Estimation for Linear and Semi-linear Infinite-dimensional Systems

by

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Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Estimating the state of a system that is not fully known or that is exposed to noise has been an intensely studied problem in recent mathematical history. Such systems are often modelled by either ordinary differential equations, which evolve in finite-dimensional state-spaces, or partial differential equations, the state-space of which is infinite-dimensional.
The Kalman filter is a minimal mean squared error estimator for linear finite-dimensional and linear infinite-dimensional systems disturbed by Wiener processes, which are stochastic processes representing the noise. For nonlinear finite-dimensional systems the extended Kalman filter is a widely used extension thereof which relies on linearization of the system. In all cases the Kalman filter consists of a differential or integral equation coupled with a Riccati equation, which is an equation that determines the optimal estimator gain.

This thesis proposes an estimator for semi-linear infinite-dimensional systems. It is shown that under some conditions such a system can also be coupled with a Riccati equation.
To motivate this result, the Kalman filter for finite-dimensional and infinite-dimensional systems is reviewed, as well as the corresponding theory for both stochastic processes and infinite-dimensional systems. Important results concerning the infinite-dimensional Riccati equation are outlined and existence of solutions for a class of semi-linear infinite-dimensional systems is established. Finally the well-posedness of the coupling between a semi-linear infinite-dimensional system with a Riccati equation is proven using a fixed point argument.

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Chapter 1

Introduction

Mathematics has always been interested in describing phenomena that are subject to change, be it the trajectory of celestial bodies, the propagation of waves, or, in recent history, the dynamics of financial markets. The motivation to study these dynamic systems comes from various disciplines and in mathematics they are addressed rigorously. In the past centuries a powerful tool has been developed to efficiently describe dynamic behaviour and has become a rich mathematical field: differential equations (DEs). In general there are two distinct types of DEs.

The first type involves systems that depend solely on one independent variable, for instance time, and are described by *ordinary differential equations* (ODEs). Geometrically, they evolve in finite-dimensional *state-spaces*, which is why they are equivalently referred to as *finite-dimensional systems*. Examples include a swinging pendulum, whose motion, once initiated, can be represented by a function of time only; or the growth of a population, which in the simplest of cases, follows a rate of change that only depends on the current population size.

The second type of systems exhibits more involved dynamics and is governed by *partial differential equations* (PDEs) that allow for simultaneous temporal and spatial dependencies. Mathematically, PDEs evolve in infinite-dimensional state spaces and are hence a special case of *infinite-dimensional systems*. The reader unfamiliar with such systems may imagine water waves in a basin, which, when looked at more closely, seem to consist of smaller waves themselves.

Although DEs have proven to be a very efficient tool to model various phenomena, it is intuitively clear that the exact behaviour of a real-life system cannot perfectly be known, and that the behaviour cannot perfectly be observed or measured. The mathematical field of *estimation* attempts to compensate for this by developing methods to accurately estimate a system's *state* by making use of the observations available.

The function $x(t)$ to be estimated is the solution of a *state-space equation*. It is most often posed as a DE or sometimes as an integral equation. At a given time t, the vector $x(t)$ is referred to as the system's state. An observation function $y(t)$, often a lower-dimensional linear function of the original system's state vector $x(t)$, yields the *observation* $Y_t = \{y(s) \mid 0 \leq s \leq t\}$ that can be obtained from the system up to time t. The aim, given the observation Y_t, is to maximize the information of the state of the system at time t.

A second system, denoted by $\hat{x}(t)$ and referred to as *estimator* is introduced with the purpose of approximating the system's state $x(t)$. At a fixed time t the state of the estimator is called an *estimate*.

There are two main domains of application for estimators: deterministic systems and stochastic systems. The latter contain a *noise* component modelled by a *stochastic process* that changes in time, but spatially exhibits a Gaussian distribution for each individual time t. Estimation for stochastic systems of a state $x(t)$ given Y_t is called *filtering* and estimators in that context are referred to as *filters*.

In deterministic systems, under some conditions an estimator can be introduced such that the error $\|x(t) - \hat{x}(t)\|$ converges to zero over time. In this case the estimator is called *observer*.

In the presence of noise, zero asymptotic error is generally not possible. Instead, the filtering objective often becomes to minimize the mean-squared error between estimator and original state.

A frequently used method in both the deterministic and stochastic cases is to define the estimator $\hat{x}(t)$ via an equation that is similar to the state-space equation and to inject the error $(x(t) - \hat{x}(t))$ as input into the estimator equation.

For linear stochastic ODEs and linear stochastic PDEs, the Kalman filter has been a widely used method for state estimation as it provides minimal mean-squared error estimates. Due to its success it has been extended to nonlinear ODEs via linearization. This is known as the extended Kalman filter (EKF).

For continuous-time systems, that is, systems where the time variable changes continuously on the real line, the KF and the EKF lead to an estimator equation coupled with a Riccati equation. The latter can be expressed in a differential or integral form and determines the optimal gain for the error injection.

Based on the EKF for finite-dimensional systems, this thesis proposes an estimator for semi-linear infinite-dimensional systems. It is proven that under some conditions, the coupling of a semi-linear PDE with the same Riccati equation that arises in optimal estimation for linear PDEs is well-posed.

To serve as motivation and provide the necessary background other estimation techniques are reviewed first, with particular emphasis on the Kalman filter for linear stochastic PDEs

The organization of this thesis is as follows:

Chapter 2 introduces the mathematical theory of stochastic processes. In Section 2.1 integration with respect to stochastic processes is defined and Section 2.2 outlines how solutions to stochastic differential equations can be understood.

Chapter 3 first gives a historical overview over the development of estimation and filtering theory. Consequently, to familiarize the reader with the estimation problem, Section 3.1 presents a widely used class of estimators for linear ODEs. Then, Section 3.2 derives the KF for continuous-time stochastic DEs and states a widely-used algorithm for the discrete-time case. Section 3.3 outlines how this method can be extended to nonlinear systems and Section 3.4 states conditions for the estimator to converge to the original system.

In Chapter 4 the theory for infinite-dimensional systems is introduced, where, more precisely, Section 4.1 covers *semigroup* theory, which is necessary in treating abstract PDEs, and Section 4.2 covers *evolution operators*, which are a generalization of semigroups.

Chapter 5 is an exposition of the filtering problem for linear infinite-dimensional stochastic systems. For that purpose a historical review over past research thereof is given. Section 5.1 then derives the Kalman filter for linear stochastic PDE systems. Section 5.2 recapitulates, in a general sense, the role of the Riccati equations in linear-quadratic control and estimation in infinite dimensions. The results therein will be of use in later chapters.

In Chapter 6 the existence of solutions for a class of semi-linear systems is investigated.

Chapter 7 proposes a novel estimator: the extended Kalman filter for deterministic semi-linear infinite-dimensional systems. Well-posedness of the coupling of the semi-linear estimator equation and the same Riccati equation arising in the linear PDE case is proven.

Finally Chapter 8 summarizes the thesis and proposes future directions of study.

Notation:

Lower-case variables like x, y refer to points in space, whereas $x(t), y(t)$ either denotes trajectories/functions in space dependent on time or stochastic processes, where the spatial dependency is not explicitly shown. In all cases, the context makes it clear what is represented.

Random variables are also denoted by lower-case variables, like f, where the spatial input is omitted as well.

The spaces and subspaces are denoted by blackboard bold letters, like the Hilbert space $\mathbb{H}$. For operator spaces $L(\mathbb{H}, \mathbb{K})$ denotes the space of bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$ or $L(\mathbb{H})$ for endomorphisms and $\mathrm{C}([a, b], \mathbb{S})$ denotes functions or operators that are continuous from $[a, b]$ into some space $\mathbb{S}$.

For operators mapping between normed linear spaces as well as for sets, upper-case letters are used, an example being the semigroup A. The domain is denoted by $\mathcal{D}(A)$. Operators mapping to or between function spaces are denoted by serif letters, an example being the contraction mapping G.
For the expected value and the covariance serif letters are used as well.

The scalar product is denoted by $\langle \cdot, \cdot \rangle$. The norm $\| \cdot \|$ will always be clear from the context. The real and imaginary parts are $\Re\{\cdot\}$ and $\Im\{\cdot\}$ respectively.

Measures are denoted by μ and μ_p stands for a probability measure. Fraktur letters, like $\mathfrak{A}$ are used for σ-algebras.

Given a measure space $(\Omega, \mathfrak{A}, \mu)$, the space of p-integrable random variables is denoted by $L^p(\Omega, \mathfrak{A}, \mu)$. When an emphasis is put on the image space, for example $f : \Omega \longrightarrow \mathbb{R}^n$, this will instead be $L^p(\Omega, \mathfrak{A}; \mathbb{R}^n)$ if the measure on the pre-image is clear and $L^p(\Omega, \mu; \mathbb{R}^n)$ if the σ-algebra is given by the context.

Chapter 2

Stochastic processes

Since part of this work is concerned with stochastic systems, it is first necessary to define these and the tools that are needed to treat them mathematically. More precisely, the linear stochastic differential equation (SDE)

$$\frac{dx(t)}{dt} = A(t)x(t) + D(t)w(t) \tag{2.1}$$

will be of importance, the exact properties of which are defined later.
The reader is assumed to be familiar with the theory of systems of ordinary differential equations.
The reason why (2.1) is referred to as SDE is that it is disturbed by *noise* represented by the term $w(t)$, which will later be introduced as *Wiener process*. Phenomenologically, this term introduces randomness to the system which, for example, should account for unknown system errors as well as disturbances that the system may be exposed to.

In order to see how the derivative in (2.1) can be understood, it is useful to first recall some basic concepts of probability theory to then rigorously define the *stochastic disturbance* $w(t)$ and introduce the required tools of stochastic calculus.

Definition 2.1 (σ-algebra)
[4, Definition 1.2] Let Ω be a set. A σ-algebra $\mathfrak{A}$ is a set of subsets of Ω satisfying

1. *Ω and $\emptyset$ belong to $\mathfrak{A}$*
2. *if $A, B \subset \mathfrak{A}$ then $A \cap B \in \mathfrak{A}$ and $A \setminus B \in \mathfrak{A}$*
3. *for any sequence A_n with $A_n \in \mathfrak{A}, n \in \mathbb{N}$ it holds that $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{A}$.*

For every set Ω, it is easy to define the coarsest and the finest possible σ-algebra. To see this, let

$$\mathfrak{A}_{triv} = \{\emptyset, \Omega\}$$

which is called the *trivial σ-algebra*. Clearly, the trivial σ-algebra is a sub-σ-algebra of any other given σ-algebra, say $\tilde{\mathfrak{A}}$, meaning that it is contained completely in $\tilde{\mathfrak{A}}$.
On the other hand, let $\mathcal{P}(\Omega)$ be the *power set*, that is, the set containing all possible subsets of Ω. Since $\mathcal{P}(\Omega)$ evidently satisfies all of the conditions in Definition 2.1, it is a σ-algebra. Since all other σ-algebras are contained in $\mathcal{P}(\Omega)$ it follows that it is the finest σ-algebra, also referred to as the *discrete σ-algebra*.
Even though these are prominent examples they are often not very useful, unlike the following cases.

Definition 2.2 (Borel σ-algebra)
[4, Definition 1.3] Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space. The smallest σ-algebra containing all open subsets of $\mathbb{X}$ is called the Borel σ-algebra, or the Borel algebra and is denoted by $\mathfrak{B}(\mathbb{X})$.

In general, the smallest σ-algebra that contains a certain set of sets $\{A_i\}_{i=1,\dots,k}$ is referred to as the σ-algebra *generated by* these sets and often written as $\sigma(\{A_i\}_{i=1,\dots,k})$.

For example, taking $\mathbb{X} = \mathbb{R}^n$, then the Borel σ-algebra of $\mathbb{R}^n$ is

$$\mathfrak{B}(\mathbb{R}^n) = \sigma(\{A | A \text{ is an open subset of } \mathbb{R}^n\}).$$

However it is also generated by the following classes of sets:

- $\{(a_1, b_1), \times \cdots \times (a_n, b_n) | -\infty \leq a_i < b_i \leq \infty, i = 1, \dots, n\}$
- $\{(a_1, b_1), \times \cdots \times (a_n, b_n) | a_i, b_i \in \mathbb{Q}, i = 1, \dots, n\}$
- $\{(-\infty, x_1), \times \cdots \times (-\infty, x_n) | x_i \in \mathbb{R}, i = 1, \dots, n\}$
- $\{(-\infty, x_1), \times \cdots \times (-\infty, x_n) | x_i \in \mathbb{Q}, i = 1, \dots, n\}$.

This concept of generated sub-σ-algebras can now be extended to mappings.

Definition 2.3
[5, Definition 2.1.4] Let Ω_1 be a set and $(\Omega_2, \mathfrak{A}_2)$ be a measurable space. Let further $\{f_i\}_{i \in \mathbb{I}}$ be a family of mappings where $f_i : \Omega_1 \longrightarrow \Omega_2$ for all i. Then

$$\sigma(\{f_i\}_{i \in \mathbb{I}}) = \sigma(\{f_i^{-1}(B) | i \in \mathbb{I}, B \in \mathfrak{A}_2\})$$

is called the σ-algebra generated by the f_i.

As a consequence of this Definition 2.3 all the f_i are measurable with respect to $\sigma(\{f_i\}_{i\in\mathbb{I}})$. This concept is illustrated with an example [5, Example 2.1.3]. For that purpose let Ω_1 be an arbitrary set and define $\Omega_2 = \mathbb{R}$, $\mathfrak{A}_2 = \mathfrak{B}(\mathbb{R})$ as well as $f = \mathbb{1}_A$ for some $A \subset \Omega_1$. This yields

$$\sigma(f) = \sigma(A) = \{\emptyset, \Omega, A, A^c\}.$$

Note that when the family of mappings $\{f_i\}_{i\in\mathbb{I}}$ or the family of sets $\{A_i\}_{i\in\mathbb{I}}$ only consist of one element, the generated σ-algebra is simply written as $\sigma(f)$ or $\sigma(A)$ instead of $\sigma(\{f\})$ or $\sigma(\{A\})$.

Definition 2.4 (σ-additivity)
[16, Definition 1.3.2] A real-valued set function μ is called σ-additive or countably additive if

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for all pairwise disjoint sets A_n in $\mathfrak{A}$.

Definition 2.5 (measure)
[4, section 1.4] Let $\mathfrak{A}$ be a σ-algebra on Ω. A function $\mu : \mathfrak{A} \to [0, \infty]$ is called a measure if

1. $\mu(\emptyset) = 0$
2. $\mu(A) \geq 0$ *for all* $A \in \mathfrak{A}$
3. μ *is σ-additive.*

A well-known measure is the *Lebesgue measure* λ which assigns each set its volume in the classical sense, or more precisely

$$\lambda((a_1, b_1) \times \cdots \times (a_n, b_n)) = \prod_1^n |a_i - b_i|. \tag{2.2}$$

It is assumed that the reader is familiar with Lebesgue measures and the corresponding L^p-spaces.

Definition 2.6 (measure space)
[4, section 1.4] Let $\mathfrak{A}$ be a σ-algebra on Ω and μ a measure on $\mathfrak{A}$.
The pair $(\Omega, \mathfrak{A})$ is called a measurable space and the tripel $(\Omega, \mathfrak{A}, \mu)$ is called a measure space.
If in addition it holds that $\mu(\Omega) = 1$ then $(\Omega, \mathfrak{A}, \mu)$ is called a probability space and μ a probability measure.

Henceforth probability measures are denoted by μ_p.

Definition 2.7 (measurability)
[4, section 2.2 & Proposition 2.1] Let $(\Omega, \mathfrak{A})$ *and* $(\Gamma, \mathfrak{F})$ *be two measurable spaces and let* $f : \Omega \to \Gamma$.
The map f *is* $\mathfrak{A}$*-measurable if* $f^{-1}(\mathfrak{F}) \subset \mathfrak{A}$*, or equivalently, if*

$$\forall F \in \mathfrak{F} : f^{-1}(F) \in \mathfrak{A}.$$

If the σ*-algebras are clear from the context,* f *is simply called measurable.*

For the reader familiar with topology it is an immediate consequence that any continuous map $f : (\Omega, \mathfrak{B}(\Omega)) \longrightarrow (\Gamma, \mathfrak{B}(\Gamma))$ is measurable, since all pre-images of open sets are open.

The construction of the integral with respect to a measure is now concisely explained. This will be useful when introducing *stochastic integrals* later in this chapter. More detail can be found in [4], [5] or[16], or any other book on measure theory and integration.

Similar to the definition of the Riemann integral via step-functions, one can start constructing the integral for measurable functions by introducing *simple functions* $f : \Omega \longrightarrow \mathbb{R}$, which can be written as

$$f(x) = \sum_{i=1,\dots,k} c_i \mathbb{1}_{A_i}(x) \tag{2.3}$$

for some $\{A_1, \dots, A_k\}$, with $A_i \in \mathfrak{A}$ and non-negative constants $\{c_1, \dots, c_k\}$. The indicator function is defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Initially these A_i can overlap, but it is possible to represent f in (2.3) as a step-function over disjoint sets. Thus, without loss of generality it can be assumed that the A_i do not intersect. It is then intuitive to define the integral with respect to a measure μ for simple functions as

$$\int_{\cup_i A_i} f d\mu = \sum_{i=1}^{k} c_i \mu(A_i).$$

This can be extended to non-negative measurable functions f by defining

$$\int_{A_i} f d\mu = \sup \Big\{ \int_{A_i} g d\mu : g \text{ is simple and } g(\omega) \leq f(\omega) \text{ for all } \omega \in A_i \Big\}. \tag{2.4}$$

The reason why the definition (2.4) only works for non-negative functions is that the supremum is taken. However, a general measurable function f can be split into a positive part $f_+(x) = \max\{f(x), 0\}$ and a negative part $f^-(x) = \max\{-f(x), 0\}$ such that $f(x) = f^+(x) - f^-(x)$. Since both f^+ and f^- are non-negative, the integral of f with respect to μ can be defined as

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

In the following let $(\Omega, \mathfrak{A}, \mu)$ and $(\Omega, \mathfrak{A}, \mu_p)$ be a measure space and a probability space respectively and let $\mathbb{H}, \mathbb{K}$ be real separable Hilbert spaces and let $\mathbb{X}$ be a Banach space.

Definition 2.8 (random variable)
[29, Definition 1.3] A map $f : \Omega \longrightarrow \mathbb{H}$ is called a random variable if it is measurable with respect to μ.

Since this thesis treats both finite-dimensional and infinite-dimensional stochastic systems, the following well known definitions are stated for the general case of infinite-dimensional spaces.

Definition 2.9 (expectation and covariance)
[29, Definition 1.3] Let $f : \Omega \longrightarrow \mathbb{H}$. If $f \in L^1(\Omega, \mu; \mathbb{H})$ the expectation (or the mean) is defined as

$$\mathsf{E}\{f\} = \int_\Omega f d\mu.$$

If $f \in L^2(\Omega, \mu; \mathbb{H})$ the covariance operator is defined as

$$\mathsf{Cov}\{f\} = \mathsf{E}\{(f - \mathsf{E}\{f\}) \circ (f - \mathsf{E}\{f\})\}$$

where the operation

$$\begin{aligned} \circ : \mathbb{H} \times \mathbb{H} &\longrightarrow L(\mathbb{H}) \\ (u, v) &\mapsto u\langle v, \cdot\rangle. \end{aligned}$$

That means for all $h \in \mathbb{H}$ it holds that $(u \circ v)(h) = u\langle v, h\rangle \in \mathbb{H}$.
Note that in finite dimensions with $u, v \in \mathbb{R}^n$ this reduces to $u \circ v = uv^T \in \mathbb{R}^{n\times n}$ and thus if $f : \Omega \longrightarrow \mathbb{R}^n$ the covariance is

$$\mathsf{Cov}\{f\} = \mathsf{E}\{(f - \mathsf{E}\{f\})(f - \mathsf{E}\{f\})^T\}.$$

An important question related to random variables is how likely they will take certain values. More precisely, given a set A in the image it is often desirable to know the probability of the function f mapping to that set, i.e. $\mu_p(\{x \in \Omega | f(x) \in A\})$, which is often written as $\mu_p(\{f \in A\})$. In some cases there is an analytic expression for this probability and one of the most prominent examples are Gaussian random variables.

Definition 2.10 (Gaussian random variable (in finite dimensions))
A random variable $f : \Omega \longrightarrow \mathbb{R}$ is Gaussian if for real σ, m

$$\mu_p(\{y | f(y) \leq x\}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-(s-m)^2}{2\sigma^2}\right) ds. \tag{2.5}$$

The constants σ^2 and m are the variance and the mean respectively.

By letting a vector be a Gaussian random variable in each component, this concept can be generalized to infinite-dimensional spaces.

Definition 2.11 (Gaussian random variable (in infinite dimensions))
[29, Definition 1.7] Let $f \in L^2(\Omega, \mu_p; \mathbb{H})$ be a random variable and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{H}$. Then f is Gaussian if $\langle f, e_i \rangle$ is a real Gaussian random variable for all $i \in \mathbb{N}$.

Definition 2.12 (independence)
[32, Definition 5.8] Let $u, v : \mathfrak{A} \longrightarrow \mathbb{X}$ be random variables.
If for all Borel sets $A, B \in \mathfrak{B}(\mathbb{X})$ the sets $\omega^{-1}(A) = \{\omega : u(\omega) \in A\}$ and $\omega^{-1}(B) = \{\omega : v(\omega) \in B\}$ are independent sets in $\mathfrak{A}$, that is

$$\mu_p(\omega^{-1}(A) \cap \omega^{-1}(B)) = \mu_p(\omega^{-1}(A))\mu_p(\omega^{-1}(B)), \tag{2.6}$$

then u and v are called independent.

Definition 2.13 (stochastic process)
[29, Definition 1.4] Let Ω be a set and $t_f, t_0 \geq 0$. A stochastic process is a map $w : [t_0, t_f] \times \Omega \longrightarrow \mathbb{H}$ that is measurable on $[t_0, t_f] \times \Omega$ where the Lebesgue measure is used on $[t_0, t_f]$.

It is very common to omit the spatial dependence of a stochastic process to allow for better readability. The same practice is used in this thesis. That means, for a stochastic process $w(t)$ will henceforth be used instead of $w(t, \omega)$.

2.1 Stochastic integration

Having reviewed the necessary concepts of stochastic analysis the stochastic disturbance in (2.1) can be defined rigorously. Integration and differentiation in the context of noise can also be understood.

Definition 2.14
A certain statement is said to be true almost everywhere, abbreviated a.e., if the set of points where the statement is false has measure zero. Similarly, if the measure is a probability measure the corresponding expression is with probability 1, abbreviated w.p.1.

Definition 2.15 (continuous sample paths)
[32, Definition 5.15] Let $(\Omega, \mathfrak{A}, \mu_p)$ be a probability space and let $w : [t_0, t_f] \times \Omega \longrightarrow \mathbb{X}$ be a stochastic process. Then $w(t)$ has continuous sample paths if

$$\lim_{\delta \to 0} \mu_p\big(\{ \sup_{t_0 \le t \le t_f} \|w(t+\delta) - w(t)\| > 0\}\big) = 0. \tag{2.7}$$

Equivalently, it is said that $w(t)$ is continuous w.p.1.

The mapping $t \mapsto w(t, \omega)$ with ω fixed is commonly referred to as *sample path*. The above definition means that almost all such sample paths are continuous.

Definition 2.16 (nuclear operator)
[49, Definitions A.1 & A.2] Let $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ denote the scalar products in the separable Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$ respectively. An operator $T \in L(\mathbb{K}, \mathbb{H})$ is nuclear if

$$Tx = \sum_{\mathbb{N}} a_i \langle b_i, x \rangle_{\mathbb{K}} \quad x \in \mathbb{K}$$

where $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{H}$ and $\{b_i\}_{i \in \mathbb{N}} \subset \mathbb{K}$ are such that

$$\sum_{\mathbb{N}} \|a_i\|_{\mathbb{H}} \|b_i\|_{\mathbb{K}} < \infty.$$

Definition 2.17 (trace)
[49, Definitions A.1 & A.2] For an operator $T : \mathbb{H} \longrightarrow \mathbb{H}$ the trace is defined as

$$\operatorname{tr} T = \sum_{\mathbb{N}} \langle T e_i, e_i \rangle$$

where $\{e_i\}_{i \in \mathbb{N}} \subset \mathbb{H}$ is an orthonormal basis.

It is an immediate consequence of Definition 2.16 that the trace of nuclear operators is finite.

Now Wiener processes can be defined.

Definition 2.18 (Wiener process)
[32, Definition 5.22] A Wiener process is a stochastic process $w : [0, t_f] \times \Omega \longrightarrow \mathbb{H}$ *if it satisfies*

1. $w(0) = 0$,
2. $w(t)$ *has continuous sample paths on* $[0, t_f]$,
3. $w(t) - w(s)$ *is a* $\mathbb{H}$*-valued Gaussian random variable with zero mean for all* $s, t \in [0, t_f]$,
4. $\mathsf{Cov}\{w(t) - w(s)\} = (t - s)W$ *where* $W \in \mathcal{L}(\mathbb{H})$ *is positive and nuclear,*
5. $w(s_4) - w(s_3)$ *and* $w(s_2) - w(s_1)$ *are independent whenever* $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq t_f$.

The Wiener process was first introduced to describe the trajectories of particles subject to seemingly random interactions with other particles. This phenomenon was first discovered by the Scottish botanist Robert Brown in 1827 observing small particles in liquids and gases.

Lemma 2.19
[32, Lemma 5.23] Let $w(t)$ *be a* $\mathbb{H}$*-valued Wiener process. There exists a complete orthonormal basis* $\{e_i\}_{i \in \mathbb{N}}$ *such that*

$$w(t) = \sum_{i \in \mathbb{N}} \beta_i(t) e_i \quad w.p.1 \tag{2.8}$$

where the $\beta_i(t)$ *are mutually independent one-dimensional real Wiener processes with covariances* λ_i *such that*

$$\sum_{i \in \mathbb{N}} \lambda_i < \infty.$$

The stochastic integral has been explored in a variety of settings and several generalizations of the integral to (infinite-dimensional) stochastic processes have been found. The following construction, although not as general as other definitions, is sufficient for the processes treated in this thesis. For more detail on this and the development of integration in filtering for infinite-dimensional processes see [28].

The construction of integrals with respect to Wiener processes is analogous to the derivation of the integral with respect to a measure. The complete construction can be found in more detail in [32, chapter 5]: The main steps are as follows.

Definition 2.20
[32, Lemma 5.26] Let $f : [t_0, t_f] \longrightarrow \mathbb{H}$ be a step function of the form

$$f(x) = \sum_{i=0}^{n-1} f_i \mathbb{1}_{[s_i, s_{i+1})}(x), \qquad t_0 = s_0 < s_1 < \cdots < s_n = t_f$$

and let $\beta(t) : [t_0, t_f] \times \Omega \longrightarrow \mathbb{H}$ be a one-dimensional Wiener process. The integral of f with respect to $\beta(t)$ is defined as

$$\int_{t_0}^{t_f} f(s) d\beta(s) = \sum_{i=0}^{n-1} f_i \cdot (\beta(s_{i+1}) - \beta(s_i)).$$

Lemma 2.21
[32, Lemma 5.26] Let the assumptions of Definition 2.20 hold and let λ be the covariance of $\beta(t)$. Then

1. $\mathsf{E}\{\int_{t_0}^{t_f} f(s) d\beta(s)\} = 0$
2. $\mathsf{E}\{\langle \int_{t_0}^{t_f} f(s) d\beta(s), \int_{t_0}^{t_f} f(s) d\beta(s) \rangle\} = \lambda \int_{t_0}^{t_f} \langle f(s), f(s) \rangle ds.$

Since the step functions are dense in $L^2([t_0, t_f], \mathbb{H})$ it is possible to define the integral for square-integrable functions as a limit.

Definition 2.22
[34, p. 137] Let $f \in L^2([t_0, t_f], \mathbb{H})$ and let $\{f_n\}_{n \in \mathbb{N}} \subset L^2([t_0, t_f], \mathbb{H})$ be a sequence of step-functions such that f_n converges to f in $L^2([t_0, t_f], \mathbb{H})$. Define

$$\int_{t_0}^{t_f} f(s) d\beta(s) = \lim_{n \to \infty} \int_{t_0}^{t_f} f_n(s) d\beta(s). \tag{2.9}$$

The right-hand side in (2.9) is independent of the choice of sequence, as $L^2([t_0, t_f], \mathbb{H})$ is a Hilbert space and the set of all step-functions is a dense subset. In fact, the space $L^2([t_0, t_f], \mathbb{H})$ is often defined as the completion of the set of step functions with respect to the L^2-norm (for more details on L^p-spaces see [5]).

Definition 2.23 (strong measurability)
[64, p. 6] Let $\mathbb{F}$ be a separable Banach space. A function $f : \Omega \longrightarrow \mathbb{F}$ is strongly measurable if there is a sequence $\{f_n\}$ of simple functions such that for all $\omega \in \Omega$

$$\lim_{n\to\infty} f_n(\omega) = f(\omega).$$

Finally it is possible to extend the integral with respect to Wiener processes, which is in [28] referred to as the *Wiener integral*, to operator-valued functions.
Define

$$\mathbb{B}^2([t_0, t_f], L(\mathbb{H}, \mathbb{K})) = \{\phi : [t_0, t_f] \to L(\mathbb{H}, \mathbb{K}), \text{ such that } \phi(t) \text{ is strongly measurable and } \int_{t_0}^{t_f} \|\phi(t)\|^2 dt < \infty\}.$$

Definition 2.24 (Wiener integral)
[32, Definition 5.25] Let $\phi \in \mathbb{B}^2([t_0, t_f], L(\mathbb{K}, \mathbb{H}))$ and let $w(t)$ be a $\mathbb{K}$-valued Wiener process with a representation as in (2.8). For $0 \leq t_0 \leq t \leq t_f$ define

$$\int_{t_0}^{t_f} \phi(s) dw(s) = \sum_{i\in\mathbb{N}} \int_{t_0}^{t_f} \phi(s) e_i d\beta_i(s)$$

whenever the right-hand side is in $L^2(\Omega, \mu_p; \mathbb{H})$.

Recall that the stochastic process $w(t)$ also has a spatial dependence, which is omitted, as explained earlier in this chapter. Note further that for a fixed time t a stochastic process $w(t)$ is a random variable. Therefore the term

$$\int_{t_0}^{t_f} \phi(s) dw(s)$$

as defined above is also a random variable.

Some important properties of integrals with respect to Wiener processes are stated.

Lemma 2.25
[32, Lemma 5.27& Lemma 5.28] Let $\phi, \phi_1, \phi_2 \in \mathbb{B}^2([t_0, t_f], L(\mathbb{K}, \mathbb{H}))$ and let $w(t), w_1(t), w_2(t)$ be $\mathbb{K}$-valued independent Wiener processes. For $0 \leq t_0 \leq t \leq t_f$ the following properties hold

1. $\int_{t_0}^{t_f} \phi(s)dw(s) \in L^2(\Omega, \mu_p; \mathbb{H})$ *has continuous sample paths,*
2. $\mathsf{E}\{\int_{t_0}^{t_f} \phi(s)dw(s)\} = 0,$
3. $\mathsf{E}\{\int_{t_0}^{t_f} \|\phi(s)dw(s)\|^2\} = \mathrm{tr}\{\int_{t_0}^{t_f} \phi(s)W\phi^*(s)ds\} \leq \mathrm{tr}\{W \int_{t_0}^{t_f} \|\phi(s)\|^2 ds\},$
4. $\mathsf{E}\{\int_{t_0}^{s_1} \phi_1(s)dw(s) \circ \int_{t_0}^{s_2} \phi_2(s)dw(s)\} = \mathsf{E}\{\int_{t_0}^{\min\{s_1,s_2\}} \phi_1(s)W\phi_2^*(s)ds\},$
5. $\mathsf{E}\{\int_{s_1}^{s_2} \phi_1(s)dw_1(s) \circ \int_{s_3}^{s_4} \phi_2(s)dw_2(s)\} = 0,$

where $s_i \in [t_0, t_f], i = 1, \ldots, 4.$

2.2 Stochastic differential equations

Having defined Wiener integrals it is now possible to determine solutions to SDEs with inputs defined by Wiener processes. To do that it is first necessary to recall two main results from the theory of linear ODEs.

Theorem 2.26
[85, p. 164] Let $A(t) \in \mathrm{C}([t_0, t_f], L(\mathbb{R}^n))$ and consider the homogeneous initial value problem (IVP)

$$\begin{cases} \dfrac{dx_{hom}(t)}{dt} = A(t)x_{hom}(t) \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases} \tag{2.10}$$

Then there exists a fundamental matrix $U(t, t_0)$ with the properties

$$\begin{aligned} U(t_0, t_0) &= I \\ U(t_0, t) &= U^{-1}(t, t_0) \\ U(t, t_0) &= U(t, s)U(s, t_0) \\ \frac{dU(t, t_0)}{dt} &= A(t)U(t, t_0) \end{aligned}$$

such that (2.10) *has the unique solution*

$$x(t) = U(t, t_0)x_0 \quad t_0 \leq t \leq t_f.$$

The reader may recall that in the case of a constant matrix, $A(t) = A$, the fundamental matrix is given by the exponential $U(t, t_0) = \exp(A(t - t_0))$.

Theorem 2.27
[85, p. 171] Let $A(t) \in \mathrm{C}([t_0, t_f], L(\mathbb{R}^n))$, $f(t) \in \mathrm{C}([t_0, t_f], \mathbb{R}^n)$ *and consider the inhomogeneous IVP*

$$\begin{cases} \dfrac{dx_{inh}(t)}{dt} = A(t)x_{inh}(t) + f(t) \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases}$$

Then the differentiable solution is

$$x_{inh}(t) = U(t, t_0)\left(x_0 + \int_{t_0}^{t} U^{-1}(s, t_0)f(s)ds\right)$$

where $U(t, t_0)$ *is the fundamental matrix corresponding to the homogeneous system* (2.10) *in Theorem 2.26.*

Finally the SDE (2.1) can be treated rigorously, which is here reformulated as an initial value problem (IVP). Let therefore

$$\begin{cases} dx(t) = A(t)x(t)dt + D(t)dw(t) \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n \end{cases} \tag{2.11}$$

where $x(t) \in \mathbb{R}^n$, $A(t) \in \mathrm{C}([t_0, t_f], L(\mathbb{R}^n))$, $D(t) \in \mathbb{B}^2([t_0, t_f], L(\mathbb{R}^k, \mathbb{R}^n))$ and $w(t)$ is a $\mathbb{R}^k$-valued Wiener process.

At the beginning of this chapter, in equation (2.1), the same SDE (2.11) was introduced with the common Leibniz notation $\frac{d}{dt}$. However, after developing the theory for Wiener process disturbances, it is clear that even though $w(t)$ has continuous sample paths, it is not necessarily differentiable in t. In fact, it is nowhere differentiable [22]. That is why the 'dt'-notation, as in (2.11), is often used in SDEs to emphasize that the disturbance, and hence the solution to an SDE, may not be differentiable.

Looking at the stochastic IVP (2.11) the question arises whether there exists a result similar to Theorem 2.27, that determines the solutions for inhomogeneous linear ODEs, however with $f(t) = D(t)dw(t)$. Indeed, the *strong solution* to (2.11) has an analogous representation.
Before that result, a more general existence theorem is given.

Definition 2.28 (strong solution)
[22, Definition 12.2] Let $w(t)$ be a $\mathbb{R}^d$-valued Wiener process and let for $t_0 \geq 0$ the functions $G : [t_0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^d$ as well as $F : [t_0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous. Consider the SDE

$$\begin{cases} dx(t) = F(t, x(t))dt + G(t, x(t))dw(t) \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases} \tag{2.12}$$

If the integrals $\int_{t_0}^t F(t, x(t))dt$ and $\int_{t_0}^t G(t, x(t))dw(t)$ exists for every $t \geq t_0$, then

$$x(t) = x_0 + \int_{t_0}^t F(t, x(t))dt + \int_{t_0}^t G(t, x(t))dw(t) \tag{2.13}$$

is called a strong solution to the SDE (2.12).

Theorem 2.29 (existence of strong solutions)
[22, Theorem 12.1] Consider the SDE given in Definition 2.28. If the functions F and G are Lipschitz in x, then the solution (2.13) exists and has continuous sample paths.

Therefore, the strong solution to (2.11) is

$$x(t) = x_0 + \int_{t_0}^t A(s)x(s)ds + \int_{t_0}^t D(s)dw(s). \tag{2.14}$$

However, an explicit representation similar to the solution for linear ODEs, exists.

Theorem 2.30
[58, pg. 150][22, Theorem 12.1] Let $U(t, t_0)$ be, as above, the fundamental matrix corresponding to the homogeneous system (2.10) in Theorem 2.26.
The unique strong solution to the stochastic IVP (2.11) is

$$x(t) = U(t, t_0) \left(x_0 + \int_{t_0}^t U^{-1}(s, t_0)D(s)dw(s) \right).$$

Furthermore, $x(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n)$, $t \geq t_0$ and $x(t)$ has continuous sample paths.

Henceforth, in order to distinguish between stochastic and deterministic systems, the "dt"-notation, like in (2.11), is used for SDEs while the common "$\frac{d}{dt}$"-notation is used for deterministic ODEs.

Finally, consider a stochastic system of the form

$$\begin{cases} d\hat{x}(t) = H(t, \hat{x}(t))dt + F(t)dx(t) \\ \hat{x}(t_0) = \hat{x}_0, \quad \hat{x}_0 \in \mathbb{R}^n, \end{cases} \tag{2.15}$$

where $H : [t_0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $F : [t_0, \infty) \to L(\mathbb{R}^n)$ are continuously differentiable in all variables and $x(t)$ is the strong solution to the linear SDE 2.11. The class (2.15) of stochastic SDEs will be encountered in the later chapters.

To determine its solution, the following integral has to be defined.

Definition 2.31
[22, Definition 11.2] Let $x(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n), t \in [t_0, t_f]$ be the strong solution to the linear SDE (2.11) and let $F(t)$ be as defined in (2.15). For $t \in [t_0, t_f]$ define the integral of $F(t)$ with respect to $x(t)$ as

$$\int_{t_0}^{t} F(s)dx(s) = \int_{t_0}^{t} F(s)A(s)x(s)ds + \int_{t_0}^{t} D(s)dw(s). \tag{2.16}$$

Note that this integral (2.16) is in $L^2(\Omega, \mu_p; \mathbb{R}^n)$ since by Theorem 2.30 $x(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n)$ and $A(t)$ as well as $F(t)$ are continuous. Furthermore, as both terms on the right-hand side of (2.16) have continuous sample paths, so does their sum. Hence the integral is well-defined.

With this definition (2.16) the strong solution to (2.32) can be defined.

Definition 2.32
If $\hat{x}(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n)$ satisfies

$$\hat{x}(t) = x_0 + \int_{t_0}^{t} H(s, \hat{x}(s))ds + \int_{t_0}^{t} F(s)dx(s)$$

then it is called a strong solution to (2.15).

Chapter 3

Estimation for finite-dimensional systems

In the case of no noise, under some conditions the algebraic properties of the state-space system and the estimator system can be used so that the output error converges to zero. In this case the estimator is an observer. David G. Luenberger in [63] was one of the first to explore this method, the main concepts of which are presented in the next section.

If there is noise in the original system, then zero asymptotic error is generally not possible. Often, the optimality condition in stochastic estimation, is to minimize the trace of the error covariance, or equivalently, to minimize the mean squared error between original state and estimate.

One of the earliest results on filtering for noisy finite-dimensional linear systems was developed by N. Wiener [86] in 1950 to deal with systems disturbed by stationary stochastic processes, i.e. processes the statistical properties of which, like variance or distribution, do not change over time. The *Wiener Filter* minimizes the mean squared error and was widely applied in estimation [11].

In the 1960s this result was extended to non-stationary stochastic processes. In [57] R. E. Kalman developed a filtering algorithm for discrete linear stochastic systems - a method that is now known as the *Kalman Filter* (KF). This result was extended by R. E. Kalman and R. S. Bucy to continuous-time linear stochastic systems [19]. The filter minimizes the mean squared error [23] and the linear output error feedback law, also referred to as the *gain*, is obtained by solving a matrix Riccati equation. Due to the optimality and the simplicity of the algorithm it has been applied innumerable times in various areas.

Shortly after its invention the KF was extended to systems of nonlinear ODEs through linearization at each timestep (for more details see [23]). This method is commonly referred to as *extended Kalman filter* (EKF). In [10] the EKF is considered as asymptotic limit of a recursive filter and in [17] local asymptotic convergence of the EKF for a nonlinear deterministic discrete time system is shown. Exponential convergence with a prescribed degree of convergence for deterministic systems has been established in [74, 75, 76].

Apart from applications of the EKF other attempts have been made to address nonlinear filtering. The authors in [14] consider a filter for a nonlinear SDE as a asymptotic limit of a family of filters. Filters for a class of nonlinear one-dimensional ODEs with scalar perturbations are derived in [13]. The *unscented Kalman filter* [20, 55] uses statistically chosen data points to estimate the state. I. Gyöngy investigated filtering for nonlinear processes that take values on a sub-manifold of $\mathbb{R}^n$ [52].
A comprehensive review of the development stochastic filtering was done by D. Crisan in [25].

Even though it is not proven that the EKF provides any kind of optimality [79] and may even diverge [72, 73] it is widely used in control and estimation, for instance in state-of-charge estimation [3], data assimilation [83], signal detection [62], reactor physics [68], robotics [35] and even in environmental science [61].

This chapter presents two well-known estimation methods for linear ODEs and another method for nonlinear ODEs. Important concepts in estimation for random processes are also outlined.

3.1 Luenberger observers

An early approach to observer design was made by David G. Luenberger in [63] where, based on algebraic properties of the system, an observer is constructed via a linear feedback law. The basic theory is outlined here and the information in this section is taken from [65].

Consider the linear ODE model

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned} \tag{3.1}$$

where $A(t) \in \mathrm{C}([0,\infty), L(\mathbb{R}^n))$ is the system matrix, $B(t) \in \mathrm{C}([0,\infty), L(\mathbb{R}^p, \mathbb{R}^n))$ is the control matrix and $C(t) \in \mathrm{C}([0,\infty), L(\mathbb{R}^n, \mathbb{R}^k), k \leq n$ is the observation matrix. In general,

$A(t)$ describes the systems dynamics or the relation between the systems components, $B(t)$ specifies which components are being controlled by which control vector component and $C(t)$ determines which system components are measured.

It is assumed that $A(t)$ and $B(t)$, as well as the continuous input $u(t)$ are known.
Hence, if the initial condition x_0 were also fully known, the trajectory of $x(t)$ would be obtained by solving (3.1), the solution of which is (see Theorem (2.27))

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-s))Bu(s)ds.$$

However, in many real-life systems x_0 cannot be perfectly determined, even if the system's dynamics are not exposed to a significant disturbance. Since x_0 is the only unknown parameter, the objective becomes to define an estimator $\hat{x}(t)$ that converges to the solution $x(t)$ of (3.1) for any initial condition.

Naturally the observation matrix $C(t)$ plays a crucial role in constructing an observer. It maps the full state of the system into a subspace of the state-space to account for the fact that, often, only part of the system's state can be observed or measured.

Since there are no random elements in the original system (3.1), an intuitive attempt in constructing an observer is to define a system with the same system matrix $A(t)$ and the same input $B(t)u(t)$ and to additionally inject the output error.

Therefore let a second system be defined as

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + B(t)u(t) + K(t)(y(t) - C(t)\hat{x}(t)) \\ \hat{x}(0) = \hat{x}_0, \quad \hat{x}_0 \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

Definition 3.1
Let $x(t)$ be the solution of (3.1)and $\hat{x}(t)$ be a solution to (3.2). The estimator (3.2) is an observer for the system (3.1) if

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$$

for any choice of initial condition $\hat{x}_0$.

Definition 3.2
Let $A(t) \in \mathrm{C}([0,\infty), L(\mathbb{C}^n)$ and let $\lambda_i(t) \in \mathbb{C}, i = 1, \dots, n$ denote the eigenvalues of $A(t)$. If $\Re\{\lambda_i(t)\} \leq c < 0$ for all i and $t \in [0,\infty)$ then $A(t)$ is Hurwitz.

Definition 3.3
The pair $(A(t), C(t))$ is detectable if there exists a matrix $K(t)$ such that the matrix $A(t) - K(t)C(t)$ is Hurwitz.

Having defined these properties, the following theorem gives the necessary and sufficient conditions for (3.2) to be an observer.

Theorem 3.4
The system (3.2) is an observer for (3.1) if $(A(t) - K(t)C(t))$ is Hurwitz.

Proof. Letting the error $e(t) = x(t) - \hat{x}(t)$

$$\frac{de(t)}{d(t)} = (A(t) - K(t)C(t))e(t)$$

and hence if $(A(t) - K(t)C(t))$ is Hurwitz the error will converge to zero regardless of $e(0)$. □

This observer (3.2) where $(A(t)-K(t)C(t))$ is Hurwitz is referred to as Luenberger observer. In general $K(t)$ can be chosen from a great set of matrices that make $(A(t) - K(t)C(t)$ Hurwitz. This is often used to design a feedback input that meets certain requirements imposed on the system, for example *stability*, *robustness*, etc. For more details see [65].

3.2 The Kalman filter

Although the Kalman filter was first introduced as an algorithm for finite-dimensional discrete stochastic systems and is mainly used as such in applications, it is useful to look at the continuous-time derivation, often also referred to as Kalman-Bucy filter, to understand its mathematical properties.

The derivation of the continuous-time Kalman filter will be outlined in this section. For more information and detailed computations the reader is referred to [19, 24, 50, 78].

The general objective of the filtering problem is to find the best estimate, at a given time t, for a stochastic process $x(t)$ based on an *observation process* $y(s), s \in [0, t]$. The latter is a (often linear) function of $x(t)$. The Kalman-Bucy filter provides a minimal mean squared error filter for linear SDEs. As this will be outlined, $x(t)$, $y(t)$ and $\hat{x}(t)$ treated in this section will be stochastic processes.

Before treating estimation for stochastic processes, the estimation theory for random variables will be outlined. For that purpose let $f, g : \Omega \longrightarrow \mathbb{R}$ be two random variables. The best estimate for f based on g will be determined. The corresponding theory will provide a better understanding of the stochastic case, as $x(t)$, $y(t)$ and $\hat{x}(t)$ are random variables for any fixed t.

Let henceforth $(\Omega, \mathfrak{A}, \mu_p)$ be a probability space and consider the Borel algebra on $\mathbb{R}$.
Let $h_0, h : \mathbb{R} \longrightarrow \mathbb{R}$ denote Borel-measurable functions. The best estimate for f *based on* g is the random variable $h_0(g)$ such that

$$\mathsf{E}\{(f - h_0(g))^2\} \leq \mathsf{E}\{(f - h(g))^2\} \text{ for all } h. \tag{3.3}$$

In other words, the best estimate for f based on g is a function $h_0(g)$ of g such that it yields the minimal mean-squared error. Note that h_0 may be nonlinear.

The reader familiar with probability theory may already expect that $h_0(g)$ is the *conditional expectation* of f based on g.

Definition 3.5 (conditional expectation)
[22, Definition 5.2] Let $\mathsf{E}\{f\} < \infty$, let f be measurable with respect to a σ-algebra $\mathfrak{A}$ and let $\mathfrak{F} \subset \mathfrak{A}$ be a sub-σ-algebra. The conditional expectation of f with respect to $\mathfrak{F}$ is a $\mathfrak{F}$-measurable random variable, denoted by $\mathsf{E}\{f|\mathfrak{F}\}$, such that

$$\int_F f d\mu = \int_F \mathsf{E}\{f|\mathfrak{F}\} d\mu \quad \text{for all } F \in \mathfrak{F}. \tag{3.4}$$

The random variable $\mathsf{E}\{f|\mathfrak{F}\}$ exists and is a.e. uniquely determined by (3.4).

This definition of the conditional expectation with respect to a sub-σ-algebra, albeit abstract, is the most common one due to its universality. It does not rely on another random variable.

Note that by (3.4) $\mathsf{E}\{f|\mathfrak{F}\}$ is defined to be identical to f almost everywhere on $\mathfrak{F}$. Two properties can immediately be observed:

1. $\mathsf{E}\{\mathsf{E}\{f|\mathfrak{A}\}\} = \mathsf{E}\{f\}$, or equivalently $\mathsf{E}\{f|\mathfrak{A}\} = f$ a.e. and
2. $\mathsf{E}\{\mathsf{E}\{f|\mathfrak{F}\}|\mathfrak{F}\}\} = \mathsf{E}\{f|\mathfrak{F}\}$.

They motivate an essential geometric property: the conditional expectation with respect to $\mathfrak{F}$ can also be defined as a projection of f onto the subspace $L^2(\Omega, \mathfrak{F}, \mu_p) \subset L^2(\Omega, \mathfrak{A}, \mu_p)$. This is formulated as theorem.

Theorem 3.6
[22, Theorem 5.4] Let again $\mathfrak{F} \subset \mathfrak{A}$ be two σ-algebras. Then the mapping

$$\mathsf{E}\{\cdot|\mathfrak{F}\} : L^2(\Omega, \mathfrak{A}, \mu_p) \longrightarrow L^2(\Omega, \mathfrak{F}, \mu_p) \subset L^2(\Omega, \mathfrak{A}, \mu_p) \tag{3.5}$$

is an orthogonal projection onto $L^2(\Omega, \mathfrak{F}, \mu_p)$.

With the L^2-scalar product this orthogonality means that

$$\int_\Omega \mathsf{E}\{f|\mathfrak{F}\}(f - \mathsf{E}\{f|\mathfrak{F}\})d\mu_p = \mathsf{E}\big\{\mathsf{E}\{f|\mathfrak{F}\}(f - \mathsf{E}\{f|\mathfrak{F}\})\big\} = 0.$$

In Hilbert spaces orthogonal projections onto subspaces have a very important geometric meaning in optimization problems: they yield the closest point on that subspace with respect to a reference point. This is formulated in the following well-known result.

Theorem 3.7 (orthogonal minimizer)
[59, Theorem 3.3-1 & Lemma 3.3-2] Let $\mathbb{H}$ be a Hilbert space and $\mathbb{Y} \subset \mathbb{H}$ be a closed subspace. Then for every $x \in \mathbb{H}$ there is a unique $y \in \mathbb{Y}$ such that

$$\inf_{\tilde{y} \in \mathbb{Y}} \|x - \tilde{y}\| = \|x - y\|.$$

Moreover $z = x - y$ is orthogonal to $\mathbb{Y}$.

The same is true for the conditional expectation with respect to a sub-σ-algebra.

Theorem 3.8
[5, Theorem 12.1.2] Let $f \in L^2(\Omega, \mathfrak{A}, \mu_p)$ and let $\mathfrak{F} \subset \mathfrak{A}$ be a sub-σ-algebra.
Then $\mathsf{E}\{f|\mathfrak{F}\}$ satisfies

$$\mathsf{E}\big\{(f - \mathsf{E}\{f|\mathfrak{F}\})^2\big\} = \inf\big\{\mathsf{E}\{(f-g)^2\} \mid g \in L^2(\Omega, \mathfrak{F}, \mu_p)\big\}. \tag{3.6}$$

The results above shows that the best estimate for a random variable $f \in L^2(\Omega, \mathfrak{A}, \mu_p)$ with respect to a given sub-σ-algebra $\mathfrak{F} \subset \mathfrak{A}$ is the conditional expectation $\mathsf{E}\{f|\mathfrak{F}\}$, or equivalently, the projection of f onto $L^2(\Omega, \mathfrak{F}, \mu_p)$.
To see how this can be used to estimate f based on another random variable g let, as in Definition 2.3, $\sigma(g)$ be the sub-σ-algebra generated by g,

$$\sigma(g) = \sigma\left(\{g^{-1}(A) \mid A \in \mathfrak{B}(\mathbb{R})\}\right). \tag{3.7}$$

Setting $\mathfrak{F} = \sigma(g)$ and looking at (3.6) makes it clear that $\mathsf{E}\{f|\sigma(g)\}$ yields the minimal mean squared-error with respect to all square-integrable real functions that are measurable with respect to $\sigma(g)$. As $h : \mathbb{R} \longrightarrow \mathbb{R}$ in (3.3) is Borel-measurable and $h(g) \in L^2(\Omega, \sigma(g), \mu_p)$, then this includes $h(g)$. Hence, the best estimate of f based on g that minimizes (3.3) is $\mathsf{E}\{f|\sigma(g)\}$. This is formulated in the following theorem.

Theorem 3.9
[5, Theorem 12.1.1] Let $f, g \in L^2(\Omega, \mathfrak{A}, \mu_p)$ be two real random variables. Then

$$\mathsf{E}\big\{(f - \mathsf{E}\{f|\sigma(g)\})^2\big\} = \inf\Big\{\mathsf{E}\big\{(f - h(g))^2\big\} \mid h : \mathbb{R} \longrightarrow \mathbb{R} \text{ is Borel measurable and } \mathsf{E}\{h(g)^2\} < \infty\Big\}.$$

This result justifies the following definition.

Definition 3.10
[22, Definition 5.3] Let $f, g : \Omega \longrightarrow \mathbb{R}$ be two random variables and let $\mathsf{E}\{f^2\} < \infty$. The conditional expectation of f with respect to g is

$$\mathsf{E}\{f|g\} = \mathsf{E}\{f|\sigma(g)\}. \tag{3.8}$$

Summarizing the above exposition verbally:

1. The best estimate of $f \in L^2(\Omega, \mathfrak{A}, \mu_p)$ given a sub-σ-algebra $\mathfrak{F} \subset \mathfrak{A}$ is the conditional expectation $\mathsf{E}\{f|\mathfrak{F}\}$, or equivalently, the projection of f onto $L^2(\Omega, \mathfrak{F}, \mu_p)$.

2. The best estimate of $f \in L^2(\Omega, \mathfrak{A}, \mu_p)$ based on $g \in L^2(\Omega, \mathfrak{A}, \mu_p)$ is the conditional expectation of $\mathsf{E}\{f|g\} = \mathsf{E}\{f|\sigma(g)\}$, or equivalently, the projection of f onto $L^2(\Omega, \sigma(g), \mu_p)$.

The results above also hold true for multidimensional random variables $f : \Omega \longrightarrow \mathbb{R}^n$ and $g : \Omega \longrightarrow \mathbb{R}^k$ with $k, n \in \mathbb{N}$ (see [37]). In that case the conditional expectation is understood component-wise as

$$\mathsf{E}\{f|g\} = \begin{pmatrix} \mathsf{E}\{f_1|\sigma(g)\} \\ \vdots \\ \mathsf{E}\{f_n|\sigma(g)\} \end{pmatrix}$$

and it minimizes the mean squared error in each component.

Now the Kalman-Bucy filter for linear SDEs can be derived.

Let $A(t) \in \mathrm{C}([t_0, \infty), L(\mathbb{R}^n))$, $C(t) \in \mathrm{C}([t_0, \infty), L(\mathbb{R}^k, \mathbb{R}^n))$, $k \leq n$ and consider the continuous-time linear stochastic state-space system

$$\begin{aligned} dx(t) &= A(t)x(t)dt + dw(t), \\ dy(t) &= C(t)x(t)dt + dv(t), \end{aligned} \tag{3.9}$$

where $w(t)$ and $v(t)$ are $\mathbb{R}^n$-valued and $\mathbb{R}^k$-valued Wiener processes respectively with covariances

$$\begin{aligned} \mathsf{E}\{w(t)w^T(\tau)\} &= W\delta(t-\tau), \\ \mathsf{E}\{v(t)v^T(\tau)\} &= R\delta(t-\tau), \\ \mathsf{E}\{w(t)v^T(\tau)\} &= 0, \end{aligned}$$

where W, R are self-adjoint and invertible and where $\delta(\cdot)$ is defined as

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The disturbances $w(t)$ and $v(t)$ account for the noise present in the system and the sensor respectively.

Recall from Section 2.1 that the "dt"-notation is used for systems exposed to stochastic noise, the solution of which is not necessarily differentiable. However, Theorem 2.30 ensures that there exist strong solutions $x(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n)$ and $y(t) \in L^2(\Omega, \mu_p; \mathbb{R}^n)$ for (3.9), $t \geq t_0$, satisfying Definition 2.28.

The objective is to find a filter $\hat{x}(t)$ that minimizes

$$\mathsf{E}\{\|x(t) - \hat{x}(t)\|^2\} \tag{3.10}$$

based on the observation $Y_t = \{y(s) \mid t_0 \leq s \leq t\}$. Recalling the results for the best estimate of a random variable outlined above as well as the fact that $x(t)$ for a fixed time t is a random variable justifies the following form of the optimal filter.

Theorem 3.11
[19, Theorem 4.1] The optimal filter for $x(t)$ given by the stochastic system (3.9) based on Y_t that minimizes (3.10) is

$$\hat{x}(t) = \mathsf{E}\{x(t) \mid \sigma(Y_t)\}. \tag{3.11}$$

In other words, the optimal filter $\hat{x}(t)$ is the projection of $x(t)$ onto $L^2(\Omega, \sigma(Y_t), \mu_p)$.

Recall that by Definition 2.3 $\sigma(Y_t)$ is the σ-algebra generated by the family $\{y(s)\}_{s\in[t_0,t]}$ of random variables.

Now, if only a theoretical expression for the best filter was sought, there would be nothing more to do. However, an explicit computational scheme can be found for the optimal filter $\hat{x}(t)$. For continuous-time systems, $\hat{x}(t)$ is the strong solution to a SDE that is similar to the differential equation for the Luenberger observer.

The following lemma is known as the orthogonal projection lemma, or equivalently, the orthogonality principle. Its proof is omitted as it is very long and technical and as in Section 5.1 a version for infinite-dimensional systems is proven. The details for the present case can be found in [19], [24] or [78].

Lemma 3.12 (orthogonal projection lemma)
[78, section 4.6] Consider again $x(t)$ and $y(t)$ in (3.9). The filter $\hat{x}(t)$ is optimal, that is, it is the projection of $x(t)$ onto $L^2(\Omega, \sigma(Y_t), \mu_p)$ if and only if for all $t_0 \leq \sigma \leq \tau \leq t_f$

$$\mathsf{E}\left\{\left(x(t) - \hat{x}(t)\right)\left(y(\tau) - y(\sigma)\right)^T\right\} = 0.$$

This lemma implies an interesting fact regarding the space $L^2(\Omega, \sigma(Y_t), \mu_p)$, though it is complicated to prove: the space $L^2(\Omega, \sigma(Y_t), \mu_p)$ is the L^2-completion of the linear hull of $Y_t = \{y(s) \mid t_0 \leq s \leq t\}$. For more details on this see [78, pages 100-102 & Theorem 4.11].

The following is known as the Wiener-Hopf equation, the solution of which will later be the optimal gain.

Lemma 3.13 (Wiener-Hopf equation)
[78, Theorem 4.13][19, p. 53] Under the assumption on the system (3.9) the integral matrix equation

$$K(t,s)R + \int_{t_0}^{t} K(t,\tau)C(\tau)\mathsf{E}\{x(\tau)x^T(s)\}C^T(s)d\tau = \mathsf{E}\{x(t)x^T(s)\}C(s) \tag{3.12}$$

has a solution $K(t,s)$ that is differentiable in t and continuous in $s \in [t_0, t]$.

Now an explicit representation for $\hat{x}(t)$ can be derived. This is the integral form of the Kalman-Bucy filter.

Lemma 3.14
[19, Chapter 4] The optimal estimator (3.11) for the system (3.9) is

$$\hat{x}(t) = \int_{t_0}^{t} K(t,s)C(s)x(s)ds + \int_{t_0}^{t} K(t,s)dv(s) \tag{3.13}$$

where $K(t,s)$ satisfies the Wiener-Hopf equation (3.12).

Proof. Define the process

$$z(t)=\int_{t_0}^{t}K(t,s)C(s)x(s)ds+\int_{t_0}^{t}K(t,s)dv(s)$$

where $K(t,s)$ is the solution to (3.12). Note that $z(t)\in L^2(\Omega,\mu_p;\mathbb{R}^n)$ as $K(t,s)$ and $C(t)$ are continuous and $x(t)\in L^2(\Omega,\mu_p;\mathbb{R}^n)$ for all t. It will now be shown that $z(t)=\hat{x}(t)$ by use of the orthogonality principle in Lemma 3.12. More precisely, it is shown that for all $t_0\leq\sigma\leq\tau\leq t$

$$\mathsf{E}\big\{\big(x(t)-z(t)\big)\big(y(\tau)-y(\sigma)\big)^T\big\}=0 \tag{3.14}$$

which then implies that $z(t)$ is equal to $\hat{x}(t)$. To do that it will be shown that

$$\mathsf{E}\big\{z(t)(y(\tau)-y(\sigma))^T\big\}=\mathsf{E}\big\{x(t)(y(\tau)-y(\sigma))^T\big\}. \tag{3.15}$$

For that purpose, fix $0\leq\sigma\leq\tau\leq t$. Then

$$\mathsf{E}\big\{z(t)(y(\tau)-y(\sigma))^T\big\} \tag{3.16}$$

$$=\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)C(s)x(s)ds+\int_{t_0}^{t}K(t,s)dv(s)\Big)\cdot\Big(\int_{\sigma}^{\tau}C(s)x(s)ds+\int_{\sigma}^{\tau}dv(s)\Big)^T\Big\}$$

$$=\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)C(s)x(s)ds\Big)\Big(\int_{\sigma}^{\tau}C(s)x(s)ds\Big)^T\Big\} \tag{3.17}$$

$$+\underbrace{\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)C(s)x(s)ds\Big)\Big(\int_{\sigma}^{\tau}dv(s)\Big)^T\Big\}}_{=0\text{ due to the Wiener integral properties in Lemma 2.25}}$$

$$+\underbrace{\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)dv(s)\Big)\Big(\int_{\sigma}^{\tau}C(s)x(s)ds\Big)^T\Big\}}_{=0\text{ due to same integral properties in Lemma 2.25}}$$

$$+\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)dv(s)\Big)\Big(\int_{\sigma}^{\tau}dv(s)\Big)^T\Big\}. \tag{3.18}$$

Using property (4.) in Lemma 2.25 for the term (3.18) and the covariance R for $v(t)$

$$\mathsf{E}\Big\{\Big(\int_{t_0}^{t}K(t,s)dv(s)\Big)\Big(\int_{\sigma}^{\tau}dv(s)\Big)^T\Big\}=\int_{\sigma}^{\tau}K(t,s)Rds. \tag{3.19}$$

Interchanging the integrals in the first term (3.17) then yields

$$\begin{aligned}\mathsf{E}\big\{z(t)(y(\tau)-y(\sigma))^T\big\} &= \int_\sigma^\tau \left(K(t,s)R + \int_{t_0}^t K(t,\tau)C(\tau)\mathsf{E}\{x(\tau)x^T(s)\}C^T(s)d\tau\right) ds \\ &= \int_\sigma^\tau \mathsf{E}\{x(t)x^T(s)\}C^T(s)ds\end{aligned}$$

where in the last step it was used that $K(t,s)$ satisfies the Wiener-Hopf equation (3.12). Finally it suffices to observe that

$$\begin{aligned}\int_\sigma^\tau \mathsf{E}\{x(t)x^T(s)\}C^T(s)ds &= \mathsf{E}\Big\{x(t)\left(\int_\sigma^\tau C(s)x(s)ds\right)^T\Big\} \\ &= \mathsf{E}\Big\{x(t)\left(\int_\sigma^\tau C(s)x(s)ds\right)^T\Big\} + \underbrace{\mathsf{E}\Big\{x(t)\left(\int_\sigma^\tau dv(s)\right)^T\Big\}}_{=0} \\ &= \mathsf{E}\Big\{x(t)\left(\int_\sigma^\tau C(s)x(s)ds + \int_\sigma^\tau dv(s)\right)^T\Big\} \\ &= \mathsf{E}\big\{x(t)(y(\tau)-y(\sigma))^T\big\}. \end{aligned} \tag{3.20}$$

Now, by equating (3.16) and (3.20) yields (3.15) and hence $z(t) = \hat{x}(t)$. □

Grönwall's lemma will be of frequent use in this thesis.

Lemma 3.15 (Grönwall's lemma)
[85, VI.] Let $\phi(t) \in \mathrm{C}([t_0, t_f], \mathbb{R})$ and $\beta \geq 0$. If $\phi(t)$ satisfies

$$\phi(t) \leq \alpha + \beta \int_{t_0}^t \phi(s)ds$$

on $[t_0, t_f]$ then

$$\phi(t) \leq \alpha \exp(\beta t), \quad t \in [t_0, t_f]. \tag{3.21}$$

Now it is possible to prove the main theorem. Since the computations are long and technical, they are not stated in detail.

Theorem 3.16
[19] The optimal filter $\hat{x}(t)$ is the strong solution to

$$\begin{aligned} d\hat{x}(t) &= A(t)\hat{x}(t)dt + K(t,t)\big(dy(t) - C(t)\hat{x}(t)dt\big) \\ \hat{x}(t_0) &= 0. \end{aligned} \tag{3.22}$$

where $K(t,s)$ solves (3.12). Furthermore

$$K(t,t) = P(t)C^T(t)R^{-1}$$

where $P(t)$ is the unique solution to

$$\begin{cases} \dfrac{dP(t)}{dt} = A(t)P(t) + P(t)A(t)^T + W - P(t)C^T(t)R^{-1}C(t)P(t) \\ P(t_0) = P_0. \end{cases} \tag{3.23}$$

Proof. Let $\hat{x}$(t) be given by (3.13). Using the relation 3.12 for the integral kernel $K(t,s)$, which is differentiable in t by Lemma 3.13, it can be computed that

$$\frac{d}{dt}K(t,s) = A(t)K(t,s) - K(t,t)C(t)K(t,s)$$

and further that with $P(t) = \mathsf{Cov}\{x(t) - \hat{x}(t)\}$,

$$K(t,t) = P(t)C^T(t)R^{-1} \tag{3.24}$$

Note that with this kernel $K(t,s)$ the filter $\hat{x}(t)$ is already optimal by Lemma 3.14. Next by Definition 2.31 of the stochastic integral, $\hat{x}(t)$ in (3.13) can be written as

$$\begin{aligned} \hat{x}(t) &= \int_{t_0}^{t} K(t,s)C(s)x(s)ds + \int_{t_0}^{t} K(t,s)dv(s) \\ &= \int_{t_0}^{t} K(t,s)dy(s) \end{aligned}$$

where $dy(s)$ is given in (3.9). Using (3.24) it can be seen that $\hat{x}(t)$ is the strong solution to (3.22).
To obtain the differential Riccati equation (3.23) for $P(t)$, observe that substracting (3.9) from (3.22) and defining $e(t) = \hat{x}(t) - x(t)$ yields the error system

$$\begin{aligned} de(t) &= \big(A(t) - P(t)C^T(t)R^{-1}C(t)\big)e(t)dt - dw(t) + K(t,t)dv(t) \\ e(t_0) &= e_0 = x_0. \end{aligned} \tag{3.25}$$

By Theorem 2.30 the strong solution to (3.25) has the form

$$e(t) = U(t,t_0)\left(e_0 + \int_{t_0}^{t} U^{-1}(t_0,s)P(s)C^T(s)R^{-1}C(s)dv(s) - \int_{t_0}^{t} U^{-1}(t_0,s)dw(s)\right) \tag{3.26}$$

where $U(t,t_0)$ is the fundamental matrix corresponsing to the homogeneous system of (3.25), that is

$$\frac{dU(t,t_0)e_0}{dt} = \big(A(t) - P(t)C^T(t)R^{-1}C(t)\big)U(t,t_0)e_0.$$

Due to property 2 in Lemma 2.25 and the Gaussian initial condition x_0 it follows that $\mathsf{E}\{e(t)\} = 0$. Hence $P(t)$ is given by

$$P(t) = \mathsf{E}\{e(t)e^T(t)\}. \tag{3.27}$$

Note that $P(t)$ is self-adjoint. Substituting $e(t)$ from (3.26) into (3.27) yields after some computation

$$P(t) = U(t,t_0)P_0U^T(t,t_0) + \int_{t_0}^{t} U(t,s)\big(W + P(s)C^T(s)R^{-1}C(s)P(s)\big)U^T(t,s)ds. \tag{3.28}$$

where $P(t_0) = P_0 = \mathsf{Cov}\{e_0\}$. Equation (3.28) is the integral form of the Riccati equation. Differentiating (3.28) now yields (3.23).

It remains to see that the solution to (3.23) is unique. To see that, suppose there is an interval $[t_0, t_f]$ such that (3.23) admits two distinct solutions $P_1(t), P_2(t)$. Their difference $\Delta P(t) = P_1(t) - P_2(t)$ satisfies

$$\begin{cases} \dfrac{d\Delta P(t)}{dt} = A(t)\Delta P(t) + \Delta P(t)A(t) - \Delta P(t)Q(t)P_1(t) - P_2(t)Q(t)\Delta P(t) \\ \Delta P(t_0) = 0, \end{cases} \tag{3.29}$$

where $Q(t) = C^T(t)R^{-1}C(t)$. Since $A(t)$, $C(t)$ and the $P_i(t)$ are continuous, define

$$c = 4\max_{[t_0,t_f]}\{\|A(t)\|, \|Q(t)P_1(t)\|, \|P_2(t)Q(t)\|\}. \tag{3.30}$$

Then integrating (3.29) and taking the norm of the four terms on the right-hand side of (3.29), as well as using (3.30) yields for $t \in [t_0, t_f]$

$$\|\Delta P(t)\| \le c\int_{t_0}^{t} \|\Delta P(s)\|ds \tag{3.31}$$

and with Grönwall's lemma 3.15 it follows that $\Delta P(t) \equiv 0$ on $[t_0, t_f]$. Hence the theorem is proven. □

The optimality criterion (3.10) can also be expressed in terms of the solution to the Riccati equation (3.23): the optimal filter $\hat{x}(t)$ minimizes

$$\mathsf{E}\{\|x(t)-\hat{x}(t)\|^2\} = \operatorname{tr}\left\{\mathsf{E}\left\{\big(x(t)-\hat{x}(t)\big)\big(x(t)-\hat{x}(t)\big)^T\right\}\right\} = \operatorname{tr}\{P(t)\}. \tag{3.32}$$

Indeed many authors prefer to first derive (a version of) the Riccati equation for $P(t)$ and to then find the optimal gain $K(t,s)$ by minimizing $\operatorname{tr}\{P(t)\}$. See for example [24] for this approach.

Since the Kalman filter is used to algorithmically filter out noise in linear time-varying systems its algorithmic form for a discrete state space system is also presented here. Consider the system

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k \\ y_k &= C_k x_k + v_k \end{aligned} \tag{3.33}$$

where $A_k \in L(\mathbb{R}^n)$, $C_k \in L(\mathbb{R}^k, \mathbb{R}^n)$ and where the sequences $\{w_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ are Gaussian noise on the input and the sensor respectively with zero mean and positive definite variance matrices $\operatorname{var}\{w_k\} = W_k$ and $\operatorname{var}\{v_k\} = R_k$ respectively.

The initial condition x_0 has a Gaussian distribution and the random variables x_0, w_k, v_j are assumed to be independent for all $k, j \geq 0$.

The Kalman filter for discrete systems is a recursive algorithm for the estimator $\hat{x}$ consisting of a prediction step $\hat{x}_{k|k-1}$ and a correction step $\hat{x}_{k|k}$.

The optimality criterion is, again, to minimize the trace of the covariance

$$P_{k|k} = \mathsf{Cov}\{x_k - \hat{x}_{k|k}\}.$$

The following algorithm is known as the Kalman filter for discrete linear stochastic systems [24]:

$$\begin{cases} P_{0|0} = \operatorname{var}\{x_0\} \\ \hat{x}_{0|0} = \mathsf{E}\{x_0\} \\ P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + W_{k-1} \\ \hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} \\ K_k = P_{k|k-1}C_k^T(C_kP_{k|k-1}C_k^T + R_k)^{-1} \\ P_{k|k} = (I - K_kC_k)P_{k|k-1} \\ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}). \end{cases}$$

3.3 The extended Kalman filter

Many physical systems contain nonlinearities as well as stochastic inputs. A more general class of systems is

$$\begin{aligned} dx(t) &= f(x(t),t)dt + dw(t) \\ dy(t) &= h(x(t),t)dt + dv(t) \end{aligned} \tag{3.34}$$

where $f \in \mathrm{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$ and $h \in \mathrm{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^k)$ and $w(t)$ and $v(t)$ are again $\mathbb{R}^n$-valued and $\mathbb{R}^k$-valued Wiener processes respectively with covariances

$$\begin{aligned} \mathsf{E}\{w(t)w^T(\tau)\} &= W\delta(t-\tau), \\ \mathsf{E}\{v(t)v^T(\tau)\} &= R\delta(t-\tau), \\ \mathsf{E}\{w(t)v^T(\tau)\} &= 0. \end{aligned}$$

The *extended Kalman filter* (EKF) relies on linearizing the nonlinear state-space system and subsequently applying the Kalman filter for linear systems. It is assumed that for f and g the linearization around a nominal state $\bar{x}(t)$

$$f(x(t),t) \approx f(\bar{x}(t),t) + \frac{df}{dx}(\bar{x}(t),t)(x(t)-\bar{x}(t)) \tag{3.35}$$

$$h(x(t),t) \approx h(\bar{x}(t),t) + \frac{dh}{dx}(\bar{x}(t),t)(x(t)-\bar{x}(t)) \tag{3.36}$$

yields a small approximation error if $\bar{x}(t)$ is close to $x(t)$.
In the EKF the current estimate is used as nominal state meaning $\bar{x}(t) = \hat{x}(t)$ and following the Kalman-Bucy filter for linear systems, the estimator is defined as

$$d\hat{x}(t) = f(\hat{x}(t),t)dt + K(t)(h(x(t),t) - h(\hat{x}(t),t))dt$$

where $K(t)$ has to be determined. Define

$$A(t) = \frac{df}{dx}(\hat{x}(t),t) \quad \text{and} \quad C(t) = \frac{dh}{dx}(\hat{x}(t),t).$$

Let the nonlinearities f and h in the state-space system (3.34) be replaced by the linearizations (3.35) and (3.36) around $\bar{x}(t) = \hat{x}(t)$ respectively. Then the error $e(t) = \hat{x}(t) - x(t)$ satisfies

$$de(t) = (A(t) - K(t)C(t))e(t)dt + K(t)dv(t) - dw(t)$$

which is identical to (3.25) and hence $P = \{e(t)e^T(t)\}$ is again (3.28).
Therefore the gain is defined as

$$K(t) = P(t)C^T(t)R^{-1} \tag{3.37}$$

where the matrix $P(t)$ satisfies the differential Riccati equation (3.23), i.e.

$$\frac{dP(t)}{dt} = A(t)P(t) + P(t)A(t)^T + W - P(t)C^T(t)R^{-1}C(t)P(t). \tag{3.38}$$

Definition 3.17
[75, Definition 6] The system (3.3) is an extended Kalman filter if the gain $K(t)$ is given by (3.37) where $P(t)$ satisfies the differential Riccati equation (3.38).

Since the EKF is based on the Taylor approximation it is hard or impossible to prove any optimality, as a first order approximation is not guaranteed to be sufficient in any way. Nevertheless the EKF is widely applied in various areas and has proven to be an efficient estimation tool for many problems.

Now, consider a time step sequence $\{t_k\}_{k\in\mathbb{N}}$. A discretized version of (3.34) is

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k \\ y_k &= C_k x_k + v_k \end{aligned}$$

where with $f_k(\cdot) = f(\cdot, t_k)$ and $h_k(\cdot) = h(\cdot, t_k)$ the approximation

$$f_k(x_k) \approx f_k(\bar{x}_k) + \frac{df_k}{dx}(\bar{x}_k)(x_k - \bar{x}_k) = f_k(\bar{x}_k) + A_k(x_k - \bar{x}_k) \tag{3.39}$$

$$h_k(x_k) \approx h_k(\bar{x}_k) + \frac{dh_k}{dx}(\bar{x}_k)(x_k - \bar{x}_k) = f_k(\bar{x}_k) + C_k(x_k - \bar{x}_k) \tag{3.40}$$

is used and where for $w_k = w(t_k)$ and $v_k = v(t_k)$ the sequences $\{w_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ are again Gaussian noise on the input and the sensor respectively with zero mean and positive definite covariance matrices W_k and R_k respectively. The initial condition x_0 has a Gaussian distribution and the random variables x_0, w_k, v_j are assumed to be independent for all $k, j \geq 0$.

Let an algorithm be given as [24]

$$
\begin{aligned}
&\text{Initialize} \quad && \begin{cases} P_{0|0} = \operatorname{var}\{x_0\} \\ \hat{x}_{0|0} = \mathsf{E}\{x_0\} \end{cases} \\
&\text{Predict} \quad && \begin{cases} P_{k|k-1} = \alpha A_{k-1} P_{k-1|k-1} A_{k-1}^T + W_k \\ \hat{x}_{k|k-1} = f_{k-1}(\hat{x}_{k-1|k-1}) \end{cases} \\
&\text{Correct} \quad && \begin{cases} K_k = P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1} \\ P_{k|k} = (I - K_k C_k) P_{k|k-1} \\ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - h_k(\hat{x}_{k|k-1})). \end{cases}
\end{aligned}
\tag{3.41}
$$

For $\alpha = 1$ this is the standard EKF algorithm. The case $\alpha > 0$ is explored below.

3.4 The extended Kalman filter as exponential observer

However, even though there is no proven optimality of the EKF, K. Reif and R. Unbehauen in [76] proved that under some conditions the EKF algorithm (3.41) produces a sequence of estimates $\{x_{k|k}\}_{k\in\mathbb{N}}$ that converge exponentially, with a prescribed rate of convergence, to the state $x(t_k)$ of the original system. An analogous result exists for the continuous-time EKF [74]. The main results for the discrete case are stated here.

The authors of [74, 76] consider a noiseless discrete dynamical system

$$
\begin{aligned}
x_{k+1} &= f(x_k, u_k) \\
y_k &= h(x_k)
\end{aligned}
\tag{3.42}
$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^q$ is the input. Furthermore $f(\cdot,\cdot)$ and $h(\cdot)$ are assumed to be continuously differentiable and with $f_k(\cdot) = f(\cdot, u_k)$ can therefore be expanded as

$$
f_k(x_k) - f_k(\hat{x}_{k|k}) = A_k(x_k - \hat{x}_{k|k}) + \varphi(x_k, \hat{x}_{k|k}, u_k) \tag{3.43}
$$

$$
h(x_k) - h(\hat{x}_{k|k-1}) = C_k(x_k - \hat{x}_{k|k-1}) + \chi(x_k, \hat{x}_{k|k-1}) \tag{3.44}
$$

for some remainder functions φ and χ and where

$$
A_k = \frac{df_k}{dx}(\hat{x}_{k|k}) \quad \text{and} \quad C_k = \frac{dh}{dx}(\hat{x}_{k|k-1}).
$$

and $\hat{x}_{k|k}$ and $\hat{x}_{k|k-1}$ are as given in the EKF algorithm (3.41). However as the system (3.42) is noiseless, there is no covariance matrix W for the noise and hence the matrix W in the the algorithm (3.41) can be chosen arbitrarily.

Defining the estimation error $e_k = x_k - \hat{x}_{k|k-1}$ results in the discrete system

$$\begin{cases} e_{k+1} = A_k(I - C_k K_k)e_k + r_k \\ e_0 = x_0 - \hat{x}_{0|-1}, \quad \hat{x}_{0|-1} \in \mathbb{R}^n, \end{cases} \tag{3.45}$$

where r_k is given by

$$r_k = \varphi(x_k, \hat{x}_{k|k}, u_k) - A_k K_k \chi(x_k, \hat{x}_{k|k-1}).$$

Definition 3.18 (exponentially stable equilibrium)
[76, Definition 1] The discrete dynamical system (3.45) has an exponentially stable equilibrium at 0 *if there exist constants* $\delta_1 \geq 1, \epsilon > 0$ *and* $\delta_2 > 1$ *such that whenever* $\|e_0\| \leq \epsilon$,

$$\|e_k\| \leq \delta_1 \exp(-k \ln(\delta_2))\|e_0\| \tag{3.46}$$

for all $k \in \mathbb{N}$.

Definition 3.19 (exponential observer)
[76, Definition 2] The discrete system given by the filtering algorithm (3.41) is an exponential observer if the error system (3.45) has an exponentially stable equilibrium at 0.

The following result shows that for small enough initial error e_0, and some assumptions on the remainder functions φ and χ, as well as the system matrices, the EKF algorithm (3.41) is an exponential observer with prescribes degree α.

Theorem 3.20
[76, Theorem 7] Consider the discrete dynamical system (3.42) and the EKF algorithm (3.41) for $\alpha > 0$.
Let the following assumptions hold:

1. *There are positive constants* $\bar{a}, \bar{c}, p_1, p_2$ *such that for all* k

$$\begin{aligned} \|A_k\| &\leq \bar{a} \\ \|C_k\| &\leq \bar{c} \\ p_1 \leq \|P_{k|k-1}\| &\leq p_2 \\ p_1 \leq \|P_{k|k}\| &\leq p_2. \end{aligned}$$

2. *A_k is nonsingular for all $k \geq 0$.*

3. *There are positive constants $\epsilon_\varphi, \epsilon_\chi, \kappa_\varphi, \kappa_\chi$ such that whenever $\|x_k - \hat{x}_{k|k}\| \leq \epsilon_\varphi$ and $\|x_k - \hat{x}_{k|k-1}\| \leq \epsilon_\chi$ the remainder functions φ and χ in (3.43) and (3.44) are bounded by*

$$\|\varphi(x_k, \hat{x}_{k|k}, u_k)\| \leq \kappa_\varphi \|x_k - \hat{x}_{k|k}\|^2$$
$$\|\chi(x_k, \hat{x}_{k|k-1})\| \leq \kappa_\chi \|x_k - \hat{x}_{k|k-1}\|^2.$$

Then the EKF algorithm (3.41) is an exponential observer for (3.42). Furthermore, the error decay constant $\delta_2 > \alpha$.

Chapter 4

Infinite-dimensional systems theory

Having reviewed the estimation theory for finite-dimensional systems, the question arises whether there are similar results available for infinite-dimensional systems. Indeed, the rest of this thesis will treat estimation for partial differential equations, which evolve in infinite-dimensional state-spaces.
This chapter presents the theory necessary to investigate infinite-dimensional systems.

Let again $\mathbb{H}$ denote a Hilbert space. An IVP for a time-invariant linear PDE can abstractly be written as

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + f(t) \\ z(t_0) = z_0 \end{cases} \tag{4.1}$$

where $A : \mathcal{D}(A) \longrightarrow \mathbb{H}$ is a linear operator and $f(t) : \mathbb{R} \longrightarrow \mathbb{H}$.

Recall that in finite dimensions, with $\mathbb{H} = \mathbb{R}^n$, the solution to (4.1) would be found by first solving the homogeneous IVP with $f(t) \equiv 0$ to obtain the homogeneous solution $z_{hom}(t) = \exp(A(t-t_0))z_0$ where $\exp(A(t-t_0))$ is the fundamental matrix. For continuous $f(t)$ the inhomogeneous solution is then (see Theorem 2.27)

$$z_{inh}(t) = \exp(A(t-t_0))z_0 + \int_{t_0}^{t} \exp(A(t-s))f(s)ds. \tag{4.2}$$

The first part of this chapter determines solutions for the linear infinite-dimensional IVP (4.1) following a similar procedure. To do that, the theory of *strongly continuous semigroups* is outlined. These serve as generalization of the fundamental matrix in linear ODE systems and display many of the same properties in infinite-dimensional spaces.

The second part of this section introduces *evolution operators*, which are a generalization of semigroups to the case of linearly time-varying IVPs

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + D(t)z(t) + f(t) \\ z(t_0) = z_0. \end{cases} \tag{4.3}$$

To allow for an intuitive understanding of C_0-semigroups and evolution operators the solutions to the abstract IVPs above will be determined gradually.

4.1 Semigroup theory

Definition 4.1 (Strongly continuous semigroup)
[34, Definition 2.1.2] An operator $T(\cdot) : \mathbb{R}_0^+ \to L(\mathbb{H})$ is a strongly continuous semigroup, or C_0-semigroup if

1. $T(t+s) = T(t)T(s)$ *for* $t, s > 0$,
2. $T(0) = I$,
3. $\|T(t)x_0 - x_0\| \to 0$ *as* $t \to 0^+$ *for all* $x_0 \in \mathbb{H}$.

The following is a useful property of C_0-semigroups that will be needed later.

Theorem 4.2
[34, Theorem 2.1.6] If $T(t)$ is a strongly continuous semigroup then there are constants $M_\omega \geq 1, \omega \geq 0$ such that

$$\|T(t)\| \leq M_\omega \exp(\omega t), \quad t \in \mathbb{R}_+.$$

Definition 4.3 (infinitesimal generator)
[34, Definition 2.1.8] The infinitesimal generator A of a C_0-semigroup is defined by

$$Ax_0 = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)x_0$$

whenever the limit exists. The domain $\mathcal{D}(A)$ is the set of all $x_0 \in \mathbb{H}$ for which the limit exists.

Definition 4.4 (closed operator)
[59, Theorem 4.13-3] Let $\mathbb{X}$ *and* $\mathbb{Y}$ *be normed linear spaces and let* $A : \mathcal{D}(A) \subset \mathbb{X} \longrightarrow \mathbb{Y}$ *be a linear operator.* A *is closed if for all sequences* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ *such that* $x_n \longrightarrow x$ *and* $Ax_n \longrightarrow y$ *it holds that* $x \in \mathcal{D}(A)$ *and* $Ax = y$.

Theorem 4.5
[34, Theorem 2.1.10][69, Theorem 2.4(d)] Let A *be the infinitesimal generator of a strongly continuous semigroup* $T(t)$ *on* $\mathbb{H}$*. Then the following properties hold:*

- $\mathcal{D}(A)$ *is dense in* $\mathbb{H}$,
- A *is a closed linear operator,*
- *for* $z_0 \in \mathcal{D}(A)$, $T(t)z_0 \in \mathcal{D}(A)$ *for all* $t \geq 0$,
- $\frac{d}{dt}T(t)z_0 = AT(t)z_0 = T(t)Az_0$ *for* $z_0 \in \mathcal{D}(A)$,
- $T(t)z - z = A\int_0^t T(s)zds$ *for all* $z \in \mathbb{H}$,
- $T(t)z_0 - z_0 = \int_0^t AT(s)z_0ds$ *for all* $z_0 \in \mathcal{D}(A)$.

With these properties it is clear that if A generates a C_0-semigroup $T(t)$ then the IVP

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) \\ z(t_0) = z_0, \quad z_0 \in \mathcal{D}(A) \end{cases} \tag{4.4}$$

has the differentiable solution $z(t) = T(t - t_0)z_0$, that is

$$\frac{dz(t)}{dt} = \frac{dT(t-t_0)z_0}{dt} = AT(t-t_0)z_0 = Az(t), \quad z(0) = T(0)z_0 = z_0.$$

Indeed, it is necessary for A to generate a C_0-semigroup in order for (4.4) to have a differentiable solution, as shows the following result.

Definition 4.6 (resolvent set)
[59, Definition 7.2-1] Let $A : \mathcal{D}(A) \longrightarrow \mathbb{H}$ *be a linear operator and let* I *denote the identity. The resolvent set is the set of all* $\lambda \in \mathbb{C}$ *such that* $(A - \lambda I)^{-1}$ *is a densely defined bounded operator.*

Theorem 4.7
[69, Theorem 4.1.3] Let $A : \mathcal{D}(A) \longrightarrow \mathbb{H}$ be densely defined and let A have non-empty resolvent set. The homogeneous IVP (4.4) has a unique continuously differentiable solution for all $z_0 \in \mathcal{D}(A)$ if and only if A is the infinitesimal generator of a C_0-semigroup.

The following is a useful criterion to determine if an operator generates a semigroup.

Theorem 4.8
[34, Corollary 2.2.3] If for a densely defined closed operator $A : \mathcal{D}(A) \subset \mathbb{H} \longrightarrow \mathbb{H}$ it holds that

$$\begin{aligned} \Re\{\langle Az, z\rangle\} &\leq \omega \|z\|^2 \quad \textit{for all } z \in \mathcal{D}(A), \\ \Re\{\langle A^* z, z\rangle\} &\leq \omega \|z\|^2 \quad \textit{for all } z \in \mathcal{D}(A^*), \end{aligned}$$

then A generates a strongly continuous semigroup $T(t)$ on $\mathbb{H}$ satisfying $\|T(t)\| \leq \exp(\omega t)$.

Example 4.9 (heat equation)
[82, Example 3.8.1] Set $\mathbb{H} = L^2(0,1)$ and consider the IVP

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2}, & t > 0, x \in \Omega = (0,1), \\ z(t,0) = z(t,1) = 0, & t \geq 0, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{4.5}$$

where for simplicity of exposition $z_0 \in \mathrm{C}([0,1], \mathbb{R})$. This so-called heat equation describes the distribution of temperature $z(t,x)$ of a bar located at $(0,1)$. Let $A = \frac{d^2}{dx^2}$ and define

$$\mathcal{D}(A) = \Big\{ \varphi \in \mathbb{H} \mid \frac{d^2}{dx^2}\varphi \in \mathbb{H} \textit{ and } \varphi(0) = \varphi(1) = 0 \Big\}.$$

Then the IVP (4.5) can be written as

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) \\ z(0) = z_0. \end{cases}$$

With the usual L^2-scalar product, integrating by parts yields for $\varphi, \psi \in \mathcal{D}(A)$

$$\langle A\varphi, \psi\rangle = \int_0^1 \left(\frac{d^2}{dx^2}\varphi(x) \right) \psi(x) dx = \int_0^1 \varphi(x) \left(\frac{d^2}{dx^2}\psi(x) \right) dx = \langle \varphi, A\psi\rangle. \tag{4.6}$$

Indeed, it can be shown that $\mathcal{D}(A) = \mathcal{D}(A^*)$ which implies that A is self-adjoint. Moreover, for $\varphi \in \mathcal{D}(A)$,

$$\langle A\varphi, \varphi \rangle = -\int_0^1 \left(\frac{d}{dx}\varphi(x) \right)^2 dx \le 0 \le \|\varphi\|_2^2$$

where $\|\cdot\|_2$ denotes the L^2-norm. By Theorem 4.8 A is the infinitesimal generator of a strongly continuous semigroup $T(t)$. It can be shown that the range of A is the whole space, that is $\mathcal{R}(A) = \mathbb{H}$ and that A has a bounded inverse. By Theorem 4.7 the unique differentiable solution of (4.5) is

$$z(t, x) = T(t)z_0(x).$$

Now consider the inhomogeneous IVP

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + f(t) \\ z(t_0) = z_0, \quad z_0 \in \mathcal{D}(A) \end{cases} \tag{4.7}$$

Recalling the solution in the finite-dimensional case (4.2) the questions arises whether a similar representation can be found for the solution of (4.7) in infinite-dimensional spaces. Indeed this is the case, as is stated in the following result.

Theorem 4.10
[69, Corollary 2.5] Let A be the infinitesimal generator of a C_0-semigroup and let $f(t) \in \mathrm{C}^1([t_0, t_f], \mathbb{H})$.
For every $z_0 \in \mathcal{D}(A)$ and $t \in [t_0, t_f]$ the unique differentiable solution to the IVP (4.7) is

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - s)f(s)ds. \tag{4.8}$$

In many cases the inhomogeneity $f(t)$ may not even be continuous. Naturally, looking at (4.7), this implies that the solution can not be differentiable.
However, there is a weaker concept of solutions, which, much like weak solutions, relies on the integral formulation (4.8).

Definition 4.11
[69, Definition 4.2.3] Let A be the infinitesimal generator of a C_0-semigroup and let $f(t) \in L^p([t_0, t_f], \mathbb{H})$, $p \ge 1$.
The function $z(t) \in \mathrm{C}([t_0, t_f], \mathbb{H})$ given by (4.8) is the mild solution to (4.1)on $[t_0, t_f]$.

A concept equivalent to mild solutions are *weak solutions*, which are only briefly presented here.

Definition 4.12
[32, Definition 2.25] Under the assumptions of Theorem 4.11 $z(t)$ is called a weak solution to (4.1) on $[t_0, t_f]$ if

1. $z(t) \in \mathrm{C}([t_0, t_f], \mathbb{H})$ *and*
2. *for all* $g(t) \in \mathrm{C}([t_0, t_f], \mathbb{H})$ *it holds that*

$$\int_{t_0}^{t_f} \langle z(t), g(t)\rangle ds - \int_{t_0}^{t_f} \langle f(t), \int_t^{t_f} T^*(s-t)g(s)ds\rangle dt - \int_{t_0}^{t_f} \langle z_0, T^*(s-t)g(s)ds\rangle = 0. \tag{4.9}$$

Proposition 4.13
[32, Proposition 2.26] Let A be the infinitesimal generator of a C_0-semigroup and let $f(t) \in L^p([t_0, t_f], \mathbb{H})$, $p \geq 1$.
For every $z_0 \in \mathcal{D}(A)$ the IVP (4.1) has a unique weak solution, which is the mild solution (4.8).

Proof. Changing the order of integration in (4.9) yields

$$\int_0^{t_f} \langle z(t) - T(t-t_0)z_0 - \int_{t_0}^{t} T(t-s)f(s)ds, g(t)\rangle dt. \tag{4.10}$$

Therefore it is clear that a mild solution to IVP (4.1) is also a weak solution.
To prove uniqueness, assume there are two weak solutions $z_1(t)$ and $z_2(t)$. Then for all $g(t) \in \mathrm{C}([t_0, t_f], \mathbb{H})$ their difference satisfies

$$\int_{t_0}^{t_f} \langle z_1(t) - z_2(t), g(t)\rangle dt = 0 \tag{4.11}$$

and so $z_1(t) = z_2(t)$. □

A useful result for linear perturbations of C_0-semigroups is stated.

Theorem 4.14
[69, Corollary 3.1.3] If A is the infinitesimal generator of a semigroup of linear operators and D is a linear bounded operator on $\mathbb{H}$, then $A + D$ is the infinitesimal generator of a semigroup of linear operators on $\mathcal{D}(A)$.

Now let $\mathbb{H}$ denote a real Hilbert space. Consider (4.7) with a constant perturbation, that is

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + Dz(t) + f(t) \\ z(t_0) = z_0, \quad z_o \in \mathcal{D}(A). \end{cases} \tag{4.12}$$

where $D \in L(\mathbb{H})$. With Theorem 4.14 and the results stated above it is clear that letting $T_D(t)$ be the C_0-semigroup generated by $A + D$ the solution for (4.12) is

$$z(t) = T_D(t - t_0)z_0 + \int_{t_0}^{t} T_D(t - s)f(s)ds,$$

where $z(t)$ is a mild solution if $f(t) \in L^p([t_0, t_f], \mathbb{H})$ and differentiable if $f(t)$ is differentiable.

4.2 Evolution operators

In order to determine solutions in for a class of time-varying systems a generalization of C_0-semigroups is needed.

Definition 4.15 (mild evolution operator)
[34, Definition 3.2.4] Defining

$$\Delta(t_f) := \{(t, s) : 0 \leq s \leq t \leq t_f\},$$

$U : \Delta(t_f) \to L(\mathbb{H})$ *is a mild evolution operator if*

1. $U(s, s) = I$ *for* $s \in [0, t_f]$, *where* I *is the identity,*
2. $U(t, r)U(r, s) = U(t, s)$ *for* $0 \leq s \leq r \leq t \leq t_f$,
3. $U(t, s)$ *is strongly continuous in* s *on* $[0, t]$ *and in* t *on* $[s, t_f]$.

Definition 4.16 (quasi-evolution operator)
[32, Definition 2.35] A quasi-evolution-operator is a mild evolution operator $U : \Delta(t_f) \longrightarrow L(\mathbb{H})$ *such that there exists* $x \in \mathbb{H}, x \neq 0$ *and a closed linear operator* $A(s)$ *on* $\mathbb{H}$ *for almost all* $s \in [0, t_f]$ *satisfying*

$$U(t, s)x - x = \int_{s}^{t} U(t, \tau)A(\tau)xd\tau. \tag{4.13}$$

The set of all $x \in \mathbb{H}$ *for which (4.13) holds is denoted by* $\mathcal{D}_A$ *and* $A(s)$ *is called the quasi generator of* $U(t, s)$.

Let $\mathbb{K}$ be another real Hilbert space. Define

$$\mathbb{B}_\infty([t_0,t_f],L(\mathbb{K},\mathbb{H})) = \{D : [t_0,t_f] \longrightarrow L(\mathbb{K},\mathbb{H}) | D(\cdot)x \text{ is strongly measurable for all } x \in \mathbb{K} \text{ and } \operatorname*{ess\,sup}_{[t_0,t_f]} \|D(t)\| < \infty\}.$$

Theorem 4.17
[31, Theorem 1.1 & Theorem 1.2] Let $U(t,s)$ be a mild evolution operator on $\Delta(t_f)$ and let $D \in \mathbb{B}_\infty([s,t],L(\mathbb{H}))$. Then there exists a unique mild evolution operator $U_D(t,s)$ solving

$$U_D(t,s)z_0 = U(t,s)z_0 + \int_s^t U(t,r)D(r)U_D(r,s)z_0 dr \quad z_0 \in \mathbb{H} \tag{4.14}$$

If $U(t,s)$ is a quasi-evolution operator, then $U_D(t,s)$ is also a quasi-evolution operator.

Theorem 4.18
[31, Corollary 1.2] The unique solution $U_D(t,s)$ to (4.14) is also the unique solution to

$$U_D(t,s)z_0 = U(t,s)z_0 + \int_s^t U_D(t,r)D(r)U(r,s)z_0 dr \quad z_0 \in \mathbb{H}.$$

Now, to see why mild evolution operators generalize strongly continuous semigroups and how mild solutions to IVPs containing time-varying operators can be defined, consider

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + D(t)z(t) \\ z(t_0) = z_0, \quad z_0 \in \mathcal{D}(A) \end{cases} \tag{4.15}$$

where A generates the C_0-semigroup $T(t)$ and $D(t) \in \mathbb{B}_\infty([t_0,t_f],\mathbb{H})$.
It follows immediately from the properties of a C_0-semigroup stated in Theorem 4.5 that $U(t,s) = T(t-s)$ is a mild evolution operator. Letting $U_D(t,s)$ be the mild evolution operator given by (4.14) and defining $z(t) = U_D(t,t_0)z_0$ yields

$$\begin{aligned} z(t) &= U_D(t,t_0)z_0 \\ &= U(t,t_0)z_0 + \int_s^t U(t,r)D(r)U_D(r,s)z_0 dr \\ &= U(t,t_0)x_0 + \int_{t_0}^t U(t,r)D(r)z(r)dr. \end{aligned}$$

Hence, $z(t)$ satisfies the integral formulation (4.8) and therefore, by Definition 4.11, it is the mild solution to the homogeneous perturbed IVP (4.15).

More regularity can be obtained in this case.

Corollary 4.19
If A is the generator of a C_0-semigroup $T(t)$, $U(t,s) = T(t-s)$ and $D \in \mathbb{B}_\infty([s,t], L(\mathbb{H}))$ then $U_D(t,s)$ given by (4.14) is a quasi-evolution operator and called the quasi-evolution operator generated by $A + D(t)$.

Finally, consider the perturbed inhomogeneous IVP

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + D(t)z(t) + f(t) \\ z(t_0) = z_0, \quad z_0 \in \mathcal{D}(A), \end{cases} \tag{4.16}$$

where A generates the C_0-semigroup $T(t)$, $f(t) \in L^p([t_0, t_f], \mathbb{H})$ for some integer $p \geq 1$ and $D(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$.
With the results outlined above it is now possible to define a mild solution for (4.16).

Definition 4.20
[34, Definition 3.2.9] Let $U_D(t,s)$ be the quasi-evolution operator generated by $A + D(t)$. The mild solution to (4.16) on $[t_0, t_f]$ is

$$z(t) = U_D(t, t_0) + \int_{t_0}^{t} U_D(t,s) f(s) ds.$$

Chapter 5

Filtering for linear stochastic infinite-dimensional systems

Due to the success of the Kalman filter for linear finite-dimensional systems, the result was gradually extended to linear PDEs. One of the first attempts to do so was made in [8], where the authors considered a second order parabolic operator and derived an estimator via a quadratic cost function. Similarly in [81] the author considered a nonlinear hyperbolic or parabolic PDE and an appropriate error functional to derive, with methods of calculus of variations, a Hamilton-Jacobi equation for the minimizer of an error functional.
In [6, 42, 84] a stochastic PDE (SPDE) is considered and, with a derivation and result analogous to the finite-dimensional KF, the linear gain operator is obtained via an integral Riccati equation. Gaussian noise is considered as disturbance and in [80] specific measurement points are taken into consideration. A detailed survey on early approaches to infinite-dimensional filtering can by found in [28].

Since for infinite-dimensional linear systems both the KF and the linear quadratic (LQ) cost problem require solving a Riccati equation analogous to the finite-dimensional case, different formulations of this equation have been studied in Banach and Hilbert spaces. As early as 1966, P. Falb and D. Kleinman in [43] investigate some properties of the infinite-dimensional Riccati equation (IDRE) using variational principles. Beginning in the 1970s a lot of research on the IDRE was done by R. Curtain and A. Pritchard, for example in [31, 30] where the IDRE was first investigated in the context of evolution operators, which serve as a generalization of semigroups for abstract (stochastic) evolution equations. The same authors established the Kalman filter in Hilbert spaces, see [26, 27, 29, 32, 34].

The filtering problem was independently solved by A. Bensoussan for a large class of PDE systems in [12]. In [15] a filter for a linear PDE system with pointwise disturbances is derived.

For numerical reasons infinite-dimensional systems are often approximated by finite-dimensional sub-systems, also referred to as *lumped* systems, where, depending on whether the approximation is made before or after solving the respective equations, it is distinguished between early and late lumping, respectively. Although there have been early convergence results of Kalman-Bucy type equations, still mainly finite-dimensional approximations are used to deal with PDEs, see for example Germani et al. [45].
J. Gibson investigated the IDRE in the context of LQ control with many contributions to their approximation properties, see for instance [47, 48]. Recent results on approximation methods for the IDRE are given in [33, 87] in the setting of Banach spaces. Only very recently, in both the finite-dimensional and the infinite-dimensional case the convergence of the discrete Kalman filter to the continuous estimator equations has been shown in [1] by A. Aalto, who therein states that “however, surprisingly, no result exists on the rate of convergence”. The KF in infinite dimensions has been applied in various areas, for example in determining the optimal sensor location [66, 87].

The first part of this chapter presents the basic filtering theory for infinite-dimensional linear systems and the results stated here can be found in [26, 29, 32], which, to the best of the authors knowledge, provide the most detailed derivation of the Kalman filter for Wiener process disturbances in infinite dimensions.
The second part outlines the most relevant results on the Riccati equations, which will be of use in later chapters.

5.1 The Kalman filter in Hilbert spaces

Let $\mathbb{H}, \mathbb{K}$ be separable Hilbert spaces and consider integral abstract signal processes

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Ddw(s) \tag{5.1}$$

where $T(t)$ is a C_0-semigroup with generator A, $w(t)$ is a $\mathbb{K}$-valued Wiener process with covariance operator $W \in L(\mathbb{K})$, $D \in L(\mathbb{K}, \mathbb{H})$ and $x_0 \in L^2(\Omega, \mu_p; \mathbb{H})$ is a Gaussian random variable with zero mean and covariance operator P_0. The observation process is

$$y(t) = \int_0^t Cx(s)ds + Fv(t) \tag{5.2}$$

where $C \in L(\mathbb{H}, \mathbb{R}^k)$, $F, F^{-1} \in L(\mathbb{R}^k)$ and $v(t)$ is a $\mathbb{R}^k$-valued Wiener process with covariance $V \in L(\mathbb{R}^k)$ such that $V^{-1} \in L(\mathbb{R}^k)$. The random variables $x_0, v(t), w(t)$ are independent for all t.

Under some conditions $x(t)$ and $y(t)$ can be defined as strong solutions of stochastic PDEs, analogous to the processes in Section 3.2, but this is more technical. Instead, $x(t)$ and $y(t)$ are treated as stochastic integral processes, meaning that for each time t, $x(t)$ and $y(t)$ are random variables given by (5.1) and (5.2) respectively.

Due to the continuous sample paths of Wiener processes (see Definition 2.18) some regularity exists.

Proposition 5.1
The processes $x(t)$ and $y(t)$ in (5.1) and (5.2) respectively have continuous sample paths. That is, the maps

$$t \mapsto x(t)$$
$$t \mapsto y(t)$$

are continuous with probability 1 (Definition 2.15).
Furthermore, $\mathsf{E}\{x(t)\} = \mathsf{E}\{y(t)\} = 0$ *for all* $t \geq 0$.

Note that, as explained in Chapter 1 and Chapter 2 the spatial dependency in stochastic processes is omitted.

Proof. By definition the semigroup $T(t)$ is continuous and hence

$$t \mapsto \int_0^t T(t-s)Ddw(s)$$

has continuous sample paths by Lemma 2.25. Therefore $x(t)$ has continuous sample paths, implying that $y(t)$, too, has continuous sample paths.
Furthermore,

$$\mathsf{E}\{x(t)\} = T(t)\underbrace{\mathsf{E}\{x_0\}}_{=0} + \mathsf{E}\Big\{\int_0^t T(t-s)Ddw(s)\Big\} = 0$$

since the second term is zero by Lemma 2.25. Similarly, $\mathsf{E}\{y(t)\} = 0$. □

Let

$$\mathbb{B}^2([t_0,t_f],L(\mathbb{K},\mathbb{H})) = \{B : [t_0,t_f] \longrightarrow L(\mathbb{K},\mathbb{H}), \text{ such that } B(t) \text{ is strongly measurable and } \int_{t_0}^{t_f} \|B(t)\|^2 dt < \infty\}.$$

The aim of the filtering problem is to find the best linear filter, that is, an estimator for the stochastic process $x(t)$ given by (5.1) based on the observation $Y_t = \{y(s) \mid 0 \le s \le t\}$ determined by (5.2).
To do that, linear estimates

$$\hat{x}(t) = \int_0^t K(t,s)Cx(s)ds + \int_0^t K(t,s)F dv(s) \tag{5.3}$$

are considered, where the integral kernel $K(t,\cdot) \in \mathbb{B}^2([0,t],L(\mathbb{R}^k,\mathbb{H}))$.

Definition 5.2
Let $y(t)$ be given by (5.2) and let $B(t) \in \mathbb{B}([0,t_f],L(\mathbb{R}^k,\mathbb{H}))$ for some $t_f > 0$. Define the integral of $B(t)$ with respect to $y(t)$ for $t \in [0,t_f]$ as

$$\int_0^t B(s)dy(s) = \int_0^t B(s)Cx(s)ds + \int_0^t B(s)F dv(s). \tag{5.4}$$

With Definition 5.2 the filter (5.3) is henceforth written as

$$\hat{x}(t) = \int_0^t K(t,s)dy(s).$$

Definition 5.3 (best linear filter)
The best linear filter of $x(t)$ given by (5.1) based on the observation process $y(t)$ given by (5.2) is

$$\hat{x}(t) = \int_0^t K(t,s)dy(s) \tag{5.5}$$

where $K(t,\cdot) \in \mathbb{B}^2([0,t],L(\mathbb{R}^k,\mathbb{H}))$ for almost all t and $K(t,s)$ is such that

$$\mathsf{E}\big\{\langle x(t) - \hat{x}(t), h\rangle^2\big\} \tag{5.6}$$

is minimized for all $h \in \mathbb{H}$.

The procedure of finding the best integral kernel $K(t,s)$ is very similar to the procedure applied for finite-dimensional systems in Section 3.2. Nevertheless there is one significant difference: the best linear filter (5.5) is not defined to be the conditional expectation. The reason for this is that, as the processes $x(t)$ and $y(t)$ map to the Hilbert spaces $\mathbb{H}$ and $\mathbb{R}^k$ respectively, the expression

$$\mathsf{E}\{\underbrace{x(t)}_{\in L^2(\Omega,\mu_p;\mathbb{H})} |\sigma(\underbrace{Y_t}_{\subset L^2(\Omega,\mu_p;\mathbb{R}^k)})\} \tag{5.7}$$

is not easy to justify. As $\mathbb{R}^k$ is not a subspace of $\mathbb{H}$, it would be necessary to introduce appropriate isomorphisms between $L^2(\Omega,\mu_p;\mathbb{R}^k)$ and $L^2(\Omega,\mu_p;\mathbb{H}_{sub})$, where $\mathbb{R}^k \cong \mathbb{H}_{sub} \subset \mathbb{H}$, that is, where $\mathbb{H}_{sub}$ is a subspace of $\mathbb{H}$ that is isomorphic to $\mathbb{R}^k$. Defining these isomorphisms is very technical and they are delicate to work with. Hence it is not shown that the best linear filter can also be understood as conditional expectation, similar to the finite-dimensional case. For this result the reader is referred to [32].

In the finite-dimensional filtering problem (see Section 3.2) the estimator is optimal if it minimizes the trace of the error covariance matrix

$$\operatorname{tr}\left\{\mathsf{Cov}\{e(t)\}\right\} = \sum_{i\in\mathbb{I}} \left\langle \mathsf{E}\{e(t)\circ e(t)\}b_i, b_i\right\rangle \tag{5.8}$$

where $e(t) = x(t) - \hat{x}(t)$ is the error and $\{b_i\}_{i\in\mathbb{I}}$ is an orthonormal basis of the state-space for some finite index set $\mathbb{I}$.
In the infinite-dimensional case, where $\mathbb{I} = \mathbb{N}$, the question arises whether the best linear filter that minimizes (5.6) also minimizes (5.8). This is indeed the case.
To see this it suffices to observe that

$$\begin{aligned}\left\langle \mathsf{E}\big\{e(t)\circ e(t)\big\}b_i, b_i\right\rangle &= \mathsf{E}\big\{\big\langle\big(e(t)\circ e(t)\big)b_i, b_i\big\rangle\big\} \\ &= \mathsf{E}\big\{\big\langle e(t), b_i\big\rangle^2\big\} \\ &\geq 0\end{aligned}$$

by the definition of the operation $\circ$. Thus, every individual term in the sum (5.8) is non-negative. Hence an estimator that minimizes (5.6) for all $h = b_i$, i.e. that minimizes all individual terms in (5.8), consequently also minimizes (5.8).
In other words, the best linear filter with respect to Definition 5.5 also minimizes the trace of the error covariance.

The best linear filter satisfies the same orthogonality condition as its finite-dimensional analogue.

Lemma 5.4 (orthogonal projection lemma)
[32, Lemma 6.2] Let $\hat{x}(t)$ be given by (5.5), $x(t)$ and $y(t)$ be given by (5.1) and (5.2) respectively and define the error $e(t) = x(t) - \hat{x}(t)$.
The function $\hat{x}(t)$ is the best linear filter for $x(t)$ if and only if

$$\mathsf{E}\{e(t) \circ \big(y(\alpha) - y(\beta)\big)\} = 0 \quad \textit{for all} \quad 0 \leq \beta \leq \alpha \leq t.$$

Proof. Given any $h \in \mathbb{H}$, by defining the space

$$\mathbb{L}^2(h) = \{\langle f, h\rangle : f \in L^2(\Omega, \mu_p; \mathbb{H})\}$$

and the scalar product

$$[\langle u, h\rangle, \langle v, h\rangle] = \mathsf{E}\{\langle u, h\rangle\langle v, h\rangle\} \tag{5.9}$$

the pair $(\mathbb{L}^2(h), [\cdot, \cdot])$ becomes a Hilbert space. Fix a time t and define the subspace $\mathbb{L}^2(h|t) \subset \mathbb{L}^2(h)$ with functions of the form (5.5) as

$$\mathbb{L}^2(h|t) = \Big\{\langle x(t), h\rangle \mid x(t) = \int_0^t B(t,s)dy(s) \text{ for some } B(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))\Big\}$$

where the integral with respect to $y(t)$ is defined in Definition 5.2. Since t is fixed, the spaces $\mathbb{L}^2(h)$ and $\mathbb{L}^2(h|t)$ both contain square-integrable $\mathbb{H}$-valued random variables.

By Definition 5.3 the best linear filter minimizes (5.6) over the set $\mathbb{L}^2(h|t)$. With the scalar product (5.9), this cost function (5.6) can be rewritten as

$$\mathsf{E}\{\langle x(t) - \hat{x}(t), h\rangle^2\} = [\langle x(t) - \hat{x}(t), h\rangle, \langle x(t) - \hat{x}(t), h\rangle].$$

Since $\langle x(t), h\rangle \in \mathbb{L}^2(h)$ and $\langle \hat{x}(t), h\rangle \in \mathbb{L}^2(h|t)$ and both spaces are Hilbert spaces, by Theorem 3.7 the minimal error

$$\langle e(t), h\rangle = \langle x(t) - \hat{x}(t), h\rangle \in \mathbb{L}^2(h)$$

has to be orthogonal to $\mathbb{L}^2(h|t)$ with respect to the $[\cdot, \cdot]$ scalar product. That means that

$$[\langle e(t), h\rangle, \langle z(t), h\rangle] = \mathsf{E}\big\{\langle e(t), h\rangle, \langle z(t), h\rangle\big\} = 0$$

for all $\langle z(t), h\rangle \in \mathbb{L}^2(h|t)$. Now, recalling from the definition of covariance (Definition 2.9) that for all $h \in \mathbb{H}$ $(u \circ v)(h) = u\langle v, h\rangle \in \mathbb{H}$ it is clear that

$$\mathsf{E}\big\{\langle u, h\rangle\langle v, h\rangle\big\} = [\langle u, h\rangle, \langle v, h\rangle] = 0 \quad \text{for all } h \in \mathbb{H} \quad \Leftrightarrow \quad \mathsf{E}\{u \circ v\} = 0.$$

So it is only necessary to show that

$$\mathsf{E}\{e(t) \circ z(t)\} = 0 \quad \text{for all } z(t) \in \mathbb{L}^2(h|t) \tag{5.10}$$

if and only if for all $0 \leq \beta \leq \alpha \leq t$

$$\mathsf{E}\big\{e(t) \circ \big(y(\alpha) - y(\beta)\big)\big\} = 0. \tag{5.11}$$

Suppose first that (5.11) holds. Then if $B_0(t,s)$ is a step function in s, that is

$$B_0(t,s) = \sum_{i=0}^{n-1} f_i(t)\mathbb{1}_{[s_i,s_{i+1})}(s) \quad \text{for} \quad 0 = s_0 \leq \cdots \leq s_n = t$$

for some $f_i(t)$ such that $B_0(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))$, then for

$$z(t) = \int_0^t B(t,s)dy(s), \quad z(t) \in \mathbb{L}(h|t),$$

(5.10) holds true since

$$\begin{aligned}\mathsf{E}\{e(t) \circ z(t)\} &= \mathsf{E}\Big\{e(t) \circ \int_0^t \sum_{i=0}^{n-1} f_i(t)\mathbb{1}_{[s_i,s_{i+1})}(s)dy(s)\Big\} \\ &= \sum_{i=1}^{n-1} f_i(t)\underbrace{\mathsf{E}\big\{e(t) \circ \big(y(s_{i+1}) - y(s_i)\big)\big\}}_{=0 \text{ since (5.11) holds}} \\ &= 0.\end{aligned}$$

Hence (5.10) holds for all step functions in s. By approximation with step functions, (5.10) is easily shown to hold for all $B(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))$.
Conversely, suppose that (5.10) holds, but (5.11) does not for some β, α. Choose

$$B_0(t,s) = \begin{cases} \mathsf{E}\big\{e(t) \circ \big(y(\alpha) - y(\beta)\big)\big\} & 0 \leq \beta \leq s \leq \alpha \leq t \\ 0 & \text{otherwise.}\end{cases}$$

For such $B_0(t,s)$ clearly $z(t) \in \mathbb{L}^2(h|t)$. However, by observing that

$$z(t) = \int_0^t B_0(t,s)dy(s) = \underbrace{\mathsf{E}\big\{e(t) \circ \big(y(\alpha) - y(\beta)\big)\big\}}_{=\text{constant in } s}\big(y(\alpha) - y(\beta)\big)$$

it follows that

$$\mathsf{E}\{e(t) \circ z(t)\} = \mathsf{E}\big\{e(t) \circ \big(y(\alpha) - y(\beta)\big\}^2 \neq 0$$

and therefore (5.10) cannot hold either. Hence the theorem is proven. □

Before proving an important result another lemma is needed, the proof of which is long and only consist of technical calculations. Therefore the proof is omitted.

Lemma 5.5
[26, Lemma 2.2] Let $x(t)$ be given by (5.1) and let $\hat{x}(t)$ be given by (5.5) where $K(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))$. Let further

$$\Lambda(t,\tau) = \mathsf{E}\{x(t) \circ x(\tau)\}.$$

Then it holds that

$$\mathsf{E}\{\hat{x}(t) \circ x(\tau)\} = \mathsf{E}\Big\{ \int_0^t K(t,s)dy(s) \circ x(\tau)\Big\} = \int_0^t K(t,s)C\Lambda(s,\tau)ds.$$

Lemma 5.6
[32, Lemma 6.7] Let $\Lambda(t,s)$ be as above and let C, F and V be as defined in (5.2). The integral equation

$$\int_0^t K(t,s)C\Lambda(s,\tau)C^* ds + K(t,\tau)FVF^* = \Lambda(t,\tau)C^*. \tag{5.12}$$

has a unique solution $K(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))$.

The following result is a key step in the derivation of the best linear filter, as it determines the optimal integral kernel to be the solution of this integral equation (5.12).

Theorem 5.7
[26, Theorem 2.3] Let the assumptions of Lemma 5.6 hold. The estimator (5.3) is the best linear filter if and only if its kernel $K(t,s)$ satisfies the integral equation (5.12).

Proof. Suppose first that $\hat{x}(t)$ is the best linear filter with $K(t,\cdot) \in \mathbb{B}^2([0,t], L(\mathbb{R}^k, \mathbb{H}))$. For $0 \leq \tau \leq t$ define

$$\rho(\tau) = \int_0^\tau Cx(s)ds = y(\tau) - Fv(\tau).$$

Recall that $x(t)$ has continuous sample paths by Proposition 5.1. In other words, the map

$$t \mapsto x(t)$$

is continuous with probability 1, or if the spatial dependence were not omitted, then for almost all $\omega \in \Omega$ the map

$$t \mapsto x(t,\omega)$$

is continuous. Consequently, for all such $\omega \in \Omega$ the map

$$t \mapsto \rho(t,\omega) = \int_0^t Cx(s,\omega)ds$$

is continuously differentiable with

$$\frac{d}{dt}\rho(t,\omega) = Cx(t,\omega),$$

or equivalently,

$$\frac{d}{dt}\rho(t) = Cx(t)$$

with probability 1. Now let the error $e(t) = x(t) - \hat{x}(t)$. Then

$$\begin{aligned}
\frac{d}{d\tau}\mathsf{E}\{e(t)\circ\rho(\tau)\} &= \mathsf{E}\{e(t)\circ Cx(\tau)\} \\
&= \mathsf{E}\{e(t)\circ x(\tau)\}C^* \\
&= \mathsf{E}\{x(t)\circ x(\tau)\}C^* - \mathsf{E}\Big\{\int_0^t K(t,s)dy(s)\circ x(\tau)\Big\}C^* \\
&= \Lambda(t,\tau)C^* - \int_0^t K(t,s)C\Lambda(s,\tau)C^*ds \qquad (5.13)
\end{aligned}$$

where in the last steps Lemma 5.5 was used. However

$$\begin{aligned}
\mathsf{E}\{e(t)\circ\rho(\tau)\} &= \mathsf{E}\big\{e(t)\circ\big(y(\tau)-y(0)-Fv(\tau)\big)\big\} \qquad (5.14)\\
&= \underbrace{\mathsf{E}\big\{e(t)\circ\big(y(\tau)-y(0)\big)\big\}}_{=0 \text{ by Theorem 5.4}} - \mathsf{E}\{e(t)\circ Fv(\tau)\} \\
&= -\mathsf{E}\{x(t)\circ Fv(\tau)\} + \mathsf{E}\Big\{\int_0^t K(t,s)Cx(s)ds\circ\int_0^t Fdv(s)\Big\} \qquad (5.15)\\
&+ \mathsf{E}\Big\{\int_0^t K(t,s)Fdv(s)\circ\int_0^\tau Fdv(\tau)\Big\} \qquad (5.16)
\end{aligned}$$

The first term in (5.15) is zero since

$$\mathsf{E}\{x(t)\circ Fv(\tau)\} = T(t)\underbrace{\mathsf{E}\{x_0\circ Fv(\tau)\}}_{=0 \text{ by independence}} + \underbrace{\mathsf{E}\Big\{\int_0^t T(t-s)Ddw(s)\circ\int_0^\tau Fdv(s)\Big\}}_{=0 \text{ by Lemma 2.25, property (5.)}} = 0.$$

Similarly, for the second term in (5.15)

$$\begin{aligned}
\mathsf{E}\Big\{\int_0^t K(t,s)Cx(s)ds \circ \int_0^t Fdv(s)\Big\} &= \mathsf{E}\Big\{\int_0^t K(t,s)CT(s)x_0 ds \circ Fv(\tau)\Big\} \\
&+ \mathsf{E}\Big\{\int_0^t K(t,s)C\int_0^s T(s-r)Ddw(r)ds \circ \int_0^\tau Fdv(s)\Big\} \\
&= \int_0^t K(t,s)CT(s) \underbrace{\mathsf{E}\{x_0 \circ Fv(\tau)\}}_{=0 \text{ by independence}} ds \\
&+ \int_0^t K(t,s)C \underbrace{\mathsf{E}\Big\{\int_0^s T(s-r)Ddw(r) \circ \int_0^\tau Fdv(s)\Big\}}_{=0 \text{ by Lemma 2.25, property (5.)}} ds \\
&= 0.
\end{aligned}$$

Using the Wiener integral property (4.) of Lemma 2.25, the third term (5.16) yields

$$\mathsf{E}\Big\{\int_0^t K(t,s)Fdv(s) \circ \int_0^\tau Fdv(\tau)\Big\} = \int_0^\tau K(t,s)FVF^* ds \tag{5.17}$$

and finally equating the initial term in (5.14) with (5.17),

$$\mathsf{E}\{e(t) \circ \rho(\tau)\} = \int_0^\tau K(t,s)FVF^* ds. \tag{5.18}$$

Taking the derivative of (5.18) results in

$$\frac{d}{d\tau}\mathsf{E}\{e(t) \circ \rho(\tau)\} = K(t,\tau)FVF^*. \tag{5.19}$$

Equating (5.13) and (5.19) yields (5.12).
Conversely assume that $K(t,s)$ satisfies (5.12) for $0 \le s \le t$. By the orthogonality projection lemma 5.4 is suffices to show that

$$\mathsf{E}\big\{e(t) \circ \big(y(\tau) - y(\sigma)\big)\big\} = 0$$

for all $0 \le \sigma \le \tau \le t$. Due to the linearity of the expectation let without loss of generality $\sigma = 0$. Then

$$\mathsf{E}\big\{e(t) \circ \big(y(\tau) - y(0)\big)\big\} = \mathsf{E}\{e(t) \circ \rho(\tau)\} + \mathsf{E}\{e(t) \circ Fv(\tau)\} = 0$$

by the previous computation (5.14). Hence, the lemma is proven. □

The proof of the following is omitted due to its length. It is very similar to the proof of Theorem 3.16.

Theorem 5.8
[32, Theorem 6.9] The unique solution to (5.12) is

$$K(t,s) = U(t,s)P(s)C^*(FVF^*)^{-1}$$

where the mild evolution operator $U(t,s)$ is the perturbation of $T(t)$ by $-P(t)C^*(FVF^*)^{-1}C$ and $P(t)$ is the unique solution to

$$\begin{cases} \frac{d}{dt}\langle P(t)h,k\rangle = \big\langle \big(AP(t) + P(t)A^* + DWD^* - P(t)C^*(FVF^*)^{-1}CP(t)\big)h, k\big\rangle \\ P(0) = P_0 \end{cases} \tag{5.20}$$

for all $h, k \in \mathcal{D}(A^*)$.
Thus, the best linear filter is

$$\hat{x}(t) = \int_0^t U(t,s)P(s)C^*(FVF^*)^{-1}dy(s).$$

Analogous to the finite-dimensional Kalman filter (see Section 3.2), the solution to the Riccati equation is given by the error covariance.

Corollary 5.9
Let the operator $P(t)$ be as above and let again the filtering error $e(t) = x(t) - \hat{x}(t)$. Then

- $P(t) = \mathsf{E}\{e(t) \circ e(t)\}$
- $\mathsf{E}\{\|e(t)\|^2\} = \mathrm{tr}\{P(t)\}$.

The question remains under what conditions the solution $P(t)$ of the Riccati equation (5.20) is nuclear. A sufficient condition requires the initial condition P_0 to be nuclear. This result is given in the following section, which treats the Riccati equations arising in optimal filtering for linear infinite-dimensional systems in the more general case of systems defined via mild evolution operators.

5.2 The infinite-dimensional Riccati equation in linear-quadratic control and linear filtering

The linear filtering problem in both finite and infinite dimensions yields Riccati operator equations for the error covariance that determine the optimal filter gain. The same holds true for the *linear quadratic* (LQ) control problem, which seeks to find the optimal control minimizing a quadratic cost function for linear systems.

In both contexts a lot of research has been done on the infinite-dimensional Riccati equation (IDRE) [7, 29, 32, 42, 46, 80, 84].

In Chapter 7 an estimator will be designed for a semi-linear system, which will be based on the solution of a IDRE under weaker conditions than in the previous Section 5.1. This section presents the most relevant results of different versions of the IDRE that will be needed later. To provide a context for these results the role of the IDRE will be outlined in the context of both LQ-control and the filtering problem in infinite dimensions.

In the following let again $\mathbb{H}$ denote a real Hilbert space and let $\mathbb{U}$ and $\mathbb{Y}$ be two Hilbert spaces serving as input and output space respectively. Let $P_{0,c} \in L(\mathbb{H})$, $W(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ and $R_c(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ be self-adjoint, nonnegative operators for all t. Let $R_c(t)$ satisfy

$$\langle y, R_c(t)y\rangle \geq \kappa_c \|y\|$$

for all t and some $\kappa_c > 0$.

In [31], a paper by R. Curtain and A. J. Pritchard on optimal control for linear infinite-dimensional systems and similarly in a later work by them [32, Chapter 4], as well as in a paper by J. S. Gibson [46], the cost functional

$$J(u; t_0, t) = \langle z(T), P_{0,c}z(T)\rangle + \int_{t_0}^{t} \big(\langle z(s), W(s)z(s)\rangle + \langle u(s), R_c(s)u(s)\rangle\big)ds$$

was considered for the control system

$$z(t) = U_c(t, t_0)z_0 + \int_{t_0}^{t} U_c(t, s)B(s)u(s)ds \tag{5.21}$$

where $U_c(t, t_0)$ is a mild evolution operator and $B(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$.

Theorem 5.10
[31, Theorem 2.3] Let the operators satisfy the assumptions stated above and let $U_c(t, t_0)$

be a mild evolution operator.
There is a unique, self-adjoint operator $P_c(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ such that $P_c(t)$ satisfies for all $x \in \mathbb{H}$

$$P_c(t)x = V_c^*(t_f, t)P_{0,c}V_c(t_f, t)x + \int_t^{t_f} V_c^*(s,t)\big(W(s) + P_c(s)B(s)R_c^{-1}(s)B^*(s)P_c(s)\big)V_c(s,t)xds \tag{5.22}$$

where $V_c(t, s)$ is the perturbation of $U_c(t, s)$ by $-B(t)R_c^{-1}(t)B^*(t)P_c(t)$, i.e.

$$V_c(t, s)x = U_c(t, s)x - \int_s^t U_c(t,r)B(r)R_c^{-1}(r)B^*(r)P_c(r)V_c(r, s)xds \quad x \in \mathbb{H}. \tag{5.23}$$

The optimal control is, similar to the linear filtering problem, given by a linear feedback law.

Theorem 5.11
[31, Theorem 2.2] The unique control minimizing $J(u; t_0, t)$ is given by the feedback law

$$u(t) = -R_c^{-1}(t)B(t)P_c(t)z(t)$$

with the minimal cost

$$\min J(u; t_0, t) = \langle z_0, P_c(t_0)z_0\rangle.$$

If the mild evolution operator $U_c(t, s)$ is the perturbation of a semigroup, the solution to the integral Riccati equation even admits a sense of differentiability. Recall first that in this case $U_c(t, s)$ is a quasi-evolution operator, ensuring that $V_c(t, s)$ as perturbation of such is also a quasi-evolution operator by Theorem 4.17.

Theorem 5.12
[31, Theorem 2.4 & Corollary 2.1] Let A be the infinitesimal generator of a C_0-semigroup, let $D(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ and let the quasi-evolution operator $U_c(t, s)$ be generated by $A + D(t)$.
Then the Riccati equation (5.22) has a unique, self-adjoint, positive, strongly continuous operator solution $P_c(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ where $V_c(t, s)$, the perturbation of $U_c(t, s)$ by $-BR_c^{-1}(t)B^*P_c(t)$ is given by (5.23).
Furthermore, for all $x, y \in \mathcal{D}(A)$ the operator $P_c(t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}\langle P_c(t)x, y\rangle + \langle P_c(t)x, (A + D(t))y\rangle + \langle (A + D(t))x, P_c(t)y\rangle \\ \qquad\qquad + \langle W(t)x, y\rangle - \langle P_c(t)B(t)R_c^{-1}(t)B^*(t)P_c(t)x, y\rangle = 0 \\ P_c(t_f) = P_{0,c} \end{cases} \tag{5.24}$$

a.e. on $[t_0, t_f]$.
If $B(t), W(t)$ *and* $R_c(t)$ *are strongly continuous on* $[t_0, t_f]$ *then* (5.24) *is satisfied everywhere on* $[t_0, t_f]$.

In a recent work on optimal sensor placement in Hilbert spaces, X. Wu and B. Jacob in [87] consider the stochastic evolution equation

$$\begin{aligned} x(t) &= U_e(t,t_0)x(t_0) + \int_{t_0}^{t} U_e(t,s)\big(B(s)u(s) + D(s)\omega(s)\big)ds \\ y(t) &= H(t)x(t) + E(t)\nu(t) \end{aligned} \tag{5.25}$$

where $U_e(t,s)$ is a mild evolution operator and $\omega(t) \in L^2([0,t_f],\mathbb{X}_1), \nu(t) \in L^2([0,t_f],\mathbb{X}_2)$ are white noise, i.e. random variables with zero mean and identity covariance and the spaces $\mathbb{X}_1$ and $\mathbb{X}_2$ are finite-dimensional. The input $u(t) \in L^2([0,t_f],\mathbb{X}_1)$. More detail on white noise disturbances can be found in [7]. The operators $B(t) \in \mathbb{B}_\infty([0,t_f], L(\mathbb{X}_1,\mathbb{H}))$, $D(t) \in \mathbb{B}_\infty([0,t_f], L(\mathbb{X}_1,\mathbb{H}))$, $H(t) \in \mathbb{B}_\infty([0,t_f], L(\mathbb{H},\mathbb{X}_2))$ and $E(t) \in \mathbb{B}_\infty([0,t_f], L(\mathbb{X}_2))$.
With the definitions

$$\begin{aligned} \hat{x}_0 &= \mathsf{E}\{x(t_0)\} & P_{e,0} &= \mathsf{Cov}\{x(t_0) - \hat{x}_0\} \\ e(t) &= x(t) - \hat{x}(t) & P_e(t) &= \mathsf{Cov}\{e(t)\} \end{aligned}$$

the objective of the filtering problem becomes to find the best estimate of the form

$$\begin{aligned} \hat{x}(t) = U_e(t,t_0)\hat{x}_0 &+ \int_{t_0}^{t} U_e(t,s)B(s)u(s)ds \\ &+ \int_{t_0}^{t} K_e(t,s)\big(y(s) - H(s)\hat{x}(s)\big)ds \end{aligned}$$

based on the observation $Y_t = \{y(s) \mid t_0 \le s \le t\}$, where $K_e(\cdot,\cdot)$ is to be found.

Definition 5.13
[87] A filter $\hat{x}(t)$ *for the state-space system* (5.25) *is optimal if it minimizes the trace of the error covariance, i.e. if* $\mathrm{tr}\{P_e(t)\}$ *is minimized by* $\hat{x}(t)$.

Theorem 5.14
[87, Theorem 4.5] Let $Q(t) = D(t)D^*(t), R_e(t) = E(t)E^*(t)$ *and let* $R_e(t)$ *be invertible for all* $t \in [t_0, t_f]$. *Considering system* (5.25), *the linear filter* $\hat{x}(t)$ *based on* Y_t *is optimal if it satisfies*

$$\hat{x}(t) = V_e(t,t_0)\hat{x}_0 + \int_{t_0}^{t} V_e(t,s)\big(K_e(s,s)y(s) + B(s)u(s)\big)ds$$

where $K_e(t,s) = U_e(t,s)P_e(s)H^*(s)R_e^{-1}(s)$ *and* V_e *is the perturbation of* U_e *by*

$$K_e(t,t) = P_e(t)H^*(t)R_e^{-1}(t)H(t)$$

such that $P_e(t)$ *satisfies*

$$\begin{aligned} P_e(t)x &= V_e(t,t_0)P_{e,0}V_e^*(t,t_0)x \\ &\quad + \int_{t_0}^{t} V_e(t,s)\big(Q(s) + P_e(s)H^*(s)R_e^{-1}(s)H(s)P_e(s)\big)V_e^*(t,s)x ds. \end{aligned} \tag{5.26}$$

for all $x \in \mathbb{H}$.

The following result exists for the covariance operator.

Theorem 5.15
[87, Theorems 4.7 & 4.8] Consider system (5.25) *with the optimal filter* (5.14).
If $P_{e,0}$ *is a nuclear operator then for* $t \in [t_0, t_f]$ *the covariance* $P(t)$ *exists and is a nuclear operator.*

Now, defining $U_c(t,s) = U_e^*(t_f - s, t_f - t)$ and

$$B(s) = H^*(t_f - s) \qquad W(s) = D(t_f - s)D^*(t_f - s)$$
$$P_{c,0} = P_{e,0} \qquad R_c(s) = R_e(t_f - s)$$

the solution to (5.26) equals that of (5.22) corresponding to the time-varying system (5.21). More on this *duality* between the IDRE in control and estimation can be found in [31].

The following theorem is an immediate consequence of the relations outlined above as well as Theorem 5.12. Let $P_0 \in L(\mathbb{H})$ be nonnegative and self-adjoint and let $R(t), R^{-1}(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{X}_2))$ and $W(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$ be nonnegative and self-adjoint for all $t \in [0, t_f]$. Let $C(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}, \mathbb{X}_2))$.

The following two results will be important in the estimator design for semi-linear systems in Chapter 7.

Theorem 5.16
[31, Theorem 3.1 & 3.3][29, Theorem 2.1]
Let $U_e(t,s)$ *be a quasi-evolution operator generated by* $A + D(t)$, *where* A *is the infinitesimal generator of a* C_0*-semigroup* $T(t)$ *and* $D(t) \in \mathbb{B}_\infty([t_0, t_f], L(\mathbb{H}))$. *Then there exists a unique,*

positive, self-adjoint operator solution $P_e(t) \in \mathrm{C}([t_0, t_f], L(\mathbb{H}))$ *to the two equivalent Riccati equations*

$$P_e(t)x = V_e(t,t_0)P_0V_e^*(t,t_0)x + \int_{t_0}^{t} V_e(t,s)\big(W(s) + P_e(s)C^*(s)R^{-1}(s)C(s)P_e(s)\big)V_e^*(t,s)xds \quad x \in \mathbb{H} \tag{5.27}$$

$$P_e(t)x = V_e(t,t_0)P_0U_e^*(t,t_0)x + \int_{t_0}^{t} V_e(t,s)W(s)U_e^*(t,s)xds \quad x \in \mathbb{H} \tag{5.28}$$

where $V_e(t,s)$ *is the perturbation of* $U_e(t,s)$ *by* $-P_e(t)C^*(t)R^{-1}(t)C(t)$*, that is*

$$V_e(t,s)x = U_e(t,s)x - \int_{s}^{t} U_e(t,r)P_e(r)C^*(r)R^{-1}(r)C(r)V_e(r,s)xdr \quad x \in \mathbb{H}.$$

Theorem 5.17
[31, Theorems 2.4 & 3.3] Under the conditions of Theorem 5.16, the unique solution $P_e(t)$ *of* (5.27) *and* (5.28) *also satisfies*

$$\begin{cases} \dfrac{d}{dt}\langle P_e(t)x, y\rangle - \langle P_e(t)x, (A+D(t))^*y\rangle - \langle (A+D(t))^*x, P_e(t)y\rangle \\ \qquad - \langle W(t)x, y\rangle + \langle P_e(t)C^*(t)R^{-1}(t)C(t)P_e(t)x, y\rangle = 0 & x, y \in \mathcal{D}(A^*) \\ P(0) = P_0 \end{cases} \tag{5.29}$$

a.e. on $[0, t_f]$.
If $W(t), C(t)$ *and* $R^{-1}(t)$ *are strongly continuous in time, then* (5.29) *is satisfied everywhere.*

An important consequence, which will be needed in the later Chapter 7, is stated as corollary.

Corollary 5.18
This means that under the conditions of Theorem 5.16, the three Riccati equations (5.27), (5.28) *and* (5.29) *are equivalent. Thus a solution* $P(t)$ *of one of the equations* (5.27), (5.28) *or* (5.29) *also satisfies the other two.*

Chapter 6

Semi-linear systems

This chapter treats existence results for solutions of semi-linear systems.

Let a system be

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + F(t, z(t)) + d(t) \\ z(0) = z_0 \end{cases} \tag{6.1}$$

where A generates a strongly continuous semigroup on the Hilbert space $\mathbb{H}$ and the following assumptions hold.

Assumption 6.1
The disturbance $d(t)$ is such that $d(t) \in L^p([0, t_f], \mathbb{H})$ for some integer $p \geq 1$ on every bounded time interval $[0, t_f]$.

Assumption 6.2
The nonlinear operator $F : \mathbb{R} \times \mathbb{H} \longrightarrow \mathbb{H}$ is continuous in t and locally Lipschitz on $\mathbb{H}$, uniformly in t on bounded intervals. That is, for every $t_f > 0$ and constant $\delta_z \geq 0$ there is a constant L such that

$$\|F(t, z_1(t)) - F(t, z_2(t))\| \leq L\|z_1(t) - z_2(t)\|$$

for all $t \in [0, t_f]$ and $\|z_1(t)\|, \|z_2(t)\| \leq \delta_z$.

The following are widely used definitions of solutions.

Definition 6.3 (classical solution)
[69, Definition 4.2.1] A function $z : [0, t_f) \to \mathbb{H}$ is called a classical solution to the initial value problem (IVP) (6.1) if it is continuous on $[0, t_f)$ and continuously differentiable on $(0, t_f)$, $z(t) \in \mathcal{D}(A)$ for $0 < t < t_f$ and equation (6.1) is satisfied on $[0, t_f)$.

Definition 6.4 (mild solution)
[69, Definition 6.1.1] Let A be the generator of a strongly continuous semigroup $T(t)$. If the function $z(t) \in \mathrm{C}([0, t_f], \mathbb{H})$ satisfies

$$z(t) = T(t)z_0 + \int_0^t T(t-s)(F(s, z(s)) + d(s))ds \tag{6.2}$$

for every $z_0 \in \mathbb{H}$ it is called a mild solution to IVP (6.1).

It has been shown [41, Theorem 3.1] that for F independent of time (6.1) has a mild solution that exists locally. However the observer dynamics contain time-varying feedback, which has been treated to some extent in [69] for (Lipschitz) continuous time-dependencies and for disturbances that are at least continuous. Here, in (6.1), this is not the case. Therefore a new result is stated that ascertains the local existence of mild solutions for (6.1).

The Banach fixed point theorem, also known as the Contraction Mapping Theorem, is used several times in this thesis.

Theorem 6.5 (Banach fixed point theorem)
[59, Theorem 5.1-2]
Let (X, d) be a complete non-empty metric space and $T : X \to X$ a contraction mapping, i.e. a mapping such that

$$\mathrm{d}(T(x), T(y)) \leq \alpha\, \mathrm{d}(x, y)$$

for all $x, y \in X$ and some $0 \leq \alpha < 1$. Then T has a unique fixed point.

The proof of the following Theorem 6.6 is similar to [69, Theorem 6.1.4] and [41, Theorem 3.1].

Theorem 6.6
Consider (6.1) where Assumption 6.1 is satisfied, as well as Assumption 6.2 for some $t_d > 0$, $\delta_d \geq 0$, that is, $\|d(t)\|_p \leq \delta_d$ with $\|\cdot\|_p$ the L^p-norm on $[0, t_d]$. Then for all $z_0 \in \mathcal{D}(A)$ there exists a final time $0 < t_f \leq t_d$ such that (6.1) has a unique mild solution in $\mathrm{C}([0, t_f], \mathbb{H})$.

Proof. Given $z_0 \in \mathbb{H}$ choose $\delta_0 > 0, t_f > 0$ with $t_f \leq t_d$ such that for all $t \in [0, t_f]$

$$\|T(t)z_0 - z_0\| \leq \delta_0.$$

By assumption $\|d(t)\|_p \leq \delta_d$ with $\|\cdot\|_p$ the L^p-norm. Define a closed and bounded subset $\mathbb{S} \subset \mathrm{C}([0,t_f],\mathbb{H})$ via

$$\mathbb{S} := \{z(t) \in \mathrm{C}([0,t_f],\mathbb{H}) | z(0) = z_0, \|z(t) - z_0\| \leq 2\delta_0\}$$

and the operator

$$\mathsf{V}(z(t)) = T(t)z_0 + \int_0^t T(t-s)F(s,z(s))ds + \int_0^t T(t-s)d(s)ds. \tag{6.3}$$

It will be shown that V is a contraction mapping on $\mathbb{S}$ for $t_f > 0$ chosen sufficiently small. Using the triangle inequality it is clear that

$$\|\mathsf{V}(z(t)) - z_0\| \leq \|T(t)z_0 - z_0\| + \|\int_0^t T(t-s)F(s,z(s))ds\| + \|\int_0^t T(t-s)d(s)ds\|. \tag{6.4}$$

By Theorem 4.2 there is $\delta_T > 0$ such that $\max_{[0,t_f]} \|T(t)\| \leq \delta_T$. Letting $\delta_{F_0} = \max_{[0,t_f]} \|F(t,0)\|$ and recalling Assumption 6.2 there is $L > 0$ such that for all $t \in [0,t_f]$

$$\begin{aligned}\|F(t,z(t))\| &\leq \|F(t,z(t)) - F(t,0)\| + \|F(t,0)\| \\ &\leq L\|z(t)\| + \delta_{F_0} \\ &\leq L\delta_z + \delta_{F_0} \\ &\underset{def.}{=} \delta_F\end{aligned}$$

for $z(t)$ in the ball of radius $\delta_z = \|z_0\| + 2\delta_0$. Therefore the second term in (6.4) is bounded by

$$\|\int_0^t T(t-s)F(s,z(s))ds\| \leq \delta_T\delta_F t_f.$$

The third term on the right in (6.4) can be bounded using Hölder's inequality

$$\|\int_0^t T(t-s)d(s)ds\| \leq \delta_T\|d(t)\|_p \left(\int_0^{t_f} 1dt\right)^{\frac{1-p}{p}} \leq \delta_T\delta_d t_f^{\frac{1-p}{p}}.$$

Therefore (6.4) is bounded by

$$\|\mathsf{V}(z(t)) - z_0\| \leq \delta_0 + \delta_T\delta_d t_f^{\frac{1-p}{p}} + \delta_T\delta_F t_f.$$

Thus for t_f small enough such that

$$\delta_T\delta_d t_f^{\frac{1-p}{p}} + \delta_T\delta_F t_f \leq \delta_0, \tag{6.5}$$

$\mathsf{V}:\mathbb{S}\to\mathbb{S}$. Now let $z_1(t), z_2(t)\in\mathbb{S}$. Then

$$\begin{aligned}\|\mathsf{V}(z_1(t))-\mathsf{V}(z_2(t))\| &\leq \|\int_0^t T(t-s)(F(s,z_1(s))-F(s,z_2(s)))ds\| \\ &\leq \delta_T L t_f \max_{[0,t_f]}\|z_1(t)-z_2(t)\|.\end{aligned} \tag{6.6}$$

It is evident that if additionally $t_f<\frac{1}{\delta_T L}$ the mapping $\mathsf{V}:\mathbb{S}\to\mathbb{S}$ is a contraction. By the contraction mapping principle (Theorem 6.5) equation (6.3) has a unique fixed point in $\mathrm{C}([0,t_f],\mathbb{H})$. As this fixed point $z(t)$ satisfies

$$z(t)=\mathsf{V}(z(t))=T(t)z_0+\int_0^t T(t-s)\big(F(s,z(s))+d(s)\big)ds$$

it is by Definition 6.4 the mild solution to (6.1) on the interval $[0,t_f]$. □

Theorem 6.6 ensures local existence of mild solutions in the case of locally Lipschitz time-dependent nonlinearities. It also implies existence of mild solutions for a range of inputs $d(t)$ up to a common local time.
More specifically, let $\delta_d>0$, $p\geq 1$ and let $\mathbb{B}^p_{t_d}(\delta_d)\subset L^p([0,t_d],\mathbb{H})$ denote the closed ball of inputs $d(t)$ such that $\|d(t)\|_p\leq\delta_d$.

Corollary 6.7
Consider system (6.1), let Assumptions 6.1 and 6.2 hold and let $\delta_d>0$.
There is a $t_f>0$ such that for any $d(t)\in\mathbb{B}^p_{t_d}(\delta_d)$ the system (6.1) has a mild solution $z(t)\in\mathrm{C}([0,t_f],\mathbb{H})$.

Proof. This can be seen easily since the inequality (6.5) holds for all $d(t)\in\mathbb{B}^p_{t_d}(\delta_d)$. □

If some further regularity assumptions are imposed the mild solution becomes a classical solution.

Theorem 6.8
[69, Theorem 6.1.5] Under Assumptions 6.1 and 6.2, if the right-hand side $F(t,x)+d(t)$ of (6.1) is continuously differentiable from $[0,t_f]\times\mathbb{H}$ into $\mathbb{H}$ then the mild solution of (6.1) with $z_0\in D(A)$ is is a classical solution.

The following Theorem 6.10, the proof of which is very similar to [69, Theorem 6.1.5], shows that if the nonlinearity F is uniformly Lipschitz on $[0,t_f]$ then mild solutions to (6.1) exist on the same interval $[0,t_f]$. First recall an important corollary of the Banach fixed point Theorem 6.5.

Corollary 6.9
[59, Lemma 5.4-3] Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be continuous. If T^n is a contraction for some integer $n \geq 1$ then T has a unique fixed point.

Theorem 6.10
[69, Theorem 6.1.2] Assume that for $t_f > 0$ $F : [0, t_f] \times \mathbb{H} \longrightarrow \mathbb{H}$ is continuous in t and uniformly Lipschitz on $\mathbb{H}$, i.e. there is a constant L such that

$$\|F(t, z_1) - F(t, z_2)\| \leq L\|z_1 - z_2\|$$

for all $t \in [0, t_f]$ and all $z_1, z_2 \in \mathbb{H}$. Let Assumption 6.1 be satisfied.
Then for every $z_0 \in D(A)$ and $d(t) \in L^p([0, t_f], \mathbb{H})$, $p \geq 1$, the initial value problem (6.1) has the unique mild solution $z(t) \in \mathrm{C}([0, t_f]), \mathbb{H})$ given by (6.2).

Proof. Given a z_0 let an operator

$$\mathsf{V} : \mathrm{C}([0, t_f], \mathbb{H}) \longrightarrow \mathrm{C}([0, t_f], \mathbb{H})$$

be defined via

$$\mathsf{V}(z(t)) = T(t)z_0 + \int_0^t T(t-s)(F(s, z(s)) + d(s))ds.$$

Let $\delta_T = \max_{[0,t_f]} \|T(t)\|$ and $\delta_d = \|d(t)\|_p$. Since F is uniformly Lipschitz on $[0, t_f]$, with the same calculation as in the proof to Theorem 6.6

$$\|F(t, x)\| \leq L\|x\| + \delta_{F,0}$$

where $\delta_{F,0} = \max_{[0,t_f]} \|F(t, 0)\|$. For any $z(t) \in \mathrm{C}([0, t_f], \mathbb{H})$

$$\|\mathsf{V}(z(t))\| \leq \delta_T\|z_0\| + \delta_T L \int_0^t \|z(s)\|ds + \delta_T \delta_{F,0} t_f + t_f^{\frac{p-1}{p}} \delta_d < \infty$$

and hence V is well defined. Now it will be proven by induction that for $z_1(t), z_2(t) \in \mathrm{C}([0, t_f], \mathbb{H})$

$$\|\mathsf{V}^n(z_1(t)) - \mathsf{V}^n(z_2(t))\| \leq \frac{(\delta_T L t)^n}{n!} \max_{[0,t_f]} \|z_1(t) - z_2(t)\|. \tag{6.7}$$

For $n = 1$ this is easily seen to hold, since for $0 \leq t \leq t_f$

$$\|\mathsf{V}(z_1(t)) - \mathsf{V}(z_2(t))\| \leq \delta_T L t \max_{[0,t_f]} \|z_1(t) - z_2(t)\|.$$

Now assume (6.7) holds for $n \geq 1$. Then

$$\begin{aligned}\|\mathsf{V}^{n+1}(z_1(t)) - \mathsf{V}^{n+1}(z_2(t))\| &\leq \delta_T L \int_0^t \|\mathsf{V}^n(z_1(s)) - \mathsf{V}^n(z_2(s))\| ds \\ &\leq \frac{(\delta_T L)^{n+1}}{n!} \max_{[0,t_f]} \|z_1(t) - z_2(t)\| \int_0^t s^n ds \\ &= \frac{(\delta_T L t)^{n+1}}{(n+1)!} \max_{[0,t_f]} \|z_1(t) - z_2(t)\|.\end{aligned}$$

Finally, taking the maximum over $[0, t_f]$ in (6.7) yields

$$\max_{[0,t_f]} \|\mathsf{V}^n(z_1(t)) - \mathsf{V}^n(z_2(t))\| \leq \frac{(\delta_T L t_f)^n}{n!} \max_{[0,t_f]} \|z_1(t) - z_2(t)\|.$$

Hence, if n is sufficiently large, then $\frac{(\delta_T L t_f)^n}{n!} < 1$ and V^n is a contraction on $\mathrm{C}([0, t_f], \mathbb{H})$. By Lemma 6.9 V has a unique fixed point which is the desired mild solution (6.2). □

Corollary 6.11
If $F : \mathbb{H} \longrightarrow \mathbb{H}$ is globally Lipschitz then for all $z_0 \in \mathcal{D}(A)$ the IVP (6.1) has a unique mild solution that exists globally.

Proof. This follows immediately since then Theorem 6.10 holds for arbitrary $t_f \in [0, \infty)$. □

The following example of a semi-linear infinite-dimensional IVP has mild solutions that exist globally.

Example 6.12
Consider again the IVP (4.5) in Example 4.9 only with the globally Lipschitz nonlinearity $\sin(\cdot)$ *and a quadratic disturbance, that is*

$$\begin{cases} \dfrac{dz(t)}{dt} = Az(t) + \sin(z(t)) + t^2, \\ z(0) = z_0, \quad z_0 \in \mathcal{D}(A), \end{cases} \tag{6.8}$$

where again $A = \frac{d^2}{dx^2}$. It was shown in Example 4.9 that A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on $\mathbb{H} = L^2(0,1)$. Corollary 6.11 now ensures that the mild solution to (6.8),

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\big(\sin(z(s)) + s^2\big) ds,$$

exists globally.

The following is an example of an ODE IVP with a locally Lipschitz nonlinearity that only permits local existence of solutions.

Example 6.13
[85] Consider the one-dimensional IVP

$$\begin{cases} \dfrac{dx(t)}{dt} = x^2(t) \\ x(1) = 1. \end{cases} \tag{6.9}$$

The solution to (6.9) *is*

$$x(t) = \frac{1}{2-t}$$

which has a blow-up at $t = 2$.

Chapter 7

A time-varying estimator for semi-linear systems

Although filtering for linear PDE systems is well explored, not as much theory has been developed for nonlinear PDEs.

One of the first approaches was done by J. H. Seinfeld in 1969 [81] by considering a nonlinear PDE and deriving an estimator via second-order expansion of a Hamilton-Jacobi equation. J. S. Baras and A. Bensoussan treat optimal sensor scheduling for a nonlinear filter of a diffusion process in [9]. Therein in the authors investigate how to best use a set of sensors by considering a cost function that penalizes the off-and-on switching of sensors as well as estimation errors. A nonlinear filter is proposed for a second-order PDE with unbounded differential operator in [56]. As in finite-dimensions, I. Gyöngy studies filters for stochastic PDEs that live on certain manifolds [53]. The same author studies partially observable stochastic PDEs motivated by the filtering problem [54] and approximations for such equations with applications in nonlinear filtering [51]. The authors in [18] obtain the optimal filter for a finite-dimensional system by solving a PDE for the filter distribution. A filter for a nonlinear second order PDE is proposed in [21] based on fuzzy interpolation of the state-space system. In [88] a nonlinear undisturbed PDE is considered and an estimator is proposed based on an inertial manifold containing all information of the long-time behaviour of the state-space system.

Other methods have been developped, for instance *backstepping* [40, 44, 60, 70] which uses a modified copy of the system as observer with the estimation error as control injection. An analogue to the finite-dimensional *sliding-mode observer* ([38, 39]) has been established in infinite dimensions [2], which is a method based on linearizing a part of the system and injecting a discontinuous error correction signal, allowing the observer to 'slide' on a linear manifold close to the reference trajectory.

The EKF in infinite dimensions has been applied computationally to particular nonlinear PDE systems, so for example a highway traffic model [77] or to state-of-charge estimation in lithium-ion batteries [3]. Versions of the KF are also applied for approximated PDEs [67].
However, no proof exists for the existence of the continuous-time EKF for nonlinear PDE systems. This thesis provides a first result in extending the EKF from finite-dimensional systems to infinite dimensions.

Before proving this result, it is useful to summarize the results leading up to this point.

In Chapter 5 the filtering problem was investigated for infinite-dimensional linear systems, which, analogous to the Kalman filter for finite-dimensional linear systems (see Section 3.2) yields a Riccati equation. The solution of this equation determines the optimal filter gain. More precisely, the trace of the error covariance $P(t) = \mathsf{Cov}\{e(t)\}$ is minimized if $P(t)$ satisfies this Riccati equation.

For finite-dimensional nonlinear systems this method has been extended on a small time interval to the EKF, which, as mentioned in Section 3.3 does not provide any optimality. The EKF proposes an estimator containing the same nonlinearity as the original system and that is coupled with the Riccati equation corresponding to the *linearized* systems. The motivation behind this is to apply the linear filtering theory to the linearizations, which, as shown in Section 3.3, yield the same differential equation for the error as in linear filtering and hence also the same integral and differential Riccati equations.
For continuous time systems the EKF leads to a set of coupled equations as the state of the nonlinear estimator $\hat{z}(t)$ depends on the solution $P(t)$ of the Riccati equation, which itself depends on the linearization along $\hat{z}(t)$ (see Definition 3.17).

To extend the EKF to semi-linear infinite-dimensional systems it has to be proven that the coupling between a semi-linear system and one of the three equivalent infinite-dimensional Riccati equations (5.27), (5.28) or (5.29) is well-posed.
This is the aim of this section.

However, there are two main differences to the exposition of the finite-dimensional EKF :

1. An additional linear perturbation by αI is introduced, where $\alpha > 0$ will be a prescribed rate of error convergence and I is the identity. This is inspired by the results in [74, 75, 76].
   Since all the results will clearly also hold for $\alpha = 0$ this is a generalization of the EKF to scalar perturbations.

2. Integral forms of the semi-linear infinite-dimensional systems are used to deal with the lack of differentiability in the case of L^p-disturbances.

7.1 Well-posedness of a semi-linear system coupled with a Riccati equation

As before, let $\mathbb{H}$ be a real Hilbert space and let $\mathbb{X}$ be a finite-dimensional Hilbert space.

Assumption 7.1
In the following let $P_0 \in L(\mathbb{H})$, $W(t) \in \mathrm{C}([0,\infty), L(\mathbb{H}))$ and $R(t), R^{-1}(t) \in \mathrm{C}([0,\infty), L(\mathbb{X}))$ be self-adjoint and positive operators for all t. Let $C \in L(\mathbb{H}, \mathbb{X})$.

Let $T(t)$ be the C_0-semigroup generated by $A : \mathcal{D}(A) \to \mathbb{H}$ and $z(t)$ a continuous function. This section treats the semi-linear integral equation

$$\begin{aligned} \hat{z}_P(t) = T(t)\hat{z}_0 &+ \int_0^t T(t-s)F(\hat{z}_P(s))ds \\ &+ \int_0^t T(t-s)P(s)C^*R^{-1}(s)C[z(s) - \hat{z}_P(s)]ds \end{aligned} \tag{7.1}$$

where $\hat{z}_0 \in \mathbb{H}$, $P(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$ and F satisfies the Assumptions below.
The subscript in $\hat{z}_P(t)$ denotes the dependence of $\hat{z}_P(t)$ on $P(t)$.

Assumption 7.2
The operator $F : \mathbb{H} \to \mathbb{H}$ has a locally Lipschitz Fréchet-derivative DF that is globally bounded in operator norm, that is for some $\delta_{DF} > 0$

$$\max_{x \in \mathbb{H}} \|DF(x)\| \leq \delta_{DF}.$$

Furthermore $F(0) = 0$.

Theorem 7.3 (Mean Value Theorem)
[36, Theorem 5.1.12] Let $\mathbb{H}$ be a Hilbert space. If $F : \mathbb{H} \to \mathbb{H}$ is continuously Fréchet-differentiable, then

$$\|F(z+h) - F(z)\|_{\mathbb{H}} \leq \sup_{t \in [0,1]} \|DF(z+th)\| \|h\|_{\mathbb{H}}$$

To rigorously prove well-posedness of an integral Riccati equation coupled with the semi-linear estimator, it is useful to express this relation via two intermediate mappings: one mapping a solution of (7.1) to a solution of a Riccati and another mapping a solution of the Riccati equation to a solution of (7.1).

Proposition 7.4
Let $t_f > 0$ and let $\hat{z}_P(t)$ be given by (7.1) where the subscript denotes the dependence on $P(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$. The mapping

$$\mathsf{G}_1 : \begin{cases} P(t) \longrightarrow \hat{z}_P(t) \\ \mathrm{C}([0, t_f], L(\mathbb{H})) \longrightarrow \mathrm{C}([0, t_f], \mathbb{H}) \end{cases}$$

is well-defined.

Proof. Since $DF(\cdot)$ is globally bounded the Mean-Value Theorem 7.3 implies that F is globally Lipschitz. Hence, Theorem 6.10 ensures that $\hat{z}_P(t) \in \mathrm{C}([0, t_f], \mathbb{H})$ for all $P(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$. □

Let $T_\alpha(t)$ denote the C_0-semigroup generated by $A + \alpha I$, where $\alpha > 0$ and I is the identity. The second auxiliary mapping is now introduced.

Given any function $y(t) \in \mathrm{C}([0, t_f], \mathbb{H})$, by Theorem 4.17 there is a unique quasi-evolution operator U_y satisfying for all $x \in \mathbb{H}$

$$U_y(t, s)x = T_\alpha(t)x + \int_s^t T_\alpha(t - r)DF(y(r))U_y(r, s)xdr. \tag{7.2}$$

By Theorem 5.16, there is a unique positive, self-adjoint operator $P_y(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$ such that for all $x \in \mathbb{H}$

$$U_\pi(t, s)x = U_y(t, s)x - \int_s^t U_y(t, r)P_y(r)C^* R^{-1}(r)CU_\pi(r, s)xdr \tag{7.3}$$

$$P_y(t)x = U_\pi(t, 0)P_0 U_y^*(t, 0)x + \int_0^t U_\pi(t, s)W(s)U_y^*(t, s)xds. \tag{7.4}$$

The above procedure justifies the following.

Proposition 7.5
Let $t_f > 0$ and let $P_y(t)$ be the solution to (7.4), where $U_y(t, s)$ and $U_\pi(t, s)$ are as defined above in (7.2) and (7.3) respectively and where the subscript in $U_y(t, s)$ denotes the dependence on $y(t) \in \mathrm{C}([0, t_f], \mathbb{H})$. There is a well-defined mapping G_2 such that

$$\mathsf{G}_2 : \begin{cases} y(t) \mapsto P_y(t) \\ \mathrm{C}([0, t_f], \mathbb{H}) \longrightarrow \mathrm{C}([0, t_f], L(\mathbb{H})). \end{cases}$$

Finally a mapping $\mathsf{G} = \mathsf{G}_2 \circ \mathsf{G}_1$ can be defined. Note that

$$\begin{aligned}\mathsf{G} : \mathrm{C}([0, t_f], L(\mathbb{H})) &\longrightarrow \mathrm{C}([0, t_f], L(\mathbb{H}))\\ P(t) &\mapsto P_{\hat{z}_P}(t).\end{aligned}$$

In this section it is proved that the semi-linear system (7.1) coupled with the Riccati equation (7.2)–(7.4) is well-posed by proving that G has a fixed point. This means that there is a pair $(\hat{z}_P(t), P(t))$ with $\hat{z}_P(t)$ satisfying the semi-linear system (7.1) and $P(t)$ the Riccati equation (7.4) (with $y(t) = \hat{z}_P(t)$).

For $\delta_p, t_f > 0$ let $\mathbb{P}_{t_f}(\delta_p) \subset \mathrm{C}([0, t_f], L(\mathbb{H}))$ be the closed ball where $\max_{[0,t_f]} \|P(t)\| \leq \delta_p$.

Before the proof of the main theorem two lemmas are needed.
For that recall that if $P(t)$ satisfies (7.4), by Theorem 5.16 and Theorem 5.17 it also satisfies a Riccati equation of the form (5.27).

Lemma 7.6
[31, Lemma 2.2] Let Assumption 7.1 hold and let U_y be given by (7.2). If $P_y(t)$ satisfies the Riccati equation (5.27) on $[0, t_f]$ then

$$\|P_y(t)\| \leq \|P_0\| \|U_y(t, 0)\|^2 + \delta_W \int_0^t \|U_y(t, s)\|^2 ds$$

for all $t \in [0, t_f]$ with $\delta_W = \max_{[0,t_f]} \|W(t)\|$.

The following inequality is known as Bellman inequality and is used several times in the proof of the main theorem.

Lemma 7.7 (Bellman inequality)
[71, Theorem 1.1.4] Let $\phi(t)$ be a real continuous function on $[t_0, t_f]$ and let $\phi(t)$ satisfy

$$\phi(t) \leq \alpha(t) + \int_0^t \beta(\tau)\phi(\tau) d\tau \qquad \text{for } t \in [t_0, t_f],$$

where $\beta(t)$ and $\alpha(t)$ are nonnegative and continuous and $\alpha(t)$ is non-decreasing. Then it holds that

$$\phi(t) \leq \alpha(t) \exp\left(\int_0^t \beta(\tau) d\tau\right) \qquad \text{for } t \in [t_0, t_f].$$

Theorem 7.8
Let Assumptions 7.1 and 7.2 be satisfied and let $t_f > 0$. Then for all $\hat{z}_0 \in \mathbb{H}$ the mapping G has a unique fixed point $P(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$.

The proof is divided into three steps:

1. It is shown that for δ_p large enough the mapping G maps the ball $\mathbb{P}_{t_f}(\delta_p)$ to itself, i.e. $\mathsf{G} : \mathbb{P}_{t_f}(\delta_p) \longrightarrow \mathbb{P}_{t_f}(\delta_p)$.

2. G^n is contractive on $\mathbb{P}_{t_f}(\delta_p)$ for large enough n.

3. A fixed point theorem concludes the proof.

Proof. **Step 1:** For $\delta_p > 0$ large enough $\mathsf{G} : \mathbb{P}_{t_f}(\delta_p) \longrightarrow \mathbb{P}_{t_f}(\delta_p)$.
As in Assumption 7.2 let $\delta_{DF} = \max_{x\in\mathbb{H}} \|DF(x)\|$. Fix $t_f > 0$ and choose $P(t) \in \mathrm{C}([0, t_f], L(\mathbb{H}))$.
From (7.2) by using the Grönwall inequality 3.15

$$\begin{aligned} \max_{0\leq s\leq t} \|U_{\hat{z}_P}(t,s)\| &\leq \delta_{T,\alpha} + \delta_{T,\alpha}\delta_{DF} \int_0^t \max_{0\leq s\leq r} \|U_{\hat{z}_P}(r,s)\| dr \\ &\leq \delta_{T,\alpha} \exp(\delta_{T,\alpha}\delta_{DF} t_f) \\ &\leq \delta_{U_{\hat{z}_P}} \end{aligned} \tag{7.5}$$

for some $\delta_{U_{\hat{z}_P}} > 0$, where $\delta_{T,\alpha} = \max_{[0,t_f]} \|T_\alpha(t)\|$.
Now let $P(t) \in \mathbb{P}_{t_f}(\delta_p)$. Then with Lemma 7.6 and $\delta_W = \max_{[0,t_f]} \|W(t)\|$

$$\begin{aligned} \max_{[0,t_f]} \|\mathsf{G}(P(t))\| &\leq \|P_0\| \max_{[0,t_f]} \|U_{\hat{z}_P}(t,0)\|^2 + t_f \delta_W \max_{[0,t_f]} \|U_{\hat{z}_P}(t,s)\|^2 \\ &\leq (\|P_0\| + t_f\delta_W)\delta^2_{U_{\hat{z}_P}} \leq \delta_p \end{aligned}$$

where the last inequality is true if δ_p is sufficiently large.
Then, for all such δ_p

$$\mathsf{G} : \mathbb{P}_{t_f}(\delta_p) \longrightarrow \mathbb{P}_{t_f}(\delta_p).$$

Step 2: G^n is a contraction on $\mathbb{P}_{t_f}(\delta_p)$ for large enough $n \in \mathbb{N}$.
First recall the notation outline above. For $i = 1, 2$ let $P_i(t) \in \mathbb{P}_{t_f}(\delta_p)$. Then:

- $\hat{z}_{P_i}(t) = \mathsf{G}_1(P_i(t))$ is given by (7.1) with $P(t) = P_i(t)$
- $U_{\hat{z}_P,i}(t,s)$ is the quasi-evolution operator given by (7.2) with $y(t) = \hat{z}_{P_i}(t)$
- $U_{\pi,i}(t,s)$ is the perturbation of $U_{\hat{z}_P,i}(t,s)$ by $-\mathsf{G}(P_i(t))C^*R^{-1}(t)$ given by (7.3).

Using these relations define for convenience

$$\Delta\hat{z}_P(t) = \hat{z}_{P_1}(t) - \hat{z}_{P_2}(t) \qquad \Delta U_{\hat{z}_P}(t,s) = U_{\hat{z}_P,1}(t,s) - U_{\hat{z}_P,2}(t,s)$$
$$\Delta P(t) = P_1(t) - P_2(t) \qquad \Delta U_\pi(t,s) = U_{\pi,1}(t,s) - U_{\pi,2}(t,s).$$

Step 2.1: In a first step, bounds for $\Delta U_\pi(t,s)$ and $\Delta U_{\hat{z}_P}(t,s)$ will be derived.

Let ι_F be the Lipschitz constant of F. For $\hat{z}_P(t)$ in (7.1), with Assumption 7.2 and Grönwall's lemma, it is easily computed that there is a bound $\delta_{\hat{z}_P} > 0$ such that

$$\|\hat{z}_P(t)\| \leq \delta_{\hat{z}_P} \quad \text{for all } P(t) \in \mathbb{P}_{t_f}(\delta_p). \tag{7.6}$$

By writing the difference as

$$\begin{aligned}\Delta\hat{z}_P(t) = &\int_0^t T(t-s)\big(F(\hat{z}_{P_1}(s)) - F(\hat{z}_{P_2}(s))\big)ds \\ &- \int_0^t T(t-s)\big(\Delta P(s)C^*R^{-1}(s)C\hat{z}_{P_1}(s) + P_2(s)C^*R^{-1}(s)C\Delta\hat{z}_P(s)\big)ds. \\ &+ \int_0^t T(t-s)\Delta P(s)C^*R^{-1}(s)Cz(s)ds\end{aligned}$$

it is clear that there are constants $\delta_1, \delta_2 > 0$ and $c_1 > 0$ such that

$$\begin{aligned}\|\Delta\hat{z}(t)\| &\leq \delta_1 \max_{0\leq s\leq t} \|\Delta P(s)\| + \delta_2 \int_0^t \|\Delta\hat{z}(s)\|ds. \\ &\leq c_1 \max_{0\leq s\leq t} \|\Delta P(s)\|\end{aligned} \tag{7.7}$$

where the last inequality was obtained by applying the Bellman inequality 7.7.

Similarly, from (7.2) and with Assumption 7.2

$$\begin{aligned}\Delta U_{\hat{z}_P}(t,s)x = &\int_s^t T_\alpha(t-r)\big(DF(\hat{z}_{P_1}(r)) - DF(\hat{z}_{P_2}(r))\big)U_{\hat{z}_P,1}(r,s)xdr \\ &+ \int_s^t T_\alpha(t-r)DF(\hat{z}_{P_2}(r))\Delta U_{\hat{z}_P}(r,s)xdr.\end{aligned}$$

Recall that by (7.6) $\max_{[0,t_f]} \|\hat{z}_P(t)\| \leq \delta_{\hat{z}_P}$ for all $P(t) \in \mathbb{P}_{t_f}(\delta_p)$. Hence, letting ι_{DF} denote the Lipschitz constant of DF on the ball $\|x\| \leq \delta_{\hat{z}_P}$

$$\begin{aligned}\max_{0\leq s\leq t} \|\Delta U_{\hat{z}_P}(t,s)\| \leq &\,\delta_{T,\alpha}\iota_{DF}\delta_{U_{\hat{z}_P}} \int_0^t \|\Delta\hat{z}(s)\|ds \\ &+ \delta_{T,\alpha}\delta_{DF} \int_0^t \max_{0\leq s\leq r} \|\Delta U_{\hat{z}_P}(r,s)\|dr.\end{aligned}$$

Employing (7.5) along with the Bellman inequality yields a $c_2 > 0$ such that

$$\|\Delta U_{\hat{z}_P}(t,s)\| \leq c_2 \int_0^t \|\Delta \hat{z}(s)\| ds$$

and using (7.7)

$$\|\Delta U_{\hat{z}_P}(t,s)\| \leq c_2 c_1 \int_0^t \max_{0\leq\tau\leq s} \|\Delta P(\tau)\| ds. \tag{7.8}$$

By using the obtained bound for $U_{\hat{z}_P,i}(t,s), i=1,2$ in (7.5), as well as Grönwall's lemma, the perturbed evolution operators $U_{\pi,i}(t,s), i=1,2$ can, similarly to $U_{\hat{z}_P}(t,s)$, be bounded by

$$\max_{0\leq s\leq t} \|U_{\pi,i}(t,s)\| \leq \delta_{U_\pi} \tag{7.9}$$

for some $\delta_{U_\pi} > 0$.

Let $Q(t) = C^* R^{-1}(t) C$. For $\Delta U_\pi(t,s)$ it can be derived that

$$\begin{aligned}\Delta U_\pi(t,s)x = \Delta U_{\hat{z}_P}(t,s)x &- \int_s^t \Delta U_{\hat{z}_P}(t,r)\mathsf{G}(P_1(r))Q(r)U_{\pi,1}(r,s)x dr \\ &- \int_s^t U_{\hat{z}_P,2}(t,r)\big(\mathsf{G}(P_1(r)) - \mathsf{G}(P_2(r))\big)Q(r)U_{\pi,1}(r,s)x dr \\ &- \int_s^t U_{\hat{z}_P,2}(t,r)\mathsf{G}(P_2(r))Q(r)\Delta U_\pi(r,s)x dr.\end{aligned}$$

Using the bounds in (7.5) and (7.9), as well as Bellman's inequality yields

$$\max_{0\leq s\leq t} \|\Delta U_\pi(t,s)\| \leq c_3 \max_{0\leq s\leq t} \|\Delta U_{\hat{z}_P}(t,s)\| + c_4 \int_0^t \|\mathsf{G}(P_1(s)) - \mathsf{G}(P_2(s))\| ds$$

for some constants $c_3, c_4 > 0$. By inserting (7.8)

$$\|\Delta U_\pi(t,s)\| \leq c_3 c_2 c_1 \int_0^t \max_{0\leq\tau\leq s} \|\Delta P(\tau)\| ds + c_4 \int_0^t \|\mathsf{G}(P_1(s)) - \mathsf{G}(P_2(s))\| ds \tag{7.10}$$

Step 2.2: Now the obtained bounds are used to bound

$$\mathsf{G}(P_1(t)) - \mathsf{G}(P_2(t)).$$

Since the operators $\mathsf{G}(P_i(t)), i=1,2$ satisfy for all $x \in \mathbb{H}$

$$G(P_i(t))x = U_{\pi,i}(t,0)P_0 U^*_{\hat{z}_P,i}(t,0)x + \int_0^t U_{\pi,i}(t,s)W(s)U^*_{\hat{z}_P,i}(t,s)x ds$$

their difference can be computed as

$$\big(\mathsf{G}(P_1(t)) - \mathsf{G}(P_2(t))\big)x = \Delta U_\pi(t,0)P_0 U^*_{\hat{z}_P,1}(t,0)x + U_{\pi,2}(t,0)P_0\Delta U^*_{\hat{z}_P}(t,0)x + \int_0^t \big(\Delta U_\pi(t,s)W(s)U^*_{\hat{z}_P,1}(t,s) + U_{\pi,2}(t,s)W(s)\Delta U^*_{\hat{z}_P}(t,s)\big)x ds \quad x \in \mathbb{H}.$$

By using (7.5) and (7.9) and by taking the maximum, for some constant c_5

$$\|\mathsf{G}(P_1(t)) - \mathsf{G}(P_2(t)\| \leq c_5 \max_{0\leq s\leq t} \|\Delta U_\pi(t,s)\| + c_5 \max_{0\leq s\leq t} \|\Delta U_{\hat{z}_P}(t,s)\|.$$

By substituting (7.8) and (7.10)

$$\|\mathsf{G}(P_1(t)) - \mathsf{G}(P_2(t)\| \leq (c_5c_2c_1)(c_3+1)\int_0^t \max_{0\leq\tau\leq s}\|\Delta P(\tau)\|ds + c_5c_4\int_0^t \|\mathsf{G}(P_1(s)) - \mathsf{G}(P_2(s))\|ds$$

and finally by Bellman's inequality for some $c > 0$

$$\|\mathsf{G}(P_1(t)) - \mathsf{G}(P_2(t))\| \leq c\int_0^t \max_{0\leq\tau\leq s}\|P_1(\tau) - P_2(\tau)\|ds. \tag{7.11}$$

To show that G^n is a contraction for large enough n an induction argument will be used to prove

$$\|\mathsf{G}^n(P_1(t)) - \mathsf{G}^n(P_2(t))\| \leq \frac{(ct)^n}{n!}\max_{0\leq\tau\leq t}\|P_1(\tau) - P_2(\tau)\|. \tag{7.12}$$

For $n = 1$ the argument holds by (7.11). Now let $n \geq 1$ and assume (7.12) holds for $k \leq n-1$. By (7.11)

$$\begin{aligned}
\|\mathsf{G}^n(P_1(t)) - \mathsf{G}^n(P_2(t))\| &\leq c\int_0^t \max_{0\leq\tau\leq s}\|\mathsf{G}^{n-1}(P_1(\tau)) - \mathsf{G}^{n-1}(P_2(\tau))\|ds \\
&\leq \frac{c^n}{(n-1)!}\int_0^t s^{n-1}\max_{0\leq\tau\leq s}\|P_1(\tau) - P_2(\tau)\|ds \\
&\leq \frac{c^n}{(n-1)!}\max_{0\leq\tau\leq t}\|P_1(\tau) - P_2(\tau)\|\int_0^t s^{n-1}ds \\
&\leq \frac{(ct)^n}{n!}\max_{0\leq\tau\leq t}\|P_1(\tau) - P_2(\tau)\|
\end{aligned}$$

which proves (7.12). By taking the maximum on $[0, t_f]$ it follows that

$$\max_{[0,t_f]}\|\mathsf{G}^n(P_1(t)) - \mathsf{G}^n(P_2(t))\| \leq \frac{(ct_f)^n}{n!}\max_{[0,t_f]}\|P_1(t) - P_2(t)\|.$$

Hence for n large enough such that $\frac{(ct_f)^n}{n!} < 1$, G^n is a contraction on $\mathbb{P}_{t_f}(\delta_p)$. Lemma 6.9 ensures that there is a unique fixed point on $\mathbb{P}_{t_f}(\delta_p)$. Hence the Theorem is proven. □

Corollary 7.9
For all $\hat{z}_0 \in \mathbb{H}$ and all $t_f > 0$ there exist $\hat{z}_P(t) \in \mathrm{C}([0,t_f],\mathbb{H})$ and $P(t) \in \mathrm{C}([0,t_f],L(\mathbb{H}))$ such that $\hat{z}_P(t)$ solves (7.1) and $P(t)$ satisfies the Riccati equation (7.4) with $y(t) = \hat{z}_P(t)$.

Recall that by Corollary 5.18 $P(t)$ also satisfies two other Riccati equations, (5.27) and (5.29). Note further that (7.1) is the mild solution to (see Definition 6.4)

$$\begin{cases} \dfrac{d\hat{z}_P(t)}{dt} = A\hat{z}_P(t) + F(\hat{z}_P(t)) + P(t)C^*R^{-1}(t)C\big(z(t) - \hat{z}_P(t)\big) \\ \hat{z}_P(0) = \hat{z}_0, \quad \hat{z}_0 \in \mathbb{H}. \end{cases} \tag{7.13}$$

There is a solution to a semi-linear PDE coupled with a differentiated Riccati equation. This is stated by the following result.

Recall before, that $U_{\hat{z}_P}(t,s)$ in (7.2) is the perturbation of $A + \alpha I$ by $DF(\hat{z}_P(t))$, where $\alpha > 0$.

Corollary 7.10
For all $\hat{z}_0 \in \mathbb{H}$, $\alpha > 0$ and $t_f > 0$ there is a mild solution $\hat{z}_P(t) \in \mathrm{C}([0,t_f],\mathbb{H})$ to (7.13) such that $P(t) \in \mathrm{C}([0,t_f],L(\mathbb{H}))$ solves for all $x,y \in \mathcal{D}(A^*)$

$$\begin{cases} \dfrac{d}{dt}\langle P(t)x,y\rangle - \langle P(t)x, \big(A+\alpha I + DF(\hat{z}_P(t))\big)^*y\rangle - \langle \big(A+\alpha I + DF(\hat{z}_P(t))\big)^*x, P(t)y\rangle \\ \qquad\qquad - \langle W(t)x,y\rangle + \langle P(t)C^*R^{-1}(t)CP(t)x,y\rangle = 0 \\ P(0) = P_0 \end{cases} \tag{7.14}$$

everywhere on $[0,t_f]$.
Furthermore, $P(t)$ is the unique, positive, self-adjoint solution to (7.14).

Chapter 8

Conclusion and future work

This thesis establishes the well-posedness of a semi-linear infinite-dimensional system coupled with a Riccati equation.

For that purpose, the stochastic theory necessary to treat disturbed systems has been introduced in infinite dimensions. Wiener processes have been defined as stochastic disturbance and some properties thereof have been outlined. It has been shown how integration and differentiation can be understood with respect to Wiener processes and some existence results to stochastic differential equations have been given.

Thereafter estimation for finite-dimensional systems has been reviewed. The Luenberger observer has been presented for deterministic continuous-time systems. To provide the reader with a good understanding of the estimation of stochastic processes, optimal estimation for random variables has been outlined. In particular, it was concluded that given a random variable f, the minimal mean squared error estimate based on another random variable g is the conditional expectation $\mathsf{E}\{f|g\}$. Following that, the Kalman-Bucy filter has been derived as minimal mean squared error estimator for differential equations disturbed by Wiener processes. It has been proven that, given a signal process $x(t)$ and an observation process $y(t)$ the optimal filter is the conditional expectation $\hat{x}(t) = \mathsf{E}\{x(t)|\sigma(Y_t)\}$, where $\sigma(Y_t)$ is the σ-algebra generated by $Y_t = \{y(s) \mid 0 \leq s \leq t\}$, and that $\hat{x}(t)$ can be obtained by solving a stochastic initial value problem along with a Riccati equation.
For nonlinear ODEs the extended Kalman filter (EKF) was outlined and some conditions were stated under which the EKF is an observer.

Semigroups have been introduced as a way of analyzing linear abstract partial differential equations (PDEs), as well as evolution operators for linear PDEs perturbed by time-varying operators. It has been shown how these can be used to define and obtain mild solutions to initial value problems of abstract PDEs.

Consequently the Kalman filter was derived as the minimal mean squared error filter for linear stochastic PDEs by considering linear estimators. It was shown that an infnite-dimensional analogue of the Wiener-Hopf equation still determines the optimal integral kernel for the filter. It was concluded that the error covariance is minimal if it satisfies a Riccati equation.

Following that, more general results on the infinite-dimensional Riccati equations were presented. For reasons of readability the corresponding context in linear-quadratic control and linear filtering theory was also outlined briefly.

Existence of solutions to semi-linear PDEs was investigated and some local and global existence results were obtained.

Finally, the well-posedness of the above-mentioned coupled equations has been shown for semi-linear infinite-dimensional systems. More precisely it has been proven that, under some conditions, there is a global solution $\hat{z}_P(t)$ to a semi-linear PDE system with a linear feedback error $P(t)C^*R^{-1}(t)C(z(t) - \hat{z}_P(t))$ such that $P(t)$ satisfies a Riccati equation.

It remains to see in what sense $\hat{z}_P(t)$ in Corollary 7.10 is indeed an observer for a corresponding semi-linear state-space system. The author has already obtained some preliminary results and is investigating under which conditions local exponential convergence for zero disturbance and bounded error in the case of an L^p-disturbance is possible.

Apart from that, numerical investigations could be undertaken to test the theory on specific systems, which has already been done in several cases. It would also be interesting to try to extend the obtained results to fully-nonlinear systems or to obtain similar results under weaker conditions, i.e. local existence of solutions. It would also be interesting to see if extensions are possible to spaces of lesser regularity, for instance Banach spaces.

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