Quantum Walks on Oriented Graphs

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis extends results about periodicity and perfect state transfer to oriented graphs. We prove that if a vertex a is periodic, then elements of the eigenvalue support lie in $\mathbb{Z}\left(\sqrt{\Delta}\right)$ for some squarefree negative integer Δ. We find an infinite family of orientations of the complete graph that are periodic. We find an example of a graph with both perfect state transfer and periodicity that is not periodic at an integer multiple of the period, and we prove and use Gelfond-Schneider Theorem to show that every oriented graph with perfect state transfer between two vertices will have both vertices periodic. We find a complete characterization of when perfect state transfer can occur in oriented graphs, and find a new example of a graph where one vertex has perfect state transfer to multiple other vertices.

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Chapter 1

Introduction

1. INTRODUCTION

The motion of a quantum particle in a crystal lattice can be modeled as a continuous quantum walk, e^{-itH} where t is the time operator and H is the Hamiltonian representing the system. Hamiltonian matrices store information about the energy and position of quantum particles, although in practice computing the Hamiltonian is a non-trivial task. More information about the physical interpretation of a quantum walk can be found in quantum mechanics references, such as Feynman [10].

We can model this mathematically by thinking about the lattice as a graph, and the Hamiltonian as some Hermitian matrix related to the graph. Common choices of matrix are the adjacency matrix, the Laplacian matrix, and the unsigned Laplacian matrix.

Continuous quantum walks on graphs are also relevant in quantum computing as algorithms as discussed in Ambainis [2]. Two key kinds of algorithms are hitting algorithms and search algorithms. In the first type, we have a graph, a starting vertex, and an ending vertex, and we are looking at the earliest time we hit the ending vertex via random walk. For certain kinds of graphs, the hitting time of a continuous quantum walk is exponentially faster than for a classical random walk. In search algorithms, we are looking for a marked vertex on a graph. Search is one of the algorithms that is known to be faster on quantum computers than classical computers, so the ability of continuous quantum walks to implement fast search suggests their usefulness in creating new algorithms.

Outside of their relevance to quantum mechanics and quantum computing, quantum walks are also mathematically interesting. They can be analysed using algebraic graph theory and reveal new ways of studying and classifying graphs. For example, a quantum walk has perfect state transfer from vertex a to vertex b if at a certain time, the quantum walk starting at vertex a will be at vertex b with probability one. Vertices with perfect state transfer have a relationship to each other that is similar but distinct from known ways that vertices are related, such as cospectrality. The study of quantum walks is also related to other areas of math outside of algebraic graph theory, since number theoretic conditions keep appearing in the study of graphs with certain special properties.

Quantum walks have been studied on graphs using the adjacency matrix, Laplacian matrix, and unsigned Laplacian. All of these graphs have been necessarily undirected, since a directed graph will not have a Hermitian adjacency matrix. However, if we allow allow oriented graphs where every single edge has a unique direction, then we can define a Hermitian matrix based off these directed edges by assigning every arc a weight of i in the direction of the arc and a weight of $-i$ in the opposite direction. This gives us a new kind of matrix to analyse, and opens the door to studying quantum walks on directed graphs.

Quantum walks using Hermitian non-symmetric matrices are not just a minor variation of the study of quantum walks. They have some dramatically different behavior. It was shown by Kay that on an undirected graph, perfect state transfer is monogomous, meaning a vertex can have perfect state transfer to at most one other vertex [16], but Cameron et.al found an example of an oriented graph on three vertices which had perfect state transfer between every two pairs of vertices [4]. Despite interesting behavior like this, quantum walks on oriented graphs have barely been studied, especially in comparison with what is known about quantum walks with the symmetric adjacency matrix.

This thesis extends results about non-oriented graphs to prove similar characterizations for oriented graphs. We find a complete characterization of the eigenvalues of graphs that have perfect state transfer to themselves, as well as draw on number theory to find an infinite family of orientations of the complete graph where every vertex is periodic. We prove several necessary conditions for when perfect state transfer can occur between distinct vertices. We also find counterexamples to properties that quantum walks on non-oriented graphs have, oriented graphs do not. We prove the Gelfond-Schneider Theorem, a classic result from transcendental number theory with repeated applications to quantum walks on graphs, both oriented and non-oriented. Finally, we find a characterization of vertex transitive graphs with perfect state transfer, as well as another graph that breaks the monogomous principle by having three partners in perfect state transfer.

Chapter 2

Background

The study of quantum walks on graphs draws on techniques in linear algebra, graph theory, group theory, field theory, and number theory, among other areas. Although this reveals interesting connections between different areas of mathematics, it also means that the amount of background that is needed is substantial. In this chapter, we develop enough of the theory to start, and will develop more as the need arises in the future.

Our primary tool in analysing quantum walks is spectral decomposition, so we will begin by looking at the spectral decomposition of normal matrices before turning our attention to oriented graphs and the properties of the adjacency matrices we define to go with them. Once we do that, we are in a position to define quantum walks on oriented graphs, and to draw attention to the particular kinds of behavior that we are interested in studying. Next, we draw a connection between oriented graphs and bipartite undirected graphs, then explore some of the basic properties of quantum walks on oriented graphs. We then go more algebraic, first by looking how quantum walks interact with automorphisms, then by developing basic facts about Cayley graphs and circulant graphs to help us with further examples. Finally, we introduce some basic number theory, and then some results from field theory to characterize the spectral idempotents.

2.1 Spectral Decomposition

An important tool in algebraic graph theory, and the study of quantum walks, is spectral decomposition. We will begin with a brief introduction. Our first theorem is a famous result in linear algebra, which can be found in standard textbooks, such as Zhang [20].

2.1.1 Theorem (Spectral Theorem)**.** *Let A be a matrix. Then A is normal if and only if there exists a unitary matrix L and a diagonal matrix D such that*

$$A = L^* DL.$$

We may use this to find a spectral decomposition, as in Godsil and Royle [14].

2.1.2 Theorem. *Let A be a normal matrix and consider the set $\theta_1, \dots, \theta_d$ of distinct eigenvalues in A. Then for each eigenvalue θ_r there exist pairwise* *orthogonal idempotent projections into the θ_r eigenspace, denoted E_r such that*

$$A = \sum_{r=1}^{d} \theta_r E_r.$$

Proof. Because A is normal, we know that we may write

$$A = L^* D L$$

where D is a diagonal matrix with eigenvalues as entries and L is unitary.

Now, for all r we define the diagonal matrix

$$(D_r)_{j,j} = \begin{cases} 1 & D_{j,j} = \theta_r \\ 0 & \text{otherwise} \end{cases}.$$

Let $E_r = L^* D_r L$.

Then we see that

$$A = L^* D L = L^* \left(\sum_r D_r \right) L = \sum_r \theta_r E_r.$$

Now, let θ_r and θ_s be eigvenvalues. Then we compute that

$$E_r E_s = L^* D_r L L^* D_s L = L^* D_r D_s L = \begin{cases} E_r & r = s \\ 0 & r \neq s \end{cases}.$$

Therefore, the E_r matrices are pairwise orthogonal idempotents.

Finally, since

$$A E_r = \left(\sum_s \theta_s E_s \right) E_r = \theta_r E_r$$

and E_r is idempotent, we can see that E_r represents projection into the θ_r eigenspace. □

For an eigenvalue θ_r, the corresponding idempotent E_r is called the *spectral idempotent.* The spectral idempotents have some nice properties, also found in [14].

2.1.3 Theorem. *Let θ_r be an eigenvalue of a normal matrix A with corresponding idempotent E_r. Then the following properties hold:*

(a) *The idempotent E_r is Hermitian.*

(b) *The idempotent E_r is positive semidefinite.*

(c) *The sum of spectral idempotents is the identity.*

Proof. We begin by noting that

$$E_r^* = (L^* D_r L)^* = L^* D_r L = E_r,$$

so the spectral idempotent is Hermitian.

Next, we see that

$$E_r = (D_r L)^* \left(D_r L\right),$$

so E_r is a positive semidefinite matrix.

Finally, we have that

$$\sum_r E_r = \sum_r L^* D_r L = \text{Ł}^* \left(\sum_r D_r \right) L = L^* I L = I,$$

by the construction of our idempotents. □

Part of the power of spectral decomposition comes in evaluating functions that take matrices as arguments. In particular, since the spectral idempotents are pairwise orthogonal, when we evaluate any polynomial in a matrix A, it is equivalent to evaluating the polynomial at the eigenvalues and then multiplying by the idempotents. Using power series, we can expand this to other functions.

2.1.4 Theorem. *Let A be a normal matrix with spectral decomposition*

$$A = \sum_{r=1}^{d} \theta_r E_r,$$

and let f be a univariate function whose Taylor series converges to f on the spectrum of A. Then

$$f(A) = \sum_{r=1}^{d} f(\theta_r) E_r.$$

We now turn our focus to a particular kind of normal matrix associated with oriented graphs.

2.2 Skew Symmetric Matrices and Oriented Graphs

An *oriented graph* is a simple graph where every edge has a single direction. We will usually use X to refer to an oriented graph.

The adjacency matrix of an oriented graph is the matrix $A(X)$ indexed by vertices of X where

$$(A(X))_{a,b} = \begin{cases} 1 & \text{there is an edge from } a \text{ to } b \\ -1 & \text{there is an edge from } b \text{ to } a \\ 0 & \text{there are no edges between } a \text{ and } b. \end{cases}$$

If the graph is clear from context, we will usually just write the adjacency matrix as A.

Let X be a graph with n vertice. If a is a vertex in X, then the *characteristic vector* of a, denoted $\mathbf{e}_a$, is the $n \times 1$ vector with a one in the row that indexes vertex a and a zero in every other row.

A real matrix A is said to be *skew symmetric* if $A^T = -A$.

Skew symmetric matrices are closely related to Hermitian matrices. The following result about Hermitian matrices is standard, and can be found in [20].

2.2.1 Theorem. *Let A be a Hermitian matrix. Then all the eigenvalues of A are real, and A has an orthonormal basis of eigenvectors.*

This can then be adapted to skew symmetric matrices, as noted by Godsil [6].

2.2.2 Corollary. *Let A be a skew symmetric matrix with real entries. Then the real part of every eigenvalue of A is zero, and A has an orthonormal basis of eigenvectors.*

Proof. Observe that for a skew skymmetric matrix A, we have

$$(-iA)^* = iA^T = -iA$$

so $-iA$ is Hermitian. Let θ be an eigenvalue of $-iA$ with corresponding eigenvector v. Then

$$Av = (i\theta)\, v,$$

so $i\theta$ is an eigenvalue of A with eigenvector v. Since $-iA$ has an orthonormal basis of eigenvectors, A must as well, and since every eigenvalue of $-iA$ was real, every eigenvalue of A must be zero or imaginary. □

In fact, skew symmetric matrices have a symmetry to their eigenvalues that real symmetric matrices do not.

2.2.3 Lemma. *Let X be an oriented graph with skew-symmetric adjacency matrix A. If θ is an eigenvalue with idempotent E, then $-\theta$ is also an eigenvalue with corresponding idempotent $\overline{E}$.*

Proof. Because A has real entries and θ has no real part, we have

$$A\overline{E} = \overline{AE} = \overline{\theta E} = -\theta\overline{E},$$

so $-\theta$ is an eigenvalue with corresponding idempotent $\overline{E}$. □

This symmetry will prove especially useful in our study of quantum walks.

2.3 Quantum Walks

Now that we know how to evaluate functions on mattrices associated to oriented graphs, we may define a continuous quantum walk.

Although the adjacency matrix A for an oriented graph is not Hermitian, iA is. Letting iA be the Hamiltonian, we get that the *quantum walk on an oriented graph* is

$$U(t) = e^{-itiA} = e^{tA} = \sum_r e^{t\theta_r} E_r.$$

At time t, the matrix $U(t)$ is sometimes referred to as the *transition matrix.* The transition matrix is unitary, and the sum of the squared norms of any row or column will be one.

For a vertex a, we can think of $U(t)\,\mathbf{e}_a$ as telling us the probabilities of where a particle starting at vertex a will be at time t. Depending on the graph and the time chosen, the probabilities could display several kinds of interesting behavior.

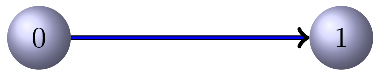

Figure 2.1: A path on two vertices.

2.3.1 Example. Consider the graph in Figure 2.1.

The adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and it has spectral decomposition

$$A = i\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} - i\frac{1}{2}\begin{pmatrix} 1 & i \\ -1 & 1 \end{pmatrix}.$$

The transition matrix is given by

$$U(t) = \begin{pmatrix} \frac{1}{2}\left(e^{it} - e^{-it}\right) & -i\frac{1}{2}\left(e^{it} + e^{-it}\right) \\ i\frac{1}{2}\left(e^{it} + e^{-it}\right) & \frac{1}{2}\left(e^{it} - e^{-it}\right) \end{pmatrix}.$$

Recalling Euler's identity gives us

$$U(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Now, we can use the transition matrix to see what is happening at a few key times. For example, at time $\frac{\pi}{2}$ we have transition matrix

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \\ -\sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

At time π we have the transition matrix

$$\begin{pmatrix} \cos(\pi) & \sin(\pi) \\ -\sin(\pi) & \cos(\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

And for our final example, at time $\frac{\pi}{4}$ the transition matrix is

$$\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Not only is K_2 a simple example to work with, it has the additional nice property that it demonstrates three different kinds of interesting behavior that graphs can have when they are analysed in terms of their quantum walks.

A graph has *perfect state transfer* from vertex a to vertex b if there exists some time τ and some complex phase γ with $|\gamma| = 1$ such that

$$U(\tau)\,\mathbf{e}_a = \gamma \mathbf{e}_b.$$

We can see that K_2 has perfect state transfer from vertex 0 to vertex 1 at time $\frac{\pi}{2}$. Intuitively, we can think of this as beginning our quantum walk at some vertex a and, some time τ later, finding ourselves inevitably at some other vertex b.

Perfect state transfer requires two distinct vertices, which leads us to a similar definition for when there is only one vertex in question. A graph is *periodic* at vertex a if there exists some time τ and some complex phase γ with $|\gamma| = 1$ such that

$$U(\tau)\,\mathbf{e}_a = \gamma \mathbf{e}_a.$$

We can see that K_2 is periodic at both vertices 0 and 1 at time π.

Perfect state transfer and periodicity both represent one kind of extreme behavior where a row of the transition matrix has one entry of norm one and every other entry is zero, so a quantum walk beginning at one state is guaranteed to end at a single other state. The opposite kind of extreme can also occur where a quantum walk starting at some vertex has an equal probability of being at any of the vertices in the graph at a certain time.

A vector or matrix is said to be *flat* if the absolute value of every entry is the same. A graph has *local uniform mixing* at vertex a if there exists some time τ such that $U(\tau)\,\mathbf{e}_a$ is flat. We can see that K_2 is flat at both vertex 0 and vertex 1 at time $\frac{\pi}{4}$.

2.4 Bipartite Graphs

We observe that every bipartite graph has a natural orientation by directing all of the edges from one colour class towards the second colour class. A quantum walk on the original graph and on this natural orientation will have similar spectral decompositions.

2.4.1 Theorem. *Let G be an undirected bipartite graph with adjacency matrix*

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some matrix B with entries either 0 or 1. Let θ be an eigenvalue with spectral idempotent

$$E = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

Then X, the oriented graph with adjacency matrix

$$\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

has eigenvalue $i\theta$ and spectral idempotent

$$\hat{E} = \begin{pmatrix} M_1 & -iM_2 \\ iM_3 & M_4 \end{pmatrix}.$$

Proof. Since E is idempotent we know that

$$E^2 = \begin{pmatrix} M_1^2 + M_2M_3 & M_1M_2 + M_2M_4 \\ M_3M_1 + M_4M_3 & M_3M_2 + M_4^2 \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

We now compute

$$\hat{E}^2 = \begin{pmatrix} M_1^2 + M_2M_3 & -iM_1M_2 - iM_2M_4 \\ iM_3M_1 + iM_4M_3 & M_3M_2 + M_4^2 \end{pmatrix} = \begin{pmatrix} M_1 & -iM_2 \\ iM_3 & M_4 \end{pmatrix} = \hat{E},$$

so we know that $\hat{E}$ is idempotent.

We also have that

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} BM_3 & BM_4 \\ B^TM_1 & B^TM_2 \end{pmatrix} = \theta \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

and can compute that

$$\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} M_1 & -iM_2 \\ iM_3 & M_4 \end{pmatrix} = \begin{pmatrix} iBM_3 & BM_4 \\ -B^TM_1 & iB^TM_2 \end{pmatrix} = i\theta \begin{pmatrix} M_1 & -iM_2 \\ iM_3 & M_4 \end{pmatrix},$$

and therefore $\hat{E}$ represents projection into the $i\theta$ eigenspace.

Finally, if E_r and E_s are distinct spectral idempotents of G, then they are orthogonal. Thus, using natural notation, we havee

$$\begin{pmatrix} M_{r_1} & M_{r_2} \\ M_{r_3} & M_{r_4} \end{pmatrix} \begin{pmatrix} M_{s_1} & M_{s_2} \\ M_{s_3} & M_{s_4} \end{pmatrix} = \begin{pmatrix} M_{r_1}M_{s_1} + M_{r_2}M_{s_3} & M_{r_1}M_{s_2} + M_{r_2}M_{s_4} \\ M_{r_3}M_{s_1} + M_{r_4}M_{s_3} & M_{r_3}M_{s_2} + M_{r_4}M_{s_4} \end{pmatrix} = \mathbf{0}$$

and therefore

$$\hat{E}_r\hat{E}_s = \begin{pmatrix} M_{r_1}M_{s_1} + M_{r_2}M_{s_3} & -iM_{r_1}M_{s_2} - iM_{r_2}M_{s_4} \\ iM_{r_3}M_{s_1} + iM_{r_4}M_{s_3} & M_{r_3}M_{s_2} + M_{r_4}M_{s_4} \end{pmatrix} = \mathbf{0},$$

so $\hat{E}_r$ and $\hat{E}_s$ are orthogonal.

From this we see that the idempotents $\hat{E}_r$ represent orthogonal projection into the $i\theta_r$ eigenspaces, giving us our eigenvalues and idempotents. □

The spectral decompositions are sufficiently similar that the quantum walks will have similar behavior.

2.4.2 Theorem. *If G is an unoriented bipartite graph with adjacency matrix*

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some matrix B with entries either 0 or 1 and X is the oriented graph with adjacency matrix

$$\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

then perfect state transfer, periodicity, and local uniform mixing occur in G if and only if they occur in X.

Proof. By the previous theorem, for an eigenvalue θ_r of G we may write the spectral decomposition

$$E_r = \begin{pmatrix} M_{r,1} & M_{r,2} \\ M_{r,3} & M_{r,4} \end{pmatrix},$$

and we know that X will have an eigenvalue of $\theta_r i$ with spectral idempotent

$$E_r = \begin{pmatrix} M_{r,1} & -iM_{r,2} \\ iM_{r,3} & M_{r,4} \end{pmatrix}.$$

Suppose that a and b are in the same colour class, without loss of generality assume they both lie in the first colour class. Then perfect state transfer from a to b occurs on G if and only if for all r we have

$$\begin{pmatrix} M_{r_1} & M_{r_2} \end{pmatrix} \mathbf{e}_a = \frac{\gamma}{e^{it\theta_r}} \begin{pmatrix} M_{r_1} & M_{r_2} \end{pmatrix} \mathbf{e}_b.$$

We can see that this is equivalent to

$$\begin{pmatrix} M_{r_1} & -iM_{r_2} \end{pmatrix} \mathbf{e}_a = \frac{\gamma}{e^{ti\theta_r}} \begin{pmatrix} M_{r_1} & -iM_{r_2} \end{pmatrix} \mathbf{e}_b,$$

so perfect state transfer occurs from a to b on G if and only if it occurs from a to b on X. Note that this case also covers periodic vertices.

Otherwise, suppose that a is in the first colour class and b is in the second. Then perfect state transfer from a to b occurs on G if and only if for all r we have

$$\begin{pmatrix} M_{r_1} & M_{r_2} \end{pmatrix} \mathbf{e}_a = \frac{\gamma}{e^{it\theta_r}} \begin{pmatrix} M_{r_3} & M_{r_4} \end{pmatrix} \mathbf{e}_b.$$

We can see that this is equivalent to

$$\begin{pmatrix} M_{r_1} & -iM_{r_2} \end{pmatrix} \mathbf{e}_a = \frac{\gamma}{e^{ti\theta_r}} \begin{pmatrix} M_{r_3} & -iM_{r_4} \end{pmatrix} \mathbf{e}_b$$

or

$$\begin{pmatrix} M_{r_1} & -iM_{r_2} \end{pmatrix} \mathbf{e}_a = \frac{-i\gamma}{e^{ti\theta_r}} \begin{pmatrix} iM_{r_3} & M_{r_4} \end{pmatrix} \mathbf{e}_b,$$

so perfect state transfer occurs from a to b on G if and only if it occurs from a to b on X.

Now, suppose that there is local uniform mixing on a in G where, without loss of generality, a is in the first colour class. This means that for all r, the vectors $e^{it\theta_r} M_{r,1}\mathbf{e}_a$ and $e^{it\theta_r} M_{r,2}\mathbf{e}_a$ are flat and, moreover, the norm of every entry in either vector is the same. This is true if and only if $e^{ti\theta_r} M_{r,1}\mathbf{e}_a$ and $-ie^{ti\theta_r} M_{r,2}\mathbf{e}_a$ are flat and the entries have the same norm. Therefore, we have local uniform mixing on a in G if and only if we have local uniform mixing on a in X. $\square$

From this we can see that any characterization of perfect state transfer or local uniform mixing will apply to non-oriented bipartite graphs up to a phase factor difference of $-i$ in the case of perfect state transfer. Therefore, understanding perfect state transfer on oriented graphs can give us a deeper understanding of perfect state transfer on a special class of undirected graphs.

2.5 Basic Properties of Quantum Walks

We now turn our attention to some straightforward but useful properties about when quantum walks exhibit special behavior of perfect state transfer, periodicity, or local uniform mixing.

We begin by considering the phase factor. In the non-oriented case, there is very little information about the phase factor and what it might be. However, because of the symmetry of the eigenvalues of skew symmetric matrices, we know much more in the oriented case.

2.5.1 Theorem. *If vertex a is either periodic or has perfect state transfer to vertex b, then $\gamma = \pm 1$.*

Proof. Suppose there is perfect state transfer from a to b. By definition, we have some time τ such that

$$\sum_r e^{\tau\theta_r} E_r \mathbf{e}_a = \gamma \mathbf{e}_b.$$

Taking the conjugate of both sides gives us

$$\sum_r e^{-\tau\theta_r} \overline{E_r} \mathbf{e}_a = \overline{\gamma} \mathbf{e}_b.$$

By Lemma 2.2.3 we know that $\overline{E_r}$ is the corresponding idempotent for $-\theta_r$, so

$$\overline{\gamma}\mathbf{e}_b = \sum_r e^{-\tau\theta_r} \overline{E_r} \mathbf{e}_a = \sum_r e^{\tau\theta_r} E_r \mathbf{e}_a = \gamma \mathbf{e}_b$$

and therefore $\overline{\gamma} = \gamma$, telling us that γ is real. Since $|\gamma| = 1$ it follows that $\gamma = \pm 1$. If a is periodic, then mathematically it can be thought of as having perfect state transfer to itself, so the phase factor will also be plus/minus one. □

In future statements about perfect state transfer or periodicity, we will use ± 1 in lieu of γ to represent the phase factor.

If a vertex is periodic at a certain time, we can also draw some immediate conclusions about when else the vertex can be periodic.

2.5.2 Lemma. *If vertex a is periodic at times τ_1, τ_2, then the following hold:*

(a) *Vertex a is periodic at time $-\tau_1$.*

(b) *Vertex a is periodic at time $\tau_1 + \tau_2$.*

Proof.

(a) By definition, we have

$$U(\tau_1)\,\mathbf{e}_a = \pm\mathbf{e}_a.$$

Multiplying both sides by $U(-\tau_1)$ we get

$$\mathbf{e}_a = U(-\tau_1)\,U(\tau_1)\,\mathbf{e}_a = \pm U(-\tau_1)\,\mathbf{e}_a,$$

and then we may multiply by the phase factor to get

$$U(-\tau_1)\,\mathbf{e}_a = \pm\mathbf{e}_a.$$

(b) We use the definition of periodicity to compute that

$$\begin{aligned} U(\tau_1 + \tau_2)\,\mathbf{e}_a &= U(\tau_1)\,U(\tau_2)\,\mathbf{e}_a \\ &= \pm U(\tau_1)\,\mathbf{e}_a \\ &= \pm\mathbf{e}_a. \end{aligned}$$

□

We can also see that perfect state transfer is not entirely one-way.

2.5.3 Lemma. *If there is perfect state transfer from vertex a to vertex b at time τ, then there is perfect state transfer from b to a at time $-\tau$.*

Proof. By definition, we have

$$U(\tau)\,\mathbf{e}_a = \pm\mathbf{e}_b.$$

Multiplying both sides by $U(-\tau)$ we get

$$\mathbf{e}_a = U(-\tau_1)\,U(\tau_1)\,\mathbf{e}_a = \pm U(-\tau_1)\,\mathbf{e}_b.$$

We multiply by the phase factor and get

$$U(-\tau_1)\,\mathbf{e}_b = \pm\mathbf{e}_a.$$

□

Finally, local uniform mixing also has special results on oriented graphs.

2.5.4 Lemma. *Let X be an oriented graph on n vertices with local uniform mixing at time τ at vertex a. Then the entries of $U(\tau)\mathbf{e}_a$ are $\pm\frac{1}{\sqrt{n}}$.*

Proof. Since A has real entries, the transition matrix

$$U(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

must always have real entries, so the transition matrix is always real.

If local uniform mixing occurs at time τ for vertex a, then the entries of $U(t)\mathbf{e}_a$ will always have the same norm, and the sum of the norms squared will be one, or equivalently, n times the square of the norm will be one. Thus the norm of each entry of $U(\tau)\mathbf{e}_a$ must be $\frac{1}{\sqrt{n}}$, and since $U(\tau)$ is real, the entries of $U(\tau)\mathbf{e}_a$ are $\pm\frac{1}{\sqrt{n}}$. □

2.6 Switching Isomorphisms

In seeking to understand quantum walks, particularly when graphs have perfect state transfer or periodic vertices, we would like to look at quantum walks that are different from each other. For this, we need a way of knowing when multiple oriented graphs will have the same quantum walk associated to them.

Let X and Y be two oriented graphs. They are *isomorphic* if there is a permutation matrix P such that

$$A(Y) = P^T A(X) P.$$

Note that this definition matches the expected definition that there is a mapping of the vertices of X to the vertices of Y that preserves the oriented adjacency.

Although this is the standard definition of isomorphism, it turns out to not be the most useful definition to take when studying quantum walks.

A *signed permutation matrix* $\tilde{P}$ is the product of permutation matrix with a digonal matrix with diagonal entries ± 1.

We say that graphs X and Y are *switching isomorphic* if there is a signed permutation matrix $\tilde{P}$ such that

$$A(Y) = \tilde{P}^T A(X) \tilde{P}.$$

Intuitively, we can think of a switching isomorphism of taking an isomorphism, then choosing a set of vertices and, for each vertex in the set, switching the orientation of all incident arcs. The permutation matrix is the normal isomorphism, and the diagonal entries of -1 signal that the corresponding vertex is in the switching set.

Switching isomorphisms are the isomorphisms that we care about because two graphs that are switching isomorphic will generate the same quantum walk.

2.6.1 Lemma. *Let X and Y be switching isomorphic graphs. Then the transition matrices for the quantum walk at time t for X and Y will be similar.*

Proof. Observe that, since X and Y are switching isomorphic we have

$$A(Y) = (PD)^T A(X)(PD)$$

for some permutation matrix P and some ± 1 diagonal matrix D. Then since $P^TP = I$ and $DD = I$ we will have

$$\left(DP^T A(X) PD\right)^n = DP^T (A(X))^n PD,$$

giving us that

$$(A(Y))^n = (PD)^T (A(X))^n (PD).$$

Let $U_Y(t)$ denote the transition matrix for the quantum walk on graph Y at time t, and let $U_X(t)$ be defined the same way. We can see that

$$\begin{aligned}
U_Y(t) &= \sum_{n=0}^{\infty} \frac{(itA(Y))^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\left(itDP^T A(X) PD\right)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(it)^n DP^T (A(x))^n PD}{n!} \\
&= DP^T \left(\sum_{n=0}^{\infty} \frac{(it)^n (A(x))^n}{n!}\right) PD \\
&= (PD)^T U_X(t) PD
\end{aligned}$$

so the transition matrices at time t are similar. $\square$

In particular, this tells us that interesting behavior for one graph translates to interesting behavior for the other.

2.6.2 Corollary. *Let φ be a switching isomorphism from X to Y. If X has perfect state transfer, periodicity, or local uniform mixing at vertex a, then Y has that same property at vertex $\varphi(a)$.*

Proof. Suppose there is perfect state transfer from vertex a to vertex b at time τ in X. Then

$$(PD)^T U(\tau)(PD)\mathbf{e}_a = \pm\mathbf{e}_b$$

or

$$U(\tau)(PD)\mathbf{e}_a = \pm(PD)\mathbf{e}_b$$

Because $PD\mathbf{e}_a = \mathbf{e}_{\varphi(a)}$ and $PD\mathbf{e}_b = \mathbf{e}_{\varphi(b)}$, this shows that

$$U(\tau)\mathbf{e}_{\varphi(a)} = \pm\mathbf{e}_{\varphi(b)},$$

thus there is perfect state transfer from $\varphi(a)$ to $\varphi(b)$. Letting $b = a$, this proof shows periodicity as well.

Finally, suppose there is local uniform mixing at a at time τ in X. Then

$$(PD)^T U(\tau)\mathbf{e}_{\varphi(a)} = (PD)^T U(\tau)(PD)\mathbf{e}_a$$

is flat. Since P can only permute the order of the entry and D can only change the sign, $U(\tau)\mathbf{e}_{\varphi(a)}$ is also flat, and thus local uniform mixing is preserved. □

Because switching isomorphic graphs generate the same quantum walks, when we try to create examples or talk about graphs being unique, there is an implicit understanding that we mean up to switching isomorphism. We may also talk about the switching automorphism group instead of just the automorphism group, and we say that a graph is *switching vertex transitive* if the switching automorphism group acts transitively on the vertices. Note that any vertex transitive graph will be switching vertex transitive.

2.7 Cayley Graphs

A large class of graphs that are studied in algebraic graph theory are Cayley graphs. The definitons and theorem are from Godsil and Royle, but are adapted here for oriented rather than undirected graphs. [14]

Let G be a group and let C be a subset of elements of G that does not contain the identity. The *Cayley graph* of G with connection set C, denoted $\mathrm{Cay}(G, C)$, is the graph with vertex set G where there is an edge from g to h if and only if $hg^{-1} \in C$.

For our purposes of finding oriented graphs, it is sufficient to demand that the connection set C cannot contain both an element and its inverse. In particular, our connection set cannot contain an element of order two.

2.7.1 Theorem. *Cayley graphs are vertex-transitive.*

Proof. Let G be a group and C be a connection set, and let g and h be vertices in $\mathrm{Cay}(G, C)$.

Let φ be the permutation of vertices defined by

$$a \mapsto ag^{-1}h.$$

Then for vertices a, b we may observe that

$$\left(ag^{-1}h\right)\left(bg^{-1}h\right)^{-1} = ag^{-1}hh^{-1}gb^{-1} = ab^{-1},$$

so $ag^{-1}h$ is adjacent to $bg^{-1}h$ if and only if a is adjacent to b, so φ is an automorphism.

We can see that

$$\varphi(g) = h,$$

so clearly φ is an automorphism taking g to h. This is true for any choice of vertices and any choice of Cayley graphs, so we may conclude that Cayley graphs are vertex transitive. □

2.8 Circulants

Cayley graphs are a large class of graphs that are easier to understand than arbitrary graphs, but computing their spectral decompositions or making general statements about quantum walks on Cayley graphs can still be challenging. Restricting ourselves to a single group can help with that.

A *circulant graph* is a Cayley graph where the group is cyclic. As stated previously, we will assume that the connection set never contains an element and its inverse. The signed adjacency matrix of a circulant graph is a *circulant matrix*.

We will derive the equation for eigenvalues of circulant matrices, following the proof in Davis [9]. First, though, we will need a few more matrices related to circulants.

Let A be an $n \times n$ circulant matrix. We define π_n to be the $n \times n$ matrix

$$(\pi_n)_{j,k} = \begin{cases} 1 & j \equiv i - 1 \bmod (n) \\ 0 & \text{otherwise} \end{cases}.$$

Equivalently,

$$\pi_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This matrix is fundamental to any circulant matrix.

2.8.1 Lemma. *We may write A as a polynomial in π_n with coefficients from $\{1, -1, 0\}$.*

Proof. Since A is the signed adjacency matrix of a Cayley graph of $\mathbb{Z}/n\mathbb{Z}$, we know there is some connection set $\mathcal{C}$ that contains no inverses. For $j = 1, \ldots, n-1$, we define

$$c_j := \begin{cases} 1 & j \in \mathcal{C} \\ -1 & -j \in \mathcal{C} \\ 0 & j, -j \notin \mathcal{C} \end{cases}.$$

Then

$$A = c_1 \pi_n + c_2 \pi_n^2 + \cdots + c_{n-1} \pi_n^{n-1},$$

giving us the desired polynomial. Note that this is equivalent to

$$A = A_{1,2} \pi_n + A_{1,3} \pi_n^2 + \cdots + A_{1,n} \pi_n^{n-1}.$$

□

Let ω_n be a primitive nth root of unity, and let Ω_n be the diagonal matrix with powers of ω_n along the diagonal, that is,

$$\Omega_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega_n & \cdots & 0 \\ 0 & 0 & \cdots & \omega_n^{n-1} \end{pmatrix}.$$

We may also define the *Fourier matrix* of order n, denoted $\mathcal{F}_n$, by

$$(\mathcal{F}_n)_{j,k} = \frac{1}{\sqrt{n}}\omega_n^{(j-1)(k-1)},$$

or equivalently

$$\mathcal{F}_n = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-2} & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{n-4} & \omega_n^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega_n^{n-2} & \omega_n^{n-4} & \cdots & \omega_n^4 & \omega_n^2 \\ 1 & \omega_n^{n-1} & \omega_n^{n-2} & \cdots & \omega_n^2 & \omega_n \end{pmatrix}.$$

2.8.2 Theorem. *The Fourier matrix diagonalizs π, that is, for any n,*

$$\pi_n = \mathcal{F}_n^*\Omega_n\mathcal{F}_n.$$

Proof. Because n is fixed, we may omit it without loss of clarity.

By definition, we have that for all $1 \le k, r \le n$,

$$\mathcal{F}_{r,k} = \frac{1}{\sqrt{n}}\omega^{(r-1)(k-1)}.$$

We compute that, for all $1 \le j, r \le n$, we have

$$(\mathcal{F}^*\Omega) = \sum_{s=1}^{n}\mathcal{F}_{j,s}^*\Omega_{s,r} = \mathcal{F}_{j,r}^*\Omega_{r,r} = \frac{1}{\sqrt{n}}\omega^{j(j-1)(r-1)}\omega^{r-1} = \omega^{-j(r-1)}.$$

Combining these, we get that

$$(\mathcal{F}^*\Omega\mathcal{F})_{j,k} = \sum_{r=0}^{n-1}\frac{1}{n}\omega^{-jr}\omega^{(k-1)r} = \frac{1}{n}\sum_{r=0}^{n-1}\omega^{r(k-1-j)}.$$

Because ω is a root of unity, we can see that

$$\sum_{r=0}^{n-1}\omega^{r(k-1-j)}$$

will be zero unless j is equivalent to $k-1$ modulo n, in which case it will be n. This gives us the desired result that

$$\mathcal{F}^*\Omega\mathcal{F} = \pi.$$

$\square$

Now we are ready to find the eigenvalues of A.

2.8.3 Corollary. *The eigenvalues of A are given by*

$$\lambda_k = \sum_{j=0}^{n-1} A_{1,j+1}\omega^{jk}.$$

Proof. Using the previous two results, we may write

$$\begin{aligned}\mathcal{F}^* A \mathcal{F} &= \mathcal{F}^* \left(c_1\pi + c_2\pi^2 + \cdots + c_{n-1}\pi^{n-1}\right) \\ &= c_1\Omega + c_2\Omega^2 + \cdots + c_{n-1}\Omega^{n-1}.\end{aligned}$$

Since powers of Ω are diagonal matrices, we can see that $\mathcal{F}^* A \mathcal{F}$ must be diagonal as well, and the kth eigenvalue will be given by

$$\sum_{j=1}^{n-1} c_j \omega^{jk}.$$

□

2.9 Number Theory

Results from number theory show up a reasonable number of times in the study of quantum walks, so it is worth some attention here.

An *algebraic integer* is the root of a monic polynomial with integer coefficients. Note that the eigenvalues of a matrix with integer entries will always be algebraic integers.

A *transcendental number* is any number that cannot be written as the root of a monic polynomial with rational coefficients.

2.9.1 Lemma. *Rational algebraic integers are integers.*

Proof. Consider a rational algebraic integer $\frac{p}{q}$ with p and q coprime. Then we know that $\frac{p}{q}$ is a root of the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Since zero is an integer, we may assume without loss of generality that $\frac{p}{q}$ is nonzero, and let k be the smallest integer such that a_k is nonzero. Therefore

$$\left(\frac{p}{q}\right)^k \left(\left(\frac{p}{q}\right)^{n-k} + a_{n-1}\left(\frac{p}{q}\right)^{n-k-1} + \cdots + a_{k+1}\left(\frac{p}{q}\right) + a_k\right) = 0$$

or

$$p^{n-k} + a_{n-k-1}qp^{n-k-1} + \cdots + a_{k+1}q^{n-k-1}p = -q^{n-k}a_k.$$

We can then conclude that q divides

$$p^{n-k} + a_{n-k-1}qp^{n-k-1} + \cdots + a_{k+1}q^{n-k-1}p,$$

so it must be the case that q divides p. Since p and q are coprime, this means that $q = 1$, so our rational algebraic integer is in fact an integer. □

We will need more results from number theory to understand particular examples and impossible behavior of quantum walks. Howeer, these results are case-specific, and are best dealt with as the need arises.

2.10 Field Theory

Finally, we need some results from field theory to better understand the relationship between eigenvalues and their spectral idempotents. The following results are based off of comments in Godsil and Coutinho [6] and standard field theory as in Cox [8].

We want a way of translating between field automorphisms of the eigenvalues and ring automorphisms of the spectral idempotents. Our translation will be via the entrywise application of the field autormphism.

2.10.1 Lemma. *Let $\varphi : \mathbb{F} \to \mathbb{F}$ b a field automorphism and let R be the ring of $n \times n$ matrices with entries from $\mathbb{F}$. Then the mapping $\hat{\varphi}$ defined as the entry-wise application of φ to the matrix is a ring automorphism.*

Proof. Since φ must preserve the additive and multiplicative identities, and the only entries of the identity matrix I are these two elements, it must be the case that

$$\hat{\varphi}(I) = I,$$

and so $\hat{\varphi}$ preserves the identity.

Let A, B be matrices in R. We compute that, for all k, j with $1 \leq k, j \leq n$,

$$\begin{aligned}
(\hat{\varphi}(A+B))_{k,j} &= \varphi\left((A+B)_{k,j}\right) \\
&= \varphi(A_{k,j} + B_{k,j}) \\
&= \hat{\varphi}(A)_{k,j} + \hat{\varphi}(B)_{k,j} \\
&= (\hat{\varphi}(A) + \hat{\varphi}(B))_{k,j}.
\end{aligned}$$

From this, we can see that $\hat{\varphi}$ preserves matrix addition.

To establish matrix multiplication, we let A, B be matrices in R and consider k, j with $1 \leq k, j \leq n$. One way of considering matrix multiplication is that

$$(AB)_{k,j} = \sum_{i=1}^{n} A_{k,i} B_{i,j}.$$

Then we have

$$\begin{aligned}
(\hat{\varphi}(AB))_{k,j} &= \varphi\left((AB)_{k,j}\right) \\
&= \varphi\left(\sum_{i=1}^{n} A_{k,i} B_{i,j}\right) \\
&= \sum_{i=1}^{n} \varphi(A_{k,i})\, \varphi(B_{i,j}) \\
&= (\hat{\varphi}(A)\, \hat{\varphi}(B))_{k,j}
\end{aligned}$$

Thus $\hat{\varphi}$ preserves matrix multiplication, the last necessary condition to make it a ring isomorphism. □

2.10.2 Corollary. *Let M, N be matrices such that MN is defined. Then*

$$\hat{\varphi}(MN) = \hat{\varphi}(M)\, \hat{\varphi}(N).$$

We can now characterize the entries of the spectral idempotents.

2.10.3 Lemma. *Let θ_r be an eigenvalue of an oriented graph X. Then all the entries of the spectral idempotent E_r lie in $\mathbb{Q}(\theta_r)$.*

Proof. We begin by defining the Lagrange interpolation polynomial,

$$\ell_r(t) = \prod_{s \neq r} \frac{t - \theta_s}{\theta_r - \theta_s}.$$

Note that $\ell_r(\theta_r) = 1$ and $\ell_r(\theta) = 0$ for all other eigenvalues θ. This lets us see that

$$\ell_r(A) = \sum_s \ell_r(\theta_s) E_s = E_r.$$

Now, let $\mathbb{F}$ be the splitting field of of the minimal polynomial of X, and let φ be an automorphism of $\mathbb{F}$ which fixes θ_r. Let $\hat{\varphi}$ be the entrywise application of φ in Lemma 2.10.1. Because $\hat{\varphi}$ preserves matrix addition and multiplication,

$$\hat{\varphi}(E_r) = \hat{\varphi}\left(\prod_{s \neq r} \frac{A - \theta_s I}{\theta_r - \theta_s}\right) = \prod_{s \neq r} \frac{\hat{\varphi}(A) - \varphi(\theta_s) I}{\varphi(\theta_r) - \varphi(\theta_s)}.$$

We know that φ fixes θ_r, so $\hat{\varphi}$ must fix θ_r as well. Since A is a matrix with integer multiples of i as its entries, we can see that A will be fixed by $\hat{\varphi}$. Thus the above expression simplifies to

$$\prod_{s \neq r} \frac{A - \varphi(\theta_s) I}{\theta_r - \varphi(\theta_s)}.$$

Because the product is polynomial in A, the terms commute, which, combining with the fact that φ is an automorphism and therefore bijective, gives us

$$\hat{\varphi}(E_r) = \prod_{s \neq r} \frac{A - \varphi(\theta_s) I}{\theta_r - \varphi(\theta_s)} = \prod_{s \neq r} \frac{A - \theta_s}{\theta_r - \theta_s} = \ell_r(A) = E_r,$$

so $\hat{\varphi}$ fixes all the entries in E_r. Since φ was an arbitrary field automorphism of $\mathbb{F}$ fixing θ_r, it follows that every entry of E_r must be in $\mathbb{Q}(\theta_r)$. □

Using this characterization, we are now able to translate between automorphisms of eigenvalues and automorphisms of spectral idempotents.

2.10.4 Lemma. *Let E be a spectral idempotent for eigenvalue θ. Then given a field automorphism φ, $\hat{\varphi}(E)$ is a spectral idempotent for $\varphi(\theta)$.*

Proof. We compute that

$$(\hat{\varphi}(E))^2 = \hat{\varphi}\left(E^2\right) = \hat{\varphi}(E),$$

so $\hat{\varphi}(E)$ is idempotent.

We also have

$$A\hat{\varphi}(E) = \hat{\varphi}(AE) = \hat{\varphi}(\theta E) = \varphi(\theta)\,\hat{\varphi}(E).$$

Finally, for spectral idempotents E_r, E_s we have

$$\hat{\varphi}(E_r)\,\hat{\varphi}(E_s) = \hat{\varphi}(E_r E_s) = \hat{\varphi}(\mathbf{0}) = \mathbf{0}$$

and so the $\hat{\varphi}(E_r)$s are orthogonal. Therefore we can see that $\hat{\varphi}(E)$ represents orthogonal projection into the $\varphi(\theta)$ eigenspace, so $\hat{\varphi}(E)$ is a spectral idempotent for $\varphi(\theta)$. □

We now have enough of a background to investigate some of the special properties that quantum walks can have in more detail.

Chapter 3

Periodic Vertices

If a vertex is periodic, then it can be thought of as having perfect state transfer to itself, so in some ways periodicity is just a special case of perfect state transfer. From this perspective, it is easy to see that studying periodicity will be no harder, and possibly easier, than studying perfect state transfer.

In other ways, periodicity is far more important than merely a subcase of perfect state transfer. As Godsil showed, in oriented graphs perfect state transfer from vertex a to vertex b imply that both vertices a and b are periodic and local uniform mixing at a vertex also implies that vertex is periodic [12]. So if we would like to understand and characterize special behaviors of quantum walks on graphs, we need to also understand perodicity.

Since periodic vertices are more straightfoward than perfect state transfer or local uniform mixing to study, but are necessary for our understanding of both, we will begin our study of oriented graphs here. We will find a complete characterization of when a vertex in graph will be periodic, as well as find an infinite family of graphs where every vertex is periodic.

3.1 Eigenvalue Support

When talking about quantum walks, not every eigenvalue or spectral idempotent will contribute to the behavior of the quantum walk. We only care about those eigenvalues that do.

Let a be a vertex in a graph X with adjacency matrix $A = \sum_r \theta_r E_r$. As in Godsil [11], we define the *eigenvalue support* of a, denoted Φ_a, as the set

$$\{\theta_r : E_r \mathbf{e}_a \neq \mathbf{0}\}.$$

Note that, for a connected graph on least two vertices, $A\mathbf{e}_a$ will be nonzero and therefore

$$A\mathbf{e}_a = \sum_r \theta_r E_r \mathbf{e}_a \neq \mathbf{0},$$

so in particular the eigenvalue support must contain a nonzero eigenvalue.

The eigenvalue support has some nice closure properties.

Let $\mathbb{F}$ be the splitting field of a monic polynomial in $\mathbb{Q}$. Elements r and s of $\mathbb{F}$ are *conjugates* if there exists an automorphism φ in $\operatorname{Gal}(\mathbb{F}/\mathbb{Q})$ such that

$$r = \varphi(s).$$

3.1.1 Lemma. *For every vertex a, the eigenvalue support Φ_a is closed under conjugates.*

Proof. Let $\theta \in \Phi_a$ with corresponding spectral idempotent E and $\varphi(\theta)$ be a conjugate eigenvlaue. By Corollary 2.10.2 we have

$$\hat{\varphi}(E)\,\mathbf{e}_a = \hat{\varphi}(E\mathbf{e}_a) \neq \hat{\varphi}(\mathbf{0}),$$

and so $\hat{\varphi}(E)\,\mathbf{e}_a \neq 0$ and we can apply Lemma 2.10.4 to conclude that $\varphi(\theta) \in \Phi_a$. □

Because the eigenvalues of oriented graphs have no real part, this implies that the eigenvalue support is closed under additive inverses as well.

3.2 Ratio Condition

For quantum walks on non-oriented graphs, the eigenvalue support of periodic vertices have a ratio-based necessary condition, as shown in [11].

3.2.1 Theorem. *Let X be a non-oriented graph and let a be a periodic vertex in X. If $\theta_k, \theta_\ell, \theta_r$, and $\theta_s \in \Phi_a$ and $\theta_r \neq \theta_s$, then*

$$\frac{\theta_k - \theta_\ell}{\theta_r - \theta_s} \in \mathbb{Q}.$$

We can adapt the proof to prove a similar, but stronger and cleaner, result for quantum walks on oriented graphs.

3.2.2 Theorem. *Let X be an oriented graph. Then a is periodic if for all θ_r and $\theta_s \in \Phi_a$ with $\theta_s \neq 0$ we have*

$$\frac{\theta_r}{\theta_s} \in \mathbb{Q}.$$

Proof. Since a is periodic, we know that there exists a time τ such that

$$\sum_r e^{\tau\theta_r} E_r \mathbf{e}_a = U(\tau)\,\mathbf{e}_a = \pm\mathbf{e}_a = \pm \sum_r E_r \mathbf{e}_a.$$

By nature of the spectral idempotents, this is equivalent to saying that for all r, we have

$$e^{\tau\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_a$$

so

$$e^{\tau\theta_r} = \pm 1.$$

This tells us that for all r,

$$\tau\theta_r = k_r \pi i$$

for some integer k_r.

From this, we can see that for all r, s with $\theta_s \neq 0$,

$$\frac{\theta_r}{\theta_s} = \frac{\tau\theta_r}{\tau\theta_s} = \frac{k_r \pi i}{k_s \pi i} = \frac{k_r}{k_s} \in \mathbb{Q}.$$

Therefore, periodicity in oriented graphs implies a new, stronger ratio condition. □

3.3 A Complete Characterization

The ratio condition is a nice start to a characterization of periodic vertices. However, we would really like a necessary and sufficient condition to test for periodicity. Even more, we would rather this be a condition dependent only on the elements of the eigenvalue support, and not pairs of elements in the eigenvalue support. Fortunately, we are able to adapt the proof given in Godsil and Coutinho [6] to find such a criterion.

3.3.1 Theorem. *Let X be a connected oriented graph with at least two vertices. Then the following are equivalent:*

- *(i) The vertex a is periodic.*
- *(ii) For all r, s with $\theta_r \in \Phi_a$ and $\theta_s \neq 0$, the ratio $\frac{\theta_r}{\theta_s}$ is rational.*
- *(iii) There exists a square-free positive integer Δ such that all eigenvlaues in Φ_a are in $\mathbb{Z}\left(\sqrt{-\Delta}\right)$.*

Proof. We have shown in Theorem 3.2.2 that (i) implies (ii).

Now, suppose that the ratio condition holds on the eigenvalue support of a. Since X is a connected graph on at least two vertices, we know that the eigenvalue support contains some nonzero eigenvalue, call it θ_1. Let $\delta = |\Phi_a|$.

By Theorem 3.2.2, we know that for all r corresponding to an eigenvalue θ_r, there exists some rational a_r such that $\theta_r = a_r\theta_1$. Therefore,

$$\prod_r \theta_r = \theta_1^\delta \prod_r a_r.$$

By Lemma 3.1.1, we know that the eigenvalue support is closed under conjugates, so if $\mathbb{F}$ is the splitting field of the characteristic polynomial of X in $\mathbb{Q}$, then every automorphism in Gal $(\mathbb{F}/\mathbb{Q})$ will fix

$$\prod_r \theta_r.$$

Therefore

$$\theta_1^\delta \prod_r a_r = \prod_r \theta_r \in \mathbb{Q}.$$

Since each a_r is rational, $\theta_1^\delta \in \mathbb{Q}$, and since eigenvalues are algebraic integers, by Lemma 2.9.1 we see that θ_1^δ is an integer. We may let m be the smallest positive integer such that $\theta_1^m \in \mathbb{Z}$.

For any $k = 0, \ldots, m-1$, θ_1 has conjugate eigenvalues

$$\theta_1 e^{\frac{2\pi ik}{m}}$$

and, since all eigenvalues have no real part, θ_1 is imaginary and it follows that m must be one or two. Therefore $\theta_1^2 \in \mathbb{Z}$, and so we can write

$$\theta_1 = m_1\sqrt{-\Delta}$$

where m_1 is an integer and Δ and is a square-free integer, possibly one.

Now, for all r, we have

$$\theta_r = a_r m_1\sqrt{-\Delta} = m_r\sqrt{-\Delta}$$

where m_r is some rational number.

Then

$$\theta_r^2 = -m_r^2\Delta \in \mathbb{Z},$$

and since Δ is square-free, it must be the case that $m_r^2 \in \mathbb{Z}$, making m_r a rational algebraic integer, otherwise known as an integer by Lemma 2.9.1.

Thus we can conclude that every eigenvalue in the support of a must lie in $\mathbb{Z}\left(\sqrt{-\Delta}\right)$, so (ii) implies (iii).

Finally, assume that a is a vertex of X such that all eigenvalues in Φ_a are in $\mathbb{Z}\left(\sqrt{-\Delta}\right)$ for some square-free integer Δ.

We can define

$$g := \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right).$$

Now, letting $\tau = \frac{2\pi}{g\sqrt{\Delta}}$, we can see that for all θ_r in the eigenvalue support of a, we have

$$e^{\tau\theta_r} = e^{2\pi i\left(\frac{\theta_r}{g\sqrt{-\Delta}}\right)} = 1.$$

From this, we conclude that

$$\begin{aligned} U(\tau)\,\mathbf{e}_a &= \sum_r e^{\tau\theta_r} E_r \mathbf{e}_a \\ &= \sum_{r, \theta_r \in \Phi_a} e^{\tau\theta_r} E_r \mathbf{e}_a \\ &= \sum_r E_r \mathbf{e}_a \\ &= \mathbf{e}_a, \end{aligned}$$

so at time τ, X is periodic at vertex a. □

In field theory, the *degree of an extension* is the dimension of the vector space E over the field F. An immediate consequence of the above theorem is that all eigenvalues in the support of a periodic vertex, and therefore all entries of their spectral idempotents, lie in a field extension of degree at most two.

3.4 Minimum Period

Using this characterization, we are able to learn more about the first time periodicity will occur during a quantum walk.

3.4.1 Lemma. *Let X be a graph that is periodic at vertex a, and let*

$$g := \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \theta_r \text{ is an odd multiple of } i \text{ for all } r \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \text{otherwise} \end{cases}.$$

Then the minimum period is $\frac{2\pi}{\sqrt{\Delta}g}$.

Proof. Let τ be the minimum period at a. We may write

$$\tau = \frac{2\pi}{\sqrt{\Delta}m}$$

for some $m \in \mathbb{R}_+$.

We will first consider the case where the phase factor is negative one, that is, for all r such that $\theta_r \in \Phi_a$,

$$e^{\tau\theta_r} = -1.$$

This tells us that there exists some odd integer k_r such that

$$ik_r\pi = \tau\theta_r = \frac{2\pi\theta_r}{\sqrt{\Delta}m},$$

so $\frac{2\theta_r}{\sqrt{-\Delta}m}$ is an odd integer for all r.

By Theorem 3.3.1, we have that $\frac{\theta_r}{\sqrt{-\Delta}}$ is an integer, and therefore m must be rational, so we may write $m = \frac{p}{q}$ where p and q are relatively prime.

We know that $\frac{\theta_r q}{\sqrt{-\Delta}p}$ is an odd integer, and since p and q are relatively prime, it follows that $\frac{\theta_r}{\sqrt{-\Delta}p}$ is always a positive integer, which would make $\frac{\theta_r \pi}{\sqrt{-\Delta}p}$ a strictly smaller period unless $q = 1$. This means that m is an integer, in fact, the largest integer dividing every $\frac{2\theta_r}{\sqrt{-\Delta}}$. Therefore, if for all $\theta_r \in \Phi_a$, we have that $\frac{\theta_r}{\sqrt{-\Delta}g}$ is odd, then $m = g$ and the minimum period is $\frac{\pi}{\sqrt{\Delta}g}$.

Otherwise the phase factor must be one, so

$$\frac{\theta_r}{\sqrt{-\Delta}m} = k_r$$

for some integer k_r. As before, we can conclude that $m = \frac{p}{q}$ is a rational number, and if $q > 1$ then we have a strictly smaller period. Therefore, we know that $m = p$ is an integer, specifically, the greatest integer dividing $\frac{\theta_r}{\sqrt{-\Delta}}$, so $m = g$.

This allows us to conclude the desired result: $\frac{2\pi}{g\sqrt{\Delta}}$ must be the minimum period. $\square$

Now that we know when the first time that a vertex can be periodic, we narrow the options for when a vertex can ever be periodic.

3.4.2 Corollary. *If a vertex a is periodic at time τ, then τ is an integer multiple of $\frac{\pi}{g\sqrt{\Delta}}$.*

Proof. Let T be the minimum period of a, and let k be the greatest integer such that $\tau - kT$ is positive. By Lemma 2.5.2 we have that $\tau - kT$ is periodic, and therefore

$$\tau - kT \geq T,$$

and by construction of k we can conclude that

$$\tau - kT = T,$$

and so any period must be an integer multiple of the minimum period $\frac{\pi}{g\sqrt{\Delta}}$. □

3.5 Periodic Graphs

Similar to the notion of periodic vertices, we can consider graphs where every single vertex is periodic. We say that X is a *periodic graph* if there is a time τ such that

$$U(\tau) = \pm I.$$

We can extend our results on periodic vertices to apply to periodic graphs to find an equivalent statement for oriented graph to the nonoriented Corollary 3.3 in Godsil [11].

3.5.1 Theorem. *Let X be a connected oriented graph with at least two vertices. Then X is periodic if and only if there is a square-free integer Δ such that all eigenvlaues of X are in $\mathbb{Z}\left(\sqrt{-\Delta}\right)$.*

Proof. We can define

$$g := \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right).$$

Now, letting $\tau = \frac{2\pi}{g\sqrt{\Delta}}$, we can see that for all θ_r,

$$e^{\tau\theta_r} = e^{2\pi i\left(\frac{\theta_r}{g\sqrt{\Delta}}\right)} = 1.$$

From this, we conclude that

$$U(\tau) = \sum_r e^{\tau\theta_r} E_r = \sum_r E_r \mathbf{e}_a = I$$

so at time τ, X is periodic.

Conversely, assume that X is periodic. In particular, this means that there is some time τ such that every vertex is periodic.

Every eigenvalue must be in the support of some vertex, so for any two eigenvalues θ_r and θ_s we may select two vertices a, b such that $\theta_r \in \Phi_a$ and $\theta_s \in \Phi_b$. By Theorem 3.3.1 and Corollary 3.4.2, we know that there exist square-free integers Δ_1, Δ_2 and integers g_1, g_2, k_1, k_2 such that

$$\frac{k_1 \pi}{g_1 \sqrt{\Delta_1}} = t = \frac{k_2 \pi}{g_2 \sqrt{\Delta_2}}.$$

We arrange to see that

$$\frac{\sqrt{\Delta_1}}{\sqrt{\Delta_2}} = \frac{k_2 g_1}{k_1 g_2},$$

and therefore $\frac{\sqrt{\Delta_1}}{\sqrt{\Delta_2}}$ is rational. Since Δ_1 and Δ_2 are both square-free, it follows that $\Delta_1 = \Delta_2$, and therefore every eigenvalue must be an integer multiple of the same square root of a square-free integer Δ. □

Using this characterization, we wish to find a family of periodic graphs. We can find one using an orientation of the complete graph and a mixture of algebraic graph theory, linear algebra, and number theory.

3.6 Number Theory from Gauss and Beyond

The following definitions and results are standard results from number theory, as found in Lemmermeyer [17].

We say that a is a *quadratic residue* modulo p if there exists some x such that $x^2 \equiv a \pmod{p}$. If no such x exists, then a is a *quadratic nonresidue*.

3.6.1 Lemma. *For an odd prime p, there are $\frac{p-1}{2}$ quadratic residues.*

Proof. Let a be a quadratic residue. Since p is prime, we know that $\mathbb{Z}/p\mathbb{Z}$ is a field, and therefore the quadratic equation $x^2 - a = 0$ can have at most two solutions. We know that it has a solution y, and we compute that

$$(p-y)^2 = p^2 - 2py + y^2 \equiv y^2 \pmod{p} \equiv a \pmod{p},$$

so $p-y$ is also a solution. Since p is odd, $-y$ must be distinct from y. These give us our only two solutions to

$$x^2 \equiv a \pmod{p}.$$

From this, we can conclude that $1^2, 2^2, \ldots, \frac{p-1}{2}^2$ give us $\frac{p-1}{2}$ unique quadratic residues. Moreover, these are the only quadratic residues, so there must be $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic nonresidues. □

For an integer a and odd prime p, the *Legendre symbol*, denoted $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & a \text{ is a quadratic residue modulo } p \\ -1 & a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

3.6.2 Theorem. *[Euler's Criterion] Let a be an integer and p be an odd prime not dividing a. Then*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}}.$$

3.6.3 Lemma. *Let a, b be integers and p be an odd prime not dividing a or b. Then*

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

Proof. Using Theorem 3.6.2, we can see that

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = a^{\frac{p-1}{2}} b^{\frac{p-1}{2}} = (ab)^{\frac{p-1}{2}} = \left(\frac{ab}{p}\right).$$

□

We are now ready for a remarkable result. It is first credited to Gauss, but the proof that follows is due to Lemmermeyer, [17].

3.6.4 Theorem. *Let p be an odd prime and let ω be a pth root of unity. Then*

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \omega^a = \begin{cases} \pm\sqrt{p} & p \equiv 1 \pmod{4} \\ \pm i\sqrt{p} & p \equiv 3 \pmod{4} \end{cases}.$$

Proof. Let

$$\tau = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \omega^a.$$

We compute that

$$\tau^2 = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \omega^a \sum_{b=1}^{p-1} \left(\frac{b}{p}\right) \omega^b = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{ab}{p}\right) \omega^{a+b}.$$

We may write $b = ac$ for some c and get

$$\tau^2 = \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \left(\frac{a^2c}{p}\right) \omega^{a+ac} = \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \left(\frac{c}{p}\right) \left(\omega^{1+c}\right)^a.$$

We note that ω^{1+c} is a primitive pth root of unity for all $c \neq -1$, and so for all such c we have

$$\sum_{a=1}^{p-1} \left(\omega^{1+c}\right)^a = -1.$$

Thus we can simplify

$$\tau^2 = -\sum_{c=1}^{p-2} \left(\frac{c}{p}\right) + \left(\frac{-1}{p}\right) \sum_{a=1}^{p-1} 1 = -\sum_{c=1}^{p-1} \left(\frac{c}{p}\right) + \left(\frac{-1}{p}\right) p.$$

Note that there are $\frac{p-1}{2}$ quadratic residues with Legendre symbol 1, and $\frac{p-1}{2}$ quadratic nonresidues with Legendre symbol -1, so the sums cancel each other out and

$$\sum_{c=1}^{p-1} \left(\frac{c}{p}\right) = 0.$$

This leaves us

$$\tau^2 = \left(\frac{-1}{p}\right) p.$$

We note that

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv 3 \pmod 4, \end{cases}$$

and so

$$\tau^2 = \begin{cases} p & p \equiv 1 \pmod 4 \\ -p & p \equiv 3 \pmod 4 \end{cases}.$$

It follows immediately that

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \omega^a = \begin{cases} \pm\sqrt{p} & p \equiv 1 \pmod 4 \\ \pm i\sqrt{p} & p \equiv 3 \pmod 4 \end{cases}.$$

□

We conclude with a final number theoretic result, also usually credited to Gauss.

3.6.5 Lemma. *If $p \equiv 3 \pmod 4$, then $\left(\frac{a}{p}\right) = 1$ if and only if $\left(\frac{-a}{p}\right) = -1$.*

Proof. Let $a \in \mathbb{Z}/p\mathbb{Z}^*$. We compute

$$\left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{a}{p}\right) = -\left(\frac{a}{p}\right)$$

and so a is a quadratic residue if and only if $-a$ is not. □

3.7 Circulant Complete Graphs

We now know that there is a specific signed sum that will add to the square root of a prime. Now we just need to combine this with our earlier development of circulant graphs to prove that we can find a family of graphs that fits our requirements.

3.7.1 Theorem. *Let $p \equiv 3 \pmod 4$ be prime, and let $\mathcal{C}$ be the set of quadratic residues modulo p. Then Cay $(\mathbb{Z}/p\mathbb{Z}, \mathcal{C})$ is an oriented periodic graph.*

Proof. We know from Lemma 3.6.5 that the connection set for $\mathbb{Z}/p\mathbb{Z}$ is anti-inverse closed, so it is a connection set. Lemma 3.6.1 shows that each vertex has in-degree $\frac{p-1}{2}$ and out-degree $\frac{p-1}{2}$, and therefore the Cayley graph is an orientation of the complete graph.

Because $\mathbb{Z}/p\mathbb{Z}$ is cyclic, the graph is a circulant and therefore the eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{n-1} A_{1,j+1}\omega^{jk}.$$

Since the in-degree and out-degree are equal, we see that $\lambda_0 = 0$. If $k \neq 0$, then ω^k is a primitive pth root of unity, so by Theorem 3.6.4 the eigenvalues are $\pm i\sqrt{p}$. Thus we conclude that the only eigenvalues are 0 and $\pm i\sqrt{p}$, so by Theorem 3.5.1, Cay $(\mathbb{Z}/p\mathbb{Z}_+, \mathcal{C})$ is periodic. $\square$

Periodic vertices are easy to characterize, and relatively easy to find, since we simply need to compute the spectral decomposition and look to see if the eigenvalues are always quadratic integers. Periodicity is nevertheless an interesting property of the graph, both for the interpretation it has with a quantum walk and for the blend of mathematics it involves, as seen in our example of oriented complete graphs. We can generalize the results of periodicity from non-oriented graphs to apply to oriented graphs, and we end up with a slightly cleaner ratio condition and condition on the eigenvalue support than we have for non-oriented graphs. This is not true of every behavior that a quantum walk can have, and when we introduce a second vertex, generalizing results from non-oriented graphs becomes more complicated.

Chapter 4

Perfect State Transfer

Now that we have established when a vertex in a graph can be periodic, the next logical step is to see when a vertex can have perfect state transfer to some other vertex. This is interesting from a graph theoretic perspective, because it is a special property that only certain graphs and certain vertices have. Perfect state transfer is less common than periodic vertices, and suggests a new relationship between distinct vertices in a graph.

Perfect state transfer is also important from a physical perspective. The no-cloning theorem proves that it is impossible to make a copy of a quantum state [19], but if we have a quantum walk that has perfect state transfer to some other vertex, then we can create not an exact copy, but a state that was in a sense transferred from a pre-existing state. This makes perfect state transfer of interest in the implementation of quantum walks inside quantum algoritms.

Quantum walks on non-oriented graphs have been studied significantly, and a number of theorems have been proven about them. Some of these results hold for oriented graphs with minor changes to the proofs, some results can be adapted to prove a similar statement for oriented graphs, and some results that are true for non-oriented graphs are false in the oriented case. In this chapter we present some known results about perfect state transfer on graphs with symmetric adjacency matrices, and provide counterexamples or proof adaptations when the graph has a skew symmetric adjacency matrix.

4.1 A Useful Example

In many cases, we do not need to look far to find a counterexample. When oriented, the complete graph on three vertices is a counterexample to several results about perfect state transfer that are known to be true in the non-oriented case. We begin by working out the quantum walk on this graph.

4.1.1 Example. Consider the graph shown in Figure 4.1.

We may find the spectral decomposition and see that the eigenvalues are $0, \sqrt{3}i, -\sqrt{3}i$ with corresponding spectral idempotents

$$\frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix},\ \frac{1}{3}\begin{pmatrix}1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}}\\ e^{\frac{2\pi i}{3}} & 1 & e^{\frac{4\pi i}{3}}\\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1\end{pmatrix},\ \text{and}\ \frac{1}{3}\begin{pmatrix}1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}}\\ e^{\frac{4\pi i}{3}} & 1 & e^{\frac{2\pi i}{3}}\\ e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} & 1\end{pmatrix}.$$

We then compute that

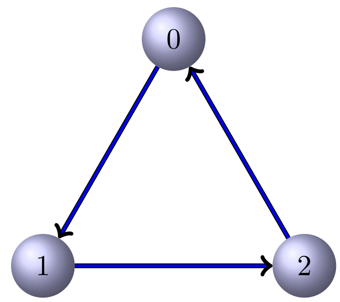

Figure 4.1: One orientation of the complete graph on 3 vertices.

$$
\begin{aligned}
U(t) &= \frac{1}{3}\begin{pmatrix} 1+e^{it\sqrt{3}}+e^{-it\sqrt{3}} & 1+e^{i\frac{4\pi}{3}t\sqrt{3}}+e^{-i\frac{4\pi}{3}t\sqrt{3}} & 1+e^{i\frac{2\pi}{3}t\sqrt{3}}+e^{-i\frac{2\pi}{3}t\sqrt{3}} \\ 1+e^{i\frac{2\pi}{3}t\sqrt{3}}+e^{-i\frac{2\pi}{3}t\sqrt{3}} & 1+e^{it\sqrt{3}}+e^{-it\sqrt{3}} & 1+e^{i\frac{4\pi}{3}t\sqrt{3}}+e^{-i\frac{4\pi}{3}t\sqrt{3}} \\ 1+e^{i\frac{2\pi}{3}t\sqrt{3}}+e^{-i\frac{2\pi}{3}t\sqrt{3}} & 1+e^{i\frac{4\pi}{3}t\sqrt{3}}+e^{-i\frac{4\pi}{3}t\sqrt{3}} & 1+e^{it\sqrt{3}}+e^{-it\sqrt{3}} \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1+2\cos\left(t\sqrt{3}\right) & 1+2\cos\left(\frac{4\pi}{3}+t\sqrt{3}\right) & 1+2\cos\left(\frac{2\pi}{3}+t\sqrt{3}\right) \\ 1+2\cos\left(\frac{2\pi}{3}+t\sqrt{3}\right) & 1+2\cos\left(t\sqrt{3}\right) & 1+2\cos\left(\frac{4\pi}{3}+t\sqrt{3}\right) \\ 1+2\cos\left(\frac{4\pi}{3}+t\sqrt{3}\right) & 1+2\cos\left(\frac{2\pi}{3}+t\sqrt{3}\right) & 1+2\cos\left(t\sqrt{3}\right). \end{pmatrix}
\end{aligned}
$$

We can use the transition matrix to gain information about what is happening in the quantum walk at various times. For example, at time $\frac{2\pi}{3\sqrt{3}}$ we have

$$
U\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos(2\pi) & 1+2\cos\left(\frac{4\pi}{3}\right) \\ 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos(2\pi) \\ 1+2\cos(2\pi) & 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right). \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

As another example, at time $\frac{4\pi}{3\sqrt{3}}$ we have

$$
U\left(\frac{4\pi}{\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos(2\pi) \\ 1+2\cos(2\pi) & 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right) \\ 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos(2\pi) & 1+2\cos\left(\frac{4\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
$$

Finally, at time $\frac{2\pi}{\sqrt{3}}$ we have

$$U\left(\frac{2\pi}{3\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} 1+2\cos\left(2\pi\right) & 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right) \\ 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos\left(2\pi\right) & 1+2\cos\left(\frac{4\pi}{3}\right) \\ 1+2\cos\left(\frac{4\pi}{3}\right) & 1+2\cos\left(\frac{2\pi}{3}\right) & 1+2\cos\left(2\pi\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This example will come up several times as we study properties of perfect state transfer in oriented graphs.

4.2 Multiple Perfect State Transfer

For non-oriented graphs with real symmetric adjacency matrices, a vertex can have perfect state transfer with at most one other vertex as shown in Kay [16]. This is not the case for oriented graphs with skew-symmetric adjacency matrices, as first shown in Cameron et.al [4].

4.2.1 Example. Consider K_3. Using our earlier calculation of the transition matrix, we can see that at time $\frac{2\pi}{3\sqrt{3}}$ we have perfect state transfer from vertex 0 to vertex 1 and at time $\frac{4\pi}{3\sqrt{3}}$ we have perfect state transfer from vertex 0 to vertex 2. Therefore, vertex 0 has perfect state transfer between more than two vertices.

We say that a graph which has perfect state transfer from one vertex to multiple other vertices has *multiple state transfer*.

4.2.2 Lemma. *Let X be an oriented graph with distinct periodic vertices a, b, and c. If there is perfect state transfer from vertex a to b and from vertex a to c, then there is perfect state transfer from vertex b to c.*

Proof. Let τ_1 be the minimum period of a, let τ_2 be the first time that there is perfect state transfer from a to b, and let τ_3 be the first time there is perfect state transfer from a to c.

We compute that

$$\begin{aligned} U\left(\tau_1 - \tau_2 + \tau_3\right)\mathbf{e}_b &= U\left(\tau_3\right)U\left(\tau_1\right)U\left(-\tau_2\right)\mathbf{e}_b \\ &= \pm U\left(\tau_3\right)U\left(\tau_1\right)\mathbf{e}_a \\ &= \pm U\left(\tau_3\right)\mathbf{e}_a \\ &= \pm \mathbf{e}_c, \end{aligned}$$

so there is perfect state transfer from b to c. □

A graph where every pair of vertices have perfect state transfer has *universal state transfer*.

Circulant graphs with universal state transfer were studied in [4], and several necessary conditions were found; however, further examples of graphs with universal state transfer were not. The similar question of graphs that exhibit multiple state transfer has remained unstudied.

Since any graph that exhibits multiple state transfer also has perfect state transfer, understanding this phenomenom requires understanding when perfect state transfer can occur in an oriented graph. For this, we need to look at other results about non-oriented graphs.

4.3 Cospectrality, Weak and Strong

Two vertices a and b are said to be *strongly cospectral* if, for all spectral idempotents E, we have

$$E\mathbf{e}_a = \pm E\mathbf{e}_b.$$

For non-oriented graphs with real adjacency matrices, we know that perfect state transfer between vertices a and b implies that a and b are strongly cospectral, as shown in Coutinho [5]. However, the same does not hold for oriented graphs.

4.3.1 Example. Consider K_3. We have previously established that the spectral idempotents are

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 1 & e^{\frac{4\pi i}{3}} \\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \\ e^{\frac{4\pi i}{3}} & 1 & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} & 1 \end{pmatrix},$$

and that there is perfect state transfer from vertex 0 to vertex 1.

However,

$$\begin{pmatrix} 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 1 & e^{\frac{4\pi i}{3}} \\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \pm \begin{pmatrix} 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 1 & e^{\frac{4\pi i}{3}} \\ e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The problem with strong cospectrality is that it was defined from the viewpoint of graphs with symmetric adjacency matrices. In order to address this, we need a definition that generalizes to oriented graphs and their skew symmetric matrices.

To this end, we need to consider the weaker, but broader-reaching, notion of cospectrality.

We say that vertices a and b are *cospectral* if, for $\varphi(X,t)$ the characteristic polynomial, we have

$$\varphi(X \setminus a, t) = \varphi(X \setminus b, t).$$

In the symmetric case, cospectrality has a long list of equivalent conditions, as shown in Godsil and Smith [15]. Here, we will content ourselves with two equivalent conditions.

4.3.2 Theorem. *Let a and b be vertices in an oriented graph X with adjacency matrix A. Then a and b are cospectral if and only if, for all spectral idempotents E we have*

$$(E)_{a,a} = (E)_{b,b}.$$

Proof. We begin by noting that the adjacency matrix for $X \setminus a$ is A with the ath row and column deleted, and similarly for $X \setminus b$. Then using the determinant and cofactor computation of inverse we get that

$$(tI - A)^{-1}_{a,a} = \frac{\varphi(X \setminus a, t)}{\varphi(X,t)}.$$

We then use spectral decomposition to see that

$$(tI - A)^{-1} = \sum_r \frac{1}{t - \theta_r} E_r.$$

If, for all r we have

$$(E_r)_{a,a} = (E_r)_{b,b}$$

then we can combine the above two equations to get

$$\frac{\varphi(X \setminus a, t)}{\varphi(X,t)} = \sum_r \frac{1}{t - \theta_r} (E_r)_{a,a} = \sum_r \frac{1}{t - \theta_r} (E_r)_{b,b} = \frac{\varphi(X \setminus b, t)}{\varphi(X,t)}$$

and so

$$\varphi(X \setminus a, t) = \varphi(X \setminus b, t)$$

and therefore a and b are cospectral.

Conversely, suppose that a and b are cospectral. We note that the denominators of

$$\sum_r \frac{1}{t-\theta_r}\left(E_r\right)_{a,a}$$

are different, and therefore the $\left(E_r\right)_{a,a}$ terms represent the numerators of a partial fraction decomposition of

$$\frac{\varphi\left(X\setminus a,t\right)}{\varphi\left(X,t\right)}.$$

At the same time, because a and b are cospectral, we also have

$$\frac{\varphi\left(X\setminus a,t\right)}{\varphi\left(X,t\right)}=\frac{\varphi\left(X\setminus b,t\right)}{\varphi\left(X,t\right)}=\sum_r \frac{1}{t-\theta_r}\left(E_r\right)_{b,b},$$

and so the $\left(E_r\right)_{b,b}$ terms also represent the numerators of a partial fraction decomposition of

$$\frac{\varphi\left(X\setminus a,t\right)}{\varphi\left(X,t\right)}.$$

Since partial fraction decomposition is unique, we therefore have that for all r,

$$\left(E_r\right)_{a,a}=\left(E_r\right)_{b,b}. \qquad \square$$

We define two vertices a and b to be *parallel* if, for all spectral idempotents r, $E_r\mathbf{e}_a$ is a scalar multiple of $E_r\mathbf{e}_b$.

Lemma 4.1 in [15] states that two vertices are strongly cospectral if and only if they are cospectral and parallel. Taking this as our definition, we may now come up with an equivalent definition of strongly cospectral vertices in oriented graphs.

4.3.3 Lemma. *Vertices a and b are strongly cospectral if and only if, for all spectral idempotents E_r, there exists a complex scalar α_r with $|\alpha_r|=1$ such that*

$$E_r\mathbf{e}_a=\alpha_r E_r\mathbf{e}_b.$$

Proof. For both directions, we may assume that for all r, there exists a complex α_r such that

$$E_r\mathbf{e}_a=\alpha_r E_r\mathbf{e}_b.$$

We will show that cospectrality occurs if and only if $|\alpha_r| = 1$ for all r.

We have

$$\begin{aligned}
(E_r)_{a,a} &= \mathbf{e}_a^T E_r \mathbf{e}_a \\
&= \alpha_r \mathbf{e}_a^T E_r \mathbf{e}_b \\
&= \alpha_r \left(\mathbf{e}_b^T \overline{E_r} \mathbf{e}_a\right)^T \\
&= \alpha_r \overline{\mathbf{e}_b^T E_r \mathbf{e}_a} \\
&= \alpha_r \overline{\alpha_r \mathbf{e}_b^T E_r \mathbf{e}_b} \\
&= \alpha_r \overline{\alpha_r} \overline{\mathbf{e}_b^T E_r \mathbf{e}_b} \\
&= |\alpha_r| \, \mathbf{e}_b^T E_r \mathbf{e}_b \\
&= |\alpha_r| \, (E_r)_{b,b} \, ,
\end{aligned}$$

so by Theorem 4.3.2 vertices a and b are cospectral if and only if $|\alpha_r| = 1$. □

Now, we are able to show that for our new and improved definition of strongly cospectral, perfect state transfer does in fact imply strong cospectrality.

4.3.4 Lemma. *Let X be an oriented graph with perfect state transfer from vertex a to vertex b. Then a and b are strongly cospectral.*

Proof. Since there is perfect state transfer from vertex a to vertex b, we know that there is some $t \in \mathbb{R}$ such that for all r we have

$$e^{t\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b,$$

so

$$E_r \mathbf{e}_a = \pm e^{-it\theta_r} E_r \mathbf{e}_b,$$

and, since $\left|\pm e^{-it\theta_r}\right| = 1$, we know that a and b are strongly cospectral. □

We now have a few results about the eigenvalue support of strongly cospectral vertices.

4.3.5 Lemma. *Let a and b be strongly cospectral vertices. Then the following statements hold:*

(i) The eigenvalue support of a is the same as the eigenvalue support of b.

(ii) If $(E_r)_{a,b} = 0$*, then* $\theta_r \notin \Phi_a$*.*

Proof. For (i), we note that since a and b are strongly cospectral, there exists α_r with $|\alpha_r| = 1$ such that

$$E_r \mathbf{e}_a = \alpha_r E_r \mathbf{e}_b.$$

Then $E_r \mathbf{e}_a$ will be zero if and only if $E_r \mathbf{e}_b$ is zero, so $\Phi_a = \Phi_b$.

For (ii), suppose that we use strong cospectrality to conclude that

$$\mathbf{e}_a^T E_r \mathbf{e}_a = \alpha_r \mathbf{e}_a^T E_r \mathbf{e}_b,$$

so if $(E_r)_{a,b} = 0$, then $(E_r)_{a,a} = 0$ as well. But since E_r is positive semidefinite, this implies that that $E_r \mathbf{e}_a = \mathbf{0}$, so $\theta_r \notin \Phi_a$. □

4.4 Robustly Cospectral

One of the reasons that strongly cospectral vertices are studied in quantum walks on undirected graphs is that they are useful in forming a complete characterization of the graphs where perfect state transfer can occur, as in Coutinho [5]. This characterization relied on dividing the eigenvalue support into partitions

$$\Phi_{ab}^+ = \{\theta_r : \mathbf{e}_a = E_r \mathbf{e}_b\}$$

and

$$\Phi_{ab}^- = \{\theta_r : \mathbf{e}_a = -E_r \mathbf{e}_b\}.$$

We are not able to do this with skew symmetric matrices without having infinitely many sets. There is hope, however, because we can define a new kind of cospectral property.

Given vertices a and b, for any r such that $(E_r)_{a,b}$ is nonzero, we define the *quarrel* from a to b in r, denoted $q_r(a, b)$ to be the unique number between -1 and 1 such that, for some positive scalar s_r,

$$(E_r)_{a,b} = s_r e^{i\pi q_r(a,b)}.$$

Let a, b be strongly cospectral vertices. We say that a and b are *robustly cospectral* if, for all r such that $\theta_r \in \Phi_a$ we have

$$E_r \mathbf{e}_b = e^{i\pi q_r(a,b)} E_r \mathbf{e}_a.$$

In fact, the condition that $\theta_r \in \Phi_a$ is equivalent to saying that the quarrel is defined, as seen with Lemma 4.3.3.

We conclude this section by proving that perfect state transfer implies our stronger cospectrality condition.

4.4.1 Lemma. *Let X be an oriented graph with perfect state transfer from vertex a to vertex b. Then a and b are robustly cospectral.*

Proof. Since there is perfect state transfer from a to b, we know there is some time t such that for all r we have

$$e^{t\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b.$$

This tells us that

$$e^{t\theta_r} \mathbf{e}_a^T E_r \mathbf{e}_a = \pm \mathbf{e}_a^T E_r \mathbf{e}_b$$

or

$$\left(E_r\right)_{a,a} = \pm e^{-t\theta_r} \left(E_r\right)_{a,b}.$$

We may write

$$\left(E_r\right)_{a,b} = s_r e^{i\pi q_r(a,b)}$$

for some real-valued s_r. Since the spectral idempotents are Hermitian, we know $\left(E_r\right)_{a,a}$ is real, and therefore

$$e^{i\pi q_r(a,b) - it\theta_r}$$

is real. In other words,

$$e^{i\pi q_r(a,b)} = \pm e^{t\theta_r},$$

where the plus/minus that appears is the same phase factor as in our perfect state transfer.

From this, we see that

$$e^{\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b$$

becomes

$$e^{i\pi q_r(a,b)} E_r \mathbf{e}_a = E_r \mathbf{e}_b,$$

so vertices a and b are robustly cospectral. $\square$

In fact, a similar argument can be adapted for non-oriented graphs with real spectral idempotents.

4.4.2 Lemma. *Let G be a non-oriented graph with perfect state transfer from vertex a to vertex b. Then a and b are robustly cospectral.*

Proof. Since there is perfect state transfer from a to b, we know there is some time τ and some complex phase factor γ with $|\gamma| = 1$ such that for all r we have

$$e^{t\theta_r} E_r \mathbf{e}_a = \gamma E_r \mathbf{e}_b,$$

or

$$\frac{e^{t\theta_r}}{\gamma} E_r \mathbf{e}_a = E_r \mathbf{e}_b.$$

Since the adjacency matrix of G is symmetric, the spectral idempotents are real, so $\frac{e^{t\theta_r}}{\gamma}$ is real; specifically, it must always be ± 1. Then

$$\frac{e^{t\theta_r}}{\gamma} \mathbf{e}_a^T E_r \mathbf{e}_a = \mathbf{e}_a^T E_r \mathbf{e}_b,$$

and since E_r is positive semidefinite, $(E_r)_{a,a}$ is positive. Therefore $(E_r)_{a,b}$ has the same sign as $\frac{e^{t\theta_r}}{\gamma}$, so

$$E_r \mathbf{e}_b = e^{i\pi q_r(a,b)} E_r \mathbf{e}_a$$

a and b are robustly cospectral. □

From this, we see that what is happening in oriented graphs is not completely detached from what is happening in non-oriented graphs with symmetric adjacency matrices. This gives us hope of being able determine when perfect state transfer can occur in oriented graphs the same way we can for non-oriented graphs.

4.5 A First Characterization

Coutinho proved an equivalent characterization of perfect state transfer using strong cospectrality and a partition of the eigenvalue support. [5] Letting θ_0 be the largest eigenvalue, he proved the following theorem.

4.5.1 Theorem. *[5]Let G be a non-oriented graph and let $a, b \in V(G)$. Then perfect state transfer occurs from a to b at time τ if and only if:*

(i) Vertices a and b are strongly cospectral.

(ii) For all eigenvalues $\theta_r \in \Phi_{ab}^{+}$, there exists a k_r such that $\tau(\theta_0 - \theta_r) = 2k_r\pi$.

(iii) For all eigenvalues $\theta_r \in \Phi_{ab}^{-}$, there exists a k_r such that $\tau(\theta_0 - \theta_r) = (2k_r + 1)\pi$.

Using robust cospectrality, we may rewrite this in a new way.

4.5.2 Theorem. *Let G be a non-oriented graph and let $a, b \in V(G)$. Then perfect state transfer occurs from a to b at time τ if and only if:*

(i) Vertices a and b are robustly cospectral.

(ii) For all eigenvalues $\theta_r \in \Phi_{ab}$, there exists a k_r such that $\tau(\theta_0 - \theta_r) = (2k_r + q_r(a, b))\pi$.

Although the end result is the same, the process is slightly different, since robust cospectrality implies both strong cospectrality and the partition Φ_{ab}^{+} and Φ_{ab}^{-}. It is useful for our purposes, because it suggests a similar characterization for oriented graphs. We prove that characterization, taking advantage of the phase factor necessarily being ± 1 to simplify the expression.

4.5.3 Theorem. *Let X be an oriented graph and let a and b be vertices in X. Then perfect state transfer occurs from vertex a to vertex b at time τ if and only if the following conditions hold:*

(i) Vertices a and b are robustly cospectral.

(ii) For all r with $\theta_r \in \Phi_a$, there exists some integer k_r such that

$$\frac{\tau\theta_r}{i\pi} + q_r(a, b) = k_r.$$

Proof. By Lemma 4.4.2 we know that perfect state transfer between a and b implies a and b are robustly cospectral. Therefore, it is sufficient to prove that, for robustly cospectral vertices a and b, perfect state transfer occurs from a to b at time τ if and only if there exists some integer k_r such that

$$\frac{\tau\theta_r}{i\pi} + q_r(a,b) = k_r.$$

Given robust cospectrality, perfect state transfer from a to b is equivalent to saying that, for all r,

$$e^{\tau\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b = \pm e^{i\pi q_r(a,b)} E_r \mathbf{e}_a.$$

This is true if and only if there exists some integer k_r such that

$$\tau\theta_r - i\pi q_r(a,b) = k_r i\pi,$$

and dividing both sides by $i\pi$ gives us the desired result. $\square$

This gives us our desired first characterization of when perfect state transfer can occur, similar to what we had in the non-oriented case, but it also gives us a slightly different characterization of non-oriented graphs. Although this change is mainly cosmetic, it was a change that became easier to see when expanding our focus to look at oriented graphs. Quantum walks on non-oriented graphs have been better studied than quantum walks on oriented graphs, and so for the most part we are interested in taking results from the non-oriented case and seeing which we can generalize to oriented graphs. However, there is the hope that, as with robust cospectrality, we might be able to turn this around and use new knowledge about what is happening in the oriented case to deepen our understanding of what is going on in the general case of quantum walks on undirected graphs.

4.6 Perfect State Transfer and Periodicity

For non-oriented graphs, perfect state transfer between two vertices implies that both vertices are periodic, as shown in Coutinho [5].

4.6.1 Theorem. *If a graph X admits perfect state transfer between vertices a and b at time τ, then*

(i) The graph X admits perfect state transfer between vertices b and a at time τ.

(ii) The graph X is periodic at both vertices a and b at time 2τ.

The proof is short, simple, and does not generalize to oriented graphs. In fact, the statement does not generalize to oriented graphs.

4.6.2 Example. Consider the graph in Figure 4.1. We know that the graph has perfect state transfer from vertex 0 to vertex 1 at time $\frac{2\pi}{3\sqrt{3}}$, but we can also see that at time $\frac{4\pi}{3\sqrt{3}}$ vertex 0 is not periodic. In fact, at that time it has perfect state transfer to vertex 1. Vertex 0 is not periodic until time $\frac{2\pi}{\sqrt{3}}$.

Although this example shows that neither part of Theorem 4.6.1 is true for all oriented graphs, it is still true that the graph is periodic, and at an integer multiple of the time to perfect state transfer. This raises the question of whether there exists an analogue of this theorem showing that perfect state transfer between two vertices implies both vertices are periodic. It turns out that there is, at least for some graphs.

4.6.3 Theorem. *Let X be a graph and φ be a switching automorphism of order n. Then if perfect state transfer occurs from a to $\hat{\varphi}(a)$ at time τ then a is periodic at time $n\tau$.*

Proof. We compute that

$$U(2\tau)\,\mathbf{e}_a = \pm U(\tau)\,\mathbf{e}_{\varphi(a)} = \pm\mathbf{e}_{\varphi^2(a)},$$

so we have perfect state transfer from a to $\varphi^2(a)$ at time 2τ. In general, if we have perfect state transfer from $\varphi(a)$ to $\varphi^k(a)$ at time $k\tau$, then we must have perfect state transfer from a to $\varphi^{k+1}(a)$ at time $(k+1)\tau$. Let n be the order of φ, and we can see that a must be periodic at time $n\tau$. □

This theorem applies to vertex transitive graphs, so for any Cayley graph we know that perfect state transfer implies periodicity at an integer multiple of the time for perfect state transfer. However, most graphs do not have automorphisms between vertices, and without a switching automorphism, our proof technique would not work. In fact, it is not always true that perfect state transfer implies periodicity at an integer multiple of the time to perfect state transfer.

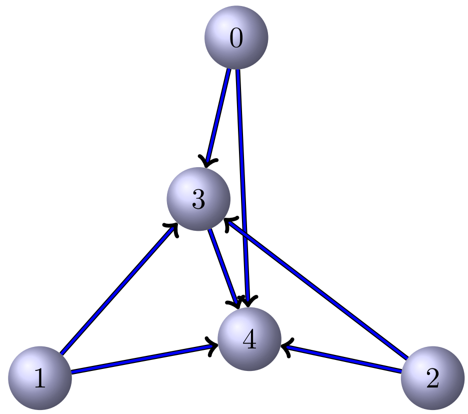

Figure 4.2: A new counterexample

4.7 A Different Counterexample

In the last chapter, we were able to develop characterizations of not only which graphs have periodic vertices, but at what times periodicity must occur. In particular, we know that the time must be a rational multiple of π divided by a certain square-free integer. So to find a counterexample, we are interested in perfect state transfer occuring at some time that is not of that form.

To help us provide such an example, we will need a result about trigonometric functions from Niven [18].

4.7.1 Lemma. *For θ a rational multiple of π, the only rational values $\cos(\theta)$ can take are $0, \pm\frac{1}{2}$, and ± 1.*

Conversely, if $\arccos(\theta)$ is a rational that is not any of $0, \pm\frac{1}{2}$, or ± 1, then we know that θ cannot be a rational multiple of π. All we need to do is find an example that makes use of this fact.

4.7.2 Example. Consider the graph shown in Figure 4.7. We have an eigenalue $\sqrt{7}$ with idempotent

$$\frac{1}{7}\begin{pmatrix} 1 & 1 & 1 & \frac{1+\sqrt{7}i}{2} & \frac{-1+\sqrt{7}i}{2} \\ 1 & 1 & 1 & \frac{1+\sqrt{7}i}{2} & \frac{-1+\sqrt{7}i}{2} \\ 1 & 1 & 1 & \frac{1+\sqrt{7}i}{2} & \frac{-1+\sqrt{7}i}{2} \\ \frac{1-\sqrt{7}i}{2} & \frac{1-\sqrt{7}i}{2} & \frac{1-\sqrt{7}i}{2} & 2 & \frac{3+\sqrt{7}i}{2} \\ \frac{-1-\sqrt{7}i}{2} & \frac{-1-\sqrt{7}i}{2} & \frac{-1-\sqrt{7}i}{2} & \frac{3-\sqrt{7}i}{2} & 2 \end{pmatrix}.$$

We also have an eigenvalue $-\sqrt{7}$ with idempotent

$$\frac{1}{7}\begin{pmatrix} 1 & 1 & 1 & \frac{1-\sqrt{7}i}{2} & \frac{-1-\sqrt{7}i}{2} \\ 1 & 1 & 1 & \frac{1-\sqrt{7}i}{2} & \frac{-1-\sqrt{7}i}{2} \\ 1 & 1 & 1 & \frac{1-\sqrt{7}i}{2} & \frac{-1-\sqrt{7}i}{2} \\ \frac{1+\sqrt{7}i}{2} & \frac{1+\sqrt{7}i}{2} & \frac{1+\sqrt{7}i}{2} & 2 & \frac{3-\sqrt{7}i}{2} \\ \frac{-1+\sqrt{7}i}{2} & \frac{-1+\sqrt{7}i}{2} & \frac{-1+\sqrt{7}i}{2} & \frac{3+\sqrt{7}i}{2} & 2 \end{pmatrix},$$

and finally we have eigenvlaue 0 with idempotent

$$\frac{1}{7}\begin{pmatrix} 5 & -2 & -2 & -1 & 1 \\ -2 & 5 & -2 & -1 & 1 \\ -2 & -2 & 5 & -1 & 1 \\ -1 & -1 & -1 & 3 & -3 \\ 1 & 1 & 1 & -3 & 3 \end{pmatrix}.$$

We can write

$$\frac{3}{2}+\frac{\sqrt{7}i}{2} = 2e^{i\arccos(frac34)},$$

and so the quarrel for the $\sqrt{-7}$ eigenvalue from vertex 3 to vertex 4 is $\frac{\arccos\left(\frac{3}{4}\right)}{\pi}$.

Similarly, we have

$$\frac{3}{2}-\frac{\sqrt{7}i}{2} = 2e^{i-\arccos(frac34)},$$

and so the quarrel for the $-\sqrt{-7}$ eigenvalue from vertex 3 to vertex 4 is $\frac{-\arccos\left(\frac{3}{4}\right)}{\pi}$.

Finally, we have that the quarrel for the 0 eigenvalue from vertex 3 to vertex 4 will be 1.

Now, let

$$\tau = \frac{\pi - \arccos\left(\frac{3}{4}\right)}{\sqrt{7}}.$$

Then

$$\frac{0 \cdot \tau}{i\pi} + 1 = 1.$$

We also have

$$\frac{\sqrt{7}i\tau}{i\pi} + q_{\sqrt{7}i}\left(3,4\right) == 1 - \frac{\arccos\left(\frac{3}{4}\right)}{\pi} + \frac{\arccos\left(\frac{3}{4}\right)}{\pi} = 1$$

and

$$\frac{-\sqrt{7}i\tau}{i\pi} + q_{-\sqrt{7}i}\left(3,4\right) = -1 + \frac{\arccos\left(\frac{3}{4}\right)}{\pi} - \frac{\arccos\left(\frac{3}{4}\right)}{\pi} = -1.$$

In all cases, $\frac{\tau\theta_r}{\pi} + q_r\left(a,b\right)$ is an integer, so from Theorem 4.5.3, we can see that our graph has perfect state transfer from vertex 3 to vertex 4 at time τ.

However, from Lemma 4.7.1, we know that $\arccos\left(\frac{3}{4}\right)$ is not a rational multiple of π, so there is no integer k such that $k\tau$ is an integer multiple of $\frac{\pi}{\sqrt{7}}$. It follows from Corollary 3.4.2 that vertex 3 cannot be periodic at an integer multiple of τ.

In general, therefore, the time that perfect state transfer occurs does not give us any information about when periodicity occurs. In fact, it is not even clear that there is any connection between perfect state transfer and periodic vertices in oriented graphs. Although the graph in the example was periodic, it is conceivable that could have just been a coincidence and there are cases where perfect state transfer occurs, but neither vertex is periodic.

It is not a coincidence, and perfect state between two vertices does in fact imply that both vertices are periodic. However, to prove this requires more advanced tools from number theory.

Chapter 5

The Gelfond-Schneider Theorem

Up until now, the number theory that we have used has been elementary, usually results that Gauss or a close contemporary proved. Our next result, from transcendental number theory, is neither. It was posed as Hilbert's Seventh problem, and was proven independently by Aleksandr Gelfond and Theodor Schneider in 1934.

5.0.1 Theorem (Gelfond-Schneider)**.** *If α and β are algebraic numbers with $\alpha \neq 0, 1$ and β irrational, then α^β is transcendental.*

At first glance, it might seem a bit surprising that the Gelfond-Schneider Theorem is both true and relevant to our study of quantum walks. But the theorem is true, and can be used as a tool when it is easier to determine that numbers α, β, and α^β are algebraic than it is to determine that β is rational. When dealing with exponential functions and eigenvalues, as we are when we work with quantum walks, this situation arises with relative frequency.

The Gelfond-Schneider Theorem was first applied to quantum walks to prove conditions on when cycles could not exhibit uniform mixing in Adamczak et.al [1]. It has also been used to place restrictions on when the transition matrix for a quantum walk can be algebraic, as in Godsil, Mullin, and Roy [13]. Godsil also applied the Gelfond-Schneider Theorem to oriented graphs to observe that both perfect state transfer and local uniform mixing imply a ratio condition on the eigenvalues [12].

Because the Gelfond-Schneider Theorem is such a powerful tool, and because of the repeated appearance of number theory in the study of quantum walks, it is worth understanding the proof. It is sufficient for our purposes to assume that β is real, so we will only prove the theorem for that simpler case.

5.1 Equivalent Forms

The canonical way of presenting the Gelfond-Schneider Theorem is that an algebraic number raised to an irrational algebraic number is transcendental. Although this has a certain visual and intuitive appeal, it is often more useful to think about it in other ways. For this reason, we present several alternate forms, taken from Burger and Tubbs [3] and Niven [18].

5.1.1 Theorem. *The following are equivalent:*

(i) If α and β are algebraic numbers with $\alpha \neq 0, 1$ and β irrational, then α^β is transcendental.

(ii) If α and γ are nonzero algebraic numbers and $\alpha \neq 1$, then $\frac{\ln \gamma}{\ln \alpha}$ is either rational or transcendental.

(iii) If β is irrational and ζ is nonzero, then at least one of $\beta, e^\zeta, e^{\zeta\beta}$ is transcendental.

Proof. Assume that (i) holds, and let α, γ be nonzero algebraic numbers with $\alpha \neq 1$. For $\beta = \frac{\ln \gamma}{\ln \alpha}$,

$$\alpha^\beta = \alpha^{\frac{\ln \gamma}{\ln \alpha}} = \alpha^{\log_\alpha \gamma} = \gamma.$$

Since γ is algebraic, β does not satisfy the hypotheses of (i). In particular, this means that β must be either transcendental or rational, so (i) implies (ii).

Next, assume that (ii) holds, and let β be irrational and $\zeta \neq 0$. Let $\alpha = e^\zeta$ and $\gamma = e^{\zeta\beta}$. If either of α or γ are transcendental, we are done, so we may assume α and β are both algebraic. Thus by (ii), we know that

$$\frac{\ln \gamma}{\ln \alpha} = \frac{\zeta\beta}{\zeta} = \beta$$

is either rational or transcendental. Since β was assumed to be irrational, it must be transcendental, and therefore (ii) implies (iii).

Finally, assume (iii) and let α, β be algebraic with $\alpha \neq 0, 1$ and β irrational. Let $\zeta = \ln \alpha$. Then by (iii), we know that

$$e^{\zeta\beta} = \left(e^\zeta\right)^\beta = \alpha^\beta.$$

is transcendental. Thus (iii) implies (i), so all three statements are equivalent. □

To prove the Gelfond-Schneider Theorem, we will follow the proof in [3], which models itself after Gelfond's version of the proof. The general idea is to use the third formulation, assume that β, e^ζ, and $e^{\zeta\beta}$ are all algebraic, and use this to find the obvious contradiction of a positive integer that is less than one.

5.2 Symmetric Functions

We begin with some standard definitions in symmetric functions, taken from Cox [8] and Burger and Tubbs [3].

We say that a function $F(x_1, x_2, \ldots, x_L)$ is *symmetric* if every permutation of the variables $x_1, x_2, \ldots, x_L$ results in the same function.

5.2.1 Example. Given variables $x_1, x_2, \ldots, x_n$, the functions

$$\begin{aligned}
\sigma_1 &= x_1 + x_2 + \cdots + x_n \\
\sigma_2 &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j \\
&\vdots \\
\sigma_r &= \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_r=i_{r-1}}^{n} x_{i_1} x_{i_2} \cdots x_{i_r} \\
&\vdots \\
\sigma_n &= x_1 x_2 \cdots x_n
\end{aligned}$$

are all symmetric. In fact, these functions are such an important class of symmetric functions that they are called *elementary symetric functions*.

It is useful to have a way to compare monomial terms of functions. For this, we will use *lexicographic order*, where

$$c x_1^{n_1} x_2^{n_2} \cdots x_L^{n_L} > c' x_1^{n'_1} x_2^{n'_2} \cdots x_L^{n'_L}$$

exactly when the first nonzero term in

$$n_1 - n'_1, n_2 - n'_2, \cdots, n_L - n'_L$$

is positive. Because the monomials have finite length, this is a well-ordering.

We prove the next famous result by merging ideas from the proofs in [8] and [3].

5.2.2 Theorem (Fundamental Theorem of Symmetric Polynomials)**.** *Let R be a ring and let $P(x_1, x_2, \ldots, x_L) \in R[x_1, x_2, \ldots, x_L]$ be a symmetric polynomial. Then there exists a polynomial F with coefficients from R such that*

$$P(x_1, x_2, \ldots, x_L) = F(\sigma_1, \sigma_2, \ldots, \sigma_L).$$

Proof. Let $P(x_1, x_2, \ldots, x_L)$ be a counterexample with minimal greatest monomial, denoted
$$\mathcal{M} = cx_1^{k_1}x_2^{k_2}\cdots x_L^{k_L}.$$

Because $P(x_1, x_2, \ldots, x_L)$ is a symmetric function and $\mathcal{M}$ is the greatest monomial term, we may observe that $k_1 \geq k_2 \geq \cdots \geq k_L$.

We next observe that the greatest monomial term of
$$\sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_L^{n_L}$$
will be
$$x_1^{n_1+n_2+\cdots+n_L}x_2^{n_2+\cdots+n_L}\cdots x_L^{n_L}.$$

Combining these two observations, we can see that the
$$c\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\cdots\sigma_L^{k_L}$$
has greatest monomial $\mathcal{M}$. Therefore,
$$P(x_1, x_2, \ldots, x_L) - c\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\cdots\sigma_L^{k_L}$$
is a symmetric monomial with greatest monomial less than $\mathcal{M}$. By the minimality of $P(x_1, x_2, \ldots, x_L)$, we know that there exists $F(\sigma_1, \sigma_2, \ldots \sigma_L) \in R[\sigma_1, \sigma_2, \ldots \sigma_L]$ such that
$$P(x_1, x_2, \ldots, x_L) - c\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\cdots\sigma_L^{k_L} = F(\sigma_1, \sigma_2, \ldots \sigma_L).$$

Since $c \in R$, it follows that $P(x_1, x_2, \ldots, x_L)$ can also be written as a polynomial of $\sigma_1, \sigma_2, \ldots, \sigma_L$ with coefficients from R. □

In our quest for an impossibly small integer, finding a function that necessarily evaluates to an integer is an important first step. As shown in [3], symmetric functions can have this desired property.

5.2.3 Corollary. *Let $\alpha_1, \alpha_2, \ldots, \alpha_L$ be all the conjugates of an algebraic integer α_1, and let $P(x_1, x_2, \ldots, x_L)$ be a symmetric polynomial with integer coefficients. Then $P(\alpha_1, \alpha_2, \ldots, \alpha_L)$ is an integer.*

Proof. Since α_1 is an algebraic integer and $\alpha_1, \alpha_2, \ldots, \alpha_L$ comprise all the conjugates, we know there exists a monic polynomial with integer coefficients $a_0, a_1, \ldots, a_{L-1}$ satisfying
$$z^L + a_{L-1}z^{L-1} + \cdots + a_1 z + a_0 = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_L).$$

Expanding out the right side, we get

$$(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_L) = z^L - \sigma_1 z^{L-1} + \sigma_2 z^{L-2} - \cdots + (-1)^L \sigma_L,$$

so by transitivity

$$z^L + a_{L-1}z^{L-1} + \cdots + a_1 z + a_0 = z^L - \sigma_1 z^{L-1} + \sigma_2 z^{L-2} - \cdots + (-1)^L \sigma_L.$$

This tells us that

$$\begin{aligned}\sigma_1 &= a_{L-1}\\ \sigma_2 &= a_{L-2}\\ &\vdots\\ \sigma_L &= \pm a_0\end{aligned}$$

must all be integers. By Theorem 5.2.2 we know that we can write $P(\alpha_1, \alpha_2, \ldots, \alpha_L)$ as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_L$ with integer coefficients, so therefore we can write $P(\alpha_1, \alpha_2, \ldots, \alpha_L)$ as a linear combination of integers, otherwise known as an integer. □

To find an integer, we may find a symmetric function evaluated at every conjugate of some algebraic integer. If we take the absolute value, then provided the function is not zero where we evaluate it we will end up with a positive integer. If we can then bound the function sufficiently well, we will end up with the desired contradictory integer between zero and one.

With this rough map, we are now ready to develop the additional tools we will need to prove the Gelfond-Schneider Theorem.

5.3 Primitive Element

For Corollary 5.2.3, we want to consider the conjugates of a single algebraic integer. If we were to try and prove the Gelfond-Schneider Theorem right now, we would have three equally important algebraic numbers in possibly distinct fields. The next result is standard field theory, allow us to deal with that. [3]

5.3.1 Theorem. *If α and β are algebraic integers, then there exists an algebraic integer θ such that $\mathbb{Q}(\theta) = \mathbb{Q}(\alpha, \beta)$.*

Proof. Let $\alpha = \alpha_1, \alpha_, \ldots, \alpha_M$ be the list of all conjugates of α, and $\beta = \beta_1, \beta_, \ldots, \beta_N$ be all conjugates of β. Let

$$\mathcal{L} := \left\{ \frac{\alpha_m - \alpha_1}{\beta_1 - \beta_n} : m = 1, \ldots, M, n = 2, \ldots, N \right\}.$$

Since $\mathcal{L}$ is finite, we can choose γ to be one of infinitely many integers not in $\mathcal{L}$. Let $\theta = \alpha + \beta\gamma$. Observe that θ is an algebraic integer, and by construction $\theta \in \mathbb{Q}(\alpha, \beta)$. It remains to show that $\beta \in \mathbb{Q}(\theta)$, which would imply $\alpha \in \mathbb{Q}(\theta)$ and thus $\mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\theta)$.

Denote the minimal polynomials in $\mathbb{Q}$ of α and β by $f_\alpha(x)$ and $f_\beta(x)$.

Note that the minimal polynomial of β in $\mathbb{Q}(\theta)$, denoted $g_\beta(x)$ must divide $f_\beta(x)$, so the roots of $g_\beta(x)$ will be a subset of $\beta_1, \ldots, \beta_N$.

Next, we observe that

$$f_\alpha(\theta - \gamma\beta) = f_\alpha(\alpha) = 0,$$

so $g_\beta(x)$ factors $f_\alpha(\theta - \gamma x)$. In particular, for any root β_n, there exists some $1 \leq m \leq M$ such that

$$\alpha_m = \theta - \gamma\beta_n = \alpha_1 + \gamma\beta_1 - \gamma\beta_n.$$

If $n \neq 1$, then we may rewrite this

$$\gamma = \frac{\alpha_m - \alpha_1}{\beta_1 - \beta_n},$$

which is a contradiction of our choice of γ, and therefore the only root of $g_\beta(x)$ is β. In other words,

$$g_\beta(x) = x - \beta \in \mathbb{Q}(\theta)[x],$$

so $\beta \in \mathbb{Q}(\theta)$. □

This theorem lends itself immediately to the specialized case that we need.

5.3.2 Corollary. *Let α, β, γ be three algebraic numbers. Then there exists an algebraic integer θ such that $\mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\theta)$.*

Suppose that θ is the primitive element of degree d. Then any element in $\mathbb{Z}(\theta)$ can be written as

$$r_0 + r_1\theta + \cdots + r_{d-1}\theta^{d-1}$$

for integers $r_0, r_1, \ldots, r_{d-1}$. If $\theta = \theta_1, \theta_2, \ldots, \theta_d$ are all the conjugates of θ, then

$$\prod_{i=1}^{d} \left(r_0 + r_1\theta_i + \cdots + r_{d-1}\theta_i^{d-1}\right)$$

is symmetric in $\theta_1, \ldots, \theta_d$. By finding a small enough element in $\mathbb{Z}(\theta)$, we will be able to use symmetric functions to find a small integer. We will find this sufficiently small element by constructing another small function, this time not symmetric, and evaluating it at a point where it is nonzero.

5.4 A Nonzero Function

We turn our attention now to building such a function. We want it to lie in the field $\mathbb{Q}\left(\beta, e^{\zeta}, e^{\beta\zeta}\right)$ at known points. Observe that for some polynomial $P(x, y) \in \mathbb{Z}[x, y]$, we can see that for all integers m and n we will have

$$P\left(m + n\beta, e^{\zeta(m+n\beta)}\right) \in \mathbb{Q}\left(\beta, e^{\zeta}, e^{\beta\zeta}\right).$$

This gives us a solid starting place for building our desired function, provided that it will not be zero for every possible $m + n\beta$.

As in [3], we start by resolving the simpler potential problem that our function might be zero everywhere.

5.4.1 Lemma. *Let ζ be a nonzero complex number. Then for any nonzero polynomial $P(x, y) \in \mathbb{Z}[x, y]$, the function $P\left(z, e^{\zeta z}\right)$ is not identically zero.*

Proof. Let $P(x, y) \in \mathbb{Z}[x, y]$ be a nonzero polynomial. We may write

$$P(x, y) = \sum_{m=0}^{D_1} \sum_{n=0}^{D_2} a_{m,n} x^m y^n.$$

For $0 \leq m \leq D_1$, we define

$$P_m(y) = \sum_{n=0}^{D_2} a_{m,n} y^n$$

and note that

$$P(x,y) = \sum_{m=0}^{D_1} P_m(y)\, x^m.$$

Since $P(x,y)$ is not identically zero, there must be some m between 0 and D_1 such that $P_m(y)$ is not identically zero. Then $P_m(y)$ has at most D_2 roots, so we may let y_0 be one of the infinitely many non-roots. Since ζ is nonzero

$$z_0 = \frac{\ln y_0}{\zeta}$$

is defined. The coefficient $P_m(y_0)$ of x^m in

$$P\left(z, e^{\zeta z_0}\right) = P(z, y_0)$$

is not zero, so $P(z, y_0)$ is not identically zero, thus it has at most D_1 roots. In particular, there are at most D_1 integers k such that

$$P\left(\frac{2\pi i k}{\zeta} + z_0, e^{\zeta z_0}\right) = 0.$$

Let k_0 be one of the infintely many other integers, and let

$$z_1 = \frac{2\pi i k_0}{\zeta} + z_0.$$

We can see that

$$P\left(z_1, e^{\zeta z_1}\right) = P\left(z_1, e^{2\pi i k_0} e^{\zeta z_0}\right) = P\left(\frac{2\pi i k_0}{\zeta} + z_0, e^{\zeta z_0}\right) \neq 0,$$

as required. □

Because we are looking for nonzero elements in $\mathbb{Q}\left(\beta, e^{\zeta}, e^{\beta\zeta}\right)$, we need an even stronger nonzero property. Recalling basic results from analysis and basing our proof on the sketch in [3], we can show that our to-be-developed function must have this property.

5.4.2 Theorem. *If β is a real and negative irrational number and $F(z)$ is a nonzero analytic function, then there exist positive integers m, n such that $F(m + n\beta) \neq 0$.*

Proof. For an integer n, let

$$\beta_n = n\beta - \lfloor n\beta \rfloor .$$

Note that for disinct n, m, if $\beta_n = \beta_m$ then

$$\beta (n - m) = \lfloor n\beta \rfloor - \lfloor m\beta \rfloor ,$$

which contradicts the irrationallity of β. Therefore, the sequence $(\beta_n)_{n \in \mathbb{N}}$ is infinite, and lies within $[0, 1]$, so by compactness it must have a subsequence (β_{n_ℓ}) converging to $\beta^* \in [0, 1]$.

Since $F(z)$ is nonzero, it has a nonzero power series expansion around β^*, that is, there exists an $M \geq 0$ such that b_M is not zero and

$$F(z) = \sum_{m=M}^{\infty} b_m (z - \beta^*)^m .$$

We now define a new function

$$G(z) = \sum_{m=M}^{\infty} b_m (z - \beta^*)^{m-M}$$

which is, like $F(z)$, continuous, but unlike $F(z)$, is necesarily nonzero at β^*. From this we conclude that there exists $\varepsilon > 0$ such that, for all z with $0 < |z - \beta^*| < \varepsilon$, the evaluation $G(z) \neq 0$.

Because the sequence (β_{n_ℓ}) converges to β^*, we know that there exits an L such that $|\beta_{n_L} - \beta^*| < \varepsilon$.

Now suppose that $F(m + n\beta) = 0$ for all positive integers m, n. Since β is negative, we know that n and $-\lfloor n\beta \rfloor$ are both positive integers, and therefore

$$F(\beta_{n_L}) = F(-\lfloor n\beta \rfloor + n\beta) = 0.$$

It follows that

$$G(\beta_{n_L}) = \frac{F(\beta_{n_L})}{(z - \beta^*)^M} = 0,$$

which is a contradiction. This shows that $F(m + n\beta)$ cannot be zero for all positive integers n, m. $\square$

Given our purposes in using the Gelfond-Schneider Theorem, it is worth drawing attention to the fact that this is the first time we have used the assumption that β is irrational. We will use that assumption once more, but without it the previous proof falls apart, and our subsequent attempts to build a small function might just result in a function that can only evaluate to zero in our field.

5.5 Smaller is Better

It is not enough that we have a function that evaluates to some nonzero element in $\mathbb{Q}\left(\beta, e^{\zeta}, e^{\beta\zeta}\right)$, we need our function to evaluate to some extremely small element of that field.

We begin by recalling a standard result from analysis. The formulation below is from [3].

5.5.1 Theorem. *[Maximum Modulus Principle] Let $D \subseteq \mathbb{C}$ be an open disk and let $\overline{D}$ denote the union of D and its boundary. If $f : \overline{D} \to \mathbb{C}$ is a continuous function that is analytic on D, then $|f(z)|$ attains its maximum value at a point on the boundary of D.*

Having established that we can construct a function $F(z)$ such that $F(k_1 + k_2\beta)$ will not be zero for every pair of integers k_1 and k_2, we now aim to show that if $F(k_1 + k_2\beta)$ is zero for a lot of integer pairs k_1 and k_2, then we can find integers k_1^* and k_2^* where $F(k_1^* + k_2^*\beta)$ must be relatively small. The next lemma, with proof from [3], formalizes this idea.

5.5.2 Lemma. *Let β be algebraic and irrational and let M be a positive integer. Suppose there exists an analytic function $F(z)$ and nonnegative integers k_1^*, k_2^* at most M such that $F(k_1 + k_2\beta) = 0$ for all $0 \leq k_1, k_2 < M$ but $F(k_1^* + k_2^*\beta)$ does not equal zero. Then*

$$F(k_1^* + k_2^*\beta) < |F(z)|_{M^{\frac{3}{2}} + M(1+|\beta|)} (1 + |\beta|)^{M^2} \left(e^{\frac{1}{2}}\right)^{-M^2 \ln M}.$$

Proof. We define

$$G(z) = \frac{F(z)}{\prod_{k_1=0}^{M-1} \prod_{k_2=0}^{M-1} |z - (k_1 + k_2\beta)|}.$$

Clearly

$$\left|F\left(k_1^*+k_2^*\beta\right)\right|=\left|G\left(k_1^*+k_2^*\beta\right)\right|\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}\left|k_1^*-k_1+\left(k_2^*-k_2\right)\beta\right|.$$

Considering $G(z)$, we may observe that for all valid k_1 and k_2, the number $k_1+k_2\beta$ will be a zero of denominator, and by the irrationallity of β they will be distinct, so every zero of the denominator is of this form. Then by construction of $F(z)$, we have every zero of the denominator must be a zero of the numerator as well, so $G(z)$ is entire.

Let

$$R=M^{\frac{3}{2}}+M\left(1+|\beta|\right).$$

Then since

$$\left|k_1^*+k_2^*\beta\right|\leq\left|k_1^*\right|+\left|k_2^*\beta\right|\leq M\left(1+|\beta|\right)<R,$$

we know that $k_1^*+k_2^*\beta$ lies in the disk of radius R. Then by Theorem 5.5.1, we know that

$$\begin{aligned}F\left(k_1^*+k_2^*\beta\right)&<\left|G(z)\right|_R\prod_{k_2=0}^{M-1}\left|k_1^*-k_1+\left(k_2^*-k_2\beta\right)\right|\\&=\frac{\left|F(z)\right|_R}{\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}\left|z-\left(k_1+k_2\beta\right)\right|_R}\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}\left|k_1^*-k_1+\left(k_2^*-k_2\beta\right)\right|\end{aligned}$$

From this, we see that we can bound $F\left(k_1^*+k_2^*\beta\right)$ by bounding each of these terms.

For the denominator, we need to find a lower bound on

$$\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}\left|z-\left(k_1+k_2\beta\right)\right|_R.$$

Using reverse triangle inequality and the fact that $R>|k_1*+k_2^*\beta|$, we get

$$\left|R-\left(k_1+k_2\beta\right)\right|\geq|R|-\left|k_1+k_2\beta\right|,$$

and then we may use triangle inequality to see that, because $k_1,k_2\geq 0$,

$$|R|-\left|k_1+k_2\beta\right|\geq R-\left(k_1+k_2\left|\beta\right|\right).$$

Therefore,

$$\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}|z-(k_1+k_2\beta)|_R \geq \prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}[R-(k_1+k_2|\beta|)]$$
$$\geq (R-M(1+|\beta|))^{M^2}$$
$$= \left(M^{\frac{3}{2}}\right)^{M^2},$$

giving us our desired bound.

The bound for the double product is easy:

$$\prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}|k_1^*-k_1+(k_2^*-k_2)\beta| \leq \prod_{k_1=0}^{M-1}\prod_{k_2=0}^{M-1}(|k_1^*-k_1|+(|k_2^*-k_2|)|\beta|)$$
$$\leq (M(1+|\beta|))^{M^2}.$$

Putting this all together gives us

$$|F(k_1^*+k_2^*\beta)| < |F(z)|_{M^{\frac{3}{2}}+M(1+|\beta|)}\left(\frac{M(1+|\beta|)}{M^{\frac{3}{2}}}\right)^{M^2}$$
$$= |F(z)|_{M^{\frac{3}{2}}+M(1+|\beta|)}\left(M^{-\frac{1}{2}}(1+|\beta|)\right)^{M^2}$$
$$= |F(z)|_{M^{\frac{3}{2}}+M(1+|\beta|)}(1+|\beta|)^{M^2}\left(e^{\frac{1}{2}}\right)^{-M^2\ln M}$$

□

Now suppose that there were to exist some constant c in no way dependent on M and some function $h(x)$ that grows more slowly than $x^2\ln x - x^2$ such that

$$|F(z)|_{M^{\frac{3}{2}}+M(1+|\beta|)} < c^{h(M)}.$$

Then there would exist an M sufficiently large that we could guarantee $|F(k_1^*+k_2^*)|$ is however small we want it to be. This gives us a new goal of proving that we can in fact create such a function for M arbitrarily large.

5.6 Bounding Coefficient Size

In order to proceed in trying to find a bound, we need to have a clear idea of what exactly we are bounding. To this end, we may define the *height* of an algebraic integer θ, denoted $\mathcal{H}(\theta)$, as the maximal absolute value of the coefficients of the minimum polynomial of θ.

If $\beta \in \mathbb{Q}(\theta)$, then we may write

$$\beta = b_0 + b_1\theta + \cdots + b_{d-1}\theta^{d-1}.$$

We say the *height* of β in $\mathbb{Q}(\theta)$, denoted $\mathcal{H}_\theta(\beta)$ is

$$\max_{0\le j\le d-1} \{|b_j|\}.$$

With these definitions, we are ready to prove two useful results, adapted from [3].

5.6.1 Lemma. *Let θ be an algebraic integer of degree d. Then for all $n \ge d$, we have*

$$\mathcal{H}_\theta^n(\theta) \le (1 + \mathcal{H}(\theta))^{n+1-d}.$$

Proof. We will proceed by induction on n. If $n = d$, then because θ is an algebraic integer, there exist integers $-c_{0,d}, -c_{1,d}, \ldots, -c_{d-1,d}$ such that

$$\theta^d - c_{d-1,d}\theta^{d-1} - \cdots - c_{1,d}\theta - c_{0,d} = 0,$$

so we may write

$$\theta^d = c_{d-1,d}\theta^{d-1} + \cdots + c_{1,d}\theta + c_{0,d},$$

and by definition,

$$\mathcal{H}_\theta\left(\theta^d\right) \le \mathcal{H}(\theta) \le 1 + \mathcal{H}(\theta).$$

For the induction step, choose k such that there exist integers $c_{0,k}, c_{1,k}, \ldots, c_{d-1,k}$ satisfying

$$\theta^k = c_{0,k} + c_{1,k}\theta + \cdots + c_{d-1,k}\theta^{d-1}$$

and

$$\max_{0\le j\le d-1} \{|c_{j,k}|\} \le (1 + \mathcal{H}(\theta))^{k+1-d}.$$

Then we may write

$$\begin{aligned}
\theta^{k+1} &= \theta\left(\theta^k\right) \\
&= \theta\left(c_{0,k} + c_{1,k}\theta + \cdots + c_{d-1,k}\theta^{d-1}\right) \\
&= c_{0,k}\theta + c_{1,k}\theta^2 + \cdots + c_{d-2,k}\theta^{d-1} + c_{d-1,k}\left(c_{d-1,d}\theta^{d-1} + \cdots + c_{1,d}\theta + c_{0,d}\right) \\
&= c_{0,d}c_{d-1,k} + (c_{0,k} + c_{d-1,k}c_{1,d})\,\theta + \cdots + (c_{d-2,k} + c_{d-1,k}c_{d-1,d})\,\theta^{d-1}.
\end{aligned}$$

This shows us that, for all $0 \le i \le d-1$ define

$$c_{i,k+1} = \begin{cases} c_{d-1,k}c_{0,d} & i = 0 \\ c_{i-1,k} + c_{d-1,k}c_{i,d} & i \ge 1 \end{cases}.$$

By the inductive hypothesis, $c_{i,k+1}$ is an integer and

$$\begin{aligned} |c_{i,k+1}| &\le (1+\mathcal{H}(\theta))^{k+1-d} + \mathcal{H}(\theta)(1+\mathcal{H}(\theta))^{k+1-d} \\ &= (1+\mathcal{H}(\theta))^{k+1-d}(1+\mathcal{H}(\theta)) \\ &= (1+\mathcal{H}(\theta))^{k+2-d}. \end{aligned}$$

For any $n \ge d$, we can therefore conclude that θ^n is a polynomial with bounded integer coefficients. □

This allows us to prove our next, even more useful, lemma.

5.6.2 Lemma. *Let θ be an algebraic integer of degree d and let $\beta_1, \beta_2, \ldots, \beta_L$ be elements of $\mathbb{Z}(\theta)$. Then*

$$\mathcal{H}_\theta(\beta_1\beta_2\cdots\beta_L) \le d^L \mathcal{H}_\theta(\beta_1)\mathcal{H}_\theta(\beta_2)\cdots\mathcal{H}_\theta(\beta_L)(2\mathcal{H}(\theta))^{dL}.$$

Proof. We proceed by induction. If $L = 1$, then the claim follows immediately from the definition of $\mathcal{H}_\theta(\beta_1)$. Suppose the result holds for some $k \ge 1$. Then

$$\beta_1\beta_2\cdots\beta_k = s_0 + s_1\theta + \cdots + s_{d-1}\theta^{d-1}$$

with integer coefficients $s_0, s_1, \ldots, s_{d-1}$ satisfying

$$\max_{0 \le j \le d-1}\{|s_j|\} \le d^k \mathcal{H}_\theta(\beta_1)\mathcal{H}_\theta(\beta_2)\cdots\mathcal{H}_\theta(\beta_k)(2\mathcal{H}(\theta))^{dk}.$$

We consider

$$\begin{aligned} \beta_1\beta_2\cdots\beta_k\beta_{k+1} &= \left(s_0 + s_1\theta + \cdots + s_{d-1}\theta^{d-1}\right)\left(b_0 + b_1\theta + \cdots + b_{d-1}\theta^{d-1}\right) \\ &= \sum_{i=0}^{2d-2}\left(\sum_{j=0}^{i} s_j b_{i-j}\right)\theta^i. \end{aligned}$$

Using Lemma 5.6.1, we can see that

$$\beta_1\beta_2\cdots\beta_k\beta_{k+1} = \sum_{i=0}^{d-1}\left(\sum_{j=0}^{i} s_j b_{i-j} + \sum_{k=d}^{2d-2} c_{i,k}\left(\sum_{j=0}^{k} s_j b_{i-j}\right)\right)\theta^i.$$

Let

$$r_i = \sum_{j=0}^{i} s_j b_{i-j} + \sum_{k=d}^{2d-2} c_{i,k} \left(\sum_{j=0}^{k} s_j b_{i-j} \right).$$

Note that since for all i, j, and k, $c_{i,k}$ is an integer and, for all i, we have

$$\begin{aligned}
|r_i| &\leq \mathcal{H}_\theta(\beta_{k+1}) \max\{|s_j|\} + (d-1)\mathcal{H}_\theta(\beta_{k+1}) \max\{|s_j|\}(1+\mathcal{H}(\theta))^d \\
&= \max\{|s_j|\} \mathcal{H}_\theta(\beta_{k+1}) \left(1 + (d-1)(1+\mathcal{H}(\theta))^d\right) \\
&\leq d^k \mathcal{H}_\theta(\beta_1) \mathcal{H}_\theta(\beta_2) \cdots \mathcal{H}_\theta(\beta_k) (2\mathcal{H}(\theta))^{dk} \mathcal{H}_\theta(\beta_{k+1}) \left(d(2\mathcal{H}(\theta))^d\right) \\
&= d^{k+1} \mathcal{H}_\theta(\beta_1) \mathcal{H}_\theta(\beta_2) \cdots \mathcal{H}_\theta(\beta_k) \mathcal{H}_\theta(\beta_{k+1}) (2\mathcal{H}(\theta))^{d(k+1)}.
\end{aligned}$$

So by induction, we may conclude that

$$\mathcal{H}_\theta(\beta_1\beta_2\cdots\beta_L) \leq d^L \mathcal{H}_\theta(\beta_1) \mathcal{H}_\theta(\beta_2) \cdots \mathcal{H}_\theta(\beta_L) (2\mathcal{H}(\theta))^{dL}.$$

□

5.7 Siegel's Lemma

Recall that, for arbitrarily large M, we want to create a function that is zero at $F(k_1 + k_2\beta)$ for all integers $0 \leq k_1, k_2 < M$. To help us with that, we will need an important result from transcendental number theory, following the proof in [3].

Given a vector v, the *height* of v, denoted $\mathcal{H}(v)$, is the absolute value of the largest entry of v.

5.7.1 Lemma. *[Siegel's Lemma] Let $\mathcal{C}$ be an $m \times n$ nonzero integer matrix, and let*

$$c = \left\{ \max_{1\leq i\leq m, 1\leq j\leq n} |\mathcal{C}_{i,j}| \right\}.$$

If $m < n$, there exists a nonzero vector $v \in \mathbb{Z}^n$ such that

$$\mathcal{C}v = \mathbf{0} \quad \textit{and} \quad \mathcal{H}(v) \leq (cn)^{\frac{m}{n-m}}.$$

Proof. We begin by defining

$$a = \lfloor (cn)^{\frac{m}{n-m}} \rfloor.$$

Consider the $(1+a)^n$ vectors in $\mathbb{Z}^n$ with entries between 0 and a. Taking the set of these vector as our domain and multiplication by $\mathcal{C}$ as our function, we are interested in the number of possible elements in our range.

Elements of the range will be vectors of length m with entries $y_1, y_2, \ldots, y_m$. For a fixed i between 1 and m, we note that $\sum_{j=1}^{n} \mathcal{C}_{i,j}$ will either be nonnegative or negative. If it is nonnegative, then

$$0 \leq y_i = \sum_{j=1}^{n} \mathcal{C}_{i,j} x_j \leq \sum_{j=1}^{n} ca = nca,$$

and otherwise

$$0 > y_i = \sum_{j=1}^{n} \mathcal{C}_{i,j} x_j \geq -\sum_{j=1}^{n} ca = -nca.$$

In either case, for each m we have at most $(1+acn)$ different possibilities, which means there are at most $(1+acn)^m$ posible vectors in the range of $\mathcal{C}$ for the given domain.

By our choice of a,

$$1 + a = 1 + \lfloor (cn)^{\frac{m}{n-m}} \rfloor > (cn)^{\frac{m}{n-m}},$$

so

$$(1+a)^n = (1+a)^{n-m} (1+a)^m > (cn)^m (1+a)^m.$$

Recalling our choice of c and n as positive integers,

$$(cn)^m (1+a)^m = (cn + acn)^m \geq (1+acn)^m,$$

so

$$(1+a)^n > (1+acn)^m.$$

In particular, this tells us that the number of elements in the domain is strictly larger than the number of elements in the range, and so by pigeonhole principle there exist distinct vectors x_1, x_2 such that $\mathcal{C}x_1 = \mathcal{C}x_2$. Let $v = x_1 - x_2$. Clearly v is nonzero, and since each entry of x_1, x_2 is between 0 and a, the diference must be between $-a$ and a and therefore $\mathcal{H}(v) \leq (cn)^{\frac{m}{n-m}}$. $\square$

5.8 A Polynomial Construction

With these tools, we are almost ready to build our not-too-big function that is zero at our chosen points. First, though, we need to formalize the notion of a function being not-too-big.

Given a polynomial, we say that the *height* is the maximum absolute value of the coefficients.

Now, we pull together our previous work to find our desired function. The proof is based off [3].

5.8.1 Lemma. *Given an algebraic integer θ and $\beta, e^{\zeta}, e^{\beta\zeta} \in \mathbb{Q}(\theta)$, there exists some c such that, for K arbitrarily large, there exists a nonzero polynomial $P(x,y)$ with height less than $c^{K^{\frac{3}{2}}\ln K}$ and $P\left(k_1 + k_2\beta, e^{\zeta k_1} e^{\beta\zeta k_2}\right) = 0$ for all nonnegative integers k_1, k_2 less than K.*

Proof. Let d be the degree of the minimal polynomial for θ. Note that we can square any large integer and multiply it by $2d$ to create an arbitrarily large integer $K \geq 3$. We now define

$$D_1 = \sqrt{2dK^3} \quad \text{and} \quad D_2 = \sqrt{2dK}.$$

By the way that we chose K, we know that D_1 and D_2 will both be integers.

Since $\beta, e^{\zeta}, e^{\beta\zeta} \in \mathbb{Q}(\theta)$, we may let $\delta_1, \delta_2, \delta_3$ be the smallest integers such that $\delta_1\beta, \delta_2 e^{\zeta}, \delta_3 e^{\beta\zeta} \in \mathbb{Z}(\theta)$.

Next, for all integers m, n, k_1, k_2 such that $0 \leq m \leq D_1 - 1, 0 \leq n \leq D_2 - 1$, and $0 \leq k_1, k_2, < K$, Lemma 5.6.2 shows that

$$(\delta_1(k_1 + k_2\beta))^m \left(\delta_2 e^{\zeta}\right)^{nk_1} \left(\delta_3 e^{\zeta\beta}\right)^{nk_2}$$

will have height in $\mathbb{Q}(\theta)$ at most

$$d^{m+nk_1+nk_2} (\delta_1 \mathcal{H}_\theta(k_1 + k_2\beta))^m \left(\delta_2 \mathcal{H}_\theta\left(e^{\zeta}\right)\right)^{nk_1} \left(\delta_3 \mathcal{H}_\theta\left(e^{\zeta\beta}\right)\right)^{nk_2} (2\mathcal{H}(\theta))^{d(m+nk_1+nk_2)}.$$

Because k_1 and k_2 are integers bounded above by K,

$$\mathcal{H}_\theta(k_1 + k_2\beta) \leq 2K\mathcal{H}_\theta(\beta),$$

which combines with our other bounds to give us height at most

$$d^{D_1+2D_2K} (\delta_1 2K\mathcal{H}_\theta(\beta))^{D_1} \left(\delta_2 \mathcal{H}_\theta\left(e^{\zeta}\right)\right)^{D_2K} \left(\delta_3 \mathcal{H}_\theta\left(e^{\zeta\beta}\right)\right)^{D_2K} (2\mathcal{H}(\theta))^{d(D_1+2D_2K)}$$
$$= \left(dKH_\theta(\beta)(2\mathcal{H}(\theta))^d\right)^{D_1} \left(d^2\mathcal{H}_\theta\left(e^{\zeta}\right)\mathcal{H}_\theta\left(e^{\zeta\beta}\right)(2\mathcal{H}(\theta))^{2d}\right)^{D_2K}$$

For the sake of clarity, we can combine together similar terms into constants. These constants will depend directly and indirectly on $\beta, e^{\zeta}, e^{\zeta\beta}$, and θ, but as long as they do not depend on K or the related D_1 or D_2, we will not be overly concerned with their exact values.

With this in mind, let

$$c_1 = 2\delta_1 d\mathcal{H}_\theta(\beta)(2\mathcal{H}(\theta))^d$$

and

$$c_2 = d^2\delta_2\mathcal{H}_\theta\left(e^{\zeta}\right)\delta_3\mathcal{H}_\theta\left(c^{\zeta\beta}\right)(2\mathcal{H}(\theta))^{2d}.$$

We can also write $K = e^{\ln K}$ and, since $K \geq 3$, we get

$$\mathcal{H}_\theta\left((\delta_1(k_1+k_2\beta))^m\left(\delta_2 e^{\zeta}\right)^{nk_1}\left(\delta_3 e^{\zeta\beta}\right)^{nk_2}\right) \leq c_1^{D_1}c_2^{D_2K}e^{D_1\ln K} \leq (ec_1)^{D_1\ln K}c_2^{D_2K}.$$

Then letting

$$c_3 = \max\{ec_1, c_2\},$$

we see that we may write

$$(\delta_1(k_1+k_2\beta))^m\left(\delta_2 e^{\zeta}\right)^{nk_1}\left(\delta_3 e^{\zeta\beta}\right)^{nk_2} = \sum_{j=0}^{d-1}(r_j(k_1,k_2,m,n))\,\theta^j$$

where, for every k_1, k_2, m, and n we have integers $r_j(k_1,k_2,m,n)$ such that

$$|r_j(k_1,k_2,m,n)| \leq c_3^{D_1\ln K + D_2K}.$$

Create a $dK^2 \times D_1D_2$ matrix $\mathcal{C}$ whose $((j,k_1,k_2),(m,n))$-entry is $r_j(k_1,k_2,m,n)$. By our choice of D_1 and D_2,

$$D_1D_2 = 2dK^2 > dK^2,$$

so by Lemma 5.7.1 there exists a vector $\mathbf{x} \in \mathbb{Z}^{D_1D_2}$ such that $\mathcal{C}\mathbf{x} = 0$ and

$$\mathcal{H}(\mathbf{x}) < \left(c_3^{D_1\ln K + D_2K}D_1D_2\right)^{\frac{dK^2}{D_1D_2 - dK^2}}.$$

Applying the definitions of D_1 and D_2 gives us

$$\begin{aligned}\mathcal{H}(\mathbf{x}) &< \left(c_3^{\sqrt{2dK^3}(\ln K+1)}dK^2\right)\\ &\leq c_3^{2\sqrt{2d}K^{\frac{3}{2}}\ln K}\left(2de^2\right)^{\ln K}\\ &\leq c_3^{2\sqrt{2d}K^{\frac{3}{2}}\ln K}\left(2de^2\right)^{K^{\frac{3}{2}}\ln K}\end{aligned}$$

Then letting $c = c_3^{2\sqrt{2d}} de^2$, we have that

$$\mathcal{H}\left(\mathbf{x}\right) < c^{K^{\frac{3}{2}} \ln K}.$$

For all integers m, n with $0 \leq m \leq D_1 - 1$ and $0 \leq n \leq D_2 - 1$, we define $a_{m,n}$ to be the integer indexed by row m, n of $\mathbf{x}$. We have just shown that, for all valid m and n, we have $|a_{m,n}| < c^{K^{\frac{3}{2}} \ln K}$. Since $\mathbf{x}$ is nonzero,

$$P\left(x, y\right) = \sum_{m=0}^{D_1-1} \sum_{n=0}^{D_2-1} a_{m,n} x^m y^n$$

is a nonzero polynomial, and by Lemma 5.4.1 we conclude that $F\left(z\right)$ is not everywhere zero. However, for all integers k_1, k_2 with $0 \leq k_1, k_2 < K$ we have

$$\begin{aligned} P\left(k_1 + k_2\beta, e^{\zeta(k_1+k_2\beta)}\right) &= \sum_{m=0}^{D_1-1} \sum_{n=0}^{D_2-1} a_{m,n} \left(k_1 + k_2\beta\right)^m \left(e^{\zeta}\right)^{k_1 n} \left(e^{\zeta\beta}\right)^{k_2 n} \\ &= \sum_{m=0}^{D_1-1} \sum_{n=0}^{D_2-1} a_{m,n} \left(\sum_{j=0}^{d-1} r_j\left(k_1, k_2, m, n\right) \theta^j\right) \\ &= \sum_{j=0}^{d-1} \left(\sum_{m=0}^{D_1-1} \sum_{n=0}^{D_2-1} a_{m,n} r_j\left(k_1, k_2, m, n\right)\right) \theta^j. \end{aligned}$$

This is equivalent to saying that, for all $0 \leq j \leq d-1$, the coefficient of θ^j is the product of the row of $\mathcal{C}$ indexed by k_1, k_2, and j and the vector x, which will by construction be zero for all such j. So, for all integers k_1, k_2 between 0 and K, $F\left(k_1 + k_2\beta\right) = 0$, showing that $P\left(x, y\right)$ is a function with our desired properties. $\square$

5.9 A Small Number

Now that we have our function, we are ready to carry out our goal of finding a small nonzero element in our field.

5.9.1 Lemma. *Let θ be an algebraic integer and let $\beta, e^{\zeta}, e^{\beta\zeta} \in \mathbb{Q}\left(\theta\right)$ with β irrational. Then there exists some real number c and $\mathcal{F} \in \mathbb{Q}\left(\theta\right)$ such that, for integer $M \geq \left(1 + |\beta|\right)^2$ arbitrarily large we have*

$$\mathcal{H}_\theta\left(\mathcal{F}\right) < c^{M^{\frac{3}{2}} \ln M}$$

and

$$|\mathcal{F}| < c^{M^{\frac{3}{2}\ln M} + M^2 - M^2 \ln M}.$$

Proof. We use Lemma 5.8.1 to find K arbitrarily large and the promised polynomial

$$P(x,y) = \sum_{m=0}^{D_1-1}\sum_{n=0}^{D_2-1} a_{m,n} x^m y^n$$

with height at most $c_1^{K^{\frac{3}{2}}\ln K}$ for some constant c_1. Let

$$F(z) = \sum_{m=0}^{D_1-1}\sum_{n=0}^{D_2-1} a_{m,n} z^m \left(e^{\zeta z}\right)^n.$$

Because $F(z)$ is not everywhere zero, we know by Theorem 5.4.2 that $F(k_1 + k_2\beta)$ is not zero for every pair of integers k_1, k_2. We may therefore choose a pair k_1^*, k_2^* such that $\max\{k_1^*, k_2^*\}$ is minimal. It follows that there exists some $M \geq K$ such that for every pair of nonnegative integers k_1, k_2 strictly less than M, $F(k_1 + k_2\beta) = 0$, but $k_1^*, k_2^* \leq M$.

Letting δ_1, δ_2, and δ_3 be the smallest integers such that $\delta_1\beta, \delta_2 e^{\zeta}, \delta_3 e^{\beta\zeta} \in \mathbb{Z}(\theta)$ and $\delta^* = \delta_1\delta_2\delta_3$, we define

$$c_2 = \max\left\{2d\delta_1\mathcal{H}_\theta(\beta)(2\mathcal{H}(\theta))^d e, d^2\delta_2\mathcal{H}_\theta\left(e^{\zeta}\right)\delta_3\mathcal{H}_\theta\left(e^{\zeta\beta}\right)(2\mathcal{H}(\theta))^{2d}\right\}^{\sqrt{2d}}.$$

Then we can repeat similar calculations to the previous theorem to see that

$$\begin{aligned}
\delta^* F(k_1^* + k_2^*\beta) &= \sum_{m=0}^{D_1-1}\sum_{n=0}^{D_2-1} a_{m,n}\left(\delta_1 k_1^* + \delta_1 k_2^*\beta\right)^m \left(\delta_2 e^{\zeta}\right)^{nk_1^*}\left(\delta_3 e^{\zeta\beta}\right)^{nk_2^*} \\
&= \sum_{m=0}^{D_1-1}\sum_{n=0}^{D_2-1} a_{m,n}\sum_{j=0}^{d-1} r_j(k_1^*, k_2^*, m, n)\,\theta^j
\end{aligned}$$

where for all m, n, and j we have

$$|r_j(k_1^*, k_2^*, m, n)| \leq c_2^{K^{\frac{3}{2}}\ln M + K^{\frac{1}{2}}M}.$$

We can therefore conclude that

$$\begin{aligned}\mathcal{H}_\theta\left(\delta^* F\left(k_1^* + k_2^*\beta\right)\right) &\leq c_2^{M^{\frac{3}{2}}\ln M} \\ &\leq 2dK^2 c_1^{K^{\frac{3}{2}}\ln K} c_2^{K^{\frac{3}{2}}\ln M + K^{\frac{1}{2}}M} \\ &\leq \left(c_2\right)^{K^{\frac{3}{2}}\ln M + K^{\frac{1}{2}}M}\left(2de^2\right)^{\ln K} c_1^{K^{\frac{3}{2}}\ln K} \\ &\leq \left(c_2\right)^{2M^{\frac{3}{2}}\ln M}\left(2de^2 c_1\right)^{M^{\frac{3}{2}}\ln M}.\end{aligned}$$

So if we let

$$c_3 = c_2^2 2de^2 c_1$$

we get

$$\mathcal{H}_\theta\left(\delta^* F\left(k_1^* + k_2^*\beta\right)\right) < c_3^{M^{\frac{3}{2}}\ln M.}$$

At the same time, we observe that $F(z)$ satisfies all the requirements of Lemma 5.5.2, so

$$\left|F\left(k_1^* + k_2^*\beta\right)\right| < \left|F(z)\right|_{M^{\frac{3}{2}}+M(1+|\beta|)}\left(1+|\beta|\right)^{M^2}\left(e^{\frac{1}{2}}\right)^{-M^2\ln M}$$

and

$$\left|\delta^* F\left(k_1^* + k_2^*\beta\right)\right| < \left|\delta^* F(z)\right|_{M^{\frac{3}{2}}+M(1+|\beta|)}\left(1+|\beta|\right)^{M^2}\left(e^{\frac{1}{2}}\right)^{-M^2\ln M}.$$

Let $R = M^{\frac{3}{2}} + M(1+|\beta|)$. We have

$$\left|\delta^* F(z)\right|_R \leq c_3^{M^{\frac{3}{2}}\ln M}\left|z\right|_R^{D_1}\left|e^{\zeta z}\right|_R^{D_2}.$$

We then have $|z|_R = R$ and

$$\begin{aligned}\left|e^{\zeta z}\right|_R &= \left|e^{\mathrm{Re}(\zeta)\mathrm{Re}(z)-\mathrm{Im}(\zeta)\mathrm{Im}(z)}e^{i(\mathrm{Re}(\zeta)\mathrm{Im}(z)+\mathrm{Im}(\zeta)\mathrm{Re}(z))}\right|_R \\ &= \left|\frac{e^{\mathrm{Re}(\zeta)\mathrm{Re}(z)}}{e^{\mathrm{Im}(\zeta)\mathrm{Im}(z)}}\right|_R\left|e^{i(\mathrm{Re}(\zeta)\mathrm{Im}(z)+\mathrm{Im}(\zeta)\mathrm{Re}(z))}\right|_R \\ &= \left|\frac{e^{\mathrm{Re}(\zeta)\mathrm{Re}(z)}}{e^{\mathrm{Im}(\zeta)\mathrm{Im}(z)}}\right|_R\end{aligned}$$

From this we can see that $e^{\zeta z}$ will be maximized on the disk of radius R when z is real, giving us

$$\left|e^{\zeta z}\right|_R \leq \left(e^{\mathrm{Re}(\zeta)}\right)^R.$$

So letting

$$c_4 = 2dec_3^2 \quad \text{and} \quad c_5 = \max\left\{e^{\sqrt{2d}}, e^{\mathrm{Re}(\zeta)\sqrt{2d}}\right\}$$

we have

$$\left|\delta^* F(z)\right|_R \leq c_4^{M^{\frac{3}{2}}\ln M} c_5^{M^{\frac{3}{2}}\ln R + M^{\frac{1}{2}}R}.$$

Recalling that $M \geq (1+|\beta|)^2$,

$$2M^{\frac{3}{2}} \geq M^{\frac{3}{2}} + M(1+|\beta|) = R.$$

This gives

$$\begin{aligned}
\left|\delta^* F(z)\right|_R &\leq c_4^{M^{\frac{3}{2}}\ln M} c_5^{M^{\frac{3}{2}}\ln\left(2M^{\frac{3}{2}}\right)+2M^2} \\
&= c_4^{M^{\frac{3}{2}}\ln M} c_5^{M^{\frac{3}{2}}\ln 2 + \frac{3}{2}M^{\frac{3}{2}}\ln M + 2M^2} \\
&= c_4^{M^{\frac{3}{2}}\ln M}\left(c_5^{\ln 2}\right)^{M^{\frac{3}{2}}}\left(c_5^{\frac{3}{2}}\right)^{M^{\frac{3}{2}}\ln M}\left(c_4^2\right)^{M^2}.
\end{aligned}$$

Lemma 5.5.2 yields

$$\delta^*\left|F\left(k_1^* + k_2^*\beta\right)\right| < \left(c_4 c_5^{\ln 2} c_5^{\frac{3}{2}}\right)^{M^{\frac{3}{2}}\ln M}\left(c_4(1+|\beta|)\right)^{M^2}\left(e^{\frac{1}{2}}\right)^{-M^2\ln M}$$

Then, letting

$$c = \max\left\{c_3, c_4 c_5^{\ln 2} c_5^{\frac{3}{2}}, c_4^2(1+|\beta|), e^{\frac{1}{2}}\right\}$$

and

$$\mathcal{F} = \delta^* F\left(k_1^* + k_2^*\beta\right)$$

we have

$$\mathcal{H}_\theta(\mathcal{F}) \leq c^{M^{\frac{3}{2}}\ln M}$$

and

$$|\mathcal{F}| < c^{M^2 + M^{\frac{3}{2}}\ln M - M^2\ln M}.$$

$\square$

5.10 A Proof of the Gelfond-Schneider Theorem

Now that we have all of our pieces, we are ready to put them together for a proof.

5.10.1 Theorem. *[Gelfond-Schneider] Given two nonzero numbers β and ζ with β real and irrational, at least one of β, e^{ζ}, or $e^{\beta\zeta}$ is transcendental.*

Proof. Suppose by way of contradiction that β, e^{ζ}, and $e^{\beta\zeta}$ are all algebraic. By Lemma 5.3.1 we know there exists an algebraic integer θ such that $\mathbb{Q}\left(\beta, e^{\zeta}, e^{\beta\zeta}\right) = \mathbb{Q}(\theta)$. Let d be the degree of the minimal polynomial of θ, and let $\theta_1 = \theta, \theta_2, \ldots, \theta_{d-1}$ be all the conjugates of θ.

By Lemma 5.9.1 we know there exists c dependent on our choice of $\beta, e^{\zeta}, e^{\beta\zeta}$, and θ such that for M arbitrarily large, we may find a number $\mathcal{F} \in \mathbb{Z}(\theta)$ such that

$$\mathcal{H}_\theta(\mathcal{F}) \leq c^{M^{\frac{3}{2}} \ln M}$$

and

$$|\mathcal{F}| < c^{M^2 + M^{\frac{3}{2}} \ln M - M^2 \ln M}.$$

Let

$$c^* = c^2 d^2 \max\{1, |\theta_2|, \ldots, |\theta_d|\}^{d-1}$$

Since the negative $M^2 \ln M$ term grows faster than the positive $M^2 + M^{\frac{3}{2}} \ln M$ term, we may choose M large enough that

$$M^2 + M^{\frac{3}{2}} \ln M - M^2 \ln M < \log_{c^*} 1.$$

Now, we may write

$$\mathcal{F} = b_0 + b_1\theta + \cdots + b_{d-1}\theta^{d-1}$$

and define

$$\mathcal{N} = \prod_{i=1}^{d} \left(b_0 + b_1\theta_i + \cdots + b_{d-1}\theta_i^{d-1}\right).$$

Note that the function

$$\prod_{i=1}^{d} \left(b_0 + b_1 x_i + \cdots + b_{d-1} x_i^{d-1}\right)$$

is a symmetric function with integer coefficients, so by Corollary 5.2.3 $\mathcal{N}$ will be an integer.

If $\mathcal{N} = 0$, then there would exist some conjuate θ_j such that

$$b_0 + b_1\theta_j + \cdots + b_{d-1}\theta^{d-1} = 0.$$

This implies that either there is a minimal polynomial for θ_j with degree at most $d-1$, or that $b_0 = b_1 = \cdots = b_{d-1} = 0$, either of which are contradictions. Therefore, $\mathcal{N}$ is a nonzero integer.

We wish to find a bound on $|\mathcal{N}|$. We already have a very good bound for θ_1, so we may deal more coarsely with the other conjugates.

For $2 \leq j \leq d$, we can see that

$$\begin{aligned}
\left|b_0 + b_1\theta_j + \cdots + b_{d-1}\theta_j^{d-1}\right| &\leq d \max_{0\leq k\leq d-1} \{|b_j|\} \max\{|\theta_j|, 1\}^{d-1} \\
&\leq dc^{M^{\frac{3}{2}} \ln M} \max\{1, |\theta_2|, \ldots, |\theta_d|\}^{d-1} \\
&\leq \left(dc \max\{1, |\theta_2|, \ldots, |\theta_d|\}^{d-1}\right)^{M^{\frac{3}{2}} \ln M}.
\end{aligned}$$

Then we have

$$\begin{aligned}
|N| &\leq |\mathcal{F}| \left|\prod_{j=2}^{2} b_0 + b_1\theta_j + \cdots + b_{d-1}\theta_j^{d-1}\right| \\
&< c^{M^2 + M^{\frac{3}{2}} \ln M - M^2 \ln M} \left(d^2 c \max\{1, |\theta_2|, \ldots, |\theta_d|\}^{d-1}\right)^{M^{\frac{3}{2}} \ln M} \\
&\leq (c^*)^{M^2 + M^{\frac{3}{2}} \ln M - M^2 \ln M} \\
&< (c^*)^{\log_{c^*} 1} \\
&= 1
\end{aligned}$$

So we have that $|\mathcal{N}|$ is an integer with

$$0 < |\mathcal{N}| < 1,$$

which is an obvious contradiction. Therefore we conclude that our assumption that β, e^{ζ}, and $e^{\beta\zeta}$ were all algebraic must have been wrong, and at least one of them must be transcendental. $\square$

5.11 Perfect State Transfer and Periodicity

Now that we have proven the Gelfond-Schneider Theorem, we can show that perfect state transfer implies periodicity. The observation was first shown in Godsil [12], although the proof here is new.

5.11.1 Theorem. *Let X be an oriented graph with perfect state transfer from a to b. Then both a and b are periodic.*

Proof. Suppose there is perfect state transfer from vertex a to vertex b at time τ. Then, for all $\theta_r \in \Phi_a$ we have

$$e^{\tau\theta_r} E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b.$$

By Lemma 2.10.3, the entries of E_r lie in $\mathbb{Q}(\theta_r, i)$. Thus, every entry of $E_r\mathbf{e}_a$ and $E_r\mathbf{e}_b$ is algebraic, so in particular, $e^{\tau\theta_r}$ is algebraic.

Therefore, for all θ_r, θ_s in Φ_a with $\theta_s \neq 0$, $e^{\tau\theta_r}$ and $e^{\tau\theta_s}$ are algebraic. The Gelfond-Schneider Theorem shows that

$$\frac{\ln e^{\tau\theta_r}}{\ln e^{\tau\theta_s}} = \frac{\tau\theta_r}{\tau\theta_s} = \frac{\theta_r}{\theta_s}$$

is either rational or transcendental. As the ratio of two eigenvalues it must be algebraic, and that means that it is rational. By Theorem 3.3.1, a and b are periodic. □

5.12 Local Uniform Mixing and Periodicity

We can also show that in oriented graphs, local uniform mixing implies that the vertex is periodic. This was also shown in [12], although this proof is new.

5.12.1 Theorem. *Let X be an oriented graph with local uniform mixing at vertex a. Then a is periodic.*

Proof. Suppose there is local uniform mixing at vertex a at time τ. Then $U(\tau)\mathbf{e}_a$ is flat and by Lemma 2.5.4 the entries of $U(\mathbf{e}_a)$ are $\pm\frac{1}{\sqrt{V(X)}}$, which is algebraic. Then, for all $\theta_r \in \Phi_a$,

$$e^{\tau\theta_r} E_r \mathbf{e}_a = E_r U(\tau)\mathbf{e}_a.$$

Since the entries of E_r and the entries of $U(\tau)\,\mathbf{e}_a$ are algebraic, $e^{\tau\theta_r}$ is algebraic.

Therefore, for all θ_r, θ_s in Φ_a with $\theta_s \neq 0$, $e^{\tau\theta_r}$ and $e^{\tau\theta_s}$ are algebraic. The Gelfond-Schneider Theorem shows that

$$\frac{\ln e^{\tau\theta_r}}{\ln e^{\tau\theta_s}} = \frac{\tau\theta_r}{\tau\theta_s} = \frac{\theta_r}{\theta_s}$$

is either rational or transcendental. As the ratio of two eigenvalues is algebraic, it is rational. Thus by Theorem 3.3.1, a and b are periodic. $\square$

Chapter 6

Perfect State Transfer Revisited

The Gelfond-Schneider Theorem is a powerful tool in quantum walks, and allows us to prove that periodicity is a necessary condition for both local uniform mixing and perfect state transfer to occur. Ultimately, however, it fails to give insight as to when perfect state transfer will occur in relation to the period. Coutino and Godsil were able to prove a polynomial time algorithm for testing for perfect state transfer in a graph. [7] We would like to be able to come up with a similar characterization for perfect state transfer in oriented graphs, but we cannot without knowing more about the time of perfect state transfer.

Consequently, we turn our attention to switching vertex transitive graphs, which do have a relationship between the time of perfect state trnasfer and the time of periodicity. We are able to prove a stronger characterization of when perfect state transfer can occur, as well as a first characterization of multiple state transfer. We are then able to place restriction on the size of the sets for multiple state transfer, and show another example of a graph with multiple state transfer.

6.1 First Perfect State Transfer

In the non-oriented case, we know that if perfect state transfer occurs, it first occurs at half the period. Although we do not have that result for oriented graphs, we can still use the minimum period to give us a bound as to when perfect state transfer can occur.

6.1.1 Lemma. *If perfect state transfer occurs, it first occurs at a time less than the minimum period.*

Proof. Suppose that perfect state transfer occurs from vertex a to vertex b at time τ. Let σ be the minimum period of a, and suppose $\tau > \sigma$. Then we have, for $\tau' = \tau - \sigma$,

$$\pm \mathbf{e}_b = U(\tau)\,\mathbf{e}_a = U(\tau')\,U(\sigma)\,\mathbf{e}_a = \pm U(\tau')\,\mathbf{e}_a,$$

which tells us that there is perfect state transfer from a to b at time $\tau' < \tau$, which is a contradiction. □

For graphs with switching isomorphisms, we might have a stronger result of when perfect state transfer must occur.

6.1.2 Theorem. *Let X be an oriented graph and φ a switching isomorphism of order n with perfect state transfer from a to $\varphi(a)$ first occuring at time τ. Then letting*

$$g := \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \theta_r \text{ is always odd} \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \text{otherwise} \end{cases}$$

we know that there exists an integer $m < n$ such that

$$\tau = \frac{2m\pi}{ng\sqrt{\Delta}}.$$

Proof. By Theorem 4.6.3, we know that $n\tau$ will be a period. By Lemma 2.5.2 we know that

$$n\tau = \frac{2m}{g\sqrt{\Delta}},$$

and by Lemma 6.1.1, we know that

$$\tau = \frac{2m\pi}{ng\sqrt{\Delta}} < \frac{2\pi}{g\sqrt{\Delta}},$$

so $m < n$. □

This also extends to multiple state transfer.

6.1.3 Corollary. *Let X be an oriented graph and φ a switching isomorphism of order n with perfect state transfer from a to $\varphi(a)$ first occuring at time τ. Then letting*

$$g := \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \theta_r \text{ is always odd} \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \text{otherwise} \end{cases}$$

the vertex a has perfect state transfer to elements of $\{\varphi(a), \ldots, \varphi^{n-1}(a)\}$ at the set of times

$$\left\{\frac{2\pi}{ng\sqrt{\Delta}}, \frac{4\pi}{ng\sqrt{\Delta}}, \ldots, \frac{2(n-1)\pi}{ng\sqrt{\Delta}}\right\}.$$

6.2 Characterizing State Transfer

With our knowledge of at what perfect state transfer will occur, we can improve our characterization, similar to Coutinho [5].

6.2.1 Theorem. *Let X be an oriented graph and let φ be a switching automorphism of order n. Then there is perfect state transfer from vertex a to vertex $\varphi(a)$ if and only if:*

(i) Vertices a and $\varphi(a)$ are robustly cospectral.

(ii) Every eigenvalue in Φ_a is an integer multiple of $\sqrt{-\Delta}$ for some square-free integer Δ.

(iii) Let

$$g := \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \theta_r \text{ is always odd} \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \text{otherwise.} \end{cases}$$

Then there exists some $m < n$ such that, for all r with $\theta_r \in \Phi_a$,

$$\frac{2m\theta_r}{ng\sqrt{-\Delta}} + q_r(a, \varphi(a))$$

is an integer.

Proof. From Lemma 4.4.2 we know that a and $\varphi(a)$ must be robustly cospectral, and by Theorem 5.11.1 a and $\varphi(a)$ are periodic, so we may assume that (i) and (ii) hold.

Suppose that (iii) holds. Let

$$\tau = \frac{m\pi}{ng\sqrt{\Delta}}.$$

For all r,

$$\frac{\tau\theta_r}{i\pi} + q_r(a, \varphi(a))$$

is an integer by (iii), so by Theorem 4.5.3 we know that perfect state transfer occurs from a to $\varphi(a)$ at time τ.

Conversely, suppose that perfect state transfer occurs from a to $\varphi(a)$. Then by Theorem 6.1.2 we can see that perfect state transfer will occur at time

$$\frac{m\pi}{ng\sqrt{-\Delta}}$$

for some $m < n$. So, by the characterization in 4.5.3 we have

$$\frac{m\pi}{ng\sqrt{-\Delta}} + q_r$$

is an integer. □

This characterization leaves something to be desired, because it both requires there be an automorphism from a to the vertex where perfect state transfer occurs, and that we guess the right integer m. The second complaint is related to the possibility of multiple state transfer, which directly influences at what time perfect state transfer can occur for a known minimum period. In fact, if we look for multiple state transfer instead of perfect state transfer, some of the requirements become more transparent, provided that we are looking for multiple state transfer on the right sized set.

6.3 Switching Vertex Transitive Graphs

We say that a set of vertices is *closed* under perfect state transfer at time τ if for every vertex a in the set that has perfect state transfer at time τ, the vertex that a has perfect state transfer to is also in the set. A set of vertices is *saturated* if for every time that perfect state transfer occurs between elements of the set, the set is closed at that time.

With that additional condition, we are ready to extend our characterization to multiple state transfer.

6.3.1 Theorem. *Let X be an oriented swtiching vertex transitive graph, and let S be a subset of vertices. Then S is saturated and has multiple state transfer if and only if all of the following hold.*

(i) For some vertex $a_0 \in S$, every vertex in S is robustly cospectral to a_0.

(ii) For some vertex $a_0 \in S$, every eigenvalue in Φ_{a_0} is an integer multiple of $\sqrt{-\Delta}$ for some squarefree integer Δ.

(iii) For

$$g := \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r\in\Phi_a}\right) & \theta_r \text{ is always odd} \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r\in\Phi_a}\right) & \text{otherwise,} \end{cases}$$

there exists an ordering of the vertices in S denoted by $\{a_0, a_1, \ldots, a_{n-1}\}$ such that, for all r with $\theta_r \in \Phi_{a_0}$,

$$\frac{2j\theta_r}{ng\sqrt{-\Delta}} + q_r(a_0, a_j)$$

is an integer.

Proof. Suppose that (i), (ii), and (iii) all hold. For all integers j between one and n, define

$$\tau_j := \frac{2j\pi}{ng\sqrt{\Delta}}.$$

By Theorem 4.5.3, there is perfect state transfer from a_0 to a_j. Then repeated application of Lemma 4.2.2, shows that the set S has multiple state transfer.

Let b be a vertex in S that has pefect state transfer at time $\frac{2m\pi}{ng\sqrt{\Delta}}$ for some integer m. Then there exists a vertex c such that

$$U\left(\frac{2m\pi}{ng\sqrt{\Delta}}\right)\mathbf{e}_b = \pm\mathbf{e}_c.$$

Since b is in S, there is another integer k such that

$$U\left(\frac{2k\pi}{ng\sqrt{\Delta}}\right)\mathbf{e}_{a_0} = \pm\mathbf{e}_b,$$

and therefore

$$U\left(\frac{2(k+m)\pi}{ng\sqrt{\Delta}}\right)\mathbf{e}_{a_0} = \pm\mathbf{e}_c,$$

and so c must be in our set S.

Conversely, suppose that multiple state transfer occurs on S. This means that for a fixed vertex a_0, there is perfect state transfer from every vertex in S to a_0, and by Lemma 4.4.2 every vertex in S is robustly cospectral to a_0. By Theorem 5.11.1 we also know that a_0 must be periodic, so we may assume that (i) and (ii) hold.

Fix some vertex a_0 and order the remaining vertices such that the minimum periods of perfect state transfer from vertex a_0 to a_j is strictly increasing as j increases. Now, since X is vertex-transitive, there exists some switching isomorphism φ such that $\varphi(a_0) = a_1$. Let m denote the order of φ. Because a_1 is the first vertex that a_0 has perfect state transfer to, it follows that m has maximal order for switching automorphisms of X. We wish to show that, for all $j \leq n$, we have $\varphi^j(a_0) = a_j$.

Assume to the contrary that there were some vertex $b \in S$ such that for all j we had $b \neq \varphi^j(a_0)$. Since X is vertex-transitive, there is some other automorphism $\tilde{\varphi}$ with order $\tilde{m}$ such that $\tilde{\varphi}(a_0) = b$. By Corollary 6.1.3 perfect state transfer occurs from a_0 to elements of the set $\{\varphi(a), \ldots, \varphi^{n-1}(a)\}$ at times

$$\left\{\frac{2\pi}{mg\sqrt{\Delta}}, \frac{4\pi}{mg\sqrt{\Delta}}, \ldots, \frac{2(m-1)\pi}{mg\sqrt{\Delta}}\right\},$$

and from a_0 to elements of the set $\{\tilde{\varphi}(a), \ldots, \tilde{\varphi}^{n-1}(a)\}$ at times

$$\left\{\frac{2\pi}{\tilde{m}g\sqrt{\Delta}}, \frac{4\pi}{\tilde{m}g\sqrt{\Delta}}, \ldots, \frac{2(\tilde{m}-1)\pi}{\tilde{m}g\sqrt{\Delta}}\right\}.$$

Now, there is perfect state transfer from a_0 to a_1 at time $\frac{2\pi}{mg\sqrt{\Delta}}$, and there is perfect state transfer from a_0 to b at, without loss of generality, time $\frac{2\pi}{\tilde{m}g\sqrt{\Delta}}$. Since a_0 is periodic at time $\frac{2\pi}{g\sqrt{\Delta}}$ we can see that

$$U\left(\frac{2\pi(m-1)}{mg\sqrt{\Delta}}\right)\mathbf{e}_{a_1} = U\left(\frac{-2\pi}{mg\sqrt{\Delta}}\right)U\left(\frac{2\pi}{g\sqrt{\Delta}}\right)\mathbf{e}_{a_1} = \pm U\left(\frac{-2\pi}{mg\sqrt{\Delta}}\right)\mathbf{e}_{a_1} = \pm\mathbf{e}_{a_0},$$

so there is perfect state transfer from a_1 to a_0 at time $\frac{2\pi(m-1)}{mg\sqrt{\Delta}}$. Then

$$U\left(\frac{2\pi}{\tilde{m}g\sqrt{\Delta}} + \frac{2\pi(m-1)}{mg\sqrt{\Delta}}\right)\mathbf{e}_{a_1} = \pm U\left(\frac{2\pi}{\tilde{m}g\sqrt{\Delta}}\right)\mathbf{e}_{a_0} = \pm\mathbf{e}_b$$

and there is perfect state transfer from a_1 to b at time $\frac{2\pi(\tilde{m}(m-1)+m)}{m\tilde{m}g\sqrt{\Delta}}$.

This tells us that there is perfect state transfer from a_0 to $\tilde{\varphi}(\varphi^{-1}(a_0))$ at time $\frac{2\pi(\tilde{m}(m-1)+m)}{m\tilde{m}g\sqrt{\Delta}}$, so by Corollary 6.1.3 there must be some other vertex c such that a_0 has perfect state transfer at time $\frac{2\pi}{\mathrm{lcm}(m\tilde{m})g\sqrt{\Delta}}$. Because S is saturated, we know that $c \in S$. Since the time that perfect state transfer occurs from a_0 to c is strictly less than the time for perfect state transfer from a_0 to a_1, this is a contradiction of our ordering. Therefore, we know that for all $j \leq n$, it must be the case that $\varphi^j(a_0) = a_j$ and therefore the order of φ is n.

From here, (iii) follows directly from our ordering of the vertices in S, Corollary 6.1.3, and the characterization of perfect state transfer in Theorem 4.5.3. $\square$

A consequence of this is that, for a vertex transitive graph with perfect state transfer, all the quarrels are rational. In particular, one of the entries of every idempotent matrix will be a root of unity times some real rational number. We can use this to characterize the size of saturated sets with multiple state transfer.

6.4 Multiple State Transfer

We need to know more about roots of unity and the extension field that they lie in. This has been studied by algebraists and number theorists, and the following definition and result is standard for the study of cyclotomic polynomials and can be found in Cox [8].

Let n be a positive integer. Then the *Euler totient function* of n, denoted $\phi(n)$ is the number of nonnegative integers less than n that are relatively prime to n.

6.4.1 Theorem. *Let ω_n be a primitive nth root of unity. Then the degree of the extension field $\mathbb{Q}(\omega_n)$ over the rationals is $\phi(n)$.*

This leads immediately to the following corollary.

6.4.2 Corollary. *Let ω_n be a primitive nth root of unity. If there exists some square-free integer Δ such that $\omega_n \in \mathbb{Q}\left(\sqrt{\Delta}\right)$, then $n = 1, 2, 3, 4,$ or 6.*

We can now use this with our knowledge of when multiple state transfer can occur to place restriction on the size of saturated sets of multiple state transfer.

6.4.3 Theorem. *Let X be an oriented graph and let S be a saturated set of vertices with multiple state transfer. Then $|S| = 2, 3, 4, 6, 8$ or 12.*

Proof. Let $n = |S|$ and

$$g = \begin{cases} 2\gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \theta_r \text{ is always odd} \\ \gcd\left(\left\{\frac{\theta_r}{\sqrt{-\Delta}}\right\}_{\theta_r \in \Phi_a}\right) & \text{otherwise} \end{cases}.$$

Since S is a set with perfect state transfer, we may assume $n > 1$.

By Theorem 6.3.1 there exist vertices a, b robustly cospectral in S such that, for every $\theta_r \in \Phi_a$,

$$\frac{2\theta_r}{ng\sqrt{-\Delta}} + q_r(a, b)$$

is an integer.

At the same time, by Theorem 3.3.1 and Lemma 2.10.3

$$e^{i\pi q_r(a,b)} \in \mathbb{Q}\left(\sqrt{\Delta}\right),$$

so it follows from Corollary 6.4.2 that the denominator of $q_r(a, b)$ can only be $1, 2, 3, 4$, or 6. Then the denominator of

$$\mu = \frac{2\theta_r}{ng\sqrt{-\Delta}}$$

can only be $1, 2, 3, 4$, or 6.

If n is divisible by a prime number p other than 2, by construction of g the denominator of μ will be divisible by p. Similarly, if n is divisbile by 2^k, then the denominator of μ will be divisible by 2^{k-1}. Then because of the limits of the denominator, we may conclude that $n = 2, 3, 4, 6, 8$, or 12. □

This extends immediately to a result about universal state transfer.

6.4.4 Corollary. *Let X be a vertex-transitive oriented graph on n vertices. If X has universal state transfer, then $n = 2, 3, 4, 6, 8,$ or 12.*

There are oriented graphs on two and three vertices that have universal state transfer. We have ruled out the possibility of a switching vertex transitive graph with universal state transfer on four, six, or even eight vertices by brute force and computer calculations, but at twelve vertices the computations are infeasible. Thus, if there is a third example of a switching vertex transitive graph with universal state transfer, it must have twelve vertices, and the question as to whether such a graph exists remains open.

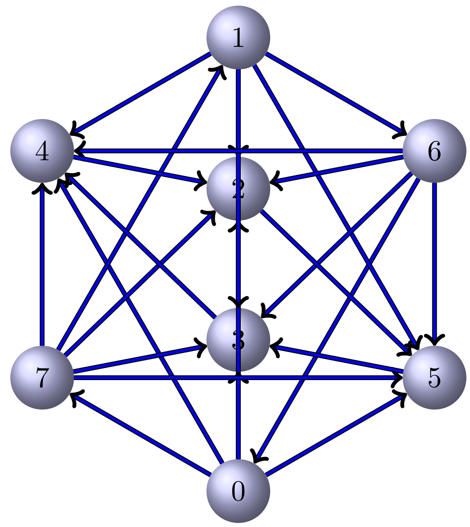

Figure 6.1: More Multiple State Transfer

6.5 Multiple State Transfer on Four Vertices

So far, we only have one example of a graph with multiple state transfer. However, unlike universal state transfer, we can find a second example of multiple state transfer. Even better, we can find an example on a set of vertices with size greater than three.

6.5.1 Example. The graph in Figure 6.5 is switching vertex transitive, even though it is not vertex transitive in the traditional sense.

The eigenvalue of 0 has spectral idempotent

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The eigenvalue of 2 has spectral idempotent

$$\frac{1}{8}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -i & i \\ -1 & 1 & 0 & 0 & 0 & 0 & i & -i \\ 0 & 0 & 1 & -1 & i & -i & 0 & 0 \\ 0 & 0 & -1 & 1 & -i & i & 0 & 0 \\ 0 & 0 & -i & i & 1 & -1 & 0 & 0 \\ 0 & 0 & i & -i & -1 & 1 & 0 & 0 \\ i & -i & 0 & 0 & 0 & 0 & 1 & -1 \\ -i & i & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

while -2 is an eigenvalue with spectral idempotent

$$\frac{1}{8}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & i & -i \\ -1 & 1 & 0 & 0 & 0 & 0 & -i & i \\ 0 & 0 & 1 & -1 & -i & i & 0 & 0 \\ 0 & 0 & -1 & 1 & i & -i & 0 & 0 \\ 0 & 0 & i & -i & 1 & -1 & 0 & 0 \\ 0 & 0 & -i & i & -1 & 1 & 0 & 0 \\ -i & i & 0 & 0 & 0 & 0 & 1 & -1 \\ i & -i & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

We also have an eigenvalue of 4 with spectral idempotent

$$\frac{1}{8}\begin{pmatrix} 1 & 1 & i & i & i & i & 1 & 1 \\ 1 & 1 & i & i & i & i & 1 & 1 \\ -i & -i & 1 & 1 & 1 & 1 & -i & -i \\ -i & -i & 1 & 1 & 1 & 1 & -i & -i \\ -i & -i & 1 & 1 & 1 & 1 & -i & -i \\ -i & -i & 1 & 1 & 1 & 1 & -i & -i \\ 1 & 1 & i & i & i & i & 1 & 1 \\ 1 & 1 & i & i & i & i & 1 & 1 \end{pmatrix}$$

and its partner eigenvalue of -4 with spectral idempotent

$$\frac{1}{8}\begin{pmatrix} 1 & 1 & -i & -i & -i & -i & 1 & 1 \\ 1 & 1 & -i & -i & -i & -i & 1 & 1 \\ i & i & 1 & 1 & 1 & 1 & i & i \\ i & i & 1 & 1 & 1 & 1 & i & i \\ i & i & 1 & 1 & 1 & 1 & i & i \\ i & i & 1 & 1 & 1 & 1 & i & i \\ 1 & 1 & -i & -i & -i & -i & 1 & 1 \\ 1 & 1 & -i & -i & -i & -i & 1 & 1 \end{pmatrix}.$$

Clearly all of the eigenvalues are integers, and it is straightforward to verify that vertices $0, 1, 6$ and 7 are cospectral. If we let $a_0 = 0, a_1 = 6, a_2 = 1$ and $a_3 = 7$ then, for all five of our spectral idempotents and $j = 1, 2, 3$, we have

$$\frac{j\theta_r}{4\sqrt{-1}} + q_r\left(a_0, a_j\right).$$

Therefore, the set of vertices $\{0, 1, 6, 7\}$ has multiple state transfer.

From this we see that multiple state transfer is not a phenomena isolated to a single graph, or even a single set size.

6.6 Further Questions

Orienting the edges of graphs dramatically changes the properties of the continuous quantum walk on that graph. However, outside of this thesis, the paper of Cameron et. al [4] and Godsil [12], quantum walks on oriented graphs have barely been studied. This leaves many open questions that go beyond the topics in this thesis. However, based on the work here, some related questions arise.

Although we now have two examples of graphs where multiple state transfer can occur, more examples would be nice. Can we build infinite families of graphs with multiple state transfer? Can we find examples of graphs with multiple state transfer on sets of size six, eight, and twelve?

As has been stated previously, our characterization of switching vertex transitive graphs applies immediately to any Cayley graph. However, of the examples in this thesis of perfect state transfer, only K_2 and K_3 are Cayley graphs. Studying Cayley graphs to find additional examples and non-examples would be interesting both for the additional information on vertex transitive graphs that it would give us and for its own sake. Quantum walks on Cayley graphs where the connection sets are inverse-closed have been studied so quantum walks on Cayley graphs where the connection set never contains an element and its inverse is a closely-related area that might reveal interesting parallels.

On the other hand, most graphs are not vertex-transitive. In fact, most graphs do not have any automorphisms at all, and as we have shown, there are examples of perfect state transfer from a to b occuring when there is no switching automorphism taking a to b. We currently have only a weak characterization of the graphs and vertices which have perfect state transfer without a relevant automorphism. A stronger characterization, ideally one that also tells us when perfect state transfer will first occur, would be helpful for further study of perfect state transfer in graphs.

Relatedly, without a general understanding of when perfect state transfer can occur between two vertices, we lack information of when multiple state transfer can occur in general. Is there an example of perfect state transfer between more than two vertices without a switching automorphism between the vertices? Can there be multiple state transfer on a set of vertices of any size? Is there another example of universal state transfer apart from the complete graphs on two and three vertices? The methods that were used for non-oriented graphs and switching vertex transitive do not apply here, so we need a new approach to what is happening in general.

Finally, the topic of local uniform mixing on oriented graphs remains largely unexplored. We have the single necessary condition that any vertex with local uniform mixing must be periodic, which is not a condition that holds for all non-oriented graphs. This gives oriented graphs an advantage in studying uniform mixing, and any results in characterizing when uniform mixing occurs would translate directly to bipartite graphs. It is feasible that further study of uniform mixing would yield results which could then be translated back to the non-oriented case to better understand when local uniform mixing can occur in general.

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