[(b_j - \rho \sigma u i + d) \tau - 2 \ln \left(\frac{1 - ge^{d\tau}}{1 - g} \right) \right],$$

$$F(u, \tau) = \frac{b_j - \rho \sigma u i + d}{\sigma^2} \left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}} \right],$$

for $j = 1, 2$, $a = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\sigma$, $b_2 = \kappa + \lambda$, $g = \frac{b_j - \rho\sigma ui + d}{b_j - \rho\sigma ui - d}$, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$ and $d_j = \sqrt{(\rho\sigma ui - b_j)^2 - \sigma^2(2u_j ui - u^2)}$.

2.4 Pricing a GMAB with State-dependent Fee

In this section we discuss the problem of pricing a guaranteed minimum accumulation benefit (GMAB) with state-dependent fees. In Section 2.4.1, we formulate the pricing problem with state-dependent fees and find the corresponding characteristic function of the logarithm of the fund value under the Heston-type stochastic volatility model. In Section 2.4.2, the Black-Scholes Fourier method is introduced for the option pricing using the characteristic function. In Section 2.4.3, we price the GMAB with a state-dependent fee structure using the characteristic function derived in Section 2.4.1.

2.4.1 Heston Model with State-dependent Fee

Motivated by Polaris Choice IV, we tie the fee rate with the market condition. We expect that inclusion of a state-dependent fee will lead to a more adequate coverage of the true hedging cost of VA guarantees. In general, a state-dependent fee $c_t(F_t, v_t, t)$ may depend on the time from inception, the value of the fund F_t and value of "variance" or squared volatility v_t. We consider a simple case where the fee rate is a linear function of the current market variance, i.e., $c_t = a + bv_t$, a and $b > 0$ are constant and v_t is the squared volatility. We define $F_t(a, b) := F_t$ assuming that a fee is continuously deducted from the fund value at the rate $c_t(v_t) := c_t = a + bv_t$. With a constant risk-free rate r, the drift term of the VA fund, $F_t(a, b)$, is given by

$$\mu_t = r - c_t = r - a - bv_t.$$

Then, assuming a Heston model for volatility and using (2.8), we arrive at the following description of the dynamic of the log price of the fund value

$$
\begin{cases}
dx_t = \left(r - c_t(v_t) - \dfrac{1}{2}v_t\right)dt + \sqrt{v_t}dW_x(t) = (r^* - b^*v_t)\,dt + \sqrt{v_t}dW_x(t), \\
\qquad\qquad dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_v(t), \qquad v_0 > 0,
\end{cases}
\tag{2.12}
$$

where $r^* = r - a$, $b^* = b + \frac{1}{2}$ and the correlation is given by $dW_x(t)dW_v(t) = \rho_{x,v}dt$. Due to the linear fee structure with respect to the variance, we have a model with an affine structure.

The corresponding symmetric instantaneous covariance matrix for the stochastic process in (2.12) is given by

$$
\Sigma_t := \begin{bmatrix} v_t & \rho_{x,v}\gamma v_t \\ \rho_{x,v}\gamma v_t & \gamma^2 v_t \end{bmatrix}.
\tag{2.13}
$$

Checking (2.2), (2.3) and (2.4), we can verify that the process in (2.12) is affine and thus the characteristic function of x_T conditional on x_t and v_t has the following form

$$
\phi_{\text{VA}}(u, x_t, v_t, \tau) = \exp\left(A(u,\tau) + B(u,\tau)x_t + C(u,\tau)v_t\right).
\tag{2.14}
$$

The functions of $A(u,\tau) =: A(\tau)$, $B(u,\tau) =: B(\tau)$ and $C(u,\tau) =: C(\tau)$ for $u \in \mathbb{R}$ and $\tau \geq 0$ in (2.14) for the Heston model with state-dependent fee must satisfy the following system of ODEs (a similar result can be found in Grzelak and Oosterlee (2011)):

$$
\begin{aligned}
B'(\tau) &= 0, & B(u,0) &= iu, \\
C'(\tau) = \frac{1}{2}B(B - 2b^*) + (\rho_{x,v}\gamma B - \kappa)\,C + \frac{1}{2}\gamma^2 C^2, & C(u,0) &= 0, \\
A'(\tau) = r^*B - r + \kappa\bar{v}C, & A(u,0) &= 0,
\end{aligned}
\tag{2.15}
$$

where κ, $\bar{v}$, γ and $\rho_{x,v}$ are defined in (2.12).

The following proposition provides a closed-form representation of $\phi_{\text{VA}}(u, x_t, v_t, \tau)$ in (2.14).

Proposition 4. The solution of the ODE system in (2.15) is given by

$$
B(u,\tau) = iu,
$$

$$
C(u,\tau) = \frac{-a_1 - C_1}{2a_2\left(1 - Ge^{-C_1\tau}\right)}\left(1 - e^{-C_1\tau}\right),
$$

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$$A(u, \tau) \;=\; \int_0^\tau (r^* iu - r) ds + \kappa \bar{v} \int_0^\tau C(u, s) ds = (r^* iu - r)\tau + \kappa \bar{v} I_C(\tau),$$

where the parameters $a_0 = \frac{1}{2}(iu)(iu - 2b^*)$, $a_1 = \rho_{x,v}\gamma(iu) - \kappa$, $a_2 = \frac{1}{2}\gamma^2$, $C_1 = \sqrt{a_1^2 - 4a_0 a_2}$, $G = \frac{-a_1 - C_1}{-a_1 + C_1}$. and $I_C(\tau) = \frac{1}{2a_2}\left[(-a_1 - C_1)\tau - 2\ln\left(\frac{1 - Ge^{-C_1 \tau}}{1 - G}\right)\right]$. Thus, the characteristic function for x_T admits the following closed form

$$\phi_{\mathrm{VA}}(u, x_t, v_t, \tau) = \exp\left(A(u, \tau) + B(u, \tau)x_t + C(u, \tau)v_t\right),$$

Proof. Due to the terminal condition $B(u, 0) = iu$ and $\frac{\partial B(u,\tau)}{\partial \tau} = 0$, we have $B(u, \tau) = iu$. For the second ODE, we obtain the following Riccati differential equation with constant complex-valued coefficients in the form of

$$\frac{\partial C(u, \tau)}{\partial \tau} = a_0 + a_1 C(u, \tau) + a_2 C^2(u, \tau), \qquad C(u, 0) = 0. \tag{2.16}$$

The above Riccati equation can be solved by using a computation software such as Mathematica, and we thus have the explicit formula for $C(u, \tau)$ in Proposition 4 with the terminal condition $C(u, 0) = 0$.

The solution to the third ODE of $A(u, \tau)$ follows directly from the solutions of the first and the second ODEs. $\qquad\square$

2.4.2 Pricing of Options using Fourier Analysis

This section employs the Black-Scholes Fourier method for the valuation for a GMAB on the fund. We first begin with some basic definitions.

Definition 2.4.1. The Fourier Transform $\mathcal{F}\{\cdot\}$ and inverse Fourier transformation $\mathcal{F}^{-1}\{\cdot\}$ of an integrable function, $q(x)$, are

$$\mathcal{F}\{q(x)\} = \int_{-\infty}^{\infty} e^{iux} q(x) dx = \phi(u), \tag{2.17}$$

$$\mathcal{F}^{-1}\{\phi(u)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \phi(u) du = q(x). \tag{2.18}$$

In our framework, we define $q_T(x)$ as the risk neutral density function of the logarithm of the underlying fund, $x_T = \ln F_T$, for its corresponding characteristic function $\phi_{\mathrm{VA}}(u) :=$

$\phi_{\text{VA}}(u, x_t, v_t, \tau)$. We next consider pricing a GMAB using the Black-Scholes Fourier method such as Ding (2012). Under appropriately modified equivalent probability measures and letting $x_T = \ln F_T(a, b)$, the price of a European call with spot fund value F_t and log-strike price $g = \ln G_T$ is given by

$$
\begin{aligned}
C_t(F_T, G_T, T) &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(F_T - G_T)^+ \right] \\
&= e^{-r(T-t)} \int_0^\infty (e^x - G_T)^+ q_T(x) dx \\
&= e^{-r(T-t)} \int_g^\infty e^x q_T(x) dx - e^{-r(T-t)} G_T \int_g^\infty q_T(x) dx \\
&= F_t \mathcal{P}_1 - e^{-r(T-t)} G_T \mathcal{P}_2,
\end{aligned}
\tag{2.19}
$$

where the integral $e^{-r(T-t)} \int_g^\infty e^x q(x) dx$ can be written as $F_t \mathcal{P}_1$ under the risk-neutral probability $\mathbb{Q}$. $\mathcal{P}_2 = \int_g^\infty q_T(x) dx$ is the probability $\mathbb{Q}(x_T \geq g)$. The characteristic function for $x_T = \ln F_T$ is $\phi_{\text{VA}}(u) = \int_{-\infty}^\infty e^{iux} q_T(x) dx$, and hence

$$
\begin{aligned}
\mathcal{P}_2 &= \int_g^\infty \left(\frac{1}{2\pi} \int_{-\infty}^\infty e^{-iux} \phi_{\text{VA}}(u) du \right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{\text{VA}}(u) \left(\int_g^\infty e^{-iux} dx \right) du \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-iug} \phi_{\text{VA}}(u)}{iu} \right] du.
\end{aligned}
$$

Since the fund value serves as numéraire in $\mathcal{P}_1$, we introduce a change of measure from $\mathbb{Q}$ to an equivalent measure $\tilde{\mathbb{Q}}$ by a Radon-Nikodym derivative $z = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{e^{x_T}}{\mathbb{E}^{\mathbb{Q}}[e^{x_T}]}$. It is easy to check that $\mathbb{E}^{\mathbb{Q}}[z] = 1$ and $z > 0$. With this new measure $\tilde{\mathbb{Q}}$, the Fourier transform of $\mathcal{P}_1$ is defined as

$$
\tilde{\phi}_{\text{VA}}(u) = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[e^{iux_T} \right] = \int_{-\infty}^\infty e^{iux} d\tilde{\mathbb{Q}}(x) = \int_{-\infty}^\infty e^{iux} \frac{e^x}{\mathbb{E}^{\mathbb{Q}}[e^{x_T}]} d\mathbb{Q}(x) = \frac{\mathbb{E}^{\mathbb{Q}}[e^{x_T} e^{iux_T}]}{\mathbb{E}^{\mathbb{Q}}[e^{x_T}]} = \frac{\phi_{\text{VA}}(u-i)}{\phi_{\text{VA}}(-i)}.
$$

Due to the no arbitrage condition $\mathbb{E}^{\mathbb{Q}}[F_T] = F_t e^{r(T-t)}$, we get $\phi_{\text{VA}}(-i)$ as its characteristic function and for $\mathbb{E}^{\mathbb{Q}}[e^{x_T} e^{iux_T}]$ we get $\phi_{\text{VA}}(u-i)$. We treat $\mathbb{E}^{\tilde{\mathbb{Q}}}[e^{iux_T}]$ as a characteristic

function and invert it accordingly to evaluate the probability

$$\mathcal{P}_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-iuk}\phi_{\text{VA}}(u-i)}{iu\phi_{\text{VA}}(-i)} \right] du.$$

Note that the integrals in $\mathcal{P}_1$ and $\mathcal{P}_2$ can be approximated by Matlab functions using recursive adaptive Lobatto quadrature to within an error of 10^{-6} (see for example Gander and Gautschi (2000)).

2.4.3 Option Pricing in GMAB with State-dependent Fee

In the model (2.12), the dynamic of the fund value F_t in a guaranteed minimum accumulation benefit (GMAB) depends on the market volatility and the state-dependent fee rate. Given a maturity guarantee G_T, the payoff of the GMAB at maturity T is in the form of $\max(F_T, G_T)$. The price of the GMAB can be represented either as a sum of a European call and a discount guaranteed amount or a sum of European put and the initial fund value. These decompositions are given respectively by

$$P_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\max(F_T, G_T)] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(F_T - G_T, 0)] + e^{-rT}G_T, \tag{2.20}$$

and

$$P_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(G_T - F_T, 0)] + e^{-rT}\mathbb{E}^{\mathbb{Q}}[F_T]. \tag{2.21}$$

In Proposition 5 below, we present the Black-Scholes Fourier formula based on the decompositions in (2.20).

Proposition 5. The Black-Scholes Fourier price of a GMAB with the payoff $\max(F_T, G_T)$ at maturity T and the state-dependent fee structure in (2.12) has the form of

$$P_0 = F_0\mathcal{P}_1 - e^{-rT}G_T\mathcal{P}_2 + e^{-rT}G_T, \tag{2.22}$$

where

$$\mathcal{P}_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-iuk}\phi_{\text{VA}}(u-i)}{iu\phi_{\text{VA}}(-i)} \right] du,$$

$$\mathcal{P}_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-iuk}\phi_{\text{VA}}(u)}{iu} \right] du,$$

and ϕ_{VA} is the characteristic function given in (2.14).

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2.4.4 Numerical Results

In previous sections, we have developed a pricing method for the GMAB with a state-dependent fee structure of $c_t = a + bv_t$ using the Heston model. In the limit when $a \to 0$ and $b \to 0$, the call price in (2.22) with a state-dependent fee then degenerate to the pricing with no state-dependent fee in (2.11). By implementing the method we find that the call price in (2.22) is equal to 32.3 by letting $a = b = 0$, which is the same as the Heston call price in (2.11) with the parameters $\rho_{xv} = -30\%$, $\kappa = 0.3$, $\gamma = 0.5$, $\bar{v} = v_0 = 0.05$, $r = 0.5\%$, and $F_0 = G_T = 100$. Figure 2.1 and Figure 2.2 illustrate the patterns of the VA call prices and the fund values for different strikes G_T and maturities T by specifying a fee rate in the form of $c_t = 0.01 + 0.25v_t$ respectively.
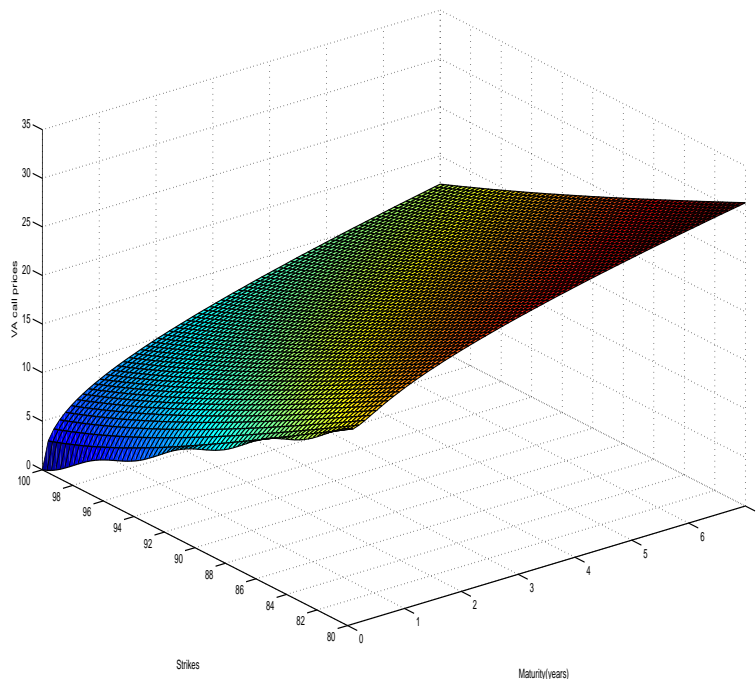
Figure 2.1: VA call prices (in-the-money) obtained by the Black-Scholes Fourier method in the Heston model with the state-dependent fee structure with the parameters $\rho_{xv} = -30\%$, $\kappa = 0.3$, $\gamma = 0.5$, $\bar{v} = v_0 = 0.05$, $r = 0.5\%$, $F_0 = 100$, $a = 0.01$ and $b = 0.25$.
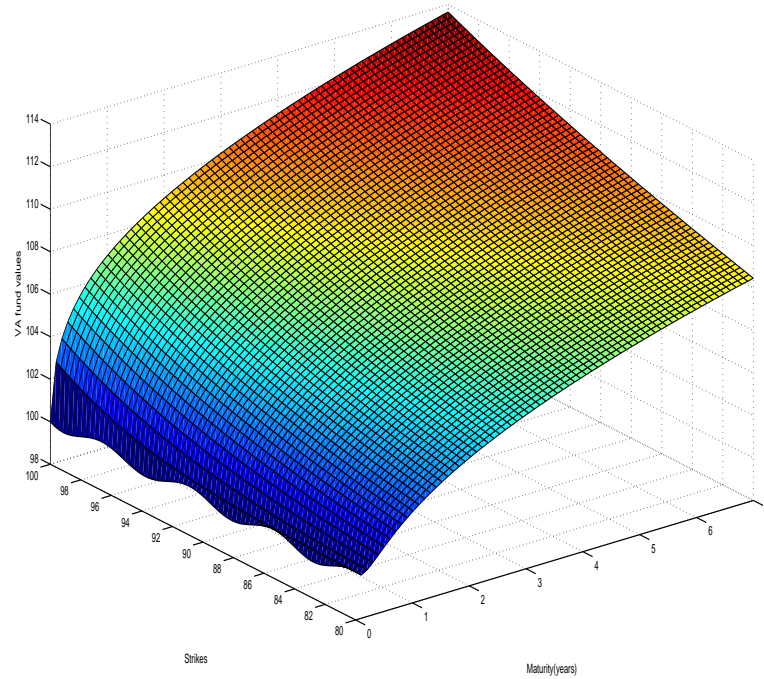
Figure 2.2: VA fund values evaluated at time 0 obtained by the Black-Scholes Fourier method in the Heston model with the state-dependent fee structure with the parameters $\rho_{xv} = -30\%$, $\kappa = 0.3$, $\gamma = 0.5$, $\bar{v} = v_0 = 0.05$, $r = 0.5\%$, $F_0 = 100$, $a = 0.01$ and $b = 0.25$.

2.5 Fee Rate Determination in GMAB

In this section, we study the benefits of charging state-dependent fees in a GMAB contract when compared to constant fees. We formulate the criteria for the fee rate determination and consider measuring the risk related to writing a GMAB contract with a state-dependent fee by the expected shortfall and the Value-at-Risk (VaR) of the difference between the GMAB liability and the fees to be collected. Charging higher fees from the VA fund value could be less attractive for the policyholder. We thus study the effect of the fee rate selection on the fund value.

Unlike standard exchange traded options, most insurance companies charge for the downside protection by deducting an ongoing fraction of invested assets instead of an upfront fee (Milevsky and Salisbury, 2006). In other words, a standard guarantee in the form of an option is financed upfront with a premium which is paid by the buyer of the option at the inception, whereas a guarantee embedded in a variable annuity is financed with fees paid by the policyholder during the lifetime of the contract:

$$P^*(T, F_T, v_T; a, b) \sim \text{fee}(0, T, F_{0,T}, v_{0,T}; T), \tag{2.23}$$

where $P^*(T, F_T, v_T; a, b) := (G_T - F_T)^+$ denotes the payoff of the GMAB liability put option at maturity T. In (2.23), $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$ denotes the accumulated value at time T of the fees collected over the entire period $[0, T]$, in which $F_{0,T}$ and $v_{0,T}$ are the paths of the fund and the variance in $[0, T]$ respectively. " $\sim$ " denotes the similarity in amount that $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$ can be financed with state-dependent fees for the put payoff $P^*(T, F_T, v_T; a, b)$ at T.

The $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$ in (2.23) is financed for the purpose of covering the costs of the GMAB liability put option $P^*(T, F_T, v_T; a, b)$ at maturity T. Any mismatch between the costs of the liability put option and the fees leads to either overcharging the policyholder or increasing the hedging difficulty of the issuer when the fees are insufficient. At time T, we observe that the entire fees collected from 0 to T, $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$, can be decomposed as

$$\text{fee}\left(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T\right) + \text{fee}\left(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T\right),$$

where $\text{fee}\left(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T\right)$ denotes the accumulated fees at time T, which are financed from 0 to Δ. $\text{fee}\left(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T\right)$ denotes the fees financed from Δ to T and accumulated to T. The time-T value of fee collected over $[0, \Delta]$ can be defined as

$$\text{fee}\left(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T\right) := \int_0^\Delta e^{r(T-s)}(a + bv_s)F_s ds, \tag{2.24}$$

Similarly, we define the time-T value of fee collected over $[\Delta, T]$ as

$$\text{fee}\left(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T\right) := \int_{\Delta}^{T} e^{r(T-s)}(a + bv_s)F_s ds. \tag{2.25}$$

We need the above decomposition of fee$(0, T, F_{0,T}, v_{0,T}; T)$ for presenting the risk related to writing a GMAB contract with the state-dependent fees at any Δ, $\Delta \in [0, T]$. Later we will use this when formulating a criterion for the fee rate determination.

Definition 2.5.1. Suppose that the fee rate is of the form $c_t = a + bv_t$. The time-Δ value of expected fee collected from Δ to time T can be represented as

$$\xi\left(\Delta, T, F_{\Delta}, v_{\Delta}; \Delta\right) := \mathbb{E}^{\mathbb{P}}\left[\int_{\Delta}^{T} e^{-r(s-\Delta)}(a + bv_s)F_s ds | \mathcal{F}_{\Delta}\right], \tag{2.26}$$

where the sigma-field $\mathcal{F}_{\Delta}$ encodes information up to time Δ.

Both the expected fees defined in (2.26) and the corresponding costs of the GMAB liability put option depend on time, the VA fund value and the market volatility. At $\Delta \in [0, T]$, we are interested in the risk related to writing a GMAB contract, measured by the proxy liability $d(\Delta)$ as a difference between the liability put option price and the expected fees. According to the definition below, $d(\Delta)$ measures the degree of the expected fees collected for covering the liability put option at time Δ. Later we will formulate some criterion on $d(\Delta)$ to determine the state-dependent fee structure when formulating some criterion on $d(\Delta)$. The proxy liability $d(\Delta)$ is defined in the following way

$$
\begin{aligned}
d(\Delta) \quad &:= \quad e^{-r(T-\Delta)}\mathbb{E}_{\Delta}^{\mathbb{P}}\left[P^*(T, F_T, v_T; a, b) - \text{fee}(0, T, F_{0,T}, v_{0,T}; T)\right] \\
&= \quad e^{-r(T-\Delta)}\mathbb{E}_{\Delta}^{\mathbb{P}}\left[P^*(T, F_T, v_T; a, b) - \underbrace{\text{fee}\left(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T\right)}_{\text{known at } \Delta} - \text{fee}\left(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T\right)\right] \\
&= \quad \underbrace{P^*(\Delta, F_{\Delta}, v_{\Delta}; a, b) - \text{fee}\left(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta\right)}_{\text{cost of put reduced by the fee collected from 0 to } \Delta} - \underbrace{\xi\left(\Delta, T, F_{\Delta}, v_{\Delta}; \Delta\right)}_{\text{expected fee from } \Delta \text{ to } T \text{ in money at } \Delta} \quad , \tag{2.27}
\end{aligned}
$$

where we define $\mathbb{E}_{\Delta}^{\mathbb{P}}$ as the expectation evaluated at time $\Delta \in [0, T]$ under the physical measure $\mathbb{P}$.

In particular, when $\Delta = 0$

$$d(0) = P^*\left(0, F_0, v_0; a, b\right) - \xi(0, T, F_0, v_0; 0). \tag{2.28}$$

We propose to use Monte Carlo methods to find the representation of $\xi\left(\Delta, T, F_\Delta, v_\Delta; \Delta\right)$ in (2.27), and below we derive an equivalent formula that reduces the computational burden. Recall that the payoff of a GMAB at maturity can be decomposed as

$$F_T + (G_T - F_T)^+ = G_T + (F_T - G_T)^+. \qquad (2.29)$$

Finding values of each component in (2.29) at time $\Delta \in [0, T]$, one obtains

$$F_\Delta - \xi\left(\Delta, T, F_\Delta, v_\Delta; \Delta\right) + P^*\left(\Delta, F_\Delta, v_\Delta; a, b\right) = G_T e^{-r(T-\Delta)} + C^*\left(\Delta, F_\Delta, v_\Delta; a, b\right), \quad (2.30)$$

where $C^*\left(\Delta, F_\Delta, v_\Delta; a, b\right)$ is the corresponding time-Δ call option price, which can be evaluated by Equation (2.19). Note that Equation (2.30) is the well-known put-call parity relationship.

By (2.30), the proxy liability $d(\Delta)$ in (2.27) can be represented as

$$d(\Delta) \;=\; C^*(\Delta, F_\Delta, v_\Delta; a, b) + G_T e^{-r(T-\Delta)} - F_\Delta - \text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta).$$

Both (2.27) and (2.31) provide a representation of the proxy liability $d(\Delta)$. By measuring the difference between the liability pt options and the corresponding fees, the third line in (2.27) appears more straightforward to the nature of $d(\Delta)$ when compared to the second line in (2.31). The second line in (2.31) reduces the computational burden without simulating the expected fees in (2.26), while $\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$ has been realized from the single path of the fund jointly with the corresponding path of the variance by $\Delta \in [0, T]$. Later we use (2.31) in the Monte Carlo simulation of the numeric analysis.

Consistency between (2.27) **and** (2.31)**.** The issuer charges an ongoing fee for the GMAB policyholder that pays the GMAB put option out. The net liability of a GMAB at time Δ should be the cash inflow (fees collected in $[\Delta, T]$) minus that of cash outflows (reduced costs of put option), which implies (2.27). For the policyholder, the cash outflow at Δ is the investment in VA fund F_Δ and the realized fees in $[0, \Delta]$ paying to the issuer before Δ. The cash inflow that the policyholder receives is the guarantee amount and the bonus if the fund value at maturity is positive, which implies (2.31). Under the fair condition, both of the net liabilities at time Δ to the insurer and the policyholder should be equal. In particular, $d(0)$ is set to be zero in both representations.

2.5.1 Comparison between State-dependent Fees and Constant Fees

Indicated by Polaris Choice IV, the state-dependent fee rate structure of $c_t = a + bv_t$ is pre-selected for all new policies. This means the policy issuer will determine the parameters of a and b prior to the contract sale to the policyholder. Both parameters in the fee rate structure are assumed to remain the same to each new policyholder whenever his/her contract comes into effect and whatever the market condition is.

Our goal is to determine the pre-selected fee rate c_t under some fair conditions. To show the advantage of charging the state-dependent fee rate instead of the constant one, we first simplify the state-dependent fee rate as $c_t = bv_t$, that is, we take $a = 0$. We then determine the sensitivity parameter b and the constant fee rate c by letting a market-consistent expected proxy liability, $d(0)$, be zero at time 0. Since the contract has to come into effect with the pre-selected fee structure even when the market condition changes, we then compare the effects of charging state-dependent fees and constant ones respectively against different market conditions. We conclude that the volatility-dependent fee outperform its constant counterpart when the average volatility is high and vice versa.

Fee rate structure of the form bv_t. We first examine a fee rate structure of $c_t = bv_t$ by directly associating the fee rate as a fraction of the current variance. We thus define the market-consistent proxy liability under the state-dependent fee rate of $c_t = bv_t$ as

$$
\begin{aligned}
d_b(0) : \quad &= \quad P_b^*\left(0, F_0, v_0; a = 0, b\right) - \xi(0, T, F_0, v_0; 0) \\
&= \quad C_b^*(0, F_0, v_0; a = 0, b) + G_T e^{-rT} - F_0,
\end{aligned}
\tag{2.31}
$$

where P_b^* denotes the time-0 price of the GMAB put option under the state-dependent fee rate $c_t = bv_t$. C_b^* is the corresponding time-0 call price, which can be evaluated by Equation (2.19).

Applying the fair condition that the expected fees collected in $[0, T]$ are equal to the liability put at time 0, we obtain the value of b^* that solves (2.31). This condition identifies b^* as a root of the equation from

$$
\begin{aligned}
d_b(0) \quad &= \quad P_b^*\left(0, F_0, v_0; a = 0, b^*\right) - \xi(0, T, F_0, v_0; 0) \\
&= \quad C_b^*(0, F_0, v_0; a = 0, b^*) + G_T e^{-rT} - F_0 = 0,
\end{aligned}
\tag{2.32}
$$

There exists a root in (2.32) since the expected fees $\xi(0, T, F_0, v_0; 0)$ are charged from the fund by matching the costs of the liability put option $P_b^*\left(0, F_0, v_0; a = 0, b^*\right)$ on the

fund. An increasing b^* is expected to raise the level of average fees tied with the market variance and to reduce the fund value during the contract life. When b^* increases, both the expected fees charged from the fund and the costs of the put option will increase until $P_b^* (0, F_0, v_0; a = 0, b^*)$ equals $\xi(0, T, F_0, v_0; 0)$. In (2.32), we solve for b^* with the call option $C_b^* (0, F_0, v_0; a = 0, b^*)$ using put-call parity, instead of simulating $\xi(0, T, F_0, v_0; 0)$ directly with the put option by Monte Carlo.

To better illustrate the advantage of charging the state-dependent fees, similarly, we define a market-consistent proxy liability $d_c(0)$ under a constant fee rate c as

$$
\begin{aligned}
d_c(0) : &= P_c^* (0, F_0, v_0; a = c, b = 0) - \xi(0, T, F_0, v_0; 0) \\
&= C_c^*(0, F_0, v_0; a = c, b = 0) + G_T e^{-rT} - F_0,
\end{aligned}
\tag{2.33}
$$

where P_c^* denotes the time-0 price of the GMAB put option under the constant fee rate $c_t = c$. C_c^* is the corresponding time-0 call price, which can be evaluated by Equation (2.19).

At time 0, a fair constant fee rate c^* solves the equation

$$
\begin{aligned}
d_c(0) &= P_c^* (0, F_0, v_0; a = c^*, b = 0) - \xi(0, T, F_0, v_0; 0) \\
&= C_c^*(0, F_0, v_0; a = c^*, b = 0) + G_T e^{-rT} - F_0 = 0,
\end{aligned}
\tag{2.34}
$$

We used the equations (2.32) and (2.34) to get b^* and c^* for the following values of the parameters: $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $\bar{v} = v_0 = 0.015$, $r = 2\%$, $F_0 = 100$, $G_T = 100$. We find the roots for both equations by using the Matlab "fzero" function. For the state-dependent fee structure in (2.32), we obtained $b^* = 1.543$, while $c^* = 0.0196$ for the constant fee rate. Note that (2.32) and (2.34) imply the same costs of a GMAB to the policyholder under both fee rate structures.

Next, we show how the proxy liabilities in (2.31) and (2.33) can vary under the constant and the state-dependent fees respectively for new policies. Since a new policyholder may purchase/enter the contract at any time, the market conditions may vary from the time when the pre-selected parameters were set. A desirable pre-selected fee rate structure is expected to be less sensitive to market conditions so that the pre-selected structure can provide the policy issuer with the consistent valuation, even under new market conditions. For example, the proxy liabilities, $d_b(0)$ and $d_c(0)$, can vary from the mean level of market squared volatility/variance $\bar{v}$. For the pre-determined parameters of b^* and c^*, Figure 2.3 illustrates the effect of the proxy liabilities on the change of the mean levels of market variance in the long term, given the levels of current variance v_0. We find that the proxy liability $d_c(0)$ under the constant fee rate increases as a function of the long-term variance

level $\bar{v}$ and then becomes positive for the larger mean level to which the volatility reverts. Under the state-dependent fees, on the contrary, the proxy liability $d_v(0)$ turns negative for the larger mean level of market variance. When $d_b(0)$ or $d_c(0)$ is greater than zero, the expected fees cannot sufficiently mitigate the costs of the corresponding liability put option for the issuer at time 0. The above findings suggest that if the long-term mean variance $\bar{v}$ is lower than the value of $\bar{v}$ that we used for solving (2.32) and (2.34), then the constant fee structure is better for covering the costs of the GMAB liability put option, while the state-dependent fee structure is better for a higher $\bar{v}$. All the illustrations in Figure 2.3 have quite similar shapes and positions whatever the levels of v_0 are. Based on our findings, both the costs of the GMAB liability put option and the total fees financed for the entire life of the contract are mainly dependent on the level of market mean variance $\bar{v}$. Assuming three levels of $\bar{v}$, the initial variance v_0 has very minor impacts on the difference between the liability guarantee and the fees for a long-term contract life of $T = 20$ years.

Figure 2.3: Comparison between the market-consistent proxy liabilities and the current variance level under the constant fee rate (red line) and state-dependent fee rate (blue line) respectively, given the levels of current variance $v_0 = 0.01$ (left panel), 0.015 (middle panel) and 0.03 (right panel) with the parameters of $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $r = 2\%$, $F_0 = G_T = 100$ and $T = 20$.

Figure 2.4 illustrates the effect that a change of v_0 has on the proxy liability. In these graphs, we consider three different levels of the mean variance, and the equations (2.31) and (2.33) are solved for v_0 with the fee structures that we determined in (2.32) and (2.34). The illustrations in Figure 2.4 are consistent with the ones in Figure 2.3, in which the state-dependent fee rate is better than the constant one in the scenario with a higher $\bar{v}$ and vice versa. In Figure 2.4, $d_b(0)$ decreases as a function of the current variance v_0 and turns negative for a higher v_0 under the state-dependent fees. This also suggests that the

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state-dependent fee rate can be better than its counterpart when the current variance is higher than the one we used to calculate the fee structure.
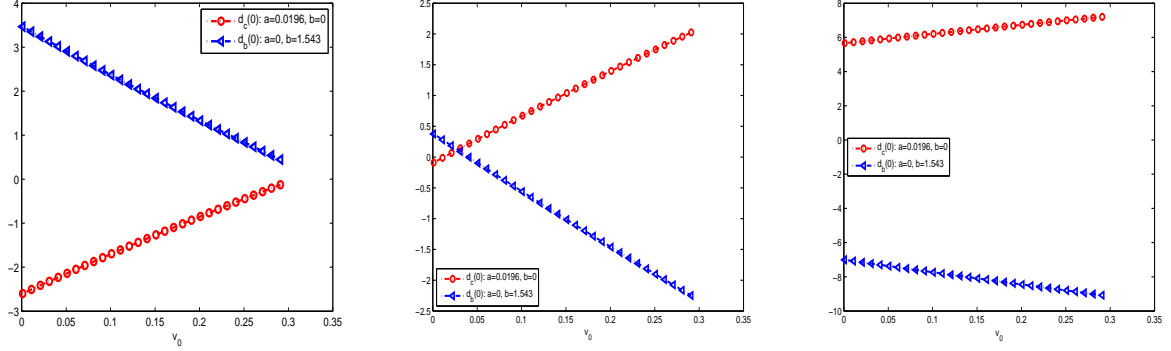
Figure 2.4: Comparison between the market-consistent proxy liabilities and the current variance level under the constant fee rate (red line) and state-dependent fee rate (blue line) respectively, given the levels of average variance $\bar{v} = 0.01$ (left panel), 0.015 (middle panel) and 0.03 (right panel) with the parameters of $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $r = 2\%$, $F_0 = G_T = 100$ and $T = 20$.

In Figure 2.5, we illustrate the patterns of the proxy liabilities on the change of maturity T under different mean levels of market variance. For a long-term liability, a higher mean level of the market variance raises the expected state-dependent fees faster, while the embedded option payoff can be pulled down by a growing fund value by T. Therefore, either $d_b(0)$ or $d_c(0)$ turns smaller for a larger T. As Figure 2.5 suggests if the mean level of market variance is lower than the one we used to calculate the fee structure, the constant fee rate is better than the state-dependent one and vice versa.

According to the previous findings, the mean level of the market variance has a significant impact on the associated state-dependent fees in the long run. This can also be shown from the distributions of the variance process in the Heston model. It is well known that the transition distribution of $v_t | v_u$, where $t > u$, follows a non-central chi-squared distribution

$$v_t | v_u = v \sim \frac{\gamma^2 (1 - \exp(-\kappa(t - u)))}{4\kappa} \chi_d^2(\lambda), \nu := \frac{4\bar{v}\kappa}{\gamma^2}, \lambda := \frac{4\kappa \exp(-\kappa(t - u))}{\gamma^2 (1 - \exp(-\kappa(t - u)))} v,$$

where $\chi_\nu^2(\lambda)$ denotes the noncentral chi-squared random variable with ν degrees of freedom and noncentrality parameter λ. For $t \to \infty$, v_t converges to the stationary distribution
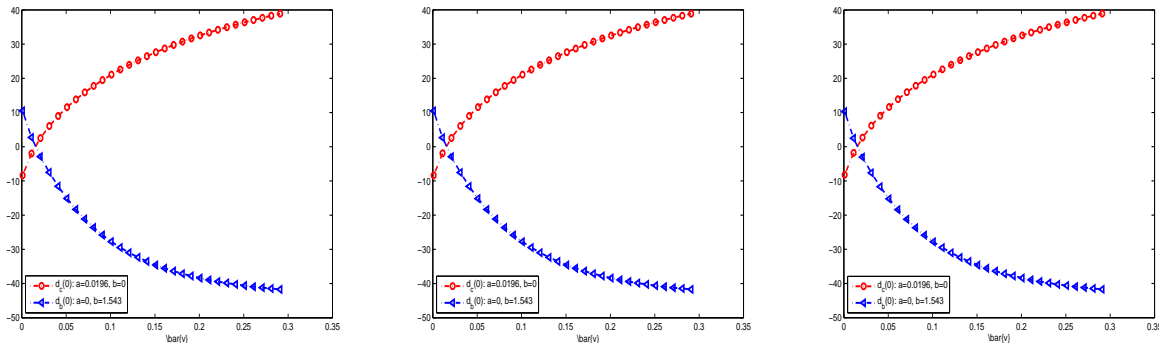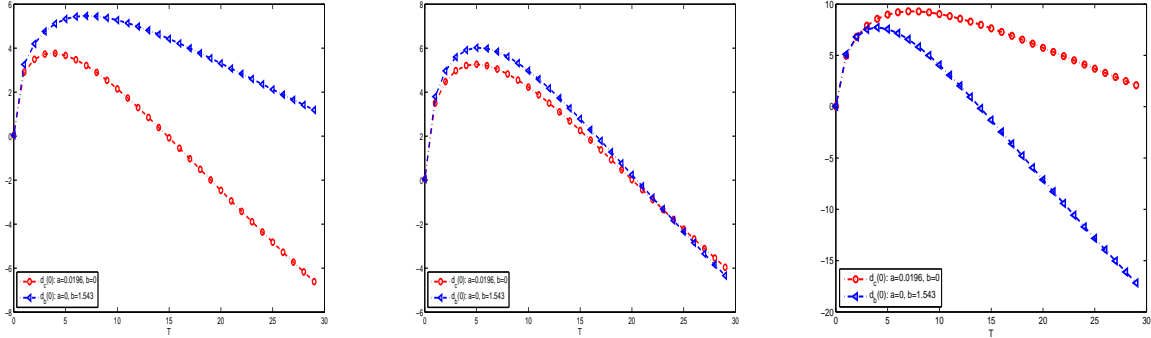
Figure 2.5: Comparison between the market-consistent proxy liabilities and the maturity T under the constant fee rate (red line) and state-dependent fee rate (blue line) respectively, given the levels of average variance $\bar{v} = 0.01$ (left panel), 0.015 (middle panel) and 0.03 (right panel) with the parameters of $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $r = 2\%$, $F_0 = G_T = 100$ and $v_0 = 0.015$.

(see for example, Drăgulescu and Yakovenko (2002))

$$\Pi_*(v) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \frac{v^{\alpha-1}}{\bar{v}^\alpha} e^{-\alpha v/\bar{v}}, \quad \alpha = \frac{2\kappa\bar{v}}{\gamma^2},$$

which is the probability density function of a Gamma distribution.

In Figure 2.6, we present both the one-year transition distribution for instantaneous variance v_t and the stationary distribution for the mean level of market variance $\bar{v}$. We find that both distributions are much more skewed to the right for higher mean variance level than those for lower mean variance level. The right skewness of the above variance distributions suggests that the expected state-dependent fees increase as a function of the mean level of the market variance $\bar{v}$.

2.5.2 Determination of a and b

We now generalize the fee rate structure by considering $c_t = a + bv_t$, where $a \neq 0$ and $b > 0$. At time $\Delta \in [0, T]$, we want to determine c_t by minimizing the risk of the proxy liability quantified by its expected shortfall $\mathbb{E}[d^+(\Delta)]$, where $d^+(\Delta) := \max(d(\Delta), 0)$. By the definition of the proxy liability $d(\Delta)$ in (2.27), this ensures that the issuers can optimally
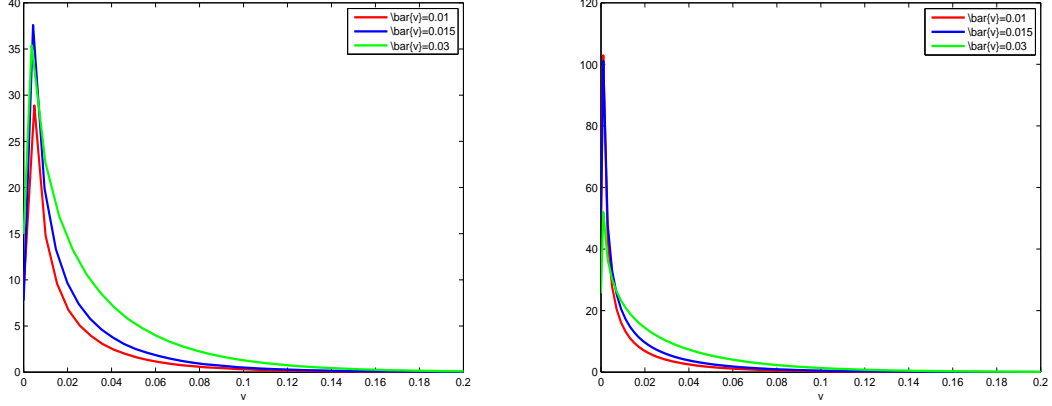
Figure 2.6: One-year transition probability densities of $v_t|v_0$ (left panel) and the corresponding stationary distribution $\Pi_*(v)$ (right panel).

cover the ongoing GMAB liability put option by collecting expected fees under a given fee rate structure c_t, at a fixed Δ, $\Delta \in [0, T]$. In practice, the issuers select Δ in need for the optimization purpose of c_t at a target time. The optimization problem is governed by the global constraint $d(0) = 0$ and that leads to the following optimization problem

$$(a^*, b^*) = \arg\min_{a,b} \mathbb{E}^{\mathbb{P}} \left[d^+(\Delta; a, b) \right] \tag{2.35}$$

with a nonlinear constraint

$$d(0; a, b) = 0.$$

Note that we add the arguments of a and b to the notation of expected shortfall $d^+(\Delta)$ and proxy liability $d(0)$ as $d^+(\Delta; a, b)$ and $d(0; a, b)$ respectively. The expected shortfall is a coherent risk measure that has good theoretical properties (see, for example, Artzner, Delbaen, Eber, and Heath (1999) and Acerbi, Nordio, and Sirtori (2001)).

We next determine the pair of (a, b) by solving the problem (2.35). In practice, the issuer may be faced with minimizing the expected shortfall at different time periods over the life of the contract. We therefore investigate the fee rate pattern and the corresponding shortfall risk $\mathbb{E}^{\mathbb{P}} \left[d^+(\Delta; a, b) \right]$ at times $\Delta = 0.2, 0.8, 2.0, 2.2, 3.8, 4.0, 4.2, 4.4, 5.0$ and 6.0 respectively.

We now explain the numerical procedure that we have used to determine the optimal pair of a and b for each fixed Δ:

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1. Using the Euler scheme (see for example, Andersen (2008)), simulate a path of the fund prices F_t (in the form of log-return of the fund value x_t) jointly with the variance process v_t, for $t \in [0, T]$, by discretizing the time by $N = 100$ intervals. We fix the random numbers for simulating both the underlying and the variance paths under different state-dependent fees.

2. For a fixed $\Delta \in [0, T]$, determine $d(\Delta; a, b)$ in (2.31) using the path of the fund value jointly with the path of the corresponding variance from the previous step. We approximate fee$(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$ in (2.31) by integrating along the fund-value and volatility paths according to (2.24).

3. Repeat Steps 1 - 2 for $M = 50,000$ times and find $\mathbb{E}^{\mathbb{P}}[d^+(\Delta); a, b]$, as well as the Value-at-Risk of $d(\Delta; a, b)$, based on M numbers of $d(\Delta; a, b)$.

4. Use Matlab "fmincon" function (function-minimization-with constraint) to minimize $\mathbb{E}^{\mathbb{P}}[d^+(\Delta; a, b)]$ with the global constraint $d(0; a, b) = 0$, and thus obtain the optimal pair (a^*, b^*) at Δ.

5. Given the optimal pair of (a^*, b^*) at Δ, compute the corresponding $\mathbb{E}^{\mathbb{P}}[d^+(\Delta); a^*, b^*]$ and the Value-at-Risk of $d(\Delta; a^*, b^*)$ based on M numbers of $d(\Delta; a^*, b^*)$.

Given the mean levels of market variance, Figure 2.7 illustrates the feasible pairs of (a, b) only satisfying the condition $d(0; a, b) = 0$ at time 0. It shows that the sensitivity parameter b increases as a function of the location parameter a in the linear fee rate structure of c_t. In the Polaris contract, the location parameter a determines the initial fee rate, which is then adjusted by the degree of the market variance change associated with the sensitivity parameter b. Therefore, the larger sensitivity of the state-dependent fees to the market variance leads to lower initial fees in the contract.

In Table 2.1, the optimal pairs of (a^*, b^*) have been determined for the issuer who wants to minimize the shortfall risk $\mathbb{E}^{\mathbb{P}}[d^+(\Delta; a, b)]$ at a specific $\Delta \in [0, T]$, under the condition that the embedded option is fairly priced at time 0. We observe that the Value-at-Risk on the proxy liability $d(\Delta; a^*, b^*)$ turns smaller and becomes negative for the reason that the state-dependent fee charged in percentage from the fund value can be high in the long run. As illustrated in Figure 2.8, the distribution of $d(\Delta; a^*, b^*)$ shifts to the left when Δ increases. This is because the percentage fee charged from the fund value is significantly larger than the ongoing GMAB put liability. As shown in Figure 2.9, the fund value grows well over the time under the physical measure $\mathbb{P}$. By the definition of $d(\Delta; a, b)$ in (2.27), the increase of F_Δ devalues the reduced liability put option $P^*(\Delta, F_\Delta, v_\Delta; a, b) -$ fee $(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$ and increases the expected fees $\xi(\Delta, T, F_\Delta, v_\Delta; \Delta)$ at $\Delta \in [0, T]$.
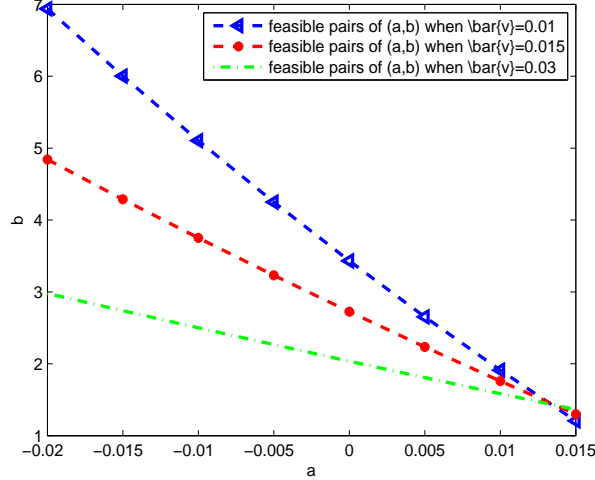
Figure 2.7: Plot of feasible pairs of (a, b) constrained by $d(0) = 0$ with the parameters of $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $v_0 = 0.01$, $r = 2\%$, $F_0 = 100$, $G_T = 100$, and $T = 20$, under $\bar{v} = 0.01$, 0.03 and 0.05 respectively.

This implies that the issuer's ongoing risk related to writing the GMAB contract by Δ can be significantly reduced by charging percentage fees.

The pairs of optimal (a^*, b^*) in Table 2.1 are determined at different Δ that satisfy the issuer's need. Alternatively, the issuer may consider minimizing the proxy liability $d(\Delta; a, b)$ over a period of time but not limited to a specific Δ. We thus formulate the optimization problem on averaging the expected shortfalls in the sense that

$$(a^*, b^*) = \arg\min_{a,b} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}} \left[d^+(\Delta_i; a, b) \right] \tag{2.36}$$

with a nonlinear constraint

$$d(0; a, b) = 0,$$

where $\Delta_i \in [0, T]$ for $i = 1, 2, \cdots, n$.

For example, we take the average of the expected shortfalls in (2.36) based on $n = 11$ numbers of Δ_i, from the time set of $\{0.2, 0.8, 2, 2.2, 3.8, 4, 4.2, 4.4, 4.6, 5, 6\}$ with the same parameters as in Table 2.1. This is a subjective choice of the time set, but it illustrates the feasibility of such an approach. We find the optimal pair of $a^* = 0.0043$ and $b^* = 1.2133$.

Δ	optimal (a^*, b^*)	$\mathbb{E}^{\mathbb{P}}\left[d^+(\Delta; a^*, b^*)\right]$	$\text{VaR}_{0.90}\left(d(\Delta)\right)$	$\text{VaR}_{0.95}\left(d(\Delta)\right)$	$\text{VaR}_{0.99}\left(d(\Delta)\right)$
0.2	$(-0.0023, 1.7673)$	0.1883	0.4424	0.5394	0.7493
0.8	$(-0.0692, 8.2517)$	0.1210	0.4557	0.8786	1.7410
2.0	$(0.0066, 1.0283)$	0.1834	0.7372	1.3302	2.3067
2.2	$(0.0069, 1)$	0.2394	0.5052	0.6249	0.8791
3.8	$(0.0088, 0.8481)$	0.0076	-2.6014	-1.8181	-0.6339
4.0	$(0.0074, 0.9622)$	0.0073	-3.2129	-2.2966	-0.9134
4.2	$(0.0057, 1.0958)$	0.0056	-3.8236	-2.7442	-1.1093
4.4	$(0.0046, 1.1935)$	0.0050	-4.6303	-3.3984	-1.5361
4.6	$(0.0031, 1.3090)$	0.0042	-5.3149	-3.9097	-1.7734
5.0	$(0.0012, 1.4669)$	0.0034	-7.0832	-5.3604	-2.7361
6.0	$(-0.0036, 1.8744)$	0.0016	-12.7982	-10.0377	-5.7922

Table 2.1: Optimal pairs of (a^*, b^*) that solve the problem (2.35) with the parameters of $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $v_0 = 0.015$, $\bar{v} = 0.015$, $\mu = 15\%$, $\lambda = -0.02$, $r = 2\%$, $F_0 = 100$, $G_T = 100$, and $T = 20$. $N = 100$ intervals and $M = 50,000$ paths.

The corresponding minimum expected average shortfall is $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^{\mathbb{P}}\left[d^+(\Delta_i; a^*, b^*)\right] = 0.0707$.

Under the higher mean levels of market variance, charging the volatility-dependent fees reduces the risk of the proxy liability for the issuer by providing more cash flows to cover the GMAB liability put option. We have determined the optimal state-dependent fee rate by the optimization on the expected shortfall of the proxy liability. In practice, the optimal fee rate can be determined by minimizing the expected shortfall of the proxy liability $d(\Delta; a, b)$ at a selected Δ by the issuer. An optimization based on averaging the expected shortfalls of the proxy liability can provide the issuer with the optimal state-dependent fee rate over a period of time.

2.5.3   Expected Present Value (EPV) of the VA fund

Although the state-dependent fees vary when the market volatility changes, they reduce the VA fund value. In a GMAB, the policyholder can receive either the guarantee or the fund value at maturity, whichever is larger. Therefore, charging higher fees from the fund value could be less attractive for the policyholder. Thus, we are motivated to study the effect of the fee rate selection on the fund value.
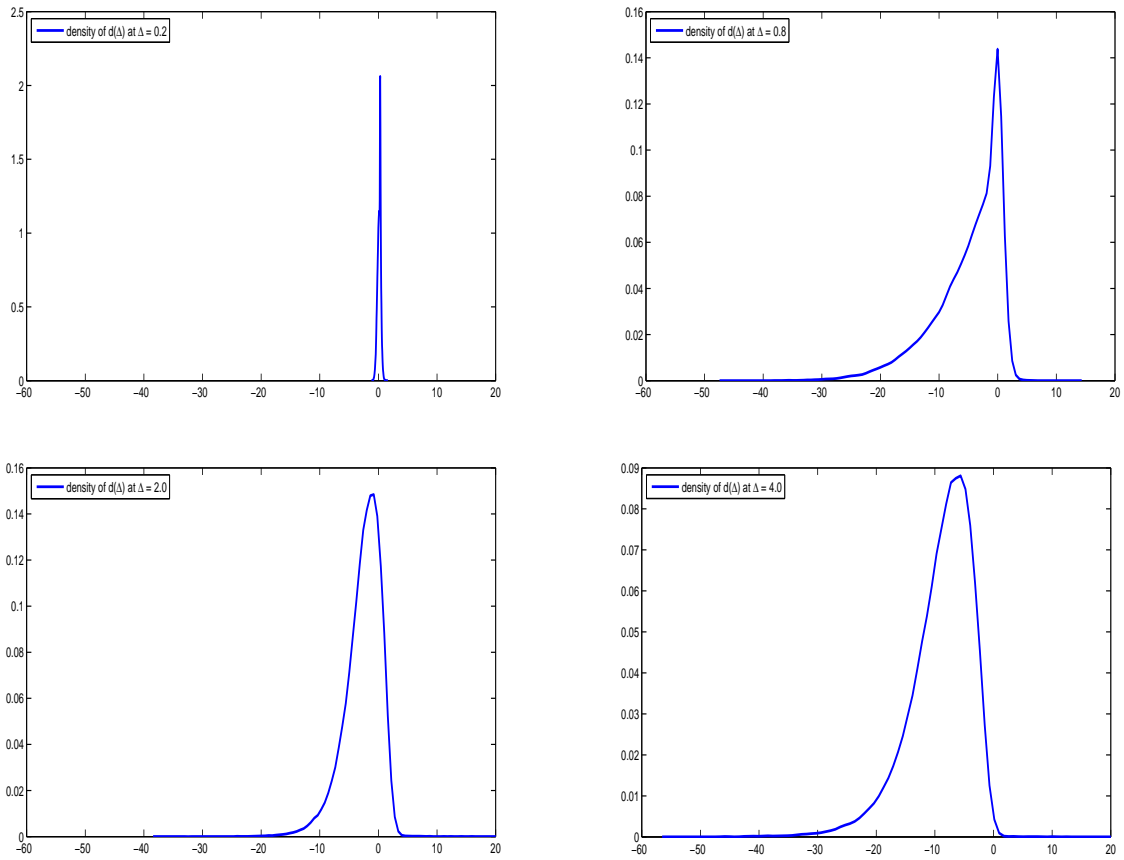
Figure 2.8: Probability densities of $d(\Delta)$ at $\Delta = 0.2$, 0.8, 2.0, and 4.0 with the corresponding optimal (a^*, b^*).
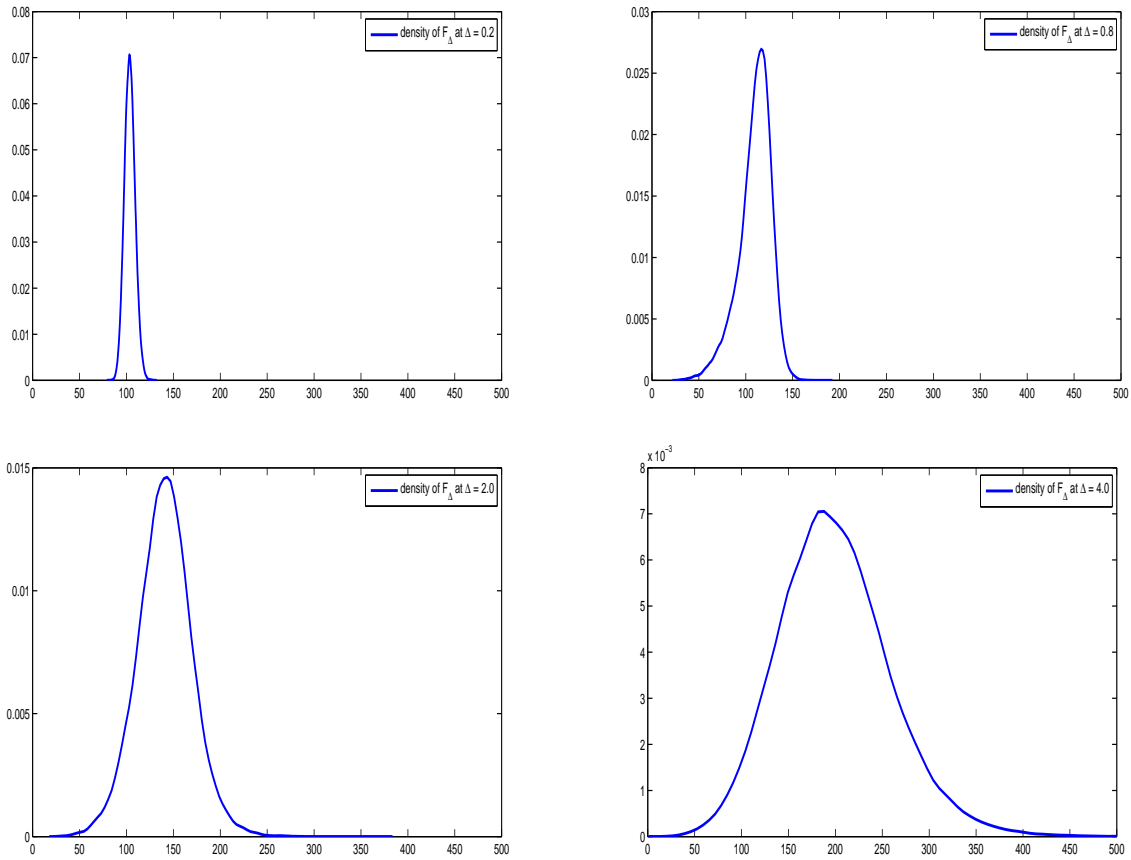
Figure 2.9: Probability densities of F_Δ at $\Delta = 0.2$, 0.8, 2.0, and 4.0 with the corresponding optimal (a^*, b^*).

Given the characteristic function $\phi_{\mathrm{VA}}(u)$ in (2.14), we can evaluate the EPV of the GMAB fund with the state-dependent fees. The probability density function of the log-return of the fund value is given by $p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi_{\mathrm{VA}}(u)du$. The expected present value of the fund value at time 0 is

$$
\begin{aligned}
e^{-rT}\mathbb{E}^{\mathbb{Q}}[F_T(a,b)] &= \int_{-\infty}^{\infty} g(x)p(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)e^{-iux}\phi_{\mathrm{VA}}(u)dudx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} se^{-iu(\ln s)}\phi_{\mathrm{VA}}(u)du\frac{1}{s}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iu(\ln s)}\phi_{\mathrm{VA}}(u)duds.
\end{aligned} \tag{2.37}
$$

In Table 2.2, we use the optimal pairs of (a^*, b^*) from Table 2.1 to calculate the EPV of the fund at $\Delta = 0$ by (2.37). For a twenty-year lifelong contract, the Heston constant fee rate of 1.96% returns the EPV of the fund value as 67.57. By charging the volatility-dependent fees, we observe that the fund value gets lower when the sensitivity parameter b is larger. Since the market volatility reverts to the mean level, a significant sensitivity of the state-dependent fees to the volatility enlarges the amount of average fees collected from the VA fund and that pulls down the fund value fast when the market volatility is high. To keep the VA contract more attractive, the issuer needs to control the level of the sensitivity parameter b in his fee rate structure without dampening the fund value too much to the policyholder. For example, a pair of (a, b) can be selected for matching the corresponding expected fund value with the one charged by the constant fee rate.

(a^*, b^*)	$e^{-rT}\mathbb{E}^{\mathbb{Q}}[F_T(a^*, b^*)]$
$(0.0069, 1)$	65.80
$(0.0066, 1.0283)$	65.67
$(0.0031, 1.3090)$	65.18
$(-0.0023, 1.7673)$	64.05
$(-0.0692, 8.2517)$	46.73
No fee : $(0, 0)$	100
Heston constant fee : $(0.0196, 0)$	67.57

Table 2.2: EPV of the fund value, $e^{-rT}\mathbb{E}[F_T(a,b)]$, with the parameters $\rho_{xv} = -30\%$, $\kappa = 3$, $\gamma = 0.5$, $\bar{v} = v_0 = 0.015$, $r = 2\%$, $F_0 = 100$, $G_T = 100$ and $T = 20$.

2.6 Concluding Remarks

In this chapter, we propose a new product design that allows to better align the costs of the VA guarantees and the corresponding fees collected from the policyholder under the market-consistent conditions. The proposed design has a state-dependent fee linked to the market volatility and is inspired by designs of recent VAs issued in the U.S. insurance market.

We have formulated an affine model to tackle the problem of modeling and pricing VAs with fees tied to market volatility. Then, we have studied the benefits of charging the state-dependent fee rate in a GMAB contract when compared to its counterpart, the constant fee rate. Our findings suggest that the mean level of the market variance has a significant impact on the state-dependent fees in the long run. If the long-term mean variance is low, the constant fee structure is better for covering the costs of the GMAB liability put option, while the state-dependent fee structure is better for the higher mean variance. To determine the state-dependent fee rate, we have formulated the criteria for the fee rate determination and considered measuring the risk related to writing a GMAB contract with the state-dependent fee by the expected shortfall and the Value-at-Risk (VaR). Charging higher fees from the VA fund value could be less attractive for the policyholder. We thus have studied the effect of the fee rate selection on the fund value. Our numerical results have shown that, when market volatility becomes high, a significant sensitivity of the state-dependent fees to the market volatility enlarges the amount of expected fees collected from the VA fund and that pulls down the fund value very fast. To keep the VA contract more attractive, the issuer needs to control the level of the sensitivity parameter in his fee rate structure without dampening the fund value too much to the policyholder.

The proposed state-dependent fee tied to the market volatility aims to facilitate the risk management of the guarantees but cannot replace a hedging program. It allows the VA products to automatically "re-price" themselves over time. This is an important advantage and allows to avoid recent stress of insurers that constantly update the guarantees they offer and corresponding fees as market conditions change. Our formulation with the proposed VA products is under the stochastic volatility environment. Stochastic volatility typically leads to an incomplete market in which perfect hedging strategies do not exist for many contingent claims. To tackle the hedging issues in the VA program, an optimal static hedging strategy can be developed as a benchmark to compare to other hedging strategies. The main idea behind the theory of efficient hedging introduced by Föllmer and Leukert (2000) is to find a hedge that can minimize the expected shortfall from replication where the shortfall is weighted by some loss function, given either limited budget or in the incomplete market. Kolkiewicz (2016) developed a general method of constructing static hedging

strategies for path-dependent options that minimize the shortfall risk for a given time interval. In Chapter 4, the optimal static hedging options in the Heston-type stochastic volatility model will be developed to fulfill the needs for hedging VA products.

Chapter 3

Reverse Mortgage with Default Risk Models

3.1 Introduction

A reverse mortgage (RM) is essentially a financial product designed to allow elder home-owners aged 62 or over to convert the home equity to either a lump sum, annuity payments, a line of credit or any combination of these. The homeowners retain full ownership of their property for the whole life of the loan. A reverse mortgage is different from a traditional mortgage in that it does not require the borrower to make monthly payments to the lender to repay the loan. Instead, loan proceeds are periodically paid out from the collateral house value to the borrower. In the United Sates, reverse mortgages are offered by the Home Equity Conversion Mortgage (HECM) program, which is issued by the Federal Housing Administration. According to the recent brief by Moulton and Haurin (2015), only about two percent of eligible seniors have reverse mortgages, but the demand for reverse mortgages has generally been rising since 2005, and many anticipate that this trend will continue as more people reach retirement with inadequate income from traditional sources.

One important feature of reverse mortgage loans is the "non-recourse" property, which means that the lender can only reclaim the loan by seizing the collateral house but can never seek out the borrower for any compensation when the house value does not cover the loans they already paid. With the non-recourse feature, when the loan is terminated, the borrower only needs to repay the loan amount or proceeds from the sale of the house, whichever is less. From the lender's perspective, the non-recourse feature in reverse mortgage loans incurs the risk that can be summarized as the "crossover risk" as illustrated in

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Figure 3.1. In the case that the loan value exceeds the collateral house value, the lender is limited to recover only the proceeds from the sale of the house when a reverse mortgage loan is terminated. Any excess of the loan value over the house value is then considered as a loss to the lender. On the other hand, if the loan is terminated before the crossover, any excess of the proceeds from the sale will revert back to the borrower, rather than becoming the lender's gain. Thus, the crossover risk is unilaterally taken by the lender. The crossover risk is a combination of three major underlying risks: termination risk, interest rate risk and house price risk. The termination risk indicates that the borrower could live in the house too long so that the loan value accumulates to a point where it exceeds the house value. In the literature, the termination risk typically accounts for a combination of mortality and mobility. In this chapter, we also consider the risk source from default and it is directly related to the termination of reverse mortgages. It is obvious that the loan repayment is capped by the house price. A high interest rate environment and a depressed real estate market can obviously exacerbate the crossover risk.
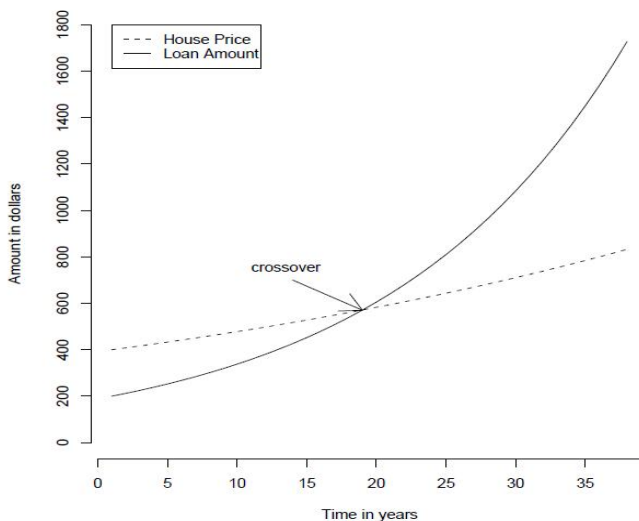
Figure 3.1: Illustration of "crossover risk".

Recently, HECM programs confront a rising default risk in the wake of the financial crisis, which is jeopardising the financial soundness of reverse mortgages. In reverse mortgages, the default risk is the risk that the borrower does not pay the property taxes or the homeowner's insurance, or fails to maintain the home in "saleable" condition that causes the contract termination. This constitutes a violation of the mortgage and the lender can call the loan due. According to Moulton, Haurin, and Shi (2016), ten percent of all active

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HECM loans were in default, which affected more that $54,000$ senior homeowners in 2013. The major consequence of defaults on reverse mortgage is the foreclosure. If the foreclosure is approved, the bank becomes the owner and sells the property through the traditional route to recoup its loss. After that, the borrower receives the remaining balance amount from the bank. The borrower must recover the default and pay off the debt in order to avoid a foreclosure. By law, the lender must allow the borrower to recover the default to prevent or stop a foreclosure. The U.S. Department of Housing and Urban Development (HUD) now begins to require lenders to verify that borrowers can afford to pay property taxes and insurance before selling reverse mortgage (i.e. a mandatory "financial assessment"). If not, the borrowers may be turned down for the loan. The rising default risk is a serious concern to the social economy. First, the high default rates place more seniors at risk of foreclosure, creating significant personal and community impact. Second, it creates mismatch between the cash income and the guarantee to the lenders, and that causes negative impact on their solvency. In the extreme case, systemic defaults may impair the soundness of lending institutions and it is costly to the borrower. To mitigate the default rate, HUD has responded by restricting initial withdrawals and introducing underwriting criteria. According to the analysis of Moulton, Haurin, and Shi (2016), the combined impact of the policy changes could reduce property tax and insurance default by as much as 50 percent.

In this chapter, we are motivated to determine the reverse mortgage payments according to borrowers' individual credit and default risk. We thus propose a new pricing scheme for the HECM program that allows the loan payments to better reflect the individual borrower's default level. From the borrower's perspective, the newly-designed payment can be interesting compared to the flat one that is applicable to all the borrowers in the current HECM program. Such pricing scheme is also expected to provide a risk sharing mechanism between the lender and the borrower by shifting default risk to the insured. Due to the complexity of reverse mortgages, pricing these contracts typically involves statistical and stochastic models in analytic studies. For example, Chen, Cox, and Wang (2010) price the non-recourse provision of reverse mortgages and compare it with the insurance premiums. They use a generalized Lee-Carter model with asymmetric jump effects to fit the mortality data and model the house price index via an ARIMA-GARCH process. Lee, Wang, and Huang (2012) propose a valuation framework with mortality risk, interest rate risk and housing price risk to determine the premiums when the present value of premiums equals the present value of contingent losses. Ji, Hardy, and Li (2012) develop a semi-Markov multiple state model to determine the premiums for the No-Negative-Equity-Guarantee in reverse mortgages. Kogure, Li, and Kamiya (2014) use a Bayesian multivariate framework that involves several risks of mortality rates, interest rates and house prices to determine the

insurance premiums in the reverse mortgages. Shao, Hanewald, and Sherris (2015) analyze the combined impact of house price risk and longevity risk on the pricing and risk profile of reverse mortgage loans in a stochastic multi-period model. These studies determine the insurance premiums or the fair loan payments in the absence of modelling the default risk of the borrower. However, as a credit product, the scheduled payment (pricing) on a reverse mortgage is expected to reflect the borrower's default risk. In practice, the lender is willing to enroll the borrowers with low-default risk to secure the solvency of the program. Some recent empirical studies begin to investigate the financial demand and outcomes of reverse mortgage loans in the presence of default risk and mortality risk. For example, Moulton, Haurin, and Shi (2016) identify factors associated with reverse mortgage default, including large upfront withdrawals from the HECM, lower initial credit scores, high property taxes relative to income, low levels of available revolving credit, and a prior history of delinquency on the mortgage or property taxes. Nakajima and Telyukova (2017) find that poor health will be one of the most influential factors that drive seniors to take out a reverse mortgage, with the intent of using home equity to help offset current and future medial expenses. Moulton, Loibl, and Haurin (2017) estimate potential demand in reverse mortgages based on their survey data of the senior population. Loibl, Haurin, Brown, and Moulton (2018) analyze the long-term borrowers' outcomes of reverse mortgage contracts as a financing tool. By considering an annuity reverse mortgage with default risk, the borrower will receive annuity payments when she survives without default and a non-negative balance when she is in foreclosure or dies before default. Our pricing scheme initiates to customize the fair annuity loan payments according to the level of the borrower's default risk and provides the lender with a better payment arrangement from the perspective of risk management. This pricing method can also be extended to find the fair annuity loan payments in accordance with the level of borrower's mortality risk and health conditions.

The contributions of this research are threefold. First, we propose a pricing approach for deriving a fair level of periodic annuity payments under the prevalent HECM program in the presence of default risk from the borrower. This pricing allows HECM to provide the market fair payments depending on the borrower's default risk. Second, the proposed pricing formula achieves an analytic solution for reverse annuity mortgages in the presence of default risk. This allows to investigate the relation between fair payments and risk factors. Third, we introduce a rating scheme for the reverse mortgage contract to enhance the application of our pricing scheme in practice. Our model suggested in this chapter offers a great degree of flexibility regarding the assumptions about the risk factors while keeping analytical tractability.

The remaining body of this chapter is organized as follows. In Section 3.2, we formulate the pricing problem through a different perspective from the existing literature, in which

the modelling of default risk is naturally implemented in Section 3.3. We specify various risks involved in the HECM program and provide the model assumptions in Section 3.4. In Section 3.5, we characterize the pricing of defaultable reverse mortgage and investigate the interaction among fair loan payments and risk factors such as mortality rate, interest rate and hazard rate. In Section 3.6, we introduce a rating scheme to the current HECM program for enhancing the application of the pricing method. Section 3.7 concludes the chapter.

3.2 Problem Setup

In this section, we consider a reverse mortgage (RM) contract, in which the collateral house value can be converted into initial withdrawals, periodic annuity payments and a non-negative balance as a loan payable to the borrower during the lifetime of the contract. Given an initial withdrawal, the pricing of an RM contract is equivalent to finding the rate for the periodic annuity loan payments. We use the following notations: $H(t)$ is the spot price of the collateral house price at time t, $t \in [0, T]$. $L(t)$ is the accumulated annuity loan payments of an RM contract at the same date t, $t \in [0, T]$, prior to the maturity T. $\pi(0)$ is the initial lump-sum withdrawal by the borrower at inception of the contract. The outstanding RM contract value after the immediate initial withdrawal is defined as $\tilde{H}(0) := H(0) - \pi(0)$. For $t \in [0, T]$, we define the outstanding RM contract value as $\tilde{H}(t)$, which fully tracks the price change of the collateral house $H(t)$. The non-negative balance at time t is thus defined as $Bal(t) := \left(\tilde{H}(t) - L(t) \right)^{+}$, $t \in [0, T]$.

Since the collateral house is the only asset that lenders may use to reclaim the loan, the lender faces a so-called "crossover risk" when the accumulated annuity loan payments $L(t)$ exceed the value of the outstanding RM contract $\tilde{H}(t)$. For $t \in [0, T]$, the crossover risk is defined as

$$Loss(t) := \left(L(t) - \tilde{H}(t) \right)^{+}, \tag{3.1}$$

which is the loss that the lender faces at time t.

Many lenders are unwilling to enter the market, for fear of crossover risk caused by the non-recourse clause. To encourage the lenders to participate and offer RMs, premium charges will be collected to compensate for the crossover risk unilaterally taken by lenders. In accordance with the principle that the present value of an insurance premium equals the present value of the expected loss, we assume that the RM contract will be terminated

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by its maturity T. Then, we obtain

$$\text{EPV of premium charges} = \text{EPV of } \left(L(T) - \tilde{H}(T) \right)^+. \tag{3.2}$$

The premiums for the crossover risk in (3.2) are charged by taking values from the borrower's collateral house to the lender until the maturity T. In the HECM program, an initial charge equals $100p_0\%$ of the initial house value $H(0)$ at the inception of the contract, and the annual premium is $100p_a\%$ of the outstanding contract value $\tilde{H}(t)$. It is clear that more premium charges should be taken for more significant crossover risk.

The non-recourse clause benefits the RM borrower when the accumulated annuity loan payments $L(T)$ exceeds the outstanding contract value $\tilde{H}(T)$. In such a case, at maturity T, the borrower only needs to repay the amount of $\tilde{H}(T)$ at most. When the crossover risk does not occur, the borrower pays back the value of $L(T)$ and keeps the remaining positive balance $\tilde{H}(T) - L(T)$ after the sale of the collateral house by the lender. At maturity T, the borrower repays the RM lender with the amount

$$\min \left(L(T), \tilde{H}(T) \right),$$

and thus the borrower's remaining non-negative balance value is given by

$$Bal(T) = \tilde{H}(T) - \min \left(L(T), \tilde{H}(T) \right) = \left(\tilde{H}(T) - L(T) \right)^+. \tag{3.3}$$

Due to the non-recourse feature, after the initial withdrawal $\pi(0)$, the borrower will receive either the amount of accumulated annuity loan payments or the outstanding contract value at maturity, whichever is more:

$$\max \left(L(T), \tilde{H}(T) \right) = L(T) + \left(\tilde{H}(T) - L(T) \right)^+ = L(T) + Bal(T). \tag{3.4}$$

The initial contract value of RM, $\mathbb{H}(0, T)$, is thus given by

$$\begin{aligned} \mathbb{H}(0, T) : \ &= \ \pi(0) + \text{EPV of max} \left(L(T), \tilde{H}(T) \right) + \text{EPV of premium charges} \\ &= \ \pi(0) + \text{EPV of } L(T) + \text{EPV of } Bal(T) + \text{EPV of } Loss(T). \end{aligned} \tag{3.5}$$

As illustrated in Figure 3.2, the right-hand side of the second equation in (3.5) represents

the expected values of discounted cash flow claims during the lifetime of a defautable RM contract. The first three terms include the borrower's loan reception of the initial withdrawals, the annuity loan payments and the non-negative balance amount when the contract is due. These terms indicate that the RM contract can be treated as a scheduled long-term debt in the form of loan payments by the policy issuer (lender) to the borrower. We thus can formulate the pricing issue on reverse mortgages in the spirit of Madan, Bakshi, and Zhang (2006), where the defaultable long-term corporate bond has been formulated and priced. The fourth term of the second equation in (3.5) represents the costs of the crossover risk defined in (3.1), which are paid as premiums to the lender by taking values from the borrower's collateral house. In particular, the crossover risk can be caused by the borrower's default, which results in the early termination of the RM contract. Therefore, the costs of the crossover risk depend on both the payoff and the conditioning event (default), and have to be priced accordingly. In reverse mortgages, the default risk is the one that the borrower does not pay the property taxes or homeowner's insurance, or fails to maintain the home in "saleable" condition that causes the contract termination by foreclosure. If the foreclosure is approved, the bank becomes the owner and sells the property through the traditional route to recoup its loss. To encourage the borrowers out of default, the borrowers are expected to be rewarded when they maintain the RM contract well without default. On the contrary, the lenders are expected to penalize the borrowers for the default at which time the lenders' hedging becomes more difficult. By the introduction of default probability, we propose to find a fair annuity loan payment schedule that provides a better alignment between the borrower's default risk and the annuity loan payment in a defaultable RM contract.

3.3 Reverse Mortgage with Default Risk

In this section, we propose a model for default risk in reverse mortgage contracts with the pricing scheme in (3.5). As we demonstrate below, our model leads to analytical pricing formulae.

Let τ denote the random time when default occurs before the termination of the contract at T. We then associate the unit step function $\chi(t)$ with τ:

$$\chi(t) = \begin{cases} 1 & t \geq \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$
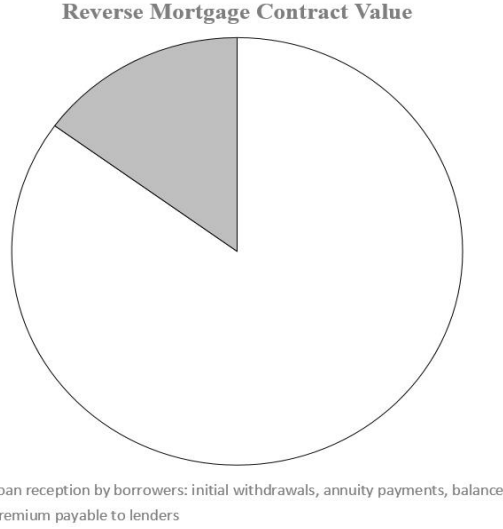
Figure 3.2: Composition of a reverse mortgage contract value.

For spot interest rate $r(t)$, let $b(t) := \exp\left(\int_0^t r(s)ds\right)$ be the accumulation of the money market account. Denote by $\{c(t) : t > 0\}$ the rate of continuous-annuity payment. Then, for $t \in [0, T]$, the accumulated annuity loan payments can be represented as $L(t) = \int_0^t \frac{b(t)}{b(s)} c(s)ds$. Under the risk-neutral valuation, the time-0 contract value of the defaultable reverse mortgage with a random default time τ and contract duration T from time 0 is then given by

$$
\begin{aligned}
\mathbb{H}(0, T) = \pi(0) \;\; + \;\; & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T {}_u p_x \frac{1}{b(u)} c(u) \mathbb{1}_{\{\tau \geq u\}} du \right] \\
+ \;\; & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T {}_u p_x \mu_u \frac{1}{b(u)} Bal(u) \mathbb{1}_{\{\tau \geq u\}} du \right] \\
+ \;\; & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T {}_u p_x \mu_u \frac{1}{b(u)} Loss(u) \mathbb{1}_{\{\tau \geq u\}} du \right] \\
+ \;\; & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T {}_u p_x \frac{1}{b(u)} (1 - w(u)) Bal(u) f_\tau(u) du \right] \\
+ \;\; & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T {}_u p_x \frac{1}{b(u)} (1 - w(u)) Loss(u) f_\tau(u) du \right], \quad (3.7)
\end{aligned}
$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation operator under the pricing measure $\mathbb{Q}$. $f_\tau(t)$ is the probability density function of τ. $\pi(0)$ is the initial lump-sum withdrawal amount at time 0. $_tp_x$ is the survival probability that age (x) survives until time t. μ_t is the force of mortality of age (x) at time t. $w(t)$ is the recovery rate that enables the borrowers to recover their defaults before foreclosure. In practice, the maturity T is the limiting year, at which time the survival probability, $_Tp_x$, for age (x) is zero.

The first three expectations in (3.7) refer to the cash flows before the default occurs, while the cash flows at the occurrence of the default can be represented by the last two expectations in (3.7). In the absence of default, the cash flows include the receipts of annuity payments when the borrower is alive, the non-negative balance amount when the contract is due, and the corresponding premiums for the crossover risk at borrower's death. To be specific, the first expectation accounts for the borrower's receipt of annuity payments prior to default when he is alive. The second and third expectations amount to the EPV of the non-negative balance and the corresponding premiums for the crossover risk at borrower's death prior to his default respectively. In the presence of default at which time the borrower is still alive, the foreclosure (due) payout to the borrower and the corresponding premiums for the crossover risk at his default are respectively represented by the fourth and fifth expectations. We assume that the due payouts for both death and default by the borrower amount to the same values as the non-negative balance value $Bal(t)$.

For a surprise default time that is a stopping time, there exists a positive process $h(t)$, called the hazard rate process, such that $\chi(t) - \int_0^t (1 - \chi(s)) h(s) ds$ is a martingale (see Bielecki and Rutkowski (2013) for example). We refer to $h(t)$ as a risk-neutral hazard rate process under $\mathbb{Q}$. For simplicity, we assume that the default does not depend on the loan status, the house price, mortality and interest rate but rather only depends on the borrower's ability to maintain the house out of foreclosure. And thus, the hazard rate process $h(t)$ is independent of other variables in our model. Heuristically speaking, $h(t)dt$ models the probability of default in the interval $(t, t + dt)$, the instantaneous likelihood of default. We thus have

$$P\left(\tau \geq t\right) = \mathbb{E}^{\mathbb{Q}}\left[1 - \chi(t)\right] = \exp\left(-\int_0^t h(s)ds\right), \tag{3.8}$$

and the probability density function for τ

$$f_\tau(t) = -\frac{dP\left(\tau \geq t\right)}{dt} = h(t)\exp\left(-\int_0^t h(s)ds\right). \tag{3.9}$$

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In particular, assuming that $h(t) = h > 0$ is constant, one obtains

$$P\left(\tau \geq t\right) = e^{-ht} \quad \text{and} \quad f_\tau(t) = he^{-ht}. \tag{3.10}$$

By substituting (3.8) into (3.7), it follows that

$$
\begin{aligned}
\mathbb{H}(0,T) &= \pi(0) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T {}_u p_x \frac{1}{b(u)} P\left(\tau \geq u\right) c(u) du\right] \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T {}_u p_x \frac{1}{b(u)} P\left(\tau \geq u\right) + {}_u p_x \frac{1}{b(u)}(1 - w(u))f_\tau(u)\right) Bal(u) du\right] \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T {}_u p_x \frac{1}{b(u)} P\left(\tau \geq u\right) + {}_u p_x \frac{1}{b(u)}(1 - w(u))f_\tau(u)\right) Loss(u) du\right] \\
&= \pi(0) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T {}_u p_x \frac{1}{b(u)} \exp\left(-\int_0^u h(s)ds\right) c(u) du\right] \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (h(u)(1 - w(u)) + \mu_u) \, {}_u p_x \frac{1}{b(u)} \exp\left(-\int_0^u h(s)ds\right) (Bal(u) + Loss(u)) \, du\right].
\end{aligned}
\tag{3.11}
$$

The pricing equation (3.11) reduces the problem of pricing a defaultable reverse mortgage contract to that of pricing non-defaultable one with an altered discount rate and cash flow claims. In particular, the corresponding expression for a non-defaultable reverse mortgage contract can be simplified as

$$
\begin{aligned}
\mathbb{H}(0,T) &= \pi(0) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T {}_u p_x \frac{1}{b(u)} c(u) du\right] \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\int_0^T {}_u p_x \mu_u \frac{1}{b(u)} Bal(u) du\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T {}_u p_x \mu_u \frac{1}{b(u)} Loss(u) du\right]. \tag{3.12}
\end{aligned}
$$

The three expectations in (3.12) represent the cash flows for a non-defaultable RM contract. The first expectation accounts for the borrower's receipt of the annuity loan payment when he is alive. The second expectation represents the return of the borrower's non-negative balance amount for his death and the corresponding premiums, which are payable to the issuer for the crossover risk due to the contract termination, can be represented by the third expectation.

3.4 Model Specifications and Assumptions

In this section, we propose a model to evaluate the defaultable RM contract in (3.11). In our problem, RM lenders face four main sources of risk: house price risk, interest rate risk, mortality risk and default risk in a reverse annuity mortgage. We incorporate these risk factors in the valuation of an RM contract. Next, we specify the corresponding models for these risk factors to provide an analytical formula of our defaultable RM contract.

3.4.1 House Price Dynamics

The existing literature pertaining to house price modelling suggests two perspectives. First, discrete time models assume that the house price returns exhibit autocorrelation (Case and Shiller (1989) and Ito and Hirono (1993) for example) and generalized autoregressive conditional heteroscedasticity (e.g. Nothaft, Gao, and Wang (1995), Chen, Cox, and Wang (2010) and Li, Hardy, and Tan (2010)). Second, continuous time models employ traditional geometric Brownian motion or a jump diffusion model to capture the dynamics of house prices (see Kau, Keenan, Muller, and Epperson (1992), Bardhan, Karapandža, and Urošević (2006) and Lee, Wang, and Huang (2012)).

We assume that the logarithm of outstanding RM contract value $x(t) = \ln \tilde{H}(t)$ follows a geometric Brownian motion. Under the risk-neutral measure $\mathbb{Q}$, the model is then written in the following way:

$$dx(t) = \left(r - \delta - \frac{1}{2}\sigma_x^2 \right) dt + \sigma_x dW_x(t), \tag{3.13}$$

where r is the constant risk-free rate, δ is the constant rental rate (or maintenance yield) for the house, σ_x is the volatility of house prices, and $W_x(t)$ is a Wiener process under the risk-neutral measure $\mathbb{Q}$. We model $\tilde{H}(t)$ as a geometric Brownian motion, which seems a simple and analytically tractable yet adequate choice commonly used in the literature such as Bardhan, Karapandža, and Urošević (2006).

3.4.2 Mortality Models

As mortality rates continue to improve, longevity risks are critical. In recent years, financial markets have produced solutions by providing a variety of securities with payoffs tied to certain mortality or longevity indexes and reverse mortgage is typically a good example. To implement the no arbitrage approach on pricing a mortality-linked product, the first

step is to estimate the distribution of future mortality rates in the real-world probability measure. Then the real-world distribution is transformed to its risk-neutral counterpart. Modelling for the mortality rate and the survival probability can be widely selected. For instance, in the constant force of mortality model we assume the mortality rate $\mu_x = \mu$, $x > 0$, and thus the survival probability $_tp_x = e^{-\mu t}$. In the Makeham model, $\mu_x = A + BC^x$ and $_tp_x = S^t g^{C^x(C^t-1)}$, in which A, B and C are constant, $S = e^{-A}$ and $g = \exp(-B/\ln C)$. In the Gompertz model, the mortality $\mu_x = \alpha\beta^x$, such that the survival probability $_tp_x = \exp\left(\frac{-\alpha}{\ln\beta}\beta^x(\beta^t - 1)\right)$. In this chapter, we focus on static mortality models such as constant force of mortality and the Makeham model for the purpose of illustration. Moreover, our pricing of defaultable reverse mortgages can also be extended to use any stochastic mortality models such as the Lee-Carter model and the Cairns-Blake-Dowd (CBD) stochastic mortality model, which are defined in the real-world measure and fitted to past data.

3.4.3 Other Assumptions and Pricing Formula

In the literature, the hazard rate $h(t)$ can be modeled as a stochastic process, either a mean-reverting process or a function associated with other stochastic variables. In particular, the default follows a time-homogeneous Poisson process when the hazard rate h is constant. A time-inhomogeneous Poisson process is generalized by assuming that the hazard rate takes a deterministic form of $h(t)$, which is commonly used for the term structure of credit spreads. Cox process is usually used for modeling the stochastic hazard rate. Our goal is to find the annuity payment $c(t)$ of reverse mortgages in the presence of default risk (refers to the level of hazard rate $h(t)$). For the purpose of illustration, we investigate with a constant annuity payment rate $c(t) = c$ under the assumption of a constant hazard rate $h(t) = h$. With the constant annuity payment rate c, the accumulated annuity loan payment at time t is given by

$$L(t) = c \times \int_0^t e^{\int_s^t (r+\pi_r)du}ds = c \times \left(\frac{e^{(r+\pi_r)t} - 1}{r + \pi_r}\right),$$

where r is the risk-free rate and π_r is the interest rate spread. We also assume that the recovery rate satisfies $w(t) = w$ for the defaultable reverse mortgage contracts and there is no rental rate or maintenance yield of the house so that $\delta = 0$. We hereby revise the

pricing formula in (3.11) in the following way

$$
\begin{aligned}
\mathbb{H}(0,T) \;=\; & \pi(0) + c \times \int_0^T {}_u p_x e^{-(h+r)u} du \\
+\; & \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (h(1-w)+\mu_u)\,{}_u p_x e^{-hu} e^{-ru}\left(Bal(u)+Loss(u)\right)du\right], \quad (3.14)
\end{aligned}
$$

where ${}_t p_x$ and μ_t follow the selected law of mortality.

3.5 Characterizing Defaultable Reverse Mortgage Prices

In this section, we further develop the pricing formula in (3.14) and determine the fair annuity payment by introducing default risk for the existing HECM program. Then, we study the effect of the fair annuity payment on the risk factors using the proposed formula.

Based on the model assumptions in Section 3.4, the payoffs of $Bal(u)$ and $Loss(u)$, independent of other model parameters in (3.14), are the only random sources. Thus, the expectations on the discounted payoffs of $Bal(u)$ and $Loss(u)$ can be characterized as call and put options respectively, with the underlying $\tilde{H}(u)$ and the deterministic strike $L(u)$ in the Black-Scholes setting. We then evaluate the corresponding prices of call and put options as follows:

$$
\begin{aligned}
\mathcal{C}\left(\tilde{H}(0), L(u), 0, u\right) : \;=\; & e^{-ru}\mathbb{E}^{\mathbb{Q}}\left[Bal(u)\right] = e^{-ru}\mathbb{E}^{\mathbb{Q}}\left[\left(\tilde{H}(u)-L(u)\right)^+\right] \\
=\; & \tilde{H}(0)e^{-\delta u}\Phi\left(d_1\right) - c \times \left(\frac{e^{\pi_r u} - e^{-ru}}{r+\pi_r}\right)\Phi\left(d_2\right), \quad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{P}\left(\tilde{H}(0), L(u), 0, u\right) : \;=\; & e^{-ru}\mathbb{E}^{\mathbb{Q}}\left[Loss(u)\right] = e^{-ru}\mathbb{E}^{\mathbb{Q}}\left[\left(L(u)-\tilde{H}(u)\right)^+\right] \\
=\; & c \times \left(\frac{e^{\pi_r u} - e^{-ru}}{r+\pi_r}\right)\Phi(-d_2) - \tilde{H}(0)e^{-\delta t}\Phi(-d_1), \quad (3.16)
\end{aligned}
$$

where $d_1 = \frac{\ln\left(\frac{\tilde{H}(0)}{L(u)}\right)+(r+\frac{\sigma_x^2}{2})u}{\sigma_x\sqrt{u}}$ and $d_2 = d_1 - \sigma_x\sqrt{u}$. Φ is the standard normal cumulative distribution function.

We thus have proven the following Proposition 6, which provides the solution for evaluating the initial contract value $\mathbb{H}(0,T)$. Note that the annuity payment rate c also appears inside the pricing formulas for the calls and puts defined in (3.15) and (3.16) respectively. By equating $\mathbb{H}(0,T)$ with the house price $H(0)$ at time 0, we find out the fair annuity payment c given the level of default risk for the reverse annuity mortgage contract.

Proposition 6. The following pricing result of the reverse mortgage $\mathbb{H}(0,T)$ is obtained by (3.11), equation (3.15) and equation (3.16):

$$
\begin{aligned}
\mathbb{H}(0,T) &= \pi(0) + c \times \int_0^T {}_u p_x e^{-(h+r)u} du \\
&\quad + \int_0^T (h(1-w) + \mu_u)_u p_x e^{-hu} \left(\mathcal{C}(\tilde{H}(0), L(u), 0, u) + \mathcal{P}(\tilde{H}(0), L(u), 0, u) \right) du.
\end{aligned}
$$
(3.17)

The fair regular annuity payment rate c is determined by equating the contract value $\mathbb{H}(0,T)$ with the house price $H(0)$ at the inception of the contract at time 0.

Based on Proposition 6, we determine the fair RM annuity payment c in accordance with the borrower's level of default risk (refer to the level of the hazard rate h). We use the parameters $r = 0.01$, $\pi_r = 0.02$, $\delta = 0$, $\sigma_x = 0.8$, $H(0) = 1,000,000$, $T = 25$, $\mu = 0.01$, $\pi(0) = 100,000$, $w = 0.1$, $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $C = 1.124$ and age $x = 62$. The above set of parameters is used for the purpose of illustration. We can verify the results by the Implicit Function Theorem with a broad range of parameters in the Appendix. In Figure 3.3, we find that in both the constant force of mortality and Makeham models, the fair payment c first decreases and then increases by h. Note that we will ignore the increasing phase because the increasing amount of payment inspires the moral hazard from the borrower with high-default risk and he will stay in the program longer to get more loans. In practice, the lender will not accept the high-default profiles to the program. In the decreasing phase, the lender offers a smaller amount of loan payment by reducing the annuity payment rate to the borrower in higher-default profile.

Empirical studies such as Moulton and Haurin (2015) show that larger initial withdrawals leads to higher default risk in the RM contract. To secure the HECM program's solvency, HUD now restricts the amount that a borrower can withdraw as a lump sum at the inception of RM loans to 60 percent of the initial collateral house value. Our results are consistent with the evidence and provide further implications among the default risk, loan payment and other risk factors. We summarize the findings as follows:

(1) In Figure 3.4, the hazard rate h increases by the amount of initial withdrawal $\pi(0)$ given the level of annuity loan payments satisfying Proposition 6.

(2) In Figure 3.5, the hazard rate h is decreasing in the level of the interest rate r given the level of annuity loan payments satisfying Proposition 6.

(3) In Figure 3.6, we take the constant force of mortality as an example and investigate the relation between the mortality and the annuity payment given the levels of the hazard rate. The annuity payment c is first decreasing and then increasing with respect to the force of mortality μ.

From the above observation (1), the larger initial withdrawal implies a higher level of potential default risk by the borrower. This verifies that the policy restriction on the initial withdrawals by HUD is an effective approach to secure a certain level of the default risk in the HECM program. The outstanding RM contract $\tilde{H}(t)$ becomes less valuable after a larger initial withdrawal. Then, given a fixed annuity loan payment schedule, the payoff of the crossover risk in (3.1) becomes more costly, which encourages the borrower to terminate the contract early by default, for reducing the premiums periodically taken from his collateral house. The observation (2) implies that the default risk is potentially low when the market risk-free rate hikes. In the market of high interest rate, the loan payments from reverse mortgages will accumulate fast. The borrower is willing to stay longer in the program to receive stable income under the insufficient market liquidity. This potentially reduces the chance of default by the borrower, but also it may increase the crossover risk to the insurer at maturity. On the other hand, the borrower is willing to default when the interest rates are low, at which time the loan payments from the reverse mortgages will accumulate slowly. Therefore, the lender needs to be more cautious on the solvency of the HECM program when the default occurs in this period. For the observation (3), we ignore the increasing phase assuming that the lender is unwilling to offer the RM to the high-mortality risk profile. Based on the findings by Nakajima and Telyukova (2017) that the mortality risk (refers to health condition) is one of the most influential factors that drive seniors to take out a reverse mortgage, the decreasing phase indicates that the borrower will be authorized smaller annuity loan payments when the borrower is more likely to die or in poor health condition. Such loan arrangement relieves the hedging difficulty for the lender in the event of the borrower's death or poor health condition. The mortality works the same pattern as the hazard rate to the annuity payment rate because both of them can be counted as the risk sources of termination in reverse mortgages.
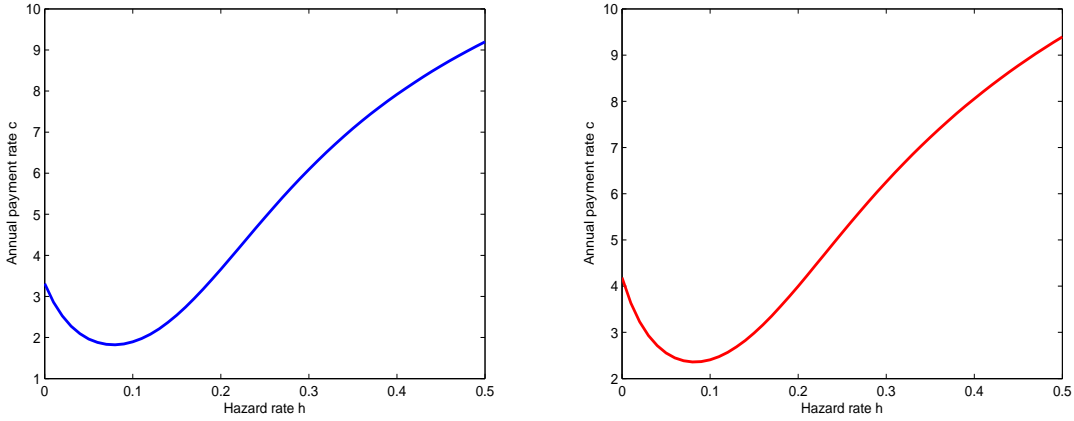
Figure 3.3: Relations between annual payment c in ten thousands and hazard rate h under the constant force of mortality (left panel) and the Makeham model (right panel) respectively.
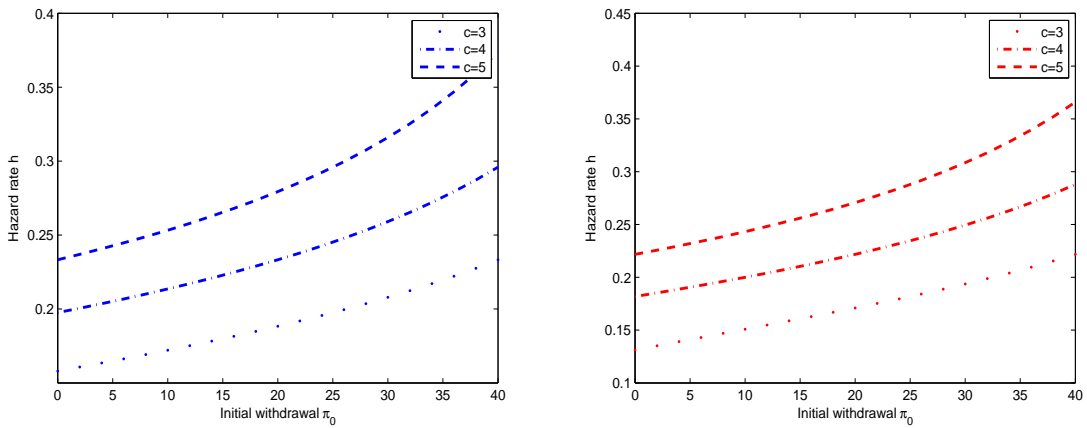
Figure 3.4: Relations between hazard rate h and initial withdrawal $\pi(0)$ in ten thousands with different annual payment level c under the constant force of mortality (left panel) and the Makeham model (right panel) respectively with the parameters $\pi_r = 0.02$, $\delta = 0$, $\sigma_x = 0.8$, $H(0) = 1,000,000$, $T = 25$, $\mu = 0.01$, $w = 0.1$, $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $C = 1.124$ and age $x = 62$.
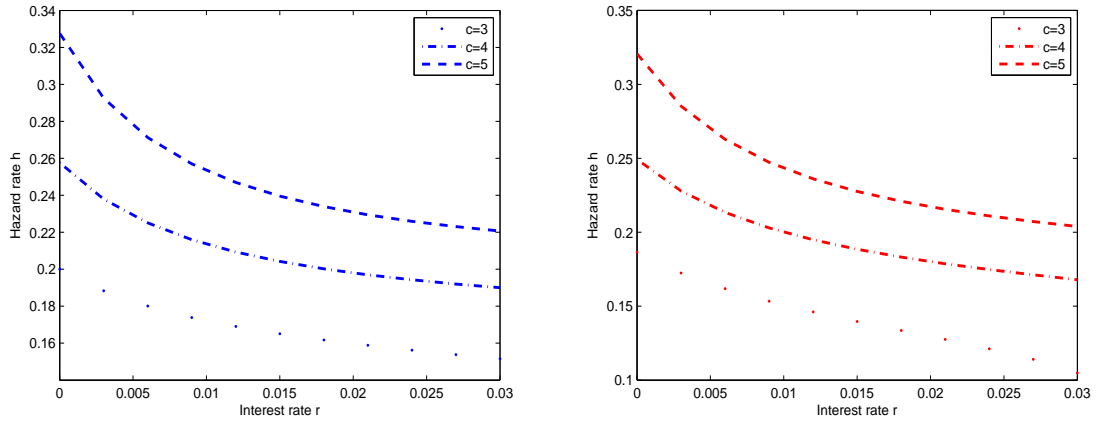
Figure 3.5: Relations between hazard rate h and interest rate r with different annual payment level c in ten thousands under the constant force of mortality (left panel) and the Makeham model (right panel) respectively with the parameters $\pi_r = 0.02$, $\delta = 0$, $\sigma_x = 0.8$, $H(0) = 1,000,000$, $T = 25$, $\mu = 0.01$, $\pi(0) = 100,000$, $w = 0.1$, $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $C = 1.124$ and age $x = 62$.
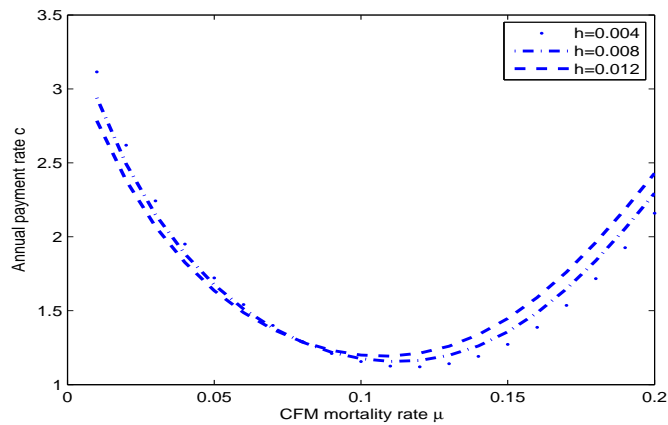
Figure 3.6: Relations between constant force of mortality μ and annual payment level c in ten thousands with different default levels h with the parameters $r = 0.01$, $\pi_r = 0.02$, $\delta = 0$, $\sigma_x = 0.8$, $H(0) = 1,000,000$, $T = 25$, $\pi(0) = 100,000$ and $w = 0.1$.

3.6 Determination of the Hazard Rates

In the previous sections, we have described a new product design for the reverse annuity mortgages with fixed default probabilities under the risk-neutral probability measure $\mathbb{Q}$. To enhance the application on the proposed pricing, we design a default rating scheme of projecting the actual hazard rate in Section 3.6.1. In Section 3.6.2, we propose a parametric estimation to calibrate the actual hazard rate through the market experience. In Section 3.6.3, we make use of a distortion operator such as the Wang transform to create a risk-neutral measure $\mathbb{Q}$ from the physical measure $\mathbb{P}$ and thus obtain a risk-neutral hazard rate that will be used for pricing from the calibrated actual one.

3.6.1 Design for Default Rating

Denote by $h^{\mathbb{P}}$ and $h^{\mathbb{Q}}$ the hazard rates under the physical and risk-neutral measures respectively. We propose to determine $h^{\mathbb{P}}$ in reverse mortgage contracts from certain rating categories by the lender. This means the lender can group similar defaultable reverse mortgage contracts based on a thorough credit rating of the various elements of the transaction to assess the borrowers' default risks. Then, we find $h^{\mathbb{P}}$ based on the default history under the specific category that the target borrower's rating lies in. A credit rating is an evaluation of the credit risk of a prospective debtor (an individual, a business, company or a government), predicting their ability to pay back the debt, and an implicit forecast of the likelihood of the debtor defaulting. The current credit ratings in reverse mortgage markets mainly focus on assessing the likelihood of full repayment from the lender of the original loans and any accrued interest when the home is due (for example, see the DBRS Rating process[1] and Moody's global methodology[2] for rating reverse mortgage securitizations). As HECM program begins improving the underwriting criteria for selection of the default risk profile, it is also possible for rating agencies to serve the needs of market participants by default ratings for the chance of foreclosure, depending on their credit profile with relevant risk components such as credit score, prior delinquency on mortgage debt, the presence of a prior tax lien, and the property tax burden as suggested by Moulton and Haurin (2015).

The reverse mortgage lender may collect the default experience from a similar risk category and then calibrate the default probabilities based on the market data. In Table

---

[1]The introduction to DBRS rating system is available at http://www.dbrs.com/research/230124/rating-reverse-mortgage-transactions-in-canada-archived.pdf.

[2]The introduction to Moody's rating system is available at https://www.moodys.com/research/Moodys-publishes-global-methodology-for-rating-reverse-mortgage-securitizations--PR_325140.

[3.1](), we have simulated ten censored observations in 25 years life-long reverse mortgage contracts under a rating category "B". The durations of both uncensored and censored observations are presented, in which "1" accounts for the status that the default occurs before the end of the contract, and otherwise, the status is noted by "0".

Reverse Mortgage Contracts in Rating Category "B"		
Contract Number	Duration	Default Status
1	25	0
2	21	1
3	23	1
4	25	0
5	20	1
6	25	0
7	23	1
8	25	0
9	25	0
10	25	0

Table 3.1: Simulated censored observations on reverse mortgage contracts in rating category "B". "1" for contract with default and "0" for one without default.

3.6.2 Maximum Likelihood Estimation for $h^{\mathbb{P}}$

Next we use the maximum likelihood estimation (MLE) method for estimation on the constant hazard rate $h^{\mathbb{P}}$. The MLE method is well known in the literature of survival analysis such as McLachlan and McGiffin (1994). Given the observations, it aims to find the parameter values that maximize the likelihood function. For uncensored observations, the probability of a contract that defaults at the end of its duration is given by the probability density function $f(t)$ evaluated at time t_i. For censored observations, the probability of a duration greater than t_i is given by the survival function $S(t)$ evaluated at time t_i. Now suppose that we have r uncensored observations and $n - r$ censored observations with parameters θ. Then, we write the likelihood function as

$$\mathcal{L}(\theta; t_1, \cdots, t_n) = \prod_{i=1}^{r} f(t_i|\theta) \prod_{i=r+1}^{n} S(t_i|\theta).$$

In this section, we estimate solely the hazard rate by MLE, which is independent of the borrower's mortality rate to be estimated in Section 3.6.3. For a constant hazard rate $h(t) = h$, the default density function is $f(t) = he^{-ht}$ and the survival function is $S(t) = e^{-ht}$. The likelihood function for the data in Table 3.1 is $\mathcal{L}(h) = e^{-h\cdot 25} \times he^{-h\cdot 21} \times \cdots \times e^{-h\cdot 25} = h^4 e^{-h\cdot 237}$. By solving for equation $d\mathcal{L}/dh = 0$, we obtain $h^{\mathbb{P}} = 0.0169$ that maximizes the likelihood function.

3.6.3 From $h^{\mathbb{P}}$ to $h^{\mathbb{Q}}$

In Section 3.6.2, we discuss a method of estimation for the hazard rate $h^{\mathbb{P}}$. To address the pricing issue with the hazard rate $h^{\mathbb{Q}}$ under the risk-neutral measure, we propose to make use of a distortion operator such as the Wang transform (Wang (2000)) to create a risk-neutral measure $\mathbb{Q}$, under which the mortality and default-linked reverse mortgages can be priced. Let $\Phi(x)$ be the standard normal cumulative distribution function with probability density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

for all x. Wang (2000) define the distortion operator as

$$g_\lambda(t) = \Phi\left[\Phi^{-1}(t) - \lambda\right],$$

for $0 < t < 1$. Given a distribution with cumulative distribution function $F(x)$, a "distorted" distribution $F^*(x)$ is determined by λ according to the equation

$$F^*(x) = g_\lambda\left(F(x)\right),$$

where the parameter λ is called the market price of risk, reflecting the systematic risk of an insurer's liability X. Thus, the Wang transform will produce a "risk-adjusted" distribution F^* for a given liability X.

To life insurers, an adverse mortality experience means the insureds die earlier than expected. In order to account for a risk premium, Lin and Cox (2005) first employ the Wang transform to adjust the estimated physical survival probabilities for the risk-neutral ones by

$$_tq_x^{\mathbb{Q}} := \Phi\left(\Phi^{-1}\left(_tq_x^{\mathbb{P}}\right) - \lambda_m\right),$$

where $_tq_x := 1 - _tp_x$ is the t-year mortality probability for a x-year old, i.e. the probability for an individual aged x to die within the next t years, and λ_m is the market price of mortality risk from the Wang transform. For the transformation to be of good use, the

mortalities have to shift downwards, meaning that under the distorted mortalities, people live longer. This is obtained for $\lambda_m > 0$.

In order to find a suitable transform, Lin and Cox (2005) derive λ_m from market prices of annuities: using the current yield curve and physical survival probabilities, the hypothetical value $\ddot{a}$ of an annuity contract paying an amount of K annually is derived as a function of the transform parameter λ_m, i.e.

$$\ddot{a}_x(\lambda_m) = K \sum_{t=1}^{\infty} {}_t p_x^{\mathbb{Q}} P(0,t) = K \sum_{t=1}^{\infty} \left(1 - \Phi \left(\Phi^{-1} \left({}_t q_x^{\mathbb{P}} \right) - \lambda_m \right) \right) P(0,t).$$

Here, $P(0,t)$ denotes the current (time 0) value of a zero coupon bond with maturity t. λ_m is determined by equating the hypothetical value and the actual market price of the annuity. For example, Lin and Cox (2005) use the 1995 market quotes of immediate annuities based on the 1995 U.S. Buck Annuity Mortality Tables and the 1995 Treasury yield curve to get the market price of mortality risk λ_m for the insured aged (65). The market prices of mortality risk are 0.1479 for male annuitants and 0.2024 for female annuitants. Bauer and Ruß (2006) use synthetic German annuity data based on the mortality tables provided by the German society of actuaries and the German yield curve of December 2005 to get $\lambda_m \approx 0.42$ for aged (50). By the estimated λ_m, we thus obtain ${}_t q_x^{\mathbb{Q}}$ and ${}_t p_x^{\mathbb{Q}}$. Assuming we fit the real-world mortality rate with a constant one $\mu^{\mathbb{P}}$, the risk-neutral mortality rate

$$\mu_t^{\mathbb{Q}} = -\frac{1}{{}_t p_x^{\mathbb{Q}}} \frac{d}{dt} {}_t p_x^{\mathbb{Q}} = \frac{\mu^{\mathbb{P}} e^{-\mu^{\mathbb{P}} t} \phi(A_1)}{[1 - \Phi(A_1)] \phi [\Phi^{-1}(B_1)]}, \tag{3.18}$$

where $A_1 = \Phi^{-1}(B_1) - \lambda_m$ and $B_1 = {}_t q_x^{\mathbb{P}} = 1 - e^{-\mu^{\mathbb{P}} t}$.

Similarly to the survival probabilities, we introduce the market price of default risk by applying the Wang transform to the physical default probabilities by

$$_t \tilde{q}_x^{\mathbb{Q}} := \Phi \left(\Phi^{-1} \left({}_t \tilde{q}_x^{\mathbb{P}} \right) + \lambda_d \right),$$

where ${}_t \tilde{q}_x := 1 - {}_t \tilde{p}_x$ is the t-year default probability for a x-year old, i.e. the probability for an individual aged x to default within the next t years, and λ_d is the market price of the default risk in the Wang transform. We expect that a positive risk premium λ_d should be rewarded to the lender for bearing the default risk so that the default probability under the risk-neutral measure $\mathbb{Q}$ is larger than the one under the real-world measure $\mathbb{P}$. By the above transform, we also obtain ${}_t \tilde{q}_x^{\mathbb{Q}}$ and ${}_t \tilde{p}_x^{\mathbb{Q}}$. Given a constant real-world hazard rate $h^{\mathbb{P}}$,

the risk-neutral hazard rate is given by

$$h_t^{\mathbb{Q}} = -\frac{1}{{}_t\tilde{p}_x^{\mathbb{Q}}}\frac{d}{dt}{}_t\tilde{p}_x^{\mathbb{Q}} = \frac{h^{\mathbb{P}}e^{-h^{\mathbb{P}}t}\phi(A_2)}{[1 - \Phi(A_2)]\,\phi\left[\Phi^{-1}(B_2)\right]}, \tag{3.19}$$

where $A_2 = \Phi^{-1}(B_2) + \lambda_d$ and $B_2 = {}_t\tilde{q}_x^{\mathbb{P}} = 1 - e^{-h^{\mathbb{P}}t}$.

To determine the proposed λ_d, we plug ${}_tq_x^{\mathbb{Q}}$, ${}_tp_x^{\mathbb{Q}}$, ${}_t\tilde{q}_x^{\mathbb{Q}}$, ${}_t\tilde{p}_x^{\mathbb{Q}}$, $\mu_t^{\mathbb{Q}}$, $h_t^{\mathbb{Q}}$ in equation (3.17), and thus it can be rewritten as

$$
\begin{aligned}
\mathbb{H}(0, T) &= \pi(0) + c_d \cdot \int_0^T {}_up_x^{\mathbb{Q}}\left[1 - \Phi\left(\Phi^{-1}\left({}_u\tilde{q}_x^{\mathbb{P}}\right) + \lambda_d\right)\right]e^{-ru}du \\
&+ \int_0^T (h_u^{\mathbb{Q}}(1 - w)) + \mu_u^{\mathbb{Q}}){}_up_x^{\mathbb{Q}}\left[1 - \Phi\left(\Phi^{-1}\left({}_u\tilde{q}_x^{\mathbb{P}}\right) + \lambda_d\right)\right] \\
&\times \left(\mathcal{C}(\tilde{H}(0), L(u), 0, u) + \mathcal{P}(\tilde{H}(0), L(u), 0, u)\right)du,
\end{aligned}
\tag{3.20}
$$

where ${}_t\tilde{q}_x^{\mathbb{P}} = 1 - e^{-h^{\mathbb{P}}t}$, the constant risk-free rate $r = -\frac{\ln P(0,T)}{T}$ and $P(0, T)$ is the price of a zero-coupon bond with maturity T. c_d is the market-agreed annuity payment rate, reflecting the implied market price of default risk λ_d for the defaultable RM contract.

We assume a constant market price of default risk λ_d. By equating the initial contract value of the RM, $\mathbb{H}(0, T)$, with the initial (market) house price $H(0)$, one obtains λ_d that corresponds to a particular hazard rate $h^{\mathbb{P}}$ with a single market-agreed annuity payment rate c_d. When λ_d is determined, we can project a more customized market-agreed annuity payment c_d to the borrower by matching different hazard rates $h^{\mathbb{P}}$ obtained by MLE in Section 3.6.2. For example, we obtain $\lambda_d = 1.1463$ by using a single market-agreed annuity payment rate of $c_d = 25,000$ when $h^{\mathbb{P}} = 0.0169$, with the parameters $r = 0.01$, $\pi_r = 0.02$, $\delta = 0$, $\sigma_x = 0.8$, $H(0) = 1,000,000$, $T = 25$, $\pi(0) = 100,000$ and $w = 0.1$. One may doubt if the resulting constant λ_d is adequate for pricing products over different default-risk profiles, of which λ_d may change with the level of $h^{\mathbb{P}}$. If this is the case, we can modify λ_d as a convex and decreasing function of $h^{\mathbb{P}}$ to capture some economic relation between the risk-neutral and actual default probabilities (see for example Heynderickx, Cariboni, Schoutens, and Smits (2016)).

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3.7 Concluding Remarks

Reverse mortgages carry both benefits and drawbacks for homeowners. Senior homeowners who do not intend to stay in their house over the long term or who have difficulty in maintaining their home may not be ideal candidates for a reverse mortgage. In this chapter, we approach the pricing problem from a different angle by considering the reverse annuity mortgage as a long-term credit product to the borrower that allows the introduction of modelling the default risk in pricing. We thus have developed a valuation model for determining the annuity loan payment to the borrowers based on the levels of their default risk. To enhance the application of pricing, we have also introduced a default rating scheme and proposed techniques for no-arbitrage valuation on default risk using the real market data. The pricing is expected to reduce the unqualified candidates with high default risks from the current HECM program. We illustrate with numerical examples the effect of the proposed pricing formula on the default risk and the other risk factors. Further implications based on the findings are summarized as follows:

1. The proposed pricing will encourage the lenders to offer more RM products in accordance with the borrower's default risk profile. It is designed to reduce the lender's hedging difficulty for the crossover risk by shifting the default risk to the insured.

2. Our new product design will also be attractive for the reverse mortgage borrower with a better default-risk profile compared to the current market offers that do not differentiate the loan payment from the levels of borrowers' risk profiles.

3. Our pricing scheme supports the policy effectiveness from HUD by limiting the initial withdrawals from the HECM program. It also provides policy implications in terms of market interest rates and mortality levels.

4. HUD now begins to require the lender to verify that the borrower can afford the program by a financial assessment. Our proposed default rating scheme aims to be a supplement assessment and to enhance the application of our proposed pricing.

5. The mechanism of the proposed annuity loan payment in this chapter can take effect voluntarily to the market. Therefore, HUD may use the "pricing" to secure the default levels of the HECM program.

Despite our pricing scheme designed for reducing the lender's hedging difficulty, it cannot replace the existing hedging program. The defaultable reverse mortgage contracts

have the payoffs conditional on the borrower's credit risks. An analysis of optimal hedging will be required for such contracts in a setting where the payoff and the conditioning event(s) are dependent and have to be priced accordingly. The optimal static hedging option in Chapter 4 can be a natural solution for hedging the RM contracts embedded with more complex dependency.

APPENDIX

Verification of the Patterns in Figure 3.3, Figure 3.4, Figure 3.5, and Figure 3.6

We now study the relation among the annuity loan payment rate, initial withdrawal, hazard rate, force of mortality and interest rate. The relations have been illustrated by the patterns from Figure 3.3 to Figure 3.6, which provide important implications of the proposed pricing formula in Proposition 6. Below we use the Implicit Function Theorem to verify that such patterns are consistent with a broad range of parameters. For the purpose of illustration, we assume a constant force of mortality, $\mu_u = \mu$ and $_up_x = e^{-\mu u}$. We are interested in the derivatives $\frac{\partial c}{\partial h}$, $\frac{\partial h}{\partial \pi(0)}$, $\frac{\partial c}{\partial \mu}$, and $\frac{\partial h}{\partial r}$. According to the pricing formula in Proposition 6, denote by the implicit equation

$$
\begin{aligned}
0 \;=\;& f\left(h, c, \pi(0), \mu\right) := -\mathbb{H}(0, T) + \pi(0) + c \times \int_0^T {}_up_x e^{-(h+r)u} du \\
&+ \int_0^T (h(1-w) + \mu)_u p_x e^{-hu} \left(\mathcal{C}(\tilde{H}(0), L(u), 0, u) + \mathcal{P}(\tilde{H}(0), L(u), 0, u) \right) du.
\end{aligned}
$$

Then, we calculate the partial derivatives of the function f with respect to h, c, $\pi(0)$, μ and r:

$$
\begin{aligned}
\frac{\partial f}{\partial h} \;=\;& -c \times \int_0^T {}_up_x e^{-(h+r)u} u\, du \\
&+ \int_0^T \left[1 - w - (h(1-w) + \mu)\, u \right] e^{-hu}{}_up_x \left(\mathcal{C}(\tilde{H}(0), L(u), 0, u) + \mathcal{P}(\tilde{H}(0), L(u), 0, u) \right) du.
\end{aligned}
$$

$$\frac{\partial f}{\partial c} = -\int_0^T {}_u p_x e^{-(h+r)u} du + \int_0^T \left(h(1-w)+\mu\right) e^{-hu} {}_u p_x \left(\frac{\partial \mathcal{C}}{\partial L}\frac{\partial L}{\partial c} + \frac{\partial \mathcal{P}}{\partial L}\frac{\partial L}{\partial c}\right) du$$

$$= -\int_0^T {}_u p_x e^{-(h+r)u} du - \int_0^T \left(h(1-w)+\mu\right) e^{-hu} {}_u p_x \left(\left(\frac{\tilde{H}(0)}{L}-1\right)\left(\frac{1}{\sigma_x \sqrt{T}}\right)\right) du.$$

where

$$\frac{\partial \mathcal{C}}{\partial L} = \tilde{H}(0)e^{-\delta u}\phi(d_1)\frac{\partial d_1}{\partial L} - \Phi(d_2) - L\phi(d_2)\frac{\partial d_2}{\partial L},$$

$$\frac{\partial \mathcal{P}}{\partial L} = \Phi(-d_2) - L\phi(-d_2)\frac{\partial d_2}{\partial L} + \tilde{H}(0)e^{-\delta u}\phi(-d_1)\frac{\partial d_1}{\partial L},$$

with $d_1 = \frac{\ln\left(\frac{\tilde{H}(0)}{L(u)}\right)+(r+\frac{\sigma_x^2}{2})u}{\sigma_x\sqrt{u}}$, $d_2 = d_1 - \sigma_x\sqrt{u}$, and Φ and ϕ are the standard normal cumulative distribution function and probability density function respectively.

$$\frac{\partial f}{\partial \pi(0)} = 1 + \int_0^T \left(h(1-w)+\mu\right) e^{-hu} {}_u p_x \left(\frac{\partial \mathcal{C}}{\partial \tilde{H}(0)}\frac{\partial \tilde{H}(0)}{\partial \pi(0)} + \frac{\partial \mathcal{P}}{\partial \tilde{H}(0)}\frac{\partial \tilde{H}(0)}{\partial \pi(0)}\right) du$$

$$= -\int_0^T {}_u p_x e^{-(h+r)u} du - \int_0^T \left(h(1-w)+\mu\right) e^{-hu} {}_u p_x \left(-2\phi(d_1) + \frac{2L}{\tilde{H}(0)}\phi(-d_2)\right) du.$$

where

$$\frac{\partial \mathcal{C}}{\partial \pi(0)} = \frac{\partial \mathcal{C}}{\partial \tilde{H}(0)}\frac{\partial \tilde{H}(0)}{\partial \pi(0)} = -\Phi(d_1) + \tilde{H}(0)e^{-\delta u}\phi(d_1)\frac{\partial d_1}{\partial \pi(0)} - L\phi(d_2)\frac{\partial d_2}{\partial \pi(0)},$$

$$\frac{\partial \mathcal{P}}{\partial \pi(0)} = \frac{\partial \mathcal{P}}{\partial \tilde{H}(0)}\frac{\partial \tilde{H}(0)}{\partial \pi(0)} = -L\phi(-d_2)\frac{\partial d_2}{\partial \pi(0)} + \Phi(-d_1) + \tilde{H}(0)e^{-\delta u}\phi(-d_1)\frac{\partial d_1}{\partial \pi(0)}.$$

$$\frac{\partial f}{\partial \mu} = -c \times \int_0^T e^{-(h+\mu+r)u} u\, du$$

$$+ \int_0^T \left[1 - w - \left(h(1-w)+\mu\right)u\right] e^{-(h+\mu)u} \left(\mathcal{C}(\tilde{H}(0), L(u), 0, u) + \mathcal{P}(\tilde{H}(0), L(u), 0, u)\right) du.$$

$$\frac{\partial f}{\partial r} \;=\; -c \times \int_0^T {}_u p_x e^{-(h+r)u} u\, du + \int_0^T \left(h(1-w) + \mu \right) e^{-hu} {}_u p_x \left(\frac{\partial \mathcal{C}}{\partial r} + \frac{\partial \mathcal{P}}{\partial r} \right) du.$$

For any $T \geq 5$, we find that the signs of all the above derivatives hold for a broad range of parameters with $0 \leq h \leq 0.2$, $0 \leq \mu \leq 0.2$, $0 \leq r \leq 0.1$, $0 \leq \pi_r \leq 0.3$, $0 \leq \delta \leq 0.1$, $0 < \sigma_x \leq 1$, $0 \leq \pi(0) \leq 500,000$. We have

$$\frac{\partial f}{\partial h} < 0, \quad \frac{\partial f}{\partial c} \;\; \text{first} < 0 \text{ and then} > 0, \quad \frac{\partial f}{\partial \pi(0)} > 0, \quad \frac{\partial f}{\partial \mu} < 0, \quad \frac{\partial f}{\partial r} < 0,$$

and

$$\frac{\partial c}{\partial h} \;=\; -\frac{\partial f}{\partial h} \Big/ \frac{\partial f}{\partial c} \quad \text{first} < 0 \text{ and then} > 0, \qquad \frac{\partial h}{\partial \pi(0)} \;=\; -\frac{\partial f}{\partial \pi(0)} \Big/ \frac{\partial f}{\partial h} > 0,$$

$$\frac{\partial c}{\partial \mu} \;=\; -\frac{\partial f}{\partial \mu} \Big/ \frac{\partial f}{\partial c} \quad \text{first} < 0 \text{ and then} > 0, \qquad \frac{\partial h}{\partial r} \;=\; -\frac{\partial f}{\partial r} \Big/ \frac{\partial f}{\partial h} < 0,$$

which verify the patterns in Figure 3.3, Figure 3.4, Figure 3.5, and Figure 3.6.

Chapter 4

Efficient Hedging of Path-dependent Options under the Heston-type Stochastic Volatility

In an incomplete market, a hedger is faced with the problem of searching for strategies which reduce the risk as much as possible. In practice, a superhedging strategy can often be too expensive. For this reason, Föllmer and Leukert (2000) investigate the possibility of investing less capital than the superhedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized. Föllmer and Leukert (2000) used the so-called quantile hedging to determine a portfolio strategy which minimizes the probability of loss and that leads to partial hedges, using the expected loss function as risk measure. The resulting dynamic optimization problem of finding a self-financing strategy that minimizes the shortfall risk can be split into a static optimization problem and a representation problem. The optimal strategy consists in superhedging a modified claim $\tilde{\varphi}H$, where H is the payoff of the claim and $\tilde{\varphi}$ is the solution of the statistical optimization problem, the optimal randomized test.

Kolkiewicz (2016) has developed a general method of constructing static hedging strategies for path-dependent options that minimize the shortfall risk for a given time interval. Assuming that prices are given by the Black-Scholes model, the author first described the hedging risk for a path-dependent option using only a European option. Then, Kolkiewicz (2016) find the hedging option that minimizes the shortfall risk using the expectation of the shortfall weighted by some loss function. The methodology is applicable to the path-dependent contracts but not limited to the Black-Scholes setting, in which the underlying

volatility is assumed to be constant over the life of the derivative. In this Chapter, we construct the optimal static hedging option under the Heston stochastic volatility models, which is well known for well capturing the features of the implied volatility surface and resolving the shortcoming of the Black-Scholes model. Markets are incomplete with respect to payoffs that are not entirely determined by market prices. The presence of the stochastic volatility typically leads to an incomplete market in which perfect hedging strategies do not exist for many contingent claims, depending on the available trading opportunities. In this chapter, the optimal static hedging options adapted to the stochastic volatility will be developed in the spirit of Kolkiewicz (2016), as a benchmark to compare other strategies, such as traditional dynamic hedging and short-dated static hedging. Our hedging option uses a more general assumption for the asset price process that admits the Heston-type stochastic volatility to capture the empirical observations. The corresponding optimal hedging option can also be expected to fulfill the hedging needs for long-term and path-dependent liabilities such as most VA liabilities, reverse mortgages and simplified hedges for path-dependent options that require working with more general forms of dependency among risk factors.

In this chapter, we construct the optimal hedging strategy for a particular GMAB contract with the volatility-dependent fees in the market of Heston-type stochastic volatility. Our method relies on the Broadie-Kaya approach that can be used to improve the theoretical tractability and accuracy for construction of the optimal hedging strategy. The remaining body of this chapter is organized as follows. In Section 4.1, we formulate the optimal hedging problem for a path-dependent GMAB contract by minimizing the shortfall risk at maturity, the fee payment of which is tied with a stochastic equity volatility. In Section 4.2, a framework of Heston-type stochastic volatility is then introduced. In Section 4.2.1, we characterize the path-dependency for the GMAB contract with the equity index and the path of the equity volatility in the Heston model. In Section 4.2.2, we take the GMAB liability put option as an example and present the theoretical results for the optimal hedging strategy for path-dependent options under the Heston model. In Section 4.3.1, we propose to model such path-dependency using the method of Broadie and Kaya (2006). The proposed method enables us to sample the residual risk naturally for the GMAB liability put option conditional on the benchmark index. The likelihood ratio for characterization of the proposed optimal hedging option is derived from the formula of Drăgulescu and Yakovenko (2002) in Section 4.3.2. Numerical examples are illustrated in Section 4.4. Section 4.5 concludes the chapter.

4.1 Static Hedging of a GMAB Contract

We use the following notation: S_t is the spot price of the equity index at time t, $t \in [0, T]$, and F_t is the fund value of a GMAB contract that expires at T, $t \in [0, T]$. The GMAB contract has its liability payoff

$$L := (G_T - F_T)^+ \tag{4.1}$$

which is in the form of a European put option on the fund F_T with the strike G_T at maturity T. Assume that the GMAB fund F fully tracks a non-dividend paying market equity index S with stochastic volatility. The presence of the stochastic volatility of the equity index thus makes the market incomplete. Given that the market volatility-dependent fees are charged from the fund F during the entire life of the contract, we want to hedge the GMAB liability put option in (4.1). The GMAB liability put option L on the fund F is path-dependent because it is jointly linked with the index S and the path of the equity volatility, or equivalently, the path of the equity variance $v_{0,T}$ in $[0, T]$. Our objective is to hedge the path-dependent liability L using only path-independent European-style options on the equity index S with fixed maturity T.

For a given payoff function h, the performance of the corresponding hedging option can be measured by the expected shortfall risk in the form of a power function:

$$\mathbb{E}^{\mathbb{P}} \left[\left((L - h(S_T))^+ \right)^p \right], \tag{4.2}$$

with $p \geq 1$. The hedging option h will be considered optimal for minimizing the expected shortfall risk in (4.2).

Since the GMAB fund F fully tracks the market index S, the liability L and S are dependent. We thus define

$$L(s) := (G_T - F_T^s)^+ \, |S_T = s, \qquad s \in \mathbb{R}^+, \tag{4.3}$$

where we rewrite the fund value as F_T^s, which is conditional on the terminal market index $S_T = s$. We shall refer to these variables $L(s)$ as the residual risk. Let us also define two sets

$$\mathcal{S}_L := \{s \in \mathbb{R}^+ : \text{interior of supp}(L(s)) \text{ is nonempty}\}$$

and

$$\mathcal{H}^0 := \{\text{functions } h \text{ on } \mathcal{S}_L \text{ such that } h(s) \in \overline{\text{supp}\,(L(s))}\},$$

with $\bar{A}$ denoting the closure of a set A and supp() denoting the support of a random variable.

To make the definition of the criterion complete, we define the set of admissible functions h for a given initial budget V_I, which satisfies the following budget constraint

$$\mathcal{H} := \{h \in \mathcal{H}^0 : \mathbb{E}^{\mathbb{Q}}[h(S_T)] \leq V_0\}, \tag{4.4}$$

where $V_0 := V_I \exp(rT)$ and r is the risk-free rate. Note that h is not restricted to be a linear function of the market index S. In practice, the hedger typically will not be able to buy or sell an option whose payoff is equal to the hedging option h for the liability put in (4.1). However, we can approximate a smooth function h closely on bounded intervals by a piecewise linear function in the form

$$\alpha s + \beta + \sum_{i=1}^{m} \alpha_i \left(\gamma_i - s\right)^+,$$

for suitably chosen constants α, β, α_i and γ_i, $i = 1, \cdots, m$. Therefore, a GMAB liability put option in (4.1) can be hedged by the strategy h that is created synthetically through a portfolio of vanilla put options on the market index.

4.2 Optimal Static Hedge under the Heston Model

In this section a framework that admits the Heston-type stochastic volatility is considered for hedging the GMAB contract with volatility-dependent fees. It extends the results of hedging path-dependent options from the Black-Scholes framework to the Heston model in the spirit of Kolkiewicz (2016). In Section 4.2.1, we characterize the path-dependency of a GMAB liability put option with the equity index and the path of equity volatility under the Heston model. In Section 4.2.2, an optimal static hedge is then generalized under the Heston model.

4.2.1 GMAB fund under the Heston Model

As motivated in Chapter 2, here we propose the Heston stochastic volatility model as a natural candidate that accommodates both the pricing and hedging needs for a GMAB contract with volatility-dependent fees. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t, t \geq 0\}$, where $\mathbb{P}$ is the physical probability measure. In the framework

of Heston (1993), the equity index S follows the dynamics

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t d\tilde{W}_x(t), & S_0 > 0, \\ dv_t = \kappa^* \left(\bar{v}^* - v_t \right) dt + \sigma \sqrt{v_t} d\tilde{W}_v(t), & v_0 > 0, \end{cases} \tag{4.5}$$

where μ represents the physical return, and $\tilde{W}_x(t)$ and $\tilde{W}_v(t)$ are two correlated standard Brownian motions under $\mathbb{P}$ with correlation ρ. The constant κ^*, $\kappa^* > 0$, determines the speed of adjustment of the volatility towards its long-run mean $\theta^* > 0$, and $\sigma > 0$ is the volatility of the volatility.

The Heston market is incomplete and thus there exists an infinite number of equivalent martingale measures. For pricing purposes, the risk-neutral measure is obtained by specifying the market price of volatility risk λ. The budget for the optimal hedging problem relies on the pricing of the GMAB contract in the Heston-type stochastic volatility setting. Under the risk-neutral probability $\mathbb{Q}$, the Heston model follows the system of stochastic differential equations

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_x(t), \\ dv_t &= \kappa \left(\bar{v} - v_t \right) dt + \sigma \sqrt{v_t} dW_v(t), \end{aligned}$$

where $W_x(t)$ and $W_v(t)$ are Brownian motions under $\mathbb{Q}$ with $d\tilde{W}_t^1 d\tilde{W}_t^2 = \rho dt$. The process S models the underlying asst price dynamics and v the (stochastic) variance of S. Here, r is the risk-free rate of interest, $\kappa = \kappa^* + \lambda$ is the mean-reversion speed of v, $\bar{v} = \kappa^* \bar{v}^* / (\kappa^* + \lambda)$ is the long-term average variance, and σ is the volatility of v.

In Chapter 2, we determined a linear fee rate structure of $c_t := a + bv_t$ at the stage of pricing a GMAB contract with volatility-dependent fees. Given a volatility-dependent fee rate, the hedger considers how to hedge with a path-dependent GMAB liability put option under its budget constraint. We thus want to characterize such path-dependency for a GMAB liability put option, for which the fund F is fully tracking the equity index S. In the Heston framework of (4.5), the non-dividend equity index at time $t > 0$, given the initial values of S_0 and v_0, can be written as

$$S_t = S_0 \exp \left[\mu t - \frac{1}{2} \int_0^t v_s ds + \rho \int_0^t \sqrt{v_s} d\tilde{W}_v(s) + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} d\tilde{W}_x(s) \right] \tag{4.6}$$

and the variance at time t is given by

$$v_t = v_0 + \kappa^* \bar{v}^* t - \kappa^* \int_0^t v_s ds + \sigma \int_0^t \sqrt{v_s} d\tilde{W}_v(s). \tag{4.7}$$

With the volatility-dependent fee rate $c_t = a + bv_t$, one obtains a GMAB fund at time t

$$F_t = F_0 \exp \left[\left(\mu t - \int_0^t c_s ds \right) - \frac{1}{2} \int_0^t v_s ds + \rho \int_0^t \sqrt{v_s} d\tilde{W}_v(s) + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} d\tilde{W}_x(s) \right]. \tag{4.8}$$

By (4.6) and (4.8), F_t can be represented as

$$F_t = S_t \exp \left(- \int_0^t c_s ds \right) = S_t \exp \left[- \left(at + b \int_0^t v_s ds \right) \right]. \tag{4.9}$$

Note that $F_0 = S_0$. The equation (4.9) indicates that the price of the fund F at time t can be modified by subtracting continuously the volatility-dependent dividend (fee) payments in $[0, t]$ from the equity index price S. It shows that the GMAB contract has path-dependency with the equity index, as well as the path of the variance in $[0, t]$.

By (4.9), we can represent the residual risk L for a GMAB liability put option, conditional on $S_T = s$, as

$$L(s) = (G_T - F_T^s)^+ | S_T = s \tag{4.10}$$

and

$$F_T^s = s \cdot \exp \left(- \int_0^T c_s ds \right) = s \cdot \exp \left[- \left(aT + b \int_0^T v_t^s dt \right) \right], \tag{4.11}$$

where the integrated variance $\int_0^T v_t^s dt$ is conditional on $S_T = s$. We use the symbol v_t^s to account for the variance processes that evolve with the fixed terminal value of the underlying $S_T = s$. F_T^s denotes the terminal fund value F_T conditional on the fixed underlying $S_T = s$, which jointly evolves with the variance path v_t^s in $[0, T]$.

4.2.2   Optimal Hedging Option

To define the optimal hedging option, we first define the set of admissible payoff function h. Let

$$\mathcal{H}_{h_L,h_U} := \{h \in \mathcal{H} : h_L(s) \le h(s) \le h_U(s) \text{ for } s \in \mathcal{S}_L\}, \tag{4.12}$$

where h_L and h_U are given bounding functions. By introducing them, we are able to solve our optimization problem by using the Neyman-Pearson lemma. Because of the budget requirement, the functions h_U and h_L do not form binding constraints in our optimization problem, and as such will have no impact on the optimal h_{opt}. Then, the optimal hedging problem can be regarded as an optimization problem over functions that take values in a bounded interval, by representing each admissible function h from $\mathcal{H}_{h_L,h_U}$ as

$$h = h_L + \gamma\left(h_U - h_L\right),$$

where γ is a function on $\mathcal{S}_L$ with values in $[0, 1]$. The optimal hedging option h_{opt} is then determined as we find out the corresponding optimal ratio $\tilde{\gamma}$. The bounding functions h_L and h_U are given functions that satisfy:

(C1) h_L and h_U belong to $\mathcal{H}^0$ and are continuous.

(C2) $h_L(s) < h_U(s)$ for $s \in \operatorname{supp}(S_T) \cap \mathcal{S}_L$.

(C3) $\mathbb{E}^{\mathbb{Q}}[h_L(S_T)] \le V_0 \le \mathbb{E}^{\mathbb{Q}}[h_U(S_T)] < \infty$.

(C4) $\mathbb{E}^{\mathbb{P}}[(h_U(S_T) - h_L(S_T))^p] < \infty$, where p is defined in the optimization criterion.

Our problem is to find the optimal hedging strategy h_{opt} that minimizes the hedging risk quantified by using the expected shortfall. Thus, the optimal hedging option h_{opt} (corresponding to $\tilde{\gamma}$) is the solution to the optimization problem

$$h_{opt} : = \arg\inf_{h \in \mathcal{H}_{h_L,h_U}} \mathbb{E}^{\mathbb{P}}\left[\left((L(S) - h(S))^+\right)^p\right]. \tag{4.13}$$

The above minimization problem can be rewritten as follows

$$h_{opt} = \arg\min_{\gamma} \mathbb{E}^{\mathbb{P}}\left[(h_U(S) - h_L(S))^p g\left(S, 1 - \gamma(S); p\right)\right], \tag{4.14}$$

where

$$g(s, z; p) := \frac{g_0(s, z(h_U(s) - h_L(s)) - h_U(s))}{(h_U(s) - h_L(s))^p}, \qquad (s, z) \in \mathcal{S}_L \times z \in [0, 1]$$

and

$$g_0(s, z; p) = \mathbb{E}\left[\left((L(s) + z)^+\right)^p\right], \qquad (s, z) \in \mathcal{S}_L \times z \in [-h_U(s), -h_L(s)].$$

We are also assuming the following conditions:

(C5) For each $s \in \mathcal{S}_L$, the function $z \to g(s, z; p)$, is continuously differentiable at any $z \in [0, 1]$, where at the end points $z = 0$ and $z = 1$ we consider one-sided derivatives, and its derivative is strictly increasing.

(C6) For $g_{0,z}(s, z; p) := \partial g_0(s, z; p)/\partial z$, we have

$$\mathbb{E}^{\mathbb{P}}\left[(h_U(S_T) - h_L(S_T)) \, g_{0,z}\left(S_T, -h_L(S_T); p\right)\right] < \infty.$$

It can be easily verified that $g(s, z; p)$ is convex and non-decreasing for each $s \in \mathcal{S}_L$. Under the assumptions (C5) and (C6), the inverse of the function $\partial g(s, z; p)/\partial z$ is well defined and we denote it by

$$l_e(s, y) := \frac{h_U(s) - F^{-1}(1 - y)}{h_U(s) - h_L(s)} 1_{[1-\alpha_H, 1-\alpha_L]}(y) + 1_{1-\alpha_L, \infty}(y). \tag{4.15}$$

We now extend the result of optimal hedging option under the Heston-type stochastic volatility in the spirit of Kolkiewicz (2016), in which the optimal hedging option has been developed in the Black-Scholes framework. We have the following characterization of the optimal option in Theorem 7.

Theorem 7. Suppose that the assumptions $C1 - C6$ are satisfied. Then, in the Heston setting, the solution to the optimization problem in (4.13) can be represented in the form

$$h_{opt}(s) = h_L(s) + \tilde{\gamma}(s) \left[h_U(s) - h_L(s)\right], \tag{4.16}$$

where

$$\tilde{\gamma}(s) = 1 - l_e\left(s, c \cdot (h_U(s) - h_L(s))^{1-p} \frac{d\mathbb{Q}^*}{d\mathbb{P}^*}(s)\right) \tag{4.17}$$

and a unique c is selected so that

$$\mathbb{E}^{\mathbb{Q}}\left[h_{opt}(S_T)\right] = V_0. \tag{4.18}$$

The continuous likelihood ratio $d\mathbb{Q}^*/d\mathbb{P}^*$ is given in (4.25).

The structure of the optimal test $\tilde{\gamma}(s)$ is given by the Neyman-Pearson lemma. The proofs of Theorem 7 and the uniqueness of the constant c follow the characterization of the optimization problem by Kolkiewicz (2016).

4.3 Determination of the Optimal Static Hedge

In practice, one challenge to implement the theoretical result in Theorem 7 is how to find the distribution of conditional residual risk $L(s)$ that we have defined in (4.3). In the Black-Scholes setting, a Brownian bridge technique can be developed so that it is possible to obtain an analytical form of the optimal static hedge for the path-dependent options or simulate the distribution of $L(s)$ with the Brownian bridge technique, conditional on a fixed benchmark value, i.e., $S_T = s$ (see Kolkiewicz (2016) for example). In the Heston framework, the bridge techniques for approximating the distribution of $L(s)$ can be hardly derived due to the complexity of joint distribution between the equity process and its stochastic volatility, and thus the simulation method is expected to be applied. Another challenge for constructing the optimal hedging strategy in the stochastic volatility setting is how to find the likelihood ratio, $d\mathbb{Q}^*/d\mathbb{P}^*$, which reconciles the distributions of the equity index S under both the physical and risk-neutral measures, which is desirable for determining the optimal ratio $\tilde{\gamma}$ by the Neyman-Pearson lemma with the Heston model.

To address these challenges, we propose the following methods:

(1) We generally use the forward simulations such as the Euler scheme to produce the distribution of the residual risk $L(s)$, conditional on $S_T = s$, defined in (4.3). Using Euler discretization, we simulate M paths of the underlying process, jointly with the corresponding process of stochastic variance. Based on M paths of the underlying process jointly with the variance process by forward simulation, we collect a sufficient number of the underlying paths such that the selected values of the terminal underlying S_T are located in a small interval $[s, s + \delta)$, $\delta \to 0$. We then approximate the conditional terminal values F_T^s in (4.11) by taking sufficient numbers of the terminal underlying $S_T \in [s, s + \delta)$, which are associated with the corresponding paths v_t^s for $S_T \in [s, s + \delta)$ in $[0, T]$. Finally we can approximate the distribution of $L(s)$ in (4.10) based on the sufficient number of F_T^s.

(2) To improve the theoretical tractability and accuracy, the method of Broadie and Kaya (2006) is proposed for exact simulation of $\int_0^T v_s ds$ so that the volatility-dependent fees of the GMAB contract can be exactly simulated. We thus produce the vector in (4.22)

that enables us to simulate the residual risk for a GMAB liability put option jointly with its benchmark on the market index S_T. Then, we can approximate the distribution of residual risk by sampling a sufficient number of paths within a small range centered at a benchmark underlying value $S_T = s$. The sampling by the Broadie-Kaya approach is introduced in Section 4.3.1.

(3) Drăgulescu and Yakovenko (2002) derived an analytical formula (D-Y formula) for computing the probability density function of stock log-returns based on the Heston model. In Section 4.3.2, we use the D-Y formula to derive the likelihood ratio $d\mathbb{Q}^*/d\mathbb{P}^*$ as needed for the optimal test for the construction of the proposed optimal hedging option.

4.3.1 Sampling Path-Dependency by the Broadie-Kaya Approach

In the previous section, we characterized the dependency of a GMAB fund tied with the market index and its variance process. In our problem, the joint distribution of these variables is required to construct the proposed optimal hedging strategy for the path-dependent GMAB liability put option conditional on the equity index. Due to the complexity of product designs, Monte Carlo simulations can often be used to sample the residual risk L, which can be time-consuming for path-dependent payoffs, and somewhat inaccurate because of sampling errors and biases. For example, we consider the forward simulations by using the Euler discretization that can be directly used to approximate the paths of the asset price and variance processes on a discrete time grid. Let $[0 = t_0 < t_1 < \cdots < t_N = T]$ be a partition of a time interval into N equal segments of length Δt, i.e., $t_i = iT/N$ for each $i = 0, 1, \cdots, N$. The discretization for the asset price process is

$$S_{t_i} = S_{t_{i-1}} + \mu S_{t_{i-1}} \Delta t + \sqrt{v_{t_{i-1}}} S_{t_{i-1}} \left(\rho \Delta \tilde{W}_v(t_i) + \sqrt{1 - \rho^2} \Delta \tilde{W}_x(t_i) \right), \qquad (4.19)$$

where $\Delta \tilde{W}_x(t_i) = \tilde{W}_x(t_i) - \tilde{W}_x(t_{i-1})$ and $\Delta \tilde{W}_v(t_i) = \tilde{W}_v(t_i) - \tilde{W}_v(t_{i-1})$. The discretization for the variance process is

$$v_{t_i} = v_{t_{i-1}} + \kappa^* \left(\bar{v}^* - v_{t_{i-1}} \right) \Delta t + \sigma \sqrt{v_{t_{i-1}}} \Delta \tilde{W}_v(t_i), \qquad (4.20)$$

where v_{t_i} is strictly positive when the Feller condition satisfies, i.e., $2\kappa^* \bar{v}^* > \sigma^2$.

There are two types of error associated with the Euler scheme: the standard error of the simulation estimator at the rate $\mathcal{O}(1/\sqrt{N})$ and the discretization bias. At the mean

time, the convergence rate of the error for the Euler discretization is $\mathcal{O}(s^{-1/3})$.

To improve the theoretical tractability and accuracy, we propose to construct the dependency between the residual risk L and the corresponding benchmark on the equity index by the method of Broadie and Kaya (2006), which works out a theoretically exact simulation for the Heston model. Using the method of exact simulation, we can obtain the joint distribution of S_t and $\int_0^t v_s ds$. We then sample the joint distribution of S_t, $\int_0^t v_s ds$ and F_t, which is desirable for approximating the distribution of conditional residual risk on the equity index S for the GMAB liability put option as shown in equations (4.10) and (4.11).

We now recall the exact simulation approach for (4.5) given by Broadie and Kaya (2006) as follows:

(B1) Simulate v_t given $v_0 = x$. It is well known that for $t > 0$ the distribution of $v_t|v_0$ follows a non-central chi-squared distribution

$$v_t|v_0 = x \sim \frac{\sigma^2(1 - \exp(-\kappa^* t)))}{4\kappa^*} \chi_d^2 \left(\frac{4\kappa^* \exp(-\kappa^* t)}{\sigma^2(1 - \exp(-\kappa^* t))} x \right), d := \frac{4\kappa^* \bar{v}^*}{\sigma^2},$$

where $\chi_\nu^2(\eta)$ denotes the noncentral chi-squared random variable with ν degrees of freedom and noncentrality parameter η.

(B2) Generate a sample from the distribution of $\int_0^t v_s ds$ given $v_0 = x$ and $v_t = y$. This is the most expensive step. Broadie and Kaya (2006) obtained its distribution function by computing the conditional Laplace transform and performing a numerical inversion of the characteristic function $\Phi(a) := \mathbb{E}\left[\exp\left(ia \int_0^t v_s ds|v_0, v_t\right)\right]$. Φ is given explicitly as

$$\Phi(a) = \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa^*)t}\left(1 - e^{-\kappa^* t}\right)}{\kappa^*\left(1 - e^{-\gamma(a)t}\right)} \frac{A(a)}{B} \exp(C(a)),$$

where the terms $A(a)$, B, $C(a)$ and $\gamma(a)$ are

$$A(a) \ : \ = I_{d/2-1}\left(\sqrt{v_0 v_t}\frac{4\gamma(a)e^{-\frac{1}{2}\gamma(a)t}}{\sigma^2\left(1-e^{-\gamma(a)t}\right)}\right),$$

$$B \ : \ = I_{d/2-1}\left(\sqrt{v_0 v_t}\frac{4\kappa^* e^{-\frac{1}{2}\kappa^* t}}{\sigma^2\left(1-e^{-\kappa^* t}\right)}\right),$$

$$C(a) \ : \ = \frac{v_0 + v_t}{\sigma^2}\left(\frac{\kappa^*\left(1+e^{-\kappa^* t}\right)}{1-e^{-\kappa^* t}} - \frac{\gamma(a)\left(1+e^{-\gamma(a)t}\right)}{1-e^{-\gamma(a)t}}\right),$$

$$\gamma(a) \ : \ = \sqrt{\kappa^{*2}-2\sigma^2 ia},$$

and $I_\nu(x)$ denotes the modified Bessel function of the first kind and $d = \frac{4\kappa^*\bar{v}^*}{\sigma^2}$. The cumulative distribution function $F(x)$ can be obtained by the inversion of Φ as

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(zx)}{z}\Re\left(\Phi(z)\right) dz.$$

(B3) Recover $\int_0^t \sqrt{v_s}d\tilde{W}_v(s)$ from (4.7) given $v_t = y$, $v_0 = x$ and $\int_0^t v_s ds = z$ as

$$\int_0^t \sqrt{v_s}d\tilde{W}_v(s) = \frac{1}{\sigma}(y - x + \kappa^*\bar{v}^* t - z).$$

(B4) Conditional on $\int_0^t \sqrt{v_s}d\tilde{W}_v(s)$ and the integrated variance $\int_0^t v_s ds$, we have

$$\ln S_t \bigg| \int_0^t v_s ds, \int_0^t \sqrt{v_s}d\tilde{W}_v(s) \sim \mathcal{N}\left(\phi, \varphi\right),$$

where $\phi := \ln S_0 + \mu t - \frac{1}{2}\int_0^t v_s ds + \rho\int_0^t \sqrt{v_s}d\tilde{W}_v(s)$ and $\varphi := (1-\rho^2)\int_0^t v_s ds$.

From Step (B2) to Step (B4), we can sample the joint distribution of S_t and $\int_0^t v_s ds$ as

$$\left(S_t, \int_0^t v_s ds\right). \tag{4.21}$$

Furthermore, given a specific fee rate structure of $c_t = a + bv_t$, we can also sample the joint distribution of S_t and $\int_0^t v_s ds$, together with F_t in (4.9), as

$$\left(S_t, \int_0^t v_s ds, F_t \right). \tag{4.22}$$

Based on the joint distribution in (4.22), we want to find the distribution of the residual risk $L(s)$ in (4.10) given the distribution of terminal fund value F_T^s conditional on $S_T = s$. Since it is not explicit to obtain the conditional fund value F_T^s under the Heston framework, we thus approximate F_T^s by collecting sufficient numbers of the underlying path such that the selected values of the terminal underlying S_T are located in a small interval $[s, s + \delta)$, $\delta \to 0$. In (4.22), we jointly simulate S_t, $\int_0^t v_s ds$ and F_t by the method of Broadie and Kaya (2006). We take sufficient numbers of terminal underlying S_T in $[s, s+\delta)$ and jointly collect the corresponding conditional integrated variances $\int_0^T v_t^s dt$ for $S_T \in [s, s + \delta)$. Finally, we obtain the corresponding numbers of F_T^s by (4.11) for the approximate distribution of $L(s)$. Using the exact simulation of Broadie and Kaya (2006), the volatility-dependent fee payment can be exactly simulated by a sample of $\int_0^t v_s ds$, instead of using Monte Carlo approximation.

4.3.2   Likelihood Ratio in the Heston Model

Drăgulescu and Yakovenko (2002) enables us to find the likelihood ratio reconciling the distributions of the equity index S under both the physical and risk-neutral measures, which is desirable for characterization of our proposed optimal hedging option in the Heston-type stochastic volatility setting. Drăgulescu and Yakovenko (2002), hereafter D-Y, derived a closed-form analytical solution for the probability density function of equity index returns based on the Heston model. It has shown that the Heston (1993) model adequately describes the probability density function of empirical security returns and the model can therefore assist in the valuation of path-dependent options. By setting $y_t = \ln(S_t/S_0) - \mu t$, we express the Heston model in terms of the centered log-return y_t and v_t such that

$$dy_t = -\frac{v_t}{2} dt + \sqrt{v_t} d\tilde{W}_x(t).$$

The process is then characterized by the transition probability $P_t[y, v|v_0]$ to have log-return y and variance v at time t given the initial log-return $y = 0$ and variance v_0 at time $t = 0$. The time evolution of $P_t[y, v|v_0]$ is governed by the following Fokker-Planck (or forward Kolmogorov) equation:

$$\frac{\partial}{\partial t}P \;=\; \kappa^* \frac{\partial}{\partial v}\left[(v - \bar{v}^*)\, P\right] + \frac{1}{2}\frac{\partial}{\partial y}\left(vP\right) + \rho\sigma\frac{\partial^2}{\partial y \partial v}\left(vP\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(vP\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial v^2}\left(vP\right).$$

(4.23)

Solving this equation (4.23) yields the following semi-analytical formula for the density of centered returns y, given a time lag t of the price changes (D-Y formula)

$$f_Y(y) := P_t(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[i\xi y + F_t\left(\xi\right)\right] d\xi,$$

(4.24)

which is an expression for the probability distribution of the centered log-returns y for a frequency t where

$$F_t\left(\xi\right) = \frac{\kappa^* \bar{v}^*}{\sigma^2}\Gamma t - \frac{2\kappa^* \bar{v}^*}{\sigma^2}\ln\left[\cosh\frac{\Omega t}{2} + \frac{\Omega^2 - \Gamma^2 + 2\kappa^*\Gamma}{2\kappa^*\Omega}\sinh\frac{\Omega t}{2}\right],$$

in which $\Gamma = \kappa^* + i\rho\sigma\xi$ and $\Omega = \left[\Gamma^2 + \sigma^2\left(\xi^2 - i\xi\right)\right]^{1/2}$.

Let us denote $\mathbb{P}^*$ and $\mathbb{Q}^*$ as the distributions of the underlying asset S under the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$, respectively. Given the density of centered returns in equation (4.24), we determine the likelihood ratio $d\mathbb{Q}^*/d\mathbb{P}^*$ in the Heston model

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}^*}(s) = \frac{g_S^{\mathbb{Q}}(s)}{g_S^{\mathbb{P}}(s)} = \frac{f_Y^{\mathbb{Q}}\left(y(s)\right)}{f_Y^{\mathbb{P}}\left(y(s)\right)},$$

(4.25)

where f and g are the probability density functions for the centered return y and the underlying S respectively. By the change of variable, we obtain $g_S(s) = f_Y(y(s))/s$.

4.3.3 Numerical Procedures

In the following, we show how to determine the optimal hedging option numerically for the GMAB liability put option defined in (4.1). For comparison, we are interested in the minimization of the mean-square error of the loss/profit ignoring the budget constraint, in which case the optimal mean-square hedging option h_{ms} solves

$$h_{ms}(s) := \arg\inf_{h \in \mathcal{L}^2(S(T))}\mathbb{E}^{\mathbb{P}}\left[\left(L(s) - h(s)\right)^2\right],$$

(4.26)

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with $\mathcal{L}^2(S(T))$ denoting the set of measurable and square integrable functions of $S(T)$. h_{ms} admits a representation given by

$$h_{ms}(s) := \mathbb{E}^{\mathbb{P}}[L(s)|S(T) = s]. \qquad (4.27)$$

Using the method of Broadie and Kaya (2006) in Section 4.3.1, the jointly-distributed vector in (4.22) leads to a continuous distribution of the residual risk $L(s)$ defined in (4.3). For $\alpha_H > \alpha_L$, we take $h_U(s) = q_{\alpha_H}^L(s)$ and $h_L(s) = q_{\alpha_L}^L(s)$, in which $q_{\alpha_H}^L(s)$ and $q_{\alpha_L}^L(s)$ denote the α_H and α_L-quantiles of the residual risk $L(s)$ respectively.

We present the numerical procedure for how to determine the optimal hedging strategy h_{opt} for the GMAB liability put option in (4.1) by Theorem 7.

(S1) Determine the feasible pairs of (a, b) constrained by the condition in (2.28). For a selected pair of (a, b) determined in the pricing stage of a GMAB contract, calculate the price of a liability put option in (4.1) at time 0 as initial budget V_I and that $V_0 = e^{rT}V_I$ at maturity T.

(S2) Use the exact simulation of Broadie and Kaya (2006) to simulate sufficient sets of S_T, $\int_0^T v_s ds$ and F_T. Then, take sufficient numbers of terminal underlying S_T in $[s, s + \delta)$ and collect the corresponding conditional integrated variances $\int_0^T v_t^s dt$ for $S_T \in [s, s + \delta)$. Finally, obtain the corresponding numbers of F_T^s under the fee rate structure of $c_t = a + bv_t$ by (4.11) for the approximate distribution of $L(s) = (G_T - F_T^s)^+$ in $[s, s + \Delta)$, $\Delta \to 0$. The mean-square hedging option h_{ms} can be approximated using the following unbiased and asymptotically consistent sample means

$$h_{ms}(s) \approx \frac{1}{M}\sum_{i=1}^{M}(G_T - F_T^s)^+.$$

(S3) For selected values α_L and α_H from the interval $(0, 1)$ such that $\alpha_L < \alpha_H$, find the corresponding empirical quantiles of the distributions of $L(s)$ at each $S_T = s$. Then, obtain the lower and upper bounding functions h_L and h_U as the corresponding quantiles of $L(s)$, $s \in \mathcal{S}_L$.

(S4) To determine h_{opt} in Theorem 7, we need to approximate $l_e(s, y)$ and $\tilde{\gamma}(s)$. By con-

structing the mesh of points

$$\mathcal{M} := \{(s_m(i), z_m(j)) : s_m(i) \;=\; s_{\min} + i\frac{\Delta_s}{K_s}, i = 0, \cdots, K_s$$

$$z_m(j) \;=\; -h_U(s_m(i)) + j\frac{\Delta_z(i)}{K_z}, j = 0, \cdots, K_z\},$$

where $\Delta_s := s_{\max} - s_{\min}$ and $\Delta_z(i) := h_U(s_m(i)) - h_L(s_m(i))$.

(i) Select $\hat{s}(l)$, $l = 1, \cdots, k$, in each small interval $(s_m(i-1), s_m(i)]$ for approximating the distribution at $s = s_m(i)$. Then, obtain a sufficient number of residual risks of $L(s_m(i)) = \left(G_T - F_T^{\hat{s}(l)}\right)^+ |S_T = s_m(i)$ in each small interval $(s_m(i-1), s_m(i)]$. From these points, use a kernel density estimator to find an estimate $\hat{s}_i^L$ of the density of $L(s_m(i))$, which will be used to evaluate the conditional expectation of $\mathbb{E}\left[(L(s_m(i)) + z_m(j))^+ | S_T = s_m(i)\right]$ for each equally spaced mesh of points $z_m(j)$ from $\mathcal{M}$. This gives the approximation of $\hat{g}_0(s_m(i), z_m(j); 1)$. For the bounds $h_L(s_m(i))$ and $h_U(s_m(i))$, take the 5th and 95th quantiles of the distribution of $L(s_m(i))$ for evaluating $g(s_m(i), z_m(j); p)$ over $\mathcal{M}$.

(ii) Use a central finite difference to approximate $\hat{l}_e(s, y)$ in (4.15) by inverting the derivative of $\hat{g}$.

(iii) Use the bisection method to fit the value of c by the budget constraint in (4.18). $\hat{\tilde{\gamma}}(s)$ and $\hat{h}_{opt}(s_m(i))$ are then determined by all the elements obtained by the aforementioned steps with the likelihood ratio $d\mathbb{Q}^*/d\mathbb{P}^*$ derived in (4.25).

(S5) Repeat the above processes and get all $h_{opt}(s_m(i))$ for $i = 1, \cdots, K_s$. Finally, we find the optimal option h_{opt} on $S_T = s$ by Theorem 7.

4.4 Numerical Examples

We illustrate the result presented in Theorem 7 on two examples of path-dependent options. In Section 4.4.1, we find the optimal hedging option for an arithmetic Asian option by generally using the forward simulations such as Euler scheme. In Section 4.4.2, the optimal static hedge is implemented for the GMAB liability put option as discussed in the previous sections.

4.4.1 Asian Option under the Heston Model

In this section, we find the optimal static hedging for a path-dependent Asian option under the Heston stochastic volatility model. Consider a K-strike call option on arithmetic average sampled at discrete time intervals:

$$(A_N - K)^+ , \tag{4.28}$$

where $A_N = \left(\sum_{i=1}^{N} S(t_i) \right) / N$ and $\{t_1, \cdots, t_N = T\}$ is a set of monitoring dates. In (4.28), we are interested in the underlying S and it is directly known from financial data, whereas variance is a hidden stochastic variable. We define the distributions of conditional residual risks $L(s)$ by

$$L_{AC}(s, T) := (A_N^s - K)^+ \,|S_T = s, \qquad s \in \mathbb{R}^+, \tag{4.29}$$

where A_N^s is the arithmetic average conditional on $S_T = s$.

The main steps in our procedure to find the optimal static hedge for the Asian option are similar to the ones for the GMAB liability put options in Section 4.3.3, and can be outlined as follows:

(P1) Use Monte Carlo to simulate the price of an Arithmetic Asian call option in (4.28) at time 0, which is the initial budget V_I and that $V_0 = e^{rT}V_I$ at maturity T.

(P2) Use the Euler Scheme to simulate M paths of the underlying S. Based on M paths of the underlying S, collect sufficient number of the underlying paths such that the selected values of the terminal underlying S_T are located in a small interval $[s, s+\delta)$, $\delta \to 0$. Then, approximate the conditional arithmetic Asian A_N^s in (4.29) by taking sufficient numbers of the underlying paths with the terminal underlying values $S_T \in [s, s+\delta)$. Finally, approximate the distribution of $L_{AC}(s, T)$ in $[s, s+\delta)$ based on sufficient numbers of A_N^s. The mean-square hedging option h_{ms} can be approximated using the following unbiased and asymptotically consistent sample means

$$h_{ms}(s) \approx \frac{1}{M} \sum_{i=1}^{M} (A_N^s - K)^+ .$$

(P3) The next steps are similar to the ones in steps (S3) − (S5).

In Figure 4.1, we implement the optimal static hedge for an arithmetic Asian call in (4.28) with the parameters: $S_0 = K = 100$, $\rho = -0.3$, $\kappa^* = 3$, $\bar{v}^* = 0.02$, $\mu = 0.15$,

$r = 0.05$, $\sigma = 0.5$, $\lambda = -0.15$, $T = 1$, $\alpha_L = 0.05$, $\alpha_H = 0.95$, $M = 50,000$, $K_s = K_z = 150$, $\delta = 0.2$ and $c = 0.80$. Depending on the model parameters, the optimal hedging option h_{opt} significantly outperforms the mean-square one h_{ms}. We find that the expected shortfalls from h_{opt} and h_{ms} are 0.702 and 1.701 respectively. The shortfall risk is thus reduced by 58% with h_{opt}, while the corresponding standard deviation is reduced by 56%.
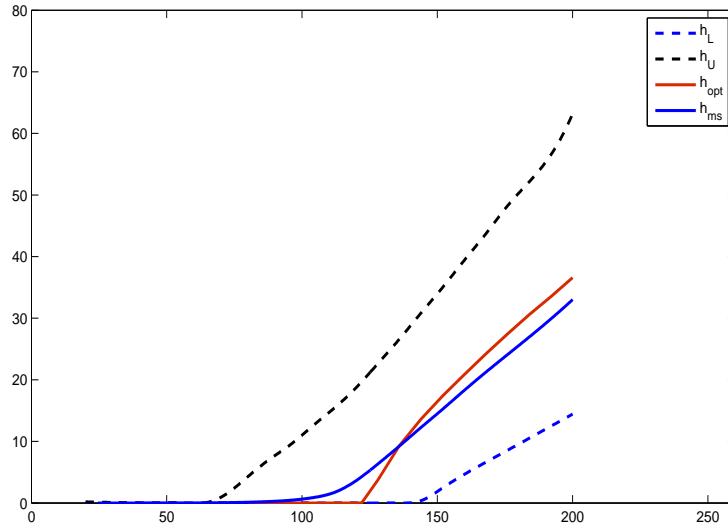

Figure 4.1: The optimal static hedging option for the arithmetic Asian call $(A_N^s - K)^+$ conditional on $S_T = s$ with the Heston-type stochastic volatility.

4.4.2 Implementation for GMAB

Figure 4.2 illustrates the optimal hedging strategy for the GMAB liability put option in (4.1) and the distributions of the market index under the Heston-type stochastic volatility, given the market parameters: $S_0 = F_0 = G_T = 100$, $\rho = -0.3$, $\kappa^* = 3$, $\bar{v}^* = 0.02$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.5$, $\lambda = -0.15$, $T = 1$, $\alpha_L = 0.05$, $\alpha_H = 0.95$, $M = 50,000$, $K_s = K_z = 150$ with the selected pair of $a = 0.1513$ and $b = 1.5$. The constant c determined in Step (S4) is 0.225. The numerical results show that the optimal hedging option h_{opt} outperforms its counterpart, the mean-square one h_{ms}. For $\mu = 0.15$, the expected shortfalls from h_{opt} and h_{ms} are 0.6746 and 0.9663 respectively, which is a reduction of 30.2% in shortfall risk. The

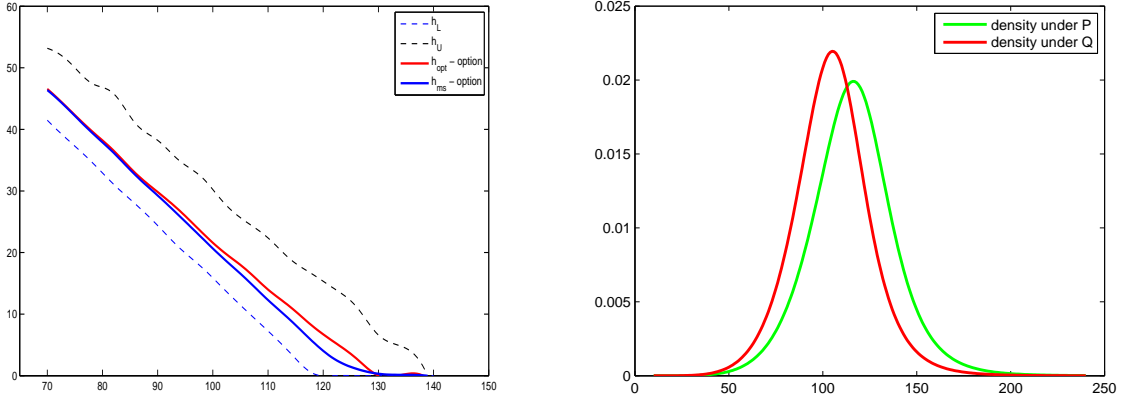standard deviation of the shortfall risk from h_{opt} is reduced by 23.72% when compared to h_{ms}.

Figure 4.2: The optimal static hedging option for the GMAB liability put option $(G_T - F_T^s)^+$ (red line) conditional on $S_T = s$ in the left panel and the probability densities of the underlying S_T with the Heston-type stochastic volatility in the right panel.

4.5 Concluding Remarks

The main objective of this chapter is to extend a theoretical result that characterizes an optimal hedging strategy for a path-dependent GMAB liability in the incomplete Heston market. The presented numerical results demonstrate both the feasibility of the optimal hedging method. The optimal static hedging strategy can be considered as a benchmark, when compared to other strategies with the introduction of the stochastic volatility and the path-dependent feature on the payoff. We take the GMAB liability put option as an example but the proposed hedging strategy is not limited to that. For example, the construction of simplified hedges and most VA liabilities discussed in this thesis can result in the hedging problem for the path-dependent contracts with the stochastic volatility. For the reverse mortgage contracts, the proposed optimal hedging options can be developed for the complexity of the crossover risk, such as hedging with the conditioning event(s) under the environment of either the stochastic volatility or the stochastic interest rate.

Chapter 5

Conclusions and Future Research

In this thesis, we explored pricing and hedging issues on various innovative products in finance and insurance. The innovations of pricing fulfilled the rising market demands for the products with path-dependent features, such as simplified derivatives, variable annuities and reverse mortgage contracts. To hedge these products, we relied on the method by Kolkiewicz (2016) and extended the results that can be applied in different angles. In this chapter, we propose several lines of future research.

The first direction lies in developing an optimal static hedging strategy for the simplified derivatives. In Chapter 1, we extended the work of Bernard, Moraux, Rüschendorf, and Vanduffel (2015) to construct a simplified alternative that resembles a given highly path-dependent derivative by adding multiple conditional benchmarks. A natural extension is to construct the simplified alternatives under different criteria in the context of minimizing its expected shortfall and the Value-at-Risk to the original derivative. Due to the nature of the simplified derivative, we can generalize the corresponding construction by conditioning on more benchmark states for improving the performance to meet the given criteria. Since the simplified derivative is constructed by only preserving certain distributional properties of the original payoff, the proposed construction can be expected to provide a better hedging performance to the original payoff than the simplified hedge only keeping its distributional property. For hedging the simplified derivative itself, we will compare the hedging performance of the optimal static hedging strategy to the simplified alternative with that of the semi-static hedging as proposed in Chapter 1, which only involves a limited number of basis options. The optimal static hedging option for the simplified derivative can be constructed either in the Black-Scholes framework or more general markets with stochastic volatility.

The second area of future research might consider the effect of state-dependent fees or the state-dependent structure on more complex path-dependent VA products, such as guaranteed minimum withdrawal benefits (GMWBs) and flexible premium in variable annuities (FPVAs) introduced by Bernard, Cui, and Vanduffel (2017). Both these contracts allow the policyholder to make withdrawals or additional payment contributions to the underlying account of the variable annuity depending on the path of the fund value. In Chapter 2, the benefits of a GMAB contract tied with the volatility-dependent fees have been examined. In general, the affine models and dynamic programming framework offer the possibility to involve more path-dependent features such as the state-dependent fees tied with the market volatility, underlying processes or surrender strategies in most recent VA contracts. Charging a state-dependent fee cannot exclude the possibility of the lapse of the contract, but we expect that it will make the optimal surrender boundary lower than that of charging a fixed fee rate. In the study of GMWBs, we can consider the strategy of optimal withdrawals in the presence of state-dependent fees. In the study of FPVAs, we will study the optimal flexible premiums depending on market conditions such as the underlying fund and the interest rates, while most current academic research assume that there is a single premium payment independent of market conditions. We expect that such flexible premiums can be optimized for covering the costs of liabilities. By modifying the state-dependent premiums to the fund return in FPVAs, this problem is comparable to the problem of optimal withdrawals by the policyholder in a GMWB where the withdrawal rate is subtracted from the fund return. Adding the state-dependency to the VA contracts leads to the complexity of the hedging strategy. Thus, the corresponding optimal hedging strategy will be developed for accommodating the hedging needs.

The third direction is to design the defaultable reverse mortgage (RM) contracts in more general markets and to study the application of the optimal static hedging on the reverse mortgage markets. Our goal is not limited to hedging a particular defaultable contract: more general credit-risk derivatives can be considered. In Chapter 3, a novel pricing of the RM contract was proposed for improving the solvency of the current HECM program in the presence of the borrower's default risk. Despite our pricing scheme designed for reducing the lender's hedging difficulty, it cannot replace the existing hedging program. To hedge a defaultable RM contract, we require an analysis of optimal hedging in a setting where the payoff of the crossover risk and the corresponding conditioning event are dependent and have to be priced and hedged accordingly. Since the methods by Kolkiewicz (2016) allow the generosity of dependency, it is possible to build the optimal static hedging option on the conditioning events, at which the crossover risk arises from the events of default and death. Existing literature such as Bertus, Hollans, and Swidler (2008) and Fabozzi, Shiller, and Tunaru (2010) has shown that the house price risk is difficult to hedge due

to the basis risk between house price index and the corresponding future contracts offered by the Chicago Mercantile Exchange (CME). The optimal hedging strategy, allowing the inclusion of basis risk, is then motivated for a solution to minimize the expected shortfall for the crossover risk using the CME housing future contracts. In Chapter 3, we formulated the RM pricing problem under the Black-Scholes framework but it can also be extended to more complex settings by involving stochastic volatility and stochastic interest rates.

For problems in Chapters 2 - 4, we will consider other models, like those based on Lévy processes. Similarly, hedging problems can also be considered for Lévy processes. For example, Alonso-García, Wood, and Ziveyi (2018) extend the Fourier-cosine (COS) method to the pricing and hedging of variable annuities embedded with GMWB riders. They demonstrate superior computational efficiency of the COS method to price and hedge the GMWB riders when the underlying fund dynamics evolve under the influence of the general class of Lévy processes. The framework developed is general enough to incorporate complex policyholder behavior decisions and sophisticated contract features such as the reset provision. They further use the framework to investigate the risk minimisation hedging strategies under the concepts outlined in Kolkiewicz and Liu (2012). Our problems in Chapters 2 - 4 can be further generalized in at least two ways. The first one is to search for the optimal solutions in both the stage of pricing, such as allowing the static and dynamic policyholder withdrawal behaviour in GMWBs, and the stage of building the optimal static hedging strategies in the spirit of Kolkiewicz (2016), for more complex path-dependent VA products. To tackle the complexity, we will adopt a backward recursive dynamic programming algorithm in aid of the COS method as proposed by Alonso-García, Wood, and Ziveyi (2018). The second generalization is to model the defaultable reverse mortgage contracts in the Lévy markets and to price and hedge accordingly with the conditioning events under more realistic credit-risk modeling. In Chapter 3, we have developed a valuation model for determining the annuity loan payment based on the levels of default risk, assuming that the default probability is modeled with an average hazard rate. Further, more realistic assumptions in Mei (2016) can be considered for modeling the default risk by assuming that the hazard rate can be stochastically tied with the market conditions such as loan status, interest rate and house price. And then, the annuity loan payment rate can be determined accordingly based on the market conditions. Since the RM contracts allow the lapse by default, it arises the optimal determination for the annuity loan payment in accordance with the default risk based on market conditions, which is in the same spirit of the optimal withdrawals for GMWBs. Pricing and hedging such defaultable RM contracts can be extended in the framework of dynamic programing with the aid of the COS method by Alonso-García, Wood, and Ziveyi (2018).

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