Transition to light-like trajectories in thin shell dynamics

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Abstract

It was recently shown that a massive thin shell that is sandwiched between a flat interior and an exterior geometry given by the outgoing Vaidya metric becomes null in a finite proper time. We investigate this transition for a general spherically-symmetric metric outside the shell and find that it occurs generically. Once the shell is null its persistence on a null trajectory can be ensured by several mechanisms that we describe. Using the outgoing Vaidya metric as an example we show that if a dust shell acquires surface pressure on its transition to a null trajectory it can evade the Schwarzschild radius through its collapse. Alternatively, the pressureless collapse may continue if the exterior geometry acquires a more general form.
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1. Introduction

Hypersurfaces of discontinuity are idealizations of narrow transitional regions between space-time domains with different physical properties. The thin shell formalism [1–3] makes this idealization consistent by prescribing joining rules for the solutions of the Einstein equations on both sides of the hypersurface. These rules — junction conditions — determine dynamics of

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the shell. The resulting joined geometry is a solution of the Einstein equations with an additional distributional stress-energy tensor that is concentrated on the hypersurface.

The thin shell formalism plays a role in studies of cosmological phase transitions [4], impulsive gravitational waves [5], gravastars and other non-singular substitutes of black holes [6], traversable wormholes [7], and gravitationally-induced decoherence [8]. A massive thin shell separating a flat interior from a curved exterior spacetime region provides the simplest model of collapse. Classically the exterior spherical geometry is described by a Schwarzschild metric and the shell collapses into a black hole in finite proper time.

Such models has also been used in investigations of collapse-induced radiation [9,10] anticipated before formation of the event horizon. The basic idea is that the process of gravitational collapse excites fields in the spacetime, giving rise to asymptotically thermal radiation [9]. We shall refer to this as pre-Hawking radiation [11–13]. The consequences it might have for black hole formation and the information paradox have been a subject of interest in recent years [9,10,14–22]. While it has been argued that such effects are too small to prevent the formation of an event horizon [10,22], others contend that such approximations are not reliable and that horizons may not form if pre-Hawking radiation is properly taken into account [16,19]. Indeed, it has been posited that this should be a generic feature of quantum gravity, with the black hole interior and accompanying singularity replaced with a genuine quantum geometry where the notion of event horizon ceases to be useful [23].

A number of researchers have argued [15–18] that there are two options for the evolution of a thin shell in a spacetime with pre-Hawking radiation. One possibility is that an event horizon never forms: either the shell does not cross its Schwarzschild radius r_g before complete evaporation or a manifest breakdown of semiclassical dynamics, such as violation of the adiabatic condition [26], or formation of some quantum geometry [23] occurs. The other alternative is that evaporation stops producing a macroscopic or a Planck-scale remnant [25], forever preventing a distant observer at late times from detecting Hawking radiation. An outgoing Vaidya metric [27] is often used as an example of the exterior geometry of this process despite its known limitations [28].

This result applies to a broad class of possible exterior geometries. It is based on an implicit assumption that through their evolution a massive shell remains timelike and a massless shell remains null. However, it was recently demonstrated by Chen, Unruh, Wu and Yeom (CUWY) [29] that this assumption is unwarranted. Via an explicitly regular coordinate system, while CUWY confirmed horizon avoidance for an exterior geometry described by the outgoing Vaidya metric, they also demonstrated that it comes with a price.

Indeed, it was shown that if the exterior metric outside is outgoing Vaidya, a massive dust shell sheds its rest mass in finite proper time (while still outside its Schwarzschild radius), becoming null. It was further argued that if the evaporation continues the shell becomes superluminal. The choice is evidently between eventual tachyonic behavior or switching off the radiation. In the latter case the subsequent development is classical and the shell crosses r_g at a finite value of a suitable affine parameter [29].

Our goal is to understand in this context the limits of the thin shell formalism in various geometries and validity of the semiclassical approximation taking pre-Hawking radiation into account. The detailed description of the basic assumptions of this approximation and their application to thin shells is given in [17]. In practical terms, the standard curvature terms of the left hand side of the Einstein equations are equated to the expectation value of the renormalized stress-energy tensor. We assume its existence and consistency, but make no assumptions beyond that of spherical symmetry and certain regularity conditions that are described below. We leave

aside the conceptual implications of radiation suppression and/or horizon avoidance [17,24]. The question of the origin of the pre-Hawking quanta is open, as it has been posited that this takes place at or near the surface of the collapsing body [13,16] or within a region $\Delta r \sim r_g$ outside the horizon [30].

The flat geometry inside is given in the outgoing Eddington–Finkelstein (EF) coordinates,

$$ds_-^2 = -du_-^2 - 2du_-dr + r^2d\Omega, \tag{1}$$

where $u_- = t - r$, and the most general spherically-symmetric geometry outside is

$$ds_+^2 = -e^{2h(u_+,r)}f(u_+,r)du_+^2 - 2e^{h(u_+,r)}du_+dr + r^2d\Omega, \tag{2}$$

extending the set-up of the previous studies [17,29]. In the following we omit the subscript "+" from the exterior quantities when it does not lead to confusion. Here

$$f(u,r) = 1 - C(u,r)/r, \tag{3}$$

the Schwarzschild radius is implicitly defined via $r_g = C(u, r_g)$, where in the Schwarzschild geometry $r_g = C = 2M$.

To represent evaporation we assume that $\partial_u C \leq 0$, and the evaporation stops at some u_* either at $f \equiv 1$ or with some finite value of the mass function $C(r) > 0$. The null coordinates $u_\pm$ are distinct and the relation between them is determined by the first junction condition, while the radial coordinate is continuous across the surface [3].

We consider the transition of a massive evaporating shell to a null trajectory and investigate the circumstances under which such a shell continues along a null trajectory. We find this does not take place only if $h(u, r) \equiv 0$ and the absence of surface tension or pressure is imposed on the shell when it becomes null. Provided that the metric at the Schwarzchild radius is regular — specifically, the function $h(u, r_g(u))$ is finite, we show by reworking the arguments of [18] that the subsequent null trajectory will never cross the ever-shrinking $r_g(u)$. In this sense the massive-to-null-to-superluminal case considered by CUWY is exceptional among the metrics with the finite functions $h(u, r)$. We find that a collapsing evaporating shell can undergo a number of possible endstates. We discuss the physical implications of the various possible cases in the concluding section of our paper, noting that which, if any, scenario is realized can be decided only by performing explicit analysis of the coupled matter-gravity system.

To simplify the notation in the following we use $w := u_-$ and refer the quantities on the shell Σ by capital letters, such as $R := r_{|\Sigma}$, $F := f(U, R)$. The jump of some quantity A across the shell is $[A] := A|_{\Sigma_+} - A|_{\Sigma_-}$. All derivatives are explicitly indicated by subscripts, as in $A_R = \partial_R A(U, R)$. The total proper time derivative $dA/d\tau$ is denoted as $\dot{A}$, and the total derivative over some parameter λ is $A_\lambda := A_R R_\lambda + A_U U_\lambda$.

2. Transition to massless shell

The metric across the two domains can be represented as the continuous distributional tensor [3]

$$\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}^+ \Theta(z) + \bar{g}_{\mu\nu}^- \Theta(-z), \tag{4}$$

using the set of special coordinates $\bar{x}^\mu = (w, z, \theta, \phi)$. Here $\Theta(z)$ is the step function and the interior and exterior metrics $\bar{g}^\pm(\bar{x})$ are continuously joined at $z = 0$. The coordinate z is defined in Appendix A. A mathematically equivalent (and sometimes easier to implement) approach

is the thin shell formalism. We will use it in most of our analysis, both for the consistency with [17,18] and because it makes structure of the distributional stress-energy tensor more transparent.

The explicit form of the interpolating metric $\bar{g}_{\mu\nu}$ when the exterior geometry that is modelled by the outgoing Vaidya metric is given in [29]. We treat a general spherically-symmetric exterior geometry and provide the resulting metric $\bar{g}_{\mu\nu}$ in Appendix A.

In discussing the timelike-to-null transition it is particularly convenient to have a unified description that is applicable to both types of shells [31]. Unlike the proper time τ that stops increasing at the transition to the null trajectory, two additional parameterizations are regular there. When discussing the null shell it is convenient to use

$$\lambda := -R, \tag{5}$$

while the Minkowski retarded coordinate w is used in calculations that involve the interpolating metric. We will primarily use the thin shell formalism in τ and λ parameterizations.

We first consider an initially massive thin shell Σ and assume that the exterior geometry is described by the outgoing Vaidya metric (eq. (2) with $f(u, r) = 1 - C(u)/r$, for some decreasing function $C(u)$, $h(u, r) = 0$. While the shell is timelike its four-velocity is given by

$$v_{\pm}^{\mu} = \dot{\lambda} k_{\pm}^{\mu}, \tag{6}$$

where

$$k_{+}^{\mu} := (U_{\lambda}, -1, 0, 0), \qquad k_{-}^{\mu} := (W_{\lambda}, -1, 0, 0), \tag{7}$$

and

$$\dot{\lambda} = (-k^{\mu} k_{\mu})^{-1/2}. \tag{8}$$

The first junction condition identifies the induced metric on the two sides of Σ,

$$W_{\lambda}^2 - 2W_{\lambda} = FU_{\lambda}^2 - 2U_{\lambda}, \tag{9}$$

where $F = f(U, R)$. Similarly, the condition $\bar{k}^{\mu} \equiv \bar{k}_{-}^{\mu} = \bar{k}_{+}^{\mu}$ (see Appendix A) holds both for a massive shell and in the lightlike regime, where $k_{\pm}^2 = 0$. For a massive shell we also have

$$\dot{U} = \frac{-\dot{R} + \sqrt{F + \dot{R}^2}}{F}, \tag{10}$$

that approximately becomes $\dot{U} \approx -2\dot{R}/F$ for large $-\dot{R}$, and for the null shell

$$U_{\lambda} = 2/F. \tag{11}$$

An important auxiliary quantity for a massive shell is the outward pointing (unit) spacelike normal,

$$\hat{n}^{\mu} = \dot{\lambda} n^{\mu}, \qquad n_{\mu} = (1, U_{\lambda}, 0, 0), \tag{12}$$

and as the shell approaches the null trajectory $n_{\mu} \to -k_{\mu}$. Both $\bar{k}^{\mu}$ and $\bar{n}^{\mu}$ are continuous across the shell (when written in the interpolating coordinates $\bar{x}^{\mu}$).

The second junction condition relates the jump in extrinsic curvature

$$K_{ab} := \hat{n}_{\mu;\nu} e_a^{\mu} e_b^{\nu}, \tag{13}$$

to the surface stress-energy tensor. Here we use the surface coordinates y^a, $a = 1, 2, 3$, the shell is given via parametric expressions $x_\pm^\mu(y)$, and $e_a^\mu = \partial x^\mu / \partial y^a$. In this case the optimal choice of the surface coordinates is $(\tau, \Theta := \theta|_\Sigma, \Phi := \phi|_\Sigma)$.

Assuming a general relationship between the proper mass density (that is related to the shell's rest mass via $m_0 = 4\pi R^2 \sigma$) and the tension/pressure $p(\sigma)$, we obtain equations that govern their evolution,

$$8\pi p(\sigma) = \frac{2\ddot{R} + F_R}{2\sqrt{F + \dot{R}^2}} - \frac{\ddot{R}}{\sqrt{1 + \dot{R}^2}} + \frac{F_U(1 - 2\dot{R}\dot{U})}{2F\sqrt{F + \dot{R}^2}}$$
$$+ \frac{\sqrt{F + \dot{R}^2} - \sqrt{1 + \dot{R}^2}}{R}, \tag{14}$$

and

$$-4\pi\sigma = \frac{\sqrt{F + \dot{R}^2} - \sqrt{1 + \dot{R}^2}}{R}. \tag{15}$$

The details of the derivation can be found in [17] and in Appendix C. The system can be solved for $\ddot{R}$ providing the basis for numerical integration. In the limit of large $\dot{R}$ the pressure is negligible, and the asymptotic expression becomes [17]

$$\ddot{R} \approx \frac{4C_U \dot{R}^4}{RF^2} \approx \frac{4CC_U \dot{R}^4}{X^2}, \tag{16}$$

where we defined the gap between the shell and the Schwarzschild radius,

$$X := R - C, \tag{17}$$

and the second equality in Eq. (16) holds for $X \ll C$.

To illustrate the timelike-to-null transition we set $p(\sigma) = 0$ and adopt the law

$$\frac{dC}{dU} = -\frac{\alpha}{C^2}. \tag{18}$$

The results of numerical integration of (14) with $p = 0$ are presented in Fig. 1; using the same initial conditions as in [29] we obtain the same result.

Henceforth we use

$$F_R = \frac{C}{R^2}, \qquad F_U = -\frac{C_U}{R} = +\frac{\alpha}{C^2 R}, \tag{19}$$

while

$$F_\lambda = F_R R_\lambda + F_U U_\lambda = -F_R + F_U U_\lambda. \tag{20}$$

We establish the divergence in $\dot{R}$ at finite proper time by considering the Taylor series for R (and thus $\dot{R}$) at some regular point τ and showing that its radius of convergence goes to zero as τ increases. The third and higher derivatives $R^{(r)}(\tau)$ are calculated from the knowledge of the coordinates $R(\tau)$ and $U(\tau)$, velocity $\dot{R}$ and the function $C(U)$. We estimate the radius of convergence of the series using Eqs. (16) (since for large values of $|\dot{R}|$ it is the dominant contribution to the derivative of $\dot{R}$) and (18).

In the leading term the $(r + 1)$-th derivative pulls down the exponent from $\dot{R}$ (which equals to $3r - 2$), increases the power of $\dot{R}$ by $3 = -1 + 4$ and increases the power of x in the denominator by 2, when we substitute $\ddot{R}$. As a result, the leading term in the derivatives scales as
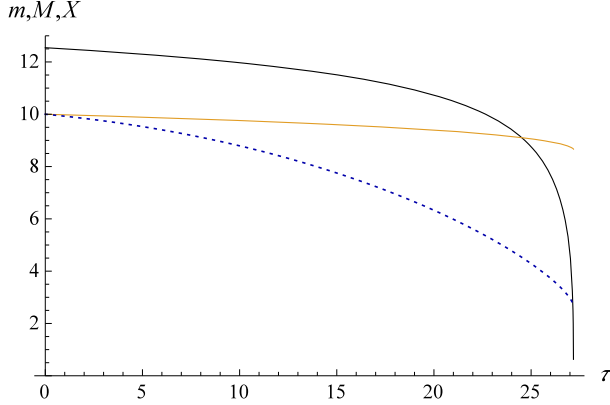
Fig. 1. The orange line represents $M(\tau) \equiv C(\tau)/2$. The rest mass $m_0(\tau)$ is shown as the thin black line, and the gap $X(\tau) = R(\tau) - C(\tau)$ as the blue dotted line. The initial conditions are as in [29], where $R_w(0) = -0.1$ is translated into $\dot{R}(0) = -0.111803$ using Eq. (A.15). Other initial data is $C_0 \equiv C(0) = 20$ and $R(0) = 30$. The coefficient in the evaporation law Eq. (18) is $\alpha = 8$ that corresponds to $\alpha = 1$ of [29]. The evaporation ends at time is $u_* = C_0^3/3\alpha = 1000/3$, but the system breaks down at approximately $\tau_0 = 27.179869$, indicating transition to the null trajectory. Already at $\tau_m = 27.1796$ ($u_m = 116.431$, $w_m = 39.6574$) the timelike condition $v^\mu v_\mu = -1$ is satisfied only with the precision of 8.93×10^{-9}, while Eq. (23) gives the estimate of the radius of convergence $\varrho \approx 0.00049$. At the transition most of the gravitational mass is still contained within the shell: $C(\tau_m)/C(0) = 0.867$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$|R^{(r+1)}| \sim \frac{|\dot{R}|^{(3r+1)}}{(R-C)^{2r}} \left(\frac{4\alpha}{C}\right)^r \prod_{n=2}^{r} (3n-2), \tag{21}$$

and the r-th coefficient in the Taylor series is

$$|c_{r+1}| = \frac{|R^{(r+1)}|}{(r+1)!} \sim \frac{12^r}{\Gamma(\frac{1}{3})r^{-5/3}} \frac{|\dot{R}|^{(3r+1)}}{(R-C)^{2r}} \left(\frac{\alpha}{C}\right)^r. \tag{22}$$

Then the radius of convergence ϱ is

$$\varrho = \left(\lim_{r\to\infty} |c_r|^{1/r}\right)^{-1} \approx \frac{CX^2}{12\alpha|\dot{R}|^3} \tag{23}$$

which goes to zero with decreasing C and X and increasing $|\dot{R}|$.

On the other hand, if the shell is still timelike the gap X begins to increase after reaching approximately $X \approx \epsilon_* = 2\alpha/C$ ([17]; in Section 4 we revisit this estimate taking into account the timelike-to-null transition). Since the acceleration is negative and increasing in absolute value the transition to a null trajectory can occur only for $X > 0$, i.e. outside the Schwarzschild radius. While the above result does not allow identification of the transition point τ_0, it shows that it exists.

Lightlike matter has a vanishing rest mass. Eq. (15) ensures that when the shell becomes null [i.e. $\dot{R} \to -\infty$], its surface density goes to zero,

$$\sigma = \frac{C}{8\pi|\dot{R}|R^2} + \mathcal{O}(1/|\dot{R}^3|), \tag{24}$$

causing the rest mass $m_0 = 4\pi\sigma R^2$ to vanish. The rate of shedding of the rest mass at large velocities is

$$\dot{m}_0 \sim -\frac{\dot{C}}{2\dot{R}} + \frac{C\ddot{R}}{2\dot{R}^2} \approx -\frac{2|C_U|C^2\dot{R}^2}{X^2}, \tag{25}$$

where only the term proportional to $\ddot{R} < 0$, approximated by using Eq. (16), appreciably contributes to the final result. We see that this rate is much higher than the rate of decrease of C. Indeed, for macroscopic shells $C \gg 1$ the fraction of the gravitational mass lost before the transition to the null trajectory goes to zero slower than $1/C$ (Appendix B).

For a general exterior geometry of Eq. (2) we find (Appendix C) that a sufficient condition for the transition to a light-like null trajectory in finite proper time is the requirement that the metric function h is finite everywhere and, in particular $h(u, r_g(u)) < \infty$.

3. Preserving the null condition

For the shell becoming null at $\lambda = \lambda_0$ we now investigate the conditions necessary to keep it null in a general spherically symmetric metric (2). First we recall a few properties of the surface stress-energy tensor in the null case. Since the normal n_μ "declines into tangency with Σ" [31] an alternative auxiliary vector is used, defined by

$$N_\mu^\pm N_\pm^\mu = 0, \qquad N_\mu^\pm k_\pm^\mu = -1 \tag{26}$$

on both sides of the shell. When the shell becomes null at some $\lambda = \lambda_0$

$$N_+^\mu = (0, \tfrac{1}{2}F, 0, 0), \qquad N_-^\mu = (0, \tfrac{1}{2}, 0, 0). \tag{27}$$

Analogously to the vector $\bar{k}^\mu$ it is continuous at Σ, $[\bar{N}^\mu] = 0$. On the null hypersurface that is orthogonal and tangent to the 4-velocity of the null shell we install the coordinates y^a that consist of $\lambda \geq \lambda_0$, possibly non-affine parameter of the hypersurface generators, and the transversal y^A that are most conveniently taken to be (Θ, Φ). If the shell remains null it traverses the hypersurface. From the bulk point of view the shell is described by parametric relations $x^\mu(y^a)$, and the set of three tangent vector fields is formed by two spacelike vectors

$$e_A^\mu := \frac{\partial x_\pm^\mu}{\partial y^A}\bigg|_\Sigma, \tag{28}$$

transverse to it that are also continuous across the surface, and the null vector $e_\lambda^\mu := k^\mu$. The transverse inner product $\sigma_{AB} := e_A^\mu e_{B\mu}$ is continuous as well. For a spherical shell it is

$$\sigma_{AB}dy^A dy^B = \lambda^2(d\Theta^2 + \sin^2\Theta d\Phi^2). \tag{29}$$

The surface stress-energy tensor of a massless shell depends on the observer. However, all such objects are derived from

$$S^{\mu\nu} = \varsigma k^\mu k^\nu + p\sigma^{AB} e_A^\mu e_B^\nu + j^A(k^\mu e_A^\nu + e_A^\mu k^\nu), \tag{30}$$

where ς is the shell's surface energy density, p is an isotropic surface pressure, and j^A is surface current that is zero in the spherical case. From the extrinsic perspective these quantities are related to the discontinuity of the derivative of the continuous interpolating metric via $\gamma_{\mu\nu} := [\bar{g}_{\mu\nu,\alpha}]\bar{N}^\alpha$ where the calculation is performed using the coordinates $\bar{x}^\mu$. From the intrinsic perspective the key quantity is a transverse curvature $\mathfrak{C}_{ab}$,

$$\mathfrak{C}_{ab} := -N_\mu e_{a;\nu}^\mu e_b^\nu. \tag{31}$$

Density, pressure and current are obtained from its discontinuity, $[\mathfrak{C}_{ab}] = \frac{1}{2}\gamma_{ab}e_a^\mu e_b^\nu$. A straightforward calculation identifies

$$\varsigma = \frac{C}{8\pi R^2}, \tag{32}$$

while the pressure is directly connected to the preservation of the null condition $k_\mu k^\mu = 0$,

$$p = -\frac{[\kappa]}{8\pi}, \tag{33}$$

where $\kappa = \mathfrak{C}_{\lambda\lambda} = -N_\mu k^\mu_{;\nu}k^\nu$ measures the failure of λ to be the affine parameter, with acceleration $a^\mu_\pm := k^\mu_{\pm;\nu}k^\nu_\pm$ being

$$a^\mu_\pm = \kappa_\pm k^\mu_\pm, \tag{34}$$

on either side of the shell. Note that unlike its massive counterpart, a massless shell moves on a geodesic, possibly non-affinely parameterized. The shell that becomes null at $\lambda = \lambda_0$ and continues as as null for $\lambda > \lambda_0$ should satisfy Eq. (34) already at $\lambda = \lambda_0$.

At this stage two options are possible. One is that the spacetime outside the shell is still described by the outgoing Vaidya metric. The other is that the form of the metric changes at the null transition. We shall explore each in turn.

Suppose the spacetime outside the shell retains its outgoing Vaidya form. The shell will continue on a null trajectory for $\lambda > \lambda_0$, provided it acquires a surface pressure. Indeed, the acceleration of the shell expressed in the outside and the inside coordinates is

$$a^\mu_+ = \begin{pmatrix} U_{\lambda\lambda} - 2F_R/F^2 \\ 2F_U/F^2 \end{pmatrix}, \qquad a^\mu_- = \begin{pmatrix} W_{\lambda\lambda} \\ 0 \end{pmatrix}, \tag{35}$$

respectively, where we suppressed the trivial angular components. Then from Eqs. (7) and (35) it follows that

$$\kappa_- = 0, \qquad \kappa_+ = -2F_U/F^2, \tag{36}$$

resulting in the surface pressure

$$p = -\frac{C_U}{4\pi R(1 - C/R)^2} > 0 \tag{37}$$

upon using (33). Furthermore, (as we shall see in Sec. 4) $R \geq C + \epsilon_*$, and so the pressure is finite for all finite values of C_U. The shell moves on a null geodesic with a consistent value of the second derivative $U_{\lambda\lambda} \equiv d(2/F)/d\lambda$ for $\lambda \geq \lambda_0$.

Alternatively, one could impose the requirement $p \equiv 0$ as was done by CUWY [29]. A combination of the null shell property $W_\lambda = 2$ (Eq. (A.16)) and

$$F_\lambda = -F_R + F_U U_\lambda = -F_R + 2F_U/F, \tag{38}$$

then ensures that the only consistent solution for $\lambda > \lambda_0$ that is not superluminal is

$$F_U \equiv 0, \qquad U_{\lambda\lambda} = \frac{2F_R}{F^2} = -\frac{2F_\lambda}{F^2}, \qquad W_\lambda = 2. \tag{39}$$

In other words, emission of any radiation terminates and (as C_u must drop to zero at $\lambda = \lambda_0$) the metric has a discontinuity in the first derivative, reducing it to the Schwarzschild metric expressed in Eddington–Finkelstein coordinates. We discuss the superluminal solution in Appendix E.

We therefore see that it is impossible to have both a pressureless null shell and a shrinking Schwarzschild radius $dr_g/du < 0$ if the exterior metric is the outgoing Vaidya.

The surface stress-energy tensor of Eq. (30) satisfies the weak energy condition for positive values of ς and p. Indeed, for an arbitrary timelike vector $t^\mu = (\dot{u}, \dot{r}, \dot{\theta}, \dot{\phi})$ we have

$$S_{\mu\nu}t^\mu t^\nu = \varsigma(k_\mu t^\mu)^2 + pR^2(\dot{\theta}^2 + \sin^2\Theta\dot{\phi}^2) \geq 0. \tag{40}$$

We discuss other energy conditions and additional properties of thin shells elsewhere [32]. The transverse pressure p, however, may not be a benign feature of the solution. Having normal matter is not sufficient to rule out the superluminal propagation of disturbances, i.e. to guarantee that the speed of sound is less then the speed of light [33,34]. However for the case at hand the motion of the shell determines the surface quantities (p, ς) via the junction conditions; there is no equation of state and thus no well-defined speed of sound. Note that for the most part the shell is super-stiff, i.e., $p > \varsigma$ [34].

This leads us to the other option: the metric has a different discontinuity at the null transition allowing both zero pressure and a shrinking r_g. For a general metric (2) the two first components of Eq. (34) become

$$U_{\lambda\lambda} + \left(-\tfrac{1}{2}e^H F_R - e^H F H_R + H_U\right)\frac{4e^{-2H}}{F^2} = \kappa\frac{2e^{-H}}{F}, \tag{41}$$

$$\frac{2e^{-H}F_U}{F^2} + H_R = -\kappa, \tag{42}$$

where $H = h(U, R)$. Given the functions f and h this pair of equations yields κ and $U_{\lambda\lambda}$. The shells continues on a null trajectory, with the velocity $U_\lambda = 2e^{-H}/F$ in a general metric of Eq. (2) satisfying

$$U_{\lambda\lambda} = d(2e^{-H}/F)/d\lambda. \tag{43}$$

Since $\kappa_- \equiv 0$ it is enough to require $\kappa = 0$ to ensure that the shell remains pressureless while continuing to move on a null geodesic. In this case Eq. (42) will serve as a constraint on the exterior metric: $H_R = -2e^{-H}F_U/F^2$. The evaporation continues with $F_U > 0$. There is a discontinuity in the derivative of the metric, $h(U(\lambda_0), R(\lambda_0)) = 0$, $\partial_R h(U(\lambda_0), R(\lambda_0)) = -2F_U/F^2$, and for $\lambda > \lambda_0$ the metric is not of the Vaidya form, but rather of the form (2).

The question concerning which scenario is actually realized can be answered only through detailed study involving matching of the bulk stress-energy tensor that results in a self-consistent analysis of evaporation.

4. Horizon and singularity avoidance

We have seen that the shell, once it becomes null, continues as such. Here we show how the event horizon is avoided if we still model the exterior geometry by the outgoing Vaidya metric. In this case the shell must have a surface pressure (37). A general analysis, including comparison of the evolution described in different coordinate systems will be presented elsewhere.

The key quantity is the gap

$$X = R - r_g = R - C, \tag{44}$$

where the last equality holds only for the Vaidya metric. This quantity can be viewed as either function of w or λ, via the relationships $R(w)$, $U(w)$, or $R(\lambda)$, $U(\lambda)$ respectively. Evaluating its derivative over, e.g., λ, we have

$$X_\lambda = R_\lambda - \frac{dC}{dU} U_\lambda = -1 + \left| \frac{dC}{dU} \right| \frac{2}{F} > -1 + \frac{2|C_U|C}{X}. \tag{45}$$

As a result the gap decreases only until $X \approx \epsilon_* := 2C|C_U|$, and crossing of the Schwarzschild radius is possible only if the evaporation completely stops. A detailed evaluation of the approach to ϵ_* is given in Appendix B.

We pause to comment that some solutions exhibiting collapsing matter that does not cross its Schwarzschild radius have been obtained, but these still have unavoidable singularities; for example, a $(3 + 1)$ scenario based on the outgoing Vaidya metric [35] or a $(2 + 1)$ solution that ends in a conical singularity [36].

Nevertheless, regular scenarios may arise in the outgoing Vaidya metric with finite evaporation retarded time, $C(u_E) = 0$. We note first that if $\lim_{u \to u_E} \epsilon_* > 0$ then $R > 0$ at the last moment of evaporation. Since this limit implies that C_U is divergent and $F = 0$, Eq. (37) results in divergent surface pressure.

On the other hand, if

$$0 < K := -\lim_{u \to u_E} C_u < \infty, \tag{46}$$

Eq. (45) does not prevent the gap at u_E being zero, i.e. collapse to the point $R = C(u_E) = 0$. In this scenario the surface density $\varsigma \propto C/R^2$ diverges. However, as the following necessary condition shows such collapse is possible only for some values of K. Consider Eq. (11) that using $R(\lambda) = r_0 - \lambda$ we write as

$$\frac{dU}{d\lambda} = 2 \left(1 - \frac{C(U(\lambda))}{r_0 - \lambda} \right)^{-1}, \tag{47}$$

where the evaporation lasts from $\lambda = 0$ (and thus $U(0) = 0$) until $\lambda \le r_0$, and the singularities can appear if $u_E = U(r_0)$. Assume that this is the case. Then by taking the limit $\lambda \to r_0$ on both sides of Eq. (47) we obtain the equation

$$\mathfrak{u}' = \frac{2}{1 - K\mathfrak{u}'}, \tag{48}$$

that $\mathfrak{u}' := \lim_{\lambda \to r_0} U_\lambda$ should satisfy. It has real solutions only if $K \le 1/8$. Hence if the limit of evaporation rate at the end of the process is $K > 1/8$ the shell disappears at finite radius $R > 0$ without singularity, as

$$\varsigma \to 0, \qquad p \to 0. \tag{49}$$

Conversely, if $K \le 1/8$ we expect the semi-classical approximation to break down before a singularity is reached.

5. Discussion

If the spacetime outside a massive thin shell is modelled by an outgoing Vaidya metric or by an arbitrary spherically symmetric metric with $h(u, r_g(u)) < \infty$ then initially massive thin shells shed their rest mass and become null.

Once the shell becomes null there are several options, that are best illustrated by the evolution of a massive dust shell with the exterior geometry given by the outgoing Vaidya metric:

1. The metric retains its Vaidya form but has a derivative discontinuity, though this perhaps could be ameliorated if the process halts sufficiently smoothly. The pre-Hawking radiation either halts abruptly or smoothly if the latter situation holds. Collapse to a black hole proceeds classically on a null geodesic (as a hypersurface separating Minkowski and Schwarzschild geometries).
2. The metric retains its Vaidya form and the shell remains pressureless, in which case it must become superluminal. This option can be discarded as unphysical [29].
3. The shell acquires surface pressure discontinuously (though one could consider modelling this as a smooth but rapid transition) and propagates on a null geodesic (as a hypersurface separating Minkowski and outgoing Vaidya geometries). In this case an horizon does not form, as shown in Section 4. A transient naked singularity may form both in the case of the shell collapsing to a point at the last moment of evaporation and as ring singularity at finite radius. Alternatively, the shell may just evaporate away. What happens depends on the precise limiting form of the evaporation rate.
4. At the opposite extreme the shell remains pressureless, for a part or the entire duration of its evolution. There is a derivative discontinuity in the metric, but the exterior Vaidya form is not retained. The shell propagates on a null geodesic (as a hypersurface separating Minkowski and a generalized outgoing Vaidya geometries). It is possible to show by modifying the analysis of [18] along the lines of Sec. 4 that if $h\big(u, r_g(u)\big) < \infty$ the shell does not cross its Schwarzschild radius. A more plausible option is a combination of some surface pressure and $h(u, r) \neq 0$, ensuring subluminal propagation of density perturbations. We will address the questions of regularity of this scenario in future work [32].

In any version of scenarios 3 and 4 there three possibilities for the final fate of the shell: it may smoothly fade away, or may produce a point-like transient singularity with $R = 0$ or else shell-like with $R > 0$. Which possibility is realized depends on the metric and will be discussed in [32].

It is clear that radiation can modify thin shell collapse in a variety of ways. In particular, it can lead to horizon avoidance or to evaporation suppression. The avoidance scale ϵ_* is sub-Planckian for a macroscopic shell. Since the Schwarzschild radius r_g is inside the interior flat domain, this is also the physical scale of apparent horizon avoidance. Hence even a consistent scenario invokes sub-Planckian distances, and it may indicate an early breakdown of the semiclassical approximation at the scale of the Schwarzschild radius (as, for example, was argued in [23]) — well before the collapsing object is compressed to the Planck scale. Only a more explicit analysis of the coupled matter-gravity systems will allow use to determine whether avoidance or suppression are universal features, along with consistency of the resulting scenarios. Such investigations should point the way toward a more complete description of gravitational collapse that goes beyond the semi-classical approximation.

Appendix A. Interpolating metric

Extending the construction of [29] we adapt the coordinates $\bar{x}^{\mu} = (w, z, \theta, \phi)$, where $w := u_{-}$ and

$$r =: \begin{cases} R(w) + z, & z \leq 0 \\ R(w) + z \exp[-h(U(w), R(w))]/U_{w}, & z > 0. \end{cases} \tag{A.1}$$

We abbreviate $h(U(w), R(w))$ as $H(w)$. For general exterior point (u, r) the coordinates (w, z) are obtained by identifying $u_{w} \equiv U_{w}$. Equivalently, after finding the radial coordinate of the shell at the moment of the retarded time w, one obtains the equation $R_{-}(w) = R_{+}(U) \equiv R_{+}(u)$ that can be solved for $u(w)$. We explicitly use the subscripts indicating the spacetime domain because even if the radial coordinate is continuous, the functional dependence on the relevant retarded time is different in each region. In addition we note that

$$R_{w} := \frac{dR_{-}}{du_{-}} = \frac{dR_{+}}{du_{+}} U_{w}. \tag{A.2}$$

We will need the explicit form of the Jacobian on the shell:

$$\frac{\partial u}{\partial w}\Big|_{\Sigma} = U_{w}, \qquad \frac{\partial u}{\partial z}\Big|_{\Sigma} = 0, \qquad \frac{\partial r}{\partial w}\Big|_{\Sigma} = R_{w}, \qquad \frac{\partial r}{\partial z}\Big|_{\Sigma} = \frac{e^{-H}}{U_{w}}. \tag{A.3}$$

The first junction condition in the form

$$1 + 2R_{w} = FU_{w}^{2} + 2U_{w}R_{w}, \tag{A.4}$$

and the requirement that both u_{-} and u_{+} increase together result in the explicit expression

$$U_{w} = \frac{-R_{w} + \sqrt{R_{w}^{2} + F(1 + 2R_{w})}}{F}. \tag{A.5}$$

The tangent vector is continuous in the coordinates $\bar{x}^{\mu}$ across the shell. Both

$$k_{+}^{\mu} = (U_{\lambda}, -1, 0, 0) = W_{\lambda}(U_{w}, R_{w}, 0, 0), \qquad k_{-}^{\mu} = (W_{\lambda}, -1, 0, 0) = W_{\lambda}(1, R_{w}, 0, 0), \tag{A.6}$$

become

$$\bar{k}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x_{\pm}^{\alpha}} k_{\pm}^{\alpha} = (W_{\lambda}, -R_{w}W_{\lambda} - 1, 0, 0) = (W_{\lambda}, 0, 0, 0), \tag{A.7}$$

where we used Eqs. (A.1), (A.2) and (A.3).

In these coordinates the metric inside the shell is written as

$$ds_{-}^{2} = -(1 + 2R_{w})dw^{2} - 2dwdz + (R + z)^{2}d\Omega_{2}. \tag{A.8}$$

Outside the shell we have

$$du = U_{w}dw, \tag{A.9}$$

and

$$dr = \left[R_{w} - \frac{z}{e^{H}U_{w}}\left(\frac{U_{ww}}{U_{w}} + H_{w}\right)\right]dw + \frac{dz}{e^{H}U_{w}}, \tag{A.10}$$

where

$$H_w = h_R(U, R)R_w + h_U(U, R)U_w. \tag{A.11}$$

As a result the metric is

$$ds_+^2 = -\left[\bar{f}e^{2\bar{h}}U_w^2 + 2e^{\bar{h}}U_w R_w - 2ze^{\bar{h}-H}\left(\frac{U_{ww}}{U_w} + H_w\right)\right]dw^2$$
$$- 2e^{\bar{h}-H}dwdz + r^2(w,z)d\Omega_2, \tag{A.12}$$

where

$$\bar{f} = 1 - \frac{C\big(u(w,z), r(w,z)\big)}{r(w,z)}, \qquad \bar{h}(w,z) = h\big(u(w,z), r(w,z)\big). \tag{A.13}$$

While the shell is timelike the normalization $v_\mu v^\mu = -1$ implies

$$\dot{W}^2(1 + 2R_w) = 1, \tag{A.14}$$

resulting in

$$\dot{W} = -\dot{R} + \sqrt{\dot{R}^2 + 1} \approx -2\dot{R} = 2\dot{\lambda}, \tag{A.15}$$

where the approximate equality holds for the large values of $|\dot{R}|$. When the shell becomes null the first junction condition leads to

$$W_\lambda = 2, \qquad R_w = -\tfrac{1}{2}, \qquad U_w = \frac{e^{-H}}{F}, \qquad U_{ww} = -\frac{e^{-H}}{F}\left(H_w + \frac{F_w}{F}\right). \tag{A.16}$$

The interior metric becomes

$$ds_-^2 = -2dwdz + (R+z)^2 d\Omega_2, \tag{A.17}$$

and the exterior metric simplifies to

$$ds_+^2 = -\left[\frac{\bar{f}}{F^2}e^{2(\bar{h}-H)} - \frac{e^{\bar{h}-H}}{F} + 2ze^{\bar{h}-H}\frac{F_w}{F}\right]dw^2$$
$$- 2e^{\bar{h}-H}dwdz + r^2(w,z)d\Omega_2, \tag{A.18}$$

with

$$r(w,z) = R(w) + zF(w). \tag{A.19}$$

Appendix B. Estimate of gravitational mass loss and the closest approach to the Schwarzschild radius

First we show that from the moment the evaporation becomes important (or switched-on in the numerical simulation) and until the shell loses all of its rest mass only a relatively small fraction of the Bondi–Sachs mass $C/2$ evaporates. Equivalently, the elapsed interval of the EF coordinate u is much smaller then the evaporation time, $\Delta U \ll u_E$. If the evaporation is governed by Eq. (18) then

$$C = \big(C_0^3 - 3\alpha u\big)^{1/3}, \tag{B.1}$$

where the evaporation time is given by $u_E = C_0^3/3\alpha$.

For $C_0 \gg 1$ we can assume $C = C_0$ up to the transition as the first approximation. In this case using Eqs. (5) and (10) we have

$$U_\lambda \lesssim \frac{2}{F} = \frac{2\lambda}{\lambda + C} \approx -\frac{2C_0}{\lambda + C_0}, \tag{B.2}$$

where we also assume that $X \ll C$.

The integration from the "initial" R_i (a quantity sharply defined in the simulation and approximately in, e.g., adiabatic approximation) to the radial coordinate R_0 where the shell becomes null results in

$$\Delta U \approx 2C \ln \frac{X_i}{X_0} \approx 2C \ln \frac{X_i}{2C|C_U|} \approx 2C_0 \ln \frac{C_0 X_i}{2\alpha}. \tag{B.3}$$

The second equality is obtained by assuming that $X_0 = \epsilon_*$. Numerical simulations indicates the actual value is different by a factor 2–10, but this precision is sufficient for our estimate. Substituting this result into Eq. (B.1) we find that the relative reduction of the rest mass is

$$\frac{|\Delta C|}{C_0} = \frac{\alpha}{C_0^2} \ln \frac{C_0 X_i}{2\alpha} \ll 1. \tag{B.4}$$

We now provide a better estimate of the mass loss that also demonstrates how the shell radius R approaches the Schwarzschild radius. We assume that $X \ll C$, but take into account Eq. (B.1). Hence the approach of the shell to the Schwarzschild radius is governed by the equation

$$\frac{dU}{d\lambda} \approx -\frac{2C(U)}{\lambda + C(U)} \quad \Leftrightarrow \quad \frac{d\lambda}{dU} = -\frac{1}{2} - \frac{\lambda}{2C}. \tag{B.5}$$

The first equation is exact for the null shell. Solution of the approximate equation is

$$-X \equiv \lambda + C = e^{C^2/4\alpha} \left(L + \sqrt{\alpha\pi}\, \mathrm{Erf}(C/2\sqrt{\alpha}) \right), \tag{B.6}$$

where $\mathrm{Erf}(z)$ is the error function, and L is determined by the initial conditions.

It allows us to find the minimal gap X between a (massive or null) shell and its Schwarzschild radius. Setting $U(\lambda_0) = 0$ and $\lambda_0 = -C_0 - X_0$, and approximating the error function of a large argument as 1, we find that

$$L = -X_0 e^{-C_0^2/4\alpha} - \sqrt{\pi\alpha}, \tag{B.7}$$

where we suppressed the exponentially small correction terms, and for $C_0 \geq C \gg \sqrt{\alpha}$ we find

$$X = X_0 e^{-(C_0^2 - C^2)/4\alpha} + \frac{2\alpha}{C} \approx \epsilon_*, \tag{B.8}$$

where the last equality holds for $C_0 \gg C$, in agreement with the discussion in Section 4. Note that the condition $C^2/\alpha \gg 1$ corresponds to the adiabatic condition of [26].

Appendix C. Equations of motion and transition to a null trajectory in a general case

Dynamics of the shell is obtained via the second junction condition [3] that equates the discontinuity of the extrinsic curvature K_{ab} with the surface energy-momentum tensor S_{ab},

$$S_{ab} = -([K_{ab}] - [K]h_{ab})/8\pi, \tag{C.1}$$

where $K := K_a^a$, and $[K] := K|_{\Sigma^+} - K|_{\Sigma^-}$ is the discontinuity of the extrinsic curvature K across the two sides $\Sigma^\pm$ of the surface.

We derive the equation of motion for a massive thin shell following the steps that are outlined in [17]. For a massive shell separating the flat Minkowski interior from a generic spherically-symmetric geometry of Eq. (2) outside, components of the four-velocity $v^\mu = (\dot{U}, \dot{R}, 0, 0)$ satisfy

$$\dot{U} = \frac{-\dot{R} + \sqrt{F + \dot{R}^2}}{\bar{E} F}, \tag{C.2}$$

where $H := h(U, R)$ and $\bar{E} := \exp(H)$. The outward-pointing unit spacelike normal is

$$\hat{n}_\mu = \bar{E}(-\dot{R}, \dot{U}, 0, 0), \tag{C.3}$$

and the most convenient coordinates to use on the surface are (τ, θ, ϕ).

Here we calculate the extrinsic curvature outside the shell (and to reduce the clutter suppress the subscript '+'). Expressions for the interior are analogous. Components of the extrinsic curvature are given by

$$K_{\tau\tau} = -\hat{n}_\alpha v^\alpha_{;\beta} v^\beta = -\hat{n}_\alpha a^\alpha, \qquad K_{\theta\theta} = \hat{n}_{\theta\,;\theta}, \qquad K_{\phi\phi} = \hat{n}_{\phi\,;\phi}. \tag{C.4}$$

Straightforward evaluation results in

$$K_{\tau\tau} = \frac{2\ddot{R} + F_R}{2\sqrt{F + \dot{R}^2}} + \frac{F_U}{2F\sqrt{F + \dot{R}^2}}\left(\bar{E}^{-1} - 2\dot{R}\dot{U}\right) + H_R\sqrt{F + \dot{R}^2}, \tag{C.5}$$

and

$$K^\theta_\theta = K^\phi_\phi = \sqrt{\dot{R}^2 + F}/R. \tag{C.6}$$

The two independent equations are

$$S^\tau_\tau = -\sigma = \frac{1}{4\pi}\left(\frac{\sqrt{F + \dot{R}^2} - \sqrt{1 + \dot{R}^2}}{R}\right), \tag{C.7}$$

that determines the evolution of the rest mass $m = 4\pi R \sigma$, and

$$8\pi p(\sigma) = \frac{2\ddot{R} + F_R}{2\sqrt{F + \dot{R}^2}} - \frac{\ddot{R}}{\sqrt{1 + \dot{R}^2}} + \frac{\sqrt{F + \dot{R}^2} - \sqrt{1 + \dot{R}^2}}{R}$$
$$+ \frac{F_U}{2F\sqrt{F + \dot{R}^2}}\left(\bar{E}^{-1} - 2\dot{R}\dot{U}\right) + H_R\sqrt{F + \dot{R}^2} \tag{C.8}$$

that upon substitution of $\dot{U}$ from Eq. (C.2) results in the equation of motion for R. The first three terms correspond to the classical collapse, the fourth term appears if the spacetime outside the shell is described by an outgoing Vaidya metric, and the full expression for a general spherically-symmetric geometry outside. It reduces to Eq. (14) when $h \equiv 0$ and $C(u, r) \to C(u)$. Note that in the general case $F_R = C/R^2 - \partial_R C/R$.

The asymptotic expression for the radial acceleration is

$$\ddot{R} \approx -\frac{2\dot{R}^2 F_U}{F\bar{E}}\left(1 + \frac{2\dot{R}^2}{F}\right) \approx \frac{4C_U \dot{R}^2}{F^2 \bar{E}}. \tag{C.9}$$

It is valid if both the classical contribution to acceleration and the term proportional to H_R become negligible close to the Schwarzschild radius r_g, $f(u, r_g) = 0$. This is possible if $fe^h \to 0$ when $r \to r_g$. If the function h is finite, i.e. $h < \mathfrak{h} < \infty$ for some constant $\mathfrak{h}$ for all $r \geq r_g(u)$ then $\bar{E} < e^{\mathfrak{h}} < \infty$ then this condition is satisfied and the shell becomes null in a finite proper time.

Appendix D. Alternative expressions for the surface pressure

There are additional methods to calculate the surface pressure. One is based on Raychaudhuri's equation, giving

$$[\kappa]\theta = [G_{\mu\nu}k^{\mu}k^{\nu}], \tag{D.1}$$

where θ is expansion of the geodesic congruences. In our case $\kappa_{-} \equiv 0$, for an incoming spherical null shell $\theta = -2/R$, and the non-zero components of the Einstein tensor outside the shell are

$$G_{uu} = e^{2h_1}\frac{\partial_r C}{r^2}\left(1 - \frac{C}{r}\right) - e^{h_1}\frac{\partial_u C}{r^2}, \tag{D.2}$$

$$G_{ur} = e^{h_1}\frac{\partial_r C}{r^2}, \tag{D.3}$$

$$G_{rr} = \frac{2\partial_r h_1(u,r)}{r}. \tag{D.4}$$

A different method is based on the direct use of the discontinuity

$$p = -\frac{1}{16\pi}\gamma_{\mu\nu}\bar{k}^{\mu}\bar{k}^{\nu}, \qquad \gamma_{\mu\nu} := [\bar{g}_{\mu\nu,\alpha}]\bar{N}^{\alpha}. \tag{D.5}$$

For the interpolating metric of Appendix A the tangent vector at the timelike-to-null transition becomes

$$\bar{k}^{\mu} = (2,0,0,0), \tag{D.6}$$

and the auxiliary null vector

$$\bar{N}^{\mu} = (0,\tfrac{1}{2},0,0). \tag{D.7}$$

Hence

$$p = -\frac{1}{16\pi}2 \times 2 \times \tfrac{1}{2}\bar{g}_{00,z}^{+} = \frac{1}{8\pi}\left(H_R + \frac{2F_U}{e^H F^2}\right), \tag{D.8}$$

in agreement with Eq. (42).

Appendix E. Tachyons and superluminality

The appearance of tachyonic behavior is most easily observed by expressing the equation of motion for the shell in terms of the parameter λ. While it can be done starting with the definition of K_{ab} in Eq. (13), it is easier to substitute

$$\dot{R} = -\dot{\lambda} = -\frac{1}{\sqrt{U_{\lambda}(FU_{\lambda} - 2)}}, \tag{E.1}$$

and

$$\ddot{R} = -\frac{d\dot{\lambda}}{d\lambda}\dot{\lambda} = \tfrac{1}{2}\left(FU_{\lambda}^2 - 2U_{\lambda}\right)^{-2}\left(2U_{\lambda\lambda}(FU_{\lambda} - 1) + F_{\lambda}U_{\lambda}^2\right), \tag{E.2}$$

into Eq. (14) and set $p \equiv 0$. We move directly to the asymptotic expression Eq. (16). In λ-parameterization it becomes

$$2U_{\lambda\lambda}(FU_\lambda - 1) + F_\lambda U_\lambda^2 = \frac{8C|C_U|}{(R-C)^2}. \tag{E.3}$$

At the timelike-to-null transition at λ_0 we have $U_{\lambda_0} = 2/F(\lambda_0) \equiv 2/F_0$. Hence we have

$$U_{\lambda_0\lambda_0} + \frac{2F_{\lambda_0}}{F_0^2} = -\frac{4C_0|C_{U_0}|}{(R_0 - C_0)^2}. \tag{E.4}$$

If the right hand side of this equation is non-zero, then the null consistency condition $U_{\lambda\lambda} = d(2/F_\lambda)/d\lambda$ is violated and the shell must become tachyonic. However, this happens solely because the equation lacks the pressure contribution that appears in the correct equation of motion Eq. (41).

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