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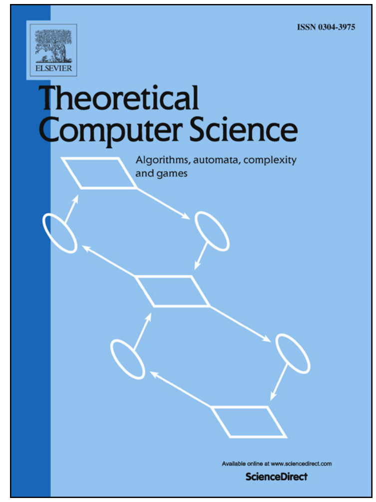

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The number of valid factorizations of Fibonacci prefixes

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Abstract

We establish several recurrence relations and an explicit formula for $V(n)$, the number of factorizations of the length-n prefix of the Fibonacci word into a (not necessarily strictly) decreasing sequence of standard Fibonacci words. In particular, we show that the sequence $V(n)$ is the shuffle of the ceilings of two linear functions of n.

Keywords: numeration systems, Fibonacci numeration system, Fibonacci word

2010 MSC: 68R15, 11B39

1. Introduction

In the classical Fibonacci, or Zeckendorf, numeration system [6, 11], a positive integer is represented as a sum of Fibonacci numbers:

$$n = F_{m_k} + F_{m_{k-1}} + \cdots + F_{m_0},$$

where $m_k > m_{k-1} > \cdots > m_0 \geq 2$ and, as usual, $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. For example, $16 = 13 + 3 = F_7 + F_4 = [100100]_F$, where a digit in brackets is 1 if the respective Fibonacci number appears in the sum, and 0 otherwise. Here a representation ends by the digits corresponding to $F_4 = 3$, $F_3 = 2$ and $F_2 = 1$.

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Under the condition that m_i and m_{i+1} are never consecutive, that is, $m_{i+1} - m_i \geq 2$, or, equivalently, that the Fibonacci numbers F_i are chosen greedily, such a *canonical* representation is unique, and the language L_V of all canonical representations is given by the regular expression $\epsilon + 1(0 + 01)^*$, where the empty word ϵ is the representation of 0. At the same time, if consecutive Fibonacci numbers are allowed, but at most once each, the number of such *legal* representations of n is the well-known integer sequence A000119 from the Online Encyclopedia of Integer Sequences (OEIS) [8]. Its values oscillate between 1 (on numbers of the form $F_i - 1$) and $\sqrt{n+1}$ (on numbers of the form $n = F_i^2 - 1$) [10].

For example, since

$$\begin{aligned} 16 &= 13 + 3 = 8 + 5 + 3 = 8 + 5 + 2 + 1 = 13 + 2 + 1 \\ &= [100100]_F = [11100]_F = [11011]_F = [100011]_F, \end{aligned}$$

the number of legal representations of 16 is 4. Each legal representation of n can be obtained from a canonical one by a series of replacements

$$\cdots 100 \cdots \longleftrightarrow \cdots 011 \cdots ,$$

corresponding to the replacement of a Fibonacci number F_{m+2} by $F_{m+1} + F_m$.

In this paper, we allow even more freedom in Fibonacci representations of n, allowing the transformations

$$\cdots k0l \cdots \longleftrightarrow \cdots (k-1)\ 1(l+1) \cdots \tag{1}$$

for all $k > 0$, $l \geq 0$. Note that the introduced transformation corresponds to passing from a sum of the form $kF_{m+1} + lF_{m-1}$ to the sum $(k-1)F_{m+1} + F_m + (l+1)F_{m-1}$, and, in particular, does not change the represented number.

The representations that can be obtained from the canonical one by a series of transformations as in (1) are called *valid*, and were introduced in [4] in a more general setting because of their link to the Fibonacci word and factorizations of its prefixes, as explained below. Clearly, each legal representation is valid, but

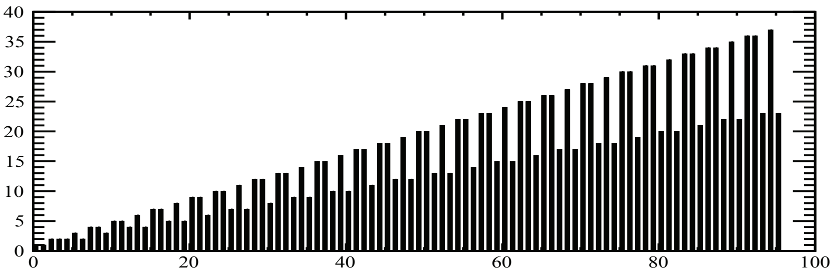

Figure 1: First 100 values of $V(n)$

the opposite is not true. For example, starting from the legal representation $16 = [11011]_F$, we can find two more valid representations

$$16 = [10121]_F = [1221]_F,$$

and starting from the legal representation $16 = [11100]_F$, we find a new representation

$$16 = [20000]_F,$$

so that the total number of valid representations of 16 is 7.

Let $V(n)$ denote the number of valid representations of n. The goal of this paper is to prove a precise formula for $V(n)$, given below in Theorem 1. Our formula demonstrates that the values of $V(n)$ are determined by the shuffle of two straight lines of irrational slope; see Fig. 1.

2. Notation and Sturmian representations

We use notation common in combinatorics on words; the reader is referred, for example, to [3] for an introduction. Given a finite word u, we denote its length by $|u|$. The power u^k just means the concatenation $u^k = \underbrace{u \cdots u}_{k}$. The i'th symbol of a finite or infinite word u is denoted by $u[i]$, so that $u = u[1]u[2]\cdots$. A factor $w[i+1]w[i+2]\cdots w[j]$ of a finite or infinite word w, or, more precisely, its occurrence starting from position $i+1$ of w, is denoted by $w(i..j]$. In particular, for $j \geq 0$, the word $w(0..j]$ is the prefix of w of length j.

The *standard Fibonacci sequence* (f_n) of words over the binary alphabet $\{a, b\}$ is defined as follows:

$$f_{-1} = b, \qquad f_0 = a, \qquad f_{n+1} = f_n f_{n-1} \text{ for all } n \geq 0. \tag{2}$$

The word f_n is called also the *standard word of order* n. In particular, $f_1 = ab$, $f_2 = aba$, $f_3 = abaab$, $f_4 = abaababa$, and so on. From the definition, we easily see that the length of f_n is the Fibonacci number F_{n+2}.

The infinite word

$$\mathbf{f} = \lim_{n\to\infty} f_n = abaababaabaababaababa\cdots$$

is called the Fibonacci infinite word. Here we index it starting with $\mathbf{f}[1] = a$.

In the *Fibonacci*, or *Zeckendorf numeration system*, a non-negative integer $N < F_{n+3}$ is represented as a sum of Fibonacci numbers

$$N = \sum_{0\leq i\leq n} k_i F_{i+2}, \tag{3}$$

where $k_i \in \{0, 1\}$ for $i \geq 0$. In the canonical version of the definition, the following condition holds:

$$\text{for } i \geq 1, \text{ if } k_i = 1, \text{ then } k_{i-1} = 0. \tag{4}$$

Under this nonadjacency condition, the representation of N is unique up to leading zeros. However, by removing the nonadjacency condition, we can get multiple representations: for example, $14 = F_7 + F_2 = F_6 + F_5 + F_2 = F_6 + F_4 + F_3 + F_2$. We call such representations *legal* and denote a representation $N = \sum_{0\leq i\leq n} k_i F_{i+2}$ by $N = [k_n \cdots k_0]_F$. If the condition (4) holds, we call the representation *canonical*.

Let $L(n)$ denote the number of legal representations of n. The sequence $(L(n))$ is well-studied (see, e.g., [2]) and listed in the OEIS as sequence A000119. In particular, $1 \leq L(n) \leq \sqrt{n+1}$, and both bounds are precise [10].

The following lemma is a particular case of [4, Prop. 2].

Lemma 1. *For all $k_0, \ldots, k_n$ such that $k_i \in \{0, 1\}$, the word $f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$ is a prefix of the Fibonacci word $\mathbf{f}$.*

So $L(n)$ is also the number of ways to factor the prefix $\mathbf{f}(0..n]$ of the Fibonacci word as a sequence of standard words in strictly decreasing order.

To expand this definition, in this note we consider all factorizations of Fibonacci prefixes $\mathbf{f}(0..n]$ as a concatenation of standard words in (non-strictly) decreasing order. We write $N = [k_n \cdots k_0]_F$ and call this representation of N *valid* if $k_i \geq 0$ for all i and $\mathbf{f}(0..N] = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$. Note that according to the previous lemma, every legal representation is valid, but not the other way around. For example, $\mathbf{f}(0..14] = (abaab)(aba)(aba)(aba)$, making the representation $14 = [1300]_F$ valid. Theorem 1 of [4] says, in particular, that valid representations are exactly those that can be obtained from the canonical one by a series of transformations (1).

Note that a digit of a valid representation cannot exceed 3 since the Fibonacci word does not contain a factor of the form u^4 for any non-empty word u [5].

The number of valid representations of N is denoted by $V(N)$, and this note is devoted to the study of the sequence $(V(n))$, recently listed in the OEIS as sequence A300066. Clearly, $V(n) \geq L(n)$, and moreover, we prove an explicit formula for $V(n)$ that implies its linear growth.

3. Result

As is well-known, the Fibonacci infinite word

$$\mathbf{f} = abaababa \cdots$$

is the fixed point of the Fibonacci morphism $\mu : a \to ab, b \to a$; moreover, for each $n \geq 1$, we have $f_n = \mu(f_{n-1})$. Consequently, if $N = [k_n \cdots k_0]_F$, then Lemma 1 implies that

$$\mu(\mathbf{f}(0..N]) = \mu(\mathbf{f}(0..[k_n \cdots k_0]_F]) = \mu(f_n^{k_n} \cdots f_0^{k_0}) = f_{n+1}^{k_n} \cdots f_1^{k_0} = \mathbf{f}(0..[k_n \cdots k_0 0]_F].$$

Let φ denote the golden ratio: $\varphi = \frac{1+\sqrt{5}}{2}$. It is important that the Fibonacci word is a Sturmian word of slope $1/(\varphi + 1) = 1/\varphi^2$ and zero intercept (see

Example 2.1.24 of [3]), that is, for all n, we have

$$\mathbf{f}[n] = \begin{cases} a, & \text{if } \{n/\varphi^2\} < 1 - 1/\varphi^2; \\ b, & \text{otherwise.} \end{cases} \tag{5}$$

Here $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Proposition 1. *If $\mathbf{f}[n] = a$, all valid representations of n end with an even number of 0s. If $\mathbf{f}[n] = b$, all of them end with an odd number of 0s.*

Proof. It suffices to consult the definition of a valid representation and notice that f_i ends with a if and only if i is even. □

We now state our main result.

Theorem 1. *If $\mathbf{f}[n] = a$, then $V(n) = \lceil n/\varphi^2 \rceil$, or, equivalently, $V(n)$ is equal to the number of occurrences of b in $\mathbf{f}(0..n]$, plus one. If $\mathbf{f}[n] = b$, then $V(n) = \lceil n/\varphi^3 \rceil$, or, equivalently, $V(n)$ is equal to the number of occurrences of aa in $\mathbf{f}(0..n]$, plus one.*

To prove the theorem, we will need several more propositions.

Proposition 2.

(a) *$V([r0]_F) \geq V([r]_F)$ for all $r \in \{0,1\}^*$.*

(b) *For all $k \geq 0$ and all $r' \in \{0,1\}^*$, we have $V([r'10^{2k+1}]_F) = V([r'10^{2k}]_F)$.*

Proof. (a): Consider a factorization $\mathbf{f}(0..[r]_F] = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$. Applying the Fibonacci morphism μ to both sides, we get the factorization $\mathbf{f}(0..[r0]_F] = f_{n+1}^{k_n} f_n^{k_{n-1}} \cdots f_1^{k_0}$. So the number of factorizations of $\mathbf{f}(0..[r0]_F]$ (which is equal to $V([r0]_F)$) is at least as large as the number of factorizations of $\mathbf{f}(0..[r]_F]$ (which is equal to $V([r]_F)$).

(b) If, in addition $r = r'10^{2k}$ for some $k \geq 0$, we see that $\mathbf{f}(0..[r]_F]$ ends with f_{2k} and $\mathbf{f}(0..[r0]_F]$ ends with f_{2k+1}, which in turn ends with b. From Proposition 1, no factorization of $\mathbf{f}(0..[r0]_F]$ ends with f_0; that is, such a factorization must be of the form $\mathbf{f}(0..[r0]_F] = f_{n+1}^{k_n} f_n^{k_{n-1}} \cdots f_1^{k_0}$. Taking the μ-preimage, we get the factorization $\mathbf{f}(0..[r]_F] = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$, thus establishing a bijection and the equality $V([r'10^{2k+1}]_F) = V([r'10^{2k}]_F)$. □

Proposition 3. *We have*

$$V([z10^{2k}]_F) = V([z10^{2k-2}]_F) + V([z(01)^k]_F).$$

for all $z \in \{0,1\}^*$ *and all* $k \geq 1$.

Proof. Proposition 1 tells us that $\mathbf{f}[[z10^{2k}]_F] = a$, and moreover, since $k > 0$, the prefix of length $[z10^{2k}]_F$ of $\mathbf{f}$ ends with aba, which is a suffix of f_{2k}. Consider a valid factorization $\mathbf{f}(0..[z10^{2k}]_F] = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$. If $k_0 = 0$, then $k_1 = 0$ since f_1 ends with b, so the factorization is of the form $f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_2^{k_2}$. Taking the μ^2-preimage, we get a factorization $f_{n-2}^{k_n} f_{n-3}^{k_{n-1}} \cdots f_0^{k_2}$ of $\mathbf{f}(0..[z10^{2k-2}]_F]$. Moreover, μ^2 is a bijection between all the factorizations of $\mathbf{f}(0..[z10^{2k-2}]_F]$ and the factorizations of $\mathbf{f}(0..[z10^{2k}]_F]$ with $k_0 = k_1 = 0$.

On the other hand, if $k_0 \neq 0$, then $k_0 = 1$ since the word that we factor ends with aba. Removing this last occurrence of $f_0 = a$, we get the prefix of $\mathbf{f}$ of length $[z10^{2k}]_F - 1 = [z(01)^k 0]_F$. From Proposition 2, the number of valid factorizations of $\mathbf{f}(0..[z(01)^k 0]_F]$ is equal to that of $\mathbf{f}(0..[z(01)^k]_F]$. Combining the two possibilities, we get the statement of the proposition. □

Proposition 4. *For all* $z \in \{0,1\}^*$ *and for all* $k \geq 1$, *we have*

$$V([z10^k 1]_F) = \begin{cases} V([z10^{k+1}]_F), & \text{if } k \text{ is odd;} \\ V([z10^k]_F) + V([z(01)^{k/2}]_F), & \text{if } k \text{ is even.} \end{cases}$$

Proof. If k is odd, then $[z10^k 1]_F = [z10^{k+1}]_F + 1$, and the prefix $\mathbf{f}(0..[z10^{k+1}]_F]$ was considered in the previous proposition. It ends with aba, and the symbol added to get $\mathbf{f}(0..[z10^k 1]_F]$ is also a. So $\mathbf{f}(0..[z10^k 1]_F]$ ends with $abaa$, and all valid factorizations end with f_0. This means that the number of valid factorizations of $\mathbf{f}(0..[z10^k 1]_F]$ is equal to that of $\mathbf{f}[0..[z10^{k+1}]_F]$; that is, $V([z10^k 1]_F) = V([z10^{k+1}]_F)$.

If k is even, $k > 0$, then $\mathbf{f}(0..[z10^k 1]_F]$ ends with $f_3 f_0 = abaaba$. In particular, the last factor of any valid factorization of $\mathbf{f}(0..[z10^k 1]_F]$ is either $f_0 = a$, or $f_2 = aba$. Indeed, $f_4 = abaababa$ and thus for all $l > 2$ the f_{2l} do not have a common suffix with $\mathbf{f}(0..[z10^k1]_F]$. So, letting $V_2(n)$ denote the number of factorizations of $\mathbf{f}(0..n]$ of the form $f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_2^{k_2}$, we get

$$\begin{aligned} V([z10^k1]_F) &= V([z10^k1]_F - 1) + V_2([z10^k1]_F - 3) \\ &= V([z10^{k+1}]_F) + V_2([z(01)^{k/2}00]_F) \\ &= V([z10^k]_F) + V([z(01)^{k/2}]_F). \end{aligned}$$

Here the last equality follows from Proposition 2 (for the first addend) and by taking μ^{-2} of each factorization (for the second one). $\square$

Propositions 2 to 4 give a full list of recurrence relations sufficient to compute $V(n)$ for every $n > 1$, starting from $V(1) = 1$. Before using them to prove the main theorem, we consider two particular cases.

Corollary 1. *For all $k \geq 1$ we have*

$$V(F_{2k+1} - 1) = V(F_{2k+1} - 2) = F_{2k-1}$$

and

$$V(F_{2k+2} - 2) = F_{2k}$$

Proof. For $k = 1$, the equalities can be easily checked: $V(F_3 - 1) = V(1) = V(F_3 - 2) = V(0) = 1 = F_1$, and $V(F_4 - 2) = V(1) = 1 = F_2$. We also observe that $F_{2k+1} - 1 = [(10)^{k-1}1]_F$, $F_{2k+1} - 2 = [(10)^{k-1}0]_F$, and $F_{2k+2} - 2 = [(10)^{k-1}01]_F$. Now we assume that the equalities hold for k, and use Propositions 3 and 4 to prove they hold for $k + 1$:

$$\begin{aligned} V(F_{2k+3} - 2) &= V([(10)^k0]_F) = V([(10)^{k-1}1]_F) + V([(10)^{k-1}01]_F) \\ &= V(F_{2k+1} - 1) + V(F_{2k+2} - 2) = F_{2k-1} + F_{2k} = F_{2k+1}, \\ V(F_{2k+3} - 1) &= V([(10)^k1]_F) = V([(10)^k0]_F) = V(F_{2k+3} - 2) = F_{2k+1}, \\ V(F_{2k+4} - 2) &= V([(10)^k01]_F) = V([(10)^k0]_F) + V([(10)^{k-1}01]_F) \\ &= V(F_{2k+3} - 2) + V(F_{2k+2} - 2) = F_{2k+1} + F_{2k} = F_{2k+2}. \end{aligned}$$

$\square$

Corollary 2. *For all $k \geq 1$, we have*

$$V(F_{2k}) = V(F_{2k+1}) = F_{2k-2} + 1.$$

Proof. For $k = 1$, the equalities can be easily checked: $V(F_2) = V(1) = V(F_3) = V(2) = 1 = F_0 + 1$. Suppose the equalities hold for k; let us prove them for $k + 1$. With Proposition 3, we have

$$V(F_{2k+2}) = V([10^{2k}]_F) = V([10^{2k-2}]_F) + V([(10)^{k-1}1]_F) = F_{2k-2} + 1 + F_{2k-1} = F_{2k} + 1,$$

and with Proposition 2, we have

$$V(F_{2k+3}) = V([10^{2k+1}]_F) = V([10^{2k}]_F) = V(F_{2k+2}) = F_{2k} + 1.$$

□

Proposition 5. *Let $n = [z]_F$ and $n' = [z0]_F$ be such that $\mathbf{f}[n] = a$. Then $\lceil n/\varphi^2 \rceil = \lceil n'/\varphi^3 \rceil$.*

Proof. Let us write the canonical Fibonacci representation of n as $\sum_{1 \leq i \leq l} F_{m_i}$, where $2 \leq m_1 < m_2 < \cdots < m_l$. Since $\mathbf{f}[n] = a$, from Proposition 1 we get that m_1 is even.

Now $F_k = \frac{1}{\sqrt{5}}(\varphi^k - \psi^k)$, where $\psi = \frac{1-\sqrt{5}}{2}$, $-1 < \psi < 0$. So

$$n = \sum_{1 \leq i \leq l} F_{m_i} = \frac{1}{\sqrt{5}} \left(\sum_{1 \leq i \leq l} \varphi^{m_i} - \sum_{1 \leq i \leq l} \psi^{m_i} \right)$$

and

$$n' = \sum_{1 \leq i \leq l} F_{m_i+1} = \frac{1}{\sqrt{5}} \left(\sum_{1 \leq i \leq l} \varphi^{m_i+1} - \sum_{1 \leq i \leq l} \psi^{m_i+1} \right),$$

implying that

$$\frac{n'}{\varphi} = \frac{1}{\sqrt{5}} \left(\sum_{1 \leq i \leq l} \varphi^{m_i} - \frac{1}{\varphi} \sum_{1 \leq i \leq l} \psi^{m_i+1} \right).$$

The difference between the two values is

$$\frac{n'}{\varphi} - n = \frac{1}{\sqrt{5}} \left(1 - \frac{\psi}{\varphi} \right) S,$$

where

$$S = \sum_{1\le i\le l} \psi^{m_i} = \psi^{m_1} \sum_{1\le i\le l} \psi^{m_i - m_1}.$$

Let us estimate S. Since $m_1 \ge 2$, m_1 is even and $0 < \psi^{m_1} < \psi^2$, an upper bound for S is

$$S < \psi^{m_1} \sum_{k=0}^{\infty} \psi^{2k} = \frac{\psi^{m_1}}{1-\psi^2} \le \frac{\psi^2}{1-\psi^2},$$

whereas a lower bound is

$$S > \psi^{m_1}\left(1 + \sum_{k=1}^{\infty} \psi^{2k+1}\right) > \psi^{m_1}\left(1 + \sum_{k=0}^{\infty} \psi^{2k+1}\right) = \psi^{m_1}\left(1 + \frac{\psi}{1-\psi^2}\right) = 0.$$

So

$$0 < \frac{n'}{\varphi} - n < \frac{\psi^2}{\sqrt{5}}\left(1 - \frac{\psi}{\varphi}\right)\frac{1}{1-\psi^2} = \frac{1}{\varphi^2}.$$

Dividing by φ^2, we get

$$0 < \frac{n'}{\varphi^3} - \frac{n}{\varphi^2} < \frac{1}{\varphi^4} < \frac{1}{\varphi^2}.$$

Together with (5), meaning that $\{n/\varphi^2\} < 1 - 1/\varphi^2$, the last inequality implies the statement of the Proposition. □

Proof of Theorem 1. Let us start with the case of $\mathbf{f}[n] = a$ and proceed by induction starting with $V(1) = 1$. For $n > 1$, there are three subcases:

(a) $n = [z10^{2k}]_F$, $k > 0$;

(b) $n = [z10^k1]_F$, k odd;

(c) $n = [z10^k1]_F$, k even.

From now on we suppose that the statement of the theorem holds for all $n', n'' < n$.

(a) Since $n = [z10^{2k}]_F$ and $k > 0$, Proposition 3 gives $V(n) = V([z10^{2k}]_F) = V([z10^{2k-2}]_F) + V([z(01)^k]_F)$. Write $[z10^{2k-2}]_F = n'$ and $[z(01)^k]_F = n''$. Note that Proposition 1 gives $\mathbf{f}[n'] = \mathbf{f}[n''] = a$. At the same time, $n''+1 = [z10^{2k-1}]_F$ and thus $\mathbf{f}[n''+1] = b$. Now (5) implies that $\{n'/\varphi^2\} \in (0, 1 - 1/\varphi^2)$ and $\{(n''+1)/\varphi^2\} \in (1 - 1/\varphi^2, 1)$. Also, the Fibonacci representation of n' is obtained from that of $n''+1$ by a one-symbol shift to the left. So, summing up n' and $n''+1$, due to the Fibonacci recurrence relation, we get the number with the same representation but shifted to the left yet another position, meaning that $n'+n''+1=n$.

Let us consider the sum $t=\{n'/\varphi^2\}+\{(n''+1)/\varphi^2\}$. From the inclusions above, we see that t belongs to the interval $(1-1/\varphi^2, 2-1/\varphi^2)$. But we also know that $\{n/\varphi^2\}=\{(n'+n''+1)/\varphi^2\}\in(0,1-1/\varphi^2)$, since $\mathbf{f}[n]=a$. So

$$\{n/\varphi^2\}=\{n'/\varphi^2\}+\{(n''+1)/\varphi^2\}-1,$$

which is equivalent to $\lfloor n/\varphi^2\rfloor=\lfloor n'/\varphi^2\rfloor+\lfloor(n''+1)/\varphi^2\rfloor+1$ and to $\lfloor n/\varphi^2\rfloor=\lfloor n'/\varphi^2\rfloor+\lfloor n''/\varphi^2\rfloor+1$ (since $\lfloor n''/\varphi^2\rfloor=\lfloor(n''+1)/\varphi^2\rfloor$). Since all the numbers under consideration are irrational, and thus every ceiling is just the floor plus 1, we get

$$\lceil n/\varphi^2\rceil=\lceil n'/\varphi^2\rceil+\lceil n''/\varphi^2\rceil.$$

To establish the statement of the theorem for this subcase, it is sufficient to use Proposition 3 and the induction hypothesis: $V(n')=\lceil n'/\varphi^2\rceil$ and $V(n'')=\lceil n''/\varphi^2\rceil$.

(b): Here $n=[z10^{2k-1}1]_F$ and $k>0$. It suffices to refer to the previous subcase and to Proposition 4: $V(n)=V(n-1)=V([z10^{2k}]_F)=\lceil(n-1)/\varphi^2\rceil$. It remains to notice that $\lceil(n-1)/\varphi^2\rceil=\lceil n/\varphi^2\rceil$, since $\mathbf{f}[n-1]=a$.

(c): Here $n=[z10^{2k}1]_F$ and $k>0$. We use Proposition 4: $V([z10^{2k}1]_F)=V([z10^{2k}]_F)+V([z(01)^k]_F)$. As above, write $n'=[z10^{2k}]_F$ and $n''=[z(01)^k]_F$; then $n=n'+n''+2$, whereas $V(n)=V(n')+V(n'')$. By the induction hypothesis, $V(n')=\lceil n'/\varphi^2\rceil$ and $V(n'')=\lceil n''/\varphi^2\rceil$.

We have $\mathbf{f}[n]=a$ and $\mathbf{f}[n-1]=b$, implying from (5) that $\{(n-1)/\varphi^2\}\in(1-1/\varphi^2,1)$ and thus $\{n/\varphi^2\}\in(0,1/\varphi^2)$. At the same time, $\mathbf{f}[n']=\mathbf{f}[n'']=a$ implies $\{n'/\varphi^2\},\{n''/\varphi^2\}\in(0,1-1/\varphi^2)$ and thus

$$\{n'/\varphi^2\}+\{n''/\varphi^2\}+\{2/\varphi^2\}\in(2/\varphi^2,2).$$

Comparing it to $\{n/\varphi^2\} = \{(n' + n'' + 2)/\varphi^2\} \in (0, 1/\varphi^2)$, we see that

$$\{n/\varphi^2\} = \{n'/\varphi^2\} + \{n''/\varphi^2\} + \{2/\varphi^2\} - 1.$$

But since $n = n' + n'' + 2$ and $x = \lfloor x \rfloor + \{x\}$ for every x, this also means that

$$\lfloor n/\varphi^2 \rfloor = \lfloor n'/\varphi^2 \rfloor + \lfloor n''/\varphi^2 \rfloor + 1.$$

Finally, since k/φ^2 is not an integer for any integer $k > 0$, we have $\lceil k/\varphi^2 \rceil = \lfloor k/\varphi^2 \rfloor + 1$, so that

$$\lceil n/\varphi^2 \rceil = \lceil n'/\varphi^2 \rceil + \lceil n''/\varphi^2 \rceil.$$

It remains to use the induction hypothesis to establish

$$V(n) = V(n') + V(n'') = \lceil n'/\varphi^2 \rceil + \lceil n''/\varphi^2 \rceil = \lceil n/\varphi^2 \rceil,$$

which was to be proved.

To complete the part of the proof concerning $\mathbf{f}[n] = a$, it remains to notice that $\lceil n/\varphi^2 \rceil$ is equal to the number of bs in $\mathbf{f}(0..n]$ plus one, due to (5).

Now for $\mathbf{f}[n] = b$, it is sufficient to combine Propositions 1, 2 and 5: if $\mathbf{f}[n] = b$, then $n = [r0]_F$, where $m = [r]_F$ and $\mathbf{f}[m] = a$. Then

$$V(n) = V(m) = \lceil m/\varphi^2 \rceil = \lceil n/\varphi^3 \rceil.$$

Here $\mathbf{f}(0..n] = \mu(\mathbf{f}(0..m])$, and so the occurrences of aa in $\mathbf{f}(0..n]$ correspond exactly to occurrences of b in $\mu(\mathbf{f}(0..m])$. The theorem is proved. □

The theorem ensures that the sequence $(V(n))$ grows as depicted in Fig. 1. The two visible straight lines correspond to the symbols of the Fibonacci word equal to a (the upper line) or b (the lower line).

4. Fibonacci-regular representation

A sequence $(s(n))_{n\geq 0}$ is said to be *Fibonacci-regular* if there exist an integer k, a row vector v of dimension k, a column vector w of dimension k, and a $k \times k$ matrix-valued morphism ρ on $\{0, 1\}^*$ such that

$$s([z]_F) = v\rho(z)w$$

for all canonical Fibonacci representations $z \in L_V$. The triple (v, ρ, w) is called a *linear representation*; see, for example, [7].

Berstel [2] gave the following linear representation for the function $L(n)$ we mentioned previously in Section 2:

$$v = [1\,0\,0\,0], \quad \rho(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \rho(1) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $L(n)$ is Fibonacci-regular.

We can find a similar representation for the function $V(n)$. For technical reasons it is easier to deal with the reversed Fibonacci representation; one can then obtain the ordinary linear representation by interchanging the roles of the vectors and taking the transposes of the matrices.

Theorem 2. *$V(n)$ has the reversed linear representation (t, γ, u), where*

$$t = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0], \qquad u = [1\ 1\ 1\ 1\ 1\ 2\ 1\ 4]^T$$

$$\gamma(0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -3 & 3 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 & 3 & 0 & 1 & 0 \end{bmatrix}, \quad \gamma(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. Define $g(x) = V([x]_F)$ if x is a valid canonical representation (that is, containing no leading zeroes, and no two consecutive 1's), and 0 otherwise. It suffices to show, for all $x \in \{0,1\}^*$ and $i \in \{0,1\}$, that

$$\begin{bmatrix} g(xi) \\ g(xi0) \\ g(xi1) \\ g(xi00) \\ g(xi000) \\ g(xi100) \\ g(xi0000) \\ g(xi10000) \end{bmatrix} = \gamma(i) \begin{bmatrix} g(x) \\ g(x0) \\ g(x1) \\ g(x00) \\ g(x000) \\ g(x100) \\ g(x0000) \\ g(x10000) \end{bmatrix}. \tag{6}$$

Once we prove this, it is then easy to see (using induction on $|z|$) that, if z is the Fibonacci representation of n, then $t\gamma(z^R)u = V(n)$, where z^R is the reversal of z.

Thus it suffices to verify Eq. (6). This is equivalent to proving the following identities for x.

$$g(x01) = -g(x) + g(x0) + g(x00) \tag{7}$$

$$g(x10) = g(x1) \tag{8}$$

$$g(x0100) = -g(x) + 2g(x00) + g(x000) \tag{9}$$

$$g(x1000) = g(x100) \tag{10}$$

$$g(x010000) = -g(x) - g(x0) + 2g(x00) + 3g(x000) + g(x0000) \tag{11}$$

$$g(x00000) = g(x) - g(x0) - 3g(x00) + 3g(x000) + g(x0000). \tag{12}$$

Identities (8) and (10) are particular cases of Proposition 2 (b).

To prove (7), consider separately two cases: if x ends with an even number of zeros, then $g(x) = g(x0)$ due to Proposition 2 (b) and $g(x00) = g(x01)$ due to Proposition 4, so the identity holds. If x ends with an odd number of zeros, $x = z10^{2k+1}$, $k \geq 0$, then due to Proposition 4,

$$g(x01) = g(z10^{2k+2}1) = g(z10^{2k+2}) + g(z(01)^{k+1}) = g(x0) + g(z(01)^{k+1}).$$

On the other hand, due to Propositions 2 and 3,

$$g(x00) = g(x0) = g(z10^{2k+2}) = g(z10^{2k}) + g(z(01)^{k+1}) = g(x) + g(z(01)^{k+1}).$$

Comparing these equalities, we get (7).

To prove (9), it is sufficient to use Proposition 3 to get

$$g(x0100) = g(x01) + g(x001),$$

and then to use (7) twice, for $g(x01)$ and for $g(x001)$.

To prove (11), it is sufficient to use Propositions 3 and 2 to get

$$g(x010000) = g(x0100) + g(x00101) = g(x0100) + g(x00100).$$

Now (11) is obtained immediately by summing up (9) applied to x and to $x0$.

Finally, to prove (12), we again have to consider two cases. If $x = z10^{2k}$, $k \geq 0$, then due to Proposition 2, $g(x0^5) = g(x0000)$, $g(x000) = g(x00)$, $g(x0) = g(x)$, and the equality holds. If now $x = z10^{2k+1}$, $k \geq 0$, then (12) immediately simplifies with Proposition 2 as

$$g(z10^{2k+6}) - g(z10^{2k+4}) = 3[g(z10^{2k+4}) - g(z10^{2k+2})] - [g(z10^{2k+2}) - g(z10^{2k})].$$

Applying Proposition 3, we reduce it to

$$g(z(01)^{k+3}) = 3g(z(01)^{k+2}) - g(z(01)^{k+1}),$$

or, writing $y = z(01)^{k+1}$ and applying Proposition 4 again,

$$g(y0100) = 3g(y00) - g(y).$$

But this is exactly (9) since $g(y00) = g(y000)$.

□

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