Sidon and Kronecker-like Sets in Compact Abelian Groups

by

Xu Yang

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Mahmoud Filali University Lecturer in Mathematics, Dept. of Mathematical Sciences, University of Oulu
Supervisor:	Kathryn Hare Professor, Dept. of Pure Mathematics, University of Waterloo
Internal Members:	Brian Forrest Professor, Dept. of Pure Mathematics, University of Waterloo Nico Spronk Professor, Dept. of Pure Mathematics, University of Waterloo
Internal-External Member:	David Siegel Professor, Dept. of Applied Mathematics, University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let G be a compact abelian group and Γ be its discrete dual group. In this thesis we study various types of interpolation sets.

A subset $E \subset \Gamma$ is Sidon if every bounded function on E can be interpolated by the Fourier transform of a finite complex measure. Sidon sets have been extensively studied, and one significant breakthrough, that Sidonicity is equivalent to proportional quasi-independence, was proved by Bourgain and Pisier during the early 80s. In this thesis we will give a detailed demonstration of Pisier's approach. We also seek for possible extensions of Pisier's theorems. Based on Pisier's techniques, we will show Sidonicity is equivalent to proportional independence of higher degrees and minimal constants.

A subset $E \subset \Gamma$ is ε-Kronecker if every function on E with range in the unit circle can be interpolated by a continuous character on Γ with an error of ε. We will prove some interesting properties of ε-Kronecker sets and give an estimation of the Sidon constant of such sets. Generalizations of Kronecker sets include binary Kronecker sets and N-pseudo-Rademacher sets. We compute the binary Kronecker constants of some interesting examples. For N-pseudo-Rademacher sets, we give a characterization of such sets, describe their structures and prove the existence of large N-pseudo-Rademacher sets.

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Dedication

This is dedicated to my parents and my girlfriend Xinrui Zhang.

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Chapter 1

Introduction

We begin by introducing the basic concepts and notations that will be used throughout the thesis.

1.1 Basic harmonic analysis background

Let G be a locally compact abelian group with its Haar measure m. The **dual group** of G, $\widehat{G}$ or Γ, contains all the continuous multiplicative characters on G. When we identify Γ as a subset in $L^\infty(G) \cong L^1(G)^*$ equipped with the weak* topology, Γ becomes a locally compact, abelian topological group with its Haar measure.

There is a natural embedding, ϕ, between G and the dual group of Γ, $\phi : G \to \widehat{\Gamma}$, defined by $\phi(x)(\gamma) = \gamma(x)$ for $x \in G$ and $\gamma \in \Gamma$.

Theorem 1.1.1 (Pontryagin duality theorem)**.** *The embedding given above between G and $\widehat{\Gamma}$ is a homeomorphism of G onto $\widehat{\Gamma}$.*

Proposition 1.1.2. *G is compact if and only if Γ is discrete.*

When G is compact, the Haar measure m is normalized so that $m(G) = 1$. The Haar measure on the discrete group Γ is the counting measure.

Example 1.1.3. (1) When $G = \mathbb{T}$, the unit circle group in $\mathbb{C}$, Γ can be identified as the group of integers $\mathbb{Z}$.
(2) When $G = \mathbb{Z}_n$ for $n \in \mathbb{N}$, the group of n-th roots of unity, Γ is also $\mathbb{Z}_n$.
(3) When $G = \prod_{i\in I} G_i$ is a product of locally compact abelian groups G_i, $\Gamma = \bigoplus_{i\in I} \Gamma_i$ is the direct sum of the dual groups $\Gamma_i = \widehat{G_i}$.

Notation 1.1.4. Suppose $\Gamma = \bigoplus_{j\in I} \Gamma_j$. For $j \in I$ we define the projection $\mathrm{Proj}_j : \Gamma \to \Gamma_j$ by letting $\mathrm{Proj}_j(\gamma)$ be the j-th coordinate of γ.

Definition 1.1.5. (1) The group Γ is **torsion-free** if Γ has no elements of finite order (except the identity).
(2) The group G is **divisible** if for all $x \in G$ and $n \in \mathbb{N}$ there exists $y \in G$ such that $y^n = x$.

Proposition 1.1.6. *Suppose G is compact. The following are equivalent:*
(1) Γ is torsion-free.
(2) G is connected.
(3) G is divisible.

For $f \in L^1(G)$, the **Fourier transform** is given by $\widehat{f} : \Gamma \to \mathbb{C}$,

$$\widehat{f}(\gamma) := \int_G f(x)\overline{\gamma(x)}\ dm.$$

The **Fourier-Stieltjes transform** (or just Fourier transform) of a finite complex measure $\mu \in M(G)$ is defined similarly:

$$\widehat{\mu}(\gamma) := \int_G \overline{\gamma(x)}\ d\mu(x).$$

Example 1.1.7. (1) For $\gamma \in \Gamma$, the Haar measure m of a compact abelian group G satisfies

$$\widehat{m}(\gamma) = \begin{cases} 0 & \text{if } \gamma \neq 1 \\ m(G) & \text{if } \gamma = 1 \end{cases}.$$

(2) Let $a \in G$ and $\gamma \in \Gamma$. The point mass measure δ_a has

$$\widehat{\delta_a}(\gamma) = \overline{\gamma(a)}.$$

Definition 1.1.8. For $f_1, f_2 \in L^1(G)$, the **convolution** $f_1 * f_2 \in L^1(G)$ is defined as

$$f_1 * f_2(x) = \int_G f_1(t) f_2(x - t)\ dm(t).$$

For $\mu_1, \mu_2 \in M(G)$, $\mu_1 * \mu_2 \in M(G)$ is defined as

$$\int_G f\ d\mu_1 * \mu_2 = \int_G \int_G f(x + y)\ d\mu_1(x) d\mu_2(y)$$

for simple functions f.

Definition 1.1.9. Suppose G is compact.
(1) For $f \in L^1(G)$, the **Fourier series** of f is the series $\sum_{\gamma \in \Gamma} \widehat{f}(\gamma)\gamma$.
(2) The set of **trigonometric polynomials**, $\mathrm{Trig}(G)$, is defined as the set of all $f \in L^1(G)$ such that $\widehat{f}$ is supported on a finite set. Furthermore, for $E \subset \Gamma$ we define $\mathrm{Trig}_E(G)$ to be the set of all $f \in \mathrm{Trig}(G)$ such that $\widehat{f}$ is supported on E.

Theorem 1.1.10. *(1) For all $\mu_1, \mu_2 \in M(G)$,*

$$\widehat{\mu_1 + \mu_2} = \widehat{\mu_1} + \widehat{\mu_2}$$

and

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1}\widehat{\mu_2}.$$

Moreover, $||\mu_1 * \mu_2||_{M(G)} \leq ||\mu_1||_{M(G)}\, ||\mu_2||_{M(G)}$.

(2) For $f \in L^1(G)$ and $\mu \in M(G)$, we have

$$\sup_{\gamma\in\Gamma} |\widehat{f}(\gamma)| \leq ||f||_1$$

and

$$\sup_{\gamma\in\Gamma} |\widehat{\mu}(\gamma)| \leq ||\mu||_{M(G)} .$$

(3) [Riemann-Lebesgue Lemma] If $f \in L^1(G)$, then $\widehat{f} \in C_0(\Gamma)$.
(4) [Parserval's Theorem] If $f \in L^2(G)$, then $\hat{f} \in L^2(\Gamma)$ and

$$||f||_{L^2(G)} = \left|\left|\widehat{f}\right|\right|_{L^2(\Gamma)} .$$

Let $H \subset G$ be a closed subgroup. We denote by $H^\perp$ the annihilator of H. We have the following relation between dual groups and quotient groups.

Proposition 1.1.11. *$H^\perp = \widehat{G/H}$ and $\widehat{H} = \Gamma/H^\perp$.*

In this thesis, the unit circle group $\mathbb{T}$ is identified in multiple ways. When $\mathbb{T}$ is the unit circle in the complex plane, $\mathbb{T} = \{e^{ix} : x \in [0, 2\pi]\}$, the group operation is multiplication and duality is given by $n(e^{ix}) = e^{inx}$ for $n \in \mathbb{Z}$. The trigonometric polynomials have the form $p(z) = \sum_{j=-N}^{N} a_j z^j$.

When $\mathbb{T}$ is identified as an interval $[0, 1]$ in chapters 4 and 5, then the group operation is addition mod integers and the duality is multiplication mod integers. The trigonometric polynomials have the form $p(x) = \sum_{j=-N}^{N} a_j e^{2\pi ijx}$.

Notice that when $\mathbb{T}$ is the unit circle group in $\mathbb{C}$, $\mathbb{Z}_n$, the group of n-th roots of unity, is

$$\mathbb{Z}_n = \left\{e^{2\pi ij/n} : 0 \leq j \leq n-1\right\} .$$

When $\mathbb{T} = [0,1]$,

$$\mathbb{Z}_n = \{k/n : 0 \leq k \leq n-1\}.$$

Next we recall the Bohr compactification of Γ. Consider the group G equipped with the discrete topology, G_d, and the dual group of G_d, $\overline{\Gamma}$. Clearly $\Gamma \subset \overline{\Gamma}$. By Proposition 1.1.2, $\overline{\Gamma}$ is compact.

Definition 1.1.12. The compact group $\overline{\Gamma}$ is called the **Bohr compactification** of Γ.

Proposition 1.1.13. Γ *is a dense subgroup in* $\overline{\Gamma}$.

Example 1.1.14. In the case that $G = \mathbb{T}$ and $\Gamma = \mathbb{Z}$, basic open neighborhoods around 0 in $\overline{\mathbb{Z}}$ are the sets $\left\{\beta \in \overline{\mathbb{Z}} : |1 - \beta(x_i)| < \varepsilon \ \forall 1 \leq i \leq N\right\}$ for finite collections $\{x_1, ..., x_N\} \subset \mathbb{T}$ and $\varepsilon > 0$.

More details on this background information can be found in [35].

1.2 Independent sets and the Riesz product

From here on G will denote a compact abelian group with identity element e, and Γ will be its discrete dual group with identity element 1. The books by Lopez and Ross [25] and Graham and Hare [9] are good references for the material discussed in the rest of this chapter and include the proofs of any results not otherwise referenced.

Definition 1.2.1. A set $E \subset \Gamma \backslash \{1\}$ is **independent** if for all $j \in \mathbb{N}$, distinct $\gamma_1, ..., \gamma_j \in E$ and $m_1, ..., m_j \in \mathbb{Z}$, $\prod_{n=1}^{j} \gamma_n^{m_n} = 1$ implies $\gamma_n^{m_n} = 1$ for all $1 \leq n \leq j$. This equivalent to saying $m_n = 0$ for all n if Γ is torsion-free.

The notion of independence is classical. One example of an independent set is the **Rademacher set** defined as follows. Let $G = \prod_1^\infty \mathbb{Z}_2$ and $\Gamma = \bigoplus_1^\infty \mathbb{Z}_2 = \mathbb{Z}_2^\infty$. The Rademacher set is $\{\pi_j : j \geq 1\} \subset \Gamma$, where each π_j has only a non-trivial entry at coordinate j.

Independent sets are known to have good interpolation properties. For example, in the case that Γ is torsion-free and $E \subset \Gamma$ is independent, for every $\varphi : E \to \mathbb{T}$ there exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$. More specifically, we will prove the following.

Proposition 1.2.2. *Let $E \subset \Gamma$. The following are equivalent:*

(1) For all $\varphi : E \to \mathbb{T}$ with $\varphi(\gamma) \in \mathrm{Range}(\gamma)$ there exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for $\gamma \in E$.

(2) The set E is independent.

Proof. Assume E is independent and that $\varphi : E \to \mathbb{T}$ satisfies $\varphi(\gamma) \in \mathrm{Range}(\gamma)$ for $\gamma \in E$. We let $\langle E \rangle$ be the subgroup generated by E. By Proposition 1.1.11, we can deduce that if we can find $x \in \widehat{\langle E \rangle}$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$, then there exists $x' \in G$ such that $\varphi(\gamma) = \gamma(x')$. Thus we may assume $\Gamma = \langle E \rangle = \bigoplus_{\gamma \in E} \langle \gamma \rangle$. For each $\gamma \in E$, there exists $x_\gamma \in \widehat{\langle \gamma \rangle}$ such that $\gamma(x_\gamma) = \varphi(\gamma)$. If E is finite, we let $x = \prod_{\gamma \in E} x_\gamma$ and x can interpolate φ exactly. For the case that E is infinite, since G is compact, we let $x \in G$ be a cluster point of the following set,

$$\left\{ \prod_{\gamma \in F} x_\gamma : F \subset E, |F| < \infty \right\},$$

and such an x can interpolate φ exactly.

Conversely, if E is not independent, then there exist $k \in \mathbb{N}$, $\gamma_1, ..., \gamma_k \in E$ and $m_1, ..., m_k \in \mathbb{Z}$ such that $\gamma_1^{m_1} ... \gamma_k^{m_k} = 1$, but $\gamma_i^{m_i} \neq 1$ for all $1 \leq i \leq k$. Consider the function $\varphi : E \to \mathbb{T}$ such that $\varphi(\gamma_i) = 1$ for all $i > 1$, $\varphi(\gamma_1)^{m_1} \neq 1$ and $\varphi(\gamma) \in \mathrm{Range}(\gamma)$ for all $\gamma \in E$. Notice that such φ exists, because $\gamma_1^{m_1} \neq 1$ means $\mathrm{Range}(\gamma_1^{m_1}) \neq \{1\}$ and hence there exists $x \in G$ such that $\gamma_1(x)^{m_1} \neq 1$. Let $\varphi(\gamma_1) = \gamma_1(x)$. This function φ cannot be interpolated by any $x \in G$. $\square$

Remark 1.2.3. Suppose $E \subset \Gamma$ is independent and Γ is torsion-free. Notice that $\gamma(x) = \widehat{\delta_{x^{-1}}}(\gamma)$, and therefore using Proposition 1.2.2 we see that for all bounded $\varphi : E \to \mathbb{T}$ there exists $\mu \in M(G)$ such that $\varphi = \widehat{\mu}$ on E.

However, in the case that $G = \mathbb{T}$ and $\Gamma = \mathbb{Z}$, there are no independent subsets other than singleton sets. Thus weaker notions of independence have been introduced.

Definition 1.2.4. (1) A set $E \subset \Gamma$ is **dissociate** if for all $j \in \mathbb{N}$, distinct $\gamma_1, ..., \gamma_j \in E$ and $m_1, ..., m_j \in \{\pm 2, \pm 1, 0\}$, $\prod_{n=1}^{j} \gamma_n^{m_n} = 1$ implies $m_n = 0$ for all $1 \leq n \leq j$.
(2) A set $E \subset \Gamma$ is **quasi-independent** if the same statement holds for $m_1, ..., m_j \in \{\pm 1, 0\}$.

Important examples in $\mathbb{Z}$ include the lacunary sets.

Definition 1.2.5. A positive integer sequence $(n_j)_{j=0}^{\infty}$ is a **lacunary** set of **ratio** $q > 1$ if $n_j/n_{j-1} \geq q$ for all $j \geq 1$.

Example 1.2.6. The lacunary integer sequence $\{n_j : j \geq 0\}$ of ratio q is quasi-independent if $q \geq 2$ and is dissociate if $q \geq 3$.

These weaker notions of independence also have the interesting interpolation properties. This can be shown by what is known as a Riesz product construction.

Assume $E \subset \Gamma$ is a dissociate set and Γ has no elements of order two. Suppose we have a function $\varphi : E \to \mathbb{C}$ with $||\varphi||_{\infty} \leq 1/2$. For each $\gamma \in E$ we can define a trigonometric polynomial p_γ by

$$p_\gamma := 1 + \varphi(\gamma)\gamma + \overline{\varphi(\gamma)}\gamma^{-1} = 1 + 2\Re(\varphi(\gamma)\gamma).$$

Notice that each p_γ is positive-valued.

For a finite subset $F \subset E$, we form $p_F := \prod_{\gamma \in F} p_\gamma$. Since p_F is positive-valued and E is dissociate,

$$||p_F||_1 = \int p_F = 1.$$

The dissociate property similarly implies that for $\gamma \in F$, $\widehat{p_F}(\gamma) = \varphi(\gamma)$. Hence, we may obtain a measure μ as the unique weak* limit of the family

$$\{p_F : F \subset E \text{ is finite}\} \subset L^1(G) \subset M(G) \cong C(G)^*,$$

with the property that $||\mu||_{M(G)} = 1$ and $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$. The measure μ is called a **Riesz product**.

Hence, if E is dissociate and Γ has no elements of order two, for every $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1/2$, there exists $\mu \in M(G)$ such that $\widehat{\mu} = \varphi$ on E and $||\mu||_{M(G)} = 1$.

This construction is the **Riesz product construction**. Over 100 years ago M. Riesz used this construction to show the set $\{4^j : j \geq 0\}$ has this interpolation property.

Later in the next section we will see the Riesz product construction also works to show quasi-independent sets have this same interpolation property. Moreover, when Γ has elements of order two, a modification of the Riesz product construction shows any dissociate or quasi-independent set still has the interpolation property.

1.3 Sidon sets

The notion of Sidon sets is motivated by this interpolation property of independent and dissociate sets. Sidon sets are named after the Hungarian mathematician Simon Sidon who did preliminary work in this area.

Definition 1.3.1. A set $E \subset \Gamma$ is **Sidon** if for every bounded $\varphi : E \to \mathbb{C}$, there exists $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.

If E is Sidon, an open mapping theorem argument further implies that there exists a constant $C \geq 1$ such that for all $\varphi : E \to \mathbb{C}$ there exists $\mu \in M(G)$ with $||\mu||_{M(G)} \leq C\,||\varphi||_\infty$, such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$. The smallest such constant is called the **Sidon constant** of E, denoted by $S(E)$.

Example 1.3.2. (1) As we have seen with the Riesz product construction in Section 1.2, if $E \subset \Gamma$ is dissociate and Γ has no elements of order two, then E is Sidon with Sidon constant $S(E)$ bounded by 2.
(2) If $F \subset \Gamma$ is finite, then F is Sidon with $S(F) \leq \sqrt{|F|}$.
(3) All lacunary sets are Sidon.
(4) No infinite group Γ itself is Sidon.

Sidon sets are plentiful. In fact, every infinite set contains an infinite Sidon set. Furthermore, every infinite set contains a translate of an infinite dissociate set [25].

There are many equivalent descriptions of Sidon sets. For example, rather than the precise interpolation, we only need to interpolate ± 1-valued functions within a small error.

Theorem 1.3.3. *Let $E \subset \Gamma$ be a subset. The following are equivalent.*
(1) The set E is Sidon.
(2) For each $\varphi : E \to \mathbb{C}$, with $||\varphi||_\infty \leq 1$, there exists $\mu \in M(G)$ such that

$$\sup_{\gamma \in E} |\varphi(\gamma) - \widehat{\mu}(\gamma)| < 1.$$

(3) For each $\varphi : E \to \{\pm 1\}$ there exists $\mu \in M(G)$ such that

$$\sup_{\gamma \in E} |\varphi(\gamma) - \widehat{\mu}(\gamma)| < 1.$$

Proposition 1.3.4. *Let $E \subset \Gamma$. Suppose there exists a constant K and $\varepsilon < 1$ such that for all $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1$, there exists $\mu \in M(G)$ such that $||\mu||_{M(G)} \leq K$ and $\sup_{\gamma \in E} |\varphi(\gamma) - \widehat{\mu}(\gamma)| \leq \varepsilon$. Then E is Sidon and $S(E) \leq K/(1-\varepsilon)$.*

Proof. The proof of this statement involves an iterative argument, which we include to illustrate the technique. Let $\phi : E \to \mathbb{C}$ satisfy $||\phi||_\infty \le 1$. Choose $\mu_1 \in M(G)$ such that $||\mu_1||_{M(G)} \le K$ and $|\phi(\gamma) - \widehat{\mu_1}(\gamma)| < \varepsilon$ for all $\gamma \in E$. Note that the function $(\phi - \widehat{\mu_1})/\varepsilon$ satisfies $||(\phi - \widehat{\mu_1})/\varepsilon||_\infty \le 1$, and therefore we can choose $\mu_2 \in M(G)$ with $||\mu_2||_{M(G)} < K$ such that

$$|(\phi(\gamma) - \widehat{\mu_1}(\gamma))/\varepsilon - \widehat{\mu_2}(\gamma)| < \varepsilon$$

for all $\gamma \in E$. Iterating in this way, for each $n \in \mathbb{N}$ we can find $\mu_n \in M(G)$ such that $||\mu_n||_{M(G)} < K$ and

$$\left|\left|\phi - \widehat{\sum_{j=1}^{n} \varepsilon^{j-1}\mu_j}\right|\right|_\infty < \varepsilon^n.$$

Let $\mu = \sum_{j=1}^{\infty} \varepsilon^{j-1}\mu_j$. Then we have $\widehat{\mu} = \phi$ on E and because

$$||\mu|| \le \sum_{j=1}^{\infty} \varepsilon^{j-1} ||\mu_j|| \le \sum_{j=1}^{\infty} \varepsilon^{j-1} K = K/(1-\varepsilon),$$

we have $S(E) \le K/(1-\varepsilon)$. □

There is another characterization based on the duality of $C(G)$ and $M(G)$.

Theorem 1.3.5. *A set $E \subset \Gamma$ is Sidon with Sidon constant $S(E)$ if and only if for all $f \in \mathrm{Trig}_E(G)$,*

$$\sum_{\gamma \in E} |\widehat{f}(\gamma)| \le S(E)\, ||f||_\infty .$$

Remark 1.3.6. (1) Theorem 1.3.5 still holds if we replace $f \in \mathrm{Trig}_E(G)$ by all the bounded functions $f : G \to \mathbb{C}$ such that $\widehat{f}$ is supported on E.
(2) Since not every continuous function on $\mathbb{T}$ has absolutely convergent Fourier series, Theorem 1.3.5 implies that $\mathbb{Z}$ itself is not Sidon.

We will use the Riesz product construction and Proposition 1.3.4 to show quasi-independent sets are Sidon.

Proposition 1.3.7. *Suppose $E \subset \Gamma$ is quasi-independent and Γ has no elements of order two. Then E is Sidon with $S(E) \leq 3\sqrt{3}$.*

Proof. For a finite subset $F \subset E$ and $\beta \in \Gamma$, we define

$$R(F, \beta) := \left| \left\{ \omega \in \{-1, 0, 1\}^F : \prod_{\gamma \in F} \gamma^{\omega(\gamma)} = \beta \right\} \right|.$$

For $n \geq 1$ and $\beta \in \Gamma$, we define

$$R(E, n, \beta) := \sum_{F \subset E, |F| = n} R(F, \beta).$$

We first claim that for all $\beta \in \Gamma$,

$$\sum_{n \geq 1} \left(\frac{1}{2} \right)^n R(E, n, \beta) \leq 1.$$

Indeed, consider the Riesz product $\nu := \prod_{\gamma \in E} (1 + \gamma/2 + \gamma^{-1}/2)$. Then $||\nu||_{M(G)} = 1$ and for $\beta \in \Gamma$,

$$\widehat{\nu}(\beta) = \sum_{n \geq 1} \left(\frac{1}{2} \right)^n R(E, n, \beta).$$

Hence, we have

$$\sum_{n \geq 1} \left(\frac{1}{2} \right)^n R(E, n, \beta) \leq 1.$$

Notice that, in particular, if $\gamma \in E$, then

$$\sum_{n \geq 2} \left(\frac{1}{2} \right)^n R(E, n, \gamma) \leq 1/2,$$

because $R(E, 1, \gamma) = 1$.

Fix a function $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1$. For $a \in (0,1)$, we let

$$\mu_1 := \prod_{\gamma \in E} \left(1 + (a\varphi(\gamma)/2)\gamma + (\overline{a\varphi(\gamma)}/2)\gamma^{-1}\right),$$
$$\mu_2 := \prod_{\gamma \in E} \left(1 - (a\varphi(\gamma)/2)\gamma - (\overline{a\varphi(\gamma)}/2)\gamma^{-1}\right).$$

Then $||\mu_1||_{M(G)} = ||\mu_2||_{M(G)} = 1$. For $\gamma \in E$,

$$\begin{aligned}
|\frac{1}{a}(\widehat{\mu_1}(\gamma) - \widehat{\mu_2}(\gamma)) - \varphi(\gamma)| &\leq \frac{2}{a} \sum_{n \geq 3, n \text{ odd}} \sum_{F \subset E, |F| = n} R(F, \gamma) \left(\frac{a}{2}\right)^n \prod_{\beta \in F} |\varphi(\beta)| \\
&\leq 2a^2 \sum_{n \geq 3} \left(\frac{1}{2}\right)^n R(E, n, \gamma) \\
&\leq a^2.
\end{aligned}$$

Hence, by Proposition 1.3.4, $S(E)$ is bounded by $\frac{1}{1-a^2} \cdot \frac{2}{a}$ and this obtains its minimum $3\sqrt{3}$ when $a = 1/\sqrt{3}$. □

One of the classical properties of lacunary sets is the following: If $(n_k)_{k \geq 1} \subset \mathbb{N}$ is lacunary and $f(x) := \sum_{k \geq 1} a_k e^{i n_k x}$ is integrable, then $f \in L^p$ for all $p < \infty$. This is known as the $\Lambda(p)$ property and is true for independent sets of characters (and known as the Khintchine's inequality).

Rudin [36] showed in the 1960s that if E is Sidon, then E is $\Lambda(p)$ for all $p \geq 2$.

Theorem 1.3.8. *Suppose $E \subset \Gamma$ is Sidon with Sidon constant $S(E)$. Then for all $f \in \mathrm{Trig}_E(G)$, $||f||_p \leq 2S(E)\sqrt{p}\,||f||_2$ for all $2 \leq p < \infty$.*

Sidon sets also have the important property that a union of two Sidon sets is still Sidon.

Theorem 1.3.9. *Suppose E_1 and E_2 are two Sidon sets. Then $E_1 \cup E_2$ is also a Sidon set.*

This theorem was first proved by Drury [6] using carefully constructed and complicated Riesz products. Later, another proof was given by Rider using his probabilistic characterization of Sidonicity (Theorem 2.2.19, [33]).

In light of the union result, it is natural to ask if every Sidon set is the finite union of "nicer" or "simpler" sets. For example, is every Sidon set a finite union of lacunary sets or quasi-independent sets? The answer to the first question is no, while the second is an open problem.

Of course, if every Sidon set is a finite union of sets of some special type, then every finite subset of every Sidon set will contain proportional sized subsets of that special type. Because of the poor progress in characterizing Sidon sets as a finite union of nicer sets, researchers are interested in studying the weaker problem of characterizing Sidon sets by proportionality properties.

Major progress was made by Bourgain and Pisier in the early 1980s when they independently proved that Sidon sets can be characterized by the property of being "proportionally independent" ([4], [5], [29]). The precise statement is the following:

Theorem 1.3.10. *A set $E \subset \Gamma$ is Sidon if and only if there exists $\delta > 0$ such that for every finite subset $F \subset E$, there is a quasi-independent subset $F' \subset F$ with $|F'| \geq \delta |F|$.*

Both directions of this theorem are deep, difficult and important. The theorem was a break-through contribution to the study of Sidon sets as it is (still) the only algebraic characterization of Sidon sets known. This characterization gives new ways to build examples of Sidon sets and has been used to prove a number of other results about Sidon sets. For example, the union result, Theorem 1.3.9, is an immediate consequence since a union of two proportional quasi-independent sets is still proportional quasi-independent. The machinery constructed for their proofs was also used to prove the converse to Theorem 1.3.8, so that Sidon sets can also be characterized in terms of the $\Lambda(p)$ condition.

Bourgain's proof is combinatorial and analytic. The details can be found in [24].

Chapters 2 and 3 of this thesis will focus on Pisier's approach to the Theorem. The first step of Pisier's proof is to show that "proportional Sidonicity with bounded Sidon constants" implies the set is Sidon. As we have seen in Proposition 1.3.7, every quasi-independent set is Sidon with Sidon constant bounded by $3\sqrt{3}$; hence this first step proves one direction of the Theorem, namely proportional quasi-independence implies Sidonicity.

Pisier's proof involves analytical, topological, combinatorial and probabilistic arguments. The original proofs are scattered across a number of mainly unpublished manuscripts and the details are often omitted. In chapter 2, we will give a complete and detailed proof of this first step. In addition, we determine an upper bound for the Sidon constant of the set in terms of the proportionality data. That bound will then be used in chapter 4 to give an upper bound for the Sidon constant of Kronecker sets. (Kronecker sets and their relationship to Sidon sets will be introduced in section 1.5 of this chapter.)

Pisier's proof of the other direction of Theorem 1.3.10, that Sidon sets are proportionally quasi-independent, is primarily a probabilistic argument. In chapter 3, we will upgrade this argument to prove that Sidonicity implies not only proportional quasi-independence, but, in fact, a higher level of proportional independence. Using this stronger condition, we then prove that any Sidon set in a torsion-free group is proportionally Sidon with Sidon constants arbitrarily close to 1, the minimum possible value. This is one of our main new contributions to the study of Sidon sets.

1.4 I_0 sets

Since every measure has a unique decomposition as a sum of a discrete and a continuous measure, it is natural to extend the notion of Sidon and further ask the question of whether we can do interpolation with discrete or continuous measures.

In the case of a continuous measure, the answer is yes and was proven independently by Hartman and Wells ([17], [38]).

Proposition 1.4.1. *If $E \subset \Gamma$ is Sidon, then for every bounded $\varphi : E \to \mathbb{C}$ there* *exists a continuous measure μ such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.*

Interpolating with discrete measures motivates the notion of I_0 sets.

Definition 1.4.2. A set $E \subset \Gamma$ is $\boldsymbol{I_0}$ if for every bounded $\varphi : E \to \mathbb{C}$ there exists a discrete measure $\mu \in M_d(G)$ such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.

The $\boldsymbol{I_0}$ **constant** is defined similarly to the Sidon constant. If E is I_0, the open mapping theorem implies there exists a constant C such that for all $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1$ there exists $\mu \in M_d(G)$ with $||\mu||_{M(G)} \leq C$ such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$. The $\boldsymbol{I_0}$ **constant** of E is the infimum of these constants.

Naturally, every I_0 set is Sidon and the I_0 constant is greater than or equal to the Sidon constant. Grow [13] showed that if E is finite, then the I_0 constant of E is the same as the Sidon constant of E.

Similar to Sidon sets, I_0 sets also have the following characterization through approximating ± 1-valued functions.

Theorem 1.4.3. *A set $E \subset \Gamma$ is I_0 if and only if there exists $\varepsilon < 1$ such that for all $\varphi : E \to \{\pm 1\}$ there exists $\mu \in M_d(G)$ such that $|\widehat{\mu}(\gamma) - \varphi(\gamma)| \leq \varepsilon$ for all $\gamma \in E$.*

Finite sets are I_0. Kunen and Rudin ([23]) showed that all lacunary sets are I_0. However, the Riesz product construction does not help prove this result because the Riesz product measure that arises as the weak* limit is not a discrete measure. Because of Proposition 1.2.2, independent sets, such as the Rademacher set, are I_0.

We have mentioned that Sidon sets are plentiful. In fact, I_0 sets are also plentiful: every infinite set in Γ contains an I_0 subset of the same cardinality (Corollary 4.5.3 in [9]).

Topologically, I_0 sets are closely related to almost periodic functions, the continuous functions that are periodic within any desired level of accuracy. The original definition of an I_0 set was a set E such that ‘every bounded function on $E \subset \mathbb{Z}$ can be extended to an almost periodic function’. But these extensions could always be found in the space consisting of Fourier transforms of discrete measures restricted to E ([18]). Thus, the modern definition is that I_0 sets are the sets such that ‘every bounded function is a Fourier transform of discrete measures’.

Unlike Sidon sets, I_0 sets have an elegant topological characterization due to Hartman and Ryll-Nardzewski [18].

Theorem 1.4.4. *A set $E \subset \Gamma$ is I_0 if and only if any two disjoint subsets in E have disjoint closures in the Bohr compactification, $\overline{\Gamma}$.*

Using Theorem 1.4.4 we can show that a union of two I_0 sets may not be I_0. Hence, the class of I_0 sets is indeed a proper subset of the class of Sidon sets.

Example 1.4.5. [27] Consider $E_1 = \{10^j : j \geq 1\}$ and $E_2 = \{10^j + j : j \geq 1\}$. The sets E_1 and E_2 are disjoint and lacunary. Hence, both E_1 and E_2 are I_0. Because $\mathbb{N}$ is dense in $\overline{\mathbb{Z}}$, there exists a net $(n_\alpha)_\alpha$ that clusters at 0. If we let $\beta \in \overline{\mathbb{Z}}$ be a cluster point of $(10^{n_\alpha})_\alpha$ in $\overline{\mathbb{Z}}$ then we have $\beta \in \overline{E_1} \cap \overline{E_2}$. Theorem 1.4.4 implies $E_1 \cup E_2$ is not I_0.

Since the class of I_0 sets is not closed under finite unions it is natural to ask if every Sidon set is a finite union of I_0 sets. This is open. However, it was proven by Ramsey ([32]), using the proportional quasi-independent characterization of Sidonicity, that every Sidon set is proportionally I_0.

Ramsey ([32]) also used Theorem 1.4.4 to prove the fact that an I_0 set cannot cluster at a continuous character and hence cannot be dense in the Bohr compactification. A long standing open problem is whether Sidon sets can be dense in the Bohr compactification of Γ. If Sidon sets were finite unions of I_0 sets, the answer would be no. Partial progress was made by Ramsey, who showed that if there exists a Sidon set clustering at $1 \in \Gamma$, then there exists a dense Sidon set [32]. This also used the proportional quasi-independent characterization of Sidonicity.

1.5 Kronecker sets

Another interesting family of interpolation sets are the ε-Kronecker sets.

Definition 1.5.1. Let $0 < \varepsilon \leq 2$ and $E \subset \Gamma$. The set E is **ε-Kronecker** if for all $\varphi : E \to \mathbb{T}$ there exists $x \in G$ such that $|\varphi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in E$.

The **Kronecker constant** of E, $\kappa(E)$, is defined by

$$\kappa(E) := \inf \{\varepsilon : E \text{ is } \varepsilon\text{-Kronecker}\}.$$

Definition 1.5.2. A set $E \subset \Gamma$ is **Kronecker** if $\kappa(E) < 2$.

Historically, the notion of ε-Kronecker sets was inspired in part by the classical approximation theorem of Kronecker, with early work done by Hewitt and Kakutani [19]. The terminology was introduced by Varapolous in [37].

There are many known examples of ε-Kronecker sets.

Example 1.5.3. (1) In $\mathbb{Z}$ any singleton set other than $\{0\}$ is 0-Kronecker.
(2) The set $\{-1, 1\}$ has Kronecker constant $\sqrt{2}$.
(3) Any finite set in $\mathbb{Z}$ excluding 0 is Kronecker.
(4) The Radamacher set has Kronecker constant $\sqrt{2}$. More generally, independent sets in Γ with large orders have small Kronecker constants. An independent set in a torsion-free group has zero Kronecker constant, as we have seen in Proposition 1.2.2.

Lacunary integer sequences with large ratios have small Kronecker constants. To be precise, the following is known.

Proposition 1.5.4. *(1) Suppose $E = (n_j)_{j=1}^{\infty} \subset \mathbb{N}$ is lacunary with ratio $q > 2$. Then $\kappa(E) \leq |1 - e^{i\pi/(q-1)}|$.*
(2) The geometric sequences, $E_n := \{n^j : j \geq 0\}$ for integer $n \geq 2$, have $\kappa(E_n) = |1 - e^{i\pi/n}|$ ([15]).

Remark 1.5.5. It is an open problem whether a lacunary sequence with ratio between 1 and 2 is Kronecker.

Kronecker sets exist universally. Indeed, given any infinite set $E \subset \Gamma$, there exists a Kronecker set $E_1 \subset E$ with $|E_1| = |E|$ ([11]).

Kronecker sets are closely related to the other interpolation notions we have already introduced. From Theorem 1.4.3 we immediately see that $(1-\varepsilon)$-Kronecker sets are I_0. In fact, a more careful iterative argument shows that a Kronecker set with Kronecker constant less than $\sqrt{2}$ is I_0 and this bound is known to be sharp.

Hare and Ramsey in [14] proved that any Kronecker set is Sidon. Their proof is based on another characterization of Sidonicity due to Pisier, called the ε-net condition. For details of the ε-net condition we direct the reader to chapter 3. In chapter 4 we will give a quantitative estimation of the Sidon constants for $(2-\varepsilon)$-Kronecker sets based on further generalizations that we develop of Pisier's work on the connection between the ε-net condition and Sidon sets.

While there are Sidon sets in some groups that are not Kronecker, it is unknown if every Sidon set in $\mathbb{Z}$ is Kronecker. Furthermore, it also unknown if every Sidon set is a finite union of Kronecker sets. Thus again it is of interest to investigate the weaker notion of proportionality. In chapter 4 we will show Sidon sets are proportional ε-Kronecker for any $\varepsilon > 1$. This improves the previous result that only obtained $\varepsilon \geq \sqrt{2}$. The problem is open for arbitrarily small $\varepsilon > 0$.

There are other interesting open problems about Kronecker sets. For example, unlike Sidon sets and I_0 sets, it is unknown if a union of two Kronecker sets in $\mathbb{Z}$ remains Kronecker. In chapter 4, the partial result that the union of an integer Kronecker set and a finite set excluding 0 is still Kronecker will be proved. Using this we will show that a translation of a Kronecker set in $\mathbb{Z}$ away from the identity remains Kronecker.

1.6 Generalizations of Kronecker sets

Sidon sets and I_0 sets have characterizations based on the interpolation of ± 1-valued functions (Theorem 1.3.3 and Theorem 1.4.3). This motivates the notion of binary Kronecker sets.

When investigating binary Kronecker sets, it is convenient to identify the unit circle $\mathbb{T}$ as $\mathbb{T} = [0, 1]$, with 0 as the identity and addition mod 1 as the group operation. The metric on $\mathbb{T}$ is

$$d_{\mathbb{T}}(x, y) = \min \{|x - y|, 1 - |x - y|\}.$$

The duality is given by the following: for $n \in \mathbb{Z}$ and $x \in \mathbb{T}$, the duality is multiplication nx mod integers.

Definition 1.6.1. Let $\varepsilon > 0$. A subset $E \subset \mathbb{Z}$ is called **binary ε-Kronecker** if for all $\varphi : E \to \{0, 1/2\}$ there exists $x \in G$ such that $d_{\mathbb{T}}(\varphi(n), nx) < \varepsilon$ for all $n \in E$.

The **binary Kronecker constant** of E, $\beta(E)$, is defined by

$$\beta(E) := \inf \{\tau : E \text{ is binary } \tau\text{-Kronecker}\}.$$

Using this identification of $\mathbb{T}$, an **angular ε-Kronecker set** $E \subset \mathbb{Z}$ is defined by requiring that for all $\varphi : E \to [0, 1]$ there exists $x \in \mathbb{T}$ such that $d_{\mathbb{T}}(\varphi(n), nx) < \varepsilon$. The angular Kronecker constant of $E \subset \mathbb{Z}$ using the identification $\mathbb{T} = [0, 1]$ is denoted by $\alpha(E)$ and is called the **angular Kronecker constant** of E.

Remark 1.6.2. Let $E \subset \Gamma$. (1) We have $\alpha(E), \beta(E) \in [0, 1/2]$.
(2) The relation between the angular Kronecker constant $\alpha(E)$ and the Kronecker constant $\kappa(E)$ is that $\kappa(E) = |1 - e^{2\pi i \alpha(E)}|$.

Binary Kronecker sets are different from Kronecker sets. For example, we will see in chapter 5 that the set of odd integers is binary 1/4-Kronecker, but it is not a Sidon set and not a Kronecker set either.

Clearly for $E \subset \Gamma$, $\alpha(E) \geq \beta(E)$. When E has small binary Kronecker constant, its angular Kronecker constant is also small because it is known that $\beta(E) \leq \alpha(E) \leq 2\beta(E)$ (Theorem 2.5.1 in [9]). In chapter 5 we will see examples showing that this inequality is sharp. When $\beta(E) < 1/3$, Theorem 1.4.3 implies E is I_0. Thus it is very interesting to study binary Kronecker sets with small binary Kronecker constant.

Because it is difficult to compute the Kronecker constant of the set $\{1, ..., n\}$, Hare and Ramsey [16] computed the binary Kronecker constant

$$\beta(\{1, ..., n\}) = 1/2 - 1/(n+1).$$

This is the best known lower bound for the angular Kronecker constant of the set $\{1, ..., n\}$.

In chapter 5 we will show that the binary Kronecker constant of a symmetric set is particularly easy to compute and is always greater or equal to 1/4. We will compute the binary Kronecker constants for such sets as $\{\pm 1, ..., \pm n\}$, $\left\{\pm n^k : k \geq 0\right\}$ and $\{\pm(ak+b) : k \in \mathbb{N}\}$ for $a, b \in \mathbb{N}$.

We will also show that the binary Kronecker constant of the lacunary sequence $\left\{n^k : k \geq 0\right\}$ is the same as its angular Kronecker constant, $1/(2k)$.

Another way to generalize Kronecker sets based on targeting ± 1-valued functions is to ask for exact interpolation of these functions. This is the notion of pseudo-Rademacher sets.

Definition 1.6.3. A set $E \subset \Gamma$ is **pseudo-Rademacher** if for every $\varphi : E \to \{\pm 1\}$ there exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$.

These sets are called pseudo-Rademacher because the Rademacher set, being an independent set in $\mathbb{Z}_2^\infty$, has this property. Pseudo-Rademacher sets are I_0.

In chapter 6, we introduce a more general notion where we ask to exactly interpolate $\mathbb{Z}_N$-valued functions.

Definition 1.6.4. Let $N \geq 2$ be an integer. A set $E \subset \Gamma$ is called **N-pseudo-Rademacher** if for every $\varphi : E \to \mathbb{Z}_N$ there exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$.

Naturally, N-pseudo-Rademacher sets are I_0 and are ε-Kronecker for $\varepsilon = |1 - e^{i\pi/N}|$.

As we have seen in Proposition 1.2.2, the property of independence is equivalent to exact interpolation of all functions with proper range by characters. In chapter 6 we will prove N-pseudo-Rademacher sets can be characterized by a weaker independence property.

Using this characterization, we will explore the structure of N-pseudo-Rademacher sets. Our main result is to prove, similar to the case for Sidon, I_0 and ε-Kronecker sets, that large (with regard to cardinality) N-pseudo-Rademacher subsets always exist (in the appropriate settings) and, in particular, this gives a new proof that all uncountable sets contain I_0 subsets of the same cardinality.

Chapter 2

Pisier's Characterization of Sidon Sets

2.1 Introduction

Sidon sets have been extensively studied since the 1920s, when Sidon proved that lacunary series have the property that every bounded function with Fourier transform supported on the series has absolutely convergent Fourier series, and hence are Sidon sets, as we have seen in chapter 1 (Theorem 1.3.5, Remark 1.3.6). Many important results, including the union theorem (Theorem 1.3.9) and the characterization by proportional quasi-independence (Theorem 1.3.10), were found during the 1970s and 1980s by Drury, Rider, Bourgain and Pisier, among others.

In Pisier's approach to the "proportional quasi-independent" property, one crucial step is to show Sidon sets are equivalent to being "proportional Sidon". In this chapter we explore Pisier's proof of that result. We start in section 2.2 with the necessary definitions, preliminary probabilistic results and facts about Orlicz spaces and entropy numbers that will be needed. We begin section 2.3 by stating the main theorem, which consists of six equivalent statements. To prove it, we first translate the "proportional Sidon" condition into various Orlicz norm comparisons (section 2.3.1). The hardest step (section 2.3.2) is to relate these Orlicz norm characterizations of Sidonicity to relevant properties involving entropy. We finally obtain the Sidon property through probabilistic results of Dudley, Fernique and Rider (section 2.3.3). The proof presented in this chapter has all the necessary details and is slightly simpler than Pisier's argument.

While proving the equivalence of the statements in the main theorem, we will also trace the constants involved and obtain a bound for the final Sidon constant, which will be used in later chapters.

In the next chapter we will upgrade Pisier's proportional quasi-independence characterization of Sidonicity and show that Sidon sets are "proportional Sidon" with arbitrary levels of independence and "minimal Sidon constants".

2.2 Definitions and preliminary results

In this section we give the definitions and preliminary results needed to state and prove the main result. Throughout this chapter, we let (Ω, P) be a probability space and $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ be a collection of independent random variables on (Ω, P) indexed by Γ, such that each ε_γ takes only values 1 and -1 with equal probability $1/2$. We let (Ω_1, P_1) be another probability space and $(g_\gamma)_{\gamma\in\Gamma}$ be a collection of independent standard Gaussian random variables on (Ω_1, P_1), indexed by Γ. (X, μ) is any universal probability space.

2.2.1 Special Orlicz Spaces and the $\ell_{p,1}$ space

In this subsection, we collect basic facts about Orlicz spaces and the $\ell_{p,1}$ space that will be used in the main proof of this chapter, Pisier's proportionality result.

Definition 2.2.1. For $q > 1$ and $x \in \mathbb{R}$, we define

$$\phi_q(x) := \exp(|x|^q) - 1$$

and

$$\varphi_q(x) = |x|(1 + \log(1 + |x|))^{1/q}.$$

Notice that ϕ_q and φ_q are convex, increasing on $\mathbb{R}^+$ and pass through the origin.

Definition 2.2.2. Suppose β_q is either ϕ_q or φ_q for $q \in (1, \infty)$. We let $L_{\beta_q}(X)$, the Orlicz space, be the set of all measurable $f : X \to \mathbb{C}$ such that there exists $c > 0$ with $\int_X \beta_q \left(|\frac{f}{c}|\right) < \infty$.

Remark 2.2.3. We note that $L_{\beta_q}(X)$ defined above is a vector space.

Note that

$$L^\infty(X) \subset L_{\phi_q}(X) \subset \bigcap_{p \geq 1} L^p(X).$$

Definition 2.2.4. On the Orlicz space $L_{\phi_q}(X)$, we may define the Orlicz norm as the following: for $f \in L_{\phi_q}(X)$,

$$||f||_{(\phi_q)} := \sup \left\{ \left| \int_X fg \right| : \int_X \varphi_q(|g|) \leq e - 1 \right\}.$$

For β_q being ϕ_q or φ_q, we define the Luxemburg norm by

$$||f||_{\beta_q} := \inf \left\{ c > 0 : \int_X \beta_q \left(\left| \frac{f}{c} \right| \right) < \beta_q(1) \right\}$$

for $f \in L_{\beta_q}(X)$, and we define the dual norm via

$$||f||_{\beta_q^*} := \sup_{||g||_{\beta_q} \leq 1} |\int_X fg|.$$

Remark 2.2.5. Notice that by the convexity of ϕ_q and the fact that $\phi_q(0) = 0$, if for some $c > 0$ we have that

$$\int \phi_q \left(\left| \frac{f}{c} \right| \right) \leq K(e-1)$$

for some $K > 1$, then

$$\int \phi_q \left(\left| \frac{f}{Kc} \right| \right) \leq \int \frac{1}{K} \phi_q \left(\left| \frac{f}{c} \right| \right) \leq e - 1.$$

Hence, $||f||_{\phi_q} \leq Kc$.

Theorem 2.2.6. *[22] The norms given above are well-defined and are equivalent. Indeed, for all $f \in L_{\phi_q}(X)$ we have*

$$||f||_{\phi_q} \leq ||f||_{(\phi_q)} = ||f||_{\varphi_q^*} \leq 2 \, ||f||_{\phi_q} .$$

Moreover, the Orlicz space, equipped with any norm given above, is a Banach space.

Theorem 2.2.7. *[22] If $f \in L_{\phi_2}(G)$ and $g \in L_{\varphi_2}(G)$, then $f * g \in C(G)$ and*

$$||f * g||_\infty \leq C \, ||f||_{\phi_2} \, ||g||_{\varphi_2}$$

for some constant C.

The following is an interpolation result.

Proposition 2.2.8. *Suppose $1 < q_1 < q < \infty$ and let $\theta := q_1/q$. Then for any $f \in L_\infty(X)$,*

$$||f||_{\phi_q} \leq ||f||_{\phi_{q_1}}^{\theta} \, ||f||_\infty^{1-\theta} .$$

Proof. It suffices to show for every $c > 0$,

$$\int_X \phi_q \left(\left| \frac{f}{c^\theta \, ||f||_\infty^{1-\theta}} \right| \right) \leq \int_X \phi_{q_1} \left(\left| \frac{f}{c} \right| \right) .$$

By the Riesz-Thorin interpolation theorem,

$$||f||_{kq} \leq ||f||_{kq_1}^{\theta} \, ||f||_{\infty}^{1-\theta}$$

for all $k \geq 1$. Thus, using the definition of $\phi_q(x) = \exp(|x|^q) - 1$ we have

$$\begin{aligned}\int_X \phi_q \left(\left| \frac{f}{c^\theta \, ||f||_\infty^{1-\theta}} \right| \right) = \sum_{k\geq 1} \frac{1}{k!} \left|\left| \frac{f}{c^\theta \, ||f||_\infty^{1-\theta}} \right|\right|_{kq}^{kq} &\leq \sum_{k\geq 1} \frac{1}{k!} \cdot \frac{||f||_{kq_1}^{kq_1} \, ||f||_\infty^{(1-\theta)kq}}{(c^\theta \, ||f||_\infty^{1-\theta})^{kq}} \\ &= \sum_{k\geq 1} \frac{1}{k!} \left|\left| \frac{f}{c} \right|\right|_{kq_1}^{kq_1} = \int_X \phi_{q_1} \left(\left| \frac{f}{c} \right| \right).\end{aligned}$$

□

Remark 2.2.9. From the proof above, we can also see that if $f, g \in L_{\phi_q}(X)$ and $||f||_s \leq ||g||_s$ for all $1 \leq s < \infty$, then $||f||_{\phi_q} \leq ||g||_{\phi_q}$.

Next, we define the space $\ell_{p,1}$.

Definition 2.2.10. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \alpha_n = 0$. We define the complex sequence $(\alpha_n^*)_{n\geq 1}$ as the re-arrangement of $(\alpha_n)_{n\geq 1}$ in the way that if $i \leq j$ then $|\alpha_i^*| \geq |\alpha_j^*|$.

Definition 2.2.11. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers. We define $(\alpha_n)_{n\geq 1} \in \ell_{p,1}$ if and only if

$$||(\alpha_n)||_{p,1} := \sum_{n=1}^{\infty} |\alpha_n^*| n^{-1/q} < \infty,$$

where $1/p + 1/q = 1$ and $1 < p \leq 2 \leq q < \infty$.

Example 2.2.12. Consider a finite set $A \subset \Gamma$. We let 1_A be the indicator function of A. Then $||1_A||_{p,1} = \sum_{i=1}^{|A|} i^{-1/q} \geq |A||A|^{-1/q} = |A|^{1/p}$.

Remark 2.2.13. Here is a useful convex decomposition of finitely supported positive sequences in $\ell_{p,1}$. Assume $\alpha := (\alpha_\gamma)_\gamma$ is supported on a finite set $A \subset \Gamma$ and each $\alpha_\gamma \in \mathbb{R}^+$ for $\gamma \in A$. Assume further that $||\alpha||_{p,1} = 1$. Order A such that $\alpha_{\gamma_1} \geq \alpha_{\gamma_2} ... \geq \alpha_{\gamma_n} > 0$, where $|A| = n$. For $1 \leq i \leq n$, define $A_i := \{\gamma_j : 1 \leq j \leq i\}$ and let 1_{A_i} be the indicator function of A_i. We put $\lambda_i := (\alpha_{\gamma_i} - \alpha_{\gamma_{i+1}}) ||1_{A_i}||_{p,1}$ for $1 \leq i \leq n-1$ and $\lambda_n := \alpha_{\gamma_n}$. We note that each

$$\alpha_\gamma = \sum_{i=1}^{n} \frac{\lambda_i}{||1_{A_i}||_{p,1}} 1_{A_i}(\gamma)$$

for $\gamma \in A$. Thus, if we define $\beta_i(\gamma) = \frac{1_{A_i}(\gamma)}{||1_{A_i}||_{p,1}}$, then $\alpha = \sum_{i=1}^{n} \lambda_i \beta_i$ with each $||\beta_i||_{p,1} = 1$. We further notice that $1 = ||\alpha||_{p,1} = \sum_{i=1}^{n} \lambda_i$. Thus, α is a convex combination of norm one sequences.

If $1 < r < p < 2$, then $\ell_r \subset \ell_{p,1} \subset \ell_2$ ([22]). Furthermore, we have the following interpolation result.

Proposition 2.2.14. *[1] Let $1 < r < p < 2$. Define $0 < \theta < 1$ by $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$. Then, there exists a constant K_θ such that for all $\alpha \in \ell_1$,*

$$||\alpha||_{p,1} \leq K_\theta ||\alpha||_r^{1-\theta} ||\alpha||_2^{\theta}.$$

2.2.2 Probabilistic results

Lemma 2.2.15. *Let $(\xi_n)_{n=1}^k$ be real valued, non-zero almost everywhere, independent, symmetrical and identically distributed random variables on the probability space (X, μ). Let $(\alpha_n)_{n=1}^k$ be real numbers. Suppose $(x_n)_{n=1}^k$ are from a Banach space Y and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable function with $\Phi(0) = 0$. For all ± 1-valued $(\beta_n)_{n=1}^k$, we have*

$$\int_X \Phi\left(\left|\left|\sum_{n=1}^{k} \alpha_n \xi_n x_n\right|\right|_Y\right) = \int_X \Phi\left(\left|\left|\sum_{n=1}^{k} \beta_n \alpha_n \xi_n x_n\right|\right|_Y\right).$$

Proof. We first consider the case that every ξ_n is ±1-valued. For each $l \in \{-1,1\}^k$ we consider the set

$$X_l := \{x \in X : \xi_n(x) = l(n)\ \forall 1 \leq n \leq k\},$$

and let

$$a_l := \Phi\left(\left\|\sum_{n=1}^{k} \alpha_n l(n) x_n\right\|_Y\right).$$

Since ξ_n, $1 \leq n \leq k$, are independent and identically distributed, $\mu(X_l) = 1/2^k$ for all $l \in \{-1,1\}^k$. Hence,

$$\int_X \Phi\left(\left\|\sum_{n=1}^{k} \alpha_n \xi_n x_n\right\|_Y\right) = \sum_{l\in\{\pm1\}^k} a_l \cdot \mu(X_l) = \sum_{l\in\{\pm1\}^k} \frac{a_l}{2^k}.$$

If we define $F : \{\pm1\}^k \to \{\pm1\}^k$ by $F(l)(n) = l(n)\beta(n)$, we note that F is a bijection. Moreover,

$$\int_X \Phi\left(\left\|\sum_{n=1}^{k} \alpha_n \beta_n \xi_n x_n\right\|_Y\right) = \sum_{l\in\{\pm1\}^k} a_{F(l)}\mu(X_l) = \sum_{l\in\{\pm1\}^k} \frac{a_l}{2^k}.$$

Hence,

$$\int_X \Phi\left(\left\|\sum_{n=1}^{k} \alpha_n \beta_n \xi_n x_n\right\|_Y\right) = \int_X \Phi\left(\left\|\sum_{n=1}^{k} \alpha_n \xi_n x_n\right\|_Y\right).$$

Notice that this can be generalized to deal with the case that each ξ_n is a symmetric simple function and therefore the Lemma 2.2.15 follows. $\square$

Proposition 2.2.16. *Let $(\xi_n)_{n=1}^k$ be real valued, non-zero almost everywhere, independent, identically distributed, symmetric random variables on the probability space (X,μ) and $(x_n)_{n=1}^k$ be in a Banach space Y. Suppose $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex, increasing function and $\Phi(0) = 0$.*

(1) Suppose $(\alpha_n)_{n=1}^k$ are real numbers. Then

$$\int_X \Phi\left(\min|\alpha_n| \left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right) \le \int_X \Phi\left(\left\|\sum_{n=1}^k \alpha_n \xi_n x_n\right\|_Y\right)$$
$$\le \int_X \Phi\left(\max|\alpha_n| \left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right).$$

(2) Suppose $(\alpha_n)_{n=1}^k$ are complex numbers such that $|\alpha_n| = 1$ for $1 \le n \le k$. Then

$$\int_X \Phi\left(\frac{1}{2}\left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right) \le \int_X \Phi\left(\left\|\sum_{n=1}^k \alpha_n \xi_n x_n\right\|_Y\right) \le \int_X \Phi\left(2\left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right).$$

Proof. (1) Assume that $\max|\alpha_n| = 1$. Notice that the function $F : [-1,1]^k \to \mathbb{R}$,

$$(\alpha_1, ..., \alpha_k) \to \int_X \Phi\left(\left\|\sum_{n=1}^k \alpha_n \xi_n x_n\right\|_Y\right).$$

is convex. Hence, it will obtain its maximum on the extreme points in $[-1,1]^k$, namely $(\beta_1, ..., \beta_k)$ where each $\beta_n = \pm 1$. By Lemma 2.2.15, for all $\beta_n = \pm 1$,

$$\int_X \Phi\left(\left\|\sum_{n=1}^k \beta_n \xi_n x_n\right\|_Y\right) = \int_X \Phi\left(\left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right) = \int_X \Phi\left(\max|\alpha_n| \left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right),$$

proving the second inequality.

To see the first inequality in (1), we suppose $\min|\alpha_n| > 0$. Let $\eta_n = \frac{1}{\alpha_n}$ and $y_n = (\min|\alpha_n|)\alpha_n x_n$. The first inequality follows as

$$\int_X \Phi\left(\min|\alpha_n| \left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right) = \int_X \Phi\left(\left\|\sum_{n=1}^k \eta_n \xi_n y_n\right\|_Y\right)$$
$$\le \int_X \Phi\left(\max|\eta_n| \left\|\sum_{n=1}^k \xi_n y_n\right\|_Y\right) = \int_X \Phi\left(\left\|\sum_{n=1}^k \alpha_n \xi_n x_n\right\|_Y\right).$$

(2) Since Φ is convex and increasing, notice that

$$\int_X \Phi\left(\left\|\sum_{n=1}^k \alpha_n\xi_n x_n\right\|_Y\right) \leq \int_X \Phi\left(\left\|\sum_{n=1}^k \Re(\alpha_n)\xi_n x_n\right\|_Y + \left\|\sum_{n=1}^k \Im(\alpha_n)\xi_n x_n\right\|_Y\right)$$

$$\leq \frac{1}{2}\int_X \Phi\left(2\left\|\sum_{n=1}^k \Re(\alpha_n)\xi_n x_n\right\|_Y\right) + \frac{1}{2}\int_X \Phi\left(2\left\|\sum_{n=1}^k \Im(\alpha_n)\xi_n x_n\right\|_Y\right)$$

$$\leq \int_X \Phi\left(2\left\|\sum_{n=1}^k \xi_n x_n\right\|_Y\right),$$

by (1) as each $|\Re(\alpha_n)|, |\Im(\alpha_n)| \leq 1$. The other inequality follows via replacing x_n by $\overline{\alpha_n} x_n$. ☐

Proposition 2.2.17. *Suppose $A \subset \Gamma$ is a finite subset. For $|\alpha_\gamma| = 1$,*

$$\sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} \alpha_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} \leq 2\sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q}.$$

Proof. We first show that if $h_\gamma \in [-1,1]$ for $\gamma \in A$, then

$$\sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} h_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} \leq \sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q}.$$

Indeed, the map $F : [-1,1]^{|A|} \to \mathbb{R}$ given by

$$(h_\gamma)_{\gamma\in A} \to \sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} h_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q}$$

is convex. Hence, it obtains its maximum on the extreme points of $[-1,1]^{|A|}$, which are $\{-1,1\}^{|A|}$. Since ε_γ are ± 1-valued, we have that for all $b_\gamma \in \{-1,1\}$,

$$\sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} b_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} = \sup_{\omega\in\Omega}\left\|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q}.$$

Hence, for some $(b_\gamma)_{\gamma\in A} \in \{-1,1\}^{|A|}$,

$$\sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} h_\gamma\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q} \leq \sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} b_\gamma\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q} = \sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q}.$$

Now assume $\alpha_\gamma \in \mathbb{C}$ with $|\alpha_\gamma| = 1$ and we write $\alpha_\gamma = d_\gamma + c_\gamma i$ for $d_\gamma, c_\gamma \in [-1,1]$. We have

$$\begin{aligned}\sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} \alpha_\gamma\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q} &\leq \sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} d_\gamma\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q} + \sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} c_\gamma\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q}\\ &\leq 2\sup_{\omega\in\Omega}\left|\left|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_q}.\end{aligned}$$

□

Proposition 2.2.18. *If $h \in \mathrm{Trig}(G)$ and $k \in L_{\varphi_2}(G)$, then*

$$\int_\Omega \left|\left|\sum_{\gamma\in\Gamma} \varepsilon_\gamma \widehat{h}(\gamma)\widehat{k}(\gamma)\gamma\right|\right|_{C(G)} \leq C\,||h||_2\,||k||_{\varphi_2}$$

for some constant C.

Proof. From Khintchine's inequality ([21]), it is well-known that

$$\left|\left|\sum_{\gamma\in\Gamma} \widehat{h}(\gamma)\varepsilon_\gamma\right|\right|_{L^{2k}(\Omega)} \leq 2\sqrt{k}\left|\left|\sum_{\gamma\in\Gamma} \widehat{h}(\gamma)\varepsilon_\gamma\right|\right|_{L^2(\Omega)}.$$

We first claim that it follows from this that

$$\int_\Omega \left|\left|\sum_{\gamma\in\Gamma} \widehat{h}(\gamma)\varepsilon_\gamma\gamma\right|\right|_{\phi_2} \leq 12\,||h||_2\,.$$

We note that for all $t \in G$,

$$\int_\Omega \phi_2\left(\frac{|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma\gamma(t)|}{6\,||h||_2}\right) = \sum_{j=1}^\infty \frac{\left|\left|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma\gamma(t)\right|\right|_{L^{2j}(\Omega)}^{2j}}{6^{2j}\,||h||_2^{2j}\,j!}$$
$$\leq \sum_{j=1}^\infty \frac{2^{2j}j^j\left|\left|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma\gamma(t)\right|\right|_{L^2(\Omega)}^{2j}}{6^{2j}\,||h||_2^{2j}\,j!} \leq \sum_{j=1}^\infty \frac{j^j}{3^{2j}j!} < 1.$$

Thus, if we put

$$S(\omega) := \int_G \phi_2\left(\frac{|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma(\omega)\gamma(t)|}{6\,||h||_2}\right)\,dt,$$

then $\int_\Omega S \leq 1$.

We let $K(\omega) := \left|\left|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma(\omega)\gamma\right|\right|_{\phi_2}$. By the definition we have

$$\int_G \phi_2\left(\frac{|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma(\omega)\gamma(t)|}{K(\omega)}\right)\,dt = e - 1 > 1.$$

If $a \geq 1$ and $f(x) = x^{a^2} - ax + a - 1$, then $f(x) \geq 0$ for all $x \geq 1$. Applying this with $x = \exp(b^2)$, for $b \geq 0$ we have $a\phi_2(b) \leq \phi_2(ab)$ for $a \geq 1$. Thus, if $K(\omega) \geq 6\,||h||_2$,

$$\int_G \frac{K(\omega)}{6\,||h||_2}\phi_2\left(\frac{|\sum_\gamma \widehat{h}(\gamma)\varepsilon_\gamma(\omega)\gamma(t)|}{K(\omega)}\right)\,dt \leq S(\omega).$$

Hence, if $K(\omega) \geq 6\,||h||_2$, $K(\omega) \leq 6\,||h||_2\,S(\omega)$.

By considering the set

$$\{\omega \in \Omega : K(\omega) \geq 6\,||h||_2\}$$

and its complement separately, we have

$$\int_\Omega K \leq 6\,||h||_2 + 6\,||h||_2 \int_\Omega \, S(\omega) \leq 12\,||h||_2\,,$$

proving the claim.

By Theorem 2.2.7, we have

$$\int_\Omega \left|\left|\sum_{\gamma\in\Gamma} \varepsilon_\gamma \widehat{h}(\gamma)\widehat{k}(\gamma)\gamma\right|\right|_\infty \leq C \int_\Omega \left|\left|\sum_{\gamma} \widehat{h}(\gamma)\varepsilon_\gamma\gamma\right|\right|_{\phi_2} ||k||_{\varphi_2} \leq 12C\,||h||_2\,||k||_{\varphi_2}$$

for some constant C. □

Finally, we have an important result from Rider which is a key ingredient in the proof of the main result of this chapter. The proof can be found in Rider's original paper [33] and a detailed proof can be found in [24].

Theorem 2.2.19 (Rider)**.** *[33] Consider the probability space $(\mathbb{T}^\Gamma, P_2)$, where P_2 is the Haar measure on $\mathbb{T}^\Gamma$. For each $t = (t_\gamma)_{\gamma\in\Gamma} \in \mathbb{T}^\Gamma$ and $\gamma \in \Gamma$, let $\omega_\gamma : \mathbb{T}^\Gamma \to \mathbb{T}$ be given by $\omega_\gamma(t) = t_\gamma$. Let $\Lambda \subset \Gamma$ and assume there exists a constant C such that for every $f \in \mathrm{Trig}_\Lambda(G)$,*

$$\left|\left|\hat{f}\right|\right|_{\ell_1} \leq C \int_{\mathbb{T}^\Gamma} \left|\left|\sum_{\gamma\in\Gamma} \widehat{f}(\gamma)\omega_\gamma(t)\gamma(x)\right|\right|_{C(G)} \, dt.$$

Then Λ is Sidon. Moreover, there exists a constant K such that the Sidon constant of Λ is bounded by KC^3.

2.2.3 Entropy numbers

Entropy integrations will be involved as an intermediate step during the proof of the main theorem. We start with definitions.

Definition 2.2.20. Suppose d is a pseudo-metric on G.
(1) The entropy number $N_d(\varepsilon)$ is the minimum of the number of open ε-balls in metric d needed to cover G.
(2) We let $M_d(\varepsilon)$ be the minimum integer n such that there exists a partition of G, $P_1, ..., P_n$, with each P_i having diameter at most ε.
(3) We say d is translation invariant if $d(s,t) = d(sx, tx)$ for all $s, t, x \in G$.

Lemma 2.2.21. *Let d, d_1, d_2, d_3 be pseudo-metrics on G.*
(1) For all $\varepsilon > 0$, we have $M_d(2\varepsilon) \leq N_d(\varepsilon) \leq M_d(\varepsilon)$.
(2) If there is some $\theta \in (0,1)$ with

$$d_3(s,t) \leq d_1(s,t)^{1-\theta} d_2(s,t)^{\theta}$$

for all $s, t \in G$, then for all $\varepsilon_1, \varepsilon_2 > 0$, $M_{d_3}(\varepsilon_1^{1-\theta}\varepsilon_2^{\theta}) \leq M_{d_1}(\varepsilon_1) M_{d_2}(\varepsilon_2)$.

Proof. (1) It is clear from the definitions that $M_d(2\varepsilon) \leq N_d(\varepsilon)$. To see that $N_d(\varepsilon) \leq M_d(\varepsilon)$, we let $P_1, ..., P_{M_d(\varepsilon)}$ be a partition of G, each of which has diameter at most ε. Pick $x_i \in P_i$ for $1 \leq i \leq M_d(\varepsilon)$. For $\varepsilon > 0$ and $x \in G$ we let $b_\varepsilon(x)$ be the open ball of radius ε centered at x. Then, $b_\varepsilon(x_i) \supset P_i$ and hence $\bigcup_{i=1}^{M_d(\varepsilon)} b_\varepsilon(x_i) \supset G$. Thus, $N_d(\varepsilon) \leq M_d(\varepsilon)$.

(2) Let $P_1, ..., P_{M_{d_1}(\varepsilon_1)}$ be a partition of (G, d_1) where each P_i has diameter at most ε_1. Let $Q_1, ..., Q_{M_{d_2}(\varepsilon_2)}$ be a partition of (G, d_2) where each Q_j has diameter at most ε_2. Let $s, t \in P_i \bigcap Q_j$. We have

$$d_3(s,t) \leq d_1(s,t)^{1-\theta} d_2(s,t)^{\theta} \leq \varepsilon_1^{1-\theta}\varepsilon_2^{\theta}.$$

Thus, each $P_i \bigcap Q_j$ in (G, d_3) has diameter at most $\varepsilon_1^{1-\theta}\varepsilon_2^{\theta}$ for $1 \leq i \leq M_{d_1}(\varepsilon_1)$ and $1 \leq j \leq M_{d_2}(\varepsilon_2)$, proving (2). $\square$

Definition 2.2.22. Suppose d is a translation invariant pseudo-metric on G and e is the identity in G. We let $\mu_d : \mathbb{R}^+ \to [0,1]$ be given by

$$\mu_d(\varepsilon) := m(\{x \in G : d(x,e) < \varepsilon\}).$$

We also let $\sigma_d : [0,1] \to \mathbb{R}^+$ be the increasing rearrangement of $d(x,e)$, that is,

$$\sigma_d(t) := \sup\{y : \mu_d(y) < t\}.$$

Lemma 2.2.23. *Suppose d, d_1, d_2, d_3 are translation invariant pseudo-metrics on G. Put $\sigma_i := \sigma_{d_i}$ for $i = 1, 2, 3$.*
(1) For $\varepsilon > 0$,

$$\mu_d(\varepsilon)^{-1} \leq N_d(\varepsilon) \leq \mu_d(\varepsilon/2)^{-1};$$

(2) If there exists $\theta \in (0, 1)$ such that

$$d_3(x, y) \leq d_1(x, y)^{1-\theta} d_2(x, y)^{\theta}$$

for all $x, y \in G$, then

$$\sigma_3(ts) \leq 4\sigma_1(t)^{1-\theta} \sigma_2(s)^{\theta}$$

for all $s, t \in [0, 1]$.

Proof. (1) From the definitions, it is clear that $N_d(\varepsilon)\mu_d(\varepsilon) \geq 1$. Moreover, we note that if we let $S_d(\varepsilon)$ be the maximum number of points $x_1, ..., x_{S_d(\varepsilon)}$ such that $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$, then $N_d(\varepsilon) \leq S_d(\varepsilon)$. Thus,

$$N_d(\varepsilon)\mu_d(\varepsilon/2) \leq S_d(\varepsilon)\mu_d(\varepsilon/2) \leq 1,$$

because the balls $b_{\varepsilon/2}(x_i)$, $1 \leq i \leq S_d(\varepsilon)$, are pairwise disjoint.

(2) We first show that for all $\varepsilon_1, \varepsilon_2 > 0$, we have

$$\mu_{d_1}(\varepsilon_1/4)\mu_{d_2}(\varepsilon_2/4) \leq \mu_{d_3}(\varepsilon_1^{1-\theta}\varepsilon_2^{\theta}).$$

Indeed, from previous results, we have

$$\begin{aligned} \mu_{d_1}(\varepsilon_1/4)^{-1}\mu_{d_2}(\varepsilon_2/4)^{-1} &\geq N_{d_1}(\varepsilon_1/2)N_{d_2}(\varepsilon_2/2) \geq M_{d_1}(\varepsilon_1)M_{d_2}(\varepsilon_2) \\ &\geq M_{d_3}(\varepsilon_1^{1-\theta}\varepsilon_2^{\theta}) \geq N_{d_3}(\varepsilon_1^{1-\theta}\varepsilon_2^{\theta}) \geq \mu_{d_3}(\varepsilon_1^{1-\theta}\varepsilon_2^{\theta})^{-1}. \end{aligned}$$

Now we suppose that for some $s, t \in [0, 1]$, we have

$$\sigma_3(ts) > (4\sigma_1(t))^{1-\theta}(4\sigma_2(s))^{\theta}.$$

Let $y_1, y_2 \in [0,1]$ be such that $y_1 > (4\sigma_1(t))^{1-\theta}$, $y_2 > (4\sigma_2(s))^{\theta}$ and $y_1 y_2 < \sigma_3(ts)$. Then,

$$ts > \mu_{d_3}(y_1 y_2) \geq \mu_{d_1}(y_1^{1/(1-\theta)}/4)\mu_{d_2}(y_2^{1/\theta}/4) \geq ts,$$

which is a contradiction. $\square$

Definition 2.2.24. Let d be a translation invariant pseudo-metric on G. For $\alpha > 1$ we let

$$\begin{aligned} J_\alpha(d) &:= \int_0^\infty (\log N_d(r))^{1/\alpha}\ dr \\ K_\alpha(d) &:= \int_0^\infty |\log \mu_d(r)|^{1/\alpha}\ dr \\ I_\alpha(d) &:= \int_0^1 \frac{\sigma_d(t)}{t|\log t|^{1-1/\alpha}}\ dt. \end{aligned}$$

Lemma 2.2.25. *(1) $K_\alpha(d) \leq J_\alpha(d) \leq 2K_\alpha(d)$.*
(2) $K_\alpha(d) < \infty$ if and only if $I_\alpha(d) < \infty$. If $I_\alpha(d) < \infty$, then $I_\alpha(d) = \alpha K_\alpha(d)$.

Proof. (1) is clear from Lemma 2.2.23 (1). To see (2), we use integration by parts. Since σ_d is the distribution function of μ_d,

$$K_\alpha(d) = \int_0^\infty |\log \mu_d(r)|^{1/\alpha}\ dr = \int_0^1 |\log t|^{1/\alpha}\ d\sigma_d(t).$$

From integration by parts, for $\delta > 0$ we have

$$\int_\delta^1 |\log t|^{1/\alpha}\ d\sigma_d(t) = \left[\sigma_d(t)|\log t|^{1/\alpha}\right]_\delta^1 + \frac{1}{\alpha}\int_\delta^1 \frac{\sigma_d(t)}{t|\log t|^{1-1/\alpha}}\ dt.$$

We let

$$F_1(\delta) := \int_\delta^1 |\log t|^{1/\alpha}\ d\sigma_d(t)$$
$$F_2(\delta) := \sigma_d(\delta)|\log \delta|^{1/\alpha}$$
$$F_3(\delta) := \int_\delta^1 \frac{\sigma_d(t)}{t|\log t|^{1-1/\alpha}}\ dt.$$

If $I_\alpha(d) < \infty$, then $\lim_{\delta\to 0} F_3(\delta) < \infty$. Thus,

$$K_\alpha(d) = \lim_{\delta\to 0} F_1(\delta) \le \lim_{\delta\to 0} F_3(\delta)/\alpha < \infty.$$

Moreover, we cannot have $\lim_{\delta\to 0} F_2(\delta) > 0$, for then $\lim_{\delta\to 0} F_3(\delta) = \infty$. Therefore we have $K_\alpha(d) = I_\alpha(d)/\alpha$.

Next, we suppose $K_\alpha(d) < \infty$. We note that for any $\delta > 0$,

$$\begin{aligned} F_3(\delta)/\alpha &= F_1(\delta) + F_2(\delta) \\ &\le K_\alpha(d) + \int_0^{\sigma_d(\delta)} |\log \mu_d(r)|^{1/\alpha}\ dr \\ &\le 2K_\alpha(d) < \infty. \end{aligned}$$

Thus, $I_\alpha(d) < \infty$. □

Definition 2.2.26. Let $1 < p < \infty$ and q be the dual index to p. Given $f \in L^2(G)$ with $\widehat{f} \in \ell_{p,1}$, we define a pseudo-metric on G by

$$d^f_{p,1}(s,t) := \left|\left|(\widehat{f}(\gamma)(\gamma(s) - \gamma(t))_{\gamma\in\Gamma}\right|\right|_{p,1}.$$

We let $N_{p,1}(x,f) := N_{d^f_{p,1}}(x)$ and

$$J_{p,1}(f) := \int_0^\infty (\log N_{p,1}(x,f))^{1/q}\ dx.$$

Similarly, for $\widehat{f} \in \ell_p$, we define

$$d_p^f(s,t) := \left(\sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)|^p |\gamma(s) - \gamma(t)|^p \right)^{1/p}.$$

We let $N_p(x,f) := N_{d_p^f}(x)$ and let

$$J_p(f) := \int_0^\infty (\log N_p(x,f))^{1/q}\, dx.$$

Remark 2.2.27. Let $1 < p < 2$ with dual index q and $f \in L^2(G)$ with $\widehat{f} \in \ell_{p,1}$. Using notations introduced in Definition 2.2.24 we have that

$$\begin{aligned} J_{p,1}(f) &= J_q(d_{p,1}^f) \\ J_p(f) &= J_q(d_p^f). \end{aligned}$$

We next prove a result about the convergence of the entropy integration $J_p(f)$.

Proposition 2.2.28. *Suppose $f \in C(G)$ with $\widehat{f} \in \ell_1(\Gamma)$. Let $1 < p < \infty$. Then $J_p(f) < \infty$ and, furthermore, there exists a constant C_p such that*

$$J_p(f) \le C_p \sum_{\gamma \neq 0} |\widehat{f}(\gamma)|.$$

In order to prove Proposition 2.2.28 we first prove a lemma.

Lemma 2.2.29. *[26] Let $1 < p < \infty$ and $(a_k)_{k \ge 1} \in \ell_1$ be a non-increasing, non-negative sequence. Define the section*

$$B((a_k)) := \{(b_k)_{k\ge 1} : |b_k| \le a_k\} \subset \ell_p.$$

For $\varepsilon > 0$, we let $H(\varepsilon, (a_k))$ be the minimum number of ε-balls in ℓ_p needed to cover $B((a_k))$. There exists a constant C_p, dependent on p, such that

$$\int_0^\infty \log(H(r,(a_k))^{1/q}\ dr \leq C_p \sum_{k\geq 0} a_k,$$

where $1/p + 1/q = 1$.

Proof. Throughout the proof C_p will denote a constant, only dependent on p, that may change from one occurrence to another. The main strategy for this proof is to build epsilon dense sets of points for a rich set of epsilons for the section. Actually, we will produce these for a larger section, $B((c_k))$, defined similarly, whose properties are easier to work with. The sequence $(c_k) \in \ell_1$ will be non-increasing, non-negative and satisfy $a_k \leq c_k$, $c_k \leq 2c_{k+1}$ for $k \geq 2$, and $c_1 \geq \left(\frac{1}{2}\sum_{k=2}^\infty c_k^p\right)^{1/p}$. Moreover, $\sum_{k=1}^\infty c_k \leq 5\sum_{k=1}^\infty a_k$.

Indeed, we can construct such a sequence by putting $c_2 := \max\{a_2, a_1/2\}$ and for $k > 2$ define c_k to be $c_{k-1}/2$ if $a_k < c_{k-1}/2$ and otherwise define c_k to be a_k. Notice that

$$\sum_{k\geq 2} c_k \leq \sum_{k=1}^\infty a_k + \sum_{k=1}^\infty \left(\sum_{j=1}^\infty \frac{a_k}{2^j}\right) = 2\sum_{k=1}^\infty a_k.$$

We put

$$c_1 := a_1 + (\frac{1}{2}\sum_{k=2}^\infty c_k^p)^{1/p}.$$

It is easy to see (c_k) has the specified properties.

Since $c_k \leq 2c_{k+1}$, for $n \geq 2$ we have

$$\left(\sum_{k=n}^\infty c_k^p\right)^{1/p} \leq 2\left(\sum_{k=n+1}^\infty c_k^p\right)^{1/p}. \tag{2.0}$$

For $n \geq 0$, we define

$$\delta_n := \left(\frac{1}{n+1} \sum_{k=n+1}^{\infty} c_k^p \right)^{1/p}, \ \varepsilon_n := \left(2 \sum_{k=n+1}^{\infty} c_k^p \right)^{1/p},$$
$$M(n) := \min \{n, \max \{k : c_k \geq \delta_n\}\}.$$

Observe that δ_n is a decreasing sequence, $M(n)$ is increasing and

$$\delta_n \leq \varepsilon_n / n^{1/p} \leq C_p \delta_n \tag{2.1}$$

for $n \geq 1$. Moreover, since we have $c_1 \geq \delta_1$, $\{k : c_k \geq \delta_n\} \neq \emptyset$ for all $n \geq 1$.

For a fixed $n \geq 1$, consider the following subset $S((c_k), n)$ in $B((c_k))$, which consists of the complex sequences (b_k) with

$$b_k = \begin{cases} j_k \delta_n + l_k \delta_n i \text{ for some } j_k, l_k \in \mathbb{Z} & \text{if } 1 \leq k \leq M(n) \\ 0 & \text{if } k > M(n) \end{cases}.$$

If $(b_k) \in S((c_k), n)$, say $b_k = j_k \delta_n + l_k \delta_n i$, then for $k \leq M(n)$, $|j_k|, |l_k| \leq \lfloor c_k / \delta_n \rfloor$. Hence,

$$|S((c_k), n)| \leq \prod_{k=1}^{M(n)} 4 \left(\frac{c_k}{\delta_n} + 1 \right)^2.$$

Moreover, for every $x \in B((c_k))$, since $c_k \leq \delta_n$ if $M(n) + 1 \leq k \leq n$, there exists $y \in S((c_k), n)$ such that

$$\begin{aligned} ||x - y||_p^p &\leq M(n) \left(\frac{\sqrt{2}\delta_n}{2} \right)^p + \sum_{k=M(n)+1}^{n} c_k^p + \sum_{k=n+1}^{\infty} c_k^p \\ &\leq n \delta_n^p + \sum_{k=n+1}^{\infty} c_k^p \leq 2 \sum_{k=n+1}^{\infty} c_k^p. \end{aligned}$$

Hence, $||x-y||_p \leq \varepsilon_n$ and therefore,

$$\log H(\varepsilon_n, (c_k)) \leq \log \left(\prod_{k=1}^{M(n)} 4 \left(\frac{c_k}{\delta_n} + 1 \right)^2 \right) \leq \log \left(16^n \prod_{k=1}^{M(n)} \left(\frac{c_k}{\delta_n} \right)^2 \right) =: T(n).$$

We have

$$\begin{aligned}
\int_0^\infty (\log(H(r,(c_k)))^{1/q}\ dr &= \int_0^{\delta_0} (\log(H(r,(c_k)))^{1/q}\ dr \\
&\leq \sum_{n=1}^{\infty} (\log(H(\varepsilon_n,(c_k)))^{1/q}(\varepsilon_{n-1}-\varepsilon_n) \\
&\leq \sum_{n=1}^{\infty} T(n)^{1/q}(\varepsilon_{n-1}-\varepsilon_n) \\
&= \varepsilon_0 T(1)^{1/q} + \sum_{n=1}^{\infty} \varepsilon_n (T(n+1)^{1/q} - T(n)^{1/q}). \qquad (2.2)
\end{aligned}$$

We note that $\varepsilon_0 \leq 2\,||(c_k)||_p \leq 2\sum_{k=1}^{\infty} c_k$ and one can check that

$$\begin{aligned}
T(1) = \log \left(\frac{16 c_1^2}{\delta_1^2} \right) &= \log \left(16 \left(\frac{a_1}{\delta_1} + 1 \right)^2 \right) \\
&\leq \log \left(16 \left(\frac{2^{1/p} a_1}{c_2} + 1 \right)^2 \right) \leq \log(16(2^{1/p} \cdot 2 + 1)^2).
\end{aligned}$$

Since $T(n) \geq \log(16^n) \geq n$, we have

$$\begin{aligned}
\varepsilon_n (T(n+1)^{1/q} - T(n)^{1/q}) &= \varepsilon_n \left(\frac{T(n+1)}{T(n+1)^{1/p}} - \frac{T(n)}{T(n)^{1/p}} \right) \\
&\leq \varepsilon_n \frac{T(n+1)-T(n)}{n^{1/p}} \\
&\leq C_p \delta_n (T(n+1)-T(n)), \qquad (2.3)
\end{aligned}$$

because of Eq. (2.1). Also,

$$T(n+1)-T(n) = \log\left(16\left(\frac{\delta_n}{\delta_{n+1}}\right)^{2M(n)}\left(\prod_{k=M(n)+1}^{M(n+1)}\frac{c_k^2}{\delta_{n+1}^2}\right)\right)$$
$$= \log(16) + 2M(n)\log\left(\frac{\delta_n}{\delta_{n+1}}\right) + 2\sum_{k=M(n)+1}^{M(n+1)}\log\left(\frac{c_k}{\delta_{n+1}}\right).$$

Observe that, if $M(n) < n$, we have

$$\sum_{k=M(n)+1}^{M(n+1)}\log\left(\frac{c_k}{\delta_{n+1}}\right) \le (M(n+1)-M(n))\log\left(\frac{c_{M(n)+1}}{\delta_{n+1}}\right)$$
$$\le (M(n+1)-M(n))\log\left(\frac{\delta_n}{\delta_{n+1}}\right).$$

Otherwise, if $M(n) = n$, then

$$\sum_{k=M(n)+1}^{M(n+1)}\log\left(\frac{c_k}{\delta_{n+1}}\right) \le \frac{c_{n+1}}{\delta_{n+1}}.$$

Hence, in general, we have

$$\sum_{k=M(n)+1}^{M(n+1)}\log\left(\frac{c_k}{\delta_{n+1}}\right) \le (M(n+1)-M(n))\log\left(\frac{\delta_n}{\delta_{n+1}}\right) + \frac{c_{n+1}}{\delta_{n+1}}.$$

Therefore,

$$T(n+1)-T(n) \le \log(16) + 2M(n+1)\log\left(\frac{\delta_n}{\delta_{n+1}}\right) + \frac{2c_{n+1}}{\delta_{n+1}}$$
$$\le \log(16) + 2M(n+1)\left(\frac{\delta_n}{\delta_{n+1}} - 1\right) + \frac{2c_{n+1}}{\delta_{n+1}}. \tag{2.4}$$

Eq. (2.0) implies $\delta_n \le C_p\delta_{n+1}$ and since $M(n) \le n$, we have

$$\delta_n\left(2M(n+1)\left(\frac{\delta_n}{\delta_{n+1}} - 1\right) + \frac{2c_{n+1}}{\delta_{n+1}}\right) \le C_p(n(\delta_n - \delta_{n+1}) + c_{n+1}). \tag{2.5}$$

Hence, Eq. (2.4) and Eq. (2.5) imply

$$\begin{aligned}\sum_{n=1}^{\infty}\delta_n(T(n+1)-T(n)) &\leq \log(16)\sum_{n=1}^{\infty}\delta_n + C_p\sum_{n=1}^{\infty}n(\delta_n-\delta_{n+1}) + C_p\sum_{n=1}^{\infty}c_{n+1}\\ &\leq C_p\sum_{n=1}^{\infty}(\delta_n+c_n).\end{aligned}$$

It is known (see [2], Theorem 2) that

$$\sum_{n=1}^{\infty}\delta_n \leq C_p\sum_{k=1}^{\infty}c_k,$$

and hence,

$$\sum_{n=1}^{\infty}\delta_n(T(n+1)-T(n)) \leq C_p\sum_{k=1}^{\infty}c_k. \tag{2.6}$$

Finally, by Eq. (2.2), Eq. (2.3), Eq. (2.6) and the definition of ε_0, we have

$$\begin{aligned}\int_0^{\infty}\log(H(r,(a_k))^{1/q}\,dr &\leq \int_0^{\infty}\log(H(r,(c_k))^{1/q}\,dr\\ &\leq \varepsilon_0 T(1)^{1/q} + \sum_{n=1}^{\infty}\varepsilon_n(T(n+1)^{1/q}-T(n)^{1/q})\\ &\leq \varepsilon_0 T(1)^{1/q} + C_p\sum_{n=1}^{\infty}\delta_n(T(n+1)-T(n))\\ &\leq C_p\sum_{k=1}^{\infty}c_k \leq C_p\sum_{k=1}^{\infty}a_k.\end{aligned}$$

□

Proof of Proposition 2.2.28. The proof is based on Lemma 2.2.29. Let $(a_k)_{k\geq 0}$ be the non-increasing re-arrangement of $(|\widehat{f}(\gamma)|)_{\gamma\in\Gamma}$ and for $\varepsilon > 0$, let $H(\varepsilon,(a_k))$ and

$B((a_k))$ be defined as in Lemma 2.2.29. Lemma 2.2.29 gives a constant C_p such that

$$\int_0^\infty \log(H(\varepsilon,(a_k))^{1/q}\, d\varepsilon \leq C_p \sum_{k\geq 0} a_k,$$

where $1/p + 1/q = 1$.

Let $(\gamma_k)_{k\geq 0}$ be a rearrangement of Γ such that $|\widehat{f}(\gamma_k)| = a_k$ for $k \geq 0$. Define $F : G \to \ell_p(\mathbb{C})$ by $F(x) = (\widehat{f}(\gamma_k)\gamma_k(x))_{k\geq 0}$. Notice that F is an isometry from (G, d_p^f) onto $F(G) \subset \ell_p(\mathbb{C})$.

We claim that for $\varepsilon > 0$, $N_p(2\varepsilon, f) \leq H(\varepsilon,(a_k))$. Indeed, suppose $\varepsilon > 0$ and $b_1, ..., b_n$ are balls of radius ε in ℓ_p such that $\bigcup_{i=1}^n b_i \supset B((a_k)) \supset F(G)$. We choose (may reorder if necessary) $b_1, ..., b_m$ to be the ones with $b_i \bigcap F(G) \neq \emptyset$ for $1 \leq i \leq m$. Let $x_i \in b_i \bigcap F(G)$ and let $(B_i)_{1\leq i\leq m}$ be the balls of radius 2ε and centered at x_i. We have that $\bigcup_{i=1}^m B_i \supset F(G)$. Since $F : G \to F(G)$ is an isometry, we can cover G by m balls of radius 2ε as well. Hence, $N_p(2\varepsilon, f) \leq H(\varepsilon,(a_k))$.

Thus, we have

$$J_p(f) \leq \frac{C_p}{2} \sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)|.$$

Finally, we let $g \in C(G)$ such that $\widehat{g}(\gamma) = \widehat{f}(\gamma)$ for all $\gamma \neq 0$ and $\widehat{g}(1) = 0$. We notice that $d_p^g = d_p^f$ and therefore we have

$$J_p(f) = J_p(g) \leq \frac{C_p}{2} \sum_{\gamma\in\Gamma} |\widehat{g}(\gamma)| = \frac{C_p}{2} \sum_{\gamma\neq 0} |\widehat{f}(\gamma)|.$$

□

Next we prove an interpolation type result.

Proposition 2.2.30. *Let $1 < r < p < 2$. Let θ be given by $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$. There* *exists a constant C_θ such that for all $f \in \mathrm{Trig}(G)$ we have*

$$J_{p,1}(f) \le K_\theta J_2(f)^\theta J_r(f)^{1-\theta}.$$

Proof. Let q, s be given by $1/p + 1/q = 1$ and $1/s + 1/r = 1$. Since $f \in \mathrm{Trig}(G)$, Proposition 2.2.28 implies $J_r(f), J_2(f) < \infty$ and therefore by Lemma 2.2.25, $sK_s(d_r^f) = I_s(d_r^f) < \infty$ and $2K_2(d_2^f) = I_2(d_2^f) < \infty$.

From Proposition 2.2.14, for all $s, t \in G$ and some constant C_θ,

$$d_{p,1}^f(s,t) \le C_\theta d_r^f(s,t)^{1-\theta} d_2^f(s,t)^\theta.$$

Thus, by Lemma 2.2.23, for all $s, t > 0$,

$$\sigma_{d_{p,1}^f}(ts) \le 4C_\theta \sigma_{d_r^f}(t)^{1-\theta} \sigma_{d_2^f}(s)^\theta.$$

Using the previous inequality, the change of variable $x = t^2$ and Holder's inequality,

$$\begin{aligned}
I_q(d_{p,1}^f) =& \int_0^1 \frac{\sigma_{d_{p,1}^f}(x)}{x|\log x|^{1/p}}\, dx = \int_0^1 \frac{\sigma_{d_{p,1}^f}(t^2)}{t^2|\log t^2|^{1/p}} 2t\, dt \\
\le& 4 \cdot 2^{1/q} C_\theta \int_0^1 \frac{\left(\sigma_{d_r^f}(t)\right)^{1-\theta}\left(\sigma_{d_2^f}(t)\right)^\theta}{t|\log t|^{\frac{1-\theta}{r}+\frac{\theta}{2}}}\, dt \\
\le& 4 \cdot 2^{1/q} C_\theta \left(\int_0^1 \frac{\sigma_{d_r^f}(t)}{t|\log t|^{1/r}}\, dt\right)^{1-\theta} \left(\int_0^1 \frac{\sigma_{d_2^f}(t)}{t|\log t|^{1/2}}\, dt\right)^\theta \\
=& 4 \cdot 2^{1/q} C_\theta \left(I_s(d_r^f)\right)^{1-\theta} \left(I_2(d_2^f)\right)^\theta < \infty.
\end{aligned}$$

Combined with the relationships of Lemma 2.2.25, this gives

$$\begin{aligned}
J_{p,1}(f) = J_q(d_{p,1}^f) &\le \frac{2}{q} I_q(d_{p,1}^f) \le \frac{8}{q} \cdot 2^{1/q} C_\theta (I_s(d_r^f))^{1-\theta} (I_2(d_2^f))^\theta \\
&= \frac{8}{q} \cdot 2^{1/q+\theta} s^{1-\theta} C_\theta (K_s(d_r^f))^{1-\theta} (K_2(d_2^f))^\theta \\
&\le \frac{8}{q} \cdot 2^{1/q+\theta} s^{1-\theta} C_\theta (J_s(d_r^f))^{1-\theta} (J_2(d_2^f))^\theta \\
&= \frac{8}{q} \cdot 2^{1/q+\theta} s^{1-\theta} C_\theta (J_r(f))^{1-\theta} (J_2(f))^\theta,
\end{aligned}$$

which shows the desired result with the constant in the statement of the Proposition 2.2.30 being $\frac{8}{q} \cdot 2^{1/q+\theta} s^{1-\theta} C_\theta$. $\square$

Finally, we will need to use Fernique's classical inequality. A proof can be found in [24] for example. Recall that $(g_\gamma)_{\gamma \in \Gamma}$ is a collection of independent standard Gaussian random variables on (Ω_1, P_1) indexed by Γ.

Theorem 2.2.31 (Fernique's inequality)**.** *There exists a constant C such that for all $f \in \mathrm{Trig}(G)$,*

$$J_2(f) + |\widehat{f}(0)| \leq C \int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g_\gamma \gamma \right\|_{C(G)} .$$

2.3 The main theorem

We state and prove the main theorem. Recall that the definitions of the entropy integrals $J_p(f)$ and $J_{p,1}(f)$ are given in Definition 2.2.26. Recall that $(\varepsilon_\gamma)_{\gamma \in \Gamma}$ is a collection of independent random variables on (Ω, P) indexed by Γ such that each ε_γ takes only values 1 and -1 with equal probability $1/2$.

Theorem 2.3.1. *Let $\Lambda = \{\gamma_n : n \in \mathbb{N}\}$ be a subset in Γ. The following are equivalent.*

(1) The set Λ is proportionally Sidon with bounded Sidon constant, which means there exist constants C_1 and $\delta_1 > 0$ such that for all finite subsets $A \subset \Lambda$, there exists a Sidon set $B \subset A$ such that $S(B) \leq C_1$ and $|B| \geq \delta_1 |A|$.

(2) There exists $\delta_2 > 0$ such that for every finite subset $A \subset \Lambda$ we have

$$\int_\Omega \left\| \sum_{\gamma \in A} \varepsilon_\gamma \gamma \right\|_{C(G)} \geq \delta_2 |A|.$$

(3) For all $p \in (1,2]$ there exists some constant C_3 such that for all finite $A \subset \Lambda$,

$$\left\| \left| \sum_{\gamma \in A} \gamma \right| \right\|_{\phi_q} \leq C_3 |A|^{1/p}.$$

(4) For all $p \in (1,2]$ there exists some constant C_4 such that for all $f \in \mathrm{Trig}_\Lambda(G)$ we have

$$\sum_{\gamma \in \Lambda \setminus \{0\}} |\widehat{f}(\gamma)| \leq C_4 J_{p,1}(f).$$

(5) There exists some constant C_5 such that for all $f \in \mathrm{Trig}_\Lambda(G)$ we have

$$\sum_{\gamma \in \Lambda \setminus \{0\}} |\widehat{f}(\gamma)| \leq C_5 J_2(f).$$

(6) The set Λ is Sidon with Sidon constant bounded by C_6.

Remark 2.3.2. The constants stated in Theorem 2.3.1 will be used later in this thesis. In particular, given that (2) holds for δ_2, we need to know a bound C_6 for the Sidon constant in (6) (in terms of δ_2).
Assume (2) holds for δ_2. During the proof of Theorem 2.3.1, we will show (3), (4), (5) and (6) hold and the quantities C_3, C_4, C_5 and C_6 can be chosen in the following manner (in terms of δ_2) for some constants $K(p)$ (or $K(\xi)$ depending on ξ) independent from δ_2:

$$\begin{aligned}
C_3 &= K(p)\delta_2^{-2/q} \quad (q \text{ is the dual index to } p \text{ in } (3)) \\
C_4 &= K(p)\delta_2^{-2/q} \quad (q \text{ is the dual index to } p \text{ in } (4)) \\
C_5 &= K(\xi)\delta_2^{-(1+\xi)} \\
C_6 &= K(\xi)\delta_2^{-(3+\xi)}.
\end{aligned}$$

Note that (6) $\to$ (1) is obvious and therefore we will prove (1) $\to$ (2) $\to$ (3) $\to$ (4) $\to$ (5) $\to$ (6) of Theorem 2.3.1.

2.3.1 Proof of Theorem 2.3.1: (1) $\to$ (2) $\to$ (3)

Proof of Theorem 2.3.1 (1) $\to$ (2) $\to$ (3): (1) $\to$ (2): Fix a finite set $A \subset \Lambda$. From (1), there exists a Sidon set $B \subset A$ with $S(B) \leq C_1$ and $|B| \geq \delta_1 |A|$. Hence, for $\omega \in \Omega$, there exists $\mu_\omega \in M(G)$ such that $\widehat{\mu_\omega}(\gamma) = \varepsilon_\gamma(\omega)$ for $\gamma \in B$ and $||\mu_\omega||_{M(G)} \leq C_1$. We have

$$\delta_1|A| \leq |B| = |\sum_{\gamma \in B} \gamma(e)| = \left|\left|\sum_{\gamma\in B}\gamma\right|\right|_{C(G)} = \int_\Omega \left|\left|\mu_\omega * (\sum_{\gamma\in B}\varepsilon_\gamma\gamma)\right|\right|_{C(G)},$$

where the last equality is because of the definition of $\widehat{\mu_\omega}$. As $||\mu_\omega|| \leq C_1$,

$$\int_\Omega \left|\left|\mu_\omega * (\sum_{\gamma\in B}\varepsilon_\gamma\gamma)\right|\right|_{C(G)} \leq \int_\Omega ||\mu_\omega||_{M(G)} \left|\left|\sum_{\gamma\in B}\varepsilon_\gamma\gamma\right|\right|_{C(G)} \leq C_1 \int_\Omega \left|\left|\sum_{\gamma\in B}\varepsilon_\gamma\gamma\right|\right|_{C(G)}.$$

Notice that from Lemma 2.2.15 (take $\Phi(x) = x$, $X = (\Omega, P)$ and $Y = (C(G), ||\cdot||_\infty)$ in Lemma 2.2.15) and the triangle inequality, we have

$$\begin{aligned}\int_\Omega \left|\left|\sum_{\gamma\in A}\varepsilon_\gamma\gamma\right|\right|_{C(G)} &= \frac{1}{2}\left(\int_\Omega \left|\left|\sum_{\gamma\in A}\varepsilon_\gamma\gamma\right|\right|_{C(G)} + \int_\Omega \left|\left|\sum_{\gamma\in B}\varepsilon_\gamma\gamma + \sum_{\gamma\in A\setminus B}(-1)\varepsilon_\gamma\gamma\right|\right|_{C(G)}\right)\\ &\geq \int_\Omega \left|\left|\sum_{\gamma\in B}\varepsilon_\gamma\gamma\right|\right|_{C(G)}.\end{aligned}$$

Putting everything together, we get (2).

(2) $\to$ (3): Assume (2). We first show (3) in the case that $p = 2$. From Proposition 2.2.18, for all $g \in L_{\varphi_2}(G)$ and any finite subset $A \subset \Lambda$, we have

$$\int_\Omega \left|\left|\sum_{\gamma\in A}\varepsilon_\gamma\widehat{g}(\gamma)\gamma\right|\right|_{C(G)} \leq C|A|^{1/2}\,||g||_{\varphi_2}.$$

By assumption (2) and Proposition 2.2.16 (2) (take $\Phi(x) = x$, $X = (\Omega, P)$ and

$Y = (C(G), ||\cdot||_\infty)$ as in Proposition 2.2.16), there exists $\delta_2 > 0$ such that

$$\delta_2 |A| \min_{\gamma \in A} |\widehat{g}(\gamma)| \leq \min_{\gamma \in A} |\widehat{g}(\gamma)| \int_\Omega \left|\left| \sum_{\gamma \in A} \varepsilon_\gamma \gamma \right|\right|_{C(G)}$$

$$\leq 2 \int_\Omega \left|\left| \sum_{\gamma \in A} \varepsilon_\gamma \widehat{g}(\gamma) \gamma \right|\right|_{C(G)} \leq 2C|A|^{1/2} \, ||g||_{\varphi_2} .$$

Hence,

$$|A|^{1/2} \min_{\gamma \in A} |\widehat{g}(\gamma)| \leq 2C\delta_2^{-1} \, ||g||_{\varphi_2} .$$

Next, we reorder A so that $A = \{\gamma_j : 1 \leq j \leq |A|\}$ and $|\widehat{g}(\gamma_{j_1})| \geq |\widehat{g}(\gamma_{j_2})|$ if $j_1 \leq j_2$. Applying the argument above to each $A_j := \{\gamma_k : 1 \leq k \leq j\}$, we have

$$j^{1/2} |\widehat{g}(\gamma_j)| \leq 2C\delta_2^{-1} \, ||g||_{\varphi_2}$$

for all $1 \leq j \leq |A|$. Hence,

$$\sum_{j=1}^{|A|} |\widehat{g}(\gamma_j)| \leq 2C\delta_2^{-1} \left(\sum_{j=1}^{|A|} j^{-1/2} \right) ||g||_{\varphi_2} \leq 4C\delta_2^{-1} |A|^{1/2} \, ||g||_{\varphi_2} .$$

Hence, by Theorem 2.2.6 and the definition of the φ_2^* norm,

$$\left|\left| \sum_{\gamma \in A} \gamma^{-1} \right|\right|_{\phi_2} \leq \left|\left| \sum_{\gamma \in A} \gamma^{-1} \right|\right|_{\varphi_2^*} = \sup_{||g||_{\varphi_2} \leq 1} \left| \int_G (\sum_{\gamma \in A} \gamma^{-1}) g \right|$$

$$= \sup_{||g||_{\varphi_2} \leq 1} \left| \sum_{\gamma \in A} \widehat{g}(\gamma) \right| \leq 4C\delta_2^{-1} |A|^{1/2},$$

proving the case for $p = 2$.

Next, we deal with $p \in (1, 2)$. We let $p \in (1, 2)$ and q be such that $1/p + 1/q = 1$. By the interpolation property of Orlicz norms (Proposition 2.2.8) and taking $\theta = 2/q$,

$$\left|\left|\sum_{\gamma\in A}\gamma\right|\right|_{\phi_q} \leq \left|\left|\sum_{\gamma\in A}\gamma\right|\right|_{\phi_2}^{\theta} \left|\left|\sum_{\gamma\in A}\gamma\right|\right|_{C(G)}^{1-\theta} \leq (4C\delta_2^{-1}|A|^{1/2})^{\theta}|A|^{1-\theta} = (4C\delta_2^{-1})^{\theta}|A|^{1/p},$$

proving (3) with $C_3 = K(p)\delta_2^{-2/q}$. □

2.3.2 Proof of Theorem 2.3.1: (3) → (4)

Next we prove (3) → (4). This is a critical step because we are going to transit to the entropy integration. We need the following variation of Dudley's Theorem [7].

Proposition 2.3.3. *Let d be a pseudo-metric on G, $2 \leq q < \infty$ and $f \in \mathrm{Trig}(G)$. Suppose that $\int_0^\infty (\log N_d(r))^{1/q}\, dr < \infty$ and for all $s, t \in G$, $||f_t - f_s||_{\phi_q} \leq d(s,t)$, where $f_t(x) = f(x+t)$. Then,*

$$\sup_{x,y\in G} |f(x) - f(y)| \leq D_q \int_0^\infty (\log N_d(r))^{1/q}\, dr$$

for some constant D_q.

Proof. Let D be the diameter of G with respect to d and $\delta_n := D4^{-n}$ for $n \geq 0$. Let $N_n := N_d(\delta_n)$. For each $n \in \mathbb{N}$, there exist a partition $(A_{j,n})_{j\leq N_n}$ of G and $z_{j,n} \in A_{j,n}$ such that the diameter of each $A_{j,n}$ is at most $2\delta_n$.

We let

$$f_t^n(x) := \sum_{j\leq N_n} 1_{A_{j,n}}(t) f_{z_{j,n}}(x)$$

and notice that $f_t^0(x) = f_{z_{1,0}}(x)$. Since f is continuous, $f_t^n(x) \to f_t(x)$ as n tends to infinity for each $t, x \in G$ and therefore

$$f_t(x) = f_t^0(x) + \sum_{n\geq 1} \left(f_t^n(x) - f_t^{n-1}(x)\right).$$

Hence, for all $a \in G$,

$$\begin{aligned}
\sup_{x,y\in G} |f(x) - f(y)| &= \sup_{t,s\in G} |f_t(a) - f_s(a)| \\
&\leq 2 \sum_{n\geq 1} \sup_{t\in G} |f_t^n(a) - f_t^{n-1}(a)| \\
&= 2 \sum_{n\geq 1} \sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}}(a) - f_{z_{j,n-1}}(a)|,
\end{aligned}$$

where $\Lambda_n \subset \{1, ..., N_n\} \times \{1, ..., N_{n-1}\}$ and $(i, j) \in \Lambda_n$ exactly when $A_{i,n} \bigcap A_{j,n-1}$ is non-empty.

We next claim that

$$\int_G \sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}} - f_{z_{j,n-1}}| \leq \phi_q^{-1}(2|\Lambda_n|) \sup_{(i,j)\in\Lambda_n} \left|\left|f_{z_{i,n}} - f_{z_{j,n-1}}\right|\right|_{\phi_q}.$$

Indeed, we may assume $\sup_{(i,j)\in\Lambda_n} \left|\left|f_{z_{i,n}} - f_{z_{j,n-1}}\right|\right|_{\phi_q} \leq 1$, which means

$$\int_G \phi_q(|f_{z_{i,n}} - f_{z_{j,n-1}}|) \leq e - 1.$$

Since ϕ_q is convex and increasing, by Jensen's inequality we have

$$\begin{aligned}
\phi_q\left(\int_G \sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}} - f_{z_{j,n-1}}|\right) &\leq \int_G \phi_q\left(\sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}} - f_{z_{j,n-1}}|\right) \\
&= \int_G \sup_{(i,j)\in\Lambda_n} \phi_q\left(|f_{z_{i,n}} - f_{z_{j,n-1}}|\right) \\
&\leq \int_G \sum_{(i,j)\in\Lambda_n} \phi_q(|f_{z_{i,n}} - f_{z_{j,n-1}}|) \\
&\leq |\Lambda_n|(e-1) \leq 2|\Lambda_n|.
\end{aligned}$$

Hence, $\int_G \sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}} - f_{z_{j,n-1}}| \leq \phi_q^{-1}(2|\Lambda_n|)$, proving the claim.

Thus, we have

$$
\begin{aligned}
\sup_{x,y\in G} |f(x)-f(y)| &= \int_G \sup_{t,s\in G} |f_t - f_s| \\
&\le 2\int_G \sum_{n\ge 1} \sup_{(i,j)\in\Lambda_n} |f_{z_{i,n}} - f_{z_{j,n-1}}| \\
&\le 2\sum_{n\ge 1} \phi_q^{-1}(2|\Lambda_n|) \sup_{(i,j)\in\Lambda_n} \left|\left|f_{z_{i,n}} - f_{z_{j,n-1}}\right|\right|_{\phi_q}.
\end{aligned}
$$

We also note that

$$
\phi_q^{-1}(2|\Lambda_n|) \le \phi_q^{-1}(2N_n^2) \le 4^{1/q}(\log N_n)^{1/q}.
$$

For all $(i,j)\in\Lambda_n$ and $t\in A_{i,n}\bigcap A_{j,n-1}$, we have

$$
\begin{aligned}
\left|\left|f_{z_{i,n}} - f_{z_{j,n-1}}\right|\right|_{\phi_q} &\le \left|\left|f_{z_{i,n}} - f_t\right|\right|_{\phi_q} + \left|\left|f_{z_{j,n-1}} - f_t\right|\right|_{\phi_q} \\
&\le d(z_{i,n},t) + d(z_{j,n-1},t) \le 2D4^{-(n-1)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sup_{x,y\in G} |f(x)-f(y)| &\le 4\cdot 4^{1/q}\sum_{n\ge 1} D4^{-(n-1)}(\log N_n)^{1/q} \\
&= \frac{4^4}{3}\cdot 4^{1/q}\sum_{n\ge 1}\frac{3D}{4}4^{-(n+1)}(\log N_d(D4^{-n}))^{1/q} \\
&\le \frac{4^4}{3}\cdot 4^{1/q}\sum_{n\ge 1}\int_{D4^{-(n+1)}}^{D4^{-n}} (\log N_d(r))^{1/q}\, dr \\
&\le \frac{4^4}{3}\cdot 4^{1/q}\int_0^\infty (\log N_d(r))^{1/q}\, dr.
\end{aligned}
$$

□

We are now ready to prove (3) → (4) in Theorem 2.3.1.

Proof of Theorem 2.3.1: (3) → (4). Let $p\in(1,2]$ and $C=C_3$ be as given in (3). The triangle inequality and (3) gives that for all finite sets $A\subset\Lambda$ and for $h_\gamma=\pm 1$, we have

$$\left\|\sum_{\gamma\in A} h_\gamma \gamma\right\|_{\phi_q} \leq \left\|\sum_{h_\gamma=1} \gamma\right\|_{\phi_q} + \left\|\sum_{h_\gamma=-1} -\gamma\right\|_{\phi_q} \leq 2C|A|^{1/p.}$$

By Proposition 2.2.17, for $|\alpha_\gamma| = 1$,

$$\sup_{\omega\in\Omega} \left\|\sum_{\gamma\in A} \alpha_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} \leq 2 \sup_{\omega\in\Omega} \left\|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q}.$$

Thus, whenever $|\alpha_\gamma| = 1$ and $A \subset \Lambda$, then

$$\left\|\sum_{\gamma\in A} \alpha_\gamma \gamma\right\|_{\phi_q} \leq \sup_{\omega\in\Omega} \left\|\sum_{\gamma\in A} \alpha_\gamma \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} \leq 2 \sup_{\omega\in\Omega} \left\|\sum_{\gamma\in A} \varepsilon_\gamma(\omega)\gamma\right\|_{\phi_q} \leq 4C|A|^{1/p}. \tag{2.7}$$

Let $f \in \mathrm{Trig}_A(G)$ and $\xi_\gamma \in \mathbb{T}$ be such that $\xi_\gamma \hat{f}(\gamma) = |\hat{f}(\gamma)|$. We denote

$$g := \sum_{\gamma\in A} \xi_\gamma \hat{f}(\gamma)\gamma = \sum_{\gamma\in A} |\hat{f}(\gamma)|\gamma.$$

Since $(\xi_\gamma \hat{f}(\gamma))_{\gamma\in A}$ is a positive sequence, we decompose it, as explained in Remark 2.2.13, by

$$\xi_\gamma \hat{f}(\gamma) = \sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} 1_{A_i}(\gamma),$$

so that

$$||\widehat{g}||_{p,1} = \sum_{i=1}^{|A|} \lambda_i,$$

where here A_i are suitable subsets in A and λ_i are positive real numbers. (For the definition of the $||\cdot||_{p,1}$ norm we direct the reader to Definition 2.2.11.)

As shown in Remark 2.2.13 and Example 2.2.12, we have

$$\left|\left|\widehat{f}\right|\right|_{p,1} = ||\widehat{g}||_{p,1} = \sum_{i=1}^{|A|} \lambda_i = \sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} ||1_{A_i}||_{p,1} \geq \sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} |A_i|^{1/p}.$$

Thus, by (2.7),

$$\left|\left|\widehat{f}\right|\right|_{p,1} \geq (4C)^{-1} \left(\sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} \left|\left| \sum_{\gamma \in A_i} \xi_\gamma^{-1} \gamma \right|\right|_{\phi_q} \right).$$

The triangle inequality implies

$$\begin{aligned} ||f||_{\phi_q} &= \left|\left| \sum_{\gamma \in A} \sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} \xi_\gamma^{-1} 1_{A_i}(\gamma)\gamma \right|\right|_{\phi_q} \\ &\leq \left(\sum_{i=1}^{|A|} \frac{\lambda_i}{||1_{A_i}||_{p,1}} \left|\left| \sum_{\gamma \in A_i} \xi_\gamma^{-1} \gamma \right|\right|_{\phi_q} \right) \leq 4C \left|\left|\widehat{f}\right|\right|_{p,1}. \end{aligned} \tag{2.8}$$

We recall that for $a \in G$, $g_a(x) = g(x+a)$ and we direct the reader to Definition 2.2.26 for the definitions of $J_{p,1}(f)$ and the $d^f_{p,1}$ pseudo metric, which we will now use.

The inequality Eq. (2.8) above holds for all $f \in \mathrm{Trig}_A(G)$. Since $g \in \mathrm{Trig}_A(G)$, it is easy to see that for all $s, t \in G$ we have $g_s - g_t \in \mathrm{Trig}_A(G)$. Applying Eq. (2.8) to $g_s - g_t$ gives

$$||g_s - g_t||_{\phi_q} \leq 4C \, ||\widehat{g}_s - \widehat{g}_t||_{p,1} = 4C d^f_{p,1}(s,t).$$

Since $f \in \mathrm{Trig}(G)$, Proposition 2.2.28 implies $J_u(f) < \infty$ for all $u > 1$. Hence, we have $J_{p,1}(f) < \infty$ from Proposition 2.2.30. Thus, by Proposition 2.3.3, we have

$$\sup_{x,y \in G} |g(x) - g(y)| \leq 4CD_q J_{p,1}(f).$$

Notice that

$$\sum_{\gamma\in\Lambda\setminus\{0\}} |\widehat{f}(\gamma)| = \sum_{\gamma\in\Lambda\setminus\{0\}} \widehat{g}(\gamma) = \sum_{\gamma\in\Gamma} \widehat{g}(\gamma) - \widehat{g}(0) = g(e) - \int_G g(y)\ dm(y)$$
$$\leq \sup_{x\in G} \left| g(x) - \int g(y)\ dm(y) \right| \leq \sup_{x,y\in G} |g(x) - g(y)|.$$

Hence,

$$\sum_{\gamma\in\Lambda\setminus\{0\}} |\widehat{f}(\gamma)| \leq 4CD_q J_{p,1}(f),$$

proving (4) with $C_4 = 4D_q C_3 = K(p)\delta_2^{-2/q}$. $\square$

It remains to show (4) implies (5) and (5) implies (6) because (6) implies (1) is trivial.

2.3.3 Proof of Theorem 2.3.1: (4) → (5) → (6)

We first prove a lemma. Recall that $(g_\gamma)_{\gamma\in\Gamma}$ is a collection of independent standard Gaussian random variables indexed by Γ on a probability space (Ω_1, P_1). We let ω_γ on $(\mathbb{T}^\Gamma, P_2)$ be given as in Rider's theorem (Theorem 2.2.19). We continue to let $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ be a collection of independent random variables on (Ω, P) indexed by Γ such that each ε_γ takes only values 1 and -1 with equal probability 1/2.

Lemma 2.3.4. *(1) Given $C > 0$ and $\xi > 0$, there exists a constant $C_1 = K(\xi)C^\xi$ such that*

$$\int_{\Omega_1} \left\| \sum_{\gamma\in\Gamma} \widehat{f}(\gamma) g_\gamma \gamma \right\|_{C(G)} dP_1 \leq \frac{\left\|\widehat{f}\right\|_{\ell_1}}{C} + C_1 \int_\Omega \left\| \sum_{\gamma\in\Gamma} \widehat{f}(\gamma)\varepsilon_\gamma\gamma \right\|_{C(G)} dP$$

for all $f \in \mathrm{Trig}(G)$.
(2) For all $f \in \mathrm{Trig}(G)$,

$$\int_\Omega \left\| \sum_{\gamma\in A} \varepsilon_\gamma \widehat{f}(\gamma)\gamma \right\|_{C(G)} dP \leq 2 \int_{\mathbb{T}^\Gamma} \left\| \sum_{\gamma\in A} \omega_\gamma \widehat{f}(\gamma)\gamma \right\|_{C(G)} dP_2.$$

Proof. (1) Given $C > 0$, we let $C_1 > 0$ be large enough that

$$\int_{\Omega_1} |g_\gamma| 1_{\{|g_\gamma| > C_1\}} \, dP_1 \leq 1/C$$

for all $\gamma \in \Gamma$. Later we will see such C_1 exists and consider the size of C_1.

For each $\gamma \in \Gamma$, we put $g'_\gamma := g_\gamma 1_{\{|g_\gamma| > C_1\}}$ and $g''_\gamma := g_\gamma - g'_\gamma$. From the symmetry of the Gaussian distribution we note that

$$\begin{aligned}
\int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g''_\gamma \gamma \right\|_{C(G)} dP_1 &= \int_{\Omega} \int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) \varepsilon_\gamma g''_\gamma \gamma \right\|_{C(G)} dP_1 \, dP \\
&= \int_{\Omega_1} \left(\int_{\Omega} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) \varepsilon_\gamma g''_\gamma \gamma \right\|_{C(G)} dP \right) dP_1.
\end{aligned}$$

As $|g''_\gamma| \leq C_1$, Proposition 2.2.16 (1) implies

$$\begin{aligned}
\int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g''_\gamma \gamma \right\|_{C(G)} dP_1 &\leq \int_{\Omega_1} \left(\int_{\Omega} C_1 \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) \varepsilon_\gamma \gamma \right\|_{C(G)} dP \right) dP_1 \\
&= C_1 \int_{\Omega} \left\| \sum_{\gamma \in A} \varepsilon_\gamma \widehat{f}(\gamma) \gamma \right\|_{C(G)} dP.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g_\gamma \gamma \right\|_{C(G)} &\leq \int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g'_\gamma \gamma \right\|_{C(G)} + \int_{\Omega_1} \left\| \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g''_\gamma \gamma \right\|_{C(G)} \\
&\leq \sum_{\gamma \in A} |\widehat{f}(\gamma)| \int_{\Omega_1} |g'_\gamma| + C_1 \int_{\Omega} \left\| \sum_{\gamma \in A} \varepsilon_\gamma \widehat{f}(\gamma) \gamma \right\|_{C(G)} \\
&\leq \frac{\left\| \hat{f} \right\|_{\ell_1}}{C} + C_1 \int_{\Omega} \left\| \sum_{\gamma \in A} \varepsilon_\gamma \widehat{f}(\gamma) \gamma \right\|_{C(G)}.
\end{aligned}$$

We now consider the size of C_1. Indeed, we have

$$\int_{\Omega_1} |g_\gamma| 1_{\{|g_\gamma|>C_1\}}\, dP_1 = C_1 P_1(\{|g_\gamma| > C_1\}) + \int_{C_1}^{\infty} P_1(\{|g_\gamma| > t\})\, dt.$$

Since g_γ is Gaussian, it is well-known that for $x \in \mathbb{R}^+$, $P_1(\{|g_\gamma| > x\}) \leq 2e^{-x^2/2}$. Hence,

$$\int_{\Omega_1} |g_\gamma| 1_{\{|g_\gamma|>C_1\}}\, dP_1 \leq 2C_1 e^{-C_1^2/2} + 2\int_{C_1}^{\infty} e^{-t^2/2}\, dt.$$

Since both $e^{-x^2/2}$ and $\int_x^\infty e^{-t^2/2}\, dt$ decay more rapidly than any power of x^{-1}, for all $\xi > 0$ there exists $K(\xi)$ such that

$$2xe^{-x^2/2} + 2\int_x^{\infty} e^{-t^2/2}\, dt \leq K(\xi) x^{-1/\xi}$$

for all $x > 0$. Hence, we can make $C_1 = K(\xi)^\xi C^\xi$ and

$$\int_{\Omega_1} |g_\gamma| 1_{\{|g_\gamma|>C_1\}}\, dP_1 \leq K(\xi) C_1^{-1/\xi} = 1/C.$$

(2) From Proposition 2.2.16 and the symmetry of $(\omega_\gamma)_\gamma$, we have

$$\int_\Omega \left|\left| \sum_{\gamma\in A} \varepsilon_\gamma \widehat{f}(\gamma)\gamma \right|\right|_{C(G)} dP \leq 2\int_{\mathbb{T}^\Gamma} \left|\left| \sum_{\gamma\in A} \omega_\gamma \widehat{f}(\gamma)\gamma \right|\right|_{C(G)} dP_2.$$

□

Proof of Theorem 2.3.1 (4) → (5) → (6)*:* (4) → (5): Let $p \in (1,2)$ and $C_4 > 0$ be given in (4). We invoke Proposition 2.2.30 and Proposition 2.2.28. From (4) and Proposition 2.2.30, for $f \in \mathrm{Trig}_\Lambda(G)$ we have

$$\sum_{\gamma\in\Lambda\setminus\{0\}} |\widehat{f}(\gamma)| \leq C_4 J_{p,1}(f) \leq C_4 K_\theta J_2(f)^\theta J_r(f)^{1-\theta},$$

where $1 < r < p < 2$ and $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$. As $\widehat{f} \in \ell_1$, Proposition 2.2.28 then shows

$$\sum_{\gamma \in \Lambda \setminus \{0\}} |\widehat{f}(\gamma)| \leq C_4 K_\theta J_2(f)^\theta (C_r \sum_{\gamma \in \Lambda \setminus \{0\}} |\widehat{f}(\gamma)|)^{1-\theta}.$$

Thus,

$$\sum_{\gamma \in \Lambda \setminus \{0\}} |\widehat{f}(\gamma)| \leq (C_4 K_\theta C_r^{1-\theta})^{1/\theta} J_2(f).$$

Recall that $C_4 = K(p)\delta_2^{-2/q}$, and therefore we have that the constant in (5), C_5, is $K(p,r)(\delta_2)^{-2/(q\theta)}$, where q is the dual index to p and, as stated above, θ is given by $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$ for $r \in (1,p)$. We note

$$\frac{2}{q\theta} = \frac{(p-1)(2-r)}{p-r}.$$

One can easily check that if we let $r \to 1$, $\frac{2}{q\theta}$ decreases to 1.
Hence, C_5 is $K(\xi)(\delta_2)^{-(1+\xi)}$ for $\xi > 0$.

To show (5) $\to$ (6) we start from Fernique's inequality (Theorem 2.2.31). Then we use probabilistic arguments to change the Gaussian random variables to the random variables on $\mathbb{T}^\Gamma$ so that the Rider's Theorem (Theorem 2.2.19) can be applied.

(5) $\to$ (6): Fix some $f \in \mathrm{Trig}_\Lambda(G)$ and let $A \subset \Lambda$ be the support of $\widehat{f}$. From (5) and Fernique's inequality (Theorem 2.2.31) we have that

$$\left|\left|\hat{f}\right|\right|_{\ell_1} \leq C_5 J_2(f) + |\hat{f}(0)| \leq C_5 \int_{\Omega_1} \left|\left|\sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g_\gamma \gamma\right|\right|_\infty \tag{2.9}$$

for the constant C_5 coming from (5).

From Lemma 2.3.4 (1), we obtain a constant c, depending on C_5, such that

$$\int_{\Omega_1} \left|\left|\sum_{\gamma \in \Gamma} \widehat{f}(\gamma) g_\gamma \gamma\right|\right|_\infty \leq \frac{\left|\left|\widehat{f}\right|\right|_{\ell_1}}{2C_5} + c \int_{\Omega} \left|\left|\sum_{\gamma \in \Gamma} \widehat{f}(\gamma) \varepsilon_\gamma \gamma\right|\right|_\infty. \tag{2.10}$$

Combining Eq. (2.9) and Eq. (2.10) with Lemma 2.3.4 (2), we have

$$\begin{aligned}
\left\|\hat{f}\right\|_{\ell_1} &\leq C_5 \int_{\Omega_1} \left\|\sum_{\gamma\in\Gamma} \widehat{f}(\gamma) g_\gamma \gamma\right\|_\infty \\
&\leq \frac{1}{2}\left\|\hat{f}\right\|_{\ell_1} + C_5 c \int_{\Omega} \left\|\sum_{\gamma\in A} \varepsilon_\gamma \widehat{f}(\gamma)\gamma\right\|_{C(G)} \\
&\leq \frac{1}{2}\left\|\hat{f}\right\|_{\ell_1} + 2C_5 c \int_{\mathbb{T}^\Gamma} \left\|\sum_{\gamma\in A} \omega_\gamma \widehat{f}(\gamma)\gamma\right\|_{C(G)}.
\end{aligned}$$

Therefore, we have

$$\left\|\hat{f}\right\|_{\ell_1} \leq 4C_5 c \int_{\mathbb{T}^\Gamma} \left\|\sum_{\gamma\in A} \omega_\gamma \widehat{f}(\gamma)\gamma\right\|_{C(G)}$$

and hence by Rider's theorem (Theorem 2.2.19), Λ is Sidon.

Lastly, Rider's Theorem gives that the Sidon constant is bounded by $K(C_5c)^3$. Recall that, for $\xi > 0$, C_5 is $K(\xi)(\delta_2)^{-(1+\xi)}$ and from Lemma 2.3.4 (1), c can be made of size $K(\xi)C_5^\xi$. Hence, we can make $C_6 = K(\xi)\delta_2^{-(3+\xi)}$, as we claimed in Remark 2.3.2. □

Finally, we repeat this quantitative corollary from the proof of Theorem 2.3.1 for later use.

Corollary 2.3.5. *Suppose $\Lambda \subset \Gamma$, and for some $\delta > 0$ we have that for any finite $A \subset \Lambda$,*

$$\int_{\Omega} \left\|\sum_{\gamma\in A} \varepsilon_\gamma \gamma\right\|_{C(G)} \geq \delta|A|.$$

Then Λ is Sidon with Sidon constant of $K(\xi)\delta^{-(3+\xi)}$ for any $\xi > 0$.

Chapter 3

New Characterizations of Sidon Sets

3.1 Introduction

One of the central open problems about Sidon sets is whether Sidon sets can be decomposed into a finite union of more special sets. As we mentioned in chapter 1, there has not been much progress towards solving this problem. The best positive result is due to Bourgain, who proved that every Sidon set is a finite union of n-length independent sets [3].

Definition 3.1.1. Let $n \in \mathbb{N}$ and $E \subset \Gamma$. We say that E is **n-length independent** if whenever $k \in \mathbb{N}$, $\gamma_1, ..., \gamma_k \in E$ are distinct and $m_1, ..., m_k \in \{0, \pm 1\}$ satisfy $\sum_{i=1}^{k} |m_i| \leq n$, then $\prod_{i=1}^{k} \gamma_i^{m_i} = 1$ implies $\gamma_i^{m_i} = 1$ for all $1 \leq i \leq k$.

Theorem 3.1.2. *For each $n \in \mathbb{N}$, a Sidon set $E \subset \Gamma$ is a finite union of n-length independent sets.*

Unfortunately, Theorem 3.1.2 is not very profound because the notion of n-length independence is relatively weak in nature.

Researchers have therefore considered the notion of proportionality. Bourgain and Pisier ([4], [5], [29]) proved one of the most important results about Sidon sets:

Sidonicity can be characterized by proportional quasi-independence. In light of this, it is natural to wonder whether a Sidon set is related to stronger notions of independence in terms of proportionality. One way to strengthen quasi-independence is the notion of n-degree independence.

Definition 3.1.3. Let $n \in \mathbb{N}$ and $E \subset \Gamma$. We say that E is **n-degree independent** if whenever $k \in \mathbb{N}$, $\gamma_1, ..., \gamma_k \in E$ are distinct and $m_1, ..., m_k$ are integers with $|m_i| \leq n$, then $\prod_{i=1}^{k} \gamma_i^{m_i} = 1$ implies $\gamma_i^{m_i} = 1$ for all $1 \leq i \leq k$.

Remark 3.1.4. (1) 1-degree independence and 2-degree independence are usually known as quasi-independence and dissociateness.
(2) A set is independent if it is n-degree independent for all $n \geq 1$.
(3) n-degree independence is much stronger than n-length independence. In fact, even a quasi-independent set (1-degree independent) is n-length independent for all $n \geq 1$.

One of our main results in this thesis, proven in section 3.2, is the following.

Theorem 3.1.5. *Suppose $E \subset \Gamma \backslash \{1\}$ is Sidon and $n \in \mathbb{N}$. Suppose Γ contains no non-trivial elements of order less or equal to n. There exists $\delta > 0$ such that for all finite $A \subset E$, there exists an n-degree independent subset $A' \subset A$ with $|A'| \geq \delta |A|$.*

Remark 3.1.6. (1) Taking $n = 1$ in Theorem 3.1.5 gives Bourgain and Pisier's quasi-independence characterization of Sidon sets.
(2) When Γ is torsion-free, Theorem 3.1.5 implies there are proportional n-degree independent subsets for all n.

Pisier [29] introduced probabilistic techniques to prove Sidon sets are proportional quasi-independent. We will upgrade his techniques to prove Theorem 3.1.5. Once we obtain this, we will use a Riesz product construction to prove another one of the main results of this thesis (Theorem 3.3.1), that a Sidon set in a torsion-free group is even proportional Sidon with Sidon constants arbitrarily close to 1, the minimum possible value.

In the case that Γ has torsion, it is not realistic to expect a Sidon set is proportional Sidon with Sidon constants arbitrarily close to 1 because in this case even an independent set of two elements can have Sidon constant above 1 (Proposition 3.3.6).

Pisier also proved that the Sidon property is equivalent to the "ε-net condition".

Definition 3.1.7. Let $\varepsilon_1, \varepsilon_2 > 0$. We say the subset $E \subset \Gamma$ satisfies the $(\varepsilon_1, \varepsilon_2)$**-net condition** if for any finite subset $F \subset E$ there exists $A \subset G$ such that $|A| \geq 2^{\varepsilon_1 |F|}$ and

$$\sup_{\gamma \in F} |\gamma(x) - \gamma(y)| \geq \varepsilon_2$$

for all $x \neq y \in A$. When $\varepsilon_1 = \varepsilon_2 = \varepsilon$, E is said to satisfy the ε**-net condition**.

Notice that the $(\varepsilon_1, \varepsilon_2)$-net condition is more demanding when ε_1 and ε_2 increase. Naturally, the $(\varepsilon_1, \varepsilon_2)$-net condition is not achievable when $\varepsilon_2 > 2$.

The following theorem is proved by Pisier in [31].

Theorem 3.1.8. *A subset $E \subset \Gamma$ is Sidon if and only if E satisfies the ε-net condition for some $\varepsilon > 0$.*

We will strengthen this by proving that in the torsion-free case, a Sidon set can satisfy the $(\varepsilon_1, \varepsilon_2)$-condition for ε_2 arbitrarily close to 2 (Theorem 3.4.2). This uses our Theorem 3.1.5. In the torsion case it is not realistic to expect this result. For example, in the group $\mathbb{Z}_3^\infty$ any subset can only satisfy the $(\varepsilon_1, \varepsilon_2)$-condition for ε_2 up to $\sqrt{3}$.

Finally, we will use the techniques introduced in chapter 2 to estimate an upper bound of the Sidon constant of a set satisfying the ε-net condition (Corollary 3.4.7). Our estimation greatly improves the result in [24].

3.2 Sidon sets are proportionally n-degree independent

In this section we will show Sidon sets in a torsion-free group, for example, are proportionally n-degree independent for all $n \in \mathbb{N}$. We start with some preliminary results.

Lemma 3.2.1. *Suppose $E \subset \Gamma \setminus \{1\}$ is Sidon. There exists a constant K, depending only on E, such that for all finite $A \subset E$ and real numbers $(\alpha_\gamma)_{\gamma \in A}$, we have*

$$\int \exp\left(\sum_{\gamma\in A} \alpha_\gamma \Re(\gamma)\right) \leq \exp\left(K \sum_{\gamma \in A} \alpha_\gamma^2\right).$$

Proof. Let $A \subset E$ be a finite set. We let $f := \sum_{\gamma\in A} \alpha_\gamma \gamma$ and $M := 2S(E)$ where $S(E)$ is the Sidon constant of E.

By Theorem 1.3.8, $||f||_k \leq M\sqrt{k}\,||f||_2$ for all $k \geq 2$. Using this gives

$$\begin{aligned}
\int_G \exp\left(\sum_{\gamma\in A} \alpha_\gamma \Re(\gamma)\right)\, dm &= \sum_{k\geq 0} \int \frac{\left(\sum_{\gamma\in A} \alpha_\gamma \Re(\gamma)\right)^k}{k!}\, dm \\
&= \sum_{k\geq 2} \int \frac{\left(\sum_{\gamma\in A} \alpha_\gamma \Re(\gamma)\right)^k}{k!}\, dm + 1 \leq \sum_{k\geq 2} \int \frac{|f|^k}{k!}\, dm + 1 \\
&= \sum_{k \geq 2} \frac{||f||_k^k}{k!} + 1 \leq \sum_{k\geq 2} \frac{(M\sqrt{k}\,||f||_2)^k}{k!} + 1 \\
&= \sum_{p\geq 1} \frac{(M\sqrt{2p}\,||f||_2)^{2p}}{(2p)!} + \sum_{p \geq 1} \frac{(M\sqrt{2p+1}\,||f||_2)^{2p+1}}{(2p+1)!} + 1.
\end{aligned}$$

We let $L := \max\{M+1, 4\}$. Then, since $p^p \leq (2p)(2p-1)\ldots(p+1)$,

$$\frac{(M\sqrt{2p}\,||f||_2)^{2p}}{(2p)!} \leq \frac{(2LM^2\,||f||_2^2)^p}{(M+1)p!}.$$

Thus,

$$\sum_{p\geq 1} \frac{(M\sqrt{2p}\,||f||_2)^{2p}}{(2p)!} \leq \frac{1}{M+1}\exp(2LM^2\,||f||_2^2) - \frac{1}{M+1}.$$

Moreover, we also have

$$\frac{(M\sqrt{2p+1}\,||f||_2)^{2p+1}}{(2p+1)!} \leq \frac{M\,||f||_2}{M+1}\frac{(2LM^2\,||f||_2^2)^p}{p!}.$$

Hence,

$$\sum_{p\geq 1} \frac{(M\sqrt{2p+1}\,||f||_2)^{2p+1}}{(2p+1)!} \leq \frac{M\,||f||_2}{M+1}\exp(2LM^2\,||f||_2^2) - \frac{M\,||f||_2}{M+1}.$$

We therefore have

$$\int \exp\left(\sum_{\gamma\in A}\alpha_\gamma \Re(\gamma)\right)\,dm \leq \frac{1+M\,||f||_2}{M+1}\exp(2LM^2\,||f||_2^2) + \frac{M-M\,||f||_2}{M+1}.$$

Hence, if $||f||_2 \geq 1$,

$$\int \exp\left(\sum_{\gamma\in A}\alpha_\gamma \Re(\gamma)\right)\,dm \leq (1+M\,||f||_2)\exp(2LM^2\,||f||_2^2) \leq \exp(4LM^2\,||f||_2^2).$$

Otherwise $||f||_2 < 1$, and we have

$$\begin{aligned}\int \exp\left(\sum_{\gamma\in A}\alpha_\gamma \Re(\gamma)\right)\,dm &\leq \left(\frac{1+M\,||f||_2}{M+1} + \frac{M-M\,||f||_2}{M+1}\right)\exp(2LM^2\,||f||_2^2)\\ &= \exp(2LM^2\,||f||_2^2).\end{aligned}$$

Hence, in general we have

$$\int \exp\left(\sum_{\gamma\in A}\alpha_\gamma \Re(\gamma)\right)\,dm \leq \exp(4LM^2\,||f||_2^2).$$

$\square$

Notation 3.2.2. Let $E \subset \Gamma$ be a subset. For $k \geq 1$ we let

$$E_k := \left\{\gamma^k : \gamma \in E\right\}$$

and

$$E^{(k)} := \bigcup_{i=1}^{k} E_i.$$

Lemma 3.2.3. *Suppose $E \subset \Gamma$ is a Sidon set.*
(1) Let $n \in \mathbb{N}$. If Γ has no elements of order less or equal to n, then for all $1 \leq k \leq n$, E_k is Sidon with the same Sidon constant as E.
(2) If Γ is torsion-free, then for all $k \in \mathbb{N}$ the set E_k is also Sidon with the same Sidon constant as E.

Proof. (1) We first claim that for all $1 \leq k \leq n$ and $x \in G$ there exists $y \in G$ such that $y^k = x$. Indeed, consider the map $f_k : G \to G$ given by $f_k(x) = x^k$. If f_k is not onto, then $f_k(G)$ is a compact subgroup of G and $\widehat{G/f_k(G)}$ is non-trivial. This means $f_k(G)$ has non-trivial annihilator, which contradicts that Γ has no elements of order less or equal to n.

Since E is Sidon with Sidon constant C, $\sum_{\gamma\in E} |\widehat{f}(\gamma)| \leq C\,||f||_\infty$ for all $f \in \mathrm{Trig}_E(G)$. Let $g \in \mathrm{Trig}_{E_k}(G)$. By the claim above, if we let $g_1 \in \mathrm{Trig}_E(G)$ be given by $g_1 = \sum_{\gamma\in E} \widehat{g}(\gamma^k)\gamma$, then $||g_1||_\infty = ||g||_\infty$. Hence,

$$\sum_{\gamma\in E_k} |\widehat{g}(\gamma)| = \sum_{\gamma\in E} |\widehat{g_1}(\gamma)| \leq C\,||g_1||_\infty = C\,||g||_\infty .$$

This shows E_k is also a Sidon set with Sidon constant bounded by C. It is even easier to see E_k has Sidon constant at least C and hence we have equality.

(2) This follows immediately from (1). $\square$

Lemma 3.2.4. *Suppose $E \subset \Gamma \backslash \{1\}$ is Sidon, $n \in \mathbb{N}$ and Γ contains no non-trivial elements of order less than or equal to n. Then there exists a constant K_n, depending* *only on n and the Sidon constants of E_k, $1 \le k \le n$, such that for all $\lambda \in (0, 1/n)$ and finite $A \subset E$,*

$$\int \prod_{\gamma \in A} \left(1 + \lambda \sum_{k=1}^{n} \Re(\gamma^k)\right) \le \exp(K_n |A| n^3 \lambda^2).$$

Proof. Let $A \subset E$ be a finite set and $\lambda \in (0, 1/n)$. Since $0 < 1 + x \le \exp(x)$ for $x \in (-1, \infty)$, we have

$$\int \prod_{\gamma \in A} \left(1 + \lambda \sum_{k=1}^{n} \Re(\gamma^k)\right) \le \int \exp\left(\lambda \sum_{\gamma \in A} \sum_{k=1}^{n} \Re(\gamma^k)\right).$$

Using the notations in Notation 3.2.2, we write

$$\sum_{\gamma \in A} \sum_{1 \le k \le n} \Re(\gamma^k) = \sum_{\beta \in A^{(n)}} a_\beta \Re(\beta).$$

Note that the coefficients a_β satisfy $0 \le a_\beta \le 2n$, since the assumption that Γ contains no elements of order $\le n$ ensures that $\Re(\gamma^k) = \Re(\chi^k)$ for $\gamma, \chi \in A$ and $k \le n$ only if $\gamma = \chi$ or $\overline{\chi}$. As a finite union of Sidon sets is Sidon (Theorem 1.3.9) with Sidon constant depending only on the Sidon constants of the individual sets and the number of sets in the union, Lemma 3.2.3 implies $A^{(n)}$ is Sidon with Sidon constant bounded by that of $E^{(n)}$, which depends only on the Sidon constant of E and n.

We invoke Lemma 3.2.1 to see there exists a constant K_n such that

$$\begin{aligned} \int \exp(\lambda \sum_{\gamma \in A} \sum_{k=1}^{n} \Re(\gamma^k)) &= \int \exp(\lambda \sum_{\beta \in A^{(n)}} \alpha_\beta \Re(\beta)) \\ &\le \exp(K_n |A^{(n)}| n^2 \lambda^2) \le \exp(K_n |A| n^3 \lambda^2). \end{aligned}$$

□

We now prove our main result.

Proof of Theorem 3.1.5. Firstly, we notice that we only need to prove Theorem 3.1.5 for subsets A with $|A| > C$ for some large C and obtain the proportion δ', because we can then prove Theorem 3.1.5 by making $\delta := \min\{1/C, \delta'\}$.

Fix $n \in \mathbb{N}$. For a finite subset $F \subset E$, we let

$$R_n(F) := \left|\left\{(\xi_\gamma)_{\gamma\in F} \in \{-n, ..., -1, 0, 1, ..., n\}^F : \prod_{\gamma\in F} \gamma^{\xi_\gamma} = 1\right\}\right|.$$

We first claim that if E is Sidon, then there exist constants $\delta_n > 0$ and $\alpha_n > 0$ such that for all finite subsets $A \subset E$, there exists $A' \subset A$ with $|A'| \geq \delta_n|A|$ and $R_n(A') \leq 2 \cdot 2^{\alpha_n|A'|}$.

To prove the claim, we fix an arbitrary finite subset $A \subset E$ and $\lambda \in (0, 1/n)$. Let $(\tau_\gamma)_{\gamma\in A}$ be a collection of independent random variables on a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}\{\tau_\gamma = 1\} = \lambda/2$ and $\mathbb{P}\{\tau_\gamma = 0\} = 1 - \lambda/2$. From Fubini's Theorem, independence and Lemma 3.2.4, we have

$$\begin{aligned}\int_\Omega\int_G \prod_{\gamma\in A}\left(1 + \tau_\gamma\sum_{k=1}^n(\gamma^k + \overline{\gamma^k})\right) &= \int_G\int_\Omega \prod_{\gamma\in A}\left(1 + \tau_\gamma\sum_{k=1}^n(\gamma^k + \overline{\gamma^k})\right)\\ &= \int_G \prod_{\gamma\in A}\left(1 + \lambda\sum_{k=1}^n \Re(\gamma^k)\right) \leq \exp(K_n|A|n^3\lambda^2).\end{aligned}$$

If we let $A(\omega) := \{\gamma \in A : \tau_\gamma(\omega) = 1\}$, then

$$\int_\Omega R_n(A(\omega)) = \int_\Omega\int_G \prod_{\gamma\in A}\left(1 + \tau_\gamma\sum_{k=1}^n(\gamma^k + \overline{\gamma^k})\right) \leq \exp(K_n|A|n^3\lambda^2).$$

By Markov's inequality, with probability at least a half we have

$$R_n(A(\omega)) \leq 2\exp(K_n|A|n^3\lambda^2).$$

Meanwhile,

$$\mathbb{E}(|A(\omega)| - \mathbb{E}|A(\omega)|)^2 = \mathbb{E}(\sum_{\gamma\in A}(\tau_\gamma - \mathbb{E}\tau_\gamma))^2.$$

Notice that if $\gamma_1 \neq \gamma_2$, then $\mathbb{E}((\tau_{\gamma_1} - \mathbb{E}\tau_{\gamma_1})(\tau_{\gamma_2} - \mathbb{E}\tau_{\gamma_2})) = 0$. Hence,

$$\mathbb{E}(|A(\omega)| - \mathbb{E}|A(\omega)|)^2 = \sum_{\gamma \in A} \mathbb{E}(\tau_\gamma - \mathbb{E}\tau_\gamma)^2 = |A|(\lambda/2 - \lambda^2/4) \leq |A|\lambda/2.$$

Since $\mathbb{E}|A(\omega)| = |A|\lambda/2$, it follows from Chebyshev's inequality that

$$\begin{aligned} \mathbb{P}\{|A(\omega)| \leq |A|\lambda/4\} &\leq \mathbb{P}\left\{(|A(\omega)| - \mathbb{E}|A(\omega)|)^2 \geq \frac{|A|^2\lambda^2}{16}\right\} \\ &\leq \frac{\mathbb{E}((|A(\omega)| - \mathbb{E}|A(\omega)|)^2)}{|A|^2\lambda^2/16} = \frac{|A|\lambda/2}{|A|^2\lambda^2/16} = \frac{8}{|A|\lambda}. \end{aligned}$$

Choose $\lambda_n \in (0, 1/n)$ small enough that $\exp(4K_n n^3 \lambda_n) < 2$ and let $\alpha_n \in (0,1)$ be given by $2^{\alpha_n} = \exp(4K_n n^3 \lambda_n)$. The probability that $|A(\omega)| > |A|\lambda_n/4$ is at least $1 - \frac{8}{|A|\lambda_n} > \frac{1}{2}$ if A is sufficiently large.

Hence, there is a positive probability that both $|A(\omega)| > |A|\lambda_n/4$ and

$$R_n(A(\omega)) \leq 2\exp(K_n|A|n^3\lambda_n^2) \leq 2 \cdot 2^{\alpha_n|A(\omega)|},$$

which proves the claim.

We now show that E is proportionally n-degree independent. We call a finite set $F \subset E$ an n-relation set if there exists $(\xi_\gamma)_{\gamma \in F} \in \{-n, ..., -1, 1, ..., n\}^F$ such that $\prod_{\gamma \in F} \gamma^{\xi_\gamma} = 1$. For a finite set $F \subset E$, we let $M(F)$ be a maximal (with respect to inclusion) subset in F that is an n-relation set. The maximality gives that $F \backslash M(F)$ is an n-degree independent set.

It only remains to verify the following claim: If $A \subset \Lambda$ is a finite set that is large enough, and for some $\alpha > 0$ we have $R_n(A) \leq 2 \cdot 2^{\alpha|A|}$, then there exist a constant θ, only depending on α, and a subset $H \subset A$ with $|H| \geq |A|/2$ and $|M(H)| \leq \theta|H|$.

Once this claim is established the proof is complete since $H \backslash M(H) \subset A$ is n-degree independent with $|H \backslash M(H)| \geq (1-\theta)|A|/2$.

Before proving this, we first notice that if we let

$$\chi(\theta) := \left(\frac{1-\theta}{2}\right)\log_2\left(\frac{2e}{1-\theta}\right)$$

for $\theta \in (0,1)$, then $\lim_{\theta\to 1}\chi(\theta) = 0$ and, since $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$, we have

$$\binom{n}{\frac{n(1-\varepsilon)}{2}} \le 2^{\chi(\varepsilon)n}.$$

We let θ be sufficiently close to one that $1-\chi(\theta) > \alpha$.

Assume, otherwise, that the claim is false. Without loss of generality we can assume A has an even number of elements. Then for all $H \subset A$ with $|H| = |A|/2$ we have $|M(H)| \ge \theta|H|$.

If $H_0 \subset A$ and $\frac{\theta|A|}{2} < |H_0| < \frac{|A|}{2}$, then

$$|\{H_1 \subset A : |H_1| = |A|/2, H_0 \subset H_1\}| = \binom{|A|-|H_0|}{|A|/2-|H_0|} \le \binom{|A|}{|A|(1-\theta)/2} \le 2^{\chi(\theta)|A|}.$$

We let $\mathcal{F}(A)$ be the collection of all subsets $H_0 \subset A$ such that there exists $H_1 \subset A$, $|H_1| = |A|/2$ and $M(H_1) = H_0$. Naturally, we have $R_n(A) \ge |\mathcal{F}(A)|$.

We thus have

$$\begin{aligned}\binom{|A|}{|A|/2} &= |\{H_1 \subset A : |H_1| = |A|/2\}| \\ &= \sum_{H_0\subset A} |\{H_1 \subset A : |H_1| = |A|/2, M(H_1) = H_0\}| \\ &\le \sum_{H_0\in\mathcal{F}(A)} |\{H_1 \subset A : |H_1| = |A|/2, H_1 \supset H_0\}| \\ &\le |\mathcal{F}(A)|2^{|A|\chi(\theta)} \le R_n(A)2^{|A|\chi(\theta)}.\end{aligned}$$

We therefore have a contradiction for A large enough:

$$R_n(A) \geq \binom{|A|}{|A|/2} 2^{-|A|\chi(\theta)} \sim \frac{1}{\sqrt{|A|}} 2^{|A|} 2^{-|A|\chi(\theta)} > 2 \cdot 2^{\alpha|A|},$$

because we have chosen $1 - \chi(\theta) > \alpha$, which finishes the verification of the claim.

Theorem 3.1.5 is therefore proved. $\square$

Corollary 3.2.5. *Suppose $E \subset \Gamma$ is Sidon. Then E is proportionally quasi-independent. Furthermore, if Γ contains no elements of order 2, then E is proportionally dissociate.*

Corollary 3.2.6. *Suppose Γ is torsion-free and $E \subset \Gamma$ is Sidon. For all $n \in \mathbb{N}$, there exists $\delta > 0$ such that for all finite $A \subset E$, there exists an n-degree independent subset $A' \subset A$ with $|A'| \geq \delta|A|$.*

3.3 Sidon sets are proportionally Sidon with small Sidon constants

3.3.1 Sidon sets in a torsion-free group

In this subsection we assume G is a connected compact abelian group and Γ is the discrete torsion-free dual group of G.

It is an immediate consequence of Theorem 3.1.5 that Sidon sets in a torsion-free group are proportionally n-degree independent for all $n \geq 1$, as stated in Corollary 3.2.6.

Our next main result is the following.

Theorem 3.3.1. *Suppose $E \subset \Gamma$ is Sidon. For all $\xi > 1$ there exists $\delta > 0$ such that for every finite $F \subset E$ there is a Sidon set $F' \subset F$ with Sidon constant bounded by ξ and $|F'| \geq \delta |F|$.*

Before proving Theorem 3.3.1, we establish an elementary lemma.

Lemma 3.3.2. *For any $\varepsilon > 0$, there exists a continuous even function f on $\mathbb{T} = [-1/2, 1/2]$ such that $f \geq 0$, $\widehat{f}(0) = 1$, $\widehat{f}(1) = \widehat{f}(-1)$ and $\widehat{f}(\pm 1) \geq 1 - \varepsilon$*

Proof. We identify $\mathbb{T} = [-1/2, 1/2]$ in this lemma and consider the maps $f_n : \mathbb{T} \to \mathbb{R}^+$, $n \geq 2$, given by $f_n(0) = n$, $f_n(x) = n - n|x|$ for $x \in [-1/n, 1/n]$ and $f_n(x) = 0$ elsewhere. It is clear that f_n is even, $f_n \geq 0$ on $\mathbb{T}$ and $\widehat{f_n}(0) = ||f_n||_1 = 1$.

Fix $\varepsilon > 0$. Choose N large enough that $|e^{2\pi i t} - 1| < \varepsilon$ for $t \in [-1/N, 1/N]$. Then,

$$\begin{aligned} |\widehat{f_N}(-1) - 1| &= |\int_{\mathbb{T}} f_N(t) e^{2\pi i t}\, dt - \int_{\mathbb{T}} f_N(t)\, dt| \\ &\leq \int_{\mathbb{T}} f_N(t) |e^{2\pi i t} - 1|\, dt \leq \varepsilon. \end{aligned}$$

Thus, $\widehat{f_N}(-1) \geq 1 - \varepsilon$. Moreover, as f_N is even, $\widehat{f_N}(1) = \widehat{f_N}(-1)$. □

We prove Theorem 3.3.1 using a Riesz product style of construction.

Proof of Theorem 3.3.1. We first claim that for all $\varepsilon > 0$, there exists a real-valued, non-negative trigonometric polynomial $p \in \mathrm{Trig}(\mathbb{T})$ such that $\widehat{p}(0) = 1$, $\widehat{p}(1) = \widehat{p}(-1)$, and $\widehat{p}(\pm 1) > 1 - \varepsilon$.

Indeed, fix $\varepsilon > 0$ and choose $\delta > 0$ such that $\frac{1-\delta}{1+\delta} > 1 - \varepsilon$.

From Lemma 3.3.2, we find a real-valued, continuous, even function f on $\mathbb{T}$ such that $\widehat{f}(0) = 1$, $\widehat{f}(1) = \widehat{f}(-1)$, $\widehat{f}(\pm 1) > 1 - \delta/2$ and $f \geq 0$. By the Stone-Weierstrass

Theorem, we find a real $q \in \mathrm{Trig}(\mathbb{T})$ such that $||q - f||_\infty < \delta/2$, $\widehat{q}(0) \in \mathbb{R}$ and $\widehat{q}(1) = \widehat{q}(-1) \in \mathbb{R}$. Put

$$p := \frac{q + \delta/2}{\widehat{q}(0) + \delta/2}.$$

It is clear that $\widehat{p}(0) = 1$, $\widehat{p}(1) = \widehat{p}(-1) \in \mathbb{R}$ and $p \geq 0$. We also note that

$$\widehat{p}(\pm 1) = \frac{\widehat{q}(\pm 1)}{\widehat{q}(0) + \delta/2} \geq \frac{1 - \delta}{1 + \delta} > 1 - \varepsilon.$$

Next, we suppose $\varepsilon > 0$ is arbitrary and we let the polynomial p be given as above. Put $n := \deg(p)$. By Corollary 3.2.6, there exists $\delta > 0$ such that for all finite $A \subset E$, there exists an $(n+1)$-degree independent $A' \subset A$ with $|A'| \geq \delta |A|$.

We will use generalized Riesz products to show A' has Sidon constant bounded by $1/(1-\varepsilon)$. Let $\varphi : A' \to \mathbb{C}$ and $||\varphi||_\infty \leq 1 - \varepsilon$. We claim that for each $\gamma \in A'$, there exists $P_\gamma \in \mathrm{Trig}(G)$ such that $\widehat{P_\gamma}(1) = 1$, $P_\gamma \geq 0$, $\widehat{P_\gamma}(\gamma) = \varphi(\gamma)$ and $\deg(P_\gamma) \leq n$.

For this we let $u_\gamma := \frac{\varphi(\gamma)}{|\varphi(\gamma)|} \in \mathbb{T}$ and P_γ be given by

$$P_\gamma := \frac{|\varphi(\gamma)|}{\widehat{p}(1)} p\left(u_\gamma \gamma\right) + \left(1 - \frac{|\varphi(\gamma)|}{\widehat{p}(1)}\right),$$

where we identify $p \in \mathrm{Trig}(\mathbb{T})$ as $p(z) = \sum_{k=-N}^{N} a_k z^k$.

Using properties of p, it is easy to see $\widehat{P_\gamma}(1) = 1$ and $\widehat{P_\gamma}(\gamma) = \varphi(\gamma)$. We also note that $P_\gamma \geq 0$ because $|\varphi(\gamma)| \leq 1 - \varepsilon \leq \widehat{p}(1)$.

Finally, we let $F := \prod_{\gamma \in A'} P_\gamma \in L^1(G) \subset M(G)$. Since A' is $(n+1)$-degree independent, we have that $\widehat{F} = \varphi$ on A' and $||F||_{M(G)} = ||F||_1 = 1$. This shows A' has Sidon constant bounded by $1/(1-\varepsilon) \leq \xi$, if ε is chosen suitably, and finishes the proof of Theorem 3.3.1. $\square$

3.3.2 Sidon sets in a torsion group

We first consider the case $\Gamma = \bigoplus_{i=1}^{N} \mathbb{Z}_{p_i}^{\mathbb{N}}$ where $\{p_1, ..., p_N\}$ is a finite collection of prime numbers. We need two results from Bourgain.

Theorem 3.3.3. *[3] Let p be a prime number. Any Sidon set in $\Gamma = \mathbb{Z}_p^{\mathbb{N}}$ is a finite union of independent sets.*

Theorem 3.3.4. *[3] Suppose $G = G_1 \times G_2$ and $\Gamma = \widehat{G} = \Gamma_1 \bigoplus \Gamma_2$. Recall that $\mathrm{Proj}_i : \Gamma \to \Gamma_i$, $i \in \{1, 2\}$, are the projections defined in Notation 1.1.4. Suppose $S \subset \Gamma$ is a Sidon set. Then there is a finite set F so that S can be decomposed as $S = \bigcup_{\alpha \in F} S_\alpha$. For each S_α, $\alpha \in F$, there exists $i \in \{1, 2\}$ such that $\mathrm{Proj}_i(S_\alpha)$ is a Sidon set in Γ_i and Proj_i is one-to-one on S_α.*

It is not hard to see that Theorem 3.3.4 can be extended to a finite product of groups. Based on these two results we have the following corollary.

Corollary 3.3.5. *Suppose $\Gamma = \bigoplus_{i=1}^{N} \mathbb{Z}_{p_i}^{\mathbb{N}}$. Let $l = \min\{p_1, ..., p_N\}$. Then any Sidon set in Γ is a finite union of sets that are at least $(l-1)$-independent.*

Proof. Let $S \subset \Gamma$ be a Sidon set and the projections $\mathrm{Proj}_j : \Gamma = \bigoplus_{i=1}^{N} \mathbb{Z}_{p_i}^{\mathbb{N}} \to \mathbb{Z}_{p_j}^{\mathbb{N}}$ for $1 \le j \le N$. Extending Theorem 3.3.4 to Γ gives that there exists a finite set F and a decomposition of S, $S = \bigcup_{\alpha \in F} S_\alpha$, satisfying that for each $\alpha \in F$ there exists $j \in \{1, ..., N\}$ such that Proj_j is one-to-one on S_α and $\mathrm{Proj}_j(S_\alpha)$ is a Sidon set in $\mathbb{Z}_{p_j}^{\mathbb{N}}$.

Theorem 3.3.3 implies $\mathrm{Proj}_j(S_\alpha)$ is a finite union of independent sets in $\mathbb{Z}_{p_j}^{\mathbb{N}}$. Thus, we have that there exists a decomposition for S_α, $S_\alpha = \bigcup_{\beta \in F_\alpha} S_{\alpha,\beta}$ for some finite sets F_α satisfying Proj_j is one-to-one on $S_{\alpha,\beta}$ and $\mathrm{Proj}_j(S_{\alpha,\beta})$ is an independent set in $\mathbb{Z}_{p_j}^{\mathbb{N}}$. It is easy to check that $S_{\alpha,\beta} \subset \Gamma$ is $(p_j - 1)$-degree independent and therefore the decomposition $S = \bigcup_{\alpha \in F} \bigcup_{\beta \in F_\alpha} S_{\alpha,\beta}$ proves the corollary. □

Unfortunately, even an independent set in a torsion group does not necessarily have Sidon constant 1.

Proposition 3.3.6. *Suppose $\Gamma = \mathbb{Z}_p^{\mathbb{N}}$ for some prime number p. Suppose $E = \{a, b\} \subset \Gamma$ is an independent set of two elements. The Sidon constant of E is at least $1/\cos(\pi/2p)$.*

Proof. The Sidon constant of E is the supremum of $\left|\left|\hat{f}\right|\right|_1 / \left|\left|f\right|\right|_\infty$ as f ranges over $\mathrm{Trig}_E(G)$. Thus, as E is independent, the Sidon constant of E is the following:

$$\sup_{\alpha_1,\alpha_2\in\mathbb{C}} \frac{|\alpha_1| + |\alpha_2|}{\max_{\xi_1,\xi_2\in\mathbb{Z}_p} |\alpha_1\xi_1 + \alpha_2\xi_2|}. \tag{3.1}$$

Notice that for all $\alpha_1, \alpha_2 \in \mathbb{C}$, the maximum of $|\alpha_1\xi_1 + \alpha_2\xi_2|$ for $\xi_1, \xi_2 \in \mathbb{Z}_p$ can be obtained as $|\alpha_1 + \alpha_2\xi_0|$, where $\xi_0 \in \mathbb{Z}_p$ is such that

$$|\operatorname{Arg}(\alpha_1) - \operatorname{Arg}(\alpha_2\xi_0)| \le \pi/p.$$

Hence, it is not hard to see Eq. (3.1) is at least $1/\cos(\pi/2p)$, as this value is obtained by the quotient when $\alpha_1 = 1$ and $\alpha_2 = e^{\pi i/p}$. $\square$

Remark 3.3.7. Thus Theorem 3.3.1 does not extend to these groups.

We next consider the case $\Gamma = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i}$, where $(p_i)_{i=1}^{\infty}$ is a sequence of increasing prime numbers.

Proposition 3.3.8. *Suppose $\Gamma = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i}$ where $(p_i)_{i=1}^{\infty}$ is a sequence of increasing prime numbers, and $E \subset \Gamma$ is Sidon. For all $\xi > 1$ there exists $\delta = \delta(E, \xi) > 0$ such that for all finite $F \subset E$, there exists $H \subset F$ such that H has Sidon constant bounded by ξ and $|H| \ge \delta|F|$.*

Proof. Fix $\xi = 1/(1-\varepsilon) > 1$ and suppose F is a finite subset of E. Let p be the polynomial defined in the proof of Theorem 3.3.1 with $\widehat{p}(0) = 1$, $\widehat{p}(1) = \widehat{p}(-1) > 1-\varepsilon$ and $p \ge 0$. Put $N = \deg p$. Choose n_0 such that $p_i > N + 1$ for all $i > n_0$. Let $\Gamma_1 = \bigoplus_{i=1}^{n_0} \mathbb{Z}_{p_i}$ and $M = |\Gamma_1|$. Choose $F_1 \subseteq F$ such that $F_1 = \gamma Y$ where $\gamma \in \Gamma_1$, $Y \subseteq \bigoplus_{i>n_0} \mathbb{Z}_{p_i}$ and $|F_1| \ge |F|/M$. Since translation preserves Sidon constants, Y is a Sidon set with constant at most that of E.

Now consider $Y_k = \{\chi^k : \chi \in Y\}$ for $k \leq N$. Since the elements of $\mathbb{Z}_{p_i}$ for $i > n_0$ have prime order exceeding N, Lemma 3.2.3 shows that each Y_k is Sidon with Sidon constant the same as E.

Applying Theorem 3.1.5 we see there is a constant $\delta > 0$ (depending on N) and an $(N+1)$-degree independent set $Y_0 \subseteq Y$ such that $|Y_0| \geq \delta\,|Y|$. For Y_0, being $(N+1)$-degree independent is the same as saying $\prod_{i=1}^{k} \gamma_i^{m_i} = 1$ for $|m_i| \leq N+1$ only if $\gamma_i = 1$ for all i. That fact allows us to apply the Riesz product construction of the proof of Theorem 3.3.1 (with the polynomial p) and as in that proof we deduce that the Sidon constant of Y_0 is at most ξ. Of course, this is also a bound on the Sidon constant of $H = \gamma Y_0$ and this subset of F has cardinality at least $(\delta/M)\,|F|$, completing the proof. $\square$

3.4 The ε-net condition

The main result of this section is that if Γ is torsion-free, then any Sidon set E in Γ will satisfy the $(\varepsilon_1, \varepsilon_2)$-condition for ε_2 arbitrarily close to 2 and some $\varepsilon_1 > 0$. We direct the reader to Definition 3.1.7 for the definition of the $(\varepsilon_1, \varepsilon_2)$-condition.

We start with a lemma.

Lemma 3.4.1. *Let Γ be a torsion-free discrete abelian group. Suppose $E \subset \Gamma$ is a Sidon set. Given any $\tau \in (-1, 0)$, there exist constants $\delta = \delta(E, \tau) > 0$ and $a = a(E, \tau) > 0$ such that for any finite $F \subset E$, there exists $F' \subset F$ with $|F'| \geq \delta |F|$, with the property that whenever integer $N \leq |F'|$, $\{\gamma_1, ..., \gamma_N\} \subset F'$, and $c_1, ..., c_N \in \mathbb{T}$, then the set*

$$X_{N,\tau} := \left\{ x \in G : \inf_{1 \leq n \leq N} \Re(c_n \gamma_n(x)) > \tau \right\}$$

has Haar measure less than $(1+a)^{-N}$.

Proof. Fix $\tau \in (-1, 0)$. There exists $a = a(\tau) > 0$ such that the function $h : \mathbb{T} \to \mathbb{C}$

defined by

$$h(z) = \begin{cases} 1+a & \text{if } \Re(z) > \tau \\ 0 & \text{else} \end{cases}$$

satisfies $\int_{\mathbb{T}} h = 1$ and $h \geq 0$. Notice that h is continuous except at two points and thus by the Stone-Weierstrass Theorem, there exists a real-valued polynomial $p \in \mathrm{Trig}(\mathbb{T})$ such that $\widehat{p}(0) = 1$, $p \geq 0$ and $p(z) \geq 1+a/2$ for all $z \in \mathbb{T}$ with $\Re(z) > \tau$.

Let $\deg(p) = K$. Since Γ is torsion-free, by Corollary 3.2.6 there exists $\delta > 0$ such that every finite set $F \subset E$ has a subset $F' \subset F$ with $|F'| \geq \delta|F|$ and F' is K-degree independent.

For any $\gamma_1, ..., \gamma_N \in F'$ and $c_1, ..., c_N \in \mathbb{T}$, we define a product in $\mathrm{Trig}(G)$ as

$$P(x) = \prod_{1 \leq n \leq N} p(c_n \gamma_n(x)).$$

We let

$$X_{N,\tau} := \left\{ x \in G : \inf_{1 \leq n \leq N} \Re(c_n \gamma_n(x)) > \tau \right\}.$$

Since F' is K-degree independent, the constant term in the product P is 1 and therefore $1 = \int_G P\, dm$. On the other hand, by the definition of the set $X_{N,\tau}$, we also have

$$\int_G P\, dm \geq (1+a/2)^N m(X_{N,\tau}).$$

Hence, $m(X_{N,\tau}) < (1+a/2)^{-N}$. $\square$

The main theorem of this section is the following.

Theorem 3.4.2. *Let Γ be a torsion-free discrete abelian group. Suppose $E \subset \Gamma$ is Sidon. For all $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that E satisfies the $(\varepsilon_1, 2-\varepsilon)$-net condition.*

Proof. Fix small $\varepsilon > 0$. Put $\tau = -\cos\varepsilon$ so that $|z| = 1$ and $\Re(z) < \tau$ imply $|z-1| > 2-\varepsilon$. Obtain δ, a as in Lemma 3.4.1; without loss of generality $a < 1$. Pick N_0 large enough that $4 > (1+a)^{N_0} > 3$. We will show E satisfies the $(\varepsilon_1, 2-\varepsilon)$-net condition with $\varepsilon_1 = \frac{\delta}{2}\log_3(1+a)$.

Suppose $F \subset E$ is a finite set. Notice that we may assume $|F| \geq N_0/\delta$, because otherwise

$$2^{\varepsilon_1|F|} = 2^{\delta|F|\log_3(1+a)/2} < 2$$

and we can take A to be a singleton to trivially satisfy the requirement.

From Lemma 3.4.1, we pick $F' \subset F$ with $|F'| \geq \delta|F| > N_0$ with the specified properties of the lemma. We put $K := \lfloor \log_3((1+a)^{|F'|}) \rfloor \geq 1$. For any fixed $x_1, ..., x_{K-1} \in G$, we have

$$m\left(\left\{x \in G : \inf_{\gamma\in F'} \Re\left(\gamma(\prod_1^{K-1} x_i)\gamma(x)\right) > \tau\right\}\right) < (1+a)^{-|F'|}.$$

Hence, if for $n = (n_1, ..., n_K) \in \{-1, 0, 1\}^K$ we denote

$$X_n := \left\{(x_1, ..., x_K) \in G^K : \inf_{\gamma\in F'} \Re\left(\gamma(\prod_1^K x_i^{n_i})\right) > \tau\right\},$$

we have $m^K(X_n) < (1+a)^{-|F'|}$, where m^K is the product measure on G^K.

The definition of K ensures that $3^K \leq (1+a)^{|F'|}$, and therefore,

$$m^K(\bigcup_{n\neq 0} X_n) < (3^K - 1)(1+a)^{-|F'|} < 1.$$

We let $(x_1, ..., x_K) \in G^K \backslash (\bigcup_{n\neq 0} X_n)$ and define the set

$$A := \left\{x = \prod_1^K x_i^{s_i} : s_i \in \{0, 1\}\right\}.$$

We have

$$|A| = 2^K \geq 2^{(\log_3(1+a))\delta|F|/2}.$$

Finally, we verify this set A has the desired property. If $x \neq y \in A$, $xy^{-1} = \prod_1^K x_i^{n_i}$ for some $n = (n_1, ..., n_K) \in \{-1, 0, 1\}^K \setminus \{0\}$. Since $(x_1, ..., x_K) \in G^K \setminus (\bigcup_{n \neq 0} X_n)$, there exists $\gamma \in F'$ such that $\Re(\gamma(\prod_1^K x_i^{n_i})) \leq \tau$, which implies

$$|\gamma(x) - \gamma(y)| = |\gamma(xy^{-1}) - 1| > 2 - \varepsilon.$$

$\square$

Finally, we give a numerical bound for the Sidon constant based on the $(\varepsilon_1, \varepsilon_2)$-net condition.

Notation 3.4.3. Let $A \subset \Gamma$ be a finite set and $f = \sum_{\gamma \in A} \gamma \in \mathrm{Trig}_A(G)$. We recall that the pesudo-metric d_2^f on G is given by

$$d_2^f(s,t) := \left(\sum_{\gamma \in A} |\gamma(s) - \gamma(t)|^2 \right)^{1/2}.$$

We also define

$$d_A(s,t) := \sup_{\gamma \in A} |\gamma(s) - \gamma(t)|.$$

Let N_A be the entropy number associated with d_A. For simplicity we let $N_{2,A} = N_{d_2^f}$ be the entropy number associated with d_2^f.

Remark 3.4.4. Suppose $E \subset \Gamma$ satisfies the $(\varepsilon_1, \varepsilon_2)$-net condition for some $\varepsilon_1, \varepsilon_2 > 0$. For any finite set $A \subset E$ we have $N_A(\varepsilon_2/2) \geq 2^{\varepsilon_1 |A|}$.

The following lemma shows the entropy numbers N_A and $N_{2,A}$ are comparable (Proposition V.10. in Chapter 13 of [24]).

Lemma 3.4.5. *Let $A \subset \Gamma \backslash \{1\}$ be a finite set. There exists a constant $K \geq 1$, not depending on A, such that for all $\varepsilon > 0$ and $\varepsilon' \in (0,1]$, we have*

$$N_A(2\varepsilon) \leq K N_{2,A}(\varepsilon\varepsilon'\sqrt{|A|}) e^{K\sqrt{\varepsilon'}|A|}.$$

Theorem 3.4.6. *Let $E \subset \Gamma \backslash \{1\}$ satisfy the $(\varepsilon_1, \varepsilon_2)$-net condition for some $\varepsilon_1, \varepsilon_2 \in (0,1)$. Then E is Sidon with Sidon constant bounded by*

$$\frac{K(\xi)}{\varepsilon_1^{7.5(1+\xi)}\varepsilon_2^{3(1+\xi)}},$$

where $\xi > 0$ and $K(\xi)$ is a constant only depending on ξ.

Proof. Let C be a constant that may change according to the context. Fix a finite set $A \subset E$ and put $f_A := \sum_{\gamma \in A} \gamma \in \mathrm{Trig}(G)$. Let K be the constant in Lemma 3.4.5.

We first assume A is large enough that $|A| \geq 4(\ln K)/\varepsilon_1$. From the $(\varepsilon_1, \varepsilon_2)$-net condition and Lemma 3.4.5, for all $\varepsilon' \in (0,1]$, we have

$$K N_{2,A}\left(\frac{\varepsilon_2\varepsilon'}{4}\sqrt{|A|}\right)\exp(K\sqrt{\varepsilon'}|A|) \geq N_A(\varepsilon_2/2) \geq \exp(\varepsilon_1|A|\ln 2).$$

Hence,

$$N_{2,A}\left(\frac{\varepsilon_2\varepsilon'}{4}\sqrt{|A|}\right) \geq K^{-1}\exp(\varepsilon_1|A|\ln 2 - K\sqrt{\varepsilon'}|A|).$$

(The entropy integral will now be involved and we direct the reader to Definition 2.2.26 for the definitions.) Since $N_{2,A}(t_1) \geq N_{2,A}(t_2)$ if $t_1 \leq t_2$, we get

$$\begin{aligned}
J_2(f_A) = \int_0^\infty (\log N_{2,A}(t))^{1/2}\, dt &\geq \int_0^{\varepsilon_2\varepsilon'\sqrt{|A|}/4} (\log N_{2,A}(t))^{1/2}\, dt \\
&\geq \frac{\varepsilon_2}{4}\varepsilon'\sqrt{|A|}\left(\varepsilon_1|A|\ln 2 - K\sqrt{\varepsilon'}|A| - \ln K\right)^{1/2} \\
&\geq \frac{\varepsilon_2}{4}\varepsilon'\sqrt{|A|}\left(\varepsilon_1|A|\ln 2 - K\sqrt{\varepsilon'}|A| - \varepsilon_1|A|/4\right)^{1/2}.
\end{aligned}$$

Put $\varepsilon' = \frac{\varepsilon_1^2}{16K^2}$ and then we have $J_2(f_A) \geq C\varepsilon_1^{2.5}\varepsilon_2|A|$.

Recall that $(g_\gamma)_{\gamma\in\Gamma}$ are the independent standard Gaussian random variables indexed by Γ and $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ are independent ±1-valued random variables.

From Fernique's inequality (Theorem 2.2.31),

$$J_2(f_A) \leq C\mathbb{E}\left\|\sum_{\gamma\in A} g_\gamma\gamma\right\|_\infty .$$

Hence,

$$C\varepsilon_1^{2.5}\varepsilon_2|A| \leq \mathbb{E}\left\|\sum_{\gamma\in A} g_\gamma\gamma\right\|_\infty .$$

From Lemma 2.3.4, for all $M > 0$ and $\xi > 0$, there exists a constant $M_1 = C(\xi)M^\xi$ such that

$$\mathbb{E}\left\|\sum_{\gamma\in A} g_\gamma\gamma\right\|_\infty \leq \frac{|A|}{M} + M_1\mathbb{E}\left\|\sum_{\gamma\in A} \varepsilon_\gamma\gamma\right\|_\infty .$$

Put $M = 2/(C\varepsilon_1^{2.5}\varepsilon_2)$ and we have that for all $\xi > 0$, there exists a constant $K(\xi)$ such that

$$\mathbb{E}\left\|\sum_{\gamma\in A} \varepsilon_\gamma\gamma\right\|_\infty \geq K(\xi)\varepsilon_1^{2.5(1+\xi)}\varepsilon_2^{1+\xi}|A|.$$

Moreover, we may assume $K(\xi) \in (0, \frac{1}{4\ln K})$.

Next, we deal with the case $|A| \leq 4(\ln K)/\varepsilon_1$. If $|a_\gamma| = 1$, then

$$\sum_{\gamma\in A} |a_\gamma| = |A| \leq S(A)\left\|\sum_{\gamma\in A} a_\gamma\gamma\right\|_{C(G)} ,$$

where we recall that $S(A)$ denotes the Sidon constant of A. As the Sidon constant of A is at most $\sqrt{|A|}$, in this case we have

$$\mathbb{E}\left\|\sum_{\gamma\in A}\varepsilon_\gamma\gamma\right\|_\infty \geq \frac{|A|}{S(A)} \geq \left(\frac{\varepsilon_1}{4\ln(K)}\right)^{1/2}|A| \geq K(\xi)\varepsilon_1^{2.5(1+\xi)}\varepsilon_2^{1+\xi}|A|.$$

Corollary 2.3.5 gives the desired result. □

Taking $\varepsilon_1 = \varepsilon_2$ gives the following corollary.

Corollary 3.4.7. *Let $E \subset \Gamma\backslash\{1\}$ satisfy the ε-net condition for some $\varepsilon \in (0,1)$. Then E is Sidon with Sidon constant bounded by $\frac{K(\xi)}{\varepsilon^{10.5+\xi}}$, where $\xi > 0$ and $K(\xi)$ is a constant only depending on ξ.*

Remark 3.4.8. Corollary 3.4.7 improves Proposition V.12. in Chapter 13 of [24] by reducing the degree from 63 to 10.5.

Chapter 4

Kronecker Sets

4.1 Introduction

In this chapter we explore another notion of interpolation sets: ε-Kronecker sets. We recall that $E \subset \Gamma$ is **ε-Kronecker** for some $\varepsilon > 0$ if every $\varphi : E \to \mathbb{T}$ can be interpolated by some $x \in G$ within a difference of ε (Definition 1.5.1). The **Kronecker constant** of E, $\kappa(E)$, is defined as the infimum of ε such that E is ε-Kronecker.

Definition 4.1.1. Let $\varepsilon > 0$ and $E \subset \Gamma$. We say E is **weak ε-Kronecker** if for all $\varphi : E \to \mathbb{T}$ there exists $x \in G$ such that $|\varphi(\gamma) - \gamma(x)| \leq \varepsilon$ for all $\gamma \in E$.

Remark 4.1.2. Since G is compact, a compactness argument shows $E \subset \Gamma$ is weak $\kappa(E)$-Kronecker.

In some cases, the angular notion, viewing $\mathbb{T} = [0, 1]$, is convenient, and we also have the definition of angular Kronecker sets.

Definition 4.1.3. (1) We define the **angle metric** $d_{\mathbb{T}}$ on $\mathbb{T} = [0, 1]$ by

$$d_{\mathbb{T}}(x, y) = \min \{|x - y|, 1 - |x - y|\}.$$

(2) We define $\mathrm{Arg} : \left\{e^{2\pi i\theta} : \theta \in [0,1)\right\} \to [0,1)$ by

$$\mathrm{Arg}\, e^{2\pi i\theta} = \theta.$$

(3) $E \subset \Gamma$ is (**weak**) **angular θ-Kronecker** if for all $f : E \to \mathbb{T} = [0,1]$ there exists $x \in G$ such that $d_{\mathbb{T}}(f(\gamma), \mathrm{Arg}\,\gamma(x)) < \theta$ (or $d_{\mathbb{T}}(f(\gamma), \mathrm{Arg}\,\gamma(x)) \leq \theta$) for all $\gamma \in E$.

(4) The **angular Kronecker constant** of E, $\alpha(E)$, is defined by

$$\alpha(E) := \inf\{\theta : E \text{ is angular } \theta\text{-Kronecker}\}$$

Recall that we say $E \subset \Gamma$ is **Kronecker** if $\kappa(E) < 2$ or, equivalently, $\alpha(E) < 1/2$.

Clearly, we have $\kappa(E) = |1 - e^{2\pi i\alpha(E)}|$.

Example 4.1.4. As noted in Proposition 1.5.4 (2), for $E_n := \left\{n^k : k \geq 0\right\}$, $n \geq 2$, we have $\alpha(E_n) = 1/(2n)$.

Kronecker sets are closely related to other interpolation notions. As we mentioned in chapter 1, ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are I_0. Hare and Ramsey ([14]) proved that any Kronecker set is Sidon and, as a partial converse, Graham and Hare proved Sidon sets are proportionally weak $\sqrt{2}$-Kronecker (Theorem 9.3.2 in [9]). In general there are non-Kronecker, Sidon sets ([14]), but this is unknown for $\mathbb{Z}$.

In this chapter, we first give an estimation of the magnitude of the Sidon constant of a $(2-\varepsilon)$-Kronecker set (Theorem 4.2.1). This uses Theorem 3.4.6. We improve Theorem 9.3.2 in [9] by proving Sidon sets are proportionally $(1+\xi)$-Kronecker for any $\xi > 0$ (Theorem 4.2.2). It is still open if a Sidon set (even in $\mathbb{Z}$) can be proportional ε-Kronecker for any arbitrarily small ε. Moreover, when $\varepsilon < \sqrt{2}$, an explicit relation between ε-Kronecker sets and I_0 sets will be given (Proposition 4.2.8).

Unlike Sidon sets, it is open if a union of two Kronecker sets in $\mathbb{Z}$ remains Kronecker, even in $\mathbb{Z}$. We will show that a union of two Kronecker sets is still Kronecker in some special cases (Corollary 4.3.7). We will also show a translation of a Kronecker set away from 0 in $\mathbb{Z}$ remains Kronecker (Corollary 4.3.8).

4.2 Interpolation properties of Kronecker sets

In this section we show various relations among Kronecker sets, Sidon sets and I_0 sets.

4.2.1 Kronecker sets and Sidon sets

We give an upper bound for the Sidon constant of a Kronecker set.

Theorem 4.2.1. *Let $\varepsilon > 0$ and $\xi > 0$. Suppose $E \subset \Gamma$ is $(2-\varepsilon)$-Kronecker. Then E is Sidon with $S(E) \leq \frac{K(\xi)}{\varepsilon^{27/4+\xi}}$, where $K(\xi)$ is a constant depending on ξ.*

Proof. Let $F \subset E$ be a finite subset. Consider the pseudo-metric on G given by $d_F(x, y) := \sup_{\gamma \in F} |\gamma(x) - \gamma(y)|$ and the collection

$$\mathcal{C} := \{A \subset G : \forall x \neq y \in A,\ d_F(x, y) \geq \varepsilon/2\}.$$

Zorn's lemma gives a maximal element $M \in \mathcal{C}$ and furthermore, the compactness of G implies M is finite. We will see that the set M is large enough to satisfy the ε-net condition.

For $h \in G$ and $\lambda > 0$, we let

$$U(h, \lambda) := \{g \in G : d_F(h, g) < \lambda\},$$

which is the ball centered at h of radius λ. By the maximality of M, for all $g \in G$ there is $h \in M$ such that $g \in U(h, \varepsilon/2)$.

Let $\phi : F \to \mathbb{T}$. By the definition of the Kronecker constant, there is some $g \in G$ such that $\sup_{\gamma \in F} |\gamma(g) - \phi(\gamma)| \leq \kappa(E) = 2 - \varepsilon$. Since $g \in U(h, \varepsilon/2)$ for some $h \in M$, we have $\phi \in W(h)$, where

$$W(h) := \left\{ \varphi : F \to \mathbb{T} : \sup_{\gamma \in F} |\gamma(h) - \varphi(\gamma)| \leq 2 - \varepsilon/2 \right\}.$$

Hence, $\mathbb{T}^F = \cup_{h \in M} W(h)$. Notice that, if we identify $\mathbb{T}$ with $[0, 2\pi]$, each

$$W(h) \subset \prod_{\gamma \in F} [\gamma(h) - \eta, \gamma(h) + \eta] \subset \mathbb{T}^F$$

for some angle $\eta \in [0, \pi]$, and therefore have

$$\bigcup_{h \in M} \prod_{\gamma \in F} [\gamma(h) - \eta, \gamma(h) + \eta] = \mathbb{T}^F.$$

By the cosine law, $\cos(\eta) = 1 - \frac{(2-\varepsilon/2)^2}{2}$, which implies $\eta = \eta(\varepsilon) < \pi$. Comparing the volumes, we have $|M| \geq \left(\frac{2\pi}{2\eta}\right)^{|F|} = 2^{\varepsilon_1 |F|}$.

Hence, E satisfies the $(\varepsilon_1, \varepsilon_2)$-condition for $\varepsilon_2 = \varepsilon/2$ and $\varepsilon_1 = \log_2\left(\frac{\pi}{\eta(\varepsilon)}\right) := f(\varepsilon)$. It is easy to check $\lim_{\varepsilon \to 0} f(\varepsilon)/\sqrt{\varepsilon} = \sqrt{2}/(\pi \log(2))$ and therefore $\varepsilon_1 \sim \sqrt{\varepsilon}$.

Hence, by Theorem 3.4.6, $S(E) \leq \frac{K(\xi)}{\varepsilon^{27/4+\xi}}$, where $\xi > 0$ and $K(\xi)$ is a constant depending on ξ. □

Next, we show Sidon sets in a group with no elements of order two are proportional angular $(1/6 + \xi)$-Kronecker (or equivalently, $(1 + \xi)$-Kronecker).

Theorem 4.2.2. *Assume Γ has no elements of order 2. Suppose $E \subset \Gamma$ is Sidon. For all $\xi > 0$ there exists $\delta > 0$ such that for any finite subset $F \subset E$, there exists a weak angular $(1/6 + \xi)$-Kronecker set $F' \subset F$ with $|F'| \geq \delta |F|$.*

Before proving Theorem 4.2.2, we state a few preliminary results. The first one is a lemma from [9] (Lemma 9.2.5 in [9]).

Lemma 4.2.3. *Let $F \subset \Gamma$ be a finite set. Let $\varepsilon > 0$ and assume $Y \subset G$ satisfies $|Y| \geq 2^{\varepsilon|F|}$ and*

$$\sup_{\gamma \in F} |\gamma(x) - \gamma(y)| \geq \varepsilon$$

for all $x \neq y \in Y$. Then there exist constants $0 < \beta, \delta, \lambda \leq 1/8$, depending only on ε, a subset $F' \subset F$ and two arcs $I_1, I_2 \subset \mathbb{T}$ satisfying the following:
(1) $|F'| \geq \delta|F|$,
(2) The lengths of I_1 and I_2 are equal and bounded by λ,
(3) The gap between I_1 and I_2 is at least β,
(4) For any $A \subset F'$ there exists $g \in Y$ such that

$$\gamma(g) \in \begin{cases} I_1 & \text{if } \gamma \in A \\ I_2 & \text{if } \gamma \in F' \backslash A \end{cases} .$$

We also need a combinatorial result from A. Pajor.

Lemma 4.2.4. *[28] Suppose $X = X^+ \bigcup X^-$, where $|X^+| = p \geq 1$, $|X^-| = q \geq 1$ and $X^+ \bigcap X^- = \emptyset$. For $N \geq 1$ we define $\pi : X^N \to \mathbb{Z}_2^N$ by*

$$\pi(x)_n = \begin{cases} 1 & \text{if } x_n \in X^+ \\ -1 & \text{if } x_n \in X^- \end{cases}$$

for $x \in X^N$. There exists $\delta > 0$, depending only on p and q, such that whenever $S \subset X^N$ is large enough that $\pi(S) = \mathbb{Z}_2^N$, then there exist $t \in X^+$, $u \in X^-$ and $I \subset \{1, ..., N\}$ satisfying:
(1) $|I| \geq \delta N$,
(2) For all $x \in \{t, u\}^I$ there exists $y \in S$ such that the restriction of y to I, $y|_I = x$.

Finally, we need a known result about ε-Kronecker sets (Corollary 2.5.5 in [9]).

Lemma 4.2.5. *Assume Γ has no elements of order 2 and let $E \subset \Gamma$. Suppose there are two disjoint intervals $I, J \subset \mathbb{T} = [0, 1]$ of the same length strictly less than $1/2$*

such that for all $E' \subset E$ there exists $g \in G$ such that

$$\gamma(g) \in \left\{ \begin{array}{ll} I & \text{if } \gamma \in E' \\ J & \text{if } \gamma \in E \backslash E' \end{array} \right. .$$

Then E is weak angular $(1/2 - m)$-Kronecker, where m is the length of the smaller of the two gaps separating I and J.

We now prove Theorem 4.2.2.

Proof of Theorem 4.2.2. We fix $1/200 > \xi > 0$ and assume E is Sidon. Choose N large enough that $1/N < \frac{3\xi}{4}$. Since E is Sidon, there exists $\delta_1 > 0$ such that E satisfies the δ_1-net condition. Hence, for every finite $F \subset E$, there exists a set $Y \subset G$ with $|Y| \geq 2^{\delta_1 |F|}$ and satisfying

$$\sup_{\gamma \in F} |\gamma(x) - \gamma(y)| \geq \delta_1$$

for all $x \neq y \in Y$.

Lemma 4.2.3 thus gives a $\delta_2 > 0$, $F' \subset F$ with $|F'| \geq \delta_2 |F|$, and two intervals I_1 and I_2 of equal length at most λ and separated by a gap of length at least $\beta \leq 1/8$, where δ_2, λ and β only depend on E, having the property that for all $A \subset F'$ there is some $g \in Y$ such that $\gamma(g) \in I_1$ for $\gamma \in A$ and $\gamma(g) \in I_2$ for $\gamma \in F' \backslash A$. We define

$$Y' := \{g \in Y : \gamma(g) \in I_1 \cup I_2 \ \forall \gamma \in F'\}.$$

Partition each I_1 and I_2 into s disjoint subintervals $I'_1, ..., I'_s$ and $I'_{s+1}, ..., I'_{2s}$, respectively, having equal lengths at most $\rho := \beta/N$. Let $X^+ := \{1, ..., s\}$ and $X^- := \{s+1, ..., 2s\}$ and $X := X^+ \cup X^-$.

View Y' as a subset of $X^{F'}$ by identifying $g \in Y'$ with $(g_\gamma)_{\gamma \in F'}$ where $\gamma(g) \in I'_{g_\gamma}$. Define $\pi : X^{F'} \to \{-1, 1\}^{F'}$ by

$$\pi((g_\gamma)_{\gamma \in F'}) = (r_\gamma)_{\gamma \in F'},$$

where $r_\gamma = 1$ if $g_\gamma \in X^+$ and $r_\gamma = -1$ if $g_\gamma \in X^-$.

Lemma 4.2.4 shows that there exist $t \in X^+$, $u \in X^-$, $\delta_3 > 0$ and $F_1' \subset F'$ with $|F_1'| \geq \delta_3 |F'|$ such that for every $A \subset F_1'$, there exists $g \in Y'$ such that $\gamma(g) \in I_t'$ if $\gamma \in A$ and $\gamma(g) \in I_u'$ if $\gamma \in F_1' \backslash A$.

Let $b \geq \beta$ be the length of the smaller gap between I_t' and I_u'. If b exceeds $1/3$, Lemma 4.2.5 implies F_1' is weak angular $1/6$-Kronecker.

For an interval $I = [a, b] \subset \mathbb{T}$ and a factor $c \in \mathbb{R}^+$, we let $cI = [ca, cb]$ identified in $\mathbb{T} = [0, 1]$. We note that when intervals I and J are small and a gap T between I and J is so small that $|cT| < 1 - |cI| - |cJ|$, the gap between cI and cJ will be cT. Thus, if $b \in [1/6, 1/3)$, the two gaps between the intervals $2I_t'$ and $2I_u'$ have lengths at least $1/3$ and

$$1 - 2/3 - 4\rho \geq 1/3 - 4\beta/N \geq 1/3 - 3\beta\xi > 1/3 - \xi.$$

Hence, by Lemma 4.2.5, F_1' is weak angular $(1/6 + \xi)$-Kronecker. Similarly, if $b \in [5/36, 1/6)$, the two gaps between the intervals $3I_t'$ and $3I_u'$ have lengths at least $5/12$ and

$$1 - 1/2 - 6\rho > 1/3.$$

Again, by Lemma 4.2.5, F_1' is weak angular $1/6$-Kronecker. Otherwise,

$$b \in \frac{1}{6}\left[\frac{1}{k+1} + \frac{1}{k}, \frac{1}{k} + \frac{1}{k-1}\right)$$

for some $k \geq 3$. Notice that if $k \geq 3$,

$$(k+1)\left(\frac{1}{k} + \frac{1}{k-1}\right) \leq \frac{10}{3},$$

with equality obtained at $k = 3$. Hence, the two gaps between the intervals $(k+1)I_t'$

and $(k+1)I'_u$ have lengths at least $(k+1)b \geq 1/3$ and

$$\begin{aligned}
1 - 2(k+1)\rho - (k+1)b &\geq 1 - 2(k+1)\frac{\beta}{N} - \frac{k+1}{6}\left(\frac{1}{k} + \frac{1}{k-1}\right) \\
&\geq 1 - 2(k+1)\frac{3\xi}{4}\beta - \frac{5}{9} \\
&\geq \frac{4}{9} - 2(k+1)\frac{3\xi}{4}b \\
&\geq \frac{4}{9} - 2(k+1)\frac{1}{6}\left(\frac{1}{k} + \frac{1}{k-1}\right)\frac{3\xi}{4} \\
&\geq \frac{4}{9} - \frac{5\xi}{6} > 1/3.
\end{aligned}$$

Thus, Lemma 4.2.5 still implies F'_1 is weak angular 1/6-Kronecker. □

Remark 4.2.6. The condition that Γ has no elements of order 2 is necessary, as in [9] (Example 9.3.1) a set E is constructed such that every element in E has infinite order, but E is not proportional ε-Kronecker for any $\varepsilon < \sqrt{2}$.

4.2.2 Kronecker sets and I_0 sets

Definition 4.2.7. Let $N \in \mathbb{N}$ and $\delta > 0$. A set $E \subset \Gamma$ is $I_0(N,\delta)$ if for every $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1$ there exists $\mu = \sum_{n=1}^{N} c_n \delta_{x_n}$, where $|c_n| \leq 1$ and $x_n \in G$, such that $|\widehat{\mu}(\gamma) - \varphi(\gamma)| \leq \delta$ for all $\gamma \in E$.

From the iterative argument, which is used in Proposition 1.3.4, we have that any $I_0(N,\delta)$ set with $\delta < 1$ is I_0. A significant fact is that the converse is also true: every I_0 set is $I_0(N,\delta)$ for some $N \in \mathbb{N}$ and $\delta \in (0,1)$ (Proposition 3.2.12 in [9]).

We will show an angular θ-Kronecker set with $\theta < 1/4$ is $I_0(1,\delta)$ for some $\delta < 1$.

Proposition 4.2.8. *Let $E \subset \Gamma$ be a subset with $\alpha(E) = \theta < 1/4$. Then E is $I_0(1, \delta)$ for*

$$\delta = \begin{cases} \sin(2\pi\theta) & \text{if } \theta \geq 1/8 \\ \sec(2\pi\theta)/2 & \text{if } \theta < 1/8 \end{cases}.$$

Before we prove this, we first prove a technical lemma.

Notation 4.2.9. Let $\theta \in (0, 1/2)$ and

$$A_\theta := \left\{ e^{2\pi i\alpha} : \alpha \in (-\theta, \theta) \right\} \subset \mathbb{T}.$$

Lemma 4.2.10. *Let $\theta \in (0, 1/4)$. Then*

$$\inf_{\lambda \in (0,1)} \sup_{\substack{z \in [0,1] \\ w \in A_\theta}} |z - \lambda w| = \begin{cases} \sin(2\pi\theta) & \text{if } \theta \geq 1/8 \\ \sec(2\pi\theta)/2 & \text{if } \theta < 1/8 \end{cases}.$$

Proof. We first compute $\sup_{\substack{z \in [0,1] \\ w \in A_\theta}} |z - \lambda w|$ as a function of λ and θ.

Write $w = e^{2\pi i\alpha}$ for $\alpha \in (-\theta, \theta)$ and we have

$$|z - \lambda e^{2\pi i\alpha}|^2 = z^2 - 2\lambda \cos(2\pi\alpha) z + \lambda^2.$$

For $\alpha \in (-\theta, \theta)$ the maximum is obtained at $z = 0$ or $z = 1$, and therefore

$$\begin{aligned} \sup_{\substack{z \in [0,1] \\ \alpha \in (-\theta,\theta)}} |z - \lambda e^{2\pi i\alpha}|^2 &= \sup_{\alpha \in (-\theta,\theta)} \max\left\{ \lambda^2, 1 - 2\lambda\cos(2\pi\alpha) + \lambda^2 \right\} \\ &= \begin{cases} \lambda^2 & \text{if } \lambda \geq \sec(2\pi\theta)/2 \\ 1 - 2\lambda\cos(2\pi\theta) + \lambda^2 & \text{if } \lambda < \sec(2\pi\theta)/2 \end{cases}. \end{aligned}$$

Thus,

$$\inf_{\lambda \in (0,1)} \sup_{\substack{z \in [0,1] \\ w \in A_\theta}} |z - \lambda w|^2 = \inf_{\substack{\lambda \in (0,1) \\ \lambda < \sec(2\pi\theta)/2}} (1 - 2\lambda\cos(2\pi\theta) + \lambda^2).$$

Notice that the minimum above is obtained either at $\lambda = \cos(2\pi\theta)$, if

$$\cos(2\pi\theta) < \sec(2\pi\theta)/2,$$

or at the end point $\lambda = \sec(2\pi\theta)/2$. Hence,

$$\inf_{\substack{\lambda\in(0,1)\\ \lambda<\sec(2\pi\theta)/2}} (1 - 2\lambda\cos(2\pi\theta) + \lambda^2) = \begin{cases} \sec^2(2\pi\theta)/4 & \text{if } \theta < 1/8 \\ \sin^2(2\pi\theta) & \text{if } \theta \geq 1/8 \end{cases}.$$

□

Next, we prove Proposition 4.2.8.

Proof of Proposition 4.2.8. We first deal with the case that $\theta \in [0, 1/8)$. Let $\varphi : E \to \mathbb{C}$ be a function such that $||\varphi||_\infty \leq 1$. Define $\phi : E \to \mathbb{T}$ by

$$\phi(\gamma) = \begin{cases} \frac{\varphi(\gamma)}{|\varphi(\gamma)|} & \text{if } \varphi(\gamma) \neq 0 \\ 1 & \text{if } \varphi(\gamma) = 0 \end{cases}.$$

Since E has angular Kronecker constant θ, we can find $x \in G$ such that $\frac{\gamma(x)}{\phi(\gamma)} \in A_\theta$ for all $\gamma \in E$.

Let $\lambda = \sec(2\pi\theta)/2$. Notice that $|\varphi(\gamma)| \in [0, 1]$ and $\frac{\gamma(x)}{\phi(\gamma)} \in A_\theta$ and therefore, Lemma 4.2.10 implies

$$\begin{aligned} |\lambda\widehat{\delta_{x^{-1}}}(\gamma) - \varphi(\gamma)| &= |\lambda\gamma(x) - \varphi(\gamma)| = |\lambda\frac{\gamma(x)}{\phi(\gamma)} - |\varphi(\gamma)|| \\ &\leq \sup_{\substack{z\in[0,1]\\ w\in A_\theta}} |z - \lambda w| = \sec(2\pi\theta)/2. \end{aligned}$$

Hence, E is $I_0(1, \sec(2\pi\theta)/2)$.

Similarly, if $\theta \in [1/8, 1/4)$, E is $I_0(1, \sin(2\pi\theta))$ □

Remark 4.2.11. Suppose E satisfies $\alpha(E) < 1/4$. As we have shown, for every $\varphi : E \to \mathbb{C}$ with $||\varphi||_\infty \leq 1$, there exists $\lambda > 0$ and $x \in G$ such that

$$|\lambda \widehat{\delta_x}(\gamma) - \varphi(\gamma)| < \delta < 1.$$

If we apply the iterative argument (Proposition 1.3.4, for example), we have that every bounded $\varphi : E \to \mathbb{C}$ can be interpolated by the Fourier transform of not just a discrete measure, but even a positive discrete measure.

4.3 Union of Kronecker sets

In this section we obtain some partial results about unions of Kronecker sets. We first note that in general, it is false that a union of two Kronecker sets is still Kronecker. In fact, as the following proposition shows, even a finite set excluding the identity element may not be Kronecker.

Proposition 4.3.1. *Let $G = \Gamma = \mathbb{Z}_n$ for some composite $n \in \mathbb{N}$. Then the set $\mathbb{Z}_n$ excluding the identity element has Kronecker constant* 2.

Proof. During this proof, we identify

$$\Gamma = \mathbb{Z}_n = \{\gamma_0, \gamma_1, ..., \gamma_{n-1}\},$$

where γ_0 is the identity element, $\gamma_j = \gamma_1^j$ and

$$G = \mathbb{Z}_n = \{x_0, x_1, ..., x_{n-1}\} \subset \mathbb{T}$$

with identity element $x_0 = 1 \in \mathbb{T}$ and $x_j = e^{2\pi i j/n}$. The duality is given by $\gamma_j(x_k) = x_{kj}$ for $0 \leq j, k \leq n-1$. We let $E_n := \Gamma \setminus \{\gamma_0\}$.

Suppose $n = ap$ where p is prime and $a \geq 2$. We have $\gamma_a(x_p) = \gamma_p(x_a) = x_0 = 1$. We define a function $\phi : E_n \to \mathbb{T}$ by the following:

$$\phi(\gamma_k) := \begin{cases} -\gamma_k(x_k) & k \neq a \text{ and } k \neq p \\ -1 & k = p \text{ or } k = a \end{cases}.$$

Notice that for all $x_k \in G$, $0 \leq k \leq n-1$, there is some $\gamma_j \in E_n$, $1 \leq j \leq n-1$, such that $|\phi(\gamma_j) - \gamma_j(x_k)| = 2$. Indeed, this can be seen by pairing x_0 with γ_p, x_k with γ_k for $k \neq a, p$, x_a with γ_p and x_p with γ_a.

This implies E_n is at most weak 2-Kronecker and therefore E_n has Kronecker constant 2. $\square$

As we mentioned before, it is still open if a union of even two integer Kronecker sets remains Kronecker. We will prove the partial result that a union of an integer Kronecker set and a finite set, excluding 0, is still Kronecker.

Notation 4.3.2. For $n \in \mathbb{Z}$ and $\varepsilon > 0$, we define

$$B_{n,\varepsilon} := \{x \in \mathbb{T} : x^n \in A_\varepsilon\}$$

using the notation introduced in Notation 4.2.9.

Lemma 4.3.3. *Let $E \subset \mathbb{Z}$.*
(1) E is weak $|1 - e^{i(\pi-\varepsilon)}|$-Kronecker if and only if for any $(x_n)_{n\in E} \subset \mathbb{T} \subset \mathbb{C}$,

$$\bigcup_{n\in E} x_n B_{n,\varepsilon} \neq \mathbb{T} \subset \mathbb{C}.$$

(2) E is not Kronecker if and only if for any $\varepsilon > 0$ there exists a sequence $(x_n)_{n\in E} \subset \mathbb{T} \subset \mathbb{C}$ such that

$$\bigcup_{n\in E} x_n B_{n,\varepsilon} = \mathbb{T} \subset \mathbb{C}.$$

Proof. (1) Fix $\varepsilon > 0$. Suppose E is not weak $|1 - e^{i(\pi-\varepsilon)}|$-Kronecker. Let $x \in \mathbb{T}$. There exist $f : E \to \mathbb{T}$ and $n = n(x) \in E$ such that

$$|f(n) - x^n| > |1 - e^{i(\pi-\varepsilon)}|.$$

For each $n \in E$, choose $x_n \in \mathbb{T}$ such that $x_n^n = -f(n)$. Then $(xx_n^{-1})^n \in A_\varepsilon$, so $x \in x_n B_{n,\varepsilon}$. Therefore $\bigcup_{n\in E} x_n B_{n,\varepsilon} = \mathbb{T}$. This can be reversed to give the converse.

(2) The proof of (2) follows similarly. $\square$

Definition 4.3.4. (1) Let $E \subset \mathbb{Z}$, $X = (x_n)_{n\in E}$ be a sequence in $\mathbb{T} \subset \mathbb{C}$ and $\varepsilon > 0$. We define the covering

$$C(E, X, \varepsilon) := \cup_{n\in E}^{\infty} x_n B_{n,\varepsilon} \subset \mathbb{T}.$$

(2) Let $E \subset \mathbb{Z}$ and $\varepsilon > 0$. We define the maximum covering length

$$L(E, \varepsilon) := \sup_{X=(x_n)_{n\in E}\subset\mathbb{T}} \{\lambda(l) : l \subset C(E, X, \varepsilon) \text{ is a closed arc in } \mathbb{T}\},$$

where $\lambda(l)$ is the arc length of l.
(3) Given $n \in \mathbb{Z}$, $x \in \mathbb{T}$ and $\varepsilon > 0$, we call a point $y \in \mathbb{T}$ a center in $xB_{n,\varepsilon}$ if y is the center of one of the branches of $xB_{n,\varepsilon}$, that is $y = xe^{2\pi ij/n}$ for some $j \in \{0, ..., n-1\}$.

Remark 4.3.5. Notice that $L(E, \varepsilon)$ is non-decreasing with respect to ε.

Proposition 4.3.6. *A subset $E \subset \mathbb{Z}$ is Kronecker if and only if* $\lim_{\varepsilon\to 0} L(E, \varepsilon) = 0$.

Proof. We first assume that if $\lim_{\varepsilon\to 0} L(E, \varepsilon) > 0$ and prove that E is not Kronecker. Pick $M \in \mathbb{N}$ such that

$$L := 2\pi/M < \lim_{\varepsilon\to 0} L(E, \varepsilon).$$

Suppose, for a contradiction, that E has Kronecker constant $|1 - e^{i(\pi-\varepsilon_0)}| < 2$ for some $\varepsilon_0 > 0$. Since

$$L < \lim_{\varepsilon\to 0} L(E, \varepsilon) \leq L(E, \varepsilon_0/(2M)),$$

there exist a finite set $\{n_1, ..., n_k\} \subset E$ and $\{x_1, ..., x_k\} \subset \mathbb{T}$ such that

$$\bigcup_{j=1}^{k} x_j B_{n_j,\varepsilon_0/(2M)} \supset \left\{e^{i\theta} : \theta \in [0, L]\right\} := I_L. \tag{4.1}$$

As $L = 2\pi/M$, given any $z \in \mathbb{T}$, we can find $z_0 \in I_L$ such that $z_0^M = z$.

By Eq. (4.1), there exist $1 \leq s \leq k$ and $y_0 \in \mathbb{T}$ such that y_0 is a center of $x_s B_{n_s,\varepsilon_0/(2M)}$

and

$$z_0 y_0^{-1} \in A_{\varepsilon_0/(2Mn_s)}.$$

Clearly, y_0^M is a center of $x_s^M B_{n_s,\varepsilon_0/2}$. Moreover,

$$y_0^M z^{-1} = (y_0 z_0^{-1})^M \in A_{\varepsilon_0/(2n_s)}.$$

Thus, $z \in x_s^M B_{n_s,\varepsilon_0/2}$ and this shows

$$\bigcup_{j=1}^{k} x_j^M B_{n_j,\varepsilon_0/2} \supset \mathbb{T},$$

which contradicts Lemma 4.3.3 (1).

Conversely assume $\lim_{\varepsilon\to 0} L(E,\varepsilon) = 0$. Then, for ε small enough and for any choice $X = (x_n)_{n\in E} \subset \mathbb{T}$, $C(E, X, \varepsilon) \neq \mathbb{T}$. Lemma 4.3.3 (2) implies that E is Kronecker.

□

We thus have the following corollary.

Corollary 4.3.7. *If $E \subset \mathbb{Z}$ is Kronecker and $F \subset \mathbb{Z}\setminus\{0\}$ is a finite set, then $E \cup F$ is Kronecker.*

Proof. We may assume $F = \{n\}$ and $n \neq 0$. Since E is Kronecker, by Proposition 4.3.6, $\lim_{\varepsilon\to 0} L(E,\varepsilon) = 0$. Choose $\delta > 0$ small enough that

$$L(E,\delta) < \frac{2\pi - 2\delta}{|n|}.$$

As the arcs between the branches of $yB_{n,\delta}$ have length $(2\pi - 2\delta)/|n|$ for any $y \in \mathbb{T}$, there is no choice of y and $(x_k)_{k\in E} \subset \mathbb{T}$ such that

$$yB_{n,\delta} \bigcup \bigcup_{k\in E} x_k B_{k,\delta} = \mathbb{T}.$$

Hence, Lemma 4.3.3 (1) implies $E \cup F$ is weak $|1 - e^{i(\pi-\delta)}|$-Kronecker . □

Next, we deduce that a translation of a Kronecker set in Z away from 0 is Kronecker. This is open for Kronecker sets in other groups.

Corollary 4.3.8. *Suppose $E \subset \mathbb{Z}$ is Kronecker, $n \in \mathbb{Z}$ and $-n \notin E$. Then $n + E$ is Kronecker.*

Proof. This follows from Corollary 4.3.7 and the fact proven in [9] (Corollary 2.2.15) that says if $E \subset \mathbb{Z}$ is Kronecker and $n \in \mathbb{Z}$, there exists a finite subset $F \subset E$ such that $n + (E \backslash F)$ is Kronecker. $\square$

Finally, we have a result regarding a Kronecker set union its inverse.

Proposition 4.3.9. *Suppose $E \subset \Gamma$ and $\alpha(E) < 1/4$. Then, $E \cup E^{-1}$ is also Kronecker.*

Proof. Let $\varphi : E \cup E^{-1} \to \mathbb{T} = [0, 1]$. For $\gamma \in E$, we let $\phi_\gamma : \{1, -1\} \to \mathbb{T}$ be given by $\phi_\gamma(1) = \varphi(\gamma)$ and $\phi_\gamma(-1) = \varphi(\gamma^{-1})$.

It is not hard to see $\alpha(\{-1, 1\}) = 1/4$ (see [16]) and therefore for each γ there exists $x_\gamma \in \mathbb{T}$ such that

$$\max_{j \in \{-1,1\}} d_{\mathbb{T}}(jx_\gamma, \phi_\gamma(j)) \leq 1/4.$$

Since $\alpha(E) < 1/4$, there exists $y \in G$ such that

$$\sup_{\gamma \in E} d_{\mathbb{T}}(\operatorname{Arg} \gamma(y), x_\gamma) < 1/4.$$

Putting these together, we have

$$\begin{aligned} \sup_{j \in \{-1,1\}, \gamma \in E} d_{\mathbb{T}}(\operatorname{Arg} \gamma^j(y), \varphi(\gamma^j)) &\leq \sup_{j \in \{-1,1\}, \gamma \in E} d_{\mathbb{T}}(\gamma^j(y), jx_\gamma) + d_{\mathbb{T}}(jx_\gamma, \varphi(\gamma^j)) \\ &< 1/4 + 1/4 = 1/2, \end{aligned}$$

which implies $E \cup E^{-1}$ is Kronecker. $\square$

Chapter 5

Binary Kronecker Sets

5.1 Introduction

Recall that binary Kronecker sets are weakened versions of Kronecker sets. Instead of interpolating functions with range $\mathbb{T}$, in the binary Kronecker setting we only require to interpolate ± 1-valued functions.

In this chapter we identify $\mathbb{T} = [0, 1]$ with 0 being the identity and the group operation being addition mod integers. The metric on $\mathbb{T}$ in this chapter is $d_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \to [0, 1/2]$ given by $d_{\mathbb{T}}(x, y) = \min \{|x - y|, 1 - |x - y|\}$. The dual group is the integer group $\mathbb{Z}$ and, for $n \in \mathbb{Z}$ and $x \in \mathbb{T}$, we identify the duality by the multiplication nx mod integers.

Since we identify $\mathbb{T} = [0, 1]$, ± 1-valued functions become $\{0, 1/2\}$-valued functions. We recall the definition of binary ε-Kronecker sets in this notation.

Definition 5.1.1. Let $\varepsilon > 0$. A subset $E \subset \mathbb{Z}$ is called **binary ε-Kronecker** if for all $\varphi : E \to \{0, 1/2\}$ there exists $x \in \mathbb{T}$ such that $d_{\mathbb{T}}(\varphi(n), nx) < \varepsilon$ for all $n \in E$.

The **binary Kronecker constant** of E, $\beta(E)$, is defined by

$$\beta(E) := \inf \{\tau : E \text{ is binary } \tau\text{-Kronecker}\}.$$

Remark 5.1.2. The notion of a binary Kronecker set can be defined more generally for $E \subset \Gamma$. But our interest is in $E \subset \mathbb{Z}$.

Example 5.1.3. Consider $E \subset \mathbb{Z}$, the set of odd integers. We note that if $n \in E$ and $x_0 = 1/4$, then nx_0 is either $1/4$ or $3/4$. Hence, $\beta(E) \leq 1/4$. Later in the chapter we will see, being a symmetric set, $\beta(E) \geq 1/4$ from Proposition 5.2.4. Hence, $\beta(E) = 1/4$. We also note that E is not a Sidon set because it is known that Sidon sets cannot contain arbitrarily long arithmetic progressions, and the angular Kronecker constant is $\alpha(E) = 1/2$.

As we mentioned in chapter 1, for a subset $E \subset \Gamma$ we have $\beta(E) \leq \alpha(E) \leq 2\beta(E)$ ([10]). Example 5.1.3 shows if E is the set of odd integers, $2\beta(E) = \alpha(E)$. On the other hand, $\beta(\mathbb{Z}) = \alpha(\mathbb{Z}) = 1/2$. Proposition 5.3.1 will show that for geometric sequences E we also have $\alpha(E) = \beta(E)$.

Since I_0 sets can be characterized by interpolating ± 1-valued functions within error 1 (measuring on the unit circle on the complex plane) (Theorem 1.4.3), if a set has binary Kronecker constant less than $1/6$, it is an I_0 set. Moreover, if a set has binary Kronecker constant less than $1/4$, then its angular Kronecker constant is less than $1/2$ and therefore it is a Sidon set. Note that the example of odd integers implies the bound $1/4$ is sharp.

It is known that if a set has angular Kronecker constant less than $1/2$, the set does not cluster at any continuous character in the Bohr topology. The binary Kronecker sets have similar properties.

Proposition 5.1.4. *If $\beta(E) < 1/2$, then $E \subset \mathbb{Z}$ does not cluster at 0.*

Proof. Since $\beta(E) < 1/2$, we can find $\varepsilon > 0$ and $x \in G$ such that $d_{\mathbb{T}}(nx, 1/2) < 1/2 - \varepsilon$ for all $n \in E$. That means $d_{\mathbb{T}}(nx, 0) \geq \varepsilon$ for all $n \in E$ and therefore E does not cluster at 0. $\square$

As with Kronecker sets, the binary Kronecker constant of a set is the supremum of the binary Kronecker constants of its finite subsets.

Lemma 5.1.5. *Let $E \subset \mathbb{Z}$ be a subset. Then $\beta(E) = \sup_{F \subset E, F \text{ finite}} \beta(F)$.*

Proof. Let $\sup_{F \subset E, F \text{ finite}} \beta(F) := a$. We only need to show $a \geq \beta(E)$. Let $\varphi : E \to \{\pm 1\}$ be a function. For each finite subset $F \subset E$ we let $x_F \in G$ be such that $d_{\mathbb{T}}(nx_F, \varphi(n)) \leq a$ for all $n \in F$. We partially order the subsets of E by inclusion. Since $\mathbb{T}$ is compact, we let x be a cluster point of the net $(x_F)_{\text{finite } F \subset E}$. Then for all $n \in E$ we have $d_{\mathbb{T}}(nx, \varphi(n)) \leq a$. This proves $a \geq \beta(E)$. □

Thus it is of interest to compute the binary Kronecker constants of finite sets of integers and this is what we will do in the remainder of the chapter for particular examples. The structure of symmetrized sets often makes this problem tractable and most of our examples are of this type.

Since $\mathbb{Z} \backslash \{0\}$ is dense in $\overline{\mathbb{Z}}$, Proposition 5.1.4 implies $\beta(\mathbb{Z} \backslash \{0\}) = 1/2$. Hence, it is of interest to know the growth rate of the binary Kronecker constant of $E_n := \{\pm 1, ..., \pm n\}$ as a function of n. In Proposition 5.2.5 we will prove $\beta(E_n) = n/(2(n+1))$. In comparison, the binary Kronecker constant of $\{1, ..., n\}$ is known to be $(n-1)/(2(n+1))$ ([16]).

In Proposition 5.2.6 we study the symmetric powers $H_n := \left\{\pm n^k : k \geq 1\right\}$. One interesting phenomenon here is that $\beta(H_n) = \beta\{\pm 1, \pm n\}$. Another is that the answer depends on whether n is even or odd.

Proposition 5.2.8 concerns cosets in $\mathbb{Z}$. Since Sidon sets cannot contain arbitrarily long arithmetic progressions, these cosets are never Sidon sets and it is interesting to study their binary Kronecker interpolation properties.

We also study one class of non-symmetric examples in section 5.3, which is the non-symmetrized geometric sequences. As we mentioned before, this is a non-trivial example of a set whose binary Kronecker constant coincides with its angular Kronecker constant.

5.2 Symmetrized sets

In this section we will see symmetry can greatly reduce the complexity of computing the binary Kronecker constant. A subset $E \subset \mathbb{Z}$ is **symmetrized** if $E = -E$. We need a few notations.

Notation 5.2.1. Let $f : E \subset \mathbb{Z} \to \{0, 1/2\}$ be a function.
(1) Let $x \in \mathbb{T}$. We define

$$D(E, f, x) := \sup_{n \in E} d_{\mathbb{T}}(nx, f(n)).$$

(2) $D(E, f) := \inf_{x \in \mathbb{T}} D(E, f, x)$.
(3) We define $\varphi_E : E \to \{0, 1/2\}$ as $\varphi_E(n) = \begin{cases} 0 & \text{if } n \geq 0 \\ 1/2 & \text{if } n < 0 \end{cases}$.
(4) For $x \in \mathbb{T}$ we define $\mathcal{D}(E, x) := \sup_{n \in E} d_{\mathbb{T}}(nx, \varphi_E(n))$.

We establish a few preliminary results.

Lemma 5.2.2. *(1) Suppose $n_0 \in E$ is a positive integer. Assume $f : E \to \{0, 1/2\}$ satisfies $f(n_0) = f(-n_0)$. We let $g : E \to \{0, 1/2\}$ be given by $g(n_0) = 1/2 - f(n_0)$ and $g = f$ elsewhere. Then,*

$$D(E, g) \geq D(E, f).$$

(2) Suppose $f : E \to \{0, 1/2\}$ and $\pm n_0 \in E$. We define $g : E \to \{0, 1/2\} \subset \mathbb{T}$ by

$$g(n_0) = f(-n_0)$$
$$g(-n_0) = f(n_0)$$

and $g = f$ elsewhere. Then, $D(E, f) = D(E, g)$.

Proof. (1) Define $E_0 := E \setminus \{-n_0\}$. Since $f(n_0) = f(-n_0) \in \{0, 1/2\}$, for all $x \in \mathbb{T}$, $d_{\mathbb{T}}(n_0 x, f(n_0)) = d_{\mathbb{T}}(-n_0 x, f(-n_0))$ and therefore $D(E, f) = D(E_0, f|_{E_0})$. As $E_0 \subset E$,

$$D(E, f) = D(E_0, f|_{E_0}) = D(E_0, g|_{E_0}) \leq D(E, g).$$

(2) If $f(n_0) = f(-n_0)$, we are done. Hence, we assume $f(n_0) \neq f(-n_0)$ and without lose of generality, we further assume $f(n_0) = 0$ and $f(-n_0) = 1/2$. Then, for any $x \in \mathbb{T}$,

$$D(\{\pm n_0\}, f|_{\{\pm n_0\}}, x) = \max\{d_{\mathbb{T}}(n_0 x, 0), d_{\mathbb{T}}(-n_0 x, 1/2)\} = D(\{\pm n_0\}, g|_{\{\pm n_0\}}, x).$$

Since $f = g$ off $\{\pm n_0\}$, we have $D(E, f, x) = D(E, g, x)$ for all $x \in \mathbb{T}$. Hence, $D(E, f) = D(E, g)$. □

Corollary 5.2.3. *Let $E \subset \mathbb{Z}$ be symmetrized. Then $\beta(E) = D(E, f)$ for all $f : E \to \{0, 1/2\}$ satisfying $f(n) \neq f(-n)$ for all $n \in E$.*

Proposition 5.2.4. *(1) $\beta(E) = D(E, \varphi_E) = \inf_{x \in \mathbb{T}} \mathcal{D}(E, x)$.*
(2) For any symmetrized set $E \subset \mathbb{Z}$ we have $\beta(E) \geq 1/4$.

Proof. (1) This follows from Corollary 5.2.3.
(2) Notice that for $n \geq 0$ in E and any $x \in \mathbb{T}$,

$$\mathcal{D}(E, x) \geq \mathcal{D}(\{\pm n\}, x) = \max\{d_{\mathbb{T}}(nx, 0), d_{\mathbb{T}}(-nx, 1/2)\} \geq 1/4.$$

Thus, $\beta(E) \geq 1/4$. □

We compute the binary Kronecker constants for three types of symmetrized integer sets.

Proposition 5.2.5. *Let $S_n := \{\pm 1, \pm 2, ..., \pm n\}$. Then $\beta(S_n) = n/(2n+2)$.*

Proof. We first note that if we let $x_0 = 1/(2n+2)$, then $d_{\mathbb{T}}(x_0, 1/2) = d_{\mathbb{T}}(nx_0, 0) = n/(2n+2)$. Hence,

$$\mathcal{D}(S_n, x_0) = \max_{1 \leq k \leq n} \{d_{\mathbb{T}}(kx_0, 0), d_{\mathbb{T}}(-kx_0, 1/2)\} = n/(2n+2).$$

Thus, $\beta(S_n) \leq n/(2n+2)$. It remains to prove $\beta(S_n) \geq n/(2n+2)$.

Suppose, otherwise, that there exists $y \in \mathbb{T}$ such that $\mathcal{D}(S_n, y) < n/(2n+2)$. Clearly, for all $1 \leq k \leq n$,

$$\pm ky \notin \left[0, \frac{1}{2n+2}\right] \bigcup \left[\frac{1}{2} - \frac{1}{2n+2}, \frac{1}{2} + \frac{1}{2n+2}\right] \bigcup \left[1 - \frac{1}{2n+2}, 1\right].$$

In other words, for $1 \leq k \leq n$, $\pm ky$ stays in two open intervals:

$$\pm ky \in \left(\frac{1}{2n+2}, \frac{n}{2n+2}\right) \bigcup \left(\frac{n+2}{2n+2}, \frac{2n+1}{2n+2}\right).$$

Notice that the open intervals have equal length $(n-1)/(2n+2)$.

For $1 \leq k \leq n-1$, let

$$I_k := \left(\frac{k}{2n+2}, \frac{k+1}{2n+2}\right].$$

Notice that $(I_k)_{1 \leq k \leq n-1}$ is a partition of $(\frac{1}{2n+2}, \frac{n}{2n+2}]$. Also, for $1 \leq k \leq n-1$, we let

$$J_k := \left(\frac{n+k+1}{2n+2}, \frac{n+k+2}{2n+2}\right].$$

Then, $(J_k)_{1 \leq k \leq n-1}$ forms a partition for $(\frac{n+2}{2n+2}, \frac{2n+1}{2n+2}]$.

For $1 \leq k \leq n-1$, we let $P_k := I_k \cup J_k$. Since $\{y, 2y, ..., ny\} \subset \bigcup_{k=1}^{k=n-1} P_k$, there exist $1 \leq m \leq n-1$ and $1 \leq a, b \leq n$ such that $ay, by \in P_m = I_m \bigcup J_m$.

If

$$ay \in I_m = \left(\frac{m}{2n+2}, \frac{m+1}{2n+2}\right]$$

and

$$by \in J_m = \left(\frac{n+m+1}{2n+2}, \frac{n+m+2}{2n+2}\right],$$

then

$$(a-b)y \in \left(\frac{n}{2n+2}, \frac{n+2}{2n+2}\right).$$

But $(a-b)y = \pm ky$ for some $k = 1, ..., n$ and therefore we get a contradiction. Similarly, if $ay \in J_m$ and $by \in I_m$, we also have a contradiction.

It remains to consider the case $ay, by \in I_m$ or $ay, by \in J_m$. Since I_m and J_m have length $1/(2n+2)$, either $ay, by \in I_m$ or $ay, by \in J_m$ implies

$$(a-b)y \in \left[0, \frac{1}{2n+2}\right] \bigcup \left[1 - \frac{1}{2n+2}, 1\right],$$

which is again a contradiction. Thus $\beta(S_n) \geq n/(2n+2)$. $\square$

Proposition 5.2.6. *Let $H_n := \left\{\pm n^k : k \geq 0\right\}$. Then $\beta(H_n) = \frac{n+2}{4n+4}$ for even n and $\beta(H_n) = 1/4$ for odd n.*

Proof. First, if n is odd, n^k is odd for all $k \geq 0$ and by Example 5.1.3, $\beta(H_n) \leq 1/4$. Since H_n is a symmetric set, Proposition 5.2.4 gives $\beta(H_n) \geq 1/4$. Thus, for odd n, $\beta(H_n) = 1/4$.

If n is even, we denote $x_1 = n/(4n+4)$, $x_2 = (n+2)/(4n+4)$, $x_3 = -x_2$ and $x_4 = -x_1$. Notice that x_1 and x_2 are symmetric about the point $1/4$.
We first deal with the case $n = 4m$ for some $m \in \mathbb{Z}$. We note that

$$nx_1 - x_4 = n^2/(4n+4) + n/(4n+4) = n/4 \in \mathbb{Z}$$

and therefore $nx_1 = x_4$. Likewise,

$$nx_2 - x_1 = (n^2+2n)/(4n+4) - n/(4n+4) = n/4 \in \mathbb{Z}$$

implies $nx_2 = x_1$. Similarly, $nx_3 = x_4$ and $nx_4 = x_1$ follow by symmetry.

Based this observation we can show $\beta(H_n) \leq (n+2)/(4n+4)$. This is because

$nx_1 = x_4$ and $n^2 x_1 = nx_4 = x_1$, therefore

$$\begin{aligned} \mathcal{D}(H_n, x_1) &= \sup_{k \geq 0} \max \left\{ d_{\mathbb{T}}(n^k x_1, 0), d_{\mathbb{T}}(-n^k x_1, 1/2) \right\} \\ &= \max \left\{ d_{\mathbb{T}}(x_1, 0), d_{\mathbb{T}}(-x_1, 1/2), d_{\mathbb{T}}(x_4, 0), d_{\mathbb{T}}(-x_4, 1/2) \right\} \\ &= (n+2)/(4n+4). \end{aligned}$$

Next we show $\beta(H_n) \geq (n+2)/(4n+4)$. Suppose for some y we have

$$\mathcal{D}(H_n, y) < (n+2)/(4n+4).$$

Then, $n^k y \in (x_1, x_2) \cup (x_4, x_3)$ for all $k \geq 0$. So, $y \in (x_1, x_2) \cup (x_4, x_3)$ and

$$ny \in (nx_1, nx_2) \cup (nx_4, nx_3).$$

But

$$(nx_1, nx_2) = (nx_3, nx_4) = (x_4, x_1),$$

which is a contradiction. Hence, $\beta(H_n) = (n+2)/(4n+4)$.

Otherwise, $n = 2(2m+1)$ for some $m \in \mathbb{Z}$. We can show that $nx_1 = x_2$, $nx_2 = x_3$, $nx_3 = x_2$ and $nx_4 = x_3$ and the argument is similar to the above. □

Remark 5.2.7. We have $\beta(\{\pm 1, \pm n\}) = \beta(\left\{\pm n^k : k \geq 0\right\})$ in the n-even case.

Proposition 5.2.8. *Let $A_{a,b} := \{\pm(an+b) : n \in \mathbb{N}\}$ for positive integers $a, b \in \mathbb{N}$. Then $\beta(A_{a,b}) = 1/4$ if $a/\gcd(a,b)$ is even, and $\beta(A_{a,b}) = 1/4 + \gcd(a,b)/(4a)$ if $a/\gcd(a,b)$ is odd.*

Proof. We first notice that we may assume $\gcd(a,b) = 1$ by replacing a and b with $a/\gcd(a,b)$ and $b/\gcd(a,b)$ because

$$\beta(mE) = \beta(E) \ \forall m \in \mathbb{Z} \setminus \{0\}.$$

Case 1: a is even. In this case, since $\gcd(a,b) = 1$, b is odd and hence $A_{a,b}$ only contains odd numbers. Thus, $\beta(A_{a,b}) = 1/4$.

Case 2: a is odd. If $a = 1$, clearly we have $\beta(A_{a,b}) = 1/2$. Hence, we assume $a \geq 3$. Since $\beta(A_{a,b}) = \inf_{x\in\mathbb{T}} \mathcal{D}(A_{a,b}, x)$, we will find the optimal x to minimize $\mathcal{D}(A_{a,b}, x)$. For $x \in \mathbb{T}$ we let

$$G_x := \{(an + b)x : n \geq 0\}.$$

It is not hard to see

$$\mathcal{D}(A_{a,b}, x) = \sup_{y\in G_x} \max \{d_{\mathbb{T}}(y, 0), d_{\mathbb{T}}(y, 1/2)\}.$$

If $|G_x| = \infty$, G_x is dense in $\mathbb{T}$ and thus $\mathcal{D}(A_{a,b}, x) = 1/2$. If $|G_x| = N < \infty$, it is a translation of a finite subgroup of order N in $\mathbb{T}$ and we can observe $\mathcal{D}(A_{a,b}, x) \geq (1 - 1/N)/2$.

We have that the optimal x is a rational number in $\mathbb{T} = [0, 1]$, because irrational x implies $|G_x| = \infty$. We let $x = k/m$ for some integers k, m with $\gcd(k, m) = 1$. Since $\gcd(k, m) = 1$, we have that $|G_x| = m/\gcd(m, ak) = m/\gcd(m, a)$.

If $|G_x| \geq 3$, $\mathcal{D}(A_{a,b}, x) \geq 1/3$. Moreover, if $|G_x| = 1$, then m divides a and hence $\gcd(b, m) \leq \gcd(b, a) = 1$. Since $G_x = \{bk/m\}$, we have that

$$\mathcal{D}(A_{a,b}, x) = \max \left\{ d_{\mathbb{T}}\left(\frac{bk}{m}, 0\right), d_{\mathbb{T}}\left(\frac{bk}{m}, \frac{1}{2}\right) \right\}.$$

Since $\gcd(b, m) = 1$, as k ranges over the integers, bk/m can range over all m-th roots of unity. Thus, the optimal k and m are such that bk/m (mod integers) is closest to $1/4$ (or $3/4$). Hence, the optimal choices are $m = a$ and k satisfies

$$bk = \begin{cases} (a-1)/4 & \text{if } a \equiv 1 \bmod 4 \\ (a+1)/4 & \text{if } a \equiv 3 \bmod 4 \end{cases}.$$

Thus, the minimal $\mathcal{D}(A_{a,b}, x)$ in the case that $|G_x| = 1$ is $1/4 + 1/(4a)$. Notice that $1/4 + 1/(4a) \leq 1/3$ when $a \geq 3$.

It remains to consider x such that $|G_x| = 2$. Then, $m/\gcd(m, a) = 2$ and hence m

divides $2a$. The set G_x is $\{bk/m, bk/m+1/2\}$. Notice that

$$\max\left\{d_{\mathbb{T}}\left(\frac{bk}{m},0\right), d_{\mathbb{T}}\left(\frac{bk}{m},\frac{1}{2}\right)\right\} = \max\left\{d_{\mathbb{T}}\left(\frac{bk}{m}+\frac{1}{2},0\right), d_{\mathbb{T}}\left(\frac{bk}{m}+\frac{1}{2},\frac{1}{2}\right)\right\}$$

and therefore

$$\mathcal{D}(A_{a,b},x) = \max\left\{d_{\mathbb{T}}\left(\frac{bk}{m},0\right), d_{\mathbb{T}}\left(\frac{bk}{m},\frac{1}{2}\right)\right\}.$$

Similar to above, the optimal choice for m is $m = 2a$ and k is such that bk/m is closest to $1/4$. We have that the minimal $\mathcal{D}(A_{a,b},x)$ is $1/4 + 1/(4a)$.

Hence, to summarize, when a is odd, $\beta(A_{a,b}) = 1/4 + 1/(4a)$. $\square$

5.3 A non-symmetrized example

Lastly, we compute one non-symmetrized example.

Proposition 5.3.1. *Let $A_k := \{k^n : n \geq 0\}$, $k \geq 2$. Then $\beta(A_k) = 1/(2k)$.*

Proof. Since the angular Kronecker constant for A_k is $1/(2k)$ ([15]), we have $\beta(A_k) \leq 1/(2k)$. We will show $\beta(A_k) \geq 1/(2k)$.

We write each $x \in [0,1] = \mathbb{T}$ in digits of base k: $x = 0.d_0 d_1 ... d_n ....$ We need to find optimal digits to minimize $D(A_k, f, x)$ for functions $f : A_k \to \{0, 1/2\}$. We have

$$k^m x = d_m/k + d_{m+1}/k^2 + ...$$

mod integers.

First, we assume k is odd. Consider the function $f : A_k \to \{0, 1/2\}$ given by $f(1) = 0$ and $f(k^m) = 1/2$ for all $m \geq 1$.

We claim that if we were to have $D(A_k, f, x) < 1/(2k)$, then d_m could only be $(k-1)/2$ for $m \geq 1$. Indeed, for $m \geq 1$, if we have $d_m/k \geq 1/2 + 1/(2k)$, then

$$D(A_k, f, x) \geq d_{\mathbb{T}}(k^m x, f(k^m)) \geq d_m/k - 1/2 \geq 1/(2k).$$

Or, if $d_m/k \leq 1/2 - 3/(2k)$, we have

$$D(A_k, f, x) \geq d_{\mathbb{T}}(f(k^m), k^m x) = 1/2 - k^m x \geq 3/(2k) - (k-1)(\sum_{j \geq 2} 1/k^j) = 1/(2k).$$

Thus, we only need to consider

$$1/2 - 3/(2k) < d_m/k < 1/2 + 1/(2k)$$

for $m \geq 1$, equivalently, $d_m = (k-1)/2$. Then $x = d_0/k + 1/(2k)$ and it is easy to see $d_{\mathbb{T}}(f(1), x) \geq 1/(2k)$ for all choices of d_0. This shows $D(A_k, f, x) \geq 1/(2k)$ for all $x \in [0, 1]$ and therefore, when k is odd, $\beta(A_k) = 1/(2k)$.

Otherwise k is even. Consider the target function $f(1) = 0$, $f(k) = 1/2$ and $f(k^m) = 0$ for all $m \geq 2$. We claim that if $D(A_k, f, x) < 1/(2k)$, either $d_2 = d_3 = ... = k-1$ or $d_2 = d_3 = ... = 0$. Indeed, similar to above, since $f(k^m) = 0$ for all $m \geq 2$, if $1 \leq d_m \leq k-2$ for some $m \geq 2$, then

$$1/k \leq k^m x \leq \sum_{j \geq 2} (k-1)/k^j + (k-2)/k = (k-1)/k.$$

Hence, $d_{\mathbb{T}}(k^m x, f(k^m)) \geq 1/k > 1/(2k)$.

If $d_2 = 0$ and $d_3 = k-1$, then

$$D(A_k, f, x) \geq d_{\mathbb{T}}(k^2 x, f(k^2)) \geq (k-1)/k^2 \geq 1/(2k)$$

as $k \geq 2$. Hence, if $d_2 = 0$ then $d_3 = 0$. Continuing this way, we have that $d_2 = d_3 = d_4 = ... = 0$. Similarly, if $d_2 = k-1$, then $d_m = k-1$ for all $m \geq 2$.

If $d_m = 0$ for all $m \geq 2$, then $D(A_k, f, x) \leq 1/(2k)$ implies $d_1 = k/2$. Hence, $x = d_0/k + 1/(2k)$ and one can check that $D(A_k, f, x) \geq 1/(2k)$ for all choices of d_0. It is a similar situation for $d_m = k-1$ for all $m \geq 2$. $\square$

Remark 5.3.2. The odd integer set has binary Kronecker constant 1/4 and the angular Kronecker constant is 1/2. In the example of Proposition 5.3.1, the binary Kronecker constant is the same as the angular Kronecker constant. Hence, the bounds, $\beta(E) \leq \alpha(E) \leq 2\beta(E)$, are sharp.

Chapter 6

N-pseudo-Rademacher Sets

6.1 Introduction

Independence is a property prevalent throughout mathematics and has motivated this thesis. Recall that the Rademacher set in the dual of the infinite direct product of infinitely many copies of $\mathbb{Z}_2$ is an independent set of characters. These functions have the property that every ± 1-valued function defined on the set is evaluation at some x in the group. A similar interpolation property holds for all independent sets.

Graham and Hare in [11] introduced the weaker notion of pseudo-Rademacher sets, sets of characters where every ± 1-valued function is point-wise evaluation, in order to study the problem of the existence of Kronecker sets. Galindo and Hernandez in [8] and Graham and Lau in [12] both consider interpolation sets of characters of finite order. For other references and further background information on pseudo-Rademacher sets we refer the reader to [9].

In this chapter, we generalize pseudo-Rademacher sets to N-pseudo-Rademacher sets (or N-PR sets for short), sets of characters with the property that every $\mathbb{Z}_N$-valued function on the set is point-evaluation. Specifically, we recall the following definition that we introduced in chapter 1.

Definition 6.1.1. A set $E \subset \Gamma$ is **N-pseudo-Rademacher** (or N-PR) if for all $\varphi : E \to \mathbb{Z}_N$ there exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$.

Example 6.1.2. As we have seen in chapter 1, typical examples of N-PR sets are the independent sets in $\mathbb{Z}_N^\infty$ (Proposition 1.2.2). More generally, if E is independent and $\mathbb{Z}_N \subset \mathrm{Range}(\gamma)$ for all $\gamma \in E$, then E is N-PR.

Of course, a pseudo-Rademacher set is a 2-PR set. Moreover, if a set is N-PR, it is also M-PR if M divides N.

We first observe that N-PR sets are ε-Kronecker sets for $\varepsilon = \varepsilon(N) = |1 - e^{\frac{\pi i}{N}}|$. Note this tends to 0 as $N \to \infty$. Later in this chapter, the Example 6.2.3 will show that not every ε-Kronecker set is N-PR.

Proposition 6.1.3. *N-PR sets are weak ε-Kronecker for $\varepsilon = |1 - e^{\frac{\pi i}{N}}|$.*

Proof. Assume $E \subset \Gamma$ is N-PR. Let $\varphi : E \to \mathbb{T}$. Define $\varphi_N : E \to \mathbb{Z}_N$ by $\varphi_N(\gamma) = t$, where $t \in \mathbb{Z}_N$ satisfies $|t - \varphi(\gamma)| \le |1 - e^{\frac{\pi i}{N}}|$. Let $x \in G$ be such that $\gamma(x) = \varphi_N(\gamma)$ for $\gamma \in E$. We have $|\gamma(x) - \varphi(\gamma)| \le |1 - e^{\frac{\pi i}{N}}|$. $\square$

The 2-PR sets are I_0 by Theorem 1.4.3. An N-PR set for $N \ge 3$ is at least 1-Kronecker and therefore is also I_0.

In this chapter, we give an arithmetic characterization of N-PR sets (Theorem 6.2.2), describe their structures (Proposition 6.3.5) and prove the existence of large N-PR sets (Theorem 6.4.3). Theorem 6.4.3 gives a new proof that any uncountable subset in Γ contains a large ε-Kronecker set.

For this chapter, important groups are the group of all p^n-th roots of unity, $\mathcal{C}(p^\infty)$, and the group of n-th roots of unity, $\mathbb{Z}_n$.

Notation 6.1.4. Let p be a prime number. The group $\mathcal{C}(p^\infty)$ is the group of all p^n-th roots of unity for $n \ge 1$. That is,

$$\mathcal{C}(p^\infty) = \left\{x \in \mathbb{T} : x^{p^n} = 1 \text{ for some } n \in \mathbb{N}\right\} = \left\{e^{2\pi i j/p^n} : j, n \in \mathbb{N}\right\}.$$

Notice that when we identify $\mathbb{T} = [0, 1]$, the group $\mathcal{C}(p^\infty)$ is identified as

$$\mathcal{C}(p^\infty) = \{k/p^n : n \in \mathbb{N}, 0 \leq k \leq p^n - 1\} \subset [0, 1] = \mathbb{T},$$

and $\mathbb{Z}_n$ is

$$\mathbb{Z}_n = \{k/n : 0 \leq k \leq n - 1\} \subset [0, 1] = \mathbb{T}.$$

6.2 Characterization of N-PR sets

In this section we give an algebraic characterization of N-PR sets. We first establish some useful lemmas.

Lemma 6.2.1. *Let $E \subset \Gamma$, $N \in \mathbb{N}$, and $\Lambda \subset \Gamma$ be a subgroup.*

(1) Let $q : \Gamma \to \Gamma/\Lambda$ be the quotient map. If q is one-to-one on E and $q(E)$ is N-PR, then E is N-PR.

(2) Suppose $E \subset \Lambda$. Then E is N-PR as a subset of Γ if and only if E is N-PR as a subset of Λ.

Proof. (1) Suppose $q : \Gamma \to \Gamma/\Lambda$ is one-to-one on E and $q(E)$ is N-PR. Let $\varphi : E \to \mathbb{Z}_N$ be a function. Because q is one-to-one on E, for each $\gamma, \beta \in E$, if $\beta \neq \gamma$, then $\gamma\beta^{-1} \notin \Lambda$. Thus, we can define $\varphi' : q(E) \to \mathbb{Z}_N$ via $\varphi'(\gamma\Lambda) = \varphi(\gamma)$ for $\gamma \in E$. Since $q(E)$ is N-PR, there exists $x \in \Lambda^\perp = \widehat{\Gamma/\Lambda}$ such that $\varphi'(\gamma\Lambda) = x(\gamma\Lambda)$ for all $\gamma \in E$. As $x \in \Lambda^\perp$, $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$. This means E is N-PR.

(2) We first suppose E is an N-PR subset of Γ. Let $\varphi : E \to \mathbb{Z}_N$ be a function. There exists $x \in G$ such that $\varphi(\gamma) = \gamma(x)$ for all $\gamma \in E$. Let $x\Lambda^\perp \in G/\Lambda^\perp = \widehat{\Lambda}$. Since $E \subset \Lambda$, for all $\gamma \in E$ we have

$$\varphi(\gamma) = \gamma(x) = (x\Lambda^\perp)(\gamma).$$

This means E is N-PR as a subset of Λ. The proof of the converse part of (2) is similar. $\square$

Next, we introduce our main result in this section, a characterization of N-PR sets.

Theorem 6.2.2. *The following are equivalent:*

(1) $E \subset \Gamma$ is N-PR.

(2) If $\gamma_i \in E$ for $1 \leq i \leq n$ are distinct and $\prod_{i=1}^{n} \gamma_i^{m_i} = 1$ for some $m_i \in \mathbb{Z}$, then N divides m_i for all i.

Proof. Suppose (2) fails. Then there exist distinct $\gamma_i \in E$ and integers $m_i \in \mathbb{Z}$ with $\prod_{i=1}^{n} \gamma_i^{m_i} = 1$, while N does not divide m_1. Let $f : E \to \mathbb{Z}_N$ be given by $f(\gamma_1) = e^{2\pi i/N}$ and $f(\gamma) = 1$ for all $\gamma \neq \gamma_1$. For any $x \in G$, $1 = \prod_{i=1}^{n} \gamma_i(x)^{m_i}$, while

$$\prod_{i=1}^{n} f(\gamma_i)^{m_i} = f(\gamma_1)^{m_1} \neq 1.$$

Thus, this function f cannot be interpolated by any $x \in G$ and therefore (1) fails.

Conversely, suppose (2) holds. Let $E_N = \langle \gamma^N : \gamma \in E \rangle$, the subgroup generated by $\left\{\gamma^N : \gamma \in E\right\}$, and $\pi : \Gamma \to \Gamma / E_N$ be the quotient map. Elements in E_N have the form $\gamma_1^{k_1 N} ... \gamma_m^{k_m N}$ for $\gamma_1, ..., \gamma_m \in E$ and $k_1, ..., k_m \in \mathbb{Z}$. We claim that $\pi(E) \subset \pi(\Gamma)$ is independent. Indeed, suppose there are $\gamma_i \in E$ with distinct $\pi(\gamma_i)$ and $m_i \in \mathbb{Z}$ such that $\prod_{i=1}^{n} \gamma_i^{m_i} \in \ker(\pi)$. Then there exists $k_j \in \mathbb{Z}$, $1 \leq j \leq s$ and $s \geq n$ such that

$$\prod_{i=1}^{n} \gamma_i^{m_i} = \prod_{j=1}^{s} \gamma_j^{k_j N}$$

with distinct $\gamma_j \in E$. Hence,

$$\prod_{i=1}^{n} \gamma_i^{m_i - k_i N} \prod_{j=n+1}^{s} \gamma_j^{-k_j N} = 1 \in \Gamma.$$

By (2), $N \mid m_i - k_i N$ for all $1 \leq i \leq n$ and hence $N \mid m_i$ for all $1 \leq i \leq n$. This means each $\gamma_i^{m_i} \in \ker(\pi)$ and therefore the set $\pi(E)$ is independent.

Moreover, we claim that π is one-to-one on E. Suppose, otherwise, that there are $\gamma_1 \neq \gamma_2 \in E$ and $\gamma_1 \gamma_2^{-1} \in \ker(\pi)$. Then there exists $k_j \in \mathbb{Z}$, $1 \leq j \leq s$ and $s \geq 2$

such that

$$\gamma_1\gamma_2^{-1} = \prod_{j=1}^{s} \gamma_j^{k_j N}$$

with distinct $\gamma_j \in E$. We have

$$\gamma_1^{1-k_1N}\gamma_2^{-1-k_2N}\prod_{j=3}^{s}\gamma_j^{-k_jN} = 1.$$

Again, (2) gives $N \mid (1-k_1N)$ and $N \mid (-1-k_2N)$, which are not possible.

Clearly, the elements of $\pi(E)$ have order N. Thus $\pi(E)$ is N-PR from Proposition 1.2.2. Because π is also one-to-one on E, Lemma 6.2.1 gives that E is N-PR, proving (1). $\square$

Now we can deduce that not every ε-Kronecker set in a torsion group is N-PR. Here is an example.

Example 6.2.3. For every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists an infinite ε-Kronecker set E with every $\gamma \in E$ having finite order a multiple of N, but E is not N-PR.

Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $(p_i)_{i=1}^{\infty}$ be an increasing sequence of primes coprime to N. Let $\Gamma = \mathbb{Z}_N \bigoplus \bigoplus_{i\geq 1} \mathbb{Z}_{p_i}$. Let

$$S := \{(1,0,0,...),(1,1,0,0,...),(1,1,1,0,0,...),...\}.$$

One can easily see that elements in S have orders a multiple of N. It is not hard to see that if we exclude finitely many elements of small orders, we have a co-finite ε-Kronecker set $E \subset S$ (See [11] where a similar idea is used). From Theorem 6.2.2 no subsets of S other than singletons are N-PR and therefore E is not N-PR.

We also note that not every p-PR set is an independent set. The following example shows a p-PR set whose only independent subsets are singletons.

Example 6.2.4. Consider $E = \{\gamma_n : n \geq 2\} \subseteq \bigoplus_{i\geq 1} \mathcal{C}(p^\infty) \subset [0,1]^{\mathbb{N}}$, where $\mathrm{Proj}_1(\gamma_n) = 1/p^2$, $\mathrm{Proj}_n(\gamma_n) = 1/p$ and $\mathrm{Proj}_k(\gamma_n) = 0$ for all $k \neq 1, n$. (Recall that the projection is defined in Notation 1.1.4 and here we use additive group operation.) The set E is p-PR by Theorem 6.2.2, but E does not contain any independent subsets other than singletons because for all $i \neq j$, $p\gamma_i = p\gamma_j \neq 1$.

Corollary 6.2.5. *Let $a, b \in \mathbb{N}$ be co-prime. The subset $E \subset \Gamma$ is (ab)-PR if and only if E is both a-PR and b-PR.*

Proof. Since $\mathbb{Z}_a, \mathbb{Z}_b \subset \mathbb{Z}_{ab}$, if E is (ab)-PR, E is both a-PR and b-PR. To see the converse, we assume E is both a-PR and b-PR. Consider $\gamma_1, ..., \gamma_n \in E$ and $m_1, ..., m_n \in \mathbb{Z}$ such that $\prod_{i=1}^n \gamma_i^{m_i} = 1$. Since E is both a-PR and b-PR, $a|m_i$ and $b|m_i$ for all $1 \leq i \leq n$. Since a and b are co-prime, $ab|m_i$ for all $1 \leq i \leq n$. Hence, by Theorem 6.2.2, E is (ab)-PR. □

Similarly to Kronecker sets in $\mathbb{Z}$ (Corollary 4.3.8), N-PR sets are closed under translation assuming suitable hypotheses.

Corollary 6.2.6. *Suppose $E \subset \Gamma$ is N-PR and $\langle E \rangle \cap \langle \gamma \rangle = \{1\}$. Then γE is also N-PR.*

Proof. Let $\gamma_1, ..., \gamma_n \in E$ and $m_1, ..., m_n \in \mathbb{Z}$ be such that $\prod_{i=1}^n (\gamma\gamma_i)^{m_i} = 1$. Since $\langle E \rangle \cap \langle \gamma \rangle = \{1\}$, we have $\prod_{i=1}^n \gamma_i^{m_i} = 1$ and therefore $N|m_i$ for all $1 \leq i \leq n$. This shows γE is N-PR by Theorem 6.2.2. □

6.3 Structure of N-PR sets

In this section, we investigate the structure of N-PR sets. We rely heavily on the following structure theorem for general abelian groups.

Theorem 6.3.1. *[34] Every abelian group Γ is isomorphic to a subgroup of*

$$\bigoplus_{\alpha} \mathbb{Q}_\alpha \bigoplus \bigoplus_{\beta} \mathcal{C}(p_\beta^\infty),$$

where $\mathbb{Q}_\alpha$ are copies of $\mathbb{Q}$ and p_β are prime numbers.

Notation 6.3.2. (1) We let Γ_0 be the torsion subgroup of Γ and $\pi_0 : \Gamma \to \Gamma/\Gamma_0$ be the quotient map. In the notation of Theorem 6.3.1,

$$\pi_0 : \bigoplus_{\alpha} \mathbb{Q}_\alpha \bigoplus \bigoplus_{\beta} \mathcal{C}(p_\beta^\infty) \to \bigoplus_{\alpha} \mathbb{Q}_\alpha$$

is the quotient map.

(2) Let $p \in \mathbb{N}$ be a prime number and $n \in \mathbb{N}$. We let Γ_{p^n} be the subgroup of Γ_0 containing elements whose orders are not a multiple of p^n, or equivalently, whose orders are not divisible by p^n. Let $\pi_{p^n} : \Gamma \to \Gamma/\Gamma_{p^n}$ be the quotient map.

(3) If $\Gamma = \bigoplus_i \mathcal{C}(p_i^\infty)$ with p_i distinct, the map π_{p_i} can be viewed as $\pi_{p_i} = \mathrm{Proj}_i$.

Remark 6.3.3. Notice Γ_{p^n} in (2) above is indeed a subgroup because if γ_1 and γ_2 have orders m_1 and m_2, neither a multiple of p^n, then $\mathrm{lcm}(m_1, m_2)$ is not a multiple of p^n. Hence, the order of $\gamma_1\gamma_2$, which divides $\mathrm{lcm}(m_1, m_2)$, is not a multiple of p^n either.

Example 6.3.4. (1) We identify $\mathbb{T}$ as $\mathbb{T} = [0, 1]$. In the case that $\Gamma = \mathcal{C}(p^\infty) \subset \mathbb{T}$ and $n \in \mathbb{N}$, we note that $\Gamma_p = \{0\}$ and $\Gamma_{p^n} = \langle 1/p^{n-1} \rangle$. Take an element $x = (p + 1)/p^n = 1/p^{n-1} + 1/p^n$. Then, $\pi_{p^n}(x)$ has order p and can be identified as $1/p$. In general, we can identify

$$\begin{aligned} \pi_{p^n} : \mathcal{C}(p^\infty) &\to \mathcal{C}(p^\infty)/\Gamma_{p^n} \cong \mathcal{C}(p^\infty) \\ \pi_{p^n}(x) &= p^{n-1}x. \end{aligned}$$

(2) We still identify $\mathbb{T}$ as $\mathbb{T} = [0, 1]$. In the case that $\Gamma = \mathbb{Q} \bigoplus \mathcal{C}(p^\infty) \bigoplus \mathcal{C}(q^\infty)$ for another prime number $q \neq p$, since π_{p^n} does not affect elements of infinite order, π_{p^n} can be understood as

$$\begin{aligned} \pi_{p^n} : \mathbb{Q} \bigoplus \mathcal{C}(p^\infty) \bigoplus \mathcal{C}(q^\infty) &\to \mathbb{Q} \bigoplus \mathcal{C}(p^\infty) \bigoplus \mathcal{C}(q^\infty) \\ (x, y, z) &\to (x, p^{n-1}y, 0). \end{aligned}$$

(3) If we identify $\mathbb{T}$ as the unit circle group in $\mathbb{C}$, the map

$$\pi_{p^n} : \mathcal{C}(p^\infty) \to \mathcal{C}(p^\infty)/\Gamma_{p^n} \cong \mathcal{C}(p^\infty)$$

is $x \to x^{p^{n-1}}$, where $x = e^{2\pi i j/p^k}$ for some $j, k \in \mathbb{N}$.

Proposition 6.3.5. *Let $E \subset \Gamma$, $n \in \mathbb{N}$ and p be a prime number. The following are equivalent:*

(1) E is p^n-PR.

(2) $\pi_{p^k}(E)$ is p^{n+1-k}-PR and π_{p^k} is one-to-one on E for all $1 \leq k \leq n$.

(3) $\pi_{p^k}(E)$ is p^{n+1-k}-PR and π_{p^k} is one-to-one on E for some $1 \leq k \leq n$.

Proof. We first show (1) implies (2). Fix $1 \leq k \leq n$ and suppose $E \subset \Gamma$ is p^n-PR. To see that π_{p^k} is one-to-one on E, note that if $\gamma_1, \gamma_2 \in E$ have $\gamma_1\gamma_2^{-1} \in \Gamma_{p^k}$, then $\gamma_1\gamma_2^{-1}$ has finite order that is not divisible by p^k. But from Theorem 6.2.2, this implies E is not even p^k-PR and therefore contradicts that E is p^n-PR.

Now, we show that $\pi_{p^k}(E)$ is p^{n+1-k}-PR. We let $\gamma_1, ..., \gamma_s \in E$ and $\beta_i := \pi_{p^k}(\gamma_i)$ for $1 \leq i \leq s$. Suppose for some $m_1, ..., m_s \in \mathbb{Z}$ we have $\prod_{i=1}^s \beta_i^{m_i} = 1$. This means $\prod_{i=1}^s \gamma_i^{m_i} \in \Gamma_{p^k}$ and therefore there exists $l \in \mathbb{N}$, which is not divisible by p^k, such that $\prod_{i=1}^s \gamma_i^{lm_i} = 1$. By Theorem 6.2.2, $p^n | lm_i$ and hence $p^{n+1-k} | m_i$. Theorem 6.2.2 implies $\pi_{p^k}(E)$ is p^{n+1-k}-PR.

Since (2) implies (3) is obvious, it remains to show (3) implies (1). We assume (3) holds for some $1 \leq k \leq n$. Let $\gamma_1, ..., \gamma_s \in E$ be distinct and $m_1, ..., m_s \in \mathbb{Z}$ such that $\prod_{i=1}^s \gamma_i^{m_i} = 1$. Then $\prod_{i=1}^s \pi_{p^k}(\gamma_i)^{m_i} = 1$ and the injectivity of π_{p^k} implies $\pi_{p^k}(\gamma_i)$ are distinct. Since $\pi_{p^k}(E)$ is p^{n+1-k}-PR, we have $p^{n+1-k} | m_i$ for all $1 \leq i \leq s$.

If $n + 1 - k \geq k - 1$, then m_i/p^{k-1} is an integer. We note that order of the product $\prod_{i=1}^s \gamma_i^{m_i/p^{k-1}}$ divides p^{k-1} and therefore $\prod_{i=1}^k \gamma_i^{m_i/p^{k-1}} \in \Gamma_{p^k}$. This means

$$\prod_{i=1}^k \pi_{p^k}(\gamma_i)^{m_i/p^{k-1}} = 1 \in \Gamma/\Gamma_{p^k}.$$

Hence, by Theorem 6.2.2, we have p^{n+1-k} divides m_i/p^{k-1} and this gives p^n divides m_i for all $1 \leq i \leq s$. Theorem 6.2.2 gives (1).

Otherwise, we have $n + 1 - k < k - 1$. Notice that the order of the product

$\prod_{i=1}^{s} \gamma_i^{m_i/p^{n+1-k}}$ divides p^{n+1-k}. Since $n+1-k < k-1$,

$$\prod_{i=1}^{k} \gamma_i^{m_i/p^{n+1-k}} \in \Gamma_{p^k}.$$

As above, we have p^{n+1-k} divides m_i/p^{n+1-k}, which means $p^{2(n+1-k)}|m_i$. We continue doing this until we reach $r(n+1-k) \geq k-1$ for some $r \in \mathbb{N}$. The previous case gives (1). □

Corollary 6.3.6. *Let $N = p_1^{n_1}...p_k^{n_k}$ for distinct primes p_i and $n_i \in \mathbb{N}$. Then E is N-PR if and only if $\pi_{p_i^{n_i}}(E)$ is p_i-PR and $\pi_{p_i^{n_i}}$ is one-to-one on E for all $1 \leq i \leq k$.*

Proof. This follows from Corollary 6.2.5 and Proposition 6.3.5. □

We thus have the following result about the structure of N-PR sets in the torsion subgroup.

Proposition 6.3.7. *Let $E \subset \Gamma_0 \subset \bigoplus_{p \text{ prime}} \mathcal{C}(p^\infty)^{\alpha_p}$ be an N-PR set (α_p are some cardinals) and $N = p_1^{n_1}...p_k^{n_k}$ where the p_i are distinct prime numbers. There exist $p_i^{n_i}$-PR sets $E_i \subset \mathcal{C}(p_i^\infty)^{\alpha_{p_i}}$ for $1 \leq i \leq k$, and bijections $f_2 : E_1 \to E_2$, ..., $f_k : E_1 \to E_k$ such that*

$$E = \{(\gamma, f_2(\gamma), ..., f_k(\gamma), \beta_\gamma) : \gamma \in E_1\}$$

for some $\beta_\gamma \in \bigoplus_{p \neq p_i \ \forall 1 \leq i \leq k} \mathcal{C}(p^\infty)^{\alpha_p}$.

Proof. For $1 \leq i \leq k$, we let $E_i := \pi_{p_i}(E)$ and the quotient map

$$\pi_{p_i} : \bigoplus_{p \text{ prime}} \mathcal{C}(p^\infty)^{\alpha_p} \to \mathcal{C}(p_i^\infty)^{\alpha_{p_i}}$$

will be understood as in Example 6.3.4 (2). Since E is N-PR, from Proposition 6.3.5, each E_i is $p_i^{n_i}$-PR and the maps $\pi_{p_i} : E \to E_i$ are injective. For $2 \leq i \leq k$, we define f_i on E_1 as $f_i(\pi_{p_1}(\gamma)) := \pi_{p_i}(\gamma)$ for $\gamma \in E$. The injectivity of π_{p_1} on E ensures the maps are well-defined. Moreover, the injectivity of π_{p_i} implies f_i is injective. Each f_i

is clearly surjective by its construction and therefore a bijection. Since each $\gamma \in E$ can be represented as

$$\gamma = (\pi_{p_1}(\gamma), ..., \pi_{p_k}(\gamma), \beta_\gamma)$$

for some $\beta_\gamma \in \bigoplus_{p \neq p_i \ \forall 1 \leq i \leq k} \mathcal{C}(p^\infty)^{\alpha_p}$, Proposition 6.3.7 follows. □

We also have the following result about the maximum size of a p-PR set inside a product group.

Proposition 6.3.8. *Suppose $E \subseteq \bigoplus_{i \in B_1} \mathbb{Q} \bigoplus \bigoplus_{i \in B_2} \mathcal{C}(p^\infty)$ is p-PR. Then $|E| \leq |B_1| + |B_2|$.*

Proof. Recall that we can identify each element in $\mathcal{C}(p^\infty)$ in the form of a/p^n for some $a, n \in \mathbb{N}$ with additive group operation. Hence, we can identify $\bigoplus_{i \in B_1} \mathbb{Q} \bigoplus \bigoplus_{i \in B_2} \mathcal{C}(p^\infty)$ as a subset of the real vector space $\mathbb{R}^{|B_1|+|B_2|}$.

Suppose that E is not linearly independent in $\mathbb{R}^{|B_1|+|B_2|}$. Then there exist $\{\gamma_i : 1 \leq i \leq k\} \subseteq E$ and $a_1, ..., a_k \in \mathbb{R}$ such that $a_1\gamma_1 + ... + a_k\gamma_k = 0$, while not all a_i's are zero and each $\gamma_i \in \mathbb{R}^{|B_1|+|B_2|}$ is identified as above. Since the entries of each γ_i are in $\mathbb{Q}$, we may assume $a_i \in \mathbb{Q}$ for all $1 \leq i \leq k$. Furthermore, we may assume $a_i \in \mathbb{Z}$. Notice that if $p|a_i$ for all i, we may replace a_i by a_i/p. Hence, we may find a choice of such a_i's, not all divisible by p. This contradicts that E is p-PR by Theorem 6.2.2.

Thus, $|E| \leq |B_1| + |B_2|$. □

6.4 Existence of N-PR Sets

In this section, we show some existence results about N-PR sets and that large N-PR sets are plentiful. We first notice that if $E \subset \Gamma$ is countable, there may not exist non-trivial N-PR subsets.

Example 6.4.1. Let $E = \mathcal{C}(p^\infty)$, so that E is countably infinite. By Proposition 6.3.8, E only contains p-PR sets that are singletons.

The more interesting case is that $E \subset \Gamma$ is uncountable. We first prove a lemma.

Lemma 6.4.2. *Assume $E \subseteq \bigoplus_{\beta \in B} \Gamma_\beta$ is uncountable.*

(1) Let p be a prime number. If $\Gamma_\beta = \mathcal{C}(p^\infty)$ for all $\beta \in B$, then E contains a p-PR subset of the same cardinality.

(2) If $\Gamma_\beta = \mathbb{Q}$ for all $\beta \in B$, then E contains an independent subset of the same cardinality.

Proof. Without loss of generality, we further assume for each $\beta \in B$ there exists $\gamma \in E$ such that $\mathrm{Proj}_\beta(\gamma)$ is non-trivial. Since each $\gamma \in E$ can only have finitely many $\beta \in B$ such that $\mathrm{Proj}_\beta(\gamma)$ is non-trivial, our assumption implies $|E| = |B|$.

(1) We first prove (1) in the special case that every $\gamma \in E$ has order p. We consider the collection $\mathcal{C}$ of subsets of E defined as $A \in \mathcal{C}$ if for all finite subsets $F \subseteq A$ there exists an arrangement $F = \{\gamma_1, ..., \gamma_n\}$ such that for each $1 \leq k \leq n$ there is some $\beta \in B$ with $\mathrm{Proj}_\beta(\gamma_k)$ non-trivial, but $\mathrm{Proj}_\beta(\gamma_j)$ trivial for all $1 \leq j < k$. We partially order $\mathcal{C}$ by inclusion and Zorn's Lemma gives a maximal $S \in \mathcal{C}$.

We claim $|S| = |E|$. Indeed, if $|S| < |E|$, we let $B_1 \subset B$ be given by $\beta \in B_1$ if there exists $\gamma \in S$ such that $\mathrm{Proj}_\beta(\gamma)$ is non-trivial. We thus have $|B_1| = |S| < |E| = |B|$. Let $\beta_0 \in B \backslash B_1$ and $\gamma_0 \in E$ be such that $\mathrm{Proj}_{\beta_0}(\gamma_0)$ is non-trivial. Since $\beta_0 \in B \backslash B_1$, $\gamma_0 \notin S$ and we form the set $S_1 := S \cup \{\gamma_0\}$. It is easy to see $S_1 \in \mathcal{C}$ and this contradicts the maximality of S.

Moreover, the construction of S, the assumption that every element has order p and Theorem 6.2.2 imply S is p-PR. Indeed, take any finite set $\{\gamma_1, ..., \gamma_n\} \subset S$ and order the elements such that for each $1 \leq k \leq n$ there is some $\beta \in B$ with $\mathrm{Proj}_\beta(\gamma_k)$ non-trivial, but $\mathrm{Proj}_\beta(\gamma_j)$ trivial for all $1 \leq j \leq k$. Suppose for some integers $m_1, ..., m_n \in \mathbb{Z}$ we have $\prod_{k=1}^n \gamma_k^{m_k} = 1$. Let $\beta_n \in B$ satisfy that $\mathrm{Proj}_{\beta_n}(\gamma_n)$

is non-trivial but $\mathrm{Proj}_{\beta_n}(\gamma_k)$ is trivial for all $1 \le k < n$. Therefore,

$$\big(\mathrm{Proj}_{\beta_n}(\gamma_n)\big)^{m_n} = \mathrm{Proj}_{\beta_n}(\gamma_n^{m_n}) = \mathrm{Proj}_{\beta_n}\left(\prod_{k=1}^{n} \gamma_k^{m_k}\right) = \mathrm{Proj}_{\beta_n}(1) = 1.$$

Since γ_n has order p and $\mathrm{Proj}_{\beta_n}(\gamma_n)$ is non-trivial, $\mathrm{Proj}_{\beta_n}(\gamma_n)$ also has order p. Hence, p divides m_n.

We next find $\beta_{n-1} \in B$ such that $\mathrm{Proj}_{\beta_{n-1}}(\gamma_{n-1})$ is non-trivial but $\mathrm{Proj}_{\beta_{n-1}}(\gamma_k) = 1$ for all $1 \le k < n-1$. Since γ_n has order p, $\mathrm{Proj}_{\beta_{n-1}}(\gamma_n)$ has order p or 1. As p divides m_n,

$$\big(\mathrm{Proj}_{\beta_{n-1}}(\gamma_n)\big)^{m_n} = \mathrm{Proj}_{\beta_{n-1}}(\gamma_n^{m_n}) = 1.$$

Hence,

$$\begin{aligned}\big(\mathrm{Proj}_{\beta_{n-1}}(\gamma_{n-1})\big)^{m_{n-1}} &= \mathrm{Proj}_{\beta_{n-1}}(\gamma_{n-1}^{m_{n-1}})\,\mathrm{Proj}_{\beta_{n-1}}(\gamma_n^{m_n}) \\ &= \mathrm{Proj}_{\beta_{n-1}}\left(\prod_{k=1}^{n} \gamma_k^{m_k}\right) = 1.\end{aligned}$$

Since γ_{n-1} has order p, we again have p divides m_{n-1}.

Continue this way and we deduce that p divides m_k for all $1 \le k \le n$. Theorem 6.2.2 therefore implies S is p-PR, which finishes the proof for the special case.

For the general case, we let E_k be defined as the subset in E containing elements whose order divides p^k. Then $E = \bigcup_{k \ge 0} E_k$ and hence there exists a positive integer $K \in \mathbb{N}^+$ such that $|E_K| = |E| = |B|$. If $K = 1$, the special case finishes the proof, and therefore we suppose $K > 1$.

We also recall that the quotient map π_{p^n} on $\bigoplus \mathcal{C}(p^\infty)$ will be understood as $\pi_{p^n} : \bigoplus_B \mathcal{C}(p^\infty) \to \bigoplus_B \mathcal{C}(p^\infty)$ in the manner of Example 6.3.4 (2). That is, if we identify $\mathbb{T} = [0,1]$, $\beta \in B$ and $\gamma \in E$,

$$\mathrm{Proj}_\beta(\pi_{p^n}(\gamma)) = p^{n-1}\,\mathrm{Proj}_\beta(\gamma).$$

We have two cases. The first case is that there exists $1 \leq n_0 < K$ such that $|\pi_{p^{n_0}}(E_K)| = |E_K|$, while $|\pi_{p^{n_0+1}}(E_K)| < |E_K|$. We let $B_1 \subset B$ be given by $\beta \in B_1$ if there exists $\gamma \in E_K$ such that $\mathrm{Proj}_\beta(\gamma)$ has order greater or equal to p^{n_0+1}, and therefore $\pi_{p^{n_0+1}}(\gamma)$ has non-trivial entry at β. Hence,

$$|B_1| \leq \aleph_0 |\pi_{p^{n_0+1}}(E_K)| < |E_K|$$

because $|\pi_{p^{n_0+1}}(E_K)| < |E_K|$ and $|E_K| = |E|$ is uncountable.

For a subset $C \subset B$, we define the projection $\mathrm{Proj}_C : \bigoplus_{\beta \in B} \mathcal{C}(p^\infty) \to \bigoplus_{\beta \in C} \mathcal{C}(p^\infty)$. Since $|B_1| < |E_K| = |\pi_{p^{n_0}}(E_K)|$ and $\mathcal{C}(p^\infty)$ is countable,

$$|\mathrm{Proj}_{B_1}(\pi_{p^{n_0}}(E_K))| \leq |B_1|\aleph_0 < |E_K| = |\pi_{p^{n_0}}(E_K)|.$$

Moreover, we note that

$$|\pi_{p^{n_0}}(E_K)| \leq |\mathrm{Proj}_{B_1}(\pi_{p^{n_0}}(E_K))||\mathrm{Proj}_{B\setminus B_1}(\pi_{p^{n_0}}(E_K))|.$$

Hence, we have

$$|\mathrm{Proj}_{B\setminus B_1}(\pi_{p^{n_0}}(E_K))| = |\pi_{p^{n_0}}(E_K)| = |E_K|.$$

Furthermore, by the construction of the set B_1, $\mathrm{Proj}_{B\setminus B_1}(\gamma)$ has order p^n for some $n \leq n_0$ for each $\gamma \in E_K$. As a result, since we identify the quotient map $\pi_{p^{n_0}}$ as in Example 6.3.4 (2),

$$\mathrm{Proj}_{B\setminus B_1}(\pi_{p^{n_0}}(E_K)) = \pi_{p^{n_0}}(\mathrm{Proj}_{B\setminus B_1}(E_K))$$

only contains elements of order p or 1. Thus, the special case gives a p-PR set

$$E' \subset \mathrm{Proj}_{B\setminus B_1}(\pi_{p^{n_0}}(E_K))$$

such that

$$|E'| = |\mathrm{Proj}_{B\setminus B_1}(\pi_{p^{n_0}}(E_K))| = |E_K| = |E|.$$

Then, Lemma 6.2.1 (1) implies there exists a p-PR subset in $\pi_{p^{n_0}}(E_K)$ of the same cardinality as E (as a one-to-one choice for the pre-image). Proposition 6.3.5 finishes the proof for this case.

The other case is that $|\pi_{p^K}(E_K)| = |E_K|$. Since E_K consists of elements whose order divides p^K, $\pi_{p^K}(E_K)$ only contains elements of order p or 1, and therefore satisfies the special case. A similar argument to above finishes the proof.

(2) The proof of (2) is similar to the first part of the argument of (1). We use Zorn's lemma to obtain a maximal subset $S \subset E$ such that for all finite subsets $F \subseteq S$ there exists an arrangement $F = \{\gamma_1, ..., \gamma_n\}$ such that for each $1 \leq k \leq n$ there is some $\beta \in B$ with $\mathrm{Proj}_\beta(\gamma_k)$ non-trivial, but $\mathrm{Proj}_\beta(\gamma_j)$ trivial for all $1 \leq j < k$. Then, $|S| = |E|$ and it is easy to verify S is independent, since the elements of S have infinite order. $\square$

Theorem 6.4.3. *Let $E \subset \Gamma$ be uncountable. Then there exists a prime number p such that E contains a p-PR set of the same cardinality.*

Proof. Embed $E \subset \bigoplus_{i\in B_0} \mathbb{Q} \bigoplus \bigoplus_{j=1}^{\infty} \bigoplus_{i\in B_j} \mathcal{C}(p_j^\infty)$, where $(B_j)_{j=0}^\infty$ are index sets and p_j are distinct primes. We assume that for each index $i \in \bigcup_{j=0}^\infty B_j$, there exists $\gamma \in E$ such that $\mathrm{Proj}_i(\gamma)$ is non-trivial. Since E is uncountable and the groups $\mathbb{Q}$ and $\mathcal{C}(p_j^\infty)$ are countable, there exists $K \in \mathbb{N}$ such that $|B_K| = |E|$.

If $K = 0$, from Lemma 6.4.2 (2) we may extract an independent set $F \subset \pi_0(E)$ with $|F| = |E|$. If we choose $E' \subset E$ such that π_0 is one-to-one on E' and $\pi_0(E') = F$, then E' is N-PR for all $N \geq 2$ by Lemma 6.2.1 and Proposition 1.2.2.

Similarly, if $K \geq 1$, by Lemma 6.4.2 (1) we extract a p_K-PR subset $F \subset \pi_{p_K}(E)$ with $|F| = |E|$ and therefore obtain a p_K-PR subset in E of the same cardinality. $\square$

Remark 6.4.4. Since any p-PR set is an I_0 set, Theorem 6.4.3 implies that any uncountable subset in Γ contains a large I_0 set of the same cardinality.

Theorem 6.4.5. *Let $E \subset \Gamma$ be uncountable, p be a prime number and $n \in \mathbb{N}$. Then E contains a p^n-PR subset of the same cardinality if and only if $|\pi_{p^n}(E)| = |E|$.*

Proof. If E contains a p^n-PR subset E_1 of the same cardinality, by Proposition 6.3.5 $|\pi_{p^n}(E)| \geq |\pi_{p^n}(E_1)| = |E_1| = |E|$.

Now assume $|\pi_{p^n}(E)| = |E|$. Recall that the quotient map π_{p^n} is understood as in the Example 6.3.4 (2) and we represent

$$\pi_{p^n}(\Gamma) \subset \bigoplus_{i\in B_0} \mathbb{Q} \bigoplus \bigoplus_{i\in B_1} \mathcal{C}(p^\infty)$$

for some index sets B_0 and B_1. We define the projection

$$\pi_1 : \bigoplus_{i\in B_0} \mathbb{Q} \bigoplus \bigoplus_{i\in B_1} \mathcal{C}(p^\infty) \to \bigoplus_{i\in B_1} \mathcal{C}(p^\infty).$$

Either $|\pi_0(\pi_{p^n}(E))| = |E|$ or $|\pi_1(\pi_{p^n}(E))| = |E|$, because otherwise E being uncountable implies

$$|\pi_{p^n}(E)| \leq |\pi_0(\pi_{p^n}(E))||\pi_1(\pi_{p^n}(E))| < |E|,$$

which is a contradiction.

If $|\pi_0(\pi_{p^n}(E))| = |E|$, we appeal to (2) in Lemma 6.4.2 to get a p^n-PR set in $\pi_0(\pi_{p^n}(E))$ and Lemma 6.2.1 finishes the proof. If $|\pi_1(\pi_{p^n}(E))| = |E|$, then (1) in Lemma 6.4.2 similarly finishes the proof. $\square$

Proposition 6.4.6. *Suppose Γ is an uncountable infinite group and $N = p_1^{m_1}...p_k^{m_k}$ is an integer with distinct prime numbers p_i, $1 \leq i \leq k$. Then Γ contains an N-PR set E with $|E| = |\Gamma|$ if and only if $|\pi_{p_i^{m_i}}(\Gamma)| = |\Gamma|$ for all $1 \leq i \leq k$.*

Proof. If Γ contains an N-PR set E with $|E| = |\Gamma|$, then by Corollary 6.3.6 $|\pi_{p_i^{m_i}}(\Gamma)| = |\Gamma|$ for all $1 \leq i \leq k$.

To see the converse, first, suppose $|\pi_0(\Gamma)| = |\Gamma|$. Lemma 6.2.1 implies we may assume Γ is torsion-free. Since Γ is uncountable, the conclusion follows by Lemma 6.4.2 (2), because independent sets in a torsion-free group are N-PR.

Thus we may assume $|\pi_0(\Gamma)| < |\Gamma|$. Hence $|\Gamma_0| = |\Gamma|$ and we may further assume Γ is a torsion group.

Since $|\pi_{p_i^{m_i}}(\Gamma)| = |\Gamma|$ for all $1 \leq i \leq k$, by Lemma 6.4.2 (1) (thinking of $\pi_{p_i^{m_i}}(\Gamma) \subset \bigoplus \mathcal{C}(p_i^\infty)$), we let S_i be a subset in $\pi_{p_i^{m_i}}(\Gamma)$ such that S_i is p_i-PR with $|S_i| = |\Gamma|$. We let $J_i \subset \Gamma$ be such that $\pi_{p_i^{m_i}}$ is one-to-one on J_i and $\pi_{p_i^{m_i}}(J_i) = S_i$.

For each $\gamma \in J_i$, there exists an integer n_γ, only containing prime factors p_j with $j \neq i$, such that the order of γ^{n_γ} is a power of p_i. We let $J_i' := \{\gamma^{n_\gamma} : \gamma \in J_i\}$ and $S_i' = \pi_{p_i^{m_i}}(J_i')$.

As $\pi_{p_i^{m_i}}$ is one-to-one on J_i, if $\gamma_1 \neq \gamma_2 \in J_i$, then $\pi_{p_i^{m_i}}(\gamma_1) \neq \pi_{p_i^{m_i}}(\gamma_2)$. Since p_i does not divide n_{γ_1} and n_{γ_2}, and S_i is p_i-PR, Theorem 6.2.2 implies

$$\pi_{p_i^{m_i}}(\gamma_1^{n_{\gamma_1}}) = \left(\pi_{p_i^{m_i}}(\gamma_1)\right)^{n_{\gamma_1}} \neq \left(\pi_{p_i^{m_i}}(\gamma_2)\right)^{n_{\gamma_2}} = \pi_{p_i^{m_i}}(\gamma_2^{n_{\gamma_2}})$$

and therefore $\gamma_1^{n_{\gamma_1}} \neq \gamma_2^{n_{\gamma_2}}$. This implies $|J_i'| = |J_i|$ and $\pi_{p_i^{m_i}}$ is injective on J_i'.

Moreover, since for $\gamma \in J_i$, p_i does not divide n_γ, Theorem 6.2.2 also implies S_i' is p_i-PR. Hence, by replacing J_i with J_i' and S_i with S_i', we may further assume that for all $\gamma \in J_i$, $1 \leq i \leq k$, the order of γ is a power of p_i.

Since $|J_i| = |\Gamma|$ for all $1 \leq i \leq k$, we let $f_i : J_1 \to J_i$, for $2 \leq i \leq k$, be bijections and we form the set

$$E := \{\gamma f_2(\gamma)...f_k(\gamma) : \gamma \in J_1\}.$$

Then $|E| = |\Gamma|$. For all $\gamma \in J_i$, $1 \leq i \leq k$, the order of γ is a power of p_i. Thus, if $\gamma \in J_1$, then $f_i(\gamma) \in \Gamma_{p_1^{m_1}}$ for all $2 \leq i \leq k$ and hence

$$\pi_{p_1^{m_1}}(\gamma f_2(\gamma)...f_k(\gamma)) = \pi_{p_1^{m_1}}(\gamma).$$

Similarly,

$$\pi_{p_i^{m_i}}(\gamma f_2(\gamma)...f_k(\gamma)) = \pi_{p_i^{m_i}}(f_i(\gamma)),$$

for all $2 \leq i \leq k$. Hence, $\pi_{p_i^{m_i}}(E) = \pi_{p_i^{m_i}}(J_i) = S_i$ for $1 \leq i \leq k$. By Corollary 6.3.6, E is N-PR. $\square$

In [11] the terminology "N-large" sets is introduced.

Definition 6.4.7. Let $N \in \mathbb{N}$, $H_N \subset \Gamma$ be the subgroup of elements of orders dividing N and $Q_N : \Gamma \to \Gamma/H_N$ be the quotient map. A set $E \subset \Gamma$ is N**-large** if $|Q_N(E)| < |E|$.

Example 6.4.8. Let B_1 and B_2 be two countable infinite index sets. Consider $\Gamma = \bigoplus_{j\in B_1} \mathbb{Z}_2 \bigoplus \bigoplus_{j\in B_2} \mathbb{Z}_3$.
(1) Suppose $E_1 \subset \Gamma$ is given by $\gamma \in E_1$ if and only if $\mathrm{Proj}_j(\gamma)$ is non-trivial for one $j \in B_1$ and no $j \in B_2$. Since $Q_2(E_1) = \{1\}$ and $Q_3(E_1) = E_1$, the set E_1 is 2-large, but not 3-large.
(2) If we define E_2 analogously and put $E = E_1 \cup E_2$, we have that E is neither 2-large nor 3-large.
(3) Finally, we note that no infinite set E in Γ can be both 2-large and 3-large. This is because $|E| \leq |Q_2(E)||Q_3(E)|$ and hence either $|Q_2(E)| = |E|$ or $|Q_3(E)| = |E|$.

The following is one of the main theorems (Theorem 2.2 (2)) in [11].

Theorem 6.4.9. *If $E \subset \Gamma$ and N is the smallest integer for which E is N-large, then for all primes powers p^n dividing N there exists a weak $|1 - e^{\pi i/p^n}|$-Kronecker subset $F \subset E$ with $|F| = |E|$.*

Suppose $E \subset \Gamma$ is uncountable. We recall that in Theorem 6.4.5 we prove that if $|\pi_{p^n}(E)| = |E|$ for some prime number p and $n \in \mathbb{N}$, then E contains a p^n-PR subset of the same cardinality. We claim this result is stronger than Theorem 6.4.9 for uncountable sets.

First, we note that the assumption made in Theorem 6.4.5 is weaker than the assumption in Theorem 6.4.9: specifically, if E is infinite and N-large for minimal $N = p^m p_1^{m_1} ... p_k^{m_k}$, and if $n \leq m$ for some integer n, then $|\pi_{p^n}(E)| = |E|$. To see this, we argue by contradiction and assume $|\pi_{p^n}(E)| < |E|$. Let $M := p^{n-1} p_1^{m_1} ... p_k^{m_k} < N$. If $\gamma \in \Gamma_{p^n} \cap H_N$ and k is the order of γ, then k divides N, while p^n does not divide k. This implies k divides M and hence $\gamma \in H_M$. Thus $\Gamma_{p^n} \cap H_N = H_M$. Consider the map

$$T : Q_M(E) \to Q_N(E) \times \pi_{p^n}(E)$$
$$T(Q_M(\gamma)) = (Q_N(\gamma), \pi_{p^n}(\gamma))$$

This map is well-defined and injective, because $Q_M(\gamma_1) = Q_M(\gamma_2)$ if and only if $\gamma_1\gamma_2^{-1} \in H_M$, which is equivalent to

$$\gamma_1\gamma_2^{-1} \in \Gamma_{p^n} \cap H_N.$$

Hence, $Q_M(\gamma_1) = Q_M(\gamma_2)$ is equivalent to

$$(Q_N(\gamma_1), \pi_{p^n}(\gamma_1)) = (Q_N(\gamma_2), \pi_{p^n}(\gamma_2)).$$

Thus, if $|\pi_{p^n}(E)| < |E|$, then

$$|Q_M(E)| \leq |Q_N(E)||\pi_{p^n}(E)| < |E||E| = |E|$$

for infinite E. Thus E is M-large and this contradicts the assumption that N is minimal.

Moreover, recall that p^n-PR sets are special weak $|1 - e^{\pi i/p^n}|$-Kronecker sets. This shows Theorem 6.4.5 improves Theorem 6.4.9 when E is uncountable.

Bibliography

[1] C. Bennett, R. C. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.

[2] R. P. Boas, *Inequalities for monotonic series*, J. Mathematical Analysis and Applications, 1: 121-126, 1960.

[3] J. Bourgain, *Proprietes de decomposition pour les ensembles de Sidon.*, Bull. Soc. Math. France, 111(4):421–428, 1983.

[4] J. Bourgain, *Subspaces of ℓ_N^∞, arithmetical diameter and Sidon sets*, In Probability in Banach spaces, V (Medford, Mass., 1984), volume 1153 of Lecture Notes in Math., pages 96–127, Berlin, Heidelberg, New York, 1985.

[5] J. Bourgain, *Sidon sets and Riesz products*, Ann. Inst. Fourier(Grenoble), 35(1):137-148, 1985.

[6] S. Drury, *Sur les ensembles de Sidon*, C. R. Acad. Sci. Paris, 271A: 162-163, 1970.

[7] R. M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Functional Analysis, 1: 290–330, 1967.

[8] J. Galindo, S. Hernandez, *The concept of boundedness and the Bohr compactification of a MAP Abelian group*, Fundamenta Mathematicae, 159(3):195–217, 2010.

[9] C. Graham, K. Hare, *Interpolation and Sidon Sets for Compact Groups*, CMS Books in Math. Series, Springer, New York, 2013.

[10] C. Graham, K. Hare, *Characterizing Sidon sets by interpolation properties of subsets*, Colloq. Math., 112(2):175–199, 2008.

[11] C. Graham, K. Hare, *Existence of Large ε-Kronecker and FZI_0 sets in discrete abelian groups*, Colloq. Math., 127(1): 1-15, 2012.

[12] C. Graham, A. Lau, *Relative weak compactness of orbits in Banach spaces associated with locally compact groups*, Transactions Amer. Math. Soc., 359(3): 1129-1160, 2007.

[13] D. Grow, *Sidon Sets and I_0 Sets*, Colloq. Math., 53: 269-270, 1987.

[14] K. Hare, T. Ramsey, *The Relationship Between ε-Kronecker Sets and Sidon Sets*, Canad. Math. Bull., 59: 521-527, 2016.

[15] K. Hare, T. Ramsey, *Exact Kronecker Constants of Hadamard Sets*, Colloq. Math., 130: 39-49, 2013.

[16] K. Hare, T. Ramsey, *Kronecker Constants of Arithmetic Progressions*, Experimental Math., 23(4): 414-422, 2014.

[17] S. Hartman, *Interpolation par les mesures diffuses*, Colloq. Math., 26: 339–343, 1972.

[18] S. Hartman, C. Ryll-Nardzewski, *Almost periodic extensions of functions*, Colloq. Math., 12: 23–39, 1964.

[19] E. Hewitt and S. Kakutani, *A class of multiplicative linear functionals on the measure algebra of a locally compact Abelian group*, Illinois J. Math., 4: 553-574, 1960.

[20] J. Kahane, *Some Random Series of Functions*, Cambridge University Press, Cambridge, 1993.

[21] A. Khintchine, *Uber dyadische Bruche*, Math. Z., 18: 109-116, 1923.

[22] M. A. Krasnoselskii, Y. B. Rutitskii, *Convex functions and Orlicz spaces*, P. Noordhoff, Groningen Batavia, 1961.

[23] K. Kunen, W. Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Camb. Phil. Soc., 126: 117-137, 1999.

[24] D. Li, H. Queffelec, *Introduction à l'étude des espaces de Banach : Analyse et probabilités*, SMF, Paris, 2005.

[25] J. Lopez, K. Ross, *Sidon Sets*, Marcel Dekker Inc, New York, 1975.

[26] M. B. Marcus, *The ε-entropy of some compact subsets of ℓ_p*, J. of Approximation Theory, 10: 304-312, 1974.

[27] J. Mela, *Sur les ensembles d'interpolation de C. Ryll-Nardzewski et de S. Hartman*, Studia Math., 29: 168–193, 1968.

[28] A. Pajor, *Plongement de ℓ_1^n dans les espaces de Banach complexes*, C. R. Acad. Sci. Paris, 296: 741–743, 1983.

[29] G. Pisier, *Arithmetic characterizations of Sidon sets*, Bull. Amer. Math. Soc., 8(1): 87-89, 1983.

[30] G. Pisier, *Ensembles de Sidon et process Gaussiens*, C. R. Acad. Sci. Paris, 286(15): A671-A674, 1978.

[31] G. Pisier, *Conditions dentropie et caracterisations arithmetique des ensembles de Sidon*, In Proc. Conf. on Modern Topics in Harmonic Analysis, 911–941, 1982.

[32] T. Ramsey, *Comparisons of Sidon and I_0 sets*, Colloq. Math., 70(1): 103–132, 1996.

[33] D. Rider, *Randomly continuous functions and Sidon sets*, Duke Mathematical J., 42(4), 1975.

[34] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Grad. Texts in Math. 148, Springer, New York, 1995.

[35] W. Rudin, *Fourier Analysis on Groups*, Wiley Interscience, New York, N. Y., 1962.

[36] W. Rudin, *Trigonometric Series with Gaps*, Indiana Univ. Math. J., 9(2): 203–227, 1960.

[37] N. Varopoulus, *Tensor algebras over discrete spaces*, J. Functional Anal., 3: 321-335, 1969.

[38] B. Wells, *Restrictions of Fourier transforms of continuous measures*, Proc. Amer. Math. Soc., 38: 92–94, 1973.