# Matrix analytic methods for computations in risk theory

by

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## Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

**Abstract**

The introduction of matrix analytic methods in risk theory has marked a significant progress in computations in risk theory. Matrix analytic methods have proven to be powerful computational tools for numerically analyzing complex risk models that traditional methods often had difficulty with. This is particularly noteworthy in the modern age of advanced computing and big data. Moving away from the traditional view of collective risk theory, we can now consider risk models that comprise of many stochastic processes of which data are abundant. These models may fall under the existing class of risk models; however, these more realistic risk models involve a large number of variables which increases the computational complexity significantly. Matrix analytic methods can provide reliable computing algorithms for risk models of such computational complexity, which have not been numerically feasible to analyze with the traditional computational tools in risk theory.

This thesis is dedicated to improving the accessibility of the matrix analytic methodology in risk theory and developing further generalizations of the existing matrix analytic methods in risk theory in the attempt to promote its computational use. Although the literature of matrix analytic methods in risk theory is in its early stage, it is believed that the advancement in computations in risk theory brought by the matrix analytic methods will broaden the spectrum of problems in the risk theory literature in the direction of more realistic and practical risk models and computational analyses of these models. This will make risk theory as a whole more appealing to practitioners and those who are looking for more advanced actuarial risk management tools.

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## Dedication

*To my parents and Soyeon Moon, for their unconditional love and support*

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# Notation and definitions

The list given here is not comprehensive. The intention in providing this list is to guide the readers through the thesis by highlighting some of the notation and important definitions pertaining to each chapter/section, as there are many stochastic processes introduced and many matrices defined in this document.

**Mathematical notation**:

$\mathbb{N}$: the set of natural numbers.

$\mathbb{Z}$: the set of integers.

$\mathbb{Z}^+$: the set of positive integers.

$\mathbb{Z}^-$: the set of negative integers.

$\mathbb{C}$: the set of complex numbers.

$\mathbb{R}$: the set of real numbers.

$\mathbb{R}^+$: the set of nonnegative real numbers.

$\Re(s)$: the real part of the complex number $s$.

$|\nu|$: the modulus of $\nu$.

Pr: the probability function.

$E$: the expectation function.

$\mathcal{I}[A]$: the indicator function of event $A$.

$\lfloor x \rfloor$: the nearest integer less than or equal to $x$.

$\boldsymbol{A}^\intercal$: the transpose of the matrix $\boldsymbol{A}$.

$\boldsymbol{A}^{-1}$: the inverse of the matrix $\boldsymbol{A}$.

$\otimes$: the Kronecker product operator.

$vec$: the vectorization operator for a matrix.

$\mathrm{Sp}(\boldsymbol{A})$: the spectral radius of the matrix $\boldsymbol{A}$.

$\|\boldsymbol{A}\|_{\max}$: the max norm of the matrix $\boldsymbol{A}$.

**Stochastic processes in Chapter 2**:

$U_t$: the surplus process of the G/M/1-type discrete-time risk model.

$\boldsymbol{R}_t$: the external process of $U_t$.

$X_t$: the level process of the dual G/M/1-type chain of $(U_t, \boldsymbol{R}_t)$.

$\boldsymbol{J}_t$: the phase process of the dual G/M/1-type chain of $(U_t, \boldsymbol{R}_t)$.

$\tau$: the time of ruin defined as $\inf\{t \in \mathbb{Z}^+ : U_t < 0\}$.

$\tau_i^-$: $\inf\{t \in \mathbb{Z}^+ : X_t < i\}$.

$\tau_i$: $\inf\{t \in \mathbb{Z}^+ : X_t = i\}$.

$U_t^{(b)}$: the surplus process of the MAP risk model with a dividend barrier $b$.

$J_t$: the phase process of the associated MAP of the MAP risk model with a dividend barrier $b$.

$X_t^{(b)}$: the level process of the dual G/M/1-type chain of $(U_t^{(b)}, J_t)$.

$(V_t^{(b)}, J_t)$: the bivariate phase process of the dual G/M/1-type chain of $(U_t^{(b)}, J_t)$, where $V_t^{(b)} = \lfloor \frac{U_t^{(b)}}{c} \rfloor$.

$D_t$: the dividend amount paid at time $t$.

$D_T^{\mathrm{Tot}}(\nu)$: the total discounted dividends paid up to time $T$.

$\tau_{\mathcal{S}_B}^-$: the first return time of the dual G/M/1-type chain of the MAP risk model without a dividend barrier to levels $\{0, 1, \ldots, B\}$.

**Matrices in Chapter 2**:

$\boldsymbol{I}_n$: an identity matrix of size $n \times n$ (we drop the subscript $n$ when it is obvious).

$\boldsymbol{e_j}$: a row vector whose $\boldsymbol{j}$-th entry is 1 and all the others are 0.

${}^{\mathcal{A}^c}\boldsymbol{P}$: a block component of the TPM of $(X_t, \boldsymbol{J}_t)$ pertaining to levels in $\mathbb{N}$.

${}^{\mathcal{A}}\boldsymbol{P}$: a block component of the TPM of $(X_t, \boldsymbol{J}_t)$ pertaining to levels in $\mathbb{Z}^-$.

$\boldsymbol{A}_{i,l}$: a block component of the TPM of $(X_t, \boldsymbol{J}_t)$ corresponding to levels $i$ and $l$ in the general case.

$\boldsymbol{A}_i$: a block component of the TPM of $(X_t, \boldsymbol{J}_t)$ when $(X_t, \boldsymbol{J}_t)$ is level-independent.

${}^{\nu}\boldsymbol{H}_{i,l}$: a block component of the discounted fundamental matrix ${}^{\nu}\boldsymbol{H}$ which records the time discounted pre-$\tau$ occupation measure of $(X_t, \boldsymbol{J}_t)$ to level $l$ given that the chain starts in level $i$.

$\{{}^{\nu}\boldsymbol{R}_{i,l}\}_{i=0,l\geq i}^{\infty}$: the set of discounted rate matrices which record the time discounted pre-$\tau_{i+1}^-$ occupation measure of $(X_t, \boldsymbol{J}_t)$ to level $l$ given that the chain starts in level $i$.

$\{{}^{\nu}\boldsymbol{G}_{i,l}\}_{i,l\geq 0}$: the set of discounted fundamental period matrices which record the time discounted first passage time probabilities of $(X_t, \boldsymbol{J}_t)$ to level $l$ given that the chain starts in level $i$.

${}^{\nu}\boldsymbol{R}$: denotes ${}^{\nu}\boldsymbol{R}_{i,i+1}$ when $(X_t, \boldsymbol{J}_t)$ is level-independent.

$\{{}^{\nu}\boldsymbol{Q}_{i,z}\}_{i=0,z\leq i}^{\infty}$: the set of discounted ladder height distribution matrices which record the time discounted ladder height distributions of $(X_t, \boldsymbol{J}_t)$ to level $z$ given that the chain starts in level $i$.

$\{{}^{\nu}\boldsymbol{Q}_l\}_{l=0}^{\infty}$: the set of discounted ladder height distribution matrices which record the time discounted ladder height distributions of $(X_t, \boldsymbol{J}_t)$ to level $z$ given that the chain starts in level $i$ when the chain is level-independent and $i - z = l$.

${}^{\mathcal{A}^c}_{\ b}\boldsymbol{P}$: a block component of the TPM of $(X_t^{(b)}, V_t^{(b)}, J_t)$ pertaining to levels in $\{0, 1, \ldots, B\}$.

${}^{\mathcal{A}}_{b}\boldsymbol{P}$: a block component of the TPM of $(X_t^{(b)}, V_t^{(b)}, J_t)$ pertaining to levels in $\mathbb{Z}^-$.

$\{{}^{b,\nu}\boldsymbol{H}_{i,l}\}_{i,l=0}^{B}$, $\{{}^{b,\nu}\boldsymbol{R}_{i,l}\}_{i=0,l\geq i}^{B}$, $\{{}^{b,\nu}\boldsymbol{G}_{i,l}\}_{i,l=0}^{B}$, and $\{{}^{b,\nu}\boldsymbol{Q}_{i,l}\}_{i=0,l\leq i}^{B}$: the set of discounted fundamental, rate, fundamental period, and ladder height distribution matrices of $(X_t^{(b)}, V_t^{(b)}, J_t)$ corresponding to levels $i$ and $l$.

${}^{\mathcal{S}_B}\boldsymbol{P}$: a block component of the TPM of the dual G/M/1-type chain of the MAP risk model without a dividend barrier pertaining to levels $\{0, 1, \ldots, B\}$.

${}^{\mathcal{S}_{B+}}\boldsymbol{P}$: a block component of the TPM of the dual G/M/1-type chain of the MAP risk model without a dividend barrier pertaining to levels $\{B+1, B+2, \ldots\}$.

${}^{\mathcal{S}_B:\mathcal{S}_{B+}}\boldsymbol{P}$: a block component of the TPM of the dual G/M/1-type chain of the MAP risk model without a dividend barrier pertaining to transitions from levels $\{0, 1, \ldots, B\}$ to levels $\{B+1, B+2, \ldots\}$.

${}^{\mathcal{S}_{B+}:\mathcal{S}_B}\boldsymbol{P}$: a block component of the TPM of the dual G/M/1-type chain of the MAP risk model without a dividend barrier pertaining to transitions from levels $\{B+1, B+2, \ldots\}$ to levels $\{0, 1, \ldots, B\}$.

${}_{B,B+}^{\ \ \nu}\boldsymbol{R}$: a matrix which records the time discounted pre-$\tau_{\mathcal{S}_B}^-$ occupation measure of the dual G/M/1-type chain of the MAP risk model without a dividend barrier to levels $\{B+1, B+2, \ldots\}$ given that the chain starts in levels $\{0, 1, \ldots, B\}$.

${}_{B,B+}^{\ \ \nu}\boldsymbol{Q}$: a matrix which records the time discounted first return time probabilities of the dual G/M/1-type chain of the MAP risk model without a dividend barrier to levels $\{0, 1, \ldots, B\}$ given that the chain starts in levels $\{0, 1, \ldots, B\}$.

${}_{B,B}^{\nu}\boldsymbol{H}$, ${}_{B,B^+}^{\nu}\boldsymbol{H}$, ${}_{B^+,B}^{\nu}\boldsymbol{H}$, and ${}_{B,B^+}^{\nu}\boldsymbol{H}$: block components of the discounted fundamental matrix of the dual G/M/1-type chain of the MAP risk model without a dividend barrier partitioned according to the levels $\{0, 1, \ldots, B\}$ and $\{B+1, B+2, \ldots\}$.

**Stochastic processes in Chapter 3**:

$U_t$: the surplus process of the MAP risk model with phase-dependent premium rates and phase-type claim size distributions.

$J_t$: the phase process of the associated MAP of $U_t$.

$X_t$: the level process of the dual pre-QBD process of $(U_t, J_t)$.

$W_t$: the phase process of the dual pre-QBD process of $(U_t, J_t)$.

$L_t$: the level process of the dual QBD process of $(U_t, J_t)$.

$(V_t, W_t)$: the bivariate phase process of the dual QBD process $(U_t, J_t)$, where $V_t = \lfloor \frac{X_t}{c_{max}} \rfloor$.

$\tau$: the time of ruin defined as $\inf\{t \in \mathbb{Z}^+ : U_t < 0\}$.

$\kappa$: $\inf\{t \in \mathbb{Z}^+ : X_t < 0\} = \inf\{t \in \mathbb{Z}^+ : L_t < 0\}$.

$s_1([h,k])$: denotes the total number of times $W_t$ is in $\mathcal{S}_1$ in the time interval $[h,k]$, $h, k \in \mathbb{N}$, where $s_1([h,k]) = 0$ when $k < h$ ($h$ and $k$ may be nonnegative integer-valued random variables as well).

$\eta(v)$: $\inf\{t \in \mathbb{N} : L_t = v\}$.

$\kappa_v^-$: $\inf\{t \in \mathbb{Z}^+ : L_t < v\}$.

**Matrices in Chapter 3**:

$\mathbf{1}$: a row vector of ones.

$\boldsymbol{Q}$: the TPM of $(X_t, W_t)$.

$\{\boldsymbol{A}_i\}_{i=0}^{c_{max}}$: block components of $\boldsymbol{Q}$ corresponding to transitions with the increase of $i$ units in $X_t$.

$\boldsymbol{B}$: a block component of $\boldsymbol{Q}$ corresponding to transitions with the decrease of 1 unit in $X_t$.

$\boldsymbol{Q}'$: the TPM of $(L_t, V_t, W_t)$.

$\boldsymbol{D}_0$, $\boldsymbol{D}_1$, and $\boldsymbol{D}_2$: block components of $\boldsymbol{Q}'$ corresponding to transitions with the change of 1, 0, and -1 units in $L_t$, respectively.

${}^{\nu}\boldsymbol{G}$: a matrix which records the time discounted (discounted by the time $(L_t, V_t, W_t)$ spends in $\mathcal{S}_1$) first passage time probabilities of $(L_t, V_t, W_t)$ to level $i-1$ given that $(L_t, V_t, W_t)$ starts in level $i$ for all $i \in \mathbb{Z}^+$.

${}^{\nu}\boldsymbol{R}$: a matrix which records the time discounted (discounted by the time $(L_t, V_t, W_t)$ spends in $\mathcal{S}_1$) pre-$\kappa_{i+1}^-$ occupation measure of $(L_t, V_t, W_t)$ to level $i+1$ given that $(L_t, V_t, W_t)$ starts in level $i$ for all $i \in \mathbb{N}$.

${}^{\nu}\boldsymbol{\Xi}_z$: a matrix which records the time discounted (discounted by the time $(L_t, V_t, W_t)$ spends in $\mathcal{S}_1$) pre-$\kappa$ occupation measure of $(L_t, V_t, W_t)$ to level $z$ given that $(L_t, V_t, W_t)$ starts in level $z$ for all $z \in \mathbb{N}$.

**Stochastic processes in Section 4.5**:

$N_t$: the number of active contracts at time $t$.

$A_t$: the age process.

$N$: the maximum number of active contracts the insurance firm can hold at any given time.

$K$: the maximum age of the age process.

$\boldsymbol{N}_t^+$: the CTMC describing $\boldsymbol{N}_t = (N_t, A_t)$ when the surplus process is above level 0.

$\boldsymbol{N}_t^-$: the CTMC describing $\boldsymbol{N}_t = (N_t, A_t)$ when the surplus process is below level 0.

$(L_t^+, \boldsymbol{N}_t^+)$: the claims arrival MAP when the surplus process is above level 0.

$(L_t^-, \boldsymbol{N}_t^-)$: the claims arrival MAP when the surplus process is below level 0.

$U_t$: the surplus process of the dynamic individual risk model.

$\tau$: the time of ruin defined as $\inf\{t > 0 : U_t < 0 \text{ and } \boldsymbol{N}_t = \mathbf{0}\}$.

$(F_t^+, \boldsymbol{J}_t^+)$: a fluid flow process whose sample paths can be connected to those of the surplus process when it is above level 0.

$(F_t^-, \boldsymbol{J}_t^-)$: a fluid flow process whose sample paths can be connected to those of the surplus process when it is below level 0.

$(F_t, \boldsymbol{J}_t)$: a level-independent fluid flow process with the dynamics of $(F_t^+, \boldsymbol{J}_t^+)$ when it is above level 0 and with the dynamics of $(F_t^-, \boldsymbol{J}_t^-)$ when it is below level 0.

$\kappa$: $\inf\{t > 0 : F_t < 0 \text{ and } \boldsymbol{J}_t \in \mathcal{W}_0^-\}$.

$O_t$: the shift process which keeps track of the time $(F_t, \boldsymbol{J}_t)$ spends in $\mathcal{W}_0^+ \cup \mathcal{W}_1^+ \cup \mathcal{W}_1^-$.

$\sigma$: the last epoch the surplus level falls below 0 prior to the time of ruin.

$\eta$: the time at which the last descent of $F_t$ into the negative real line prior to $\kappa$ ends.

$\kappa(y)$: $\inf\{t > 0 : F_t = y\}$.

$\tilde{U}_t$: the time-reversed version of $U_t$.

$\tilde{\boldsymbol{N}}_t$: the time-reversed version of $\boldsymbol{N}_t$.

$(R_t, \boldsymbol{E}_t)$: a fluid flow process whose sample paths can be linked to those of $(\tilde{U}_t, \tilde{\boldsymbol{N}}_t)$ reflected on the time axis.

$H_t$: a shift process which keeps track of the time $\boldsymbol{E}_t$ spends in $\mathcal{W}_1^-$.

$\theta(y)$: $\inf\{t > 0 : R_t = y\}$.

**Matrices in Section 4.5**:

$\boldsymbol{T}^+$: the infinitesimal rate matrix of $\boldsymbol{J}_t^+$.

$\boldsymbol{T}_{11}^+$, $\boldsymbol{T}_{12}^+$, $\boldsymbol{T}_{22}^+$, $\boldsymbol{T}_{21}^+$, $\boldsymbol{T}_{00}^+$, $\boldsymbol{T}_{10}^+$, $\boldsymbol{T}_{01}^+$, $\boldsymbol{T}_{20}^+$, and $\boldsymbol{T}_{02}^+$: block components of $\boldsymbol{T}^+$ partitioned according to $\mathcal{W}_0^+$, $\mathcal{W}_1^+$, and $\mathcal{W}_2^+$.

$\boldsymbol{T}^-$: the infinitesimal rate matrix of $\boldsymbol{J}_t^-$.

$\boldsymbol{T}_{11}^-$, $\boldsymbol{T}_{12}^-$, $\boldsymbol{T}_{22}^-$, $\boldsymbol{T}_{21}^-$, $\boldsymbol{T}_{00}^-$, $\boldsymbol{T}_{10}^-$, $\boldsymbol{T}_{01}^-$, $\boldsymbol{T}_{20}^-$, and $\boldsymbol{T}_{02}^-$: block components of $\boldsymbol{T}^-$ partitioned according to $\mathcal{W}_0^-$, $\mathcal{W}_1^-$, and $\mathcal{W}_2^-$.

$\widehat{\boldsymbol{\Psi}}^+(s)$: a matrix which records the LST of $O_t$ during the journey of $(F_t, \boldsymbol{J}_t)$ from level 0 to level 0 given that $\boldsymbol{J}_t$ starts in $\mathcal{W}_1^+$.

$\widehat{\boldsymbol{G}}^+(s, y)$: a matrix which records the LST of $O_t$ during the journey of $(F_t, \boldsymbol{J}_t)$ from level $y > 0$ to level 0 given that $\boldsymbol{J}_t$ starts in $\mathcal{W}_2^+$.

$\widehat{\boldsymbol{\Psi}}^-(s)$: a matrix which records the LST of $O_t$ during the journey of $(F_t, \boldsymbol{J}_t)$ from level 0 to level 0 given that $\boldsymbol{J}_t$ starts in $\mathcal{W}_2^-$.

$\widehat{\boldsymbol{G}}^-(s, y)$: a matrix which records the LST of $O_t$ during the journey of $(F_t, \boldsymbol{J}_t)$ from level $y < 0$ to level 0 given that $\boldsymbol{J}_t$ starts in $\mathcal{W}_1^-$.

$\widehat{\boldsymbol{K}}^-(s, dy|x)$: a matrix which records the pre-$\kappa(0)$ and pre-$\kappa$ occupation measure with respect to $O_t$ of $(F_t, \boldsymbol{J}_t)$ to $(dy, \mathcal{W}_1^-)$ given that $(F_t, \boldsymbol{J}_t)$ starts in $(x, \mathcal{W}_1^-)$, $x, y < 0$.

$\widehat{\boldsymbol{\Upsilon}}^-(s, dx)$: a matrix which records the pre-$\kappa(0)$ and pre-$\kappa$ occupation measure with respect to $O_t$ of $(F_t, \boldsymbol{J}_t)$ to $(dx, \mathcal{W}_1^-)$ given that $(F_t, \boldsymbol{J}_t)$ starts in $(0, \mathcal{W}_2^-)$, $x < 0$.

$\boldsymbol{B}$: the infinitesimal rate matrix of $\boldsymbol{E}_t$.

$\boldsymbol{B}_{11}$, $\boldsymbol{B}_{22}$, $\boldsymbol{B}_{12}$, and $\boldsymbol{B}_{21}$: block components of $\boldsymbol{B}$ partitioned according to $\mathcal{W}_1^-$ and $\mathcal{W}_2^-$.

$\widehat{\mathbf{\Theta}}(s)$: a matrix which records the LST of $H_t$ during the journey of $(R_t, \boldsymbol{E}_t)$ from level 0 to level 0 given that $\boldsymbol{E}_t$ starts in $\mathcal{W}_1^-$.

$\widehat{\boldsymbol{Q}}(s, y)$: a matrix which records the LST of $H_t$ during the journey of $(R_t, \boldsymbol{E}_t)$ from level $y > 0$ to level 0 given that $\boldsymbol{E}_t$ starts in $\mathcal{W}_2^-$.

# Chapter 1

# Introduction and preliminaries

## 1.1 Introduction

The general form of an insurance risk reserve process (i.e., risk process) $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{T}\}$, for an arbitrary index set $\mathbb{T}$ (i.e., continuous or discrete), is given by

$$U_t = u + C_t - L_t, \;\; t \in \mathbb{T},$$

and some external (possibly multi-dimensional) process $\{\boldsymbol{R}_t, t \in \mathbb{T}\}$, where $u \geq 0$ is the initial surplus level, $L_t$ is the total claims amount up to time $t$, and $C_t$ is the total premiums received up to time $t$. Characterizations of stochastic processes $\{\boldsymbol{R}_t, t \in \mathbb{T}\}$, $\{L_t, t \in \mathbb{T}\}$, and $\{C_t, t \in \mathbb{T}\}$, including their dependence structure, are the determinants of the dynamics of $\{U_t, t \in \mathbb{T}\}$. Herein, we write $R_t$ for $\boldsymbol{R}_t$ whenever $\boldsymbol{R}_t$ is univariate.

Of many problems actuarial researchers have studied in relation to the risk process defined above is the time of ruin $\tau = \inf\{t \in \mathbb{T} : U_t < 0\}$. Hitting time random variables such as the time of ruin have long been the subjects of interest in applied probability, many times purely motivated by the mathematical complexity inherent in them. The time of ruin analysis in actuarial science is no exception. Its complexity and probabilistic nature have intrigued many researchers from various areas of applied probability.

On the other hand, ruin-related problems are also of practical importance for the time-dependent analysis of the risk process it enables. By taking into account the dynamic nature of the cash flow affecting the risk process, the time of ruin analysis measures the impact of the timing of claims on the risk process and hence provides a more refined picture of the financial stability of an insurance entity.

Since the problem was first mathematically formulated by Lundberg (1903), the analysis of the time of ruin has been considered to be a difficult problem. Only in a few simple models are the explicit formulas for the infinite-time ruin probabilities available, and in the case of the finite-time ruin probabilities, explicit formulas are even rarer. Nonetheless, actuarial researchers have ventured into numerous different paths at analyzing the time of ruin from various numerical methods such as the transform inversion method, matrix analytic methods, and differential and integral equation methods to approximations and simulations. As a result, the literature has now matured enough to include a discussion of vast scope on more realistic and sophisticated risk models than the earlier simple risk models such as the Cramér-Lundberg model (see e.g., Lundberg (1903, 1926), Cramér (1930), Seal (1969, 1972), and Prabhu (1961) for earlier works in risk theory, and Albrecher and Asmussen (2010) for a recent survey on the literature).

While much of the earlier works in risk theory focused on the time of ruin distribution, Gerber and Shiu (1998) introduced a functional that would become known as the *Gerber-Shiu function*. The Gerber-Shiu function collectively analyzes the time of ruin, surplus prior to ruin, and deficit at ruin, where the surplus prior to ruin and deficit at ruin random variables are defined as $U_{\tau^-}$ and $|U_\tau|$, respectively. The introduction of the Gerber-Shiu function initiated substantial advances in risk theory. Actuarial researchers began to analyze other ruin-related quantities than just the time of ruin, and the mathematical analysis in risk theory also took a great leap forward.

Since the introduction of the Gerber-Shiu function, Gerber-Shiu functions in more complex risk models other than the compound Poisson risk model have been studied and even some generalizations of the Gerber-Shiu function have also been introduced where ruin-related quantities other than the surplus prior to ruin and deficit at ruin are considered (see e.g., Albrecher and Asmussen (2010), Cheung et al. (2010), and Woo (2012)). This aggregate effort of researchers in risk theory has resulted in forming a strong literature on the mathematical analysis of the stochastic evolution of insurance risk processes today.

In comparison to the maturity of analytical solutions in risk theory however, the computational aspect in risk theory seems to have room for more discussion. For continuous-time risk models, the most prevalent method of choice is the integro differential equation (IDE) method. This method is used for computing the Laplace-Stieltjes transform (LST) of some risk process related functionals with respect to the time variable. One can then numerically invert the computed transform values to evaluate the functionals of interest. The method involves the derivation of an IDE, and in solving this IDE, finding the roots of what is known as the *generalized Lundberg fundamental equation* (or generalized Lundberg equation, for short) plays a key role, as the computable expressions of the transforms under consideration are expressed in terms of the roots of the generalized Lundberg equation. However, this root finding process can be numerically unstable for some complex models, thus hindering the computational tractability of the IDE method.

For discrete-time risk models, there are several computational methods that have been widely implemented in the literature. The first is the difference equation method which is the discrete-time counterpart of the IDE method for the continuous-time risk models. As it is the case in the continuous-time risk models, this difference equation method also involves solving for the roots of the generalized Lundberg equation, leading to the same numerical issues the IDE method has (see e.g., Willmot (1993) and Landriault (2008a,b)). For evaluating some transient solutions (e.g., finite-time ruin probabilities), a well-known method is the recursive method introduced by De Vylder and Goovaerts (1988), which is obtained essentially by conditioning on the one-step transition of the risk process. It is a simple yet a powerful computational algorithm for computing the transient distributions of the discrete-time risk processes. The core idea of conditioning on the one-step transition behind this recursive method have since then been widely adopted by many researchers for the numerical analyses of more complex risk models and used for producing meaningful numerical results (see e.g., Dickson and Waters (1991, 1992), Dickson et al. (1995), Cossette et al. (2004a,b), Drekic and Mera (2011), and Kim and Drekic (2016)). In addition, Alfa and Drekic (2007) introduced a discrete-time Markov chain (DTMC) representation of a discrete-time risk process (a Sparre Andersen risk model to be specific) and derived a matrix representation for the joint probability mass function (pmf) of the time of ruin, surplus prior to ruin and deficit at ruin (see e.g., Alfa and Drekic (2007) and Drekic and Mera (2011) for more details).

Despite their simple computational implementations however, the computational times of both the recursive method and the DTMC method by Alfa and Drekic (2007) grow nearly quadratically in the time unit of interest, and the memory consumption rates of both methods grow linearly in the time unit of interest. Therefore, these methods may not be suitable for large scale problems where large time units are of interest.

Meanwhile, in other areas of applied probability, matrix analytic methods have been extensively employed in evaluating the transforms (with respect to the time variable) of functionals that are similar in their probabilistic interpretations to the transforms of some transient distributions of risk processes. In contrast to the IDE and difference equation methods, matrix analytic methods do not rely on the roots of the generalized Lundberg equation, and instead, involves numerically more stable matrix equation solving problems.

Recently in risk theory, a series of papers have applied matrix analytic methods in the computation of the transforms of some risk process related functionals with respect to the time variable (see e.g., Badescu et al. (2005a,b), Ramaswami (2006), Ahn and Badescu (2007), and Kim et al. (2008)). In the continuous-time case, the enhanced numerical stability of the matrix analytic methods compared to the traditional IDE method have made the computational analysis of more complex risk models more feasible. In the discrete-time case, the difference between the algorithmic procedures of the matrix analytic approach compared to the recursive method and DTMC method by Alfa and Drekic (2007), offers us hope in achieving superior computational times and memory consumption rates. Furthermore, the probabilistic interpretations of underlying matrix analytic methods have opened up the doors to different perspectives than the more analytic approaches taken in the IDE method at approaching problems in risk process analyses.

The introduction of matrix analytic methods in risk theory has marked a significant progress in computations in risk theory. Matrix analytic methods have proven to be powerful computational tools for numerically analyzing complex risk models that traditional methods often had difficulty with. This is particularly noteworthy in the modern age of advanced computing and big data. Moving away from the traditional view of collective risk theory, we can now consider risk models that comprise of many stochastic processes of which data are abundant. These models may fall under the existing class of risk models; however, these more realistic risk models involve a large number of variables which increases the computational complexity significantly. Matrix analytic methods can provide reliable computing algorithms for risk models of such computational complexity, which have not been numerically feasible to analyze with the traditional computational tools in risk theory.

This thesis is dedicated to improving the accessibility of the matrix analytic methodology in risk theory and developing further generalizations of the existing matrix analytic methods in the attempt to promote its computational use in risk theory. Although the literature of matrix analytic methods in risk theory is in its early stage, it is believed that the advancement in computations in risk theory brought by the matrix analytic methods will broaden the spectrum of problems in the risk theory literature in the direction of more realistic and practical risk models and computational analyses of these models. This will make risk theory as a whole more appealing to practitioners and those who are looking for more advanced actuarial risk management tools.

## 1.2 Mathematical preliminaries

In this section, we give a brief discussion on the basic mathematical tools to be used throughout the thesis.

### 1.2.1 DTMC

Let $\{J_k, k \in \mathbb{N}\}$ be a discrete-time stochastic process defined on the countable state space $\mathcal{S}$. $\{J_k, k \in \mathbb{N}\}$ is said to be a (homogeneous) DTMC if

$$\Pr\{J_k = i_k | J_{k-1} = i_{k-1}, J_{k-2} = i_{k-2}, \ldots, J_0 = i_0\} = \Pr\{J_k = i_k | J_{k-1} = i_{k-1}\}$$

and

$$\Pr\{J_k = i_k | J_{k-1} = i_{k-1}\} = q_{i_{k-1}, i_k} \ \forall \ k \in \mathbb{Z}^+, \ i_{k-1}, i_k \in \mathcal{S}.$$

Let $\boldsymbol{\alpha}$ be the initial probability row vector of $\{J_k, k \in \mathbb{N}\}$ (i.e., the $i$-th entry of $\boldsymbol{\alpha}$ is equal to $\Pr\{J_0 = i\}$, $i \in \mathcal{S}$). Assuming $\mathcal{S}$ is expressable as $\mathcal{S} = \{0, 1, 2, \ldots\}$, the transition probability matrix (TPM) of $\{J_k, k \in \mathbb{N}\}$ is given by

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} q_{0,0} & q_{0,1} & q_{0,2} & \cdots \\ q_{1,0} & q_{1,1} & q_{1,2} & \cdots \\ q_{2,0} & q_{2,1} & q_{2,2} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}.$$

Due to the Markov and homegeneity properties of DTMCs, the quantity $\Pr\{J_k = j | J_0 = i\}$, $k \in \mathbb{N}$, is given by the $(i, j)$-th entry of $\boldsymbol{P}^k$.

The state space $\mathcal{S}$ can be decomposed into disjoint communicating classes, of which there are two types: open and closed. An open class consists of a set of states the probability of returning to which, once left, is zero. A closed class consists of a set of states in which the probability of leaving, given that the chain starts in that class, is zero. If the chain has both open and closed classes, we sometimes refer to the open classes as transient classes and the closed classes as absorbing (or recurrent) classes. If the chain consists of one class, it is said to be irreducible.

Another important quantity is the stationary vector. A stationary (row) vector $\boldsymbol{\theta}$ is a vector that satisfies the equation $\boldsymbol{\theta} P = \boldsymbol{\theta}$. If $\boldsymbol{\theta}$ is a probability vector (i.e., the entries of $\boldsymbol{\theta}$ are nonnegative and sum to 1), we refer to $\boldsymbol{\theta}$ as the stationary probability vector. Further details on Markov chains can be found in various stochastic processes reference texts (see e.g., Resnick (2002)).

### 1.2.2 G/M/1-type Markov chain

Let $\{(X_k, \boldsymbol{J}_k), k \in \mathbb{N}\}$ be a multivariate DTMC on the state space $\mathcal{S} = \mathbb{Z} \times \mathcal{G}$ for some finite set $\mathcal{G}$, where $X_k \in \mathbb{Z}$ denotes the *level* of the process and $\boldsymbol{J}_k \in \mathcal{G}$ the *phase* of the process. We will write $J_k$ for $\boldsymbol{J}_k$ whenever $\boldsymbol{J}_k$ is univariate. Suppose that the TPM of $\{(X_k, \boldsymbol{J}_k), k \in \mathbb{N}\}$ is expressable as

$$\boldsymbol{P} = \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \cdots & -1 & 0 & 1 & 2 & \cdots \end{array} \\ \left(\begin{array}{cccccc} \ddots & \ddots & & & & \\ \cdots & \boldsymbol{A}_{-1,-1} & \boldsymbol{A}_{-1,0} & & & \\ \cdots & \boldsymbol{A}_{0,-1} & \boldsymbol{A}_{0,0} & \boldsymbol{A}_{0,1} & & \\ \cdots & \boldsymbol{A}_{1,-1} & \boldsymbol{A}_{1,0} & \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \end{array},$$

where the empty blocks are zero-block matrices and $\{\boldsymbol{A}_{i,j}\}$ are block matrices of appropriate size corresponding to the number of phases at levels $i$ and $j$ of the chain. Markov chains having TPMs with the same structure as that of $\boldsymbol{P}$ are known as the G/M/1-type Markov chains (usually defined on levels in $\mathbb{N}$, but for our purposes, the above representation is more suitable).

### 1.2.3 Discrete QBD process

Let $\{(X_k, \boldsymbol{J}_k), k \in \mathbb{N}\}$ be a multivariate DTMC on the state space $\mathcal{S} = \mathbb{Z} \times \mathcal{G}$ for some finite set $\mathcal{G}$, where $X_k \in \mathbb{Z}$ denotes the *level* of the process and $\boldsymbol{J}_k \in \mathcal{G}$ the *phase* of the process. Again, we will write $J_k$ for $\boldsymbol{J}_k$ whenever $\boldsymbol{J}_k$ is univariate. Suppose that the TPM of $\{(X_k, \boldsymbol{J}_k), k \in \mathbb{N}\}$ is expressable as

$$\boldsymbol{P} = \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \boldsymbol{A}_{-1,-2} & \boldsymbol{A}_{-1,-1} & \boldsymbol{A}_{-1,0} & & & \\ & & \boldsymbol{A}_{0,-1} & \boldsymbol{A}_{0,0} & \boldsymbol{A}_{0,1} & & \\ & & & \boldsymbol{A}_{1,0} & \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array},$$

where the empty blocks are zero-block matrices and $\{\boldsymbol{A}_{i,j}\}$ are block matrices of appropriate size corresponding to the number of phases at levels $i$ and $j$ of the chain. Then, we call $\{(X_k, \boldsymbol{J}_k), k \in \mathbb{N}\}$ a discrete QBD process.

In the literature, the definition of the discrete QBD process is usually restricted to the level-independent QBD process with its levels defined on the natural number set. However, in this work, we consider the above more general definition as it is more suitable for the context of risk theory.

### 1.2.4 Discrete phase-type distribution

A random variable $X$ is said to follow a discrete phase-type distribution of order $m$ if and only if its pmf takes the form

$$f(x) = \begin{cases} \boldsymbol{\alpha}\boldsymbol{U}^{x-1}\boldsymbol{\gamma}^{\intercal}, & x \in \mathbb{Z}^{+}, \\ \alpha_0, & x = 0, \end{cases}$$

where $\alpha_0 \geq 0$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a row vector of size $m$ with $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$, and $\alpha_0 + \sum_{i=1}^{m} \alpha_i = 1$, $\boldsymbol{U} = (u_{i,j})_{i,j \in \{1,2,\ldots,m\}}$ is an $m \times m$ substochastic matrix, and

$\boldsymbol{\gamma}^\intercal = (\gamma_1, \gamma_2, \ldots, \gamma_m)^\intercal = \mathbf{1}^\intercal - \boldsymbol{U}\mathbf{1}^\intercal$. Here, $\intercal$ denotes the transpose operator and $\mathbf{1}^\intercal$ is an $m \times 1$ column vector of ones.

Another interpretation of $X$ is to consider $X$ as the time until absorption of a DTMC defined on the state space $\mathcal{S} = \{0, 1, 2, \ldots, m\}$ with state 0 being the absorbing state of the chain and the rest being transient states. The initial probability vector of the chain is given by $(\alpha_0, \boldsymbol{\alpha})$ and the portion of the TPM governing the transient states of the chain is given by $\boldsymbol{U}$.

We note that the class of phase-type distributions is large and includes many different families of discrete distributions defined on the natural number set.

### 1.2.5 Discrete-time MAP

Let $\{(N_k, J_k), k \in \mathbb{N}\}$ be a bivariate DTMC on the state space $\mathcal{S} = \mathbb{N} \times \{0, 1, 2, \ldots, m-1\}$ where $m \in \mathbb{Z}^+$. Here, $N_k$ represents the number of arrivals up to and including time $k$ and $J_k$ represents the so-called phase of the process at time $t$. Let

$$p_{0;i,j} = \Pr\{(N_{k+1}, J_{k+1}) = (n, j) | (N_k, J_k) = (n, i)\}$$

and

$$p_{1;i,j} = \Pr\{(N_{k+1}, J_{k+1}) = (n+1, j) | (N_k, J_k) = (n, i)\}$$

denote the one-step transition probabilities without arrivals and with arrivals, respectively. Furthermore, let $\boldsymbol{P}_0$ be an $m \times m$ matrix whose $(i, j)$-th entry is $p_{0;i,j}$ and $\boldsymbol{P}_1$ an $m \times m$ matrix whose $(i, j)$-th entry is $p_{1;i,j}$. As a result, the TPM associated with $\{(N_k, J_k), k \in \mathbb{N}\}$

is given by

$$
\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{pmatrix}
\boldsymbol{P}_0 & \boldsymbol{P}_1 & & & & \\
 & \boldsymbol{P}_0 & \boldsymbol{P}_1 & & & \\
 & & \boldsymbol{P}_0 & \boldsymbol{P}_1 & & \\
 & & & \boldsymbol{P}_0 & \boldsymbol{P}_1 & \\
 & & & & \ddots & \ddots
\end{pmatrix}
\end{array},
$$

where the empty spots in $\boldsymbol{P}$ are $m \times m$ zero matrices. Whenever $N_k$ increases, we say there is an arrival.

The counting process $\{(N_k, J_k), t \in \mathbb{N}\}$ is called the discrete-time Markovian arrival process (MAP). As can be seen from the phase-dependent structure, a MAP can be used to model non identical and independently distributed (i.i.d.) inter-arrival times. Although the dependence structure that can be incorporated is restricted to the underlying Markov chain $\{J_k, k \in \mathbb{N}\}$, a risk model operating under a MAP is undoubtedly a step forward from a Sparre Andersen risk process in modelling for correlation. For further details on MAPs, we refer the reader to He (2014).

### 1.2.6 CTMC

Let $\{J_t, t \in \mathbb{R}^+\}$ be a continuous-time stochastic process defined on the countable state space $\mathcal{J}$, and let $\lambda_i$, $i \in \mathcal{J}$, be a strictly positive real number which we refer to as the rate parameter. Furthermore, let $\{\xi_k, k \in \mathbb{N}\}$ with $\xi_0 = 0$ be the jump epochs of $\{J_t, t \in \mathbb{R}^+\}$ and let $\{\sigma_k, k \in \mathbb{Z}^+\}$ be the inter-arrival times of $\{\xi_k, k \in \mathbb{N}\}$ (i.e., $\sigma_k = \xi_k - \xi_{k-1}$ $\forall$ $k \in \mathbb{Z}^+$) which is exponentially distributed with rate $\lambda_{J_{\xi_{k-1}}}$. Now, let the embedded discrete-time stochastic process $\{J_{\xi_k}, k \in \mathbb{N}\}$ form a DTMC, where for $i \neq j \in \mathcal{J}$, we let $q_{i,j} = \Pr\{J_{\xi_k} = j | J_{\xi_{k-1}} = i\}$ $\forall$ $k \in \mathbb{Z}^+$ such that $\sum_{j \neq i, j \in \mathcal{J}} q_{i,j} = 1$ $\forall$ $i \in \mathcal{J}$. Lastly, set $J_t = J_{\xi_{k-1}}$ for $t \in [\xi_{k-1}, \xi_k)$ and $J_0$ is determined by the initial probability vector $\boldsymbol{\alpha} = (\alpha_i)_{i \in \mathcal{J}}$. Then, $\{J_t, t \in \mathbb{R}^+\}$ is a continuous-time Markov chain (CTMC).

If $\mathcal{J}$ is expressable as $\mathcal{J} = \{0, 1, 2, \ldots\}$, then the so-called infinitesimal rate matrix of $\{J_t, t \in \mathbb{R}^+\}$ is given by

$$\boldsymbol{R} = (r_{i,j})_{i,j \in \mathcal{J}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} -\lambda_0 & \lambda_0 q_{0,1} & \lambda_0 q_{0,2} & \cdots \\ \lambda_1 q_{1,0} & -\lambda_1 & \lambda_1 q_{1,2} & \cdots \\ \lambda_2 q_{2,0} & \lambda_2 q_{2,1} & -\lambda_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}.$$

Decomposition of the state space for DTMCs applies to CTMCs as well, and thus, we do not discuss it further here. However, analyses of some other aspects of CTMCs are different from those of DTMCs. In particular, $\Pr\{J_t = j | J_0 = i\}$, $t \geq 0$, is given by the $(i, j)$-th entry of the matrix exponential $E(t)$, where

$$E(t) = e^{\boldsymbol{R}t} = \sum_{n=0}^{\infty} \frac{(\boldsymbol{R}t)^n}{n!}.$$

Moreover, the stationary probability (row) vector $\boldsymbol{\theta} = (\theta_i)_{i \in \mathcal{J}}$ now satisfies the equation $\boldsymbol{\theta} R = \mathbf{0}$ (subject to the entries of $\boldsymbol{\theta}$ being nonnegative and summing to 1), where $\mathbf{0}$ denotes a row vector of zeros.

### 1.2.7 Time-reversed CTMC

If the CTMC $\{J_t, t \in \mathbb{R}^+\}$ is initialized with $\boldsymbol{\alpha} = \boldsymbol{\theta}$, we can consider its time-reversed version. In what follows, we denote the stationary version of $\{J_t, t \in \mathbb{R}^+\}$ by $\{J_t^*, t \in \mathbb{R}^+\}$ and the time-reversed version of $\{J_t^*, t \in \mathbb{R}^+\}$ by $\{\tilde{J}_t^*, t \in \mathbb{R}^+\}$.

The time-reversed version of $\{J_s^*, s \in [0,t]\}$ is defined as $\{\tilde{J}_s^*, s \in [0,t]\} = \{J_{t-s}^*, s \in [0,t]\}$. Then, $\{\tilde{J}_t^*, t \in \mathbb{R}^+\}$ is also a CTMC defined on the same state space $\mathcal{J}$ as that of $\{J_t^*, t \in \mathbb{R}^+\}$ with infinitesimal rate matrix given by

$$\tilde{\boldsymbol{R}} = (\tilde{r}_{i,j})_{i,j\in\mathcal{J}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} -\lambda_0 & \lambda_0\tilde{q}_{0,1} & \lambda_0\tilde{q}_{0,2} & \cdots \\ \lambda_1\tilde{q}_{1,0} & -\lambda_1 & \lambda_1\tilde{q}_{1,2} & \cdots \\ \lambda_2\tilde{q}_{2,0} & \lambda_2\tilde{q}_{2,1} & -\lambda_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array},$$

where $\tilde{q}_{i,j} = \frac{\theta_j}{\theta_i} q_{j,i}$, $i, j \in \mathcal{J}$.

### 1.2.8 Continuous phase-type distribution

A random variable $X$ is said to follow a continuous phase-type distribution of order $m$ if and only if its probability density function (pdf) takes the form

$$f(x) = \begin{cases} \boldsymbol{\alpha} e^{\boldsymbol{U}x} \boldsymbol{\gamma}^\intercal, \ x > 0, \\ \alpha_0, \ x = 0, \end{cases}$$

where $\alpha_0 \geq 0$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a row vector of size $m$ with $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$, and $\alpha_0 + \sum_{i=1}^m \alpha_i = 1$, $\boldsymbol{U} = (u_{i,j})_{i,j\in\{1,2,\ldots,m\}}$ is an $m \times m$ substochastic generator matrix, and $\boldsymbol{\gamma}^\intercal = (\gamma_1, \gamma_2, \ldots, \gamma_m)^\intercal = -\boldsymbol{U}\mathbf{1}^\intercal$.

Another interpretation of $X$ is to consider $X$ as the time until absorption of a CTMC defined on the state space $\mathcal{S} = \{0, 1, 2, \ldots, m\}$ with state 0 being the absorbing state of the chain and the rest being transient states. The initial probability vector of the chain is given by $(\alpha_0, \boldsymbol{\alpha})$ and the portion of the generator matrix restricted to the transient states of the chain is given by $\boldsymbol{U}$.

Similar to its discrete counterpart, the class of continuous phase-type distributions is large and shows versatility in modelling, including many families of continuous distributions defined on the set $[0, \infty)$ as special cases.

### 1.2.9 Continuous-time MAP

Let $\{(N_t, J_t), t \in \mathbb{R}^+\}$ be a bivariate CTMC on the state space $\mathcal{J} = \mathbb{N} \times \{0, 1, 2, \ldots, m-1\}$ where $m \in \mathbb{Z}^+$. In an analogous fashion to its discrete counterpart, $N_t$ represents the number of arrivals up to and including time $t$ and $J_t$ represents the so-called phase of the process at time $t$. Let $d_{0,i,j}$ and $d_{1,i,j}$ denote the transition rates into state $j$ from state $i$ without arrivals and with arrivals, respectively. Furthermore, let $\boldsymbol{D}_0$ be an $m \times m$ matrix whose $(i, j)$-th entry is $d_{0,i,j}$ and $\boldsymbol{D}_1$ an $m \times m$ matrix whose $(i, j)$-th entry is $d_{1,i,j}$. Then, the infinitesimal matrix of $\{(N_t, J_t), t \in \mathbb{R}^+\}$ is given by

$$\boldsymbol{D} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{D}_0 & \boldsymbol{D}_1 & & & & \\ & \boldsymbol{D}_0 & \boldsymbol{D}_1 & & & \\ & & \boldsymbol{D}_0 & \boldsymbol{D}_1 & & \\ & & & \boldsymbol{D}_0 & \boldsymbol{D}_1 & \\ & & & & \ddots & \ddots \end{pmatrix} \end{array},$$

where the empty spots in $\boldsymbol{D}$ are $m \times m$ zero matrices. The counting process $\{(N_t, J_t), t \in \mathbb{R}^+\}$ is called the continuous-time MAP.

### 1.2.10 Continuous-time MAP risk model with phase-type claim size distributions

A continuous-time MAP risk model $\{U_t, t \in \mathbb{R}^+\}$ is comprised of a continuous-time MAP $\{(N_t, J_t), t \in \mathbb{R}^+\}$ defined on $\mathbb{N} \times \mathcal{J}$, $\mathcal{J} = \{1, 2, \ldots, m\}$, $m \in \mathbb{Z}^+$, rate matrices $(\boldsymbol{D}_0, \boldsymbol{D}_1) = ((d_{0,i,j})_{i,j \in \mathcal{J}}, (d_{1,i,j})_{i,j \in \mathcal{J}})$, and the conditionally i.i.d. claim amount sequence $\{Y_k, k \in \mathbb{Z}^+\}$ (conditional on the phase process $\{J_t, t \in \mathbb{R}^+\}$ of the MAP). In particular, $Y_k$ denotes the amount of the $k$-th claim to be made and the distribution of $Y_k$ depends only on the type of the phase transition that the claim is accompanied by. In other words, let $f^{(i,j)}(y)$, $i, j \in \mathcal{J}$, $y \geq 0$, denote the pdf of $Y^{(i,j)} = Y_k | (J_{\xi_k^-} = i, J_{\xi_k} = j)$, where $\{\xi_k, k \in \mathbb{Z}^+\}$ denotes the arrival epochs of the associated MAP. We further assume that $Y^{(i,j)}$ follows a continuous-time phase-type distribution of order $n^{(i,j)} \in \mathbb{Z}^+$ with pdf $f^{(i,j)}(y) = \boldsymbol{\alpha}^{(i,j)} e^{(\boldsymbol{U}^{(i,j)})y} (\boldsymbol{\gamma}^{(i,j)})^\intercal$, $y \geq 0$, $i, j \in \mathcal{J}$, and that the premium rate is constant at $C_t = ct$, $c > 0$. Then, we can write

$$U_t = u + ct - \sum_{k=1}^{N_t} Y_k, \ t \in \mathbb{R}^+, \, u \in \mathbb{R}^+.$$

### 1.2.11 Fluid flow process

Consider a bivariate continuous-time process $\{(F_t, W_t), t \in \mathbb{R}^+\}$, where $\{W_t, t \in \mathbb{R}^+\}$ is a finite-state CTMC whose state space is given by $\mathcal{W}$. Let $r_i \in \mathbb{R}$, $i \in \mathcal{W}$, denote the flow rates of the process $\{F_t, t \in \mathbb{R}^+\}$ where $F_t$ evolves at the flow rates $r_{W_t}$. Then, we refer to $\{(F_t, W_t), t \in \mathbb{R}^+\}$ as the fluid flow process and $\{W_t, t \in \mathbb{R}^+\}$ as the phase process. Unless otherwise specified, we assume $F_0 = 0$.

Usually, fluid flow processes have boundaries at level 0, meaning that the processes do not fall below level 0. However, for the contents of this thesis, we leave the definition of a fluid flow process to be that of the unbounded fluid flow process, where the fluid flow process can fall below level 0.

## 1.3 Matrix analytic methods

In applied probability, the constitution of the definition of solutions to mathematical problems has been predominantly analytical. In spite of the mathematical beauty associated with the analytical solutions however, often these solutions are not easily computable. In the spirit of developing more easily computable forms of solutions, Dr. Marcel F. Neuts initiated the building of the theory of matrix analytic methods. This movement led to the emergence of both mathematically beautiful and computationally superior probability theories such as the theory of matrix-geometric distributions, phase-type distributions, and MAPs (see e.g., Neuts (1981, 1989), Latouche and Ramaswami (1999), and He (2014) for comprehensive textbooks on matrix analytic methods). In this section, we briefly discuss matrix analytic methods for discrete QBD processes and some of the key matrices appearing therein as we will reference them in this thesis.

Consider a discrete QBD process $\{(X_t, J_t), t \in \mathbb{N}\}$ defined on $\mathbb{N} \times \{0, 1, \ldots, m-1\}$, $m \in \mathbb{Z}^+$, with the TPM

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{B} & \boldsymbol{A}_0 & & & \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & & \\ & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}.$$

Let $\boldsymbol{G}$ and $\boldsymbol{R}$ be $m \times m$ square matrices whose $(i, j)$-th entries are given by

$$(\boldsymbol{G})_{i,j} = \Pr\{\eta(l-1) < \infty, J_{\eta(l-1)} = j | (X_0, J_0) = (l, i)\} \ \forall \ l \in \{1, 2, \ldots\},$$

and

$$(\boldsymbol{R})_{i,j} = \sum_{k=1}^{\infty} \Pr\{\kappa_l^- > k-1, (X_{k-1}, J_{k-1}) = (l, j) | (X_0, J_0) = (l-1, i)\} \ \forall \ l \in \{1, 2, \ldots\},$$

where $\eta(l) = \inf\{k \in \mathbb{N} : X_k = l\}$ and $\kappa_l^- = \inf\{k \in \mathbb{Z}^+ : X_k = l-1\}$. (Note that both the definitions of $\boldsymbol{G}$ and $\boldsymbol{R}$ do not depend on the value of $l$ due to the level independence of $\boldsymbol{P}$.) Furthermore, assuming that the QBD process is irreducible and positive recurrent, let $\boldsymbol{\pi}$ denote the stationary probability vector of the QBD process and $\boldsymbol{\pi}_l$ the section of $\boldsymbol{\pi}$ corresponding to level $l$. Then,

**Lemma 6.3.2, Latouche and Ramaswami (1999)**

$$\boldsymbol{\pi}_l = \boldsymbol{b}(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}^l \ \forall \ l \in \mathbb{N},$$

where $\boldsymbol{I}$ is an identity matrix of appropriate size and $\boldsymbol{b}$ is the unique solution of the system $\boldsymbol{b} = \boldsymbol{b}\boldsymbol{A}$, $\boldsymbol{b}\boldsymbol{1}^\intercal = 1$, with $\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}_1 + \boldsymbol{A}_2$.

The matrix $\boldsymbol{G}$ and $\boldsymbol{R}$ are referred to as the *fundamental period* and *rate* matrices, respectively. As can be seen from the above lemma, the rate matrix is of primary interest in identifying the steady-state distribution of the QBD process under consideration. However, often the computational algorithms for computing the matrix $\boldsymbol{G}$ are more stable, and by exploiting the connection between $\boldsymbol{G}$ and $\boldsymbol{R}$, one first computes the matrix $\boldsymbol{G}$ and then $\boldsymbol{R}$ via the following relation (see Eq. (8.2) in Latouche and Ramaswami (1999)):

$$\boldsymbol{R} = \boldsymbol{A}_0(\boldsymbol{I} - \boldsymbol{A}_1 - \boldsymbol{A}_0\boldsymbol{G})^{-1}.$$

Of several algorithms available in the literature for computing $\boldsymbol{G}$, one that is quadratically convergent (given the QBD process is positive recurrent) was given by Latouche and Ramaswami (1993), namely the Logarithmic-Reduction (L-R) algorithm. First of all, let

$$\boldsymbol{H}^{(0)} = (\boldsymbol{I} - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_0,$$
$$\boldsymbol{L}^{(0)} = (\boldsymbol{I} - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2,$$

and for $k \in \mathbb{Z}^+$, recursively define

$$\boldsymbol{H}^{(k)} = (\boldsymbol{I} - \boldsymbol{U}^{(k-1)})^{-1}(\boldsymbol{H}^{(k-1)})^2,$$
$$\boldsymbol{L}^{(k)} = (\boldsymbol{I} - \boldsymbol{U}^{(k-1)})^{-1}(\boldsymbol{L}^{(k-1)})^2,$$

where

$$\boldsymbol{U}^{(k)} = \boldsymbol{H}^{(k)}\boldsymbol{L}^{(k)} + \boldsymbol{L}^{(k)}\boldsymbol{H}^{(k)},\ k \in \mathbb{N}. \tag{1.3.1}$$

Then, we have

$$\boldsymbol{G} = \sum_{k=0}^{\infty}\left(\prod_{i=0}^{k-1}\boldsymbol{H}^{(i)}\right)\boldsymbol{L}^{(k)}, \tag{1.3.2}$$

and if the QBD process is positive recurrent, the sequence $\{\boldsymbol{G}^{(k)} = \sum_{l=0}^{k}(\prod_{i=0}^{l-1}\boldsymbol{H}^{(i)})\boldsymbol{L}^{(l)}\}_{k=0}^{\infty}$ quadratically converges to $\boldsymbol{G}$.

The proof of the L-R algorithm is purely probabilistic and quite elegant. To keep the discussion short, we refer the reader to Latouche and Ramaswami (1999), pp. 187-197, for a complete proof.

On a short note, one may directly compute $\boldsymbol{R}$ by setting $\boldsymbol{R}(0) = \boldsymbol{0}$ and

$$\boldsymbol{R}(k+1) = \boldsymbol{A}_0 + \boldsymbol{R}(k)\boldsymbol{A}_1 + (\boldsymbol{R}(k))^2\boldsymbol{A}_2, \ k \in \mathbb{N}.$$

Then, the sequence $\{\boldsymbol{R}(k), k \in \mathbb{N}\}$ converges to $\boldsymbol{R}$ (see e.g., Eq. (3.36) in He (2014)).

## 1.4 Matrix analytic methods in risk theory

Most of the papers in the literature of matrix analytic methods in risk theory seem to focus on continuous-time risk models. There is one paper by Kim et al. (2008) on matrix analytic methods applied to a discrete-time risk model, but the development there seems premature compared to the literature on matrix analytic methods for continuous-time risk models, as some of the important quantities such as the surplus prior to ruin and the transient distribution of the surplus process are not studied. Therefore, in this section, we focus on the matrix analytic methods for continuous-time risk models.

The matrix analytic methods for continuous-time risk models stem from matrix analytic methods for fluid flow processes. For a continuous-time MAP risk model, one can draw a sample paths connection between the risk process and a fluid flow process. From this sample paths connection, matrix analytic methods for fluid flow processes can be applied to MAP risk processes. The very first paper in risk theory (to our knowledge) to exploit such a sample paths connection and employ matrix analytic methods in analyzing a MAP risk model was Badescu et al. (2005a), where the authors derived an elegant, computable matrix exponential expression for the LST of the time of ruin. Following this first paper in 2005, another paper Badescu et al. (2005b) was published, where the authors derived a computable matrix exponential expression for the joint pdf of the surplus prior to ruin and deficit at ruin. However, what these two papers did not include was the joint pdf of the time of ruin, surplus prior to ruin, and deficit at ruin, and studying the three random variables simultaneously seemed to be a nontrivial work.

In 2004, Ahn and Ramaswami published a paper on the transient distribution of fluid flow processes, where they derived a computable matrix exponential representation of the LST of the transient distribution of a fluid flow process with respect to the time variable based on the novel idea of coupled queues and stochastic limits. (The original work in Ahn and Ramaswami (2004) is highly nontrivial, and as a result, they published another paper in 2006 which presents the materials in their original work via a more elementary level-crossing argument while hiding the complex mathematical ideas of coupled queues and stochastic limits originally shown in their 2004 paper.) Ramaswami (2006) then first applied the matrix analytic methods developed in Ahn and Ramaswami (2004) to MAP risk processes by exploiting a sample paths connection between a MAP risk process and a fluid flow process, where they derived a computable matrix exponential expression for the LST of the joint pdf of the time of ruin, surplus prior to ruin, and deficit at ruin with respect to the time variable. Initiated by Ramaswami (2006), fluid flow process based matrix analytic methods have since then been applied to many other problems in risk theory (see e.g., Ahn and Badescu (2007), Badescu et al. (2007a,b, 2009), and Badescu and Landriault (2009)).

In what follows, we present some of the key results in Ramaswami (2006) to demonstrate how the fluid flow based matrix analytic methods can be applied to analyzing MAP risk models. The discussion here will focus only on the key ideas. More interested readers are referred to Ramaswami (2006).

Consider a continuous-time MAP risk model $\{U_t, t \in \mathbb{R}^+\}$ comprised of the MAP $\{(N_t, J_t), t \in \mathbb{R}^+\}$ defined on $\mathbb{N} \times \mathcal{J}$, $\mathcal{J} = \{1, 2, \ldots, m\}$, $m \in \mathbb{Z}^+$, rate matrices $(\boldsymbol{D}_0, \boldsymbol{D}_1) = ((d_{0,i,j})_{i,j\in\mathcal{J}}, (d_{1,i,j})_{i,j\in\mathcal{J}})$, and the phase-type claim size distributions of order $n^{(i,j)} \in \mathbb{Z}^+$ with pdf $f^{(i,j)}(y) = \boldsymbol{\alpha}^{(i,j)} e^{(\boldsymbol{U}^{(i,j)})y} (\boldsymbol{\gamma}^{(i,j)})^{\intercal}$, $y \geq 0$, $i, j \in \mathcal{J}$. Here, without loss of generality (w.l.o.g.), we assume that the premium rate $c = 1$.

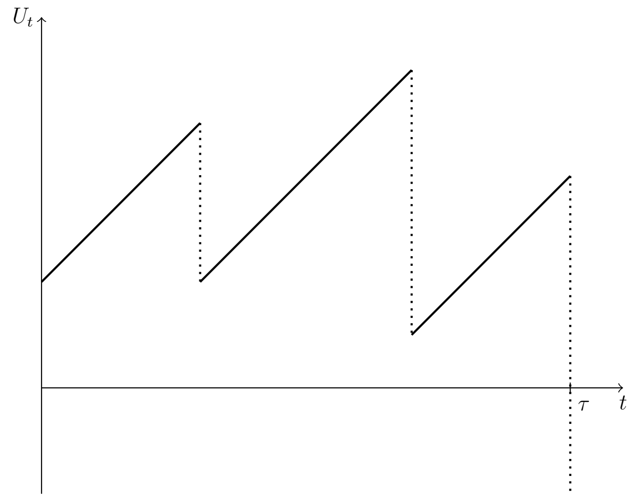


Figure 1.1: Sample path of $\{U_t, t \in \mathbb{R}^+\}$

By stretching the downward jumps of claim amounts into linear downward journeys, we can manipulate the sample paths of the MAP risk process such that the sample paths of the risk process resemble those of a fluid flow process. A formal mathematical construction of such a fluid flow process is a standard exercise and well detailed in Ramaswami (2006). As our intention in this section is to give the reader a snapshot of how the fluid flow based matrix analytic methods are applied to risk models, we assume that such a fluid flow process $\{(F_t, W_t), t \in \mathbb{R}^+\}$ has been well defined and give a pictorial description of the connection between the risk process $\{U_t, t \in \mathbb{R}^+\}$ and the fluid flow process $\{(F_t, W_t), t \in \mathbb{R}^+\}$ below.

First of all, in Figure 1.1, $\tau$ denotes the time of ruin of the risk process, and in Figure 1.2, $\kappa$ denotes the time that the fluid flow process first reaches level 0, $\eta$ denotes the last descent before $\kappa$ initiates, and $\rho$ denotes the time that the last descent initiated at $\eta$ ends. Now, let $\sigma(0, t, x, y)$, $t, x, y > 0$, denote the amount of time the fluid flow process $\{(F_t, W_t), t \in \mathbb{R}^+\}$ is in its upward journeys in the time interval $(0, t)$ given that $F_0 = x$ and $F_t = y$. Then, since both the upward journeys and downward journeys of the fluid

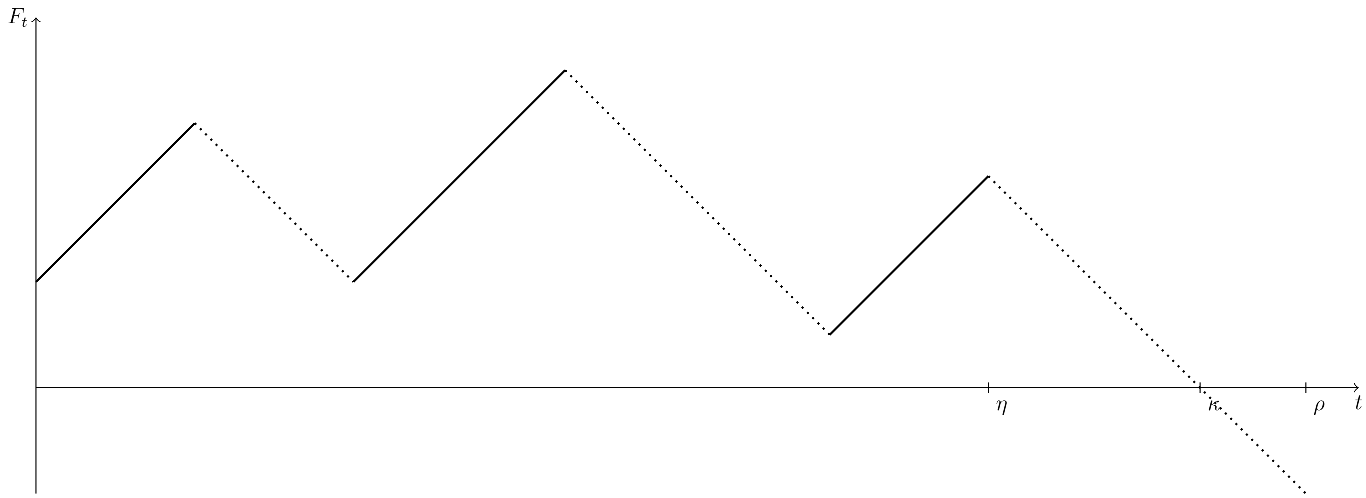


Figure 1.2: Sample path of $\{F_t, t \in \mathbb{R}^+\}$

flow process are at unit rates, one can easily deduce that

$$\sigma(0,t,x,y) = \begin{cases} \frac{t-(x-y)}{2}, \ x \geq y, \\ \frac{t-(y-x)}{2} + (y-x) = \frac{t+(y-x)}{2}, \ y > x. \end{cases} \tag{1.4.1}$$

Next, consider the joint conditional pdf of $(\tau, U_{\tau^-}, J_{\tau^-}, |U_\tau|)$ given that $(U_0, J_0) = (u, i)$, which is denoted by $h(t, x, j, y|u, i)$, $i, j \in \mathcal{J}$, $t, u, x, y > 0$. Then, noting that $(\tau, U_{\tau^-}, J_{\tau^-}, |U_\tau|) = (\sigma(0, \eta, F_0, F_{\eta^-}), F_{\eta^-}, W_{\eta^-}, |F_\rho|)$ with probability (w.p.) 1, one may rewrite the LST of $h(t, x, j, y|u, i)$ with respect to the time variable as

$$\begin{aligned} h(s,x,j,y|u,i) &= \int_0^\infty e^{-st} h(t,x,j,y|u,i)dt = \int_0^\infty e^{-s\sigma(0,t,u,x)} g(t,x,j,y|u,i)dt \\ &= \begin{cases} e^{\frac{s(u-x)}{2}} \int_0^\infty e^{-\frac{s}{2}t} g(t,x,j,y|u,i)dt, \ u \geq x, \\ e^{\frac{-s(x-u)}{2}} \int_0^\infty e^{-\frac{s}{2}t} g(t,x,j,y|u,i)dt, \ x > u, \end{cases} \end{aligned}$$

by (1.4.1), where $g(t, x, j, y|u, i)$ is the joint conditional pdf of $(\eta, F_{\eta^-}, W_{\eta^-}, |F_\rho|)$ given $(F_0, W_0) = (u, i)$, and $s \in \mathbb{C}$, $\Re(s) \geq 0$. Finally, the evaluation of the integral term $\int_0^\infty e^{-\frac{s}{2}t} g(t, x, j, y|u, i)dt$ can be established by applying the matrix analytic procedure for fluid flow processes developed in Ahn and Ramaswami (2004), and therefore, also does the evaluation of $h(s, x, j, y|u, i)$.

Note that when the premium rates depend on the phase process $\{J_t, t \in \mathbb{R}^+\}$, the simple sample paths relation (1.4.1) no longer exists. Hence, the matrix analytic methods in Ahn and Ramaswami (2004) cannot be directly applied to computing $h(s, x, j, y|u, i)$. To remedy this, Ahn (2009) proposed matrix analytic methods for computing the first passage time LSTs of the processes obtained by observing a fluid flow process only when it is either increasing or staying at level, or when it is either decreasing or staying at level. However, the resulting processes lose the skip-free sample paths of their original fluid flow process and thus the analyses become even more complex than in the original analysis in Ahn and Ramaswami (2004). Therefore, applying the matrix analytic methods in Ahn (2009) to evaluate $h(s, x, j, y|u, i)$ for the MAP risk models with phase-dependent premium rates would require more complex probabilistic analysis. Ahn (2009) does not discuss this problem.

## 1.5 Organization of the thesis

Much of the existing literature on matrix analytic methods in risk theory is based on the extension of Ahn and Ramaswami's matrix analytic methodology (Ahn and Ramaswami (2004)) for fluid flow processes to continuous-time risk models. However, the mathematics behind Ahn and Ramaswami's methodology is highly nontrivial, and this mathematical barrier hinders the accessibility of the methodology by practitioners and also makes it difficult to extend the methodology to problems that are not yet treated in the literature of matrix analytic methods in risk theory. As a way to circumvent this problem, in Chapter 2, we propose a matrix analytic methodology for a certain class of discrete-time risk models. The exposition of the methodology in Chapter 2 is more elementary than that of Ahn and Ramaswami's methodology, and hence, more accessible in many respects. Moreover, the model class that we consider in Chapter 2 is a fairly general class of risk models. Thus, we hope that the accessibility and the generality of the model classes that our methodology treats together will serve well in promoting the use of matrix analytic techniques to handle computational concerns in risk theory.

In Chapter 3, we introduce the discrete-time version of a generalization of Ahn and Ramaswami's methodology. The original adaptation of Ahn and Ramaswami's methodology in risk theory does not allow for the analysis of risk models with phase-dependent premium rates. Ahn (2009) later gave another matrix analytic formulation to remedy this issue through the analysis of the fluid flow process with downward jumps, but at the expense of losing the simple level-crossing structure of the skip-free sample paths of the fluid flow process without jumps, which often simplifies the relevant analysis greatly. Our methodology is the discrete-time version of a generalization of Ahn and Ramaswami's methodology in the sense that it is built directly on a sample paths connection between the risk process and a QBD process without downward jumps, even with the phase-dependent premium rates. Hence, our methodology can exploit the skip-free nature of the QBD process even with the phase-dependent premium rates, unlike the alternative methodology introduced by Ahn (2009) involving fluid flow processes with downward jumps.

In Chapter 4, we discuss an adaptation of the matrix analytic methodology for fluid flow processes developed by Bean and O'Reilly (2013) in risk theory. In contrast to Ahn and Ramaswami's methodology, the adaptation of Bean and O'Reilly's matrix analytic methodology in risk theory enables the analysis of continuous-time risk models with phase-dependent premium rates based on sample paths connections between the risk processes and fluid flow processes without jumps. In the first part of Chapter 4, we demonstrate how Bean and O'Reilly's methodology can be extended to continuous-time MAP risk models with phase-dependent premium rates. In the second part of Chapter 4, we apply the adaptation of Bean and O'Reilly's methodology in risk theory developed in the first part of Chapter 4 to a new risk model which takes into account the stochastic dynamics of the customers' arrivals and departures. This new risk model takes a more microscopic perspective on the evolution of an insurance risk process than the view of traditional collective risk theory. In this particular risk model, premium rates depend on certain variables and level-crossings at level 0 must be considered. Thus, the adaptation of Bean and O'Reilly's procedures is a suitable choice of methodology to employ.

# Chapter 2

# A matrix analytic methodology for a class of discrete-time risk models

## 2.1 Introduction

The general form of an insurance risk reserve process (i.e., risk process) $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{T}\}$, for an arbitrary index set $\mathbb{T}$ (i.e., continuous or discrete), is given by

$$U_t = u + C_t - L_t, \;\; t \in \mathbb{T},$$

and some external (possibly multi-dimensional) process $\{\boldsymbol{R}_t, t \in \mathbb{T}\}$, where $u \geq 0$ is the initial surplus level, $L_t$ is the total claims amount up to time $t$, and $C_t$ is the total premiums received up to time $t$. Characterizations of stochastic processes $\{\boldsymbol{R}_t, t \in \mathbb{T}\}$, $\{L_t, t \in \mathbb{T}\}$, and $\{C_t, t \in \mathbb{T}\}$, including their dependence structure, are the determinants of the dynamics of $\{U_t, t \in \mathbb{T}\}$. Herein, we write $R_t$ for $\boldsymbol{R}_t$ whenever $\boldsymbol{R}_t$ is univariate.

Recently in risk theory, a series of papers have applied a fluid flow process based matrix analytic methodology introduced by Ahn and Ramaswami (2004) in the computation of some transient distributions in continuous-time risk models by exploiting the duality between the risk processes and fluid flow processes (see, e.g., Badescu et al. (2005a,b), Ramaswami (2006), and Ahn and Badescu (2007)). The enhanced numerical stability of the matrix analytic methodology compared to the traditional IDE method have made the computational analysis of more complex risk models more feasible. Moreover, the probabilistic interpretation of the matrix analytic methodology has opened up the doors to different perspectives than the more analytic approaches taken in the IDE method at approaching problems in risk process analyses.

Despite the advantages of the methodology however, the mathematics behind the fluid flow process based matrix analytic methodology is highly nontrivial, and this mathematical barrier makes it difficult to extend the methodology to problems that are not yet treated in the literature of matrix analytic methods in risk theory. As a way to circumvent this problem, and in the hopes of highlighting its computational effectiveness in risk theory, we propose in this work a matrix analytic methodology for a class of discrete-time risk models. The computational analysis of discrete-time risk models in general relies on more elementary mathematics than that of continuous-time risk models, and discrete-time risk models can also be used as approximations to continuous-time risk models via the process of discretization (see, e.g., Cossette et al. (2004b)). In actual fact, the exposition of the methodology in this work is more elementary than that of the fluid flow process based methodology by Ahn and Ramaswami (2004) for continuous-time risk models, and hence, more accessible. Moreover, the model class that we consider in this work is a fairly general class of risk models. Thus, we hope that the accessibility of our methodology and the generality of the model class that our methodology treats together will serve well in promoting its computational use in risk theory.

While much of the attention is paid to continuous-time risk models, there is only one (to our knowledge) relevant paper on matrix analytic methods in discrete-time risk models. Kim et al. (2008) develop a QBD process based matrix analytic methodology for computing the infinite-time ruin probabilities and deficit at ruin distribution of a discrete-time risk model with a randomized dividend paying strategy. However, the QBD process based methodology developed by Kim et al. (2008) does not include the analysis of the surplus prior to ruin. Besides the importance of the surplus prior to ruin itself, the analysis of the surplus prior to ruin is equivalent to the analysis of what is known as the occupation measure. The occupation measure plays an important role in the development of a matrix analytic methodology for risk models in that it provides the means to analyze other quantities of interest such as the expected total discounted dividends paid prior to ruin. Furthermore, the QBD process based matrix analytic methodology by Kim et al. (2008) assumes that the claim size distributions are of phase-type, which is a family of light-tailed distributions.

In this work, we develop a matrix analytic methodology for computing the joint conditional pmf of the time of ruin, surplus prior to ruin, and deficit at ruin of a risk process belonging to a model class that we refer to as the G/M/1-type discrete-time risk model (G/M/1 DTRM) class. As we show later in this work, the G/M/1 DTRM class is a fairly large class of risk models and is not restricted to risk models with phase-type claim size distributions, rendering a matrix analytic methodology for risk models with general claim size distributions including heavy-tailed distributions. We first develop a matrix analytic methodology for the general risk models belonging to the G/M/1 DTRM class and for certain special cases of the G/M/1 DTRM class, we will be able to substantially reduce the computational complexity of the methodology, compared to the general case, by exploiting the special structures in these risk models.

The rest of Section 2.1 discusses some known discrete-time risk models, and introduces the Gerber-Shiu function and so-called discounted joint pmfs. In Section 2.2, we develop a matrix analytic methodology for the G/M/1 DTRM class. In Section 2.3, we consider the MAP risk model, which is a subclass of the G/M/1 DTRM class. In Section 2.4, we consider the MAP risk model with a dividend barrier, which is also a subclass of the G/M/1 DTRM class. In Section 2.5, numerical examples are provided.

### 2.1.1 Discrete-time risk models

In discrete-time risk models, $\mathbb{T} = \mathbb{N}$, $U_t \in \mathbb{Z}$, and the aggregate claims amount process $\{L_t, t \in \mathbb{N}\}$ can be expressed in two ways. Firstly, we can write $L_t = \sum_{k=1}^{t} Y_k$, where $\{Y_k, k \in \mathbb{Z}^+\}$ is a sequence of nonnegative integer-valued random variables denoting the claim amount at time $k$. Secondly, $L_t$ can be written in terms of a random sum—namely, $L_t = \sum_{k=1}^{N_t} \mathcal{Y}_k$, where $\{N_t, t \in \mathbb{N}\}$ is a counting process corresponding to the inter-arrival time sequence of claims $\{\eta_k, k \in \mathbb{Z}^+\}$ and $\{\mathcal{Y}_k, k \in \mathbb{Z}^+\}$ denotes the (positive) amount of the $k$-th claim. Below, we give some of the examples of discrete-time risk models in the literature.

**Compound binomial risk model:** One of the very first discrete-time risk models to be introduced was the compound binomial risk model. In the compound binomial risk model, $\{R_t, t \in \mathbb{N}\}$ is simply a constant (i.e., nonstochastic) process independent of $\{U_t, t \in \mathbb{N}\}$, and hence, irrelevant. $\{Y_k, k \in \mathbb{Z}^+\}$ forms an i.i.d. sequence of random variables with $\Pr\{Y_k = 0\} = 1 - p$ and $\Pr\{Y_k = l\} = pf(l)$, $l \in \mathbb{Z}^+$, where $0 < p < 1$ and $f(l)$ is a proper pmf on $\mathbb{Z}^+$. Furthermore, it is assumed that $C_t = ct$, $t \in \mathbb{N}$, $c \in \mathbb{Z}^+$ (see e.g., Gerber (1988), Shiu (1989), and Willmot (1993)).

**Compound binomial model in a Markovian environment:** Cossette et al. (2004b) introduced the compound binomial risk model situated in a Markovian environment as an extension of the above compound binomial model. Let $\{R_t, t \in \mathbb{N}\}$ be a DTMC on a finite state space $\mathcal{S}$. In this risk model, $\{Y_t, t \in \mathbb{Z}^+\}$ forms an i.i.d. sequence of nonnegative integer-valued random variables, conditional on $\{R_t, t \in \mathbb{N}\}$. More precisely, $\Pr\{Y_t = 0|R_{t-1} = i\} = 1 - \alpha_i$ and $\Pr\{Y_t = y|R_{t-1} = i\} = \alpha_i f_i(y)$ $\forall$ $t \in \mathbb{Z}^+$, $y \in \mathbb{Z}^+$, $i \in \mathcal{S}$, where $0 < \alpha_i < 1$ and $f_i(y)$ is a proper pmf on $\mathbb{Z}^+$. It is also assumed that $C_t = ct$, $t \in \mathbb{N}$, $c \in \mathbb{Z}^+$.

**Sparre Andersen risk model:** Another extension to the compound binomial risk model is to relax the distributional assumption imposed on the inter-arrival time sequence of claims $\{\eta_k, k \in \mathbb{Z}^+\}$. In the discrete-time Sparre Andersen risk model, $\{\eta_k, k \in \mathbb{Z}^+\}$ forms an i.i.d. sequence of random variables but is assumed to follow a (general) positive integer-valued distribution unlike the geometric distribution of the compound binomial risk model. Here, $C_t$ is usually assumed to take the form $C_t = ct$, $t \in \mathbb{N}$, $c \in \mathbb{Z}^+$ (see e.g., Pavlova and Willmot (2004), Li (2005a, 2005b), Wu and Li (2009), and Woo (2012)).

Some variations of these models have been proposed as well, which include, for example, incorporating level-dependency and random premium processes (see e.g., Landriault (2008), Drekic and Mera (2011), and Kim and Drekic (2016)). Although we cannot list all of the discrete-time risk models in the literature here, the above models do provide a good summary of the types of discrete-time risk models that are generally studied in the field.

### 2.1.2 Gerber-Shiu function and discounted pmfs

Here, we specify the definitions of the Gerber-Shiu function and the so-called discounted pmfs. For a discrete-time risk process $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ defined on $\mathbb{Z} \times \mathcal{H}$ for some finite set $\mathcal{H}$ and a nonnegative (well-behaved) function $w(x, \boldsymbol{r}_1, y, \boldsymbol{r}_2)$, the Gerber-Shiu function is defined as

$$\phi(\boldsymbol{u}) = E\{\nu^\tau w(U_{\tau-1}, \boldsymbol{R}_{\tau-1}, |U_\tau|, \boldsymbol{R}_\tau)\mathcal{I}[\tau < \infty]|(U_0, \boldsymbol{R}_0) = \boldsymbol{u}\},$$
$$\boldsymbol{u} \in \mathbb{N} \times \mathcal{H},\ \nu \in \mathbb{C},\ |\nu| \leq 1, \qquad (2.1.1)$$

where $\mathcal{I}[A]$ is the indicator function of $A$ (i.e., $\mathcal{I}[A] = 1$ if $A$ is true and $\mathcal{I}[A] = 0$ if $A$ is false). The so-called discounted pmfs can be regarded as special cases of the Gerber-Shiu function, and they are essentially the generating functions of the joint distributions of the time of ruin, surplus prior to ruin, and deficit at ruin. Hence, these can be numerically inverted to obtain transient solutions or with the time variable taking values on $(0, 1]$, to obtain discounted nontransient solutions on the time of ruin, surplus prior to ruin, and deficit at ruin.

The discounted joint conditional pmf of $\{(U_{\tau-1}, \boldsymbol{R}_{\tau-1}), (U_\tau, \boldsymbol{R}_\tau)\}$ is defined as

$$h_\nu(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{u}) = \sum_{n=1}^{\infty} \nu^n \Pr\{\tau = n, (U_{n-1}, \boldsymbol{R}_{n-1}) = \boldsymbol{x}, (U_n, \boldsymbol{R}_n) = \boldsymbol{y}|(U_0, \boldsymbol{R}_0) = \boldsymbol{u}\},$$
$$\boldsymbol{u}, \boldsymbol{x} \in \mathbb{N} \times \mathcal{H},\ \boldsymbol{y} \in \mathbb{Z}^- \times \mathcal{H},\ \nu \in \mathbb{C},\ |\nu| \leq 1, \qquad (2.1.2)$$

the discounted joint conditional pmf of $(U_{\tau-1}, \boldsymbol{R}_{\tau-1})$ is defined as

$$h_\nu(\boldsymbol{x}|\boldsymbol{u}) = \sum_{n=1}^{\infty} \nu^n \Pr\{\tau = n, (U_{n-1}, \boldsymbol{R}_{n-1}) = \boldsymbol{x}|(U_0, \boldsymbol{R}_0) = \boldsymbol{u}\},$$
$$\boldsymbol{u}, \boldsymbol{x} \in \mathbb{N} \times \mathcal{H},\ \nu \in \mathbb{C},\ |\nu| \leq 1, \qquad (2.1.3)$$

and the discounted joint conditional pmf of $(U_\tau, \boldsymbol{R}_\tau)$ is defined as

$$h_\nu(\boldsymbol{y}|\boldsymbol{u}) = \sum_{n=1}^{\infty} \nu^n \Pr\{\tau = n, (U_n, \boldsymbol{R}_n) = \boldsymbol{y} | (U_0, \boldsymbol{R}_0) = \boldsymbol{u}\},$$
$$\boldsymbol{u} \in \mathbb{N} \times \mathcal{H},\ \boldsymbol{y} \in \mathbb{Z}^- \times \mathcal{H},\ \nu \in \mathbb{C},\ |\nu| \leq 1. \tag{2.1.4}$$

Our primary quantity of interest in this work is the functional $h_\nu(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{u})$.

## 2.2 G/M/1-type discrete-time risk model

### 2.2.1 Model class definition

Here, we introduce a model class named the G/M/1 DTRM class. A discrete-time risk process $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ defined on $\mathbb{Z} \times \mathcal{H}$, for some finite set $\mathcal{H}$, belongs to the G/M/1 DTRM class if $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ has a dual G/M/1-type chain $\{(X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$. In particular, for a G/M/1-type Markov chain $\{(X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$ to be a dual G/M/1-type chain of $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ in the G/M/1 DTRM context, $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ and $\{(X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$ must possess a one-to-one relationship (i.e., there exists a one-to-one mapping $\mathcal{W} : \mathbb{Z} \times \mathcal{H} \rightarrow \mathcal{S} = \mathbb{Z} \times \mathcal{G}$ such that $\{\mathcal{W}(U_t, \boldsymbol{R}_t) = (X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$ forms a G/M/1-type Markov chain). Since the two processes have a one-to-one relationship, we can analyze the dual G/M/1-type chain and subsequently convert the results in terms of $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$.

The model class definition of the G/M/1 DTRM class allows us to analyze the risk models under consideration in the context of G/M/1-type Markov chains. This change in perspective gives us an opportunity to leverage the matrix analytic methods developed for G/M/1-type Markov chains in analyzing a fairly large class of risk models. The G/M/1 DTRM class includes all the models discussed in Section 2.1.1 and more. For example, we describe below the G/M/1 DTRM representation of the compound binomial risk model introduced in Section 2.1.1.

As before, let $\Pr\{Y_k = 0\} = 1 - p$, $0 < p < 1$, and $\Pr\{Y_k = y\} = pf(y)$, $y \in \mathbb{Z}^+$, where $f(y)$ is a proper pmf on $\mathbb{Z}^+$. Let $c \in \mathbb{Z}^+$ denote the per-period constant premium rate in the compound binomial risk model setting. Consider the risk process $\{U_t, t \in \mathbb{N}\}$ given by

$$U_t = u + ct - \sum_{k=1}^{t} Y_k.$$

We remark that $\{R_t, t \in \mathbb{N}\}$ is an independent constant process and hence we focus on $\{U_t, t \in \mathbb{N}\}$ only. Now, define $\mathcal{W}(U_t) \equiv \left(\lfloor \frac{U_t}{c} \rfloor, U_t \bmod c\right) = (X_t, J_t)$, where $\lfloor x \rfloor$ denotes the nearest integer less than or equal to $x$. Clearly, $\mathcal{W}$ is a one-to-one mapping and $\mathcal{W}^{-1}(X_t, J_t) = cX_t + J_t = U_t$. Furthermore, note that $\Pr\{U_t = j | U_{t-1} = i\} = pf(i + c - j) + (1-p)\mathcal{I}[i + c - j = 0]$. Writing the same equation in terms of $X_t$ and $J_t$, we have

$$\begin{aligned}\Pr\{(X_t, J_t) =& (l, m) | (X_{t-1}, J_{t-1}) = (a, b)\} = \\ & pf(ca + b + c - (cl + m)) + (1-p)\mathcal{I}[ca + b + c - (cl + m) = 0], \quad (2.2.1)\end{aligned}$$

where $ca+b = i$ and $cl+m = j$. Note that (2.2.1) gives the one-step transition probabilities of the bivariate Markov chain $\{(X_t, J_t), t \in \mathbb{N}\}$. In particular, let $\boldsymbol{A}_{a,l}$ be a $c \times c$ matrix whose $(b, m)$-th entry is given by (2.2.1). In other words,

$$\left(\boldsymbol{A}_{a,l}\right)_{b,m} = \Pr\{(X_t, J_t) = (l, m) | (X_{t-1}, J_{t-1}) = (a, b)\}.$$

Clearly, $\{(X_t, J_t), t \in \mathbb{N}\}$ is a G/M/1-type Markov chain with state space $\mathcal{S} = \mathbb{Z} \times \{0, 1, \ldots, c-1\}$. Hence, $\{(X_t, J_t), t \in \mathbb{N}\}$ is a dual G/M/1-type chain of $\{U_t, t \in \mathbb{N}\}$ and the compound binomial risk model belongs to the G/M/1 DTRM class.

### 2.2.2 Time of ruin, surplus prior to ruin, and deficit at ruin

Alfa and Drekic (2007) first introduced a DTMC representation for the risk process of a discrete-time Sparre Andersen risk model. Although they did not identify the G/M/1 structure in their analysis, the core idea in deriving the joint pmf of the time of ruin, surplus prior to ruin, and deficit at ruin is identical to what we present here. Also, we remark that the G/M/1 DTRM class includes the risk model considered by Alfa and Drekic (2007) as a special case.

Consider a risk process $\{(U_t, \boldsymbol{R}_t), t \in \mathbb{N}\}$ and its dual G/M/1-type chain $\{(X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$. Then, the time of ruin can be defined alternatively as $\tau = \inf\{t \in \mathbb{Z}^+ : (X_t, \boldsymbol{J}_t) \in \mathcal{A}\}$ for some $\mathcal{A} \subset \mathcal{S}$ due to the one-to-one relationship between the risk process and its dual G/M/1-type chain. Thus, we derive the joint distribution of $\{\tau, (X_{\tau-1}, \boldsymbol{J}_{\tau-1}), (X_\tau, \boldsymbol{J}_\tau)\}$ instead of $\{\tau, (U_{\tau-1}, \boldsymbol{R}_{\tau-1}), (U_\tau, \boldsymbol{R}_\tau)\}$. Usually, $\mathcal{A} = \mathbb{Z}^- \times \mathcal{G}$ and we will assume this is the case unless specified otherwise. We furthermore assume that the level process $\{X_t, t \in \mathbb{N}\}$ of the dual G/M/1-type chain is irreducible.

We proceed to decompose the state space $\mathcal{S}$ into $\mathcal{A}$ and $\mathcal{A}^c$, where $\mathcal{A}^c = \mathbb{N} \times \mathcal{G}$. Let

$$
{}_{\mathcal{A}^c}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{pmatrix}
\boldsymbol{A}_{0,0} & \boldsymbol{A}_{0,1} & & & \\
\boldsymbol{A}_{1,0} & \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & & \\
\boldsymbol{A}_{2,0} & \boldsymbol{A}_{2,1} & \boldsymbol{A}_{2,2} & \boldsymbol{A}_{2,3} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \end{array}
$$

denote the TPM corresponding to state transitions from $\mathcal{A}^c$ to $\mathcal{A}^c$ and let

$$
{}^{\mathcal{A}}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccc} \cdots & -3 & -2 & -1 \end{array} \\
\begin{pmatrix}
\cdots & \boldsymbol{A}_{0,-3} & \boldsymbol{A}_{0,-2} & \boldsymbol{A}_{0,-1} \\
\cdots & \boldsymbol{A}_{1,-3} & \boldsymbol{A}_{1,-2} & \boldsymbol{A}_{1,-1} \\
\cdots & \boldsymbol{A}_{2,-3} & \boldsymbol{A}_{2,-2} & \boldsymbol{A}_{2,-1} \\
\iddots & \vdots & \vdots & \vdots
\end{pmatrix} \end{array}
$$

denote the TPM corresponding to state transitions from $\mathcal{A}^c$ to $\mathcal{A}$. Furthermore, let ${}^{\mathcal{A}^c}\boldsymbol{P}^n_{i,z}$ be a block portion of ${}^{\mathcal{A}^c}\boldsymbol{P}^n$ whose $(\boldsymbol{j},\boldsymbol{x})$-th entry is given by

$$\left({}^{\mathcal{A}^c}\boldsymbol{P}^n_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}} = \Pr\{(X_n,\boldsymbol{J}_n)=(z,\boldsymbol{x})|(X_0,\boldsymbol{J}_0)=(i,\boldsymbol{j})\},\ n\in\mathbb{N}.$$

Similarly, let ${}^{\mathcal{A}}\boldsymbol{P}^n_{i,l}$ be a block portion of ${}^{\mathcal{A}}\boldsymbol{P}^n$ whose $(\boldsymbol{j},\boldsymbol{m})$-th entry is given by

$$\left({}^{\mathcal{A}}\boldsymbol{P}^n_{i,l}\right)_{\boldsymbol{j},\boldsymbol{m}} = \Pr\{(X_n,\boldsymbol{J}_n)=(l,\boldsymbol{m})|(X_0,\boldsymbol{J}_0)=(i,\boldsymbol{j})\},\ n\in\mathbb{N}.$$

Then, using straightforward DTMC theory, we obtain

$$\begin{aligned}&\Pr\left\{\tau=n,(X_{\tau-1},\boldsymbol{J}_{\tau-1})=(z,\boldsymbol{x}),(X_\tau,\boldsymbol{J}_\tau)=(l,\boldsymbol{m})|(X_0,\boldsymbol{J}_0)=(i,\boldsymbol{j})\right\}= \\ &\qquad \left({}^{\mathcal{A}^c}\boldsymbol{P}^{n-1}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}}\left({}^{\mathcal{A}}\boldsymbol{P}_{z,l}\right)_{\boldsymbol{x},\boldsymbol{m}},\ n\in\mathbb{Z}^+,\ (i,\boldsymbol{j}),(z,\boldsymbol{x})\in\mathcal{A}^c,\ (l,\boldsymbol{m})\in\mathcal{A}.\end{aligned} \tag{2.2.2}$$

Note that our methodology does not target to compute (2.2.2) directly. Instead, our methodology sets out computational algorithms for computing the discounted joint pmf which can be numerically inverted to retrieve (2.2.2). For interested readers, direct computation of (2.2.2) can be carried out in a similar fashion as how the joint conditional pmf of the time of ruin, surplus prior to ruin, and deficit at ruin in Alfa and Drekic (2007) is computed.

### 2.2.3 Fundamental matrix in risk theory

In this subsection, we show that the discounted joint conditional pmf of $\{(U_{\tau-1}, \boldsymbol{R}_{\tau-1}), (U_\tau, \boldsymbol{R}_\tau)\}$ can be written in terms of a matrix which we refer to as the *discounted fundamental matrix*. Consider the dual G/M/1-type chain $\{(X_t, \boldsymbol{J}_t), t \in \mathbb{N}\}$ defined on $\mathbb{Z} \times \mathcal{G}$, as before. Let ${}^{\nu}\boldsymbol{H}$ denote the discounted fundamental matrix corresponding to the dual G/M/1-type chain and ${}^{\nu}\boldsymbol{H}_{i,l}$ a block component of ${}^{\nu}\boldsymbol{H}$ whose $(\boldsymbol{j}, \boldsymbol{m})$-th entry is given by

$$\left({}^{\nu}\boldsymbol{H}_{i,l}\right)_{\boldsymbol{j},\boldsymbol{m}} = \sum_{n=0}^{\infty} \nu^n p^{(n)}_{(i,\boldsymbol{j}),(l,\boldsymbol{m})}, (i, \boldsymbol{j}), (l, \boldsymbol{m}) \in \mathcal{A}^c, \tag{2.2.3}$$

where $p^{(n)}_{(i,\boldsymbol{j}),(l,\boldsymbol{m})} = \Pr\{(X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\}$. Observe that the series in (2.2.3) converges for $\nu \in \mathbb{C}$, $|\nu| \leq 1$, since every state in $\mathcal{A}^c$ is transient.

To see how the discounted fundamental matrix appears in the discounted joint conditional pmf of $\{(U_{\tau-1}, \boldsymbol{R}_{\tau-1}), (U_\tau, \boldsymbol{R}_\tau)\}$, let us write the (defective) discounted joint conditional pmf of $\{(X_{\tau-1}, \boldsymbol{J}_{\tau-1}), (X_\tau, \boldsymbol{J}_\tau)\}$ as

$$\begin{aligned} &f_\nu\big((z, \boldsymbol{x}), (l, \boldsymbol{m}) | (i, \boldsymbol{j})\big) \\ &\qquad = \sum_{n=1}^{\infty} \nu^n \left({}^{\mathcal{A}^c}\boldsymbol{P}^{n-1}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}} \left({}^{\mathcal{A}}\boldsymbol{P}_{z,l}\right)_{\boldsymbol{x},\boldsymbol{m}} \\ &\qquad = \nu \left({}^{\nu}\boldsymbol{H}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}} \left({}^{\mathcal{A}}\boldsymbol{P}_{z,l}\right)_{\boldsymbol{x},\boldsymbol{m}}, \ (i, \boldsymbol{j}), (z, \boldsymbol{x}) \in \mathcal{A}^c, \ (l, \boldsymbol{m}) \in \mathcal{A}, \ \nu \in \mathbb{C}, \ |\nu| \leq 1. \end{aligned} \tag{2.2.4}$$

Using the duality between the risk process and its dual G/M/1-type chain, we can write

$$h_\nu(\boldsymbol{x}, \boldsymbol{y} | \boldsymbol{u}) = f_\nu\big(\mathcal{W}(\boldsymbol{x}), \mathcal{W}(\boldsymbol{y}) | \mathcal{W}(\boldsymbol{u})\big), \ \boldsymbol{u}, \boldsymbol{x} \in \mathbb{N} \times \mathcal{H}, \ \boldsymbol{y} \in \mathbb{Z}^- \times \mathcal{H}. \tag{2.2.5}$$

Summing over all $\boldsymbol{x} \in \mathbb{N} \times \mathcal{H}$ and $\boldsymbol{y} \in \mathbb{Z}^- \times \mathcal{H}$ in (2.2.5), we can also obtain $h_\nu(\boldsymbol{x}|\boldsymbol{u})$ and $h_\nu(\boldsymbol{y}|\boldsymbol{u})$, respectively.

### 2.2.4 Computational procedure for discounted fundamental matrix

We next outline two computational procedures for calculating the discounted fundamental matrix ${}^{\nu}\boldsymbol{H}$.

**Method I:** Let $\tau_i^- = \inf\{t \in \mathbb{Z}^+ : X_t < i\}$. Then, we can write

$$\begin{aligned}
&\left({}^{\nu}\boldsymbol{H}_{i,l}\right)_{\boldsymbol{j},\boldsymbol{m}} \\
&= \sum_{n=0}^{\infty} \nu^n p_{(i,\boldsymbol{j}),(l,\boldsymbol{m})}^{(n)} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{(X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- \le n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{n=0}^{\infty} \sum_{k=1}^{n} \nu^n \Pr\left\{\tau_{i+1}^- = k, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{n=0}^{\infty} \sum_{k=1}^{n} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^n \Pr\left\{\tau_{i+1}^- = k, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}), (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, J_0) = (i, \boldsymbol{j})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^n \Pr\left\{\tau_{i+1}^- = k, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}), (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^n \\
&\qquad \cdot \Pr\left\{\tau_{i+1}^- = k, (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \Pr\left\{(X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^n \\
&\qquad \cdot \Pr\left\{\tau_{i+1}^- = k, (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \Pr\left\{(X_{n-k}, \boldsymbol{J}_{n-k}) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (z, \boldsymbol{x})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{k=1}^{\infty} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^k \Pr\left\{\tau_{i+1}^- = k, (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\qquad\qquad \cdot \sum_{n=k}^{\infty} \nu^{n-k} \Pr\left\{(X_{n-k}, \boldsymbol{J}_{n-k}) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (z, \boldsymbol{x})\right\} \\
&= \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^- > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\quad + \sum_{k=1}^{\infty} \sum_{z=0}^{i} \sum_{\boldsymbol{x} \in \mathcal{G}} \nu^k \Pr\left\{\tau_{i+1}^- = k, (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\qquad\qquad\qquad \cdot \sum_{n=0}^{\infty} \nu^n \Pr\left\{(X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (z, \boldsymbol{x})\right\}, \qquad (2.2.6)
\end{aligned}
$$

where the seventh and eighth equalities follow from the Markov and stationarity properties, respectively. Now, let

$$ {}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})} = \sum_{n=0}^{\infty} \nu^n \Pr\left\{\tau_{i+1}^{-} > n, (X_n, \boldsymbol{J}_n) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \tag{2.2.7} $$

and

$$ {}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})} = \sum_{k=1}^{\infty} \nu^k \Pr\left\{\tau_{i+1}^{-} = k, (X_k, \boldsymbol{J}_k) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\}. \tag{2.2.8} $$

Then, from (2.2.6), we ultimately have

$$ \left({}^{\nu}\boldsymbol{H}_{i,l}\right)_{\boldsymbol{j},\boldsymbol{m}} = \begin{cases} {}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})} + \sum_{z=0}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})} \left({}^{\nu}\boldsymbol{H}_{z,l}\right)_{\boldsymbol{x},\boldsymbol{m}}, & \text{if } l \geq i, \\ \sum_{z=0}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})} \left({}^{\nu}\boldsymbol{H}_{z,l}\right)_{\boldsymbol{x},\boldsymbol{m}}, & \text{if } l < i. \end{cases} \tag{2.2.9} $$

The computational procedure for calculating ${}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})}$ essentially follows that of Ramaswami (1982), where similar quantities to ${}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})}$ were discussed in relation to a queueing system. Although the computational procedure for our problem and the proofs are very much similar to those of Ramaswami (1982), the quantities discussed in Ramaswami (1982) are not exactly the same as ${}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})}$. Therefore, we provide for the sake of completeness the procedure for computing ${}^{\nu}r_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})}$ and the proofs here.

Let

$$ g_{(i,\boldsymbol{j}),(l,\boldsymbol{m})}^{(k)} = \Pr\left\{\tau_{i+1}^{-} > k, (X_k, \boldsymbol{J}_k) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\}, \ k \in \mathbb{N}. \tag{2.2.10} $$

Then, we can write

$$
\begin{aligned}
&g^{(k)}_{(i,\boldsymbol{j}),(i+1,\boldsymbol{m})}\\
&= \begin{cases} 0, & \text{if } k=0,\\ \big(\boldsymbol{A}_{i,i+1}\big)_{\boldsymbol{j},\boldsymbol{m}}, & \text{if } k=1,\\ \sum_{z=i+1}^{\infty}\sum_{\boldsymbol{x}\in\mathcal{G}} \Pr\big\{\tau^-_{i+1} > k-1, (X_{k-1},\boldsymbol{J}_{k-1})=(z,\boldsymbol{x}) | (X_0,\boldsymbol{J}_0)=(i,\boldsymbol{j})\big\} & \\ \qquad\qquad \cdot \Pr\big\{\tau^-_{i+1} > 1, (X_1,\boldsymbol{J}_1)=(i+1,\boldsymbol{m}) | (X_0,\boldsymbol{J}_0)=(z,\boldsymbol{x})\big\}, & \text{if } k>1, \end{cases}\\
&= \begin{cases} 0, & \text{if } k=0,\\ \big(\boldsymbol{A}_{i,i+1}\big)_{\boldsymbol{j},\boldsymbol{m}}, & \text{if } k=1,\\ \sum_{z=i+1}^{\infty}\sum_{\boldsymbol{x}\in\mathcal{G}} g^{(k-1)}_{(i,\boldsymbol{j}),(z,\boldsymbol{x})}\big(\boldsymbol{A}_{z,i+1}\big)_{\boldsymbol{x},\boldsymbol{m}}, & \text{if } k>1. \end{cases}
\end{aligned}
\tag{2.2.11}
$$

Noting that ${}^{\nu}r^{(i+1,\boldsymbol{m})}_{(i,\boldsymbol{j})} = \sum_{k=0}^{\infty}\nu^k g^{(k)}_{(i,\boldsymbol{j}),(i+1,\boldsymbol{m})}$, multiplying (2.2.11) by $\nu^k$, and summing over $k$ gives

$$
\begin{aligned}
{}^{\nu}r^{(i+1,\boldsymbol{m})}_{(i,\boldsymbol{j})} &= \nu\big(\boldsymbol{A}_{i,i+1}\big)_{\boldsymbol{j},\boldsymbol{m}} + \sum_{z=i+1}^{\infty}\sum_{\boldsymbol{x}\in\mathcal{G}}\sum_{k=2}^{\infty}\nu^k g^{(k-1)}_{(i,\boldsymbol{j}),(z,\boldsymbol{x})}\big(\boldsymbol{A}_{z,i+1}\big)_{\boldsymbol{x},\boldsymbol{m}}\\
&= \nu\big(\boldsymbol{A}_{i,i+1}\big)_{\boldsymbol{j},\boldsymbol{m}} + \sum_{z=i+1}^{\infty}\sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}r^{(z,\boldsymbol{x})}_{(i,\boldsymbol{j})}\nu\big(\boldsymbol{A}_{z,i+1}\big)_{\boldsymbol{x},\boldsymbol{m}}.
\end{aligned}
\tag{2.2.12}
$$

Let ${}^{\nu}\boldsymbol{R}_{i,l}$ be a matrix whose $(\boldsymbol{j},\boldsymbol{m})$-th entry is given by ${}^{\nu}r^{(l,\boldsymbol{m})}_{(i,\boldsymbol{j})}$. Then, (2.2.12) reduces to

$$
{}^{\nu}\boldsymbol{R}_{i,i+1} = \sum_{z=i}^{\infty} {}^{\nu}\boldsymbol{R}_{i,z}\nu\boldsymbol{A}_{z,i+1}, \tag{2.2.13}
$$

where ${}^{\nu}\boldsymbol{R}_{i,i} = I$ for all $i \in \mathbb{N}$ from the definition of ${}^{\nu}r^{(l,\boldsymbol{m})}_{(i,\boldsymbol{j})}$. Herein, we will refer to $\{{}^{\nu}\boldsymbol{R}_{i,l}\}_{i=0,l\geq i}^{\infty}$ as the set of *discounted rate matrices*. Now, by conditioning on the last time the chain visits level $i+1$ without having fallen below level $i+1$ and the phase at that instance, we can also write

$$
\begin{aligned}
g_{(i,\boldsymbol{j}),(i+n,\boldsymbol{m})}^{(k)}
&= \sum_{l=0}^{k} \sum_{\boldsymbol{x}\in\mathcal{G}} \Pr\left\{\tau_{i+1}^{-} > l, (X_l, \boldsymbol{J}_l) = (i+1, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} \\
&\qquad \cdot \Pr\left\{\tau_{i+2}^{-} > k-l, (X_{k-l}, \boldsymbol{J}_{k-l}) = (i+n, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i+1, \boldsymbol{x})\right\} \\
&= \sum_{l=0}^{k} \sum_{\boldsymbol{x}\in\mathcal{G}} g_{(i,\boldsymbol{j}),(i+1,\boldsymbol{x})}^{(l)} g_{(i+1,\boldsymbol{x}),(i+n,\boldsymbol{m})}^{(k-l)}, \ n \geq 2. \qquad (2.2.14)
\end{aligned}
$$

Once again, multiplying (2.2.14) by $\nu^k$ and summing over $k$ yields

$$
\begin{aligned}
{}^{\nu}r_{(i,\boldsymbol{j})}^{(i+n,\boldsymbol{m})} &= \sum_{k=0}^{\infty} \nu^k g_{(i,\boldsymbol{j}),(i+n,\boldsymbol{m})}^{(k)} \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{\boldsymbol{x}\in\mathcal{G}} \nu^k g_{(i,\boldsymbol{j}),(i+1,\boldsymbol{x})}^{(l)} g_{(i+1,\boldsymbol{x}),(i+n,\boldsymbol{m})}^{(k-l)} \\
&= \sum_{\boldsymbol{x}\in\mathcal{G}} \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \nu^k g_{(i,\boldsymbol{j}),(i+1,\boldsymbol{x})}^{(l)} g_{(i+1,\boldsymbol{x}),(i+n,\boldsymbol{m})}^{(k-l)} \\
&= \sum_{\boldsymbol{x}\in\mathcal{G}} \sum_{l=0}^{\infty} \nu^l g_{(i,\boldsymbol{j}),(i+1,\boldsymbol{x})}^{(l)} \sum_{k=l}^{\infty} \nu^{k-l} g_{(i+1,\boldsymbol{x}),(i+n,\boldsymbol{m})}^{(k-l)} \\
&= \sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}r_{(i,\boldsymbol{j})}^{(i+1,\boldsymbol{x})} {}^{\nu}r_{(i+1,\boldsymbol{x})}^{(i+n,\boldsymbol{m})}, \ n \geq 2. \qquad (2.2.15)
\end{aligned}
$$

Writing (2.2.15) in matrix form, we obtain

$$
{}^{\nu}\boldsymbol{R}_{i,i+n} = {}^{\nu}\boldsymbol{R}_{i,i+1} {}^{\nu}\boldsymbol{R}_{i+1,i+n}, \ n \in \mathbb{Z}^{+}. \qquad (2.2.16)
$$

Recursively expanding (2.2.16) immediately leads to

$$
{}^{\nu}\boldsymbol{R}_{i,i+n} = \prod_{k=i}^{i+n-1} {}^{\nu}\boldsymbol{R}_{k,k+1}, \ n \in \mathbb{Z}^{+}. \qquad (2.2.17)
$$

Depending on the specific structure of the dual G/M/1-type chain, one may be able to solve for the set of discounted rate matrices $\{{}^{\nu}\boldsymbol{R}_{i,l}\}_{i=0,l\geq i}^{\infty}$ from (2.2.13) and (2.2.17). However, in the general case given here, (2.2.13) and (2.2.17) are not enough to solve for the set of discounted rate matrices. Hence, we defer the discussion to the subsequent sections which discuss specific risk models in which (2.2.13) and (2.2.17) give a way of solving for the set of discounted rate matrices. In the remaining part of this subsection, we assume that the set of discounted rate matrices have been computed and proceed to compute ${}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})}$.

Let

$$q_{(i,\boldsymbol{j}),(a,\boldsymbol{b})}^{(k)} = \Pr\left\{\tau_{i+1}^{-} = k, (X_k, \boldsymbol{J}_k) = (a, \boldsymbol{b}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\},\ a \leq i,\ \boldsymbol{b} \in \mathcal{G}.$$

Then,

$$\begin{aligned} &q_{(i,\boldsymbol{j}),(z,\boldsymbol{x})}^{(k)} \\ &= \begin{cases} 0, & \text{if } k = 0 \\ \left(A_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}}, & \text{if } k = 1 \\ \sum_{a=i+1}^{\infty} \sum_{\boldsymbol{b}\in\mathcal{G}} \Pr\left\{\tau_{i+1}^{-} > k-1, (X_{k-1}, \boldsymbol{J}_{k-1}) = (a, \boldsymbol{b}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\} & \\ \qquad \times \Pr\left\{\tau_{i+1}^{-} = 1, (X_1, \boldsymbol{J}_1) = (z, \boldsymbol{x}) | (X_0, \boldsymbol{J}_0) = (a, \boldsymbol{b})\right\}, & \text{if } k > 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } k = 0 \\ \left(\boldsymbol{A}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}}, & \text{if } k = 1 \\ \sum_{a=i+1}^{\infty} \sum_{\boldsymbol{b}\in\mathcal{G}} g_{(i,\boldsymbol{j}),(a,\boldsymbol{b})}^{(k-1)} \left(\boldsymbol{A}_{a,z}\right)_{\boldsymbol{b},\boldsymbol{x}}, & \text{if } k > 1. \end{cases} \end{aligned} \tag{2.2.18}$$

Multiplying (2.2.18) by $\nu^k$ and summing over $k$ yields

$$\begin{aligned}
{}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})} &= \sum_{k=0}^{\infty} \nu^k q_{(i,\boldsymbol{j}),(z,\boldsymbol{x})}^{(k)} \\
&= \nu\big(\boldsymbol{A}_{i,z}\big)_{\boldsymbol{j},\boldsymbol{x}} + \sum_{k=1}^{\infty}\sum_{a=i+1}^{\infty}\sum_{\boldsymbol{b}\in\mathcal{G}} \nu^k g_{(i,\boldsymbol{j}),(a,\boldsymbol{b})}^{(k-1)} \big(\boldsymbol{A}_{a,z}\big)_{\boldsymbol{b},\boldsymbol{x}} \\
&= \nu\big(\boldsymbol{A}_{i,z}\big)_{\boldsymbol{j},\boldsymbol{x}} + \sum_{a=i+1}^{\infty}\sum_{\boldsymbol{b}\in\mathcal{G}}\sum_{k=1}^{\infty} \nu^{k-1} g_{(i,\boldsymbol{j}),(a,\boldsymbol{b})}^{(k-1)} \nu\big(\boldsymbol{A}_{a,z}\big)_{\boldsymbol{b},\boldsymbol{x}} \\
&= \sum_{a=i}^{\infty}\sum_{\boldsymbol{b}\in\mathcal{G}} {}^{\nu}r_{(i,\boldsymbol{j})}^{(a,\boldsymbol{b})} \nu\big(\boldsymbol{A}_{a,z}\big)_{\boldsymbol{b},\boldsymbol{x}}. \qquad (2.2.19)
\end{aligned}$$

Let ${}^{\nu}\boldsymbol{Q}_{i,z}$ be a matrix whose $(\boldsymbol{j},\boldsymbol{x})$-th entry is given by ${}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})}$, and let us refer to $\{{}^{\nu}\boldsymbol{Q}_{i,z}\}_{i=0,z\leq i}^{\infty}$ as the set of *discounted ladder height distribution matrices.* Then, writing (2.2.19) in matrix form yields

$$ {}^{\nu}\boldsymbol{Q}_{i,z} = \sum_{a=i}^{\infty} {}^{\nu}\boldsymbol{R}_{i,a}\nu\boldsymbol{A}_{a,z}. \qquad (2.2.20)$$

Returning to the fundamental matrix ${}^{\nu}\boldsymbol{H}$, from (2.2.9), we obtain

$$ {}^{\nu}\boldsymbol{H}_{i,l} = \begin{cases} {}^{\nu}\boldsymbol{R}_{i,l} + \sum_{z=0}^{i} {}^{\nu}\boldsymbol{Q}_{i,z}\,{}^{\nu}\boldsymbol{H}_{z,l}, & \text{if } l \geq i, \\ \sum_{z=0}^{i} {}^{\nu}\boldsymbol{Q}_{i,z}\,{}^{\nu}\boldsymbol{H}_{z,l}, & \text{if } l < i. \end{cases} \qquad (2.2.21)$$

Solving for ${}^{\nu}\boldsymbol{H}_{i,l}$ in (2.2.21), we have

$$ {}^{\nu}\boldsymbol{H}_{i,l} = \begin{cases} \Big(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_{i,i}\Big)^{-1}\Big({}^{\nu}\boldsymbol{R}_{i,l} + \sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i,z}\,{}^{\nu}\boldsymbol{H}_{z,l}\Big), & \text{if } l \geq i, \\ \Big(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_{i,i}\Big)^{-1}\Big(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i,z}\,{}^{\nu}\boldsymbol{H}_{z,l}\Big), & \text{if } l < i. \end{cases} \qquad (2.2.22)$$

Thus, by way of probabilistic reasoning, (2.2.22) affords us with a way to compute the discounted fundamental matrix.

**Method II:** To begin, we shall define a quantity which in the queueing theoretic literature is referred to as the $\boldsymbol{G}$ matrix (see e.g., He (2014)). Before we formally apply the concept of the $\boldsymbol{G}$ matrix to our problem, we should first tailor the definition of the $\boldsymbol{G}$ matrix to suit our problem.

Let $\tau_i = \inf\{t \in \mathbb{Z}^+ : X_t = i\}$,

$$w^{(k)}_{(i,\boldsymbol{j}),(l,\boldsymbol{m})} = \Pr\left\{\tau > k, \tau_l = k, (X_k, \boldsymbol{J}_k) = (l, \boldsymbol{m}) | (X_0, \boldsymbol{J}_0) = (i, \boldsymbol{j})\right\},\ i, l \geq 0,\ k \in \mathbb{Z}^+, \tag{2.2.23}$$

and

$$^{\nu}w^{(l,\boldsymbol{m})}_{(i,\boldsymbol{j})} = \sum_{k=1}^{\infty} \nu^k w^{(k)}_{(i,\boldsymbol{j}),(l,\boldsymbol{m})},\ i, l \geq 0,\ \nu \in \mathbb{C},\ |\nu| \leq 1. \tag{2.2.24}$$

Let $^{\nu}\boldsymbol{G}_{i,l}$ be a matrix whose $(\boldsymbol{j}, \boldsymbol{m})$-th entry is given by $^{\nu}w^{(l,\boldsymbol{m})}_{(i,\boldsymbol{j})}$. We refer to $\{^{\nu}\boldsymbol{G}_{i,l}\}_{i,l\geq 0}$ as the set of *discounted fundamental period matrices*, and next, show how the discounted fundamental period matrices can be used to solve for the discounted fundamental matrices.

We first make the following observation. Since the dual G/M/1-type chain can move up at most by one level, conditioning on $\tau_{i+1}$ and the phase at $\tau_{i+1}$, we obtain

$$w^{(k)}_{(i,\boldsymbol{j}),(i+n,\boldsymbol{m})} = \sum_{z=1}^{k} \sum_{\boldsymbol{x} \in \mathcal{G}} w^{(z)}_{(i,\boldsymbol{j}),(i+1,\boldsymbol{x})} w^{(k-z)}_{(i+1,\boldsymbol{x}),(i+n,\boldsymbol{m})},\ n \in \mathbb{Z}^+. \tag{2.2.25}$$

Multiplying (2.2.25) by $\nu^k$ and summing over $k$ leads to

$$ {}^{\nu}w_{(i,\boldsymbol{j})}^{(i+n,\boldsymbol{m})} = \sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}w_{(i,\boldsymbol{j})}^{(i+1,\boldsymbol{x})}\ {}^{\nu}w_{(i+1,\boldsymbol{x})}^{(i+n,\boldsymbol{m})},\ n \in \mathbb{Z}^+, $$

which, in matrix form, becomes

$$ {}^{\nu}\boldsymbol{G}_{i,i+n} = {}^{\nu}\boldsymbol{G}_{i,i+1}\,{}^{\nu}\boldsymbol{G}_{i+1,i+n},\ n \in \mathbb{Z}^+. \tag{2.2.26} $$

Proceeding inductively, we can ultimately conclude that

$$ {}^{\nu}\boldsymbol{G}_{i,i+n} = \prod_{k=i}^{i+n-1} {}^{\nu}\boldsymbol{G}_{k,k+1},\ n \in \mathbb{Z}^+. \tag{2.2.27} $$

To solve for ${}^{\nu}\boldsymbol{G}_{i,i+n}$, $n \in \mathbb{Z}^+$, we condition on the one-step transition of the dual G/M/1-type chain. In particular, we have

$$ w_{(i,\boldsymbol{j}),(i+1,\boldsymbol{m})}^{(k)} = \begin{cases} 0, & k = 0, \\ \left(\boldsymbol{A}_{i,i+1}\right)_{\boldsymbol{j},\boldsymbol{m}}, & k = 1, \\ \sum_{z=0}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} \left(\boldsymbol{A}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}} w_{(z,\boldsymbol{x}),(i+1,\boldsymbol{m})}^{(k-1)}, & k \geq 2. \end{cases} \tag{2.2.28} $$

Again, by multiplying (2.2.28) by $\nu^k$ and summing over $k$, we have

$$ {}^{\nu}w_{(i,\boldsymbol{j})}^{(i+1,\boldsymbol{m})} = \nu\left(\boldsymbol{A}_{i,i+1}\right)_{\boldsymbol{j},\boldsymbol{m}} + \nu \sum_{z=0}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} \left(\boldsymbol{A}_{i,z}\right)_{\boldsymbol{j},\boldsymbol{x}}\,{}^{\nu}w_{(z,\boldsymbol{x})}^{(i+1,\boldsymbol{m})}. $$

In matrix form, this gives rise to

$$
\begin{aligned}
{}^{\nu}\boldsymbol{G}_{i,i+1} &= \nu\boldsymbol{A}_{i,i+1} + \sum_{z=0}^{i} \nu\boldsymbol{A}_{i,z}\,{}^{\nu}\boldsymbol{G}_{z,i+1} \\
&= \nu\boldsymbol{A}_{i,i+1} + \sum_{z=0}^{i} \nu\boldsymbol{A}_{i,z} \prod_{k=z}^{i} {}^{\nu}\boldsymbol{G}_{k,k+1},
\end{aligned}
\tag{2.2.29}
$$

where the second equality follows from (2.2.27). Solving for ${}^{\nu}\boldsymbol{G}_{i,i+1}$ from (2.2.29) yields

$$
{}^{\nu}\boldsymbol{G}_{i,i+1} = \left(\boldsymbol{I} - \sum_{z=0}^{i-1} \nu\boldsymbol{A}_{i,z} \prod_{k=z}^{i-1} {}^{\nu}\boldsymbol{G}_{z,z+1}\right)^{-1} \nu\boldsymbol{A}_{i,i+1}. \tag{2.2.30}
$$

Hence, by initially solving for ${}^{\nu}\boldsymbol{G}_{0,1}$, one can recursively obtain ${}^{\nu}\boldsymbol{G}_{i,i+1}$, $i \in \mathbb{Z}^{+}$, using (2.2.30).

For $0 \le l \le i$, by conditioning on $\tau_{i+1}^{-}$ and the phase of the chain at $\tau_{i+1}^{-}$, we obtain

$$
w_{(i,\boldsymbol{j}),(l,\boldsymbol{m})}^{(k)} = \begin{cases} 0, & k = 0, \\ \sum_{n=1}^{k-1} \sum_{\substack{z=0\\ z\neq l}}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} q_{(i,\boldsymbol{j}),(z,\boldsymbol{x})}^{(n)} w_{(z,\boldsymbol{x}),(l,\boldsymbol{m})}^{(k-n)} + q_{(i,\boldsymbol{j}),(l,\boldsymbol{m})}^{(k)}, & k > 0. \end{cases} \tag{2.2.31}
$$

Multiplying (2.2.31) by $\nu^{k}$ and summing over $k$ leads to

$$
{}^{\nu}w_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})} = \sum_{\substack{z=0\\ z\neq l}}^{i} \sum_{\boldsymbol{x}\in\mathcal{G}} {}^{\nu}q_{(i,\boldsymbol{j})}^{(z,\boldsymbol{x})}\,{}^{\nu}w_{(z,\boldsymbol{x})}^{(l,\boldsymbol{m})} + {}^{\nu}q_{(i,\boldsymbol{j})}^{(l,\boldsymbol{m})},
$$

which in matrix form, reduces to

$$
{}^{\nu}\boldsymbol{G}_{i,l} = \sum_{\substack{z=0\\ z\neq l}}^{i} {}^{\nu}\boldsymbol{Q}_{i,z}\,{}^{\nu}\boldsymbol{G}_{z,l} + {}^{\nu}\boldsymbol{Q}_{i,l}. \tag{2.2.32}
$$

Solving for ${}^{\nu}\boldsymbol{G}_{i,l}$ from (2.2.32) and combining it with (2.2.27), we ultimately have

$$
{}^{\nu}\boldsymbol{G}_{i,l} = \begin{cases} \sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i,z}{}^{\nu}\boldsymbol{G}_{z,l} + {}^{\nu}\boldsymbol{Q}_{i,i}, & \text{if } i = l, \\ \left(I - {}^{\nu}\boldsymbol{Q}_{i,i}\right)^{-1}\left(\sum_{\substack{z=0 \\ z\neq l}}^{i-1} {}^{\nu}\boldsymbol{Q}_{i,z}{}^{\nu}\boldsymbol{G}_{z,l} + {}^{\nu}\boldsymbol{Q}_{i,l}\right), & \text{if } l < i, \\ \prod_{k=i}^{l-1} {}^{\nu}\boldsymbol{G}_{k,k+1}, & \text{if } l > i. \end{cases} \tag{2.2.33}
$$

Thus, we can compute $\{{}^{\nu}\boldsymbol{G}_{i,l}\}_{i,l\geq 0}$ recursively via (2.2.33).

Returning to the discounted fundamental matrix, by conditioning on $\tau_l$ and the phase of the chain at $\tau_l$, we can write

$$
{}^{\nu}\boldsymbol{H}_{i,l} = \begin{cases} \boldsymbol{I} + {}^{\nu}\boldsymbol{G}_{l,l}{}^{\nu}\boldsymbol{H}_{l,l}, & \text{if } l = i, \\ {}^{\nu}\boldsymbol{G}_{i,l}{}^{\nu}\boldsymbol{H}_{l,l}, & \text{if } l \neq i. \end{cases} \tag{2.2.34}
$$

Solving for ${}^{\nu}\boldsymbol{H}_{i,l}$ in the first line of (2.2.34), we ultimately have

$$
{}^{\nu}\boldsymbol{H}_{i,l} = \begin{cases} \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{G}_{l,l}\right)^{-1}, & \text{if } l = i, \\ {}^{\nu}\boldsymbol{G}_{i,l}\left(\boldsymbol{I} - {}^{\nu}\boldsymbol{G}_{l,l}\right)^{-1}, & \text{if } l \neq i. \end{cases} \tag{2.2.35}
$$

**Remark 2.2.1:** Typically, Method I will be a more preferable choice to Method II for computing the discounted fundamental matrix, since the computation of the discounted fundamental period matrices is generally more computationally intense. However, note that Method II provides a rather straightforward way to compute the discounted fundamental period matrices with (2.2.35) once the discounted fundamental matrix is already available.

**Remark 2.2.2:** The discounted fundamental period matrices can be used to study various hitting times of the risk process prior to ruin, and hence, are useful quantities in analyzing the risk process.

**Remark 2.2.3:** Although we laid out the computational procedures for the discounted fundamental matrices for general G/M/1 DTRMs above, the actual implementation of these computational procedures will vary depending on the specific models that are considered.

## 2.3 MAP risk model

A MAP risk model is comprised of a discrete-time MAP $\{(N_t, J_t), t \in \mathbb{N}\}$ with $m$ phases, TPMs $(\boldsymbol{P}_0, \boldsymbol{P}_1)$, and the conditionally i.i.d. claim amount per period sequence $\{Y_t, t \in \mathbb{Z}^+\}$ (conditional on the phase process $\{J_t, t \in \mathbb{N}\}$ of the MAP). In particular, let $f_{i,j}(y)$, $y \in \mathbb{Z}^+$, denote the pmf of $Y^{(i,j)} = Y_t|(I_t = 1, J_t = j, J_{t-1} = i)$ $\forall$ $t \in \mathbb{Z}^+$, where $\{I_t, t \in \mathbb{Z}^+\}$ is a sequence of Bernoulli random variables which are equal to 1 when there is an arrival at time $t$ in the underlying MAP. (Note that $Y_t|(I_t = 0, J_t = j, J_{t-1} = i)$ is equal to 0 w.p. 1 $\forall$ $i, j$ and $t \in \mathbb{Z}$.) Furthermore, we assume that premiums are received at a constant (deterministic) rate $c \in \mathbb{Z}^+$ per unit time. Then, for $u \in \mathbb{N}$, we can express the surplus process as

$$U_t = u + ct - \sum_{k=1}^{t} Y_k,\ t \in \mathbb{N}\ .$$

We now show that $\{(U_t, J_t), t \in \mathbb{N}\}$ belongs to the G/M/1 DTRM class.

Let $X_t = \lfloor \frac{U_t}{c} \rfloor$ and $V_t = U_t \bmod c$. Letting $X_t$ be the level of the process, $\{(X_t, V_t, J_t), t \in \mathbb{N}\}$ is clearly a dual G/M/1-type chain on the state space $\mathcal{S} = \mathbb{Z} \times \{0, 1, 2, \ldots, c-1\} \times$

$\{0,1,2,\ldots,m-1\}$ with (one-step) transition probabilities given by

$$\begin{aligned}
\left(\boldsymbol{A}_{i,l}\right)_{(j,v),(a,b)} &= \Pr\{(X_1,V_1,J_1)=(l,a,b)|(X_0,V_0,J_0)=(i,j,v)\} \\
&= \Pr\{(U_1,J_1)=(cl+a,b)|(U_0,J_0)=(ci+j,v)\} \\
&= \Pr\{(Y_1,J_1)=(ci+j+c-(cl+a),b)|J_0=v\} \\
&= p_{0;v,b}\mathcal{I}[ci+j+c-(cl+a)=0] \\
&\qquad + p_{1;v,b}f_{v,b}(ci+j+c-(cl+a)). \qquad (2.3.1)
\end{aligned}$$

Furthermore, observing the structure of these transition probabilities, we notice that $\boldsymbol{A}_{i,l}$ is not dependent on specific values of $i$ and $l$, but rather on the difference $i-l$. We refer to such a model as the level-independent G/M/1 DTRM. Thus, w.l.o.g., we will henceforth write $\boldsymbol{A}_{i-l+1}$ for $\boldsymbol{A}_{i,l}$.

### 2.3.1 Time of ruin, surplus prior to ruin, and deficit at ruin

Due to the definition of $(X_t,V_t,J_t)$, the time of ruin can be written as $\tau=\inf\{t\in\mathbb{Z}^+ : X_t<0\}$. Therefore, the absorbing class is $\mathcal{A}=\mathbb{Z}^-\times\{0,1,2,\ldots,c-1\}\times\{0,1,2,\ldots,m-1\}$. Let

$$
{}^{\mathcal{A}^c}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{pmatrix}
\boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots & \cdots & \cdots \\
\boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots & \cdots \\
\boldsymbol{A}_3 & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array}
$$

denote the TPM of the open class of $\{(X_t, V_t, J_t), t \in \mathbb{N}\}$, and let

$$
{}^{\mathcal{A}}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \cdots & -3 & -2 & -1 \end{array} \\ \begin{pmatrix} \cdots & \boldsymbol{A}_4 & \boldsymbol{A}_3 & \boldsymbol{A}_2 \\ \cdots & \boldsymbol{A}_5 & \boldsymbol{A}_4 & \boldsymbol{A}_3 \\ \cdots & \boldsymbol{A}_6 & \boldsymbol{A}_5 & \boldsymbol{A}_4 \\ \iddots & \vdots & \vdots & \vdots \end{pmatrix} \end{array}
$$

denote the TPM corresponding to state transitions from $\mathcal{A}^c$ to $\mathcal{A}$. Then, from the G/M/1 DTRM theory, the conditional joint pmf of the time of ruin, surplus prior to ruin, and deficit at ruin is given by

$$
\begin{aligned}
&\Pr\{\tau = t, (U_{\tau-1}, J_{\tau-1}) = (a, v), (U_\tau, J_\tau) = (l, b) | (U_0, J_0) = (u, j)\} = \\
&\quad \Pr\Big\{\tau = t, (X_{\tau-1}, V_{\tau-1}, J_{\tau-1}) = \big(\lfloor \frac{a}{c} \rfloor, a \bmod c, v\big), \\
&\qquad (X_\tau, V_\tau, J_\tau) = \big(\lfloor \frac{l}{c} \rfloor, l \bmod c, b\big) \big| (X_0, V_0, J_0) = \big(\lfloor \frac{u}{c} \rfloor, u \bmod c, j\big)\Big\} = \\
&\qquad\qquad \big({}^{\mathcal{A}^c}\boldsymbol{P}^{t-1}_{\lfloor \frac{u}{c} \rfloor, \lfloor \frac{a}{c} \rfloor}\big)_{(u \bmod c, j),(a \bmod c, v)} \big({}^{\mathcal{A}}\boldsymbol{P}_{\lfloor \frac{a}{c} \rfloor, \lfloor \frac{l}{c} \rfloor}\big)_{(a \bmod c, v),(l \bmod c, b)}. \quad (2.3.2)
\end{aligned}
$$

Again, note that our methodology does not target to compute (2.3.2) directly. Instead, our methodology sets out computational algorithms for computing the discounted joint pmf which can be numerically inverted to retrieve (2.3.2).

### 2.3.2 Fundamental matrix via Method I: General claim size distribution

We first describe the procedure for computing the discounted rate matrices. Due to the level independence of the dual G/M/1-type chain of the MAP risk model, the discounted rate matrices do not have level dependency either. In other words, ${}^{\nu}\boldsymbol{R}_{i,l}$ can be written as ${}^{\nu}\boldsymbol{R}_{i-l}$, $i \le l$. Moreover, by (2.2.13) and (2.2.17), we can conclude that ${}^{\nu}\boldsymbol{R}_{i-l} = {}^{\nu}\boldsymbol{R}^{i-l}$, where the matrix ${}^{\nu}\boldsymbol{R}$ is the coefficient-matrix-wise minimal nonnegative solution to

$$\boldsymbol{X} = \sum_{n=0}^{\infty} \boldsymbol{X}^{n} \nu \boldsymbol{A}_{n}. \tag{2.3.3}$$

We omit the proof as it is very similar to the proof shown in Neuts (1981). Of many existing algorithms for finding ${}^{\nu}\boldsymbol{R}$, a method for which convergence is guaranteed comes by setting ${}^{\nu}\boldsymbol{R}(0) = \boldsymbol{0}$, and recursively computing ${}^{\nu}\boldsymbol{R}(k+1)$ via the following iteration:

$${}^{\nu}\boldsymbol{R}(k+1) = \sum_{n=0}^{\infty} {}^{\nu}\boldsymbol{R}(k)^{n} \nu \boldsymbol{A}_{n}, \; k \in \mathbb{N}. \tag{2.3.4}$$

Note that $\|{}^{\nu}\boldsymbol{R}(k)^{n} \nu \boldsymbol{A}_{n}\|_{\max} \le \|{}^{1}\boldsymbol{R}^{n} \boldsymbol{A}_{n}\|_{\max}$ for all $k, n \in \mathbb{N}$, $\nu \in \mathbb{C}$, $|\nu| \le 1$, where $\|\boldsymbol{A}\|_{\max} = \max_{i,j} |a_{i,j}|$ denotes the max norm of a matrix $\boldsymbol{A} = (a_{i,j})$. Hence, the series on the right hand side of (2.3.4) converges for all $k \in \mathbb{N}$. In practice, one needs to truncate the summation by truncating $\{\boldsymbol{A}_i, i \in \mathbb{N}\}$ to $\{\boldsymbol{A}_i, i \in \{0, 1, \ldots, N\}\}$ where $\boldsymbol{A}_N = \sum_{k=N}^{\infty} \boldsymbol{A}_i$.

Once we have computed ${}^{\nu}\boldsymbol{R}$, the discounted ladder height distribution matrices can be computed via (2.2.20), namely

$${}^{\nu}\boldsymbol{Q}_{i,z} = \sum_{a=i}^{\infty} {}^{\nu}\boldsymbol{R}^{a-i} \nu \boldsymbol{A}_{a-z+1}. \tag{2.3.5}$$

Similar to the case of the discounted rate matrices, the discounted ladder height distribution matrices are also level-independent in the sense that we can write ${}^{\nu}\boldsymbol{Q}_{i-z}$ instead of ${}^{\nu}\boldsymbol{Q}_{i,z}$.

To see this from (2.3.5), we write

$$ {}^{\nu}\boldsymbol{Q}_{i,z} = \sum_{a=i}^{\infty} {}^{\nu}\boldsymbol{R}^{a-i} \nu \boldsymbol{A}_{a-z+1} = \sum_{a=0}^{\infty} {}^{\nu}\boldsymbol{R}^{a} \nu \boldsymbol{A}_{i-z+a+1}. \tag{2.3.6} $$

Clearly, ${}^{\nu}\boldsymbol{Q}_{i,z}$ does not depend on the specific pair of values of $i$ and $z$, but rather on the difference $i - z$. Hence, from (2.3.6), we can recursively compute $\{{}^{\nu}\boldsymbol{Q}_z\}_{z=0}^{\infty}$ as

$$ {}^{\nu}\boldsymbol{R}^{\nu}\boldsymbol{Q}_z = \sum_{a=0}^{\infty} {}^{\nu}\boldsymbol{R}^{a+1} \nu \boldsymbol{A}_{z+a+1} = \sum_{a=0}^{\infty} {}^{\nu}\boldsymbol{R}^{a} \nu A_{z+a} - \nu \boldsymbol{A}_z = {}^{\nu}\boldsymbol{Q}_{z-1} - \nu \boldsymbol{A}_z, $$

which implies that

$$ {}^{\nu}\boldsymbol{Q}_{z-1} = {}^{\nu}\boldsymbol{R}^{\nu}\boldsymbol{Q}_z + \nu \boldsymbol{A}_z. \tag{2.3.7} $$

If one truncates $\{\boldsymbol{A}_i, i \in \mathbb{N}\}$ to $\{\boldsymbol{A}_i, i \in \{0, 1, \ldots, N\}\}$ as in (2.3.4), then one can set ${}^{\nu}\boldsymbol{Q}_{N-1} = \nu \boldsymbol{A}_N$ and perform the recursion given by (2.3.7).

We can also compute $\{{}^{\nu}\boldsymbol{H}_{i,l}\}_{i,l \geq 0}$ quite efficiently with a different recursion than (2.2.22). We first rewrite (2.2.22) with the level independence incorporated in as

$$ {}^{\nu}\boldsymbol{H}_{i,l} = \begin{cases} \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left({}^{\nu}\boldsymbol{R}^{l-i} + \sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l}\right), & \text{if } l \geq i, \\ \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l}\right), & \text{if } l < i. \end{cases} \tag{2.3.8} $$

By observing (2.3.8) carefully, we find that the following relation holds:

$$ {}^{\nu}\boldsymbol{H}_{i,l} = {}^{\nu}\boldsymbol{H}_{i,i} {}^{\nu}\boldsymbol{R}^{l-i}, \ l \geq i \ . \tag{2.3.9} $$

We can prove (2.3.9) using mathematical induction. From (2.3.8), we clearly have

$$ {}^{\nu}\boldsymbol{H}_{0,l} = \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} {}^{\nu}\boldsymbol{R}^{l} = {}^{\nu}\boldsymbol{H}_{0,0} {}^{\nu}\boldsymbol{R}^{l}, \ l \geq 0, $$

and (2.3.9) holds true for $i = 0$ and $l \geq 0$. Next, assume that (2.3.9) holds true for levels $k = 0, 1, 2, \ldots, i$ and for all $l \geq k$. Then, for some $n \geq 0$, we have

$$
\begin{aligned}
{}^{\nu}\boldsymbol{H}_{i+1,i+1+n} &= \left(I - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left({}^{\nu}\boldsymbol{R}^n + \sum_{z=0}^{i+1-1} {}^{\nu}\boldsymbol{Q}_{i+1-z}{}^{\nu}\boldsymbol{H}_{z,i+1+n}\right) \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left({}^{\nu}\boldsymbol{R}^n + \sum_{z=0}^{i+1-1} {}^{\nu}\boldsymbol{Q}_{i+1-z}{}^{\nu}\boldsymbol{H}_{z,z}{}^{\nu}\boldsymbol{R}^{i+1+n-z}\right) \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left(\boldsymbol{I} + \sum_{z=0}^{i+1-1} \boldsymbol{Q}_{i+1-z}{}^{\nu}\boldsymbol{H}_{z,z}{}^{\nu}\boldsymbol{R}^{i+1-z}\right){}^{\nu}\boldsymbol{R}^n \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left(\boldsymbol{I} + \sum_{z=0}^{i+1-1} {}^{\nu}\boldsymbol{Q}_{i+1-z}\boldsymbol{H}_{z,i+1}\right){}^{\nu}\boldsymbol{R}^n \\
&= {}^{\nu}\boldsymbol{H}_{i+1,i+1}{}^{\nu}\boldsymbol{R}^n.
\end{aligned}
$$

Thus, we have proven that (2.3.7) holds true for all $i \geq 0$.

With (2.3.9), we can derive another recursive formula for ${}^{\nu}\boldsymbol{H}_{i,l}$, $l \leq i$, which is given by

$$
{}^{\nu}\boldsymbol{H}_{i,l} = {}^{\nu}\boldsymbol{H}_{i-l,0} + {}^{\nu}\boldsymbol{H}_{i,l-1}{}^{\nu}\boldsymbol{R}. \tag{2.3.10}
$$

To prove (2.3.10), we first have from (2.3.8) and (2.3.9)

$$
\begin{aligned}
{}^{\nu}\boldsymbol{H}_{i,i} &= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left(\boldsymbol{I} + \sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z}{}^{\nu}\boldsymbol{H}_{z,i}\right) \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} + \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1}\left(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z}{}^{\nu}\boldsymbol{H}_{z,i-1}\right){}^{\nu}\boldsymbol{R} \\
&= {}^{\nu}\boldsymbol{H}_{0,0} + {}^{\nu}\boldsymbol{H}_{i,i-1}{}^{\nu}\boldsymbol{R},\ i \geq 0. \qquad (2.3.11)
\end{aligned}
$$

Now, suppose that for fixed $l$, (2.3.10) holds true for levels $k = l, l+1, \ldots, i-1$. Then, we have

$$
\begin{aligned}
{}^{\nu}\boldsymbol{H}_{i,l} &= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l}\right) \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=l}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z}\left({}^{\nu}\boldsymbol{H}_{z-l,0} + {}^{\nu}\boldsymbol{H}_{z,l-1} {}^{\nu}\boldsymbol{R}\right)\right) + \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{l-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l-1} {}^{\nu}\boldsymbol{R}\right) \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=l}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z-l,0}\right) + \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l-1}\right) {}^{\nu}\boldsymbol{R} \\
&= \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{i-l-1} {}^{\nu}\boldsymbol{Q}_{i-l-z} {}^{\nu}\boldsymbol{H}_{z,0}\right) + \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{Q}_0\right)^{-1} \left(\sum_{z=0}^{i-1} {}^{\nu}\boldsymbol{Q}_{i-z} {}^{\nu}\boldsymbol{H}_{z,l-1}\right) {}^{\nu}\boldsymbol{R} \\
&= {}^{\nu}\boldsymbol{H}_{i-l,0} + {}^{\nu}\boldsymbol{H}_{i,l-1} {}^{\nu}\boldsymbol{R},
\end{aligned}
\tag{2.3.12}
$$

where the first equality follows from (2.3.8), and the second equality from the induction hypothesis and (2.3.9). Therefore, with (2.3.11) and (2.3.12), we have proven that (2.3.10) holds true for all $i \geq 0$.

In most cases, we are interested in computing ${}^{\nu}\boldsymbol{H}_{i,l}$, $\forall\, l \geq 0$, for a given value of $i \geq 0$. To do this in a computationally more efficient way than (2.3.8), one can compute ${}^{\nu}\boldsymbol{R}$, $\{{}^{\nu}\boldsymbol{Q}_z\}_{z=0}^{\infty}$, and $\{{}^{\nu}\boldsymbol{H}_{l,0}\}_{l=0}^{\infty}$ (with truncation), and subsequently apply (2.3.9) and (2.3.10). This results in a greatly improved computational procedure compared to directly applying (2.3.8) for all values of $l \geq 0$.

### 2.3.3 Fundamental matrix via Method I: Matrix-geometric claim size distribution

Assume that $Y^{(i,j)}|(I_t = 1, J_t = j, J_{t-1} = i)$ is independent of $j$ and let $Y^{(i)}|(I_t = 1, J_{t-1} = i)$ denote the claim size random variable at time $t$ which follow a matrix-geometric distribution. In particular, $f_i(y) = \boldsymbol{\alpha}_i \boldsymbol{\Gamma}_i^{y-1} \boldsymbol{\gamma}_i^{\intercal}$, $i \in \{0, 1, 2, \ldots, m-1\}$, where $\boldsymbol{\alpha}_i$ is a $1 \times m_i$ row vector, $\boldsymbol{\Gamma}_i$ is a $m_i \times m_i$ square matrix, $\boldsymbol{\gamma}_i$ is a $1 \times m_i$ row vector, $\boldsymbol{x}^{\intercal}$ denotes the transpose operator, and $m_i \in \mathbb{Z}^+$. We remark that $\boldsymbol{\alpha}_i \boldsymbol{\Gamma}_i^{y-1} \boldsymbol{\gamma}_i^\intercal$ is a valid and proper pmf for $y \in \mathbb{Z}^+$. We still leave the premium rate as general, i.e., $c \in \mathbb{Z}^+$.

Then, we have

$$\boldsymbol{A}_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ c-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & c-1 \end{array} \\ \begin{pmatrix} \boldsymbol{P}_0 & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \mathrm{diag}(1)\boldsymbol{P}_1 & \boldsymbol{P}_0 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \mathrm{diag}(2)\boldsymbol{P}_1 & \mathrm{diag}(1)\boldsymbol{P}_1 & \boldsymbol{P}_0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathrm{diag}(c-1)\boldsymbol{P}_1 & \mathrm{diag}(c-2)\boldsymbol{P}_1 & \mathrm{diag}(c-3)\boldsymbol{P}_1 & \cdots & \boldsymbol{P}_0 \end{pmatrix} \end{array},$$

where diag$(k)$ denotes an $m \times m$ diagonal matrix whose $(i,i)$-th element is given by $\boldsymbol{\alpha}_i \boldsymbol{\Gamma}_i^{k-1} \boldsymbol{\gamma}_i^\intercal$, $i \in \{0, 1, 2, \ldots, m-1\}$. Similarly,

$$\boldsymbol{A}_n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ c-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & c-1 \end{array} \\ \begin{pmatrix} \mathrm{diag}(cn)\boldsymbol{P}_1 & \mathrm{diag}(cn-1)\boldsymbol{P}_1 & \mathrm{diag}(cn-2)\boldsymbol{P}_1 & \cdots & \mathrm{diag}(cn+1-c)\boldsymbol{P}_1 \\ \mathrm{diag}(cn+1)\boldsymbol{P}_1 & \mathrm{diag}(cn)\boldsymbol{P}_1 & \mathrm{diag}(cn-1)\boldsymbol{P}_1 & \cdots & \mathrm{diag}(cn+2-c)\boldsymbol{P}_1 \\ \mathrm{diag}(cn+2)\boldsymbol{P}_1 & \mathrm{diag}(cn+1)\boldsymbol{P}_1 & \mathrm{diag}(cn)\boldsymbol{P}_1 & \cdots & \mathrm{diag}(cn+3-c)\boldsymbol{P}_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathrm{diag}(cn+c-1)\boldsymbol{P}_1 & \mathrm{diag}(cn+c-2)\boldsymbol{P}_1 & \mathrm{diag}(cn+c-3)\boldsymbol{P}_1 & \cdots & \mathrm{diag}(cn)\boldsymbol{P}_1 \end{pmatrix} \end{array}, \; n \in \mathbb{Z}^+.$$

It is possible to rewrite $\boldsymbol{A}_n$ in such a way that the infinite series in (2.3.3) can be avoided. To this end, we introduce a number of matrices. Let

$$\boldsymbol{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ m-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & m-1 \end{array} \\ \begin{pmatrix} \boldsymbol{\alpha}_1 & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\alpha}_2 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\alpha}_3 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\alpha}_{m-1} \end{pmatrix} \end{array}$$

and

$$\boldsymbol{\Xi} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ c-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & c-1 \end{array} \\ \begin{pmatrix} \boldsymbol{\xi}_c \boldsymbol{P}_1 & \boldsymbol{\xi}_{c-1} \boldsymbol{P}_1 & \boldsymbol{\xi}_{c-2} \boldsymbol{P}_1 & \cdots & \boldsymbol{\xi}_1 \boldsymbol{P}_1 \\ \boldsymbol{\xi}_{c+1} \boldsymbol{P}_1 & \boldsymbol{\xi}_c \boldsymbol{P}_1 & \boldsymbol{\xi}_{c-1} \boldsymbol{P}_1 & \cdots & \boldsymbol{\xi}_2 \boldsymbol{P}_1 \\ \boldsymbol{\xi}_{c+2} \boldsymbol{P}_1 & \boldsymbol{\xi}_{c+1} \boldsymbol{P}_1 & \boldsymbol{\xi}_c \boldsymbol{P}_1 & \cdots & \boldsymbol{\xi}_3 P_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{\xi}_{2c-1} \boldsymbol{P}_1 & \boldsymbol{\xi}_{2c-2} \boldsymbol{P}_1 & \boldsymbol{\xi}_{2c-3} \boldsymbol{P}_1 & \cdots & \boldsymbol{\xi}_c \boldsymbol{P}_1 \end{pmatrix} \end{array},$$

where

$$\boldsymbol{\xi}_k = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ m-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & m-1 \end{array} \\ \begin{pmatrix} \boldsymbol{\Gamma}_1^{k-1} \boldsymbol{\gamma}_1^{\intercal} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_2^{k-1} \boldsymbol{\gamma}_2^{\intercal} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_3^{k-1} \boldsymbol{\gamma}_3^{\intercal} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Gamma}_{m-1}^{k-1} \boldsymbol{\gamma}_{m-1}^{\intercal} \end{pmatrix} \end{array}, \; k \in \mathbb{Z}^+.$$

Finally, let

$$\operatorname{diag}(\Gamma) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ m-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & m-1 \end{array} \\ \begin{pmatrix} \boldsymbol{\Gamma}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_3 & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Gamma}_{m-1} \end{pmatrix} \end{array}.$$

Note that the size of $\boldsymbol{A}$ is $m \times m^*$, $\boldsymbol{\xi}_k$ is $m^* \times m$, $\boldsymbol{\Xi}$ is $cm^* \times cm$, and $\operatorname{diag}(\boldsymbol{\Gamma})$ is $m^* \times m^*$, where $m^* = \sum_{i=0}^{m-1} m_i$. Letting $\boldsymbol{I}_k$ denote the identity matrix of size $k$, we can rewrite $\boldsymbol{A}_n$

as

$$\boldsymbol{A}_n = (\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \text{diag}(\boldsymbol{\Gamma})^{c(n-1)})\boldsymbol{\Xi},\ n \in \mathbb{Z}^+, \tag{2.3.13}$$

where $\otimes$ denotes the well-known matrix Kronecker product operator (see e.g., Bernstein (2005)).

Revisiting (2.3.3), we can rewrite the matrix equation as

$$vec(\boldsymbol{X}) = vec(\boldsymbol{A}_0)\nu + \sum_{n=1}^{\infty} vec(\boldsymbol{X}^n \boldsymbol{A}_n)\nu, \tag{2.3.14}$$

where $vec(\boldsymbol{X})$ denotes the vectorization operator (see e.g., Bernstein (2005)). Applying (2.3.13) to (2.3.14) ultimately yields, following some matrix algebra:

$$\begin{aligned}
&vec(\boldsymbol{X})\\
&= vec(\boldsymbol{A}_0)\nu + \sum_{n=1}^{\infty} vec(\boldsymbol{X}^n(\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \text{diag}(\boldsymbol{\Gamma})^{c(n-1)})\boldsymbol{\Xi})\nu\\
&= vec(\boldsymbol{A}_0)\nu + \sum_{n=1}^{\infty} \Big(\big((\boldsymbol{I}_c \otimes \text{diag}(\boldsymbol{\Gamma}))^{c(n-1)}\boldsymbol{\Xi}\big)^{\intercal} \otimes \boldsymbol{X}^{n-1}\Big) vec\big(\boldsymbol{X}(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu\\
&= vec(\boldsymbol{A}_0)\nu + \sum_{n=1}^{\infty} \Big(\big(\boldsymbol{\Xi}^{\intercal}(\boldsymbol{I}_c \otimes \text{diag}(\boldsymbol{\Gamma}^{\intercal}))^{c(n-1)}\big) \otimes \boldsymbol{X}^{n-1}\Big) vec\big(\boldsymbol{X}(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu\\
&= vec(\boldsymbol{A}_0)\nu + \sum_{n=1}^{\infty} \big(\boldsymbol{\Xi}^{\intercal} \otimes \boldsymbol{I}_{cm}\big)\big((\boldsymbol{I}_c \otimes \text{diag}(\boldsymbol{\Gamma}^{\intercal}))^{c(n-1)} \otimes X^{n-1}\big) vec\big(\boldsymbol{X}(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu\\
&= vec(\boldsymbol{A}_0)\nu + \big(\boldsymbol{\Xi}^{\intercal} \otimes \boldsymbol{I}_{cm}\big) \sum_{n=1}^{\infty} \big((\boldsymbol{I}_c \otimes \text{diag}(\Gamma^{\intercal}))^{c} \otimes \boldsymbol{X}\big)^{n-1} vec\big(\boldsymbol{X}(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu,
\end{aligned} \tag{2.3.15}$$

where the second equality follows from a property of the vectorization operator, the third from the transposition property of matrix products and Kronecker products, and the fourth and fifth from the mixed-product property of Kronecker products (see e.g., Bernstein (2005)). Therefore, as in (2.3.4), the iterative scheme to solve for the discounted rate matrix follows as

$$
\begin{aligned}
&vec({}^{\nu}\boldsymbol{R}(k+1)) \\
&= vec(\boldsymbol{A}_0)\nu + \big(\boldsymbol{\Xi}^{\intercal} \otimes \boldsymbol{I}_{cm}\big) \sum_{n=1}^{\infty} \big((\boldsymbol{I}_c \otimes \mathrm{diag}(\Gamma^{\intercal}))^c \otimes {}^{\nu}\boldsymbol{R}(k)\big)^{n-1} vec\big({}^{\nu}\boldsymbol{R}(k)(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu \\
&= vec(\boldsymbol{A}_0)\nu + \big(\boldsymbol{\Xi}^{\intercal} \otimes \boldsymbol{I}_{cm}\big)\big(\boldsymbol{I}_{c^2mm^*} - (\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma}^{\intercal}))^c \otimes {}^{\nu}\boldsymbol{R}(k)\big)^{-1} vec\big({}^{\nu}\boldsymbol{R}(k)(\boldsymbol{I}_c \otimes \boldsymbol{A})\big)\nu,
\end{aligned}
\tag{2.3.16}
$$

where we set ${}^{\nu}\boldsymbol{R}(0) = \boldsymbol{0}$. Once again, noting that $\|{}^{\nu}\boldsymbol{R}^n(k)\|_{\max} \leq \|{}^{1}\boldsymbol{R}^n\|_{\max}$ for all $n, k \in \mathbb{N}$, the infinite series on the first line of (2.3.16) converges for all $k \in \mathbb{N}$, and hence, the inverse matrix on the second line of (2.3.16) is valid.

Now, recalling (2.3.6) and proceeding similarly as in (2.3.15), we obtain

$$
\begin{aligned}
&vec({}^{\nu}\boldsymbol{Q}_n) \\
&= \nu \sum_{a=0}^{\infty} vec\big({}^{\nu}\boldsymbol{R}^a \boldsymbol{A}_{n+1+a}\big) \\
&= \nu \sum_{a=0}^{\infty} vec\big({}^{\nu}\boldsymbol{R}^a (\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{c(n+a)})\boldsymbol{\Xi}\big) \\
&= \nu \sum_{a=0}^{\infty} vec\big({}^{\nu}\boldsymbol{R}^a (\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{cn})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{ca})\boldsymbol{\Xi}\big) \\
&= \nu \sum_{a=0}^{\infty} \Big(\big((\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma}))^{ca}\boldsymbol{\Xi}\big)^{\intercal} \otimes {}^{\nu}\boldsymbol{R}^a\Big) vec\big((\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{cn})\big) \\
&= \nu \sum_{a=0}^{\infty} \Big(\big(\boldsymbol{\Xi}^{\intercal}(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma}^{\intercal}))^{ca}\big) \otimes {}^{\nu}\boldsymbol{R}^a\Big) vec\big((\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{cn})\big) \\
&= \nu\big(\boldsymbol{\Xi}^{\intercal} \otimes I_{cm}\big) \sum_{a=0}^{\infty} \big((\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma}^{\intercal}))^c \otimes {}^{\nu}\boldsymbol{R}\big)^a vec\big((\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{cn})\big) \\
&= \nu\big(\boldsymbol{\Xi}^{\intercal} \otimes \boldsymbol{I}_{cm}\big)\big(\boldsymbol{I}_{c^2mm^*} - (\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma}^{\intercal}))^c \otimes {}^{\nu}\boldsymbol{R}\big)^{-1} vec\big((\boldsymbol{I}_c \otimes \boldsymbol{A})(\boldsymbol{I}_c \otimes \mathrm{diag}(\boldsymbol{\Gamma})^{cn})\big), \ n \in \mathbb{N}.
\end{aligned}
\tag{2.3.17}
$$

Hence, one has explicit expressions for $\{{}^{\nu}\boldsymbol{Q}_n\}_{n\in\mathbb{N}}$. Finally, for computing the discounted fundamental matrices, one can follow the same procedure as outlined in Subection 2.3.2.

**Remark 2.3.1:** An explicit expression for the discounted fundamental matrix in terms of $\{{}^{\nu}\boldsymbol{Q}_n\}_{n\in\mathbb{N}}$ and ${}^{\nu}\boldsymbol{R}$ is also available by recursively solving (2.3.8). However, this explicit solution is not computationally more advantageous, but rather, it adds complexity to the implementation of the computational procedure. Further simplification of the explicit expression for the discounted fundamental matrix is perhaps possible, but we have not found one yet.

## 2.4 MAP risk model with dividend barrier

The so-called dividend problem in insurance risk theory was first introduced by de Finetti (1957), where de Finetti introduced a barrier-based dividend payment strategy. In this section, we study the MAP risk model with the same barrier strategy introduced by de Finetti, and show that the MAP risk model with the constant dividend barrier belongs to the G/M/1 DTRM class.

A MAP risk model with a constant dividend barrier $b \in \mathbb{N}$ is comprised of a discrete-time MAP $\{(N_t, J_t), t \in \mathbb{N}\}$ with $m$ phases, TPMs $(\boldsymbol{P}_0, \boldsymbol{P}_1)$, and the conditionally i.i.d. claim amount per period sequence $\{Y_t, t \in \mathbb{Z}^+\}$ (conditional on the phase process $\{J_t, t \in \mathbb{N}\}$ of the MAP). In particular, let $f_{i,j}(y)$, $y \in \mathbb{Z}^+$, denote the pmf of $Y^{(i,j)} = Y_t|(I_t = 1, J_t = j, J_{t-1} = i)\ \forall\ t \in \mathbb{Z}^+$, where $\{I_t, t \in \mathbb{Z}^+\}$ is a sequence of Bernoulli random variables which are equal to 1 when there is an arrival at time $t$ in the underlying MAP. Furthermore, we assume that premiums are received at a constant (deterministic) rate $c \in \mathbb{Z}^+$ per unit time. Then, for $u \in \{0, 1, 2, \ldots, b\}$, we can recursively define the surplus process as

$$U_t^{(b)} = \min\big(U_{t-1}^{(b)} + c - Y_t, b\big),\ t \in \mathbb{Z}^+,$$

where $U_0^{(b)} = u$.

Let $X_t^{(b)} = \lfloor \frac{U_t^{(b)}}{c} \rfloor$ and $V_t^{(b)} = U_t^{(b)} \bmod c$. Letting $X_t^{(b)}$ represent the level of the chain, $\{(X_t^{(b)}, V_t^{(b)}, J_t), t \in \mathbb{N}\}$ is clearly a dual G/M/1-type chain of $\{(U_t^{(b)}, J_t), t \in \mathbb{N}\}$ with (one-step) transition probabilities given by

$$\begin{aligned}
&(\boldsymbol{A}_{i,l})_{(j,v),(a,x)} \\
&\quad = \Pr\{(X_1^{(b)}, V_1^{(b)}, J_1) = (l, a, x) | (X_0^{(b)}, V_0^{(b)}, J_0) = (i, j, v)\} \mathcal{I}[0 \le ci + j \le b, 0 \le cl + a \le b] \\
&\quad = \Pr\{(U_1^{(b)}, J_1) = (cl + a, x) | (U_0^{(b)}, J_0) = (ci + j, v)\} \mathcal{I}[0 \le ci + j \le b, 0 \le cl + a \le b] \\
&\quad = \Pr\{(Y_1, J_1) = (ci + j + c - (cl + a), x) | J_0 = v\} \mathcal{I}[0 \le ci + j \le b, 0 \le cl + a < b] \\
&\qquad + \sum_{k=0}^{ci+j+c-b} \Pr\{(Y_1, J_1) = (k, x) | J_0 = v\} \mathcal{I}[0 \le ci + j \le b, cl + a = b] \\
&\quad = p_{0;v,x} \mathcal{I}[0 \le ci + j \le b, 0 \le cl + a \le b, ci + j + c - (cl + a) = 0] \\
&\qquad + p_{1;v,x} f_{v,x}(ci + j + c - (cl + a)) \mathcal{I}[0 \le ci + j \le b, 0 \le cl + a < b] \\
&\qquad + p_{1;v,x} \sum_{k=1}^{ci+j+c-b} f_{v,x}(k) \mathcal{I}[0 \le ci + j \le b, cl + a = b].
\end{aligned} \tag{2.4.1}$$

Unlike the MAP risk model without the dividend barrier, the one-step transition probabilities of the dual G/M/1-type chain of the MAP risk model with the dividend barrier are level-independent through levels 0 up to $B - 2$ and becomes level-dependent at level $B - 1$, where we set $B = \lfloor \frac{b}{c} \rfloor$. More specifically, we can write $\boldsymbol{A}_{i,j} = \boldsymbol{A}_{i-j+1}$ (due to the level independence) for $i = 0, 1, 2, \ldots, B$ and $j$ such that $|i - j| \ge 0$, $\boldsymbol{A}_{i,i+1} = \boldsymbol{A}_0$ for $i = 0, 1, 2, \ldots, B - 2$, and $\boldsymbol{A}_{B-1,B}$ and $\boldsymbol{A}_{B,B}$ are not necessarily the same substochastic matrices as $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$.

### 2.4.1 Time of ruin, surplus prior to ruin, and deficit at ruin

As in the MAP risk model without the dividend barrier, the time of ruin can be written as $\tau = \inf\{t \in \mathbb{Z}^+ : X_t^{(b)} < 0\}$. Therefore, the absorbing class is again $\mathcal{A} = \mathbb{Z}^- \times \{0, 1, 2, \ldots, c-1\} \times \{0, 1, 2, \ldots, m-1\}$. Let

$$
{}^{\mathcal{A}^c}_{\;b}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ B-1 \\ B \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & B \end{array} \\
\begin{pmatrix}
\boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots & \cdots & \cdots \\
\boldsymbol{A}_3 & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots & \ddots & \vdots \\
\boldsymbol{A}_B & \boldsymbol{A}_{B-1} & \cdots & \cdots & \cdots & \boldsymbol{A}_{B-1,B} \\
\boldsymbol{A}_{B+1} & \boldsymbol{A}_B & \cdots & \cdots & \cdots & \boldsymbol{A}_{B,B}
\end{pmatrix}
\end{array}
$$

denote the TPM of the open class of $\{(X_t^{(b)}, V_t^{(b)}, J_t), t \in \mathbb{N}\}$, and let

$$
{}^{\mathcal{A}}_{b}\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ B \end{array}
\begin{array}{c}
\begin{array}{cccc} \cdots & -3 & -2 & -1 \end{array} \\
\begin{pmatrix}
\cdots & \boldsymbol{A}_4 & \boldsymbol{A}_3 & \boldsymbol{A}_2 \\
\cdots & \boldsymbol{A}_5 & \boldsymbol{A}_4 & \boldsymbol{A}_3 \\
\cdots & \boldsymbol{A}_6 & \boldsymbol{A}_5 & \boldsymbol{A}_4 \\
\iddots & \vdots & \vdots & \vdots \\
\cdots & \boldsymbol{A}_{B+4} & \boldsymbol{A}_{B+3} & \boldsymbol{A}_{B+2}
\end{pmatrix}
\end{array}
$$

denote the TPM corresponding to state transitions from $\mathcal{A}^c$ to $\mathcal{A}$. Then, employing the same derivation used in (2.3.2), the conditional joint pmf of the time of ruin, surplus prior to ruin, and deficit at ruin is given by

$$\Pr\{\tau = t, (U_{\tau-1}^{(b)}, J_{\tau-1}) = (a, v), (U_{\tau}^{(b)}, J_{\tau}) = (l, j) | (U_0^{(b)}, J_0) = (u, i)\} = \\ \left({}^{\mathcal{A}^c}_{b}\boldsymbol{P}^{t-1}_{\lfloor \frac{u}{c} \rfloor, \lfloor \frac{a}{c} \rfloor}\right)_{(u \bmod c, j), (a \bmod c, v)} \left({}^{\mathcal{A}}_{b}\boldsymbol{P}_{\lfloor \frac{a}{c} \rfloor, \lfloor \frac{l}{c} \rfloor}\right)_{(a \bmod c, v), (l \bmod c, b)}. \quad (2.4.2)$$

## 2.4.2 Total discounted dividends paid

One of the key quantities of interest in this subsection is the total discounted dividends paid prior to a deterministic time point or the time of ruin. First of all, we assume that the insurance company no longer operates once it enters into the ruined state, and hence, no further dividend payments are made after the company is ruined.

We first consider the expected total discounted dividends paid prior to either a fixed (finite) time point or ruin, whichever happens first, and let this fixed finite time point be denoted by $T \in \mathbb{Z}^+$. Let

$$\mathcal{D}_t = \max\left(U_{t-1}^{(b)} + c - Y_t - b, 0\right), \ t \in \mathbb{Z}^+,$$

and

$$\mathcal{D}_T^{\text{Tot}}(\nu) = \sum_{t=1}^{T} \nu^t \mathcal{D}_t \mathcal{I}[\tau > t]$$

denote the total discounted dividends paid up to $T$ for a discount factor $\nu \in (0, 1)$. Writing $\mathcal{D}_t$ in terms of the dual G/M/1-type chain, we have

$$\mathcal{D}_t = \max\left(cX_{t-1}^{(b)} + V_{t-1}^{(b)} + c - Y_t - b, 0\right), \ t \in \mathbb{Z}^+.$$

Following this, we have

$$E\big\{\mathcal{D}_T^{\text{Tot}}(\nu)\big|(U_0^{(b)},J_0)=(u,i)\big\} = \sum_{t=1}^{T}\nu^t E\big\{\mathcal{D}_t\mathcal{I}[\tau>t]\big|(U_0^{(b)},J_0)=(u,i)\big\}$$
$$= \sum_{t=1}^{T}\nu^t E\big\{\mathcal{D}_t\mathcal{I}[\tau>t]\big|(X_0^{(b)},V_0^{(b)},J_0)=(\lfloor\frac{u}{c}\rfloor, u \bmod c, j)\big\}.$$

From (2.3.1) and (2.4.1), and by also conditioning on $(X_{t-1}^{(b)},V_{t-1}^{(b)},J_{t-1})$, we obtain

$$E\big\{\mathcal{D}_t\mathcal{I}[\tau>t]\big|(X_0^{(b)},V_0^{(b)},J_0)=(\lfloor\frac{u}{c}\rfloor, u \bmod c, j)\big\}$$
$$= \sum_{z=B-1}^{B}\sum_{x=0}^{c-1}\sum_{v=0}^{m-1}\sum_{l=0}^{c-1}\sum_{w=0}^{m-1}\max(cB+l-b,0)\Big({}^{\mathcal{A}^c}_{\;b}\boldsymbol{P}^{t-1}_{\lfloor\frac{u}{c}\rfloor,z}\Big)_{(u \bmod c,j),(x,v)}\Big({}^{\mathcal{A}^c}_{\;b}\boldsymbol{P}_{z,B}\Big)_{(x,v),(l,w)}. \tag{2.4.3}$$

Therefore, we ultimately arrive at

$$E\big\{\mathcal{D}_T^{\text{Tot}}(\nu)\big|(U_0^{(b)},J_0)=(u,i)\big\}$$
$$= \sum_{t=1}^{T}\nu^t\sum_{z=B-1}^{B}\sum_{x=0}^{c-1}\sum_{v=0}^{m-1}\sum_{l=0}^{c-1}\sum_{w=0}^{m-1}\max(cB+l-b,0)\Big({}^{\mathcal{A}^c}_{\;b}\boldsymbol{P}^{t-1}_{\lfloor\frac{u}{c}\rfloor,z}\Big)_{(u \bmod c,j),(x,v)}\Big({}^{\mathcal{A}^c}_{\;b}\boldsymbol{P}_{z,B}\Big)_{(x,v),(l,w)}. \tag{2.4.4}$$

Secondly, consider

$$\mathcal{D}^{\text{Tot}}(\nu) = \sum_{t=1}^{\tau-1}\nu^t\mathcal{D}_t\mathcal{I}[\tau<\infty] = \sum_{t=1}^{\infty}\nu^t\mathcal{D}_t\mathcal{I}[\tau>t],$$

which actually represents the total discounted dividends paid prior to ruin. (Note that the event of ruin is certain (w.p. 1) to occur with the dividend barrier $b$ in place.) By the dominated convergence theorem (DCT) (see e.g., Resnick (2005)), we have

$$E\big\{\mathcal{D}^{\text{Tot}}(\nu)\big|(U_0^{(b)},J_0)=(u,i)\big\} = \sum_{t=1}^{\infty}\nu^t E\big\{\mathcal{D}_t\mathcal{I}[\tau>t]\big|(U_0^{(b)},J_0)=(u,i)\big\}.$$

Then, from (2.4.3), it follows that

$$
\begin{aligned}
&E\{\mathcal{D}^{\mathrm{Tot}}(\nu)\big|(U_0^{(b)}, J_0) = (u,i)\} \\
&= \sum_{t=1}^{\infty} \nu^t \sum_{z=B-1}^{B} \sum_{x=0}^{c-1} \sum_{v=0}^{m-1} \sum_{l=0}^{c-1} \sum_{w=0}^{m-1} \max(cB+l-b,0) \left( {}^{\mathcal{A}^c}_{b}\boldsymbol{P}^{t-1}_{\lfloor \frac{u}{c} \rfloor, z} \right)_{(u \bmod c, j),(x,v)} \left( {}^{\mathcal{A}^c}_{b}\boldsymbol{P}_{z,B} \right)_{(x,v),(l,w)} \\
&= \nu \sum_{z=B-1}^{B} \sum_{x=0}^{c-1} \sum_{v=0}^{m-1} \sum_{l=0}^{c-1} \sum_{w=0}^{m-1} \max(cB+l-b,0) \sum_{t=0}^{\infty} \nu^t \left( {}^{\mathcal{A}^c}_{b}\boldsymbol{P}^{t}_{\lfloor \frac{u}{c} \rfloor, z} \right)_{(u \bmod c, j),(x,v)} \left( {}^{\mathcal{A}^c}_{b}\boldsymbol{P}_{z,B} \right)_{(x,v),(l,w)} \\
&= \nu \sum_{z=B-1}^{B} \sum_{x=0}^{c-1} \sum_{v=0}^{m-1} \sum_{l=0}^{c-1} \sum_{w=0}^{m-1} \max(cB+l-b,0) \left( {}^{b,\nu}\boldsymbol{H}_{\lfloor \frac{u}{c} \rfloor, z} \right)_{(u \bmod c, j),(x,v)} \left( {}^{\mathcal{A}^c}_{b}\boldsymbol{P}_{z,B} \right)_{(x,v),(l,w)},
\end{aligned}
\tag{2.4.5}
$$

where

$$
{}^{b,\nu}\boldsymbol{H}_{\lfloor \frac{u}{c} \rfloor, z} = \sum_{t=0}^{\infty} \nu^t \, {}^{\mathcal{A}^c}_{b}\boldsymbol{P}^{t}_{\lfloor \frac{u}{c} \rfloor, z}
$$

and ${}^{b,\nu}\boldsymbol{H}$ is the discounted fundamental matrix of the dual G/M/1-type chain of the MAP risk model with the dividend barrier $b$.

As can be seen from (2.4.5), the discounted fundamental matrix plays, yet again, a critical role in calculating the expected total discounted dividends paid prior to ruin. Computing the discounted fundamental matrix for the dual G/M/1-type chain of the MAP risk model with the dividend barrier, however, is not as straightforward as in the MAP risk model without the dividend barrier. As a result of level dependency in the one-step transition probabilities of the dual G/M/1-type chain of the MAP risk model with the dividend barrier, the discounted rate and ladder height matrices are also level dependent, unlike the MAP risk model without the dividend barrier. This presence of level dependency really adds on the computational cost. As such, Methods I and II discussed earlier do not necessarily produce the most efficient algorithms to use. Therefore, we introduce a computationally more superior (in most cases) algorithm later on in this section. Nevertheless, we first provide brief discussions on Methods I and II to demonstrate how the algorithms developed for the general G/M/1-type discrete-time risk model can be applied to the MAP risk model with a dividend barrier.

### 2.4.3 Fundamental matrix via Method I

Let $\{{}^{b,\nu}\boldsymbol{R}_{i,l}\}_{i=0,l\geq i}^{B}$, $\{{}^{b,\nu}\boldsymbol{Q}_{i,l}\}_{i=0,l\leq i}^{B}$, and $\{{}^{b,\nu}\boldsymbol{G}_{i,l}\}_{i,l=0}^{B}$ denote the discounted rate, ladder height distribution, and fundamental period matrices of the dual G/M/1-type chain of the MAP risk model with a constant dividend barrier $b \in \mathbb{N}$. From Section 2.2,

$$\begin{aligned} {}^{b,\nu}\boldsymbol{R}_{i,i+1} &= \sum_{n=0}^{\infty} {}^{b,\nu}\boldsymbol{R}_{i,i+n}\nu\boldsymbol{A}_{i+n,i+1} \\ &= \sum_{n=0}^{B-i} {}^{b,\nu}\boldsymbol{R}_{i,i+n}\nu\boldsymbol{A}_{i+n,i+1},\ 0 \leq i \leq B-1. \end{aligned} \tag{2.4.6}$$

Solving for ${}^{b,\nu}\boldsymbol{R}_{i,i+1}$ in (2.4.6) yields

$${}^{b,\nu}\boldsymbol{R}_{i,i+1} = \begin{cases} \nu\boldsymbol{A}_{i,i+1}\Big(\boldsymbol{I} - \sum_{n=1}^{B-i} {}^{b,\nu}\boldsymbol{R}_{i+1,i+n}\nu\boldsymbol{A}_{i+n,i+1}\Big)^{-1}, & 0 \leq i \leq B-1, \\ \boldsymbol{0}, & i \geq B. \end{cases}$$

Noting that ${}^{b,\nu}\boldsymbol{R}_{i,i+n} = \prod_{k=i}^{i+n-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1}$, we have

$${}^{b,\nu}\boldsymbol{R}_{i,i+1} = \begin{cases} \nu\boldsymbol{A}_{i,i+1}\Big(\boldsymbol{I} - \sum_{n=1}^{B-i}\prod_{k=i+1}^{i+n-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1}\nu\boldsymbol{A}_{i+n,i+1}\Big)^{-1}, & 0 \leq i \leq B-1, \\ \boldsymbol{0}, & i \geq B. \end{cases} \tag{2.4.7}$$

Thus, we can recursively compute the discounted rate matrices by first computing ${}^{b,\nu}\boldsymbol{R}_{B-1,B}$. Note that unlike the MAP risk model without the dividend barrier, we can compute the discounted rate matrices exactly and without having to truncate the matrices $\{\boldsymbol{A}_{i,l}\}_{l\leq i+1}$.

Once we have computed the discounted rate matrices, we can compute the discounted ladder height distribution matrices $\{{}^{b,\nu}\boldsymbol{Q}_{i,l}\}_{i,l=0,l\leq i}^{B}$ via

$$\begin{aligned}
{}^{b,\nu}\boldsymbol{Q}_{i,l} &= \sum_{n=0}^{B-i} {}^{b,\nu}\boldsymbol{R}_{i,i+n}\nu\boldsymbol{A}_{i+n,l} \\
&= \sum_{n=0}^{B-i}\prod_{k=i}^{i+n-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1}\nu\boldsymbol{A}_{i+n,l}, \; 0 \leq i \leq B-1,
\end{aligned} \tag{2.4.8}$$

and

$${}^{b,\nu}\boldsymbol{Q}_{B,l} = \nu\boldsymbol{A}_{B,l}. \tag{2.4.9}$$

Once again, we exploit a recursive relationship that holds for ${}^{b,\nu}\boldsymbol{Q}_{i,l}$. From (2.4.8), we have

$$\begin{aligned}
{}^{b,\nu}\boldsymbol{Q}_{i,l} &= \nu\boldsymbol{A}_{i,l} + {}^{b,\nu}\boldsymbol{R}_{i,i+1}\sum_{n=1}^{B-i}\prod_{k=i+1}^{i+n-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1}\nu\boldsymbol{A}_{i+n,l} \\
&= \nu\boldsymbol{A}_{i,l} + {}^{b,\nu}\boldsymbol{R}_{i,i+1}\sum_{n=0}^{B-i}\prod_{k=i+1}^{i+1+n-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1}\nu\boldsymbol{A}_{i+1+n,l} \\
&= \nu\boldsymbol{A}_{i,l} + {}^{b,\nu}\boldsymbol{R}_{i,i+1}\,{}^{b,\nu}\boldsymbol{Q}_{i+1,l}, \; 0 \leq i \leq B-1.
\end{aligned} \tag{2.4.10}$$

As a result, one can compute ${}^{b,\nu}\boldsymbol{Q}_{B,l}$ using (2.4.9) and then apply the recursive rule of (2.4.10) to compute the remaining discounted ladder height distribution matrices.

Finally, the block components of the discounted fundamental matrix are given by

$${}^{b,\nu}\boldsymbol{H}_{i,l} = \begin{cases} \left(I - {}^{b,\nu}\boldsymbol{Q}_{i,i}\right)^{-1}\left(\prod_{k=i}^{l-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1} + \sum_{z=0}^{i-1} {}^{b,\nu}\boldsymbol{Q}_{i,z}\,{}^{b,\nu}\boldsymbol{H}_{z,l}\right), & \text{if } l \geq i, \\ \left(I - {}^{b,\nu}\boldsymbol{Q}_{i,i}\right)^{-1}\left(\sum_{z=0}^{i-1} {}^{b,\nu}\boldsymbol{Q}_{i,z}\,{}^{b,\nu}\boldsymbol{H}_{z,l}\right), & \text{if } l < i. \end{cases} \tag{2.4.11}$$

Similarly, as in the case of the MAP risk model without the dividend barrier, we can recursively compute ${}^{b,\nu}\boldsymbol{H}_{i,l}$ from ${}^{b,\nu}\boldsymbol{H}_{i,0}$. Using the same mathematical inductive arguments employed in proving (2.3.9) and (2.3.10), we can show that

$$ {}^{b,\nu}\boldsymbol{H}_{i,l} = {}^{b,\nu}\boldsymbol{H}_{i,i} \prod_{k=i}^{l-1} {}^{b,\nu}\boldsymbol{R}_{k,k+1},\ l \geq i, \tag{2.4.12} $$

and

$$ {}^{b,\nu}\boldsymbol{H}_{i,l} = {}^{b,\nu}\boldsymbol{N}_{i,l} + {}^{b,\nu}\boldsymbol{H}_{i,l-1}\, {}^{b,\nu}\boldsymbol{R}_{l-1,l},\ l \leq i, \tag{2.4.13} $$

where

$$ {}^{b,\nu}\boldsymbol{N}_{i,l} = \left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{Q}_{i,i}\right)^{-1} \left(\sum_{z=l}^{i-1} {}^{b,\nu}\boldsymbol{Q}_{i,z}\, {}^{b,\nu}\boldsymbol{N}_{z,l}\right),\ l < i, \tag{2.4.14} $$

with ${}^{b,\nu}\boldsymbol{N}_{i,i} = (\boldsymbol{I} - {}^{b,\nu}\boldsymbol{Q}_{i,i})^{-1}$. Once again, note that (2.4.12) and (2.4.13) yield a more computationally efficient algorithm than (2.4.11) when computing ${}^{b,\nu}\boldsymbol{H}_{i,l}$ for more than one value of $l$.

### 2.4.4 Fundamental matrix via Method II

A close inspection of (2.4.5) reveals that one only needs to compute ${}^{b,\nu}\boldsymbol{H}_{\lfloor \frac{u}{c} \rfloor, B-1}$ and ${}^{b,\nu}\boldsymbol{H}_{\lfloor \frac{u}{c} \rfloor, B}$ to compute the expected total discounted dividends paid out prior to ruin. In this case, Method II can deliver a more efficient algorithm in comparison to Method I. Hence, for brevity in this subsection, we only discuss Method II for computing the expected total discounted dividends paid out prior to ruin.

First of all, from (2.2.30) and (2.2.33), we recursively compute $\{{}^{b,\nu}\boldsymbol{G}_{i,i+1}\}_{i=0}^{B-1}$ according to

$$ {}^{b,\nu}\boldsymbol{G}_{i,i+1} = \left(I - \sum_{z=0}^{i-1} \nu \boldsymbol{A}_{i,z} \prod_{k=z}^{i-1} {}^{b,\nu}\boldsymbol{G}_{k,k+1}\right)^{-1} \nu \boldsymbol{A}_{i,i+1},\ 0 \leq i \leq B-1, \tag{2.4.15} $$

and compute ${}^{b,\nu}\boldsymbol{G}_{B,B-1}$ via

$$ {}^{b,\nu}\boldsymbol{G}_{B,B-1} = \Big(\boldsymbol{I} - \nu\boldsymbol{A}_{B,B}\Big)^{-1}\Big(\sum_{z=0}^{B-2}\nu\boldsymbol{A}_{B,z}\prod_{k=z}^{B-2}{}^{b,\nu}\boldsymbol{G}_{k,k+1} + \nu\boldsymbol{A}_{B,B-1}\Big), \tag{2.4.16} $$

where we note that ${}^{b,\nu}\boldsymbol{Q}_{B,z} = \nu\boldsymbol{A}_{B,z}$, $z = 0, 1, 2, \ldots, B$. We next compute ${}^{b,\nu}\boldsymbol{G}_{B-1,B-1}$ and ${}^{b,\nu}\boldsymbol{G}_{B,B}$. For ${}^{b,\nu}\boldsymbol{G}_{B,B}$, from (2.2.33) and (2.4.9), we have

$$ \begin{aligned} {}^{b,\nu}\boldsymbol{G}_{B,B} &= \sum_{z=0}^{B-1}{}^{b,\nu}\boldsymbol{Q}_{B,z}\,{}^{b,\nu}\boldsymbol{G}_{z,B} + {}^{b,\nu}\boldsymbol{Q}_{B,B} \\ &= \sum_{z=0}^{B-1}\nu\boldsymbol{A}_{B,z}\prod_{k=z}^{B-1}{}^{b,\nu}\boldsymbol{G}_{k,k+1} + \nu\boldsymbol{A}_{B,B}. \end{aligned} \tag{2.4.17} $$

For ${}^{b,\nu}\boldsymbol{G}_{B-1,B-1}$, we first compute ${}^{b,\nu}\boldsymbol{R}_{B-1,B}$ via (2.4.7), which in this particular case reduces to

$$ {}^{b,\nu}\boldsymbol{R}_{B-1,B} = \nu\boldsymbol{A}_{B-1,B}\big(\boldsymbol{I} - \nu\boldsymbol{A}_{B-1,B}\big)^{-1}. \tag{2.4.18} $$

Now, from (2.4.10) and (2.4.18), we can easily compute $\{{}^{b,\nu}\boldsymbol{Q}_{B-1,z}\}_{z=0}^{B-1}$ using

$$ {}^{b,\nu}\boldsymbol{Q}_{B-1,z} = \nu\boldsymbol{A}_{B-1,z} + {}^{b,\nu}\boldsymbol{R}_{B-1,B}\nu\boldsymbol{A}_{B,z}, \tag{2.4.19} $$

and from (2.2.33), compute ${}^{b,\nu}\boldsymbol{G}_{B-1,B-1}$ via

$$ {}^{b,\nu}\boldsymbol{G}_{B-1,B-1} = \sum_{z=0}^{B-2}{}^{b,\nu}\boldsymbol{Q}_{B-1,z}\,{}^{b,\nu}\boldsymbol{G}_{z,B-1} + {}^{b,\nu}\boldsymbol{Q}_{B-1,B-1}. \tag{2.4.20} $$

Finally, we have

$$ {}^{b,\nu}\boldsymbol{H}_{\lfloor\frac{u}{c}\rfloor,B-1} = \begin{cases} \big(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{G}_{B-1,B-1}\big)^{-1}, & \text{if } \lfloor\frac{u}{c}\rfloor = B-1, \\ {}^{b,\nu}\boldsymbol{G}_{\lfloor\frac{u}{c}\rfloor,B-1}\big(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{G}_{B-1,B-1}\big)^{-1}, & \text{if } \lfloor\frac{u}{c}\rfloor \neq B-1, \end{cases} \tag{2.4.21} $$

and

$$ {}^{b,\nu}\boldsymbol{H}_{\lfloor \frac{u}{c} \rfloor, B} = \begin{cases} \left(I - {}^{b,\nu}\boldsymbol{G}_{B,B}\right)^{-1}, & \text{if } \lfloor \frac{u}{c} \rfloor = B, \\ {}^{b,\nu}\boldsymbol{G}_{\lfloor \frac{u}{c} \rfloor, B}\left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{G}_{B,B}\right)^{-1}, & \text{if } \lfloor \frac{u}{c} \rfloor \neq B. \end{cases} \tag{2.4.22} $$

With Method I, we had to compute $\{{}^{b,\nu}\boldsymbol{R}_{i,i+1}\}_{i=0}^{B-1}$, which is comparable to computing $\{{}^{b,\nu}\boldsymbol{G}_{i,i+1}\}_{i=0}^{B-1}$ in terms of the computation time. The computational procedures for ${}^{b,\nu}\boldsymbol{G}_{B,B-1}$, ${}^{b,\nu}\boldsymbol{G}_{B,B}$, and ${}^{b,\nu}\boldsymbol{G}_{B-1,B-1}$ are straightforward without involving recursions as Method I does. Therefore, it is clear at this point that Method II is computationally superior in comparison to Method I for computing the expected total discounted dividends paid out prior to ruin.

### 2.4.5 Fundamental matrix via Method III

One advantage of Methods I and II for the MAP risk model with the dividend barrier is that we are able to compute the discounted rate matrices (and hence other related matrices) without resorting to an (approximating) iterative algorithm such as (2.3.4). However, the computational time required for the discounted rate, ladder height distribution, and fundamental period matrices when implementing Methods I and II in the case of the dividend barrier model can be much longer than that of the former model. If the iterative algorithm (2.3.4) converges relatively fast with good enough accuracy (which is true in most cases), then we can exploit a connection between these two models and write the discounted fundamental matrix of the dual G/M/1-type chain of the MAP risk model with the dividend barrier in terms of the discounted fundamental matrix of dual G/M/1-type chain of the model without the dividend barrier.

Consider the dual G/M/1-type chain of the MAP risk model without the dividend barrier $\{(X_t, V_t, J_t), t \in \mathbb{N}\}$, as defined in Section 1.3. Let us first partition the state space of $\{(X_t, V_t, J_t), t \in \mathbb{N}\}$ into $\mathcal{S}_B = \{0, 1, \ldots, B\} \times \{0, 1, \ldots, c-1\} \times \{0, 1, \ldots, m-1\}$, $\mathcal{S}_{B^+} = \{B+1, B+2, \ldots\} \times \{0, 1, \ldots, c-1\} \times \{0, 1, \ldots, m-1\}$, and $\mathcal{A} = \mathbb{Z}^- \times \{0, 1, \ldots, c-1\} \times \{0, 1, \ldots, m-1\}$. Let ${}^{\mathcal{S}_B}\boldsymbol{P}$ and ${}^{\mathcal{S}_{B^+}}\boldsymbol{P}$ be the corresponding TPM's within $S_B$ and $S_{B^+}$, respectively. Also, let ${}^{\mathcal{S}_B:\mathcal{S}_{B^+}}\boldsymbol{P}$ and ${}^{\mathcal{S}_{B^+}:\mathcal{S}_B}\boldsymbol{P}$ be the TPM's corresponding to transitions from $\mathcal{S}_B$ to $\mathcal{S}_{B^+}$ and from $\mathcal{S}_{B^+}$ to $\mathcal{S}_B$, respectively.

It is possible to define the discounted rate and ladder height distribution matrices according to the new partitioned state space. Let $\tau^-_{\mathcal{S}_B} = \inf\{t \in \mathbb{Z}^+ : (X_t, V_t, J_t) \in \mathcal{S}_B\}$ and

$$
{}^{\nu}r_{(i,v,j),(l,w,a)} = \sum_{k=0}^{\infty} \nu^k \Pr\left\{\tau^-_{\mathcal{S}_B} > k, (X_k, V_k, J_k) = (l, w, a) \middle| (X_0, V_0, J_0) = (i, v, j)\right\},
$$
$$
(i, v, j) \in \mathcal{S}_B,\ (l, w, a) \in \mathcal{S}_{B^+}. \tag{2.4.23}
$$

Let ${}_{B,B^+}^{\ \nu}\boldsymbol{R}$ be a $|\mathcal{S}_B| \times |\mathcal{S}_{B^+}|$ matrix whose $\{(i, v, j), (l, w, a)\}$-th entry is given by ${}^{\nu}r_{(i,v,j),(l,w,a)}$. Similarly, let

$$
{}^{\nu}q_{(i,v,j),(z,y,x)} = \sum_{k=1}^{\infty} \nu^k \Pr\left\{\tau^-_{\mathcal{S}_B} = k, (X_k, V_k, J_k) = (z, y, x) \middle| (X_0, V_0, J_0) = (i, v, j)\right\},
$$
$$
(i, v, j), (z, y, x) \in \mathcal{S}_B, \tag{2.4.24}
$$

and let ${}_{B,B}^{\ \nu}\boldsymbol{Q}$ be a $|\mathcal{S}_B| \times |\mathcal{S}_B|$ matrix whose $\{(i, v, j), (z, y, x)\}$-th entry is given by ${}^{\nu}q_{(i,v,j),(z,y,x)}$.

Let us now partition the discounted fundamental matrix ${}^{\nu}\boldsymbol{H}$ into

$$
{}^{\nu}\boldsymbol{H} = \begin{pmatrix} {}_{B,B}^{\ \nu}\boldsymbol{H} & {}_{B,B^+}^{\ \nu}\boldsymbol{H} \\ {}_{B^+,B}^{\ \nu}\boldsymbol{H} & {}_{B^+,B^+}^{\ \nu}\boldsymbol{H} \end{pmatrix}, \tag{2.4.25}
$$

where ${}_{B,B}^{\ \nu}\boldsymbol{H}$ is a $|\mathcal{S}_B| \times |\mathcal{S}_B|$ matrix and the dimensions of the other blocks of (2.4.25) are given likewise. Then, using the same arguments employed to obtain (2.2.21), we see that

$$\begin{aligned} {}_{B,B}^{\ \nu}\boldsymbol{H} &= \boldsymbol{I} + {}_{B,B}^{\ \nu}\boldsymbol{Q}\, {}_{B,B}^{\ \nu}\boldsymbol{H} \\ &= \left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)^{-1}. \end{aligned} \tag{2.4.26}$$

Now, by noting that ${}^{b,\nu}\boldsymbol{H}$, the discounted fundamental matrix of $\{(X_t^{(b)}, V_t^{(b)}, J_t), t \in \mathbb{N}\}$, is equal to $\left(\boldsymbol{I} - \nu^{\mathcal{A}^c}_{\ b}\boldsymbol{P}\right)^{-1}$, we can write

$$\begin{aligned} {}^{b,\nu}\boldsymbol{H} &= \left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q} - \left(\nu^{\mathcal{A}^c}_{\ b}\boldsymbol{P} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)\right)^{-1} \\ &= \left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q} - {}^{b,\nu}\boldsymbol{K}\right)^{-1} \\ &= \left(\left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}\left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)^{-1}\right)\left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)\right)^{-1} \\ &= \left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)^{-1}\left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}\left(\boldsymbol{I} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)^{-1}\right)^{-1} \\ &= {}_{B,B}^{\ \nu}\boldsymbol{H}\left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}\, {}_{B,B}^{\ \nu}\boldsymbol{H}\right)^{-1}, \end{aligned} \tag{2.4.27}$$

where ${}^{b,\nu}\boldsymbol{K} = \left(\nu^{\mathcal{A}^c}_{\ b}\boldsymbol{P} - {}_{B,B}^{\ \nu}\boldsymbol{Q}\right)$. Since we know how to compute ${}_{B,B}^{\ \nu}\boldsymbol{H}$ from Section 2.3, we focus our efforts in computing $\left(\boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}\, {}_{B,B}^{\ \nu}\boldsymbol{H}\right)^{-1}$.

From (2.2.20), we deduce that

$${}_{B,B}^{\ \nu}\boldsymbol{Q} = \nu^{\mathcal{S}_B}\boldsymbol{P} + \nu\, {}_{B,B^+}^{\ \ \nu}\boldsymbol{R}^{\mathcal{S}_{B^+}:\mathcal{S}_B}\boldsymbol{P}.$$

Recalling the original level-block representation, we also deduce that

$${}_{B,B}^{\ \nu}\boldsymbol{Q}_{i,z} = \begin{cases} \nu\boldsymbol{A}_{i-z+1}, & 0 \le i, z \le B-1,\ z \le i+1, \\ {}^{\nu}\boldsymbol{Q}_{i-z}, & i = B,\ 0 \le z \le B, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{2.4.28}$$

Therefore, we have

$$
{}^{b,\nu}\boldsymbol{K}_{i,l} = \begin{cases} \nu(\boldsymbol{A}_{B-1,B} - \boldsymbol{A}_0), & i = B-1,\, l = B, \\ \nu\boldsymbol{A}_{B,B} - {}^{\nu}\boldsymbol{Q}_0, & i,l = B, \\ \nu\boldsymbol{A}_{B-l} - {}^{\nu}\boldsymbol{Q}_{B-l}, & i = B,\, 0 \le l \le B-1, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{2.4.29}
$$

Now, letting ${}^{b,\nu}\boldsymbol{L} = \boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}_{B,B}\,{}^{\nu}\boldsymbol{H}$, we have

$$
{}^{b,\nu}\boldsymbol{L}_{i,l} = \begin{cases} \boldsymbol{I}, & i = l,\, 0 \le i,l \le B-2, \\ \boldsymbol{I} - {}^{b,\nu}\boldsymbol{K}_{B-1,B}\,{}^{\nu}\boldsymbol{H}_{B,B}, & i = l = B-1, \\ -{}^{b,\nu}\boldsymbol{K}_{B-1,B}\,{}^{\nu}\boldsymbol{H}_{B,l}, & i = B-1,\, l \in \{0,1,\ldots,B\} \setminus \{B-1\}, \\ \boldsymbol{I} - \sum_{z=0}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z,l}, & i = l = B, \\ -\sum_{z=0}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z,l}, & i = B,\, 0 \le l \le B-1, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{2.4.30}
$$

As can be seen from (2.4.30), due to the rather simple structure of ${}^{b,\nu}\boldsymbol{L}$, we can find its inverse by hand. First of all, let

$$
\begin{pmatrix} {}^{b,\nu}\boldsymbol{L}_{B-1,B-1} & {}^{b,\nu}\boldsymbol{L}_{B-1,B} \\ {}^{b,\nu}\boldsymbol{L}_{B,B-1} & {}^{b,\nu}\boldsymbol{L}_{B,B} \end{pmatrix}^{-1} = \begin{pmatrix} {}^{b,\nu}\boldsymbol{O}_{B-1,B-1} & {}^{b,\nu}\boldsymbol{O}_{B-1,B} \\ {}^{b,\nu}\boldsymbol{O}_{B,B-1} & {}^{b,\nu}\boldsymbol{O}_{B,B} \end{pmatrix}.
$$

Then, from the standard row reduction procedure, we obtain

$$ {}^{b,\nu}\boldsymbol{L}_{i,l}^{-1} = \begin{cases} \boldsymbol{I}, & i=l,\ 0 \le i,l \le B-2, \\ {}^{b,\nu}\boldsymbol{O}_{i,l}, & i,l = B-1, B, \\ -{}^{b,\nu}\boldsymbol{O}_{B-1,B-1}\,{}^{b,\nu}\boldsymbol{L}_{B-1,l} - {}^{b,\nu}\boldsymbol{O}_{B-1,B}\,{}^{b,\nu}\boldsymbol{L}_{B,l}, & i=B-1,\ 0 \le l \le B-2, \\ -{}^{b,\nu}\boldsymbol{O}_{B,B-1}\,{}^{b,\nu}\boldsymbol{L}_{B-1,l} - {}^{b,\nu}\boldsymbol{O}_{B,B}\,{}^{b,\nu}\boldsymbol{L}_{B,l}, & i=B,\ 0 \le l \le B-2, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{2.4.31} $$

Hence, we have for $0 \le i \le B$,

$$ {}^{b,\nu}\boldsymbol{H}_{i,l} = \begin{cases} {}^{\nu}\boldsymbol{H}_{i,l} + {}^{\nu}\boldsymbol{H}_{i,B-1}\,{}^{b,\nu}\boldsymbol{L}_{B-1,l}^{-1} + {}^{\nu}\boldsymbol{H}_{i,B}\,{}^{b,\nu}\boldsymbol{L}_{B,l}^{-1}, & 0 \le l \le B-2, \\ {}^{\nu}\boldsymbol{H}_{i,B-1}\,{}^{b,\nu}\boldsymbol{L}_{B-1,l}^{-1} + {}^{\nu}\boldsymbol{H}_{i,B}\,{}^{b,\nu}\boldsymbol{L}_{B,l}^{-1}, & l = B-1, B. \end{cases} \tag{2.4.32} $$

As can be seen from (2.4.32), computing $\{{}^{b,\nu}\boldsymbol{L}_{i,l}\}_{i\in\{B-1,B\},l\in\{0,1,\ldots,B\}}$ is the key to computing the discounted fundamental matrix of the dual G/M/1-type chain of the MAP risk model with the dividend barrier. Therefore, we develop a recursion for $\{{}^{b,\nu}\boldsymbol{L}_{B,l}\}_{l\in\{0,1,\ldots,B\}}$ to speed up the computation. From (2.3.9) and (2.3.10), we can write

$$ \begin{aligned} \sum_{z=0}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z,l} &= \sum_{z=0}^{l-1} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z,l-1}\,{}^{\nu}\boldsymbol{R} + \sum_{z=l}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\big({}^{\nu}\boldsymbol{H}_{z-l,0} + {}^{\nu}\boldsymbol{H}_{z,l-1}\,{}^{\nu}\boldsymbol{R}\big) \\ &= \sum_{z=0}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z,l-1}\,{}^{\nu}\boldsymbol{R} + \sum_{z=l}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\,{}^{\nu}\boldsymbol{H}_{z-l,0},\ l > 0. \end{aligned} \tag{2.4.33} $$

Therefore, by combining (2.4.30) and (2.4.33), we have

$$
{}^{b,\nu}\boldsymbol{L}_{B,l} = \begin{cases} -\sum_{z=0}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\, {}^{\nu}\boldsymbol{H}_{z,0}, & l = 0, \\ {}^{b,\nu}\boldsymbol{L}_{B,l-1}\, {}^{\nu}\boldsymbol{R} - \sum_{z=l}^{B} {}^{b,\nu}\boldsymbol{K}_{B,z}\, {}^{\nu}\boldsymbol{H}_{z-l,0}, & 1 \le l \le B-1, \\ \boldsymbol{I} + {}^{b,\nu}\boldsymbol{L}_{B,B-1}\, {}^{\nu}\boldsymbol{R} - {}^{b,\nu}\boldsymbol{K}_{B,B}\, {}^{\nu}\boldsymbol{H}_{0,0}, & l = B. \end{cases} \tag{2.4.34}
$$

To summarize the computational procedure for the discounted fundamental matrix of the dual G/M/1-type chain of the MAP risk model with the dividend barrier and initial level $i$, we perform the following steps:

(i) Compute $\{{}^{\nu}\boldsymbol{Q}_z\}_{z\in\{0,1,2,\dots,B\}}$, $\{{}^{\nu}\boldsymbol{H}_{z,0}\}_{z\in\{0,1,2,\dots,B\}}$, and $\{{}^{\nu}\boldsymbol{H}_{i,l}\}_{l\in\{0,1,2,\dots,B\}}$ using the methods developed in Section 1.3.

(ii) Compute $\{{}^{b,\nu}\boldsymbol{K}_{i,l}\}_{i,l\in\{0,1,2,\dots,B\}}$ and $\{{}^{b,\nu}\boldsymbol{L}_{i,l}\}_{i,l\in\{0,1,2,\dots,B\}}$ according to (2.4.29) and (2.4.34).

(iii) Compute $\{{}^{b,\nu}\boldsymbol{L}_{i,l}^{-1}\}_{i,l\in\{0,1,2,\dots,B\}}$ according to (2.4.31).

(iv) Compute $\{{}^{b,\nu}\boldsymbol{H}_{i,l}\}_{l\in\{0,1,2,\dots,B\}}$ according to (2.4.32).

## 2.5 Numerical analysis

The first two numerical examples we analyze establish that our method yields the same results as those produced in some of the earlier works found in the literature. We selected one example from Cossette et al. (2004a) and the other from Wu and Li (2012). In the third example we study, we implement our algorithm for a risk model having a discretized Pareto claim size distribution, which belongs to the class of heavy-tailed distributions.

**Example 1** Our first example is chosen directly from Cossette et al. (2004a). The risk model in consideration belongs to the class of the discrete-time MAP risk models with matrix-geometric claim size distributions discussed in Section 2.3.3. Hence, we can apply the matrix analytic methodology developed in Section 2.3.3 to analyze the risk model on hand. Let us first show how the risk model to be discussed here can be put into the matrix analytic methodology framework introduced in Section 2.3.3.

The risk process under consideration, denoted by $\{U_t, t \in \mathbb{N}\}$, is comprised of the claims arrival MAP $\{(N_t, J_t), t \in \mathbb{N}\}$ with the associated TPMs

$$\boldsymbol{P}_0 = \begin{pmatrix} (1-q)+\pi q & 0 \\ (1-q)-\pi(1-q) & 0 \end{pmatrix}$$

and

$$\boldsymbol{P}_1 = \begin{pmatrix} 0 & q-\pi q \\ 0 & q+\pi(1-q) \end{pmatrix},$$

where $\pi \in [0,1)$ and $q = 0.07$. Furthermore, the sequence of claim size random variables $\{Y_t, t \in \mathbb{Z}^+\}$ form an i.i.d. sequence of random variables with a zero-truncated geometric distribution with pmf $f(y) = (1-7/8)(7/8)^{y-1}$, $y \in \mathbb{Z}^+$, independently of $\{(N_t, J_t), t \in \mathbb{N}\}$. Lastly, $c = 1$.

We first note that $f(y)$ can be rewritten as $f(y) = \boldsymbol{\alpha}\boldsymbol{\Gamma}^{y-1}\boldsymbol{\gamma}^\intercal$, where $\boldsymbol{\alpha} = (1)$, $\boldsymbol{\Gamma} = (7/8)$, and $\boldsymbol{\gamma} = (1-7/8)$. Certainly, $f(y)$ belongs to the class of matrix-geometric distributions. Putting the above parameters in terms of the matrix notations given in the framework of our methodology, let $\boldsymbol{A}_0 = \boldsymbol{P}_0$ and $\boldsymbol{A}_n = f(n)\boldsymbol{P}_1$, $n \in \mathbb{Z}^+$. Then, by (2.3.13), we have

$$\boldsymbol{A}_n = f(n)\boldsymbol{P}_1 = \boldsymbol{\alpha}\boldsymbol{\Gamma}^{n-1}\boldsymbol{\gamma}^\intercal\boldsymbol{P}_1 = (\boldsymbol{I}_1 \otimes \boldsymbol{A})(\boldsymbol{I}_1 \otimes \boldsymbol{\Gamma}^{n-1})\boldsymbol{\Xi}, \; n \in \mathbb{Z}^+,$$

| | $\psi(u)$ | | $\psi(u)$ | | $\psi(u)$ | |
|---|---|---|---|---|---|---|
| | $\pi = 0$ | | $\pi = 0.05$ | | $\pi = 0.2$ | |
| $u$ | M | A | M | A | M | A |
| 10 | 0.286394 | 0.2864 | 0.295877 | 0.2959 | 0.32644 | 0.3264 |
| 20 | 0.155674 | 0.1557 | 0.165662 | 0.1657 | 0.199863 | 0.1999 |
| 30 | 0.0846187 | 0.0846 | 0.092755 | 0.0928 | 0.122365 | 0.1224 |
| 40 | 0.0459957 | 0.046 | 0.051934 | 0.0519 | 0.0749173 | 0.0749 |
| 50 | 0.0250016 | 0.025 | 0.029078 | 0.0291 | 0.0458678 | 0.0459 |
| 60 | 0.01359 | 0.0136 | 0.016281 | 0.0163 | 0.0280823 | 0.0281 |
| 70 | 0.00738704 | 0.0074 | 0.009116 | 0.0091 | 0.0171933 | 0.0172 |
| 80 | 0.00401533 | 0.004 | 0.005104 | 0.0051 | 0.0105265 | 0.0105 |
| 100 | 0.00118638 | 0.0012 | 0.0016 | 0.0016 | 0.0039458 | 0.0039 |

Table 2.1: Infinite-time ruin probabilities

where $\boldsymbol{A} = \boldsymbol{\alpha}$ and $\Xi = \boldsymbol{\gamma}^{\intercal}\boldsymbol{P}_1$. We can now follow the rest of the procedure given in Section 2.3.3 to compute the discounted fundamental matrix.

In Table 2.1, the infinite-time ruin probabilities, denoted by $\psi(u) = \Pr\{\tau < \infty | U_0 = u\}$, for various values of the initial surplus $u$ and of $\pi$ are computed. Under the columns labeled M, the values computed via our method are given. Under the columns labeled A are the values given in Cossette et al. (2004a). From the values computed, we can see that the infinite-time ruin probabilities computed via our method match those of Cossette et al. (2004a).

**Example 2** This example is taken from Wu and Li (2012), in which we compare the expected total discounted dividends paid prior to ruin (denoted by $V(u)$ for initial surplus $u \in \mathbb{N}$) computed via our method and that of Wu and Li (2012). The risk model in consideration belongs to the class of discrete-time MAP risk model with dividend barrier, and we have three different matrix analytic methods (discussed in Section 2.4) at our disposal. We have chosen Method III discussed in Section 2.4.5 as our tool here and will see that the results computed via Method III match those given in Wu and Li (2012).

In this risk model, claims are assumed to arrive following a compound binomial process with $\Pr\{I_t = 1 | I_{t-1} = j\} = p = 0.35$ and $\Pr\{I_t = 0 | I_{t-1} = j\} = 1 - p = 0.65$, $j = 0, 1$, and $Y_t | (I_t = 1)$ follows a mixed geometric distribution with pmf

$$f(y) = \begin{cases} \frac{\gamma^{y-2}(1-(\beta/\gamma)^{y-1})((1-\beta)(1-\gamma)+\alpha)}{1-\beta/\gamma}, & \text{if } y = 2, \\ \frac{\gamma^{y-2}(1-(\beta/\gamma)^{y-1})((1-\beta)(1-\gamma)+\alpha)-\gamma^{y-3}(1-(\beta/\gamma)^{y-2})\alpha}{1-\beta/\gamma}, & \text{if } y = 3, 4, \dots, \end{cases}$$

where $\beta = 0.8$, $\gamma = 0.6$, and $\alpha = 0.24$. The dividend barrier is set equal to $b = 9$ and the premium received per unit time is equal to $c = 1$.

Once again, putting the above parameters in the matrix analytic methodology framework, let

$$\boldsymbol{P}_0 = \begin{pmatrix} 1 - p \end{pmatrix}$$

and

$$\boldsymbol{P}_1 = \begin{pmatrix} p \end{pmatrix}.$$

After setting $\boldsymbol{A}_{b-1,b} = \boldsymbol{P}_0$, $\boldsymbol{A}_{b,b} = \boldsymbol{P}_0 + f(1)\boldsymbol{P}_1$, $\boldsymbol{A}_0 = \boldsymbol{P}_0$, and $\boldsymbol{A}_n = f(n)\boldsymbol{P}_1$, $n \in \mathbb{Z}^+$, we can now apply the rest of the procedure presented in Section 1.4.5 to compute the expected total discounted dividends paid prior to ruin.

In Table 2.2, we list our results in the column labeled M and those of Wu and Li (2012) in the column labeled A. Once again, we obtain agreement between the two methods.

| | $V(u)$ | |
|---|---|---|
| $u$ | M | A |
| 0 | 0.132372 | 0.13237 |
| 1 | 0.214368 | 0.21437 |
| 2 | 0.324345 | 0.32435 |
| 3 | 0.473493 | 0.47349 |
| 4 | 0.677086 | 0.67709 |
| 5 | 0.956084 | 0.95608 |
| 6 | 1.33932 | 1.33932 |
| 7 | 1.86651 | 1.86651 |
| 8 | 2.59237 | 2.59237 |
| 9 | 3.59237 | 3.59237 |

Table 2.2: Expected total discounted dividends paid prior to ruin

**Example 3** In this example, we consider the same risk model as in Example 2 with the exception that now $c = 2$, there is no dividend barrier, and the claim sizes follow a discretized Pareto distribution with pmf given by

$$f(y) = \left(1 + \frac{y-1}{30}\right)^{-8} - \left(1 + \frac{y}{30}\right)^{-8},\ y \in \mathbb{Z}^+.$$

The risk model described above is a discrete-time MAP risk model with a discretized Pareto claim size distribution. Therefore, we can apply the matrix analytic methodology developed in Section 2.3.2. To this end, let

$$\boldsymbol{P}_0 = \begin{pmatrix} 1-p \end{pmatrix}$$

and

$$\boldsymbol{P}_1 = \begin{pmatrix} p \end{pmatrix}.$$

Now, let

$$\boldsymbol{A}_0 = \begin{pmatrix} \boldsymbol{P}_0 & 0 \\ f(1)p & \boldsymbol{P}_0 \end{pmatrix},$$

and

$$\boldsymbol{A}_n = \begin{pmatrix} f(cn)\boldsymbol{P}_1 & f(cn-1)\boldsymbol{P}_1 \\ f(cn+1)\boldsymbol{P}_1 & f(cn)\boldsymbol{P}_1 \end{pmatrix}, \; n \in \mathbb{Z}^+.$$

Here, we truncate $\{\boldsymbol{A}_n\}_{n\in\mathbb{N}}$ at $N = 1000$. Hence, we can follow the rest of the matrix analytic methodology procedure in Section 2.3.2 with $\{\boldsymbol{A}_n, n \in \{0, 1, 2, \dots, N\}\}$ defined above.

In Table 2.3, the infinite-time ruin probabilities are computed, and in Table 2.4, the first and second unconditional and conditional moments of the surplus prior to ruin are computed, where $r_i(u) = E\{U^i_{\tau-1}\mathcal{I}[\tau < \infty]|U_0 = u\}$ and $\bar{r}_i(u) = E\{U^i_{\tau-1}|\tau < \infty, U_0 = u\}$, $i = 1, 2$. In contrast to the nontransient results computed in Table 2.3 and 2.4, in Table 2.5, we compare the values of the joint conditional pmf of the time of ruin, surplus prior to ruin, and deficit at ruin given the initial surplus, $\phi(n, x, y|u) = \Pr\{\tau = n, U_{\tau-1} = x, |U_\tau| = y|U_0 = u\}$, $x, u \in \mathbb{N}$, $n, y \in \mathbb{Z}^+$, computed via our matrix analytic methodology and the standard recursive method. The recursion method relies on the following recursion:

$$\Pr\{\tau > t, U_t = x|U_0 = u\} = \sum_{j=1}^{u+c} pf(j) \Pr\{\tau > t-1, U_{t-1} = x|U_0 = u + c - j\}$$
$$+ (1-p) \Pr\{\tau > t-1, U_{t-1} = x|U_0 = u + c\}, \; t \in \mathbb{Z}^+.$$

The values of $\phi(n, x, y|u)$ computed via the matrix analytic methodology are numerically inverted via the algorithm known as the *Lattice-Poisson* algorithm in Abate and Whitt (1992). The error bound used for the Lattice-Poisson algorithm is $10^{-8}$.

| $u$ | $\psi(u)$ |
|---|---|
| 0 | 0.785963 |
| 20 | 0.358841 |
| 50 | 0.118775 |
| 100 | 0.0193046 |
| 150 | 0.0031749 |
| 200 | 0.000526861 |

Table 2.3: Infinite-time ruin probabilities

| $u$ | $r_1(u)$ | $\bar{r}_1(u)$ | $r_2(u)$ | $\bar{r}_2(u)$ |
|---|---|---|---|---|
| 0 | 3.39801 | 4.32336 | 43.9685 | 55.9422 |
| 20 | 4.2208 | 11.7623 | 87.7342 | 244.493 |
| 50 | 1.64291 | 13.8321 | 46.1847 | 388.843 |
| 100 | 0.292654 | 15.1598 | 10.668 | 552.616 |
| 150 | 0.0514237 | 16.197 | 2.37295 | 747.41 |
| 200 | 0.00926308 | 17.5816 | 0.575782 | 1092.85 |

Table 2.4: Unconditional and conditional moments of the surplus prior to ruin

| | Matrix analytic methodology | Recursion |
|---|---|---|
| $n$ | $\phi(n, 50, 1\|50)$ | $\phi(n, 50, 1\|50)$ |
| 10 | $6.03994 \times 10^{-6}$ | $6.03994 \times 10^{-6}$ |
| 20 | $3.35333 \times 10^{-6}$ | $3.35333 \times 10^{-6}$ |
| 30 | $2.15238 \times 10^{-6}$ | $2.15235 \times 10^{-6}$ |
| 40 | $1.46541 \times 10^{-6}$ | $1.46494 \times 10^{-6}$ |
| 50 | $1.02747 \times 10^{-6}$ | $1.02623 \times 10^{-6}$ |
| 100 | $2.02416 \times 10^{-7}$ | $2.00622 \times 10^{-7}$ |

Table 2.5: Joint conditional pmf of time of ruin, surplus prior to ruin, and deficit at ruin

As can be seen from Table 2.5, the difference in the values of $\phi(n, x, y|u)$ computed via the recursion (which are the true values) and the values computed via the matrix analytic method are within the error bound used for the Lattice-Poisson numerical inversion algorithm (i.e., $10^{-8}$), suggesting that the matrix analytic methodology used to compute the values of $\phi(n, x, y|u)$ produce errors that are negligible up to the precision of the inversion algorithm for the values given in Table 2.5.

To comment on the computational times and memory consumption rates of both methods, we first note that the computation time required for our matrix analytic methodology was noticeably longer than the standard recursion method for all values computed in Table 2.5. However, noting that the bulk of the computation time of the matrix analytic methodology was attributable to the summation (truncated by $N$) involved in the computation of ${}^{\nu}\boldsymbol{R}$ and ${}^{\nu}\boldsymbol{Q}_k$, $k \in \mathbb{N}$, we anticipate that the overall computation time for the matrix analytic methodology can be improved greatly if the claim size distribution is of matrix-geometric type (see Section 2.3.3 for more details). Moreover, as stated in Section 1.1, the recursion method's computation time increases rapidly (nearly quadratic) as $n$ increases. On the other hand, the computation time of our methodology for computing a single value of the generating function of $\phi(n, x, y|u)$ has an upper bound (at $\nu = 1$). Therefore, the computational complexity of our methodology only depends on the inversion algorithm that is used (as per its dependency on $n$). Noting that the Lattice-Poisson algorithm we implemented here is $O(n)$, we can say that the computation time of our matrix analytic methodology grows linearly in $n$. Hence, for very large values of $n$, the matrix analytic methodology will outperform the standard recursion method in terms of computation time. Again, as stated in Section 1.1, the recursion method's computer memory consumption rate grows linearly in $n$. On the other hand, by the procedural structure of our methodology, the computer memory consumption rate of our matrix analytic methodology stays constant in $n$. Hence, the recursion method is limited by the computer memory for a large value of $n$, whereas the matrix analytic methodology is not.

For infinite-time related quantities of interest, such as the infinite-time ruin probabilities, the matrix analytic methodology seems superior in most cases compared to the standard recursion method. However, for transient results, depending on the problem at hand, one method can outperform the other in terms of the computation time. Nonetheless, we note that the matrix analytic methodology provides a viable alternative to the standard recursion method when the time horizon of interest is long (quite often the case when the discrete-time risk model of interest is an approximation of its continuous-time counterpart) and the computer memory is the limitation.

# Chapter 3

# A matrix analytic methodology for the discrete-time MAP risk model with phase-dependent premium rates and phase-type claim size distributions

## 3.1 Introduction

The work here is motivated by the fluid flow process based matrix analytic methodology developed for the analysis of the continuous-time MAP risk model with phase-type claim size distributions by Ramaswami (2006). Ahn and Ramaswami (2004, 2005) developed efficient matrix-based algorithms for some transient solutions of fluid flow models, and via a sample paths connection between the MAP risk process with phase-type claim size distributions and a fluid flow process, Ramaswami (2006) later gave a comprehensive matrix analytic methodology for computing the discounted joint pdf of the surplus prior to ruin and deficit at ruin of the MAP risk process in terms of the relevant transient solutions of the fluid flow process. (The phase-type claim size distribution assumption is necessary for this methodology, but we note that the class of phase-type distributions is dense on the nonnegative real line and therefore can be used to approximate almost all claim size distributions.) In fact, around the time when Ramaswami (2006) was published, a trend of the use of fluid flow process based matrix analytic methods in risk theory was initiated by Badescu et al. (2005a,b), and the advent of the fluid flow process based matrix analytic methodology brought forward a powerful alternative to the traditional IDE method typically employed in risk theory.

Three notable advantages of the fluid flow process based matrix analytic methodology over the IDE method are the probabilistic interpretation of the derivation of the algorithms involved, the numerical stability of the algorithms even when the number of phases in the associated MAP is large, and the exploitation of the skip-free nature of the fluid flow process. The second point enables extensive numerical analyses of the MAP risk models with many phases which the traditional IDE method often had difficulty with due to the numerical instability arising from the sensitivity of the methodology to the accurate evaluation of the roots of Lundberg's fundamental equation. The first and third points together afford an alternative perspective in the ways of solving problems in risk theory through the way of probabilistic interpretations and the exploitation of the skip-free nature of the sample paths of the fluid flow process. This alternative perspective is especially useful when we consider risk models such as the multi-threshold MAP risk models due to the much simplified analysis based on the simple level-crossing structure of the skip-free sample paths of the fluid flow process (see, e.g., Badescu et al. (2007)). Despite such advantages, however, the methodology is not without a flaw. It does not allow for the analysis of risk models with phase-dependent premium rates.

The fluid flow process based matrix analytic methodology by Ahn and Ramaswami builds on a sample paths connection between the risk process of interest and a particular fluid flow process. However, as noted in Ahn (2009), the sample paths connection between the MAP risk process with phase-dependent premium rates and the fluid flow process cannot be easily established and hence one cannot simply apply the results from the fluid flow process to the risk process as it is the case with the MAP risk processes with phase-independent premium rates. Due to this lack of the simple sample paths connection, Ahn (2009) proposed an alternative matrix analytic methodology via a sample paths connection between the MAP risk process of interest (includes MAP risk models with phase-dependent premium rates as special cases) and a fluid flow process with downward jumps. This alternative methodology, however, nullifies the exploitation of the skip-free nature of the fluid flow process in the original methodology developed by Ahn and Ramaswami (see Ahn (2009) and Baek and Ahn (2014) for more detailed discussion on this topic).

In this work, we introduce the discrete-time version of a generalization of Ahn and Ramaswami's methodology. Instead of the fluid flow process, we exploit a sample paths connection between the discrete-time MAP risk process and a discrete-time QBD process. Our methodology is the discrete-time version of a generalization of Ahn and Ramaswami's methodology in the sense that it is built directly on a sample paths connection between the MAP risk process and a QBD process without downward jumps, even when the premium rates depend on the phase process of the associated discrete-time MAP. Hence, our methodology can exploit the skip-free nature of the QBD process even when the premium rates depend on the phase process, unlike the alternative methodology introduced by Ahn (2009) involving fluid flow processes with downward jumps.

It is our hope that with the insight learned while developing the discrete-time version of the generalization of Ahn and Ramaswami's methodology, we can further extend our work to the generalization of Ahn and Ramaswami's methodology directly under the continuous-time setting. Until then, we note that the work here, besides its original function of studying the discrete-time MAP risk models, provides a powerful numerical algorithm for the discrete-time approximation to the continuous-time counterpart.

We also note that there is another fluid flow process based methodology that was introduced by Breuer (2008, 2010). While Breuer's methodology includes the analysis of the MAP risk models with phase-dependent premium rates as a special case, it differs from the methodology developed by Ahn and Ramaswami in many ways. Although we do not intend to compare the numerical stability or efficiency of the two methodologies here, it appears that Ahn and Ramaswami's methodology has a more extensive analysis on its numerical stability and efficiency available. Moreover, Ahn and Ramaswami's methodology yields a quadratically convergent algorithm for one of the key matrices in the methodology which has proven to be very fast (see Ahn and Ramaswami (2005) for more details). As the derivation of our methodology follows the footsteps of that of Ahn and Ramaswami's, we note that the numerical analysis of their methodology naturally extends to our methodology in the discrete-time setting, including the aforementioned quadratic convergence.

This chapter is organized as follows. In Section 3.2, the mathematical definition of the MAP risk model with phase-dependent premium rates and phase-type claim size distributions is given, along with a method of construction of the QBD process necessary for the development of our methodology. In Section 3.3, we develop our matrix analytic methodology for the discrete-time MAP risk model with phase-dependent premium rates and phase-type claim size distributions. In Section 3.4, a numerical example is studied.

## 3.2 Discrete-time MAP risk model with phase-dependent premiums and phase-type claim size distributions

### 3.2.1 Model description

Consider a discrete-time MAP risk model

$$U_t = u + \sum_{k=0}^{t-1} c(J_k) - \sum_{k=1}^{t} Y_k,\ t \in \mathbb{N},\, u \in \mathbb{N},$$

comprised of a discrete-time MAP $\{(N_t, J_t), t \in \mathbb{N}\}$ defined on $\mathbb{N} \times \mathcal{J}$, $\mathcal{J} = \{1, 2, \ldots, m\}$, $m \in \mathbb{Z}^+$, TPMs $(\boldsymbol{P}_0, \boldsymbol{P}_1) = ((p_{0,i,j})_{i,j\in\mathcal{J}}, (p_{1,i,j})_{i,j\in\mathcal{J}})$, and the conditionally i.i.d. claim amount per period sequence $\{Y_t, t \in \mathbb{Z}^+\}$ (conditional on the phase process $\{J_t, t \in \mathbb{N}\}$ of the MAP). In particular, let $\{I_t, t \in \mathbb{Z}^+\}$ be a sequence of Bernoulli random variables which are equal to 1 when there is an arrival at time $t$ in the underlying MAP and let $f^{(i,j)}(y)$, $i, j \in \mathcal{J}$, $y \in \mathbb{Z}^+$, denote the pmf of $Y^{(i,j)} = Y_t|(I_t = 1, J_t = j, J_{t-1} = i)$ $\forall$ $t \in \mathbb{Z}^+$. (Note that $Y_t|(I_t = 0, J_t = j, J_{t-1} = i)$ is equal to 0 with probability 1 $\forall$ $i, j$ and $t \in \mathbb{Z}^+$.) We further assume that the premium rates depend on the phase process $\{J_t, t \in \mathbb{N}\}$, i.e., $c_t = c(J_t)$, and that $Y^{(i,j)}$ follows a discrete-time phase-type distribution of order $n^{(i,j)} \in \mathbb{Z}^+$ with pmf $f^{(i,j)}(y) = \boldsymbol{\alpha}^{(i,j)}(\boldsymbol{U}^{(i,j)})^{y-1}(\boldsymbol{\gamma}^{(i,j)})^\intercal$, $y \in \mathbb{Z}^+$, $i, j \in \mathcal{J}$.

### 3.2.2 Construction of dual pre-QBD process

Herein, we outline the method of construction of the QBD process to be used in the development of the matrix analytic methodology. To this end, we first construct a DTMC $\{(X_t, W_t), t \in \mathbb{N}\}$ by transforming the premiums received per unit time into linear upward journeys (with slope equal to the premiums received during the time period) and the downward jumps of claim arrivals into linear downward journeys (with slope 1). This is analogous to how a fluid flow process of which sample paths can be connected to those of the risk process under consideration is constructed in continuous time (see, e.g., Ramaswami (2006)). However, the resulting process $\{(X_t, W_t), t \in \mathbb{N}\}$ does not necessarily form a QBD process since the increase rate can be greater than 1 in our problem. In such cases, we can apply the well-known blocking technique to $\{(X_t, W_t), t \in \mathbb{N}\}$ to obtain a QBD process. More specifically, if we let $L_t = \ell_1(X_t) = \lfloor \frac{X_t}{c_{max}} \rfloor$ and $V_t = \ell_2(X_t) = X_t \bmod c_{max}$, where $c_{\max} = \max\{c(j), j \in \mathcal{J}\}$, then $\{(L_t, V_t, W_t), t \in \mathbb{N}\}$ forms a QBD process. In what follows, we refer to $\{(X_t, W_t), t \in \mathbb{N}\}$ as the dual pre-QBD process and $\{(L_t, V_t, W_t), t \in \mathbb{N}\}$ as the dual QBD process of the risk process under consideration.

First of all, let $\mathcal{S}_1 = \mathcal{J}$ and $\mathcal{S}_2 = \cup_{i,j \in \mathcal{J}} \mathcal{J}^{(i,j)}$, where $\mathcal{J}^{(i,j)} = \{(i,j,1), (i,j,2), \ldots, (i,j,n_{i,j})\}$, $i, j \in \mathcal{J}$, are the transient states corresponding to the phase-type claim size distribution resulting from claims accompanied by phase transitions from $i$ to $j$. Now, let $\mathcal{W} = \mathcal{S}_1 \cup \mathcal{S}_2$ be the state space of $\{W_t, t \in \mathbb{N}\}$. We then set $X_{t+1} = X_t + c(W_t)$, if $W_t \in \mathcal{S}_1$, and $X_{t+1} = X_t - 1$, if $W_t \in \mathcal{S}_2$. Furthermore, $\forall\ t \in \mathbb{N}$, let the one-step transition probabilities be given by

$$\Pr\{W_{t+1} = j | W_t = i\} = \begin{cases} p_{0,i,j},\ i, j \in \mathcal{S}_1, \\ 0,\ \text{otherwise}, \end{cases}$$

$$\Pr\{W_{t+1} = (l,j,w) | W_t = i\} = \begin{cases} p_{1,i,j}\alpha_w^{(i,j)},\ i \in \mathcal{S}_1,\ (l,j,w) \in \mathcal{S}_2,\ i = l, \\ 0,\ \text{otherwise}, \end{cases}$$

$$\Pr\{W_{t+1} = (l,j,w) | W_t = (i,z,x)\} = \begin{cases} u_{x,w}^{(i,z)},\ (i,z,x), (l,j,w) \in \mathcal{S}_2,\ (i,z) = (l,j), \\ 0,\ \text{otherwise}, \end{cases}$$

and

$$\Pr\{W_{t+1} = j | W_t = (i,z,x)\} = \begin{cases} \gamma_x^{(i,z)},\ (i,z,x) \in \mathcal{S}_2,\ j \in \mathcal{S}_1,\ z = j, \\ 0,\ \text{otherwise}. \end{cases} \tag{3.2.1}$$

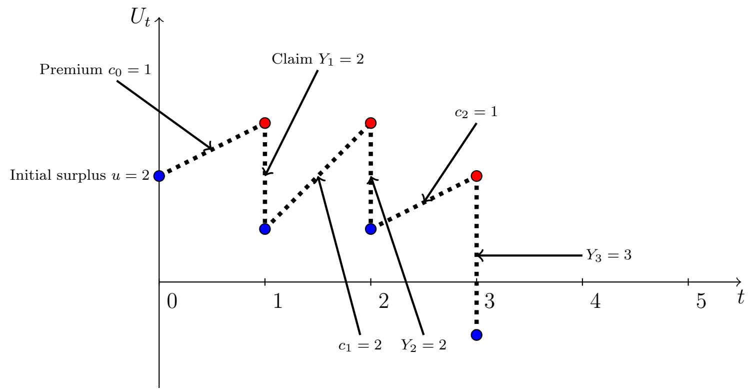


Figure 3.1: Sample path of $\{U_t, t \in \mathbb{N}\}$

As can be seen from the above construction, the times $W_t$ is in $\mathcal{S}_1$ correspond to the "real" times and the times $W_t$ is in $\mathcal{S}_2$ correspond to the "artificial" times that are created to account for the linear discounting of the claims' amount. Therefore, the number of times $W_t$ is in $\mathcal{S}_1$ before the dual pre-QBD process falls below 0 is equal to the time of ruin of the corresponding risk process (see, e.g., Figures 3.1 and 3.2, where the sample paths of the risk process and its dual pre-QBD process are depicted, and the times $W_t$ is in $\mathcal{S}_1$ are marked by the blue dots and the times $W_t$ is in $\mathcal{S}_2$ are marked by the red dots). Then, we can study the transient solutions of the original risk process simply by studying the transient solutions of the dual pre-QBD process while only tracking the times $W_t$ is in $\mathcal{S}_1$ before the dual pre-QBD process falls below 0.

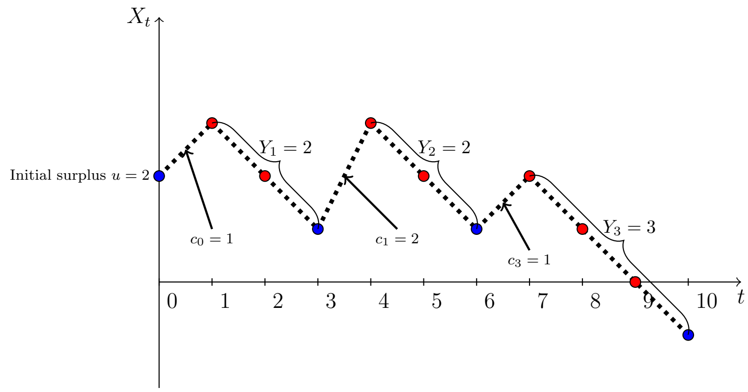


Figure 3.2: Sample path of $\{X_t, t \in \mathbb{N}\}$

## 3.3 Discounted joint conditional pmf

First of all, let the discounted joint conditional pmf of the surplus prior to ruin and deficit at ruin of $\{(U_t, J_t), t \in \mathbb{N}\}$ be given by

$$\begin{aligned}
&h_\nu\big((x,l),(y,j)\big|(u,i)\big) \\
&= \sum_{k=1}^{\infty} \nu^k \Pr\{\tau = k, (U_{k-1}, J_{k-1}) = (x,l), (|U_k|, J_k) = (y,j)|(U_0, J_0) = (u,i)\} \\
&= \sum_{k=1}^{\infty} \nu^k \Pr\{\tau > k-1, (U_{k-1}, J_{k-1}) = (x,l)|(U_0, J_0) = (u,i)\} \\
&\qquad\qquad\qquad \times \Pr\{\tau = 1, (|U_1|, J_1) = (y,j)|(U_0, J_0) = (x,l)\}, \\
&\qquad\qquad\qquad\qquad x, u \in \mathbb{N},\ y \in \mathbb{Z}^+,\ i,j,l \in \mathcal{J},\ \nu \in \mathbb{C},\ |\nu| \le 1,
\end{aligned} \tag{3.3.1}$$

where $\tau = \inf\{t \in \mathbb{Z}^+ : U_t < 0\}$ is the random variable denoting the time of ruin of the risk process and the second equality follows from the Markov property and stationarity.

Now, let $\kappa = \inf\{t \in \mathbb{Z}^+ : X_t < 0\}$ denote the time $\{(X_t, W_t), t \in \mathbb{N}\}$ for the first time falls below 0 and $s_1([h,k])$ be the random variable denoting the total number of times $W_t$ is in $\mathcal{S}_1$ in the time interval $[h,k]$, $h,k \in \mathbb{N}$. (We let $s_1([h,k]) = 0$, when $k < h$.) Then, by noting that $\kappa$ corresponds to the time of ruin $\tau$ of the risk process, we can condition on the value of $\kappa$ and track only the times $W_t$ is in $\mathcal{S}_1$ in the interval $[0, \kappa - 1]$, and rewrite (3.3.1) as

$$
\begin{aligned}
&h_\nu\big((x,l),(y,j)\big|(u,i)\big) \\
&= \sum_{k=1}^{\infty} \nu E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa > k-1, (X_{k-1}, W_{k-1}) = (x,l)]|(X_0, W_0) = (u,i)\} \\
&\qquad\qquad\qquad\qquad \times \Pr\{\tau = 1, (|U_1|, J_1) = (y,j)|(U_0, J_0) = (x,l)\} \\
&= \xi_\nu\big((x,l)\big|(u,i)\big)\nu \Pr\{\tau = 1, (|U_1|, J_1) = (y,j)|(U_0, J_0) = (x,l)\}, \\
&\qquad\qquad\qquad\qquad x, u \in \mathbb{N},\ y \in \mathbb{Z}^+,\ i, j, l \in \mathcal{S}_1,\ \nu \in \mathbb{C},\ |\nu| \le 1,
\end{aligned} \tag{3.3.2}
$$

where

$$
\xi_\nu\big((x,l)\big|(u,i)\big) = \sum_{k=1}^{\infty} E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa > k-1, (X_{k-1}, W_{k-1}) = (x,l)]|(X_0, W_0) = (u,i)\}.
$$

Now, it remains to evaluate $\xi_\nu\big((x,l)\big|(u,i)\big)$.

To compute $\xi_\nu\big((x,l)\big|(u,i)\big)$, it is more convenient to work with the dual QBD process rather than the dual pre-QBD process. Then, we can compute $\xi_\nu\big((x,l)\big|(u,i)\big)$ via the well-known sample paths dissection method shown in Ramaswami (2006), Breuer (2008, 2010), and references therein, and a generalization of the famous Neuts' matrix geometric methods (see, e.g., Neuts (1981, 1989), Latouche and Ramaswami (1999) and He (2014)). To this end, we first give the matrix representation of the one-step transition probabilities of the dual QBD process, and then, discuss the recurrence and transience of the dual QBD process as its discussion is important for the development of the algorithm.

### 3.3.1 TPM of dual QBD process

First, let $\boldsymbol{Q}$ denote the TPM of $\{(X_t, W_t), t \in \mathbb{N}\}$ and let $\boldsymbol{Q}'$ denote the TPM of $\{(L_t, V_t, W_t), t \in \mathbb{N}\}$. Let the portion of the TPM $\boldsymbol{Q}$ which governs the transition probabilities corresponding to the increase of $i$ units be denoted by $\boldsymbol{A}_i$, $i \in \{c(j), j \in \mathcal{J}\}$. Let $c^{-1}(i)$ denote the inverse function which outputs the phases $\{j\}$ such that $c(j) = i$, i.e., $c^{-1}(i) = \{j : j \in \mathcal{J}, c(j) = i\}$. Then, according to (3.2.1), $\boldsymbol{A}_i$ is given by the following block representation:

$$\boldsymbol{A}_i = \begin{pmatrix} \boldsymbol{A}_{i,1,1} & \boldsymbol{A}_{i,1,2} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{A}_{i,1,1}$ is an $m \times m$ matrix whose rows are 0 row vectors except the $j$-th rows given by the $j$-th rows of $P_0$, for $j \in c^{-1}(i)$, $\boldsymbol{A}_{i,1,2}$ is an $m \times \sum_{i=1}^{m}\sum_{j=1}^{m} n^{(i,j)}$ matrix whose rows are 0 row vectors except the $j$-th rows given by

$$(\mathbf{0}, p_{1,j,1}\boldsymbol{\alpha}^{(j,1)}, p_{1,j,2}\boldsymbol{\alpha}^{(j,2)}, \ldots, p_{1,j,m-1}\boldsymbol{\alpha}^{(j,m-1)}, p_{1,j,m}\boldsymbol{\alpha}^{(j,m)}, \mathbf{0}),$$

for $j \in c^{-1}(i)$, and $\mathbf{0}$ is a conformable matrix (vector) of zeros.

Now, let the $\sum_{i=1}^{m}\sum_{j=1}^{m} n^{(i,j)} \times \sum_{i=1}^{m}\sum_{j=1}^{m} n^{(i,j)}$ square block diagonal matrix

$$\boldsymbol{B}_{2,2} = \begin{pmatrix} \boldsymbol{U}^{(1,1)} & & & & & & & & \\ & \boldsymbol{U}^{(1,2)} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \boldsymbol{U}^{(1,m)} & & & & & \\ & & & & \boldsymbol{U}^{(2,1)} & & & & \\ & & & & & \boldsymbol{U}^{(2,2)} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \boldsymbol{U}^{(m,m-1)} & \\ & & & & & & & & \boldsymbol{U}^{(m,m)} \end{pmatrix}$$

denote the portion of $\boldsymbol{Q}$ governing the transition probabilities within $\mathcal{S}_2$, and let the $\sum_{i=1}^{m}\sum_{j=1}^{m} n^{(i,j)} \times m$ block matrix

$$\boldsymbol{B}_{2,1} = \begin{pmatrix} (\boldsymbol{\gamma}^{(1,1)})^\intercal & & & \\ & (\boldsymbol{\gamma}^{(1,2)})^\intercal & & \\ & & \ddots & \\ & & & (\boldsymbol{\gamma}^{(1,m)})^\intercal \\ (\boldsymbol{\gamma}^{(2,1)})^\intercal & & & \\ & (\boldsymbol{\gamma}^{(2,2)})^\intercal & & \\ & & \ddots & \\ & & & (\boldsymbol{\gamma}^{(2,m)})^\intercal \\ \vdots & \vdots & \vdots & \vdots \\ (\boldsymbol{\gamma}^{(m,1)})^\intercal & & & \\ & (\boldsymbol{\gamma}^{(m,2)})^\intercal & & \\ & & \ddots & \\ & & & (\boldsymbol{\gamma}^{(m,m)})^\intercal \end{pmatrix}$$

denote the portion of $\boldsymbol{Q}$ governing the transition probabilities from $\mathcal{S}_2$ to $\mathcal{S}_1$. Then, the portion of $\boldsymbol{Q}$ corresponding to the decrease of 1 unit is given by

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{B}_{2,1} & \boldsymbol{B}_{2,2} \end{pmatrix}.$$

Finally, the TPM $\boldsymbol{Q}$ of $\{(X_t, W_t), t \in \mathbb{N}\}$ can be written as

$$\boldsymbol{Q} = \begin{array}{c} \\ \vdots \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} \cdots & -1 & 0 & 1 & \cdots & c_{max} & c_{max}+1 & c_{max}+2 & \cdots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \boldsymbol{B} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{c_{max}} & \boldsymbol{0} & \boldsymbol{0} & \cdots \\ \cdots & \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{A}_0 & \cdots & \boldsymbol{A}_{c_{max}-1} & \boldsymbol{A}_{c_{max}} & \boldsymbol{0} & \cdots \\ \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B} & \cdots & \boldsymbol{A}_{c_{max}-2} & \boldsymbol{A}_{c_{max}-1} & \boldsymbol{A}_{c_{max}} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Then, by the definition of the dual QBD process $\{(L_t, V_t, W_t), t \in \mathbb{N}\}$, its TPM $\boldsymbol{Q}'$ can be written as

$$\boldsymbol{Q}' = \begin{array}{c} \\ \vdots \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \cdots & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \boldsymbol{D}_2 & \boldsymbol{D}_1 & \boldsymbol{D}_0 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots \\ \cdots & \boldsymbol{0} & \boldsymbol{D}_2 & \boldsymbol{D}_1 & \boldsymbol{D}_0 & \boldsymbol{0} & \boldsymbol{0} & \cdots \\ \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_2 & \boldsymbol{D}_1 & \boldsymbol{D}_0 & \boldsymbol{0} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array},$$

where

$$\boldsymbol{D}_0 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ c_{max}-1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & c_{max}-1 \end{array} \\ \begin{pmatrix} \boldsymbol{A}_{c_{max}} & & & \\ \boldsymbol{A}_{c_{max}-1} & \boldsymbol{A}_{c_{max}} & & \\ \vdots & \ddots & \ddots & \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{c_{max}} \end{pmatrix} \end{array},$$

$$
\boldsymbol{D}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ c_{max}-1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & c_{max}-3 & c_{max}-2 & c_{max}-1 \end{array} \\ \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_{c_{max}-3} & \boldsymbol{A}_{c_{max}-2} & \boldsymbol{A}_{c_{max}-1} \\ \boldsymbol{B} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{c_{max}-4} & \boldsymbol{A}_{c_{max}-3} & \boldsymbol{A}_{c_{max}-2} \\ \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{A}_0 & \cdots & \boldsymbol{A}_{c_{max}-5} & \boldsymbol{A}_{c_{max}-4} & \boldsymbol{A}_{c_{max}-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \cdots & \cdots & \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{A}_0 \end{pmatrix} \end{array},
$$

and

$$
\boldsymbol{D}_2 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ c_{max}-1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & c_{max}-1 \end{array} \\ \begin{pmatrix} & & & \boldsymbol{B} \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}.
$$

### 3.3.2 Recurrence and transience of dual QBD process

In this subsection, we briefly discuss the recurrence and transience of the dual QBD process, as it plays a key role in the computation of certain matrices to be used in calculating the discounted joint conditional pmf. For simplicity, assume that $\{W_t, t \in \mathbb{N}\}$ forms an irreducible finite-state Markov chain. Let $\boldsymbol{\theta}$ denote the stationary distribution vector of the process $\{W_t, t \in \mathbb{N}\}$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_m)$ is the unique positive solution to

$$
\boldsymbol{\theta}\left(\sum_{i=0}^{c_{max}} \boldsymbol{A}_i + \boldsymbol{B}\right) = \boldsymbol{\theta},
$$

$$
\boldsymbol{\theta}\boldsymbol{1}^{\intercal} = 1.
$$

Then, the dual pre-QBD process is transient if and only if

$$\boldsymbol{\theta}\left(\sum_{i=1}^{c_{max}} i\boldsymbol{A}_i\right)\mathbf{1}^\intercal \neq \boldsymbol{\theta}\boldsymbol{B}\mathbf{1}^\intercal. \tag{3.3.3}$$

Moreover, if

$$\boldsymbol{\theta}\left(\sum_{i=1}^{c_{max}} i\boldsymbol{A}_i\right)\mathbf{1}^\intercal > \boldsymbol{\theta}\boldsymbol{B}\mathbf{1}^\intercal, \tag{3.3.4}$$

then

$$\Pr\{\kappa < \infty\} < 1.$$

Note that the above statements apply equally to the dual QBD process. Furthermore, by noting that the events $\{\tau < \infty\}$ and $\{\kappa < \infty\}$ are equal in probability, we can see that the security loading condition of the original risk process $\{(U_t, W_t), t \in \mathbb{N}\}$ is also given by (3.3.4).

Proofs for the above statements are available in many textbooks on the theory of Markov chains and matrix analytic methodology (see, e.g., Latouche and Ramaswami (1999), pp. 155-158).

### 3.3.3 Key matrices

With the block representation of the TPM of the dual QBD process given, we now define some key matrices used in the algorithm for computing the discounted joint conditional pmf. First of all, let $\eta(v)$ denote the random time that the dual QBD process visits level $v$ for the first time and $\kappa_v^-$ denote the time the process falls below level $v$ for the first time after time 0. We also extend the definition of $s_1([h,k])$ to the nonnegative integer-valued random variables $h$ and $k$. Now, for $v, u, x \in \mathbb{N}$, $v < u, x$, let ${}^{\nu}\boldsymbol{G}_{u,v}$ and ${}^{\nu}\boldsymbol{R}_{v,x}$ be

$(|\mathcal{W}|\times c_{max})\times(|\mathcal{W}|\times c_{max})$ square matrices whose $\big((r_1,\boldsymbol{i}_1),(r_2,\boldsymbol{i}_2)\big)$-th and $\big((r_2,\boldsymbol{i}_2),(r_3,\boldsymbol{i}_3)\big)$-th entries are given by

$$\begin{aligned}&({}^{\nu}\boldsymbol{G}_{u,v})_{(r_1,\boldsymbol{i}_1),(r_2,\boldsymbol{i}_2)}\\&\quad=E\{\nu^{s_1([0,\eta(v)-1])}\mathcal{I}[\eta(v)<\infty,(L_{\eta(v)},V_{\eta(v)},W_{\eta(v)})=(v,r_2,\boldsymbol{i}_2)]|(L_0,V_0,W_0)=(u,r_1,\boldsymbol{i}_1)\}\end{aligned}$$

and

$$\begin{aligned}&({}^{\nu}\boldsymbol{R}_{v,x})_{(r_3,\boldsymbol{i}_3),(r_4,\boldsymbol{i}_4)}\\&\quad=\sum_{k=1}^{\infty}E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa_{v+1}^{-}>k-1,(L_{k-1},V_{k-1},W_{k-1})=(x,r_3,\boldsymbol{i}_3)]|(L_0,V_0,W_0)=(v,r_2,\boldsymbol{i}_2)\},\\&\qquad r_1,r_2,r_3\in\{0,1,\ldots,c_{max}-1\},\ \boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\in\mathcal{W},\ u,v,x\in\mathbb{N},\ v<u,x.\end{aligned}$$

Also, let ${}^{\nu}\boldsymbol{\Xi}_z$, $z\in\mathbb{N}$, be a $(|\mathcal{W}|\times c_{max})\times(|\mathcal{W}|\times c_{max})$ square matrix whose entries are given by

$$\begin{aligned}&({}^{\nu}\boldsymbol{\Xi}_z)_{(r_3,\boldsymbol{i}_3),(r_4,\boldsymbol{i}_4)}\\&\quad=\sum_{k=1}^{\infty}E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa>k-1,(L_{k-1},V_{k-1},W_{k-1})=(z,r_4,\boldsymbol{i}_4)]|(L_0,V_0,W_0)=(z,r_3,\boldsymbol{i}_3)\},\\&\qquad r_3,r_4\in\{0,1,\ldots,c_{max}-1\},\ \boldsymbol{i}_3,\boldsymbol{i}_4\in\mathcal{W},\ z\in\mathbb{N}.\end{aligned}$$

In Section 3.3.4, it will be shown that the discounted joint conditional pmf can be written in terms of the key matrices ${}^{\nu}\boldsymbol{G}_{u,v}$, ${}^{\nu}\boldsymbol{\Xi}_z$, and ${}^{\nu}\boldsymbol{R}_{v,x}$. Therefore, it only remains to compute these key matrices.

First of all, the level independence and the skip-free nature of the dual QBD process implies that ${}^{\nu}\boldsymbol{G}_{u,v}={}^{\nu}\boldsymbol{G}^{u-v}$ and ${}^{\nu}\boldsymbol{R}_{v,x}={}^{\nu}\boldsymbol{R}^{x-v}$, where ${}^{\nu}\boldsymbol{G}={}^{\nu}\boldsymbol{G}_{i,i-1}$ and ${}^{\nu}\boldsymbol{R}={}^{\nu}\boldsymbol{R}_{i,i+1}$, $\forall$ $i\in\mathbb{N}$. In fact, ${}^{\nu}\boldsymbol{G}$ and ${}^{\nu}\boldsymbol{R}$ are generalizations of the fundamental period and rate matrices that appear in Neuts' matrix geometric methods, and we can adopt the algorithms for computing the fundamental period and rate matrices in Neuts' matrix geometric methodology to our problem with a very minor alteration. Moreover, as it is the case in Neuts' matrix geometric methodology, ${}^{\nu}\boldsymbol{R}$ and ${}^{\nu}\Xi_0$ are completely determined by ${}^{\nu}\boldsymbol{G}$, and therefore, ${}^{\nu}\Xi_z$, $z \in \mathbb{N}$, as well. Hence, the computation of ${}^{\nu}\boldsymbol{G}$ leads us to the computation of all the other key matrices, and ultimately, the discounted joint conditional pmf. We defer the discussion on the algorithm for computing ${}^{\nu}\boldsymbol{G}$ to Section 3.3.5 as it requires separate attention. In the remaining part of this subsection, we assume that ${}^{\nu}\boldsymbol{G}$ has been computed and thus proceed to compute the remaining key matrices.

Define a $(|\mathcal{W}| \times c_{max}) \times (|\mathcal{W}| \times c_{max})$ diagonal matrix ${}^{\nu}\Lambda$ as

$$
{}^{\nu}\boldsymbol{\Lambda} = \begin{array}{c} \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_1 \\ \vdots \\ \mathcal{S}_2 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_1 & \cdots & \mathcal{S}_2 & \mathcal{S}_1 & \mathcal{S}_2 \end{array} \\
\begin{pmatrix}
\mathrm{diag}(\nu) & & & & & & \\
& \boldsymbol{I} & & & & & \\
& & \mathrm{diag}(\nu) & & & & \\
& & & \ddots & & & \\
& & & & \boldsymbol{I} & & \\
& & & & & \mathrm{diag}(\nu) & \\
& & & & & & \boldsymbol{I}
\end{pmatrix}
\end{array},
$$

where $\mathrm{diag}(\nu)$ denotes a diagonal matrix of an appropriate dimension with its diagonal entries set equal to $\nu$. (Essentially, the diagonal elements of ${}^{\nu}\boldsymbol{\Lambda}$ are set equal to $\nu$ if the diagonal entry corresponds to phases in $\mathcal{S}_1$ and set equal to 1 if the entry corresponds to phases in $\mathcal{S}_2$.) Then, following the same line of probabilistic reasoning used for proving, e.g., Latouche and Ramaswami (1999), Eqs. (8.2), (8.5), and (8.6), we have

$$
{}^{\nu}\Xi_0 = \left(\boldsymbol{I} - {}^{\nu}\boldsymbol{\Lambda}(\boldsymbol{D}_1 + \boldsymbol{D}_0 {}^{\nu}\boldsymbol{G})\right)^{-1} \tag{3.3.5}
$$

and

$$ {}^{\nu}\boldsymbol{R} = {}^{\nu}\boldsymbol{\Lambda}\boldsymbol{D}_0{}^{\nu}\boldsymbol{\Xi}_0. \tag{3.3.6} $$

Also, let $a$ be the minimum level visited by the dual QBD process before $\kappa$. Then, by conditioning on the minimum level $a$ that the dual QBD process visits before $\kappa$, the values of the dual QBD process at $\eta(a)$, and the last time the process visits level $a$ before $\kappa$, we can write

$$
\begin{aligned}
&({}^{\nu}\boldsymbol{\Xi}_z)_{(r_1,\boldsymbol{i}_1),(r_4,\boldsymbol{i}_4)} \\
&= \sum_{k=1}^{\infty} E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa > k-1, (L_{k-1}, V_{k-1}, W_{k-1}) = (z, r_4, \boldsymbol{i}_4)]|(L_0, V_0, W_0) = (z, r_1, \boldsymbol{i}_4)\}, \\
&= \sum_{a=0}^{z} \sum_{r_2=0}^{c_{max}-1} \sum_{\boldsymbol{i}_2\in\mathcal{W}} \sum_{r_3=0}^{c_{max}-1} \sum_{\boldsymbol{i}_3\in\mathcal{W}} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \\
&\qquad \times E\{\nu^{s_1([0,\eta(z)-1])}\mathcal{I}[\eta(a) < \infty, (L_{\eta(a)}, V_{\eta(a)}, W_{\eta(z)}) = (a, r_2, \boldsymbol{i}_2)]|(L_0, V_0, W_0) = (z, r_1, \boldsymbol{i}_1)\} \\
&\qquad \times E\{\nu^{s_1([0,l-2])}\mathcal{I}[\kappa > l-1, (L_{l-1}, V_{l-1}, W_{l-1}) = (0, r_3, \boldsymbol{i}_3)]|(L_0, V_0, W_0) = (0, r_2, \boldsymbol{i}_2)\} \\
&\qquad \times E\{\nu^{s_1([0,k-2])}\mathcal{I}[\kappa_1^- > k-1, (L_{k-1}, V_{k-1}, W_{k-1}) = (z-a, r_4, \boldsymbol{i}_4)]|(L_0, V_0, W_0) = (0, r_3, \boldsymbol{i}_3)\},
\end{aligned} \tag{3.3.7}
$$

where the second and the last equalities follow from the strong Markov property and stationarity, respectively, and the last two conditional expectations appearing in the last equality follow from the fact that $a$ is the minimum level the dual QBD process visits as well as the level independence of the dual QBD process. Finally, writing (3.3.7) in matrix equation form, we obtain

$$ {}^{\nu}\boldsymbol{\Xi}_z = \sum_{a=0}^{z} {}^{\nu}\boldsymbol{G}^{z-a}\,{}^{\nu}\boldsymbol{\Xi}_0\,{}^{\nu}\boldsymbol{R}^{z-a}, \ z \in \mathbb{N}. \tag{3.3.8} $$

An efficient way of computing (3.3.8) was suggested by Ramaswami (2006) in the continuous-time version of a similar problem. Based on the vectorization operator and the identity that

$$vec(\boldsymbol{AXB}) = (\boldsymbol{B}^{\intercal} \otimes \boldsymbol{A})vec(\boldsymbol{X})$$

for conformable matrices (see, e.g., Bernstein (2005)), one can rewrite (3.3.8) as

$$\begin{aligned} vec({}^{\nu}\Xi_z) &= \sum_{a=0}^{z} vec({}^{\nu}\boldsymbol{G}^{z-a}\,{}^{\nu}\Xi_0\,{}^{\nu}\boldsymbol{R}^{z-a}) \\ &= \sum_{a=0}^{z}(({}^{\nu}\boldsymbol{R}^{z-a})^{\intercal} \otimes {}^{\nu}\boldsymbol{G}^{z-a})vec({}^{\nu}\Xi_0) \\ &= \sum_{a=0}^{z}({}^{\nu}\boldsymbol{R}^{\intercal} \otimes {}^{\nu}\boldsymbol{G})^{z-a}vec({}^{\nu}\Xi_0) \\ &= (\boldsymbol{I} - ({}^{\nu}\boldsymbol{R}^{\intercal} \otimes {}^{\nu}\boldsymbol{G}))^{-1}(I - ({}^{\nu}\boldsymbol{R}^{\intercal} \otimes {}^{\nu}\boldsymbol{G})^{z+1})vec({}^{\nu}\Xi_0), \end{aligned} \tag{3.3.9}$$

where the invertibility in the last line is justified if and only if the spectral radius of ${}^{\nu}\boldsymbol{R}^{\intercal} \otimes {}^{\nu}\boldsymbol{G}$ is strictly less than 1. Indeed, by Corollary 7.1.2 of Latouche and Ramaswami (1999), if the dual QBD process is transient in the negative direction, we have that $\mathrm{sp}({}^{1}\boldsymbol{R})$, the spectral radius of ${}^{1}\boldsymbol{R}$, is strictly less than 1. Then, from the definition of ${}^{\nu}\boldsymbol{R}$, we can deduce that $\|{}^{\nu}\boldsymbol{R}^{k}\|_{\max} \leq \|{}^{1}\boldsymbol{R}^{k}\|_{\max}$, for all $k \in \mathbb{N}$, $\nu \in \mathbb{C}$, $|\nu| \leq 1$. Then, by Gelfand's formula (see, e.g., Kozyakin (2009)), we have

$$\mathrm{sp}({}^{\nu}\boldsymbol{R}) = \lim_{k\to\infty} \|{}^{\nu}\boldsymbol{R}^{k}\|_{\max}^{\frac{1}{k}} \leq \lim_{k\to\infty} \|{}^{1}\boldsymbol{R}^{k}\|_{\max}^{\frac{1}{k}} = \mathrm{sp}({}^{1}\boldsymbol{R}) < 1.$$

If the dual QBD process is transient in the positive direction, we find that ${}^{1}\boldsymbol{G}$ is a sub-stochastic matrix, and hence, $\mathrm{sp}({}^{1}\boldsymbol{G}) < 1$. Then, similar to the case of ${}^{\nu}\boldsymbol{R}$ , this implies that for $\nu \in \mathbb{C}$, $|\nu| \leq 1$, $\mathrm{sp}({}^{\nu}\boldsymbol{G}) \leq \mathrm{sp}({}^{1}\boldsymbol{G}) < 1$. Noting that $\mathrm{sp}({}^{1}\boldsymbol{R}) \leq 1$ (see, e.g., Proposition 3.2.5 of He (2014)) and $\mathrm{sp}({}^{1}\boldsymbol{G}) \leq 1$ in any case (recurrent or transient), and that the spectral radii of Kronecker products are bounded above by the products of the spectral radii of the respective matrices, we indeed have the spectral radius of ${}^{\nu}\boldsymbol{R}^{\intercal} \otimes {}^{\nu}\boldsymbol{G}$ is strictly less than 1 when the dual QBD process is transient. (The case where the dual QBD process is recurrent is rare in insurance risk theory.)

### 3.3.4 Formulas for $\xi_\nu\big((x,l)\big|(u,i)\big)$

Let $\boldsymbol{e}_j$ be a row vector whose $j$-th entry is 1 and all the others are 0. The size of $\boldsymbol{e}_j$ will be determined to be conformable where it appears. Then, once we have the key matrices computed, we can proceed to evaluate $\xi_\nu\big((x,l)\big|(u,i)\big)$ by considering the following two cases:

**Case 1:** $\ell_1(u) \le \ell_1(x)$ By conditioning on the last time the dual QBD process visits level $\ell_1(u)$ before $\kappa$ and the value of $W_t$ at that particular epoch, we have

$$\xi_\nu\big((x,l)\big|(u,i)\big) = \boldsymbol{e}_{(\ell_2(u),i)}\,{}^{\nu}\boldsymbol{\Xi}_{\ell_1(u)}\,{}^{\nu}\boldsymbol{R}^{\ell_1(x)-\ell_1(u)}\boldsymbol{e}^{\intercal}_{(\ell_2(x),l)}. \tag{3.3.10}$$

**Case 2:** $\ell_1(u) > \ell_1(x)$ By conditioning on the time the dual QBD process reaches level $\ell_1(x)$ for the first time and the value of $W_t$ at that particular epoch, we obtain

$$\xi_\nu\big((x,l)\big|(u,i)\big) = \boldsymbol{e}_{(\ell_2(u),i)}\,{}^{\nu}\boldsymbol{G}^{\ell_1(u)-\ell_1(x)}\,{}^{\nu}\boldsymbol{\Xi}_{\ell_1(x)}\boldsymbol{e}^{\intercal}_{(\ell_2(x),l)}. \tag{3.3.11}$$

### 3.3.5 Algorithm for ${}^{\nu}\boldsymbol{G}$

The algorithm for ${}^{\nu}\boldsymbol{G}$ to be introduced here is a generalization of the Logarithmic-Reduction (L-R) algorithm by Latouche and Ramaswami (1999). It is quadratically convergent when the dual QBD process is transient. We use the same notation as in Latouche and Ramaswami (1999), so that the interested readers can readily refer to the textbook for more details.

First of all, let

$$
\begin{aligned}
{}^{\nu}\boldsymbol{H}^{(0)} &= (\boldsymbol{I} - {}^{\nu}\boldsymbol{\Lambda}\boldsymbol{D}_1)^{-1}\,{}^{\nu}\boldsymbol{\Lambda}\boldsymbol{D}_0, \\
{}^{\nu}\boldsymbol{L}^{(0)} &= (\boldsymbol{I} - {}^{\nu}\boldsymbol{\Lambda}\boldsymbol{D}_1)^{-1}\,{}^{\nu}\boldsymbol{\Lambda}\boldsymbol{D}_2,
\end{aligned}
$$

and for $k \in \mathbb{Z}^+$, recursively define

$$
\begin{aligned}
{}^{\nu}\boldsymbol{H}^{(k)} &= (\boldsymbol{I} - {}^{\nu}\boldsymbol{U}^{(k-1)})^{-1}({}^{\nu}\boldsymbol{H}^{(k-1)})^2, \\
{}^{\nu}\boldsymbol{L}^{(k)} &= (\boldsymbol{I} - {}^{\nu}\boldsymbol{U}^{(k-1)})^{-1}({}^{\nu}\boldsymbol{L}^{(k-1)})^2,
\end{aligned}
$$

where

$$
{}^{\nu}\boldsymbol{U}^{(k)} = {}^{\nu}\boldsymbol{H}^{(k)}\,{}^{\nu}\boldsymbol{L}^{(k)} + {}^{\nu}\boldsymbol{L}^{(k)}\,{}^{\nu}\boldsymbol{H}^{(k)}, \;\; k \in \mathbb{N}. \tag{3.3.12}
$$

Then, we have

$$
{}^{\nu}\boldsymbol{G} = \sum_{k=0}^{\infty} \left( \prod_{i=0}^{k-1} {}^{\nu}\boldsymbol{H}^{(i)} \right) {}^{\nu}\boldsymbol{L}^{(k)}, \tag{3.3.13}
$$

and if the dual QBD process is transient, the sequence $\{{}^{\nu}\boldsymbol{G}^{(k)} = \sum_{l=0}^{k}(\prod_{i=0}^{l-1} {}^{\nu}\boldsymbol{H}^{(i)})\,{}^{\nu}\boldsymbol{L}^{(l)}\}_{k=0}^{\infty}$ quadratically converges to ${}^{\nu}\boldsymbol{G}$, for $\nu \in \mathbb{C}$, $|\nu| \le 1$.

The proof of the quadratic convergence of the above algorithm follows the exact same line of probabilistic reasoning used for proving that of the L-R algorithm. However, the probabilistic interpretations of the matrices $\{({}^{\nu}\boldsymbol{H}^{(k)}, {}^{\nu}\boldsymbol{L}^{(k)}), k \in \mathbb{N}\}$ here and $\{(\boldsymbol{H}^{(k)}, \boldsymbol{L}^{(k)}), k \in \mathbb{N}\}$ in Latouche and Ramaswami (1999) are slightly different.

Recalling the definition of $\eta(i) = \inf\{t \in \mathbb{N} : L_t = i\}$, $i \in \mathbb{Z}$, the $((r_1, \boldsymbol{j}_1), (r_2, \boldsymbol{j}_2))$-th entries of ${}^{\nu}\boldsymbol{H}^{(k)}$ and ${}^{\nu}\boldsymbol{L}^{(k)}$ are given by

$$
\begin{aligned}
&({}^{\nu}\boldsymbol{H}^{(k)})_{(r_1,\boldsymbol{j}_1),(r_2,\boldsymbol{j}_2)} \\
&= E\{\nu^{s_1[0,\eta(2^{k+1}-1)-1]}\mathcal{I}[\eta(2^{k+1}-1) < \kappa, (L_{\eta(2^{k+1}-1)}, V_{\eta(2^{k+1}-1)}, W_{\eta(2^{k+1}-1)}) = (2^{k+1}-1, r_2, \boldsymbol{j}_2)]| \\
&\qquad\qquad (L_0, V_0, W_0) = (2^k - 1, r_1, \boldsymbol{j}_1)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&({}^{\nu}\boldsymbol{L}^{(k)})_{(r_1,\boldsymbol{j}_1),(r_2,\boldsymbol{j}_2)} \\
&= E\{\nu^{s_1[0,\kappa-1]}\mathcal{I}[\kappa < \eta(2^{k+1}-1), (L_\kappa, V_\kappa, W_\kappa) = (-1, r_2, \boldsymbol{j}_2)]|(L_0, V_0, W_0) = (2^k - 1, r_1, \boldsymbol{j}_1)\}
\end{aligned}
$$

respectively. Now, with the above definition, one can directly follow the proof presented in Latouche and Ramaswami (1999), pp. 187-197, with ${}^{\nu}\boldsymbol{H}^{(k)}$ and ${}^{\nu}\boldsymbol{L}^{(k)}$ in place of $\boldsymbol{H}^{(k)}$ and $\boldsymbol{L}^{(k)}$ therein.

## 3.4 Numerical analysis

In this example, we examine the impact of implementing phase-dependent premium rates in a MAP risk model. The risk model we consider here is comprised of the claims arrival MAP $\{(N_t, J_t), t \in \mathbb{N}\}$ with the respective TPMs without arrival and with arrival given by

$$
\boldsymbol{P}_0 = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} p_{0,0,0} & p_{0,0,1} \\ p_{0,1,0} & p_{0,1,1} \end{pmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \end{pmatrix} \end{matrix}
$$

and

$$\boldsymbol{P}_1 = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} p_{1,0,0} & p_{1,0,1} \\ p_{1,1,0} & p_{1,1,1} \end{pmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.24 & 0.16 \\ 0.16 & 0.24 \end{pmatrix} \end{array}.$$

Concerning the claim size distributions, $Y^{(0,0)}$ and $Y^{(0,1)}$ follow zero-truncated geometric distributions with pmf $f^{(0,0)}(y) = f^{(0,1)}(y) = (1-\rho_0)^{y-1}\rho_0$, and $Y^{(1,0)}$ and $Y^{(1,1)}$ follow zero-truncated geometric distributions with pmf $f^{(1,0)}(y) = f^{(1,1)}(y) = (1-\rho_1)^{y-1}\rho_1$, $y \in \mathbb{Z}^+$, where $\rho_0 = 0.6$ and $\rho_1 = 0.2$. To investigate the impact of setting premium rates different for phases 0 and 1, we consider two different cases. In the first case, we set the premium rates equal to $c(0) = 1$ and $c(1) = 3$, which are chosen specifically to reflect the difference in the expected claim sizes when the phase process is in either state. In the second case, we set the premium rates equal to $c(0) = c(1) = 2$, which is the average of the premium rates in the first case. Let $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$ denote the risk processes of the first and second case, respectively. We next give the dual QBD representation of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and the algorithmic procedure of the matrix analytic methodology. The dual QBD representation of $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$ can be established in a similar fashion.

Let $\{(X_t, W_t), t \in \mathbb{N}\}$ denote the dual pre-QBD process of the risk model. Let $\mathcal{S}_1 = \{0, 1\}$ and $\mathcal{S}_2 = \{3, 4\}$. Recalling Section 3.2.2, set $X_{t+1} = X_t + c(W_t)$, $t \in \mathbb{N}$, when $W_t \in \mathcal{S}_1$, and $X_{t+1} = X_t - 1$, $t \in \mathbb{N}$, when $W_t \in \mathcal{S}_2$. Then, following the matrix notation given in Section 3.3.1, we write

$$A_3 = \begin{array}{c} \\ 0 \\ 1 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$A_2 = \mathbf{0},$$

$$A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.4 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and

$$B = \begin{array}{c} \\ 0 \\ 1 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (0.6)(0.6) & (0.6)(0.4) & 0.4 & 0 \\ (0.2)(0.4) & (0.2)(0.6) & 0 & 0.8 \end{pmatrix} \end{array},$$

and follow the rest of the procedure described in Section 3.3.1 to construct $D_0$, $D_1$, and

$\boldsymbol{D_2}$ with the above-defined $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, $\boldsymbol{A}_3$, and $\boldsymbol{B}$. Finally, letting

$$^{\nu}\boldsymbol{\Lambda} =$$

$$\begin{array}{c} \\ (0,0) \\ (0,1) \\ (0,3) \\ (0,4) \\ (1,0) \\ (1,1) \\ (1,3) \\ (1,4) \\ (2,0) \\ (2,1) \\ (2,3) \\ (2,4) \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} (0,0) & (0,1) & (0,3) & (0,4) & (1,0) & (1,1) & (1,3) & (1,4) & (2,0) & (2,1) & (2,3) & (2,4) \end{array} \\
\left(\begin{array}{cccccccccccc}
\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array},$$

for $\nu \in \mathbb{C}$, $|\nu| \leq 1$, we have all the ingredients to compute the key matrices introduced in Section 3.3.3.

In what follows next, we investigate the impact of incorporating phase-dependent premium rates in risk modeling by comparing the infinite-time ruin probabilities, finite-time ruin probabilities, and another set of time-dependent quantities of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$. For computing the above quantities of interest, we implement the matrix analytic methodology developed in this chapter.

Before we proceed to our investigation, however, we provide a table which compares the values of the joint conditional pmf of the time of ruin, surplus prior to ruin, and

| | Matrix analytic methodology | | Recursion | |
|---|---|---|---|---|
| $n$ | $\phi(n, 50, 1\|50, 0)$ | $\phi(n, 50, 1\|50, 1)$ | $\phi(n, 50, 1\|50, 0)$ | $\phi(n, 50, 1\|50, 1)$ |
| 10 | $2.16995 \times 10^{-6}$ | $2.1849 \times 10^{-6}$ | $2.16995 \times 10^{-6}$ | $2.1849 \times 10^{-6}$ |
| 20 | $1.37253 \times 10^{-6}$ | $1.46487 \times 10^{-6}$ | $1.37253 \times 10^{-6}$ | $1.46487 \times 10^{-6}$ |
| 30 | $1.03762 \times 10^{-6}$ | $1.12762 \times 10^{-6}$ | $1.03762 \times 10^{-6}$ | $1.12762 \times 10^{-6}$ |
| 40 | $8.37005 \times 10^{-7}$ | $9.18475 \times 10^{-7}$ | $8.37005 \times 10^{-7}$ | $9.18475 \times 10^{-7}$ |
| 50 | $6.96102 \times 10^{-7}$ | $7.69737 \times 10^{-7}$ | $6.96102 \times 10^{-7}$ | $7.69737 \times 10^{-7}$ |
| 100 | $3.23488 \times 10^{-7}$ | $3.6867 \times 10^{-7}$ | $3.23488 \times 10^{-7}$ | $3.6867 \times 10^{-7}$ |

Table 3.1: Joint conditional pmf of time of ruin, surplus prior to ruin, and deficit at ruin

deficit at ruin, denoted by $\phi(n, x, y|u, i) = \Pr\{\tau = n, U_{\tau-1} = x, |U_\tau| = y|U_0 = u, J_0 = i\}$, $x, u \in \mathbb{N}$, $n, y \in \mathbb{Z}^+$, $i = 0, 1$, computed via our matrix analytic methodology and the standard recursive method discussed in Example 3 in Section 2.5 to check our matrix analytic methodology's soundness. The values of $\phi(n, x, y|u, i)$ computed via the matrix analytic methodology are numerically inverted via the Lattice-Poisson algorithm in Abate and Whitt (1992). The error bound used for the Lattice-Poisson algorithm is $10^{-8}$.

First of all, in Table 3.1, we see that the differences in the computed values are within the error bound used for the inversion algorithm. Secondly, our matrix analytic methodology performed slower than the standard recursion method for smaller values of $n$ in Table 3.1, but for larger values of $n$ in Table 3.1, our methodology started to outperform the recursion method. As stated in Section 1.1, the recursion method's computation time increases rapidly (nearly quadratic) as $n$ increases. On the other hand, as it was the case in Example 3 in Section 2.5, the computation time of our methodology in this chapter for computing a single value of the generating function of $\phi(n, x, y|u)$ has an upper bound at $\nu = 1$. Therefore, the computational complexity of our methodology only depends on the inversion algorithm that is used (as per its dependency on $n$). Noting that the Lattice-Poisson algorithm we implemented here is $O(n)$, the computation time of our matrix analytic methodology grows linearly in $n$. Unlike Example 3 in Section 2.5 however, here we are seeing the advantage of the lower computational complexity of our methodology much earlier than in Example 3 in Section 2.5, due to the much simplified

| $n$ | $\Psi_1(n|50,0)$ | $\Psi_1(n|50,1)$ | $\Psi_2(n|50,0)$ | $\Psi_2(n|50,1)$ |
|---|---|---|---|---|
| 50 | 0.00116113 | 0.00133148 | 0.00179486 | 0.00268845 |
| 100 | 0.00182098 | 0.00199151 | 0.00283758 | 0.00396366 |
| 200 | 0.00203042 | 0.00219702 | 0.00319861 | 0.00438187 |
| 300 | 0.00204346 | 0.00220974 | 0.0032258 | 0.00441244 |
| 500 | 0.00204442 | 0.00221068 | 0.0032283 | 0.00441522 |
| 1000 | 0.00204443 | 0.00221069 | 0.00322833 | 0.00441525 |
| $\lim_{n\to\infty}$ | 0.00204443 | 0.00221069 | 0.00322833 | 0.00441525 |

Table 3.2: Time of ruin distribution

algorithmic procedure arising from the simple skip-free sample paths structure of the QBD process. Furthermore, as stated in Section 1.1, the recursion method's computer memory consumption rate grows linearly in $n$. On the other hand, by the procedural structure of our methodology, the computer memory consumption rate of our matrix analytic methodology stays constant in $n$. Therefore, for large scale problems where larger values of $n$ are of interest, the matrix analytic methodology introduced in this chapter provides an excellent alternative to the standard recursion method. Moreover, as we will see in the following discussions, for certain quantities of interest, the matrix analytic methodology outperforms the recursion method in terms of both speed and memory consumption by a significant margin.

First of all, we compare the infinite-time ruin probabilities and finite-time ruin probabilities of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$. Let $\Psi_1(n|u,i) = \Pr\{\tau_1 \leq n | U_0^{(1)} = u, J_0 = i\}$ and $\Psi_2(n|u,i) = \Pr\{\tau_2 \leq n | U_0^{(2)} = u, J_0 = i\}$ denote the finite-time ruin probabilities of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$, where $\tau_1$ and $\tau_2$ denote the time of ruin of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$, respectively. For the finite-time ruin probabilities, we only need to compute ${}^{\nu}\boldsymbol{G}$ and evaluate ${}^{\nu}\boldsymbol{G}^{\ell_1(u)}$, for the initial surplus level $u \in \mathbb{N}$ and for the values $\nu \in \mathbb{C}$ that are required for the numerical inversion algorithm. We note that this is significantly more computationally efficient than the recursion method.

As can be seen in Table 3.2, the ruin probabilities are higher for the risk model with

| $x$ | $F_1(500, x\|50, 0)$ | $F_1(500, x\|50, 1)$ | $F_2(500, x\|50, 0)$ | $F_2(500, x\|50, 1)$ |
|---|---|---|---|---|
| 60 | 0.997955 | 0.997789 | 0.99677 | 0.995583 |
| 150 | 0.99774 | 0.997578 | 0.996435 | 0.995218 |
| 300 | 0.90592 | 0.907642 | 0.902033 | 0.896299 |
| 360 | 0.652863 | 0.657865 | 0.659253 | 0.648321 |
| 420 | 0.283029 | 0.288669 | 0.29859 | 0.288677 |
| 480 | 0.0567086 | 0.0589945 | 0.0633911 | 0.0599531 |
| 600 | $9.54045 \times 10^{-5}$ | 0.000105992 | $8.86556 \times 10^{-5}$ | $7.9621 \times 10^{-5}$ |

Table 3.3: Transient distribution of surplus process

a single premium rate. Moreover, we can see that the ruin probabilities of the risk model with phase-dependent premium rates are less sensitive to the initial phase than those of the risk model with a single premium rate. To take a closer look at the root of such results, we compare the transient distributions of the risk processes of both cases.

Let $F_1(n, x|u, i) = \Pr\{U_n^{(1)} > x, \tau_1 > n | U_0^{(1)} = u, J_0 = i\}$ and $F_2(n, x|u, i) = \Pr\{U_n^{(2)} > x, \tau_2 > n | U_0^{(2)} = u, J_0 = i\}$ denote the transient tail distributions of $\{(U_t^{(1)}, J_t), t \in \mathbb{N}\}$ and $\{(U_t^{(2)}, J_t), t \in \mathbb{N}\}$, respectively. What these quantities will reveal is how the risk processes of both cases behave over time. For our purposes, we choose the value of $n = 500$ and compare the values of $F_1(500, x|50, i)$ and $F_2(500, x|50, i)$, $i = 0, 1$, over some values of $x$. We compute the values of $F_1(500, x|50, i)$ and $F_2(500, x|50, i)$, $i = 0, 1$, by computing $\xi_\nu\big((x, l)|(u, i)\big)$ as in (3.3.10) and (3.3.11), summing $\xi_\nu\big((x, l)|(u, i)\big)$ over the values of $x$ of interest and $l = 0, 1$, and numerically inverting the sum via the Lattice-Poisson inversion algorithm. Once again, the error bound used for the inversion algorithm is $10^{-8}$.

From Table 3.3, we observe that the risk process is more variable when there is a single premium rate implemented than when phase-dependent premium rates are implemented. More specifically, for the time variable $n = 500$, the transient distribution of the risk process of the case with a single premium rate exhibits stronger concentration on the two extreme ends compared to that of the risk process of the case with phase-dependent premium rates. Since only the premium rates differ between the two models, it is clear that the higher ruin probabilities of the risk model with a single premium rate in Table 3.2 are attributable to the higher probabilities of its risk process staying in the danger zone (i.e., lower values of $x$ where ruin is more likely to occur) than the risk process of the case with phase-dependent premium rates. Hence, it seems that the implementation of appropriate phase-dependent premium rates reduces the variability of the risk process, and subsequently, the ruin probabilities in the context of a MAP risk model.

# Chapter 4

# A matrix analytic methodology for the continuous-time MAP risk model with phase-dependent premium rates and a dynamic individual risk model

## 4.1 Introduction

Extending the methodology developed in Chapter 2 to continuous-time MAP risk models directly through Ahn and Ramaswami's original approach may be possible. However, Ahn and Ramaswami's approach involves nonelementary mathematical tools and quite complex coupled queues, and to extend their methodology to include phase-dependent premium rates directly through their original approach is certainly not a simple task. Fortunately, we have recently come across a paper by Bean and O'Reilly (2013), which introduces an efficient matrix-based algorithm for computing some quantities of interest in multidimensional fluid flow models. As first noted in Ahn et al. (2018), Bean and O'Reilly's methodology naturally finds its application in risk theory. Although not specifically discussed in Ahn et al. (2018), application of Bean and O'Reilly's methodology in risk theory affords a fluid flow process based matrix analytic methodology applicable to risk models with phase-dependent rates while preserving the skip-free nature of the fluid flow process.

In this work, we first extend Ahn et al.'s application of Bean and O'Reilly's methodology to the analysis of the occupation measure. This affords us a powerful fluid flow process based matrix analytic methodology for computing the so-called discounted joint conditional pdf of the surplus prior to ruin and deficit at ruin of the continuous-time MAP risk model with phase-dependent premium rates and phase-type claim size distributions. Other than the derivation of some key matrices, however, the probabilistic arguments used to derive the matrix-based algorithm for the discounted joint conditional pdf of the MAP risk model with phase-dependent rates and the MAP risk model with phase-independent rates, which has already been studied by Ramaswami (2006), are identical. Therefore, we keep the discussion brief here.

Instead, we introduce a new risk model that takes a more microscopic point of view on the evolution of an insurance risk process than the view of traditional collective risk theory. In this risk model, premium rates depend on certain variables that can be modelled via a continuous-time Markov chain (CTMC). Thus, the fluid flow process based matrix analytic methodology that we introduce in this chapter can be employed. The model does not necessarily fall under the class of MAP risk models, and hence, will require additional probabilistic analysis than the probabilistic analysis typically used in the literature of fluid flow based matrix analytic methodologies in risk theory.

First, we present some of the results from Bean and O'Reilly (2013) in Section 4.2. In Section 4.3, a procedure to evaluate the occupation measure of the risk process via the methodology developed in Bean and O'Reilly (2013) is provided. Then, in Section 4.4, we briefly discuss a procedure to evaluate the joint conditional pdf of the time of ruin, surplus prior to ruin, and deficit at ruin of a continuous-time MAP risk model with phase-dependent premium rates. The procedure involves some results on the first passage time LST given in Bean and O'Reilly (2013) and the methodology for the evaluation of the occupation measure given in Section 4.3 of this thesis. Finally, in Section 4.5, we discuss the newly introduced risk model.

## 4.2 Fluid flow process and shift process

Consider a fluid flow process $\{(F_t, W_t), t \in \mathbb{R}^+\}$, where the phase process $\{W_t, t \in \mathbb{R}^+\}$ is a finite-state CTMC whose state space is given by $\mathcal{W}$. Let $r_i \in \mathbb{R}$, $i \in \mathcal{W}$, denote the flow rates of the process $\{F_t, t \in \mathbb{R}^+\}$ (Unless otherwise specified, we assume $F_0 = 0$.) Consider another fluid flow process $\{O_t, t \in \mathbb{R}^+\}$ defined on the same phase process $\{W_t, t \in \mathbb{R}^+\}$ as $\{F_t, t \in \mathbb{R}^+\}$, but with a different set of flow rates, denoted by $\{c_i, i \in \mathcal{W}\}$. We assume that both $\{F_t, t \in \mathbb{R}^+\}$ and $\{O_t, t \in \mathbb{R}^+\}$ are defined on the entire real line and that they are conditionally independent given the phase process.

Bean and O'Reilly (2013) introduced the so-called shift process $\{Z_t, t \in \mathbb{R}^+\}$ defined as $\{Z_t = O_t - O_0, t \in \mathbb{R}^+\}$, and derived an efficient matrix-based algorithm for computing the LST of the shift process stopped at certain first passage times of $\{(F_t, W_t), t \in \mathbb{R}^+\}$. Bean and O'Reilly's algorithm gives a numerically efficient and stable method to compute numerous key quantities appearing in applied probability. In this subsection, we simply present Bean and O'Reilly's results and in Sections 4.3 and 4.4, employ their results in solving the problems at our hand.

Let us first partition the state space $\mathcal{W}$ into three disjoint sets, $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_0$, such that $r_i > 0$ for $i \in \mathcal{W}_1$, $r_i < 0$ for $i \in \mathcal{W}_2$, and $r_i = 0$ for $i \in \mathcal{W}_0$. Let the partitioned infinitesimal rate matrix (partitioned according to $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_0$) of $\{W_t, t \in \mathbb{R}^+\}$ be given by

$$\boldsymbol{T} = (v_{i,j})_{i,j\in\mathcal{W}} = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \boldsymbol{T}_{10} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \boldsymbol{T}_{20} \\ \boldsymbol{T}_{01} & \boldsymbol{T}_{02} & \boldsymbol{T}_{00} \end{pmatrix}.$$

Define the diagonal matrices $\boldsymbol{R}_1 = \text{diag}(r_i)_{i\in\mathcal{W}_1}$, $\boldsymbol{R}_2 = \text{diag}(|r_i|)_{i\in\mathcal{W}_2}$, $\boldsymbol{C}_1 = \text{diag}(c_i)_{i\in\mathcal{W}_1}$, $\boldsymbol{C}_2 = \text{diag}(c_i)_{i\in\mathcal{W}_2}$, and $\boldsymbol{C}_0 = \text{diag}(c_i)_{i\in\mathcal{W}_0}$. Given $s \in \mathbb{C}$, further define a set of matrices

$$\begin{aligned}
\boldsymbol{W}_{11}(s) &= \boldsymbol{R}_1^{-1}\big((\boldsymbol{T}_{11} - s\boldsymbol{C}_1) - \boldsymbol{T}_{10}(\boldsymbol{T}_{00} - s\boldsymbol{C}_0)^{-1}\boldsymbol{T}_{01}\big),\\
\boldsymbol{W}_{22}(s) &= \boldsymbol{R}_2^{-1}\big((\boldsymbol{T}_{22} - s\boldsymbol{C}_2) - \boldsymbol{T}_{20}(\boldsymbol{T}_{00} - s\boldsymbol{C}_0)^{-1}\boldsymbol{T}_{02}\big),\\
\boldsymbol{W}_{12}(s) &= \boldsymbol{R}_1^{-1}\big(\boldsymbol{T}_{12} - \boldsymbol{T}_{10}(\boldsymbol{T}_{00} - s\boldsymbol{C}_0)^{-1}\boldsymbol{T}_{02}\big),\\
\boldsymbol{W}_{21}(s) &= \boldsymbol{R}_2^{-1}\big(\boldsymbol{T}_{21} - \boldsymbol{T}_{20}(\boldsymbol{T}_{00} - s\boldsymbol{C}_0)^{-1}\boldsymbol{T}_{01}\big),
\end{aligned}$$

provided that the maximum real part of the eigenvalues of $\boldsymbol{T}_{00} - s\boldsymbol{C}_0$ is negative. Now, define the first passage time random variable $\kappa(y) = \inf\{t > 0 : F_t = y\}$, $y \in \mathbb{R}$, and two LST matrices $\widehat{\boldsymbol{\Psi}}(s)$ and $\widehat{\boldsymbol{G}}_y(s)$ whose $(i, j)$-th entries are given by

$$\begin{aligned}
\big(\widehat{\boldsymbol{\Psi}}(s)\big)_{i,j} &= E\{e^{-sZ_{\kappa(0)}}\mathcal{I}[\kappa(0) < \infty, W_{\kappa(0)} = j]|F_0 = 0, W_0 = i\},\ i \in \mathcal{W}_1,\ j \in \mathcal{W}_2,\\
\big(\widehat{\boldsymbol{G}}(s, y)\big)_{i,j} &= E\{e^{-sZ_{\kappa(0)}}\mathcal{I}[\kappa(0) < \infty, W_{\kappa(0)} = j]|F_0 = y, W_0 = i\},\ i, j \in \mathcal{W}_2,\ y > 0.
\end{aligned}$$

Then, the following result holds true:

**Theorem 3, Bean and O'Reilly (2013)** If the maximum real parts of the eigenvalues of $\boldsymbol{T}_{00} - s\boldsymbol{C}_0$, $\boldsymbol{W}_{11}(s)$, and $\boldsymbol{W}_{22}(s)$ are negative, then the matrix $\widehat{\boldsymbol{\Psi}}(s)$ satisfies the equation

$$\boldsymbol{W}_{12}(s) + \widehat{\boldsymbol{\Psi}}(s)\boldsymbol{W}_{21}(s)\widehat{\boldsymbol{\Psi}}(s) + \boldsymbol{W}_{11}(s)\widehat{\boldsymbol{\Psi}}(s) + \widehat{\boldsymbol{\Psi}}(s)\boldsymbol{W}_{22}(s) = \boldsymbol{0}, \tag{4.2.1}$$

and furthermore, if $s$ is real, $\widehat{\boldsymbol{\Psi}}(s)$ is the minimal nonnegative solution of (4.2.1). Moreover, we have:

**Theorem 4, Bean and O'Reilly (2013)**

$$\widehat{\boldsymbol{G}}(s,y) = e^{(\boldsymbol{W}_{22}(s)+\boldsymbol{W}_{21}(s)\widehat{\boldsymbol{\Psi}}(s))y}. \tag{4.2.2}$$

## 4.3 Occupation measure with respect to shift process

Consider the same fluid flow processes $\{(F_t, W_t), t \in \mathbb{R}^+\}$ and $\{O_t, t \in \mathbb{R}^+\}$, and the shift process $\{Z_t, t \in \mathbb{R}^+\}$, introduced in Section 4.2. For $s \in \mathbb{C}$ and $y > 0$, define a matrix $\boldsymbol{N}(s, dy)$ whose $(i, j)$-th entries are given by

$$\big(\boldsymbol{N}(s,dy)\big)_{i,j} = \int_0^\infty E\{e^{-sZ_t}\mathcal{I}[\kappa(0) > t, F_t \in dy, W_t = j]|F_0 = 0, W_0 = i\}dt,$$

for $i, j \in \mathcal{W}_1$. For now, we assume that the shift process is such that the above integral exists. In order to compute the occupation measure matrix $\boldsymbol{N}(s, dy)$, we employ a well-known time-reversal argument.

First of all, let $\boldsymbol{\theta} = (\theta_i)_{i\in\mathcal{W}}$ denote a stationary probability vector of $\{W_t, t \in \mathbb{R}^+\}$ (assuming that one exists) and let $\{W_t^*, t \in \mathbb{R}^+\}$ denote the stationary version of $\{W_t, t \in \mathbb{R}^+\}$, i.e., $W_0$ distributed with $\boldsymbol{\theta}$. Let $\{\tilde{W}_t^*, t \in \mathbb{R}^+\}$ denote the time-reversed version of $\{W_t^*, t \in \mathbb{R}^+\}$, and $\{\tilde{F}_t, t \in \mathbb{R}^+\}$ and $\{\tilde{O}_t, t \in \mathbb{R}^+\}$ denote the time-reversed versions of $\{F_t, t \in \mathbb{R}^+\}$ and $\{O_t, t \in \mathbb{R}^+\}$. Then, the flow rates of the time-reversed fluid flow processes are the negatives of those of the original processes, and the shift process of the fluid flow process $\{\tilde{O}_t, t \in \mathbb{R}^+\}$ is in fact equal to $\{\tilde{Z}_t = -Z_t, t \in \mathbb{R}^+\}$. Lastly, note that the fluid flow process $\{F_t, t \in \mathbb{R}^+\}$ and the shift process $\{Z_t, t \in \mathbb{R}^+\}$ are both time and space invariant. Then, following the same probabilistic reasoning behind the proof of Theorem

2.5 of Albrecher and Asmussen (2010), we have

$$\begin{aligned}
\theta_i(\boldsymbol{N}(s,dy))_{i,j} &= \int_0^\infty \theta_i E\{e^{-sZ_t}\mathcal{I}[\kappa(0) > t, F_t \in dy, W_t = j]|F_0 = 0, W_0 = i\}dt \\
&= \int_0^\infty E\{e^{-sZ_t}\mathcal{I}[\kappa(0) > t, F_t \in dy, W_t^* = j, W_0^* = i]|F_0 = 0\}dt \\
&= \int_0^\infty E\{e^{s\tilde{Z}_t}\mathcal{I}[\tilde{F}_t < \tilde{F}_a \ \forall \ a \in [0,t) \ , \tilde{F}_t \in -dy, \tilde{W}_t^* = i, \tilde{W}_0^* = j]|\tilde{F}_0 = 0\}dt \\
&= \int_0^\infty \theta_j E\{e^{s\tilde{Z}_t}\mathcal{I}[\tilde{F}_t < \tilde{F}_a \ \forall \ a \in [0,t) \ , \tilde{F}_t \in -dy, \tilde{W}_t^* = i]|\tilde{F}_0 = 0, \tilde{W}_0^* = j\}dt \\
&= \theta_j E\{e^{s\tilde{Z}_{\tilde{\kappa}(-y)}}\mathcal{I}[\tilde{W}^*_{\tilde{\kappa}(-y)} = i]|\tilde{F}_0 = 0, \tilde{W}_0 = j\}dy, \qquad (4.3.1)
\end{aligned}$$

where $\tilde{\kappa}(y) = \inf\{t > 0 : \tilde{F}_t = y\}$ denotes the first passage times of $\{\tilde{F}_t, t \in \mathbb{R}^+\}$.

Defining a matrix $\widehat{\tilde{\boldsymbol{G}}}(-s,y)$ whose $(j,i)$-th entries are given by

$$\big(\widehat{\tilde{\boldsymbol{G}}}(-s,y)\big)_{j,i} = E\{e^{s\tilde{Z}_{\tilde{\kappa}(-y)}}\mathcal{I}[\tilde{W}^*_{\tilde{\kappa}(-y)} = i]|\tilde{F}_0 = 0, \tilde{W}_0^* = j\},$$

(4.3.1) can be written as

$$\theta_i\big(\boldsymbol{N}(s,dy)\big)_{i,j} = \theta_j\big(\widehat{\tilde{\boldsymbol{G}}}(-s,y)\big)_{j,i}dy. \qquad (4.3.2)$$

Finally, defining a diagonal matrix $\boldsymbol{\Delta} = \text{diag}(\theta_i)_{i\in\mathcal{W}}$, we can write (4.3.2) in the following matrix equation form:

$$\boldsymbol{N}(s,dy) = \boldsymbol{\Delta}^{-1}\widehat{\tilde{\boldsymbol{G}}}(-s,y)^\intercal\boldsymbol{\Delta}dy. \qquad (4.3.3)$$

Therefore, computation of $\widehat{\tilde{\boldsymbol{G}}}(-s,y)$, which can be done using the methods shown in Bean and O'Reilly (2013), leads to the computation of $\boldsymbol{N}(s,dy)$.

## 4.4 Continuous-time MAP risk model with phase-dependent premium rates

### 4.4.1 Model description

Consider a continuous-time MAP risk model

$$U_t = u + \int_0^t c(J_s)ds - \sum_{k=1}^{N_t} Y_k,\ t \in \mathbb{R}^+,\ u \in \mathbb{R}^+,$$

comprised of a continuous-time MAP $\{(N_t, J_t), t \in \mathbb{R}^+\}$ defined on $\mathbb{N}\times\mathcal{J}$, $\mathcal{J} = \{1, 2, \ldots, m\}$, $m \in \mathbb{Z}^+$, rate matrices $(\boldsymbol{D}_0, \boldsymbol{D}_1) = ((d_{0,i,j})_{i,j\in\mathcal{J}}, (d_{1,i,j})_{i,j\in\mathcal{J}})$, and the conditionally i.i.d. claim amount sequence $\{Y_k, k \in \mathbb{Z}^+\}$ (conditional on the phase process $\{J_t, t \in \mathbb{R}^+\}$ of the MAP). In particular, $Y_k$ denotes the amount of the $k$-th claim to be made and the distribution of $Y_k$ depends only on the type of the phase transition that the claim is accompanied by. In other words, let $f^{(i,j)}(y)$, $i, j \in \mathcal{J}$, $y \geq 0$, denote the pdf of $Y^{(i,j)} = Y_k|(J_{\xi_k^-} = i, J_{\xi_k} = j)$, where $\{\xi_k, k \in \mathbb{Z}^+\}$ denotes the arrival epochs of the associated MAP. We further assume that the premium rates depend on the phase process $\{J_t, t \in \mathbb{R}^+\}$, i.e., $c_t = c(J_t)$, and that $Y^{(i,j)}$ follows a continuous-time phase-type distribution of order $n^{(i,j)} \in \mathbb{Z}^+$ with pdf $f^{(i,j)}(y) = \boldsymbol{\alpha}^{(i,j)} e^{(\boldsymbol{U}^{(i,j)})y} (\boldsymbol{\gamma}^{(i,j)})^\intercal$, $y \geq 0$, $i, j \in \mathcal{J}$.

### 4.4.2 Discounted joint conditional pdf

Let $\tau = \inf\{t > 0 : U_t < 0\}$ denote the time of ruin, and let $h(t, x, y|u)$ denote the joint conditional pdf of $(\tau, U_{\tau^-}, |U_\tau|)$, given that $U_0 = u$. Then, the so-called discounted joint conditional pdf of $(\tau, U_{\tau^-}, |U_\tau|)$ is given by $h_s(x, y|u) = \int_0^\infty e^{-st} h(t, x, y|u)dt$, $s \in \mathbb{C}$, $\Re(s) \geq 0$.

Our objective here is to derive a matrix-based algorithm for computing the discounted joint conditional pdf. To this end, we first construct a fluid flow process of which sample paths can be linked to the sample paths of the risk model of interest. The method of construction of such a fluid flow process is well detailed in many references on fluid flow process based matrix analytic methodologies in risk theory (see, e.g., Ramaswami (2006)). Once such a fluid flow process is constructed, as first noted in Ahn et al. (2018), we consider a shift process which keeps track of the time the fluid flow process spends in phases with positive flow rates. Then, one can compute $\widehat{\boldsymbol{\Psi}}(s)$, $\widehat{\boldsymbol{G}}(s, y)$, and $\boldsymbol{N}(s, dy)$ of the fluid flow process and the shift process as in Sections 4.2 and 4.3 using the algorithms given in Bean and O'Reilly (2013), and follow the same sample paths argument used in Ramaswami (2006) to evaluate the discounted joint conditional pdf. We omit the details here.

## 4.5 Dynamic individual risk model

### 4.5.1 Introduction

In this work, we propose a new risk model which we refer to as the *dynamic individual risk model*. Unlike the traditional view of collective risk theory where claims arrive according to a certain point process, here we take the view that the claims are generated by the active insurance contracts (i.e., the customers) that the firm holds at a given time. To this end, we also take into consideration the arrivals and departures of the customers by incorporating an arrival process of the new customers and their departures at the ends of deterministic time intervals (i.e., think of it as a calendar year of the insurance firm). Customers are assumed to arrive to the system according to a Poisson process. At every time point $t = kT$, $k \in \mathbb{Z}^+$, $T > 0$, one calendar year of the firm is declared finished and all of the existing contracts leave the system. Until the year end is reached, the customers currently in the system are under contract and hence cannot leave the system at will. Claims are generated by each customer in the system at random epochs which are assumed to be realizations of a Poisson process. Premiums are collected continuously from each customer while he/she is in the system.

Since this risk model keeps track of the number of active insurance contracts in the system, it is only natural to assume that the premium rates should depend on the current volume of the insurance business which is represented by the number of active insurance contracts at a given time. Moreover, we assume that there are no new customers arriving when the surplus process is below level 0 due to the lack of confidence in the firm. Since, even after the surplus process falls below level 0, there may still be active contracts in the system, the insurance business continues as long as the number of active insurance contracts stays positive (even if the surplus process is below level 0). With this view, we define the time of ruin to be the time the event that the surplus process is below level 0 and the number of active contracts is 0 at the same time is observed for the first time. Then, by design, the time of ruin is one of the calendar year ends of the firm.

In practice, the value of $T$ is deterministic. However, this poses difficulty in putting the specific risk model in a mathematically tractable framework. As such, we employ the Erlangization technique to construct a risk model that approximates the dynamic individual risk model with a constant calendar year $T$. Assume that the calendar year is randomly distributed with an Erlang distribution. If we fix the expected value of the random calendar year at $T$ and increase the shape parameter, the random variable converges in distribution to $T$. We note that the Erlangization technique in the same context, i.e., approximating risk models with deterministic time intervals, has already been employed in other papers such as Stanford et al. (2005) and Albrecher et al. (2013). For brevity, we henceforth refer to the dynamic individual risk model with an Erlang calendar year as the dynamic individual risk model.

The dynamic individual risk model described above can be constructed as a continuous-time level-dependent MAP risk model with phase-dependent premium rates, since the risk process behaves differently when it is above or below level 0. Therefore, we can employ the fluid flow process based matrix analytic methodology introduced in Sections 4.2, 4.3 and 4.4 to study the risk model of interest. The fluid flow process based matrix analytic methodology is a suitable choice of methodology for studying this risk model for several reasons. First of all, to keep track of the number of active contracts, we need a methodology which behaves numerically stable even when the number of phases is large. Moreover, incorporation of phase-dependent premium rates and the level-dependent structure arising from the definition of the time of ruin given above calls for a fluid flow process based matrix analytic methodology which can analyze MAP risk models with phase-dependent rates and at the same time, exploit the skip-free nature of the fluid flow process. The fluid flow based matrix analytic methodology introduced in Sections 4.2, 4.3 and 4.4 satisfies all of the above points.

### 4.5.2 Mathematical model description

Let $N \in \mathbb{Z}^+$ denote the maximum number of active contracts that the insurance firm can hold at any given time. Assume that new customers arrive to the system according to a Poisson process with rate $\lambda_2 > 0$ (independent of everything else), as long as the number of active contracts in the system is strictly less than $N$ and the surplus process remains above level 0. Furthermore, assume that each active contract generates claims according to a Poisson process with rate $\lambda_3 > 0$, independent of everything else, and that the claim sizes are i.i.d. with a phase-type distribution having pdf $f(y) = \boldsymbol{\alpha} e^{\boldsymbol{U} y} \boldsymbol{\gamma}^\intercal$, $y > 0$, and of order $m \in \mathbb{Z}^+$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\boldsymbol{U} = (u_{i,j})_{i,j \in \mathcal{J}}$, $\mathcal{J} = \{1, 2, \ldots, m\}$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$. Lastly, let $A \sim E(K, \lambda_1)$ be a random variable denoting the calendar year of the insurance firm with Erlang pdf $a(y) = \frac{\lambda_1^K y^{K-1} e^{-\lambda_1 y}}{(K-1)!}$, $y > 0$, $K \in \mathbb{Z}^+$, $\lambda_1 > 0$. We assume that $A$ is independent of everything else.

To put the risk model of interest in the MAP framework, we first note that $A$ can be decomposed into $K$ sequential i.i.d. random intervals with each interval exponentially distributed with rate $\lambda_1$. Then, employing the method of phases, let $\{A_t, t \in \mathbb{R}^+\}$ defined on $\mathcal{P} = \{1, 2, \ldots, K\}$ denote the age process associated with $A$. Next, let $\{N_t, t \in \mathbb{R}^+\}$ denote the number of active contracts in the system at time $t$. We can then construct a bivariate CTMC $\{\boldsymbol{N}_t = (N_t, A_t), t \in \mathbb{R}^+\}$ due to the independence between the arrival process and the age process associated with $A$, and their exponential sojourn times.

The CTMC $\{\boldsymbol{N}_t, t \in \mathbb{R}^+\}$ has two possible types of transitions. The first is that a new customer arrives before the age process advances to the next age (from age $l$ to $l+1$.) The second is that the age process advances to the next age before a new customer arrives. Note, however, that $\{\boldsymbol{N}_t, t \in \mathbb{R}^+\}$ behaves differently when the surplus process is above or below level 0. The age process retiring the age $K$ when the surplus process is above level 0 resets to age 1. On the other hand, the age process retiring the age $K$ when the surplus process is below level 0 results in the event of ruin. Furthermore, no new customers arrive when the surplus process is below level 0. To distinguish between when the surplus process is above and below level 0, let $\{\boldsymbol{N}_t^+ = (N_t^+, A_t^+), t \in \mathbb{R}^+\}$ and $\{\boldsymbol{N}_t^- = (N_t^-, A_t^-), t \in \mathbb{R}^+\}$ denote the CTMCs describing $\{\boldsymbol{N}_t, t \in \mathbb{R}^+\}$ when the surplus process is above and below level 0, respectively.

Now, let $\{(L_t^+, \boldsymbol{N}_t^+), t \in \mathbb{R}^+\}$, $\{\boldsymbol{N}_t^+, t \in \mathbb{R}^+\}$ defined on $\mathcal{N}^+ = \{0, 1, 2, \ldots, N\} \times \mathcal{P}$, denote the claims arrival MAP when the surplus process is above level 0. If a claim occurs before any other event occurs, we say there is an arrival. Hence, adopting the notation used in the earlier section on MAP processes, let $(\boldsymbol{D}_0^+, \boldsymbol{D}_1^+) = ((d^+_{0,\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l} \in \mathcal{N}^+}, (d^+_{1,\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l} \in \mathcal{N}^+})$, $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{l} = (l_1, l_2)$, denote the associated rate matrices without and with arrivals. By using the standard results from independent exponential random variables, we can write

$$d^{+}_{0,\boldsymbol{n},\boldsymbol{l}} = \begin{cases} -(\lambda_1+\lambda_2+\lambda_3 n_1), \text{ if } \boldsymbol{n}=\boldsymbol{l} \text{ and } n_1<N, \\ -(\lambda_1+\lambda_3 n_1), \text{ if } \boldsymbol{n}=\boldsymbol{l} \text{ and } n_1=N, \\ \lambda_1, \text{ if } n_1=l_1,\, l_2=n_2+1, \text{ and } n_2<K, \text{ or } l_1=0,\, l_2=1, \text{ and } n_2=K, \\ \lambda_2, \text{ if } l_1=n_1+1,\, l_2=n_2, \text{ and } n_1<N, \\ 0, \text{ otherwise,} \end{cases}$$

and

$$d^{+}_{1,\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \lambda_3 n_1, \text{ if } \boldsymbol{n}=\boldsymbol{l}, \\ 0, \text{ otherwise.} \end{cases}$$

Now, let $\{(L^-_t, \boldsymbol{N}^-_t), t \in \mathbb{R}^+\}$, $\{\boldsymbol{N}^-_t, t \in \mathbb{R}^+\}$ defined on $\mathcal{N}^- = (\{1,2,\ldots,N\} \times \mathcal{P}) \cup \{(0,0)\}$, denote the claims arrival MAP process when the surplus process is below level 0. Here, the state $\boldsymbol{0} = (0,0)$ denotes an absorbing state and when state $\boldsymbol{0}$ is reached, the event of ruin is declared. Again, by using the standard results from independent exponential random variables, we have the transition rate matrices $(\boldsymbol{D}^-_0, \boldsymbol{D}^-_1) = ((d^-_{0,\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l}\in\mathcal{N}^-}, (d^-_{1,\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l}\in\mathcal{N}^-})$ defined as

$$d^{-}_{0,\boldsymbol{n},\boldsymbol{l}} = \begin{cases} -(\lambda_1+\lambda_3 n_1), \text{ if } \boldsymbol{n}=\boldsymbol{l} \text{ and } n_2 \le K, \\ \lambda_1, \text{ if } n_1=l_1,\, l_2=n_2+1, \text{ and } n_2<K, \text{ or } l_1=0,\, l_2=0,\, n_2=K, \text{ and } n_1\in\{1,2,\ldots,N\}, \\ 0, \text{ otherwise,} \end{cases}$$

and

$$d^{-}_{1,\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \lambda_3 n_1, \text{ if } \boldsymbol{n} = \boldsymbol{l} \text{ and } \boldsymbol{n} \neq \boldsymbol{0}, \\ 0, \text{ otherwise.} \end{cases}$$

With the MAP processes depicting the claims arrival process defined as above, we are now ready to give the formal mathematical definition of the surplus process of the dynamic individual risk model.

Let $\{U_t, t \in \mathbb{R}^+\}$ denote the surplus process and $\{(L_t, \boldsymbol{N}_t), t \in \mathbb{R}^+\}$ denote the MAP process which follows the same probability law as that of $\{(L_t^+, \boldsymbol{N}_t^+), t \in \mathbb{R}^+\}$ when the surplus process is above level 0 and that of $\{(L_t^-, \boldsymbol{N}_t^-), t \in \mathbb{R}^+\}$ when the surplus process is below level 0. Let $Y_k$, $k \in \mathbb{Z}^+$, denote the $k$-th claim size arising from $\{(L_t, \boldsymbol{N}_t), t \in \mathbb{R}^+\}$. Then,

$$U_t = U_0 + \int_0^t c(N_z)dz - \sum_{k=1}^{L_t} Y_k, \ t \in \mathbb{R}^+,$$

where the only restriction we impose on the premium rates is that $c(n) > 0$ for all $n \in \{1, 2, \dots, N\}$ and $c(0) = 0$.

### 4.5.3 Time of ruin and related quantities

The motivation behind the dynamic individual risk model is to take a more realistic view on the dynamics of the cash flows of an insurance business rather than the view of the more prevalent collective risk model by taking into account the number of customers that an insurance company is liable for at any given time. Under such a model setting, however, an insurer may have outstanding financial obligations and active sources of income even after the surplus level falls below 0. This is in contrast to the collective risk model in which we generally assume that once the surplus level falls below 0, the event of ruin is declared and the deficit at ruin is set equal to the final value of the surplus level immediately after ruin. In the dynamic individual risk model, we do not stop observing the surplus process when it falls below 0. Instead, we stop observing the process when it is below 0, and at the same time, there are no active contracts in the system. Therefore, we define the time of ruin to be $\tau = \inf\{t > 0 : U_t < 0 \text{ and } \boldsymbol{N}_t = \boldsymbol{0}\}$.

The main quantity of interest here is the distribution of the time of ruin. As the time of ruin is not a simple first passage time, we need to decompose the time of ruin into several time points and piece them together. To this end, let us first consider the following functional:

$$h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n}) = \int_0^\infty e^{-st} h(t, x, \boldsymbol{l}, y|u, \boldsymbol{n}) dt,$$

$$u \geq 0,\ x, y < 0,\ \boldsymbol{n} \in \mathcal{N}^+ \setminus (\{0\} \times \mathcal{P}),\ \boldsymbol{l} \in \mathcal{N}^- \setminus \{\boldsymbol{0}\},\ s \in \mathbb{C},\ \Re(s) \geq 0,$$

where $h(t, x, \boldsymbol{l}, y|u, \boldsymbol{n})$ is the joint conditional pdf of $(\tau, U_\sigma, \boldsymbol{N}_\sigma, U_\tau)$, conditional on the event that $(U_0, \boldsymbol{N}_0) = (0, \boldsymbol{n})$, with $\sigma = \sup\{t > 0 : U_t > 0\}$ denoting the last epoch that the surplus level falls below 0 prior to the time of ruin. Then, integrating and summing the functional $h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$ over the values of $x$, $\boldsymbol{l}$, and $y$ gives the LST of the time of ruin, i.e.,

$$\psi(s|u, \boldsymbol{n}) = E\{e^{-s\tau} \mathcal{I}[\tau < 0] | U_0 = 0, \boldsymbol{N}_0 = \boldsymbol{n}\} = \sum_{\boldsymbol{l} \in \mathcal{N}^- \setminus \{\boldsymbol{0}\}} \int_{-\infty}^0 \int_{-\infty}^0 h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n}) dx dy. \tag{4.5.1}$$

In what follows, we construct a fluid flow process which will be used to develop a matrix analytic methodology for computing the functional $h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$.

### 4.5.4 Formulation of fluid flow process

Consider a sample path of the surplus process $\{(U_t, \boldsymbol{N}_t), t \in \mathbb{R}^+\}$, which comprises of a series of linear upward journeys and downward jumps. We first start off by implementing the usual method of stretching the downward jumps caused by claim arrivals into linear downward journeys. Then, the resulting sample path resembles that of a fluid flow process. Since, however, the surplus process behaves differently when it is above and below level 0, we have to construct a fluid flow process which mimics such behaviour of the surplus process. To this end, we construct two fluid flow processes, denoted by $\{(F_t^+, \boldsymbol{J}_t^+), t \in \mathbb{R}^+\}$ and $\{(F_t^-, \boldsymbol{J}_t^-), t \in \mathbb{R}^+\}$, of which sample paths exhibit certain connections to the sample paths of the surplus process when it is above and below level 0, respectively.

Let $\mathcal{J} = \{1, 2, \ldots, m\}$ be the transient states of the Markov chain associated with the phase-type claim amount distribution $f(y)$. Consider a finite multi-dimensional CTMC $\{\boldsymbol{J}_t^+, t \in \mathbb{R}^+\}$ defined on $\mathcal{W}^+ = \mathcal{W}_0^+ \cup \mathcal{W}_1^+ \cup \mathcal{W}_2^+$, where $\mathcal{W}_0^+ = \{0\} \times \mathcal{P}$, $\mathcal{W}_1^+ = \mathcal{N}^+ \setminus \mathcal{W}_0^+$, and $\mathcal{W}_2^+ = \mathcal{W}_1^+ \times \mathcal{J}$. We then construct a fluid flow process $\{F_t^+, t \in \mathbb{R}^+\}$ on $\{\boldsymbol{J}_t^+, t \in \mathbb{R}^+\}$ with the flow rates given by $\{r_{\boldsymbol{n}}, \boldsymbol{n} \in \mathcal{W}^+\}$, where $r_{\boldsymbol{n}} = c(n_1)$ for $\boldsymbol{n} \in \mathcal{W}_1^+$, $r_{\boldsymbol{n}} = 0$ for $\boldsymbol{n} \in \mathcal{W}_0^+$, and $r_{\boldsymbol{n}} = -1$ for $\boldsymbol{n} \in \mathcal{W}_2^+$. Now, let the partitioned infinitesimal rate matrix of $\{\boldsymbol{J}_t^+, t \in \mathbb{R}^+\}$ (partitioned according to $\mathcal{W}_1^+$, $\mathcal{W}_2^+$, and $\mathcal{W}_0^+$) be given by

$$\boldsymbol{T}^+ = (v^+_{\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l} \in \mathcal{W}^+} = \begin{pmatrix} \boldsymbol{T}_{11}^+ & \boldsymbol{T}_{12}^+ & \boldsymbol{T}_{10}^+ \\ \boldsymbol{T}_{21}^+ & \boldsymbol{T}_{22}^+ & \boldsymbol{T}_{20}^+ \\ \boldsymbol{T}_{01}^+ & \boldsymbol{T}_{02}^+ & \boldsymbol{T}_{00}^+ \end{pmatrix}.$$

The transition rates within $\mathcal{W}_1^+$ and $\mathcal{W}_0^+$, i.e., the elements of $\boldsymbol{T}_{11}^+$ and $\boldsymbol{T}_{00}^+$, are given by the elements of $\boldsymbol{D}_0^+$. Moreover, $\boldsymbol{T}_{10}^+$ and $\boldsymbol{T}_{01}^+$ can also be identified from $\boldsymbol{D}_0^+$. For the transition rates within, to, and from $\mathcal{W}_2^+$, the core idea is that as a claim arrives, we remember the number of active contracts and the age of $\{A_t, t \in \mathbb{R}^+\}$ at the moment of the claim's arrival, keep track of the transitions within the transient states of the Markov chain associated with the phase-type pdf $f(y)$, and as the exit from the transient states of the Markov chain associated with the phase-type pdf $f(y)$ occurs, the process finally returns to the number of active contracts and the age of $\{A_t, t \in \mathbb{R}^+\}$ at the moment of the claim's arrival.

Putting it altogether, we see that for $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_1^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} -(\lambda_1 + \lambda_2 + \lambda_3 n_1), \text{ if } \boldsymbol{n} = \boldsymbol{l} \text{ and } n_1 < N, \\ -(\lambda_1 + \lambda_3 n_1), \text{ if } \boldsymbol{n} = \boldsymbol{l} \text{ and } n_1 = N, \\ \lambda_1, \text{ if } n_1 = l_1,\, l_2 = n_2 + 1, \text{ and } n_2 < K, \\ \lambda_2, \text{ if } l_1 = n_1 + 1,\, l_2 = n_2, \text{ and } n_1 < N, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_0^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} -(\lambda_1 + \lambda_2), \text{ if } \boldsymbol{n} = \boldsymbol{l}, \\ \lambda_1, \text{ if } l_2 = n_2 + 1 \text{ and } n_2 < K, \text{ or } l_2 = 1 \text{ and } n_2 = K, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_1^+$ and $\boldsymbol{l} \in \mathcal{W}_0^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \lambda_1, \text{ if } l_2 = 1 \text{ and } n_2 = K, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_0^+$ and $\boldsymbol{l} \in \mathcal{W}_1^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \lambda_2, \text{ if } l_1 = 1 \text{ and } n_2 = l_2, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_1^+$ and $\boldsymbol{l} \in \mathcal{W}_2^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \alpha_{l_3}\lambda_3 n_1, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_2^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} u_{n_3,l_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_2^+$ and $\boldsymbol{l} \in \mathcal{W}_1^+$,

$$v^+_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \gamma_{n_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

Lastly, there are no transitions from $\mathcal{W}_0^+$ to $\mathcal{W}_2^+$, and vice versa.

Similar to $\{(F_t^+, \boldsymbol{J}_t^+), t \in \mathbb{R}^+\}$, consider a finite multi-dimensional CTMC $\{\boldsymbol{J}_t^-, t \in \mathbb{R}^+\}$ defined on $\mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^- \cup \mathcal{W}_0^-$, where $\mathcal{W}_1^- = \mathcal{N}^- \setminus \{\boldsymbol{0}\}$, $\mathcal{W}_2^- = \mathcal{W}_1^- \times \mathcal{J}$, and $\mathcal{W}_0^- = \{\boldsymbol{0}\}$. We then construct a fluid flow process $\{F_t^-, t \in \mathbb{R}^+\}$ on $\{\boldsymbol{J}_t^-, t \in \mathbb{R}^+\}$ with the flow rates given by $\{r_{\boldsymbol{n}}, \boldsymbol{n} \in \mathcal{W}^-\}$, where $r_{\boldsymbol{n}} = c(n_1)$ for $\boldsymbol{n} \in \mathcal{W}_1^-$, $r_{\boldsymbol{n}} = 0$ for $\boldsymbol{n} \in \mathcal{W}_0^+$, and $r_{\boldsymbol{n}} = -1$ for $\boldsymbol{n} \in \mathcal{W}_2^-$. Let

$$\boldsymbol{T}^- = (v^-_{\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l} \in \mathcal{W}^-} = \begin{pmatrix} \boldsymbol{T}_{11}^- & \boldsymbol{T}_{12}^- & \boldsymbol{T}_{10}^- \\ \boldsymbol{T}_{21}^- & \boldsymbol{T}_{22}^- & \boldsymbol{T}_{20}^- \\ \boldsymbol{T}_{01}^- & \boldsymbol{T}_{02}^- & \boldsymbol{T}_{00}^- \end{pmatrix}$$

denote the transition rate matrix of $\{\boldsymbol{J}_t^-, t \in \mathbb{R}^+\}$ partitioned according to $\mathcal{W}_0^-$, $\mathcal{W}_1^-$, and $\mathcal{W}_2^-$. Similar to $\{\boldsymbol{J}_t^+, t \in \mathbb{R}^+\}$, the elements of $\boldsymbol{T}_{11}^-$ and $\boldsymbol{T}_{10}^-$ can be identified from the elements of $\boldsymbol{D}_0^-$. Moreover, $\boldsymbol{T}_{12}^-$, $\boldsymbol{T}_{22}^-$, $\boldsymbol{T}_{21}^-$, $\boldsymbol{T}_{20}^-$, and $\boldsymbol{T}_{02}^-$ are defined in the same manner as

$\boldsymbol{T}_{12}^{+}$, $\boldsymbol{T}_{22}^{+}$, $\boldsymbol{T}_{21}^{+}$, $\boldsymbol{T}_{20}^{+}$, and $\boldsymbol{T}_{02}^{+}$. Lastly, $\mathcal{W}_0^-$ forms a class of absorbing states (i.e., a singleton in this case), since the event of ruin is said to occur when the surplus process is below level 0 and the state $\mathbf{0}$ is reached. Hence, we let $\boldsymbol{T}_{00}^{-} = \mathbf{0}$ and $\boldsymbol{T}_{01}^{-} = \mathbf{0}$.

Putting it altogether, the transition rates for $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_1^-$ are given by

$$v_{\boldsymbol{n},\boldsymbol{l}}^{-} = \begin{cases} -(\lambda_1 + \lambda_3 n_1), \text{ if } \boldsymbol{n} = \boldsymbol{l}, \\ \lambda_1, \text{ if } n_1 = l_1 \text{ and } l_2 = n_2 + 1, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_1^-$ and $\boldsymbol{l} \in \mathcal{W}_2^-$,

$$v_{\boldsymbol{n},\boldsymbol{l}}^{-} = \begin{cases} \alpha_{l_3} \lambda_3 n_1, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_2^-$,

$$v_{\boldsymbol{n},\boldsymbol{l}}^{-} = \begin{cases} u_{n_3, l_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_2^-$ and $\boldsymbol{l} \in \mathcal{W}_1^-$,

$$v_{\boldsymbol{n},\boldsymbol{l}}^{-} = \begin{cases} \gamma_{n_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_1^-$ and $\boldsymbol{l} \in \mathcal{W}_0^-$,

$$v_{\boldsymbol{n},\boldsymbol{l}}^{-} = \begin{cases} \lambda_1, \text{ if } n_2 = K, \\ 0, \text{ otherwise.} \end{cases}$$

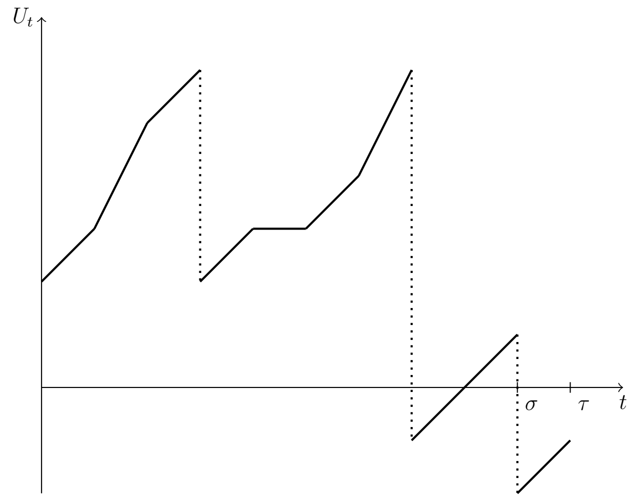


Figure 4.1: Sample path of $\{U_t, t \in \mathbb{R}^+\}$

There are no transitions from $\mathcal{W}_2^-$ to $\mathcal{W}_0^-$ and vice versa, and no transitions from $\mathcal{W}_1^-$ to $\mathcal{W}_0^-$ and vice versa.

Having constructed $\{(F_t^+, \boldsymbol{J}_t^+), t \in \mathbb{R}^+\}$ and $\{(F_t^-, \boldsymbol{J}_t^-), t \in \mathbb{R}^+\}$, define $\{(F_t, \boldsymbol{J}_t), t \in \mathbb{R}^+\}$ to be a level-dependent fluid flow process with the dynamics of $\{(F_t^+, \boldsymbol{J}_t^+), t \in \mathbb{R}^+\}$ and $\{(F_t^-, \boldsymbol{J}_t^-), t \in \mathbb{R}^+\}$, when it is above and below level 0, respectively. Then, let $\kappa = \inf\{t > 0 : F_t < 0 \text{ and } \boldsymbol{J}_t \in \mathcal{W}_0^-\}$. Let $\{O_t, t \in \mathbb{R}^+\}$, $O_t = 0$ w.p. 1, be a fluid flow process defined on $\{\boldsymbol{J}_t, t \in \mathbb{R}^+\}$ with the flow rates $\{z_{\boldsymbol{n}}, \boldsymbol{n} \in \mathcal{W}^+ \cup \mathcal{W}^-\}$, where $z_{\boldsymbol{n}} = 1$ for $\boldsymbol{n} \in \mathcal{W}_0^+ \cup \mathcal{W}_1^+ \cup \mathcal{W}_1^-$, and $z_{\boldsymbol{n}} = 0$ for $\boldsymbol{n} \in \mathcal{W}_2^+ \cup \mathcal{W}_2^- \cup \mathcal{W}_0^-$. In other words, $\{O_t, t \in \mathbb{R}^+\}$ keeps track of the time $\{\boldsymbol{J}_t, t \in \mathbb{R}^+\}$ spends in $\mathcal{W}_0^+ \cup \mathcal{W}_1^+ \cup \mathcal{W}_1^-$. Then, clearly, $\{O_t, t \in \mathbb{R}^+\}$ is a shift process of itself and $O_\kappa = \tau$ w.p. 1 (see Figures 4.1, 4.2, and 4.3). Therefore, the analysis of the functional $h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$ can be replaced with the analysis of the LST of $O_\kappa$ over the appropriate region.

More specifically, let $\eta$ denote the time at which the last descent of the fluid flow process $\{F_t, t \in \mathbb{R}^+\}$ into the negative real line prior to $\kappa$ ends. Then, from Figures 4.1, 4.2, and 4.3, we see that $(O_\kappa, F_\eta, \boldsymbol{J}_\eta, F_\kappa) = (\tau, U_\sigma, \boldsymbol{N}_\sigma, U_\kappa)$ w.p. 1 and this implies that

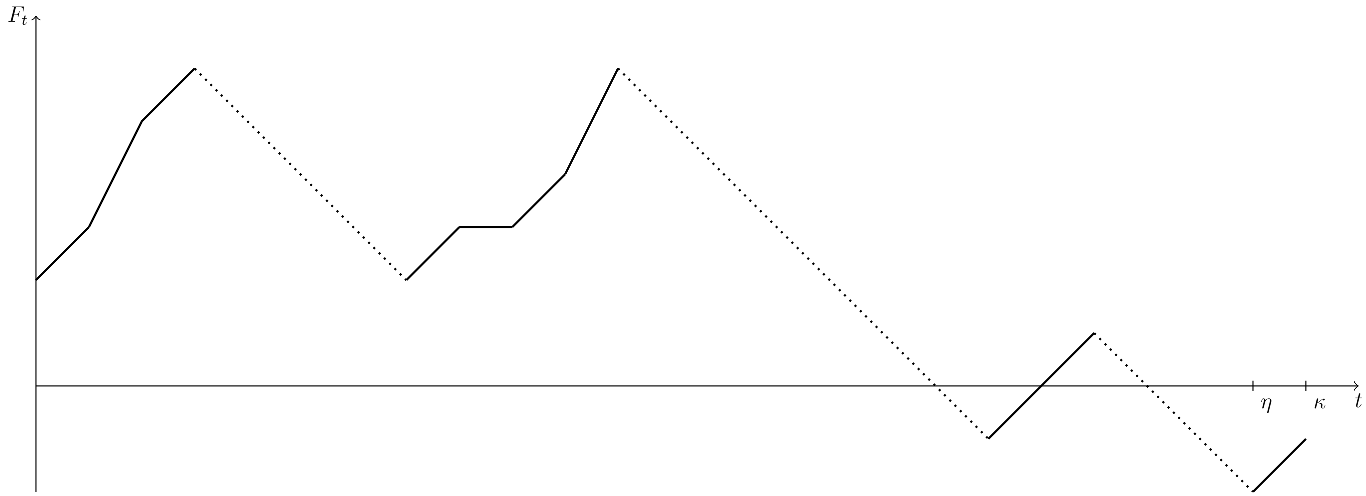


Figure 4.2: Sample path of $\{F_t, t \in \mathbb{R}^+\}$

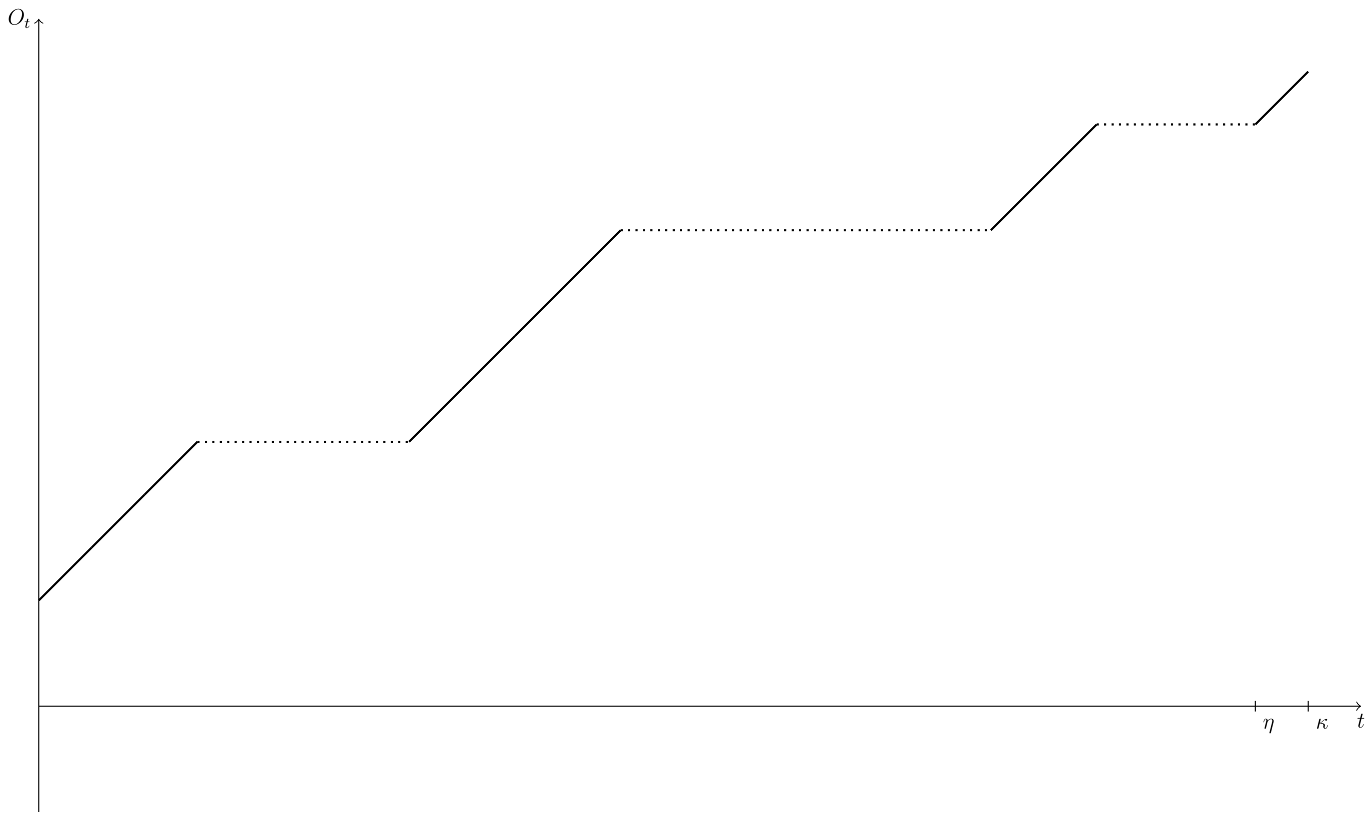


Figure 4.3: Sample path of $\{O_t, t \in \mathbb{R}^+\}$

the functional $h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$ is equal to

$$h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n}) = g_s(x, \boldsymbol{l}, y|u, \boldsymbol{n}) = \int_0^\infty e^{-st} g(t, x, \boldsymbol{l}, y|u, \boldsymbol{n}) dt, \tag{4.5.2}$$

where $g(t, x, \boldsymbol{l}, y|u, \boldsymbol{n})$ is the joint conditional pdf of $(O_\kappa, F_\eta, \boldsymbol{J}_\eta, F_\kappa)$ conditional on the event that $(F_0, \boldsymbol{J}_0) = (u, \boldsymbol{n})$.

In what follows, we present the probabilistic analysis for developing a fluid flow based matrix analytic methodology for evaluating $g_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$. Then, we can evaluate $h_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$ via (4.5.2).

### 4.5.5 Probabilistic analysis

In terms of notational consistency, let $\kappa(y) = \inf\{t > 0 : F_t = y\}$ denote the first passage time of the fluid flow process $\{(F_t, \boldsymbol{J}_t), t \in \mathbb{R}^+\}$. Then, we define a set of matrices $\widehat{\boldsymbol{\Psi}}^+(s)$, $\widehat{\boldsymbol{G}}^+(s, y)$, $\widehat{\boldsymbol{\Psi}}^-(s)$, and $\widehat{\boldsymbol{K}}^-(s, dy|x)$, where

$$\begin{aligned}
\left(\widehat{\boldsymbol{\Psi}}^+(s)\right)_{\boldsymbol{n},\boldsymbol{l}} &= E\{e^{-sO_{\kappa(0)}} \mathcal{I}[\kappa(0) < \infty, \boldsymbol{J}_{\kappa(0)} = \boldsymbol{l}] | F_0 = 0, \boldsymbol{J}_0 = \boldsymbol{n}\},\ \boldsymbol{n} \in \mathcal{W}_1^+, \boldsymbol{l} \in \mathcal{W}_2^+, \\
\left(\widehat{\boldsymbol{G}}^+(s, y)\right)_{\boldsymbol{n},\boldsymbol{l}} &= E\{e^{-sO_{\kappa(0)}} \mathcal{I}[\kappa(0) < \infty, \boldsymbol{J}_{\kappa(0)} = \boldsymbol{l}] | F_0 = y, \boldsymbol{J}_0 = \boldsymbol{n}\},\ \boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_2^+, y > 0, \\
\left(\widehat{\boldsymbol{\Psi}}^-(s)\right)_{\boldsymbol{n},\boldsymbol{l}} &= E\{e^{-sO_{\kappa(0)}} \mathcal{I}[\kappa(0) < \kappa < \infty, \boldsymbol{J}_{\kappa(0)} = \boldsymbol{l}] | F_0 = 0, \boldsymbol{J}_0 = \boldsymbol{n}\},\ \boldsymbol{n} \in \mathcal{W}_2^-, \boldsymbol{l} \in \mathcal{W}_1^-, \\
\left(\widehat{\boldsymbol{K}}^-(s, dy|x)\right)_{\boldsymbol{n},\boldsymbol{l}} &= \int_0^\infty E\{e^{-sO_t} \mathcal{I}[\kappa(0) > t, \kappa > t, F_t \in dy, \boldsymbol{J}_t = \boldsymbol{l}] | F_0 = x, \boldsymbol{J}_0 = \boldsymbol{n}\} dt, \\
&\qquad \boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_1^-,\ x, y < 0.
\end{aligned}$$

Noting that $\{(F_t, \boldsymbol{J}_t), t \in \mathbb{R}^+\}$ is level-independent within the positive and negative half lines and conditioning on the number of times the fluid flow process $\{(F_t, \boldsymbol{J}_t), t \in \mathbb{R}^+\}$

crosses level 0 prior to $\kappa$, we can rewrite $g_s(x, \boldsymbol{l}, y|u, \boldsymbol{n})$ as

$$
\begin{aligned}
g_s(x, \boldsymbol{l}, y|u, \boldsymbol{n}) &= \sum_{\boldsymbol{h}\in\mathcal{Q}} \boldsymbol{e}_{\boldsymbol{n}} \widehat{\boldsymbol{\Psi}}^{+}(s) \widehat{\boldsymbol{G}}^{+}(s,u) \sum_{n=0}^{\infty} \big(\widehat{\boldsymbol{\Psi}}^{-}(s)\widehat{\boldsymbol{\Psi}}^{+}(s)\big)^{n} e^{-\boldsymbol{T}_{22}^{-}x} \boldsymbol{T}_{21}^{-} \boldsymbol{e}_{\boldsymbol{l}}^{\intercal} \big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}} \frac{v_{\boldsymbol{h},\boldsymbol{0}}^{-}}{c(h_1)} \\
&= \sum_{\boldsymbol{h}\in\mathcal{Q}} \boldsymbol{e}_{\boldsymbol{n}} \widehat{\boldsymbol{\Psi}}^{+}(s) \widehat{\boldsymbol{G}}^{+}(s,u) \big(I - \widehat{\boldsymbol{\Psi}}^{-}(s)\widehat{\boldsymbol{\Psi}}^{+}(s)\big)^{-1} e^{-\boldsymbol{T}_{22}^{-}x} \boldsymbol{T}_{21}^{-} \boldsymbol{e}_{\boldsymbol{l}}^{\intercal} \big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}} \frac{v_{\boldsymbol{h},\boldsymbol{0}}^{-}}{c(h_1)},
\end{aligned}
\tag{4.5.3}
$$

where $\widehat{\boldsymbol{K}}^{-}(s,y|x)$ is a matrix such that $\widehat{\boldsymbol{K}}^{-}(s,dy|x) = \widehat{\boldsymbol{K}}^{-}(s,y|x)dy$ and $\mathcal{Q} = \{\boldsymbol{h} \in \mathcal{W}_1^{-} : h_2 = K\}$.

Computing the matrices $\widehat{\boldsymbol{\Psi}}^{+}(s)$, $\widehat{\boldsymbol{G}}^{+}(s,y)$, and $\widehat{\boldsymbol{\Psi}}^{-}(s)$ is a straightforward exercise, as we can simply apply Theorems 3 and 4 of Bean and O'Reilly (2013) and the algorithms therein. However, computing $\widehat{\boldsymbol{K}}^{-}(s,y|x)$ requires further attention. In what follows, we first show how to compute $\widehat{\boldsymbol{\Psi}}^{+}(s)$, $\widehat{\boldsymbol{G}}^{+}(s,y)$, and $\widehat{\boldsymbol{\Psi}}^{-}(s)$, followed by the discussion on the method of evaluating $\widehat{\boldsymbol{K}}^{-}(s,y|x)$.

To begin, consider $\{(F_t^{+}, \boldsymbol{J}_t^{+}), t \in \mathbb{R}^{+}\}$. Define diagonal matrices $\boldsymbol{R}_1 = \text{diag}(r_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_1^{+}}$, $\boldsymbol{R}_2 = \text{diag}(|r_{\boldsymbol{n}}|)_{\boldsymbol{n}\in\mathcal{W}_2^{+}}$, $\boldsymbol{Z}_1 = \text{diag}(z_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_1^{+}}$, $\boldsymbol{Z}_2 = \text{diag}(z_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_2^{+}}$, and $\boldsymbol{Z}_0 = \text{diag}(z_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_0^{+}}$, as well as a set of matrices

$$
\begin{aligned}
\boldsymbol{W}_{11}^{+}(s) &= \boldsymbol{R}_1^{-1}\big((\boldsymbol{T}_{11}^{+} - s\boldsymbol{Z}_1) - \boldsymbol{T}_{10}^{+}(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)^{-1}\boldsymbol{T}_{01}^{+}\big), \\
\boldsymbol{W}_{22}^{+}(s) &= \boldsymbol{R}_2^{-1}\big((\boldsymbol{T}_{22}^{+} - s\boldsymbol{Z}_2) - \boldsymbol{T}_{20}^{+}(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)^{-1}\boldsymbol{T}_{02}^{+}\big), \\
\boldsymbol{W}_{12}^{+}(s) &= \boldsymbol{R}_1^{-1}\big(\boldsymbol{T}_{12}^{+} - \boldsymbol{T}_{10}^{+}(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)^{-1}\boldsymbol{T}_{02}^{+}\big), \\
\boldsymbol{W}_{21}^{+}(s) &= \boldsymbol{R}_2^{-1}\big(\boldsymbol{T}_{21}^{+} - \boldsymbol{T}_{20}^{+}(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)^{-1}\boldsymbol{T}_{01}^{+}\big).
\end{aligned}
$$

Then, by Theorem 3 of Bean and O'Reilly (2013), if the maximum real parts of the eigenvalues of $\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0$, $\boldsymbol{W}_{11}^{+}(s)$, and $\boldsymbol{W}_{22}^{+}(s)$ are all negative, $\widehat{\boldsymbol{\Psi}}^{+}(s)$ is a solution to

$$\boldsymbol{W}_{12}^{+}(s) + \widehat{\boldsymbol{\Psi}}^{+}(s)\boldsymbol{W}_{21}^{+}(s)\widehat{\boldsymbol{\Psi}}^{+}(s) + \boldsymbol{W}_{11}^{+}(s)\widehat{\boldsymbol{\Psi}}^{+}(s) + \widehat{\boldsymbol{\Psi}}^{+}(s)\boldsymbol{W}_{22}^{+}(s) = \boldsymbol{0}, \tag{4.5.4}$$

and if $s$ is real, is the minimal nonnegative solution to (4.5.4). Furthermore, by Theorem 4 of Bean and O'Reilly (2013), we have

$$\widehat{\boldsymbol{G}}^{+}(s,y) = e^{(\boldsymbol{W}_{22}^{+}(s)+\boldsymbol{W}_{21}^{+}(s)\widehat{\boldsymbol{\Psi}}^{+}(s))y}. \tag{4.5.5}$$

For $\boldsymbol{W}_{11}^{+}(s)$, consider $s \in \mathbb{R}^{+}$ and

$$\big(\boldsymbol{T}_{11}^{+} - \boldsymbol{T}_{10}^{+}(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)^{-1}\boldsymbol{T}_{01}^{+}\big).$$

Since $s \geq 0$ and $\boldsymbol{Z}_0 = \boldsymbol{I}$, we can view $(\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0)$ as the rate matrix for transitions within $\mathcal{W}_0^{+}$ with the sojourn rates of the states in $\mathcal{W}_0^{+}$ increased by $s \geq 0$. Therefore, the above matrix is the rate matrix for transitions within $\mathcal{W}_1^{+}$ of the censored phase process, censoring the time spent in $\mathcal{W}_0^{+}$ with the increased sojourn rates in $\mathcal{W}_0^{+}$. Since the claims can arrive in any of the states in $\mathcal{W}_1^{+}$, we can verify that the above matrix is a substochastic matrix for all $s \in \mathbb{R}^{+}$. Then, together with the fact that $\boldsymbol{Z}_1 = \boldsymbol{I}$, we can further conclude that $\boldsymbol{W}_{11}^{+}(s)$ is a substochastic matrix for $s \in \mathbb{R}^{+}$. Hence, the maximum real part of the eigenvalues of $\boldsymbol{W}_{11}^{+}(s)$ is negative. Then, by Lemma 2 of Bean and O'Reilly (2013), we have that the maximum real part of the eigenvalues of $\boldsymbol{W}_{11}^{+}(s)$ is negative for all $s \in \mathbb{C}$, $\Re(s) \geq 0$. Similarly, noting that $\boldsymbol{Z}_2$ is a zero matrix and $\boldsymbol{T}_{22}^{+}$ is a substochastic matrix, we can also verify that the maximum real part of the eigenvalues of $\boldsymbol{W}_{22}^{+}(s)$ is negative for all $s \in \mathbb{C}$, $\Re(s) \geq 0$. Moreover, since $\boldsymbol{Z}_0 = \boldsymbol{I}$, we can again verify that the real part of the eigenvalue of $\boldsymbol{T}_{00}^{+} - s\boldsymbol{Z}_0$ is negative for all $s \in \mathbb{C}$ such that $\Re(s) \geq 0$. Therefore, we have verified that (4.5.4) and (4.5.5) are valid for all $s \in \mathbb{C}$, $\Re(s) \geq 0$.

Next, consider $\{(F_t^-, \boldsymbol{J}_t^-), t \in \mathbb{R}^+\}$, restricted to $\mathcal{W}_1^-$ and $\mathcal{W}_2^-$, (i.e., the nonruined states). Define the following matrices:

$$\begin{aligned}
\boldsymbol{W}_{11}^-(s) &= \boldsymbol{R}_1^{-1}(\boldsymbol{T}_{11}^- - s\boldsymbol{Z}_1),\\
\boldsymbol{W}_{22}^-(s) &= \boldsymbol{R}_2^{-1}(\boldsymbol{T}_{22}^- - s\boldsymbol{Z}_2),\\
\boldsymbol{W}_{12}^-(s) &= \boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^-,\\
\boldsymbol{W}_{21}^-(s) &= \boldsymbol{R}_2^{-1}\boldsymbol{T}_{21}^-.
\end{aligned}$$

Now, consider the reflection of the fluid flow process $\{(F_t^-, \boldsymbol{J}_t^-), t \in \mathbb{R}^+\}$ on the time axis. Then, considering the reversal of the slopes of the reflected fluid flow process in $\mathcal{W}_1^-$ and $\mathcal{W}_2^-$, by Theorem 3 of Bean and O'Reilly (2013), if the maximum real parts of the eigenvalues of $\boldsymbol{W}_{11}^-(s)$, and $\boldsymbol{W}_{22}^-(s)$ are negative, $\widehat{\boldsymbol{\Psi}}^-(s)$ is a solution to

$$\boldsymbol{W}_{21}^-(s) + \widehat{\boldsymbol{\Psi}}^-(s)\boldsymbol{W}_{12}^-(s)\widehat{\boldsymbol{\Psi}}^-(s) + \boldsymbol{W}_{22}^-(s)\widehat{\boldsymbol{\Psi}}^-(s) + \widehat{\boldsymbol{\Psi}}^-(s)\boldsymbol{W}_{11}^-(s) = \boldsymbol{0}, \tag{4.5.6}$$

and if $s$ is real, is the minimal nonnegative solution to (4.5.6).

Similar to the case of (4.5.4) and (4.5.5), since $\boldsymbol{Z}_1 = \boldsymbol{I}$ and claims can occur in any state in $\mathcal{W}_1^-$, $\boldsymbol{W}_{11}^-(s)$ is a substochastic matrix for $s \in \mathbb{R}^+$. Hence, we can conclude that the maximum real part of the eigenvalues of $\boldsymbol{W}_{11}^-(s)$ is negative for all $s \in \mathbb{C}$, $\Re(s) \geq 0$. Likewise, since $\boldsymbol{Z}_2$ is a zero matrix and $\boldsymbol{T}_{22}^-$ is a substochastic matrix, $\boldsymbol{W}_{22}^-(s)$ is a substochastic matrix for all $s \in \mathbb{R}^+$. Therefore, (4.5.6) is valid for all $s \in \mathbb{C}$, $\Re(s) \geq 0$.

Finally, we lay out the probabilistic arguments leading to an algorithm for evaluating $\widehat{\boldsymbol{K}}^-(s, y|x)$. Let $\widehat{\boldsymbol{G}}^-(s, y)$ denote a matrix whose entries are given by

$$\left(\widehat{\boldsymbol{G}}^-(s, y)\right)_{\boldsymbol{h},\boldsymbol{l}} = E\{e^{-sO_{\kappa(0)}}\mathcal{I}[\kappa(0) < \kappa < \infty, \boldsymbol{J}_{\kappa(0)} = \boldsymbol{l}]|F_0 = y, \boldsymbol{J}_0 = \boldsymbol{h}\},\ \boldsymbol{h}, \boldsymbol{l} \in \mathcal{W}_1^-,\ y < 0,$$

and let $\widehat{\boldsymbol{\Upsilon}}^{-}(s,dx)$ denote a matrix whose entries are given by

$$\left(\widehat{\boldsymbol{\Upsilon}}^{-}(s,dx)\right)_{\boldsymbol{h},\boldsymbol{l}} = \int_0^\infty E\{e^{-sO_t}\mathcal{I}[\kappa(0)>t,\kappa>t,F_t\in dx,\boldsymbol{J}_t=\boldsymbol{l}]|F_0=0,\boldsymbol{J}_0=\boldsymbol{h}\}dt,$$

$$\boldsymbol{h}\in\mathcal{W}_2^-,\boldsymbol{l}\in\mathcal{W}_1^-,\ x<0.$$

Then, by conditioning on the maximum value that $\{F_t^-, t\in\mathbb{R}^+\}$ attains on the sample paths in which it stays below 0, and exploiting the level independence of the fluid flow process, we can write

$$\widehat{\boldsymbol{K}}^{-}(s,y|x) = \begin{cases} \widehat{\boldsymbol{G}}^{-}(s,x-y)+\widehat{\boldsymbol{G}}^{-}(s,x-y)\int_y^0\widehat{\boldsymbol{G}}^{-}(s,y-a)\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,y-a)da,\ y>x, \\ \int_x^0\widehat{\boldsymbol{G}}^{-}(s,x-a)\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,y-a)da,\ y<x, \end{cases} \tag{4.5.7}$$

where $\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)$ is a matrix such that $\widehat{\boldsymbol{\Upsilon}}^{-}(s,da)=\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)da$.

Note that, using the reflection arguments employed in computing $\widehat{\boldsymbol{\Psi}}^{-}(s)$, we can apply Theorem 4 of Bean and O'Reilly (2013) to compute $\widehat{\boldsymbol{G}}_{\boldsymbol{n}}^{-}(s,y)$, i.e.,

$$\widehat{\boldsymbol{G}}^{-}(s,y)=e^{-(\boldsymbol{W}_{11}^{-}(s)+\boldsymbol{W}_{12}^{-}(s)\widehat{\boldsymbol{\Psi}}^{-}(s))y},\ y<0. \tag{4.5.8}$$

Then, the only quantity in (4.5.7) in which an evaluation procedure is lacking is $\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)$. One may think, from the definition of $\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)$, to apply the usual time-reversal argument to evaluate $\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)$. However, the resulting time-reversed process is unfortunately no longer a fluid flow process, and hence, the matrix analytic methodology cannot be directly applied. Therefore, unlike the usual time-reversal argument, we need to develop a time-reversal argument for $\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)$ from the perspective of the risk process.

First of all, consider $\left(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)\right)_{\boldsymbol{n},\boldsymbol{l}}$, $\boldsymbol{n},\boldsymbol{l}\in\mathcal{W}_1^-$ such that $n_1=l_1$ and $n_2\le l_2$. All the other entries are 0 since no new customers arrive when the surplus level is below

0. By the structure of $\boldsymbol{T}^-$, conditioning on the value of the claim size yields

$$\big(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^-\widehat{\boldsymbol{\Upsilon}}^-(s,a)\big)_{\boldsymbol{n},\boldsymbol{l}} = \frac{\lambda_3 n_1}{c(n_1)}\int_0^\infty f(x)\big(\widehat{\boldsymbol{K}}^-(s,a|-x)\big)_{\boldsymbol{n},\boldsymbol{l}}dx. \tag{4.5.9}$$

Now, consider the time-reversed risk process, denoted by $\{\tilde{U}_t, t \in \mathbb{R}^+\}$, on the sample paths associated with $\big(\widehat{\boldsymbol{K}}^-(s,y|-x)\big)_{\boldsymbol{n},\boldsymbol{l}}$. The time-reversed risk process travels from level $a$ to $-x$ while staying strictly below level 0 and the process $\{\tilde{\boldsymbol{N}}_t, t \in \mathbb{R}^+\}$, the time-reversed version of $\{\boldsymbol{N}_t, t \in \mathbb{R}^+\}$, transitions from $(l_1, l_2)$ to $(l_1, l_2 - 1)$ to $(l_1, l_2 - 2)$ all the way to $\boldsymbol{n}$. Since both the age process and the claims arrival process are independent Poisson processes, the time-reversed age process and claims arrival process are also independent Poisson processes. Noting that Poisson processes are invariant under time-reversion and the shift process $\{O_t, t \in \mathbb{R}^+\}$ keeps track of the time the phase process of the fluid flow process spends in $\mathcal{W}_1^-$, we have the following equality:

$$\big(\widehat{\boldsymbol{K}}^-(s,a|-x)\big)_{\boldsymbol{n},\boldsymbol{l}}dx = \int_0^\infty E\{e^{-st}\mathcal{I}[\tilde{U}_z < 0\ \forall\ z \in [0,t], \tilde{U}_t \in -dx, \tilde{\boldsymbol{N}}_t = \boldsymbol{n}]|\tilde{U}_0 = a, \tilde{\boldsymbol{N}}_0 = \boldsymbol{l}\}dt. \tag{4.5.10}$$

Substituting (4.5.10) into (4.5.9), we obtain

$$\begin{aligned}&\big(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^-\widehat{\boldsymbol{\Upsilon}}^-(s,a)\big)_{\boldsymbol{n},\boldsymbol{l}} \\ &= \int_0^\infty\int_0^\infty E\{e^{-st}\mathcal{I}[\tilde{U}_z < 0\ \forall\ z \in [0,t], \tilde{U}_t \in -dx, \tilde{\boldsymbol{N}}_t = \boldsymbol{n}]|\tilde{U}_0 = a, \tilde{\boldsymbol{N}}_0 = \boldsymbol{l}\}\frac{\lambda_3 n_1}{c(n_1)}f(x)dtdx.\end{aligned} \tag{4.5.11}$$

Next, we construct a fluid flow process and a shift process for the fluid flow based matrix analytic methodology to be employed in evaluating the right hand side of (4.5.11). Let $\{(R_t, \boldsymbol{E}_t), t \in \mathbb{R}^+\}$ denote the fluid flow process of which sample paths can be linked (as done in Figures 4.1 and 4.2) to those of the time-reversed risk process $\{(\tilde{U}_t, \tilde{\boldsymbol{N}}_t), t \in \mathbb{R}^+\}$, reflected on the time axis. We set the state space of the phase process $\{\boldsymbol{E}_t, t \in \mathbb{R}^+\}$ equal to $\mathcal{W}_1^- \cup \mathcal{W}_2^-$. We express the TPM of $\{\boldsymbol{E}_t, t \in \mathbb{R}^+\}$, partitioned according to $\mathcal{W}_1^-$ and $\mathcal{W}_2^-$, as

$$\boldsymbol{B} = (b_{\boldsymbol{n},\boldsymbol{l}})_{\boldsymbol{n},\boldsymbol{l} \in \mathcal{W}_1^- \cup \mathcal{W}_2^-} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}.$$

The transition rates for $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_1^-$ are given by

$$b_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} -(\lambda_1 + \lambda_3 n_1), \text{ if } \boldsymbol{n} = \boldsymbol{l}, \\ \lambda_1, \text{ if } n_1 = l_1 \text{ and } l_2 = n_2 - 1, \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_1^-$ and $\boldsymbol{l} \in \mathcal{W}_2^-$,

$$b_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \alpha_{l_3} \lambda_3 n_1, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_2^-$,

$$b_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} u_{n_3, l_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

For $\boldsymbol{n} \in \mathcal{W}_2^-$ and $\boldsymbol{l} \in \mathcal{W}_1^-$,

$$b_{\boldsymbol{n},\boldsymbol{l}} = \begin{cases} \gamma_{n_3}, \text{ if } (n_1, n_2) = (l_1, l_2), \\ 0, \text{ otherwise.} \end{cases}$$

We set the flow rates $\{\xi_{\boldsymbol{n}} = c(n_1), \boldsymbol{n} \in \mathcal{W}_1^-\}$ and $\{\xi_{\boldsymbol{n}} = -1, \boldsymbol{n} \in \mathcal{W}_2^-\}$. Now, let $\{H_t, t \in \mathbb{R}^+\}$ denote a fluid flow process defined on the phase process $\{\boldsymbol{E}_t, t \in \mathbb{R}^+\}$ with the flow rates $\{\chi_{\boldsymbol{n}} = 1, \boldsymbol{n} \in \mathcal{W}_1^-\}$ and $\{\chi_{\boldsymbol{n}} = 0, \boldsymbol{n} \in \mathcal{W}_2^-\}$, where $H_0 = 0$ w.p. 1. Clearly, $\{H_t, t \in \mathbb{R}^+\}$ is a shift process which keeps track of the time $\{\boldsymbol{E}_t, t \in \mathbb{R}^+\}$ spends in $\mathcal{W}_1^-$.

Define the following diagonal matrices $\boldsymbol{\Xi}_1 = \text{diag}(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_1^-}$, $\boldsymbol{\Xi}_2 = \text{diag}(|\xi_{\boldsymbol{n}}|)_{\boldsymbol{n}\in\mathcal{W}_2^-}$, $\boldsymbol{X}_1 = \text{diag}(\chi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_1^-}$, $\boldsymbol{X}_2 = \text{diag}(\chi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathcal{W}_2^-}$, and a set of matrices

$$\begin{aligned}
\boldsymbol{L}_{11}(s) &= \boldsymbol{\Xi}_1^{-1}(\boldsymbol{B}_{11} - s\boldsymbol{X}_1),\\
\boldsymbol{L}_{22}(s) &= \boldsymbol{\Xi}_2^{-1}(\boldsymbol{B}_{22} - s\boldsymbol{X}_2),\\
\boldsymbol{L}_{12}(s) &= \boldsymbol{\Xi}_1^{-1}\boldsymbol{B}_{12},\\
\boldsymbol{L}_{21}(s) &= \boldsymbol{\Xi}_2^{-1}\boldsymbol{B}_{21}.
\end{aligned}$$

Let $\widehat{\boldsymbol{\Theta}}(s)$ and $\widehat{\boldsymbol{Q}}(s,y)$ denote matrices whose entries are given by

$$\big(\widehat{\boldsymbol{\Theta}}(s)\big)_{\boldsymbol{n},\boldsymbol{l}} = E\{e^{-sH_{\theta(0)}}\mathcal{I}[\theta(0) < \infty, \boldsymbol{E}_{\theta(0)} = \boldsymbol{l}]|R_0 = 0, \boldsymbol{E}_0 = \boldsymbol{n}\},\ \boldsymbol{n} \in \mathcal{W}_1^-, \boldsymbol{l} \in \mathcal{W}_2^-,$$

and

$$\big(\widehat{\boldsymbol{Q}}(s,y)\big)_{\boldsymbol{n},\boldsymbol{l}} = E\{e^{-sH_{\theta(0)}}\mathcal{I}[\theta(0) < \infty, \boldsymbol{E}_{\theta(0)} = \boldsymbol{l}]|R_0 = y, \boldsymbol{E}_0 = \boldsymbol{n}\},\ y > 0,\ \boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_2^-,$$

where $\theta(y) = \inf\{t > 0 : R_t = y\}$ denotes the first passage time of $\{R_t, t \in \mathbb{R}^+\}$. Following the same line of logic in verifying the validity of (4.5.6) for all $s \in \mathbb{C}$, $\Re(s) \geq 0$, we can verify that the maximum real parts of the eigenvalues of $\boldsymbol{L}_{11}(s)$ and $\boldsymbol{L}_{22}(s)$ are negative for all $s \in \mathbb{C}$, $\Re(s) \geq 0$. Then, by Theorem 3 of Bean and O'Reilly (2013), for all $s \in \mathbb{C}$, $\Re(s) \geq 0$, $\widehat{\boldsymbol{\Theta}}(s)$ is a solution to

$$\boldsymbol{L}_{12}(s) + \widehat{\boldsymbol{\Theta}}(s)\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s) + \boldsymbol{L}_{11}(s)\widehat{\boldsymbol{\Theta}}(s) + \widehat{\boldsymbol{\Theta}}(s)\boldsymbol{L}_{22}(s) = \boldsymbol{0}, \tag{4.5.12}$$

and if $s$ is real, $\widehat{\boldsymbol{\Theta}}(s)$ is the minimal nonnegative solution to (4.5.12). Also, by Theorem 4 of Bean and O'Reilly (2013), we have that

$$\widehat{\boldsymbol{Q}}(s,y) = e^{(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))y}.$$

Returning to (4.5.11) and noting that $\{H_t, t \in \mathbb{R}^+\}$ is a shift process which keeps track of the time the fluid flow process $\{R_t, t \in \mathbb{R}^+\}$ spends in $\mathcal{W}_1^-$, the sample paths connection between $\{R_t, t \in \mathbb{R}^+\}$ and the reflected time-reversed risk process reveals that (4.5.11) can be rewritten as

$$\begin{aligned}
&\left(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)\right)_{\boldsymbol{n},\boldsymbol{l}} \\
&= \int_0^\infty\int_0^\infty E\{e^{-st}\mathcal{I}[\tilde{U}_z < 0 \,\forall\, z \in [0,t], \tilde{U}_t \in -dx, \tilde{\boldsymbol{N}}_t = \boldsymbol{n}] | \tilde{U}_0 = a, \tilde{\boldsymbol{N}}_0 = \boldsymbol{l}\}\frac{\lambda_3 n_1}{c(n_1)} f(x)dtdx \\
&= \int_0^\infty\int_0^\infty E\{e^{-sH_t}\mathcal{I}[R_z > 0 \,\forall\, z \in [0,t], R_t \in dx, \boldsymbol{E}_t = \boldsymbol{n}] | R_0 = -a, \boldsymbol{E}_0 = \boldsymbol{l}\}\frac{\lambda_3 n_1}{c(n_1)} f(x)dtdx.
\end{aligned} \tag{4.5.13}$$

Since the downward journey ending in phase $\boldsymbol{n}$ of the process $\{(R_t, \boldsymbol{E}_t), t \in \mathbb{R}^+\}$ must have initiated from phase $\boldsymbol{n}$ (consider the structure of $\boldsymbol{B}$), (4.5.13) can subsequently be rewritten as

$$\begin{aligned}
\left(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)\right)_{\boldsymbol{n},\boldsymbol{l}} &= \left(\widehat{\boldsymbol{\Theta}}(s)\widehat{\boldsymbol{Q}}(s,-a)\boldsymbol{B}_{21}\right)_{\boldsymbol{l},\boldsymbol{n}} \\
&= \left(\widehat{\boldsymbol{\Theta}}(s)e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))a}\boldsymbol{B}_{21}\right)_{\boldsymbol{l},\boldsymbol{n}}.
\end{aligned} \tag{4.5.14}$$

Since for all $\boldsymbol{n}, \boldsymbol{l} \in \mathcal{W}_1^-$,

$$\left(\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\widehat{\boldsymbol{\Upsilon}}^{-}(s,a)\right)_{\boldsymbol{n},\boldsymbol{l}} = 0 \implies \left(\widehat{\boldsymbol{\Theta}}(s)\widehat{\boldsymbol{Q}}(s,-a)\boldsymbol{B}_{21}\right)_{\boldsymbol{l},\boldsymbol{n}} = 0,$$

we ultimately have

$$\boldsymbol{R}_1^{-1}\boldsymbol{T}_{12}^{-}\boldsymbol{\Upsilon}(s,a) = \left(\widehat{\boldsymbol{\Theta}}(s)e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))a}\boldsymbol{B}_{21}\right)^{\intercal}. \tag{4.5.15}$$

Returning to (4.5.7) and substituting (4.5.8) and (4.5.15) into (4.5.7), we have

$$\begin{aligned}&\widehat{\boldsymbol{K}}^{-}(s,y|x)\\ &= \begin{cases} e^{-(\boldsymbol{W}_{11}^{-}(s)+\boldsymbol{W}_{12}^{-}(s)\widehat{\boldsymbol{\Psi}}^{-}(s))(x-y)} \\ \quad +e^{-(\boldsymbol{W}_{11}^{-}(s)+\boldsymbol{W}_{12}^{-}(s)\widehat{\boldsymbol{\Psi}}^{-}(s))(x-y)} \\ \quad \times \int_y^0 e^{-(\boldsymbol{W}_{11}^{-}(s)+\boldsymbol{W}_{12}^{-}(s)\widehat{\boldsymbol{\Psi}}^{-}(s))(y-a)}\boldsymbol{B}_{21}^{\intercal}e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))^{\intercal}(y-a)}\widehat{\boldsymbol{\Theta}}(s)^{\intercal}da, \ y>x, \\ \int_x^0 e^{-(\boldsymbol{W}_{11}^{-}(s)+\boldsymbol{W}_{12,\boldsymbol{n}}^{-}(s)\widehat{\boldsymbol{\Psi}}_{\boldsymbol{n}}^{-}(s))(x-a)}\boldsymbol{B}_{21}^{\intercal}e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))^{\intercal}(y-a)}\widehat{\boldsymbol{\Theta}}(s)^{\intercal}da, \ y<x, \end{cases}\end{aligned} \tag{4.5.16}$$

Therefore, we may now substitute (4.5.16) into (4.5.3) and evaluate $h_s(x,\boldsymbol{l},y|u,\boldsymbol{n}) = g_s(x,\boldsymbol{l},y|u,\boldsymbol{n})$. Finally, we can evaluate $\psi(s|u,\boldsymbol{n})$ from (4.5.1).

Unfortunately, we do not have a closed form solution for the integral equation (4.5.1) for general phase-type claim size distributions. In such cases, one may still numerically integrate (4.5.1). When the claim size distribution follows an exponential distribution (obviously a special case of the phase-type family of distributions), however, we are able to get a closed form solution for (4.5.1), as we show in the next subsection.

### 4.5.6 Special case: exponential claim size distribution

Suppose that the claim size pdf is given by $f(y) = \lambda_4 e^{-\lambda_4 y}$, $y > 0$, $\lambda_4 > 0$. Let us now revisit the integral equation (4.5.1):

$$\begin{aligned}
&\psi(s|u,\boldsymbol{n}) \\
&= \sum_{\boldsymbol{l}\in\mathcal{N}^-\setminus\{\boldsymbol{0}\}} \int_{-\infty}^{0}\int_{-\infty}^{0} h_s(x,\boldsymbol{l},y|u,\boldsymbol{n})dxdy, \\
&= \sum_{\boldsymbol{l}\in\mathcal{N}^-\setminus\{\boldsymbol{0}\}} \int_{-\infty}^{0}\int_{-\infty}^{0} \sum_{\boldsymbol{h}\in\mathcal{Q}} \boldsymbol{e_n}\widehat{\boldsymbol{\Psi}}^{+}(s)\widehat{\boldsymbol{G}}^{+}(s,u)\big(I-\widehat{\boldsymbol{\Psi}}^{-}(s)\widehat{\boldsymbol{\Psi}}^{+}(s)\big)^{-1}e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}^{-}\boldsymbol{e}_{\boldsymbol{l}}^{\intercal}\big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\frac{v_{\boldsymbol{h},\boldsymbol{0}}^{-}}{c(h_1)}dxdy.
\end{aligned}$$

Our goal in this subsection is to obtain a closed form solution for the following integral:

$$\int_{-\infty}^{0}\int_{-\infty}^{0} e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}^{-}\boldsymbol{e}_{\boldsymbol{l}}^{\intercal}\big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}}dxdy. \tag{4.5.17}$$

The exponential claim size distribution assumption implies that $\mathcal{J} = \{1\}$ and that $e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}$ forms a diagonal matrix whose diagonal entries are all equal to $\lambda_4 e^{\lambda_4 x}$. This in turn implies that the only nontrivial entry of the column vector $e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}^{-}\boldsymbol{e}_{\boldsymbol{l}}^{\intercal}$ is $\boldsymbol{e}_{(\boldsymbol{l},1)}e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}^{-}\boldsymbol{e}_{\boldsymbol{l}}^{\intercal}$. Hence, for (4.5.17), we can simply compute

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \boldsymbol{e}_{(\boldsymbol{l},1)}e^{-\boldsymbol{T}_{22}^{-}x}\boldsymbol{T}_{21}^{-}\boldsymbol{e}_{\boldsymbol{l}}^{\intercal}\big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}}dxdy = \int_{-\infty}^{0}\int_{-\infty}^{0} \lambda_4 e^{\lambda_4 x}\big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}}dxdy. \tag{4.5.18}$$

Consider now the following quantity:

$$\int_{-\infty}^{0} \left(\widehat{\boldsymbol{K}}^{-}(s,y|x)\right)_{\boldsymbol{l},\boldsymbol{h}} \frac{\lambda_3 l_1}{c(l_1)} e^{\lambda_4 x} \lambda_4 dx. \tag{4.5.19}$$

As in (4.5.10), we can establish that

$$\left(\widehat{\boldsymbol{K}}^{-}(s,y|x)\right)_{\boldsymbol{l},\boldsymbol{h}} dx = \int_{0}^{\infty} E\{e^{-st}\mathcal{I}[\tilde{U}_t \in dx, \tilde{\boldsymbol{N}}_t = \boldsymbol{l}, \tilde{U}_z < 0\,\forall\, z \in [0,t]] | \tilde{U}_0 = y, \tilde{\boldsymbol{N}}_0 = \boldsymbol{h}\} dt. \tag{4.5.20}$$

Substituting (4.5.20) into (4.5.19), (4.5.19) can now be expressed as

$$\begin{aligned}&\int_{-\infty}^{0} \left(\widehat{\boldsymbol{K}}^{-}(s,y|x)\right)_{\boldsymbol{l},\boldsymbol{h}} \frac{\lambda_3 l_1}{c(l_1)} e^{\lambda_4 x} \lambda_4 dx \\ &= \int_{-\infty}^{0}\int_{0}^{\infty} E\{e^{-st}\mathcal{I}[\tilde{U}_t \in dx, \tilde{\boldsymbol{N}}_t = \boldsymbol{l}, \tilde{U}_z < 0\,\forall\, z \in [0,t]] | \tilde{U}_0 = y, \tilde{\boldsymbol{N}}_0 = \boldsymbol{h}\} dt\, \frac{\lambda_3 l_1}{c(l_1)} e^{\lambda_4 x} \lambda_4 dx.\end{aligned} \tag{4.5.21}$$

Then, the same probabilistic reasoning used for (4.5.14) leads to

$$\int_{-\infty}^{0} \left(\widehat{\boldsymbol{K}}^{-}(s,y|x)\right)_{\boldsymbol{l},\boldsymbol{h}} \frac{\lambda_3 l_1}{c(l_1)} e^{\lambda_4 x} \lambda_4 dx = \left(\widehat{\boldsymbol{\Theta}}(s) e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))y} \boldsymbol{B}_{21}\right)_{\boldsymbol{h},\boldsymbol{l}}. \tag{4.5.22}$$

Returning to (4.5.18), the inner integral of the right hand side of (4.5.18) then becomes

$$\int_{-\infty}^{0} \left(\widehat{\boldsymbol{K}}^{-}(s,y|x)\right)_{\boldsymbol{l},\boldsymbol{h}} e^{\lambda_4 x} \lambda_4 dx = \frac{c(l_1)}{\lambda_3 l_1} \left(\widehat{\boldsymbol{\Theta}}(s) e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))y} \boldsymbol{B}_{21}\right)_{\boldsymbol{h},\boldsymbol{l}}. \tag{4.5.23}$$

Substituting (4.5.23) into (4.5.18), the double integral on the right hand side of (4.5.18) reduces to

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\big(\widehat{\boldsymbol{K}}^{-}(s,y|x)\big)_{\boldsymbol{l},\boldsymbol{h}}e^{\lambda_4 x}\lambda_4 dxdy = \int_{-\infty}^{0}\frac{c(l_1)}{\lambda_3 l_1}\big(\widehat{\boldsymbol{\Theta}}(s)e^{-(\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s))y}\boldsymbol{B}_{21}\big)_{\boldsymbol{h},\boldsymbol{l}}dy$$
$$= \frac{c(l_1)}{\lambda_3 l_1}\big(-\widehat{\boldsymbol{\Theta}}(s)\big((\boldsymbol{L}_{22}(s)+\boldsymbol{L}_{21}(s)\widehat{\boldsymbol{\Theta}}(s)\big)^{-1}\boldsymbol{B}_{21}\big)_{\boldsymbol{h},\boldsymbol{l}}. \tag{4.5.24}$$

Substituting (4.5.24) into (4.5.1) and noting that $v^{-}_{\boldsymbol{h},\boldsymbol{0}} = \lambda_1$, we ultimately have

$$\psi(s|u,\boldsymbol{n}) = \sum_{\boldsymbol{l}\in\mathcal{N}^{-}\setminus\{\boldsymbol{0}\}}\sum_{\boldsymbol{h}\in\mathcal{Q}}\boldsymbol{e}_{\boldsymbol{n}}\widehat{\boldsymbol{\Psi}}^{+}(s)\widehat{\boldsymbol{G}}^{+}(s,u)\big(I-\widehat{\boldsymbol{\Psi}}^{-}(s)\widehat{\boldsymbol{\Psi}}^{+}(s)\big)^{-1}\boldsymbol{e}^{\intercal}_{(\boldsymbol{l},1)}$$
$$\times\frac{c(l_1)\lambda_1}{c(h_1)\lambda_3 l_1}\big(-\widehat{\boldsymbol{\Theta}}_{\boldsymbol{l}}(s)\big((\boldsymbol{L}_{22,\boldsymbol{l}}(s)+\boldsymbol{L}_{21,\boldsymbol{l}}(s)\widehat{\boldsymbol{\Theta}}_{\boldsymbol{l}}(s)\big)^{-1}\boldsymbol{B}_{21}\big)_{\boldsymbol{h},\boldsymbol{l}}. \tag{4.5.25}$$

### 4.5.7 Numerical analysis

In this subsection, we examine how the variables in the dynamic individual risk model interact together to affect the time of ruin distribution. More specifically, we focus on examining how different combinations of premium rates and customer arrival rate $\lambda_2$ affect the distribution of the time of ruin while fixing the values of the claim arrival rate $\lambda_3$ and the expected value of the claim size $1/\lambda_4$ under the assumption that the claim sizes are exponentially distributed with rate $\lambda_4$. In the dynamic individual risk model, the way in which the variables interact together is not obvious. Lower premiums in principle will attract more customers at the cost of a slower initial increase in the surplus of the firm. Since in the dynamic individual risk model we assume that all customers are independent from one another, in theory, we may expect the law of large numbers to hold true for a large number of customers in the system. On the other hand, we assume that no customers arrive to the system if the surplus level is below 0. Hence, too low premiums may result in poor initial performance of the firm's surplus process, leading to a lower number of customers in the system than intended. As there are these complications to consider, it would be interesting to see how different combinations of the premium rates and $\lambda_2$ affect the infinite-time ruin probabilities of the dynamic individual risk model.

For the dynamic individual risk model under consideration, assume that the calendar year $T$ is equal to 1 unit time and let the random calendar year variable $A \sim E(K, \lambda_1)$ where $K = 3$ and $\lambda_1 = 3$ so that $E\{A\} = 1 = T$. The premium rate rule that we implement here is $c(n) = cn$ for $n \in \{0, 1, 2, \ldots, N\}$, where we set the maximum number of customers in the system equal to $N = 50$. (Note that this setup gives the number of phases equal to 150, which is an unusually large number in numerical analyses of MAP risk models in risk theory.) The claims arrival rate is set equal to $\lambda_3 = 1$ and the mean claim size is set equal to $1/\lambda_4 = 1/2$. Then, the expected value of the total claims of an individual who is in the system for the entire calendar year is equal to $\lambda_3/\lambda_4 = 1/2$. We assume that the initial surplus level is equal to 10 and the initial number of customers in the system at time 0 is equal to 1.

Before we consider different combinations of $c$ and $\lambda_2$, let us first consider fixing the value of $c$ at 0.6 and thereafter varying the value of $\lambda_2$. Our expectation from the law of large numbers is that the infinite-time ruin probabilities should decrease as $\lambda_2$ increases. Therefore, we can check whether our expectation holds true by computing the ruin probabilities for different values of $\lambda_2$ while fixing $c$ at 0.6. Since we have assumed that the claim sizes are exponentially distributed, we can compute $\psi(s|10, (1, 1))$ from (4.5.25). The matrices appearing in (4.5.25) are computed following the probabilistic analysis laid out in Section 4.5 and the algorithm used for computing $\widehat{\boldsymbol{\Psi}}^+(s)$, $\widehat{\boldsymbol{\Psi}}^-(s)$ and $\widehat{\boldsymbol{\Theta}}(s)$ is Algorithm B1 in Bean and O'Reilly (2013). For the infinite-time ruin probabilities, we can compute $\psi(0|10, (1, 1))$, and for the finite-time ruin probabilities, we numerically invert $\psi(s|10, (1, 1))$ to obtain the pdf of the time of ruin. The numerical inversion method we implemented here is the Gaver-Stehfest algorithm (see, e.g., Kuznetsov (2013)).

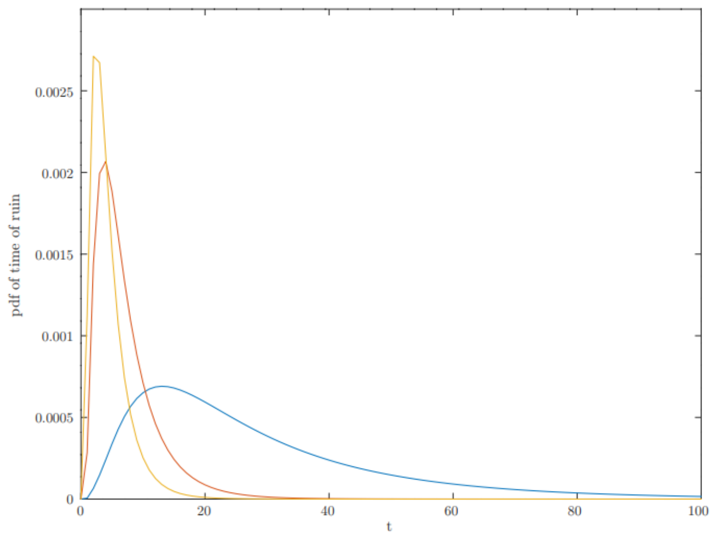


Figure 4.4: pdf of the time of ruin for $\lambda_2$ equal to 5, 25, and 50.

In Table 4.1, the infinite-time ruin probabilities for different values of $\lambda_2$, ranging from 10 to 100, are computed, and the infinite-time ruin probabilities seem to decrease as we increase the value of $\lambda_2$. In Figure 4.4, the pdfs of the time of ruin are computed for $\lambda_2$ set equal to 5, 25, and 50. The pdf corresponding to $\lambda_2 = 5$ is the blue curve, the pdf corresponding to $\lambda_2 = 25$ is the red curve, and the pdf corresponding to $\lambda_2 = 50$ is the yellow curve. As can be seen in Figure 4.4, higher values of $\lambda_2$ do not necessarily imply lower finite-time ruin probabilities on all time intervals. As $\lambda_2$ increases, the assumption that the premiums are collected continuously implies that earlier in a calendar year, the surplus process of the insurance firm is more likely to fall below level 0 due to the possibly large number of customers and lower value of the surplus process. Since no customers arrive when the surplus process is below level 0, this then in turn may result in a lower total number of customers arriving in that calendar year than for some smaller values of $\lambda_2$. However, as time goes on, the surplus process is likely to grow and when there is enough initial surplus, higher values of $\lambda_2$, as expected, are likely to result in a higher total number of customers in the given calendar year and thus the lower ruin probabilities.

| $\lambda_2$ | $\psi(0\|10,(1,1))$ |
|---|---|
| 10 | 0.0204845 |
| 20 | 0.0174097 |
| 30 | 0.0155677 |
| 40 | 0.0144086 |
| 50 | 0.0136405 |
| 60 | 0.0131027 |
| 70 | 0.0127076 |
| 80 | 0.0124059 |
| 90 | 0.0121681 |
| 100 | 0.011976 |

Table 4.1: Infinite-time ruin probabilities for different values of $\lambda_2$

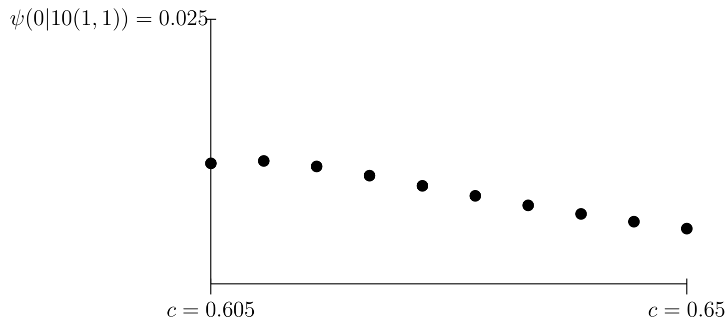


Figure 4.5: Infinite-time ruin probabilities for different values of $c$ and $\lambda_2(c)$

Now, assume that the customer arrival rate $\lambda_2$ has the following functional relation with the premium rate $c$:

$$\lambda_2(c) = \frac{1}{4(c-0.6)}, \ c \in (0.6, 0.65].$$

The function $\lambda_2(c)$ is arbitrarily chosen to describe an inverse relation between $c$ and $\lambda_2$, i.e., higher values of $c$ result in lower values of $\lambda_2$ and lower values of $c$ result in higher values of $\lambda_2$. In Figure 4.5, the infinite-time ruin probabilities are computed for values of $c$ in the increments of 0.005 over the interval $(0.6, 0.65]$. For each value of $c$, $\lambda_2$ is set equal to $\lambda_2(c)$.

As can be seen in Figure 4.5, the infinite-time ruin probability does not exhibit a linear relation with the value of $c$ when $\lambda_2$ is determined by the function $\lambda_2(c)$. Indeed, as the value of $c$ increases from 0.605 to 0.61, the value of $\lambda_2$ decreases from 50 to 25. It seems that under the given model setting, the decrease of the customer arrival rate from 50 to 25 has a more adverse impact on the solvency of the insurance firm than the positive force of the increase of the premium rate from 0.605 to 0.61. However, as $c$ increases, it seems that the higher values of $c$ result in lower values of infinite-time ruin probabilities. One note that we make here is that since we assume that there is one customer already in the system at time 0 and that customer is assumed to have agreed to pay whatever the premium rate that we set in any event, $\lambda_2$ approaching 0 would imply the dominance of the one existing contract in determining the solvency of the insurance firm. In such an extreme case, higher values of $c$ will result in lower values of infinite-time ruin probabilities.

As we initially expected, the numerical results presented in this section show that the interaction among the variables in the dynamic individual risk model has a nonlinear impact on the solvency of an insurance firm. Therefore, when we take into account the stochastic nature of the arrivals and departures of the customers in modelling the insurance risk process, it is imperative to perform a thorough analysis on how the interactions among the variables affect the evolution of the risk process over time.

# Chapter 5

# Concluding remarks and future research

In Chapter 2, we developed a matrix analytic methodology for a class of discrete-time risk models which we named the G/M/1 DTRM class. In Section 2.2, we provided a matrix analytic framework for the general risk models belonging to the G/M/1 DTRM class, and then in Sections 2.3 and 2.4, we demonstrated how the methodology developed for the general risk models in the G/M/1 DTRM class can be either directly applied to or simplified first (exploiting the special structures of the respective risk models) and then applied to the discrete-time MAP risk model with general and matrix geometric claim size distributions, and the discrete-time MAP risk model with a dividend barrier. In Section 2.5, various numerical examples were presented to demonstrate how the methodology can be applied to computing various quantities of interest and how it compares to the standard recursive method in terms of the computation time.

In Chapter 3, we developed a matrix analytic methodology for the discrete-time MAP risk model with phase-dependent premium rates and phase-type claim size distributions, based on a sample paths connection between the risk process and a discrete-time QBD process. Our methodology is built directly on a sample paths connection between the risk process and a QBD process without downward jumps, even with the phase-dependent premium rates, allowing it to exploit the skip-free nature of the QBD process and simplify the respective analysis greatly. In Section 3.3, we demonstrated how our methodology can be employed in computing the discounted joint conditional pmf of the surplus prior to ruin and deficit at ruin. In Section 3.4, a numerical example is presented to demonstrate how the methodology can be applied to computing some quantities of interest and a discussion on the impact of the phase-dependent premium rates on the evolution of the surplus process over time is provided.

In Chapter 4, we discussed an adaptation of the matrix analytic methodology for fluid flow processes developed by Bean and O'Reilly (2013) in risk theory. In Sections 4.3 and 4.4, we briefly discussed how Bean and O'Reilly's methodology can be extended to continuous-time MAP risk models with phase-dependent premium rates and phase-type claim size distributions, based on a sample paths connection between the risk process and a fluid flow process. In Section 4.5, we introduced a new type of a risk model (referred to as the dynamic individual risk model) which takes into account the stochastic dynamics of the customers' arrivals and departures, and applied Bean and O'Reilly's methodology in analyzing the LST of the time of ruin distribution. In Section 4.5.7, numerical analyses were performed to examine how the variables in the dynamic individual risk model interacted together to affect the solvency of an insurance firm. From the numerical analyses, we learned that it is imperative to perform a thorough analysis on how the interactions among the variables affect the evolution of the risk process over time when we take into account the stochastic nature of the arrivals and departures of the customers in modelling the insurance risk process.

As can be seen in this thesis, matrix analytic methods offer a computationally powerful set of methodologies for analyzing the stochastic evolution of insurance risk processes. The computational advantages of matrix analytic methods enable rigorous numerical analyses of complicated risk models such as the dynamic individual risk model introduced and studied in Chapter 4, which traditionally was considered a computationally difficult model to analyze. It is our belief that in the modern age of advanced computing and big data, the computational aspect in risk theory is ever more important, and matrix analytic methods provide powerful tools for the computations in risk theory.

For future research, there are two main directions we wish to pursue. The first is to work on a review paper which collectively presents the various matrix analytic methods in risk theory and their computational aspects. Such a review paper is already available (Badescu and Landriault (2009)), but we feel that there has been many advances in matrix analytic methods in risk theory since then. Moreover, the results on matrix analytic methods in other areas of applied probability are already available (see, e.g., Latouche and Nguyen (2018) for a most recent review), but presenting them in the context of risk theory would serve well in promoting matrix analytic methods in the field.

The second direction is to apply matrix analytic methods to more risk models that take more microscopic point of views on the risk processes than the traditional collective risk models do. As demonstrated in Ahn et al. (2018) and the dynamic individual risk model introduced in Chapter 4 of this thesis, risk models that take more microscopic point of views on the risk processes tend to result in becoming large scale problems in terms of the number of variables involved in the analyses. Traditionally, this has posed difficulties on the computational aspects of the analyses. Since matrix analytic methods have been shown to alleviate this computational difficulty, we are excited about the new types of risk models that can possibly be analyzed via matrix analytic methods.

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