On multiple random locations of stationary processes

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We generalize the concept of intrinsic location functionals to accommodate $n = 2$ random locations, which we combine together in either sets or vectors. For the set-valued case of 'intrinsic multiple-location functionals' we show that for any stationary process and compact interval, the distribution of any intrinsic multiple-location functionals is absolutely continuous on the interior of the interval, the density exists everywhere, is càdlàg, bounded at each point of the interior and satisfies certain total variation constraints. We also characterize the class of possible distributions, showing that it is a weakly closed compact set, and we find its extreme points. Moreover, we show that for almost every measure m in this class of distributions one can construct a pair comprising a stationary process and intrinsic multiple-location functional which has m as its distribution. For the vector-valued case of 'intrinsic location vectors', we identify subclasses based on the joint behaviour of the two random locations and derive results for each subclass. Some of the results connect the intrinsic location vectors back to the 'single-location case' of intrinsic location functionals.

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Dedication

I dedicate this thesis to Sasha, Théodor, Alice and Chester who always motivate and inspire me to continue discovering and conquering new frontiers.

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Chapter 1

Introduction

In this work, we look to extend the results derived by Samorodnitsky and Shen in the theory of random locations. In particular, we wish to extend the framework of intrinsic location functionals to the case of multiple locations. Examples of this case of multiple random locations include the first and second hitting times of a level k, the first two locations of the path suprema, and many others. These types of locations are clearly of interest to those working in extreme value theory, mathematical finance, quantitative risk management and so on.

In Samorodnitsky and Shen (2013a), the authors examine the probabilistic structure of the location of the supremum for stationary processes, rather than the value of the supremum. In particular, they find that the stationarity of the process has a significant effect on the distribution of the location of the path supremum. This paper was the intuitive 'base case' for the current work, as we began by considering the locations of the two path suprema.

Samorodnitsky and Shen (2013b) introduce intrinsic location functionals, which can be thought of as the generalization of path supremum locations. They are defined as measurable functionals from a space of functions closed under shifts, and the compact intervals in $\mathbb{R}$. The definition uses the assumed stationarity of the underlying processes to its advantage in a clever way, leading to the result that the stationarity of a process can actually be characterized by a sufficiently rich class of intrinsic location functionals. They also show that the possible distributions of the intrinsic location functionals over an interval form a weakly closed convex set and describe the extreme points of this set. Some simple examples of intrinsic location functionals are the location of the path infimum, the leftmost location of the largest jump, and the rightmost location of the largest slope.

The goal of this work is to expand on these results in order to accommodate both random sets and random vectors of multiple random locations. These random locations do not necessarily have to be intrinsic location functionals on their own, hence the construction is more complex than simply combining two intrinsic location functionals into a set or vector.

This generalization is a natural extension of the past theory, and now allows us to consider joint behaviours which could not be accounted for in the single location case, such as the locations of the path supremum and infimum, the two leftmost locations of the largest jumps, etc. From these two examples alone, it is clear that there are many subclasses of paired random locations to consider, which will each have different properties.

The paper is structured in the following way:

- The main results from Samorodnitsky and Shen (2013b) which we will generalize to the case of $n = 2$ locations are listed in Chapter 2.
- In Chapter 3, we introduce the set-valued intrinsic multiple-location functionals. In particular, because these functionals are set-valued, we are not distinguishing between the two locations. Hence we are concerned with the probability that one of the locations (or both) is in a Borel subset of the interval on which the functional is defined.
- In Chapter 4, we prove similar results to the 'single-location case' such as the absolute continuity of the distribution over the interior of any compact interval, the existence and boundedness of a density, the fact that the density is càdlàg everywhere, and that the density satisfies certain total variation constraints.
  - In Section 4.3, we examine the class of all measures $\mathcal{A}_T^2$ representing the distributions of the intrinsic multiple-location functionals, and show that this class is a weakly closed convex subset of the class of all measures taking values in $[0, 2]$.
  - We then identify the extreme points of the set $\mathcal{A}_T^2$ in Section 4.4, and show that for a certain subset $\mathcal{D}_T^2$ of $\mathcal{A}_T^2$, we have that for any $m \in \mathcal{D}_T^2$ one can construct a stationary process and instrinsic multiple-location functional which has distribution m on the interval $[0, T]$.
- In Chapter 5, we introduce the intrinsic location vectors, and discuss the connection with intrinsic multiple-location functionals from Chapters 3 and 4. We identify some subclasses of intrinsic location vectors, and derive some results for these subclasses.
- In Chapter 6, we give a brief overview of how the intrinsic multiple-location functionals could be extended to $n > 2$ locations.
- Lastly, in Chapter 7 we discuss some potential directions for further research.

Chapter 2

Instrinsic Location Functionals

2.1 Definitions and Examples

Let $\mathcal{H}$ be a set of functions on $\mathbb{R}$, closed under shifts. Meaning that for all $f \in \mathcal{H}, c \in \mathbb{R}$, the function $\theta_c f = f(x+c), x \in \mathbb{R}$ also belongs to $\mathcal{H}$. We equip $\mathcal{H}$ with its cylindrical σ-field:

$$\mathrm{Cyl}(\mathcal{H}) = \sigma\Big(\{f \mid f(t_1) \in A_1, f(t_2) \in A_2, \ldots, f(t_n) \in A_n\}\,, \\ n \in \mathbb{N}, t_1, \ldots, t_n \in \mathbb{R}, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})\Big),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-Algebra of $\mathbb{R}$. Let $\mathcal{I}$ be the collection of all non-degenerate compact intervals in $\mathbb{R}$: $\mathcal{I} = \{[a,b] \subseteq \mathbb{R} \mid a < b\}$.

These definitions of $\mathcal{H}$ and $\mathcal{I}$ will be used throughout the entire paper. Note that we endow the set $([0,T] \cup \{\infty\})$ with the topology obtained by treating the infinite point as an isolated point of the set.

In what we will refer to as the "single-location case", an intrinsic location functional is defined as follows:

Definition 2.1: A mapping $L : \mathcal{H} \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$ is called an *intrinsic location functional* if it satisfies all of the following conditions:

(1) For every $I \in \mathcal{I}$ the map $L(\cdot, I) : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is measurable.

(2) For every $f \in \mathcal{H}$ and $I \in \mathcal{I}$, $L(f, I) \in (I \cup \{\infty\})$.

(3) *(Shift compatibility)* For every $f \in \mathcal{H}, I \in \mathcal{I}, c \in \mathbb{R}$,

$$L(f, I) = L(\theta_c f, I - c) + c,$$

where $(I - c) = \{x - c \mid x \in I\}$, and $\infty \pm c = \infty$.

(4) *(Stability under restrictions)* For every $f \in \mathcal{H}$, and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$\text{if } L(f, I_1) \in I_2 \text{ then } L(f, I_2) = L(f, I_1).$$

(5) *(Consistency of existence)* For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$\text{if } L(f, I_2) \neq \infty \text{ then } L(f, I_1) \neq \infty.$$

Note that in defining the intrinsic location functional, we have allowed for $L(f, I)$ to take infinite value. This is to be interpreted as the random location not being found in the interval, or not well-defined. An example is the case of a hitting time of level k, where it is possible that this location is never achieved over a compact interval $I \in \mathcal{I}$. This case is explained in more detail in Example 2.3.

Example 2.2: Let $\mathcal{H}$ be the space of upper semi-continuous functions, meaning that $f \in \mathcal{H}$ has the following property for every $t \in \mathbb{R}$:

$$\limsup_{s \to t} f(s) = \lim_{\epsilon \downarrow 0} \left(\sup \{ f(s) \mid s \in B(t, \epsilon) \setminus \{t\} \} \right) \leq f(t),$$

where $B(t, \epsilon)$ is the open ball of radius ϵ with center t.

Then the leftmost location of the path supremum over the interval $[a, b] \in \mathcal{I}$ defined as

$$\tau_{f,[a,b]} := \inf \left\{ t \in [a, b] \mid f(t) = \sup_{s \in [a,b]} f(s) \right\} \tag{2.1}$$

is an intrinsic location functional. This functional was studied in detail in Samorodnitsky and Shen (2012, 2013a). It is one of the intrinsic location functionals which cannot take on an infinite value, since it always exists inside of the interval.

Some other similar intrinsic location functionals are the rightmost location of the path supremum, and the leftmost location of the path infimum:

$$\tau^{\mathrm{r}}_{f,[a,b]} := \sup\left\{t \in [a,b] \mid f(t) = \sup_{s\in[a,b]} f(s)\right\},$$
$$\alpha_{f,[a,b]} := \inf\left\{t \in [a,b] \mid f(t) = \inf_{s\in[a,b]} f(s)\right\}.$$

Example 2.3 Let $\mathcal{H}$ be the space of continuous functions $\mathcal{C}(\mathbb{R})$. Then the first hitting time of a level k over the interval $[a,b]$ defined as

$$\tau^{k}_{f,[a,b]} := \inf\{t \in [a,b] \mid f(t) = k\}$$

is an intrinsic location functional. Of course we could also use sup rather than inf, which would give us the last hitting time of the level k, which is also an intrinsic location functional.

These are intrinsic location functionals which can take on infinite value, because it is possible that f never hits the level k on $[a,b]$.

Example 2.4 Let $\mathcal{H}$ be the space of càlàg functions. Then the leftmost location of the largest jump in the path over the interval $[a,b]$ defined as

$$\tau_{f,[a,b]} := \inf\left\{t \in [a,b] \mid \left|f(t) - f(t^-)\right| = \sup_{s\in[a,b]} \left|f(s) - f(s^-)\right|\right\}$$
$$\text{where } f(t^-) = \lim_{s\uparrow t} f(s),$$

is an intrinsic location functional.

Many other examples can be thought of, such as the leftmost/rightmost location of the largest/smallest slope with $\mathcal{H} = \mathcal{C}^1(\mathbb{R})$ (the space of continuously differentiable functions), or the leftmost/rightmost location of the jump whose size is closest to a given real number for càdlàg functions.

Example 2.5: There are some functionals similar to these that we might mistakenly assume are intrinsic location functionals, but are not, such as the following:

(i) Define the first hitting time of a level k after a given time t over the interval $[a, b]$ as follows:

$$T^k_{t,f,[a,b]} := \inf \{s \in [a, b], s \geq t \mid f(s) = k\} .$$

It is easy to see that this functional satisfies stability under restrictions and consistency of existence, but it does not satisfy shift compatibility. Indeed,

$$T^k_{t,f,[a,b]} \neq T^k_{t,\theta_{2(b-a)}f,[a-2(b-a),b-2(b-a)]} + 2(b - a)$$

since the right hand side will always be ∞ if $t \in [a, b]$.

(ii) The first hitting time of a level k within a fixed distance d to the right endpoint of the interval $[a, b]$, denoted $T^{k,d}_{f,[a,b]}$ is defined as:

$$T^{k,d}_{f,[a,b]} := \inf \{s \in [a, b], s \geq b - d \mid f(s) = k\} .$$

This functional satisfies shift compatibility and stability under restrictions, but it does not satisfy consistency of existence. The hitting time may exist on a smaller interval, but once we move to a larger interval, it is less likely that we satisfy the requirement that $s \geq b - d$, since the right-endpoint could now be much larger.

(iii) The second hitting time of a level k, defined as

$$T^{k,2}_{f,[a,b]} := \inf \left\{t \in \left([a, b] \setminus T^{k,1}_{f,[a,b]}\right) \mid f(t) = k\right\},$$
$$\text{where } T^{k,1}_{f,[a,b]} := \inf \{t \in [a, b] \mid f(t) = k\}$$

is not an intrinsic location functional, because it does not satisfy consistency of existence. If we have intervals $I_2 \subseteq I_1$ such that $T^{k,1}_{f,I_1} \in (I_1 \setminus I_2)$ and $T^{k,2}_{f,I_2} \in I_2$, then $T^{k,1}_{f,I_2} = T^{k,2}_{f,I_1}$, and we could have $T^{k,2}_{f,I_2} = \infty$, a contradiction.

2.2 Previous Results

It is necessary to define some notation for the rest of this section. We denote the underlying stationary process $X = \{X_t\}_{t\in\mathbb{R}}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which has sample paths in $\mathcal{H}$.

For a compact interval $[a, b] \in \mathcal{I}$, we let $L(X, [a, b])$ denote the value of the instrinsic location functional L evaluated on the process X over the interval $[a, b]$.

Note that by Definition 2.1, $L(X, [a, b])$ is a well-defined random variable taking values in $([a, b] \cup \{\infty\})$.

It is important to note that stationarity of the process X and the shift compatibility of L clearly imply that the distribution of L on an interval depends only on the length of the interval. So we will simply consider intervals of the type $[0, b]$ and for notational simplicity denote this case as $L(X, b)$.

We denote by $F_{X,[a,b]}$ the law of $L(X, [a, b])$; a probability measure with support $([a, b] \cup \{\infty\})$. For the interval $[0, b]$ we write $F_{X,b}$.

Finally, $F_{X,[a,b]}(t)$ behaves as one would expect a cumulative distribution function to behave, i.e. $F_{X,[a,b]}(t)$ is the value $F_{X,[a,b]}$ assigns to the interval $[a, t]$ for every $a \leq t \leq b$, where again for the interval $[0, b]$, we use the notation $F_{X,b}(t)$.

The following theorem is the main result of [1] which describes the laws of the intrinsic location functionals and their properties, as well as the total variation constraints and bound on their densities.

Theorem 2.6: Let L be an intrinsic location functional and $X = \{X_t\}_{t\in\mathbb{R}}$ be a stationary process. Then the restriction of the law $F_{X,T}$ to the interior $(0, T)$ of the interval is absolutely continuous. The density, denoted by $f_{X,T}$, can be taken as the right-derivative of $F_{X,T}$, which exists at every point in the interval $(0, T)$. The density is right-continuous, has left limits, and has the following properties:

(a) The following limits exist:

$$f_{X,T}(0^+) = \lim_{t\downarrow 0} f_{X,T}(t) \quad \text{and} \quad f_{X,T}(T^-) = \lim_{t\uparrow T} f_{X,T}(t).$$

(b) The density has a universal upper bound given by

$$f_{X,T}(t) \leq \max\left(\frac{1}{t}, \frac{1}{T-t}\right) \quad \text{for every } 0 < t < T.$$

(c) The density has bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,

$$\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) \le \min(f_{X,T}(t_1), f_{X,T}(t_1^-)) + \min(f_{X,T}(t_2), f_{X,T}(t_2^-)),$$

$$\text{where} \quad \mathrm{TV}_{(t_1,t_2)}(f_{X,T}) = \sup \sum_{i=1}^{n-1} |f_{X,T}(s_{i+1}) - f_{X,T}(s_i)|$$

is the *total variation* of $f_{X,T}$ on the interval (t_1, t_2), and the supremum is taken over all choices of $t_1 < s_1 < \cdots < s_n < t_2$ with finite n.

(d) The density has bounded positive variation at the left endpoint and a bounded negative variation at the right endpoint. Furthermore, for every $0 < \epsilon < T$,

$$\mathrm{TV}^+_{(0,\epsilon)}(f_{X,T}) \le \min(f_{X,T}(\epsilon), f_{X,T}(\epsilon^-)),$$
$$\text{and} \quad TV^-_{(T-\epsilon,T)}(f_{X,T}) \le \min(f_{X,T}(T-\epsilon), f_{X,T}((T-\epsilon)^-)),$$
$$\text{where} \quad TV^{\pm}_{(a,b)}(f_{X,T}) = \sup \sum_{i=1}^{n-1} (f_{X,T}(s_{i+1}) - f_{X,T}(s_i))_{\pm}$$

is the *positive(negative) variation* of $f_{X,T}$ on the interval (a, b),where the supremum is taken over all choices of $a < s_1 < \cdots < s_n < b$ with finite n, and $(x)_+ = \max(x, 0), (x)_- = \max(-x, 0)$.

(e) The limit $f_{X,T}(0^+) < \infty$ if and only if $\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) < \infty$ for some (equivalently, any) $0 < \epsilon < T$, in which case

$$\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) \le f_{X,T}(0^+) + \min(f_{X,T}(\epsilon), f_{X,T}(\epsilon^-)).$$

Similarly, $f_{X,T}(T^-) < \infty$ if and only if $\mathrm{TV}_{(T-\epsilon,T)}(f_{X,T}) < \infty$ for some (equivalently, any) $0 < \epsilon < T$, in which case

$$\mathrm{TV}_{(T-\epsilon,T)}(f_{X,T}) \le \min(f_{X,T}(T-\epsilon), f_{X,T}((T-\epsilon)^-)) + f_{X,T}(T^-).$$

The proof of this theorem is nearly identical to the one of Theorem 3.1 in [3], with the difference that intrinsic location functionals have the possibility of an infinite value, whereas the leftmost path supremum in [3] could not. The details of this proof are found in [1].

The following are some other important results from [1] which will be expanded upon in this work.

Definition 2.7: Denote by $\mathcal{A}_T$ the class of probability measures F on $([0, T] \cup \{\infty\})$ with the following properties:

(1) The restriction of F to the interior $(0,T)$ of the interval is absolutely continuous.

(2) A version of the density is given by the right derivative of the cdf $F\left([0,t]\right), 0 < t < T$, which exists at every point in the interval $(0,T)$.

(3) This density f is right continuous, has left limits, and satisfies the total variation constraints of Theorem 2.6.

Theorem 2.8: Let $\mathcal{P}_T$ be the collection of all probability measures on $([0,T] \cup \{\infty\})$. Then $\mathcal{A}_T$ is a weakly closed convex subset of $\mathcal{P}_T$. Moreover, for any $0 < \epsilon < \frac{T}{2}$, the restrictions of the laws in $\mathcal{A}_T$ to the interval $(\epsilon, T-\epsilon)$ form a compact in total variation family of finite measures.

Theorem 2.9: The extreme points of the set $\mathcal{A}_T$ are:

(1) the measures μ_t, $t \in (0,T)$, concentrated on $(0,T)$ which are absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions

$$f_{\mu_t} = \frac{1}{t}\mathbb{I}_{(0,t)}, 0 < t < T;$$

(2) the measures ν_t, $t \in (0,T)$, concentrated on $(0,T)$ which are absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions

$$f_{\nu_t} = \frac{1}{T-t}\mathbb{I}_{(t,T)}, 0 < t < T;$$

(3) the point masses/singular measures δ_0, δ_T and δ_∞.

Note that the functions $\mathbb{I}_{(a,b)}(t)$ are indicator functions defined as

$$\mathbb{I}_{(a,b)}(t) := \begin{cases} 1 & \text{if } t \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Chapter 3

Intrinsic Multiple-Location Functionals

We want to expand the results for intrinsic location functionals to a framework which can describe $n = 2$ or more random locations. Properties such as shift compatibility from the single-location case can be preserved, but other properties will have to be generalized. In particular, stability under restrictions and consistency of existence have to be completely re-worked to fit into this new framework.

It is important to keep in mind several key intuitions for this generalization:

(1) While it is possible to simply consider two intrinsic location functionals together, this is not the only thing we want to do. Many of the locations we will work with will not be intrinsic location functionals if they were to be considered on their own in the single-case framework.

(2) The two locations can be related in several ways. They can be as dependent as the first and second hitting times of a level k, or they can be almost completely unrelated such as the first hitting time of a level k and the location of the largest slope.

(3) We can view multiple random locations as an ordered pair (vector), or as a random set where we do not distinguish between the locations. In the latter case, which we will begin with, the probabilistic question we are concerned with is whether one or more of the locations are found in a given Borel subset of the real line.

The space which our paired locations will occupy must first be defined. We are not going to concern ourselves with the ordering of the locations (until Chapter 5) so they will be described by sets in Γ_I^2 for $I \in \mathcal{I}$, which is defined as:

$$\Gamma_I^2 := \{ \text{ sets of cardinality at most 2, with elements taken from } I \} .$$

We remark that the empty set $\emptyset \in \Gamma_I^2$ as well. We interpret this construction to mean that if a random location is not found in the interval I, it will simply be excluded from the set, rather than identifying it as an infinite value as in the previous work on intrinsic location functionals by Samorodnitsky and Shen [1].

Observe that each $\xi \in \Gamma_I^2$ can be described as a point measure (as defined in [4], pg. 123-124):

$$m_\xi = \sum_{x \in \xi} \mathbb{I}_x,$$

$$\text{where } \mathbb{I}_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

Hence we can equip Γ_I^2 with the point measure σ-algebra, denoted $\mathcal{M}^2$, described in [4], pg. 124. More precisely, we let $M_p(\Gamma_I^2)$ denote the space of all point measures defined by Γ_I^2, then $\mathcal{M}^2$ is defined as the smallest σ-algebra which contains all sets of the form

$$\left\{ m \in M_p(\Gamma_I^2) \mid m(A) \in B \right\} \text{ for } A \in \mathcal{B}(I), B \in \mathcal{B}([0,2]),$$

where $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra. This $\mathcal{M}^2$ is the σ-Algebra we will use to assure the measurability of our generalized functionals. We keep the same $\mathcal{H}$ and $\mathcal{I}$ as previously defined.

Definition 3.1: A mapping $L^2 : \mathcal{H} \times \mathcal{I} \to \Gamma_{\mathbb{R}}^2$ is called an *intrinsic multiple-location functional of degree* 2 if it satisfies all of the following conditions:

(1) For every $I \in \mathcal{I}$, $L^2(\cdot, I) : \mathcal{H} \to \Gamma_I^2$ is $\text{Cyl}(\mathcal{H})/\mathcal{M}^2$-measurable.

(2) *(Shift compatibility)* For every $f \in \mathcal{H}, I \in \mathcal{I}, c \in \mathbb{R}$,

$$L^2(f, I) = L^2(\theta_c f, I - c) + c,$$

where $\xi \pm c = \{x \pm c \mid x \in \xi\}$.

(3) *(Inclusion under restriction)* For every $f \in \mathcal{H},$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$L^2(f, I_1) \cap I_2 \subseteq L^2(f, I_2).$$

(4) *(Consistency of existence)* For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$\left|L^2(f, I_1)\right| \geq \left|L^2(f, I_2)\right|,$$

where $|\cdot|$ represents the number of elements in the set.

A few things should be kept in mind about this definition for intuition:

(i) Inclusion under restriction says that when $|L^2(f, I_1)| = 2$, the elements $x_i \in L^2(f, I_1)$ which are in I_2 will 'stay there', but it may be that $(x_i \in L^2(f, I_1)) = x_j \in L^2(f, I_2)$ for $i \neq j \in \{1, 2\}$. This is merely for intuition, though, as we do not actually impose an order on the random locations explicitly until Chapter 5. Example 3.2 shows a case where $\{x_1, x_2\}$ are defined in a way that this can happen.

(ii) Also due to inclusion under restriction,

$$L^2(f, I_1) = L^2(f, I_2) \text{ if } \left|L^2(f, I_1) \cap I_2\right| = \left|L^2(f, I_1)\right|.$$

(iii) Consistency of existence can be understood in a very similar manner to the single-location case, but a key difference here is that we no longer use infinity to represent undefined locations. That is, if $\left|L^2(f, I)\right| < 2$, then one of the locations was not well-defined on I for the function f. Since $I_2 \subseteq I_1$, the locations are more likely to be well-defined on I_1.

Note that we will not always keep repeating 'of degree 2' for the intrinsic multiple-location functionals, since we will always be working with $n = 2$ until Chapter 6.

Example 3.2: The most obvious example of an intrinsic multiple-location functional of degree 2 following from the previous work of Samorodnitsky and Shen ([1],[2],[3]) describes the locations of the two largest path suprema, or locations of the first two occurences of the path supremum if it is not uniquely attained.

We let $\mathcal{H}$ be the space of upper semi-continuous functions, and let $\mathcal{I}$ be defined as before.

For any $f \in \mathcal{H}$ and $I = [a, b] \in \mathcal{I}$, we define $L^2(f, I)$ by

$$\begin{aligned} L^2(f,I) &= \left\{\tau^1_{f,I}, \tau^2_{f,I}\right\} \quad \text{where} \\ \tau^1_{f,I} &= \inf\left\{t \in M \mid f(t) = \sup_{s\in M} f(s)\right\}, \\ \text{and } \tau^2_{f,I} &= \inf\left\{t \in M \setminus \left\{\tau^1_{f,I}\right\} \mid f(t) = \sup_{s\in M\setminus\left\{\tau^1_{f,I}\right\}} f(s)\right\}, \end{aligned}$$

for $M = \{a, b\} \cup \{t \in I \mid t \text{ is a local maximum of } X\}$.

We alluded to the possibility earlier of $x_i(f, I_1) = x_j(f, I_2)$ for $I_2 \subseteq I_1, i \neq j$. This can happen in this case; if $\tau^2_{f,I_1} \in I_2$ and $\tau^1_{f,I_1} \in (I_1 \setminus I_2)$, then $\tau^2_{f,I_1} = \tau^1_{f,I_2}$. For an illustration of this scenario, see Figure 3.1 on page 14.

Example 3.3: We let $\mathcal{H}$ be the space of continuously differentiable functions: $\mathcal{H} = \mathcal{C}^1(\mathbb{R})$.

Then we define the first and second locations of the largest value of the derivative in the same way as the locations of the first two path supremum, but with the continuous derivative f' in place of f.

Example 3.4: Let $\mathcal{H}$ be the space of càdlàg functions on $\mathbb{R}$, and consider the two locations of the largest jumps of each path, or the locations of the first two occurrences of the largest jump if the magnitude of the largest jump is not uniquely attained. Then $L^2(f, [0, T])$ on the path f and interval $[0, T]$ is defined as

$$\begin{aligned} L^2(f,[0,T]) &= \left\{J^1_{f,[0,T]}, J^2_{f,[0,T]}\right\} \quad \text{where} \\ J^1_{f,[0,T]} &= \inf\left\{t \in [0,T] \;\middle|\; \left|f(t) - f(t^-)\right| = \sup_{s\in[0,T]} \left|f(s) - f(s^-)\right|\right\}, \\ \text{and } J^2_{f,[0,T]} &= \inf\left\{t \in [0,T] \setminus \left\{J^1_{f,[0,T]}\right\} \;\middle|\; \left|f(t) - f(t^-)\right| = \sup_{s\in[0,T]\setminus\left\{J^1_{f,[0,T]}\right\}} \left|f(s) - f(s^-)\right|\right\}, \end{aligned}$$

is an intrinsic multiple-location functional.

It is simple to show that the functionals defined in Examples 3.2-3.4 are intrinsic multiple-location functionals by checking the four conditions of Definition 3.1.

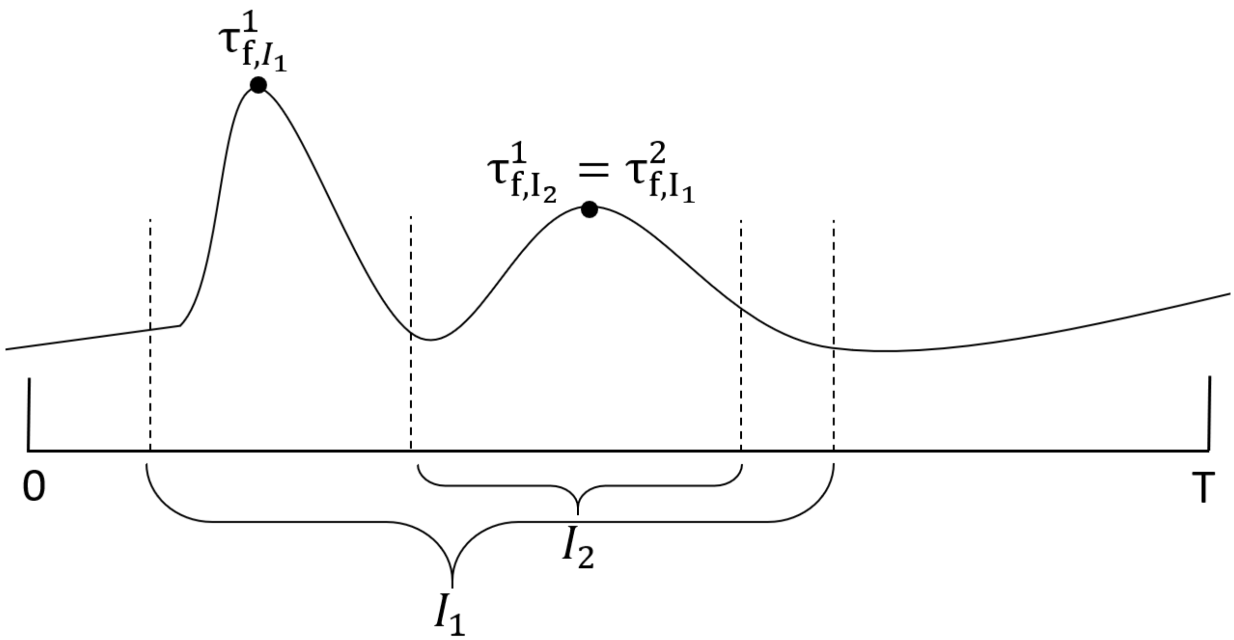

Figure 3.1: A possible behaviour of path suprema locations under restriction.

Chapter 4

Distributions of Intrinsic Multiple-Location Functionals on Stationary Processes

Now that we have defined the intrinsic multiple-location functionals of degree $n = 2$, we proceed to characterising their distributions.

In order to discuss these intrinsic multiple-location functionals in a probability setting, we consider the capacity functional $F_{X,I}(A) = \mathbb{P}\left(L^2(X,I) \cap A \neq \emptyset\right)$ (as described in [6], Section 1.2.2), which can be written as follows:

$$\begin{aligned}
F_{X,I}(A) &= \mathbb{P}\left(L^2(X,I) \cap A \neq \emptyset\right) \\
&= \sum_{j=0}^{2} \mathbb{P}\left(L^2(X,I) \cap A \neq \emptyset \mid \left|L^2(X,I)\right| = j\right) \mathbb{P}\left(\left|L^2(X,I)\right| = j\right) \\
&= 0 + \mathbb{P}\left(L^2(X,I) \in A\right) \mathbb{P}\left(\left|L^2(X,I)\right| = 1\right) \\
&\qquad\qquad + \mathbb{P}\left(\{x_1, x_2\} \cap A \neq \emptyset\right) \mathbb{P}\left(\left|L^2(X,I)\right| = 2\right) \\
&= \sum_{k=1}^{2} \left[\mathbb{P}\left(x_k \in A, \left|L^2(X,I) = 2\right|\right) + \mathbb{P}\left(x_k \in A, \left|L^2(X,I)\right| = 1\right)\right] - \mathbb{P}\left(x_1, x_2 \in A\right) \\
&= \mathbb{P}\left(x_1 \in A\right) + \mathbb{P}\left(x_2 \in A\right) - \mathbb{P}\left(x_1, x_2 \in A\right),
\end{aligned} \tag{4.1}$$

where without loss of generality, we take $x_1 = \min L^2(X,I)$ and $x_2 = \max L^2(X,I)$.

We are interested in the existence and properties of a version of the "density" of $F_{X,I}$, taken as the right-derivative, i.e.

$$f_{X,I}(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{P}\left(L^2(X,I) \cap (t, t+\epsilon] \neq \emptyset\right). \tag{4.2}$$

Intuitively, this 'density' represents the limiting probability of having at least one of the locations of $L^2(\cdot,\cdot)$ in a small neighborhood $(t, t+\epsilon]$ of $t \in (0,T)$, scaled by $\frac{1}{\epsilon}$. However, $F_{X,I}$ is not actually a measure, hence it does not have a density in the usual sense.

Therefore, rather than working directly with $F_{X,I}$, we will work with a measure $F^\star_{X,I}$, which is defined as:

$$F^\star_{X,I}(A) = \mathbb{E}\Big[\big|L^2(X,I) \cap A\big|\Big]. \tag{4.3}$$

Intuitively, we can think of $F^\star_{X,I}(A)$ as the expected number of random locations of $L^2(X,I)$ in A. In particular, when $|L^2(X,I)| = 2$, we drop the term $\mathbb{P}(x_1, x_2 \in A)$ from (4.1) to get $F^\star_{X,I}$. It will be shown later (in Section 4.2) that the right-derivative of $F^\star_{X,I}$ is equal to the 'density' defined in (4.2) for $F_{X,I}$.

For notational simplicity, when $I = [0,T]$ we write $F^\star_{X,T}$.

Lemma 4.1:

(i) For any $\Delta \in \mathbb{R}$,

$$F^\star_{X,[\Delta,T+\Delta]}(A) = F^\star_{X,T}(A - \Delta) \quad \text{for all } A \in \mathcal{B}([\Delta, T+\Delta]).$$

(ii) For intervals $[c,d] \subseteq [a,b]$,

$$F^\star_{X,[a,b]}(B) \leq F^\star_{X,[c,d]}(B) \quad \text{for all } B \in \mathcal{B}([c,d]).$$

(iii) For intervals $[c,d] \subseteq [a,b]$,

$$F^\star_{X,[a,b]}([a,b]) \geq F^\star_{X,[c,d]}([c,d]).$$

Proof:

(i)

$$
\begin{aligned}
F^{\star}_{X,[\Delta,T+\Delta]}(A) &= \mathbb{E}\Big[\big|L^2(X,[\Delta,T+\Delta])\cap A\big|\Big] \\
&= \mathbb{E}\Big[\big|(L^2(\theta_\Delta X,[0,T])+\Delta)\cap A\big|\Big] \\
&= \mathbb{E}\Big[\big|L^2(\theta_\Delta X,[0,T])\cap (A-\Delta)\big|\Big] \\
&= \mathbb{E}\Big[\big|L^2(X,[0,T])\cap (A-\Delta)\big|\Big] = F^{\star}_{X,T}(A-\Delta)
\end{aligned}
$$

by shift compatibility, and since $\{X_t\}_{t\in\mathbb{R}}$ is stationary, we have $\theta_\Delta X \overset{d}{=} X$.

(ii)

$$
\begin{aligned}
F^{\star}_{X,[a,b]}(B) &= \mathbb{E}\Big[\big|L^2(X,[a,b])\cap B\big|\Big] \\
&\leq \mathbb{E}\Big[\big|L^2(X,[c,d])\cap B\big|\Big] = F^{\star}_{X,[c,d]}(B)
\end{aligned}
$$

since $L^2(f,[a,b])\cap[c,d]\subseteq L^2(f,[c,d])$ for every $f\in\mathcal{H}$ by inclusion under restriction, and therefore $L^2(f,[a,b])\cap B\subseteq L^2(f,[c,d])\cap B$ since $B\subseteq[c,d]$.

(iii)

$$
\begin{aligned}
F^{\star}_{X,[a,b]}([a,b]) &= \mathbb{E}\Big[\big|L^2(X,[a,b])\cap [a,b]\big|\Big] \\
&= \mathbb{E}\Big[\big|L^2(X,[a,b])\big|\Big] \\
&\geq \mathbb{E}\Big[\big|L^2(X,[c,d])\big|\Big] = \mathbb{E}\Big[\big|L^2(X,[c,d])\cap [c,d]\big|\Big]
\end{aligned}
$$

since $|L^2(f,[a,b])|\geq|L^2(f,[c,d])|$ for every $f\in\mathcal{H}$ by consistency of existence. $\square$

4.1 Existence of the density and total variation constraints

Before we begin the proof that the density of $F^{\star}_{X,T}$ exists, we must make an assumption that assures the random locations of the intrinsic multiple-location functional are almost surely distinct locations. Under our framework, we would have $L^2(f, I) = \{x\}$ if the locations are both $\{x\}$, however this is supposed to mean that one of the locations was not well-defined on the interval I. This ambiguity could cause many problems.

Assumption A: The stochastic process $\{X_t\}_{t\in\mathbb{R}}$ and intrinsic multiple-location functional $L^2(\cdot,\cdot)$ are chosen such that the locations of $L^2(\cdot,\cdot)$ are almost surely distinct whenever they are both defined.

Theorem 4.2: Let L^2 be an intrinsic multiple-location functional on the space of functions $\mathcal{H}$, and $X = \{X_t\}_{t\in\mathbb{R}}$ be a stationary process with paths in $\mathcal{H}$. Then the restriction of $F^{\star}_{X,T}$ to the interior of $[0, T]$ is absolutely continuous. The density, denoted $f_{X,T}$, can be taken as the right derivative of $F^{\star}_{X,T}$, which exists at every point in the interval $(0, T)$. Moreover, $f_{X,T}$ is right-continuous, has left limits and has the following properties:

(a) The limits

$$f_{X,T}(0^+) = \lim_{t\downarrow 0} f_{X,T}(t) \quad \text{and} \quad f_{X,T}(T^-) = \lim_{t\uparrow T} f_{X,T}(t) \qquad \text{exist.} \tag{4.4}$$

(b) The density has a universal upper bound given by

$$f_{X,T}(t) \le \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \quad \text{for every } 0 < t < T. \tag{4.5}$$

(c) The density has bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,

$$\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) \le \min(f_{X,T}(t_1), f_{X,T}(t_1^-)) + \min(f_{X,T}(t_2), f_{X,T}(t_2^-)), \tag{4.6}$$

$$\text{where} \quad \mathrm{TV}_{(t_1,t_2)}(f_{X,T}) = \sup \sum_{i=1}^{n-1} |f_{X,T}(s_{i+1}) - f_{X,T}(s_i)|$$

is the *total variation* of $f_{X,T}$ on the interval (t_1, t_2), and the supremum is taken over all choices of $t_1 < s_1 < \cdots < s_n < t_2$ with finite n.

(d) The density has bounded positive variation at the left endpoint and bounded negative variation at the right endpoint. Furthermore, for every $0 < \epsilon < T$,

$$\mathrm{TV}^{+}_{(0,\epsilon)}(f_{X,T}) \leq \min(f_{X,T}(\epsilon), f_{X,T}(\epsilon^-)), \tag{4.7}$$

$$\text{and} \quad \mathrm{TV}^{-}_{(T-\epsilon,T)}(f_{X,T}) \leq \min(f_{X,T}(T-\epsilon), f_{X,T}((T-\epsilon)^-)) \tag{4.8}$$

$$\text{where} \quad TV^{\pm}_{(a,b)}(f_{X,T}) = \sup \sum_{i=1}^{n-1} (f_{X,T}(s_{i+1}) - f_{X,T}(s_i))_{\pm}$$

is the *positive(negative) variation* of $f_{X,T}$ on the interval (a,b), the supremum is taken over all choices of $a < s_1 < \cdots < s_n < b$ with finite n, and $(x)_+ = \max(x,0), (x)_- = \max(-x,0)$.

(e) The limit $f_{X,T}(0^+) < \infty$ if and only if $\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) < \infty$ for some (equivalently, any) $0 < \epsilon < T$, in which case

$$\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) \leq f_{X,T}(0^+) + \min(f_{X,T}(\epsilon), f_{X,T}(\epsilon^-)). \tag{4.9}$$

Similarly, $f_{X,T}(T^-) < \infty$ if and only if $\mathrm{TV}_{(T-\epsilon,T)}(f_{X,T}) < \infty$ for some (equivalently, any) $0 < \epsilon < T$, in which case

$$\mathrm{TV}_{(T-\epsilon,T)}(f_{X,T}) \leq \min(f_{X,T}(T-\epsilon), f_{X,T}((T-\epsilon)^-)) + f_{X,T}(T^-). \tag{4.10}$$

Once we have shown these properties for the density $f_{X,T}$ of $F^{\star}_{X,T}$, we will argue in Section 4.2 that the same properties apply to the 'density' of $F_{X,T}$ as defined in (4.2).

Lemma 4.3: $F^{\star}_{X,T}$ is absolutely continuous on $(0,T)$, and there exists a version of its density which satisfies the bound in (4.5).

Proof: We claim that for any fixed $0 < \delta < \frac{T}{2}$, and any $\delta \leq t \leq T-\delta,\ \rho \in \left(0, \frac{1}{2}\right)$, $0 < \epsilon < \frac{\rho}{1+\rho}\delta$,

$$F^{\star}_{X,T}\left((t, t+\epsilon]\right) \leq \epsilon \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right). \tag{4.11}$$

Once (4.11) is shown, we take $\delta \downarrow 0$ to show that $F^{\star}_{X,T}$ is absolutely continuous on $(0,T)$.

Towards a contradiction, assume (4.11) does not hold for some

$$\delta \leq t \leq T-\delta, \rho \in \left(0, \frac{1}{2}\right),\ 0 < \epsilon < \frac{\rho}{1+\rho}\delta.$$

Then choose $a, b \in (0, T)$ such that:

(i) $(t, t+\epsilon] \subseteq [a, b] \subseteq (0, T)$.

(ii) for some fixed $\epsilon < \theta < \frac{\rho}{1+\rho}\delta$,

$$\min(t, T-t) - \theta < \left|b - a\right| < \min(t, T-t) - \epsilon. \tag{4.12}$$

Consider

$$F^{\star}_{X,[a,b]}\left((t - i\epsilon, t - (i-1)\epsilon]\right) = \mathbb{E}\Big[\big|L^2(X, [a, b]) \cap (t - i\epsilon, t - (i-1)\epsilon]\big|\Big],$$

for $i = -\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right), \ldots, \left\lfloor \frac{t-a}{\epsilon} \right\rfloor$.

Then note that

$$\begin{aligned} &\sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} F^{\star}_{X,[a,b]}\left((t - i\epsilon, t - (i-1)\epsilon]\right) \qquad (4.13) \\ &= \sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} \mathbb{E}\Big[\big|L^2(X, [a, b]) \cap (t - i\epsilon, t - (i-1)\epsilon]\big|\Big] \\ &= \mathbb{E}\Big[\sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} \big|L^2(X, [a, b]) \cap (t - i\epsilon, t - (i-1)\epsilon]\big|\Big] \leq 2 \end{aligned}$$

since $(t - i\epsilon, t - (i-1)\epsilon] \subseteq [a, b], L^2(f, [a, b]) \subseteq [a, b]$, and the total summed expectation must be less or equal to the total number of possible points over $[a, b]$, which is 2.

By Lemma 4.1(i):

$$F^{\star}_{X,[a,b]}\left((t - i\epsilon, t - (i-1)\epsilon]\right) = F^{\star}_{X,[a+i\epsilon,b+i\epsilon]}\left((t, t+\epsilon]\right). \tag{4.14}$$

And by Lemma 4.1(ii), since $(t, t+\epsilon] \subseteq [a+i\epsilon, b+i\epsilon] \subseteq [0, T]$ for all $i = -\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right), \ldots, \left\lfloor \frac{t-a}{\epsilon} \right\rfloor$, by construction

$$F^{\star}_{X,[a+i\epsilon,b+i\epsilon]}\left((t, t+\epsilon]\right) \geq F^{\star}_{X,T}\left((t, t+\epsilon]\right). \tag{4.15}$$

Putting (4.13), (4.14), and (4.15) together we get that

$$\begin{aligned}
2 &\geq \sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} F^{\star}_{X,[a,b]}\left((t-i\epsilon, t-(i-1)\epsilon]\right) \\
&= \sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} F^{\star}_{X,[a+i\epsilon,b+i\epsilon]}\left((t, t+\epsilon]\right) \\
&\geq \sum_{i=-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)}^{\left\lfloor \frac{t-a}{\epsilon} \right\rfloor} F^{\star}_{X,T}\left((t, t+\epsilon]\right) \\
&> \left(\left\lfloor \frac{t-a}{\epsilon} \right\rfloor - \left(-\left(\left\lfloor \frac{b-t}{\epsilon} \right\rfloor - 1\right)\right) + 1\right) \epsilon \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&\geq \left(\left\lfloor \frac{t-a}{\epsilon} + \frac{b-t}{\epsilon} \right\rfloor - 1\right) \epsilon \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&= \left(\left\lfloor \frac{b-a}{\epsilon} \right\rfloor - 1\right) \epsilon \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&\geq \left(\frac{b-a}{\epsilon} - 2\right) \epsilon \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&= (b-a-2\epsilon) \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&> (\min(t, T-t) - \theta - 2\epsilon) \left(\frac{1+\rho}{1-2\rho}\right) \max\left(\frac{2}{t}, \frac{2}{T-t}\right) \\
&= \left(2 - 2\frac{\theta + 2\epsilon}{\min(t, T-t)}\right) \left(\frac{1+\rho}{1-2\rho}\right) > \left(2 - 2\frac{3\theta}{\min(t, T-t)}\right) \left(\frac{1+\rho}{1-2\rho}\right) \\
&> 2\left(1 - \frac{3\delta}{\min(t, T-t)} \frac{\rho}{1+\rho}\right) \left(\frac{1+\rho}{1-2\rho}\right) > 2\left(1 - \frac{3\rho}{1+\rho}\right) \left(\frac{1+\rho}{1-2\rho}\right) \\
&= 2\left(\frac{1-2\rho}{1+\rho}\right) \left(\frac{1+\rho}{1-2\rho}\right) = 2.
\end{aligned}$$

A contradiction, which finishes the proof of (4.11), and hence $F^{\star}_{X,T}$ is absolutely continuous on $(0, T)$. □

This result (4.11) also tells us that there exists a version of the density of $F^{\star}_{X,T}$ which satisfies the bound in (4.5), given as

$$f_{X,T}(t) = \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon} F^{\star}_{X,T}\left((t,t+\epsilon]\right) \le \max\left(\frac{2}{t}, \frac{2}{T-t}\right), \quad 0 < t < T.$$

Before finishing the rest of the proof of Theorem 4.2, we first need to prove the following useful lemma.

Lemma 4.4: Let $0 \le \Delta \le T$. Then for any $0 \le \delta \le \Delta$,

$$f_{X,T-\Delta}(t) \ge f_{X,T}(t+\delta) \text{ for almost every } t \in (0, T-\Delta), \tag{4.16}$$

where f on either side of this inequality is taken as any version of the density, which exists since we have just shown that $F^{\star}_{X,T}$ is absolutely continuous.

Moreover, for every such δ and $\epsilon_1, \epsilon_2 \ge 0$ such that $\epsilon_1 + \epsilon_2 < T - \Delta$,

$$\begin{aligned}\int_{\epsilon_1}^{T-\Delta-\epsilon_2} &\left(f_{X,T-\Delta}(t) - f_{X,T}(t+\delta)\right)\, dt \\ &\le \int_{\epsilon_1}^{\epsilon_1+\delta} f_{X,T}(t)\, dt + \int_{T-\Delta-\epsilon_2+\delta}^{T-\epsilon_2} f_{X,T}(t)\, dt.\end{aligned} \tag{4.17}$$

Proof: By Lemma 4.1, for every Borel set $B \subseteq (0, T-\Delta)$,

$$\begin{aligned}\int_B f_{X,T-\Delta}(t)\, dt &= F^{\star}_{X,T-\Delta}(B) \ge F^{\star}_{X,[-\delta,T-\delta]}(B) = F^{\star}_{X,T}(B+\delta) \\ &= \int_{B+\delta} f_{X,T}(t)\, dt = \int_B f_{X,T}(t+\delta)\, dt.\end{aligned}$$

Since this holds for arbitrary an Borel set $B \subseteq (0, T-\Delta)$, this proves (4.16).

Before we prove (4.17), observe that for any $I \in \mathcal{I}$, $F^{\star}_{X,I}$ is a measure, hence for any $A \in \mathcal{B}(I)$

$$F^{\star}_{X,I}(A) = F^{\star}_{X,I}(I) - F^{\star}_{X,I}(A^c), \tag{4.18}$$

where A^c is understood to be the complement of A in I, i.e. $A^c = I \setminus A$. Then

$$\begin{aligned}
&\int_{\epsilon_1}^{T-\Delta-\epsilon_2} \left(f_{X,T-\Delta}(t) - f_{X,T}(t+\delta)\right) \, dt \\
&= F^\star_{X,T-\Delta}\left((\epsilon_1, T-\Delta-\epsilon_2)\right) - F^\star_{X,T}\left((\epsilon_1+\delta, T-\Delta-\epsilon_2+\delta)\right) \\
&= \mathbb{E}\Big[\big|L^2(X,[0,T-\Delta]) \cap (\epsilon_1, T-\Delta-\epsilon_2)\big|\Big] - \mathbb{E}\Big[\big|L^2(X,[0,T]) \cap (\epsilon_1+\delta, T-\Delta-\epsilon_2+\delta)\big|\Big] \\
&= \mathbb{E}\Big[\big|L^2(X,[0,T]) \cap (\epsilon_1+\delta, T-\Delta-\epsilon_2+\delta)^c\big|\Big] - \mathbb{E}\Big[\big|L^2(X,[0,T-\Delta]) \cap (\epsilon_1, T-\Delta-\epsilon_2)^c\big|\Big] \\
&\quad + \Big[F^\star_{X,T-\Delta}([0,T-\Delta]) - F^\star_{X,T}([0,T])\Big] \\
&\le F^\star_{X,T}([0,\epsilon_1+\delta]) + F^\star_{X,T}([T-\Delta-\epsilon_2+\delta, T]) - F^\star_{X,T-\Delta}([0,\epsilon_1]) \\
&\quad - F^\star_{X,T-\Delta}([T-\Delta-\epsilon_2, T-\Delta]) \\
&= F^\star_{X,T}((\epsilon_1,\epsilon_1+\delta]) + \Big[F^\star_{X,T}([0,\epsilon_1]) - F^\star_{X,T-\Delta}([0,\epsilon_1])\Big] + F^\star_{X,T}([T-\Delta-\epsilon_2+\delta, T-\epsilon_2)) \\
&\quad + \Big[F^\star_{X,T}([T-\epsilon_2,T]) - F^\star_{X,[\Delta,T]}([T-\epsilon_2,T])\Big] \\
&\le F^\star_{X,T}([\epsilon_1,\epsilon_1+\delta)) + F^\star_{X,T}([T-\Delta-\epsilon_2+\delta, T-\epsilon_2]) \\
&= \int_{\epsilon_1}^{\epsilon_1+\delta} f_{X,T} \, dt + \int_{T-\Delta-\epsilon_2+\delta}^{T-\epsilon_2} f_{X,T}(t) \, dt,
\end{aligned}$$

because every term in square brackets $[\cdot]$ is non-positive by Lemma 4.1(ii) with $[0, T-\Delta] \subseteq [0,T]$, $[\Delta, T] \subseteq [0,T]$, and $\Big[F^\star_{X,T-\Delta}([0,T-\Delta]) - F^\star_{X,T}([0,T])\Big] \le 0$ by Lemma 4.1(iii). □

With this lemma in hand, we proceed to proving that $F^\star_{X,T}$ is right-differentiable at every point in $(0,T)$ in order to work with the density $f_{X,T}$ as the right-derivative.

We know that $F^\star_{X,T}$ is right-differentiable for almost every $t \in (0,T)$ since $F^\star_{X,T}$ is absolutely continuous on $(0,T)$, i.e. the set

$$A = \left\{t \in (0,T) \mid F^\star_{X,T} \text{ is not right-differentiable at } t\right\} \tag{4.19}$$

has Lebesgue measure zero. We also define

$$B = \left\{t \in A^c \mid f_{X,T} \text{ restricted to } A^c \text{ does not have right limit at } t\right\}. \tag{4.20}$$

Proposition 4.5: The set B as defined in (4.20) is at most countable.

Proof: In order to show this, we define for every $t \in A^c$

$$L(t) = \limsup_{s \downarrow t, s \in A^c} f_{X,T}(s), \qquad \ell(t) = \liminf_{s \downarrow t, s \in A^c} f_{X,T}(s).$$

The claim that B is at most countable will follow if we prove that for any $0 < \epsilon < \frac{T}{2}$ and $\theta > 0$, the following set is finite:

$$B_{\epsilon,\theta} = \{t \in A^c \cap (\epsilon, T-\epsilon) \mid L(t) - \ell(t) > \theta\}$$

since B can be written as a countable union of these $B_{\epsilon,\theta}$.

It can actually be shown that the cardinality of $B_{\epsilon,\theta}$ cannot be larger than $\frac{8}{\epsilon\theta}$. Towards a contradiction, assume that $\left|B_{\epsilon,\theta}\right| > \frac{8}{\epsilon\theta}$ for some $0 < \epsilon < \frac{T}{2}$ and $\theta > 0$. Then let $N > \frac{8}{\epsilon\theta}$ and take points $\epsilon < t_1 < t_2 < \ldots < t_N < T - \epsilon$ in $B_{\epsilon,\theta}$. Next, choose $\delta > 0$ small enough that $\delta < \frac{\epsilon}{2}$ and

$$\delta < \frac{1}{2} \min\left(t_1 - \epsilon, t_2 - t_1, \ldots, t_N - t_{N-1}, T - \epsilon - t_N\right). \tag{4.21}$$

Now for $i = 1, \ldots, N$ choose a sequence $\{s_n\}_{n=1}^{\infty} \downarrow t_i$, $s_n \in A^c$ such that $f_{X,T}(s_n) \underset{n\to\infty}{\longrightarrow} L(t_i)$.

Take n so large that $s_n - t_i < \frac{\delta}{3}$ and let $j > 0$ be an integer such that

$$j \geq \frac{1}{\delta - (s_n - t_i)}.$$

Note that

$$\bigcup_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor - 1} \left(t_i - \frac{k+1}{j}, t_i - \frac{k}{j}\right) \subseteq (t_i - \delta, t_i),$$

with $\left(t_i - \frac{k+1}{j}, t_i - \frac{k}{j}\right)$ disjoint for all k, so we have that

$$F^{\star}_{X,T-\delta}\left((t_i - \delta, t_i)\right) \geq \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor - 1} F^{\star}_{X,T-\delta}\left(\left(t_i - \frac{k+1}{j}, t_i - \frac{k}{j}\right)\right). \tag{4.22}$$

For each k in the sum, define

$$h_k := s_n - t_i + \frac{k+1}{j} \in (0, \delta],$$

where it is clear why $h_k > 0$, and $h_k \leq \delta$ since

$$\begin{aligned} h_k \leq h_{k_0} \text{ for } k_0 &= \lfloor j(\delta - (s_n - t_i)) \rfloor - 1 \\ &= s_n - t_i + \frac{\lfloor j(\delta - (s_n - t_i)) \rfloor - 1 + 1}{j} \\ &\leq s_n - t_i + \frac{j(\delta - (s_n - t_i))}{j} = \delta. \end{aligned}$$

Next, by Lemma 4.1(i),

$$\begin{aligned} &\sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor -1} F^{\star}_{X,T-\delta}\left(\left(t_i - \frac{k+1}{j}, t_i - \frac{k}{j}\right)\right) \\ &= \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor -1} F^{\star}_{X,[h_k,T-\delta+h_k]}\left(\left(t_i - \frac{k+1}{j} + h_k, t_i - \frac{k}{j} + h_k\right)\right). \end{aligned} \tag{4.23}$$

We note that $\left(t_i - \frac{k+1}{j} + h_k, t_i - \frac{k}{j} + h_k\right) \subseteq [0, T - \delta + h_k] \subseteq [0, T]$ for every k by choice of δ, and hence we can apply Lemma 4.1(ii) to (4.22), (4.23):

$$\begin{aligned} F^{\star}_{X,T-\delta}\left((t_i - \delta, t_i)\right) &\geq \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor -1} F^{\star}_{X,[h_k,T-\delta+h_k]}\left(\left(t_i - \frac{k+1}{j} + h_k, t_i - \frac{k}{j} + h_k\right)\right) \\ &\geq \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor -1} F^{\star}_{X,T}\left(\left(t_i - \frac{k+1}{j} + h_k, t_i - \frac{k}{j} + h_k\right)\right) \\ &= \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor -1} F^{\star}_{X,T}\left(\left(s_n, s_n + \frac{1}{j}\right)\right) \\ &= \lfloor j(\delta - (s_n - t_i)) \rfloor \, F^{\star}_{X,T}\left(\left(s_n, s_n + \frac{1}{j}\right)\right) \\ &\xrightarrow[j\to\infty]{} (\delta - (s_n - t_i))\, f_{X,T}(s_n). \end{aligned}$$

Letting $n \to \infty$, we can conclude that

$$F^{\star}_{X,T-\delta}\left((t_i - \delta, t_i)\right) \geq \delta L(t_i) \qquad \text{for every } i = 1, \ldots, N. \tag{4.24}$$

Similarly for $i = 1, \ldots, N$ we choose a sequence $\{w_n\}_{n=1}^{\infty} \downarrow t_i, w_n \in A^c$ such that

$f_{X,T}(w_n) \to \ell(t_i)$. For large n such that $w_n - t_i < \frac{\delta}{3}$ and integer $j \geq \frac{1}{\delta-(w_n-t_i)}$ we have

$$
\begin{aligned}
F^\star_{X,T+\delta}\left((t_i,t_i+\delta)\right) &\leq F^\star_{X,T+\delta}\left((t_i,w_n)\right) + F^\star_{X,T+\delta}\left((w_n,w_n+\delta)\right) \\
&\leq F^\star_{X,T+\delta}\left((t_i,w_n)\right) + \sum_{k=0}^{\lceil \delta j\rceil -1} F^\star_{X,T+\delta}\left(\left(w_n+\frac{k}{j}, w_n+\frac{k+1}{j}\right)\right) \qquad (4.25)
\end{aligned}
$$

since $(w_n, w_n+\delta) \subseteq \bigcup_{k=0}^{\lceil \delta j\rceil -1}\left(\left(w_n+\frac{k}{j}, w_n+\frac{k+1}{j}\right)\right)$.

Now define

$$h_k = \frac{k}{j} \in [0,\delta],$$

then because $h_k \leq \delta$, by Lemma 4.1 (i) and (4.25),

$$
\begin{aligned}
F^\star_{X,T+\delta}\left((t_i,t_i+\delta)\right) &\leq F^\star_{X,T+\delta}\left((t_i,w_n)\right) \qquad (4.26)\\
&\quad + \sum_{k=0}^{\lceil \delta j\rceil -1} F^\star_{X,[-h_k,T+\delta-h_k]}\left(\left(w_n+\frac{k}{j}-h_k, w_n+\frac{k+1}{j}-h_k\right)\right).
\end{aligned}
$$

Now by Lemma 4.1(ii) with $[0,T] \subseteq [-h_k, T+\delta-h_k]$, applied to (4.26):

$$
\begin{aligned}
F^\star_{X,T+\delta}\left((t_i,t_i+\delta)\right) &\leq F^\star_{X,T+\delta}\left((t_i,w_n)\right) + \sum_{k=0}^{\lceil \delta j\rceil -1} F^\star_{X,T}\left(\left(w_n+\frac{k}{j}-h_k, w_n+\frac{k+1}{j}-h_k\right)\right) \\
&= F^\star_{X,T+\delta}\left((t_i,w_n)\right) + \sum_{k=0}^{\lceil \delta j\rceil -1} F^\star_{X,T}\left(\left(w_n, w_n+\frac{1}{j}\right)\right) \\
&= F^\star_{X,T+\delta}\left((t_i,w_n)\right) + \lceil \delta j\rceil\, F^\star_{X,T}\left(\left(w_n, w_n+\frac{1}{j}\right)\right). \qquad (4.27)
\end{aligned}
$$

Letting $j \to \infty$ and then $n \to \infty$ in (4.27) gives

$$F^\star_{X,T+\delta}\left((t_i,t_i+\delta)\right) \leq \delta\ell(t_i) \qquad \text{for every } i=1,\ldots,N. \qquad (4.28)$$

Putting together (4.24) and (4.28), recalling that for $t \in B_{\epsilon,\theta}$, $L(t)-\ell(t) > \theta$, we get that

$$
\begin{aligned}
N\delta\theta &\leq F^\star_{X,T-\delta}\left(\bigcup_{i=1}^{N}(t_i-\delta,t_i)\right) - F^\star_{X,T+\delta}\left(\bigcup_{i=1}^{N}(t_i,t_i+\delta)\right) \\
&= \int_{\cup_{i=1}^N (t_i-\delta,t_i)} \left(f_{X,T-\delta}(t) - f_{X,T+\delta}(t+\delta)\right)\, dt. \qquad (4.29)
\end{aligned}
$$

We note that

$$\bigcup_{i=1}^{N}(t_i-\delta,t_i)\subseteq(\epsilon-\delta,T-\epsilon),$$

hence the integral in (4.29) is bounded by

$$\begin{aligned}\int_{\epsilon-\delta}^{T-\epsilon}&\left(f_{X,T-\delta}(t)-f_{X,T+\delta}(t+\delta)\right)\,dt\\&\leq\int_{\epsilon-\delta}^{\epsilon}f_{X,T+\delta}(t)\,dt+\int_{T-\epsilon+\delta}^{T-\epsilon+2\delta}f_{X,T+\delta}(t)\,dt,\end{aligned}\tag{4.30}$$

with (4.30) following from Lemma 4.4 since the integrand is non-negative almost everywhere.

We already proved the bound on the density (4.5), which we can apply here to get that

$$\begin{aligned}&\int_{\epsilon-\delta}^{\epsilon}f_{X,T+\delta}(t)\,dt+\int_{T-\epsilon+\delta}^{T-\epsilon+2\delta}f_{X,T+\delta}(t)\,dt\\&\leq 2\int_{\epsilon-\delta}^{\epsilon}\max\left(\frac{1}{t},\frac{1}{T+\delta-t}\right)\,dt+2\int_{T-\epsilon+\delta}^{T-\epsilon+2\delta}\max\left(\frac{1}{t},\frac{1}{T+\delta-t}\right)\,dt\\&=2\int_{\epsilon-\delta}^{\epsilon}\frac{1}{t}\,dt+2\int_{T-\epsilon+\delta}^{T-\epsilon+2\delta}\frac{1}{T+\delta-t}\,dt\\&\leq 2\int_{\epsilon-\delta}^{\epsilon}\frac{1}{\epsilon-\delta}\,dt+2\int_{T-\epsilon+\delta}^{T-\epsilon+2\delta}\frac{1}{\epsilon-\delta}\,dt=4\frac{\delta}{\epsilon-\delta}.\end{aligned}$$

This implies that

$$N\delta\theta\leq 4\frac{\delta}{\epsilon-\delta}\leq\frac{8\delta}{\epsilon},$$

hence $N\leq\frac{8}{\epsilon\theta}$, which contradicts our assumption that $N>\frac{8}{\epsilon\theta}$. Therefore, the set B as defined in (4.20) is at most countable. □

Proposition 4.6: $F^\star_{X,T}$ is right-differentiable at *every* point $t \in (0, T)$.

Proof: We first note that

$$\begin{aligned} f_{X,T}(t) &= \lim_{s \downarrow t} \frac{1}{s-t} F^\star_{X,T}\left((t,s]\right) \\ &= \lim_{s \downarrow t} \frac{1}{s-t} \int_t^s f_{X,T}(w)\, dt = \lim_{w \downarrow t, w \in A^c \setminus B} f_{X,T}(w) \end{aligned} \tag{4.31}$$

for every $t \in A^c \setminus B$.

We suppose to the contrary that there exists a $t \in (0, T)$ for which the right-derivative of $F^\star_{X,T}$ does not exist, and therefore take real numbers $a, b \in \mathbb{R}$ such that

$$\liminf_{\epsilon \downarrow 0} \frac{F^\star_{X,T}(t+\epsilon) - F^\star_{X,T}(t)}{\epsilon} < a < b < \limsup_{\epsilon \downarrow 0} \frac{F^\star_{X,T}(t+\epsilon) - F^\star_{X,T}(t)}{\epsilon},$$

where $F^\star_{X,T}(t) = F^\star_{X,T}\left((-\infty, t]\right)$.

This means that there exists a sequence $\{t_n\}_{n=1}^\infty \downarrow t, t_n \in (A^c \setminus B)$ for each n such that

$$f_{X,T}(t_{2n-1}) > b \quad \text{and} \quad f_{X,T}(t_{2n}) < a \qquad \text{for all } n = 1, 2, \ldots$$

Without loss of generality we can choose t_1 close to t such that $t_1 < \frac{T+t}{2}$.

By (4.31), then, for every $n = 1, 2, \ldots$ there exists a $\delta_n > 0$ such that

$$\begin{aligned} f_{X,T}(w) > b &\quad \text{for almost all } w \in (t_{2n-1}, t_{2n-1} + \delta_{2n-1}), \\ \text{and } f_{X,T}(w) < a &\quad \text{for almost all } w \in (t_{2n}, t_{2n} + \delta_{2n}). \end{aligned} \tag{4.32}$$

Now let $m \geq 1$, and consider an $s > 0$ so small that $s < \min_{n=1,\ldots,2m} \delta_n$ and $t_1 < \frac{T+t}{2} - s$. We can see that

$$\begin{aligned} &\int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s) - f_{X,T}(w)\right)_+ \, dw \\ &\geq \int_t^{t+s} \sum_{i=0}^{\left\lfloor \frac{T-t}{2s} \right\rfloor - 1} \left(f_{X,T}(w + (i+1)s) - f_{X,T}(w + is)\right)_+ \, dw \end{aligned} \tag{4.33}$$

by simple inclusion of the bounds of integration.

For every point $w \in (t, t+s)$, each of the intervals $(t_n, t_n + \delta_n)$ for $n = 1, 2, \ldots, 2m$ contains at least one of the points in the sequence $(w + is), i = 0, 1, \ldots, \left\lfloor \frac{T-t}{2s} \right\rfloor - 1$ by construction.

Intuitively, we think of this sequence as a finer partition than the one created by the given sequence of δ_n. As we can see in (4.32), for almost every $w \in (t, t+s)$, the points of the sequence $(w + is)$ which are in the odd-numbered intervals (i.e. $(t_{2n-1}, t_{2n-1} + \delta_{2n-1})$) satisfy $f_{X,T}(w+is) > b$ and those in the even-numbered intervals satisfy $f_{X,T}(w+is) < a$. Therefore we conclude that

$$\sum_{i=0}^{\lfloor \frac{T-t}{2s} \rfloor -1} \left(f_{X,T}(w+(i+1)s) - f_{X,T}(w+is)\right)_+ \geq m(b-a). \tag{4.34}$$

for almost every $w \in (t, t+s)$. And hence by (4.33) and (4.34),

$$\int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s) - f_{X,T}(w)\right)_+ \, dw \geq sm(b-a). \tag{4.35}$$

Recall that $m \geq 1$ can be taken arbitrarily large, so we finally conclude that

$$\lim_{s \downarrow 0} \frac{1}{s} \int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s) - f_{X,T}(w)\right)_+ \, dw \geq m(b-a) \text{ for every } m \geq 1,$$

$$\text{therefore } \lim_{s \downarrow 0} \frac{1}{s} \int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s) - f_{X,T}(w)\right)_+ \, dw = \infty. \tag{4.36}$$

However, we will show that (4.36) is impossible, leading to a contradiction, meaning that $F^{\star}_{X,T}$ is in fact right-differentiable at every $t \in (0, T)$.

By Lemma 4.4 (namely 4.16), for $s > 0$ small enough,

$$f_{X,T-2s}(w-s) \geq f_{X,T}(w+s) \qquad \text{almost everywhere on } (s, T-s) \supseteq \left(t, \frac{T+t}{2}\right).$$

Meaning that for such $s > 0$, we get that

$$\begin{aligned} \int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s) - f_{X,T}(w)\right)_+ \, dw &\leq \int_t^{\frac{T+t}{2}} \left(f_{X,T-2s}(w-s) - f_{X,T}(w)\right)_+ \, dw \\ &\leq \int_{t-s}^{\frac{T+t}{2}-s} \left(f_{X,T-2s}(w) - f_{X,T}(w+s)\right) \, dw, \end{aligned} \tag{4.37}$$

where we drop the $(\cdot)_+$ since the integrand is almost everywhere non-negative from another application of Lemma 4.4. Applying the second part of Lemma 4.4, we get that

$$\begin{aligned} &\int_{t-s}^{\frac{T+t}{2}-s} \left(f_{X,T-2s}(w) - f_{X,T}(w+s)\right) \, dw \\ &\leq \int_{t-s}^{t} f_{X,T}(w) \, dw + \int_{\frac{T+t}{2}}^{\frac{T+t}{2}+s} f_{X,T}(w) \, dw. \end{aligned} \tag{4.38}$$

However, using the inequalities in (4.37) and (4.38), then taking the limit as in (4.36) gives

$$
\begin{aligned}
&\lim_{s\downarrow 0}\frac{1}{s}\int_t^{\frac{T+t}{2}} \left(f_{X,T}(w+s)-f_{X,T}(w)\right)_+ \, dw \\
&\leq \lim_{s\downarrow 0}\frac{1}{s}\Big[\int_{t-s}^{t} f_{X,T}(w)\,dw + \int_{\frac{T+t}{2}}^{\frac{T+t}{2}+s} f_{X,T}(w)\,dw\Big] \\
&\leq \lim_{s\downarrow 0}\frac{1}{s}\Big[\int_{t-s}^{t} \max\left(\frac{2}{w},\frac{2}{T-w}\right)\,dw + \int_{\frac{T+t}{2}}^{\frac{T+t}{2}+s} \max\left(\frac{2}{w},\frac{2}{T-w}\right)\,dw\Big] \\
&\leq \lim_{s\downarrow 0}\frac{1}{s} s\cdot C \text{ for some } C\in\mathbb{R} \\
&= C < \infty,
\end{aligned}
$$

which contradicts (4.36), so $F^{\star}_{X,T}$ is right-differentiable for every point $t \in (0,T)$. □

Now that we have shown that $f_{X,T}$ exists everywhere in $(0,T)$ as the right-derivative of $F^{\star}_{X,T}$ in Proposition 4.6, we want to show that it is right continuous and has left limits.

Proposition 4.7: $f_{X,T}$ taken as the right-derivative of $F^{\star}_{X,T}$ is right continuous and has left limits everywhere in $(0,T)$.

Proof: Note that $F^{\star}_{X,T}$ being right-differentiable everywhere means that A as defined in (4.19) is empty.

Consequently, $f_{X,T}$ as defined in (4.31) becomes

$$f_{X,T}(t) = \lim_{s\downarrow t, s\in B^c} f_{X,T}(s), \tag{4.39}$$

where we note that there is no point $t \in (0,T)$ such that this limit on the right-hand side does not exist. Indeed, suppose to the contrary that it does not exist for some $t \in (0,T)$, then this means there is a sequence $\{t_n\}_{n=1}^{\infty} \downarrow t, t_n \in B^c$ for each n and real numbers $a < b$ such that

$$f_{X,T}(t_{2n-1}) > b \quad \text{and} \quad f_{X,T}(t_{2n}) < a \quad \text{for all } n = 1, 2, \ldots$$

as before, but we have already established that such a sequence cannot exist.

Lastly, since B was shown to be at most countable in Proposition 4.5, the restriction to $s \in B^c$ in (4.39) does not affect the right-continuity of $f_{X,T}$ at $t \in (0,T)$, so $f_{X,T}$ is right-continuous for all $t \in (0,T)$. The proof of left limits is analogous. □

We will now prove the total variation constraint of (4.6) on the density.

Proposition 4.8: If $0 < t_1 < t_2 < T$, then

$$\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) \leq \min\left(f_{X,T}(t_1), f_{X,T}(t_1^-)\right) + \min\left(f_{X,T}(t_2), f_{X,T}(t_2^-)\right).$$

Consider a sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_n \to 0$ as $n \to \infty$ and $0 < r_n < T - t_2$. Fix one such $n \in \{1, 2, \ldots\}$ and then define

$$\begin{aligned} C_+ &= \{t \in (t_1, t_2) \mid f_{X,T}(t + r_n) \geq f_{X,T}(t)\} \\ \text{and } C_- &= \{t \in (t_1, t_2) \mid f_{X,T}(t + r_n) < f_{X,T}(t)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{t_1}^{t_2} |f_{X,T}(t + r_n) - f_{X,T}(t)| \, dt = &\int_{C_+} f_{X,T}(t + r_n) - f_{X,T}(t) \, dt \\ &+ \int_{C_-} f_{X,T}(t) - f_{X,T}(t + r_n) \, dt. \end{aligned}$$

By Lemma 4.4, $f_{X,T-r_n}(t) \geq f_{X,T}(r_n + t)$ almost everywhere on $(0, T - r_n) \supseteq (t_1, t_2)$. Hence

$$\begin{aligned} \int_{C_+} f_{X,T}(t + r_n) - f_{X,T}(t) \, dt &\leq \int_{C_+} f_{X,T-r_n}(t) - f_{X,T}(t) \, dt \\ &\leq \int_{t_1}^{t_2} f_{X,T-r_n}(t) - f_{X,T}(t) \, dt. \end{aligned}$$

Now apply Lemma 4.4 with $\Delta = r_n, \delta = 0, \epsilon_1 = t_1, T - \Delta - \epsilon_2 = t_2$:

$$\int_{C_+} f_{X,T}(t + r_n) - f_{X,T}(t) \, dt \leq \int_{t_2}^{t_2 + r_n} f_{X,T}(t) \, dt,$$

which gives

$$\limsup_{n \to \infty} \frac{1}{r_n} \int_{C_+} f_{X,T}(t + r_n) - f_{X,T}(t) \, dt \leq f_{X,T}(t_2) \tag{4.40}$$

by the fundamental theorem of calculus, and $f_{X,T}$ being right-continuous.

Similarly for the C_- case, by Lemma 4.4 again, $f_{X,T}(t + r_n) \geq f_{X,T+r_n}(t + r_n)$ almost everywhere on $(0, T - r_n) \supseteq (t_1, t_2)$, and hence

$$\begin{aligned}
\int_{C_-} f_{X,T}(t) - f_{X,T}(t + r_n)\, dt &\leq \int_{C_-} f_{X,T}(t) - f_{X,T+r_n}(t + r_n)\, dt \\
&\leq \int_{t_1}^{t_2} f_{X,T}(t) - f_{X,T+r_n}(t + r_n)\, dt \\
&\leq \int_{t_1}^{t_1+r_n} f_{X,T+r_n}(t)\, dt, \qquad (4.41)
\end{aligned}$$

by using $T + r_n$ in place of T, so that $\Delta = r_n$, $(T + r_n) - \Delta = T$, and $\delta = \Delta = r_n$ in Lemma 4.4.

The integral in (4.41) is bounded by both of the following:

$$\int_{t_1}^{t_1+r_n} f_{X,T+r_n}(t)\, dt \leq \int_{t_1}^{t_1+r_n} f_{X,T}(t)\, dt, \qquad (a)$$

$$\text{and} \int_{t_1}^{t_1+r_n} f_{X,T+r_n}(t)\, dt \leq \int_{t_1}^{t_1+r_n} f_{X,T}(t - r_n)\, dt = \int_{t_1-r_n}^{t_1} f_{X,T}(t)\, dt. \qquad (b)$$

Inequality (a) follows from Lemma 4.4; $f_{X,T-\Delta}(t) \geq f_{X,T}(t + \delta)$ with $T = T + r_n$, $\delta = 0$.

Inequality (b) also follows from Lemma 4.4; $f_{X,T}(t - r_n) \geq f_{X,T+r_n}(t)$, with $T = T + r_n$, $\delta = \Delta = r_n$. Therefore

$$\limsup_{n\to\infty} \frac{1}{r_n} \int_{C_-} f_{X,T}(t) - f_{X,T}(t + r_n)\, dt \leq \min\left(f_{X,T}(t_1), f_{X,T}(t_1^-)\right). \qquad (4.42)$$

Finally, by combining (4.40) and (4.42), we get that

$$\limsup_{n\to\infty} \frac{1}{r_n} \int_{t_1}^{t_2} |f_{X,T}(t + r_n) - f_{X,T}(t)|\, dt \leq \min\left(f_{X,T}(t_1), f_{X,T}(t_1^-)\right) + f_{X,T}(t_2).$$

What remains is to apply this result to the proof of total variation. To that end, define the sequence $\{r_n\}_{n=1}^{\infty}$ as

$$r_n = \frac{|t_2 - t_1|}{n + 1} \quad \text{for all } n = 1, 2, \ldots$$

.

Then we can use (4.42) to bound the total variation, as follows:

$$\int_{t_1}^{t_2} |f_{X,T}(t+r_n) - f_{X,T}(t)|\ dt = \sum_{i=0}^{n} \int_{t_1}^{t_1+r_n} |f_{X,T}(t+(i+1)\,r_n) - f_{X,T}(t+ir_n)|\ dt,$$

and we also note that

$$\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) \le \lim_{n\to\infty} \sum_{i=0}^{n} |f_{X,T}(t+(i+1)\,r_n) - f_{X,T}(t+ir_n)|$$

uniformly for $t \in (t_1, t_1+r_n)$ since $(t+(n+1)r_n) = t+|t_2-t_1| \notin (t_1,t_2)$ and $TV_{(t_1,t_2)}(f_{X,T})$ is a non-decreasing function. Therefore

$$\begin{aligned}
&\min\left(f_{X,T}(t_1), f_{X,T}(t_1^-)\right) + f_{X,T}(t_2) \\
&\ge \limsup_{n\to\infty} \frac{1}{r_n} \int_{t_1}^{t_2} |f_{X,T}(t+r_n) - f_{X,T}(t)|\ dt \\
&\ge \limsup_{n\to\infty} \frac{1}{r_n} \int_{t_1}^{t_1+r_n} \sum_{i=0}^{n} |f_{X,T}(t+(i+1)\,r_n) - f_{X,T}(t+r_n)|\ dt \\
&\ge \mathrm{TV}_{(t_1,t_2)}(f_{X,T}).
\end{aligned}$$

Moreover, we note that

$$\begin{aligned}
\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) &= \lim_{\epsilon\downarrow 0} \mathrm{TV}_{(t_1,t_2-\epsilon)}(f_{X,T}) \\
&\le \min(f_{X,T}(t_1), f_{X,T}(t_1^-)) + \lim_{\epsilon\downarrow 0} f_{X,T}(t_2-\epsilon).
\end{aligned}$$

Therefore

$$\mathrm{TV}_{(t_1,t_2)}(f_{X,T}) \le \min\left(f_{X,T}(t_1), f_{X,T}(t_1^-)\right) + \min\left(f_{X,T}(t_2), f_{X,T}(t_2^-)\right)$$

as required. □

The proofs of (4.7) and (4.8) are identical to the proof of (4.6), with each of the proofs using one side of the two-sided argument performed for (4.6).

For (4.4), clearly the boundedness of the positive variation of the density at zero given in (4.7) gives that the limit $f_{X,T}(0^+) = \lim_{t\downarrow 0} f_{X,T}(t)$ exists. Similarly, the boundedness of the negative variation of the density at the endpoint T implies that $f_{X,T}(T^-) = \lim_{t\uparrow T} f_{X,T}(t)$ exists as well.

For (4.9), if $\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) < \infty$ for some $0 < \epsilon < T$, then the same argument used to prove (4.6) shows that for any $0 < \epsilon < T$,

$$\mathrm{TV}^{-}_{(0,\epsilon)}(f_{X,T}) \leq f_{X,T}(0^{+}).$$

Therefore with (4.7), we get that $\mathrm{TV}_{(0,\epsilon)}(f_{X,T}) < \infty$ which proves (4.9). One can prove (4.10), which is about the behaviour of the density at the right endpoint, in the same way.

This completes the proof of Theorem 4.2. $\square$

4.2 Relating Expected Number of Random Locations back to the Capacity Functional

As alluded to previously, once we had shown these results for $F^{\star}_{X,I}$, we would relate them back to $F_{X,I}$, the capacity functional. We defined a 'density' for $F_{X,I}$ in (4.2) as

$$f_{X,I}(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{P}\left(L^2(X,I) \cap (t, t+\epsilon] \neq \emptyset\right). \tag{4.43}$$

Without loss of generality, let $x_1 = \min L^2(X,I), x_2 = \max L^2(X,I)$. Then define the function $g_\epsilon(\cdot)$ as:

$$g_\epsilon(t) = \mathbb{P}\left(x_2 \in (t, t+\epsilon] \mid x_1 = t\right),\ t \in (0,T).$$

Assumption M: We assume that $g_\epsilon(t) \to 0$ as $\epsilon \downarrow 0$, uniformly for $t \in (0,T)$. Hence $g_\epsilon(t) \leq C(\epsilon)$ for every $t \in (0,T)$, for a function $C(\epsilon)$ such that $\lim_{\epsilon\downarrow 0} C(\epsilon) = 0$.

Theorem 4.9: Under Assumption M, $f_{X,I}$ as defined in (4.43) is equal to the right-derivative of $F^{\star}_{X,I}$, denoted $f^{\star}_{X,I}$.

Proof: Note that we can write $f_{X,I}(t)$ as follows for every $t \in (0,T)$:

$$\begin{aligned} f_{X,I}(t) &= \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\Big[\sum_{j=0}^{2} \mathbb{P}\left(L^2(X,I) \cap (t,t+\epsilon] \neq \emptyset \mid \left|L^2(X,I)\right| = j\right)\mathbb{P}\left(\left|L^2(X,I)\right| = j\right)\Big] \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\Big[\mathbb{P}\left(L^2(X,I) \cap (t,t+\epsilon] \neq \emptyset \mid \left|L^2(X,I)\right| = 1\right)\mathbb{P}\left(\left|L^2(X,I)\right| = 1\right) \\ &\qquad + \mathbb{P}\left(L^2(X,I) \cap (t,t+\epsilon] \neq \emptyset \mid \left|L^2(X,I)\right| = 2\right)\mathbb{P}\left(\left|L^2(X,I)\right| = 2\right)\Big]. \end{aligned}$$

Now consider $f_{X,I}(t) - f^\star_{X,I}(t)$:

$$\begin{aligned} f_{X,I}(t) - f^\star_{X,I}(t) &= \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\mathbb{P}\left(L^2(X,I)\cap(t,t+\epsilon]\neq\emptyset \mid \left|L^2(X,I)\right|=2\right)\mathbb{P}\left(\left|L^2(X,I)\right|=2\right) \\ &\quad - \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Big[\mathbb{P}\left(x_1\in(t,t+\epsilon] \mid \left|L^2(X,I)\right|=2\right)+\mathbb{P}\left(x_2\in(t,t+\epsilon] \mid \left|L^2(X,I)\right|=2\right)\Big] \\ &\qquad \times \mathbb{P}\left(\left|L^2(X,I)\right|=2\right) \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{P}\left(x_1,x_2\in(t,t+\epsilon] \mid \left|L^2(X,I)\right|=2\right)\mathbb{P}\left(\left|L^2(X,I)\right|=2\right). \end{aligned} \tag{4.44}$$

Now note that

$$\begin{aligned} \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{P}\left(x_1,x_2\in(t,t+\epsilon]\right) &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{P}\left(x_2\in(t,t+\epsilon] \mid x_1\in(t,t+\epsilon]\right)\mathbb{P}\left(x_1\in(t,t+\epsilon]\right) \\ &\leq \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_t^{t+\epsilon} g_\epsilon(y)dF_y\cdot\mathbb{P}\left(x_1\in(t,t+\epsilon]\right), \end{aligned} \tag{4.45}$$

where $F_y(A) = \mathbb{P}\left(x_1\in A \mid x_1\in(t,t+\epsilon]\right)$. Furthermore,

$$\mathbb{P}\left(x_1\in(t,t+\epsilon]\right) \leq F^\star_{X,I}\left((t,t+\epsilon]\right). \tag{4.46}$$

Hence from (4.45), (4.46) and (4.5) we get that

$$\begin{aligned} \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{P}\left(x_1,x_2\in(t,t+\epsilon]\right) &\leq \lim_{\epsilon\downarrow 0}\left(\int_t^{t+\epsilon} g_\epsilon(y)dF_y\right) f^\star_{X,I}(t) \\ &\leq \lim_{\epsilon\downarrow 0}\left(\int_t^{t+\epsilon} g_\epsilon(y)dF_y\right)\max\left(\frac{2}{t},\frac{2}{T-t}\right) \\ &\leq \lim_{\epsilon\downarrow 0}\left(\int_t^{t+\epsilon} C(\epsilon)\,dF_y\right)\max\left(\frac{2}{t},\frac{2}{T-t}\right) \\ &= \lim_{\epsilon\downarrow 0} C(\epsilon)\cdot F_y\left((t,t+\epsilon]\right)\max\left(\frac{2}{t},\frac{2}{T-t}\right) = 0. \end{aligned}$$

Hence the limit in (4.44) is zero and the 'density' of $F_{X,I}$ as defined in (4.43) is equal to the right-derivative of $F^\star_{X,I}$ and hence shares the same properties as derived in Theorem 4.2 for $f^\star_{X,I}$, as required. □

4.3 Structure of the class of measures $\mathcal{A}_T^2$

We showed in Theorem 4.2 that the distributions of intrinsic multiple-location functionals $L^2(f, [0,T])$ satisfy very specific properties, for any stationary process $\{X_t\}_{t\in\mathbb{R}}$ and $T > 0$. We will examine the structure of the class of all such distributions, and see that it has properties that we can work with to derive further results.

Definition 4.10: We denote by $\mathcal{A}_T^2$ the class of measures F on $[0,T]$ with the following properties:

(1) $F([0,T]) \leq 2$.

(2) The restriction of F to the interior $(0,T)$ of the interval is absolutely continuous.

(3) A version of the density f as the right derivative of F, given by

$$f(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(F\left((-\infty, t+\epsilon]\right) - F\left((-\infty, t]\right)\right), \quad t \in (0,T)$$

exists at every point in the interval $(0,T)$.

(4) This density f is right continuous, has left limits and satisfies the total variation restrictions (4.6) to (4.10).

Note that we do not include the requirement that f satisfies the bound (4.5) since it can be derived from the total variation constraints.

Before we prove some properties about this $\mathcal{A}_T^2$, we need the following useful lemma (Theorem 20, p. 298 in [5]).

Lemma 4.11: Let $S = \mathbb{R}$ be the real line, $\mathcal{B} = \mathcal{B}(S)$ be the Borel σ-algebra, and λ be the Lebesgue measure. Suppose $1 \leq p < \infty$ then a subset K of

$$L_p(S, \mathcal{B}, \lambda) := \left\{ f \mid \int_S \left|f(x)\right|^p d\lambda < \infty \right\}$$

is relatively compact (meaning that it has a compact closure) if and only if it is bounded and both of the following hold:

(i) $\lim_{x\to 0} \int_S \left|f(x+y) - f(y)\right|^p dy = 0$ uniformly for $f \in K$.

(ii) $\lim_{A\uparrow \sup S} \left(\int_A^{\sup S} \left|f(y)\right|^p dy + \lim_{A\downarrow \inf S} \int_{\inf S}^{A} \left|f(y)\right|^p dy \right) = 0$ uniformly for $f \in K$.

Theorem 4.12: Denote by $\mathcal{P}_T^2$ the set of all measures on $[0,T]$ which take values in $[0,2]$. Then the set $\mathcal{A}_T^2$ is a weakly closed convex subset of $\mathcal{P}_T^2$. Moreover, for any $0 < \epsilon < \frac{T}{2}$, the restrictions of the measures in $\mathcal{A}_T^2$ to the interval $(\epsilon, T-\epsilon)$ form a compact (in total variation) family of finite measures.

Proof: The convexity of $\mathcal{A}_T^2$ is clear by construction.

Fix $0 < \epsilon < \frac{T}{2}$, and let f be a version of the density for an arbitrary $F \in \mathcal{A}_T^2$ as described in Definition 4.10.

For $x > 0$ small enough, we get that

$$\begin{aligned}
&\int_\epsilon^{T-\epsilon} \left|f(x+y) - f(y)\right| \, dy \\
&\qquad = \sum_{j=1}^{\left\lfloor \frac{T-2\epsilon}{x} \right\rfloor} \int_{\epsilon+(j-1)x}^{\epsilon+jx} \left|f(x+y)-f(y)\right| \, dy + \int_{\epsilon+\left\lfloor \frac{T-2\epsilon}{x} \right\rfloor x}^{T-\epsilon} \left|f(x+y)-f(y)\right| \, dy \\
&\qquad \le \int_0^x \sum_{j=1}^{\left\lfloor \frac{T-2\epsilon}{x} \right\rfloor} \left|f(\epsilon+jx+y) - f(\epsilon+(j-1)x+y)\right| \, dy + \max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right) x \\
&\qquad \le \mathrm{TV}_{(\epsilon,T-\epsilon)}(f)x + \max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right) x \le 6 \max\left(\frac{1}{\epsilon}, \frac{1}{T-\epsilon}\right) x, \qquad (4.47)
\end{aligned}$$

by (4.5) and (4.6). Since the upper bound in (4.47) converges to 0 as $x \downarrow 0$ uniformly over the entire class $\mathcal{A}_T^2$, we conclude by Lemma 4.11 that the densities of the measures in $\mathcal{A}_T^2$ form a relatively compact family in $L_1(\epsilon, T-\epsilon)$ for each $0 < \epsilon < \frac{T}{2}$, where

$$L_1(\epsilon, T-\epsilon) := \left\{ f \;\middle|\; \int_\epsilon^{T-\epsilon} \left|f(x)\right| \, d\lambda < \infty \right\}.$$

Note that the requirement (ii) in Lemma 4.11 is trivial here since we only require

$$\lim_{A \to (T-\epsilon)} \int_A^{T-\epsilon} \left|f(y)\right| dy + \lim_{A \to \epsilon} \int_\epsilon^A \left|f(y)\right| dy = 0 \quad \text{uniformly for } f \in \partial\mathcal{A}_T^2,$$

where $\partial\mathcal{A}_T^2$ is the set of densities of measures in $\mathcal{A}_T^2$. Clearly this holds, since the Lebesgue measure is absolutely continuous and the densities f over $(\epsilon, T-\epsilon)$ are bounded by $\sup_{t\in(\epsilon,T-\epsilon)} \max\left(\frac{2}{t}, \frac{2}{T-t}\right) = \frac{2}{\epsilon}$ by (4.5).

Now let $\{F_n\}_{n=1,2,\ldots}$ be a sequence of measures in $\mathcal{A}_T^2$ such that $F_n \overset{\text{weak}}{\Longrightarrow} F$ for $F \in \mathcal{P}_T^2$. We wish to show that $F \in \mathcal{A}_T^2$ in order to prove $\mathcal{A}_T^2$ is weakly closed. To do so, we will show F satisfies the conditions of Definition 4.10.

For every $n \geq 1$, denote by f_n the version of the density of F_n as defined in Definition 4.10. Let $0 < t < T$, then for any $0 < \epsilon < \min(t, T-t)$, since F is the weak limit of F_n, we have that

$$F((t-\epsilon, t+\epsilon)) \leq \liminf_{n\to\infty} F_n((t-\epsilon, t+\epsilon)) \leq \int_{t-\epsilon}^{t+\epsilon} \max\left(\frac{2}{s}, \frac{2}{T-s}\right) ds. \tag{4.48}$$

Hence F is absolutely continuous in the interior of $[0, T]$ with a density f satisfying

$$f(t) \leq \left(\frac{2}{t}, \frac{2}{T-t}\right) \quad \text{for every } 0 < t < T.$$

Since for every $0 < \epsilon < \frac{T}{2}$, the sequence $(f_n)_{n=1,2,\ldots}$ is relatively compact in $L_1(\epsilon, T-\epsilon)$, we conclude

$$f_n \to f \quad \text{in } L_1(\epsilon, T-\epsilon). \tag{4.49}$$

From (4.49), for a fixed $0 < \epsilon < \frac{T}{2}$, there exists a subsequence $(f_{n_k})_{k=1,2,\ldots}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$f_{n_k} \to f \quad \text{almost everywhere in } (\epsilon, T-\epsilon). \tag{4.50}$$

For simplicity in the rest of the proof, we will just denote $(f_{n_k})_{k=1,2,\ldots}$ as the whole sequence $(f_n)_{n=1,2,\ldots}$, and we will define $A_\star$ as the set of $t \in (\epsilon, T-\epsilon)$ which satisfy (4.50), whose complement has zero measure.

We next claim that for every $\epsilon < t < T - \epsilon$,

$$\lim_{s\downarrow t, s\in A_\star} f(s) \text{ exists, and } \lim_{s\uparrow t, s\in A_\star} f(s) \text{ exists.} \tag{4.51}$$

We will only prove the first statement since the other is done in the same way. Suppose towards a contradiction that for some $t \in (\epsilon, T-\epsilon)$ the limit taken from the right ($s \downarrow t$) does not exist. Then there exist sequences $\{s_m\}_{m=1}^{\infty} \downarrow t$, $\{v_m\}_{m=1}^{\infty} \downarrow t$ in $A_\star$ such that

$$b := \lim_{m\to\infty} f(s_m) > a := \lim_{m\to\infty} f(v_m).$$

Without loss of generality we can take $s_1 > v_1 > s_2 > v_2 > \ldots > t$. Then we let $\tau = b - a > 0$, and take $M > 0$ sufficiently large so that

$$f(s_m) > b - \frac{\tau}{6}, \qquad f(v_m) < a + \frac{\tau}{6} \qquad \text{for all } m > M. \tag{4.52}$$

Also choose $K > 0$ sufficiently large so that

$$(2K-1)\tau > 6 \max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right).$$

Finally, choose $n > 0$ sufficiently large so that

$$\left|f_n(v_m) - f(v_m)\right| \leq \frac{\tau}{6} \ \text{ and } \ \left|f_n(s_m) - f(s_m)\right| \leq \frac{\tau}{6}, \tag{4.53}$$

for each $m = M+1, \dots, M+K$, this is possible because every s_m, v_m is in the set $A_\star$, the set on which $f_n \to f$ point-wise. It now follows from (4.52) and (4.53) that

$$f_n(s_m) > b - \frac{\tau}{3} \ \text{ and } \ f_n(v_m) < a + \frac{\tau}{3} \qquad \text{for each } m = M+1, \dots, M+K.$$

Therefore,

$$\begin{aligned}
&\sum_{m=M+1}^{M+K} \left|f_n(s_m) - f_n(v_m)\right| + \sum_{m=M+1}^{M+K-1} \left|f_n(v_m) - f_n(s_{m+1})\right| \\
&\qquad > \sum_{m=M+1}^{M+K} \left|\left(b - \frac{\tau}{3}\right) - f_n(v_m)\right| + \sum_{m=M+1}^{M+K-1} \left|f_n(s_{m+1}) - f_n(v_m)\right| \\
&\qquad > \sum_{m=M+1}^{M+K} \left|\left(b - \frac{\tau}{3}\right) - f_n(v_m)\right| + \sum_{m=M+1}^{M+K-1} \left|\left(b - \frac{\tau}{3}\right) - f_n(v_m)\right|.
\end{aligned} \tag{4.54}$$

The terms in these sums are always positive since $f_n(v_m) < a + \frac{\tau}{3} < b - \frac{\tau}{3}$ by construction, hence

$$\begin{aligned}
&\sum_{m=M+1}^{M+K} \left|f_n(s_m) - f_n(v_m)\right| + \sum_{m=M+1}^{M+K-1} \left|f_n(v_m) - f_n(s_{m+1})\right| \\
&\qquad > \sum_{m=M+1}^{M+K} b - \frac{\tau}{3} - f_n(v_m) + \sum_{m=M+1}^{M+K-1} b - \frac{\tau}{3} - f_n(v_m) \\
&\qquad = (2K-1)\left(b - \frac{\tau}{3} - f_n(v_m)\right) > (2K-1)\left(b - \frac{\tau}{3} - \left(a + \frac{\tau}{3}\right)\right) \\
&\qquad = (2K-1)\left(b - a - \frac{2\tau}{3}\right) = (2K-1)\frac{\tau}{3}.
\end{aligned}$$

We chose K such that $(2K-1)\tau > 6\max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right)$, hence we have

$$\sum_{m=M+1}^{M+K} \left|f_n(s_m) - f_n(v_m)\right| + \sum_{m=M+1}^{M+K-1} \left|f_n(v_m) - f_n(s_{m+1})\right| > 2\max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right).$$

However, by (4.5) we know that

$$\max\left(f_n(s_{M+1}), f_n(v_{M+K})\right) \leq \max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right).$$

Hence we have a contradiction of the total variation constraint (4.6) combined with the upper bound in (4.5), therefore the claim in (4.51) holds.

Next we show that the set

$$B_\star = \left\{t \in A_\star \mid f(t) \neq \lim_{s\downarrow t, s\in A_\star} f(s)\right\}$$

is at most countable. To accomplish this we show that for every $\theta > 0$, the set

$$B_\star(\theta) = \left\{t \in A_\star \mid \left|f(t) - \lim_{s\downarrow t, s\in A_\star} f(s)\right| > \theta\right\}$$

is finite, since $B_\star$ can be written as the countable union of $B_\star(\theta)$ over $\theta \in \mathbb{Q} \cap (0, \infty)$. We actually claim that

$$\left|B_\star(\theta)\right| \leq \frac{6}{\theta}\max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right). \tag{4.55}$$

To prove this, assume towards a contradiction that there are points $\epsilon < v_1 < v_2 < \ldots < v_K < T - \epsilon$ in $B_\star(\theta)$ for some

$$K > \frac{6}{\theta}\max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right).$$

Then for every $m = 1, \ldots, K$, choose $s_m \in A_\star$ with $v_m < s_m < v_{m+1}$ and $v_{K+1} = T - \epsilon$, such that

$$\left|f(v_m) - f(s_m)\right| > \theta.$$

Next choose n sufficiency large such that

$$\left|f_n(v_m) - f(v_m)\right| \leq \frac{\theta}{3}, \qquad \left|f_n(s_m) - f(s_m)\right| \leq \frac{\theta}{3}, \quad \text{for all } m = 1, \dots, K.$$

Then for every $m = 1, \dots, K$,

$$\begin{aligned}\left|f_n(v_m) - f_n(s_m)\right| &= \left|\, (f(s_m) - f(v_m)) - (f(s_m) - f_n(s_m)) - (f_n(v_m) - f(v_m)) \,\right| \\ &> \left|f(s_m) - f(v_m)\right| - \left|f_n(s_m) - f(s_m)\right| - \left|f_n(v_m) - f(v_m)\right| > \frac{\theta}{3}.\end{aligned}$$

And by our choice of K, this means

$$\sum_{m=1}^{K} \left|f_n(s_m) - f_n(v_m)\right| > \frac{K\theta}{3} > 2 \max\left(\frac{2}{\epsilon}, \frac{2}{T-\epsilon}\right).$$

However, this is again a contradiction of the total variation constraint (4.6) combined with the upper bound in (4.5). Hence $B_\star$ is at most countable.

Since we know there is a subsequence $(f_{n_k})_{k=1,2,\dots}$ with $n_k \to \infty$ as $k \to \infty$ such that $f_{n_k}(t) \to f(t)$ for almost every $t \in (\epsilon, T-\epsilon)$, we use a standard diagonal argument with sequences $\left(f^m_{n_k}\right)_{k=1,2,\dots}$ to take $\epsilon = 2^{-m} \downarrow 0$ and hence conclude that this holds for almost every $t \in (0, T)$.

We denote by $A_\star$ the set of such $t \in (0,T)$ for which this result holds, whose complement is of zero measure. We therefore conclude that (4.51) holds for every $t \in (0,T) \cap A_\star$. Since $B_\star$ (now over $(0,T)$ since $A_\star$ is redefined) is at most countable, we can now define

$$g(t) = \lim_{s \downarrow t, s \in A_\star} f(s), \quad 0 < t < T. \tag{4.56}$$

This function is clearly right-continuous by construction, and has left limits by (4.51). Additionally, g concides with f on $(A_\star \setminus B_\star)$, so g is a version of f, hence it is a density of the measure F on the interior of the interval $[0,T]$. Since g is right-continuous as the density of F on $(0,T)$, this means that F is right-differentiable at every point in $(0,T)$. Lastly, g satisfies the total variation constraints (4.6) to (4.10). Therefore $\mathcal{A}^2_T$ is weakly closed.

It remains to prove the last statement of the theorem that for any $0 < \epsilon < \frac{T}{2}$, the measures in $\mathcal{A}^2_T$ restricted to the interval $(\epsilon, T-\epsilon)$ form a compact (in total variation) family of finite measures.

Let $0 < \epsilon < \frac{T}{2}$ and let $(F_n)_{n=1,2,\dots}$ be a sequence in $\mathcal{A}_T^2$. Because $\mathcal{P}_T^2$ is weakly compact, we can choose a subsequence $(F_{n_k})_{k=1,2,\dots}$ with $n_k \to \infty$ as $k \to \infty$ which converges weakly to some F in $\mathcal{P}_T^2$. But we have just proven that $\mathcal{A}_T^2$ is weakly closed, hence this limiting measure $F \in \mathcal{A}_T^2$.

Let f be a version of the density of this F on $(0, T)$. We showed on page 37 that the densities $(f_{n_k})_{k=1,2,\dots}$ of the measures $(F_{n_k})_{k=1,2,\dots}$ form a relatively compact family in $L_1(\epsilon, T-\epsilon)$. Since f is the unique limit point of this sub-sequence, we conclude that $f_{n_k} \to f$ in $L_1(\epsilon, T-\epsilon)$.

Since $f_{n_k} \to f$ in $L_1(\epsilon, T-\epsilon)$, and we also know $f_{n_k} \to f$ almost everywhere in $(\epsilon, T-\epsilon)$, we can conclude by Scheffe's Theorem that

$$\lim_{k\to\infty} \int_{\epsilon}^{T-\epsilon} \left| f_{n_k} - f \right| \, d\lambda = 0. \tag{4.57}$$

This implies that the the measures $(F_{n_k})_{k=1,2,\dots}$ restricted to the interval $(\epsilon, T-\epsilon)$ converge in total variation to the restriction of the measure F to that same interval, since

$$\begin{aligned}
\lim_{k\to\infty} \sup_{D\in\mathcal{B}((\epsilon,T-\epsilon))} \left| F_{n_k}(D) - F(D) \right| &= \lim_{k\to\infty} \sup_{D\in\mathcal{B}((\epsilon,T-\epsilon))} \left| \int_D \left(f_{n_k} - f\right) \, d\lambda \right| \\
&\le \lim_{k\to\infty} \sup_{D\in\mathcal{B}((\epsilon,T-\epsilon))} \int_D \left| f_{n_k} - f \right| \, d\lambda \\
&\le \lim_{k\to\infty} \int_{\epsilon}^{T-\epsilon} \left| f_{n_k} - f \right| \, d\lambda = 0,
\end{aligned}$$

by (4.57) and since $\left| f_{n_k} - f \right| \ge 0$, hence we get the bound $\int_{\epsilon}^{T-\epsilon} \left| f_{n_k} - f \right| d\lambda$ for the supremum. □

4.4 Extreme points of $\mathcal{A}_T^2$

In this section, we look to analyse the convex structure of $\mathcal{A}_T^2$. In particular, we will identify the extreme points of $\mathcal{A}_T^2$, and show that nearly all of these extreme points do in fact each correspond to the distribution of a certain $L^2(\cdot,\cdot)$ on a chosen stationary $\{X_t\}_{t\in\mathbb{R}}$. In addition, we will show that for any $m \in \mathcal{A}_T^2$ satisfying certain conditions, we can construct a stationary process $\{X_t\}_{t\in\mathbb{R}}$ and intrinsic multiple-location functional $L^2(\cdot,\cdot)$ which has distribution m on $[0,T]$. We begin with some definitions and useful theorems.

Definition 4.13: (Definition 2, p. 414 in [5]) For a subset A of a linear space $\mathcal{X}$, the *closed convex hull* of A, denoted $\overline{\text{co}}(A)$, is the intersection of all closed convex sets in $\mathcal{X}$ which contain A.

Theorem 4.14: (Krein-Milman - Theorem 4, p. 440 in [5]) If K is a compact subset of a locally convex linear topological space $\mathcal{X}$, and E is the set of extremal points of K, then $\overline{\text{co}}(E) \supseteq K$, and in particular $\overline{\text{co}}(E) = K$ if K is convex.

Theorem 4.15: (Lebesgue Decomposition of Measures - Theorem 14 p. 132 in [5]) Let (S, Σ, μ) be a measure space. Then every finite countably-additive measure λ defined on Σ is *uniquely* representable as the sum $\lambda = \alpha + \beta$ where α is absolutely continuous and β is singular, both with respect to μ.

Remark 4.16: We note that the set of finite measures on $[0,T]$, equipped with the topology of weak convergence, is a locally convex linear topological space. We showed in Theorem 4.12 that $\mathcal{A}_T^2$ is a compact and convex subset of $\mathcal{P}_T^2$. Hence by Theorem 4.14, the set $\mathcal{A}_T^2$ is equal to the closed convex hull of its extreme points. This raises an obvious question: what are the extreme points of $\mathcal{A}_T^2$?

Theorem 4.17: The extreme points of $\mathcal{A}_T^2$ are:

(1) the measures $\mu_t,\ t \in (0,T)$, concentrated on $(0,T)$ which are absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions

$$f_{\mu_t} = \frac{2}{t}\mathbb{I}_{(0,t)}, 0 < t < T;$$

(2) the measures $\nu_t,\ t \in (0,T)$, concentrated on $(0,T)$ which are absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions

$$f_{\nu_t} = \frac{2}{T-t}\mathbb{I}_{(t,T)}, 0 < t < T;$$

(3) the point masses/singular measures δ_0 and δ_T defined as

$$\delta_0(A) = \begin{cases} 2 & \text{if } 0 \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_T(A) = \begin{cases} 2 & \text{if } T \in A \\ 0 & \text{otherwise} \end{cases}$$

for every Borel set $A \in \mathcal{B}([0,T])$.

(4) the null measure η on $[0,T]$, defined as $\eta(A) = 0$ for all $A \in \mathcal{B}([0,T])$.

Proof: Every measure $m \in A_T^2$ is absolutely continuous on $(0,T)$ and finite, hence by Theorem 4.13 there exists a unique decomposition $m = \alpha_1\delta_0 + \alpha_2\delta_T + \beta m_{\text{AC}} + \zeta\eta$ where $\alpha_1, \alpha_2, \beta, \zeta \geq 0$ and $\alpha_1 + \alpha_2 + \beta + \zeta = 1$ where m_{AC} is an absolutely continuous measure on $(0,T)$.

We begin by focusing on what we will call the non-trivial absolutely continuous extreme points of $\mathcal{A}_T^2$. That is, the measures which are non-zero on at least one set which has positive Lebesgue measure, which in particular excludes the null measure η.

Let f be the density of a non-trivial absolutely continuous extreme point of $\mathcal{A}_T^2$. Note that because we assume f is non-trivial, and an extreme point, we conclude that $\int_0^T f(t)dt = 2$. If this were not the case, say for $\int_0^T f(t)dt = c < 2$, then we could take the convex combination $f = \frac{c}{2}(g) + \frac{2-c}{2}(\eta')$ where η' is the density of the null measure η and g is some density of a measure in $\mathcal{A}_T^2$ such that $\int_0^T g(t)dt = 2$.

We start by showing that f must be monotone. To show this, define the functions

$$f_1(t) = \text{TV}^+_{(0,t]}(f),$$
$$f_2(t) = \text{TV}^-_{(t,T)}(f)$$

for $t \in (0,T)$. For $\text{TV}^+_{(0,t]}(f)$, we take the supremum over $0 < s_1 < \ldots < s_n \leq t$, because the interval $(0,t]$ is closed on the right. These functions are well-defined and non-negative by (4.7) and (4.8).

Note that f_1 is a non-decreasing càdlàg function with $f_1(0^+) = 0$, and f_2 is a non-increasing càdlàg function with $f_2(T^-) = 0$. Morever, from (4.7) and (4.8), we also see that

$$f(t) \geq \max(f_1(t), f_2(t)) \tag{4.58}$$

for all $0 < t < T$. Therefore, if we choose $0 < t_1 < T$, we see that for every $t_1 < t < T$,

$$\begin{aligned} f(t) &= f(t_1) + \mathrm{TV}^+_{(t_1,t]}(f) - \mathrm{TV}^-_{(t_1,t]}(f), \\ f_1(t) &= f_1(t_1) + \mathrm{TV}^+_{(t_1,t]}(f), \quad \text{and} \quad f_2(t) = f_2(t_1) - \mathrm{TV}^-_{(t_1,t]}(f). \end{aligned}$$

Therefore,

$$f(t) = f_1(t) + f_2(t) + \underbrace{(f(t_1) - f_1(t_1) - f_2(t_1))}_{:=C(t_1)} = f_1(t) + f_2(t) + C(t_1). \tag{4.59}$$

From this we can see that $C(t_1)$ is independent of t, and hence it is equal to a constant C for every $t_1 < t < T$. We also note that $C \geq -f_1(t)$ for any $0 < t < T$ by (4.58). Therefore we can let $t \downarrow 0$ to conclude that $C \geq 0$, since $f_1(0^+) = 0$. Let $f_2' = f_2 + C$, and hence $f = f_1 + f_2'$.

Now suppose towards a contradiction that f is not monotone, then clearly $\int_0^T f_1(s)\,ds > 0$ and $\int_0^T f_2'(s)\,ds > 0$. Therefore we have

$$f(t) = \frac{1}{2}\int_0^T f_1(s)\,ds \cdot \frac{f_1(t)}{\frac{1}{2}\int_0^T f_1(s)\,ds} + \frac{1}{2}\int_0^T f_2'(s)\,ds \cdot \frac{f_2'}{\frac{1}{2}\int_0^T f_2'(s)\,ds}, \quad 0 < t < T, \tag{4.60}$$

which is a convex combination of two monotone densities since

$$\frac{1}{2}\left(\int_0^T f_1(s) + \int_0^T f_2'(s)\,ds\right) = \frac{1}{2}\int_0^T f(s)\,ds = 1.$$

We note that monotone densities are automatically densities of some measures in $\mathcal{A}_T^2$, meaning that we have a contradiction of f being the density of an extreme point, therefore we conclude that f must be monotone.

We next show that f can take at most one non-zero value. Suppose that there are points $t_1, t_2 \in (0, T)$ such that $f(t_1) = a_1, f(t_2) = a_2$ for some $0 < a_1 < a_2$. Then define the functions

$$f_1(t) = \max(f(t) - a_1, 0) \;\text{ and }\; f_2(t) = f(t) - f_1(t), \quad 0 < t < T.$$

Since f is monotone, f_1 and f_2 are also monotone. As before, this allows us to represent f as

$$f(t) = \frac{1}{2}\int_0^T f_1(s)\,ds \cdot \frac{f_1(t)}{\frac{1}{2}\int_0^T f_1(s)\,ds} + \frac{1}{2}\int_0^T f_2(s)\,ds \cdot \frac{f_2(t)}{\frac{1}{2}\int_0^T f_2(s)\,ds}, \quad 0 < t < T, \quad (4.61)$$

which is again a convex combination of densities of some measures in $\mathcal{A}_T^2$, contradicting the fact that f is an extreme point of $\mathcal{A}_T^2$. Therefore, we conclude that the density f can take at most one non-zero value. This means that f is of the following form, for some $A \in \mathcal{B}([0,T])$:

$$f_A(t) = \frac{2}{\lambda(A)}\mathbb{I}_A(t), \quad 0 < t < T.$$

However, since f is monotone, this restricts A to being an interval which either begins at 0 or ends at T, which corresponds to the measures μ_t, ν_t. The last case to consider is then $A = (0,T)$, which can be represented as

$$f_{(0,T)} = \frac{1}{2}\left(2f_{\mu_{\frac{T}{2}}}\right) + \frac{1}{2}\left(2f_{\nu_{\frac{T}{2}}}\right) = \frac{1}{2}\frac{4}{T}\mathbb{I}_{\left(0,\frac{T}{2}\right)} + \frac{1}{2}\frac{4}{T}\mathbb{I}_{\left(\frac{T}{2},T\right)} \overset{\text{a.s.}}{=} \frac{2}{T}\mathbb{I}_{(0,T)},$$

which is a convex combination of densities corresponding to measures in $\mathcal{A}_T^2$, so it cannot be an extreme point. Therefore this density f of a non-trivial extreme point of $\mathcal{A}_T^2$ must be of the form μ_t or ν_t for some $0 < t < T$.

We must also show that these densities f_{μ_t}, f_{ν_t} do in fact correspond to extreme points of $\mathcal{A}_T^2$ to finish the proof. We present the proof for f_{μ_t}, with the argument for f_{ν_t} being similar.

Suppose towards a contradiction that f_{μ_t} is not an extreme point of $\mathcal{A}_T^2$. Then there exists two different measures in $\mathcal{A}_T^2$ concentrated on $(0,T)$ with respective densities g_1 and g_2 such that

$$f_{\mu_t}(s) = \rho g_1(s) + (1-\rho)g_2(s), \quad 0 < s < T \quad (4.62)$$

for some $0 < \rho < 1$. There must clearly be a point $0 < s_i < t$ such that $g_i(s_i) > \frac{2}{t}$ for each $i = 1, 2$. We also know that since $f_{\mu_t}(t) = \frac{2}{t}\mathbb{I}_{(0,t)}(t) = 0$, $g_i(t) = 0$ for each $i = 1, 2$.

Note that $\mathrm{TV}_{(0,t)}(g_i) \geq g_i(s_i)$, and since $g_i(t) = 0$, the total variation requirement in (4.9) gives us that

$$g_i(0^+) \geq \mathrm{TV}_{(0,t)}(g_i) \geq g_i(s_i) > \frac{2}{t}, \quad i = 1, 2,$$

meaning that

$$\rho g_1(0^+) + (1-\rho)g_2(0^+) > \frac{2}{t}.$$

However, this violates (4.62) in some neighborhood of the left endpoint, since $f_{\mu_t} = \frac{2}{t}\mathbb{I}_{(0,t)}$. Therefore μ_t is a non-trivial extreme point of $\mathcal{A}_T^2$.

It is easy to show that the measures δ_0, δ_T, η are also extreme points since they cannot be expressed as convex combinations of other measures in $\mathcal{A}_T^2$.

Therefore, the other measures m in $\mathcal{A}_T^2$ are convex combinations of μ_t, ν_t, δ_0, δ_T and η. □

Remark 4.18: In Theorem 4.17, we used δ_0 and δ_T as extreme points of $\mathcal{A}_T^2$, where

$$\delta_x(A) = \begin{cases} 2 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for both $x \in \{0, T\}$. However, in Assumption A (page 18) we limited our choices of $\{X_t\}_{t\in\mathbb{R}}$ and $L^2(\cdot,\cdot)$ to exclude the possibility of having two locations which coincide, which is precisely what happens with the measures δ_0, δ_T. Therefore it is necessary to look at a subset of $\mathcal{A}_T^2$ when connecting this class of measures back to the distributions of our intrinsic multiple-location functionals.

Theorem 4.19: Define

$$\mathcal{D}_T^2 := \big\{ m = (\alpha_1\delta_0 + \alpha_2\delta_T + \beta m_{\text{AC}} + \zeta\eta) \in \mathcal{A}_T^2 \;\big|\; 0 \le \alpha_1, \alpha_2 \le \frac{1}{2},\; 0 \le \beta, \zeta \le 1, \quad \alpha_1 + \alpha_2 + \beta + \zeta = 1 \big\}. \tag{4.63}$$

Then for any stationary process $\{X_t\}_{t\in\mathbb{R}}$ and intrinsic multiple-location functional $L^2(\cdot,\cdot)$, the distribution

$$F^\star_{X,T}(A) = \mathbb{E}\left[\left|L^2(X,[0,T]) \cap A\right|\right]$$

is an element of $\mathcal{D}_T^2$.

Furthermore, for every measure $m \in \mathcal{D}_T^2$, there exists a stationary process $\{X_t\}_{t\in\mathbb{R}}$ and intrinsic multiple-location functional $L^2(\cdot,\cdot)$ such that

$$m(A) = F^\star_{X,T}(A) = \mathbb{E}\left[\left|L^2(X,[0,T]) \cap A\right|\right] \qquad \text{for all } A \in \mathcal{B}([0,T]). \tag{4.64}$$

Proof: It is clear by the construction of $\mathcal{A}_T^2$ and Assumption A, which restricts the masses at the boundaries $\{0\}, \{T\}$ to be at most 1, that every $F^\star_{X,T} \in \mathcal{D}_T^2$.

Before we show the second part of the theorem, we want to show that $\mu_t, \nu_t, \delta_0, \delta_T$ and η are the distributions of a particular pair of a stochastic process and an intrinsic multiple-location functional.

We will then create a mixture of these processes and construct a new intrinsic multiple-location functional in a clever way to have distribution m.

(1) For the measures $\mu_t, t \in (0,T)$, we let $X(s) = \sin\left(\frac{2\pi s}{t} + U\right), s \in \mathbb{R}$, where U is a uniform random variable on $(0, 2\pi)$. Then let

$$
\begin{aligned}
L^2(f,[0,T]) &= \left\{\tau^1_{X,T}, \tau^2_{X,T}\right\} \\
\text{where } \tau^1_{X,T} &= \inf\left\{t \in [0,T] \mid X(t) = \sup_{s\in[0,T]} X(s)\right\}, \\
\text{and } \tau^2_{X,T} &= \inf\left\{t \in [0,T] \mid X(t) = \inf_{s\in[0,T]} X(s)\right\}.
\end{aligned}
$$

Clearly $\tau^1_{X,T}$ and $\tau^2_{X,T}$ are uniformly distributed on $(0,t)$. Therefore,

$$
\mu_t(A) = F^\star_{X,T}(A) = \sum_{i=1}^{2} \mathbb{P}\left(\tau^i_{X,T} \in A\right) = 2U_{(0,t)}(A)
$$

as required.

(2) The argument for ν_t is analogous, where we take the location of the right-most path supremum, and the location of the right-most path infimum on the same process.

(3) For $\frac{1}{2}\delta_0$, we let $X(s) = \sin\left(\frac{2\pi s}{t} + U\right)$ where U is a uniform random variable on $(0, 2\pi)$ as before. Then let

$$
\begin{aligned}
L^2(f,[0,T]) &= \left\{\tau^1_{X,T}, \ell^k_{X,T}\right\} \\
\text{where } \tau^1_{X,T} &= \{0\}, \\
\text{and } \ell^k_{X,T} &= \inf\{t \in [0,T] \mid X(t) > 2\}.
\end{aligned}
$$

Hence $L^2(f,[0,T]) = \left\{\tau^1_{X,T}\right\} = \{0\}$ for every path $f \in \mathcal{H}$.

For $\frac{1}{2}\delta_T$, we have the same argument, using $\tau^1_{X,T} = \{T\}$ with the same process used for $\frac{1}{2}\delta_0$. Hence $L^2(f,[0,T]) = \{T\}$ for every path $f \in \mathcal{H}$.

(4) Lastly, for the null measure η we let $X(s) = \sin\left(\frac{2\pi s}{t} + U\right)$ where U is a uniform random variable on $(0, 2\pi)$ and define the intrinsic multiple-location functional as the first and second occurences of hitting a level k which are never achieved:

$$\begin{aligned} L^2(f, [0,T]) &= \left\{\ell^{k,1}_{X,T}, \ell^{k,2}_{X,T}\right\} \\ \text{where } \ell^{k,1}_{X,T} &= \inf\left\{t \in [0,T] \mid f(t) = 3\right\}, \\ \text{and } \ell^{k,2}_{X,T} &= \inf\left\{t \in \left([0,T] \setminus \left\{\ell^{k,1}_{X,T}\right\}\right) \mid f(t) = 3\right\}. \end{aligned}$$

Then $L^2(f, [0,T]) = \emptyset$ for every path $f \in \mathcal{H}$, as desired.

Our goal is to construct, for a given $m \in \mathcal{D}^2_T$, a stationary process $\{X_t\}_{t\in\mathbb{R}}$ and an intrinsic multiple-location functional $L^2(X, [0,T])$ with distribution m.

To this end, we decompose $m \in \mathcal{D}^2_T$ into its absolutely continuous (and null) part, and singular point masses:

$$\begin{aligned} m &= \alpha_1\delta_0 + \alpha_2\delta_T + \beta m_{\mathrm{AC}} + \zeta\eta, \\ \text{for } 0 \le \alpha_1, \alpha_2 &\le \frac{1}{2},\ 0 \le \beta, \zeta \le 1,\ \alpha_1 + \alpha_2 + \beta + \zeta = 1. \end{aligned} \tag{4.65}$$

We first focus on what we will call the "middle part", which is comprised of m_{AC} and η. We normalize the combination of these two measures and denote this as

$$m^\star = \frac{\beta}{\beta+\zeta} m_{\mathrm{AC}} + \frac{\zeta}{\beta+\zeta}\eta. \tag{4.66}$$

We know that m_{AC} is the convex combination of measures in $\left\{\{\mu_t\}_{t\in(0,T)}, \{\nu_t\}_{t\in(0,T)}\right\}$, and $\eta = 0$ everywhere, hence $m^\star$ can be written as

$$m^\star = \int_0^T \mu_t \, dF_\mu + \int_0^T \nu_t \, dF_\nu,$$

for measures F_μ, F_ν which represent the weights of the μ_t, ν_t respectively, with

$$F_\mu\left((0,T)\right) + F_\nu\left((0,T)\right) = \frac{\beta}{\beta+\zeta}.$$

For this "middle part", we construct the following stationary processes:

$$
\begin{aligned}
Y^m_{\mu_t}(s) &= \sin\left(\frac{2\pi s}{t} + U\right), \\
Y^m_{\nu_t}(s) &= \sin\left(\frac{2\pi s}{t} + U\right) + 3, \\
Y_\eta(s) &= \sin\left(\frac{2\pi s}{t} + U\right) + 21,
\end{aligned}
$$

for every $t \in (0, T)$. We will form a mixture of these processes (and others later) in order to form our required stationary process. To this end, denote the measure on the path space of continuous functions $\mathcal{C}(\mathbb{R})$ corresponding to $Y^m_{\mu_t}$ as $\mathbb{Q}^m_{\mu_t}$, and similarly for $Y^m_{\nu_t}$ we take $\mathbb{Q}^m_{\nu_t}$, and for Y_η we take $\mathbb{Q}_\eta$.

Then define the measure $S_{\text{mid}} : \mathcal{C}(\mathbb{R}) \to [0, 1]$ as

$$
S_{\text{mid}} = \int_0^T \mathbb{Q}^m_{\mu_t} dF_\mu + \int_0^T \mathbb{Q}^m_{\nu_t} dF_\nu + \frac{\zeta}{\beta + \zeta} \mathbb{Q}_\eta,
$$

which is invariant under shifts since it is a mixture of measures that are invariant under shifts. That is, S_{mid} is a measure on $\mathcal{C}(\mathbb{R})$ corresponding to a stationary process with paths in $\mathcal{C}(\mathbb{R})$.

This is our construction for the "middle part", and we now consider the measure $m \in \mathcal{D}^2_T$, which we will decompose as follows:

$$
m = \theta_1 \left(\frac{1}{2}\delta_0 + \frac{1}{2}m^\star\right) + \theta_2 m^\star + \theta_3 \left(\frac{1}{2}\delta_T + \frac{1}{2}m^\star\right) + \theta_4 \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_T\right) \tag{4.67}
$$

$$
\text{for } \theta_i \in [0, 1] \text{ for all } i = 1, \ldots, 4, \text{ and } \sum_{i=1}^4 \theta_i = 1.
$$

We note that to match the original decomposition in (4.65), this means that:

$$
\begin{aligned}
\theta_4 &= \min\left(2\alpha_1, 2\alpha_2\right), \\
\theta_1 &= 2\alpha_1 - \theta_4, \qquad \theta_3 = 2\alpha_2 - \theta_4, \\
\text{and } \theta_2 &= 1 - \max\left(2\alpha_1, 2\alpha_2\right).
\end{aligned}
$$

As we did for the 'middle part', we define the following stationary processes to represent the 'θ_1' part of the decomposition in (4.67):

$$Y^0_{\mu_t}(s) = \sin\left(\frac{2\pi s}{t} + U\right) + 6,$$
$$Y^0_{\nu_t}(s) = \sin\left(\frac{2\pi s}{t} + U\right) + 9,$$

for every $t \in (0, T)$. We again denote the measures on the path space $\mathcal{C}(\mathbb{R})$ corresponding to $Y^0_{\mu_t}$ as $\mathbb{Q}^0_{\mu_t}$ and $Y^0_{\nu_t}$ as $\mathbb{Q}^0_{\nu_t}$.

Then define the measure $S_0 : \mathcal{C}(\mathbb{R}) \to [0, 1]$ as

$$S_0 = \int_0^T \mathbb{Q}^0_{\mu_t} dF_\mu + \int_0^T \mathbb{Q}^0_{\nu_t} dF_\nu.$$

Similarly, for the 'θ_3' part of (4.67), we define

$$Y^T_{\mu_t}(s) = \sin\left(\frac{2\pi s}{t} + U\right) + 12,$$
$$Y^T_{\nu_t}(s) = \sin\left(\frac{2\pi s}{t} + U\right) + 15,$$
$$\text{and } S_T = \int_0^T \mathbb{Q}^T_{\mu_t} dF_\mu + \int_0^T \mathbb{Q}^T_{\nu_t} dF_\nu.$$

Lastly, for the 'θ_4' term in (4.67) we define

$$Y_e(s) = \sin\left(\frac{2\pi s}{T} + U\right) + 18,$$
$$\text{and } S_e = \mathbb{Q}_e$$

Now we mix the measures S_0, S_{mid}, and S_T according to the decomposition in (4.67) in order to get the measure $S : \mathcal{C}(\mathbb{R}) \to [0, 1]$:

$$S = \theta_1 S_0 + \theta_2 S_{\text{mid}} + \theta_3 S_T + \theta_4 S_e.$$

This measure S on the path space $\mathcal{C}(\mathbb{R})$ will give us paths that will be used in conjunction with a cleverly defined $L^2(f, I)$ to achieve the desired result. In particular, we will use the fact that the path space is partitioned by S since the sinusoidal processes defined above do not ever cross paths. Then we define $L^2(f, I)$ on each of these disjoint subset of paths as we did for the extreme points, on a case-by-case basis, and it will have distribution $m \in \mathcal{D}_T^2$. For each $f \in \mathcal{C}(\mathbb{R})$ and $[a, b] \in \mathcal{I}$, the $L^2(f, I)$ we want is defined as:

$$\begin{cases} \Big\{ \inf \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, \\ \quad \inf \{t \in I \mid f(t) = \inf_{s \in I} f(s)\} \Big\} & \text{if } f(x) \in [-1, 1] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \sup \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, \\ \quad \sup \{t \in I \mid f(t) = \inf_{s \in I} f(s)\} \Big\} & \text{if } f(x) \in [2, 4] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \inf \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, a \Big\} & \text{if } f(x) \in [5, 7] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \sup \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, a \Big\} & \text{if } f(x) \in [8, 10] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \inf \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, b \Big\} & \text{if } f(x) \in [11, 13] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \sup \{t \in I \mid f(t) = \sup_{s \in I} f(s)\}, b \Big\} & \text{if } f(x) \in [14, 16] \text{ for every } x \in \mathbb{R}, \\ \Big\{ a, b \Big\} & \text{if } f(x) \in [17, 19] \text{ for every } x \in \mathbb{R}, \\ \Big\{ \inf \{t \in I \mid f(t) = 25\}, \\ \quad \sup \{t \in I \mid f(t) = 25\} \Big\} & \text{if } f(x) \in [20, 22] \text{ for every } x \in \mathbb{R}, \\ \{a, b\} & \text{otherwise.} \end{cases}$$

Then one can check that for the stationary process $\{X_t\}_{t \in \mathbb{R}}$ which corresponds to the measure S on the path space and this intrinsic multiple-location functional $L^2(\cdot, \cdot)$ we get that

$$F_{X,T}^{\star}(A) = \mathbb{E}\left[\left|L^2(X, [0, T]) \cap A\right|\right] = m(A)$$

for every Borel set $A \in \mathcal{B}([0, T])$, as required. $\square$

Chapter 5

Intrinsic Location Vectors

The ‘vector case’ we will discuss in this section is more similar to the single-location case, in the sense that we will once again allow infinite values when the random location is not well defined in the given interval. However, we now identify the two random locations, so our object of interest changes from a random set of points to a random vector.

We keep the same $\mathcal{H}$ and $\mathcal{I}$ as previously defined, and as before, we endow the set $(I \cup \{\infty\})$ with the topology obtained by treating the infinite point as an isolated point of the set, and take the Borel σ-algebra of this collection to obtain our measurability condition.

Definition 5.1: A mapping $L_v^2 : \mathcal{H} \times \mathcal{I} \to (\mathbb{R} \cup \{\infty\})^2$ is called an *intrinsic location vector of degree* 2 if it satisfies all of the following conditions:

(1) For every $I \in \mathcal{I}$ the map $L_v^2(\cdot, I) : \mathcal{H} \to (\mathbb{R} \cup \{\infty\})^2$ is measurable.

(2) For every $f \in \mathcal{H}$ and $I \in \mathcal{I}$, $L_v^2(f, I) \in (I \cup \{\infty\})^2$.

(3) *(Shift compatibility)* For every $f \in \mathcal{H}, I \in \mathcal{I}, c \in \mathbb{R}$,

$$L_v^2(f, I) = \begin{bmatrix} \ell^1_{f,I} \\ \ell^2_{f,I} \end{bmatrix} = \begin{bmatrix} \ell^1_{\theta_c f, I-c} \\ \ell^2_{\theta_c f, I-c} \end{bmatrix} + \begin{bmatrix} c \\ c \end{bmatrix} = L_v^2(\theta_c f, I - c) + \begin{bmatrix} c \\ c \end{bmatrix}$$

where $\infty \pm c = \infty$.

(4) *(Inclusion under restriction)* For every $f \in \mathcal{H},$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$, define the sets J_{f,I_1}, J_{f,I_2} as

$$J_{f,I_j} = \left\{ \ell^1_{f,I_j}, \ell^2_{f,I_j} \right\},\ j \in \{1, 2\},$$

then

$$J_{f,I_1} \cap I_2 \subseteq J_{f,I_2}.$$

(5) *(Consistency of existence)* For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$, and sets J_{f,I_1}, J_{f,I_2} as defined above,

$$\left| J_{f,I_2} \setminus \{\infty\} \right| \leq \left| J_{f,I_1} \setminus \{\infty\} \right|.$$

In the following chapter, we will not keep repeating 'of degree 2' for the intrinsic location vectors, since we will always be working with $n = 2$ until Chapter 6.

Example 5.2: In Example 3.2 on page 12 we introduced the locations of the two largest local maxima. This is of course an intrinsic location vector of degree 2, if we simply take the vector $L_v^2(f, I) = \begin{bmatrix} \tau_{f,I}^1 \\ \tau_{f,I}^2 \end{bmatrix}$ rather than the set valued $L^2(f, I)$.

Example 5.3: Consider $\mathcal{H}$ as the space of continuous functions $\mathcal{C}(\mathbb{R})$, and define the first and second hitting times of a level k, respectively, as

$$T_{f,[a,b]}^{k,1} := \inf \{ s \in [a, b] \mid f(s) = k \},$$
$$\text{and } T_{f,[a,b]}^{k,2} := \inf \left\{ s \in \left(T_{f,[a,b]}^{k,1}, b\right] \mid f(s) = k \right\}.$$

These form an intrinsic location vector of order 2:

$$L_v^2(f, [a, b]) = \begin{bmatrix} T_{f,[a,b]}^{k,1} \\ T_{f,[a,b]}^{k,2} \end{bmatrix}.$$

Theorem 5.4: For two single-location intrinsic location functionals $L_1(\cdot, \cdot), L_2(\cdot, \cdot)$ as in Definition 2.1, $L_v^2(\cdot, \cdot) = \begin{bmatrix} L_1(\cdot, \cdot) \\ L_2(\cdot, \cdot) \end{bmatrix}$ is an intrinsic location vector.

Proof: The fact that this $L_v^2(\cdot, \cdot)$ satisfies measurability is clear.

For shift compatibility, let $f \in \mathcal{H}, I \in \mathcal{I}$, and $c \in \mathbb{R}$, then

$$L_v^2(f, I) = \begin{bmatrix} L_1(f, I) \\ L_2(f, I) \end{bmatrix} = \begin{bmatrix} L_1(\theta_c f, I - c) + c \\ L_2(\theta_c f, I - c) + c \end{bmatrix} = L_v^2(\theta_c f, I - c) + \begin{bmatrix} c \\ c \end{bmatrix},$$

as required.

For inclusion under restriction, let $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$. By definition, we know that $L_j(f, I_1) \in I_2$ implies $L_j(f, I_2) = L_j(f, I_1)$ for each $j = 1, 2$. Hence

$$\{L_1(f, I_1), L_2(f, I_1)\} \cap I_2 \subseteq \{L_1(f, I_2), L_2(f, I_2)\} .$$

Lastly, to show consistency of existence, let $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$. Then by the single-location definition we know that $L_j(f, I_2) \neq \infty$ implies $L_j(f, I_1) \neq \infty$ for each $j = 1, 2$. Hence

$$\big| \{L_1(f, I_2), L_2(f, I_2)\} \setminus \{\infty\} \big| \leq \big| \{L_1(f, I_1), L_2(f, I_1)\} \setminus \{\infty\} \big|.$$

Therefore, $L_v^2(\cdot, \cdot)$ is an intrinsic location vector. □

Remark 5.5: Clearly for $L_v^2(f, I) = \begin{bmatrix} \ell^1_{f,I} \\ \ell^2_{f,I} \end{bmatrix}$, one could simply just take $L^2(f, I) = \left\{\ell^1_{f,I}, \ell^2_{f,I}\right\}$ and perform all of the prior analysis, but the vector case contains more structure which we can work with. We can now distinguish the locations from one another in a non-trivial manner, which allows us to classify the intrinsic location vectors into separate subclasses based on the relationship between the two locations.

5.1 Subclasses of Intrinsic Location Vectors

We will classify the intrinsic location vectors into subclasses based on their behaviours under restriction.

Definition 5.6: For any instrinsic location vector $L_v^2(\cdot, \cdot) = \begin{bmatrix} \ell^1_{\cdot,\cdot} \\ \ell^2_{\cdot,\cdot} \end{bmatrix}$, we say that location $\ell^j_{\cdot,\cdot}$ is *dominated by* location $\ell^i_{\cdot,\cdot}$ for $i \neq j$ if the following hold:

(1) For every $f \in \mathcal{H}$, any pair of intervals $I_1, I_2 \in \mathcal{I}$, with $I_2 \subseteq I_1$,

$$\ell^i_{f,I_1} \in (I_1 \setminus I_2) \text{ and } \ell^j_{f,I_1} \in I_2,$$

implies that $\ell^i_{f,I_2} = \ell^j_{f,I_1}$.

(2) For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$, with $I_2 \subseteq I_1$, if $\ell^i_{f,I_1} \in I_2$ then $\ell^i_{f,I_1} = \ell^i_{f,I_2}$.

(3) For every $f \in \mathcal{H}$ and $I \in \mathcal{I}$, if $\ell^i_{f,I} = \infty$ then $\ell^j_{f,I} = \infty$.

In this case, we say that the location $\ell^{i}_{\cdot,\cdot}$ *dominates* the location $\ell^{j}_{\cdot,\cdot}$, and for an $L^2_v(\cdot,\cdot)$ which has a location that dominates the other, we will say that it has a *dominating location*.

An intrinsic location vector with a dominating location can be thought of as a vector of sequential occurences. That is, the dominating location is the first occurence of some event on the path, restricted to a compact interval, and the dominated location is the second occurence.

Convention V: We will take the convention that $\ell^{1}_{\cdot,\cdot}$ will always be the dominating location, if there is one, as defined in Definition 5.6.

Example 5.7: The location of the leftmost hitting time of a level k dominates the location of the second leftmost hitting time of that same level k. For instance, we would define the leftmost hitting time as $\ell^{1}_{\cdot,\cdot}$ and the second leftmost hitting time as $\ell^{2}_{\cdot,\cdot}$ in order to satisfy Convention V.

Remark 5.8: For any intrinsic location vector $L^2_v(\cdot,\cdot) = \begin{bmatrix} \ell^{1}_{\cdot,\cdot} \\ \ell^{2}_{\cdot,\cdot} \end{bmatrix}$, it is not possible that $\ell^{1}_{\cdot,\cdot}$ dominates $\ell^{2}_{\cdot,\cdot}$, and $\ell^{2}_{\cdot,\cdot}$ dominates $\ell^{1}_{\cdot,\cdot}$ at the same time.

To see this, fix an $f \in \mathcal{H}$ and $I \in \mathcal{I}$, then take three intervals $I_1, I_2, I_3 \in \mathcal{I}$ such that $I_3 \subseteq I_2 \subseteq I_1$, and $\ell^{1}_{f,I_1} \in (I_1 \setminus I_2), \ell^{2}_{f,I_1} \in I_3, \ell^{2}_{f,I_2} \in (I_2 \setminus I_3)$. Then

$$\ell^{2}_{f,I_1} = \ell^{1}_{f,I_2}$$

since $\ell^{1}_{\cdot,\cdot}$ dominates $\ell^{2}_{\cdot,\cdot}$. However, since $\ell^{2}_{f,I_2} \in (I_2 \setminus I_3)$, then

$$\ell^{2}_{f,I_3} = \ell^{1}_{f,I_2}$$

since $\ell^{2}_{\cdot,\cdot}$ dominates $\ell^{1}_{\cdot,\cdot}$. In conclusion, this means that $\ell^{1}_{f,I_1} \in (I_1 \setminus I_3), \ell^{2}_{f,I_1} \in I_3$ and $\ell^{2}_{f,I_1} = \ell^{2}_{f,I_3}$, a contradiction that $\ell^{1}_{\cdot,\cdot}$ dominates $\ell^{2}_{\cdot,\cdot}$. See Figure (5.1) for an illustration of this scenario.

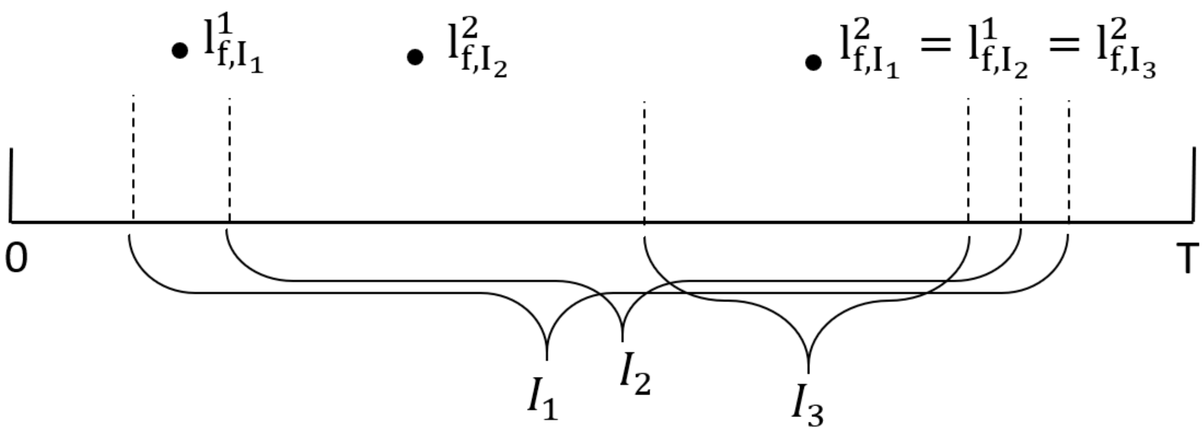

Figure 5.1: Illustration of the contradiction in Remark 5.8

Remark 5.9: For any intrinsic location vector $L_v^2(\cdot,\cdot)$, where we follow Convention V if $L_v^2(\cdot,\cdot)$ has a dominating location, we fix an $f \in \mathcal{H}$. Then when we impose a restriction from I_1 to I_2 with $I_2 \subseteq I_1$ we have two possible scenarios for $L_v^2(f,\cdot) = \begin{bmatrix} \ell^1_{f,\cdot} \\ \ell^2_{f,\cdot} \end{bmatrix}$, based on the behaviour of $\ell^2_{f,\cdot}$.

When $\ell^1_{f,I_1} \in (I_1 \setminus I_2)$, and $\ell^2_{f,I_1} \in I_2$ then there are two possibilities:

(i) $\ell^2_{f,I_1} = \ell^1_{f,I_2}$, or

(ii) $\ell^2_{f,I_1} = \ell^2_{f,I_2}$.

We will distinguish our subclasses based on the these two possibilities.

Definition 5.10: The two subclasses of intrinsic location vectors we will work with are:

(1) **Ranked Intrinsic Location Vectors:**

All $L_v^2(\cdot,\cdot)$ which have a dominating location as in Definition 5.6, which is $\ell^1_{\cdot,\cdot}$ in order to follow Convention V.

(2) **Free Intrinsic Location Vectors:**

$L_v^2(\cdot,\cdot)$ is a free intrinsic location vector if for every $f \in \mathcal{H}$, the following hold:

(i) For each $j = 1, 2$, when we have $I_1, I_2 \in \mathcal{I}$ with $I_2 \subseteq I_1$ which satisfy $\ell^j_{f,I_1} \in I_2$, we get that $\ell^j_{f,I_1} = \ell^j_{f,I_2}$.

(ii) For each $j = 1, 2$: if we take any $I_1, I_2 \in \mathcal{I}$ with $I_2 \subseteq I_1$, we get that $\ell^j_{f,I_1} = \infty$ implies $\ell^j_{f,I_2} = \infty$.

In particular, free intrinsic location vectors do not have a dominating location.

The reason we chose to take these two subclasses is that the ranked subclass represents all of the 'sequential' instrinsic location vectors we've been looking at very often, such as the first two hitting times of a level k, the two leftmost path suprema and so on. This case would be particularly important in applications.

The free subclass represents the case where one simply combines two intrinsic location functionals together into a vector. See Theorem 5.14 which proves this is indeed how the free intrinsic location vectors must be constructed.

Theorem 5.11: If the intrinsic location vector $L^2_v(\cdot,\cdot) = \begin{bmatrix} \ell^1_{\cdot,\cdot} \\ \ell^2_{\cdot,\cdot} \end{bmatrix}$ is ranked, then $\ell^1_{\cdot,\cdot}$ must be an intrinsic location functional in the sense of the 'single-location' case from Chapter 2.

Proof: By construction of $L^2_v(\cdot,\cdot)$, $\ell^1_{\cdot,\cdot}$ must satisfy the measurability and shift compatibility conditions. So we must show that it satisfies the stability under restriction and consistency of existence conditions (see Definition 2.1 on page 3).

Suppose towards a contradiction that $\ell^1_{\cdot,\cdot}$ does not satisfy the stability under restriction condition. That is, there exists $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ with $I_2 \subseteq I_1$ such that $\ell^1_{f,I_1} \in I_2$ and $\ell^1_{f,I_1} \neq \ell^1_{f,I_2}$. But this contradicts the definition of the ranked intrinsic location vector.

Now suppose towards a contradiction that $\ell^1_{\cdot,\cdot}$ does not satisfy the consistency of existence condition. That is, there exists $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ with $I_2 \subseteq I_1$ such that $\ell^1_{f,I_2} \neq \infty$ and $\ell^1_{f,I_1} = \infty$. Then by the consistency of existence condition on $L^2_v(f,\cdot)$:

$$\left| \left\{ \ell^1_{f,I_2}, \ell^2_{f,I_2} \right\} \setminus \{\infty\} \right| \leq \left| \left\{ \ell^1_{f,I_1}, \ell^2_{f,I_1} \right\} \setminus \{\infty\} \right|,$$

which is only the case if

$$\ell^2_{f,I_2} = \infty \quad \text{and} \quad \ell^2_{f,I_1} \neq \infty.$$

However, we assumed $\ell^1_{f,I_1} = \infty$, hence $\ell^2_{f,I_1} = \infty$ because $L^2_v(f,\cdot)$ is ranked. Therefore the consistency of existence condition on $L^2_v(f,\cdot)$ is violated, a contradiction.

Therefore $\ell^1_{\cdot,\cdot}$ is an intrinsic location functional. $\square$

Remark 5.12: Because we have shown that the ranked intrinsic location vector $L^2_v(\cdot,\cdot) = \begin{bmatrix} \ell^1_{\cdot,\cdot} \\ \ell^2_{\cdot,\cdot} \end{bmatrix}$ has the property that $\ell^1_{\cdot,\cdot}$ is an intrinsic location functional, we get that $\ell^1_{\cdot,\cdot}$ satisfies all of the previously shown 'single-location' results.

Hence we now also now have in this case that $\ell^2_{\cdot,\cdot}$ satisfies some nice properties. However, for a ranked intrinsic location vector, $\ell^2_{\cdot,\cdot}$ is typically *not* an intrinsic location functional. An an example, consider the second hitting time of a level k, which clearly does not satisfy stability under restriction since we can have $\ell^2_{\cdot,I_1} \in I_2$, $\ell^2_{\cdot,I_1} = \ell^1_{\cdot,I_2}$ for some $I_2 \subseteq I_1$, where $\ell^2_{\cdot,I_2}$ could be undefined, violating stability under restriction.

Corollary 5.13: Let $\{X_t\}_{t\in\mathbb{R}}$ be a stationary process and let $L^2_v(\cdot,\cdot) = \begin{bmatrix} \ell^1_{\cdot,\cdot} \\ \ell^2_{\cdot,\cdot} \end{bmatrix}$ be a ranked intrinsic location vector. Then for every $I \in \mathcal{I}$, the distribution of $\ell^2_{X,I}$, denoted

$$F^2_{X,I}(A) = \mathbb{P}\left(\ell^2_{X,I} \in A\right)$$

is absolutely continuous on the interior of the interval I. Furthermore, the version of the marginal density of $\ell^2_{X,I}$ denoted $f^2_{X,I}$, defined as the right derivative of $F^2_{X,I}$ is right continuous and has left limits.

Proof: The result follows immediately from the facts that

$$F^\star_{X,I}(A) = \mathbb{P}\left(\ell^1_{f,I} \in A\right) + \mathbb{P}\left(\ell^2_{f,I} \in A\right),$$

and $\ell^1_{\cdot,\cdot}$ is an intrinsic location functional, since $L^2_v(\cdot,\cdot)$ is ranked. $\square$

Theorem 5.14: An intrinsic location vector $L^2_v(\cdot,\cdot) = \begin{bmatrix} \ell^1_{\cdot,\cdot} \\ \ell^2_{\cdot,\cdot} \end{bmatrix}$ is free if and only if both $\ell^1_{\cdot,\cdot}$ and $\ell^2_{\cdot,\cdot}$ are intrinsic location functionals in the sense of Definition 2.1.

Proof: Part of the 'if' direction was done in Theorem 5.4: we showed that this is indeed an intrinsic location vector. To show it is free, we note that because $\ell^j_{\cdot,\cdot}$ are both intrinsic location functionals, for both $j = 1,2$ we have that for every $f \in \mathcal{H}$, whenever we have $I_1, I_2 \in \mathcal{I}$ with $I_2 \subseteq I_1$ such that $\ell^j_{f,I_1} \in I_2$, we get that $\ell^j_{f,I_1} = \ell^j_{f,I_2}$. Hence $L^2_v(X,\cdot)$ is clearly free.

For the 'only if' direction, we check the conditions required for the intrinsic location functional for $\ell^1_{\cdot,\cdot}$, and the proof of $\ell^2_{\cdot,\cdot}$ will be analogous since the definition of the free intrinsic location vector is symmetric for the two locations.

It is clear that $\ell^1_{\cdot,\cdot}$ satisfies measurability and shift compatibility by construction, and because $L^2_v(\cdot,\cdot)$ is free, stability under restrictions is also clear.

It only remains to show that consistency of existence holds. To this end, let $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$. We need to show that $\ell^1_{f,I_2} \neq \infty$ implies $\ell^1_{f,I_1} \neq \infty$.

Towards a contradiction, assume $\ell^1_{f,I_1} = \infty$. Then by definition 5.10 (2.ii) we get that $\ell^1_{f,I_2} = \infty$, a contradiction.

Therefore $\ell^1_{\cdot,\cdot}$ is an intrinsic location functional in the sense of Definition 2.1. □

Remark 5.15: We note that these subclasses are not exhaustive of all possible intrinsic location vectors. For example, we could define $L^2_v(f, I)$ on $I = [a, b]$ as:

$$L^2_v(f, I) := \begin{bmatrix} \ell^1_{f,I} \\ \ell^2_{f,I} \end{bmatrix},$$

$$\text{where } \ell^1_{f,I} = \begin{cases} b & \text{if } |b-a| \neq 1 \\ a & \text{if } |b-a| = 1 \end{cases},$$

$$\text{and } \ell^2_{f,I} = \begin{cases} b & \text{if } |b-a| = 1 \\ a & \text{if } |b-a| \neq 1 \end{cases},$$

which is clearly an intrinsic location vector. To see this, note that it trivially satisfies measurability and shift compatibility.

For inclusion under restriction, let $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$, then the only time when $\left\{\ell^1_{f,I}, \ell^2_{f,I}\right\} \cap I_2 \neq \emptyset$ is when for $I_1 = [a, b], I_2 = [c, d]$ we have $a = c$ or $b = d$. Without loss of generality, we show the $a = c$ case. If $I_1 = [a, b]$ and $I_2 = [a, d]$, then there are three possibilities:

(1) If $|b - a| = 1$, then

$$\left\{\ell^1_{f,I_1}, \ell^2_{f,I_1}\right\} \cap [a, d] = \{a, b\} \cap [a, d] = \{a\} \in \{d, a\} = \left\{\ell^1_{f,I_2}, \ell^2_{f,I_2}\right\}.$$

(2) If $|d - a| = 1$, then

$$\left\{\ell^1_{f,I_1}, \ell^2_{f,I_1}\right\} \cap [a, d] = \{b, a\} \cap [a, d] = \{a\} \in \{a, d\} = \left\{\ell^1_{f,I_2}, \ell^2_{f,I_2}\right\}.$$

(3) If neither $|b - a| = 1$ or $|d - a| = 1$, then

$$\left\{\ell^1_{f,I_1}, \ell^2_{f,I_1}\right\} \cap [a, d] = \{b, a\} \cap [a, d] = \{a\} \in \{d, a\} = \left\{\ell^1_{f,I_2}, \ell^2_{f,I_2}\right\}.$$

Therefore $L_v^2(\cdot,\cdot)$ satisfies inclusion under restriction.

Lastly, we must show that $L_v^2(\cdot,\cdot)$ satisfies consistency of existence. However, this is trivial since both locations always exist.

Hence $L_v^2(\cdot,\cdot)$ is indeed an intrinsic location vector. *However, it is neither ranked, nor free.* To show this, we give a simple counter-example where the definitions of ranked and free intrinsic location vectors are both broken.

Let $I_1 = [a,b], b > a$ and $I_2 = [c,b], a < c < b$, with $|b-c| = 1$. Therefore $\ell^1_{\cdot,I_1} \in I_2$, hence we should have that $\ell^1_{\cdot,I_1} = \ell^1_{\cdot,I_2}$ for every $f \in \mathcal{H}$ in the definitions of ranked and free intrinsic location vectors. But this is not the case:

$$L_v^2(\cdot, I_1) = \begin{bmatrix} b \\ a \end{bmatrix}, \quad \text{and } L_v^2(\cdot, I_2) = \begin{bmatrix} c \\ b \end{bmatrix}.$$

See Figure (5.2).

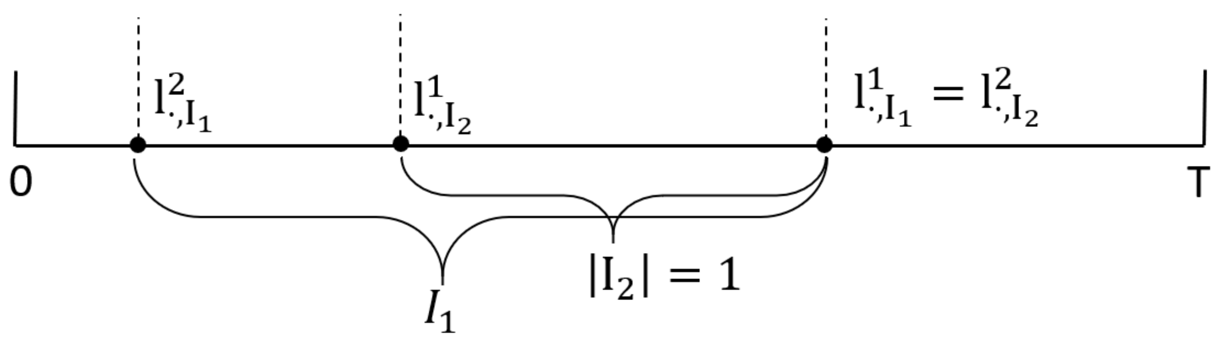

Figure 5.2: Illustration of the counterexample in Remark 5.15.

Chapter 6

Generalization to $n > 2$ locations

We have so far only considered the case of $n = 2$, but analysis can be generalized for the $n > 2$ (non-vector) locations case only by changing the definitions slightly:

(1) We change Γ_I^2 to Γ_I^n as follows:

$$\Gamma_I^n = \{\text{ sets of cardinality at most } n\text{, with elements taken from } I\}.$$

(2) Each $\xi \in \Gamma_I^n$ can be described as a point measure:

$$m_\xi = \sum_{x \in \xi} \mathbb{I}_x,$$
$$\text{where } \mathbb{I}_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

We can again equip Γ_I^n with the σ-algebra $\mathcal{M}^n$ defined as the smallest σ-algebra which contains sets of the form

$$\{m \in M_p(\Gamma_I^n) \mid m(A) \in B\} \text{ for } A \in \mathcal{B}(I), B \in \mathcal{B}([0,n]),$$

where $M_p(\Gamma_I^n)$ is the space of all point measures defined by Γ_I^n.

(3) We keep the same $\mathcal{H}$ and $\mathcal{I}$ as previously defined.

(4) A mapping $L^n : \mathcal{H} \times \mathcal{I} \to \Gamma_{\mathbb{R}}^n$ is called an *intrinsic multiple-location functional of degree* n if it satisfies all of the following conditions:

(i) For every $I \in \mathcal{I}$ the map $L^n(\cdot, I) : \mathcal{H} \to \Gamma_I^n$ is $\text{Cyl}(\mathcal{H})/\mathcal{M}^n$-measurable.

(ii) *(Shift Compatibility)* For every $f \in \mathcal{H}, I \in \mathcal{I}, c \in \mathbb{R}$,

$$L^n(f, I) = L^n(\theta_c \circ f, I - c) + c,$$

where $\xi \pm c = \{x \pm c \mid x \in \xi\}$.

(iii) *(Inclusion under restriction)* For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$L^n(f, I_1) \cap I_2 \subseteq L^n(f, I_2).$$

(iv) *(Consistency of existence)* For every $f \in \mathcal{H}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$,

$$\big|L^n(f, I_1)\big| \geq \big|L^n(f, I_2)\big|,$$

where $|\cdot|$ represents the number of elements in the set.

(5) We again look only at $F^\star_{X,I}$ because the set-function $F_{X,I}(A) = \mathbb{P}\left(L^n(f, I) \cap A \neq \emptyset\right)$ can be expanded using the inclusion-exclusion principle, where the ordering of the $x_i \in L^n(f, I)$ is again the order-statistics. Without loss of generality, we present the explanation for case where $\big|L^n(f, I)\big| = n$:

$$\begin{aligned} F_{X,I}(A) &= \mathbb{P}\left(\{x_1, x_2, \dots, x_n\} \cap A \neq \emptyset\right) \\ &= \underbrace{\sum_{i=1}^{n} \mathbb{P}\left(x_i \in A\right)}_{(\star)} - \sum_{1 \le i_1 < i_2 \le n} \mathbb{P}\left(\{x_{i_1} \in A\} \cap \{x_{i_2} \in A\}\right) \\ &\quad + \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbb{P}\left(\{x_{i_1} \in A\} \cap \{x_{i_2} \in A\} \cap \{x_{i_3} \in A\}\right) \\ &\quad + (-1)^n \sum_{1 \le i_1 < i_2 < \dots < i_{n-1} \le n} \mathbb{P}\left(\{x_{i_1} \in A\} \cap \ldots \cap \left\{x_{i_{n-1}} \in A\right\}\right) \\ &\quad + (-1)^{n+1} \mathbb{P}\left(\{x_{i_1} \in A\} \cap \ldots \cap \{x_{i_n} \in A\}\right). \end{aligned} \tag{6.1}$$

We work again with $F^\star_{X,T}$ directly in the form

$$F^\star_{X,I}(A) = \mathbb{E}\Big[\big|L^n(X, I) \cap A\big|\Big],$$

which again can be interpreted as the expected number of random locations in A. One can clearly see from these new definitions that the proofs will be analogous to the $n = 2$ case, with minor changes such as $f^n_{X,T}(t) \leq \max\left(\frac{n}{t}, \frac{n}{T-t}\right)$ which can be easily shown.

(6) In order to show that the right-derivative 'density' of the capacity functional $f_{X,I}$ is again equal to the right-derivative density of $F^{\star}_{X,I}$, denoted $f^{\star}_{X,I}$, we impose an assumption similar to Assumption M, and argue that the sum in (6.1) can be drastically simplified.

Assumption $\mathbf{M}_{\boldsymbol{n}}$: For every $i = 1, \dots, n-1$, the functions

$$g^i_\epsilon(t) = \mathbb{P}\left(x_{i+1} \in (t, t+\epsilon] \mid x_i = t\right)$$

converge to 0 uniformly on $t \in (0, T)$. More precisely, $g^i_\epsilon(t) \leq C^i(\epsilon)$ for every $t \in (0, T)$, where $C^i(\epsilon)$ is a function for each $i = 1, \dots, n-1$ such that $\lim_{\epsilon \downarrow 0} C^i(\epsilon) = 0$.

We note that by Bonferroni's inequalities,

$$\begin{aligned}
&\sum_{i=1}^{n} \mathbb{P}\left(x_i \in (t, t+\epsilon]\right) - \sum_{1 \leq i_1 < i_2 \leq n} \mathbb{P}\left(\{x_{i_1} \in (t, t+\epsilon]\} \cap \{x_{i_2} \in (t, t+\epsilon]\}\right) \\
&\leq F_{X,I}((t, t+\epsilon]) \\
&\leq \sum_{i=1}^{n} \mathbb{P}\left(x_i \in (t, t+\epsilon]\right) = F^{\star}_{X,I}\left((t, t+\epsilon]\right).
\end{aligned}$$

Hence we only need to show that

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \sum_{1 \leq i_1 < i_2 \leq n} \mathbb{P}\left(\{x_{i_1} \in (t, t+\epsilon]\} \cap \{x_{i_2} \in (t, t+\epsilon]\}\right) = 0$$

in order to conclude that $f_{X,I} = f^{\star}_{X,I}$. This clearly follows from Assumption M_n by essentially the same argument as in Section 4.2, with the additional fact that

$$\mathbb{P}\left(\{x_{i_1} \in (t, t+\epsilon]\} \cap \{x_{i_2} \in (t, t+\epsilon]\}\right) \leq \mathbb{P}\left(\{x_{i_1} \in (t, t+\epsilon]\} \cap \{x_{i_1+1} \in (t, t+\epsilon]\}\right).$$

Chapter 7

Further Research

First and foremost, Assumptions M and M_n could perhaps be proven as properties which simply follow from the definitions of intrinsic multiple-location functionals of degree $n = 2$ and $n > 2$ respectively.

The marginal density of $\ell^2_{\cdot,\cdot}$, denoted $f_2(t)$, in the ranked intrinsic location vector would also be of particular interest. One could attempt to show similar results to Theorem 4.2, such as $f_2(t) \leq c(t)$ for some function $c(\cdot)$ on the interior of $(0, T)$, as well as some bounds on the total variation and positive/negative variation at the end-points of the interval.

There are multiple ways in which one could expand on the general framework found in this work. The intrinsic location vectors of degree $n \geq 2$ could be explored. Their joint distributions would likely be of importance and interest. One could also attempt to expand the framework to encompass not only stationary processes, but stationary fields, for both the intrinsic multiple-location functionals and intrinsic location vectors.

An application in the field of risk management may be possible. In risk management, we are often concerned with extreme values. In particular, in the Peaks-Over-Threshold method (see [7]), we are concerned with losses (L_t) over a time period $[a, b]$ which exceed a given level $u > 0$. In current practice, these excess losses are used to calculate risk measures such as Value-at-Risk and Expected Shortfall, by approximating the distribution of $(L - u \mid L > u)$ with the Generalized Pareto Distribution.

If one had closed-form and tractable results for the distributions of intrinsic multiple-location functionals, or preferably the intrinsic location vectors, it may be possible to make a link to this theory.

For example, with $m > 0$ given data points representing losses $(L_{t_1}, \ldots, L_{t_m})$, one could “smooth out” the data by some method such as interpolation, and fit the data with a stationary process.

Then one could examine $L_v^k(f,[a,b])$ for $[a,b] \supseteq \{t_1,\dots,t_m\}$ and $k = 1,\dots,n$, with the random locations of $L_v^k(f,[a,b])$ being the k successive locations of exceedances of the level $u > 0$, up to n locations. A possible choice of n could be

$$n \simeq \Big| \big\{ i \in \{1,\dots,m\} \ \big| \ \mathbb{P}(\ell^i_{f,[a,b]} \text{ exists }) > \psi \in (0,1) \big\} \Big|,$$

with ψ being a “sensitivity” parameter. Then one could use the intrinsic location vectors $L_v^k(f,[0,T]), k = 1,\dots,n$, to understand the structure of dependence between the times of exceedances. If it were possible to have tractable results about the joint distribution of the intrinsic location vectors, one could then approximate the arrival time of the next exceedance, while also taking into account the dependency structure arising from the locations of past exceedances up to the current time.

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