On the Extrema of Functions in the Takagi Class

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The Takagi class is a class of fractal functions on the unit interval generalizing the celebrated Takagi function. In this thesis, we study the extrema of these functions. This is a problem that goes back to J.-P. Kahane in [12]. In this thesis, we state and prove the following new and original results on this long-standing problem. We characterize the set of all extrema of a given function in the Takagi class by means of a step condition on their binary expansions. This step condition allows us to compute the extrema and their locations for a large class of explicit examples and to deduce a number of qualitative properties of the sets of extreme points. Particularly strong results are obtained for functions in the so-called exponential Takagi class. We show that the exponential Takagi function with parameter $\nu \in (0, 1)$ has exactly two maximizers if 2ν is not the root of a Littlewood polynomial. On the other hand, we show that there exist Littlewood polynomials such that, if 2ν is a corresponding root in $(0, 1)$, the set of maximizers is a Cantor-type set with Hausdorff dimension $1/n$, where n is the degree of the polynomial. Furthermore, if ν is in $(-1, -0, 5)$, the location of the maximum is a nontrivial step function with countably many jumps. Finally, we showed that, if ν is in $(-1, -0.8)$, the minima will only attain at $t = 0.2$ and $t = 0.8$. If ν is in $(-0.8, 1)$, the only minimizer is at $t = 0.5$.

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To my family.

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List of Symbols

The next list describes several symbols that will be frequently used within the body of the thesis.

$\mathbb{N}$	The natural numbers
$\mathbb{R}$	The real numbers
$\mathbb{Z}$	The integers
$\mathbb{Q}$	The rational numbers
$\mathbb{C}$	The complex numbers
$\mathbb{T}$	The dyadic rationals
$\mathbb{T}_n$	The dyadic rationals of order n
$\mathfrak{C}$	The Takagi Class
$\mathfrak{P}$	The exponential Takagi Class
$x_{\mathbf{C}}$	The Takagi function over sequence $\mathbf{C}$
x_ν	The exponential Takagi function with parameter ν
$x_{\mathbf{C},n}$	The n^{th} order truncated Takagi function over sequence $\mathbf{C}$
$x_{\nu,n}$	The n^{th} order truncated exponential Takagi function with parameter ν
$\mathbb{Z}[x]$	The polynomial ring over integers
$\mathbb{Q}[x]$	The polynomial ring over rationals

$\mathcal{M}_{\mathbf{C}}$ The set of all maximizers of the Takagi function $x_{\mathbf{C}}$

$\tilde{\mathcal{M}}_{\mathbf{C}}$ The set of all minimizers of the Takagi function $x_{\mathbf{C}}$

$\mathcal{M}_{\mathbf{C},n}$ The set of all maximizers of the truncated Takagi function $x_{\mathbf{C},n}$

$\tilde{\mathcal{M}}_{\mathbf{C},n}$ The set of all minimizers of the truncated Takagi function $x_{\mathbf{C},n}$

$\mathcal{M}_{\nu}$ The set of all maximizers of the exponential Takagi function x_{ν}

$\tilde{\mathcal{M}}_{\nu}$ The set of all minimizers of the exponential Takagi function x_{ν}

$\mathcal{M}_{\nu,n}$ The set of all maximizers of the truncated exponential Takagi function $x_{\nu,n}$

$\tilde{\mathcal{M}}_{\nu,n}$ The set of all minimizers of the truncated exponential Takagi function $x_{\nu,n}$

$f_{\sharp}$ The mapping from parameter ν to the right-most point in $\mathcal{M}_{\nu} \cap [0, \frac{1}{2}]$

$f_{\flat}$ The mapping from parameter ν to the left-most point in $\mathcal{M}_{\nu} \cap [0, \frac{1}{2}]$

$g_{\sharp}$ The mapping from parameter ν to the right-most point in $\tilde{\mathcal{M}}_{\nu} \cap [0, \frac{1}{2}]$

$g_{\flat}$ The mapping from parameter ν to the left-most point in $\tilde{\mathcal{M}}_{\nu} \cap [0, \frac{1}{2}]$

Chapter 1

Introduction

Recent research in finance and probability requires pathwise integration theorems for integrators of various degrees of 'roughness', see e.g. [6, 7, 10]. As observed in [15], a class of generalized Takagi functions can serve this propose.

1.1 Early History

The Takagi function $x(t)$ was first introduced in 1903 by Takagi [19]. His goal was to provide an example of a continuous but nowhere differentiable function on the unit interval $[0, 1]$. The Takagi function has been discovered many times. Overviews can be found in the surveys by Allaart and Kawamura [1] and Lagarias [14]. In modern mathematical notation, the Takagi function is defined as follows:

Definition 1.1.1. The Takagi function $x : [0, 1] \to \mathbb{R}$ is defined as

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n t), \tag{1.1}$$

where $\phi(t) = \min_{z \in \mathbb{Z}} |t - z|$ is the distance from $t \in \mathbb{R}$ to its nearest integer.

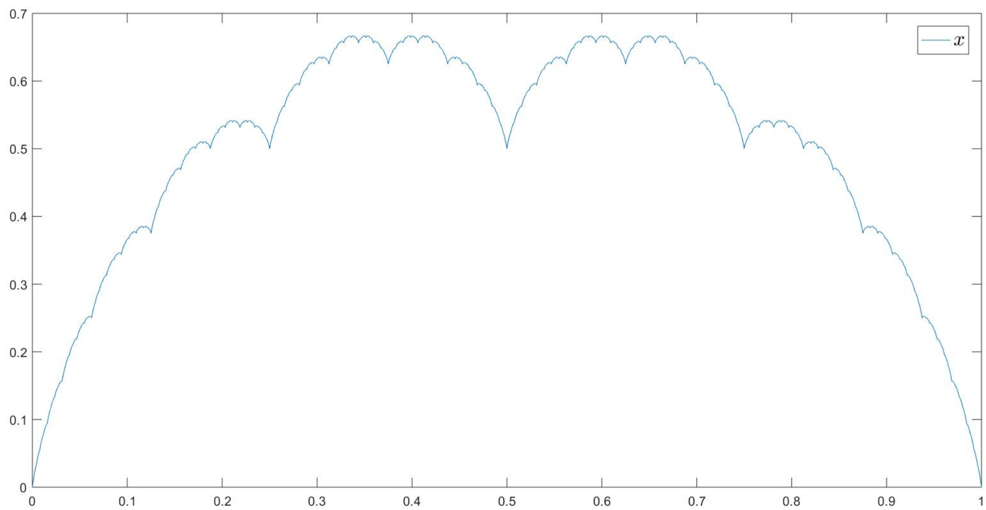

Figure 1.1: Plot of the classic Takagi function

Later, Hata and Yamaguti [11] gave a generalization of the Takagi function by replacing the coefficient $\frac{1}{2^n}$ in (1.1) with a general constant c_n. Such a collection of functions is often referred as the Takagi class $\mathfrak{C}$. Similar class were also introduced by Kahane [12]. A formal definition for the Takagi class can be given as follows:

Definition 1.1.2. The Takagi class $\mathfrak{C}$ is the collection of functions $x : [0, 1] \longrightarrow \mathbb{R}$ that can be represented as:

$$x(t) = \sum_{n=0}^{\infty} c_n \phi(2^n t), \tag{1.2}$$

where $\mathbf{C} = \{c_n\}_{n=0}^{\infty}$ is a sequence of real numbers for which x is well-defined and continuous. In order to specify the Takagi function associated with the sequence $\mathbf{C}$, we denote this function with $x_{\mathbf{C}}(t)$.

One may notice that the domain of functions in form of (1.2) can be extended from $[0, 1]$ to $\mathbb{R}$. However, it is sufficient to study the restriction of the function $x_{\mathbf{C}}$ to the unit interval $[0, 1]$, because $x_{\mathbf{C}}(t + 1) = x_{\mathbf{C}}(t)$ for all $t \in \mathbb{R}$.

In order to give a sufficient condition for the convergence of the series defined in (1.2), we introduce following definitions and theorem.

Definition 1.1.3. Let us denote the k^{th} order truncated Takagi function over an infinite sequence $\mathbf{C} = \{c_i\}_{i=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ by $x_{\mathbf{C},k}(t)$, which is written as

$$x_{\mathbf{C},k}(t) = \sum_{m=0}^{k} c_i \phi(2^m t), \quad t \in [0,1]. \tag{1.3}$$

Theorem 1.1.4. The series $x_{\mathbf{C}}(t) = \sum_{n=0}^{\infty} c_n \phi(2^n t)$ over the sequence $\mathbf{C}$ is well-defined if

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} |c_k| = 0.$$

Proof. We now write $\|\cdot\|$ for the usual sup norm on $C[0,1]$. First of all, let us prove the series $\{x_{\mathbf{C},n}\}$ is a Cauchy sequence in $C[0,1]$. Now, let us consider

$$\begin{aligned} \|x_{\mathbf{C},n} - x_{\mathbf{C},m}\| = \sup_{t\in[0,1]} |x_{\mathbf{C},n}(t) - x_{\mathbf{C},m}(t)| &= \sup_{t\in[0,1]} |\sum_{i=n+1}^{m} c_i \phi(2^i t)| \\ &\leq \frac{\sum_{i=n+1}^{m} |c_i|}{2} \leq \frac{\sum_{i=n+1}^{\infty} |c_i|}{2} \end{aligned}$$

As $\lim\limits_{n\to\infty} \sum_{k=n}^{\infty} |c_k| = 0$, hence for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n > N$,

$$\sum_{k=n}^{m} |c_k| \leq \sum_{k=n}^{\infty} |c_k| < \epsilon$$

Therefore, $\{x_{\mathbf{C},n}\}$ is a Cauchy sequence in $C[0,1]$. Hence, $x_{\mathbf{C},n}$ converges uniformly on $[0,1]$ to the function $x_{\mathbf{C}}$. □

Furthermore, Kono [13] characterized the differentiability of the Takagi class as follows:

Theorem 1.1.5. Let $x_{\mathbf{C}}$ be defined as (1.2), and let $a_n = 2^n c_n$.

1. If $\{a_n\} \in \ell^2$, then $x_{\mathbf{C}}$ is absolutely continuous and hence differentiable almost everywhere.

2. If $\{a_n\} \notin \ell^2$ but $\lim_{n\to\infty} a_n = 0$, then $x_{\mathbf{C}}$ is non-differentiable at almost every point of [0,1], but $x_{\mathbf{C}}$ is differentiable on an uncountably large set, and the range of the derivative $x'_{\mathbf{C}}$ is $\mathbb{R}$.

3. If $\limsup_{n\to\infty} |a_n| > 0$, then $x_{\mathbf{C}}$ is nowhere differentiable.

An important sub-class in the Takagi class is obtained by taking $c_n = \nu^n$ for some $\nu \in (-1, 1)$. Following Galkin and Galkina [9], we call this the *exponential Takagi class*.

Definition 1.1.6. The *exponential Takagi class* $\mathfrak{P}$ is the sub-collection of real-valued functions $x_\nu : [0, 1] \to \mathbb{R}$ in the Takagi class $\mathfrak{C}$, where x_ν can be written as

$$x_\nu(t) = \sum_{n=0}^{\infty} \nu^n \phi(2^n t). \tag{1.4}$$

The function x_ν is called the *exponential Takagi function* with parameter ν.

Moreover, Galkin and Galkina [9] gave results on the differentiability of the exponential Takagi class.

Theorem 1.1.7. Let x_ν be defined as (1.4), then

1. If $|\nu| < 1$, then the series defined in (1.4) converges uniformly in $t \in \mathbb{R}$, therefore x_ν is continuous and $|x_\nu| \leq \frac{1}{2-2|\nu|}$.

2. If $|\nu| \geq 1$, the series defined in (1.4) converges if and only if $t \in \mathbb{T}$. Furthermore, the function x_ν is discontinuous on set $\mathbb{T}$.

Proof. See Galkin and Galkina [9]. □

One may notice that Theorem 1.1.7 gives us a reason to only study the extrema of exponential Takagi function with a nature restriction for ν in $(-1, 1)$. For the following chapters, the author may directly apply this restriction on parameter ν without further notice. Furthermore, readers may see that if the parameter $\nu = \frac{1}{2}$, $x_\nu(t)$ is the classic Takagi function $x(t)$.

1.2 Previous Results

One of many important aspects of the Takagi function is the collection of extreme points for the Takagi function. Kahane [12] pioneered this research with the following theorem:

Theorem 1.2.1. The maximum value of the classical Takagi function x is $\frac{2}{3}$. Then set of maximizers is a perfect set of Hausdorff dimension $\frac{1}{2}$, and consists of all the points t with binary expansion $t = 0.\varepsilon_0\varepsilon_1\varepsilon_2\cdots$ satisfying $\varepsilon_{2n} + \varepsilon_{2n+1} = 1$ for each n.

Schied [17] and Galkin & Galkina [9] independently characterized the maximizers of the exponential Takagi function for $\nu = \frac{1}{2}$.

Theorem 1.2.2. For $\nu = \frac{1}{2}$, the maximum value of the exponential Takagi function $x_{\frac{1}{2}}$ is attained at $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$ with maximum value $\frac{2}{3}$.

Later, Mishura & Schied [15] extended this result into a larger collection in the exponential Takagi class for $\nu \in [\frac{1}{2}, 1)$.

Theorem 1.2.3. For $\nu \in [\frac{1}{2}, 1)$ the maximum value of the exponential Takagi function x_ν is attained at $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$ with maximum value $\frac{1}{3(1-\nu)}$.

For the case $\nu \in [-\frac{1}{2}, \frac{1}{4}]$, Galkin and Galkina [9] gave a conclusion as follows:

Theorem 1.2.4. The maximum value of the Exponential Takagi function x_ν, $\nu \in [-\frac{1}{2}, \frac{1}{4}]$ is attained at $t = \frac{1}{2}$ with maximum value of $\frac{1}{2}$.

In view of Theorem 1.2.1, 1.2.3, 1.2.4., it remains to analyze the maxima of x_ν for $\nu \in (\frac{1}{4}, \frac{1}{2}) \cup (-1, -\frac{1}{2})$. Tabor & Tabor [18] gave an approximation solution for the maximal value of the exponential Takagi function for certain numbers ν. Furthermore, Baba [2] characterized the maxima for generalized Takagi functions x that replace $\phi(2^n t)$ by $\phi(b^n t)$ for some $b \in \mathbb{Z}$. Besides, Fujita & Saito [8] studied an even broader class whose ϕ function could be any periodic and continuous function. From Figure 1.2, we may see that the aforementioned theorems are only able to characterize the maxima of those functions in the exponential Takagi class for which the maximum location is flat as a function of ν. On the other hand, we can specially observe that for $\nu \in (-1, -\frac{1}{2})$, the location of maximum is a nontrivial step function of ν. Furthermore, for $\nu \in (\frac{1}{4}, \frac{1}{2})$, the change of maximizers associated with ν is even more difficult to characterize and it has a fractal-like structure. Most of theorems introduced in those earlier papers are based on an induction argument for the truncated exponential Takagi functions. However, this method cannot be applied for the functions in the Takagi class, as the nature of arbitrary coefficients is not feasible for induction. Last but not least, as far as the author is aware of, only little focus has been put on the minima of the functions in the Takagi Class so far.

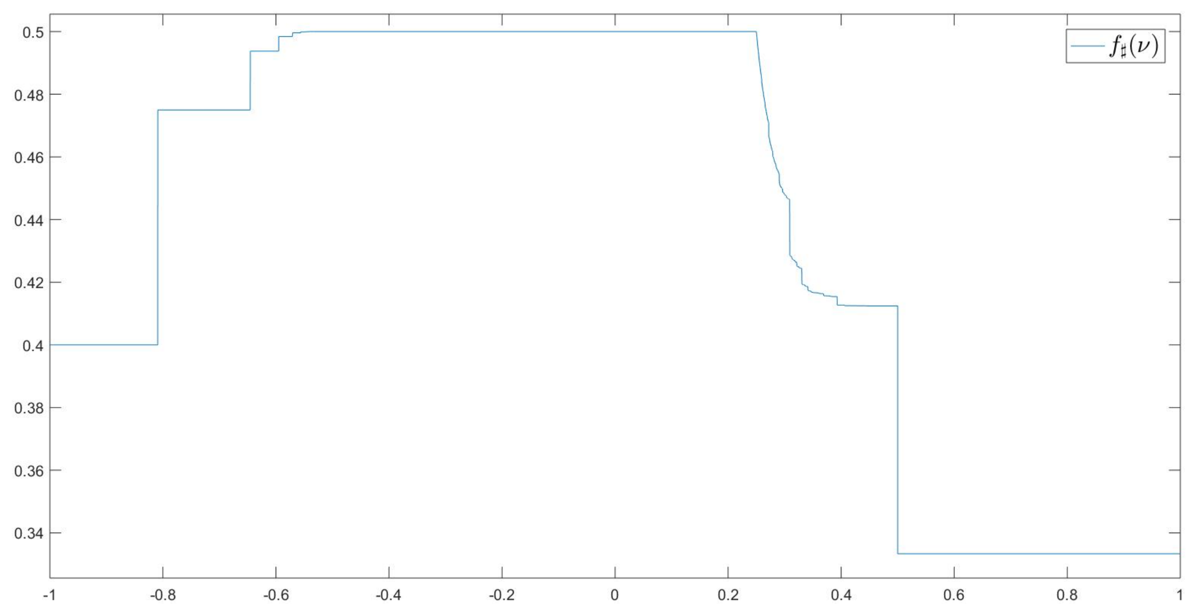

Figure 1.2: The plot for upper maxima $f_\sharp(\nu)$ for exponential Takagi function x_ν

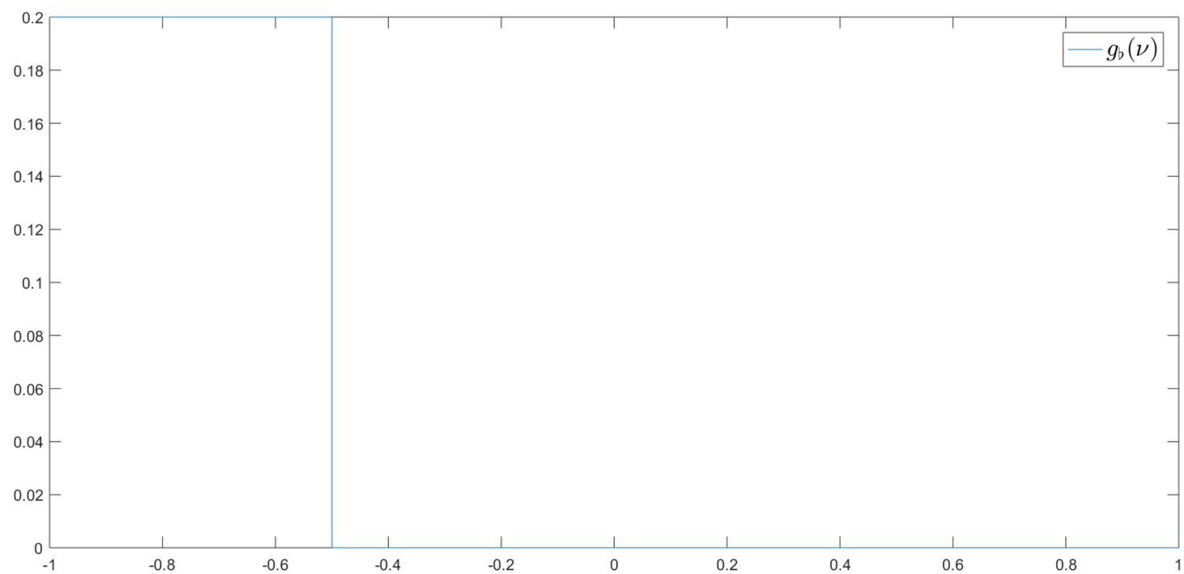

Figure 1.3: The plot for upper minima $g_\sharp(\nu)$ for exponential Takagi function x_ν

Chapter 2

Preliminaries

2.1 Facts and Fundamentals

In this section, we will introduce and partially prove some theorems and lemmas that will be applied in the proofs in Chapter 3 and Chapter 4. Those theorems are very introductory, and the author believes those mathematical statements have appeared in many references. Some proofs are given here for the sake of completeness and rigorousness of this thesis. The author does not own any credit to these result.

Theorem 2.1.1. Let $\{K_\alpha\}$ be a collection of compact sets. If the intersection of every finite sub-collection of $\{K_\alpha\}$ is non-empty, then $\bigcap_\alpha K_\alpha$ is non-empty.

Proof. We will prove this theorem by contradiction. Let us now assume that $\bigcap_\alpha K_\alpha = \emptyset$. Therefore, we must have

$$K_1 \cap (\bigcap_{\alpha \neq 1} K_\alpha) = \emptyset.$$

Furthermore, as K_α^c is an open set, we have

$$K_1 \subset \bigcup_{\alpha \neq 1} K_\alpha^c.$$

Therefore $\{K_\alpha^c\}_{\alpha \neq 1}$ is an open cover for K_1. Therefore, there exists a finite sub-cover $\{K_{\alpha_1}^c, K_{\alpha_2}^c \dots K_{\alpha_n}^c\}$ covers K_1. Therefore, we must have

$$K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \cap \quad \dots \quad \cap K_{\alpha_n} = \emptyset. \tag{2.1}$$

Hence, (2.1) results in a contradiction here. Therefore, we have

$$\bigcap_\alpha K_\alpha$$

□

Lemma 2.1.2. If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n$, then $\bigcap_{n=1}^\infty K_n$ is non-empty.

Proof. This Lemma directly comes as a corollary of Theorem 2.1.1. □

Definition 2.1.3. A polynomial $f(z) = \sum_{i=0}^n a_i z^i$ with $a_i \in \{-1, 1\}$ and $n \in \mathbb{N}$ is called a *Littlewood polynomial*. We denote $\mathcal{F}_n$ as the set of all Littlewood polynomials with order n:

$$\mathcal{F}_n = \Big\{ f_n(z) = \sum_{i=0}^n a_i z^i | a_i \in \{-1, 1\} \Big\}.$$

Moreover, let $\mathcal{F}$ be the set containing all Littlewood polynomials,

$$\mathcal{F} = \bigcup_n \mathcal{F}_n.$$

Moreover, let us denote $\mathcal{D}_\mathbb{C}$ as all the complex roots for Littlewood polynomials.

$$\mathcal{D}_\mathbb{C} = \{z \in \mathbb{C} | \exists n \in \mathbb{N}, f_n(z) = 0, \text{ for some } f_n \in \mathcal{F}_n\}. \tag{2.2}$$

And let $\mathcal{D}_\mathbb{R}$ be the set of all real roots for Littlewood polynomials, i.e.

$$\mathcal{D}_\mathbb{R} = \mathcal{D}_\mathbb{C} \cap \mathbb{R}. \tag{2.3}$$

Lemma 2.1.4. The set $\mathcal{D}_{\mathbb{C}}$ is contained in the annulus $\frac{1}{2} < |z| < 2$.

Proof. Please see [3]. □

Theorem 2.1.5 (Gauß Theorem)**.** If a non-constant polynomial $p(x) \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$, then it is also irreducible over $\mathbb{Q}$.

Lemma 2.1.6. $\mathcal{D}_{\mathbb{C}} \cap \mathbb{Q} = \mathcal{D}_{\mathbb{R}} \cap \mathbb{Q} = \{-1, 1\}$.

Proof. Let $r \in \mathbb{Q}$ be a root for $p(x)$, then we have the monic polynomial $x - r|p(x)$. By applying Gauß Theorem, we have $x - r \in \mathbb{Z}[x]$. Hence we have $r \in \mathbb{Z}$. Then, by applying Lemma 2.1.4, we have $r \in \{-1, 1\}$. □

Lemma 2.1.7. Let $\mathcal{C}$ be the Cantor set, and let $\Omega = \{w_i\}_{i=0}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}}$ be the ternary expansion of $y \in [0, 1]$:

$$y = \sum_{i=0}^{\infty} \omega_i 3^{-(i+1)} = 0.\omega_1\omega_2 \cdots .$$

Then $y \in \mathcal{C}$ if and only if there exists Ω, such that $\omega_i \in \{0, 2\}$ for all $i \in \mathbb{N}$, and $y = \sum_{i=0}^{\infty} \omega_i 3^{-(i+1)}$.

Definition 2.1.8. A binary expansion of a point $y \in [0, 1]$ is denoted by

$$y = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{2^{n+1}} = 0.\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3......$$

And we define y_n as the n^{th} order dyadic approximation for y, where

$$y_n = \sum_{i=0}^{n} \frac{\varepsilon_i}{2^i} = 0.\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3....\varepsilon_n.$$

It is well-known that the binary expansion is not unique for any real number $y \in \mathbb{T}$. For example, $0.1 = 0.0111 \cdots$. Here, we do not require the uniqueness of the binary expansion. Furthermore, in order to well distinguish the two binary expansions, we would formally define mappings $S_\sharp : [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$ and $S_\flat : [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$. First of all, let us define a mapping $s_\sharp : [0,1] \longrightarrow [0,1]$ as:

$$s_\sharp(y) = \begin{cases} 2y & \text{if} \quad y \in [0, \frac{1}{2}], \\ 2y - 1 & \text{if} \quad y \in (\frac{1}{2}, 1]. \end{cases} \tag{2.4}$$

Similarly, we define $s_\flat : [0,1] \longrightarrow [0,1]$ as:

$$s_\flat(y) = \begin{cases} 2y & \text{if} \quad y \in [0, \frac{1}{2}), \\ 2y - 1 & \text{if} \quad y \in [\frac{1}{2}, 1]. \end{cases} \tag{2.5}$$

Furthermore, the mapping $d_\sharp : [0,1] \longrightarrow \{0,1\}$ is defined as:

$$d_\sharp(y) = \begin{cases} 0 & \text{if} \quad y \in [0, \frac{1}{2}), \\ 1 & \text{if} \quad y \in [\frac{1}{2}, 1]. \end{cases} \tag{2.6}$$

And the mapping $d_\flat : [0,1] \longrightarrow \{0,1\}$ is defined as:

$$d_\flat(y) = \begin{cases} 0 & \text{if} \quad y \in [0, \frac{1}{2}], \\ 1 & \text{if} \quad y \in (\frac{1}{2}, 1]. \end{cases} \tag{2.7}$$

Then the mapping $S_\sharp$ is defined as:

$$\varepsilon_i = d_\sharp(s_\sharp^i(y)), \qquad S_\sharp(y) = \{\varepsilon_i\},$$

where $s_\sharp^i = \underbrace{s_\sharp \circ s_\sharp \cdots \circ s_\sharp}_{i \text{ times}}$. Similarly, the mapping $S_\flat$ is defined as:

$$\varepsilon_i = d_\flat(s_\flat^i(y)), \qquad S_\flat(y) = \{\varepsilon_i\}.$$

For simplicity, we refer to $S_\flat(y)$ as the lower dyadic expansion for y and to $S_\sharp(y)$ as the upper dyadic expansion for y.

2.2 Frequently Used Notation

In this section we will collect some notations that we will frequently used throughout the thesis.

Definition 2.2.1. The dyadic partition $\mathbb{T}_n$ in $[0,1]$ is defined as

$$\mathbb{T}_n := \{k2^{-n} | n \in \mathbb{N}, k = 0, 1, \dots, 2^n\}, \quad \text{for} \quad n \in \mathbb{N}.$$

Let $\mathbb{T}$ be the set of all dyadic rationals in $[0,1]$. Then we have

$$\mathbb{T} = \bigcup_{n=0}^{\infty} \mathbb{T}_n.$$

Lemma 2.2.2. For $y \notin \mathbb{T}$, we have

$$S_\sharp(y) = S_\flat(y).$$

Remark 2.2.3. For any real number that is not in the dyadic partition $\mathbb{T}$, the binary expansion is unique.

Definition 2.2.4. Let $\mathbb{S} = \{-1, 1\}$, then let $\mathbb{S}^{\mathbb{N}}$ be the collection of all infinite sequence space over $\mathbb{S}$, such a set can be represented as

$$\mathbb{S}^{\mathbb{N}} = \{(c_i)_{i=1}^{\infty} | c_i \in \mathbb{S}\}.$$

Furthermore, we may mimic some notations from abstract algebra. Let us denote $\mathbb{S}[x]$ as the collection of all polynomials $p(x)$ whose the coefficients are in $\mathbb{S}$. Similarly, we denote $\mathbb{S}[[x]]$ as the collection of all power series whose coefficients are in $\mathbb{S}$.

Definition 2.2.5. Let us denote the set of all maximizers for the Takagi function $x_{\mathbf{C}}$ as $\mathcal{M}_{\mathbf{C}}$ and the minimizer locations for the Takagi function $x_{\mathbf{C}}$ as $\tilde{\mathcal{M}}_{\mathbf{C}}$. In a more formal way, we have

$$\mathcal{M}_{\mathbf{C}} = \arg\max_{t\in[0,1]} x_{\mathbf{C}}(t) \quad \text{and} \quad \tilde{\mathcal{M}}_{\mathbf{C}} = \arg\min_{t\in[0,1]} x_{\mathbf{C}}(t).$$

Moreover, let us also denote the set of extreme location of the truncated Takagi functions as follows.

$$\mathcal{M}_{\mathbf{C},k} = \arg\max_{t\in[0,1]} x_{\mathbf{C},k}(t) \quad \text{and} \quad \tilde{\mathcal{M}}_{\mathbf{C},k} = \arg\min_{t\in[0,1]} x_{\mathbf{C},k}(t).$$

Similar notations will also be applied to the exponential Takagi function x_ν which is obtained by taking $\mathbf{C} = \{\nu^k\}_{k=0,1,2,...}$.

Definition 2.2.6. If $s \in \mathbb{T}_m$, we then denote $s^* \in \mathbb{T}_m(s)$ if and only if $|s - s^*| = 2^{-m}$. And such a s^* is called an adjoining point of s in the dyadic partition $\mathbb{T}_m$. Furthermore, we define $\bar{\mathbb{T}}_m(s) = \mathbb{T}_m(s) \cup \{s\}$ as the adjoining neighborhood of point s in the dyadic partition $\mathbb{T}_m$.

Chapter 3

Takagi Class

3.1 Global Extrema for Takagi Class

First of all, we will prove that the m^{th} order truncated extrema must be on the dyadic partition $\mathbb{T}_{m+1}$. The following lemma gives reasons why this must hold.

Lemma 3.1.1. Let $x_{\mathbf{C}} \in \mathfrak{C}$. Then for every $m \in \mathbb{N}$

$$\mathcal{M}_{\mathbf{C},m} \subset \mathbb{T}_{m+1} \quad \text{and} \quad \tilde{\mathcal{M}}_{\mathbf{C},m} \subset \mathbb{T}_{m+1}. \tag{3.1}$$

Proof. First of all, we have $x_{\mathbf{C},m}$ is linear within intervals $[t, t + 2^{-(m+1)}]$, for all $t \in \mathbb{T}_{m+1}$. Therefore, for each $y \in [0, 1]$, there exists some $t \in \mathbb{T}_{m+1}$ and $t^* \in \mathbb{T}_{m+1}(t)$, such that

$$y \in [t \wedge t^*, t \vee t^*].$$

Then due to linearity, we have

$$x_{\mathbf{C},m}(y) = 2^{m+1}|t - y|x_{\mathbf{C},m}(t^*) + 2^{m+1}|t^* - y|x_{\mathbf{C},m}(t).$$

Hence, we have

$$\min\{x_{\mathbf{C},m}(t^*), x_{\mathbf{C},m}(t)\} \leq x_{\mathbf{C},m}(y) \leq \max\{x_{\mathbf{C},m}(t^*), x_{\mathbf{C},m}(t)\}.$$

Therefore, we have

$$\mathcal{M}_{\mathbf{C},m} \subset \mathbb{T}_{m+1} \quad \text{and} \quad \tilde{\mathcal{M}}_{\mathbf{C},m} \subset \mathbb{T}_{m+1}.$$

□

Lemma 3.1.2. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every m, let $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m}$ be a maximum point of the truncated Takagi function $x_{\mathbf{C},m}$. Furthermore, let t^*_m be a point at which attains the maximum on $t \in \mathbb{T}_{m+1}(\tilde{t}_m)$. Hence, we have

$$t^*_m \in \underset{t \in \mathbb{T}_{m+1}(\tilde{t}_m)}{\arg\max}\; x_{\mathbf{C},m}(t). \tag{3.2}$$

Now, let $s \in \mathbb{T}_{m+1}$, and $s^* \in \mathbb{T}_{m+1}(s)$. Then, we have

$$x_{\mathbf{C},m}(\tilde{t}_m) + x_{\mathbf{C},m}(t^*_m) \geq x_{\mathbf{C},m}(s) + x_{\mathbf{C},m}(s^*), \tag{3.3}$$

for all $s \in \mathbb{T}_{m+1}$.

Proof. We are going to prove this Lemma using induction on m. First of all, let us consider the case when $m = 0$. Hence we have

$$x_{\mathbf{C},0}(t) = c_0 t, \quad \text{for} \quad t \in [0, \frac{1}{2}]. \tag{3.4}$$

Then if $c_0 \leq 0$, then we have $\tilde{t}_0 = \frac{1}{2}$ and $t^*_0 = 0$, otherwise, we have $\tilde{t}_0 = 0$ and $t^*_0 = \frac{1}{2}$. Also, we notice 0 and $\frac{1}{2}$ are the only choice for $s \in \mathbb{T}_1$ and $s^* \in \mathbb{T}_1(s)$.

$$x_{\mathbf{C},0}(\tilde{t}_0) + x_{\mathbf{C},0}(t^*_0) = x_{\mathbf{C},0}(s) + x_{\mathbf{C},0}(s^*) = x_{\mathbf{C},0}(0) + x_{\mathbf{C},0}(\frac{1}{2}) = \frac{c_0}{2}. \tag{3.5}$$

We can notice that whatever the choice for s and s^*, equation (**??**) must holds. Due to (3.5), we have proved that the hypothesis holds for $m = 0$. Now let us assume that (3.2) holds for all $m \leq n-1$, and we proceed to prove the case for $m = n$.

Now we will then prove the statement case by case. First of all, let us consider if $\tilde{t}_n = \tilde{t}_{n-1}$. According to Lemma 3.1.1, t^*_{n-1} is an adjoining point to $\tilde{t}_{n-1} \in \mathbb{T}_n$ on the dyadic partition $\mathbb{T}_n$. Therefore, we have

$$|\tilde{t}_{n-1} - t^*_{n-1}| = |(2\tilde{t}_{n-1} - t^*_{n-1}) - \tilde{t}_{n-1}| = 2^{-n},$$

as well as,

$$|\tilde{t}_n - \frac{\tilde{t}_{n-1} + t^*_{n-1}}{2}| = |\tilde{t}_n - \frac{3\tilde{t}_{n-1} - t^*_{n-1}}{2}| = 2^{-(n+1)}.$$

Therefore, $2\tilde{t}_{n-1} - t^*_{n-1}$ is the other adjoining point to $\tilde{t}_{n-1}$. Moreover, $\frac{3\tilde{t}_{n-1} - t^*_{n-1}}{2}$ and $\frac{\tilde{t}_{n-1} + t^*_{n-1}}{2}$ are the two adjoining points of $\tilde{t}_n$ on the dyadic partition $\mathbb{T}_{n+1}$. Then due to the linearity and inequality in (3.2), we have

$$\begin{aligned} x_{\mathbf{C},n}(\frac{\tilde{t}_{n-1} + t^*_{n-1}}{2}) &= \frac{x_{\mathbf{C},n}(\tilde{t}_{n-1}) + x_{\mathbf{C},n}(t^*_{n-1})}{2} + \frac{c_n}{2} \\ &\geq \frac{x_{\mathbf{C},n}(\tilde{t}_{n-1}) + x_{\mathbf{C},n}(2\tilde{t}_{n-1} - t^*_{n-1})}{2} + \frac{c_n}{2} \\ &= x_{\mathbf{C},n}(\frac{3\tilde{t}_{n-1} - t^*_{n-1}}{2}). \end{aligned}$$

Hence we must have

$$\frac{\tilde{t}_{n-1} + t^*_{n-1}}{2} \in \arg\max_{t \in \mathbb{T}_{n+1}(\tilde{t}_n)} x_{\mathbf{C},n}(t).$$

Now we have,

$$t^*_n = \frac{\tilde{t}_{n-1} + t^*_{n-1}}{2}. \tag{3.6}$$

Now let us assume that $s \in \mathbb{T}_n$, then we have that

$$x_{\mathbf{C},n}(s) = x_{\mathbf{C},n-1}(s) \leq x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) = x_{\mathbf{C},n}(\tilde{t}_{n-1}). \tag{3.7}$$

Furthermore, we can re-compose s^* by $s^* = \frac{s+(2s^*-s)}{2}$, then we have

$$|(2s^* - s) - s^*| = |s^* - s| = 2^{n+1}, \tag{3.8}$$

as will as,

$$|2s^* - s - s| = 2|s^* - s| = 2 * 2^{-(n+1)} = 2^{-n}. \tag{3.9}$$

Equation (3.8) and (3.9) indicates that s and $2s^* - s$ are adjoining points on the dyadic partition $\mathbb{T}_n$, and they are also the two different adjoining points of s^* on the dyadic partition $\mathbb{T}_{n+1}$. Then applying (3.2) for $n-1$, we have

$$x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + x_{\mathbf{C},n-1}(t^*_{n-1}) \geq x_{\mathbf{C},n-1}(s) + x_{\mathbf{C},n-1}(2s^* - s). \tag{3.10}$$

By applying (3.6) - (3.10), we then have

$$\begin{aligned}
x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(t^*_n) &= x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + x_{\mathbf{C},n}(\frac{t^*_{n-1} + \tilde{t}_{n-1}}{2}) \\
&= x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + \frac{1}{2}x_{\mathbf{C},n-1}(t^*_{n-1}) + \frac{1}{2}x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + \frac{c_n}{2} \\
&\geq x_{\mathbf{C},n-1}(s) + \frac{1}{2}x_{\mathbf{C},n-1}(s) + \frac{1}{2}x_{\mathbf{C},n-1}(2s^* - s) + \frac{c_n}{2} \\
&\geq x_{\mathbf{C},n}(s) + x_{\mathbf{C},n}(s^*).
\end{aligned}$$

Now, let us consider when $s \in \mathbb{T}_{n+1} - \mathbb{T}_n$. In this case, we have

$$s^* \in \mathbb{T}_{n+1}(s) \subseteq \mathbb{T}_n, \tag{3.11}$$

as well as,

$$2s - s^* \in \mathbb{T}_{n+1}(s) \subseteq \mathbb{T}_n, \tag{3.12}$$

According to (3.9), we know that $s^* \in \mathbb{T}_n$ and $2s - s^* \in \mathbb{T}_n$. Now by applying the inductive statement for order $n-1$, we have,

$$x_{\mathbf{C},n-1}(s^*) + x_{\mathbf{C},n-1}(2s - s^*) \leq x_{\mathbf{C},n-1}(t^*_{n-1}) + x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) \tag{3.13}$$

Then by applying (3.13), we have

$$\begin{aligned} x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(t_n^*) &= x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + x_{\mathbf{C},n}(\frac{t_{n-1}^* + \tilde{t}_{n-1}}{2}) \\ &= x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + \frac{1}{2}x_{\mathbf{C},n-1}(t_{n-1}^*) + \frac{1}{2}x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + \frac{c_n}{2} \\ &\geq x_{\mathbf{C},n-1}(s^*) + \frac{1}{2}x_{\mathbf{C},n-1}(s^*) + \frac{1}{2}x_{\mathbf{C},n-1}(2s - s^*) + \frac{c_n}{2} \\ &\geq x_{\mathbf{C},n}(s) + x_{\mathbf{C},n}(s^*), \end{aligned}$$

Now, let us consider the other case when $\tilde{t}_n \neq \tilde{t}_{n-1}$. For any points $s \in \mathbb{T}_n$, we have

$$x_{\mathbf{C},n}(s) = x_{\mathbf{C},n-1}(s) \leq x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) = x_{\mathbf{C},n}(\tilde{t}_{n-1}). \tag{3.14}$$

For points $s \in \mathbb{T}_{n+1} - \mathbb{T}_n$, we have $s - 2^{-(n+1)} \in \mathbb{T}_n$ and $s + 2^{-(n+1)} \in \mathbb{T}_n$. Then by applying (3.5) with $m = n - 1$, we have

$$\begin{aligned} x_{\mathbf{C},n}(\frac{\tilde{t}_{n-1} + t_{n-1}^*}{2}) &= \frac{x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) + x_{\mathbf{C},n-1}(t_{n-1}^*)}{2} + \frac{c_n}{2} \\ &\geq \frac{x_{\mathbf{C},n-1}(s - 2^{-(n+1)}) + x_{\mathbf{C},n-1}(s + 2^{-(n+1)})}{2} + \frac{c_n}{2} \quad = x_{\mathbf{C},n}(s) \end{aligned} \tag{3.15}$$

By applying (3.14) and (3.15), we get

$$\max_{t \in [0, \frac{1}{2}]} x_{\mathbf{C},n}(t) = \max\{x_{\mathbf{C},n}(\tilde{t}_{n-1}), x_{\mathbf{C},n}(\frac{\tilde{t}_{n-1} + t_{n-1}^*}{2})\}. \tag{3.16}$$

Since $\tilde{t}_n \neq \tilde{t}_{n-1}$, we have

$$\tilde{t}_n = \frac{\tilde{t}_{n-1} + t_{n-1}^*}{2}, \tag{3.17}$$

as well as,

$$t_n^* = \tilde{t}_{n-1}. \tag{3.18}$$

Hence, for all $s \in \mathbb{T}_{n+1}$, we must have either s or s^* is in the dyadic partition $\mathbb{T}_n$. Without loss of generality, let us assume that $s \in \mathbb{T}_{n+1}$. Therefore, we get $s^* \in \mathbb{T}_n$. By applying

(3.14), (3.15), (3.17) and (3.18), we have

$$\begin{aligned} x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(t_n^*) &= x_{\mathbf{C},n}(\frac{\tilde{t}_{n-1} + t_{n-1}^*}{2}) + x_{\mathbf{C},n}(\tilde{t}_{n-1}) \\ &\geq x_{\mathbf{C},n}(s) + x_{\mathbf{C},n}(s^*). \end{aligned}$$

Therefore, we finish proving the case for $m = n$, and, hence, we prove the Lemma 3.1.2. □

Corollary 3.1.3. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every m, let $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m}$. Then, we must have

$$\bar{\mathbb{T}}_{m+1}(\tilde{t}_m) \cap \mathcal{M}_{\mathbf{C},m+1} \neq \emptyset.$$

Proof. This result directly comes from (3.16) in Lemma 3.1.2. □

Corollary 3.1.4. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m \in \mathbb{N}$, let $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m}$, as well as,

$$t_m^* \in \underset{t \in \mathbb{T}_{m+1}(\tilde{t}_m)}{\arg\max}\ x_{\mathbf{C},m}(t).$$

If $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m+1}$, then

$$\frac{\tilde{t}_m + t_m^*}{2} \in \underset{t \in \mathbb{T}_{m+2}(\tilde{t}_{m+1})}{\arg\max}\ x_{\mathbf{C},m+1}(t).$$

Otherwise, we have

$$\tilde{t}_m \in \underset{t \in \mathbb{T}_{m+2}(\tilde{t}_{m+1})}{\arg\max}\ x_{\mathbf{C},m+1}(t).$$

Proof. This result directly comes from (3.6) and (3.18) in the proof for Lemma 3.1.2. □

Corollary 3.1.5. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m \in \mathbb{N}$, let $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m}$. Furthermore, we take

$$t_m^* \in \underset{t \in \mathbb{T}_{m+1}(\tilde{t}_m)}{\arg\max}\ x_{\mathbf{C},m}(t).$$

Then there must exist some $\tilde{t}_{m+1} \in \mathcal{M}_{\mathbf{C},m+1}$ such that

$$[t_{m+1}^* \wedge \tilde{t}_{m+1}, t_{m+1}^* \vee \tilde{t}_{m+1}] \subset [t_m^* \wedge \tilde{t}_m, t_m^* \vee \tilde{t}_m].$$

Proof. This results directly comes from Corollary 3.1.4. ☐

Definition 3.1.6. Let $\{\tilde{t}_n\}$ be a sequence such that $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$, and

$$[t^*_{m+1} \wedge \tilde{t}_{m+1}, t^*_{m+1} \vee \tilde{t}_{m+1}] \subset [t^*_m \wedge \tilde{t}_m, t^*_m \vee \tilde{t}_m],$$

for all $m \in \mathbb{N}$. We shall call such a sequence a *sequence of consecutive maximizers.*

Lemma 3.1.2 gives the range of a sequence of consecutive maximizers, and the following propositions will state the result how one characterize the exact location of the proceeding consecutive maximizers based previous consecutive maximizers.

Proposition 3.1.7. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every m, let $\tilde{t}_m \in \mathcal{M}_{\mathbf{C},m}$ be a maximum point of the truncated exponential Takagi function $x_{\mathbf{C},m}$. For a fixed n, we let $k = \min\{i | \tilde{t}_{n-i} \neq \tilde{t}_n\}$. Then, if $\tilde{t}_{n-k} < \tilde{t}_n$, we have

$$x_{\mathbf{C},n+1}(p - 2^{-(n+2)}) \leq x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}). \tag{3.19}$$

for every $p \in \mathbb{T}_{n+1}$.

Proof. We will prove this proposition by induction on k. Let us first consider the case $k = 1$. By Lemma 3.1.1, we have that $\tilde{t}_n$ is in the dyadic partition $\mathbb{T}_{n+1}$ and $\tilde{t}_{n-1}$ is in the dyadic partition $\mathbb{T}_n$ for any fixed $n \in \mathbb{N}$. Furthermore, we have

$$\tilde{t}_{n-k} = \tilde{t}_{n-1} < \tilde{t}_n. \tag{3.20}$$

Then due to (3.20), we can apply Lemma 3.1 in [17], then we have

$$\tilde{t}_{n-1} = \tilde{t}_n - 2^{-(n+1)}. \tag{3.21}$$

As the truncated function $x_{\mathbf{C},n}$ is linear within intervals of the form $[p - 2^{-(n+1)}, p]$, for any $p \in \mathbb{T}_{n+1}$, and the increment of the wedge has an increment of $\frac{c_{n+1}}{2}$, we get

$$x_{\mathbf{C},n+1}(p - 2^{-(n+2)}) = \frac{x_{\mathbf{C},n}(p) + x_{\mathbf{C},n}(p - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2}. \tag{3.22}$$

As $\tilde{t}_n \in \mathbb{T}_{n+1}$, we may take $p = \tilde{t}_n$ and by plugging (3.21) into (3.22), we have

$$x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}) = \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_{n-1})}{2} + \frac{c_{n+1}}{2}. \tag{3.23}$$

Since $\tilde{t}_{n-1} \in \mathcal{M}_{\mathbf{C},n-1}$, $\tilde{t}_{n-1} \in \mathbb{T}_n$, we have

$$x_{\mathbf{C},n}(\tilde{t}_{n-1}) = x_{\mathbf{C},n-1}(\tilde{t}_{n-1}). \tag{3.24}$$

As $p \in \mathbb{T}_{n+1}$, we have either $p \in \mathbb{T}_n$ or $p - 2^{-(n+1)} \in \mathbb{T}_n$.

$$\min\{x_{\mathbf{C},n}(p), x_{\mathbf{C},n}(p - 2^{-(n+1)})\} \leq x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) = x_{\mathbf{C},n}(\tilde{t}_{n-1}). \tag{3.25}$$

In addition,

$$\max\{x_{\mathbf{C},n}(p), x_{\mathbf{C},n}(p - 2^{-(n+1)})\} \leq x_{\mathbf{C},n}(\tilde{t}_n). \tag{3.26}$$

Hence, according to (3.25) and (3.26), we have

$$\frac{x_{\mathbf{C},n}(p) + x_{\mathbf{C},n}(p - 2^{-(n+1)})}{2} \leq \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_{n-1})}{2}. \tag{3.27}$$

Plugging (3.27) into (3.23) and applying (3.22), we get

$$\begin{aligned} x_{\mathbf{C},n+1}(p - 2^{-(n+2)}) &= \frac{x_{\mathbf{C},n}(p) + x_{\mathbf{C},n}(p - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2} \\ &\leq \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_{n-1})}{2} + \frac{c_{n+1}}{2} = x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}). \end{aligned} \tag{3.28}$$

This completes the proof for the case $k = 1$. For such all fixed n, we now assume the (3.19) holds true for $k \leq m$. Now we proceed to prove when $k = m + 1$, $x_{\mathbf{C},n+1}(p - 2^{-(n+2)}) \leq x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)})$ holds for $p \in \mathbb{T}_{n+1}$. As $m + 1 = \min\{i | \tilde{t}_{n-i} \neq \tilde{t}_n\}$, and $\tilde{t}_n = \tilde{t}_{n-1}$, we have

$$\min\{i | \tilde{t}_{n-1-i} \neq \tilde{t}_{n-1}\} = m. \tag{3.29}$$

As induction hypothesis holds for every $n \in \mathbb{N}$ and $k \leq m$, by applying (3.19) for $x_{\mathbf{C},n-1}$,

we have

$$x_{\mathbf{C},(n-1)+1}(\tilde{p} - 2^{-((n-1)+2)}) \leq x_{\mathbf{C},(n-1)+1}(\tilde{t}_{n-1} - 2^{-((n-1)+2)}), \tag{3.30}$$

where $\tilde{p} \in \mathbb{T}_n$. By organizing equation (3.30), we have

$$x_{\mathbf{C},n}(\tilde{p} - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}). \tag{3.31}$$

Then we will prove the statement case by case. We first consider the case $\tilde{p} \in \mathbb{T}_n$. As $\tilde{t}_n \in \mathbb{T}_{n+1}$, therefore $\tilde{t}_n - 2^{-(n+1)} \in \mathbb{T}_{n+1}$. Similarly, as $\tilde{p} \in \mathbb{T}_n$, then $\tilde{p} - 2^{-(n+1)} \in \mathbb{T}_{n+1}$. By applying equation (3.22), we get

$$x_{\mathbf{C},n+1}(\tilde{p} - 2^{-(n+2)}) = \frac{x_{\mathbf{C},n}(\tilde{p}) + x_{\mathbf{C},n}(\tilde{p} - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2}. \tag{3.32}$$

Then as $\tilde{t}_n = \tilde{t}_{n-1} \in \mathbb{T}_n$, we can replace $\tilde{p}$ with $\tilde{t}_n$ in the equation (3.32), we get

$$x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}) = \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2}. \tag{3.33}$$

As $\tilde{t}_n = \tilde{t}_{n-1} \in \mathbb{T}_n$, and $p \in \mathbb{T}_n$, then we have

$$x_{\mathbf{C},n}(\tilde{p}) = x_{\mathbf{C},n-1}(\tilde{p}) \leq x_{\mathbf{C},n-1}(\tilde{t}_{n-1}) = x_{\mathbf{C},n}(\tilde{t}_{n-1}) = x_{\mathbf{C},n}(\tilde{t}_n). \tag{3.34}$$

Then by plugging (3.34) and (3.31) into (3.32) and (3.33), we have

$$\begin{aligned} x_{\mathbf{C},n+1}(\tilde{p} - 2^{-(n+2)}) &= \frac{x_{\mathbf{C},n}(\tilde{p}) + x_{\mathbf{C},n}(\tilde{p} - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2} \\ &\leq \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2} = x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}). \end{aligned}$$

Now we discuss the case $\tilde{p} \in \mathbb{T}_{n+1} - \mathbb{T}_n$, then we have that $\tilde{p} + 2^{-(n+1)} \in \mathbb{T}_n$, therefore by applying equation (3.31) for $\tilde{p} + 2^{-(n+1)}$, we have

$$x_{\mathbf{C},n}(\tilde{p} + 2^{-(n+1)} - 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{p}) \leq x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}). \tag{3.35}$$

As the function $x_{\mathbf{C},n}$ is maximized at $\tilde{t}_n$, we have

$$x_{\mathbf{C},n}(\tilde{p} - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_n). \tag{3.36}$$

By plugging equation (3.35) and equation (3.36) into equation (3.32) and equation (3.33). We have

$$\begin{aligned} x_{\mathbf{C},n+1}(\tilde{p} - 2^{-(n+2)}) &= \frac{x_{\mathbf{C},n}(\tilde{p}) + x_{\mathbf{C},n}(\tilde{p} - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2^{n+2}} \\ &\leq \frac{x_{\mathbf{C},n}(\tilde{t}_n) + x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)})}{2} + \frac{c_{n+1}}{2^{n+2}} = x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)}). \end{aligned}$$

Therefore, we have proved that $x_{\mathbf{C},n+1}(p - 2^{-(n+2)}) \leq x_{\mathbf{C},n+1}(\tilde{t}_n - 2^{-(n+2)})$ for any $p \in \mathbb{T}_{n+1}$. Since both base case and the inductive hypothesis has been proved, then we prove this proposition. □

Proposition 3.1.8. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every m, let $\tilde{t}_m \in \arg\max_{t \in [0,1]} x_{\mathbf{C},m}(t)$ be a maximum point of the truncated function $x_{\mathbf{C},m}$. For fixed n, we let $k = \min\{i | \tilde{t}_{n-i} \neq \tilde{t}_n\}$. Then, if $\tilde{t}_{n-k} > \tilde{t}_n$, we have

$$x_{\mathbf{C},n+1}(p + 2^{-(n+2)}) \leq x_{\mathbf{C},n+1}(\tilde{t}_n + 2^{-(n+2)}),$$

for every $p \in \mathbb{T}_{n+1}$.

Proof. The proof is analogous to the proof of Proposition 3.1.7. □

Lemma 3.1.9. Let $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$, and

$$t_n^* \in \arg\max_{t \in \mathbb{T}_{n+1}(\tilde{t}_n)} x_{\mathbf{C},n}(t).$$

Then for any fixed $n \geq 1$, take $m = \inf\{i | \tilde{t}_{n-i} \neq \tilde{t}_n\}$. Then, $\tilde{t}_{n-m} < \tilde{t}_n$ if and only if $t_n^* < \tilde{t}_n$.

Proof. First of all, let us prove the only if direction. Let us assume that $\tilde{t}_{n-m} < \tilde{t}_n$, then we will discuss case by case. First of all, let us consider the case when $m = 1$. By applying

Corollary 3.1.3, we have $\tilde{t}_{n-1} = \tilde{t}_n - 2^{-(n+1)} \in \mathbb{T}_n$. Similarly, we have $\tilde{t}_n + 2^{-(n+1)} \in \mathbb{T}_n$. Then we have

$$x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}) = x_{\mathbf{C},n-1}(\tilde{t}_n - 2^{-(n+1)}) \geq x_{\mathbf{C},n-1}(\tilde{t}_n + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}).$$

We have $t_n^* = \tilde{t}_n - 2^{-(n+1)}$, and $t_n^* < \tilde{t}_n$. Now, we consider the case when $m > 1$. Moreover, we have

$$\tilde{t}_{n-1} = \tilde{t}_n.$$

Then we may apply the Proposition 3.1.7. As $\tilde{t}_{(n-1)-(m-1)} < \tilde{t}_{n-1}$, we have

$$x_{\mathbf{C},n}(p - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_{n-1} - 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}). \tag{3.37}$$

for every $p \in \mathbb{T}_n$. Because $\tilde{t}_n = \tilde{t}_{n-1} \in \mathbb{T}_n$, then by plugging $p = \tilde{t}_n + 2^{-n} \in \mathbb{T}_n$ into (3.37), we have

$$x_{\mathbf{C},n}(\tilde{t}_n + 2^{-n} - 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}).$$

Hence, we have $t_n^* = \tilde{t}_n - 2^{-(n+1)}$, and $t_n^* < \tilde{t}_n$, then we finish the proof for the only if part. Now, we aim to prove the if direction by proving its contrapositive statement through a brief discussion on m. We can notice that the contrapositive statement for the only if direction will be

If $\tilde{t}_{n-m} > \tilde{t}_n$, then $t_n^* > \tilde{t}_n$.

First of all, let us consider the case when $m = 1$. By applying Corollary 3.1.3 again, we have $\tilde{t}_{n-1} = \tilde{t}_n + 2^{-(n+1)} \in \mathbb{T}_n$. Therefore, we have $\tilde{t}_n - 2^{-(n+1)} \in \mathbb{T}_n$. Then we have

$$x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}) = x_{\mathbf{C},n-1}(\tilde{t}_n - 2^{-(n+1)}) \leq x_{\mathbf{C},n-1}(\tilde{t}_n + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}).$$

Furthermore, we have

$$x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_n).$$

Hence, we have $t_n^* = \tilde{t}_n + 2^{-(n+1)}$, and $t_n^* > \tilde{t}_n$ under the condition $m = 1$. Now, we consider the case when $m > 1$. Hence, we have

$$\tilde{t}_{n-1} = \tilde{t}_n.$$

Next, we would apply Proposition 3.1.8. As $\tilde{t}_{(n-1)-(m-1)} > \tilde{t}_{n-1}$, we have

$$x_{\mathbf{C},n}(p + 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_{n-1} + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}). \tag{3.38}$$

for every $p \in \mathbb{T}_n$. Because $\tilde{t}_n = \tilde{t}_{n-1} \in \mathbb{T}_n$, then by plugging $p = \tilde{t}_n - 2^{-n} \in \mathbb{T}_n$ into (3.38), we have

$$x_{\mathbf{C},n}(\tilde{t}_n - 2^{-n} + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tilde{t}_n - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tilde{t}_n + 2^{-(n+1)}).$$

Therefore, we have $t_n^* = \tilde{t}_n + 2^{-(n+1)}$, and $t_n^* > \tilde{t}_n$. Hence, we have proved the only if part through proving its contrapositive statement. □

Lemma 3.1.10. For $y \in [0, 1]$, let y_n be the n^{th} order dyadic approximation for y, then

$$y \in [y_n, y_n + 2^{-n}].$$

Furthermore, if we restrict all binary expansion is in the image $S_\sharp([0, 1])$, then we have

$$y \in [y_n, y_n + 2^{-n}).$$

On the other hand, if we require all binary expansion is in the image $S_\flat([0, 1])$, then we have

$$y \in (y_n, y_n + 2^{-n}].$$

Proof. Let the binary expansion of y be

$$y = \sum_{i=0}^{\infty} \frac{\varepsilon_i}{2^{i+1}} = 0.\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3......$$

Then by rewriting the binary expansion we have

$$y = \sum_{i=0}^{\infty} \frac{\varepsilon_n}{2^{i+1}} = y_n + \sum_{i=n+1}^{\infty} \frac{\varepsilon_n}{2^{i+1}}.$$

As for every $i \in \mathbb{N}$, we have $\varepsilon_i \in \{0,1\}$, then

$$0 \leq \sum_{i=n+1}^{\infty} \frac{\varepsilon_n}{2^{i+1}} \leq 2^{-(n+1)}.$$

Hence, we have

$$y \in [y_n, y_n + 2^{-n}].$$

However, if $\{\varepsilon_i\} \in S_\sharp([0,1])$, then ϵ_i cannot all be 1 for $i > n$. Therefore, we have

$$0 \leq \sum_{i=n+1}^{\infty} \frac{\varepsilon_n}{2^{i}} < 2^{-(n+1)}.$$

Hence, we have

$$y \in [y_n, y_n + 2^{-n}).$$

Similarly, if if $\{\varepsilon_i\} \in S_\flat([0,1])$, then ϵ_i cannot all be 0 for $i > n$. Therefore, we have

$$0 < \sum_{i=n+1}^{\infty} \frac{\varepsilon_n}{2^{i}} \leq 2^{-(n+1)}.$$

And this leads to

$$y \in (y_n, y_n + 2^{-n}].$$

Hence, we proved this lemma. □

Definition 3.1.11. For each n, we denote the n^{th} order upper truncated Takagi function over an infinite sequence $\mathbf{C} \in \mathbb{R}^{\mathbb{N}}$ by $x^n_{\mathbf{C}}$, which is written as

$$x^n_{\mathbf{C}}(t) = \sum_{m=n}^{\infty} c_i \phi(2^m t). \tag{3.39}$$

Lemma 3.1.12. For $y \in \mathbb{R}$, $x^n_{\mathbf{C}}(y) = x^n_{\mathbf{C}}(y + 2^{-n}k)$, for all $k \in \mathbb{Z}$.

Proof. We have,

$$x^n_{\mathbf{C}}(y + 2^{-n}k) = \sum_{i=n}^{\infty} c_i \phi(2^i(y + 2^{-n}k)) = \sum_{i=n}^{\infty} c_i \phi(2^i y + 2^{i-n}k).$$

As $i \leq n$, then $2^{i-n}k \in \mathbb{N}$, hence

$$\phi(2^i y + 2^{i-n}k) = \phi(2^i y),$$

for all i. Therefore, we have

$$x^n_{\mathbf{C}}(y) = x^n_{\mathbf{C}}(y + 2^{-n}k).$$

□

Lemma 3.1.13. For $y \in \mathbb{R}$, we have $x^n_{\mathbf{C}}(y) = x^n_{\mathbf{C}}(-y)$.

Proof. We have,

$$x^n_{\mathbf{C}}(y) = \sum_{i=n}^{\infty} c_i \phi(2^i y) = \sum_{i=n}^{\infty} c_i \phi(-2^i y) = x^n_{\mathbf{C}}(y).$$

□

Remark 3.1.14. Lemma 3.1.12 and Lemma 3.1.13 indicate that $x^n_{\mathbf{C}}$ is symmetry with respect to every points in $\mathbb{T}_{n+1}$.

Theorem 3.1.15. Let $x_{\mathbf{C}} \in \mathfrak{C}$, the following statements are equivalent:

i. $y \in \mathcal{M}_{\mathbf{C}}$.

ii. There exists a sequence $\{y_n\}_{n=0}^{\infty}$, such that $y_n \in \mathcal{M}_{\mathbf{C},n}$ for all $n \in \mathbb{N}$, and

$$y = \lim_{n \to \infty} y_n.$$

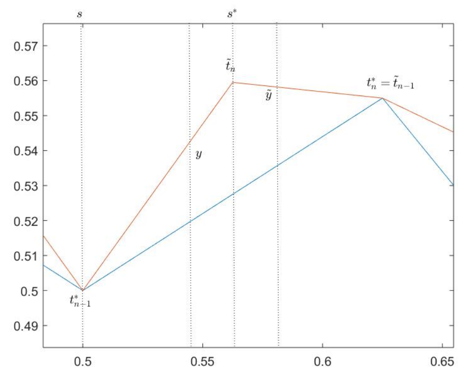

Figure 3.1: Graphical Illustration for $s = t_{n-1}^*$ and $s^* = \tilde{t}_n$.

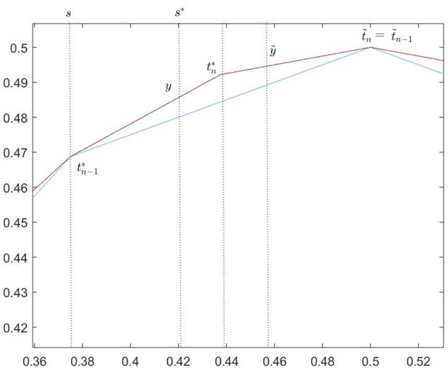

Figure 3.2: Graphical Illustration for $s = t_{n-1}^*$ and $s^* = t_n^*$.

iii. Let $\mathcal{T}_n := \{[\tilde{t}_n - 2^{-(n+1)}, \tilde{t}_n + 2^{-(n+1)}] | \tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}\}$. Furthermore, take

$$\mathcal{P}_n = \bigcup_{A \in \mathcal{T}_n} A.$$

Then,

$$y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

iv. Let $\mathcal{K}_n := \{[\tilde{t}_n \wedge t_n^*, \tilde{t}_n \vee t_n^*] \mid \tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}, t_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tilde{t}_n)}{\arg\max}\, x_{\mathbf{C},n}(\tilde{t}_n)\}$. Furthermore, take

$$\mathcal{I}_n = \bigcup_{A \in \mathcal{K}_n} A.$$

Then,

$$y \in \bigcap_{n=0}^{\infty} \mathcal{I}_n.$$

Proof. Let us prove this theorem by proving following statements in order.

- $i \implies iv$

Let us prove this statement by proving its contrapositive statement. The contrapositive statement will be

If there exists $n \in \mathbb{N}$, such that $y \notin \mathcal{I}_n$, then $y \notin \mathcal{M}_{\mathbf{C}}$.

Denote $\mathcal{N} = \{n \in \mathbb{N} | y \notin \mathcal{I}_n\}$, and denote $n = \min \mathcal{N}$. Let us assume that $y \in [s, s^*]$, where $s \in \mathbb{T}_{n+1}$ and $s^* = s + 2^{-(n+1)}$. Since $n = \min \mathcal{N}$, we must have that

$$y \in [\tilde{t}_{n-1} \wedge t^*_{n-1}, \tilde{t}_{n-1} \vee \tilde{t}^*_{n-1}].$$

for some $\tilde{t}_{n-1} \in \mathcal{M}_{\mathbf{C},n-1}$ and $t^*_{n-1} \in \arg\max_{t \in \mathbb{T}_n t} x_{\mathbf{C},n-1}(t)$. By applying Corollary 3.1.3, there exists $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n} \cap [\tilde{t}_{n-1} \wedge t^*_{n-1}, \tilde{t}_{n-1} \vee \tilde{t}^*_{n-1}]$. Furthermore, take $\tilde{t}_n \in [\tilde{t}_{n-1}, t^*_{n-1}]$, then by applying Corollary 3.1.4, we have

$$[\frac{1}{2}(\tilde{t}_{n-1} + t^*_{n-1}) \wedge \tilde{t}_{n-1}, \frac{1}{2}(\tilde{t}_{n-1} + t^*_{n-1}) \vee \tilde{t}_{n-1}] = [\tilde{t}_n \wedge t^*_n, \tilde{t}_n \vee \tilde{t}^*_n].$$

Since $y \in [\tilde{t}_{n-1} \wedge t^*_{n-1}, \tilde{t}_{n-1} \vee \tilde{t}^*_{n-1}]$, we have $\{s, s^*\} \bigcap \{\tilde{t}_n, t^*_n\} \neq \emptyset$, and $\{s, s^*\} \bigcap \{\tilde{t}_n, t^*_n\} \neq \{\tilde{t}_n, t^*_n\}$. For instance, if $t^*_{n-1} < \tilde{t}_{n-1}$, then we have

$$s = t^*_{n-1}, \quad \text{and} \quad s^* = \tilde{t}_n \wedge t^*_n.$$

As $[s, s^*] \notin \mathcal{K}_n$, therefore $s = t^*_{n-1} \notin \mathcal{M}_{\mathbf{C},n-1}$. Hence, we have

$$x_{\mathbf{C},n}(s) < x_{\mathbf{C},n}(\tilde{t}_{n-1}) \leq x_{\mathbf{C},n}(t^*_n). \tag{3.40}$$

As well as

$$x_{\mathbf{C},n}(s) \leq x_{\mathbf{C},n}(\tilde{t}_n). \tag{3.41}$$

Therefore, since $y \in [s, s^*]$,

$$\tilde{y} := 2s^* - y \in [\tilde{t}_n \wedge t^*_n, \tilde{t}_n \vee t^*_n],$$

for $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$. Then by applying Lemma 3.1.12 and Lemma 3.1.13, we have $s \in \mathbb{T}_{n+1}$,

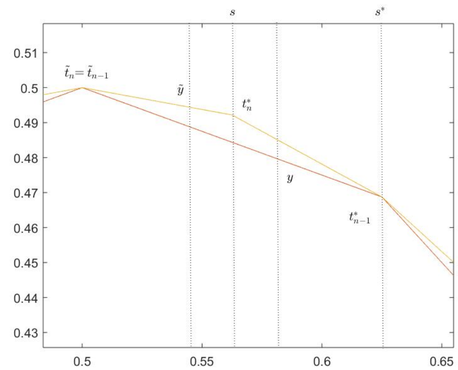

Figure 3.3: Graphical Illustration for $s = t_n^*$ and $s^* = t_{n-1}^*$.

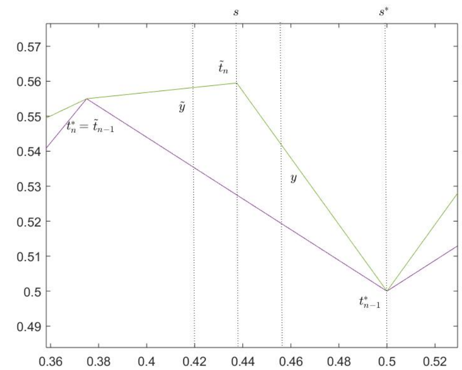

Figure 3.4: Graphical Illustration for $s = \tilde{t}_n$ and $s^* = t_{n-1}^*$.

and

$$x_{\mathbf{C}}^{n+1}(y) = x_{\mathbf{C}}^{n+1}(\tilde{y}). \tag{3.42}$$

By applying (3.40) - (3.42), we have

$$\begin{aligned} x_{\mathbf{C}}(y) = x_{\mathbf{C},n}(y) + x_{\mathbf{C}}^{n+1}(y) &= \frac{y-s}{s^*-s}x_{\mathbf{C},n}(s^*) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(s) + x_{\mathbf{C}}^{n+1}(y) \\ &= \frac{y-s}{s^*-s}x_{\mathbf{C},n}(s^*) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(s) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\ &< \frac{y-s}{s^*-s}x_{\mathbf{C},n}(\tilde{t}_n) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(t_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\ &= \frac{(2s^*-y)-\tilde{t}_n}{\tilde{t}_n-s^*}x_{\mathbf{C},n}(\tilde{t}_n) + \frac{s^*-(2s^*-y)}{\tilde{t}_n-s^*}x_{\mathbf{C},n}(t_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\ &= \frac{\tilde{y}-\tilde{t}_n}{\tilde{t}_n-s^*}x_{\mathbf{C},n}(\tilde{t}_n) + \frac{s^*-\tilde{y}}{\tilde{t}_n-s^*}x_{\mathbf{C},n}(t_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\ &= x_{\mathbf{C},n}(\tilde{y}) + x_{\mathbf{C}}^{n+1}(\tilde{y}) = x_{\mathbf{C}}(\tilde{y}). \end{aligned}$$

Hence, $y \notin \mathcal{M}_{\mathbf{C}}$. And the proof for the situation when $\tilde{t}_{n-1} < t_{n-1}^*$ is analogous to the previous proof.

- $iv \Longrightarrow iii$

First of all, let us state this statement again.

If $y \in \bigcap_{n=0}^{\infty} \mathcal{I}_n$*, then* $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$*.*

First of all, by applying Lemma 2.1.1, we have $\bigcap_{n=0}^{\infty} \mathcal{I}_n \neq \emptyset$ and $\bigcap_{n=0}^{\infty} \mathcal{P}_n \neq \emptyset$. Now, this statement is equivalent to the following inclusion, and we now aim to prove the following inclusion.

$$\bigcap_{n=0}^{\infty} \mathcal{I}_n \subset \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

As for each fixed $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$, we have

$$[\tilde{t}_n \wedge t_n^*, \tilde{t}_n \vee t_n^*] \subsetneqq [\tilde{t}_n - 2^{-(n+1)}, \tilde{t}_n + 2^{-(n+1)}]. \tag{3.43}$$

Then (3.43) directly gives,

$$\mathcal{I}_n = \bigcup_{A \in \mathcal{K}_n} A \subsetneqq \bigcup_{A \in \mathcal{T}_n} A = \mathcal{P}_n,$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\bigcap_{n=0}^{\infty} \mathcal{I}_n \subset \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

- $iii \Longrightarrow ii$

First of all, let us formally state the statement we are going to prove.

If $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$*, then there exists a sequence* $y_n \in \mathcal{M}_{\mathbf{C},n}$*, such that*

$$y = \lim_{n \to \infty} y_n$$

Since $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$, hence, for all $n \in \mathbb{N}$, we have

$$y \in \mathcal{P}_n.$$

Therefore, there exists some $A_{n,y} \in \mathcal{T}_n$ for all $n \in \mathbb{N}$, and

$$y \in A_{n,y}.$$

For each $n \in \mathbb{N}$, we take a sequence of $\{y_n\}$ for all $n \in \mathbb{N}$ such that $y_n = \mathcal{M}_{\mathbf{C},n} \cap A_{n,y}$, and then we have

$$|y_m - y| \leq 2^{-(m+1)}. \tag{3.44}$$

Hence, $\lim_{n\to\infty} y_n = y$.

- $ii \Longrightarrow i$

First of all, let us state the statement we are about to prove.

If there exists a sequence $\{y_n\}_{n=0}^{\infty}$, such that $y_n \in \mathcal{M}_{\mathbf{C},n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n$, then $y \in \mathcal{M}_{\mathbf{C}}$.

As $[0, 1]$ is a compact space, and $x_{\mathbf{C},n} \in C[0, 1]$, therefore, there exists $\beta_n = \max_{t\in[0,1]} x_{\mathbf{C},n}(t)$ for all $n \in \mathbb{N}$, as well as $\beta = \max_{t\in[0,1]} x_{\mathbf{C}}(t)$. Since, $x_{\mathbf{C},n} \to x_{\mathbf{C}}$ uniformly, therefore, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$, such that for all $n > N$,

$$x_{\mathbf{C}}(t) - \epsilon < x_{\mathbf{C},n}(t) < x_{\mathbf{C}}(t) + \epsilon. \tag{3.45}$$

for all $t \in [0, 1]$. Hence, we have

$$x_{\mathbf{C},n}(t) - \epsilon < x_{\mathbf{C}}(t) < x_{\mathbf{C},n}(t) + \epsilon. \tag{3.46}$$

Therefore (3.45) and (3.46) give us

$$\begin{cases} x_{\mathbf{C},n}(t) < \beta + \epsilon, \\ x_{\mathbf{C}}(t) - \epsilon < \beta_n, \end{cases}$$

for all $t \in [0,1]$. By taking the supremum on the left side, we have

$$\begin{cases} \beta_n \leq \beta + \epsilon, \\ \beta - \epsilon \leq \beta_n. \end{cases}$$

This leads us to

$$\beta - \epsilon \leq \beta_n \leq \beta + \epsilon.$$

Therefore, we have $\lim_{n\to\infty} \beta_n = \beta$. By uniformly convergence, as $\lim_{n\to\infty} y_n = y$, therefore,

$$\lim_{n\to\infty} x_{\mathbf{C},n}(y_n) = x_{\mathbf{C}}(y). \tag{3.47}$$

Therefore, we have $y \in \mathcal{M}_{\mathbf{C}}$.

□

Lemma 3.1.16. For all $n \in \mathbb{N}$, let $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$ and $t_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tilde{t}_n)}{\arg\max}\, x_{\mathbf{C},n}(t)$. Then

$$\lim_{n\to\infty} \tilde{t}_n = t,$$

if and only if

$$\lim_{n\to\infty} \tilde{t}_n \wedge t_n^* = t.$$

Proof. First of all, let us prove the only if direction. Now, assume that $\lim_{n\to\infty} \tilde{t}_n = t$. Then, we have for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for all $n > N$,

$$|t - \tilde{t}_n| < \frac{\epsilon}{2}.$$

Also, we have $|\tilde{t}_n - t_n^*| = 2^{-(n+1)}$. Now taking $M := \max\{N, -\log_2 \epsilon + 1\}$, then we have

$$|t - t_n^*| < |t - \tilde{t}_n| + |t - t_n^*| < \epsilon.$$

Hence,

$$\lim_{n\to\infty} t_n^* = t. \tag{3.48}$$

Therefore, we have

$$\lim_{n\to\infty} t_n^* \wedge \tilde{t}_n = t.$$

Now, let us prove the if direction. Notice that

$$\tilde{t}_n \vee t_n^* - \tilde{t}_n \wedge t_n^* = 2^{-(n+1)}.$$

Then we have

$$\lim_{n\to\infty} \tilde{t}_n \vee t_n^* = \lim_{n\to\infty} \tilde{t}_n \wedge t_n^*.$$

Furthermore, we have

$$\tilde{t}_n \vee t_n^* \geq \tilde{t}_n \geq \tilde{t}_n \wedge t_n^*.$$

By applying the sandwich theorem, we get

$$\lim_{n\to\infty} \tilde{t}_n \wedge t_n^* = \lim_{n\to\infty} \tilde{t}_n.$$

□

Corollary 3.1.17.

$$\mathcal{M}_{\mathbf{C}} = \bigcap_n \mathcal{P}_n = \bigcap_n \mathcal{I}_n$$

Proof. This corollary directly follows from Theorem 3.1.15. □

Theorem 3.1.18. For $y \in [0,1]$, let $y = 0.\varepsilon_0\varepsilon_1\varepsilon_2\cdots = \sum_{i=0}^{\infty} \varepsilon_i 2^{-(i+1)}$ be the lower binary expansion of y. Let $y_n = 0.\varepsilon_0\varepsilon_1\varepsilon_2\ldots\varepsilon_n = \sum_{i=0}^{n} \varepsilon_i 2^{-(i+1)}$. Then $y \in \mathcal{M}_{\mathbf{C}}$, if and only if there exists a sequence consecutive maximizers of $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$ and $t_n^* \in \operatorname*{arg\,max}_{t\in\mathbb{T}_{n+1}(\tilde{t}_n)} x_{\mathbf{C},n}(t)$, such that

$$y_n = \tilde{t}_n \wedge t_n^*,$$

for all $n \in \mathbb{N}$.

Proof. First of all, let us prove the if part. Now let us assume that there exists a sequence of $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$ and $t_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tilde{t}_n)}{\arg\max}\, x_{\mathbf{C},n}(t)$, such that $y_n = \tilde{t}_n \wedge t_n^*$. Instantly, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} t_n^* \wedge \tilde{t}_n = y.$$

Taking such $\tilde{t}_n$, by Lemma 3.1.16, we have

$$\lim_{n\to\infty} \tilde{t}_n = \lim_{n\to\infty} t_n^* \wedge \tilde{t}_n = \lim_{n\to\infty} y_n = y.$$

Hence, by applying Theorem 3.1.15, we have $y \in \mathcal{M}_{\mathbf{C}}$. Now let us prove the only if part by proving its contrapositive statement. Let us state the contrapositive statement first.

If for any sequence of consecutive maximizers $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$, there exists some $n \in \mathbb{N}$, such that $y_n \neq \tilde{t}_n \wedge t_n^*$, then $y \notin \mathcal{M}_{\mathbf{C}}$.

For any sequence $\{\tilde{t}_n\}$, such that $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$, define $\mathcal{N} = \{n \in \mathbf{N} | y_n \neq \tilde{t}_n\}$. Now let us take $N := \min \mathcal{N}$, therefore

$$[\tilde{t}_{N-1} \wedge t_{N-1}^*, \tilde{t}_{N-1} \vee t_{N-1}^*] = [y_{N-1}, y_{N-1} + 2^{-N}]. \tag{3.49}$$

Since $\{\tilde{t}_n\}$ is a sequence of consecutive maximizers, we have

$$[\tilde{t}_N \wedge t_N^*, \tilde{t}_N \vee t_N^*] \subsetneqq [\tilde{t}_{N-1} \wedge t_{N-1}^*, \tilde{t}_{N-1} \vee t_{N-1}^*] = [y_{N-1}, y_{N-1} + 2^{-N}], \tag{3.50}$$

as well as,

$$[y_N, y_N + 2^{-(N+1)}] \subsetneqq [\tilde{t}_{N-1} \wedge t_{N-1}^*, \tilde{t}_{N-1} \vee t_{N-1}^*]. \tag{3.51}$$

Since, we have $y_N \neq \tilde{t}_N \wedge t_N^*$, and therefore, either $y_N = \tilde{t}_N \vee t_N^*$ or $y_N + 2^{-(N+1)} = \tilde{t}_N \wedge t_N^*$. Then by (3.49) - (3.51), we have

$$[\tilde{t}_N \wedge t_N^*, \tilde{t}_N \vee t_N^*] = [\tilde{t}_{N-1} \wedge t_{N-1}^*, \tilde{t}_{N-1} \vee t_{N-1}^*].$$

Then, by applying lemma 3.1.10, we have

$$y \in [y_N, y_N + 2^{-(N+1)}).$$

Theorem 3.1.15 ($i \implies iv$) indicates

$$y \notin \mathcal{M}_{\mathbf{C}}.$$

□

Remark 3.1.19. For the following theorems and lemmas in Section 3.1, the binary expansion will only refer to the lower binary expansion.

Definition 3.1.20. The Rademacher mapping $\mathcal{H} : \{0,1\}^{\mathbb{N}} \to \mathbb{S}^{\mathbb{N}}$ is defined as

$$\{\mathcal{H}(\{d_j\})\}_i = 1 - 2d_i. \tag{3.52}$$

then for $y = 0.\varepsilon_0\varepsilon_1\varepsilon_2.....$ Take $\{\xi\}_{i=0}^{\infty} = \{\mathcal{H}(\{\varepsilon_i\})\}$. The sequence $\{\xi_i\}_{i=0}^{\infty}$ is called the quasi-binary expansion for y.

Definition 3.1.21. Let $\Xi = \{\xi_i\}_{i=0}^{\infty}$ be the quasi-binary expansion for y, and $x_{\mathbf{C}} \in \mathfrak{C}$. Take $a_i = 2^i c_i$, and $\mathbf{A} = \{a_i\}$. Now define

$$\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i.$$

Then $\Xi_n(\mathbf{A})$ will be referred as the slope series for y.

Now by using the quasi-binary expansion, we can then neatly give the most important theorem in characterizing the maximum location for the Takagi Class. Before proving the theorem, we need to prove the following lemma first, which plays an important on relating the location of a point and its truncated slope. The following lemma is closely related to Billingsley [4].

Lemma 3.1.22. Let $y = 0.\varepsilon_0\varepsilon_1\varepsilon_2\cdots = \sum_{i=0}^{n} \varepsilon_i 2^{-(i+1)}$ be the binary expansion of $y \in [0,1]$, and $\{\xi_i\}$ is the quasi-binary expansion for y. Let $y_n = 0.\varepsilon_0\varepsilon_1\varepsilon_2\ldots\varepsilon_n = \sum_{i=0}^{n} \varepsilon_i 2^{-(i+1)}$ be the n^{th} order approximation for y. Then for all $n \in \mathbb{N}$,

$$\frac{x_{\mathbf{C},n}(t_1) - x_{\mathbf{C},n}(t_2)}{t_1 - t_2} = \Xi_n(\mathbf{A}), \tag{3.53}$$

for any $t_1, t_2 \in [y_n, y_n + 2^{-(n+1)}]$.

Proof. Let us prove this statement by induction on n. First of all, let us consider the case $n = 0$. Then we have

$$x_{\mathbf{C},0} = c_0\phi(t) = c_0\{t \wedge (1-t)\}_+.$$

Also, we may notice that

$$\begin{cases} t \in [0, \frac{1}{2}] & \text{if} \quad \varepsilon_0 = 0, \\ t \in [\frac{1}{2}, 1] & \text{if} \quad \varepsilon_0 = 1. \end{cases}$$

Obviously, we have

$$\frac{x_{\mathbf{C},0}(t_1) - x_{\mathbf{C},0}(t_2)}{t_1 - t_2} = \begin{cases} c_0\frac{t_1 - t_2}{t_1 - t_2} = c_0\xi_0 = c_0 & \text{if} \quad \varepsilon_0 = 0, \\ \\ c_0\frac{(1-t_1)-(1-t_2)}{t_1 - t_2} = c_0\xi_0 = -c_0 & \text{if} \quad \varepsilon_0 = 1. \end{cases} \tag{3.54}$$

Clearly, $c_0 = a_0$, so the statement holds for $n = 0$.
Now let us assume that for an arbitrary y, (3.53) holds for all $k \leq n-1$, and we proceed to prove the case for n. Now let $t_1 \in [y_n, y_n + 2^{-(n+1)}]$ and $t_2 \in [y_n, y_n + 2^{-(n+1)}]$. By applying Lemma 3.1.5, we have

$$t_1 \in [y_{n-1}, y_{n-1} + 2^{-n}] \quad \text{and} \quad t_2 \in [y_{n-1}, y_{n-1} + 2^{-n}].$$

Hence, by applying the (3.53), we have

$$\begin{aligned} \frac{x_{\mathbf{C},n}(t_1) - x_{\mathbf{C},n}(t_2)}{t_1 - t_2} &= \frac{x_{\mathbf{C},n-1}(t_1) - x_{\mathbf{C},n-1}(t_2)}{t_1 - t_2} + c_n\frac{\phi(2^n t_1) - \phi(2^n t_2)}{t_1 - t_2} \\ &= \Xi_n(\mathbf{A}) + c_n\frac{\phi(2^n t_1) - \phi(2^n t_2)}{t_1 - t_2} \end{aligned} \tag{3.55}$$

Then by (3.55), we only remain to prove

$$c_n \frac{\phi(2^n t_1) - \phi(2^n t_2)}{t_1 - t_2} = \xi_n a_n.$$

Since $t_1 \in [y_n, y_n + 2^{-(n+1)}]$ and $t_2 \in [y_n, y_n + 2^{-(n+1)}]$, we can rewrite $t_1 = y_{n-1} + 2^{-n}\tau_1$ and $t_2 = y_{n-1} + 2^{-n}\tau_2$, where $\tau_1, \tau_2 \in [0, 1)$. Furthermore, we have

$$\begin{cases} \tau_i \in [0, \frac{1}{2}) & \text{if and only if} \quad \varepsilon_n = 0, \\ \tau_i \in [\frac{1}{2}, 1) & \text{if and only if} \quad \varepsilon_n = 1, \end{cases}$$

for $i = 1, 2$. Then we have

$$\begin{aligned} c_n \frac{\phi(2^n t_1) - \phi(2^n t_2)}{t_1 - t_2} &= c_n \frac{\phi(2^n(y_{n-1} + \tau_1)) - \phi(2^n(y_{n-1} + \tau_2)}{t_1 - t_2} \\ &= c_n \frac{\phi(2^n(\sum_{i=0}^{n-1} \varepsilon_i 2^{-(i+1)} + 2^{-n}\tau_1)) - \phi(2^n(\sum_{i=0}^{n-1} \varepsilon_i 2^{-(i+1)} + 2^{-n}\tau_2))}{t_1 - t_2} \\ &= c_n \frac{\phi(\sum_{i=0}^{n-1} \varepsilon_i 2^{n-(i+1)} + \tau_1) - \phi(\sum_{i=0}^{n-1} \varepsilon_i 2^{n-(i+1)} + \tau_2)}{2^{-n}(\tau_1 - \tau_2)} \\ &= a_n \frac{\phi(\tau_1) - \phi(\tau_2)}{\tau_1 - \tau_2}, \end{aligned}$$

because $\sum_{i=0}^{n-1} \varepsilon_i 2^{n-(i+1)} \in \mathbb{Z}$. By applying (3.54), the case $n + 1$ will be proved. And therefore, we proved the lemma. □

Definition 3.1.23. Let $\Xi = \{\xi_i\}_{i=0}^{\infty}$ be the quasi-binary expansion for y, and $x_{\mathbf{C}} \in \mathfrak{C}$. Take $a_i = 2^i c_i$, and $\mathbf{A} = \{a_i\}$. Let

$$\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i. \tag{3.56}$$

If $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ for all $n \in \mathbb{N}$, then we say that $(\Xi, \mathbf{C})$ satisfies the step condition for maxima. And if $\Xi_n(\mathbf{A})\xi_{n+1} \geq 0$ for all $n \in \mathbb{N}$, then we call $(\Xi, \mathbf{C})$ satisfies the step condition for minima.

Lemma 3.1.24. Let $x_{\mathbf{C}} \in \mathfrak{C}$, and Ξ be any quasi-binary expansion for $y \in [0,1]$. If

$$\Xi_n(\mathbf{A})\xi_{n+1} \leq 0, \tag{3.57}$$

for all $n \leq N$. Let y_{N+1} be the $N+1^{th}$ order dyadic approximation of y, then We have

$$y_{N+1} = \tilde{t}_{N+1} \wedge t^*_{N+1}, \tag{3.58}$$

where $\tilde{t}_{N+1} \in \mathcal{M}_{\mathbf{C},N+1}$ and $t^*_{N+1} = \underset{t \in \mathbb{T}_{N+1}(\tilde{t}_{N+1})}{\arg\max}\ x_{\mathbf{C},N+1}(t)$.

Proof. By applying Lemma 3.1.22, it is sufficient for us to prove the following statement.

If $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$*, for all* $n \leq N$*. We have*

$$\Xi_k(\mathbf{A}) = \frac{x_{\mathbf{C},k}(\tilde{t}_k \vee t^*_k) - x_{\mathbf{C},k}(\tilde{t}_k \wedge t^*_k)}{\tilde{t}_k \vee t^*_k - \tilde{t}_k \wedge t^*_k} \tag{3.59}$$

for all $k \leq N+1$*.*

Let us prove this lemma by induction on n. Assuming $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ now, let us first of all consider the case $n = 0$. In this case, we have

$$x_{\mathbf{C},0}(t) = c_0 \min\{t, (1-t)\}_+.$$

Hence, for all $c_0 \in \mathbb{R}$, one of the following two cases must hold:

$$\begin{cases} \tilde{t}_0 \wedge t^*_0 = 0 \quad \text{and} \quad \tilde{t}_0 \vee t^*_0 = \frac{1}{2}, \\ \tilde{t}_0 \wedge t^*_0 = \frac{1}{2} \quad \text{and} \quad \tilde{t}_0 \vee t^*_0 = 1. \end{cases} \tag{3.60}$$

Furthermore, we have

$$\begin{cases} \xi_0 = 1, \quad \text{if} \quad y \in [0, \frac{1}{2}]. \\ \xi_0 = -1, \quad \text{if} \quad y \in [\frac{1}{2}, 1]. \end{cases}$$

It is obvious that

$$\begin{cases} y_0 = 0, & \text{if and only if} \quad \xi_0 = 1. \\ y_0 = \frac{1}{2}, & \text{if and only if} \quad \xi_0 = -1. \end{cases}$$

Also, we have $a_0 = 2^0 c_0 = c_0$. Those two cases will lead us to two feasible cases for the slope:

$$\frac{x_{\mathbf{C},0}(\tilde{t}_0 \vee t_0^*) - x_{\mathbf{C},0}(\tilde{t}_0 \wedge t_0^*)}{\tilde{t}_0 \vee t_0^* - \tilde{t}_0 \wedge t_0^*} = \begin{cases} \frac{x_{\mathbf{C},0}(\frac{1}{2}) - x_{\mathbf{C},0}(0)}{\frac{1}{2} - 0} = c_0 = a_0 = \xi_0 a_0 = \Xi_0(\mathbf{A}). \\ \\ \frac{x_{\mathbf{C},0}(1) - x_{\mathbf{C},0}(\frac{1}{2})}{1 - \frac{1}{2}} = -c_0 = -a_0 = \xi_0 a_0 = \Xi_0(\mathbf{A}). \end{cases} \tag{3.61}$$

So we get (3.59) for $k = 0$ even without having to require $\Xi_0(\mathbf{A})\xi_1 \leq 0$. Hence, regardless of the choice of $\tilde{t}_0$ and t_0^*, we have

$$\frac{x_{\mathbf{C},0}(\tilde{t}_0 \vee t_0^*) - x_{\mathbf{C},0}(\tilde{t}_0 \wedge t_0^*)}{\tilde{t}_0 \vee t_0^* - \tilde{t}_0 \wedge t_0^*} = \Xi_0(\mathbf{A}).$$

Now let us assume that (3.57) implies (3.59) for all $k \leq n - 1$, and we now prove the case for $k = n$. By plugging $k = n - 1$ into (3.59), we have

$$\Xi_{n-1}(\mathbf{A}) = \frac{x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*) - x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \wedge t_{n-1}^*)}{\tilde{t}_{n-1} \vee t_{n-1}^* - \tilde{t}_{n-1} \wedge t_{n-1}^*} \tag{3.62}$$

For instance, let us assume that $\Xi_{n-1}(\mathbf{A}) > 0$. Then we have

$$x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*) - x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \wedge t_{n-1}^*) = \Xi_{n-1}(\mathbf{A})(\tilde{t}_{n-1} \vee t_{n-1}^* - \tilde{t}_{n-1} \wedge t_{n-1}^*) > 0.$$

Moreover, since $x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*) > x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*)$, we have

$$\begin{cases} \tilde{t}_{n-1} \vee t_{n-1}^* = \tilde{t}_{n-1}, \\ \tilde{t}_{n-1} \wedge t_{n-1}^* = t_{n-1}^*. \end{cases}$$

Also, by the step condition for maxima, we have

$$\xi_n = -1. \tag{3.63}$$

As $\tilde{t}_n > t_n^*$ applying Corollary 3.1.4, we have $\tilde{t}_n \wedge t_n^* = \frac{\tilde{t}_{n-1}+t_{n-1}^*}{2}$, and $\tilde{t}_n \vee t_n^* = \tilde{t}_{n-1}$. Therefore, since $\tilde{t}_{n-1} \in \mathbb{T}_n$ and $t_{n-1}^* \in \mathbb{T}_n$, we have $2^n\tilde{t}_{n-1} \in \mathbb{Z}$ as well as $2^n t_{n-1}^* = 2^n(\tilde{t}_{n-1} \pm 2^{-n}) \in \mathbb{Z}$. Hence, we have

$$\phi(2^n t_{n-1}^*) = \phi(2^n \tilde{t}_{n-1}) = 0. \tag{3.64}$$

Furthermore, we have

$$\phi(2^n \frac{\tilde{t}_{n-1} + t_{n-1}^*}{2}) = \frac{1}{2}. \tag{3.65}$$

By plugging (3.62),(3.64) and (3.65) into (3.55), we have

$$\begin{aligned}
\frac{x_{\mathbf{C},n}(\tilde{t}_n \vee t_n^*) - x_{\mathbf{C},n}(\tilde{t}_n \wedge t_n^*)}{\tilde{t}_n \vee t_n^* - \tilde{t}_n \wedge t_n^*} &= \Xi_{n-1}(\mathbf{A}) + c_n \frac{\phi(2^n \tilde{t}_{n-1} \vee t_{n-1}^*) - \phi(2^n \tilde{t}_{n-1} \wedge t_{n-1}^*)}{\tilde{t}_{n-1} \vee t_{n-1}^* - \tilde{t}_{n-1} \wedge t_{n-1}^*} \\
&= \Xi_{n-1}(\mathbf{A}) + c_n \frac{\phi(2^n(\frac{\tilde{t}_{n-1}+t_{n-1}^*}{2})) - \phi(2^n \tilde{t}_{n-1})}{\frac{\tilde{t}_{n-1}+t_{n-1}^*}{2} - \tilde{t}_{n-1}} \\
&= \Xi_{n-1}(\mathbf{A}) - 2^n c_n = \Xi_{n-1}(\mathbf{A}) + \xi_n a_n = \Xi_n(\mathbf{A}).
\end{aligned}$$

The last step holds because of (3.63). Hence, we have proved the case $\Xi_{n-1}(\mathbf{A}) > 0$. The case $\Xi_{n-1}(\mathbf{A}) < 0$ is analogous to the case $\Xi_{n-1}(\mathbf{A}) > 0$. Now let us further consider the case when $\Xi_{n-1}(\mathbf{A}) = 0$. Then we have

$$x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*) - x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \wedge t_{n-1}^*) = \Xi_{n-1}(\mathbf{A})(\tilde{t}_{n-1} \vee t_{n-1}^* - \tilde{t}_{n-1} \wedge t_{n-1}^*) = 0.$$

Then we have

$$x_{\mathbf{C},n-1}(t) = x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \vee t_{n-1}^*) = x_{\mathbf{C},n-1}(\tilde{t}_{n-1} \wedge t_{n-1}^*). \tag{3.66}$$

for all $t \in [\tilde{t}_{n-1} \wedge t_{n-1}^*, \tilde{t}_{n-1} \vee t_{n-1}^*]$. Because $\tilde{t}_{n-1} \vee t_{n-1}^* \in \mathbb{T}_n$ and $\tilde{t}_{n-1} \wedge t_{n-1}^* \in \mathbb{T}_n$, then

we have

$$\phi(2^n(\tilde{t}_{n-1} \vee t^*_{n-1})) = \phi(2^n(\tilde{t}_{n-1} \wedge t^*_{n-1})) = 0. \tag{3.67}$$

As well as,

$$\phi(\frac{1}{2}(2^n(\tilde{t}_{n-1} \vee t^*_{n-1}) + 2^n(\tilde{t}_{n-1} \wedge t^*_{n-1}))) = \frac{1}{2}. \tag{3.68}$$

Similar to (3.60), one of two following cases must be true:

$$\begin{cases} \tilde{t}_n \wedge t^*_n = \tilde{t}_{n-1} \wedge t^*_{n-1} \quad \text{and} \quad \tilde{t}_n \vee t^*_n = \tilde{t}_{n-1} \wedge t^*_{n-1} + 2^{-(n+1)}, \\ \tilde{t}_n \wedge t^*_n = \tilde{t}_{n-1} \wedge t^*_{n-1} + 2^{-(n+1)} \quad \text{and} \quad \tilde{t}_n \vee t^*_n = \tilde{t}_{n-1} \vee t^*_{n-1}. \end{cases}$$

Furthermore, by applying the inductive hypothesis and Lemma 3.1.22, we get

$$y_{n-1} = \tilde{t}_{n-1} \wedge t^*_{n-1}. \tag{3.69}$$

Therefore, (3.69) will directly lead us to

$$\begin{cases} y_n = y_{n-1} = \tilde{t}_{n-1} \wedge t^*_{n-1} \quad \text{if} \quad \xi_n = 1. \\ y_n = y_{n-1} + 2^{-(n+1)} = \tilde{t}_{n-1} \wedge t^*_{n-1} + 2^{-(n+1)} \quad \text{if} \quad \xi_n = -1. \end{cases} \tag{3.70}$$

By applying (3.66), (3.67) and (3.68). We must have one of following two cases,

$$\begin{aligned} &\frac{x_{\mathbf{C},n-1}(\tilde{t}_n \vee t^*_n) - x_{\mathbf{C},n-1}(\tilde{t}_n \wedge t^*_n)}{\tilde{t}_n \vee t^*_n - \tilde{t}_n \wedge t^*_n} \\ &= \frac{x_{\mathbf{C},n}(\tilde{t}_n \vee t^*_n) - x_{\mathbf{C},n}(\tilde{t}_n \wedge t^*_n) + \phi(2^n\tilde{t}_n \vee t^*_n) - \phi(2^n\tilde{t}_n \wedge t^*_n)}{\tilde{t}_n \vee t^*_n - \tilde{t}_n \wedge t^*_n} \\ &= \begin{cases} c_n \frac{\phi(2^n(\tilde{t}_{n-1} \wedge t^*_{n-1} + 2^{-(n+1)})) - \phi(2^n(\tilde{t}_{n-1} \wedge t^*_{n-1}))}{2^{-(n+1)}} = 2^n c_n = a_n = \Xi_{n-1}(\mathbf{A}) + a_n = \Xi_n(\mathbf{A}), \\ \\ c_n \frac{\phi(2^n\tilde{t}_{n-1} \vee t^*_{n-1}) - \phi(2^n(\tilde{t}_{n-1} \wedge t^*_{n-1} + 2^{-(n+1)}))}{2^{-(n+1)}} = -2^n c_n = -a_n = \Xi_{n-1}(\mathbf{A}) - a_n = \Xi_n(\mathbf{A}). \end{cases} \end{aligned} \tag{3.71}$$

The last step is due to (3.70). Hence, regardless of the choice of $\tilde{t}_n$ and t_n^*, we always have

$$\Xi_n(\mathbf{A}) = \frac{x_{\mathbf{C},n}(\tilde{t}_n \vee t_n^*) - x_{\mathbf{C},n}(\tilde{t}_n \wedge t_n^*)}{\tilde{t}_n \vee t_n^* - \tilde{t}_n \wedge t_n^*}.$$

Therefore, we have proved the case for n, and, we proved for this lemma. □

Theorem 3.1.25. For $x_{\mathbf{C}} \in \mathfrak{C}$, we have $y \in \mathcal{M}_{\mathbf{C}}$ if and only if there exists a quasi-binary Ξ expansion of y such that $(\Xi, \mathbf{C})$ satisfies the step condition for maxima.

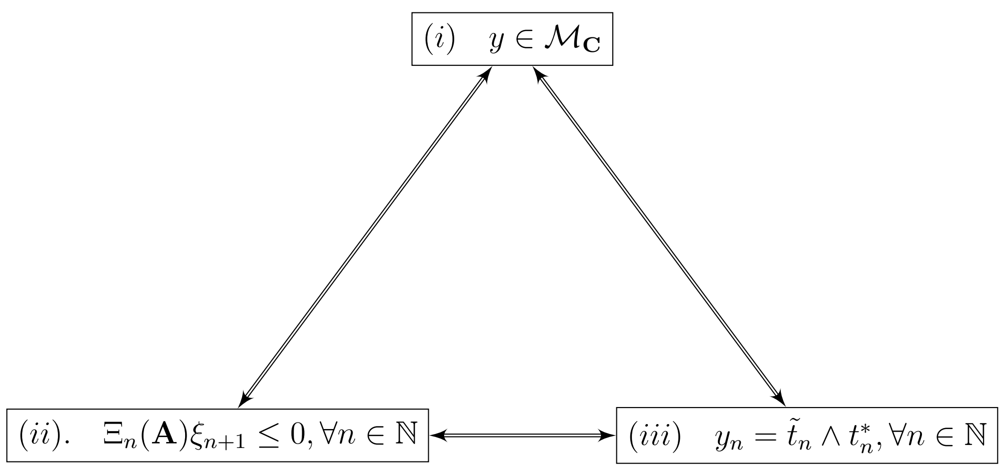

Figure 3.5: Equivalence Relation between Statements in Theorem 3.1.25

Proof. In order to prove Theorem 3.1.25, we will use Lemma 3.1.22 and Theorem 3.1.18 to establish an equivalent statement. Readers may refer Figure 3.5 as a reference to establish such a relation. First of all, we may formally write down the statement in Figure 3.5.

- (i) For $x_{\mathbf{C}} \in \mathfrak{C}$, $y \in \mathcal{M}_{\mathbf{C}}$.
- (ii) Let $y \in [0,1]$, and y_n be the n^{th} order approximation for y. There exists a sequence of consecutive maximizers $\tilde{t}_n \in \mathcal{M}_{\mathbf{C},n}$ and $t_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tilde{t}_n)}{\arg\max}\, x_{\mathbf{C},n}(t)$, such that $y_n = \tilde{t}_n \wedge t_n^*$ for all $n \in \mathbb{N}$.

- (iii) Let $\Xi = \{\xi_i\}_{i=0}^{\infty}$ be any quasi-binary expansion for y, and $x_{\mathbf{C}} \in \mathfrak{C}$. Take $a_i = 2^i c_i$, and $\mathbf{A} = \{a_i\}$, we have $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ for all $n \in \mathbb{N}$.

Theorem 3.1.25 states the equivalence between (i) and (ii). Furthermore, Theorem 3.1.18 establishes the equivalence of (i) and (iii). Therefore, it is sufficient for us to prove the equivalence between (ii) and (iii). Moreover, Lemma 3.1.22 indicates that

$$\frac{x_{\mathbf{C},n}(t_1) - x_{\mathbf{C},n}(t_2)}{t_1 - t_2} = \Xi_n(\mathbf{A}),$$

for any $t_1, t_2 \in [y_n, y_n + 2^{-(n+1)}]$. Now let us take $t_1 = y_n = \tilde{t}_n \wedge t_n^*$. Then $t_2 = y_n + 2^{-(n+1)} = \tilde{t}_n \vee t_n^*$ for all $n \in \mathbb{N}$. We now aim to prove the following statement.

$\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ *for all* $n \in \mathbb{N}$ *if and only if*

$$\Xi_n(\mathbf{A}) = \frac{x_{\mathbf{C},n}(\tilde{t}_n \vee t_n^*) - x_{\mathbf{C},n}(\tilde{t}_n \wedge t_n^*)}{\tilde{t}_n \vee t_n^* - \tilde{t}_n \wedge t_n^*}, \tag{3.72}$$

where $\{\tilde{t}_n\}$ *is some sequence of consecutive maximizers.*

Then the only if direction is directly proved by Lemma 3.1.24. Now let us consider proving if part by proving its contrapositive statement. First of all, let us state the contrapositive statement:

Let $\{\xi_i\}$ *be any quasi binary expansion for* y*. If there exist some* $n \in \mathbb{N}$ *such that* $\Xi_n(\mathbf{A})\xi_{n+1} > 0$*, then there exists* $m \in \mathbb{N}$*, such that* $y_m \neq \tilde{t}_m \wedge t_m^*$*, for any sequence of consecutive maximizers* $\{\tilde{t}_n\}$*.*

Now let us take $\mathcal{N} = \{n \in \mathbb{N} | \Xi_n(\mathbf{A})\xi_{n+1} > 0\}$, and $n = \min \mathcal{N}$. Then by Lemma 3.1.24, we have

$$y_n = \tilde{t}_n \wedge t_n^*.$$

Now let us first assume that $\Xi_n(\mathbf{A}) > 0$. Then we also have $\xi_{n+1} = 1$. Now we aim to prove that

$$y_{n+1} \neq \tilde{t}_{n+1} \wedge t_{n+1}^*.$$

We have

$$x_{\mathbf{C},n}(\tilde{t}_n \vee t_n^*) - x_{\mathbf{C},n}(\tilde{t}_n \wedge t_n^*) = \Xi_n(\mathbf{A})(\tilde{t}_n \vee t_n^* - \tilde{t}_n \wedge t_n^*) > 0,$$

by (3.72). Then we have $\tilde{t}_n = \tilde{t}_n \vee t_n^*$ and $t_n^* = \tilde{t}_n \wedge t_n^*$. By applying Corollary 3.1.4, we have

$$\tilde{t}_{n+1} \vee t_{n+1}^* = \tilde{t}_n, \quad \text{and} \quad \tilde{t}_{n+1} \wedge t_{n+1}^* = \frac{\tilde{t}_n + t_n^*}{2}.$$

Moreover, by Corollary 3.1.5, we have

$$[t_{n+1}^* \wedge \tilde{t}_{n+1}, t_{n+1}^* \vee \tilde{t}_{n+1}] \subset [t_n^* \wedge \tilde{t}_n, t_n^* \vee \tilde{t}_n].$$

Let $E = \{\varepsilon_j\}$ be the binary expansion for y. Since $\xi_{n+1} = 1$, we have

$$\varepsilon_{n+1} = \mathcal{H}^{-1}(\Xi)_{n+1} = 0.$$

Then we have

$$y_{n+1} = y_n + \varepsilon_{n+1} 2^{-(n+2)} = y_n.$$

Hence, obviously

$$y_{n+1} = \tilde{t}_n \wedge t_n^* \neq \tilde{t}_{n+1} \wedge t_{n+1}^* = \frac{\tilde{t}_n + t_n^*}{2}.$$

Hence, we have finished the proof for the case $\Xi_n(\mathbf{A}) > 0$. The case $\Xi_n(\mathbf{A}) < 0$ is analogous. Hence, we finish our proof for the statement. □

Theorem 3.1.26. For $x_{\mathbf{C}} \in \mathfrak{C}$, $y \in \tilde{\mathcal{M}}_{\mathbf{C}}$ if and only if $(y, \mathbf{C})$ satisfies the step condition for minima.

Proof. Take $\mathbf{D} = -\mathbf{C}$, then we have

$$\min_{t \in [0,1]} x_{\mathbf{C}} = -\max_{t \in [0,1]} x_{\mathbf{D}}. \tag{3.73}$$

Then we have,

$$\tilde{\mathcal{M}}_{\mathbf{C}} = \mathcal{M}_{\mathbf{D}}. \tag{3.74}$$

By Theorem 3.1.25, we have for all $n \in \mathbb{N}$,

$$\Xi_n(-\mathbf{A})\xi_{n+1} = -\Xi_n(\mathbf{A})\xi_{n+1} \leq 0.$$

Therefore, we have

$$\Xi_n(\mathbf{A})\xi_{n+1} \geq 0.$$

□

Corollary 3.1.27. Recall that $a_i = 2^i c_i$, then $x_{\mathbf{C}}(t)$ will be non-negative if $\sum_{i=0}^n a_i > 0$ for all $n \in \mathbb{N}$.

Proof. $x_{\mathbf{C}}(t)$ is non-negative if and only if

$$x_{\mathbf{C}}(t) \geq 0, \quad \text{for all } t \in [0,1].$$

Hence we have

$$0 \in \arg\min_{t\in[0,\frac{1}{2}]} x_{\mathbf{C}}(t). \tag{3.75}$$

This leads to $\xi_n = 1$ for all $n \in \mathbb{N}$. Therefore, by applying the step condition for minima, we have

$$\sum_{i=0}^n a_i > 0.$$

□

In order to have a better representation for following theorems concerning Hausdorff dimension and uniqueness of the extremum location, we will introduce following definitions and propositions.

Definition 3.1.28. Let $x_{\mathbf{C}} \in \mathfrak{C}$. Take $\mathbf{A} = \{2^i c_i\}$. Let us define a mapping $L_\sharp : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{S}^{\mathbb{N}}$, where $L_\sharp(\mathbf{C}) = \{\xi_n\}_{n=0}^{\infty}$, and for $i \geq 1$,

$$\xi_{i+1} = \begin{cases} -1 & \text{if} \quad \Xi_i(\mathbf{A}) > 0, \\ 1 & \text{if} \quad \Xi_i(\mathbf{A}) \leq 0, \end{cases} \tag{3.76}$$

with $\xi_0 = 1$, and $\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i$.

Definition 3.1.29. Let $x_{\mathbf{C}} \in \mathfrak{C}$. Take $\mathbf{A} = \{2^i c_i\}$. Let us define a mapping $L_\flat : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{S}^{\mathbb{N}}$, where $L_\flat(\mathbf{C}) = \{\xi_n\}_{n=0}^{\infty}$, and for $i \geq 1$,

$$\xi_{i+1} = \begin{cases} -1 & \text{if} \quad \Xi_i(\mathbf{A}) \geq 0, \\ 1 & \text{if} \quad \Xi_i(\mathbf{A}) < 0, \end{cases} \tag{3.77}$$

with $\xi_0 = 1$, and $\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i$.

Definition 3.1.30. Let $x_{\mathbf{C}} \in \mathfrak{C}$. Take $\mathbf{A} = \{2^i c_i\}$. Let us define a mapping $J_\sharp : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{S}^{\mathbb{N}}$, where $J_\sharp(\mathbf{C}) = \{\xi_n\}_{n=0}^{\infty}$, and for $i \geq 1$,

$$\xi_{i+1} = \begin{cases} 1 & \text{if} \quad \Xi_i(\mathbf{A}) > 0, \\ -1 & \text{if} \quad \Xi_i(\mathbf{A}) \leq 0, \end{cases} \tag{3.78}$$

with $\xi_0 = 1$, and $\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i$.

Definition 3.1.31. Let $x_{\mathbf{C}} \in \mathfrak{C}$. Take $\mathbf{A} = \{2^i c_i\}$. Let us define a mapping $J_\flat : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{S}^{\mathbb{N}}$, where $J_\flat(\mathbf{C}) = \{\xi_n\}_{n=0}^{\infty}$, and for $i \geq 1$,

$$\xi_{i+1} = \begin{cases} 1 & \text{if} \quad \Xi_i(\mathbf{A}) \geq 0, \\ -1 & \text{if} \quad \Xi_i(\mathbf{A}) < 0, \end{cases} \tag{3.79}$$

with $\xi_0 = 1$, and $\Xi_n(\mathbf{A}) = \sum_{i=0}^{n} \xi_i a_i$.

Definition 3.1.32. Let $T : \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]$, such that

$$T(\{\varepsilon_j\}) = \sum_{i=0}^{\infty} \varepsilon_i 2^{-(i+1)}. \tag{3.80}$$

Actually, T is a mapping that transforms the binary expansion back to a number in unit interval.

Definition 3.1.33. Let $F_\sharp : \mathbb{R}^{\mathbb{N}} \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$F_\sharp = T \circ \mathcal{H}^{-1} \circ L_\sharp.$$

We will say that $F_\sharp$ is the mapping for *upper maximizer* of Takagi Class on the lower half.

Definition 3.1.34. Let $F_\flat : \mathbb{R}^{\mathbb{N}} \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$F_\flat = T \circ \mathcal{H}^{-1} \circ L_\flat.$$

We will say that $F_\flat$ is the mapping for *lower maximizer* of Takagi Class on the lower half.

Definition 3.1.35. Let $G_\sharp : \mathbb{R}^{\mathbb{N}} \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$G_\sharp = T \circ \mathcal{H}^{-1} \circ J_\sharp.$$

We will say that $G_\sharp$ is the mapping for *upper minimizer* of Takagi Class on the lower half.

Definition 3.1.36. Let $G_\flat : \mathbb{R}^{\mathbb{N}} \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$G_\flat = T \circ \mathcal{H}^{-1} \circ J_\flat.$$

We will say that $G_\flat$ is the mapping for *lower minimizer* of Takagi Class on the lower half.

Theorem 3.1.37. For any $x_{\mathbf{C}}$, we have $F_\sharp(\mathbf{C}) \in \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]$ and $F_\flat(\mathbf{C}) \in \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]$.

Proof. For simplicity, let us take $\mathbf{A} = \{2^i c_i\}$, and $\Xi = L_\sharp(\mathbf{C})$. By applying (3.76), we have

$$\Xi_n(\mathbf{A})\xi_{i+1} \leq 0, \quad \text{for every} \quad n \in \mathbb{N}.$$

Hence, $(\Xi, \mathbf{A})$ satisfies the step condition for maxima. Let y be a point whose quasi-binary expansion is $\Xi = L_\sharp(\mathbf{C})$. Therefore, by applying Theorem 3.1.25, we have

$$(T \circ \mathcal{H}^{-1}) \circ L_\sharp(\nu) = F_\sharp(\mathbf{C}) = y \in \mathcal{M}_{\mathbf{C}}.$$

Furthermore, since $\xi_0 = 1$, we have

$$\mathcal{H}^{-1}(\Xi)_0 = 0.$$

Therefore, by Lemma 3.1.10, we have

$$T \circ \mathcal{H}^{-1}(\Xi) \in [0, \frac{1}{2}].$$

Hence,

$$(T \circ \mathcal{H}^{-1}) \circ L_\sharp(\mathbf{C}) = F_\sharp(\mathbf{C}) \in \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}].$$

The proof for $F_\flat(\mathbf{C}) \in \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]$ is analogous to this proof. □

Theorem 3.1.38. For any $x_{\mathbf{C}}$, we have $G_\sharp(\mathbf{C}) \in \tilde{\mathcal{M}}_{\mathbf{C}} \cap [0, \frac{1}{2}]$ and $G_\flat(\mathbf{C}) \in \tilde{\mathcal{M}}_{\mathbf{C}} \cap [0, \frac{1}{2}]$.

Proof. The proof is analogous to Theorem 3.1.37. □

Proposition 3.1.39. Let $\Gamma = \{\gamma_i\}_{i=0}^\infty \in \{0,1\}^{\mathbb{N}}$ and $\Lambda = \{\lambda_i\}_{i=0}^\infty \in \{0,1\}^{\mathbb{N}}$. Denote $\mathcal{N} = \{n \in \mathbb{N} | \gamma_n \neq \lambda_n\}$, and $n = \min \mathcal{N}$. Then $T(\Gamma) \geq T(\Lambda)$ if and only if $\gamma_n = 1$, and the equality holds if

$$\gamma_k = 0 \quad \text{and} \quad \lambda_k = 1,$$

for all $k \geq n+1$.

Proof. Now let us first of all prove the if part, according to (3.80), we have

$$T(\Gamma) - T(\Lambda) = \sum_{i=0}^\infty (\gamma_i - \lambda_i)(\frac{1}{2})^i = (\frac{1}{2})^{n+1} + \sum_{i=n+2}^\infty (\gamma_i - \lambda_i)(\frac{1}{2})^i. \tag{3.81}$$

As $\gamma_i - \lambda_i \in \{-1, 0, 1\}$ for all $i \in \mathbb{N}$, then we have

$$\inf_{\Gamma, \Lambda \in \{0,1\}^{\mathbb{N}}} \sum_{i=n+2}^\infty (\gamma_i - \lambda_i)(\frac{1}{2})^i = -\sum_{i=n+2}^\infty (\frac{1}{2})^i = -(\frac{1}{2})^{n+1}. \tag{3.82}$$

By applying (3.82) into (3.81), we have

$$L(\Gamma) - L(\Lambda) \geq 0,$$

By symmetry of the algebraic structure, we can directly obtain the only if part. Now (3.82) indicates equality holds if and only if $\sum_{i=n+2}^{\infty}(\gamma_i - \lambda_i)(\frac{1}{2})^i = -(\frac{1}{2})^{n+1}$. This condition satisfies if and only if $\gamma_k = 0$ and $\lambda_k = 1$, for all $k \geq n+2$. Hence, we finish the proof. □

Theorem 3.1.40. For any $x_{\mathbf{C}} \in \mathfrak{C}$, we have

$$F_{\sharp}(\mathbf{C}) = \sup \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}], \quad \text{and} \quad F_{\flat}(\mathbf{C}) = \inf \mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}].$$

Proof. Take $\mathbf{A} = \{2^i c_i\}$. Now let us consider any sequence $\Gamma \in \mathbb{S}^{\mathbb{N}}$ such that $(\Gamma, \mathbf{A})$ satisfies the step condition. For simplicity, let us denote $\Xi^{\sharp} = L_{\sharp}(\mathbf{A})$ and $\Xi^{\flat} = L_{\flat}(\mathbf{A})$. Denote $\mathcal{N} = \{n \in \mathbb{N} | \xi_n^{\sharp} \neq \gamma_n\}$, and $\mathcal{K} = \{n \in \mathbb{N} | \xi_n^{\flat} \neq \gamma_n\}$. Now take $n = \min \mathcal{N}$ and $k = \min \mathcal{K}$. Since we have $\xi_n^{\sharp} \neq \gamma_n$, and both $(\Xi^{\sharp}, \mathbf{A})$ and $(\Gamma, \mathbf{A})$ satisfies the step condition, we have $\Gamma_{n-1}(\mathbf{A}) = \Xi_{n-1}^{\sharp}(\mathbf{A}) = 0$, therefore, by applying (3.76), we have $\xi_n^{\sharp} = -1$, and $\gamma_n = 1$. Therefore, by applying (3.52), we have $\mathcal{H}^{-1}(\Xi^{\sharp})_n = 1$ and $\mathcal{H}^{-1}(\Gamma)_n = 0$. Therefore, by applying Proposition 3.1.39, we have

$$F_{\sharp}(\nu) \geq \tilde{t},$$

for every $\tilde{t} \in \mathcal{M}_{\nu}$. Now let us consider the lower bound for the set of maxima. Since we have $\xi_k^{\flat} \neq \gamma_k$, and both $(\Xi^{\flat}, \mathbf{A})$ and $(\Gamma, \mathbf{A})$ satisfies the step condition, we have $\Gamma_{k-1}(\mathbf{A}) = \Xi_{k-1}^{\flat}(\mathbf{A}) = 0$, therefore, by applying (4.2), we have $\xi_k^{\flat} = 1$, and $\gamma_k = -1$. Therefore, by applying (3.52), we have $\mathcal{H}^{-1}(\Xi^{\flat})_k = 0$ and $\mathcal{H}^{-1}(\Gamma)_k = 1$. Therefore, by applying Proposition 3.1.39, we have

$$F_{\flat}(\nu) \leq \tilde{t},$$

for all $\tilde{t} \in \mathcal{M}_{\nu}$. □

Theorem 3.1.41. For any $x_{\mathbf{C}} \in \mathfrak{C}$, we have

$$G_{\sharp}(\mathbf{C}) = \sup \tilde{\mathcal{M}}_{\mathbf{C}} \cap [0, \frac{1}{2}], \quad \text{and} \quad G_{\flat}(\mathbf{C}) = \inf \tilde{\mathcal{M}}_{\mathbf{C}} \cap [0, \frac{1}{2}].$$

Lemma 3.1.42. Denote $\Xi = L_\sharp(\mathbf{C})$. For a fixed $\mathbf{C} \in \mathbb{R}^{\mathbb{N}}$, if $\Xi_n(\mathbf{A}) \neq 0$ for all $n \in \mathbb{N}$, then

$$F_\sharp(\mathbf{C}) = F_\flat(\mathbf{C}).$$

Proof. Since $\Xi_n(\mathbf{A}) \neq 0$ for all $n \in \mathbb{N}$, then by applying (4.3) and (4.2), we have

$$L_\sharp(\mathbf{C}) = L_\flat(\mathbf{C}).$$

Therefore, we have

$$F_\sharp(\mathbf{C}) = (T \circ \mathcal{H}^{-1})(L_\sharp(\mathbf{C})) = (T \circ \mathcal{H}^{-1})(L_\flat(\mathbf{C})) = F_\flat(\mathbf{C}).$$

□

Corollary 3.1.43. Denote $\Xi = L_\sharp(\mathbf{C})$, and $\mathbf{A} = \{2^i c_i\}_{i=0}^{\infty}$. For a fixed $\mathbf{C}$, if $\Xi_n(\mathbf{A}) \neq 0$ for all $n \in \mathbb{N}$, then

$$|\mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]| = 1.$$

Remark 3.1.44. Corollary 3.1.43 indicates that if $\Xi_n(\mathbf{A}) \neq 0$, then the maximizer on $[0, \frac{1}{2}]$ is unique.

Proof. This directly comes as a corollary from Corollary 3.1.43. □

Lemma 3.1.45. Let $x_{\mathbf{C}} \in \mathfrak{C}$, and take $\mathbf{A} = \{2^i c_i\}_{i=0}^{\infty}$. Take $\mathcal{N} = \{n \in \mathbb{N} | \Xi_n(\mathbf{A}) = 0\}$. Suppose that $|\mathcal{N}| < \infty$ and $\mathcal{M}_{\mathbf{C}} \cap \mathbb{T} = \emptyset$, then $|\mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]| = 2^{|\mathcal{N}|}$.

Proof. We will prove this theorem by induction on $|\mathcal{N}|$. First of all, let us consider the case when $|\mathcal{N}| = 0$. As we have for all $n \in \mathbb{N}$ that $\Xi_n(\mathbf{A}) \neq 0$. Then by applying Corollary 3.1.43, we have

$$F_\sharp(\mathbf{A}) = F_\flat(\mathbf{A}).$$

Therefore, $|\mathcal{M}_{\mathbf{C}} \cap [0, \frac{1}{2}]| = 2^0 = 1$, and we have proved the assumption for $|\mathcal{N}| = 0$. Now let us assume that the induction hypothesis holds for $|\mathcal{N}| = k$, and then we proceed to prove the case for $|\mathcal{N}| = k + 1$. Take $p = \min \mathcal{N}$, $\mathcal{N}_k = \{n > k | \Xi_n(\mathbf{A}) = 0\}$. Therefore, we have $\Xi_p(\mathbf{A}) = 0$, and $|\mathcal{N}_p| = k$. Furthermore, let us denote $y = T \circ \mathcal{H}^{-1}(\Xi)$, and y_n be the n^{th} order binary approximation of y. Then we have that

$$\begin{aligned}\Xi(\mathbf{A}) &= \sum_{i=0}^{\infty} \xi_i a_i = \sum_{i=0}^{p} \xi_i a_i + \sum_{i=p+1}^{\infty} \xi_i a_i = \Xi_p(\mathbf{A}) + \sum_{i=p+1}^{\infty} \xi_i a_i \\ &= \sum_{i=0}^{\infty} \xi_{i+p+1} a_{i+p+1} =: \Xi^{p+1}(\mathbf{A}).\end{aligned} \tag{3.83}$$

Furthermore, let us take $\tau(t) = 2^{p+1}(t - y_p)$. When $t \in [y_p, y_p + 2^{-(p+1)}]$, then we have $\tau \in [0, 1]$. Moreover, let us denote $\Gamma = \{\gamma_i\}$, where $\gamma_i = \xi_{i+p+1}$ and $\mathbf{D} = \{d_i\}$, where $d_i = c_{i+p+1}$, and $\mathbf{B} = \{2^i d_i\}$. Since

$$\Xi_p(\mathbf{A}) = 0,$$

we have for $t \in [y_p, y_p + 2^{-(p+1)}]$

$$x_{\mathbf{C},p}(t) = x_{\mathbf{C},p}(y_p) = x_{\mathbf{C},p}(y_p + 2^{-(p+1)}). \tag{3.84}$$

Furthermore, for $t \in [y_p, y_p + 2^{-(p+1)}]$, we have

$$\mathcal{M}_{\mathbf{C},p} \subset [y_p, y_p + 2^{-(p+1)}],$$

and

$$\begin{aligned}x_{\mathbf{C}}(t) &= \sum_{i=0}^{\infty} c_i \phi(2^i t) = x_{\mathbf{C},p}(y_p) + \sum_{n=p+1}^{\infty} c_i \phi(2^i t) \\ &= x_{\mathbf{C},p}(y_p) + \sum_{i=0}^{\infty} c_i \phi(2^{p+1+i} t) \\ &= x_{\mathbf{C},p}(y_p) + \sum_{i=0}^{\infty} c_{i+p+1} \phi(2^{p+1+i}(t - y_p)) \quad (2^{p+1} y_p \in \mathbb{Z}) \\ &= x_{\mathbf{C},p}(y_p) + \sum_{i=0}^{\infty} d_i \phi(2^i \tau) \\ &= x_{\mathbf{C},p}(y_p) + x_{\mathbf{D}}(\tau).\end{aligned} \tag{3.85}$$

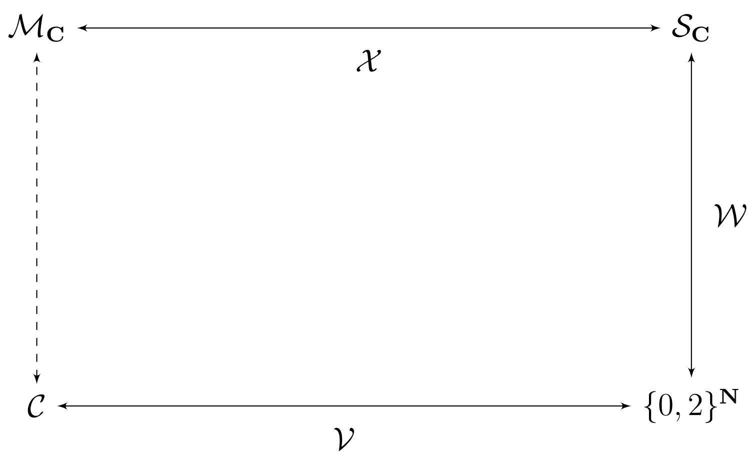

Figure 3.6: Isomophism between Cantor set $\mathcal{C}$ and $\mathcal{M}_{\mathbf{C}}$

We can do the last step due to $\tau \in [0,1]$. Furthermore, we have

$$\begin{aligned}\Xi^{p+1}(\mathbf{A}) &= \sum_{i=0}^{\infty} \xi_{i+p+1} a_{i+p+1} = \sum_{i=0}^{\infty} 2^{p+i+1} \xi_{i+p+1} c_{i+p+1} \\ &= 2^{p+1} \sum_{i=0}^{\infty} \gamma_i b_i = 2^{p+1} \Gamma(\mathbf{B}).\end{aligned} \tag{3.86}$$

Since $(\Xi, \mathbf{A})$ satisfies the step condition for maxima, then we can apply induction hypothesis and Lemma 3.1.10 here. As $\Xi_p(\mathbf{A}) = 0$, then both $\xi_{p+1} = \gamma_0 \in \{-1, 1\}$. Now for $\gamma_0 = 1$, then $x_{\mathbf{D}}(\tau)$ will have 2^k maximum points in $[0, \frac{1}{2}]$, hence then $x_{\mathbf{C}}(t)$ will have 2^k maximum points in $(y_n, y_n + 2^{-(n+2)})$. The interval is open due to $\mathcal{M}_{\mathcal{C}} \cap \mathbb{T} = \emptyset$. Similar, for $\gamma_0 = -1$, then $x_{\mathbf{D}}(\tau)$ will have 2^k maximum points in $[\frac{1}{2}, 1]$, hence then $x_{\mathbf{C}}(t)$ will have 2^k maximum points in $(y_n + 2^{-(n+2)}, y_n + 2^{-(n+1)})$. Hence, for $x_{\mathbf{C}}$, there will be 2^{k+1} maximum points. Therefore, we prove the statement for $N = k+1$, and hence, we finish proving Lemma 3.1.45. □

Lemma 3.1.46. Let $x_{\mathbf{C}} \in \mathfrak{C}$, and take $\mathbf{A} = \{2^i c_i\}_{i=0}^{\infty}$, $\Xi = L_\sharp(\mathbf{A})$. Take $\mathcal{N} = \{n \in \mathbb{N} | \Xi_n(\mathbf{A}) = 0\}$, and $\mathcal{N}_k = \{n \leq k | \Xi_n(\mathbf{A}) = 0\}$. Suppose that $|\mathcal{N}| = \infty$, then $|\mathcal{M}_{\mathbf{C}}| = 2^{\aleph_0}$.

Proof. Denote $\mathcal{C}$ as the Cantor set. A well-known fact is that $|\mathcal{C}| = 2^{\aleph_0}$. Furthermore, let us numerate the set $\mathcal{N} = \{p_0, p_1, \cdots, p_n, \cdots\}$, where $\{p_i\}_{i=0}^{\infty}$ is an strictly increasing sequence. For simplicity, we set $p_0 = -1$. Now let us denote the $\mathcal{S}_{\mathbf{C}} = \{\Xi \in \mathbb{S}^{\mathbb{N}} | (\Xi, \mathbf{C})$ satisfies step condition for maixma$\}$. According to Theorem 3.1.25, $\mathcal{X} = T \circ \mathcal{H}^{-1}$ is a bijective mapping for $\mathcal{S}_{\mathbf{C}} \to \mathcal{M}_{\mathbf{C}}$. Now, in order to prove that $|\mathcal{M}_{\mathbf{C}}| = |\mathcal{C}| = 2^{\mathbb{N}}$, we now prove they are isomorphic. Let us denote the mapping from a ternary expansion $\Omega \in \{0, 2\}^{\mathbb{N}}$ to $y \in \mathcal{C}$ as $\mathcal{V}$. By Lemma 2.1.7, this mapping is bijective. Now we will prove that there exist a mapping bijective mapping $\mathcal{Y} : \mathbb{S}^{\mathbb{N}} \to \mathcal{S}_{\mathbf{C}}$. Define $\mathcal{Y} : \mathbb{S}^{\mathbb{N}} \to \mathcal{S}_{\mathbf{C}}$, where $\mathcal{Y}(\Upsilon) = \Xi$ by:

$$\xi_{i+1} = \begin{cases} \upsilon_j \quad \text{if} \quad \Xi_i(\mathbf{A}) = 0 \quad \text{and} \quad i = p_j \\ 1 \quad \text{if} \quad i \notin \mathcal{N} \quad \text{and} \quad \Xi_i(\mathbf{A}) < 0 \\ -1 \quad \text{if} \quad i \notin \mathcal{N} \quad \text{and} \quad \Xi_i(\mathbf{A}) > 0 \end{cases} \tag{3.87}$$

where $j = \max \mathcal{N}_i$. Then, obviously, such a mapping is well-defined, as $(\Xi, \mathbf{C})$ by definition satisfies the step condition for maxima. Now let us prove that this mapping is bijective. First of all, let us prove that this mapping is injective. Let us assume that $\Upsilon \neq \Gamma$. Then we must have for some j, that $\upsilon_j \neq \gamma_j$. Then by definition, we have $\mathcal{Y}(\Upsilon)_{p_j+1} = \upsilon_j \neq \gamma_j = \mathcal{Y}(\Gamma)_{p_j+1}$. Now let us prove that this mapping is surjective. Let Ξ be any sequence in $\mathcal{S}_{\mathbf{C}}$, then take $\Upsilon = \{\upsilon_j\}$, where $\upsilon_j = \xi_{p_j+1}$. Hence, by definition $\mathcal{Y}(\Upsilon) = \Xi$. Therefore, the mapping $\mathcal{Y}$ is bijective. Now let us define a bijective mapping $\mathcal{U} : \mathbb{S}^{\mathbb{N}} \to \{0, 2\}^{\mathbb{N}}$. Let $\Upsilon \in \mathbb{S}^{\mathbb{N}}$ and $\Theta \in \{0, 2\}^{\mathbb{N}}$. Then $\mathcal{U}(\Upsilon) = \Theta$ is defined as follows:

$$\theta_i = 1 - \upsilon_i, \tag{3.88}$$

for all $i \in \mathbb{N}$. Obviously, the mapping $\mathcal{U}$ is a bijective mapping. Therefore, let us define $\mathcal{W} = \mathcal{Y}^{-1} \circ \mathcal{U}$ is a bijective mapping from $\mathcal{S}_{\mathbf{C}}$ to $\{0, 2\}^{\mathbb{N}}$. Therefore, $\mathcal{C} \cong \mathcal{M}_{\mathbf{C}}$. Hence, $|\mathcal{C}| = |\mathcal{M}_{\mathbf{C}}|$. □

3.2 Local Extrema for Takagi Class

In this section, we will analyze the behavior for the local extrema. The nature of local extrema can be regarded as a global extrema under a Takagi function adding a linear function. One may compare the following lemmas, corollaries and theorems with Section 3.1 in order to have better understanding.

Definition 3.2.1. For $t_1, t_2 \in [0,1]$, denote

$$\mathcal{M}_{\mathbf{C}}([t_1,t_2]) = \underset{t\in[t_1,t_2]}{\arg\max}\, x_{\mathbf{C}}(t).$$

Furthermore, we have

$$\mathcal{M}_{\mathbf{C},n}([t_1,t_2]) = \underset{t\in[t_1,t_2]}{\arg\max}\, x_{\mathbf{C},n}(t)$$

The following lemmas and corollaries can be regarded as a local extension of the lemmas in the previous section.

Lemma 3.2.2. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m > n$, let $\tau_m \in \mathcal{M}_{\mathbf{C},m}([z, z+2^{-n}])$ be a maximum point of the truncated Takagi function $x_{\mathbf{C},m}$ in the interval $[z, z+2^{-n}]$ for $z \in \mathbb{T}_n$. Next let τ_m^* be the larger point(s) among the two adjoining points in $\mathbb{T}_{m+1}$ of τ_m in the interval $[z, z+2^{-n}]$. Hence, we have

$$\tau_m^* \in \underset{t\in\mathbb{T}_{m+1}(\tau_m)\cap[z,z+2^{-n}]}{\arg\max} x_{\mathbf{C},m}(t). \tag{3.89}$$

Now, let $s \in \mathbb{T}_{m+1} \cap [z, z+2^{-n}]$, and $s^* \in \mathbb{T}_{m+1}(s) \cap [z, z+2^{-n}]$. Then, if $m > n$, we have

$$x_{\mathbf{C},m}(\tau_m) + x_{\mathbf{C},m}(\tau_m^*) \geq x_{\mathbf{C},m}(s) + x_{\mathbf{C},m}(s^*), \tag{3.90}$$

for all $s \in \mathbb{T}_{m+1} \cap [z, z+2^{-n}]$.

Proof. We are going to prove this Lemma using induction on m. First of all, let us consider the case when $m = n$. By applying Lemma 3.1.22,

$$x_{\mathbf{C},n}(t) = \Xi_n(\mathbf{A})t, \quad \text{for} \quad t \in [z_n, z_n + 2^{-n}]. \tag{3.91}$$

Then if $\Xi_n(\mathbf{A}) \leq 0$, then we have $\tau_n = z + 2^{-n}$ and $\tau_n^* = z$, otherwise, we have $\tau_n = z$ and $t_0^* = \tau_n^* = z + 2^{-n}$. Also, we notice z and $z+2^{-n}$ are the only choice for $s \in \mathbb{T}_k \cap [z_n, z_n + 2^{-n}]$ and s^*.

$$x_{\mathbf{C},n}(\tau_n) + x_{\mathbf{C},n}(\tau_n^*) = x_{\mathbf{C},n}(s) + x_{\mathbf{C},n}(s^*) = x_{\mathbf{C},n}(0) + x_{\mathbf{C},n}(\frac{1}{2}). \tag{3.92}$$

Due to (3.92), we have proved that the hypothesis holds for $m = n$. Now let us assume that the induction hypothesis holds for all $m \leq k-1$, and we proceed to prove the case for $m = k$.

Now we will then prove the statement case by case. First of all, consider if $\tau_k = \tau_{k-1}$. According to Lemma 3.1.1, τ_{k-1}^* is an adjoining point to τ_{k-1} on the dyadic partition $\mathbb{T}_k$ with in $[z, z + 2^{-n}]$. Therefore, we have

$$|\tau_{k-1} - \tau_{k-1}^*| = |(2\tau_{k-1} - \tau_{k-1}^*) - \tau_{k-1}| = 2^{-k}.$$

As well as,

$$|\tau_k - \frac{\tau_{k-1} + \tau_{k-1}^*}{2}| = |\tau_k - \frac{3\tau_{k-1} - \tau_{k-1}^*}{2}| = 2^{-(k+1)}.$$

Therefore, $2\tau_{k-1} - \tau_{k-1}^*$ is the other adjoining point to τ_{k-1}. Moreover, $\frac{3\tau_{k-1} - \tau_{k-1}^*}{2}$ and $\frac{\tau_{k-1} + \tau_{k-1}}{2}$ are the two adjoining points of τ_k on the dyadic partition $\mathbb{T}_{k+1}$. Then due to the linearity and inequality in (3.89), we have

$$\begin{aligned} x_{\mathbf{C},k}(\frac{\tau_{k-1} + \tau_{k-1}^*}{2}) &= \frac{x_{\mathbf{C},k}(\tau_{k-1}) + x_{\mathbf{C},k}(\tau_{k-1}^*)}{2} + \frac{c_k}{2} \\ &\geq \frac{x_{\mathbf{C},k}(\tau_{k-1}) + x_{\mathbf{C},k}(2\tau_{k-1} - \tau_{k-1}^*)}{2} + \frac{c_k}{2} \\ &= x_{\mathbf{C},k}(\frac{3\tau_{k-1} - \tau_{k-1}^*}{2}). \end{aligned}$$

Hence we must have

$$\frac{\tau_{k-1} + \tau^*_{k-1}}{2} \in \underset{t \in \mathbb{T}_{k+1}(\tau_k) \cap [z, z+2^{-n}]}{\arg\max} x_{\mathbf{C},k}(t).$$

Now we have,

$$\tau^*_k = \frac{\tau_{k-1} + \tau^*_{k-1}}{2}. \tag{3.93}$$

Now let us assume that $s \in \mathbb{T}_k$, then we have that

$$x_{\mathbf{C},k}(s) = x_{\mathbf{C},k-1}(s) \leq x_{\mathbf{C},k-1}(\tau_{k-1}) = x_{\mathbf{C},k}(\tau_{k-1}), \tag{3.94}$$

and we can re-compose s^* by $s^* = \frac{s+(2s^*-s)}{2}$. Since

$$|(2s^* - s) - s^*| = |s^* - s| = 2^{k+1}, \tag{3.95}$$

as will as,

$$|2s^* - s - s| = 2|s^* - s| = 2 \cdot 2^{-(k+1)} = 2^{-n}. \tag{3.96}$$

Equation (3.95) and (3.96) indicates that s and $2s^* - s$ are adjoining points on the dyadic partition $\mathbb{T}_k$, and they are also the two different adjoining points of s^* on the dyadic partition $\mathbb{T}_{k+1}$. Then applying the hypothesis for $k-1$, we have

$$x_{\mathbf{C},k-1}(\tau_{k-1}) + x_{\mathbf{C},k-1}(\tau^*_{k-1}) \geq x_{\mathbf{C},k-1}(s) + x_{\mathbf{C},k-1}(2s^* - s). \tag{3.97}$$

By applying (3.93) - (3.97), we then have

$$\begin{aligned}
x_{\mathbf{C},k}(\tilde{t}_k) + x_{\mathbf{C},k}(\tau^*_k) &= x_{\mathbf{C},k-1}(\tau_{k-1}) + x_{\mathbf{C},k}(\frac{\tau^*_{k-1} + \tau_{k-1}}{2}) \\
&= x_{\mathbf{C},k-1}(\tau_{k-1}) + \frac{1}{2} x_{\mathbf{C},k-1}(\tau^*_{k-1}) + \frac{1}{2} x_{\mathbf{C},k-1}(\tau_{k-1}) + \frac{c_k}{2} \\
&\geq x_{\mathbf{C},k-1}(s) + \frac{1}{2} x_{\mathbf{C},k-1}(s) + \frac{1}{2} x_{\mathbf{C},k-1}(2s^* - s) + \frac{c_k}{2} \\
&\geq x_{\mathbf{C},n}(s) + x_{\mathbf{C},n}(s^*).
\end{aligned}$$

Now, let us consider when $s \in \mathbb{T}_{k+1} - \mathbb{T}_k$. In this case, we have

$$s^* \in \mathbb{T}_{k+1}(s) \subseteq \mathbb{T}_k, \tag{3.98}$$

as well as,

$$2s - s^* \in \mathbb{T}_{k+1}(s) \subseteq \mathbb{T}_k, \tag{3.99}$$

According to (3.9), we know that $s^* \in \mathbb{T}_k$ and $2s - s^* \in \mathbb{T}_k$. Now by applying the inductive statement for order $k-1$, we have,

$$x_{\mathbf{C},k-1}(s^*) + x_{\mathbf{C},k-1}(2s - s^*) \leq x_{\mathbf{C},k-1}(\tau^*_{k-1}) + x_{\mathbf{C},k-1}(\tau_{k-1}) \tag{3.100}$$

Then by applying (3.100), we have

$$\begin{aligned} x_{\mathbf{C},k}(\tau_k) + x_{\mathbf{C},k}(\tau^*_k) &= x_{\mathbf{C},k-1}(\tau_{k-1}) + x_{\mathbf{C},k}(\frac{\tau^*_{k-1} + \tau_{k-1}}{2}) \\ &= x_{\mathbf{C},k-1}(\tau_{k-1}) + \frac{1}{2}x_{\mathbf{C},k-1}(\tau^*_{k-1}) + \frac{1}{2}x_{\mathbf{C},k-1}(\tau_{k-1}) + \frac{c_k}{2} \\ &\geq x_{\mathbf{C},k-1}(s^*) + \frac{1}{2}x_{\mathbf{C},k-1}(s^*) + \frac{1}{2}x_{\mathbf{C},k-1}(2s - s^*) + \frac{c_k}{2} \\ &\geq x_{\mathbf{C},k}(s) + x_{\mathbf{C},k}(s^*), \end{aligned}$$

Now, let us consider the other case when $\tau_k \neq \tau_{k-1}$. For any points $s \in \mathbb{T}_k$, we have

$$x_{\mathbf{C},k}(s) = x_{\mathbf{C},k-1}(s) \leq x_{\mathbf{C},k-1}(\tau_{k-1}) = x_{\mathbf{C},k}(\tau_{k-1}). \tag{3.101}$$

And for points $s \in \mathbb{T}_{k+1} - \mathbb{T}_k$, we have $s - 2^{-(k+1)} \in \mathbb{T}_k$ and $s + 2^{-(k+1)} \in \mathbb{T}_k$. Then by applying (3.5) with $m = k - 1$, we have

$$\begin{aligned} x_{\mathbf{C},n}(\frac{\tau_{k-1} + \tau^*_{k-1}}{2}) &= \frac{x_{\mathbf{C},n-1}(\tau_{k-1}) + x_{\mathbf{C},n-1}(\tau^*_{k-1})}{2} + \frac{c_n}{2} \\ &\geq \frac{x_{\mathbf{C},n-1}(s - 2^{-(n+1)}) + x_{\mathbf{C},n-1}(s + 2^{-(n+1)})}{2} + \frac{c_n}{2} \quad = x_{\mathbf{C},n}(s) \end{aligned} \tag{3.102}$$

By applying (3.101) and (3.102), we then have

$$\max_{t\in[z,z+2^{-n}]} x_{\mathbf{C},k}(t) = \max\{x_{\mathbf{C},n}(\tau_{k-1}), x_{\mathbf{C},n}(\frac{\tau_{k-1}+\tau_{k-1}^*}{2})\}. \tag{3.103}$$

And as $\tau_k \neq \tau_{k-1}$, then we have

$$\tau_k = \frac{\tau_{k-1}+\tau_{k-1}^*}{2}. \tag{3.104}$$

as well as,

$$\tau_k^* = \tau_{k-1}. \tag{3.105}$$

Then for all $s \in \mathbb{T}_{k+1}$, we must have either s or s^* is in the dyadic partition $\mathbb{T}_k$. Without loss of generality, let us assume that $s \in \mathbb{T}_{k+1}$, therefore, $s^* \in \mathbb{T}_k$. Then by applying (3.101), (3.102), (3.104) and (3.105),

$$\begin{aligned} x_{\mathbf{C},k}(\tilde{t}_k) + x_{\mathbf{C},k}(\tau_k^*) &= x_{\mathbf{C},k}(\frac{\tau_{k-1}+\tau_{k-1}^*}{2}) + x_{\mathbf{C},k}(\tau_{k-1}) \\ &\geq x_{\mathbf{C},k}(s) + x_{\mathbf{C},k}(s^*). \end{aligned}$$

Therefore, we finish proving the case for $m = k$, and, hence, we prove the Lemma 3.2.2. □

Corollary 3.2.3. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m > n$, let $\tau_m \in \mathcal{M}_{\mathcal{C},m}([t_1,t_2])$, where $t_1, t_2 \in \mathbb{T}_{n+1}$. Furthermore, let us denote

$$\tau_m^* = \operatorname*{arg\,max}_{t\in\mathbb{T}_{m+1}(\tau_m)\cap[t_1,t_2]} x_{\mathbf{C},m}(t).$$

If $\tau_m \in \mathcal{M}_{\mathcal{C},m+1}([t_1,t_2])$, then

$$\frac{\tau_m+\tau_m^*}{2} \in \operatorname*{arg\,max}_{t\in\mathbb{T}_{m+2}(\tau_{m+1})\cap[t_1,t_2]} x_{\mathbf{C},m+1}(t).$$

Otherwise, we have

$$\tau_m \in \operatorname*{arg\,max}_{t\in\mathbb{T}_{m+2}(\tau_{m+1})\cap[t_1,t_2]} x_{\mathbf{C},m+1}(t).$$

Proof. This result directly comes from (3.96) and (3.105) in the proof for Lemma 3.2.2. □

Corollary 3.2.4. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m > n$, let $\tau_m \in \mathcal{M}_{\mathcal{C},m}([t_1, t_2])$, where $t_1, t_2 \in \mathbb{T}_{n+1}$. Furthermore, let us denote

$$\tau_m^* = \underset{t \in \mathbb{T}_{m+1}(\tau_m) \cap [t_1, t_2]}{\arg\max} x_{\mathbf{C},m}(t).$$

Then there must exist some $\tau_{m+1} \in \mathcal{M}_{\mathbf{C},m+1}([t_1, t_2])$, such that

$$[\tau_{m+1} \wedge \tau_{m+1}^*] \subsetneq [\tau_m \wedge \tau_m^*].$$

We would follow Definition 3.1.6, and refer such a sequence $\{\tau_m\}$ as a sequence of consecutive local maximizers in $[t_1, t_2]$.

Remark 3.2.5. If $\{\tau_m\}$ is a sequence of consecutive local maximizers in $[0, 1]$, then $\{\tau_m\}$ is also a sequence of consecutive maximizers.

Corollary 3.2.6. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m < n$, let $\tau_m \in \mathcal{M}_{\mathbf{C},m}([t_1, t_2])$. Then, we must have

$$\bar{\mathbb{T}}_{m+1}(\tau_m) \in \mathcal{M}_{\mathbf{C},m+1}([t_1, t_2]) \neq \emptyset.$$

Proof. This result directly comes from Corollary 3.2.4. □

Proposition 3.2.7. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m > n$, let $\tau_m \in \mathcal{M}_{\mathbf{C},m}([t_1, t_2])$ be a maximum point of the truncated exponential Takagi function $x_{\mathbf{C},m}$ on the interval $[t_1, t_2]$ for $t_1, t_2 \in \mathbb{T}_{m+1}$ Furthermore, we require $|t_1 - t_2| = 2^{-(n+1)}$. For a fixed $m > n$, we let $k = \min\{i > m - n | \tau_{m-i} \neq \tau_m\}$. Then, if $\tau_{m-k} < \tau_m$ and $\tau_m - 2^{-(m+2)} \in [t_1, t_2]$, we have

$$x_{\mathbf{C},n+1}(p - 2^{-(m+2)}) \leq x_{\mathbf{C},n+1}(\tau_m - 2^{-(m+2)}). \tag{3.106}$$

for every $p \in \mathbb{T}_{m+1} \cap [t_1, t_2]$ and $p \in \mathbb{T}_{m+1} - 2^{-(m+2)} \cap [t_1, t_2]$.

Proof. Let us first consider the case $k = 1$. Since τ_m is in the dyadic partition $\mathbb{T}_{m+1}$ and τ_{m-1} is in the dyadic partition $\mathbb{T}_m$ for any fixed $m \in \mathbb{N}$, and we have

$$\tau_{m-k} = \tau_{m-1} < \tau_m. \tag{3.107}$$

Then due to (3.107), we can apply Lemma 3.1 in [17], then we have

$$\tau_{m-1} = \tau_m - 2^{-(m+1)}. \tag{3.108}$$

As the truncated function $x_{\mathbf{C},m}$ is linear within intervals of the form $[p-2^{-(m+1)}, p] \subset [t_1, t_2]$, for any $p \in \mathbb{T}_{m+1} \in [t_1, t_2]$. Furthermore, the increment of the wedge has an increment of $\frac{c_{m+1}}{2}$, we get

$$x_{\mathbf{C},m+1}(p - 2^{-(m+2)}) = \frac{x_{\mathbf{C},m}(p) + x_{\mathbf{C},m}(p - 2^{-(m+1)})}{2} + \frac{c_{m+1}}{2}. \tag{3.109}$$

As $\tau_m \in \mathbb{T}_{m+1}$, we may take $p = \tau_m$ and by plugging (3.108) into (3.109), we have

$$x_{\mathbf{C},m+1}(\tau_m - 2^{-(m+2)}) = \frac{x_{\mathbf{C},m}(\tau_m) + x_{\mathbf{C},m}(\tau_{m-1})}{2} + \frac{c_{m+1}}{2}. \tag{3.110}$$

Since $\tau_{m-1} \in \mathcal{M}_{\mathbf{C},m-1}([t_1, t_2])$, $\tau_{m-1} \in \mathbb{T}_m$, then we have

$$x_{\mathbf{C},m}(\tau_{m-1}) = x_{\mathbf{C},m-1}(\tau_{m-1}). \tag{3.111}$$

As $p \in \mathbb{T}_{m+1} \cap [t_1, t_2]$, then either $p \in \mathbb{T}_m \cap [t_1, t_2]$ or $p - 2^{-(m+1)} \in \mathbb{T}_m \cap [t_1, t_2]$.

$$\min\{x_{\mathbf{C},m}(p), x_{\mathbf{C},m}(p - 2^{-(m+1)})\} \leq x_{\mathbf{C},m-1}(\tau_{m-1}) = x_{\mathbf{C},m}(\tau_{m-1}). \tag{3.112}$$

In addition,

$$\max\{x_{\mathbf{C},m}(p), x_{\mathbf{C},m}(p - 2^{-(m+1)})\} \leq x_{\mathbf{C},m}(\tau_m). \tag{3.113}$$

Hence, according to (3.112) and (3.113), we have

$$\frac{x_{\mathbf{C},m}(p) + x_{\mathbf{C},m}(p - 2^{-(m+1)})}{2} \leq \frac{x_{\mathbf{C},m}(\tau_m) + x_{\mathbf{C},m}(\tau_{m-1})}{2}. \tag{3.114}$$

Plugging (3.114) into (3.110) and applying (3.109), we get

$$\begin{aligned} x_{\mathbf{C},m+1}(p-2^{-(m+2)}) &= \frac{x_{\mathbf{C},m}(p)+x_{\mathbf{C},m}(p-2^{-(n+1)})}{2}+\frac{c_{m+1}}{2} \\ &\leq \frac{x_{\mathbf{C},m}(\tau_m)+x_{\mathbf{C},m}(\tau_{m-1})}{2}+\frac{c_{m+1}}{2}=x_{\mathbf{C},m+1}(\tau_m-2^{-(m+2)}). \end{aligned} \tag{3.115}$$

This completes the proof for the case $k=1$. For the case $k>1$, we proceed the induction on k. For such all fixed n, we assume the inductive hypothesis holds true for $k \leq m$. Now we proceed to prove when $k=m+1$, $x_{\mathbf{C},m+1}(p-2^{-(n+2)}) \leq x_{\mathbf{C},m+1}(\tau_m-2^{-(n+2)})$ holds for $p \in \mathbb{T}_{m+1} \cap [t_1,t_2]$. As we know that $m+1=\min\{i|\tau_{m-i} \neq \tau_m\}$, and $\tau_m=\tau_{m-1}$, then we have

$$\min\{i|\tau_{m-1-i} \neq \tau_{m-1}\}=m. \tag{3.116}$$

As induction hypothesis holds for every $n \in \mathbb{N}$ and $k \leq m$, then by applying (3.106) for $x_{\mathbf{C},m-1}$, we have

$$x_{\mathbf{C},(m-1)+1}(\tilde{p}-2^{-((m-1)+2)}) \leq x_{\mathbf{C},(m-1)+1}(\tau_{m-1}-2^{-((m-1)+2)}), \tag{3.117}$$

where $\tilde{p} \in \mathbb{T}_m \cap [t_1,t_2]$. By organizing equation (3.117), we have

$$x_{\mathbf{C},m}(\tilde{p}-2^{-(m+1)}) \leq x_{\mathbf{C},m}(\tau_m-2^{-(m+1)}). \tag{3.118}$$

Then we will prove the statement case by case. We first consider the case $\tilde{p} \in \mathbb{T}_m$. As $\tau_m \in \mathbb{T}_{m+1}$, therefore $\tau_m-2^{-(m+1)} \in \mathbb{T}_{m+1}$. Similarly, as $\tilde{p} \in \mathbb{T}_m$, then $\tilde{p}-2^{-(m+1)} \in \mathbb{T}_{m+1}$. By applying equation (3.109), we get

$$x_{\mathbf{C},m+1}(\tilde{p}-2^{-(m+2)})=\frac{x_{\mathbf{C},m}(\tilde{p})+x_{\mathbf{C},m}(\tilde{p}-2^{-(m+1)})}{2}+\frac{c_{m+1}}{2}. \tag{3.119}$$

Then as $\tau_m=\tau_{m-1} \in \mathbb{T}_m$, we can replace $\tilde{p}$ with τ_m in the equation (3.119), we get

$$x_{\mathbf{C},m+1}(\tau_m-2^{-(m+2)})=\frac{x_{\mathbf{C},m}(\tau_m)+x_{\mathbf{C},m}(\tau_m-2^{-(m+1)})}{2}+\frac{c_{m+1}}{2}. \tag{3.120}$$

As $\tau_m = \tau_{m-1} \in \mathbb{T}_m$, and $p \in \mathbb{T}_m$, then we have

$$x_{\mathbf{C},m}(\tilde{p}) = x_{\mathbf{C},m-1}(\tilde{p}) \leq x_{\mathbf{C},m-1}(\tau_{m-1}) = x_{\mathbf{C},m}(\tau_{m-1}) = x_{\mathbf{C},m}(\tau_m). \tag{3.121}$$

Then by plugging (3.121) and (3.118) into (3.119) and (3.120), we have

$$\begin{aligned} x_{\mathbf{C},m+1}(\tilde{p} - 2^{-(m+2)}) &= \frac{x_{\mathbf{C},m}(\tilde{p}) + x_{\mathbf{C},m}(\tilde{p} - 2^{-(m+1)})}{2} + \frac{c_{m+1}}{2} \\ &\leq \frac{x_{\mathbf{C},m}(\tau_m) + x_{\mathbf{C},m}(\tau_m - 2^{-(m+1)})}{2} + \frac{c_{m+1}}{2} = x_{\mathbf{C},m+1}(\tau_m - 2^{-(m+2)}). \end{aligned}$$

Now we discuss the case $\tilde{p} \in \mathbb{T}_{m+1} - \mathbb{T}_m$, then we have that $\tilde{p} + 2^{-(m+1)} \in \mathbb{T}_m$, therefore by applying equation (3.118) for $\tilde{p} + 2^{-(m+1)}$, we have

$$x_{\mathbf{C},m}(\tilde{p} + 2^{-(m+1)} - 2^{-(m+1)}) = x_{\mathbf{C},m}(\tilde{p}) \leq x_{\mathbf{C},m}(\tau_m - 2^{-(m+1)}). \tag{3.122}$$

As the function $x_{\mathbf{C},m}$ is maximized at τ_m, we have

$$x_{\mathbf{C},m}(\tilde{p} - 2^{-(m+1)}) \leq x_{\mathbf{C},m}(\tau_m). \tag{3.123}$$

By plugging equation (3.122) and equation (3.123) into equation (3.119) and equation (3.120). We have

$$\begin{aligned} x_{\mathbf{C},m+1}(\tilde{p} - 2^{-(m+2)}) &= \frac{x_{\mathbf{C},m}(\tilde{p}) + x_{\mathbf{C},m}(\tilde{p} - 2^{-(m+1)})}{2} + \frac{c_{m+1}}{2} \\ &\leq \frac{x_{\mathbf{C},m}(\tau_m) + x_{\mathbf{C},m}(\tau_m - 2^{-(m+1)})}{2} + \frac{c_{m+1}}{2} = x_{\mathbf{C},m+1}(\tau_m - 2^{-(m+2)}). \end{aligned}$$

Therefore, we have proved that $x_{\mathbf{C},m+1}(p - 2^{-(m+2)}) \leq x_{\mathbf{C},m+1}(\tau_m - 2^{-(m+2)})$ for any $p \in \mathbb{T}_{m+1} \cap [t_1, t_2]$. Since both base case and the inductive hypothesis has been proved, then we prove this proposition. □

Proposition 3.2.8. Let $x_{\mathbf{C}} \in \mathfrak{C}$. For every $m > n$, let $\tau_m \in \mathcal{M}_{\mathbf{C},m}([t_1, t_2])$ be a maximum point of the truncated exponential Takagi function $x_{\mathbf{C},m}$ on the interval $[t_1, t_2]$ for $t_1, t_2 \in \mathbb{T}_{m+1}$ Furthermore, we require $|t_1 - t_2| = 2^{-(n+1)}$. For a fixed $m > n$, we let $k = \min\{i >$

$m - n|\tau_{m-i} \neq \tau_m\}$. Then, if $\tau_{m-k} > \tau_m$ and $\tau_m + 2^{-(m+2)} \in [t_1, t_2]$, we have

$$x_{\mathbf{C},m+1}(p + 2^{-(m+2)}) \leq x_{\mathbf{C},m+1}(\tau_m + 2^{-(m+2)}). \tag{3.124}$$

for every $p \in \mathbb{T}_{m+1} \cap [t_1, t_2]$ and $p + 2^{-(m+2)} \in [t_1, t_2]$.

Proof. The proof is analogous to proof for Proposition 3.2.7. □

Lemma 3.2.9. Let $t_1, t_2 \in \mathbb{T}_k$ and $|t_1 - t_2| = 2^{k+1}$. Then let $\tau_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$ for all $n > k$, and

$$\tau_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tau_n) \cap [t_1, t_2]}{\arg\max} x_{\mathbf{C},n}(t).$$

Then for any fixed $n \geq 1$, let $m = \inf\{i|\tau_{n-i} \neq \tau_n\}$. Then, $\tau_{n-m} < \tau_n$ if and only if $\tau_n^* < \tau_n$ for every $m < n - k$..

Proof. First of all, let us prove the if direction. Let us assume that $\tau_{n-m} < \tau_n$, then we will discuss case by case. First of all, let us consider the case when $m = 1$. By applying Corollary 3.1.3, we have $\tau_{n-1} = \tau_n - 2^{-(n+1)} \in \mathbb{T}_n$. Similarly, we have $\tau_n + 2^{-(n+1)} \in \mathbb{T}_n$. Then we have

$$x_{\mathbf{C},n}(\tau_n - 2^{-(n+1)}) = x_{\mathbf{C},n-1}(\tau_n - 2^{-(n+1)}) \geq x_{\mathbf{C},n-1}(\tau_n + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_n + 2^{-(n+1)}).$$

Hence, we have $\tau_n^* = \tau_n - 2^{-(n+1)}$, and $\tau_n^* < \tau_n$. Now, we consider the case when $m > 1$, then we have

$$\tau_{n-1} = \tau_n.$$

Then we can apply the Proposition 3.1.7, as $\tau_{(n-1)-(m-1)} < \tau_{n-1}$, then we have

$$x_{\mathbf{C},n}(p - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tau_{n-1} - 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_{n-1} - 2^{-(n+1)}). \tag{3.125}$$

for every $p \in \mathbb{T}_n$. Because $\tau_n = \tau_{n-1} \in \mathbb{T}_n$, then by plugging $p = \tau_n + 2^{-n} \in \mathbb{T}_n$ into (3.125), we have

$$x_{\mathbf{C},n}(\tau_n + 2^{-n} - 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_n + 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tau_n - 2^{-(n+1)}).$$

Then we have $\tau_n^* = \tau_n - 2^{-(n+1)}$, and $\tau_n^* < \tau_n$. Hence, we finish the proof for the if part. Now, we aim to prove the only if direction by proving its contrapositive statement through induction. We can notice that the contrapositive statement for the only if direction will be

$$\textit{If } \tau_{n-m} > \tau_n, \textit{ then } \tau_n^* > \tau_n.$$

Then, we will prove the contrapositive statement in a similar way as we prove the if direction. Since by assumption, we have $\tau_{n-m} > \tau_n$, then we will also discuss case by case. Now, let us consider the case when $m = 1$. By applying Corollary 3.1.3 again, we have $\tau_{n-1} = \tau_n + 2^{-(n+1)} \in \mathbb{T}_n$. Similarly, we have $\tau_n - 2^{-(n+1)} \in \mathbb{T}_n$. Then we have

$$x_{\mathbf{C},n}(\tau_n - 2^{-(n+1)}) = x_{\mathbf{C},n-1}(\tau_n - 2^{-(n+1)}) \leq x_{\mathbf{C},n-1}(\tau_n + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_n + 2^{-(n+1)}).$$

Hence, we have $\tau_n^* = \tau_n + 2^{-(n+1)}$, and $\tau_n^* > \tau_n$. Now, we consider the case when $m > 1$, then we have

$$\tau_{n-1} = \tau_n.$$

Then we can apply the Proposition 3.1.8, as $\tau_{(n-1)-(m-1)} > \tau_{n-1}$, then we have

$$x_{\mathbf{C},n}(p + 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tau_{n-1} + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_{n-1} + 2^{-(n+1)}). \tag{3.126}$$

for every $p \in \mathbb{T}_n$. Because $\tau_n = \tau_{n-1} \in \mathbb{T}_n$, then by plugging $p = \tau_n - 2^{-n} \in \mathbb{T}_n$ into (3.126), we have

$$x_{\mathbf{C},n}(\tau_n - 2^{-n} + 2^{-(n+1)}) = x_{\mathbf{C},n}(\tau_n - 2^{-(n+1)}) \leq x_{\mathbf{C},n}(\tau_n + 2^{-(n+1)}).$$

Then we have $\tau_n^* = \tau_n + 2^{-(n+1)}$, and $\tau_n^* > \tau_n$. Hence, we finish the proof for the only if part through proving its contrapositive statement. □

Lemma 3.2.10. Let $t_1, t_2 \in \mathbb{T}_k$ and $|t_1 - t_2| = 2^{k+1}$. Then let $\tau_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$ for all $n > k$, and

$$\tau_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tau_n) \cap [t_1, t_2]}{\arg\max} \; x_{\mathbf{C},n}(t).$$

Then for any fixed $n \geq 1$, let $m = \inf\{i | \tau_{n-i} \neq \tau_n\}$. Then, $\tau_{n-m} > \tau_n$ if and only if $\tau_n^* > \tau_n$ for every $m < n - k$..

Proof. This is the contrapositive statement for 3.2.9. □

Theorem 3.2.11. For $t_1, t_2 \in \mathbb{T}_n$ and $|t_1 - t_2| = 2^{-(m+1)}$ for some $m \in \mathbb{N}$. Let $x_{\mathbf{C}} \in \mathfrak{C}$, the following statements are equivalent:

i. $y \in \mathcal{M}_{\mathbf{C}}([t_1, t_2])$.

ii. There exists a sequence $\{y_n\}_{n=0}^{\infty}$, such that $y_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$ for all $n > m$, and

$$y = \lim_{n \to \infty} y_n.$$

iii. Let $\mathcal{T}_n := \{[\tau_n - 2^{-(n+1)}, \tau_n + 2^{-(n+1)}] | \tau_n \in \mathcal{M}_{\mathcal{C},n}([t_1, t_2])\}$. Furthermore, take

$$\mathcal{P}_n = \bigcup_{A \in \mathcal{T}_n} A.$$

Then,

$$y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

iv. Let $\mathcal{K}_n := \{[\tau_n \wedge \tau_n^*, \tau_n \vee \tau_n^*] \mid \tau_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2]), \tau_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tau_n) \cap [t_1, t_2]}{\arg\max} x_{\mathbf{C},n}(\tau_n)\}$. Furthermore, take

$$\mathcal{I}_n = \bigcup_{A \in \mathcal{K}_n} A.$$

Then,

$$y \in \bigcap_{n=0}^{\infty} \mathcal{I}_n.$$

Proof. Let us prove this theorem by proving following statements in order.

- $i \Longrightarrow iv$

  Let us try this statement by proving its contrapositive statement. The contrapositive statement will be

  If there exists $n > m$, such that $y \notin \mathcal{I}_n$, then $y \notin \mathcal{M}_{\mathbf{C}}([t_1, t_2])$.

  Denote $\mathcal{N} = \{k > m | y \notin \mathcal{I}_k\}$, and denote $n = \min \mathcal{N}$. Let us assume that $y \in [s, s^*] \subsetneqq [t_1, t_2]$, where $s \in \mathbb{T}_{n+1}$ and $s^* = s + 2^{-(n+1)}$. Since $n = \min \mathcal{N}$, we must have that

  $$y \in [\tau_{n-1} \wedge \tau^*_{n-1}, \tau_{n-1} \vee \tau^*_{n-1}].$$

  for some $\tau_{n-1} \in \mathcal{M}_{\mathbf{C},n-1}([t_1, t_2])$ and $\tau^*_{n-1} \in \underset{t \in \mathbb{T}_n(t) \cap [t_1, t_2]}{\arg\max} x_{\mathbf{C},n-1}(t)$. By applying Corollary 3.1.3, there exist $\tau_n \in \mathcal{M}_{\mathbf{C},n} \cap [\tau_{n-1} \wedge \tau^*_{n-1}, \tau_{n-1} \vee \tau^*_{n-1}]$. Hence, by applying Corollary 3.1.4, we have

  $$[\frac{1}{2}(\tau_{n-1} + \tau^*_{n-1}) \wedge \tau_{n-1}, \frac{1}{2}(\tau_{n-1} + \tau^*_{n-1}) \vee \tau_{n-1}] = [\tau_n \wedge \tau^*_n, \tau_n \vee \tau^*_n].$$

  Therefore, we have $\{s, s^*\} \bigcap \{\tau_n, \tau^*_n\} \neq \emptyset$, and $\{s, s^*\} \bigcap \{\tau_n, \tau^*_n\} \neq \{\tau_n, \tau^*_n\}$. For instance, if $\tau^*_{n-1} < \tau_{n-1}$, then we have

  $$s = \tau^*_{n-1}, \quad \text{and} \quad s^* = \tau_n \wedge \tau^*_n,$$

  As $[s, s^*] \notin \mathcal{K}_n$, therefore $s = \tau^*_{n-1} \notin \mathcal{M}_{\mathbf{C},n-1}([t_1, t_2])$. Hence, we have

  $$x_{\mathbf{C},n}(s) < x_{\mathbf{C},n}(\tau_{n-1}) \leq x_{\mathbf{C},n}(\tau^*_n). \tag{3.127}$$

  As well as

  $$x_{\mathbf{C},n}(s) \leq x_{\mathbf{C},n}(\tau_n). \tag{3.128}$$

  Therefore, since $y \in [s, s^*]$,

  $$\tilde{y} := 2s^* - y \in [\tau_n \wedge \tau^*_n, \tau_n \vee \tau^*_n],$$

  for $\tau_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$. Then by applying Lemma 3.1.12 and Lemma 3.1.13, as

$s \in \mathbb{T}_{n+1}$,

$$x_{\mathbf{C}}^{n+1}(y) = x_{\mathbf{C}}^{n+1}(\tilde{y}). \tag{3.129}$$

By applying (3.127) - (3.129), we have

$$\begin{aligned}
x_{\mathbf{C}}(y) = x_{\mathbf{C},n}(y) + x_{\mathbf{C}}^{n+1}(y) &= \frac{y-s}{s^*-s}x_{\mathbf{C},n}(s^*) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(s) + x_{\mathbf{C}}^{n+1}(y) \\
&= \frac{y-s}{s^*-s}x_{\mathbf{C},n}(s^*) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(s) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\
&< \frac{y-s}{s^*-s}x_{\mathbf{C},n}(\tau_n) + \frac{s^*-y}{s^*-s}x_{\mathbf{C},n}(\tau_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\
&= \frac{(2s^*-y)-\tau_n}{\tau_n-s^*}x_{\mathbf{C},n}(\tau_n) + \frac{s^*-(2s^*-y)}{\tau_n-s^*}x_{\mathbf{C},n}(\tau_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\
&= \frac{\tilde{y}-\tau_n}{\tau_n-s^*}x_{\mathbf{C},n}(\tau_n) + \frac{s^*-\tilde{y}}{\tau_n-s^*}x_{\mathbf{C},n}(\tau_n^*) + x_{\mathbf{C}}^{n+1}(\tilde{y}) \\
&= x_{\mathbf{C},n}(\tilde{y}) + x_{\mathbf{C}}^{n+1}(\tilde{y}) = x_{\mathbf{C}}(\tilde{y}).
\end{aligned}$$

Hence, $y \notin \mathcal{M}_{\mathbf{C}}([t_1, t_2])$. And the proof for the situation when $\tau_{n-1} < \tau_{n-1}^*$ is analogous to the previous proof.

- *iv* $\Longrightarrow$ *iii*

  First of all, let us state this statement again.

  If $y \in \bigcap_{n=0}^{\infty} \mathcal{I}_n$*, then* $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$.

  This statement is equivalent to the following inclusion, and we now aim to prove the following inclusion.

  $$\bigcap_{n=0}^{\infty} \mathcal{I}_n \subset \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

  As for each fixed $\tau_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$, we have

  $$[\tau_n \wedge \tau_n^*, \tau_n \vee \tau_n^*] \subsetneqq [\tau_n - 2^{-(n+1)}, \tau_n + 2^{-(n+1)}]. \tag{3.130}$$

Then (3.130) directly gives,

$$\mathcal{I}_n = \bigcup_{A \in \mathcal{K}_n} A \subsetneqq \bigcup_{A \in \mathcal{T}_n} A = \mathcal{P}_n,$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\bigcap_{n=0}^{\infty} \mathcal{I}_n \subset \bigcap_{n=0}^{\infty} \mathcal{P}_n.$$

- $iii \implies ii$
  First of all, let us formally state the statement we are going to prove.

  If $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$, then there exists a sequence $y_n \in \mathcal{M}_{\mathbf{C},n}([t_1, t_2])$, such that

  $$y = \lim_{n \to \infty} y_n$$

  Since $y \in \bigcap_{n=0}^{\infty} \mathcal{P}_n$, hence, for all $n \in \mathbb{N}$, we have

  $$y \in \mathcal{P}_n.$$

  Therefore, there exists some $A_{n,y} \in \mathcal{T}_n$ for all $n \in \mathbb{N}$, and

  $$y \in A_{n,y}.$$

  For each $n \in \mathbb{N}$, we take some $y_n = \mathcal{M}_{\mathbf{C},n} \cap A_{n,y}$, and then we have

  $$|y_m - y| \leq 2^{-(m+1)}. \tag{3.131}$$

  Hence, $\lim_{n \to \infty} y_n = y$.

- $ii \implies i$
  First of all, let us state the statement we are about to prove.

If there exists a sequence $\{y_n\}_{n=0}^{\infty}$, such that $y_n \in \mathcal{M}_{\mathbf{C},n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n$, then $y \in \mathcal{M}_{\mathbf{C}}$.

As $[t_1, t_2]$ is a compact space, and $x_{\mathbf{C},n} \in C[t_1, t_2]$, therefore, there exists $\beta_n = \max_{t\in[t_1,t_2]} x_{\mathbf{C},n}(t)$ for all $n > m$, as well as $\beta = \max_{t\in[t_1,t_2]} x_{\mathbf{C}}(t)$. Since, $x_{\mathbf{C},n} \to x_{\mathbf{C}}$ uniformly, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$, such that for all $n > N$,

$$x_{\mathbf{C}}(t) - \epsilon < x_{\mathbf{C},n}(t) < x_{\mathbf{C}}(t) + \epsilon. \tag{3.132}$$

for all $t \in [t_1, t_2]$. Hence, we have

$$x_{\mathbf{C},n}(t) - \epsilon < x_{\mathbf{C}}(t) < x_{\mathbf{C},n}(t) + \epsilon. \tag{3.133}$$

Hence, (3.132) and (3.133) give us

$$\begin{cases} x_{\mathbf{C},n}(t) < \beta + \epsilon, \\ x_{\mathbf{C}}(t) - \epsilon < \beta_n, \end{cases}$$

for all $t \in [t_1, t_2]$. By taking the supremum on the left side, we have

$$\begin{cases} \beta_n \leq \beta + \epsilon, \\ \beta - \epsilon \leq \beta_n. \end{cases}$$

Hence, we have

$$\beta - \epsilon \leq \beta_n \leq \beta + \epsilon.$$

Therefore, we have $\lim_{n\to\infty} \beta_n = \beta$. By uniformly convergence, as $\lim_{n\to\infty} y_n = y$, therefore,

$$\lim_{n\to\infty} x_{\mathbf{C},n}(y_n) = x_{\mathbf{C}}(y). \tag{3.134}$$

Therefore, $y \in \mathcal{M}_{\mathbf{C}}([t_1, t_2])$.

□

Theorem 3.2.12. For $y \in [0,1]$, let $y = 0.\varepsilon_0\varepsilon_1\varepsilon_2\cdots = \sum_{i=0}^{\infty}\varepsilon_i 2^{-(i+1)}$ be the binary expansion of y. Let $y_n = 0.\varepsilon_0\varepsilon_1\varepsilon_2\ldots\varepsilon_n = \sum_{i=0}^{n}\varepsilon_i 2^{-(i+1)}$. Then $y \in \mathcal{M}_{\mathbf{C}}[y_m, y_m + 2^{-(m+1)}]$, if and only if there exist a sequence consecutive maximizers of $\tau_n \in \mathcal{M}_{\mathbf{C},n}([y_m, y_m+2^{-(m+1)}])$ and $\tau_n^* \in \underset{t\in\mathbb{T}_{n+1}(\tau_n)\cap[y_m,y_m+2^{-(m+1)}]}{\arg\max} x_{\mathbf{C},n}(t)$, such that

$$y_n = \tau_n \wedge \tau_n^*,$$

for all $n > m$.

Proof. First of all, let us prove the if part. Now let us assume that there exists a sequence of $\tau_n \in \mathcal{M}_{\mathbf{C},n}([y_m, y_m + 2^{-(m+1)}])$ and

$$\tau_n^* \in \underset{t\in\mathbb{T}_{n+1}(\tau_n)\cap[y_m,y_m+2^{-(m+1)}]}{\arg\max} x_{\mathbf{C},n}(t),$$

such that $y_n = \tau_n \wedge \tau_n^*$. Instantly, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \tau_n^* \wedge \tau_n = y.$$

Then by taking such τ_n, by Lemma 3.1.16, then we have

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \tau_n^* \wedge \tau_n = \lim_{n\to\infty} y_n = y.$$

Hence, by applying Theorem 3.1.15, we have $y \in \mathcal{M}_{\mathbf{C}}([y_m, y_m+2^{-(m+1)}])$. Now let us prove the only if part by proving its contrapositive statement. Let us state the contrapositive statement first.

If for any sequence of consecutive maximizers $\tau_n \in \mathcal{M}_{\mathbf{C},n}([y_m, y_m + 2^{-(m+1)}])$, there exists some $n > m$, such that $y_n \neq \tau_n \wedge \tau_n^*$, then $y \notin \mathcal{M}_{\mathbf{C}}([y_m, y_m + 2^{-(m+1)}])$.

For any sequence $\{\tau_n\}$, such that $\tau_n \in \mathcal{M}_{\mathbf{C},n}([y_m, y_m + 2^{-(m+1)}])$, define $\mathcal{N} = \{n > m | y_n \neq$

$\tau_n\}$. Now let us take $N := \min \mathcal{N}$, therefore

$$[\tau_{N-1} \wedge t^*_{N-1}, \tau_{N-1} \vee t^*_{N-1}] = [y_{N-1}, y_{N-1} + 2^{-N}]. \tag{3.135}$$

Since $\{\tau_n\}$ is a sequence of consecutive maximizers, we have

$$[\tau_N \wedge \tau^*_N, \tau_N \vee \tau^*_N] \subsetneqq [\tau_{N-1} \wedge t^*_{N-1}, \tau_{N-1} \vee t^*_{N-1}] = [y_{N-1}, y_{N-1} + 2^{-N}], \tag{3.136}$$

as well as,

$$[y_N, y_N + 2^{-(N+1)}] \subsetneqq [\tau_{N-1} \wedge \tau_{N-1}, \tau_{N-1} \vee \tau_{N-1}]. \tag{3.137}$$

Since, we have $y_N \neq \tau_N \wedge \tau^*_N$, and therefore, either $y_N = \tau_N \vee \tau^*_N$ or $y_N + 2^{-(N+1)} = \tau_N \wedge \tau^*_N$. Then by (3.135) - (3.137), we have

$$[y_N, y_N + 2^{-(N+1)}) \uplus [\tau_N \wedge \tau_N, \tau_N \vee \tau_N] = [\tau_{N-1} \wedge \tau_{N-1}, \tau_{N-1} \vee \tau_{N-1}].$$

Therefore, by applying lemma 3.1.10, we have

$$y \in [y_N, y_N + 2^{-(N+1)}).$$

Theorem 3.1.15 ($i \implies iv$) indicates

$$y \notin \mathcal{M}_{\mathbf{C}}([y_m, y_m + 2^{-(m+1)}]).$$

□

Lemma 3.2.13. Let $x_{\mathbf{C}} \in \mathfrak{C}$, let Ξ be any quasi-binary expansion for $y \in [0, 1]$. If there exists $m \in \mathbb{N}$,

$$\Xi_n(\mathbf{A})\xi_{n+1} \leq 0, \tag{3.138}$$

for all $m \leq n \leq N$. We have

$$y_{N+1} = \tau_{N+1} \wedge \tau^*_{N+1}, \tag{3.139}$$

where $\tau_{N+1} \in \mathcal{M}_{\mathbf{C},N+1}([y_m, y_m + 2^{-(m+1)}])$ and

$$\tau_{N+1}^* = \underset{t \in \mathbb{T}_{N+1}(\tau_{N+1}) \cap [y_m, y_m + 2^{-(m+1)}]}{\arg\max} x_{\mathbf{C},N+1}(t).$$

Proof. By applying Lemma 3.1.22, it is sufficient for us to prove the following statement.

If there exists $m \in \mathbb{N}$, such thatz $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$, for all $m < n \leq N$. We have

$$\Xi_k(\mathbf{A}) = \frac{x_{\mathbf{C},k}(\tau_k \vee \tau_k^*) - x_{\mathbf{C},k}(\tau_k \wedge \tau_k^*)}{\tau_k \vee \tau_k^* - \tau_k \wedge \tau_k^*} \tag{3.140}$$

for all $m < k \leq N + 1$.

Let us prove this lemma by induction on n. Assuming $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ now, let us first of all consider the case $n = m$. By Lemma 3.1.22, we directly have that

$$\Xi_m(\mathbf{A}) = \frac{x_{\mathbf{C},m}(\tau_m \vee \tau_m^*) - x_{\mathbf{C},m}(\tau_m \wedge \tau_m^*)}{\tau_m \vee \tau_m^* - \tau_m \wedge \tau_m^*}.$$

due to $\{\tau_m, \tau_m^*\} = \{y_m, y_m + 2^{-(m+1)}\}$. Then it holds even without requiring (3.138). Now let us assume that (3.138) implies (3.140) for all $k \leq n - 1$, and we now prove the case for $k = n$. By hypothesis for $n - 1$, we have

$$\Xi_{n-1}(\mathbf{A}) = \frac{x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*) - x_{\mathbf{C},n-1}(\tau_{n-1} \wedge \tau_{n-1}^*)}{\tau_{n-1} \vee \tau_{n-1}^* - \tau_{n-1} \wedge \tau_{n-1}^*} \tag{3.141}$$

For instance, let us assume that $\Xi_{n-1}(\mathbf{A}) > 0$. Then we have

$$x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*) - x_{\mathbf{C},n-1}(\tau_{n-1} \wedge \tau_{n-1}^*) = \Xi_{n-1}(\mathbf{A})(\tau_{n-1} \vee \tau_{n-1}^* - \tau_{n-1} \wedge \tau_{n-1}^*) > 0.$$

Moreover, since $x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*) > x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*)$, we have

$$\begin{cases} \tau_{n-1} \vee \tau_{n-1}^* = \tau_{n-1}, \\ \tau_{n-1} \wedge \tau_{n-1}^* = \tau_{n-1}^*. \end{cases}$$

Also, by the step condition, we have

$$\xi_n = -1. \tag{3.142}$$

By applying Corollary 3.1.4, we have $\tau_n \wedge \tau_n^* = \frac{\tau_{n-1}+\tau_{n-1}^*}{2}$, and $\tau_n \vee \tau_n^* = \tau_{n-1}$. Therefore, since $\tau_{n-1} \in \mathbb{T}_n$ and $\tau_{n-1}^* \in \mathbb{T}_n$, we have $2^n\tau_{n-1} \in \mathbb{Z}$ as well as $2^n\tau_{n-1}^* = 2^n(\tau_{n-1} \pm 2^{-n}) \in \mathbb{Z}$. Hence, we have

$$\phi(2^n\tau_{n-1}^*) = \phi(2^n\tau_{n-1}) = 0. \tag{3.143}$$

Furthermore, we have

$$\phi(2^n\frac{\tau_{n-1}+\tau_{n-1}^*}{2}) = \frac{1}{2}. \tag{3.144}$$

By plugging (3.141),(3.143) and (3.144) into (3.55), we have

$$\begin{aligned}
\frac{x_{\mathbf{C},n}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n}(\tau_n \wedge \tau_n^*)}{\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*} &= \Xi_{n-1}(\mathbf{A}) + c_n\frac{\phi(2^n\tau_{n-1} \vee \tau_{n-1}^*) - \phi(2^n\tau_{n-1} \wedge \tau_{n-1}^*)}{\tau_{n-1} \vee \tau_{n-1}^* - \tau_{n-1} \wedge \tau_{n-1}^*} \\
&= \Xi_{n-1}(\mathbf{A}) + c_n\frac{\phi(2^n(\frac{\tau_{n-1}+\tau_{n-1}^*}{2})) - \phi(2^n\tau_{n-1})}{\frac{\tau_{n-1}+\tau_{n-1}^*}{2} - \tau_{n-1}} \\
&= \Xi_{n-1}(\mathbf{A}) - 2^n c_n = \Xi_{n-1}(\mathbf{A}) + \xi_n a_n = \Xi_n(\mathbf{A}).
\end{aligned}$$

The last step holds because of (3.142). Hence, we have proved the case for $\tau_n > \tau_n^*$. The case for $\Xi_{n-1}(\mathbf{A}) < 0$ is analogous to the case for $\Xi_{n-1}(\mathbf{A}) > 0$. Now let us further consider the case when $\Xi_{n-1}(\mathbf{A}) = 0$. Then we have

$$x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*) - x_{\mathbf{C},n-1}(\tau_{n-1} \wedge \tau_{n-1}^*) = \Xi_{n-1}(\mathbf{A})(\tau_{n-1} \vee \tau_{n-1}^* - \tau_{n-1} \wedge \tau_{n-1}^*) = 0.$$

Then we have

$$x_{\mathbf{C},n-1}(t) = x_{\mathbf{C},n-1}(\tau_{n-1} \vee \tau_{n-1}^*) = x_{\mathbf{C},n-1}(\tau_{n-1} \wedge \tau_{n-1}^*) \tag{3.145}$$

for all $t \in [\tau_{n-1} \wedge \tau_{n-1}^*, \tau_{n-1} \vee \tau_{n-1}^*]$. Because $\tau_{n-1} \vee \tau_{n-1}^* \in \mathbb{T}_n$ and $\tau_{n-1} \wedge \tau_{n-1}^* \in \mathbb{T}_n$, then we have

$$\phi(2^n\tau_{n-1} \vee \tau_{n-1}^*) = \phi(2^n\tau_{n-1} \wedge \tau_{n-1}^*) = 0. \tag{3.146}$$

As well as,

$$\phi(\frac{1}{2}(2^n\tau_{n-1} \vee \tau_{n-1}^* + 2^n\tau_{n-1} \wedge \tau_{n-1}^*)) = \frac{1}{2}. \tag{3.147}$$

Then, one of two following cases must be true:

$$\begin{cases} \tau_n \wedge \tau_n^* = \tau_{n-1} \wedge \tau_{n-1}^* \quad \text{and} \quad \tau_n \vee \tau_n^* = \tau_{n-1} \wedge \tau_{n-1}^* + 2^{-(n+1)}, \\ \tau_n \wedge \tau_n^* = \tau_{n-1} \wedge \tau_{n-1}^* + 2^{-(n+1)} \quad \text{and} \quad \tau_n \vee \tau_n^* = \tau_{n-1} \vee \tau_{n-1}^*. \end{cases}$$

Furthermore, by applying the inductive hypothesis and Lemma **??**, we get

$$y_{n-1} = \tau_{n-1} \wedge \tau_{n-1}^*. \tag{3.148}$$

Therefore, (3.148) will directly lead us to

$$\begin{cases} y_n = y_{n-1} = \tau_{n-1} \wedge \tau_{n-1}^* \quad \text{if} \quad \xi_n = 1. \\ y_n = y_{n-1} + 2^{-(n+1)} = \tau_{n-1} \wedge \tau_{n-1}^* + 2^{-(n+1)} \quad \text{if} \quad \xi_n = -1. \end{cases} \tag{3.149}$$

By applying (3.145), (3.146) and (3.147). Then we must have one of following two cases,

$$\begin{aligned} &\frac{x_{\mathbf{C},n-1}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n-1}(\tau_n \wedge \tau_n^*)}{\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*} \\ &= \frac{x_{\mathbf{C},n}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n}(\tau_n \wedge \tau_n^*) + \phi(2^n\tau_n \vee \tau_n^*) - \phi(2^n\tau_n \wedge \tau_n^*)}{\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*} \\ &= \begin{cases} c_n \frac{\phi(2^n(\tau_{n-1} \wedge \tau_{n-1}^* + 2^{-(n+1)})) - \phi(2^n(\tau_{n-1} \wedge \tau_{n-1}^*))}{2^{-(n+1)}} = 2^n c_n = a_n = \Xi_{n-1}(\mathbf{A}) + a_n = \Xi_n(\mathbf{A}), \\ \\ c_n \frac{\phi(2^n\tau_{n-1} \vee \tau_{n-1}^*) - \phi(2^n(\tau_{n-1} \wedge \tau_{n-1}^* + 2^{-(n+1)}))}{2^{-(n+1)}} = -2^n c_n = -a_n = \Xi_{n-1}(\mathbf{A}) - a_n = \Xi_n(\mathbf{A}). \end{cases} \end{aligned} \tag{3.150}$$

The last step is due to (3.149). Hence, regardless of the choice of τ_n and τ_n^*, we always have

$$\Xi_n(\mathbf{A}) = \frac{x_{\mathbf{C},n}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n}(\tau_n \wedge \tau_n^*)}{\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*},$$

Therefore, we have proved the case for n, and, we finish our proof for this lemma. □

Theorem 3.2.14. For $x_{\mathbf{C}} \in \mathfrak{C}$, we have $y \in \mathcal{M}_{\mathbf{C}}([y_m, y_m + 2^{-(m+1)}])$ if and only if for every quasi-binary Ξ expansions of y, $(\Xi, \mathbf{C})$ satisfies the step condition for maxima after some index $m \in \mathbb{N}$.

Proof. In order to prove Theorem 3.2.14, we will use Lemma 3.1.22 and Theorem 3.1.18 to establish an equivalent statement. First of all, let us state all those equivalent statements we are going to prove:

- (i) For $x_{\mathbf{C}} \in \mathfrak{C}$, $y \in \mathcal{M}_{\mathbf{C}}([y_m, y_m + 2^{-(m+1)}])$.
- (ii) Let $y \in [0, 1]$, and y_n be the n^{th} order approximation for y. There exists a sequence of consecutive local maximizers $\tau_n \in \mathcal{M}_{\mathbf{C},n}([y_m, y_m + 2^{-(m+1)}])$ and $\tau_n^* \in \underset{t \in \mathbb{T}_{n+1}(\tau_n)}{\arg\max}\, x_{\mathbf{C},n}(t)$, such that $y_n = \tau_n \wedge \tau_n^*$ for all $n > m$.
- (iii) Let $\Xi = \{\xi_i\}_{i=0}^{\infty}$ be any quasi-binary expansion for y, and $x_{\mathbf{C}} \in \mathfrak{C}$. Take $a_i = 2^i c_i$, and $\mathbf{A} = \{a_i\}$, we have $\Xi_n(\mathbf{A})\xi_{n+1} \leq 0$ for all $n \geq m$.

Theorem 3.2.14 states the equivalence $(i) \Leftrightarrow (ii)$. Furthermore, Theorem 3.1.18 establishes the equivalence of (i) and (iii). Therefore, it is sufficient for us to prove the equivalence between (ii) and (iii). Moreover, Lemma 3.1.22 indicates that

$$\frac{x_{\mathbf{C},n}(t_1) - x_{\mathbf{C},n}(t_2)}{t_1 - t_2} = \Xi_n(\mathbf{A}),$$

for any $t_1, t_2 \in [y_n, y_n + 2^{-(n+1)}]$. Now let us take $t_1 = y_n = \tau_n \wedge \tau_n^*$. Then $t_2 = y_n + 2^{-(n+1)} = \tau_n \vee \tau_n^*$ for all $n \in \mathbb{N}$. We now aim to prove the following statement.

$$\Xi_n(\mathbf{A})\xi_{n+1} \leq 0 \textit{ for all } n \in \mathbb{N} \textit{ if and only if}$$

$$\Xi_n(\mathbf{A}) = \frac{x_{\mathbf{C},n}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n}(\tau_n \wedge \tau_n^*)}{\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*}, \tag{3.151}$$

where $\{\tau_n\}$ is some sequence of consecutive maximizers.

Then the only if direction is directly proved by Lemma 3.2.13. Now let us consider proving if part by proving its contrapositive statement. First of all, let us state the contrapositive statement:

Let $\{\xi_i\}$ be any quasi binary expansion for y. If there exist some $n \geq m$ such that $\Xi_n(\mathbf{A})\xi_{n+1} > 0$, then there exists $m \in \mathbb{N}$, such that $y_k \neq \tau_k \wedge \tau_k^*$, for any sequence of consecutive local maximizers $\{\tau_n\}$.

Now let us take $\mathcal{N} = \{n \geq m | \Xi_n(\mathbf{A})\xi_{n+1} > 0\}$, and $n = \min \mathcal{N}$. Then by Lemma 3.2.13, we have

$$y_n = \tau_n \wedge \tau_n^*.$$

Now let us first assume that $\Xi_n(\mathbf{A}) > 0$. Then we also have $\xi_{n+1} = 1$. Now we aim to prove that

$$y_{n+1} \neq \tau_{n+1} \wedge \tau_{n+1}^*.$$

We have

$$x_{\mathbf{C},n}(\tau_n \vee \tau_n^*) - x_{\mathbf{C},n}(\tau_n \wedge \tau_n^*) = \Xi_n(\mathbf{A})(\tau_n \vee \tau_n^* - \tau_n \wedge \tau_n^*) > 0,$$

by (3.151). Then we have $\tau_n = \tau_n \vee \tau_n^*$ and $\tau_n^* = \tau_n \wedge \tau_n^*$. By applying Corollary 3.1.4, we have

$$\tau_{n+1} \vee \tau_{n+1}^* = \tau_n, \quad \text{and} \quad \tau_{n+1} \wedge \tau_{n+1}^* = \frac{\tau_n + \tau_n^*}{2}.$$

Moreover, by Corollary 3.1.5, we have

$$[\tau_{n+1}^* \wedge \tau_{n+1}, \tau_{n+1}^* \vee \tau_{n+1}] \subset [\tau_n^* \wedge \tau_n, \tau_n^* \vee \tau_n].$$

Let $E = \{\varepsilon_j\}$ be the binary expansion for y. Since $\xi_{n+1} = 1$, we have

$$\varepsilon_{n+1} = \mathcal{H}^{-1}(\Xi)_{n+1} = 0.$$

Then we have

$$y_{n+1} = y_n + \varepsilon_{n+1} 2^{-(n+2)} = y_n.$$

Hence, obviously

$$y_{n+1} = \tau_n \wedge \tau_n^* \neq \tau_{n+1} \wedge \tau_{n+1}^* = \frac{\tau_n + \tau_n^*}{2}.$$

Hence, we have finished the proof for the case $\Xi_n(\mathbf{A}) > 0$. The case $\Xi_n(\mathbf{A}) < 0$ is analogous. Hence, we finish our proof for the statement. $\square$

Chapter 4

Exponential Takagi Class

4.1 Global Extrema for Exponential Takagi Class

Let us recall the definition of the Exponential Takagi Class in Definition 1.1.6.

Definition. The exponential Takagi class $\mathfrak{P}$ is the sub-collection of real-valued functions $x_\nu : [0,1] \to \mathbb{R}$ in the Takagi class $\mathfrak{C}$, where x_ν can be written as

$$x_\nu(t) = \sum_{n=0}^{\infty} \nu^n \phi(2^n t) \tag{4.1}$$

x_ν is called the Takagi function with parameter ν.

Now we are going to set up a mapping between the parameter ν and the extremum location. In order to have well-defined mappings, we first of all formally give following definitions.

Definition 4.1.1. Let us define a mapping $l_\sharp : [-1,1] \longrightarrow \mathbb{S}^{\mathbb{N}}$,

$$l_\sharp(\nu) = L_\sharp(\mathbf{C}), \tag{4.2}$$

where $\mathbf{C} = \{\nu^i\}_{i=0}^{\infty}$.

Definition 4.1.2. Let us define a mapping $L_\flat : [-1,1] \longrightarrow \mathbb{S}^{\mathbb{N}}$, where

$$l_\flat(\nu) = L_\flat(\mathbf{C}), \tag{4.3}$$

where $\mathbf{C} = \{\nu^i\}_{i=0}^{\infty}$.

Definition 4.1.3. Let us define a mapping $j_\sharp : [-1,1] \longrightarrow \mathbb{S}^{\mathbb{N}}$,

$$j_\sharp(\nu) = J_\sharp(\mathbf{C}), \tag{4.4}$$

where $\mathbf{C} = \{\nu^i\}_{i=0}^{\infty}$.

Definition 4.1.4. Let us define a mapping $j_\flat : [-1,1] \longrightarrow \mathbb{S}^{\mathbb{N}}$, where

$$j_\flat(\nu) = J_\flat(\mathbf{C}), \tag{4.5}$$

where $\mathbf{C} = \{\nu^i\}_{i=0}^{\infty}$.

Definition 4.1.5. Let $f_\sharp : [-1,1] \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$f_\sharp = T \circ \mathcal{H}^{-1} \circ l_\sharp.$$

We will say that $f_\sharp$ is the mapping for *upper maximizer* on the lower half.

Definition 4.1.6. Let $f_\flat : [-1,1] \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$f_\flat = T \circ \mathcal{H}^{-1} \circ l_\flat.$$

We will say that $f_\flat$ is the mapping for *lower maximizer* on the lower half.

Definition 4.1.7. Let $g_\sharp : [-1,1] \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$g_\sharp = T \circ \mathcal{H}^{-1} \circ j_\sharp.$$

We will say that $g_\sharp$ is the mapping for *upper minimizer* on the lower half.

Definition 4.1.8. Let $g_\flat : [-1, 1] \longrightarrow [0, \frac{1}{2}]$ be the mapping:

$$g_\flat = T \circ \mathcal{H}^{-1} \circ j_\flat.$$

We will say that $g_\flat$ is the mapping for *lower minimizer* on the lower half.

Corollary 4.1.9. For any $\nu \in (-1, 1)$, we have $f_\sharp(\nu) \in \mathcal{M}_\nu \cap [0, \frac{1}{2}]$ and $f_\flat(\nu) \in \mathcal{M}_\nu \cap [0, \frac{1}{2}]$.

Proof. This Corollary 4.1.9 directly comes from Theorem 3.1.40. □

Corollary 4.1.10. For any ν, we have

$$f_\sharp(\nu) = \sup \mathcal{M}_\nu \cap [0, \frac{1}{2}], \quad \text{and} \quad f_\flat(\nu) = \inf \mathcal{M}_\nu \cap [0, \frac{1}{2}].$$

Proof. Corollary 4.1.10 directly comes from Lemma 3.1.46. □

Corollary 4.1.11. For any ν, we have

$$g_\sharp(\nu) = \sup \tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}], \quad \text{and} \quad g_\flat(\nu) = \inf \tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}].$$

Proof. The proof is analogous to Corollary 4.1.10. □

Lemma 4.1.12. Denote $\Xi = L_\sharp(\nu)$. For a fixed ν, if $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$, then

$$f_\sharp(\nu) = f_\flat(\nu).$$

Proof. Since $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$, then by applying (4.3) and (4.2), we have

$$L_\sharp(\nu) = L_\flat(\nu).$$

Therefore, we have

$$f_\sharp(\nu) = (T \circ \mathcal{H}^{-1})(L_\sharp(\nu)) = (T \circ \mathcal{H}^{-1})(L_\flat(\nu)) = f_\flat(\nu).$$

□

Corollary 4.1.13. Denote $\Xi = L_\sharp(\nu)$. For a fixed ν, if $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$, then

$$|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 1.$$

Proof. This directly comes as a corollary from Corollary 3.1.43. □

Corollary 4.1.14. Let $\alpha = 2\nu$, if $\alpha \in \mathcal{D}^c_{\mathbb{R}}$, then $|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 1$.

Proof. Since $\nu \notin \mathcal{D}_{\mathbb{R}}$, for every $n \in \mathbb{N}$ and $f_n \in \mathcal{F}_n$, we have $f_n(\nu) \neq 0$. Because $\Xi_n(\nu) \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, $\Xi_n(\nu) \neq 0$ for all n. Hence by applying Corollary 4.1.13, we have $|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 1$. □

Corollary 4.1.15. For $\nu \in \mathbb{Q}$, we have $|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 1$.

Proof. Let $\alpha = 2\nu$. For $\nu \in \mathbb{Q} - \{-\frac{1}{2}, \frac{1}{2}\}$, we have $\alpha \in \mathbb{Q} - \{-1, 1\}$. By applying Corollary 4.1.14, we have $|\mathcal{M}_\nu \cap [0, \frac{1}{2}] = 1$. Furthermore, for $\alpha = 1$, by applying Theorem 1.2.2, we have $\mathcal{M}_\nu \cap [0, \frac{1}{2}] = \frac{1}{3}$. Now, for $\alpha = -1$, Theorem 1.2.4 indicates that $\mathcal{M}_\nu \cap [0, \frac{1}{2}] = \frac{1}{2}$. Therefore, we finish the proof. □

Theorem 4.1.16. For $\nu \in [0, 1)$, take $\alpha = 2\nu$. Let $\Xi \in \mathbb{S}^{\mathbb{N}}$, such that (Ξ, ν) satisfies the step condition for maxima. Denote $\mathcal{N} = \{n \in \mathbb{N} | \Xi_n(\alpha) = 0\}$, and $p = \min \mathcal{N}$. Then either $|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 1$, or $\mathcal{M}_\nu$ will have the form of a Cantor-like set with Hausdorff dimension $\frac{1}{p}$.

Proof. Denote $\alpha = 2\nu$. For some $\Xi \in \mathbb{S}^{\mathbb{N}}$, (Ξ, ν) satisfies the step condition for maxima. Let us first of all, consider the case when $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$. Then by applying Lemma 4.1.12, we have that

$$f_\sharp(\nu) = f_\flat(\nu).$$

Hence, there will be only one unique maximum for x_ν. Otherwise, for any n, such that $\Xi_n(\alpha) = 0$ if and only if $n \in p\mathbb{N}$. Let $\tilde{t}_p \in \mathcal{M}_\nu \in [0, \frac{1}{2}]$, then according to Proposition (3.55), we will have

$$\frac{x_{\nu,k}(t) - x_{\nu,k}(\tilde{t}_p)}{t - \tilde{t}_p} = \Xi_p(\alpha) = 0, \tag{4.6}$$

for all $t \in (\tilde{t}_p, \tilde{t}_p + 2^{-(p+1)}]$. Therefore, for $t \in (\tilde{t}_p, \tilde{t}_p + 2^{-(p+1)}]$, we have that

$$\begin{aligned} x_\nu(t) &= \sum_{n=0}^{\infty} \nu^n \phi(2^n t) = x_{\nu,p}(\tilde{t}_p) + \sum_{n=p+1}^{\infty} \nu^n \phi(2^n t) \\ &= x_{\nu,p}(\tilde{t}_p) + \nu^{p+1} \sum_{n=0}^{\infty} \nu^n \phi(2^{p+1+n} t) \end{aligned} \tag{4.7}$$

Now, let us denote $\tau = 2^{-(1+p)}(t - \tilde{t}_p)$, then (4.7) can be re-written as

$$x_\nu(t) = x_{\nu,p}(\tilde{t}_p) + \nu^{p+1} \sum_{n=0}^{\infty} \nu^n \phi(2^n \tau) = x_{\nu,p}(\tilde{t}_p) + \nu^{p+1} x_\nu(\tau). \tag{4.8}$$

Then $\nu^{p+1} x_\nu(\tau)$ is a re-scaled exponential Takagi function with parameter ν, therefore it has the behaviour in regards to its maximizers. By symmetry. If t the truncated maximum is attained at $t \in [\tilde{t}_p, \tilde{t}_p + 2^{-(p+1)}]$, then it also attains at $t \in [1 - \tilde{t}_p - 2^{-(p+1)}, 1 - \tilde{t}_p]$. Then we can see that t achieves the maximum if and only if t lies the $\lfloor 2^p \tilde{t}_p \rfloor - th$ interval of 2^p equally divided intervals or it lies in the $2^p - \lfloor 2^p \tilde{t}_p \rfloor - th$ interval of 2^p equally divided intervals. Thus, the set $\mathcal{M}$ is a Cantor-like set constructed by keeping only the $\frac{\tilde{t}_p}{2^p}$ and $2^p - \frac{\tilde{t}_p}{2^p}$ interval of every 2^p equally divided intervals. By [20],

$$dim_H \mathcal{M} = \frac{-\log(2)}{\log(\frac{1}{2^p})} = \frac{1}{p}.$$

□

Lemma 4.1.17. Take $\Xi = l_\sharp(\alpha_1)$ and $\Gamma = l_\sharp(\alpha_2)$. If $\xi_i = \gamma_i$ for all $i \leq n$, then

$$|f_\sharp(\alpha_1) - f_\sharp(\alpha_2)| \leq 2^{-(n+1)}.$$

Proof. By applying Triangular inequality, we have

$$|f_\sharp(\alpha_1) - f_\sharp(\alpha_2)| = |\sum_{i=0}^{\infty}(1_{\{\xi_i=-1\}} - 1_{\{\gamma_i=-1\}})2^{-(i+1)}| = |\sum_{i=n+1}^{\infty}(1_{\{\xi_i=-1\}} - 1_{\{\gamma_i=-1\}})2^{-(i+1)}|$$
$$\leq \sum_{i=n+1}^{\infty}|1_{\{\xi_i=-1\}} - 1_{\{\gamma_i=-1\}}|2^{-(i+1)} \leq \sum_{i=n+1}^{\infty} 2^{-(i+1)} = 2^{-(n+1)}$$

□

Theorem 4.1.18. For any $\nu \in (-1, 1)$, let $\Xi = l_\sharp(\nu)$, if $\Xi_n(\nu) \neq 0$ for all $n \in \mathbb{N}$, then $f_\sharp(u)$ and $f_\flat(u)$ are continuous at ν.

Proof. Let $\Xi = l_\sharp(\nu)$ and let $N = \lceil -\log_2 \epsilon \rceil$ for some $\epsilon > 0$. For such Ξ, we can view $\Xi_i(u)$ as a polynomial. Then $\Xi_i(u)$ is a continuous function for all $i \in \mathbb{N}$. Hence, for each $i \leq N$, there exist $\delta_i > 0$, such that

$$|\Xi_i(u) - \Xi_i(\nu)| < \frac{|\Xi_i(\nu)|}{2}. \tag{4.9}$$

whenever

$$|u - \nu| < \delta_i.$$

By (4.9), we have

$$\Xi_i(\nu)\Xi_i(u) > 0. \tag{4.10}$$

for all $i \leq N$. Now select $\delta = \min_{i \leq N} \delta_i$. Then

$$\Xi_i(\nu)\Xi_i(u) > 0,$$

for all $i \leq N$ if $|u - \nu| < \delta$. Now let us take $\Gamma = l_\sharp(x)$. Then we have $\xi_i = \gamma_i$ for all $i \leq N$, if $|u - \nu| < \delta$. By applying Lemma 4.1.17, if $\xi_i = \gamma_i$ for all $i \leq N$, then we have

$$|f_\sharp(u) - f_\sharp(\nu)| < \epsilon.$$

Therefore, we have every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f_\sharp(u) - f_\sharp(\nu)| < \epsilon.$$

for all points x which

$$|u - \nu| < \delta,$$

where $\delta = \min_{i \leq N} \delta_i$. The proof for $f_\flat(u)$ is analogous. □

Lemma 4.1.19. Let $\nu \in (\frac{1}{4}, \frac{1}{2})$. If (Ξ, ν) satisfies step condition for maxima, then for any $n \in \mathbb{N}$ there exists $N > n$, such that

$$\Xi_N(\alpha)\Xi_{N+1}(\alpha) \leq 0.$$

Proof. Let us first of all consider the case $\xi_0 = 1$. We will prove Lemma 4.1.19 by induction on n. First of all, let us consider $n = 0$, then we aim to prove that there exists $N > 0$, such that

$$\Xi_N(\alpha)\Xi_{N+1}(\alpha) \leq 0.$$

For $\frac{1}{2} < \alpha < 1$, we have

$$\sup_{\Xi \in \tilde{\mathbb{S}}} \sum_{i=1}^{\infty} \xi_i \alpha^i = \sum_{i=1}^{\infty} \alpha^i = \frac{\alpha}{1-\alpha} > 1 = \Xi_0(\alpha). \tag{4.11}$$

By rearranging (4.11), we get

$$\Xi_0(\alpha) - \sum_{i=1}^{\infty} \alpha^i < 0.$$

Then there exists $N \in \mathbb{N}$, such that

$$\Xi_0(\alpha) - \sum_{i=1}^{N} \alpha^i < 0, \quad \text{and} \quad \Xi_0(\alpha) - \sum_{i=1}^{m} \alpha^i \geq 0, \tag{4.12}$$

for all $m < N$. Therefore, by the step condition (3.56), we have $\xi_j = -1$, for all $j \leq N$.

Hence, we get

$$\begin{cases} \Xi_N(\alpha) = \sum_{i=0}^N \xi_i \alpha^i = 1 - \sum_{i=1}^N \alpha^i < 0. \\ \Xi_{N-1}(\alpha) = \sum_{i=0}^{N-1} \xi_i \alpha^i = 1 - \sum_{i=1}^{N-1} \alpha^i \geq 0. \end{cases} \tag{4.13}$$

Hence, we have

$$\Xi_{N-1}(\alpha)\Xi_N(\alpha) \leq 0.$$

This completes the proof for the case $n = 0$. Now let us assume that for $n \in \mathbb{N}$, there exists $N > n$, such that

$$\Xi_N(\alpha)\Xi_{N+1}(\alpha) \leq 0. \tag{4.14}$$

Now, we will prove the case for $n+1$, there also exists an $M > n+1$, such that

$$\Xi_M(\alpha)\Xi_{M+1}(\alpha) \leq 0.$$

We will discuss case by case. First of all, if $N > n+1$, then we can assign $M = N > n+1$, then we have

$$\Xi_N(\alpha)\Xi_{N+1}(\alpha) = \Xi_M(\alpha)\Xi_{M+1}(\alpha) \leq 0.$$

Now let us move on to the case when $N = n+1$, then by applying (4.14), we have

$$\Xi_{n+1}(\alpha)\Xi_{n+2}(\alpha) \leq 0.$$

First of all, let us consider when $\Xi_{n+1}(\alpha) \geq 0$, then

$$\Xi_{n+2}(\alpha) = \Xi_{n+1}(\alpha) - \alpha^{n+2} \geq -\alpha^{n+2} \tag{4.15}$$

As in (4.11), using our assumption $\alpha \in (\frac{1}{2}, 1)$, we then have

$$\sup_{\Xi \in \mathbb{S}^{\mathbb{N}}} \sum_{i=n+3}^{\infty} \xi_i \alpha^i = \sum_{i=n+3}^{\infty} \alpha^i = \frac{\alpha^{n+3}}{1-\alpha} > \alpha^{n+2} \geq -\Xi_{n+2}(\alpha). \tag{4.16}$$

Rearranging (4.16), we have

$$\Xi_{n+2}(\alpha) + \sum_{i=n+3}^{\infty} \alpha^i > 0. \tag{4.17}$$

Then there exists $M > n+2$, such that

$$\Xi_{n+2}(\alpha) + \sum_{i=n+3}^{M} \alpha^i > 0. \quad \text{and} \quad \Xi_{n+2}(\alpha) + \sum_{i=n+3}^{m} \alpha^i \leq 0. \tag{4.18}$$

for all $N < m < M$. Recalling the (3.56) and arguing as for (4.13), we have

$$\Xi_{M-1}(\alpha)\Xi_M(\alpha) \leq 0.$$

As $M > n+2$, then we have $M > n+1$, hence we complete the proof for the case when $\Xi_{n+1}(\alpha) \geq 0$. Now let us consider the case when $\Xi_{n+1}(\alpha) < 0$. Then we have $\Xi_{n+2}(\alpha) \geq 0$. Since (Ξ, α) satisfies the step condition for maxima, then we have

$$\Xi_{n+2}(\alpha) = \Xi_{n+1}(\alpha) + \alpha^{n+2} \leq \alpha^{n+2}. \tag{4.19}$$

Then we have

$$\sup_{\Xi \in \mathbb{S}^{\mathbb{N}}} \sum_{i=n+3}^{\infty} \xi_i \alpha^i = \sum_{i=n+3}^{\infty} \alpha^i = \frac{\alpha^{n+3}}{1-\alpha} > \alpha^{n+2} \geq \Xi_{n+2}(\alpha). \tag{4.20}$$

By rearranging (4.20), we then can have that

$$\Xi_{n+2}(\alpha) - \sum_{i=n+3}^{\infty} \alpha^i < 0. \tag{4.21}$$

Then there exists $M > n+2$, such that

$$\Xi_{n+2}(\alpha) - \sum_{i=n+3}^{M} \alpha^i < 0. \quad \text{and} \quad \Xi_{n+2}(\alpha) - \sum_{i=n+3}^{m} \alpha^i \geq 0. \tag{4.22}$$

for all $m < M$. Then we have

$$\Xi_{M-1}(\alpha)\Xi_M(\alpha) \leq 0. \tag{4.23}$$

Hence, we complete the proof for the case $n+1$ for $\xi_0 = 1$. The cases for $\xi_0 = -1$ is analogous. Therefore, we complete the proof. □

Proposition 4.1.20. Suppose $\Xi \in \mathbb{S}^{\mathbb{N}}$, and $\nu \in (\frac{1}{4}, \frac{1}{2})$. If (Ξ, ν) satisfies the step condition for maxima, then $\Xi(\alpha) = 0$.

Proof. Let a pair (Ξ, ν) satisfy the step condition for maxima. For any $n \in \mathbb{N}$, we have

$$\Xi_{n+1}(\alpha) = \Xi_n(\alpha) + \xi_{n+2}\alpha^{n+1}.$$

As (Ξ, ν) satisfies the step condition for maxima, by applying (3.56), we have

$$|\Xi_{n+1}(\alpha)| = |\Xi_n(\alpha) + \xi_{n+2}\alpha^{n+1}| \leq \max\{\alpha^{n+1}, |\Xi_n(\alpha)|\}. \tag{4.24}$$

Furthermore, we have

$$|\Xi_{n+1}(\alpha)| \leq \alpha^{n+1}, \quad \text{if} \quad \Xi_n(\alpha)\Xi_{n+1}(\alpha) \leq 0. \tag{4.25}$$

By applying Lemma 4.1.19, for each $n \in \mathbb{N}$, there exists $m > n$, such that $\Xi_m(\alpha)\Xi_{m+1}(\alpha) \leq 0$. Then according to (4.25), we have for any $n \in \mathbb{N}$, there exists $m > n$, such that

$$|\Xi_m(\alpha)| \leq \alpha^m. \tag{4.26}$$

Let us define that $x_n = \sup_{m \geq n} |\Xi_m(\alpha)|$, as we have by (4.26)

$$\max\{\alpha^{n+2}, |\Xi_{n+1}(\alpha)|\} \leq \max\{\alpha^{n+1}, |\Xi_n(\alpha)|\}, \tag{4.27}$$

then $x_n \leq \max\{\alpha^n, |\Xi_{n-1}(\alpha)|\}$. And as x_n is a bounded monotone decreasing sequence, then there exists $x = \lim_{n\to\infty} x_n$. According to (4.26), for each fixed N_0, there exists $m > N_0$, such that $x_m \leq \alpha^m$. Then rewrite (4.24), we then have

$$0 \leq \lim_{n\to\infty} x_n = x \leq \lim_{m\to\infty} \alpha^m = 0, \tag{4.28}$$

as $\alpha < 1$. Because $\limsup_{n\to\infty} |\Xi_n(\alpha)| = 0$, therefore we can conclude

$$\lim_{n\to\infty} \Xi_n(\alpha) = \Xi(\alpha) = 0.$$

□

Lemma 4.1.21. For any $\nu \in (\frac{1}{4}, \frac{1}{2})$, and arbitrarily small $\delta > 0$, there exists some $\beta \in (\nu - \delta, \nu + \delta)$, such that $l_\sharp(\beta) \neq l_\sharp(\nu)$.

Proof. Denote $\alpha = 2\nu$. We will then prove this lemma by contradiction. Let us assume that there are $\nu \in (\frac{1}{4}, \frac{1}{2})$, there and $\delta > 0$, such that for all $u \in (\nu - \delta, \nu + \delta)$, $l_\sharp(\nu) = l_\sharp(u)$. For simplicity, let us denote $l_\sharp(\nu) = \Xi$. Now, let us regard $\Xi(u)$ as a power series centered at $u_0 = 0$. Then since

$$\limsup_{i\to\infty} \sqrt[n]{|\xi_{i+1}|} = \limsup_{i\to\infty} 1 = 1. \tag{4.29}$$

Equation (4.29) guarantees that $\Xi(u)$ is an analytic function of u with convergence radius 1. Since $\alpha \in (\frac{1}{2}, 1)$ and $(\nu - \delta, \nu + \delta) \subsetneq (-1, 1)$, then by Proposition 4.1.20, we have

$$\Xi(u) = 0 \quad \text{for all } u \in (\nu - \delta, \nu + \delta).$$

Since $(\nu - \delta, \nu + \delta)$ is connected in $\mathbb{C}$, then $\Xi(u) = 0$ for all $u \in \mathbb{R}$. However, as we have that $\xi_i \in \{-1, 1\}$, then we must have $\Xi(u) \neq 0$. We have deduced a contradiction here. Hence for any $\nu \in (\frac{1}{4} . \frac{1}{2})$, and arbitrary small $\delta > 0$, there exist some $\beta \in (\nu - \delta, \nu + \delta)$, such that $l_\sharp(\beta) \neq l_\sharp(\nu)$. □

Lemma 4.1.22. For any $\nu \in (\frac{1}{4}, \frac{1}{2})$ and $\beta \in (\frac{1}{4}, \frac{1}{2})$. We have $l_\sharp(\nu) \neq l_\sharp(\beta)$, if and only if $f_\sharp(\nu) \neq f_\sharp(\beta)$.

Proof. Let us prove the only if part of this lemma by contradiction. Let us assume there exists β such that $l_\sharp(\nu) \neq l_\sharp(\beta)$, and $f_\sharp(\nu) = f_\sharp(\beta)$. Let us denote $\Xi = l_\sharp(\nu)$ and $\Gamma = l_\sharp(\beta)$.

Then as $l_\sharp(\nu) \neq l_\sharp(\beta)$, there exists some N such that

$$\xi_N \neq \gamma_N, \quad \text{and} \quad \xi_i = \gamma_i, \quad \text{for} \quad i < N.$$

Without loss of generality, let us assume that $\xi_N = 1$, then $\gamma_N = -1$. Then according to Proposition 3.1.39, we have $f_\sharp(\nu) = f_\sharp(\beta)$ if and only if $\xi_i = -1$ and $\gamma_i = 1$ for $i > N$. Now take $\alpha = 2\nu$. By (4.26)

$$|\Xi_N(\alpha)| \leq \alpha^N.$$

However, as we have

$$\sum_{i=N+1}^{\infty} \alpha^i = \alpha^N \frac{\alpha}{1-\alpha} > \alpha^N.$$

Without loss of generality, let us assume that $\Xi_N(\alpha) > 0$. Therefore,

$$\Xi(\alpha) = \Xi_N(\alpha) - \sum_{i=N+1}^{\infty} \alpha^i \neq 0.$$

But since $\nu \in (\frac{1}{4}, \frac{1}{2})$, we must have that $\Xi(\alpha) = 0$ by Proposition 4.1.20. Hence we have proved the only if direction by contradiction. Furthermore, let us now start to prove the if part. Let us look at the contrapositive statement:

$$\textit{If } l_\sharp(\nu) = l_\sharp(\beta), \textit{ then } f_\sharp(\nu) = f_\sharp(\beta).$$

But this statement directly follows from the Definition 4.1.5. □

Theorem 4.1.23. For any $\nu \in [\frac{1}{4}, \frac{1}{2}]$, for any $\delta > 0$, $f_\sharp([\nu - \delta, \nu + \delta]) \supsetneqq f_\sharp(\{\nu\})$ and $f_\flat([\nu - \delta, \nu + \delta]) \supsetneqq f_\flat(\{\nu\})$. This means that the functions $f_\sharp$ and $f_\flat$ are nowhere flat.

Proof. Denote $\alpha = 2\nu$. As $\nu \in [\frac{1}{4}, \frac{1}{2}]$, then we can have that for any $\delta > 0$, there exists $\beta \in [\alpha - \delta, \alpha - \delta]$, such that

$$l_\sharp(\beta) \neq l_\sharp(\nu).$$

Then, by applying Lemma 4.1.22, since $\nu \in (\frac{1}{4}, \frac{1}{2})$ and $\beta \in (\frac{1}{4}, \frac{1}{2})$, we have $f_\sharp(\nu) \neq f_\sharp(\alpha)$. Hence $f_\sharp([\nu - \delta, \nu + \delta]) \supsetneqq f_\sharp(\{\nu\})$,. Furthermore, the proof for $f_\flat$ is analogous.Hence, we have proved this theorem. □

Theorem 4.1.24. For $n \geq 2$, the Littlewood polynomial

$$\mathcal{Q}_n(z) = 1 - z \cdots - z^{n-2} - z^{n-1} - z^n,$$

has a unique negative root α_{2n}. Moreover, the sequence $\{\alpha_{2n}\}$ is strictly increasing and has a limit of -1.

Proof. First of all let us prove there exists a unique root $\alpha_{2n} \in (-2, -1)$. Consider for $u < -1$,

$$\mathcal{Q}_{2n}(u) = 1 - \sum_{i=1}^{2n} u^i = 1 - u\frac{1-u^{2n}}{1-u} > 0. \tag{4.30}$$

Then all negative roots must be less than or equal to -1. Moreover, by applying Lemma 2.1.4, then we have

$$\alpha_{2n} > -2.$$

Furthermore, we have

$$\begin{aligned}\mathcal{Q}_{2n}(u) &= 1 - \sum_{i=1}^{2n} u^i = 1 - u\frac{1-u^{2n}}{1-u} \\ &= \frac{1-2u+u^{2n+1}}{1-u}.\end{aligned}$$

Letting $q_{2n}(u) = 1 - 2u + u^{2n+1}$, we have

$$q'_{2n}(u) = -2 + (2n+1)u^{2n} > 0, \tag{4.31}$$

for $u \in (-2, -1)$. We have $Q_{2n}(u) = 0$ if and only if $q_{2n}(u) = 0$. However, since q_{2n} is strictly increasing, therefore, we have that this root must be unique. Furthermore, since we have $q_{2n}(\alpha_{2n}) = 0$, we get

$$q_{2n+2}(\alpha_{2n}) = 1 - 2\alpha_{2n} + \alpha_{2n}^{2n+3} = 1 - 2\alpha_{2n} + \alpha_{2n}^{2n+1} + (\alpha_{2n}^{2n+3} - \alpha_{2n}^{2n+1}) = \alpha_{2n}^{2n+3} - \alpha_{2n}^{2n+1} < 0.$$

Then by applying (4.31), we have $\alpha_{2n+2} > \alpha_{2n}$. Hence the sequence $\{\alpha_{2n}\}$ is an increasing sequence. Then there must exist α such that

$$\lim_{n\to\infty} \alpha_{2n} = \alpha = \sup\{\alpha_{2n} | n \geq 2\}. \tag{4.32}$$

Now let us assume by contradictory that $\alpha < -1$, and because $\alpha > \alpha_{2n}$ for all $n \in \mathbb{N}$, then we have

$$q_{2n}(\alpha) = 1 - 2\alpha + \alpha^{2n+1} > 0. \tag{4.33}$$

Taking limits on both sides, then we have

$$\lim_{n\to\infty} q_{2n}(\alpha) = 1 - 2\alpha + \lim_{n\to\infty} \alpha^{2n+1} = -\infty.$$

Hence, we have contradiction, since (4.33) indicates that

$$\lim_{n\to\infty} q_{2n}(\alpha) \geq 0.$$

□

Lemma 4.1.25. Let us denote by α_k the negative real solution to $\mathcal{Q}_k(z) = 0$. Then for $u \in [\alpha_{2k}, \alpha_{2k+2})$, we have

$$\mathcal{Q}_{2k+2}(u) < 0.$$

Proof. We have

$$\begin{aligned} \mathcal{Q}_{2k+2}(u) &= 1 - \sum_{i=1}^{2k+2} u^i = 1 - u\frac{1 - u^{2k+2}}{1 - u} \\ &= \frac{1 - 2u + u^{2k+3}}{1 - u}. \end{aligned}$$

Then since $\mathcal{Q}_{2k+2}(\alpha_{2k+2}) = 0$, we have

$$q(\alpha_{2k+2}) := 1 - 2\alpha_{2k+2} + \alpha_{2k+2}^{2k+3} = 0.$$

Furthermore, we have

$$q'(u) = -2 + (2k+2)u^{2k+2} > 0,$$

for all $u \in [-2, -1]$. Then for all $u \in [\alpha_{2k}, \alpha_{2k+2}]$, we have $q(u) < 0$. Since $1 - u > 0$, then we have $\mathcal{Q}_{2k+2}(u) < 0$. □

Theorem 4.1.26. Let us denote by α_k be the negative real solution for $\mathcal{Q}_k(z)$. For $2\nu \in [\alpha_{2n}, \alpha_{2n+2})$, let us denote $\Xi = l_\sharp(\nu)$, then

$$\Xi = \{1, \underbrace{-1, -1, \cdots, -1, -1}_{2n+2}, 1, 1, -1, -1, 1, 1, -1, -1, \cdots\}$$

Proof. Take $\alpha = 2\nu$. Let us first of all prove that Ξ will have $2n + 2$ consecutive -1 in the first $2n + 3$ items. For $m \leq n$, we have

$$1 - \sum_{i=1}^{2m} \alpha^i = 1 - \alpha \frac{1 - u^{2m}}{1 - \alpha} = \frac{1 - 2\alpha + \alpha^{2m+1}}{1 - \alpha}. \tag{4.34}$$

Since $0 > \alpha \geq \alpha_{2m}$, then $\alpha - \alpha_{2m} \geq 0$ and $|\alpha^{2m+1}| \leq |\alpha_{2m}^{2m+1}|$. Then we have

$$q(\alpha) = q(\alpha) - q(\alpha_{2m}) = 2(\alpha - \alpha_{2m}) + (\alpha^{2m+1} - \alpha_m^{2m+1}) \geq 0. \tag{4.35}$$

Therefore, we have $\Xi_{2m}(\alpha) = \mathcal{Q}_{2m}(\alpha) \geq 0$ for all $m \leq n$. Furthermore, we have

$$\Xi_{2m+1}(\alpha) = \Xi_{2m}(\alpha) - \alpha^{2m+1} > 0,$$

as $\alpha^{2m-1} < 0$. Thus we have proved that Ξ will have $2n + 2$ consecutive -1 in the first $2n + 3$ items by applying step condition for maxima. Now let us prove that for all $k \in \mathbb{N}^+$, we have

$$\begin{cases} \xi_{2n+4k-1} = 1, \\ \xi_{2n+4k} = 1, \\ \xi_{2n+4k+1} = -1, \\ \xi_{2n+4k+2} = -1. \end{cases} \tag{4.36}$$

by induction on k. Now let us first of all prove the base case. By applying Lemma 4.1.25, we have $\Xi_{2n+2}(\alpha) = \mathcal{Q}_{2n+2}(\alpha) < 0$. Hence, we have $\xi_{2n+3} = 1$ by the step condition for maxima. Furthermore, we have

$$\Xi_{2n+3}(\alpha) = \Xi_{2n+2}(\alpha) + \alpha^{2n+3} < 0.$$

Therefore, we have $\xi_{2n+4} = 1$ by the step condition for maxima. For $\alpha < -1$, we have $1 + \alpha - \alpha^2 - \alpha^3 > 0$. Moreover, since $\Xi_{2n}(\alpha) \geq 0$, we get

$$\begin{aligned}\Xi_{2n+4}(\alpha) &= \Xi_{2n}(\alpha) - \alpha^{2n+1} - \alpha^{2n+2} + \alpha^{2n+3} + \alpha^{2n+4} \\ &= \Xi_{2n} - \alpha^{2n+1}(1 + \alpha - \alpha^2 - \alpha^3) > 0.\end{aligned}$$

Therefore, we have $\xi_{4n+5} = -1$. Hence, we have

$$\Xi_{2n+5}(\alpha) = \Xi_{2n+4}(\alpha) - \alpha^{4n+5} > 0.$$

Hence, we have $\xi_{4n+6} = -1$. Therefore, we complete the proof for the base case. Now let us assume that (4.36) holds for all $k \leq m$, and we further prove the case for $k = m + 1$. Then we have

$$\begin{aligned}\Xi_{2n+4m+2}(\alpha) &= \Xi_{2n+2}(\alpha) + \sum_{i=0}^{m-1}(1 + \alpha - \alpha^2 - \alpha^3)\alpha^{4i+2n+3} \\ &= \Xi_{2n+2}(\alpha) + \alpha^{2n+3}(1 + \alpha - \alpha^2 - \alpha^3)\sum_{i=0}^{m-1}\alpha^{4i} \\ &= \Xi_{2n+2}(\alpha) + \alpha^{2n+3}(1 + \alpha - \alpha^2 - \alpha^3)\frac{1 - \alpha^{4m}}{1 - \alpha^4} < 0.\end{aligned}$$

This is due to $\Xi_{2n+2}(\alpha) < 0$, $1 + \alpha - \alpha^2 - \alpha^3 > 0$, and $\frac{1-\alpha^{4m}}{1-\alpha^4} > 0$. Therefore, by the step condition for maxima, we have $\xi_{2n+4m+3} = 1$. Furthermore, we have

$$\Xi_{2n+4m+3}(\alpha) = \Xi_{2n+4m+2}(\alpha) + \alpha^{2n+4m+3} < 0.$$

Hence, we have $\xi_{2n+4m+4} = 1$. Then we have

$$\begin{aligned}\Xi_{2n+4m+4}(\alpha) &= \Xi_{2n+4m}(\alpha) - \alpha^{2n+4m+1} - \alpha^{2n+4m+2} + \alpha^{2n+4m+3} + \alpha^{2n+4m+4} \\ &= \Xi_{2n+4m}(\alpha) - \alpha^{2n+4m+1}(1 + \alpha - \alpha^2 - \alpha^3) > 0.\end{aligned}$$

This is due to $\Xi_{2n+4m}(\alpha) > 0$ and $1 + \alpha - \alpha^2 - \alpha^3 > 0$. Hence, we have $\xi_{2n+4m+5} = -1$. Then, we have

$$\Xi_{2n+4m+5}(\alpha) = \Xi_{2n+4m+4}(\alpha) - \alpha^{2n+4m+5} > 0.$$

Therefore, we have $\xi_{2n+4m+6} = -1$. Hence, we finish the inductive proof and the statement. □

Corollary 4.1.27. For any $\nu \in [\frac{\alpha_{2k}}{2}, \frac{\alpha_{2k+2}}{2})$, we have

$$f_\sharp(\nu) = f_\sharp(\frac{\alpha_{2k}}{2}).$$

Proof. Let us denote $\Xi = l_\sharp(\nu)$ and $\Gamma = l_\sharp(\frac{\alpha_{2k}}{2})$. Then by applying Theorem 4.1.26, we have

$$\Xi = \Gamma$$

Therefore, we have

$$f_\sharp(\nu) = T \circ \mathcal{H}^{-1}(\Xi) = T \circ \mathcal{H}^{-1}(\Gamma) = f_\sharp(\frac{\alpha_{2k}}{2}).$$

□

Corollary 4.1.28. For $\nu \in [-1, -\frac{1}{2}]$, we have $f_\sharp(\nu)$ is a right-continuous function.

Proof. This corollary directly comes from Corollary 4.1.27. □

Theorem 4.1.29. Let us denote α_k be the negative real solution for $\mathcal{Q}_k(z)$. For $\nu \in [\frac{\alpha_{2k}}{2}, \frac{\alpha_{2k+2}}{2})$, let us denote $\Xi = f_\flat(\nu)$, then

$$\Xi = \{1, \underbrace{-1, -1, \cdots, -1, -1}_{2n+2}, 1, 1, -1, -1, 1, 1, -1, -1, \cdots\}$$

Proof. Take $\alpha = 2\nu$. Let us first of all prove that Ξ will have $2n+2$ consecutive -1 in the first $2n+3$ items. Let us consider for $m \leq n$

$$1 - \sum_{i=1}^{2m} \alpha^i = 1 - \alpha \frac{1-u^{2m}}{1-\alpha} = \frac{1 - 2\alpha + \alpha^{2m+1}}{1-\alpha}. \tag{4.37}$$

Since $0 > \alpha > \alpha_{2m}$, then $\alpha - \alpha_2 m > 0$ and $|\alpha^{2m+1}| < |\alpha_{2m}^{2m+1}|$. Then we have

$$q(\alpha) = q(\alpha) - q(\alpha_{2m}) = 2(\alpha - \alpha_{2m}) + (\alpha^{2m+1} - \alpha_m^{2m+1}) > 0. \tag{4.38}$$

Therefore, we have $\Xi_{2m}(\alpha) = \mathcal{Q}_{2m}(\alpha) < 0$ for all $m \leq n$. Furthermore, we have

$$\Xi_{2m+1}(\alpha) = \Xi_{2m}(\alpha) - \alpha^{2m+1} > 0,$$

as $\alpha^{2m-1} < 0$. Then we have proved that Ξ will have $2n+2$ consecutive -1 in the first $2n+3$ items by applying step condition for maxima. Now let us prove the for all $k \in \mathbb{N}^+$, we have

$$\begin{cases} \xi_{2n+4k-1} = 1, \\ \xi_{2n+4k} = 1, \\ \xi_{2n+4k+1} = -1, \\ \xi_{2n+4k+2} = -1. \end{cases} \tag{4.39}$$

by induction on k. Now let us first of all prove the base case. By applying Lemma 4.1.25, we have $\Xi_{2n+2}(\alpha) = \mathcal{Q}_{2n+2}(\alpha) \leq 0$. Hence, we have $\xi_{2n+3} = 1$ by the step condition for maxima. Furthermore, we have

$$\Xi_{2n+3}(\alpha) = \Xi_{2n+2}(\alpha) + \alpha^{2n+3} < 0.$$

Therefore, we have $\xi_{2n+4} = 1$ by the step condition for maxima. For $\alpha < -1$, we have

$1 + \alpha - \alpha^2 - \alpha^3 > 0$. Moreover, since $\Xi_{2n}(\alpha) > 0$, we get

$$\begin{aligned}\Xi_{2n+4}(\alpha) &= \Xi_{2n+3}(\alpha) + \alpha^{2n+4} = \Xi_{2n}(\alpha) - \alpha^{2n+1} - \alpha^{2n+2} + \alpha^{2n+3} + \alpha^{2n+4} \\ &= \Xi_{2n} - \alpha^{2n+1}(1 + \alpha - \alpha^2 - \alpha^3) > 0.\end{aligned}$$

Therefore, we have $\xi_{4n+5} = -1$. Hence, we have

$$\Xi_{2n+5}(\alpha) = \Xi_{2n+4}(\alpha) - \alpha^{4n+5} > 0.$$

Hence, we have $\xi_{4n+6} = -1$. Therefore, we complete the proof for the base case. Now let us assume that (4.39) holds for all $k \leq m$, and we further prove the case for $k = m + 1$. Then we have

$$\begin{aligned}\Xi_{2n+4m+2}(\alpha) &= \Xi_{2n+2}(\alpha) + \sum_{i=0}^{m-1}(1 + \alpha - \alpha^2 - \alpha^3)\alpha^{4i+2n+3} \\ &= \Xi_{2n+2}(\alpha) + \alpha^{2n+3}(1 + \alpha - \alpha^2 - \alpha^3)\sum_{i=0}^{m-1}\alpha^{4i} \\ &= \Xi_{2n+2}(\alpha) + \alpha^{2n+3}(1 + \alpha - \alpha^2 - \alpha^3)\frac{1 - \alpha^{4m}}{1 - \alpha^4} < 0.\end{aligned}$$

This is due to $\Xi_{2n+2}(\alpha) < 0$, $1 + \alpha - \alpha^2 - \alpha^3 > 0$, and $\frac{1-\alpha^{4m}}{1-\alpha^4} > 0$. Therefore, by the step condition for maxima, we have $\xi_{2n+4m+3} = 1$. Furthermore, we have

$$\Xi_{2n+4m+3}(\alpha) = \Xi_{2n+4m+2}(\alpha) + \alpha^{2n+4m+3} < 0.$$

Hence, we have $\xi_{2n+4m+4} = 1$. Then we have

$$\begin{aligned}\Xi_{2n+4m+4}(\alpha) &= \Xi_{2n+4m}(\alpha) - \alpha^{2n+4m+1} - \alpha^{2n+4m+2} + \alpha^{2n+4m+3} + \alpha^{2n+4m+4} \\ &= \Xi_{2n+4m}(\alpha) - \alpha^{2n+4m+1}(1 + \alpha - \alpha^2 - \alpha^3) > 0.\end{aligned}$$

This is due to $\Xi_{2n+4m}(\alpha) > 0$ and $1 + \alpha - \alpha^2 - \alpha^3 > 0$. Hence, we have $\xi_{2n+4m+5} = -1$.

Then, we have

$$\Xi_{2n+4m+5}(\alpha) = \Xi_{2n+4m+4}(\alpha) - \alpha^{2n+4m+5} > 0.$$

Therefore, we have $\xi_{2n+4m+6} = -1$. Hence, we finish the inductive proof and the statement. □

Theorem 4.1.30. For any $\nu \in (\frac{\alpha_{2k}}{2}, \frac{\alpha_{2k+2}}{2}]$, we have

$$f_\flat(\nu) = f_\flat(\frac{\alpha_{2k}}{2}).$$

Proof. Let us denote $\Xi = l_\flat(\nu)$ and $\Gamma = l_\flat(\frac{\alpha_{2k}}{2})$. Then by applying Theorem 4.1.29, we have

$$\Xi = \Gamma$$

Therefore, we have

$$f_\flat(\nu) = T \circ \mathcal{H}^{-1}(\Xi) = T \circ \mathcal{H}^{-1}(\Gamma) = f_\flat(\frac{\alpha_{2k}}{2}).$$

□

Corollary 4.1.31. The $f_\sharp(\nu)$ and $f_\flat(\nu)$ are discontinuous at point $\frac{\alpha_{2k}}{2}$ for all $k \in \mathbb{N}$.

Proof. This directly comes as a corollary for Theorem 4.1.30 and Theorem 4.1.27. □

Corollary 4.1.32. Let us denote $\mathcal{R}$ as all the discontinuous points of $f_\sharp(\nu)$ in $[-1, -\frac{1}{2})$. Then as all $\nu \in \mathcal{R}$, we have

$$|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 2.$$

Furthermore, we have

$$|\mathcal{R}| = \aleph_0.$$

Proof. Let us first of all consider the function $f_\sharp$. According to Corollary 4.1.32 and Theorem 4.1.26, we have $f_\sharp(\nu)$ is discontinuous if and only if $\nu = \frac{\alpha_{2k}}{2}$ for some k. Now, let us denote $\Xi = l_\sharp(\nu)$. Then $\Xi_n(\nu) = 0$ if and only if $n = 2k$, then by Lemma 3.1.45,

$$|\mathcal{M}_\nu \cap [0, \frac{1}{2}]| = 2.$$

Since we have $\mathcal{R} = \{\alpha_{2k}\}_{k=1}^{\infty}$, then we have

$$|\mathcal{R}| = \aleph_0.$$

□

4.2 Local Extrema for Exponential Takagi Class

Theorem 4.2.1. Let us denote $\mathcal{V}_\nu$ as the collection of points that are a local maxima for x_ν. Formally, $\mathcal{V}_\nu$ is defined as follows:

$$\mathcal{V}_\nu = \{t \in [0,1] | \exists \delta > 0, \forall \tau \in (t-\delta, t+\delta), x_\nu(t) \geq x_\nu(\tau)\}$$

For $|\nu| \leq \frac{1}{4}$, $\mathcal{V}_\nu = \{\frac{1}{2}\}$.

Proof. Let us first of all consider the case for $|\nu| \leq \frac{1}{4}$. Let $\Gamma \in \mathbb{S}^{\mathbb{N}}$ with $\gamma_0 = 1$. We have

$$\inf_{\Gamma \in \mathbb{S}^{\mathbb{N}}} \Gamma(\alpha) = 1 - \sum_{i=1}^{\infty} |\alpha|^i = 1 - \frac{|\alpha|}{1-|\alpha|} = \frac{1-2|\alpha|}{1-|\alpha|} > 0.$$

Then we have x_ν is an strictly increasing function on $[0, \frac{1}{2}]$. Furthermore, due to symmetry, we have x_ν is an strictly decreasing function on $[\frac{1}{2}, 1]$. Then we have $\mathcal{M}_\nu = \mathcal{V}_\nu = \{\frac{1}{2}\}$. □

4.3 Earlier Results under Quasi-Binary Language

Now in this section, let us restate and prove theorems in Chapter 1 and some new particularly strong results. The theorems here are not as general as before, but those theorem can be regarded as the examples on how the step condition and Quasi-Binary Language can simplify and unify the results before.

Example 4.3.1. Let us use our general results so as to recover Theorem 1.2.3, which states for $\nu \in (\frac{1}{2}, 1)$, we have $f_\flat(\nu) = f_\sharp(\nu) = \frac{1}{3}$.

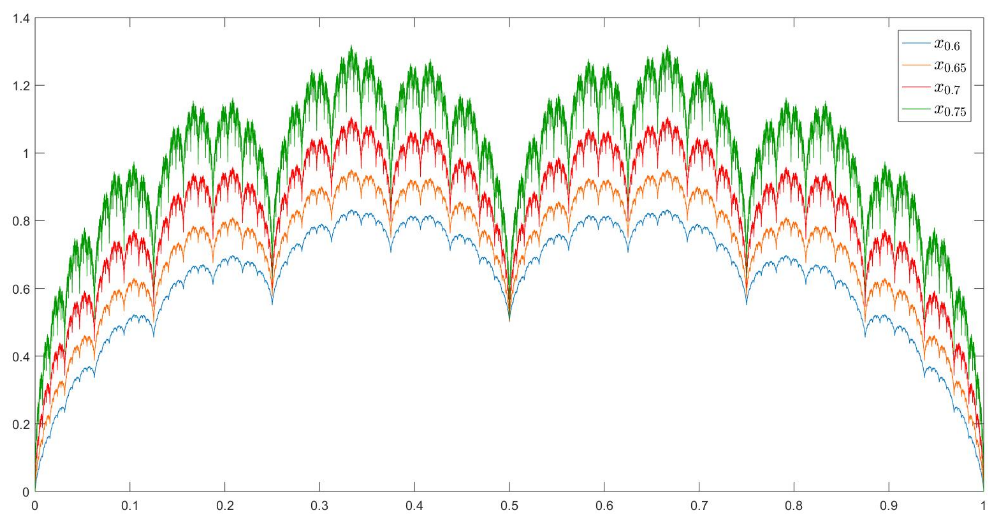

Figure 4.1: Plot for exponential Takagi function $x_{0.6}$, $x_{0.65}$, $x_{0.7}$ and $x_{0.75}$.

Proof. Take $\Xi = l_\sharp(\nu)$ and $\alpha = 2\nu$. Before proving this theorem, we first of all prove that

$$\begin{cases} \Xi_{2n}(\alpha) > 0, \\ \Xi_{2n+1}(\alpha) < 0, \end{cases}$$

for every $n \in \mathbb{N}$ by induction on n. Now let us consider the case when $n = 1$. As we have $\Xi_0(\alpha) = 1$ for all $\alpha \in (1, 2)$, then we have $\xi_1 = -1$ and $\Xi_1(\alpha) = 1 - \alpha < 0$. Hence, we finish the proof for the base case. Now let us assume that

$$\begin{cases} \Xi_{2k}(\alpha) > 0, \\ \Xi_{2k+1}(\alpha) < 0. \end{cases} \tag{4.40}$$

for all $n \leq k$, and we proceed to prove the case $k+1$. As $\Xi_{2k+1}(\alpha) < 0$, then we have $\xi_{2k+2} = 1$. By applying (4.40), we have

$$\begin{aligned}\Xi_{2k+2}(\alpha) &= \sum_{i=0}^{2k+2} \xi_i \alpha^i = \sum_{i=0}^{2k+2} (-1)^{i+1} \alpha^i = \sum_{i=0}^{k+1} \alpha^{2i} - \sum_{i=0}^{k} \alpha^{2i+1} \\ &= \frac{1-\alpha^{2k+2}}{1-\alpha^2} - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^2} = \frac{(1-\alpha)+(\alpha^{2k+1}-\alpha^{2k+2})}{1-\alpha^2} > 0.\end{aligned} \tag{4.41}$$

Then for $\Xi_{2k+3}(\alpha)$, we have

$$\Xi_{2k+3}(\alpha) = \sum_{i=0}^{2k+3} \xi_i \alpha^i = \sum_{i=0}^{2k+3} (-1)^i \alpha^i = (1-\alpha) \sum_{i=0}^{k+1} \alpha^{2i} < 0. \tag{4.42}$$

Therefore, we complete the proof for the case $k+1$, and hence we finish the proof that for every n, we have

$$\begin{cases} \Xi_{2n}(\alpha) > 0, \\ \Xi_{2n+1}(\alpha) < 0. \end{cases}$$

By applying the (3.57), we have that for every $n \in \mathbb{N}$, $\xi_{2n} = 1$ and $\xi_{2n+1} = -1$. Then

$$f_\sharp(\nu) = \sum_{i=0}^{\infty} 1_{\{\xi_i=-1\}} (\frac{1}{2})^{i+1} = \sum_{i=1}^{\infty} (\frac{1}{2})^{2i} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}. \tag{4.43}$$

as well as, by applying Lemma 4.1.12, as for all n, $\Xi_n(\alpha) \neq 0$. We then have,

$$f_\flat(\nu) = f_\sharp(\nu) = \frac{1}{3}. \tag{4.44}$$

□

Example 4.3.2. Let us use our general results so as to recover Theorem 1.2.4, which states for $\nu \in (0, \frac{1}{4})$, we have $f_\sharp(\nu) = f_\flat(\nu) = \frac{1}{2}$.

Proof. Take $\Xi = l_\sharp(\nu)$ and $\alpha = 2\nu$. First of all, we will prove for any $n \in \mathbb{N}$, $\Xi_n(\alpha) > 0$ by induction on n. Now, let us consider the case when $n = 0$. It is clear that $\Xi_0(\alpha) = 1 > 0$.

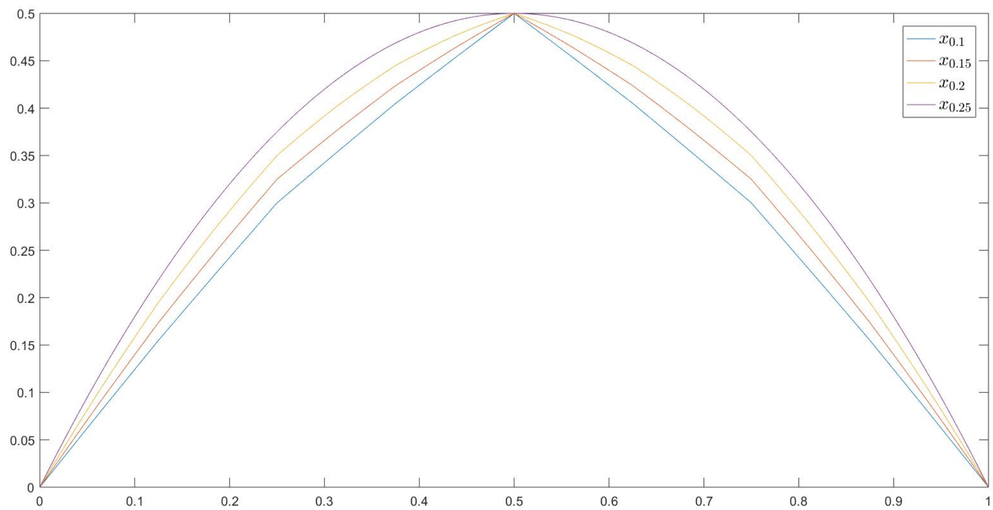

Figure 4.2: Plot for exponential Takagi function $x_{0.1}$, $x_{0.15}$, $x_{0.2}$ and $x_{0.25}$.

Now let us assume $\Xi_n(\alpha) > 0$ for all $n \leq k$. We then proceed to prove the case $k+1$, as $\Xi_k(\alpha) > 0$ and $\xi_{k+1} = -1$. Let us consider

$$\Xi_{k+1}(\alpha) = \sum_{i=0}^{k+1} \xi_i \alpha^i = 1 - \sum_{i=0}^{k+1} \alpha^i = 1 - \frac{1-\alpha^{k+1}}{1-\alpha} > 1 - \frac{\alpha}{1-\alpha} > 0. \tag{4.45}$$

Then we complete the proof for the induction. As for every $n \in \mathbb{N}$, $\Xi_n(\alpha) > 0$, then we have $\xi_n = -1$ for every $n \geq 1$. Then we have

$$f_\sharp(\nu) = \sum_{i=0}^{\infty} \xi_i (\frac{1}{2})^{i+1} = \sum_{i=2}^{\infty} (\frac{1}{2})^i = \frac{1}{2}, \tag{4.46}$$

as well as, by applying Lemma 4.1.12, we then have

$$f_\flat(\nu) = f_\sharp(\nu) = \frac{1}{2}. \tag{4.47}$$

□

Example 4.3.3. For $\nu = -\frac{1+\sqrt{5}}{4}$, the exponential Takagi function x_ν has exactly two maximum points in $[0, \frac{1}{2}]$ at 0.4 and 0.475.

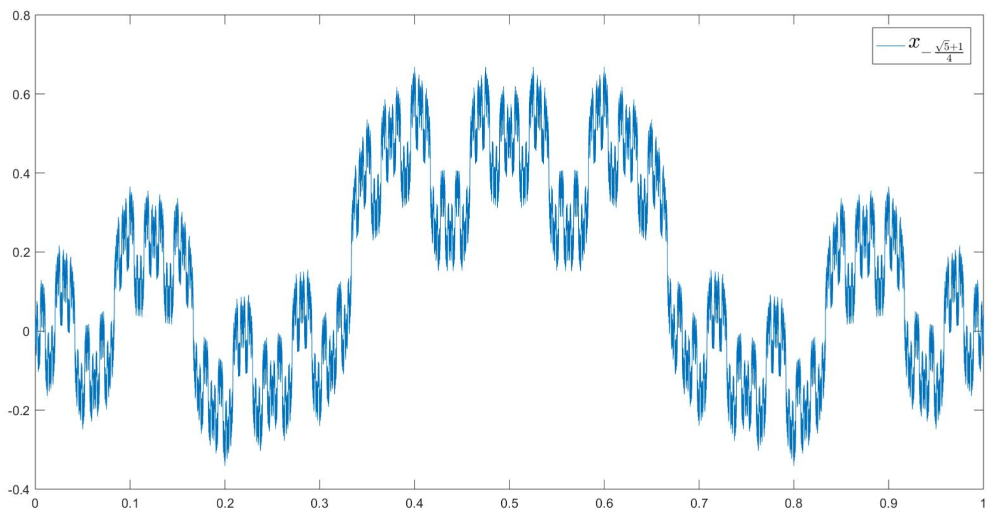

Figure 4.3: Plot for exponential Takagi function $x_{-\frac{\sqrt{5}+1}{4}}$.

Proof. Let $\alpha = 2\nu = -\frac{1+\sqrt{5}}{2}$ and take $\Xi = l_\flat(\nu)$. First of all, we will prove that for any fixed $n \geq 0$, we have

$$\begin{cases} \Xi_{4n+2}(\alpha) \leq 0, \\ \Xi_{4n+3}(\alpha) \leq 0, \\ \Xi_{4n+4}(\alpha) > 0, \\ \Xi_{4n+5}(\alpha) > 0, \end{cases} \tag{4.48}$$

by induction on n. Let us first consider the case for $\Xi_0(\alpha)$, we have

$$\Xi_0(\alpha) = 1. \tag{4.49}$$

Then according to (3.57), we then have $\xi_1 = -1$, and hence

$$\Xi_1(\alpha) = 1 - \alpha > 0. \tag{4.50}$$

By applying (3.57) again, we have for our particular choice of α

$$\Xi_2(\alpha) = 1 - \alpha - \alpha^2 = 0. \tag{4.51}$$

Then according to Definition 4.1.6, we have $\xi_3 = 1$. Therefore, we have

$$\Xi_3(\alpha) = 1 - \alpha - \alpha^2 + \alpha^3 = \alpha^3 < 0. \tag{4.52}$$

By applying (3.57) again, we have

$$\Xi_4(\alpha) = \alpha^3 + \alpha^4 = \alpha^3(1 + \alpha) > 0, \tag{4.53}$$

as well as,

$$\Xi_5(\alpha) = \alpha^3 + \alpha^4 - \alpha^5 = \alpha^3(1 + \alpha - \alpha^2) > 0. \tag{4.54}$$

Hence, by (4.51) - (4.54), we prove that the induction hypothesis holds for $n = 0$. Now let us assume that the induction hypothesis holds for $k \leq n$, then we proceed to prove the case for $k = n + 1$. Also, due to (3.57) and (4.48), we then have for all $k \leq n$,

$$\begin{cases} \xi_{4k+3} = 1, \\ \xi_{4k+4} = 1, \\ \xi_{4k+5} = -1, \\ \xi_{4k+6} = -1. \end{cases} \tag{4.55}$$

Hence, we have

$$\Xi_{4n+6}(\alpha) = \sum_{i=0}^{4n+6} \xi_i \alpha^i = (1 - \alpha - \alpha^2) + \sum_{m=0}^{n} (\alpha^3 + \alpha^4 - \alpha^5 - \alpha^6)\alpha^{4m}. \tag{4.56}$$

By applying (4.51), we have

$$\begin{aligned}\Xi_{4n+6}(\alpha) &= \sum_{i=0}^{4n+6} \xi_{i+1}\alpha^i = (1-\alpha-\alpha^2) + \sum_{m=0}^{n}(\alpha^3+\alpha^4-\alpha^5-\alpha^6)\alpha^{4m} \\ &= \sum_{m=0}^{n} \alpha^{4m+3} = \alpha^3\frac{1-\alpha^{4n+4}}{1-\alpha^4} < 0.\end{aligned} \tag{4.57}$$

Therefore, we have $\xi_{4n+7} = 1$. Hence, by applying (3.57), we have

$$\Xi_{4n+7}(\alpha) = \Xi_{4n+6}(\alpha) + \alpha^{4n+7}. \tag{4.58}$$

as $\Xi_{4n+6}(\alpha) < 0$ and $\alpha^{4n+7} < 0$, then we have $\Xi_{4n+7}(\alpha) < 0$. Hence, we have $\xi_{4n+8} = 1$, and then

$$\begin{aligned}\Xi_{4n+8}(\alpha) &= \alpha^3\frac{1-\alpha^{4n+4}}{1-\alpha^4} + \alpha^{4n+7} + \alpha^{4n+8} \\ &= \frac{\alpha^3+\alpha^{4n+8}-\alpha^{4n+11}-\alpha^{4n+12}}{1-\alpha^4} \\ &= \frac{\alpha^3+\alpha^{4n+8}-\alpha^{4n+10}+\alpha^{4n+10}-\alpha^{4n+11}-\alpha^{4n+12}}{1-\alpha^4} \\ &= \frac{\alpha^3+\alpha^{4n+8}-\alpha^{4n+10}}{1-\alpha^4} = \frac{\alpha^3+\alpha^{4n+8}+\alpha^{4n+9}-\alpha^{4n+9}-\alpha^{4n+10}}{1-\alpha^4} \\ &= \frac{\alpha^3+\alpha^{4n+9}}{1-\alpha^4} > 0,\end{aligned} \tag{4.59}$$

as $\alpha^3 < 0$, $\alpha^{4n+9} < 0$, as well as $1-\alpha^4 < 0$. Then we can have that $\xi_{4n+9} = -1$, then we have

$$\Xi_{4n+9}(\alpha) = \Xi_{4n+8}(\alpha) - \alpha^{4n+9} > 0. \tag{4.60}$$

as $\Xi_{4n+8}(\alpha) > 0$ and $\alpha^{4n+9} < 0$. Then we can notice that

$$\begin{cases} \Xi_{4n+6}(\alpha) = \Xi_{4(n+1)+2}(\alpha) \leq 0, \\ \Xi_{4n+7}(\alpha) = \Xi_{4(n+1)+3}(\alpha) \leq 0, \\ \Xi_{4n+8}(\alpha) = \Xi_{4(n+1)+4}(\alpha) > 0, \\ \Xi_{4n+9}(\alpha) = \Xi_{4(n+1)+5}(\alpha) > 0, \end{cases} \tag{4.61}$$

Hence, we finish prove the case when $k = n + 1$. Hence we finish proving the statement. Then by applying the Lemma 3.1.10, we have

$$f_\flat(\nu) = \sum_{i=1}^{\infty} 1_{\{\xi_i=-1\}}(\frac{1}{2})^i = \sum_{i=0}^{\infty} \frac{1}{2^{4n+2}} + \frac{1}{2^{4n+3}} = (\frac{1}{4} + \frac{1}{8})\frac{1}{1-2^{-4}} = \frac{2}{5}. \tag{4.62}$$

Furthermore, let us start to consider another maximum at 0.475. We will follow the same strategy for 0.4. Now we will prove that for any fixed $n \geq 0$, we have

$$\begin{cases} \Xi_{4n+2}(\alpha) \geq 0, \\ \Xi_{4n+3}(\alpha) \geq 0, \\ \Xi_{4n+4}(\alpha) < 0, \\ \Xi_{4n+5}(\alpha) < 0, \end{cases} \tag{4.63}$$

by induction on n. Let us first consider the case for $\Xi_0(\alpha)$. Now let us take $\Xi = l_\sharp(\nu)$. We have

$$\Xi_0(\alpha) = 1. \tag{4.64}$$

Then according the step condition for maxima, we then have $\xi_1 = -1$, and hence

$$\Xi_1(\alpha) = 1 - \alpha > 0. \tag{4.65}$$

Then by applying the step condition again, we have

$$\Xi_2(\alpha) = 1 - \alpha - \alpha^2 = 0. \tag{4.66}$$

Then according to the Definition 4.1.5, we have $\xi_3 = -1$. Therefore, we have

$$\Xi_3(\alpha) = 1 - \alpha - \alpha^2 - \alpha^3 = -\alpha^3 > 0. \tag{4.67}$$

By applying (3.57), we have

$$\Xi_4(\alpha) = -\alpha^3 - \alpha^4 = -\alpha^3(1+\alpha) < 0, \tag{4.68}$$

as $\alpha^3 < 0$, and $1 + \alpha < 0$, as well as,

$$\Xi_5(\alpha) = -\alpha^3 - \alpha^4 + \alpha^5 = -\alpha^3(1 + \alpha - \alpha^2) < 0. \tag{4.69}$$

Hence, by (4.66) - (4.69), we notice the induction hypothesis holds for $n = 0$. Now let us assume that the induction hypothesis holds for $k \leq n$, then we proceed to prove the case for $k = n + 1$. Also, due to (3.57) and (4.63), we then have for all $k \leq n$,

$$\begin{cases} \xi_{4k+3} = -1, \\ \xi_{4k+4} = -1, \\ \xi_{4k+5} = 1, \\ \xi_{4k+6} = 1. \end{cases} \tag{4.70}$$

Hence, we have

$$\Xi_{4n+6}(\alpha) = \sum_{i=0}^{4n+6} \xi_i \alpha^i = (1 - \alpha - \alpha^2) - \sum_{m=0}^{n} (\alpha^3 + \alpha^4 - \alpha^5 - \alpha^6)\alpha^{4m}. \tag{4.71}$$

By applying (4.66), we have

$$\begin{aligned} \Xi_{4n+6}(\alpha) &= \sum_{i=0}^{4n+6} \xi_i \alpha^i = (1 - \alpha - \alpha^2) - \sum_{m=0}^{n} (\alpha^3 + \alpha^4 - \alpha^5 - \alpha^6)\alpha^{4m} \\ &= -\sum_{m=0}^{n} \alpha^{4m+3} = -\alpha^3 \frac{1 - \alpha^{4n+4}}{1 - \alpha^4} > 0. \end{aligned} \tag{4.72}$$

Therefore, we have $\xi_{4n+7} = -1$. Hence, by applying (3.57), we have

$$\Xi_{4n+7}(\alpha) = \Xi_{4n+6}(\alpha) - \alpha^{4n+7} \tag{4.73}$$

as $\Xi_{4n+6}(\alpha) > 0$ and $\alpha^{4n+7} < 0$, then we have $\Xi_{4n+7}(\alpha) > 0$. Hence, we have $\xi_{4n+8} = -1$, and then

$$\begin{aligned}
\Xi_{4n+8}(\alpha) &= -\alpha^3 \frac{1-\alpha^{4n+4}}{1-\alpha^4} - \alpha^{4n+7} - \alpha^{4n+8} \\
&= \frac{\alpha^3 + \alpha^{4n+8} - \alpha^{4n+11} - \alpha^{4n+12}}{\alpha^4 - 1} \\
&= \frac{\alpha^3 + \alpha^{4n+8} - \alpha^{4n+10} + \alpha^{4n+10} - \alpha^{4n+11} - \alpha^{4n+12}}{\alpha^4 - 1} \\
&= \frac{\alpha^3 + \alpha^{4n+8} - \alpha^{4n+10}}{\alpha^4 - 1} = \frac{\alpha^3 + \alpha^{4n+8} + \alpha^{4n+9} - \alpha^{4n+9} - \alpha^{4n+10}}{\alpha^4 - 1} \\
&= \frac{\alpha^3 + \alpha^{4n+9}}{\alpha^4 - 1} < 0.
\end{aligned} \tag{4.74}$$

Then we can have that $\xi_{4n+9} = 1$, then we have

$$\Xi_{4n+9}(\alpha) = \Xi_{4n+8}(\alpha) + \alpha^{4n+9} < 0. \tag{4.75}$$

Then we can notice that

$$\begin{cases}
\Xi_{4n+6}(\alpha) = \Xi_{4(n+1)+2}(\alpha) \geq 0, \\
\Xi_{4n+7}(\alpha) = \Xi_{4(n+1)+3}(\alpha) \geq 0, \\
\Xi_{4n+8}(\alpha) = \Xi_{4(n+1)+4}(\alpha) < 0, \\
\Xi_{4n+9}(\alpha) = \Xi_{4(n+1)+5}(\alpha) < 0,
\end{cases} \tag{4.76}$$

Hence, we finish prove the case when $k = n + 1$. Hence we finish proving the statement.

Then by applying the Lemma 3.1.9, we have

$$\begin{aligned} f_\sharp(\nu) &= \sum_{i=1}^{\infty} 1_{\{\xi_i=-1\}}(\frac{1}{2})^i = \frac{1}{4} + \frac{1}{8} + \sum_{i=0}^{\infty}\{\frac{1}{2^{4i+4}} + \frac{1}{2^{4i+5}}\} \\ &= \frac{3}{8} + \sum_{i=0}^{\infty}\{\frac{1}{2^{4i+4}} + \frac{1}{2^{4i+5}}\} = \frac{3}{8} + (\frac{1}{16} + \frac{1}{32})\frac{1}{1-2^{-4}} = \frac{19}{40}, \end{aligned} \tag{4.77}$$

Since $\Xi_n(\alpha) = 0$ if and only if $n = 2$, then by applying Lemma 3.1.45, there will only 2 maximum points for x_ν. Hence 0.4 and 0.475 are the unique maximum points for x_ν. □

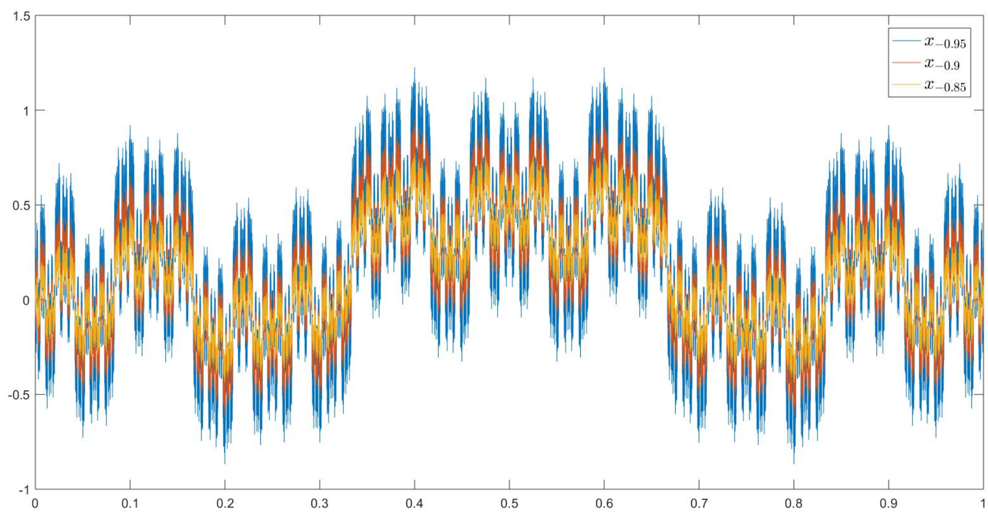

Figure 4.4: Plot for exponential Takagi function $x_{-0.85}$, $x_{-0.9}$ and $x_{-0.95}$.

Theorem 4.3.4. For $\nu \in (-1, -\frac{1+\sqrt{5}}{4})$, the exponential Takagi function x_ν has the only maximum point in $[0, \frac{1}{2}$ at $t = 0.4$.

Proof. Let $\alpha = 2\nu < -\frac{1+\sqrt{5}}{2}$. In order to prove this theorem, first of all, we will prove that for any fixed $n \geq 0$, we have

$$\begin{cases} \Xi_{4n+2}(\alpha) < 0, \\ \Xi_{4n+3}(\alpha) < 0, \\ \Xi_{4n+4}(\alpha) > 0, \\ \Xi_{4n+5}(\alpha) > 0, \end{cases} \tag{4.78}$$

by induction on n. Let us first consider the case for $\Xi_0(\alpha)$, we have

$$\Xi_0(\alpha) = 1. \tag{4.79}$$

Then according to the step condition, we have $\xi_1 = -1$, and hence

$$\Xi_1(\alpha) = 1 - \alpha > 0. \tag{4.80}$$

Then by applying the step condition again, we have

$$\Xi_2(\alpha) = 1 - \alpha - \alpha^2 =: \epsilon_\alpha. \tag{4.81}$$

As $\alpha \in (-2, -\frac{\sqrt{5}+1}{2})$, we have $\epsilon_\alpha \in (-1, 0)$. Then we have $\Xi_2(\alpha) < 0$, and $\xi_3 = 1$, and we have

$$\Xi_3(\alpha) = \epsilon_\alpha + \alpha^3 < 0, \tag{4.82}$$

then we have $\xi_4 = 1$. Next,

$$\Xi_4(\alpha) = \epsilon_\alpha + \alpha^3 + \alpha^4. \tag{4.83}$$

As $\alpha^4 - \alpha^3 > 1$, we have $\Xi_4(\alpha) > 0$, and hence $\xi_5 = -1$. Then we have

$$\Xi_5(\alpha) = \Xi_4(\alpha) - \alpha^5 > 0. \tag{4.84}$$

Now, we have proved the case when $k = 1$. Hence, by (4.81) - (4.84), we have proved that the induction hypothesis holds for $n = 0$. Now let us assume that the induction hypothesis holds for $k \leq n$. Then we proceed to prove the case for $k = n + 1$. Also, due to the step condition for maxima and (4.48), we have for all $k \leq n$,

$$\begin{cases} \xi_{4k+3} = 1, \\ \xi_{4k+4} = 1, \\ \xi_{4k+5} = -1, \\ \xi_{4k+6} = -1. \end{cases} \tag{4.85}$$

By applying (4.85), we have

$$\begin{aligned} \Xi_{4n+6}(\alpha) &= \sum_{i=0}^{4n+6} \xi_i \alpha^i = (1 - \alpha - \alpha^2) + \sum_{m=0}^{n} (\alpha^3 + \alpha^4 - \alpha^5 - \alpha^6)\alpha^{4m} \\ &= \sum_{i=0}^{4n+6} \xi_{i+1} \alpha^i = \epsilon_\alpha + \sum_{m=0}^{n} (\alpha^3 + \alpha^4 \epsilon_\alpha)\alpha^{4m} = \sum_{m=0}^{n} \alpha^{4m+3} + \epsilon_\alpha \sum_{m=0}^{n+1} \alpha^{4m} \\ &= \alpha^3 \frac{1 - \alpha^{4n+4}}{1 - \alpha^4} + \epsilon_\alpha \frac{1 - \alpha^{4n+4}}{1 - \alpha^4} = (\alpha^3 + \epsilon_\alpha) \frac{1 - \alpha^{4n+4}}{1 - \alpha^4} < 0. \end{aligned} \tag{4.86}$$

Then we have $\xi_{n+7} = 1$, and

$$\Xi_{4n+7}(\alpha) = \Xi_{4n+6}(\alpha) + \alpha^{4n+7} < 0. \tag{4.87}$$

Later, we have

$$\begin{aligned}
\Xi_{4n+8}(\alpha) &= \Xi_{4n+6}(\alpha) + \alpha^{4n+7} + \alpha^{4n+8} \\
&= (\alpha^3 + \epsilon_\alpha)\frac{1-\alpha^{4n+4}}{1-\alpha^4} + \alpha^{4n+7} + \alpha^{4n+8} \\
&= \frac{\alpha^3 - \alpha^{4n+7} + \epsilon_\alpha - \epsilon_\alpha\alpha^{4n+4} + \alpha^{4n+7} - \alpha^{4n+11} + \alpha^{4n+8} - \alpha^{4n+12}}{1-\alpha^4} \\
& \frac{\alpha^3 + \epsilon_\alpha - \epsilon_\alpha\alpha^{4n+4} + \alpha^{4n+8} + \alpha^{4n+10} - \alpha^{4n+10} - \alpha^{4n+11} - \alpha^{4n+12}}{1-\alpha^4} \\
&= \frac{\alpha^3 + \alpha^{4n+8} + \epsilon_\alpha - \alpha^{4n+4}\epsilon_\alpha - \alpha^{4n+10} + \alpha^{4n+10}\epsilon_\alpha}{1-\alpha^4} \\
&= \frac{\alpha^3 + \alpha^{4n+8} + \epsilon_\alpha - \alpha^{4}\epsilon_\alpha + \alpha^{4n+9} - \alpha^{4n+9} - \alpha^{4n+10} + \alpha^{4n+10}\epsilon_\alpha}{1-\alpha^4} \\
&= \frac{\alpha^3 + \alpha^{4n+9} + \epsilon_\alpha - \alpha^{4n+4}\epsilon_\alpha + \alpha^{4n+10}\epsilon_\alpha + \alpha^{4n+8}\epsilon_\alpha}{1-\alpha^4} \\
&= \frac{\alpha^3 + \alpha^{4n+9} + \epsilon_\alpha(1 - \alpha^{4n+4} + \alpha^{4n+10} + \alpha^{4n+8})}{1-\alpha^4}.
\end{aligned} \tag{4.88}$$

And this is strictly positive as $|\alpha| > 1$. Hence, we have $\xi_{n+9} = -1$, and

$$\Xi_{4n+9}(\alpha) = \Xi_{4n+9}(\alpha) - \alpha^{4n+9} > 0. \tag{4.89}$$

Therefore, we have completed the inductive proof. And by applying (4.62), we have $f_\sharp(\nu) = 0.4$, and as for all $n > 0$, we have $\Xi_n(\alpha) \neq 0$, then by applying Corollary 4.1.13, we have

$$f_\sharp(\nu) = f_\flat(\nu) = 0.4. \tag{4.90}$$

is the unique maximum for x_ν. □

Theorem 4.3.5. For $\nu \in (-\frac{1}{2}, 1)$, the unique minimum of x_ν in $[0, \frac{1}{2}]$ is at $t = 0$, i.e.

$$\tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}] = \{0\}.$$

Proof. Let $\alpha = 2\nu$, and take $\Xi = j_\sharp(\nu)$. First of all, let us prove that $\xi_i = 1$ for all $i \in \mathbb{N}$

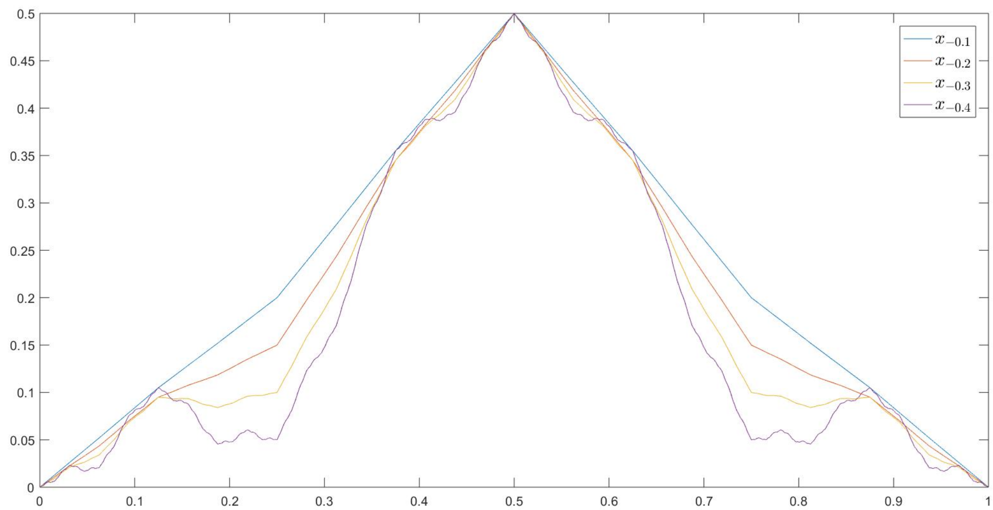

Figure 4.5: Plot for exponential Takagi function $x_{-0.1}$, $x_{-0.2}$, $x_{-0.3}$ and $x_{-0.4}$.

by induction on i. Since $\Xi_0(\alpha) = \xi_0 = 1$, we have proved the case for $i = 0$. Now let us assume that $\xi_i = 1$, for all $i \leq n$, and we aim to prove that $\xi_{n+1} = 1$. Since $\xi_i = 1$, for all $i \leq n$, then

$$\Xi_n(\alpha) = \sum_{i=0}^{n} \xi_i \alpha^i = \sum_{i=0}^{n} \alpha^i = \frac{1-\alpha^{n+1}}{1-\alpha} > 0.$$

Therefore, by applying (3.79), we have $\xi_{n+1} = 1$. Then, we have

$$g_\sharp(\nu) = \sum_{i=0}^{\infty} 1_{\{\xi=-1\}} 2^{i+1} = 0.$$

Furthermore, since $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$, we have $g_\sharp(\nu) = g_\flat(\nu) = 0$. Then 0 is the only minimizer in $[0, \frac{1}{2}]$. □

Theorem 4.3.6. For $\nu = -\frac{1}{2}$, $\tilde{\mathcal{M}}_\nu$ will be inform of a Cantor-like set with Hausdorff Dimension with $\frac{1}{2}$ with $x_\nu(\tilde{\mathcal{M}}_\nu) = \{0\}$. Furthermore, we have

$$\inf(\tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}]) = 0 \quad \text{and} \ \sup(\tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}]) = 0.25.$$

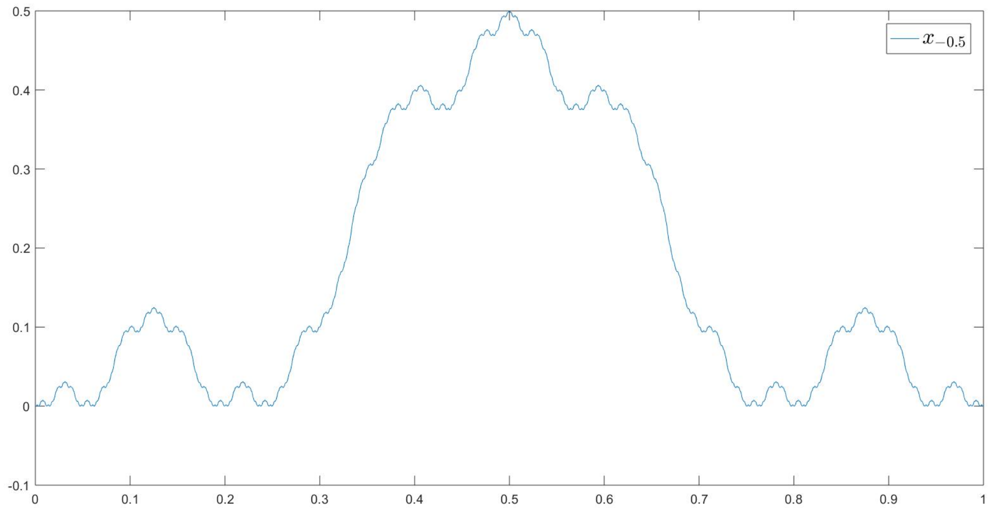

Figure 4.6: Plot for exponential Takagi function $x_{-0.5}$.

Proof. First of all, let us take $\alpha = 2\nu$, and $\Xi = j_\flat(\nu)$. Then we will prove that $\xi_i = 1$ for all $i \in \mathbb{N}$. Naturally, we have $\xi_0 = 1$, and so, we have proved the base case. Then let us assume that $\xi_i = 1$ for all $i \leq n$. Now let us prove $\xi_{n+1} = 1$. If $n \in 2\mathbb{Z}$, then

$$\Xi_n(\alpha) = \sum_{i=0}^{n} (-1)^{-i} = 1.$$

Then by the step condition for minima (3.79), we have $\xi_{n+1} = 1$. If $n \notin 2\mathbb{Z}$, then

$$\Xi_n(\alpha) = \sum_{i=0}^{n} (-1)^{-i} = 0.$$

Then by applying the step condition for minima (3.79), we have $\xi_{n+1} = 1$. Hence we have $g_\flat(\nu) = 0$, and $x_\nu(0) = 0$. Now let us take $\Xi = j_\sharp(\nu)$, then let us proceed to prove that $\xi_i = -1$ for all $i \geq 2$. First of all, we have $\Xi_0(\alpha) = 1$ and $\Xi_1(\alpha) = 1 + (-1) = 0$. Then by applying (3.78), we have $\xi_2 = -1$. Hence, we finish proving the base case for $i = 2$. Now let us assume that $\xi_i = -1$ for all $i \le n$, and we will prove that $\xi_{n+1} = -1$. Now we have

$$\Xi_n(\alpha) = \Xi_1(\alpha) - \sum_{i=2}^{n} \alpha^i = -\sum_{i=2}^{n} \alpha^i = \begin{cases} -1 & n \in 2\mathbb{Z}, \\ 0 & n \notin 2\mathbb{Z}. \end{cases}$$

Therefore, by applying (3.78), we have $\xi_{n+1} = -1$. Hence, we have

$$g_\sharp(\nu) = \sum_{i=0}^{\infty} 1_{\{\xi=-1\}} 2^{i+1} = \sum_{i=2}^{\infty} 2^{i+1} = \frac{1}{8} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{4}.$$

Take $\mathcal{N} = \{n \in \mathbb{N} | \Xi_n(\alpha) = 0\}$, then we have $\mathcal{N} = 2\mathbb{Z} - 1$. Then by applying Theorem 4.1.16, we have

$$\dim_H \tilde{\mathcal{M}}_\nu = \frac{1}{2}.$$

□

Theorem 4.3.7. For $\nu \in (-1, -\frac{1}{2})$, we have

$$\tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}] = \{0.2\}.$$

Proof. Take $\alpha = 2\nu$, and $\Xi = j_\sharp(\nu)$. Let us prove that for all $n \in \mathbb{N}$, we have

$$\begin{cases} \xi_{4n} = 1, \\ \xi_{4n+1} = 1, \\ \xi_{4n+2} = -1, \\ \xi_{4n+3} = -1, \end{cases} \tag{4.91}$$

by induction on n. Now let us first of all prove the case $n = 0$. Since $\Xi_0(\alpha) = \xi_0 = 1$, then we have $\Xi_1(\alpha) = 1 + \alpha < 0$. Then by the step condition for minima, we have $\xi_2 = -1$. Hence, we have

$$\Xi_2(\alpha) = \Xi_1(\alpha) - \alpha^2 < 0.$$

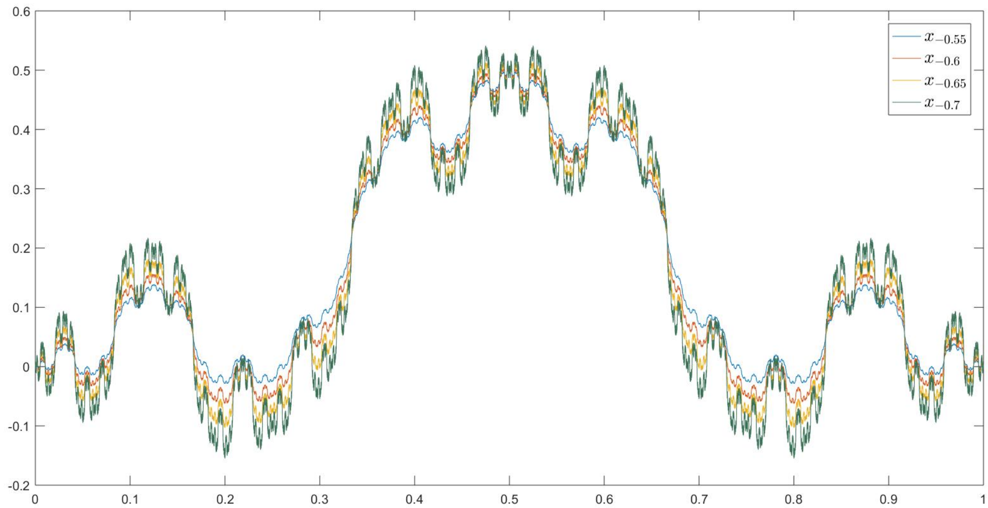

Figure 4.7: Plot for exponential Takagi function $x_{-0.55}$, $x_{-0.6}$, $x_{-0.65}$ and $x_{-0.7}$.

Hence, we have $\xi_3 = -1$. Therefore, we have proved that the inductive statement holds for $n = 0$. Now let us assume that (4.91) holds for all $n \leq k-1$, then we try to prove that (4.91) holds for $n = k$. First of all, we have

$$\Xi_{4k-1}(\alpha) = \sum_{i=0}^{4k-1} \xi_i \alpha^i = \sum_{i=0}^{k-1} (1+\alpha-\alpha^2-\alpha^3)\alpha^{4i} = (1+\alpha-\alpha^2-\alpha^3)\frac{1-\alpha^{4k}}{1-\alpha^4}.$$

Since $1-\alpha^{4k} < 0$, $1-\alpha^4 < 0$ and $1+\alpha-\alpha^2-\alpha^3 > 0$, we have $\Xi_{4k-1}(\alpha) > 0$. Therefore, we have $\xi_{4k} = 1$ by step condition for minima. Hence, we have

$$\Xi_{4k}(\alpha) = \Xi_{4k-1}(\alpha) + \xi_{4k}\alpha^{4k} = \Xi_{4k-1}(\alpha) + \alpha^{4k}.$$

Since $\Xi_{4k-1}(\alpha) > 0$ and $\alpha^{4k} > 0$, we have $\Xi_{4k}(\alpha) > 0$. Therefore, we have $\xi_{4k+1} = 1$, then we have

$$\Xi_{4k+1}(\alpha) = \sum_{i=0}^{4k+1} \xi_i \alpha^i = 1-\alpha-\sum_{i=0}^{k-1}(1+\alpha-\alpha^2-\alpha^3)\alpha^{4i+2} = (1-\alpha)-\alpha^2(1+\alpha-\alpha^2-\alpha^3)\frac{1-\alpha^{4k}}{1-\alpha^4}.$$

Since $1-\alpha < 0$, $\alpha^2(1+\alpha-\alpha^2-\alpha^3) > 0$, $1-\alpha^{4k} < 0$ and $1-\alpha^4 < 0$, we have $\Xi_{4k+1}(\alpha) < 0$ and $\xi_{4k+2} = -1$. Furthermore, we have

$$\Xi_{4k+2}(\alpha) = \Xi_{4k+1}(\alpha) + \xi_{4k+2}\alpha^{4k+2} = \Xi_{4k+1}(\alpha) - \alpha^{4k+2} < 0.$$

Hence, we have $\xi_{4k+3} < 0$. Hence, we have proved the case for $n = k$, and we proved the induction. Moreover, we have

$$g_\sharp(\nu) = \sum_{i=0}^{\infty} 1_{\{\xi_i=-1\}} 2^{-(i+1)} = \sum_{i=0}^{\infty} (\frac{1}{8} + \frac{1}{16}) 2^{-4i} = \frac{3}{16}\frac{1}{1-2^{-4}} = 0.2.$$

Since $\Xi_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$, we have $g_\sharp(\nu) = g_\flat(\nu)$. Hence,

$$\tilde{\mathcal{M}}_\nu \cap [0, \frac{1}{2}] = \{0.2\}.$$

$\square$

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