Invariant manifold theory for impulsive functional differential equations with applications

by

Kevin Church

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

   The primary contribution of this thesis is a development of invariant manifold theory for impulsive functional differential equations. We begin with an in-depth analysis of linear systems, immersed in a nonautonomous dynamical systems framework. We prove a variation-of-constants formula, introduce appropriate generalizations of stable, centre and unstable subspaces, and develop a Floquet theory for periodic systems. Using the Lyapunov-Perron method, we prove the existence of local centre manifolds at a nonhyperbolic equilibrium of nonlinear impulsive functional differential equations. Using a formal differentiation procedure in conjunction with machinery from functional analysis – specifically, contraction mappings on scales of Banach spaces – we prove that the centre manifold is smooth in the state space. By introducing a coordinate system, we are able to prove that the coefficients of any Taylor expansion of the local centre manifold are unique and sufficiently regular in the time and lag arguments that they can be computed by solving an impulsive boundary-value problem. After proving a reduction principle, this leads naturally to explorations into bifurcation theory, where we establish generalizations of the classical fold and Hopf bifurcations for impulsive delay differential equations. Aside from the centre manifold, we demonstrate the existence and smoothness of stable and unstable manifolds and prove a linearized stability theorem.

   One of the applications of the theory above is an analysis of a SIR model with pulsed vaccination and finite temporary immunity modeled by a discrete delay. We determine an analytical stability criteria for the disease-free equilibrium and prove the existence of a transcritical bifurcation of periodic solutions at some critical vaccination coverage level for generic system parameters. Then, using numerical continuation and a monodromy operator discretization scheme, we track the bifurcating endemic periodic solution until a Hopf point is identifed. A cylinder bifurcation is observed; the periodic orbit expands into a cylinder in the extended phase space before eventually contracting onto a periodic orbit as the vaccination coverage vanishes.

   The other application is an impulsive stabilization method based on centre manifold reduction and optimization principles. Assuming a cost structure on the impulsive controller and a desired convergence rate target, we prove that under certain conditions there is always an impulsive controller that can stabilize a nonhyperbolic equilibrium with a trivial unstable subspace, robustly with respect to parameter perturbation, while guaranteeing a minimal cost. We then exploit the low-dimensionality of the centre manifold to develop a two-stage program that can be implemented to compute the optimal controller. To demonstrate the effectiveness of the two-stage program, which we call the centre probe method, we use the method to stabilize a complex network of 100 diffusively coupled nodes

at a Hopf point. The cost structure is one that assigns higher cost to controlling of nodes that have more neighbours, while the jump functionals are required to be diagonal – that is, they do not introduce further coupling. We also introduce a secondary goal, which is that the number of nodes that are controlled is minimized.

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For Natalie.

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List of Symbols

Symbol	Definition
RFDE	retarded functional differential equation
CPM	centre probe method
LPCC	local probe-compatible cost
GPCC	global probe-compatible cost
$\mathcal{RCR}(I, X)$	right-continuous regulated functions from interval I to metric space X
$\mathcal{G}(I, X)$	regulated functions from interval I to metric space X
$\mathcal{RCR}$	the Banach space $\mathcal{RCR}([-r, 0], \mathbb{R}^n)$ equipped with the supremum norm
χ_s	identity-valued indicator function; see (3.6)
x_t	solution history $x_t(\theta) := x(t + \theta)$
x_{t^-}	pointwise left-limit history; see (1.4)
x_t^-	regulated left-limit history; see (2.4)
X^η	exponentially bounded space; introduced in Section 6.1
$X^{\eta, \pm s}$	one-sided exponentially bounded space; introduced in Section 9.1

Chapter 1

Introduction

1.1 Brief background on impulsive retarded functional differential equations

In this thesis we are interested in the dynamics near periodic solutions and equilibrium points of the nonlinear impulsive retarded functional differential equation (impulsive RFDE)

$$\dot{x} = f(t, x_t), \qquad\qquad\qquad t \neq t_k \qquad\qquad (1.1)$$
$$\Delta x = g(k, x_{t^-}), \qquad\qquad\qquad t = t_k. \qquad\qquad (1.2)$$

Here, $f : \mathbb{R} \times X \to \mathbb{R}^n$ and $g : \mathbb{Z} \times X \to \mathbb{R}^n$ are functions with some prescribed amount of regularity, $X \subset F_0$ is a vector subspace of F_0 defined by

$$F_0 = \left\{ \phi \in F([-r, 0], \Omega) : \lim_{s \to 0^-} \phi(s) \text{ exists} \right\} \subset F,$$

where $F([-r, 0], \Omega)$ is the set of all functions mapping the interval $[-r, 0]$ into $\Omega \subseteq R^n$, and $\{t_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ is a bi-infinite sequence or real numbers called *impulse times*. If $x : I \to \Omega$ for some interval $I \subset \mathbb{R}$ and $[t - r, t] \subseteq I$, we define the *history at time t* to be the function $x_t : [-r, 0] \to \Omega$ defined by

$$x_t(\theta) = x(t + \theta). \qquad\qquad (1.3)$$

Similarly, the *pointwise left-limit at time t* is

$$x_{t^-}(\theta) = \begin{cases} x_t(\theta), & \theta < 0 \\ \lim_{s \to 0^-} x(t + s), & \theta = 0. \end{cases} \qquad\qquad (1.4)$$

In this way, equation (1.2) is understood to mean that at times $t = t_k$,

$$\Delta x(t_k) = x(t_k) - x(t_k^-) = g(t_k, x_{t_k^-}),$$

where $x(t_k^-) = \lim_{s \to t_k^-} x(s)$.

Equations of the type (1.1)–(1.2) are useful in modelling systems that exhibit delays and memory effects in conjunction with jumps in state on very small time scales. These jumps are formally treated as occurring instantaneously. A simple class of impulsive RFDE is those with two discrete delays, which can be written in the form

$$\dot{x} = f(t, x(t), x(t - r_1), x(t - r_2)), \qquad\qquad t \neq t_k \qquad\qquad (1.5)$$
$$\Delta x = g(k, x(t^-), x(t - r_1), x(t - r_2)), \qquad\qquad t = t_k. \qquad\qquad (1.6)$$

In this way, one can see that the evolution law depends on the current state $x(t)$ as well as the states $x(t - r_1)$ and $x(t - r_2)$ at earlier times, r_1 and r_2 units of time in the past. The first equation (1.5) is often called a *differential-difference equation* in the literature. For example, the *Hutchinson equation* (1948) – see the survey [77] for this and related models – features a single discrete delay and models the growth of a single egg-laying species in which births occur τ units of time after oviposition. The result is the differential-difference equation

$$\frac{dx}{dt} = rx(t)\left[1 - \frac{x(t - \tau)}{N}\right],$$

where $r > 0$ is an intrinsic (i.e. linear-order) growth rate and $N > 0$ is the carrying capacity. If harvesting of this species occurs linearly at fixed times t_k and there is a *census delay* of $c > 0$ units of time between completing a population census and performing the harvesting, one obtains the impulsive delay differential equation

$$\dot{x} = rx(t)(1 - x(t - \tau)/N), \qquad\qquad t \neq t_k \qquad\qquad (1.7)$$
$$\Delta x = -hx(t - c), \qquad\qquad t = t_k, \qquad\qquad (1.8)$$

where $h \in [0, 1]$ is a harvesting effort representing the target proportion of the population to be harvested. Notice that there are two distinct delays: the incubation time τ and census delay c. If instead the census estimation takes the form of a time average over a period $c > 0$, then the jump in equation (1.8) would be replaced by

$$\Delta x = -\frac{h}{c} \int_{-c}^{0} x_{t^-}(s)ds. \qquad\qquad (1.9)$$

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Integral terms such as those appearing in (1.9) are usually referred to as *distributed delays*, and in general they could appear in the continuous-time part of the equation (1.1).

A function $x : [\alpha - r, \beta) \to \Omega$ could be broadly called a *solution* of the impulsive RFDE (1.1)–(1.2) if it satisfies the functional differential equation (1.1) in an appropriate sense (e.g. classically or in an integrated form) for all $t \in [\alpha, \beta)$ except possibly at times $t_k \in (\alpha, \beta)$, where it satisfies the functional difference equation (1.2).

When introducing initial-value problems, the choice of the space X becomes important and much of the early work in impulsive RFDE seems devoted to investigations into two interrelated questions.

1. What choice of X will ensure that solutions can be discontinuous enough to incorporate impulses and discontinuous initial data?

2. What specifications must be made on the vector field f so that discontinuities do not cause difficulties in establishing existence, uniqueness and continuability of solutions?

Some initial progress on this front was made by Krishna and Anokhin [53] in the context of discrete delays. The situation in that paper is far from general, however, because the vector field is of the form

$$\dot{x} = f(t, x(h(t))),$$

where $h(t) \leq t$ represents a time-varying delay. A general well-posedness result followed in 2000 with a paper by Ballinger and Liu [11]. Therein, the space X consists of functions that are continuous except at finitely many points, where they are continuous from the right and have limits on the left, while the function f is assumed to satisfy Carathéodory-type conditions and to have the property that the composition $t \mapsto f(t, x_t)$ is Lebesgue measurable for any suitable function $x : I \to \mathbb{R}^n$ that satisfies the same piecewise continuity properties. With such assumptions, they prove existence and uniqueness results for solutions defined in the extended Carathéodory sense. An earlier paper by the same authors [10] requires stronger assumptions on the function f, but guarantees a classical notion of solution. These stronger conditions include many of the types of functional dependence that are typical of applications. In both cases, the key idea is that the composition

$$t \mapsto f(t, x_t)$$

must itself satisfy a certain level of regularity whenever $x : I \to \mathbb{R}^n$ is piecewise continuous and does not have too many discontinuities.

With existence and uniqueness issues essentially resolved, development into Lyapunov stability and oscillation of impulsive RFDE (which we will not review here) continued to

develop. Despite this, our literature review indicates that dynamical systems aspects of such systems – in particular invariant manifold theory and related topics – has lagged behind. For example, analysis of specific nonlinear impulsive systems with delays appears to be mostly confined to more static notions such as well-posedness, permanence, existence of global attractors and binary stability-instability analysis of equilibrium points. There are also some results concerning permanence of compact regions of the phase space conditional on parameters — see [36, 73, 96, 103] for some recent applications to biological systems. Most dynamic bifurcation analysis at present seems restricted to numerical studies. For instance, in [105], the largest Lyapunov exponent is used to numerically investigate bifurcations to chaotic attractors in a three-species food chain model with distributed delay and impulsive control.

To understand why development of methods for the analysis of bifurcations in impulsive RFDE has been seemingly stalled, it is beneficial to first review standard methods of analysis of finite-dimensional impulsive systems.

1.2   Finite-dimensional systems

When delays and other functional dependence are absent, we have the finite-dimensional system

$$\dot{x} = f(t, x(t)), \qquad\qquad\qquad t \neq t_k \qquad\qquad (1.10)$$
$$\Delta x = g(k, x(t^-)), \qquad\qquad\qquad t = t_k, \qquad\qquad (1.11)$$

where now $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ can taken to be smooth in their second argument, while $t \mapsto D_2^j f(t, x)$ is continuous except at impulse times t_k. Some of the most highly-cited monographs [8, 78] on impulsive differential equations take the convention that solutions are continuous from the left, but here we will take the right-continuity convention since we wish to draw comparisons to systems with delays (and there, the convention is overwhelmingly that of right-continuity). Notions of existence and uniqueness of solutions in this case are very similar to those from ordinary differential equations, and we refer to the aforementioned monographs for details.

In the majority of applications, the vector field, impulse times and jump functions are periodic – that is, there exists $c > 0$ and $T > 0$ such that $f(t + T, \cdot) = f(t, \cdot)$ for all $t \in \mathbb{R}$, $t_{k+c} = t_k + T$ and $g(k + c, \cdot) = g(k, \cdot)$ for all $k \in \mathbb{Z}$. In this case, there is very little motivation to consider topics such as invariant manifold theory for these systems specifically, as one can simply reduce to a discrete time problem. Indeed, if a periodic

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solution of period T is known, one can first translate it to the origin by a time-dependent change of coordinates so that the origin becomes a fixed point. Following this, one defines a time T map on a suitable neighbourhood of $0 \in \mathbb{R}^n$ as follows: let $t \mapsto S(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}^n$ denote the solution of (1.10)–(1.11) satisfying $S(0, x) = x$. The *time T map* (also called *stroboscopic map* or *Poincaré map*) is a nonlinear map $P : U \to \mathbb{R}^n$ defined on a neighbourhood of $0 \in \mathbb{R}^n$ by the formula $P(x) = S(T, x)$. One then considers the iterated map (i.e. discrete time dynamical system)

$$x \mapsto P(x) \tag{1.12}$$

having zero as a fixed point (recall, the periodic solution was translated to the origin). Bifurcations of the fixed point can then be studied using methods from discrete time systems, thereby circumventing the need to view the impulsive system (1.10)–(1.11) as a continuous-time system. Some of the earliest applications of this idea include chemotherapy modelling [57], pest control [34], pulsed chemostat [101] and infectious disease modelling with pulsed vaccination [48]. This method was surveyed and formalized by Church and Liu [19] in 2017.

Moving away from periodicity, one must consider (1.10)–(1.11) as a fully nonautonomous object. This introduces further complications, most fundamental being that notions of hyperbolicity become harder to both deal with and verify in practice. Some progress has been made on nonautonomous bifurcations in scalar impulsive systems by Akhmet and Kashkybayev [2, 3, 5] using notions of nonautonomous bifurcation and attractor transition – we refer to the monograph [75] of Martin Rasmussen for background on these ideas. Church and Liu [18] consider bifurcations of bounded solutions from complete trajectories having double half-line exponential dichotomies, which are typical of trajectories corresponding to homoclinic orbits. Global bifurcation theory has also been used to study the structure of solutions sets of certain boundary-value problems [64, 69]. However, aside from some very special cases, the utility of these results is fairly limited. We mention them mostly to contrast with the periodic case and to emphasize that the fully nonautonomous setting continues to be mostly unexplored. It will not be the main focus of this thesis.

1.3 Infinite-dimensional systems

We have demonstrated that in the finite-dimensional case, there is little to no reason to develop invariant manifold theory. This is because in the periodic case, one can reduce to a discrete time problem. In principle, this remains true if the impulsive RFDE (1.1)–(1.2) is periodic – that is, there exists $c > 0$ and $T > 0$ such that $f(t + T, \cdot) = f(t, \cdot)$ for all

$t \in \mathbb{R}$, $t_{k+c} = t_k + T$ and $g(k + c, \cdot) = g(k, \cdot)$ for all $k \in \mathbb{Z}$. One can once again define a time T map and presumably study bifurcations by considering instead the discrete-time system (1.12). However, there are a few key elements that would be needed in order to accomplish this.

To begin, in order to consider the time T map associated to the impulsive RFDE (1.1)–(1.2), one must decide on a phase space X. It would be ideal if X were a Banach space and the resulting map $P : X \to X$ were reasonably smooth, since in a parameter-dependent setting one could proceed using a Lyapunov-Schmidt reduction to unfold bifurcations at a suitably-defined nonhyperbolic equilibrium point or periodic solution.

Next, to study the bifurcation equation it would be necessary to compute higher-order derivatives of P at zero. In the finite-dimensional setting, doing this requires solving a sequence of inhomogeneous linear impulsive differential equations. Specifically, one has an expansion of the form

$$P(x) = P_1(T)x + \frac{1}{2}P_2(T)[x, x] + \frac{1}{6}P_3(T)[x, x, x] + O(||x||^4)$$

where $P_j(t)$ is the jth Fréchet derivative of the solution map $S(t, x)$ at $x = 0$. This is a j-linear operator on $\mathbb{R}^n$ that solves a particular inhomogeneous linear impulsive differential equation, the latter of which typically contains inhomogeneities involving all $P_i(t)$ for $i < j$; see [19] for details. Generalizing appropriately to the infinite-dimensional setting, it would be necessary to solve a sequence of abstract impulsive differential equation in the spaces $\mathcal{L}^j(X)$ of j-linear maps on X. Ths would seem to be a difficult task, and as we will see it is fruitful to avoid the temptation of reducing the analysis entirely to a discrete time setting.

1.4   Dynamical systems perspective

It is the stance of the present author that the single largest reason bifurcation theory for impulsive RFDE had been undeveloped is because there was no rigorous formulation of these equations as defining dynamical systems in any appropriate sense. This claim is supported by the observation that a typical way of proving the existence of invariant manifolds – in particular the centre manifold – is by the Lyapunov-Perron method. The idea behind the Lyapunov-Perron method is one identifies an invariant manifold as a perturbation from an appropriate invariant subspace (e.g. stable manifolds are perturbations of stable subspaces). The manifold itself ends up being a fixed point of a particular nonlinear operator that is derived in part using a linear variation of constants formula. Perhaps

surprisingly, such a variation of constants formula for impulsive RFDE had yet to appear in the literature when the research associated to this dissertation began.

This said, there have been phantoms of dynamical systems ideas being applied to impulsive RFDE. For example, Bachar and Arijno [7] demonstrated that a linear delay differential equation with impulses in a general Banach Z space generates an integrated semigroup of bounded linear operators acting on the Banach space of regulated functions mapping the history interval $[-r, 0]$ into Z. The idea seems to have not caught on given the lack of citations, but it is similar in spirit to how we view impulsive RFDE in this thesis.

1.5 Outline of this thesis

Our first goal is to build a solid dynamical systems foundation on which we can explore invariant manifold theory and bifurcation theory for impulsive RFDE. Chapter 2 is devoted to some fundamental properties of what will become the phase space for our nonautonomous process (i.e. nonautonomous analogue of a semidynamical system): the space of right-continuous regulated functions. This normed vector space is introduced and we elaborate on its completeness, dense subspaces and topological dual, among other topics. We also review weak integration as it will be essential in later sections. We introduce notions of left limits of such regulated functions and provide some useful integral inequalities.

Chapter 3 is devoted to general linear systems. The fundamental result of this section is an abstract variation of constants formula for inhomogeneous systems in the phase space of right-continuous regulated functions. To arrive there, we first review existence and uniqueness of solutions as it applies to our choice of phase space. We then introduce the evolution family associated to a homogeneous system, invariant fiber bundles and spectral separation.

In Chapter 4 we explore periodic linear systems, specifically homogeneous systems. We introduce the monodromy operator and prove its compactness, thereby obtaining the Riesz formula for the projection onto its invariant subspaces. Following this we prove that homogeneous periodic systems are always spectrally separated, prove a Floquet theorem and explore notions of stability through Floquet exponents. We conclude with some computational aspects.

Chapter 5 is the beginning of our study of nonlinear systems. Through the linear variation of constants formula we define mild solutions and compare these to classical solutions. We then prove that mild solutions are smooth with respect to initial conditions,

derive a variational equation and use this result to ultimately prove a linearized stability principle.

Chapters 6 through Chapter 8 are devoted to the development of centre manifold theory for impulsive RFDE. The first, Chapter 6, provides the most general results. We prove the existence of local centre manifolds (specifically, local invariant fibre bundles) near non-hyperbolic equilibria under spectral separation assumptions for general nonautonomous impulsive RFDE. We prove that the centre manifold contains all small solutions and is attracting in the absence of unstable directions. The coordinate dynamics on the centre manifold are derived both in an integrated form and, under some mild technical assumptions, as an abstract ordinary impulsive differential equation. This dynamics equation is later put into a more concrete form in Chapter 11.

Regularity of local centre manifolds is considered in Chapter 7. We assume slightly more conditions on the evolution family and first prove that local centre manifolds are smooth in the phase space variable in Section 7.1. This is enough to perform a Taylor expansion of the centre manifold near a nonhyperbolic equilibrium, but as the centre manifold is fibre bundle and therefore time-varying, we also need to consider the regularity of the coefficients of this expansion with respect to time. This is done in Section 7.2 for systems with fixed discrete delays, but we elaborate on how one might extend the result to more general functional dependence.

Approximation of local centre manifolds is the main focus of Chapter 8. We develop the notion of a Euclidean space representation of the centre manifold and recognize that any local centre manifold must admit the same Taylor expansion at the candidate nonhyperbolic equilibrium, thereby allowing us to justify the abuse of notation earlier in this sentence where we referred to "the" centre manifold. The dynamics on the centre manifold are made concrete using the Euclidean space representation, and we derive an evolution equation satisfied by any local centre manifold, including boundary conditions. We show how one can compute the quadratic coefficient of the centre manifold using an extension of the method of characteristics, commenting briefly on higher-order terms. We provide two concrete examples in Section 8.3 to aid in visualizing the geometry of centre manifolds of impulsive RFDE.

After three chapters of content on the centre manifold, we take a break in Chapter 9 to study the stable and unstable manifold. We prove existence, smoothness, and some invariance properties. The existence of the unstable manifold will allow us to prove a converse to the linearization theorem, namely an instability theorem.

Two of the classical codimension-one bifurcations of discrete-time systems are lifted into the class of impulsive RFDE in Chapter 10. Specifically, we focus on the fold bifur-

cation and Neimark-Sacker bifurcation. In the fold case, the qualitative properties of the bifurcation remain intact except that fixed points are generically replaced with periodic solutions. In the Neimark-Sacker case, however, the discontinuities inherent to impulsive systems generically result in the birth of an invariant cylinder. For this reason, we call it a cylinder bifurcation. One example for each of these bifurcation patterns is provided.

Chapter 11 is the first of two chapters devoted to particular applications. Here we consider an extension of the delayed SIR model of Kyrychko and Blyuss [56] with pulse vaccination. From a few reasonable biological assumptions we derive the jump functional associated to the pulse vaccination. We then prove analytically that there is a unique disease-free periodic solution that undergoes a transcritical bifurcation at a critical vaccination level. Using numerical methods we continue the bifurcating endemic (i.e. infected) periodic solution with respect to the vaccination level and use monodromy operator discretization to track the dominant Floquet exponent. A cylinder bifurcation is identified and we track the evolution of the cylindrical attractor as the vaccination level is decreased. Phase locking is intermittently observed before the cylinder finally contracts onto a periodic solution of the system without vaccination.

Chapter 12 features an application of centre manifold theory to impulsive stabilization. The latter concept is reviewed before we state formally the problem we wish to solve. The initial setup is that we are given a delay differential equation that possesses a nonhyperbolic equilibrium with trivial unstable subspace. Given a specification on a class $\mathcal{B}$ of discrete delay (or without delay) jump functionals and a cost functional associated to these, we ask whether one can guarantee stabilization of the equilibrium with a specified local convergence rate using an optimal element of $\mathcal{B}$. In other words, can the (observable) bifurcation be optimally suppressed with a desired convergence rate? The result is affirmative provided the convergence is exponential with rate smaller than the spectral gap. Computing the optimal jump functional is a nontrivial problem, since the associated constraints of the optimization problem are highly nonsmooth. To reduce the complexity of the problem, we exploit the low dimensionality of the centre manifold to move the constraint satisfaction problem into a low-dimensional *probe space*, which can be thought of as set of equivalence classes of controllers with equivalent low-order impact on the local dynamics near the equilibrium. These equivalence classes are called *probe elements*. There is a natural way to associate a cost to a given probe element, and optimizing this cost in the low-dimensional probe space followed by another unconstrained convex program is guaranteed to generate an optimal solution of the original problem. We apply the two-stage *centre probe method* to the suppression of a Hopf bifurcation in a network model consisting of 200 two-dimensional oscillators coupled through a small-world network topology. We conclude with a discussion of the strengths and limitations of our method compared to more traditional Lyapunov

function-derived impulsive stabilization results.

A conclusion wraps up the thesis in Chapter 13. We review the key points and suggest a few directions to explore based on our contributions.

Chapter 2

Functional analytic setting

Many of the results from this section appear in the paper *Smooth centre manifolds for impulsive delay differential equations* [21] by K. Church and X. Liu. If a result or proof is uncited, the reader may assume it comes from this publication. Any major deviations from proofs will be mentioned. All other references will be appropriately cited.

2.1 The space $\mathcal{RCR}$ of right-continuous regulated and its topological dual

We will be working exclusively with spaces of right-continuous regulated functions; denote

$$\mathcal{RCR}(I, X) = \left\{ f : I \to X : \forall t \in I, \lim_{s \to t^+} f(s) = f(t) \text{ and } \lim_{s \to t^-} f(s) \text{ exists} \right\},$$

where $X \subseteq \mathbb{R}^n$ and $I \subseteq \mathbb{R}$. When X and I are closed,

$$\mathcal{RCR}_b(I, X) := \{ f \in \mathcal{RCR}(I, X) : ||f|| < \infty \}$$

is a Banach space with the norm $||f|| = \sup_{x \in I} |f(x)|$. We will also at times require the space $\mathcal{G}(I, X)$ of regulated functions from I into X; this is merely the set of functions $f : I \to X$ that possess left- and right limits at each point, with no continuity sidedness restriction. One may consult [44] for background on regulated functions, in particular the claim that $\mathcal{G}(I, X)$ is complete. As $\mathcal{RCR}(I, X)$ is a closed subspace thereof, its completeness follows immediately. We will write $\mathcal{RCR} := \mathcal{RCR}([-r, 0], \mathbb{R}^n)$ when there is no

ambiguity, and note that since $\mathcal{RCR}_b([-r, 0], \mathbb{R}^n) = \mathcal{RCR}([-r, 0], \mathbb{R}^n)$, we may identify $\mathcal{RCR}$ with its associated Banach space.

The step functions are dense in $\mathcal{G}(I, X)$ and by extension, the subspace $\mathcal{RCR}(I, X)$. The proof of the following proposition appears in [44].

Proposition 2.1.1. *Let I be compact. For all $f \in \mathcal{G}(I, X)$, there exists a sequence of step functions $f_n : I \to X$ such that $||f_n - f|| \to 0$.*

Adapting the aforementioned proof to the explicitly right-continuous case, one obtains a specification to $\mathcal{RCR}(I, X)$.

Lemma 2.1.1. *Let I be compact. For all $f \in \mathcal{RCR}(I, X)$, there exists a sequence of right-continuous step functions $f_n : I \to X$ such that $||f_n - f|| \to 0$.*

Regulated functions are integrable, as the following lemma guarantees.

Lemma 2.1.2. *Let $f \in \mathcal{G}(I, \mathbb{R}^n)$ for some interval I. f is locally integrable – that is, $\int_S f(x) dx$ exists for all $S \subseteq I$ compact.*

Proof. Consider the restriction $f|_S$ of f to the compact set S. For brevity, write $f = f|_S$. By Lemma 2.1.1, there exists a sequence $f_n \to f$ of step functions $f_n : S \to \mathbb{R}^n$. Since f_n converges uniformly to f and step functions are integrable, the dominated convergence theorem implies f is integrable with $\int_S f dx = \lim_{n \to \infty} \int_S f_n dx$. $\square$

We will eventually need spaces of function $f : I \to X$ that are differentiable from the right and whose right-hand derivatives are elements of $\mathcal{RCR}(I, X)$. Specifically, define the right-hand derivative by

$$d^+ f(t) = \lim_{\epsilon \to 0^+} \frac{f(t + \epsilon) - f(t)}{\epsilon}$$

and introduce the space

$$\mathcal{RCR}^1(I, X) = \{f \in \mathcal{RCR}(I, X) : d^+ f \in \mathcal{RCR}(I, X)\}.$$

This space is complete with respect to the norm $||f||_1 = ||f|| + ||d^+ f||$ when restricted to the subspace consisting of functions that are $|| \cdot |||_1$-bounded. The latter fact will, however, not be necessary in this thesis.

We will need a few convergence and boundedness results for Perron-Stieltjes integrals involving right-continuous regulated functions and functions of bounded variation. Symmetric arguments to those appearing in [85] yield the following results; see Theorem 2.8 and Corollary 2.10 therein. In what follows, $v^\intercal$ denotes the transpose of $v \in \mathbb{R}^n$. In the two lemmas below, we overload the notation and define $f^\intercal : [a, b] \to \mathbb{R}^{n*}$ by $f^\intercal(t) = [f(t)]^\intercal$.

Lemma 2.1.3. *Let $f : [a, b] \to \mathbb{R}^n$ be of bounded variation and $g \in \mathcal{RCR}([a, b], \mathbb{R}^n)$. The integral $\int_a^b f^\intercal(t)dg(t)$ exists in the Perron-Stieltjes sense, and*

$$\left| \int_a^b f^\intercal(t)dg(t) \right| \leq (|f(a)| + |f(b)| + \mathrm{var}_a^b f) \|g\|, \tag{2.1}$$

where $\mathrm{var}_a^b f$ denotes the total variation of f on the interval $[a, b]$.

Lemma 2.1.4. *Let $h_n \in \mathcal{RCR}([a, b], \mathbb{R}^n)$ and $h \in \mathcal{RCR}([a, b], R^n)$ be such that $\|h_n - h\| \to 0$ as $n \to \infty$. For any $f : [a, b] \to \mathbb{R}^n$ of bounded variation, the Perron-Stieltjes integrals $\int_a^b f^\intercal(t)dh(t)$ and $\int_a^b f^\intercal(t)dh_n(t)$ exist and*

$$\lim_{n \to \infty} \int_a^b f^\intercal(t)dh_n(t) = \int_a^b f^\intercal(t)dh(t). \tag{2.2}$$

Next, we provide a generalization of Lemma 3.2 of [11], which can itself be seen as a weakened form of the result that if $x : \mathbb{R} \to \mathbb{R}^n$ is continuous, then $F : t \mapsto x_t \in C([-r, 0], \mathbb{R}^n)$ is continuous as a function $F : \mathbb{R} \to C([-r, 0], \mathbb{R}^n)$, where the codomain is the Banach space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ equipped with the uniform norm.

Lemma 2.1.5. *Let $r > 0$ be finite and let $\phi \in \mathcal{RCR}([a, b], \mathbb{R}^n)$ for some $b \geq a + r$. With $\phi_t : [-r, 0] \to \mathbb{R}^n$ defined as in (1.3), $t \mapsto \|\phi_t\|$ is an element of $\mathcal{RCR}([a + r, b], \mathbb{R})$.*

Proof. Let $t \in [a + r, b]$ be fixed. We will only prove right-continuity, since the proof of the existence of left limits is similar. It suffices to prove that for any decreasing sequence $s_n \downarrow 0$, we have $\|\phi_{t+s_n}\| \to \|\phi_t\|$. Let $\epsilon > 0$ be given. By right-continuity of ϕ, for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < \mu < \delta$, then $|\phi(t + \mu) - \phi(t)| < \epsilon$. Therefore,

$$\|\phi_{t+s_n}\| = \sup_{\mu \in [-r,0]} |\phi(t + \mu)| \leq \sup_{\mu \in [-r,s_n]} |\phi(t + \mu)| \leq \max\{\|\phi_t\|, \sup_{\mu \in [0,s_n]} |\phi(t + \mu)|\}$$

$$\leq \max\{\|\phi_t\|, |\phi(t)| + \epsilon\} \leq \|\phi_t\| + \epsilon,$$

provided $s_n < \delta$. On the other hand, since ϕ is bounded, there exists some sequence $x_n \in [-r, 0]$ such that $|\phi_t(x_n)| \to \|\phi_t\|$. By passing to a subsequence, we may assume $x_n \to \hat{x} \in [-r, 0]$. If $\hat{x} > -r$, then we have

$$\|\phi_{t+s_n}\| \geq \sup_{\mu \in [-r+s_n,0]} |\phi(t + \mu)| = |\phi(\hat{x})| = \|\phi_t\|$$

provided $s_n < -\hat{x}$, while if $\hat{x} = -r$, we notice that the sequence $x'_n = t - r + s_n$ converges to $t + \hat{x}$, so that for all $\epsilon > 0$, there exists $N_3 > 0$ such that for $n \geq N$,

$$||\phi_{t+s_n}|| \geq |\phi(t + s_n)| \geq ||\phi_t|| - \epsilon.$$

Therefore, if we let $s_{N_1} < \delta$ and $s_{N_2} < -\hat{x}$, then by setting $N = \max\{N_1, N_2, N_3\}$, it follows by the above three inequalities that for $n \geq N$,

$$-\epsilon \leq ||\phi_{t+s_n}|| - ||\phi_t|| \leq \epsilon.$$

We conclude $||\phi_{t+s_n}||$ converges to $||\phi_t||$. $\qquad\qquad\square$

Using essentially the same argument, one can prove the following generalization. The result does not appear in [21], but as the proof is so similar to that of Lemma 2.1.5, it will be omitted.

Lemma 2.1.6. *Suppose $\phi \in \mathcal{RCR}([a,b], \mathbb{R}^n)$, $X \in \mathcal{RCR}([a,b], \mathbb{R}^{n\times m})$ and $z \in \mathcal{RCR}([a,b], \mathbb{R}^m)$ for some $b \geq a+r$. Then, the function $t \mapsto ||\phi_t + X_t z(t)||$ is an element of $\mathcal{RCR}([a+r,b], \mathbb{R})$*

The final element in our overview of right-continuous regulated functions is a characterization of the topological dual $\mathcal{RCR}^*$. A result from Tvrdy [85] provides such for the dual of the space of regulated left-continuous scalar-valued functions, and for our purposes the obvious modification that is needed is the following.

Lemma 2.1.7. *$F \in \mathcal{RCR}^*$ if and only if there exists $q \in \mathbb{R}^n$ and $p : [-r,0] \to \mathbb{R}^n$ of bounded variation such that*

$$F(x) = q^\mathsf{T} x(0) + \int_{-r}^{0} p^\mathsf{T}(t) dx(t), \qquad\qquad (2.3)$$

where the integral is a Perron-Stieltjes integral.

2.2   Pettis integration

Before introducing Pettis integration, we begin with some motivation. Recall that the ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + f(t)$$

admits the variation of constants formula

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \mu)f(\mu)d\mu,$$

where $X(t, s)$ is the *Cauchy matrix* of the homogeneous equation $\dot{u} = A(t)y$. That is, it satisfies the matrix differential equation

$$\frac{d}{dt}X(t, s) = A(t)X(t, s)$$

and the initial condition $X(s, s) = I$, where the latter denotes the identity on $\mathbb{R}^n$. We will eventually derive a similar formula for inhomogeneous linear impulsive RFDE, except that the Cauchy matrix will be replaced with a family of bounded linear oprerators $U(t, s)$: $\mathcal{RCR} \rightarrow \mathcal{RCR}$. As we will see in Section 3.2, the function $s \mapsto U(t, s)$ is generally discontinuous everywhere. Thus, even if $f : [s, t] \rightarrow \mathcal{RCR}$ is continuous, this can make it can be very difficult to prove that the integral

$$\int_s^t U(t, \mu)f(\mu)d\mu$$

exists in the strong sense. Thankfully, for our purposes, Pettis integration will suffice. We recall the following definition, which appears in [68].

Definition 2.2.1. *Let X be a Banach space and (S, Σ, μ) a measure space. We say that $f : S \rightarrow X$ is* Pettis integrable *if there exists a set function $I_f : \Sigma \rightarrow X$ such that*

$$\varphi^* I_f(E) = \int_E \varphi^* f d\mu$$

for all $\varphi^* \in X^*$ and $E \in \Sigma$. I_f is the indefinite Pettis integral *of f, and $I_f(E)$ the* Pettis integral *of f on E.*

In the above definition X^* is the topological dual of X. By abuse of notation, we will often write $I_f(E) = \int_E f d\mu$ when there is no ambiguity. For our purposes, the following proposition will be very useful. Its proof is elementary and can be found in numerous textbooks on functional analysis and integration.

Proposition 2.2.1. *The Pettis integral possesses the following properties.*

- *If f is Pettis integrable, then its indefinite Pettis integral is unique.*

- If $T : X \to X$ is a bounded linear operator, then $T \left(\int_E f d\mu \right) = \int_E (Tf) \, d\mu$ whenever one of the integrals exists.

- If $\mu(A \cap B) = 0$, then $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$.

- $|| \int_E f d\mu || \leq \int_E ||f|| d\mu$

2.3 One-point limits and regulated left-limit histories of $\mathcal{RCR}$ functions

The pointwise left-limit of a solution history was defined in (1.4). We extend this definition and introduce another now.

Definition 2.3.1. *Let $x \in \mathcal{RCR}(I, \mathbb{R}^n)$ for some interval I. The* solution history at time $t \in I$ *is the function $x_t \in \mathcal{RCR}$ defined by equation* (1.3), *provided $[t - r, t] \subset I$. The* pointwise left-limit history *is $x_{t^-} \in \mathcal{RCR}$, defined by equation* (1.4). *The* regulated left-limit history *is the function $x_t^- \in \mathcal{G}([-r, 0], \mathbb{R}^n)$ defined by*

$$x_t^-(\theta) = \lim_{\epsilon \to 0^-} x_{t+\epsilon}(\theta), \tag{2.4}$$

provided $\inf I < t - r$.

The nonuniform left-limit history is so named because while the association of x_t with x_{t^-} defines a bounded linear operator on $\mathcal{G}([-r, 0], \mathbb{R}^n)$, the same is not true of the association of x_t with x_t^-. Indeed, to compute x_t^- one requires the data from x on an interval of the form $[t - r - \epsilon, t]$ for any arbitrarily small ϵ.

2.4 Integral inequalities

We conclude this chapter with two inequalities. The first is an impulsive Gronwall-Bellman inequality for regulated functions. The result is similar to Lemma 2.3 of [8], and the proof is omitted. The second one concerns an elementary estimation of sums of continuous functions at impulses, when the sequence of impulses satisfies a separation condition. A weaker version of this result appears in [21], but the proof requires no modifications. We include it here for completeness.

Lemma 2.4.1. *Suppose $x \in \mathcal{RCR}([s, \alpha], \mathbb{R})$ satisfies the inequality*

$$x(t) \leq C + \int_s^t (p(\mu)x(\mu) + h(\mu))d\mu + \sum_{s < t_i \leq t} (b_i x(t_i^-) + g_i) \tag{2.5}$$

for some nonnegative integrable function p, integrable and bounded h, nonnegative constants b_i, g_i and c, and all $t \in [s, \alpha]$. For $t \geq s$, define

$$z(t, s) = \exp \left(\int_s^t p(\mu)d\mu \right) \prod_{s < t_i \leq t} (1 + b_i).$$

Then, $\mu \mapsto z(t, \mu)$ is integrable and the following inequality is satisfied.

$$x(t) \leq Cz(t, s) + \int_s^t z(t, \mu)h(\mu)d\mu + \sum_{s < t_i \leq t} z(t, t_i)g_i. \tag{2.6}$$

Lemma 2.4.2. *Let $f \in \mathcal{RCR}(\mathbb{R}, \mathbb{R})$ and suppose $\{t_k\}$ satisfies $t_{k+1} - t_k \geq \xi$.*

1. *If f is nondecreasing, then $\sum_{s < t_i \leq t} f(t_i) \leq \frac{1}{\xi} \int_s^{t+\xi} f(\mu)d\mu$.*

2. *If f is nonincreasing, then $\sum_{s < t_i \leq t} f(t_i) \leq \frac{1}{\xi} \int_{s-\xi}^t f(\mu)d\mu$.*

Proof. Let $\{t_0, \ldots, t_N\} = \{t_k : k \in \mathbb{Z}\} \cap (s, t]$. If f is nondecreasing, then

$$\sum_{s < t_i \leq t} f(t_i) = \sum_{i=0}^N f(t_i) = \frac{1}{\xi} \sum_{i=0}^N f(t_i)\xi \leq \frac{1}{\xi} \sum_{i=0}^N f(t_0 + i\xi)\xi \leq \frac{1}{\xi} \int_s^{t+\xi} f(\mu)d\mu.$$

The decreasing case is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Chapter 3

Linear systems theory

Nearly all results (lemmas, theorems, etc.) from this section appear in the paper *Smooth centre manifolds for impulsive delay differential equations* [21] by K. Church and X. Liu. If a result or proof is uncited, the reader may assume it comes from this publication. Any major deviations from proofs will be mentioned. All other references will be appropriately cited.

3.1 Linear impulsive RFDEs: existence and uniqueness of solutions

In this chapter we will be interested primarily in the linear impulsive RFDE

$$\dot{x} = L(t)x_t + h(t), \qquad\qquad t \neq t_k \qquad\qquad (3.1)$$
$$\Delta x = B(k)x_{t^-} + r_k, \qquad\qquad t = t_k. \qquad\qquad (3.2)$$

The following assumptions will be needed throughout.

H.1 The representation

$$L(t)\phi = \int_{-r}^{0} [d_\theta \eta(t, \theta)] \phi(\theta)$$

holds, where the integral is taken in the Lebesgue-Stieltjes sense, the function $\eta : \mathbb{R} \times [-r, 0] \to \mathbb{R}^{n \times n}$ is jointly measurable and is of bounded variation and right-continuous on $[-r, 0]$ for each $t \in \mathbb{R}$, and such that $|L(t)\phi| \leq \ell(t)||\phi||$ for some $\ell : \mathbb{R} \to \mathbb{R}$ locally integrable.

H.2 The sequence t_k is monotonically increasing with $|t_k| \to \infty$ as $|k| \to \infty$, and the representation

$$B(k)\phi = \int_{-r}^{0} [d_\theta \gamma_k(\theta)]\phi(\theta)$$

holds for $k \in \mathbb{Z}$ for functions $\gamma_k : [-r, 0] \to \mathbb{R}^{n \times n}$ of bounded variation and right-continuous, such that $|B(k)| \le b(k)$.

Remark 3.1.1. *Hypothesis H.1–H.2 could in principle be weakened. However, insofar as applied impulsive differential equations are concerned, hypothesis H.1 is sufficient. Indeed, H.1 includes the case of discrete time-varying delays: the linear delay differential equation*

$$\dot{x} = \sum_{k=1}^{m} A_k(t)x(t - r_k(t))$$

with r_k continuous, is associated to a linear operator satisfying condition H.1 with $\eta(t, \theta) = \sum A_k(t)H_{-r_k(t)}(\theta)$, where $H_z(\theta) = 1$ if $\theta \ge z$ and zero otherwise. It also obviously includes a large class of distributed delays. Moreover, each of $L(t)$ and $B(k)$ is well-defined on $\mathcal{G}([-r, 0], \mathbb{R}^n)$ and, consequently, on the subspace $\mathcal{RCR}$; see Lemma 2.1.1.

At this stage, we define integrated solutions of the linear impulsive RFDE and consider existence, uniqueness and continuability of such solutions.

Definition 3.1.1. *Let $(s, \phi) \in \mathbb{R} \times \mathcal{RCR}$. A function $x \in \mathcal{RCR}([s - r, \alpha), \mathbb{R}^n)$ for some $\alpha > s$ is an* integrated solution *of the linear impulsive RFDE (3.1)–(3.2) satisfying the initial condition (s, ϕ) if it satisfies $x_s = \phi$ and the integral equation*

$$x(t) = \begin{cases} \phi(0) + \int_s^t [L(\mu)x_\mu + h(\mu)]d\mu + \sum_{s < t_i \le t}[B(i)x_{t_i^-} + r_i], & t > s \\ \phi(t - s), & s - r \le t \le s. \end{cases} \quad (3.3)$$

Lemma 3.1.1. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, let $\{r_k : k \in \mathbb{Z}\} \subset \mathbb{R}^n$ and let hypotheses H.1–H.2 hold. For all $(s, \phi) \in \mathcal{RCR}$, there exists a unique integrated solution $x \in \mathcal{RCR}([s - r, \infty), \mathbb{R}^n)$ of (3.1)–(3.2) satisfying the initial condition (s, ϕ).*

The above lemma follows by hypotheses H.1–2, the Banach fixed-point theorem, Lemma 2.4.1 and typical continuation arguments. It could also be proven by identifying the equation with a generalized ordinary differential equation, as in [31]. Note here that h may be unbounded on the real line; however, since it is regulated it is bounded on every compact set [44]. Integrated solutions include as a special case the more classical notion of solution

appearing in, for example, [10]. As such, our definition is appropriate. From this point onward, we will drop the adjective *integrated* and merely refer to such functions as being solutions.

On the note of "classical" solutions, it will later be important that the impulsive RFDE (3.1)–(3.2) has a regularizing effect on initial conditions. Precisely, we have the following lemma.

Lemma 3.1.2. *Under the conditions of Lemma 3.1.1, the integrated solution $x : [s - r, \infty) \to \mathbb{R}^n$ is differentiable from the right on $[s, \infty)$. In particular, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution defined for all time, then $x \in \mathcal{RCR}^1(\mathbb{R}, \mathbb{R}^n)$.*

Proof. The first conclusion follows by the integral representation of solutions with the remark that observation that $\mu \mapsto L(\mu)x_\mu \in \mathcal{RCR}([s, \infty), \mathbb{R}^n)$. For the second part, one can show that the restriction of x to any interval of the form $[s, \infty)$ is differentiable from the right by applying the previous result to the restriction on $[s-r, \infty)$. Since s is arbitrary, the result is proven. □

Remark 3.1.2. *Equation (3.3) is derived by integrating the impulsive RFDE (3.1)–(3.2). From this integrated equation, one can also conclude that on the interval $[s, \infty)$, the solution can only have discontinuities at the impulse times t_k for $k \in \mathbb{Z}$ and, in particular, our definition imposes the condition that even if $s = t_k$ for some $k \in \mathbb{Z}$, the jump condition (3.2) need not be satisfied at that time. This is by design, for reasons that will become apparent in Section 3.2.*

3.2 Evolution family, processes, invariant fiber bundles and spectral separation

In this section we will specialize to the homogeneous equation

$$\dot{x} = L(t)x_t, \qquad\qquad t \neq t_k \qquad\qquad (3.4)$$
$$\Delta x = B(k)x_{t^-}, \qquad\qquad t = t_k. \qquad\qquad (3.5)$$

Definition 3.2.1. *Let hypotheses H.1–H.2 hold. For a given $(s, \phi) \in \mathbb{R} \times \mathcal{RCR}$, let $t \mapsto x(t; s, \phi)$ denote the unique solution of (3.4)–(3.5) satisfying $x_s(\cdot; s, \phi) = \phi$. The function $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ defined by $U(t, s)\phi = x_t(\cdot, s, \phi)$ for $t \geq s$ is the* evolution family *associated to the homogeneous equation (3.4)–(3.5).*

From here onward, we will take the convention that if $L : \mathcal{RCR} \to \mathcal{RCR}$ is a linear operator, then $L\phi(\theta)$ for $\phi \in \mathcal{RCR}$ and $\theta \in [-r, 0]$ should be understood as $[L(\phi)](\theta)$. Also, the symbol I_X will refer to the identity operator on X. When the context is clear, we will simply write it as I. Introduce the linear function $\chi_s : \mathbb{R}^n \to \mathcal{RCR}$ defined by

$$[\chi_s \xi](\theta) = \begin{cases} \xi, & \theta = s \\ 0, & \theta \neq s. \end{cases} \tag{3.6}$$

Lemma 3.2.1. *The evolution family satisfies the following properties.*

1) $U(t, t) = I$ *for all* $t \in \mathbb{R}$.

2) *For* $s \leq t$, $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ *is a bounded linear operator. In particular,*

$$||U(t, s)|| \leq \exp \left(\int_s^t \ell(\mu) d\mu \right) \prod_{s < t_i \leq t} (1 + b(i)). \tag{3.7}$$

3) *For* $s \leq v \leq t$, $U(t, s) = U(t, v)U(v, s)$.

4) *For all* $\theta \in [-r, 0]$, $s \leq t + \theta$ *and* $\phi \in \mathcal{RCR}$, $U(t, s)\phi(\theta) = U(t + \theta, s)\phi(0)$.

5) *For all* $t_k > s$, *one has* $U(t_k, s) = (I + \chi_0 B(k))U(t_k^-, s).$[1]

6) *Let* $C(t, s)$ *denote the evolution family on* $\mathcal{RCR}$ *associated to the "continuous" equation* $\dot{x} = L(t)x_t$. *The following factorization holds:*

$$U(t, s) = \begin{cases} C(t, s), & [s, t] \cap \{t_k\}_{k \in \mathbb{Z}} \in \{\{s\}, \emptyset\} \\ C(t, t_k) \circ (I + \chi_0 B(k)) \circ U(t_k^-, s), & t \geq t_k > s \end{cases} \tag{3.8}$$

Proof. Property 1), 3) and 4) are immediate consequences of the uniqueness assertion of Lemma 3.1.1 and the definition of the evolution family. For property 2), we obtain linearity by noticing that $\phi \mapsto x(t; s, \phi)$ is linear in ϕ for each $t \geq s$ and, consequently, $\phi \mapsto x_t(\cdot; s, \phi)$ is is also linear. To obtain boundedness, we notice that by virtue of the integral equation (3.3), $U(t, s)\phi(\theta)$ satisfies

$$|U(t, s)\phi(\theta)| \leq ||\phi|| + \int_s^{t+\theta} |L(\mu)U(\mu, s)\phi| d\mu + \sum_{s < t_i \leq t+\theta} |B(i)U(t_i^-, s)\phi|$$

$$\leq ||\phi|| + \int_s^t \ell(\mu)||U(\mu, s)\phi|| d\mu + \sum_{s < t_i \leq t} b(i)||U(t_i^-, s)\phi||.$$

---

[1]Note here that the left limit is defined by $U(t_k^-, s)\phi(\theta) = U(t_k, s)\phi(\theta)$ for $\theta < 0$, while $U(t_k^-, s)\phi(0) = U(t_k, s)\phi(0^-)$.

Since the upper bounds are independent of θ, denoting $X(t) = U(t, s)\phi$, we obtain

$$||X(t)|| \leq ||\phi|| + \int_s^t \ell(\mu)||X(\mu)||d\mu + \sum_{s < t_i \leq t} b(i)||X(t_i^-)||.$$

By Lemma 2.1.5, $t \mapsto ||X(t)|$ is an element of $\mathcal{RCR}([s - r, \infty), \mathbb{R})$. Invoking Lemma 2.4.1, we obtain the desired boundedness (3.7) of the evolution family and property 1) is proven. Finally, since

$$U(t_k, s)\phi(0) = \phi(0) + \int_s^{t_k} L(\mu)U(\mu, s)\phi d\mu + \sum_{s < t_i \leq t_k} B(i)U(t_i^-, s)\phi$$
$$= U(t_k^-, s)\phi(0) + B(k)U(t_k^-, s)\phi$$

and $U(t_k^-, s)\phi(\theta) = U(t_k, s)\phi(\theta)$ for $\theta < 0$, we readily obtain property 5). The verification of property 6) follows by existence and uniqueness of solutions and property 5). $\square$

The evolution family is an example of a more general construction in nonautonomous dynamical systems called a *process*. The following definition is borrowed from the monograph of Kloeden and Rasmussen [52], with slightly different notation.

Definition 3.2.2. *If X is a Banach space, a subset $\mathcal{M} \subseteq \mathbb{R} \times X$ is a* nonautonomous set *over X. For each $t \in \mathbb{R}$, the set*

$$\mathcal{M}(t) = \{x : (t, x) \in \mathcal{M}\}$$

is called the t-fiber of $\mathcal{M}$.

Definition 3.2.3. *A* process *on X is a pair $(S, \mathcal{M})$ where $\mathcal{M}$ is a nonautonomous set over $\mathbb{R} \times X$ and $S : \mathcal{M} \to X$, whose action we denote by $S(t, (s, x)) = S(t, s)x$, satisfies the following.*

- *$\{t\} \times X \subset \mathcal{M}(t)$ and $S(t, t) = I_X$ for all $t \in \mathbb{R}$.*

- *$S(t, s)x = S(t, v)S(v, s)x$ whenever $(s, x) \in \mathcal{M}(v)$ and $(v, S(v, s)x) \in \mathcal{M}(t)$.*

A process is forward *if for all $s \in \mathbb{R}$ and $x \in X$, $(t, S(t, s)x) \in \mathcal{M}(t)$ for all $t \geq s$. A process is* all time *if for all $t, s \in \mathbb{R}$ and $x \in X$, $(t, S(t, s)x) \in \mathcal{M}(t)$.*

Note that the above definition is slightly different, for example, than the one for process appearing in [52], where processes are defined first as (partial) mappings, independent of nonautonomous sets. The reason for our distinction here is that we want to make precise the notion that a process $S(t,s)$ need not be defined on the entire Banach space X for every pair of time arguments, the way evolution families $U(t,s)$ are. This will make things a bit more precise when nonlinear systems are considered in Chapter 5.

Central to notions of stability for linear ordinary differential equations is the dynamics invariant subspaces. These subspaces permit a decomposition of the phase space such that on each distinct factor, the dynamics admit the same qualitative behaviour. The extension of this idea to nonautonomous systems is that of *exponential trichotomy* – see [26] for a standard definition and applications to heteroclinic bifurcations. The analogue of hyperbolicity of an equilibrium point to bounded trajectories of a nonautonomous system is that of *exponential dichotomy*. For our purposes, we present a slightly modified definition.

Definition 3.2.4. *Let $U(t,s) : X \to X$ be a family of bounded linear operators defining a forward process on a Banach space X. We say that U is* spectrally separated *if there exists a triple (P_s, P_c, P_u) of bounded projection-valued functions $P_i : \mathbb{R} \to \mathcal{L}(X)$ with $P_s + P_c + P_u = I$ such that the following hold.*

1. *There exists a constant N such that $\sup_{t \in \mathbb{R}} (||P_s(t)|| + ||P_c(t)|| + ||P_u(t)||) = N < \infty$.*

2. *The projectors are mutually orthogonal; $P_i(t)P_j(t) = 0$ for $i \neq j$.*

3. *$U(t,s)P_i(s) = P_i(t)U(t,s)$ for all $t \geq s$ and $i \in \{s,c,u\}$.*

4. *Define $U_i(t,s)$ as the restriction of $U(t,s)$ to $X_i(s) = \mathcal{R}(P_i(s))$. The operators $U_c(t,s) : X_c(s) \to X_c(t)$ and $U_u(t,s) : X_u(s) \to X_u(t)$ are invertible and we denote $U_c(s,t) = U_c(t,s)^{-1}$ and $U_u(s,t) = U_u(t,s)^{-1}$ for $s \leq t$.*

5. *The operators U_c and U_u define all-time processes on the family of Banach spaces $X_c(\cdot)$ and $X_u(\cdot)$. Specifically, the following holds for all $t,s,v \in \mathbb{R}$.*

$$U_c(t,s) = U_c(t,v)U_c(v,s), \qquad U_u(t,s) = U_u(t,v)U_u(v,s).$$

6. *There exist real numbers $a < 0 < b$ such that for all $\epsilon > 0$, there exists $K \geq 1$ such that*

$$||U_u(t,s)|| \leq Ke^{b(t-s)}, \qquad\qquad t \leq s \qquad\qquad (3.9)$$
$$||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}, \qquad\qquad t,s \in \mathbb{R} \qquad\qquad (3.10)$$
$$||U_s(t,s)|| \leq Ke^{a(t-s)}, \qquad\qquad t \geq s. \qquad\qquad (3.11)$$

The definition of spectral separation above is motivated by the the spectral decomposition hypotheses associated to the centre manifold theorem for autonomous delay differential equations appearing in [27]. Definition 3.2.4 is essentially a time-varying analogue thereof.

Definition 3.2.5. *Let $U(t,s) : X \to X$ be spectrally separated. The nonautonomous sets*

$$X_i = \{(t,x) : t \in \mathbb{R}, x \in X_i(t)\}$$

for $i \in \{s,c,u\}$ are termed respectively the stable, centre, and unstable fibre bundles associated to $U(t,s)$. The evolution family is hyperbolic if X_c is trivial – that is, $X_c = \{0\}$.

Thus, if the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated, the phase space admits a direct sum decomposition

$$\mathcal{RCR} = \mathcal{RCR}_s(t) \oplus \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t) \tag{3.12}$$

for each $t \in \mathbb{R}$. If $(s,\phi) \in \mathcal{RCR}_s$, equation (3.9) implies that $U(t,s)\phi$ decays to zero exponentially as $t \to \infty$. We say that in stable fibre bundle, solutions decay exponentially in forward time. Similarly, in the unstable fibre bundle, solutions are defined for all time and decay exponentially in backward time. In the centre fibre bundle, solutions are defined for all time and grow slower than exponentially in forward and backward time. The difference between this decomposition and one more typical of autonomous or ordinary delay differential equations is that the factors of the decomposition are generally time-dependent; that is, they are determined by the t-fibres of the *invariant fibre bundles* $\mathcal{RCR}_s$, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$.

Example 3.2.1. *To aid in the understanding of Definition 3.2.4 and the notion of invariant fibre bundles, let us apply these concepts to a much simpler system: the following three-dimensional 2π-periodic ordinary differential equation*

$$\dot{x} = -y, \tag{3.13}$$
$$\dot{y} = x + \sin(t)z \tag{3.14}$$
$$\dot{z} = -z. \tag{3.15}$$

The phase space here is $\mathbb{R}^3$. One can directly check that

$$X(t) = \begin{bmatrix} \cos(t) & -\sin(t) & \frac{2}{5}\cos(t)(1-e^{-t}) - \frac{1}{5}\sin(t)(1+e^{-t}) \\ \sin(t) & \cos(t) & \frac{1}{5}\sin(t)(2-3e^{-t}) + \frac{1}{5}\cos(t)(1-e^{-t}) \\ 0 & 0 & e^{-t} \end{bmatrix} \tag{3.16}$$

is a matrix solution satisfying $X(0) = I$, so that in this context the evolution family is $U(t, s) = X(t)X^{-1}(s)$. It defines an all-time process on $\mathbb{R}^3$.

Let us identify some exceptional solutions. The first two columns of $X(t)$ define solutions that are bounded for all time:

$$X_1(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix}.$$

We further recognize that an appropriate linear combination of all three columns produces a solution that converges exponentially to the origin as $t \to \infty$. Specifically, this solution is

$$Y(t) = X_3(t) - \frac{2}{5}X_1(t) - \frac{1}{5}X_2(t) = e^{-t} \begin{bmatrix} -\frac{2}{5}\cos(t) - \frac{1}{5}\sin(t) \\ -\frac{3}{5}\sin(t) - \frac{1}{5}\cos(t) \\ 1 \end{bmatrix}.$$

The intuition is that X_1 and X_2 should span centre fibre bundle $\mathbb{R}^3_c$, while Y should span the stable fibre bundle $\mathbb{R}^3_s$. The former of the two spans is independent of time since the subspace itself is the same for each argument of t, but the second of the two is time-varying.

To realize this decomposition in terms of the definition of spectral separation, we must identify the projections $P_c(t)$ and $P_s(t)$. To do this we first apply Floquet's theorem to the fundamental matrix solution $X(t)$. There exists a nonsingular 2π-periodic continuous matrix-valued function $t \mapsto \Phi(t)$ and matrix Λ such that $X(t) = \Phi(t)e^{\Lambda t}$. Since $X(0) = I$, it follows that $\Phi(0) = I$ and, consequently, Λ satisfies the equation

$$e^{2\pi\Lambda} = \begin{bmatrix} 1 & 0 & \frac{2}{5}(1 - e^{-2\pi}) \\ 0 & 1 & \frac{1}{5}(1 - e^{-2\pi}) \\ 0 & 0 & e^{-2\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}.$$

The right-hand side is precisely the Jordan canonical form of $e^{2\pi\Lambda}$. Taking logarithms, it follows that Λ has the Jordan canonical form

$$\Lambda = QJQ^{-1} = \begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the change of variables $\begin{bmatrix} x & y & z \end{bmatrix}^\mathsf{T} = \Phi(t)Qw$ results in the autonomous system $\dot{w} = Jw$, for which the projection onto the stable subspace is precisely

$$P^w_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Pulling this projection back into the x, y, z variables, we define $P_s(0)$ by

$$P_s(0) = QP_s^w Q^{-1} = \begin{bmatrix} 0 & 0 & -\frac{2}{5} \\ 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.17}$$

Next, we define $P_s(t)$ for any t so that it satisfies the equation $U(t, 0)P_s(0) = P_s(t)U(t, 0)$. That is,

$$\begin{aligned} P_s(t) &= X(t)P_s(0)X^{-1}(t) = \Phi(t)Qe^{tJ}Q^{-1}(QP_s^w Q^{-1})Qe^{-tJ}Q^{-1}\Phi^{-1}(t) \\ &= \Phi(t)Qe^{tJ}P_s^w e^{-tJ}Q^{-1}\Phi^{-1}(t) \\ &= \Phi(t)\begin{bmatrix} 0 & 0 & -\frac{2}{5} \\ 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}\Phi^{-1}(t). \end{aligned}$$

As $t \mapsto \Phi(t)$ is periodic and nonsingular, $t \mapsto ||P_s(t)||$ is bounded. One can also verify that the equality $U(t, s)P_s(s) = P_s(t)U(t, s)$ holds for all $t \geq s$ (in fact, for all $t, s \in \mathbb{R}$). Also, we clearly have $P_s(t)^2 = P_s(t)$, thereby proving that $P_s(t)$ is indeed a projection. Finally, we have

$$\begin{aligned} ||U(t, s)P_s(s)|| &= ||X(t)X^{-1}(s)X(s)P_s(0)X^{-1}(s)|| \\ &= ||X(t)P_s(0)X^{-1}(s)|| \\ &= ||\Phi(t)Qe^{tJ}P_s^w e^{-sJ}Q^{-1}\Phi^{-1}(s)|| \\ &\leq Ke^{-(t-s)}, \end{aligned}$$

where $K = \sup_{t,s\in[0,2\pi]} ||\Phi(t)|| \cdot ||\Phi^{-1}(s)|| \cdot \kappa(Q)$ and $\kappa(Q) = ||Q|| \cdot ||Q^{-1}||$ is the condition number of Q. Therefore, the projection onto the stable fibre bundle satisfies the exponential estimate (3.11).

Next, we define $P_c(t) = I - P_s(t)$. By construction, $P_c(t)$ and $P_s(t)$ are orthogonal and sum to the identity. If we remark that $P_s(t) = X(t)[I - P_s(0)]X^{-1}(t)$, checking the exponential estimate (3.10) is fairly simple:

$$\begin{aligned} ||U(t, s)P_c(s)|| &= ||X(t)X^{-1}(S)X(s)[I - P_s(0)]X^{-1}(s)|| \\ &= ||X(t)[I - P_s(0)]X^{-1}(s)|| \\ &= ||\Phi(t)Qe^{tJ}[I - P_s^w]e^{-sJ}Q^{-1}\Phi^{-1}(s)|| \\ &\leq K, \end{aligned}$$

since $||e^{tJ}[I - P_s^w]e^{-sJ}|| = 1$. We conclude that $U(t, s) : \mathbb{R}^3 \to \mathbb{R}^3$ is spectrally separated.

Our final task is to verify our intuition that the centre and stable fibre bundles are the spans of $\{X_1, X_2\}$ and $\{Y\}$ respectively. Explicitly substituting (3.16) into $P_s(t) = X(t)P_s^w X^{-1}(t)$ and simplifying eventually yields the expression $P_s(t) = [\ 0\ \ 0\ \ Y(t)\ ]$, written in column form. Thus,

$$\mathbb{R}_s^3(t) = \mathcal{R}(P_s(t)) = \{X(t)P_s(0)X^{-1}(t)w : w \in \mathbb{R}^3\}$$
$$= \{[\ 0\ \ 0\ \ Y(t)\ ]w : w \in \mathbb{R}^3\}$$
$$= \text{span}\{Y(t)\}.$$

Conversely, explicitly computing $P_c(t)$ gives $P_c(t) = I - P_s^w$. We conclude that $\mathbb{R}_c^3(t) = \text{span}\{e_1, e_2\}$ as originally claimed.

3.2.1 Exponential dichotomy and trichotomy

For ordinary differential equations, there are several equivalent definitions of exponential dichotomy and exponential trichotomy. In this setting, exponential dichotomy is very important because it is a generalization of hyperbolicity that is sufficiently strong to provide a nonautonomous analogue of the Hartman-Grobman theorem; see [71]. The linear system $\dot{x} = A(t)x$ has *exponential dichotomy* if it has a fundamental matrix solution $X(t)$ satisfying

$$||X(t)PX^{-1}(s)|| \le Ke^{-\alpha(t-s)}, \qquad\qquad s \le t \qquad\qquad (3.18)$$
$$||X(t)(I - P)X^{-1}(s)|| \le Ke^{-\alpha(s-t)}, \qquad\qquad s \ge t, \qquad\qquad (3.19)$$

for some constants $K, \alpha > 0$ and projection P that is *independent* of time, t. If one defines $P(t) = X(t)PX^{-1}(t)$, then $P^2(t) = P(t)$ and we have the equality

$$U(t,s)P(s) = X(t)X^{-1}(s)(X(s)PX^{-1}(s))$$
$$= X(t)PX^{-1}(s) = X(t)PX^{-1}(t)(X(t)X^{-1}(s)) = P(t)U(t,s),$$

and similarly for the conjugate projector $I - P(t)$. Also, since $U(t,s)P(s) = X(t)PX^{-1}(s)$, the inequalities (3.9) and (3.11) are satisfied for $P_s(t) = P(t)$ and $P_u(t) = I - P_s(t)$. Thus, in the restricted case of exponential dichotomy, the above definition is at least as strong as spectral separation. One can similarly verify that for ordinary differential equations, if $U(t,s) : \mathbb{R}^n \to \mathbb{R}^n$ is spectrally separated and hyperbolic, then it has exponential dichotomy in the above sense. Therefore, the definitions are equivalent for ordinary differential equations.

When we move away from ordinary differential equations into infinite-dimensional systems of impulsive RFDE, we must recognize that evolution families are generally not

invertible, so it is not possible to unconditionally form the inverses in (3.18)–(3.19). These evolution families are generally only invertible upon *restriction* to the ranges of the projection operators. This is precisely why the fourth condition of spectral separation is included and why the trichotomy inequalities (3.9)–(3.11) are stated the way they are.

3.2.2 A remark on the regularity of the evolution family $U(t,s)$ with respect to the arguments t and s

It should come as no surprise that $t \mapsto U(t,s)\phi$ and $s \mapsto U(t,s)\phi$ are generally discontinuous as maps from $\mathbb{R}$ to $\mathcal{RCR}$. Indeed, the following highly trivial scalar impulsive differential equation gives us a counterexample:

$$\begin{aligned} \dot{x} &= 0, & t \neq k \in \mathbb{Z} \\ \Delta x &= x(t^-), & t = k \in \mathbb{Z}. \end{aligned}$$

The evolution family is precisely

$$U(t,s)\phi(\theta) = \begin{cases} 2^{\lfloor t+\theta-s \rfloor}\phi(0), & t+\theta-s \geq 0 \\ \phi(t+\theta-s), & t+\theta-s \leq 0 \end{cases}$$

In particular, if we make the choice $\mathcal{RCR} = \mathcal{RCR}([-1,0],\mathbb{R})$, fix s and $\phi(0) \neq 0$, then $t \mapsto U(t,s)\phi$ is discontinuous on $\mathcal{I}_s = [\lceil s \rceil, \infty)$. To see why this is the case, we notice that on this interval the solution $t \mapsto U(t,s)\phi(0)$ is piecewise constant. As consequence, in each $[k, k+1) \subset \mathcal{I}_s$ we will always have

$$||U(t,s)\phi - U(t+\epsilon,s)\phi|| \geq 2||U(k,s)\phi(0^-)|| \geq 2|\phi(0)| > 0$$

for $\epsilon > 0$ sufficiently small. Similarly, if t is fixed and $\phi(0) \neq 0$, then we can consider the regularity of $s \mapsto U(t,s)\phi$. A similar argument proves that this function is discontinuous on $\mathcal{I}^t = (-\infty, \lfloor t \rfloor)$.

To contrast, in general if s and θ are fixed, $t \mapsto U(t,s)\phi(\theta)$ is an element of $\mathcal{RCR}([s,\infty),\mathbb{R}^n)$. This follows from Lemma 3.1.1. The same is not true for $s \mapsto U(t,s)\phi(\theta)$ for t and θ fixed. Indeed, consider the scalar system

$$\begin{aligned} \dot{x} &= 0, & t \neq k \in \mathbb{Z} \\ \Delta x &= 0, & t \in k \in \mathbb{Z}. \end{aligned}$$

This generates the "translation" evolution family

$$U(t,s)\phi(\theta) = \begin{cases} \phi(0) & t + \theta - s \geq 0 \\ \phi(t + \theta - s), & t + \theta - s \leq 0. \end{cases}$$

Suppose $\phi \in \mathcal{RCR}$ has a discontinuity at some $\theta^* \in (-r, 0)$. By definition, $\theta \mapsto \phi(\theta)$ is continuous from the right. Fixing $t = 0$, we have

$$U(0,s)\phi(\theta^*) = \phi(\theta^* - s).$$

It follows that $s \mapsto U(0,s)\phi(\theta^*)$ has a discontinuity at θ^*, but is continuous from the *left*. Consequently, we cannot have $s \mapsto U(0,s)\phi(\theta^*) \in \mathcal{RCR}((\alpha, 0], \mathbb{R})$ for any $\alpha < 0$.

3.3 Variation of constants formula

Recall that a large part of our motivation for first studying linear systems of impulsive RFDE is to ultimately develop centre manifold theory for the nonlinear equation (1.1)–(1.2). We will end up using the Lyapunov-Perron method to do this. The Lyapunov-Perron method makes use the linear variation of constants formula in order to obtain an equivalence between the centre manifold and a fixed point of a particular nonlinear operator. In the fully nonautonomous context, this method was used by Chicone [16] to prove a nonautonomous centre manifold theorem by first appealing to the evolution semigroup. The evolution semigroup allows one to effectively translate the problem into an autonomous setting by enlarging the phase space. Semigroup theory then provides the requisite variation of constants formula.

To contrast to the approach of Chicone, we work directly with the evolution family associated to the linear system (3.4)–(3.5) and prove a variation of constants formula that is reminiscent of a classical formula derived by Jack Hale for functional differential equations [38]. In the aforementioned reference, Hale proves that solutions of the inhomogeneous delay differential equation $\dot{x} = Ax_t + h(t)$ satisfy the formal variation of constants formula

$$x_t = T(t-s)x_s + \int_s^t T(t-\mu)\chi_0 h(\mu)d\mu,$$

where $T(t) : X \to X$ is the strongly continuous semigroup associated to the autonomous system $\dot{x}(t) = Ax_t$, the phase space is $X = C([-r, 0], \mathbb{R}^n)$, and $\chi_0 : [-r, 0] \to \mathbb{R}^{n \times n}$ is defined by $\chi_0(0) = I$ and $\chi_0(\theta) = 0$ for $\theta < 0$. Strictly speaking, the formula is ill-defined because $\chi_0 h(\mu)$ is not in the domain of $T(t - \mu)$.

The inconsistencies in Hale's variation of constants formula can be resolved in several ways, including adjoint semigroup theory and integrated semigroup theory. See the reference [41] for an overview on these ideas. For our purposes, the use of the phase space $\mathcal{RCR}$ and the evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ will be sufficient. Ultimately, we prove that solutions of the inhomogeneous equation (3.1)–(3.2) satisfy the variation of constants formula

$$x_t = U(t, s)x_s + \int_s^t U(t, \mu)\chi_0 h(\mu)d\mu + \sum_{s < t_i \leq t} U(t, t_i)\chi_0 r(i)$$

in $\mathcal{RCR}$, where the integral is interpreted in the Pettis (weak) sense. This formula includes Hale's result as a special case.

We first prove this formula pointwise in Section 3.3.1. This means the formula is correct in the sense that if one evaluates both sides of the equation at a particular $\theta \in [-r, 0]$ and interprets the integral as a Lebesgue integral, then the equation is true. Following this, we lift the result into the phase space $\mathcal{RCR}$ in Section 3.3.2, where the integral is understood in the Pettis sense.

Despite being of fundamental importance in the representation of solutions, the fully general variation-of-constants we prove in this section was not developed until 2018 with our paper. There have appeared other variation-of-constants formulas in the literature, such as the one by Anokhin, Berezansky and Braverman [6] in 1995. However, we want both a higher level of generality (we want to include the possibility of delayed impulses, distributed delay, etc.) and a more abstract result (that is, a formula in the phase space $\mathcal{RCR}$), so it is necessary to build the formula from the ground up.

3.3.1 Pointwise variation of constants formula

Existence, uniqueness and continuability of solutions of the linear inhomogeneous equation (3.1)–(3.2) has been granted by Lemma 3.1.1. From this result we directly obtain a decomposition of solutions.

Lemma 3.3.1. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and let H.1–H.2 hold. Denote $t \mapsto x(t; s, \phi; h, r)$ the solution of the linear inhomogeneous equation (3.1)–(3.2) for inhomogeneities $h = h(t)$ and $r = r_k$, satisfying the initial condition $x_s(\cdot; s, \phi; h, r) = \phi$. The following decomposition is valid:*

$$x(t; s, \phi; h, r) = x(t; s, \phi; 0, 0) + x(t; s, 0; h, 0) + x(t; s, 0; 0, r) \tag{3.20}$$

The following lemmas prove representations of the inhomogeneous impulsive and continuous terms $x_t(\cdot; s, 0; 0, r)$ and $x_t(\cdot; s, 0; h, 0)$ respectively.

Lemma 3.3.2. *Under hypotheses H.1–H.2, one has*

$$x_t(\cdot; s, 0; 0, r) = \sum_{s < t_i \leq t} U(t, t_i)\chi_0 r_i \qquad (3.21)$$

Proof. Denote $x(t) = x(t; s, 0; 0, r)$. Clearly, for $t \in [s, \min\{t_i : t_i > s\})$, one has $x_t = 0$. Assume without loss of generality that $t_0 = \min\{t_i : t_i > s\}$. Then $x_{t_0} = \chi_0 r_0$ due to (3.3). From Lemma 3.1.1 and 3.2.1, we have $x_t = U(t, t_0)\chi_0 r_0$ for all $t \in [t_0, t_1)$, so we conclude that (3.21) holds for all $t \in [s, t_1)$. Supposing by induction that $x_t = \sum_{s < t_i \leq t} U(t, t_i)\chi_0 r_i$ for all $t \in [s, t_k)$ for some $k \geq 1$, we have

$$\begin{aligned}
x_{t_k} &= x_{t_k^-} + \chi_0 B(k)x_{t_k^-} + \chi_0 r_k \\
&= U(t_k, t_{k-1})x_{t_{k-1}} + \chi_0 r_k \\
&= U(t_k, t_{k-1}) \sum_{s < t_i \leq t_{k-1}} U(t_{k-1}, t_i)\chi_0 r_i + \chi_0 r_k \\
&= \sum_{s < t_i \leq t_k} U(t, t_i)\chi_0 r_i.
\end{aligned}$$

Equality (3.21) then holds for $t \in [t_k, t_{k+1})$ by applying Lemma 3.2.1. The result follows by induction. $\square$

Lemma 3.3.3. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Under hypotheses H.1–H.2, one has*

$$x_t(\theta; s, 0; h, 0) = \int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta)d\mu, \qquad (3.22)$$

where the integral is defined for each θ as the integral of the function $\mu \mapsto U(t, \mu)[\chi_0 h(\mu)](\theta)$ in $\mathbb{R}^n$.

Proof. The proof of this lemma is adapted from the proof of Theorem 16.3 of [38]. Let us denote $x(t; s)h = x(t; s, 0; h, 0)$. First, we note that operator $x(t, s) : \mathcal{RCR}([s, t], \mathbb{R}^n) \to \mathbb{R}^n$ is linear (a consequence of Lemma 3.1.1) for each fixed $s \leq t$, and that it admits an extension to a linear operator $\tilde{x}(t, s) : \mathcal{L}_1^{loc}([s, t], \mathbb{R}^n) \to \mathbb{R}^n$. We do not prove this claim, since the proof is essentially identical to how one would prove Lemma 3.1.1. For $w \in [s, t]$ and denoting $\tilde{x}_t = [\tilde{x}(\cdot, s)h]_t$ for brevity, we see that

$$\begin{aligned}
|\tilde{x}_w(\theta)| &\leq \int_s^{w+\theta} |L(\mu)\tilde{x}_\mu|d\mu + \int_s^{w+\theta} |h(\mu)|d\mu \\
&\leq |h|_1 + \int_s^t \ell(\mu)||x_\mu||d\mu,
\end{aligned}$$

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which implies the uniform inequality $||x_t|| \leq |h|_1 + \int_s^t \ell(\mu)||\tilde{x}||_\mu d\mu$. Applying Lemma 2.4.1 yields $||\tilde{x}_t|| \leq e^{|\ell|_1}|h|_1$, where $|\cdot|_1$ denotes the $\mathcal{L}_1[s,t]$ norm. Thus, $|\tilde{x}(t,s)h| = |\tilde{x}_t(0)| \leq e^{|\ell|_1}|h|_1$, so $\tilde{x}$ is bounded. By classical results of functional analysis, there exists an integrable, essentially bounded $n \times n$ matrix function $\mu \mapsto V(t,s,\mu)$ such that

$$\tilde{x}(t,s)h = \int_s^t V(t,s,\mu)h(\mu)d\mu. \tag{3.23}$$

First we show that $V(t,s,\mu)$ is independent of s. Let $\alpha \in [s,t]$ and let $k \in \mathcal{L}_1([s,t],\mathbb{R}^n)$ be such that $k = 0$ on $[s,\alpha]$. Then $\tilde{x}(t,s)k = x(t,\alpha)k$ and $x(t,\mu)k = 0$ for $\mu \in [s,\alpha]$. Thus,

$$\int_\alpha^t [V(t,s,\mu) - V(t,\alpha,\mu)]k(\mu)d\mu = 0$$

for all $k \in \mathcal{L}_1([\alpha,t],\mathbb{R}^n)$. Thus, $V(t,s,\mu) = V(t,\alpha,\mu)$ almost everywhere on $[\alpha,t]$. Since α is arbitrary, we have that $V(t,s,\mu)$ is independent of s.

Define $V(t,s) = V(t,s,\cdot)$ for any $t \geq s$ and $V(t,s) = 0$ for $s < t$. Let us denote $\tilde{x}(t) = \tilde{x}(t,s)h$ and $V_{t_i^-}(\theta,s) = V(t_i+\theta,s)$ when $\theta < 0$ and $V_{t_i^-}(0,s) = V(t_i^-,s)$. From the integral equation (3.3) and the representation (3.23), we have

$$\int_s^t V(t,\mu)h(\mu)d\mu$$

$$= \int_s^t L(\mu)\tilde{x}_\mu d\mu + \sum_{s<t_i\leq t} B(i)\tilde{x}_{t_i^-} + \int_s^t h(\mu)d\mu$$

$$= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)]\tilde{x}(\mu+\theta)d\mu + \sum_{s<t_i\leq t} \int_{-r}^0 [d_\theta\gamma_i(\theta)]\tilde{x}_{t_i^-}(\theta) + \int_s^t h(\mu)d\mu$$

$$= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)] \int_s^{\mu+\theta} V(\mu+\theta,\nu)h(\nu)d\nu d\mu + \sum_{s<t_i\leq t} \int_{-r}^0 [d_\theta\gamma_k(\theta)] \int_s^{t_i+\theta} V_{t_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu$$

$$= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)] \int_s^\mu V(\mu+\theta,\nu)h(\nu)d\nu d\mu + \sum_{s<t_i\leq t} \int_{-r}^0 [d_\theta\gamma_k(\theta)] \int_s^{t_i} V_{t_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu$$

$$= \int_s^t \int_\nu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\mu+\theta,\nu)h(\nu)d\mu d\nu + \sum_{s<t_i\leq t} \int_s^{t_i} \int_{-r}^0 [d_\theta\gamma_i(\theta)]V_{t_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu$$

$$= \int_s^t \left[\int_\mu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\nu+\theta,\mu)h(\mu)d\nu + \sum_{s<t_i\leq t} \chi_{(-\infty,t_i]}(\mu) \int_{-r}^0 [d_\theta\gamma_k(\theta)]V_{t_i^-}(\theta,\mu)h(\mu) + h(\mu) \right] d\mu$$

$$= \int_s^t \left[\int_\mu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\nu+\theta,\mu)d\nu + \sum_{s<t_i\leq t} \chi_{(-\infty,t_i]}(\mu) \int_{-r}^0 [d_\theta\gamma_k(\theta)]V_{t_i^-}(\theta,\mu) + I \right] h(\mu)d\mu$$

$$= \int_s^t \left[I + \int_\mu^t L(\mu)V_\nu(\cdot,\mu)d\nu + \sum_{s<t_i\leq t} B(i)V_{t_i^-}(\cdot,\mu) \right] h(\mu)d\mu.$$

Since the above holds for all $h \in \mathcal{L}_1([s,t], \mathbb{R}^n)$, we have that the *fundamental matrix* $V(t,s)$ satisfies

$$V(t,s) = \begin{cases} I + \int_s^t L(\mu)V_\mu(\cdot,s)d\mu + \sum_{s<t_i\leq t} B(i)V_{t_i^-}(\cdot,s), & t \geq s \\ 0 & t < s. \end{cases} \quad (3.24)$$

almost everywhere. By uniqueness of solutions (Lemma 3.1.1), it follows that $V(t,s)\xi = U(t,s)[\chi_0\xi](0)$ for all $\xi \in \mathbb{R}^n$. Since $\tilde{x}(t,s)$ is an extension of $x(t,s)$ to the larger space $\mathcal{L}_1([s,t], \mathbb{R}^n)$, representation (3.23) holds for $h \in \mathcal{RCR}([s,t], \mathbb{R}^n)$. Thus, for all $t \geq s$,

$$x_t(\theta; s, 0; h, 0) = x(t+\theta, s)h$$
$$= \int_s^{t+\theta} V(t+\theta,\mu)h(\mu)d\mu$$
$$= \int_s^t V(t+\theta,\mu)h(\mu)d\mu$$
$$= \int_s^t U(t+\theta,\mu)[\chi_0 h(\mu)](0)d\mu$$
$$= \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu,$$

which is what was claimed by equation (3.22). $\qquad\square$

With Lemma 3.3.1 through Lemma 3.3.3 at hand, we arrive at the variation of constants formula.

Lemma 3.3.4. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Under hypotheses H.1–H.2, one has the variation of constants formula*

$$x_t(\theta; s, \phi; h, r) = U(t,s)\phi(\theta) + \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu + \sum_{s<t_i\leq t} U(t,t_i)[\chi_0 r_i](\theta). \quad (3.25)$$

3.3.2 Variation of constants formula in the space $\mathcal{RCR}$

The goal of this section will be to reinterpret the variation of constants formula (3.25) in such a way that the integral appearing therein may be thought of as a weak integral

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in the space $\mathcal{RCR}$. Specifically, we will show that the integral may be regarded as a Gelfand-Pettis integral.

Lemma 3.3.5. *Let* $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ *and let H.1–H.2 hold. The function* $U(t, \cdot)[\chi_0 h(\cdot)] :$ $[s, t] \to \mathcal{RCR}$ *is Pettis integrable for all* $t \geq s$ *and*

$$\left[\int_s^t U(t, \mu)[\chi_0 h(\mu)] d\mu \right] (\theta) = \int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta) d\mu. \qquad (3.26)$$

Proof. By Lemma 2.1.7 and the uniqueness assertion of Proposition 2.2.1, if we can show for all $p : [-r, 0] \to \mathbb{R}^n$ of bounded variation the equality

$$\int_{-r}^0 p^\intercal(\theta) d \left[\int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta) d\mu \right] = \int_s^t \left[\int_{-r}^0 p^\intercal(\theta) d \Big[U(t, \mu)[\chi_0 h(\mu)](\theta) \Big] \right] d\mu$$

holds, then Lemma 3.3.5 will be proven. Note that the above is equivalent to

$$c \int_{-r}^0 p^\intercal(\theta) d \left[\int_s^t V(t + \theta, \mu) h(\mu) d\mu \right] = \int_s^t \left[\int_{-r}^0 p^\intercal(\theta) d \Big[V(t + \theta, \mu) h(\mu) \Big] \right] d\mu. \qquad (3.27)$$

We prove the lemma first when h is a step function. In this case, a consequence of equation (3.24) is that $\theta \mapsto V(t + \theta, \mu) h(\mu)$ and $\mu \mapsto V(t + \theta, \mu) h(\mu)$ are piecewise continuous, while Lemma 3.1.1 and Lemma 3.3.3 imply $\theta \mapsto \int_s^t V(t + \theta, \mu) h(\mu) d\mu$ is also piecewise continuous, all with at most finitely many discontinuities on any given bounded set. Consequently, both integrals in (3.27) can be regarded as a Lebesgue-Stieltjes integrals, with Fubini's theorem granting the desired equality.

When $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ is an arbitrary right-continuous regulated function, we approximate its restriction to the interval $[s, t]$ by a convergent sequence of step functions h_n by Lemma 2.1.1. Equation (3.27) is then satisfied with h replaced with h_n. Define the functions

$$J_n(\theta) = \int_s^t V(t + \theta, \mu) h_n(\mu) d\mu, \qquad K_n(\mu) = \int_{-r}^0 p^\intercal(\theta) d \Big[V(t + \theta, \mu) h_n(\mu) \Big],$$

$$J(\theta) = \int_s^t V(t + \theta, \mu) h(\mu) d\mu, \qquad K(\mu) = \int_{-r}^0 p^\intercal(\theta) d \Big[V(t + \theta, \mu) h(\mu) \Big],$$

so that $\int_{-r}^0 p^\intercal(\theta) dJ_n(\theta) = \int_s^t K_n(\mu) d\mu$. By Lemma 3.2.1 and elementary integral estimates, $J_n \to J$ uniformly. The conditions of Lemma 2.1.4 are satisfied, and we have the limit

$$\int_{-r}^0 p^\intercal(\theta) dJ_n(\theta) \to \int_{-r}^0 p^\intercal(\theta) dJ(\theta).$$

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Conversely, for each $\mu \in [s, t]$, Lemma 2.1.3 applied to the difference $K_n(\mu) - K(\mu)$ yields, together with Lemma 3.2.1,

$$|K_n(\mu) - K(\mu)| \leq (|p(0)| + |p(-r)| + \mathrm{var}_{-r}^0 p) \left(\int_s^t \exp \left(\int_y^t \ell(\nu) d\nu \right) dy \right) ||h_n - h||.$$

Thus, $K_n \to K$ uniformly, and the bounded convergence theorem implies $\int_s^t K_n(\mu) d\mu \to \int_s^t K(\mu) d\mu$. Therefore, equation (3.27) holds, and the lemma is proven. $\qquad \square$

With Lemma 3.3.4 and Lemma 3.3.5 at hand, we obtain the variation of constants formula for the linear inhomogeneous equation (3.1)–(3.2) in the Banach space $\mathcal{RCR}$.

Theorem 3.3.1. *Let H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. The unique solution $t \mapsto x_t(\cdot; s, \phi; h, r) \in \mathcal{RCR}$ of the linear inhomogeneous impulsive system (3.1)–(3.2) with initial condition $x_s(\cdot; s, \phi; h, r) = \phi$, satisfies the variation-of-constants formula*

$$x_t(\cdot; s, \phi; h, r) = U(t, s)\phi + \int_s^t U(t, \mu)[\chi_0 h(\mu)] d\mu + \sum_{s < t_i \leq t} U(t, t_i)[\chi_0 r_i], \qquad (3.28)$$

where the integral is interpreted in the Pettis sense and may be evaluated pointwise using (3.26).

As a simple corollary, if x is a solution defined on $[s - r, \infty)$, we can express $t \mapsto x_t$ defined on $[s, \infty)$ as the solution of an abstract integral equation.

Corollary 3.3.1.1. *Let H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Any solution $x : [s - r, \infty) \to \mathbb{R}^n$ of the linear inhomogeneous impulsive system (3.1)–(3.2) satisfies for all $t \geq s$ the abstract integral equation*

$$x_t = U(t, s)x_s + \int_s^t U(t, \mu)[\chi_0 h(\mu)] d\mu + \sum_{s < t_i \leq t} U(t, t_i)[\chi_0 r_i]. \qquad (3.29)$$

Equation (3.29) will be the key to defining mild solutions in Section 5.1 and, ultimately, will permit us to construct centre manifolds.

Chapter 4

Linear periodic systems

Nearly all results from this section appear in the paper *Smooth centre manifolds for impulsive delay differential equations* [21] by Church and Liu. Many of the computational aspects of Section 4.6 appear in *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [20] and *Analysis of a SIR model with pulse vaccination and temporary immunity: stability, bifurcation and a cylindrical attractor* [22], also by K. Church and X. Liu, although some additions have been made and some results have been updated or made more general to improve the flow of this thesis. The content of Section 4.6.2 is new.

4.1   Monodromy operator

We will once again be interested in the homogeneous linear system

$$\dot{x} = L(t)x_t, \qquad\qquad\qquad t \neq t_k \qquad\qquad (4.1)$$

$$\Delta x = B(k)x_{t^-}, \qquad\qquad\qquad t = t_k \qquad\qquad (4.2)$$

satisfying the conditions H.1 and H.2, but throughout this section we will assume the system is also *periodic*. That is, there exists $T > 0$ and $q \in \mathbb{N}\backslash\{0\}$ such that $L(t+T) = L(t)$ for all $t \in \mathbb{R}$, $B(k+q) = B(k)$ and $t_{k+q} = t_k + T$ for all $k \in \mathbb{Z}$.

We must discuss the interrelation between the period T and the range r of the delay. If $r < T$, it will at times be convenient to reinterpret the periodic system (4.1)–(4.2) as having the phase space $\mathcal{RCR}([-T, 0], \mathbb{R}^n)$. This can always be done, since each of $L(t)$ and

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$B(k)$ extend in an obvious, trivial way to $\mathcal{RCR}([-T, 0], \mathbb{R}^n)$. In the opposite case where $r \geq T$, we let $j \in \mathbb{N}$ satisfy $r \leq jT$ and extend the phase space to $\mathcal{RCR}([-jT, 0], \mathbb{R}^n)$. In both cases, the following proposition is true.

Proposition 4.1.1. *There exists $j \in \mathbb{N}$ minimal such that $r \leq jT$, and the evolution family $U(t, s)$ on $\mathcal{RCR}$ associated to the periodic system* (4.1)–(4.2) *extends uniquely to an evolution family $\tilde{U}(t, s)$ on $\mathcal{RCR}([-jT, 0], \mathbb{R}^n)$ satisfying the identity*

$$\tilde{U}(t, s)\phi(\theta) = U(t, s)\psi(\theta)$$

for all $\phi \in \mathcal{RCR}([-jT, 0], \mathbb{R}^n)$ and $\theta \in [-r, 0]$, where $\psi = \phi|_{[-r,0]}$. In particular, we have the representation

$$U(t, s) = \pi_\rightarrow \tilde{U}(t, s)\pi_\leftarrow$$

where the linear maps $\pi_\leftarrow : \mathcal{RCR} \rightarrow \mathcal{RCR}([-jT, 0], \mathbb{R}^n)$ and $\pi_\rightarrow : \mathcal{RCR}([-jT, 0], \mathbb{R}^n) \rightarrow \mathcal{RCR}$ are

$$\pi_\leftarrow\phi(\theta) = \begin{cases} \phi(\theta), & \theta \in [-r, 0], \\ 0, & \theta \in [-jT, -r) \end{cases} \qquad \pi_\rightarrow\phi = \phi|_{[-r,0]}.$$

Following the above proposition, we denote $\mathcal{RCR}_j = \mathcal{RCR}([-jT, 0], \mathbb{R}^n)$. For each $t \in \mathbb{R}$, define the *extended monodromy operators* $\tilde{Z}_t : \mathcal{RCR}_j \rightarrow \mathcal{RCR}_j$ and $Z_t : \mathcal{RCR} \rightarrow \mathcal{RCR}$ by

$$\tilde{Z}_t = \tilde{U}(t + jT, t), \quad Z_t = U(t + jT, t). \tag{4.3}$$

Then, define the *monodromy operators* $V_t : \mathcal{RCR} \rightarrow \mathcal{RCR}$ by

$$V_t = U(t + T, t). \tag{4.4}$$

4.2 Compactness

Recall that a linear operator $L : X \rightarrow X$ on a Banach space X is *compact* if the image under L of any bounded subset of X is relatively compact.

Lemma 4.2.1. *$\tilde{Z}_t$ is compact for each $t \in \mathbb{R}$.*

Proof. For a given $S \subset \mathbb{R}$, let PC_S denote the set of functions $f : [-jT, 0] \rightarrow \mathbb{R}^n$ that are continuous except at points $s \in S$, where they are right-continuous and possess limits on the left. If $\phi \in \mathcal{RCR}_j$, then $\tilde{Z}_t\phi$ is continuous except at times $\theta_n \in [-jT, 0]$ such that

$t + jT + \theta_n \in \{t_k : k \in \mathbb{Z}\}$. At such times, $\tilde{Z}_t\phi$ is continuous from the right and has limits on the left. Let $\Theta = \{\theta_1, \ldots, \theta_N\}$ denote the set of all such discontinuity points; note that $N = jq$ is indeed finite. Therefore, if $\mathcal{B} \subset \mathcal{RCR}_j$ is bounded, then $Y := \tilde{Z}_t(\mathcal{B}) \subset PC_\Theta$, the latter of which is complete with respect to the supremum norm.

By [8], a subset of $Y \subset PC_\Theta$ is precompact if and only if it is uniformly bounded and quasiequicontinuous – that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in [\theta_{k-1}, \theta_k) \cap [-jT, 0]$ satisfy $|t_1 - t_2| < \delta$, then $||x(t_1) - x(t_2)|| < \epsilon$ for all $x \in Y$. Uniform boundedness follows by equation (3.7). To obtain quasiequicontinuity, let $t_1 > t_2$ and $t = 0$ without loss of generality. We note that for all $\tilde{Z}_t x \in Y$,

$$
\begin{aligned}
||\tilde{Z}_t x(t_1) - \tilde{Z}_t x(t_2)|| &= ||U(jT + t_1, jT + t_2)U(jT + t_2, 0)x(0) - U(jT + t_2, 0)x(0)|| \\
&= ||\chi_0 \circ [U(jT + t_1, jT + t_2) - I]U(jT + t_2, 0)x|| \\
&\leq \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu \left(e^{\int_0^{jT} \ell(\mu)d\mu} \prod_{k=1}^{jq}(1 + b(k)) \right) C \\
&\equiv K \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu,
\end{aligned}
$$

where $||x|| \leq C$ for all $x \in \mathcal{B}$, and the inequality on the third line follows by Lemma 3.2.1 and the integral form of solutions provided by equation (3.3). Choosing δ so that $\int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu < \epsilon/K$ for $|t_1 - t_2| < \delta$ whenever $t_1, t_2 \in [-jT, 0]$ we obtain the required quasiequicontinuity of Y. Therefore, $Y = \tilde{Z}_t(\mathcal{B})$ is precompact, so $\tilde{Z}_t$ is compact. □

Lemma 4.2.2. Z_t *is compact for each* $t \in \mathbb{R}$.

Proof. By Proposition 4.1.1, we have $Z_t = \pi_\rightarrow \tilde{Z}_t \pi_\leftarrow$. The boundedness of each of $\pi_\rightarrow$ and $\pi_\leftarrow$ together with the compactness of $\tilde{Z}_t$ grants the compactness of Z_t. □

Lemma 4.2.3. $V_t^j : \mathcal{RCR} \to \mathcal{RCR}$ *is compact, where* j *is the same natural number used to define the extended monodromy operators.*

Proof. This is a direct consequence of Lemma 4.2.2, since $V_t^j = Z_t$. □

The eventual (i.e. jth power) compactness of V_t provides us with several useful results from the spectral theory of compact operators; one may consult the reference [51] for details. First, recall that if X is a real vector space, its *complexification* $X_\mathbb{C} = X \oplus X$ is a complex vector space with scalar multiplication defined by

$$(a + ib)(x_1, x_2) = (ax_1 - bx_2, bx_1 + ax_2).$$

One usually abuses notation and identifies an element $(x_1, x_2) \in X_{\mathbb{C}}$ with the formal symbol $x_1 + ix_2$. If $L : X \to X$ is a linear map, its complexification $L_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is defined by

$$L_{\mathbb{C}}(x + iy) = L(x) + iL(y).$$

Theorem 4.2.1. *Let $t \in \mathbb{R}$ and let σ_t denote the spectrum of $V_t^{\mathbb{C}} := (V_t)_{\mathbb{C}}$, the complexification of V_t.*

1. *If $\lambda \in \sigma_t$ is nonzero, then λ and $\overline{\lambda}$ are eigenvalues of $V_t^{\mathbb{C}}$.*

2. *The generalized eigenspace $M_{\lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$ associated to the eigenvalue $\lambda \in \sigma_t$ is finite-dimensional and invariant under $V_t^{\mathbb{C}}$.*

3. *The Riesz projection*

$$P_{\lambda,t} = \frac{1}{2\pi i} \int_{\gamma} (\xi I - V_t^{\mathbb{C}})^{-1} d\xi \qquad (4.5)$$

   is a projection onto $M_{\lambda,t}$, where γ is a simple continuous closed contour in $\mathbb{C}$ such that λ is the only eigenvalue of $V_t^{\mathbb{C}}$ contained in its interior.

4. *If $\Lambda \subset \sigma_t$, then*

$$P_{\Lambda,t} = \sum_{\lambda \in \Lambda} Res_{z=\lambda}(zI - V_t^{\mathbb{C}})^{-1}$$

   is a projection onto

$$M_{\Lambda,t} = \bigoplus_{\lambda \in \Lambda} M_{\lambda,t}.$$

5. *The projections $P_{\Lambda,t}$ commute with $V_t^{\mathbb{C}}$ and if Λ_1 and Λ_2 are disjoint, then $P_{\Lambda_1,t} P_{\Lambda_2,t} = 0$.*

6. *σ_t is bounded and $0 \in \sigma_t$ is the only accumulation point.*

We also have the following theorem concerning eigenvalues of distinct monodromy operators and their generalized eigenspaces, whose proof follows entirely verbatim the proof of Theorem 3.3 from [27].

Theorem 4.2.2. *Let $t, s \in \mathbb{R}$ be given with $t \geq s$ and let $\lambda \in \mathbb{C} \setminus \{0\}$.*

- *$\lambda \in \sigma_t$ if and only if $\lambda \in \sigma_s$*

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- *The restriction of $U_{\mathbb{C}}(t, s)$ to $M_{\lambda,s}$ is a topological isomorphism onto $M_{\lambda,t}$.*

Due to the uniqueness of the eigenvalues across all of the monodromy operators, the following definition is appropriate.

Definition 4.2.1. *The* Floquet multipliers *of the evolution family $U(t, s)$ are the eigenvalues $0 \neq \lambda \in \sigma_0$ of the (complexified) mondodromy operator V_0. The* multiplier spectrum *of the evolution family $U(t, s)$ is denoted $\sigma(U) := \sigma_0$.*

The projections of Theorem 4.2.1 take values in the complexified spaces $M_{\Lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$. To obtain real projections, it suffices to ensure that all conjugate multipliers are included in the set Λ. See Section IV, Theorem 2.18 and Corollary 2.19 of [27].

Corollary 4.2.2.1. *Let $0 \notin \Lambda \subset \sigma(U)$. If $\Lambda \subset \mathbb{C}$ is symmetric – that is, $\lambda \in \Lambda$ if and only if $\overline{\lambda} \in \Lambda$ – the projection $P_{\Lambda,t} : \mathcal{RCR}_{\mathbb{C}} \to \mathcal{RCR}_{\mathbb{C}}$ is the complexification of a projection operator on $\mathcal{RCR}$.*

By definition of complexification, if $x \in \mathcal{RCR}_{\mathbb{C}}$ is real (that is, $x = \xi + i0$ for some $\xi \in \mathcal{RCR}$), then $P_{\Lambda,t} x$ is also real. By abuse of notation, we will identify the complexification of said operator with itself whenever no confusion should arise. That is, we say that

$$P_{\Lambda,t} : \mathcal{RCR} \to \mathcal{RCR}$$

is also a projection, and is identified with its complexification. Similarly, we will sometimes blur the lines between a given operator $L : \mathcal{RCR} \to \mathcal{RCR}$ and its complexification $L_{\mathbb{C}} : \mathcal{RCR}_{\mathbb{C}} \to \mathcal{RCR}_{\mathbb{C}}$ whenever no confusion should result.

4.3   Spectral separation

Define the time-varying projectors

$$P_u(t) = P_{\Lambda_c, t}, \qquad P_c(t) = P_{\Lambda_c, t}, \qquad P_s(t) = I - P_u(t) - P_c(t) \qquad (4.6)$$

where $\Lambda_u = \{\lambda \in \sigma(U) : |\lambda| > 1\}$ and $\Lambda_c = \{\lambda \in \sigma(U) : |\lambda| = 1\}$. Since these sets are symmetric (about the imaginary axis), the first two operators above define, by Corollary 4.2.2.1, projections on $\mathcal{RCR}$. The third one is a complementary projector.

Lemma 4.3.1. *The projectors $P_i(t)$ for $i \in \{s, c, u\}$ are T-periodic.*

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Proof. Since $P_u(t)$ is the projector through the spectral subset Λ_u associated to the complexified operator $U_\mathbb{C}(t+T,t)$, it follows that $P_u(t+kT)$ is the projector through the same subset, associated to $U_\mathbb{C} = U_\mathbb{C}(t+T+kT,t+kT)$, for all $k \in \mathbb{Z}$. By uniqueness of solutions and the periodicity condition, the latter is equal to $U_\mathbb{C}(t+T,t)$, from which it follows that $P_u(t) = P_u(t+kT)$, and mutis mutandis for the other projectors. $\qquad\square$

Lemma 4.3.2. $\mathcal{R}(P_s(t)) = \bigcap_{\lambda \in \Lambda_c \cup \Lambda_u} \mathcal{N}(P_{\lambda,t})$.

Proof. Denote $P_{cu} = P_c + P_u$ and $\Lambda_{cu} = \Lambda_c \cup \Lambda_u$, so that $\mathcal{R}(P_s(t)) = \mathcal{N}(P_{cu}(t))$. If $x \in \bigcap_{\lambda \in \Lambda_{cu}} \mathcal{N}(P_{\lambda,t})$, then $P_{su}(t)x = \sum_{\lambda \in \Lambda_{cu}} P_{\lambda,t}x = 0$, which shows that $\bigcap_{\lambda \in \Lambda_{cu}} N(P_{\lambda,t}) \subseteq \mathcal{R}(P_s(t))$. To obtain the second inclusion, let $x \in \mathcal{RCR}_\mathbb{C}$ be such that $x \in \mathcal{N}(P_{cu}(t))$. For all $\lambda \in \Lambda_{cu}$, we have

$$P_{\lambda,t}x = P_{\lambda,t}^2 + \sum_{\mu \in \Lambda_{cu}\setminus\{\lambda\}} P_{\lambda,t}P_{\mu,t}x = P_{\lambda,t}P_{cu}(t)x = 0,$$

where the second equality is due to Theorem 4.2.1 and the conclusion is because $x \in \mathcal{N}(P_{cu}(t))$. $\qquad\square$

Lemma 4.3.3. *The restriction of $V_t^\mathbb{C}$ to the subspace $\mathcal{R}(P_s(t))$ has its spectrum contained in the ball $B_1(0) \subset \mathbb{C}$.*

Proof. With the same notation as in the previous proof, since the generalized eigenspaces $M_{\Lambda_{cu},t}$ are invariant under $V_t^\mathbb{C}$, the same is true for the (closed) complement, $\mathcal{R}(P_s(t))$. Denote $\tilde{V}_t^\mathbb{C}$ the restriction of $V_t^\mathbb{C}$ to said complement. Suppose by way of contradiction $\xi \in$ is a (generalized) eigenvector of $\tilde{V}_t^\mathbb{C}$ with eigenvalue λ with $|\lambda| \geq 1$. Then $(V_t^\mathbb{C} - \lambda I)^k \xi = (\tilde{V}_t^\mathbb{C} - \lambda I)\xi = 0$, so ξ is in fact a (generalized) eigenvector of $V_t^\mathbb{C}$ with eigenvalue λ and $|\lambda| \geq 1$. Consequently, $\xi \in \mathcal{R}(P_{cu}(t))$, which is a contradiction since $\mathcal{R}(P_s(t)) \cap \mathcal{R}(P_{cu}(t)) = \{0\}$. $\qquad\square$

Theorem 4.3.1. *The evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the periodic system* (4.1)–(4.2) *is spectrally separated, with projectors (P_s, P_c, P_u) defined as in equation* (4.6). *Also, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional.*

Proof. We prove the theorem by verifying properties 1–5 of Definition 3.2.4 explicitly.

1. Since $P_s + P_u + P_c = I$, we have the estimate $||P_s|| \leq 1 + ||P_u|| + ||P_c||$. Thus, to prove property 1, it suffices to prove that $||P_u(t)||$ and $||P_c(t)||$ are uniformly bounded. We will prove only uniform boundedness of $P_c(t)$, since the argument is identical for $P_u(t)$. Also, by periodicity (Lemma 4.3.1), it suffices to prove uniform boundedness on $[0,T]$.

Assume for the moment that property 3 and property 6 are satisfied (they will be proven later, independently of property 1). Suppose by way of contradiction that there exists $x_n \in \mathcal{RCR}$ and a sequence $t_n \in [0, T]$ with $||x_n|| = 1$ such that $||P_c(t_n)x_n|| = n$. We can then write

$$
\begin{aligned}
n = ||P_c(t_n)x_n|| &= ||U_c(t_n, T)U_c(T, t_n)P_c(t_n)x_n|| \\
&\leq ||U_c(t_n, T)|| \cdot ||P_c(T)U(T, t_n)x_n|| \\
&\leq C||U_c(t_n, T)|| \\
&\leq CKe^{\epsilon T}.
\end{aligned}
$$

for some constant $C \geq ||P_c(T)|| \cdot ||U(T, t_n)||$ (see Lemma 3.2.1) and a constant K as in property 6 of spectral separation. This is a contradiction.

2. This follows by property 5 of Theorem 4.2.1.

3. By following [27] Section XIII, Theorem 3.3, we can show that $P(t)U(t+T, s+kT) = U(t+T, s+kT)P(s)$ for some $k \in \mathbb{N}$ chosen so that $s + (k-1)T \leq t < s + kT$, for each of the projectors $P \in \{P_u, P_c, P_s\}$. This implies $P(t)U(t, s+qT) = U(t, s+qT)P(s)$ for $q = k - 1$. Thus,

$$
\begin{aligned}
P(t)U(t, s) &= P(t)U(t, s+qT)U(s+qT, s) \\
&= U(t, s+qT)P(s)U(s+T, s)^q \\
&= U(t, s+qT)P(s)^q U(s+T, s)^q \\
&= U(t, s+qT)U(s+T, s)^q P(s)^q \\
&= U(t, s)P(s),
\end{aligned}
$$

where we have used the fact that $P(s)$ is a projector and commutes with $U(s+T, s)$.

4. This follows from Theorem 4.2.2.

5. When $t \geq v \geq s$, the identity $U_c(t, s) = U_c(t, v)U_c(v, s)$ holds by properties of the evolution family U. When $t \geq s \geq v$, we find $I = U_c(t, v)^{-1}U_c(t, s)U_c(s, v)$, which implies

$$
U_c(v, s) = U_c(v, t)U_c(t, s). \tag{4.7}
$$

Also,

$$
U_c(t, s) = U_c(t, v)U_c(t, v)^{-1}U_c(t, s) = U_c(t, v)[U_c(v, t)U_c(t, s)] = U_c(t, v)U_c(v, s). \tag{4.8}
$$

Equation (4.7) implies $U_c(t, s) = U_c(t, v)U_c(v, s)$ for $v \geq s \geq t$, while (4.8) grants it for $t \geq s \geq v$. If $v \geq t \geq s$, then

$$
U_c(t, s) = U_c(t, v)U_c(t, v)^{-1}U_c(t, s) = U_c(t, v)U_c(v, t)U_c(t, s) = U_c(t, v)U_c(v, s).
$$

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If $s \geq t \geq v$, then

$$U_c(t,s) = U_c(s,t)^{-1} = [U_c(s,v)U(v,t)_c]^{-1} = U_c(t,v)U_c(v,s).$$

Similarly, the desired equality holds if $s \geq v \geq t$. We have proven that $U_c(t,s) = U_c(t,v)U_c(v,s)$ for all $t, v, s \in \mathbb{R}$. The proof is identical for U_u.

6. This section is split into two parts, where we prove the estimates for U_c and U_s separately. The proof for U_u is similar to the centre (U_c) case, and is omitted.

Centre part: U_c. Let $\epsilon > 0$ be given. Recall that $U_c(t,s)$ is the restriction of $U(t,s)$ to $\mathcal{R}(P_c(s))$, so by Lemma 3.2.1 and periodicity, there exists $K > 0$ such that, for any $s \in \mathbb{R}$, we have $||U_c(t,s)|| \leq K$ provided $t \in [s, s+T]$. As all eigenvalues of $U_c(s+T,s)$ satisfy $|\lambda| = 1$, Gelfand's (spectral radius) formula implies there exists an integer $k > 0$ such that $||U_c(s+T,s)^k|| < 1 + \epsilon T$. If we let m_t be the greatest integer such that $s + m_t kT \leq t$ and $m_t^* \in \{0, \ldots, k-1\}$ the greatest integer such that $s + m_t kT + m_t^* T \leq t$, we can write

$$U(t,s) = U(t - m_t kT - m_t^* T, s)U(s+T,s)^{m_t^* + km_t}.$$

Then, we can make the estimate

$$||U_c(t,s)|| \leq K||U_c(s+T,s)^k||^{m_t} \leq K(1+\epsilon T)^{\frac{t-s}{T}} \leq Ke^{\epsilon(t-s)}.$$

The proof is similar when $t \leq s$, and we obtain $||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}$.

Stable part: U_s. Let $t \geq s$. As in the proof for the centre part, we have $||U_s(t,s)|| \leq K$ provided $t \in [s, s+T]$. Otherwise, since $U_s(s+T,s)$ has its spectrum in the complex unit ball by Lemma 4.3.3, there exists $k > 0$ such that $||U_s(s+T,s)^k|| \leq (1+aT)$ for some $a < 0$. The rest of the proof follows by the same reasoning as the proof for the centre part, and we obtain $||U(t,s)|| \leq Ke^{a(t-s)}$ as required.

Finally, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional because Theorem 4.2.1 guarantees that the invariant subspaces $M_{\Lambda_c,t}$ and $M_{\Lambda_u,t}$ are finite-dimensional provided Λ_c and Λ_u are each finite sets — which they are because the eigenvalues of $V_t^{\mathbb{C}}$ can only accumulate at zero. The analogous result for $\mathcal{RCR}_c(t)$ and $\mathcal{RCR}_u(t)$ follows by Corollary 4.2.2.1. $\qquad\square$

Remark 4.3.1. *The centre and unstable fiber bundles $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are subsets of $\mathcal{RCR}^1$. This follows because $V_t : \mathcal{RCR} \to \mathcal{RCR}$ has range in $\mathcal{RCR}^1$ and the t-fibers of these bundles consist of eigenvectors of V_t. The same is not generally true for the stable fiber bundle, being the range of $I - P_u(t) - P_c(t)$.*

4.4   Floquet theorem

Of use in subsequent sections is the fact that, when restricted to the nonautonomous set $\mathcal{RCR}_c$, the evolution family $U(t,s)$ is essentially determined by the flow of a finite-dimensional ordinary differential equation. More generally, the same is true whenever one restricts $U(t,s)$ to one of its invariant fibre bundles. The following theorem makes this concrete; see Theorem 4.5 from Section XIII of [27] for the analogous result for delay differential equations.

Theorem 4.4.1. *Let $\Lambda \in \sigma(U)$ be finite and satisfy $\Lambda = \overline{\Lambda}$, and denote $U_\Lambda(t,s) : M_{\Lambda,s} \to M_{\Lambda,t}$ the restriction of $U(t,s)$ to $M_{\Lambda,s}$. There exists $W \in \mathcal{L}(M_{\Lambda,0})$ and $t \mapsto \alpha(t) \in \mathcal{L}(M_{\Lambda,0}, M_{\Lambda,t})$ with the following properties.*

- *α is T-periodic, $\alpha(t)$ is invertible, and there exists $\beta \geq 1$ such that for all $\phi \in M_{\Lambda,0}$,*

$$\beta^{-1}||\phi|| \leq \sup_{t \in \mathbb{R}} ||\alpha(t)\phi|| \leq \beta||\phi||.$$

- *$U_\Lambda(t,0)\phi = \alpha(t)e^{tW}\phi$ for all $\phi \in M_{\Lambda,0}$.*

Proof. Define $W = \log U_\Lambda(T,0)$, where we choose the logarithm to be branch that avoids the (finite set of nonzero) eigenvalues of $U_\Lambda(T,0)$. Defining $\alpha(t) = U_\Lambda(t,0)e^{-tW}$, one may verify (compare to Proposition 4.4 and Theorem 4.5 from Section XIII of [27]; the proofs in the present case is essentially identical) that α is periodic and $U_\Lambda(t,0)$ satisfies the claimed decomposition. Uniform boundedness of α above and below follows by its periodicity and boundedness of $U_\Lambda(t,0)$ on $[-T,T]$; see the related proof of Theorem 4.3.1. $\alpha(t)$ is clearly invertible. $\square$

The above theorem applied to $\Lambda = \Lambda_c$, the Floquet multipliers on the unit circle, gives the following corollary.

Corollary 4.4.1.1. *Denote $\mathcal{RCR}_c^{\mathbb{C}}(t)$ the complexification of $\mathcal{RCR}_c(t)$. There exists $W \in \mathcal{L}(\mathcal{RCR}_c^{\mathbb{C}}(0))$ and $\alpha : \mathbb{R} \to \mathcal{L}(\mathcal{RCR}_c^{\mathbb{C}}(0), \mathcal{RCR}_c^{\mathbb{C}})$ with the following properties.*

- *α is T-periodic, $\alpha(t) : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(t)$ is invertible, and there exists $\beta \geq 1$ such that for all $\phi \in \mathcal{RCR}_c^{\mathbb{C}}(0)$,*

$$\beta^{-1}||\phi|| \leq \sup_{t \in \mathbb{R}} ||\alpha(t)\phi|| \leq \beta||\phi||.$$

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- $U_c(t,0)\phi = \alpha(t)e^{tW}\phi$ for all $\phi \in \mathcal{RCR}_c(0)$.

Remark 4.4.1. *As we suggested we might do on occasion at the end of Section 4.2, we have been a bit imprecise with our complexifications. While true, the result of the above corollary can be made stronger. The final equation could be replaced with $U_c^{\mathbb{C}}(t,0)\phi = \alpha(t)e^{tW}\phi$, and this equality holds for all $\phi \in \mathcal{RCR}_c^{\mathbb{C}}(0)$.*

4.5 Floquet exponents and stability

The Floquet multipliers of the evolution family $U(t,s)$ determine the stability of the periodic homogeneous system (4.1)–(4.2). We give a few definitions before stating and proving the main results.

Definition 4.5.1. *We say the homogeneous impulsive RFDE (4.1)–(4.2) is*

- exponentially stable *if there exists $K > 0$ and $\epsilon > 0$ such that its evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ satisfies $||U(t,s)|| \leq Ke^{-\epsilon(t-s)}$ for all $t \geq s$;*

- stable *if there exists $K \geq 0$ such that its evolution family satisfies $||U(t,s)|| \leq K$ for all $t \geq s$;*

- unstable *if it is not stable.*

Definition 4.5.2. *Let X be a complex n-dimensional vector space. A linear operator $L : X \to X$ is* diagonalizable *if there exists a basis $B = \{x_1, \ldots, x_n\}$ such that the $n \times n$ matrix L_B of L relative to the basis B is diagonalizable.*

It is simple to verify that the above notion of diagonalizability is well-defined, in that if L_B is diagonalizable and B_2 is another basis, then L_{B_2} is also diagonalizable.

Theorem 4.5.1. *The periodic impulsive RFDE (4.1)–(4.2) is exponentially stable if and only if $\sigma(U) \subset B_1(0)$. It is stable if and only if $\sigma(U) \subset \overline{B_1(0)}$ and the linear operator $W : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(0)$ from the Floquet decomposition on the centre fibre bundle is diagonalizable.*

Proof. We first deal with exponential stability. By Theorem 4.3.1, $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated. If $\sigma(U) \subset B_1(0)$, then the sets $\Lambda_u = \{\lambda \in \sigma(U) : |\lambda| > 1\}$ and

$\Lambda_c = \{\lambda \in \sigma(U) : |\lambda| = 1\}$ are empty, so that $P_c(t) = P_u(t) = 0$. It follows that $P_s(t) = I$. Then, by spectral separation, there exists $K \geq 0$ and $a < 0$ such that

$$||U(t,s)|| = ||U(t,s)P_s(s)|| \leq Ke^{a(t-s)}, \quad t \geq s.$$

Conversely, if $||U(t,s)|| \leq Ce^{-\epsilon(t-s)}$ for some $C > 0$ and $\epsilon > 0$, then Λ_u and Λ_c must be empty. Indeed, suppose Λ_c is nonempty, so that there exists a nonzero $\xi \in \mathcal{RCR}$ such that $||U(t + kT, t)\xi|| = ||\xi||$ for all $k \in \mathbb{N}$. But then, we have the estimate $||\xi|| \leq C||\xi||e^{-\epsilon kT}$ for all positive integers k, which implies $\xi = 0$, a contradiction. Similarly, Λ_u is empty. By Lemma 4.3.3, it follows that $\sigma(U) \subset B_1(0)$.

Next we prove the claims concerning stability. Suppose $\sigma(U) \subset \overline{B_1(0)}$ and that W is diagonalizable. The former assumption implies $P_u(t) = 0$. Recall that $U_c(t,s) : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(t)$ defines an all-time process on $\mathcal{RCR}_c$ by definition of spectral separation, so we may write

$$U_c(t,s) = \alpha(t)e^{(t-s)W}\alpha^{-1}(s)$$

using Corollary 4.4.1.1. Moreover, by definition of W all of its eigenvalues have zero real part. From diagonalizability it follows that $t \mapsto ||e^{tW}||$ is bounded. Then,

$$\begin{aligned}
||U(t,s)\phi|| &\leq ||U(t,s)P_s(s)|| + ||U(t,s)P_c(s)|| \\
&\leq Ke^{a(t-s)} + ||U_c(t,s)|| \\
&\leq K + ||\alpha(t)|| \cdot ||\alpha^{-1}(s)|| \cdot ||e^{(t-s)W}||.
\end{aligned}$$

The condition $\beta^{-1}||\phi|| \leq \sup_t ||\alpha(t)\phi|| \leq \beta||\phi||$ from the Floquet theorem implies that $||\alpha^{-1}(s)||$ is bounded, from which we conclude that $||U(t,s)||$ is bounded by a constant independent of $t \geq s$.

Suppose now that $||U(t,s)|| \leq K$ for all $t \geq s$. By similar arguments to the previous case, we must have $\sigma(U) \subset \overline{B_1(0)}$. If W were not diagonalizable, then the Jordan canonical form of W_B relative to some basis B would contain a block whose exponential grows at least linearly in t. In particular, there exists $\delta > 0$ and $D > 0$ such that $||e^{tW_B}|| \geq Dt$ for $|t| \geq \delta$. If $C : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathbb{C}^c$ denotes the coordinate map satisfying $C(r_i) = e_i$ for $B = \{r_1, \ldots, r_n\}$, then we can write $W = C^{-1}W_BC$. From the convergent power series definition of the exponential, it follows that

$$||U_c(t,s)|| = ||\alpha(t)e^{(t-s)W}\alpha^{-1}(s)|| \geq \frac{1}{\beta}||C^{-1}e^{(t-s)W_B}C\alpha^{-1}(s)|| \geq \frac{D(t-s)}{||C|| \cdot ||C^{-1}||}$$

for $|t - s| \geq \delta$. If $\phi \in \mathcal{RCR}_c(s)$, the above implies $||U(t,s)\phi|| = ||U_c(t,s)\phi|| \to \infty$ as $t \to \infty$, which contradicts the uniform boundedness of $U(t,s)$. $\qquad\square$

For autonomous ordinary differential equations $\dot{x} = Ax$, the eigenvalues of A determine the stability. In this case, the analogue of Theorem 4.5.1 states that stability is equivalent to all eigenvalues of A having negative real part, while stability is equivalent to them having nonpositive real part and A being diagonalizable over $\mathbb{C}$. In some sense, the eigenvalues correspond to growth rates of solutions in the distinct invariant subspaces. With delay differential equations the statement is similar, except it is in terms of the eigenvalues of the generator of the solution semigroup; one may refer to Section IV of [27]. For periodic systems, there is a natural analogue of "eigenvalue" that captures the same idea of growth rates.

Definition 4.5.3. *The* Floquet spectrum *of the evolution family* $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ *is the set* $\lambda(U) = \{\frac{1}{T} \log(\mu) : \mu \in \sigma(U)\}$, *where the principal branch of the logarithm is taken. Its elements are called* Floquet exponents.

The Floquet exponents truly are the growth rates of solutions in the invariant Fibre bundles associated to their respective Floquet multipliers, as made precise by Theorem 4.4.1. As for stability, we can reformulate Theorem 4.5.1 in terms of Floquet exponents as follows.

Corollary 4.5.1.1. *The periodic impulsive RFDE* (4.1)–(4.2) *is exponentially stable if and only if all Floquet exponents of its evolution family have negative real parts. It is stable if and only if all Floquet exponents have zero real part and the linear operator* $W : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(0)$ *from the Floquet decomposition on the centre fibre bundle is diagonalizable.*

4.6   Computational aspects

This section is devoted to some of the computational aspects associated to Floquet exponents and the monodromy operator.

4.6.1   Floquet eigensolutions

Suppose that μ is a Floquet multiplier of $U(T,0)$. Let $B = \{\xi_1, \ldots, \xi_m\}$ be a basis for the generalized eigenspace $M_{\mu,0}$. Then,

$$U(T,0)\xi_j = \mu \xi_j$$

for $j = 1, \ldots, m$. From the Floquet Theorem 4.4.1, there exists a linear operator $W :$ $M_{\mu,0} \to M_{\mu,0}$ and $\alpha(t) : M_{\mu,0} \to M_{\mu,t}$ periodic in t such that $U_\mu(t, 0) = \alpha(t)e^{tW}$. As in the proof of Theorem 4.5.1, let $C : M_{\mu,0} \to \mathbb{C}^m$ denote the coordinate map satisfying $C(\xi_j) = e_j$. Let $W_B = PJP^{-1}$ be the Jordan canonical form of the matrix W_B of the operator W relative to the basis B. If we compute the action of $U_\lambda(t, 0)$ on the basis elements ξ_j, we get

$$U_\lambda(t, 0)\xi_j = \alpha(t)C^{-1}Pe^{tJ}P^{-1}e_j. \tag{4.9}$$

Also, since $U_\mu(T, 0) = e^{TW}$, it follows that the eigenvectors satisfy the equation $e^{TW}\xi_j = \mu\xi_j$. In terms of coordinate maps and the Jordan form of W_B, this gives

$$C^{-1}Pe^{TJ}P^{-1}C\xi_j = \mu\xi_j \implies Pe^{TJ}P^{-1}(C\xi_j) = \mu(C\xi_j).$$

It follows that $\frac{1}{T}\log\mu$ is the only eigenvalue of J. That is, J is a Jordan matrix whose only eigenvalue is the Floquet exponent $\lambda = \frac{1}{T}\log\mu$. We can then express $e^{tJ}P^{-1}e_j$ as a sum of the form

$$e^{tJ}P^{-1}e_j = \sum_{i=1}^{m} t^{i-1}e^{\lambda t}v_i$$

for some vectors $v_i \in \mathbb{R}^m$. If we now define the function $z_j(t) = U(t, 0)\xi_j(0)$, we can use the above summation formula in (4.9) to obtain the representation

$$z_j(t) = \sum_{i=1}^{m} p_i(t)t^{i-1}e^{\lambda t},$$

where $p_i(t) = [\alpha(t)C^{-1}Pv_i](0)$. By definition, $p_i : \mathbb{R} \to \mathbb{C}^n$ is periodic and is right-differentiable except at times t_k where it is continuous from the right and has a finite jump discontinuity. Moreover, z_j is a solution of the (complexified) homogeneous system (4.1)–(4.2). This proves the following theorem.

Theorem 4.6.1. *λ is a Floquet exponent of the evolution family $U(t, s)$ associated to the linear homogeneous system (4.1)–(4.2) if and only if the latter admits a solution of the form $x(t) = e^{\lambda t}p(t)$ for a nonzero T-periodic $p \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^m)$. Moreover, the m-dimensional generalized eigenspace $M_{e^{\lambda T}, t}$ is spanned by elements x_t such that $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ is a solution of (4.1)–(4.2) admitting a decomposition of the form*

$$x(t) = e^{\lambda t}\sum_{i=1}^{m} p_i(t)t^{i-1}, \tag{4.10}$$

for $p_i \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^m)$ and T-periodic.

Definition 4.6.1. *A solution of the linear homogeneous system* (4.1)–(4.2) *having a decomposition of the form* (4.10) *is a* Floquet eigensolution *with exponent* λ. *Its* rank *is* $d = \max\{i = 1, \ldots, m : p_i \neq 0\}$.

Theorem 4.6.1 states that solutions that pass through a generalized eigenspace $M_{e^{\lambda T}, t}$ for each Floquet exponent λ are linear combinations of complex vector polynomials with periodic coefficients, multiplied by the exponential growth factor $e^{\lambda t}$. This motivates the following definition.

Definition 4.6.2. *The* generalized eigenspace of $U(t, s)$ with Floquet exponent λ *is the invariant fibre bundle* E_λ *with t-fibre* $E_\lambda(t) = M_{e^{\lambda T}, t}$.

Corollary 4.6.1.1. *Considered as a subset of* $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$, *for each Floquet exponent* λ *there exists a maximal linearly independent set* $S = \{\phi^{(1)}, \ldots, \phi^{(m)}\}$ *of Floquet eigensolutions with exponent* λ *for some natural number* m. *The generalized eigenspace with Floquet exponent* λ *is given by the linear span of the histories of the elements of S; that is,* $E_\lambda(t) = \mathrm{span}\{\phi_t^{(1)}, \ldots, \phi_t^{(m)}\}$.

Proof. The existence of S follows from the finite dimensionality (Theorem 4.2.1) of $M_{e^{\lambda T}, s}$ for any s, and that these eigenspaces are isomorphic (Theorem 4.2.2). That S can be chosen to be linearly independent in $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ follows because one can impose that the histories $\{\phi_0^{(1)}, \ldots, \phi_0^{(m)}\}$ form a basis for $M_{e^{\lambda T}, 0} \subset \mathcal{RCR}([-T, 0], \mathbb{C}^n)$, which then implies the independence of S in $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$. The characterization of the t-fibre $E_\lambda(t)$ then follows from Theorem 4.6.1. $\qquad\square$

The following corollary to Theorem 4.6.1 provides a dynamical characterization of Floquet exponents. Its proof follows by substituting the Floquet eigensolution ansatz $x(t) = e^{\lambda t} p(t)$ into the periodic homogeneous system (4.1)–(4.2) and simplifying.

Corollary 4.6.1.2. *Define* $\exp_\lambda : [-T, 0] \to \mathbb{C}$ *by* $\exp_\lambda(\theta) = e^{\lambda \theta}$. λ *is a Floquet exponent of the evolution family* $U(t, s)$ *associated to the linear homogeneous system* (4.1)–(4.2) *if and only there exists a nonzero T-periodic* $p \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ *satisfying the impulsive functional differential equation*

$$\dot{p} + \lambda p = L(t)[\exp_\lambda p_t], \qquad\qquad t \neq t_k \qquad\qquad (4.11)$$
$$\Delta p = B(k)[\exp_\lambda p_{t^-}], \qquad\qquad t = t_k. \qquad\qquad (4.12)$$

From Corollary 4.6.1.2, we see that there is generally no easy way to extend the idea of the characteristic equation from autonomous ordinary differential equations or delay differential equations to the present setting. A good reference on these topics for autonomous

functional differential equations is Chapter 7 of the book by Jack Hale and Sjoerd Verduyn Lunel [39]. For an autonomous retarded functional differential equation

$$\dot{x} = Lx_t,$$

the function $v\exp_\lambda$ for some nonzero $v \in \mathbb{C}^n$ is always contained within the generalized eigenspace E_λ associated to the eigenvalue λ of the infinitesimal generator of the strongly continuous (solution) semigroup. In particular, $t \mapsto e^{\lambda t}v$ must be a solution. Substituting this into the above delay differential equation and simplifying produces the equation $\lambda v = L(\exp_\lambda v)$, which can be written equivalently as

$$(L(\exp_\lambda) - \lambda I)v = 0, \tag{4.13}$$

where we define $L(\exp_\lambda) = [\ L(e_1\exp_\lambda)\ \ \cdots\ \ L(e_n\exp_\lambda)\ ] \in \mathbb{C}^{n \times n}$. The term in parentheses in (4.13) is a complex $n \times n$ matrix – the *characteristic matrix* (see Section I.3 of [27])– and $v \in \mathbb{C}^n$. It follows that λ is an eigenvalue if and only if

$$\det(L(\exp_\lambda) - \lambda I) = 0. \tag{4.14}$$

Equation (4.14) is the *characteristic equation* for $\dot{x} = Lx_t$. It is a scalar equation that is generally transcendental in λ. It is difficult to formulate a similar equation for impulsive RFDE because the equivalent dynamical characterization for Floquet exponents is precisely given by Corollary 4.6.1.2, which contains functional terms that cannot be simplified further. For impulsive RFDE, the periodic function $p(t)$ plays the role of the constant vector v, and it is for this reason the functional terms cannot generally be simplified.

4.6.2 Characteristic equation for some special classes of linear systems

For some special classes of impulsive RFDE one can define a characteristic equation in a straightforward way, or at least reduce the problem of computing Floquet exponents to a finite-dimensional problem. The special class we will look at is introduced in the following definition.

Definition 4.6.3. *The linear periodic system* (4.1)–(4.2) *is* finitely reducible *if $L(t)$ and*

$B(k)$ can be written in the form

$$\tilde{L}(t)\phi = \sum_{j=0}^{\ell} \tilde{L}_j(t)\phi(-jT), \tag{4.15}$$

$$\tilde{B}(k)\phi = \int_{-h_k}^{0} C_k(s)\phi(s)ds + \sum_{j=0}^{\ell} \tilde{B}_j(k)\phi(-jT), \tag{4.16}$$

for some $\ell \geq 0$ and matrices $\tilde{A}_i(t)$ and $\tilde{B}_i(k)$, and continuous $C_k : [-h_k, 0] \to \mathbb{R}^{n \times n}$, with $h_k \leq t_k - t_{k-1}$.

To motivate the definition of finite reducibility, it is helpful to look at the dynamical characterization (4.11)–(4.12) of the Floquet exponents. If $L(t)$ can be written as $\tilde{L}(t)$ from (4.15), then the right-hand side of the functional differential equation (4.11) becomes

$$L(t)[\exp_\lambda p_t] = \sum_{j=0}^{\ell} \exp(-\lambda jT)\tilde{L}_j(t)p(t - jT)$$

$$= \left(\sum_{j=0}^{\ell} \exp(-\lambda jT)\tilde{L}_j(t) \right) p(t),$$

where in the second line we used the periodicity of p. As a consequence, between the impulse times the function p actually satisfies the ordinary differential equation

$$\dot{p} = \left(-\lambda I + \sum_{j=0}^{\ell} e^{-\lambda jT}\tilde{L}_j(t) \right) p \tag{4.17}$$

parameterized by the parameter λ. Let us denote $X_\lambda(t, s)$ its associated Cauchy matrix. At the impulse times, equation (4.12) becomes

$$p(t) - p(t^-) = \int_{-h_k}^{0} C_k(s)e^{\lambda s}p(t + s)ds + B_0(k)p(t^-) + \sum_{j=1}^{\ell} B_j(k)e^{-\lambda jT}p(t), \tag{4.18}$$

with $t = t_k$, after exploiting the periodicity of p. Since $h_k \leq t_k - t_{k-1}$, equation (4.17) implies $p(t_k + s) = X_\lambda(t_k + s, t_k)p(t_k^-)$ for all $s \in [-h_k, 0]$. Substituting into (4.18), it follows that

$$\Delta p(t_k) = \left(\sum_{j=1}^{\ell} B_j(k)e^{-\lambda jT} \right) p(t_k) + \left(\int_{-h_k}^{0} C_k(s)e^{\lambda s}X_\lambda(t_k + s, t_k)ds + B_0(k) \right) p(t_k^-). \tag{4.19}$$

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Assuming $x(t) = p(t)e^{\lambda t}$ is a Floquet eigensolution, equation (4.19) relates the value of $p(t_k^-)$ with that at $p(t_k)$, while the ordinary differential equation (4.17) determines the evolution of the state $p(t_k)$ to $p(t_{k+1}^-)$. Completing the argument in reverse in a similar manner, the following equivalence theorem is proven.

Theorem 4.6.2. *Let the linear periodic system* (4.1)–(4.2) *be finitely reducible.* λ *is a Floquet exponent if and only if there exist* $p_1, \ldots, p_c \in \mathbb{C}^n$ *not all zero such that*

$$\left(I + \int_{-h_k}^{0} C_k(s) X_\lambda(t_k + s, t_k) e^{\lambda s} ds + B_0(k) \right) X_\lambda(t_k, t_{k-1}) p_{k-1} = \left(I - \sum_{j=1}^{\ell} B_j(k) e^{-\lambda j T} \right) p_k, \tag{4.20}$$

for $k = 1, \ldots, c$, *where we define* $p_0 := p_c$. *If this is the case, then* $x(t) = p(t)e^{\lambda t}$ *is a Floquet eigensolution with the periodic function* $p : \mathbb{R} \to \mathbb{C}^n$ *defined by*

$$p(t) = X_\lambda(t, t_k) p_{[k]_c}, \qquad t \in [t_k, t_{k+1}).$$

To summarize, one can check whether a given λ is a Floquet exponent of a finitely reducible periodic system by solving the cyclic system of finite-dimensional equations (4.20). An explicit result is provided by the following corollary.

Corollary 4.6.2.1. *Suppose the linear periodic system* (4.1)–(4.2) *is finitely reducible with* $C_k = 0$ *and* $\det(I + B_0(k)) \neq 0$ *for* $k = 1, \ldots, c$. λ *is a Floquet exponent if and only if it satisfies the* characteristic equation

$$\det\left(I - \prod_{k=1}^{c} X_\lambda(t_{k-1}, t_k)(I + B_0(k))^{-1} \left(I - \sum_{j=1}^{\ell} B_j(k) e^{-\lambda j T} \right) \right) = 0, \tag{4.21}$$

where the product denotes composition from left to right: $\prod_{k=1}^{c} M_k = M_1 \cdots M_c$.

Proof. From (4.20) of Theorem 4.6.2, one can uniquely write

$$p_{k-1} = X_\lambda(t_{k-1}, t_k)(I + B_0(k))^{-1} M_k p_k,$$

where $M_k p_k$ is the term on the right-hand side of of (4.20) for M_k the matrix term in parentheses. It follows that

$$p_0 = \left(\prod_{k=1}^{c} X_\lambda(t_{k-1}, t_k)(I + B_0(k))^{-1} M_k \right) p_c.$$

From the cyclic condition, we must have $p_c = p_0$. It follows that there must be a nonzero solution of the equation $(I - \prod_{k=1}^{c} X_\lambda(t_{k-1}, t_k)(I + B_0(k))^{-1} M_k) p_0 = 0$, from which we obtain equation (8.38). □

Equation (4.21) provides a generalization of the characteristic equation (4.14) from autonomous functional differential equations to impulsive RFDEs with discrete delays being multiples of the period. It has minimal practical applications because delays are not usually multiples of the period. Characteristic matrices (and, consequently, characteristic equations) have been derived for periodic differential-difference equations [79, 83] under minimal assumptions, and there is certainly a possibility of extending these results to impulsive RFDE. This being said, in implementing the cited characteristic matrix construction, one must ultimately perform some kind of discretization and collocation procedure to approximate an abstract linear operator. This leads us naturally to explore related discretization schemes.

4.6.3   Monodromy operator discretization

In practice, it is efficient to discretize the monodromy operator with a high degree of precision and compute approximate Floquet multipliers μ, then obtaining the Floquet exponents by computing $\lambda = \frac{1}{T} \log \mu$. For bifurcation studies this is often sufficient. To simplify the presentation of this section, we will consider the periodic, linear homogeneous impulsive RFDE with a single discrete delay:

$$
\begin{aligned}
\dot{x} &= A(t)x(t) + B(t)x(t - \tau), \quad t \neq k\tau \\
\Delta x &= Cx(t^-) + Dx(t - \tau), \qquad t = k\tau,
\end{aligned}
\tag{4.22}
$$

where the impulses occur at periodic times $k\tau$ for $k \in \mathbb{Z}$, and $A(t)$ and $B(t)$ are periodic with period τ. The methodology of this section could be easily adapted to account for multiple delays, distributed delays and for period T incommensurate with the delay.

Recall that the Floquet exponents are independent of the choice of monodromy operator – see Theorem 4.2.2 – so we may restrict our attention to the monodromy operator $V_0 :$ $\mathcal{RCR} \to \mathcal{RCR}$. If we denote $X(t, s)$ for $t \geq s$ the Cauchy matrix associated to the linear ordinary differential equation $\dot{y} = A(t)y$ and set $X(t, s) = 0$ for $t < s$, then the variation of constants formula for ordinary differential equations implies that the solution of (4.22) passing through the initial function $\phi \in \mathcal{RCR}$ at time 0 can be written for $t \in [0, \tau]$ as follows:

$$
\begin{aligned}
x(t) = {}& X(t, 0)\phi(0) + \int_0^t X(t, s)B(s)\phi(s - \tau)ds \\
& + \chi_{\{\tau\}}(t)\left[(CX(\tau, 0) + D)\phi(0) + C\int_0^\tau X(\tau, s)B(s)\phi(s - \tau)ds\right].
\end{aligned}
$$

Evaluating at $t = \tau + \theta$ and taking $\theta \in [-\tau, 0]$ as the argument, exploiting the periodicity $X(s_1 + \tau, s_2 + \tau) = X(s_1, s_2)$ and $B(s + \tau) = B(s)$ and simplifying the expression using a few changes of variables, we can express the monodromy operator as

$$V_0 \phi(\theta) = [X(\tau + \theta, 0) + \chi_0(\theta)\,(CX(\tau, 0) + D)]\,\phi(0)$$
$$+ \int_{-\tau}^{0} [X(\theta, s) + \chi_0(\theta)CX(0, s)]\,B(s)\phi(s)ds. \tag{4.23}$$

Our next key observation is that if v is an eigenvector of V_0, then v is C^1 (in fact, C^∞) on $[-r, 0)$ with a finite jump discontinuity at zero. If we define

$$X = \{\phi : [-r, 0] \to \mathbb{C}^n : \phi|_{[-r,0)} \in C^1,\; |\phi(0) - \phi(0^-)| < \infty\},$$

then X is an invariant subspace of V_0 and it contains all of its eigenvectors. Therefore, we may consider the restriction $V_0 : X \to X$ instead of the action of the monodromy operator on the entire space $\mathcal{RCR}$.

The representation (4.23) and the description of X suggests a decomposition of ϕ into its part at $\phi = 0$ and on $[-\tau, 0)$. To do this we introduce the product space $Y = C^1([-\tau, 0], \mathbb{C}^n) \times \mathbb{C}^n$. The function $G : Y \to X$ defined by

$$G(y_1, y_2) = \chi_{[-\tau,0)}y_1 + \chi_{\{0\}}y_2$$

is an isometry, so we will make the identification $X \sim Y$. Then, for $(\phi_1, \phi_2) \in Y$, we can write

$$V_0 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} X(\tau + \cdot, 0)\phi_2 + \int_{-\tau}^{0} X(\cdot, s)B(s)\phi_1(s)ds \\ ((I + C)X(\tau, 0) + D)\phi_2 + \int_{-\tau}^{0}(I + C)X(0, s)B(s)\phi_1(s)ds \end{bmatrix}. \tag{4.24}$$

We are ready to discretize the (complexified) monodromy operator. Let $N \geq 1$ denote the number of mesh points, and let $-\tau < s_1 < \cdots < s_N < 0$ denote the Gaussian quadrature points in the interval $[-\tau, 0]$. Let $w_1, \ldots, w_N$ denote the associated weights. Then, we have the limit

$$\lim_{N \to \infty} \sum_{k=1}^{N} f(s_k)w_k = \int_{-\tau}^{0} f(s)ds$$

for any f continuous on $[-\tau, 0]$. We "sample" $\phi_1 \in C^1([-\tau, 0], \mathbb{C}^n)$ using the vector

$$\hat{\phi}_1 = [\; \phi_1(s_1) \;\; \cdots \;\; \phi_1(s_N) \;]^{\mathsf{T}} \in \mathbb{C}^{nN}$$

and, taking into account the convergence of the Gaussian quadrature and using the identity $X(\tau + x, \tau + y) = X(x, y)$, we can make the approximation $V_0 \phi \approx V_{0,N} [\ \hat{\phi}_1 \quad \phi_2\ ]^\mathsf{T}$, where

$$V_{0,N} = \begin{bmatrix} X(s_1, s_1)B(s_1)w_1 & \cdots & X(s_1, s_N)B(s_N)w_N & X(s_1, -\tau) \\ \vdots & \ddots & \vdots & \vdots \\ X(s_N, s_1)B(s_1)w_1 & \cdots & X(s_N, s_N)B(s_N)w_N & X(s_N, -\tau) \\ (I+C)X(0, s_1)B(s_1)w_1 & \cdots & (I+C)X(0, s_N)B(s_N)w_N & (I+C)X(0, -\tau) + D \end{bmatrix}. \quad (4.25)$$

Given the compactness (Lemma 4.2.2) of V_0, we should hope that any eigenvalue of V_0 is given by the limit of some eigenvalue of $V_{0,N}$ as $N \to \infty$ provided $V_{0,N} \to V_0$ pointwise (after defining $V_{0,N}$ on X by an appropriate embedding, so as to make this convergence sensical). Consequently, the spectrum of $V_{0,N}$ may be seen as an approximation of that of V_0, so we may approximate the Floquet spectrum via

$$\lambda(U) \approx \left\{ \frac{1}{\tau} \log \mu : \mu \in \sigma(V_{0,N}) \right\}.$$

We will not delve into a proof of the convergence and approximation claims at this time, as this is not the primary focus of this thesis. We refer the reader to [35], where a similar scheme is developed for the discretization of linear periodic delay differential equations and convergence results are proven.

When we use this method to approximate Floquet exponents in later sections, we use a numerically computed Cauchy matrix $X(t, s)$ generated by MATLAB's `ode45` function. The entries $X(s_i, s_j)$ appearing in the matrix $V_{0,N}$ are zero for $i < j$ and are the identity for $i = j$, so most superdiagonal entries do not need to be calculated. As for the nonzero entries, we remark that

$$X(s_i, s_j) = X(s_i, -\tau)[X(s_j, -\tau)]^{-1},$$

so it is enough to compute $X(s_k, -\tau)$ for $k = 1, \ldots, N$ mesh points and recycle these matrices to compute all others. This speeds up the process of computing the matrix $V_{0,N}$ considerably.

Chapter 5

Nonlinear systems

The content of Section 4.1 and Section 4.2 appears in *Smooth centre manifolds for impulsive delay differential equations* [21] by Church and Liu. The linearized stability result of Section 4.3 was stated without proof in the body text of *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [20]. We have formalized and proven it here.

5.1 Mild solutions

Our attention shifts now to the semilinear system

$$\dot{x} = L(t)x_t + f(t, x_t), \qquad\qquad t \neq t_k \qquad\qquad (5.1)$$
$$\Delta x = B(k)x_{t^-} + g(k, x_{t^-}), \qquad\qquad t = t_k, \qquad\qquad (5.2)$$

for nonlinearities $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ and $g : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$. Additional assumptions on the nonlinearities, evolution family and sequence of impulses may include the following.

H.4 For $j = 0, \ldots, m$, and any $\phi, \psi^{(1)}, \ldots, \psi^{(j)} \in \mathcal{RCR}([\alpha - r, \beta], \mathbb{R}^n)$, the function $t \mapsto D^j f(t, \phi_t)[\psi_1^{(1)}, \ldots, \psi_t^{(j)}]$ is an element of $\mathcal{RCR}([\alpha, \beta], \mathbb{R}^n)$.

H.5 The evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the homogeneous equation (4.1)–(4.2) is spectrally separated.

H.6 $\phi \mapsto (t, \phi)$ and $\phi \mapsto g_k(\phi)$ are C^m for some $m \geq 1$ for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, and there exists $\delta > 0$ such that for each $j = 0, \ldots, m$, there exists $c_j : \mathbb{R} \to \mathbb{R}^+$ locally bounded and a positive sequence $\{d_j(k) : k \in \mathbb{Z}\}$ such that

$$||D^j f(t, \phi) - D^j f(t, \psi)|| \leq c_j(t) ||\phi - \psi||,$$
$$||D^j g(k, \phi) - D^j g(k, \psi)|| \leq d_j(k) ||\phi - \psi||.$$

for $\phi, \psi \in B_\delta(0) \subset \mathcal{RCR}$. Also, there exists $q > 0$ such that $||D^j f(t, \phi)|| \leq qc_j(t)$ and $||D^j g_k(\phi)|| \leq qd_j(k)$ for $\phi \in B_\delta(0)$.

H.7 $f(t, 0) = g(k, 0) = 0$ and $Df(t, 0) = Dg(k, 0) = 0$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.

H.8 There exists a constant $\xi > 0$ such that $t_{k+1} - t_k \geq \xi$ for all $k \in \mathbb{Z}$.

Remark 5.1.1. *If conditions H.1–H.6 are satisfied and one has $f(t, 0) = g(k, 0) = 0$, then one can obtain a semilinear equation of the form* (5.1)–(5.2) *by defining $F(t, \phi) = f(t, \phi) - Df(t, 0)\phi$ and $G(k, \phi) = g(k, \phi) - Dg(k, 0)\phi$. Specifically, one can write*

$$\dot{x} = Df(t, 0)x_t + F(t, x_t), \qquad\qquad t \neq t_k$$
$$\Delta x = Dg(k, 0)x_{t-} + G(k, x_{t-}), \qquad\qquad t = t_k,$$

so that conditions H.1–H7 are all satisfied with $L(t) = Df(t, 0)$ and $B(k) = Dg(k, 0)$. In this sense, the linear terms should be thought of as the linear-order terms in the expansion of the vector field and jump map, near $\phi = 0$ in $\mathcal{RCR}$. For these reasons, we will refer to the linear system

$$\dot{x} = L(t)x_t, \qquad\qquad t \neq t_k$$
$$\Delta x = B(k)x_{t-}, \qquad\qquad t = t_k,$$

as the linearization *of* (5.1)–(5.2).

Definition 5.1.1. *A* mild solution *of the semilinear equation* (5.1)–(5.2) *is a function $x : [s, T] \to \mathcal{RCR}$ such that for all $s \leq t < T$*

$$x(t) = U(t, s)x(s) + \int_s^t U(t, \mu)[\chi_0 f(\mu, x(\mu))]d\mu + \sum_{s < t_i \leq t} U(t, t_i)[\chi_0 g(i, x(t_i^-))], \qquad (5.3)$$

and $x(t)(\theta) = x(t+\theta)(0)$ whenever $\theta \in [-r.0]$ satisfies $t+\theta \in [s, T]$, where U is the evolution family associated to the homogeneous equation (4.1)–(4.2) *and the integral is interpreted in the Pettis sense.*

Remark 5.1.2. *The right-hand side of equation* (5.3) *is well-posed under conditions H.1–H.3 in the sense that it naturally defines for $s \leq t < T$, a nonlinear operator from $\mathcal{RCR}([s - r, t], \mathbb{R}^n)$ into $\mathcal{RCR}$. Note also that for a function $x : [s, T] \to \mathcal{RCR}$, we denote $x(t_i^-)(\theta) = x(t_i)(\theta)$ for $\theta < 0$ and $x(t_i^-)(0) = x(t_i)(0^-)$.*

If $x : [s - r, T) \to \mathbb{R}^n$ is a *classical solution* — that is, x is differentiable from the right, continuous except at impulse times t_k, continuous from the right on $[s-r, T]$ and its derivative satisfies the differential equation (5.1)–(5.2) — then $t \mapsto x_t$ is a mild solution. This can be seen by defining the inhomogeneities $h(t) \equiv f(t, x_t)$ and $r(k) \equiv g(k, x_{t_k^-})$, solving the equivalent linear equation (3.1)–(3.2) with these inhomogeneities and initial condition $(s, x_s) \in \mathbb{R} \times \mathcal{RCR}$ in the integrated sense, and applying Corollary 3.3.1.1. For this reason, we will work with equation (5.3) exclusively from now on.

Additionally, one should note that the assumption H.5 implies that the nonlinearities are uniformly locally Lipschitz continuous. Together with the other assumptions, this implies the local existence and uniqueness of mild solutions through a given $(s, \phi) \in \mathbb{R} \times \mathcal{RCR}$. Namely, we have the following lemma, which may be seen as a local, nonlinear version of Lemma (3.1.1), with an extension of Lemma 3.1.2. Its proof is an application of the Banach fixed point theorem and is omitted. The idea is nearly identical to a portion of the proof of Proposition 5.3.1 in the following section, and the interested reader may consult it for reference.

Lemma 5.1.1. *Under assumptions H.1–H.5, for all $(s, \phi) \in \mathbb{R} \times D$, there exists a unique mild solution $x^{(s,\phi)} : [s, s + \alpha) \to \mathcal{RCR}$ of* (5.3) *for some $\alpha = \alpha(s, \phi) > 0$, satisfying $x(s) = \phi$. Moreover, the function*

$$t \mapsto y(t) := \begin{cases} x^{(s,\phi)}(t)(0), & t \in [s, s + \alpha) \\ \phi(s - t), & t \in [s, s - r) \end{cases}$$

is an element of $\mathcal{RCR}([s - r, s + \alpha), \mathbb{R}^n)$, the restriction to $[s, \alpha)$ is differentiable from the right, it is continuous except at impulse times $\{t_k : k \in \mathbb{Z}\}$, and $x^{(s,\phi)}(t) = y_t$. That is, it is a classical solution. If one defines the nonautonomous set

$$\mathcal{M} = \bigcup_{\phi \in \mathcal{RCR}} \bigcup_{s \in \mathbb{R}} \bigcup_{t \in [s, s+\alpha)} \{t\} \times \{s\} \times \{\phi\},$$

then $S : \mathcal{M} \to \mathcal{RCR}$ with $S(t, s)x = x^{(s,\phi)}(t)$ is a process on $\mathcal{RCR}$. Finally, if $x : \mathbb{R} \to \mathcal{RCR}$ is a mild solution defined for all time, then the function $y(t) = x(t)(0)$ is an element of $\mathcal{RCR}^1(\mathbb{R}, \mathbb{R}^n)$, its only discontinuities occur in $\{t_k : k \in \mathbb{Z}\}$ and $x(t) = y_t$.

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Combining the discussion following Definition 5.1.1 with Lemma 5.1.1, it follows that $S(t,s)$ satisfies the following abstract integral equation wherever it is defined.

$$S(t,s)\phi = U(t,s)\phi + \int_s^t U(t,\mu)\chi_0 f(\mu, S(\mu,s)\phi)d\mu + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 g(i, S(t_i^-,s)\phi). \quad (5.4)$$

5.2 Smoothness with respect to initial conditions

Of use later will be a result concerning the smoothness of the process $S : \mathcal{M} \to \mathcal{RCR}$ with respect to initial conditions. This result is interesting in its own right and will be useful later in proving the periodicity of centre manifolds (Theorem 7.1.2).

Definition 5.2.1. *We say the process $S(t,s) : \mathcal{M} \to \mathcal{RCR}$ guaranteed by Lemma 5.1.1 is generated by the impulsive RFDE (5.1)–(5.2).*

Theorem 5.2.1. *Under hypotheses H.1–H.6, the process $S : \mathcal{M} \to \mathcal{RCR}$ is C^m. More precisely, $\phi \mapsto S(t,s)\phi$ is C^m in a neighbourhood of ϕ, provided $(t,s,\phi) \in \mathcal{M}$. Also, $DS(t,s)\phi \in \mathcal{L}(\mathcal{RCR})$ for given $\phi \in \mathcal{RCR}$ satisfies for $t \geq s$ the abstract integral equation*

$$\begin{aligned}
DS(t,s)\phi &= U(t,s) + \int_s^t U(t,\mu)\chi_0 Df(\mu, S(\mu,s)\phi)DS(\mu,s)d\mu \\
&\quad + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 Dg(t_i, S(t_i^-,s)\phi)DS(t_i^-,s).
\end{aligned} \quad (5.5)$$

Proof. We will prove only that S is C^1, the proof of higher-order smoothness being an essentially identical albeit notationally cumbersome extension thereof. Let $s \in \mathbb{R}$ be fixed. Let $\psi \in \mathcal{RCR}$ be given. For given $\nu > 0$, denote by $B_\nu(\psi)$ the closed ball centered at ψ with radius ν in $\mathcal{RCR}$.

Introduce for given $\epsilon, \delta, \nu > 0$ the normed vector space $(X_{\epsilon,\delta,\nu}, ||\cdot||)$, where $X_{\epsilon,\delta,\nu}$ consists of the functions $\phi : [s-r, s+\epsilon] \times B_\delta(\psi) \to B_\nu(\psi)$ such that $x \mapsto \phi(t,x)$ is continuous for each t, $\phi(t,x)(\theta) = \phi(t+\theta,x)(0)$ whenever $\theta \in [-r,0]$ and $[t+\theta,t] \subset [s-r, s+\epsilon]$, and $|\phi|| < \infty$ for the norm given by

$$||\phi||_{\epsilon,\delta,\nu} = \sup_{\substack{t\in[s-r,s+\epsilon] \\ ||x-\psi||\leq\delta}} ||\phi(t,x)||.$$

It can be easily verified that $(X_{\epsilon,\delta,\nu}, ||\cdot||)$ is a Banach space. With $\mathcal{L}(\mathcal{RCR})$ the bounded linear operators on $\mathcal{RCR}$, introduce also the space $(\mathbf{X}_{\epsilon,\delta}, ||\cdot||)$ consisting of functions

$\Phi : [s - r, s + \epsilon] \times \mathcal{RCR} \to \mathcal{L}(\mathcal{RCR})$ such that $x \mapsto \Phi(t,x)$ is continuous for each t, $\Phi(t,x)h(\theta) = \Phi(t+\theta,x)h(0)$ for all $h \in \mathcal{RCR}$, and $||\Phi|| < \infty$, where the norm is $||\Phi(t,x)|| = \sup_{||h||=1} ||\Phi(t,x)h||_{\epsilon,\delta,\nu}$. Clearly, $(\mathbf{X}_{\epsilon,\delta}, || \cdot ||)$ is complete.

Define a pair of nonlinear operators

$$\Lambda_1 : X_{\epsilon,\delta,\nu} \to X_{\epsilon,\delta,\nu},$$

$$\Lambda_1(\phi)(t,x) = \chi_{[s-r,s)}(t)x(t-s) + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)x(s) + \int_s^t U(t,s)\chi_0 f(\mu, \phi(\mu,x))d\mu\right.$$

$$\left. + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 g(t_i, \phi(t_i^-,x))\right]$$

$$\Lambda_2 : X_{\epsilon,\delta} \times \mathbf{X}_{\epsilon,\delta} \to \mathbf{X}_{\epsilon,\delta}$$

$$\Lambda_2(\phi,\Phi)(t,x)h = \chi_{[s-r,s)}(t)I_{\mathcal{RCR}}h + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)h + \int_s^t U(t,\mu)\chi_0 Df(\mu, \phi(\mu,x))\Phi(\mu,x)hd\mu +\right.$$

$$\left. + \sum_{s<t_i\leq t} U(t,\mu)\chi_0 Dg(i, \phi(t_i^-,x)))\Phi(t_i^-,x)h\right],$$

where in the definition of Λ_2 we have $h \in \mathcal{RCR}$. By choosing ϵ and δ small enough, Λ_1 can be shown to be a uniform contraction. Indeed, if we denote $\kappa = \sup_{||x-\psi||\leq 2\delta} ||x||$, the mean-value theorem grants the estimate

$$||\Lambda_1(\phi) - \Lambda_1(\gamma)|| \leq \kappa \sup_{t\in[s,s+\epsilon]} \left(\int_s^t ||U(t,\mu)||c_1(\mu)d\mu + \sum_{s<t_i\leq t} ||U(t,t_i)||d_1(i)\right) ||\phi - \gamma||$$

$$\equiv \kappa L_\epsilon ||\phi - \gamma||$$

We can always obtain a uniform contraction by taking ϵ small enough. Also, note that $t \mapsto \Lambda_1(\phi)(t,x) \in \mathcal{RCR}$, $x \mapsto \Lambda_1(\phi,x)$ is continuous and $\Lambda_1(\phi)(t,x)(\theta) = \Lambda_1(\phi)(t+\theta,x)(0)$. To ensure the appropriate boundedness, if we denote $\overline{\kappa} = \sup_{||x-\psi||\leq\delta} k_0(x)$, the estimate

$$||\Lambda_1(\phi) - \psi|| \leq ||\phi - \psi|| + \overline{\kappa} \sup_{t\in[s,s+\epsilon]} \left(\int_s^t ||U(t,\mu)||c_0(\mu)d\mu + \sum_{s<t_i\leq t} ||U(t,t_i)||d_0(i)\right)$$

$$\equiv \delta + \overline{\kappa}M_\epsilon$$

implies it is sufficient to choose $\epsilon, \delta, \nu > 0$ small enough so that $\delta + \overline{\kappa}M_\epsilon < \nu$. This can always be done because $M_\epsilon \to 0$ as $\epsilon \to 0$ due to H.5 and Lemma 3.2.1.

The continuity of $\phi \mapsto \Lambda_2(\phi, \Phi)$ for fixed $\Phi \in \mathbf{X}_{\epsilon,\delta}$ follows by the estimate

$$||\Lambda_2(\phi, \Phi) - \Lambda_2(\gamma, \Phi)|| \leq \left(\int_s^{s+\epsilon} ||U(s+\epsilon, \mu)|| |c_1(\mu)| ||(\phi(\mu, x) - \gamma(\mu, x)|| d\mu \right.$$

$$\left. + \sum_{s < t_i \leq s+\epsilon} ||U(s+\epsilon, t_i)|| |d_1(i)| ||\phi(t_i^-, x) - \gamma(t_i^-, x)|| \right) ||\Phi||.$$

Also, for each $\phi \in B_\delta(\psi)$ it is readily verified that $||\Lambda_2(\phi, \Phi) - \Lambda_2(\phi, \Gamma)|| \leq \kappa L_\epsilon ||\Phi - \Gamma||$, which by previous choices of $\epsilon, \delta, \nu > 0$ indicates that $\Phi \mapsto \Lambda_2(\phi, \Phi)$ is a uniform contraction.

We are ready to prove the statement of the theorem. Denote by (x_n, x_n') the iterates of the map $\Lambda : X_{\epsilon,\delta,\nu} \times \mathbf{X}_{\epsilon,\delta,\nu} \to X_{\epsilon,\delta,\nu} \times \mathbf{X}_{\epsilon,\delta,\nu}$ defined by $\Lambda(x, x') = (\Lambda_1(x), \Lambda_2(x, x'))$ and initialized at (x_0, x_0') with $x_0(t, x) = x$ and $x_0'(t, x) = I_{\mathcal{RCR}}$. The fiber contraction theorem – see [43] for the original, more abstract result or Theorem 1.176 of [15] for a more concrete formalism – implies convergence $(x_n, x_n') \to (x, x')$. Note also that $Dx_0 = x_0'$. If we suppose $Dx_n = x_n'$ for some $n \geq 0$, then for $t \geq s$, Lemma 3.3.5 implies that for each $\theta \in [-r, 0]$,

$$Dx_{n+1}(t, \phi)(\theta) = D\left[U(t, s)x_n(s, \phi)(\theta) + \int_s^t U(t, \mu)\chi_0 f(\mu, x_n(\mu, \phi))(\theta) d\mu \right.$$

$$\left. + \sum_{s < t_i \leq t} U(t, t_i)\chi_0 g(i, x_n(t_i^-, \phi))(\theta) \right]$$

$$= D\left[U(t, s)x_n(s, \phi)(\theta) + \int_s^t V(t+\theta, \mu)f(\mu, x_{n+1}(\mu, \phi)) d\mu \right.$$

$$\left. + \sum_{s < t_i \leq t} V(t+\theta, t_i)g(i, x_{n+1}(t_i^-, \phi)) \right]$$

$$= U(t, s)x_n'(s, \phi)(\theta) + \int_s^t V(t+\theta, \mu)Df(\mu, x_n(\mu, \phi))x_n'(\mu, \phi) d\mu$$

$$+ \sum_{s < t_i \leq t} V(t+\theta, t_i)Dg(i, x_n(t_i^-, \phi))x_n'(t_i^-, \phi),$$

which is precisely $\Lambda_2(x_n, x_n')(t, \phi)(\theta) = x_{n+1}'(t, \phi)(\theta)$. For for $t < s$, it is easily checked that $Dx_{n+1}(t, \phi) = x_{n+1}'(t, \phi)$. This proves that $Dx_{n+1}(\theta) = x_{n+1}'(\theta)$ pointwise in θ. To prove the result uniformly, we note that the difference quotient can be written for $t \geq s$ as

$$\frac{1}{||h||}\left(x_{n+1}(t, \phi + h) - x_{n+1}(t, \phi) - x_{n+1}'(t, \phi)h \right)$$

$$= \int_s^t U(t, \mu)\chi_0 \frac{1}{||h||}\left(f(\mu, x_n(\mu, \phi + h)) - f(\mu, x_n(\mu, \phi)) - Df(\mu, x_n(\mu, \phi))Dx_n(\mu, \phi)h \right) d\mu$$

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$$+ \sum_{s < t_i \leq t} U(t, t_i) \chi_0 \frac{1}{||h||} \left(g(i, x_n(t_i^-, \phi + h)) - g(i, x_n(t_i^-, \phi)) - Dg(i, x_n(t_i^-, \phi)) Dx_n(t_i^-, \phi)h \right).$$

Since x_n is differentiable by the induction hypothesis, the integrand and summand converge uniformly to zero as $h \to 0$. Thus, x_{n+1} is differentiable and $Dx_{n+1} = x'_{n+1}$, so by induction $Dx_n = x'_n$ for each n. Also, by construction, x'_n is continuous for each n and, being the uniform limit of continuous functions, $x' = \lim_{n \to \infty} x'$ is continuous. By the fundamental theorem of calculus,

$$
\begin{aligned}
\frac{x(\phi + h) - x(\phi) - x'(\phi)h}{||h||} &= \lim_{n \to \infty} \frac{x_n(\phi + h) - x_n(\phi) - Dx_n(\phi)h}{||h||} \\
&= \lim_{n \to \infty} \int_0^1 \frac{1}{||h||} \left[x'_n(\phi + (\lambda - 1)h) - x'_n(\phi) \right] h d\lambda \\
&= \int_0^1 \frac{1}{||h||} \left[x'(\phi + (\lambda - 1)h) - x'(\phi) \right] h d\lambda \to 0
\end{aligned}
$$

as $h \to 0$. By definition, x is differentiable and $Dx = x'$.

If we define $y(t)\phi = x(t, \phi)$ for the fixed point $x : [s - r, s + \epsilon] \times B_\delta(\psi) \to B_\nu(\psi)$, then y satisfies $y(t)\phi = S(t, s)\phi$ for $(t, \phi) \in [s, s + \epsilon] \times B_\delta(\psi)$. This can be seen by comparing the fixed point equation $y(t) = \Lambda_1(y)(t, \phi)$ with the abstract integral equation (5.4). We conclude that S is C^1 (fibrewise). The correctness of equation (5.5) follows by comparing to the fixed point equation associated to Λ_2. $\square$

5.3 Linearized stability

A fundamental result in stability theory for ordinary differential equations is that the equilibrium point x^* of the nonlinear system

$$\dot{x} = f(x)$$

is locally exponentially stable if the eigenvalues of $Df(x^*)$ have strictly negative real part. Similar results hold for discrete-time systems. For autonomous delay differential equations, the same conclusion holds with respect to the eigenvalues of the generator of the continuous semigroup associated to the linearization; see Section VII, Corollary 5.12 of [27]. For impulsive RFDE, we have an analogous result under certain conditions. First, a definition.

Definition 5.3.1. *Let $S : \mathcal{M} \to \mathcal{RCR}$ be the process associated to the semilinear system* (5.1)–(5.2). *The fixed point $0 \in \mathcal{RCR}$ is*

- exponentially stable *if there exists $\delta, \alpha, K > 0$ such that for $t \geq s$, one has $||S(t, s)\phi|| \leq Ke^{-\alpha(t-s)}$ whenever $||\phi|| < \delta$;*

- stable *if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, s) > 0$ such that for all $t \geq s$, one has $||S(t, s)\phi|| \leq \epsilon$ whenever $||\phi|| < \delta$;*

- unstable *if it is not stable.*

Proposition 5.3.1. *Let assumptions H.1–H.7 hold. Assume for all $\delta > 0$ sufficiently small, there exists $c(\delta) \geq 0$ satisfying $\lim_{\delta \to 0^+} c(\delta) = 0$ and such that*

$$||f(t, \phi) - f(t, \psi)|| \leq c||\phi - \psi||$$
$$||g(k, \phi) - g(k, \psi)|| \leq c||\phi - \psi||,$$

for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $\phi, \psi \in B_\delta(0)$. If the evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the linearization of (5.1)–(5.2) is hyperbolic and $\mathcal{RCR}_u(t) = \{0\}$, the fixed point 0 is exponentially stable.

Proof. Since $\mathcal{RCR}_u(t) = \{0\}$ and the the linearization is hyperbolic, the evolution family satisfies $||U(t, s)|| \leq Ke^{a(t-s)}$ for some $a < 0$, for all $t \geq s$. We first prove that the fixed point is stable. Let $\epsilon > 0$ be arbitrary, and choose some $\epsilon' \leq \epsilon$ small enough so that

$$\left(1 + \frac{1}{\xi} \right) c(\epsilon') < \frac{1}{2}, \tag{5.6}$$

where c is the constant from the statement of the theorem. Next, choose $\delta > 0$ satisfying

$$\delta < \frac{\epsilon'}{2K}. \tag{5.7}$$

Let $||\phi|| < \delta$, and introduce the space of history-valued functions X_ϕ, defined by

$$X_\phi = \left\{ z : [s, \infty) \to \mathcal{RCR} \ : \ \exists y \in \mathcal{RCR}([s - r, \infty), \mathbb{R}^n), z(t) = y_t, ||z|| < \epsilon', \ z(s) = \phi \right\},$$

on which we introduce the norm $||z|| = \sup_{t \geq s} ||z(t)||$ where the latter is the typical supremum norm. X_ϕ is clearly equivalent up to isometry as a normed space to the subspace

$$\{w \in \mathcal{RCR}([s - r, \infty) : ||w|| < \epsilon, w_s = \phi\} \subset \mathcal{RCR}([s - r, \infty), \mathbb{R}^n).$$

Since the latter is complete, the same is true of X_ϕ. Consider the formal expression

$$F(z)[t] = U(t, s)\phi + \int_s^t U(t, \mu)[\chi_0 f(\mu, z(\mu))]d\mu + \sum_{s < t_i \leq t} U(t, t_i)[\chi_0 g(i, z(t_i^-))]$$

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for $z \in X_\phi$. By assumption H.3 and Lemma 3.3.5, the above defines a nonlinear map $F : X_\phi \to X$, with

$$X = \{z : [s, \infty) \to \mathcal{RCR} \ : \ \exists y \in \mathcal{RCR}([s - r, \infty), \mathbb{R}^n), z(t) = y_t, ||z|| < \infty\} \supset X_\phi.$$

We claim that $\text{im}(F) \subset X_\phi$. We can estimate the nom $||F(z)[t]||$ using $||U(t, s)|| \leq Ke^{a(t-s)}$, the fundamental theorem of calculus for Banach-space valued C^1 functions and Lemma 2.4.2. The result is

$$||F(z)[t]|| \leq Ke^{a(t-s)}\delta + \frac{K(1 - e^{a(t-s)})}{-a}c(\epsilon')\epsilon' + \frac{K(1 - e^{a(t+\xi-s)})}{-a\xi}c(\epsilon')\epsilon',$$

and from inequalities (5.6) and (5.7) together with $a < 0$, it follows that $||F(z)[t]|| < \epsilon'$ for all $t \geq s$, independent of s. Since mild solutions are precisely fixed points of F, it follows that whenever $||\phi|| < \delta$, the process satisfies $||S(t, s)\phi|| < \epsilon' \leq \epsilon$ for all $t \geq s$. That is, the fixed point is (uniformly) stable.

To get exponential stability, repeat the above argument but with the stronger condition that $\epsilon' \leq \epsilon$ is small enough to guarantee in addition to (5.6), the inequality

$$\rho := a + Kc(\epsilon')\left(1 + \frac{1}{\xi}\right) < 0. \tag{5.8}$$

We begin with the the integral equation (5.4) for the mild solution. We have the estimate

$$||S(t, s)\phi|| \leq Ke^{a(t-s)}||\phi|| + \int_s^t Ke^{a(t-\mu)}||S(\mu, s)\phi||c(\epsilon')d\mu + \sum_{s < t_i \leq t} Ke^{a(t-t_i)}||S(t_i^-, s)\phi||c(\epsilon')$$

for all $t \geq s$, provided $||\phi|| < \delta$, where $\delta > 0$ is again chosen according to (5.7). Multiplying both sides by e^{-at}, this is equivalent to

$$e^{-at}||S(t, s)\phi|| \leq Ke^{-as}||\phi|| + \int_s^t Kc(\epsilon')e^{-a\mu}||S(\mu, s)\phi||d\mu + \sum_{s < t_i \leq t} Kc(\epsilon')e^{-at_i}||S(t_i^-, s)\phi||.$$

Applying the Gronwall Lemma 2.4.1 to $t \mapsto e^{-at}||S(t, s)\phi||$, we eventually obtain

$$e^{-at}||S(t, s)\phi|| \leq Ke^{-as}||\phi|| \exp\left((t - s)Kc(\epsilon') + \frac{(t + \xi - s)}{\xi}\log(1 + Kc(\epsilon'))\right).$$

Multiplying by e^{at} and exploiting $\log(1 + x) \leq x$ for $x > 0$, we obtain

$$||S(t, s)\phi|| \leq K(1 + Kc(\epsilon'))||\phi||e^{\rho(t-s)},$$

and since $\rho < 0$ from the assumption (5.8) that ϵ' is chosen small enough, we obtain exponential stability. $\square$

We would expect the opposite result to also be true. That is, if the unstable fibre bundle $\mathcal{RCR}_u$ is non-trivial then the trivial fixed point of the semilinear system (5.1)–(5.2) is unstable. At this stage, proving such a result seems quite difficult. The reason for this is that the integral formulation of mild solutions of (5.3) is slightly less amenable to establishing lower bounds on solutions than it is on upper bounds. Indeed, in proving the linearized stability theorem we mercilessly exploited the inequality $||U(t,s)|| \leq Ke^{a(t-s)}$ which, since integrals behave well with respect to upper majorization, ultimately allowed us to get exponentially decaying estimates on (nonlinear) mild solutions. This does not work with lower bounds, even though one can exploit the fact that the all-time process $U_u(t,s) : \mathcal{RCR}_u \to \mathcal{RCR}_u$ restricted to the unstable fibre bundle admits the lower bound $||U_u(t,s)|| \geq \frac{1}{K}e^{b(t-s)}$.

If the system were periodic, we could make a slight modification to the proof of Proposition 5.10 of Section VII from [27] to prove an instability theorem, but this is not a general result. Instead, we will postpone such discussions until Chapter 9 where we will study hyperbolicity in a bit more detail.

Chapter 6

Centre manifold theory I: existence and reduction principles

The content of this chapter appears almost in its entirety in *Smooth centre manifolds for impulsive delay differential equations* [21] by Church and Liu, although there are some improvements. Specifically, the cutoff nonlinearities have been redefined so that we are later able to get smoothness in state under weaker assumptions on the linear part. There is also a minor correction to part 3 of Theorem 5.5.1 in this thesis that was not caught earlier; the errata has been submitted to the publisher. Corollary 6.6.1.1 and the contents of Section 6.8 are unpublished, as is the example in Section 6.6.1.

6.1 Spaces of exponentially weighted functions

At this stage it is appropriate to introduce several exponentially weighted Banach spaces that will be needed to construct the centre manifolds. First, denote $PC(\mathbb{R}, \mathbb{R}^n)$ the set of functions $f : \mathbb{R} \to \mathbb{R}^n$ that are continuous everywhere except for at times $t \in \{t_k : k \in \mathbb{Z}\}$ where they are continuous from the right and have limits on the left.

$$\mathcal{PC}^\eta = \left\{ \begin{array}{l} \phi : \mathbb{R} \to \mathcal{RCR} : \phi(t) = f_t \text{ for some } f \in PC(\mathbb{R}, \mathbb{R}^n), \\ \text{and } ||\phi||_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} ||\phi(t)|| < \infty \end{array} \right\}$$

$$B^\eta(\mathbb{R}, \mathcal{RCR}) = \{ f : \mathbb{R} \to \mathcal{RCR} : ||f||_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} ||f(t)|| < \infty \}$$

$$PC^\eta(\mathbb{R}, \mathbb{R}^n) = \{ f \in PC(\mathbb{R}, \mathbb{R}^n) : ||f||_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} ||f(t)|| < \infty \}$$

$$B_{t_k}^\eta(\mathbb{Z}, \mathbb{R}^n) = \{f : \mathbb{Z} \to \mathbb{R}^n : ||f||_\eta = \sup_{k \in \mathbb{Z}} e^{-\eta|t_k|}|f_k| < \infty\}.$$

Also, if $\mathcal{M} \subset \mathbb{R} \times \mathcal{RCR}$ is a nonautonomous set over $\mathcal{RCR}$, we define the space $\mathcal{PC}^\eta(\mathbb{R}, \mathcal{M})$ of piecewise-continuous functions taking values in $\mathcal{M}$ by

$$\mathcal{PC}^\eta(\mathbb{R}, \mathcal{M}) = \{f \in \mathcal{PC}^\eta : f(t) \in \mathcal{M}(t)\}.$$

If X^η is one of the above spaces, then the normed space $X^{\eta,s} = (X^\eta, || \cdot ||_{\eta,s})$ with norm

$$||F||_{\eta,s} = \begin{cases} \sup_{t \in \mathbb{R}} e^{-\eta|t-s|}||F(t)||, & \operatorname{dom}(F) = \mathbb{R} \\ \sup_{k \in \mathbb{Z}} e^{-\eta|t_k-s|}||F(k)||, & \operatorname{dom}(F) = \mathbb{Z}, \end{cases}$$

is complete.

6.2 η-bounded solutions from inhomogeneities

In this section we will characterize the η-bounded solutions of the inhomogeneous linear equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\mu)[\chi_0 F(\mu)]d\mu + \sum_{s < t_i \le t} U(t,t_i)[\chi_0 G_i], \qquad -\infty < s \le t < \infty, \tag{6.1}$$

for inhomogeneous terms F and G. As defined in Definition 3.2.4, we recall now that $\mathcal{RCR}_c(t) = \mathcal{R}(P_c(t))$, where P_c is the projection onto the centre bundle of the linear part of (5.1)–(5.2).

Lemma 6.2.1. *Let* $\eta \in (0, \min\{-a,b\})$ *and let H.1, H.2 and H.5 hold. Then,*

$$\mathcal{RCR}_c(\nu) = \{\varphi \in \mathcal{RCR} : \exists x \in \mathcal{PC}^\eta, \ x(t) = U(t,s)x(s), \ x(\nu) = \varphi\}. \tag{6.2}$$

Proof. If $\varphi \in \mathcal{RCR}_c(\nu)$, then $P_c(\nu)\varphi = \varphi$ and the function $x(t) = U(t,\nu)P_c(\nu)\varphi = U_c(t,\nu)\varphi$ is defined for all $t \in \mathbb{R}$, satisfies $x(t) = U(t,s)x(s)$, $x(\nu) = \varphi$, $x(t)(\theta) = x(t+\theta)(0)$, and by choosing $\epsilon < \eta$, there exists $K > 0$ such that

$$e^{-\eta|t|}||x(t)|| \le Ke^{\epsilon|\nu|}e^{-(\eta-\epsilon)|t|}||\varphi|| \le Ke^{\epsilon|\nu|}||\varphi||.$$

Finally, as $x(t) = [U(t,s)x(s)(0)]_t$ for all $t \in \mathbb{R}$, we conclude $x \in \mathcal{PC}^\eta$.

Conversely, suppose $\varphi \in \mathcal{RCR}$ admits some $x \in \mathcal{PC}^\eta$ such that $x(t) = U(t,s)x(s)$ and $x(\nu) = \varphi$. Let $||x||_\eta = \overline{K}$. We will show that $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, so that $\varphi = I\varphi = (P_c(\nu) + P_s(\nu) + P_u(\nu))\varphi = P_c(\nu)\varphi$, from which we will conclude $\varphi \in \mathcal{RCR}_c(\nu)$.

By spectral separation, we have for all $\rho < \nu$,

$$
\begin{aligned}
e^{-\eta|\rho|}||P_s(\nu)\varphi|| &= e^{-\eta|\rho|}||U_s(\nu,\rho)P_s(\rho)x(\rho)|| \\
&\le e^{-\eta|\rho|}Ke^{a(\nu-\rho)}||P_s(\rho)|| \cdot ||x(\rho)|| \\
&\le K\overline{K}e^{a(\nu-\rho)}||P_s(\rho)||,
\end{aligned}
$$

which implies $||P_s(\nu)\varphi|| \le K\overline{K}e^{a\nu}||P_s(\rho)|| \exp(\eta|\rho| - a\rho)$. Since $\eta < -a$ and $\rho \mapsto ||P_s(\rho)||$ is bounded, taking the limit as $\rho \to -\infty$ we obtain $||P_s(\nu)\varphi||| \le 0$. Similarly, for $\rho > \nu$, we have

$$
\begin{aligned}
e^{-\eta|\rho|}||P_u(\nu)\varphi|| &= e^{-\eta|\rho|}||U_u(\nu,\rho)P_u(\rho)x(\rho)|| \\
&\le e^{-\eta|\rho|}Ke^{b(\nu-\rho)}||P_u(\rho)|| \cdot ||x(\rho)|| \\
&\le K\overline{K}e^{b(\nu-\rho)}||P_u(\rho)||,
\end{aligned}
$$

which implies $||P_u(\nu)\varphi|| \le K\overline{K}e^{b\nu}||P_u(\rho)|| \exp(\eta|\rho| - b\rho)$. Since $\eta < b$ and $\rho \mapsto ||P_u(\rho)||$ is bounded, taking the limit $\rho \to \infty$ we obtain $||P_u(\nu)\varphi|| \le 0$. Therefore, $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, and we conclude that $P_c(\nu)\varphi = \varphi$ and $\varphi \in \mathcal{RCR}_c(\nu)$. $\square$

Lemma 6.2.2. *Let conditions H.1, H.2 and H.5 be satisfied. Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. The integrals*

$$
\int_s^t U(t,\mu)P_c(\mu)[\chi_0 h(\mu)]d\mu, \qquad \int_v^t U(t,\mu)P_u(\mu)[\chi_0 h(\mu)]d\mu
$$

are well-defined as Pettis integrals for all $s,t,v \in \mathbb{R}$, where we define $\int_b^a f d\mu = -\int_a^b f d\mu$ when $a < b$.

Proof. The nontrivial cases are where $t \le s$ and $t \le v$. For the former, defining $H(\mu) =$

$\chi_0 h(\mu)$ we have the string of equalities

$$U_c(t,s)P_c(s)\int_t^s U(s,\mu)H(\mu)d\mu = U_c(t,s)\int_t^s U_c(s,\mu)P_c(\mu)H(\mu)d\mu$$
$$= \int_t^s U_c(t,\mu)P_c(\mu)H(\mu)d\mu$$
$$= \int_t^s U(t,\mu)P_c(\mu)H(\mu)d\mu$$
$$= -\int_s^t U(t,\mu)P_c(\mu)H(\mu)d\mu.$$

The first integral on the left exists due to Lemma 3.3.5 and Proposition 2.2.1. The subsequent equalities follow by Proposition 3.3.5 and the definition of spectral separation. The case $t \leq v$ for the other integral is proven similarly.

$\square$

Define the (formal) linear operators $\mathcal{K}_s^\eta : \mathcal{PC}^{\eta,s} \oplus B_{t_k}^\eta(\mathbb{Z},\mathbb{R}^n) \to B^\eta(\mathbb{R},\mathcal{RCR})$ by the equation

$$\mathcal{K}_s^\eta(F,G)(t) = \int_s^t U(t,\mu)P_c(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu$$
$$+ \sum_s^t U(t,t_i)P_c(t_i)[\chi_0 G_i]dt_i - \sum_t^\infty U(t,t_i)P_u(t_i)[\chi_0 G_i]dt_i + \sum_{-\infty}^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i,$$
$$(6.3)$$

indexed by $s \in \mathbb{R}$, where the external direct sum $\mathcal{PC}^{\eta,s} \oplus B_{t_k}^{\eta,s}(\mathbb{Z},\mathbb{R}^n)$ is identified as a Banach space with norm $||(f,g)||_{\eta,s} = ||f||_{\eta,s} + ||g||_{\eta,s}$, and the summations are defined as follows:

$$\sum_a^b F(t_i)dt_i = \begin{cases} \displaystyle\sum_{a<t_i\leq b} F(t_i), & a \leq b \\ \displaystyle-\sum_b^a F(t_i)dt_i, & b < a. \end{cases}$$

Lemma 6.2.3. *Let H.1, H.2, H.5 and H.7 hold, and let $\eta \in (0,\min\{-a,b\})$.*

1. *The function $\mathcal{K}_s^\eta : \mathcal{PC}^{\eta,s} \oplus B_{t_k}^{\eta,s}(\mathbb{Z},\mathbb{R}^n) \to B^{\eta,s}(\mathbb{R},\mathcal{RCR})$ with $\eta \in (0,\min\{-a,b\})$ and defined by formula* (6.3) *is linear and bounded. In particular, the norm satisfies*

$$||\mathcal{K}_s^\eta|| \leq C\left[\frac{1}{\eta-\epsilon}\left(1+\frac{e^{(\eta-\epsilon)\xi}}{\xi}\right) + \frac{1}{-a-\eta}\left(1+\frac{2e^{(\eta-a)\xi}}{\xi}\right) + \frac{1}{b-\eta}\left(1+\frac{2e^{(b+\eta)\xi}}{\xi}\right)\right]$$
$$(6.4)$$

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for some constants C and ϵ independent of s.

2. $\mathcal{K}_s^\eta$ *has range in* $\mathcal{PC}^{\eta,s}$ *and* $v = \mathcal{K}_s^\eta(F,G)$ *is the unique solution of* (6.1) *in* $\mathcal{PC}^{\eta,s}$ *satisfying* $P_c(s)v(s) = 0$.

3. *The expression* $\mathcal{K}_*(F,G)(t) = (I - P_c(t))K_s^0(F,G)(t)$ *uniquely defines, independent of s, a bounded linear map*

$$\mathcal{K}_* : \mathcal{PC}^0 \oplus B_{t_k}^0(\mathbb{Z}, \mathbb{R}^n) \to \mathcal{PC}^0.$$

Proof. Let $\epsilon < \min\{\min\{-a, b\} - \eta, \eta\}$. To show that $\mathcal{K}_s^\eta$ is well-defined, we start by mentioning that all improper integrals and infinite sums appearing on the right-hand side of (6.3) can be interpreted as limits of well-defined finite integrals and sums, due to Lemma 3.3.5, Lemma 6.2.2 and Proposition 2.2.1. For brevity, write

$$\mathcal{K}_s^\eta(F,G) = \left(K_1^{u,f} - K_1^{c,F} + K_1^{u,F} \right) + \left(K_2^{u,G} - K_2^{c,G} + K_2^{s,G} \right),$$

where each term corresponds to the one in (6.3) in order of appearance.

We start by proving the convergence of the improper integrals. Denote

$$I(v) = \int_t^v U(v, \mu) P_u(\mu) [\chi_0 F(\mu)] d\mu,$$

and let $v_k \nearrow \infty$. We have, for $m > n$ and n sufficiently large so that $v_m > 0$,

$$\begin{aligned}
||I(v_m) - I(v_n)|| &\leq \int_{v_n}^{v_m} KN e^{b(t-\mu)} |F(\mu)| d\mu \\
&\leq \int_{v_n}^{v_m} KN e^{b(t-\mu)} e^{\eta\mu} ||F||_\eta d\mu \\
&= KN ||F||_\eta e^{bt} \int_{v_n}^{v_m} e^{\mu(\eta-b)} d\mu \\
&= \frac{KN ||F||_\eta}{b - \eta} e^{bt} \left(e^{-v_n(b-\eta)} - e^{-v_m(b-\eta)} \right) \\
&\leq \frac{KN ||F||_\eta}{b - \eta} e^{bt} e^{-v_n(b-\eta)}.
\end{aligned}$$

Therefore, $I(v_k) \in \mathcal{RCR}$ is Cauchy, and thus converges; namely, it converges to the improper integral $K^{u,F}(t)$. One can similarly prove that $K^{s,F}(t)$ converges. For the infinite sums, we employ similar estimates; if we denote $S = \sum_{t < t_i < \infty} ||U_u(t, t_i) [\chi_0 G_i]||$ and assume

without loss of generality that $t_0 = 0$, a fairly crude estimate (that we will later improve) yields

$$
\begin{aligned}
S &\leq \sum_{t < t_i < \infty} KN e^{b(t-t_i)} e^{\eta |t_i|} ||G||_\eta \\
&= \sum_{-|t| < t_i \leq 0} KN ||G||_\eta e^{bt} e^{|t_i|(b+\eta)} + \sum_{0 < t_k < \infty} KN ||G||_\eta e^{bt} e^{-(b-\eta)t_i} \\
&\leq KN e^{bt} \left(\frac{|t|}{\xi} e^{|t|(b+\eta)} + \frac{1}{1 - e^{-(b-\eta)\xi}} \right) ||G||_\eta.
\end{aligned}
$$

Thus, $K^{u,G}(t)$ converges uniformly. One can show by similar means that $K^{s,F}(t)$ and $K^{s,G}(t)$ both converge. Therefore, $\mathcal{K}^\eta_s(F, G)(t) \in \mathcal{RCR}$ exists. We can now unambiguously state that $\mathcal{K}^\eta_s$ is clearly linear.

Our next task is to prove that $||K^\eta_s(F, G)]||_{\eta,s} \leq Q ||(F, G)||_{\eta,s}$ for constant Q satisfying the estimate of equation (6.4). We will prove the bounds only for $||K^{u,F}||_{\eta,s}$, $||K^{u,G}||_{\eta,s}$, $||K^{c,F}||_{\eta,s}$ and $||K^{c,G}||_{\eta,s}$; the others follow by similar calculations. For $t < s$, we we have

$$
\begin{aligned}
e^{-\eta|t-s|} ||K^{u,F}(t)|| &\leq e^{-\eta|t-s|} \int_t^\infty KN e^{b(t-\mu)} |F(\mu)| d\mu \\
&\leq e^{\eta(t-s)} KN \left[\int_t^s e^{b(t-\mu)} e^{\eta|\mu-s|} ||F||_{\eta,s} d\mu + \int_s^\infty e^{b(t-\mu)} e^{\eta|\mu-s|} ||F||_{\eta,s} d\mu \right] \\
&= e^{\eta(t-s)} KN ||F||_{\eta,s} \left[\int_t^s e^{b(t-\mu)} e^{\eta(s-\mu)} d\mu + \int_s^\infty e^{b(t-\mu)} e^{\eta(\mu-s)} d\mu \right] \\
&= e^{\eta(t-s)} KN ||F||_{\eta,s} \left[e^{bt+\eta s} \frac{e^{-(b+\eta)t} - e^{-(b+\eta)s}}{b+\eta} + e^{bt-\eta s} \frac{e^{-(b-\eta)s}}{b-\eta} \right] \\
&\leq KN ||F||_{\eta,s} \frac{1}{b-\eta}
\end{aligned}
$$

The above inequality is also satisfied for $t \geq 0$, and we conclude $||K^{u,F}||_{\eta,s} \leq KN(b - \eta)^{-1} ||(F, G)||_{\eta,s}$. Next, for $t < s$,

$$
\begin{aligned}
e^{-\eta|t-s|} ||K^{u,G}(t)|| &\leq e^{-\eta|t-s|} \sum_{t < t_i < \infty} KN e^{b(t-t_i)} |G_i| \\
&\leq e^{\eta(t-s)} KN \left[\sum_{t < t_i < s} e^{b(t-t_i)} e^{\eta|t_i - s|} ||G||_{\eta,s} + \sum_{s \leq \tau_i < \infty} e^{b(t-t_i)} e^{\eta|t_i - s|} ||G||_{\eta,s} \right] \\
&\leq e^{\eta(t-s)} KN ||G||_{\eta,s} \frac{1}{\xi} \left[\int_{t-\xi}^s e^{b(t-\mu)} e^{\eta(s-\mu)} d\mu + \int_{s-\xi}^\infty e^{b(t-\mu)} e^{\eta(\mu-s)} d\mu \right]
\end{aligned}
$$

$$\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\left[e^{bt+\eta s}\frac{e^{-(b+\eta)(t-\xi)}-e^{-(b+\eta)x}}{b+\eta}+e^{bt-\eta s}\frac{e^{-(b-\eta)(s-\xi)}}{b-\eta}\right]$$

$$\leq \frac{2KN||G||_{\eta,s}}{\xi(b-\eta)}\cdot e^{(b+\eta)\xi},$$

where we have made use of Lemma 2.4.2 to estimate the sums. The same conclusion is valid for $t \geq s$, and it follows that $||K^{u,G}||_{\eta,s} \leq 2KNe^{(b+\eta)\xi}(\xi(b-\eta))^{-1}||(F,G)||_{\eta,s}$. Next, for $t \leq s$,

$$e^{-\eta|t-s|}||K^{c,G}(t)|| \leq e^{\eta(t-s)}KN||G||_{\eta,s}\sum_{t<t_i\leq s}e^{\epsilon(t_i-t)}e^{\eta(s-t_i)}$$

$$\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\int_{s-\xi}^{t}e^{\epsilon(\mu-t)}e^{\eta(s-\mu)}d\mu$$

$$= e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)}\left(e^{\epsilon(s-\xi-t)}e^{\eta\xi}-e^{-\eta(t-s)}\right)$$

$$\leq \frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)}e^{(\eta-\epsilon)\xi}$$

This estimate continues to hold for all $t, s \in \mathbb{R}$. To compare to the integral term, for $s \leq t$ we have

$$e^{-\eta|t-s|}||K^{c,F}(t)|| \leq e^{-\eta(t-s)}KN||F||_{\eta,s}\int_{s}^{t}e^{\epsilon(t-\mu)}e^{\eta(\mu-s)}d\mu$$

$$= e^{-\eta(t-s)}KN||F||_{\eta,s}\frac{1}{\eta-\epsilon}\left(e^{\eta(t-s)}-e^{\epsilon(t-s)}\right)$$

$$\leq \frac{KN||F||_{\eta,s}}{\eta-\epsilon}$$

and this estimate persists for all $t, s \in \mathbb{R}$. Similar estimates for the other integrals and sums appearing in (6.3) ultimately result in the bound appearing in (6.4). This proves part 1.

To prove part 2, denote $v = \mathcal{K}_s^{\eta}(F,G)$. It is clear from the definition of v, the orthogonality of the projection operators and Proposition 2.2.1 that $P_c(s)v(s) = 0$. Also, for all

$-\infty < z \leq t < \infty$, denoting $\overline{F} = \chi_0 F$ and $\overline{G}_i = \chi_0 G$, we have

$$U(t,z)v(z) + \int_z^t U(t,\mu)\overline{F}(\mu)d\mu + \sum_z^t U(t,t_i)\overline{G}_i dt_i$$

$$= U(t,z)v(z) + \int_z^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu - \int_t^z U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu$$

$$+ \int_z^t U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu + \sum_z^t U(t,t_i)P_c(t_i)\overline{G}_i d\tau_i$$

$$- \sum_t^z U(t,t_i)P_u(t_i)\overline{G}_i dt_i + \sum_z^t U(t,t_i)P_u(t_i)\overline{G}_i d\tau_i$$

$$= \int_s^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu - \int_t^\infty U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)\overline{F}(\mu)d\mu$$

$$+ \sum_s^t U(t,t_i)P_c(t_i)\overline{G}_i d\tau_i - \sum_t^\infty U(t,t_i)P_u(t_i)\overline{G}_i d\tau_i + \sum_{-\infty}^t U(t,t_i)P_s(t_i)\overline{G}_i d\tau_i$$

$$= v(t),$$

so that $t \mapsto v(t)$ solves the integral equation (6.1). This also demonstrates that $v \in \mathcal{PC}^\eta$. To show that it is the only solution in $\mathcal{PC}^\eta$ satisfying $P_c(s)v(s) = 0$, suppose there is another $r \in PC^\eta$ that satisfies $P_c(s)r(s) = 0$. Then the function $w := v - r$ is an element of $\mathcal{PC}^\eta$ that satisfies $w(t) = U(t,z)w(z)$ for $-\infty < z \leq t < \infty$. By Lemma 6.2.1, we have $w(s) \in \mathcal{RCR}_c(s)$. But since $P_c(s)w(s) = 0$ and $P_c(s)$ is the identity on $\mathcal{RCR}_c(s)$, we obtain $w(s) = 0$. Therefore, $w(t) = U(t,s)0 = U_c(t,s)0 = 0$ for all $t \in \mathbb{R}$, and we conclude $v = r$, proving the uniqueness assertion.

For assertion 3, we compute first

$$\mathcal{K}_*(F,G)(t) = \int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu$$

$$\sum_{-\infty}^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i - \sum_t^\infty U(t,t_i)P_u(t_i)[\chi_0 G_i]dt_i.$$

Routine estimation using inequalities (3.9)–(3.11) together with Lemma 2.4.2 produces the bound

$$||\mathcal{K}_*(F,G)(t)|| \leq KN \left(\frac{-1}{a} + \frac{1}{b} - \frac{e^{-a\xi}}{a\xi} + \frac{e^{b\xi}}{b\xi} \right) ||(F,G)||,$$

and as the bound is independent of t, s, the result is proven. $\qquad\square$

6.3 Substitution operator and modification of nonlinearities

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ bump function satisfying

  i) $\xi(y) = 1$ for $0 \leq y \leq 1$,

 ii) $0 \leq \xi(y) \leq 1$ for $1 \leq y \leq 2$,

iii) $\xi(y) = 0$ for $y \geq 2$.

We modify the nonlinearities of (5.1)–(5.2) in the centre and hyperbolic directions separately. For $\delta > 0$ and $s \in \mathbb{R}$, we let

$$F_{\delta,s}(t,x) = f(t,x)\xi\left(\frac{||P_c(s)x||}{N\delta}\right)\xi\left(\frac{||(P_s(s) + P_u(s))x||}{N\delta}\right) \tag{6.5}$$

$$G_{\delta,s}(k,x) = g(k,x_{0-})\xi\left(\frac{||P_c(s)x_{0-}||}{N\delta}\right)\xi\left(\frac{||(P_s(s) + P_u(s))x_{0-}||}{N\delta}\right), \tag{6.6}$$

Notice that $G_{\delta,s}(k,x)$ takes the pointwise left-limit in the evaluation (6.6). The proof of the following lemma and corollary will be omitted. They can be proven by emulating the proof of Lemma 6.1 from [45] and taking into account the uniform boundedness of the projectors P_i; see property 1 of Definition 3.2.4.

Lemma 6.3.1. *Let $f(t,\cdot)$ and $g(k,\cdot)$ be uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous on the ball $B_{\mathcal{RCR}}(\delta,0)$ in $\mathcal{RCR}$ with mutual Lipschitz constant $L(\delta)$, and let $f(t,0) = g_k(0) = 0$. The functions*

$$F_{\delta,s} : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n, \qquad G_{\delta,s} : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$$

are globally, uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous with mutual Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$, independent of s.

Corollary 6.3.0.1. *The substitution operator*

$$R_{\delta,s} : \mathcal{PC}^{\eta,s} \to B^{\eta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B^{\eta,s}_{t_k}(\mathbb{Z}, \mathbb{R}^n)$$

defined by $R_{\delta,s}(x)(t,k) = (F_{\delta,s}(t,x(t)), G_{\delta,s}(k,x(t_k)))$ is globally Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta$ that satisfies $\tilde{L}_\delta \to 0$ as $\delta \to 0$. Moreover, the Lipschitz constant is independent of η, s.

Corollary 6.3.0.2. $||(F_{\delta,s}(t,x), G_{\delta,s}(k,x))|| \leq 4\delta L_\delta$ *for all $x \in \mathcal{RCR}$ and $(t,k) \in \mathbb{R} \times \mathbb{Z}$.*

6.4 Fixed-point equation and existence a Lipschitz centre manifold

Let $\eta \in (\epsilon, \min\{-a, b\})$ and define a mapping $\mathcal{G}_s : \mathcal{PC}^{\eta,s} \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ by

$$\mathcal{G}_s(u, \varphi) = U(\cdot, s)\varphi + \mathcal{K}_s^\eta(R_{\delta,s}(u)). \tag{6.7}$$

Note that by Lemma 6.2.3 and Corollary 6.3.0.1, the operator is well-defined, $\mathcal{K}_s^\eta$ is bounded and R_δ is globally Lipschitz continuous for each $\delta > 0$, provided H.1–H.7 hold. Choose δ small enough so that

$$\tilde{L}_\delta \|K_s^\eta\|_\eta < \frac{1}{2}. \tag{6.8}$$

Notice that δ can be chosen so that (6.8) can be satisfied independent of s, due to Lemma 6.2.3. If $\|\varphi\| < r/(2K)$ then $\mathcal{G}_s(\cdot, \varphi)$ leaves $\overline{B(r, 0)} \subset \mathcal{PC}^{\eta,s}$ invariant. Moreover, $\mathcal{G}_s(\cdot, \varphi)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. One may notice that r is arbitrary. We can now prove the following:

Theorem 6.4.1. *Let conditions H.1–H.7 hold. If δ is chosen as in (6.8), then there exists a globally Lipschitz continuous mapping $u_s^* : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ such that $u_s = u_s^*(\varphi)$ is the unique solution in $\mathcal{PC}^{\eta,s}$ of the equation $u_s = \mathcal{G}_s(u_s, \varphi)$.*

Proof. The discussion preceding the statement of Theorem 6.4.1 indicates that $\mathcal{G}_s(\cdot, \varphi)$ is a contraction mapping on $\overline{B(r, 0)} \subset \mathcal{PC}^{\eta,s}$ for every $r > \|\varphi\|2K$. Since the latter is a closed subspace of the Banach space $\mathcal{PC}^{\eta,s}$, the contraction mapping principle implies the existence of the function u_s^*. To show that it is a Lipschitz continuous, we note

$$\|u_s^*(\varphi) - u_s^*(\psi)\|_{\eta,s} = \|\mathcal{G}_s(u_s^*(\varphi), \varphi) - \mathcal{G}_s(u_s^*(\psi), \psi))\|_{\eta,s}$$
$$\leq K\|\varphi - \psi\| + \frac{1}{2}\|u_s^*(\varphi) - u_s^*(\psi)\|_{\eta,s}.$$

Therefore, u_s^* is Lipschitz continuous with Lipschitz constant $2K$. $\qquad\square$

Definition 6.4.1 (Lipschitz centre manifold)**.** *The centre manifold, $\mathcal{W}_c \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibers for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_c(t) = \mathrm{Im}\{\mathcal{C}(t, \cdot)\}, \tag{6.9}$$

where $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{C}(t, \phi) = u_t^*(\phi)(t)$.

Remark 6.4.1. *The centre manifold depends non-canonically on the choice of cut-off function from Section 6.3. That is, the centre manifold is not unique, so we are committing an abuse of syntax by referring to such a construct generally as "the" centre manifold. One must always understand that the definition of the centre manifold is with respect to a particular cutoff function.*

The construction above implies the centre manifold is fibrewise Lipschitz. We can prove a stronger result, namely that the Lipschitz constant can be chosen independent of the given fiber.

Corollary 6.4.1.1. *There exists a constant $L > 0$ such that $||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq L||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$. Moreover, L can be chosen so that $L \to 0$ as $\delta \to 0$.*

Proof. Denote $u^\phi = u_t(\phi)$ and $u^\psi = u_t(\psi)$. A preliminary estimation appealing to the fixed-point equation (6.7) yields

$$||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq ||\phi - \psi|| + ||(\mathcal{K}_t^\eta(R_\delta u^\phi) - \mathcal{K}_t^\eta(R_\delta u^\psi))(t)||.$$

By Corollary 6.3.0.2, each of $R_\delta u^\phi$ and $R_\delta u^\psi$ are uniformly bounded, so Lemma 6.2.3 implies the existence of a constant $c > 0$ such that

$$
\begin{aligned}
||(\mathcal{K}_t^\eta(R_{\delta,t} u^\phi) - \mathcal{K}_t^\eta(R_{\delta,t} u^\psi))(t)|| &\leq c||(R_{\delta,t} u^\phi - R_{\delta,t} u^\psi)(t)|| \\
&\leq c \sup_{s \in \mathbb{R}} ||(R_{\delta,t} u^\phi - R_{\delta,t} u^\psi)(s)|| e^{-\eta|t-s|} \\
&\leq c \tilde{L}_\delta ||u^\phi - u^\psi||_{\eta,t} \\
&\leq c \tilde{L}_\delta 2K ||\phi - \psi||,
\end{aligned}
$$

and in the last line we used the Lipschitz constant from Theorem 6.4.1. Combining this result with the previous estimate for $||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)||$ yields the uniform Lipschitz constant. By Corollary 6.3.0.1, the Lipschitz constant has the claimed property. $\square$

6.4.1 A remark on centre manifold representations: graphs and images

Our initial definition of the centre manifold was as the fibre bundle whose t-fibers are the images of $\mathcal{C}(t, \cdot)$. However, sometimes one likes to think of the centre manifold as being the graph of a function. To accomplish this, one can use the hyperbolic part. Let us define

the function $\mathcal{H} : \mathcal{RCR}_c \to \mathcal{RCR}$ by $\mathcal{H}(t, \phi) = (I - P_c(t))\mathcal{C}(t, \phi)$. In this way, the centre manifold can be identified with the graph of the hyperbolic part of the centre manifold. Indeed, by part 2 of Lemma 6.2.3, we have the decomposition $\mathcal{C}(t, \phi) = \phi + (I - P_c(t))\mathcal{C}(t, \phi)$, so that

$$\mathcal{W}_c(t) = \{\phi + \mathcal{H}(t, \phi) : \phi \in \mathcal{RCR}_c(t)\} \sim \{(\phi, \mathcal{H}(t, \phi)) : \phi \in \mathcal{RCR}_c(t)\} = \mathrm{Graph}(\mathcal{H}(t, \cdot)).$$

Since $\mathcal{RCR}_c(t)$ and its complement $\mathcal{R}(I - P_c(t)) = \mathcal{RCR}_s(t) \oplus \mathcal{RCR}_u(t)$ have only 0 in their intersection, this identification makes sense. When one reduces down to ordinary differential equations, one usually thinks of precisely the function $\mathcal{H}$ as *being* the centre manifold. This ambiguity between the function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$, the fibre bundle $\mathcal{W}_c$, the hyperbolic part $\mathcal{H} : \mathcal{RCR}_c \to \mathcal{RCR}$ and its graph can sometimes make statements about centre manifolds imprecise. In this thesis, the term *centre manifold* without any additional qualifiers will always mean the fibre bundle $\mathcal{W}_c$.

6.5   Invariance and smallness properties

Recall that by Lemma 5.1.1, there is a process $(S, \mathcal{M})$ on $\mathcal{RCR}$ such that $t \mapsto S(t, s)\phi$ is the unique mild solution of (5.3) through the initial condition (s, ϕ) defined on an interval $[s, s+\alpha)$. With this in mind, the centre manifold is locally positively invariant with respect to $\mathcal{S}$.

Theorem 6.5.1 (Centre manifold: invariance and inclusion of bounded orbits). *Let conditions H.1–H.7 hold. The centre manifold $\mathcal{W}_c$ enjoys the following properties.*

1. *$\mathcal{W}_c$ is locally positively invariant: if $(s, \phi) \in \mathcal{W}_c$ and $||S(t, s)\phi|| < \delta$ for $t \in [s, T]$, then $(t, S(t, s)\phi) \in \mathcal{W}_c$ for $t \in [s, T]$.*

2. *If $(s, \phi) \in \mathcal{W}_c$, then $S(t, s)\phi = u_t^*(P_c(t)S(t, s)\phi)(t) = \mathcal{C}(t, P_c(t)S(t, s)\phi)$*

3. *If $x : \mathbb{R} \to \mathcal{RCR}$ is a mild solution of (5.3) satisfying $||x|| < \delta$, then $(t, x(t)) \in \mathcal{W}_c$ for all $t \in \mathbb{R}$.*

4. *$\mathbb{R} \times \{0\} \subset \mathcal{W}_c$ and $\mathcal{C}(t, 0) = 0$ for all $t \in \mathbb{R}$.*

Proof. Let $(s, \phi) \in \mathcal{W}_c$ and denote $x(t) = S(t, s)\phi$, with $||x|| < \delta$. Since $(s, \phi) \in \mathcal{W}_c$, there exists $\varphi \in \mathcal{RCR}_c(s)$ such that $\phi = u_s^*(\varphi)(s)$. Define $\hat{x} = u_s^*(\varphi)$. Then, it follows that

$\varphi = P_c(s)\phi$, $\hat{x}(s) = \phi = P_c(s)\phi + K_s^\eta(R(\hat{x}))(s)$, and

$$\hat{x}(t) = U(t,s)\varphi + \mathcal{K}_s^\eta(R_\delta(\hat{x}))(t)$$

$$= U(t,s)\varphi + \left[U(t,s)K_s^\eta(R_{\delta,s}(\hat{x}))(s) + \int_s^t U(t,\mu)\chi_0 F_{\delta,s}(\mu,\hat{x}(\mu))d\mu + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 G_{\delta,s}(i,\hat{x}(t_i)) \right]$$

$$= U(t,s)\hat{x}(s) + \int_s^t U(t,\mu)\chi_0 F_{\delta,s}(\mu,\hat{x}(\mu))d\mu + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 G_{\delta,s}(i,\hat{x}(t_i))$$

for all $t \in [s,T]$. But since $||x(t)|| < \delta$ on $[s,T]$, uniqueness of mild solutions (Lemma 3.1.1 with Theorem 3.3.1) implies that $x = \hat{x}|_{[s,T]}$.

Let $v \in [s,T]$ and define $z : \mathbb{R} \to \mathcal{RCR}$ by $z = \hat{x} - U(\cdot,v)P_c(v)\hat{x}(v)$. One can easily verify that

$$z(t) = U(t,v)z(v) + \int_v^t U(t,\mu)U(t,\mu)\chi_0 F_{\delta,s}(\mu,\hat{x}(\mu))d\mu + \sum_{v<t_i\leq t} U(t,t_i)\chi_0 G_{\delta,s}(i,\hat{x}(t_i))$$

for all $t \in [v,\infty)$ and that $P_c(v)z(v) = 0$. On the other hand, since $||\hat{x}|| < \delta$ we have $R_{\delta,s}(\hat{x}) = R_{\delta,v}(\hat{x})$. From these two observations and Lemma 6.2.3, $z = \mathcal{K}_v^\eta(R_{\delta,v}(\hat{x}))|_{[v,\infty)}$, so that we may write

$$\hat{x} = U(\cdot,v)P_c(v)\hat{x}(v) + \mathcal{K}_v^\eta(R_{\delta,v}(\hat{x})) = u_v^*(P_c(v)\hat{x}(v)).$$

Therefore, $\hat{x}(v) = u_v^*(P_c(v)\hat{x}(v))(v)$, and since $x(v) = \hat{x}(v)$, this proves that $(v,x(v)) \in \mathcal{W}_c$ and, through essentially the same proof, that

$$x(v) = u_v^*(P_c(v)x(v))(v) = \mathcal{C}(v,x(v))(v).$$

The proofs of the other two assertions of the theorem follow by similar arguments, and are omitted. $\square$

The modification of the nonlinearity R_δ results in the function u_s^* that defines the centre manifold having a uniformly small hyperbolic part. Namely, we have the following lemma.

Lemma 6.5.1. *Define* $\widehat{P}_c : \mathcal{PC}^\eta \to \mathcal{PC}^\eta(\mathbb{R},\mathcal{RCR}_c)$ *by* $\widehat{P}_c\phi(t) = P_c(t)\phi(t)$. *If* $\delta > 0$ *is sufficiently small, then* $||(I - \widehat{P}_c)u_s^*||_0 < \delta$.

Proof. Recall that u_s^* satisfies the fixed-point equation $u_s^* = U(\cdot,s)\varphi + \mathcal{K}_s^\eta(R_{\delta,s}(u_s^*))$. Thus, with $\widehat{P}_h = I - \widehat{P}_c$,

$$\widehat{P}_h u_s^* = \widehat{P}_h \circ \mathcal{K}_s^\eta(R_{\delta,s}(u_s^*))$$

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because $U(t, s)$ is an isomorphism of $\mathcal{RCR}_c(s)$ onto $\mathcal{RCR}_c(t)$ and $\varphi \in \mathcal{RCR}_c(s)$. By Corollary 6.3.0.2, we have for all $t \in \mathbb{R}$ that $||R_{\delta,s}(u_s^*(t))|| \leq 4\delta L_\delta$, which implies $R_{\delta,s}(u_s^*) \in B^0(\mathbb{R}, \mathbb{R}^n) \oplus B^0_{t_k}(\mathbb{Z}, \mathbb{R}^n)$. We obtain the claimed result by applying the second conclusion of Lemma 6.2.3 and taking δ sufficiently small, recalling from Corollary 6.3.0.1 that $L_\delta \to 0$ as $\delta \to 0$. $\qquad\square$

6.6 Asymptotic phase

The centre manifold has a property that is sometimes called asymptotic phase. Qualitatively, it says that the unstable and stable components of pairs of sufficiently small solutions must eventually come into phase, either backward or forward in time. This suggests that such solutions must be "close" to the centre manifold. The following theorem is inspired by Lemma 2.3 of [9] and Theorem 8.1, Chapter IX of [27].

Theorem 6.6.1 (Asymptotic phase in $\mathcal{RCR}_s$ and $\mathcal{RCR}_u$). *For $\nu > 0$ and $S^* \in \mathbb{R}$, there exist positive constants C and δ such that*

1. *If u and v are mild solutions of (5.3) on the interval $I = [s - T, s]$ for $T > 0$ and $s \leq S^*$ satisfying*

   - $(I - P_s(j))u(j) = (I - P_s(j))v(j)$ *for either $j = s$ or $j = s - T$;*
   - $||u(t)|| \leq \delta$ *and $||v(t)|| \leq \delta$ for all $t \in I$,*

   then, $||P_s(s)[u(s) - v(s)]|| \leq C||P_s(s - T)[u(s - T) - v(s - T)]||e^{(a+\nu)T}$.

2. *If u and v are mild solutions on the interval $I = [s, s + T]$ for $T > 0$ and $s \geq S^*$ satisfying*

   - $(I - P_u(j))u(j) = (I - P_u(j))v(j)$ *for either $j = s$ or $j = s + T$;*
   - $||u(t)|| \leq \delta$ *and $||v(t)|| \leq \delta$ for all $t \in I$,*

   then, $||P_u(s)[u(s) - v(s)]|| \leq C||P_u(s + T)[u(s + T) - v(s + T)]||e^{-(b-\nu)T}$.

Proof. We begin by proving the first assertion, and will prove only the case where $(I - P_s(s))(u(s) - v(s)) = 0$, as the other case is similar. Let $L = L(\delta)$ be the Lipschitz constant of the nonlinearity $\tilde{R}_\delta$, and denote

$$z_-(t) = ||P_s(t)(u - v)(t)||, \qquad z_+(t) = ||(I - P_s(t))(u - v)(t)||.$$

Let $t \in I$. Routine integral estimation with the spectral separation assumptions result in the estimate

$$z_-(s)e^{at} \leq Kz_-(t) + \int_t^s KNLe^{a(t-\mu)}(z_+(\mu) + z_-(\mu))d\mu + \sum_t KNLe^{a(t-t_i)}(z_+(t_i^-) + z_i(t_i^-))dt_i. \quad (6.10)$$

Also, spectral separation guarantees that the expression

$$(I - P_s(t))u(t) = U(t,s)[P_c(s) + P_u(s)]u(s) + \int_s^t U(t,\mu)[P_c(\mu) + P_u(\mu)]\chi_0 f(\mu, u(\mu))d\mu$$

$$+ \sum_s^t U(t,t_i)[P_c(t_i) + P_u(t_i)]\chi_0 g(i, u(t_i^-))dt_i$$

is well-defined even when $t \leq s$, and similarly for v. Using the fact that $(I - P_s(s))(u - v)(s) = 0$, we get the estimate

$$z_+(t) \leq \int_t^s KNLe^{b(t-\mu)}(z_-(\mu) + z_+(\mu))d\mu + \int_t^s KNLe^{b(t-t_i)}(z_-(t_i^-) + z_+(t_i^-))dt_i.$$

Some routine changes of variables and Lemma 2.4.1 then imply

$$z_+(t) \leq KNL\left[\int_t^s e^{(b-KL)(t-\mu)}z_-(\mu)d\mu + \sum_t^s e^{(b-KL)(t-t_i)}z_i(t_i^-)dt_i\right].$$

Substituting the above into (6.10) results in the somewhat bulky expression

$$z_-(s)e^{at} \leq Kz_-(t) + \int_t^s KNLe^{a(t-\mu)}z_-(\mu)d\mu + \sum_t^s KNLe^{a(t-t_i)}z_-(t_i^-)dt_i$$

$$+ \int_t^s (KNL)^2\left[\int_\mu^s e^{a(t-\mu)+(b-KNL)(\mu-\eta)}z_-(\eta)d\eta + \sum_\mu^s e^{a(t-\mu)+(b-KNL)(\mu-t_i)}z_-(t_i)dt_i\right]d\mu$$

$$+ \sum_t^s (KNL)^2\left[\int_\mu^s e^{a(t-t_i)+(b-KNL)(t_i-\mu)}z_-(\mu)d\mu + \sum_{t_i}^s e^{a(t-t_i)+(b-KNL)(t_i-t_k)}z_-(t_k^-)dt_k\right]dt_i.$$

Applying Fubini's Theorem and estimating sums via Lemma 2.4.2 yields

$$z_-(s)e^{at} \leq Kz_-(t) + \int_t^s (KNL + \kappa)e^{a(t-\mu)}z_-(\mu)d\mu + \sum_t^s (KNL + \kappa)e^{a(t-t_i)}z_-(t_i^-)dt_i,$$

$$\kappa = \frac{(KNL)^2}{b-a-KNL}\left(1 + \frac{e^{(b-a-KNL)\xi}}{\xi}\right),$$

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where ξ is the constant appearing in assumption H.7. Note that $\kappa = \kappa(\delta)$ is positive provided δ is chosen small enough. More changes of variables and use of Lemma 2.4.1 eventually lead us to the inequality

$$z_-(s)e^{-a(s-t)}Ke^{-as} \leq z_-(t)\exp\left(-e^{-as}(KNL+\kappa)(t-s)\right),$$

which upon substituting $t = s - T$ and rearranging grants

$$z_-(s) \leq Ke^{-as} \cdot \exp\left((a + e^{-as}(KNL+\kappa))T\right)z_-(s-T).$$

Choosing $C = Ke^{-aS^*}$ and δ small enough so that $e^{-as}(KNL(\delta) + \kappa(\delta)) \leq \nu$ results in the desired inequality.

An analogous argument proves the second assertion of the theorem. The proof is omitted. $\qquad\square$

Consider part 1 of the theorem, which concerns the stable component. If $\mathcal{RCR}_u$ is trivial, we would hope that close to the centre manifold, distinct solutions should in some sense "converge to" the centre manifold. Suppose $u : [0,\infty) \to \mathbb{R}^n$ is a mild solution that remains uniformly small; $||u|| < \delta$. As a first guess, let us define another mild solution $v : [0,\infty) \to \mathcal{RCR}$ on the centre manifold using the invariance condition:

$$v(t) = S(t,0)\mathcal{C}(0, P_c(0)u(0)).$$

Without loss of generality, we may also assume $||v|| < \delta$. One can then check using the fixed-point characterization of the centre manifold that $P_c(0)v(0) = P_c(0)u(0)$. Choose any $\eta < |a|$. For each $t > 0$, define $T = t$ and $s = t$ in the statement of Theorem 6.6.1. It follows that if δ is sufficiently small, then $||P_s(t)[u(t) - v(t)]|| \leq C||P_s(0)[u(0) - v(0)]||e^{(a+\nu)t}$. Our instinct is to take $t \to \infty$, so that we could conclude that the hyperbolic components of u and v are asymptotically equal. If one solution is on the centre manifold for all time, then we might suspect that $||u(t) - v(t)|| \to 0$ as $t \to \infty$. However, we cannot take t arbitrarily large because in principle, δ depends on the constant S^*, and we must assume $s \leq S^*$. Since $s = t$, we cannot take t arbitrarily large.

Recall that since v is a solution on the centre manifold, it can be uniquely extended to $v : \mathbb{R} \to \mathcal{RCR}$. If we instead fix $s = t$ and let $T = s' > 0$, then we get the inequality

$$||P_s(t)[u(t) - v(t)]|| \leq CN\delta e^{(a+\nu)(t-s')},$$

where N is the bound on the projection and we use the assumption that u and v are uniformly bounded by δ. If we take the limit $s' \to -\infty$, then we get the somewhat

perplexing conclusion that $P_s(t)[u(t) - v(t)] = 0$. This is similar to a concept referred to as *pullback attraction* [58]. In terms of the definition in the aforementioned reference, $v : \mathbb{R} \to \mathcal{RCR}$ is pullback attracting if for all $t \in \mathbb{R}$, there exists $\delta > 0$ such that if $\lim_{s \to -\infty} ||u(s) - v(s)|| < \delta$, then $\lim_{s \to -\infty} ||S(t,s)u(s) - v(t)|| \to 0$.

In autonomous systems, pullback attraction and the usual (forward) attraction are interchangeable concepts because in such systems, the nonlinear process will satisfy $S(t,s) = S(t - s, 0)$. In the general nonautonomous context, they are distinct. However, they are also equivalent under periodicity assumptions, and it is ultimately a consequence of this fact that we can prove the following attraction property.

Corollary 6.6.1.1. *Suppose conditions H.1–H.7 hold and, additionally, (5.1)–(5.2) is periodic with period T and $c > 0$ impulses per period. That is $L(t + T) = L(t)$, $f(t + T, \cdot) = f(t, \cdot)$, $B(k + c) = B(k)$, $g(k + c, \cdot) = g(k + c)$ and $t_{k+c} = t_k + T$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let u and v be mild solutions of (5.3) on the interval $I = [s, t]$ for some $t > s$. For all $\nu > 0$, there exist positive constants C and δ (independent of t and s) such that if u and v satisfy*

- $(I - P_s(s))u(s) = (I - P_s(s))v(s)$;

- $||u(\mu)|| \leq \delta$ and $||v(\mu)|| \leq \delta$ for all $\mu \in I$;

then, $||P_s(t)[u(t) - v(t)]|| \leq C||P_s(s)[u(s) - v(s)]||e^{(a+\nu)(t-s)}$.

Proof. Let $t = kT + r$ for some $r \in [0, T)$ and let $s = t - q$ for some $q > 0$. By uniqueness of mild solutions – Lemma 5.1.1 – and the periodicity assumption, we can write

$$S(t, s) = S(kT + r, kT + r - q) = S(r, r - q).$$

From the smallness assumption, we can write $u(t) = S(t, s)u(s)$ and $v(t) = S(t, s)v(s)$ (that is, we do not need to use the cutoff process S_δ). Also, the periodicity of the projector $P_s(\cdot)$ guaranteed by Lemma 4.3.1 implies

$$P_s(r - q) = P_s(s - kT) = P_s(s).$$

We can now apply part 1 Theorem 6.6.1 to the mild solutions $\hat{u}$ and $\hat{v}$ defined on $I = [r - q, r]$ by $\hat{u}(\mu) = S(\mu, r - q)u(s)$ and $\hat{v}(\mu) = S(\mu, r - q)v(s)$ – recall that $r \in [0, T)$, so we can apply the result with $S^* = T$. We get the inequality

$$||P_s(r)[S(r, r - q)u(s) - S(r, r - q)v(s)]|| \leq C||P_s(r - q)[u(s) - v(s)]||e^{(a+\nu)q}.$$

All of the previous observations imply that the above is equivalent to the inequality in the statement of the proposition. $\square$

Even this result is not quite strong enough to grant a *reduction principle* that is formally analogous to what there is for ordinary differential equations. We postpone more in-depth discussions until Section 6.8, but motivate some of the intricacies with an example.

6.6.1 A note on convergence rates of components in the centre fibre bundle

From the discussion preceding Corollary 6.6.1.1, one might think we are free to conclude that the centre manifold is locally attracting in the periodic case. Unfortunately, the argument is incomplete, because while it may be reasonable to suspect that the mild solution v in the centre manifold we constructed will converge to u, it generally does not do this at an exponential rate. Moreover, it is difficult to characterize this rate in general. To see this, consider the following two-dimensional autonomous example:

$$\dot{x} = (1 + y^2)x^2,$$
$$\dot{y} = -y.$$

There are a continuum of centre manifolds $\mathcal{W}(k)$ parameterized by $k \in \mathbb{R}$:

$$\mathcal{W}^{\pm}(k) = \left\{ (x, y) : \log|y| + \frac{1}{2}y^2 = \frac{1}{x} + k, \ x < 0 \right\} \cup \{(x, 0) : x \geq 0\},$$

obtained by restricting to the upper and lower half-plane. There is also an analytic centre manifold, $\mathcal{W}(-\infty) := \{(x, 0) : x \in \mathbb{R}\}$. The centre subspace X_c is precisely $\mathcal{W}(-\infty)$, whereas the stable subspace (which corresponds to the stable manifold in this case) is the y axis; $X_s = \{(0, y) : y \in \mathbb{R}\}$. The projection onto X_c is the orthogonal projection, so that $P_c(x, y) = (x, 0)$. However, notice that if x remains fixed, the solutions y of the equation

$$\log|y| + \frac{1}{2}y^2 = \frac{1}{x} + k$$

are each monotone with respect to k. It follows from the differential equation that solutions in the left half-plane flow faster on $\mathcal{W}(k_2)$ than on $\mathcal{W}(k_1)$, whenever $k_2 > k_1$. As a consequence, if $(x, y) \in \mathcal{W}(k_1)$ and we project onto X_c to a coordinate $(x, 0)$, then the map onto some $(x, \hat{y}) \in \mathcal{W}(k_2)$, the two solutions from (x, y) and $(x, \hat{y})$ will generally converge to zero at different rates. It is our claim that, in particular, the difference between these rates is on the order of $\frac{1}{t^2}$.

Let us investigate this claim. Consider the initial condition (x_0, y_0) with $x_0 < 0$ and, without loss of generality, $y_0 > 0$. We can write the component x in the centre subspace as

$$P_c u(t) = \frac{1}{\frac{1}{2}y_0^2(e^{-2t} - 1) + \frac{1}{x_0} - t}.$$

Note, $u(t)$ is contained in the centre manifold $\mathcal{W}(k')$ with $k' = \log y_0 + \frac{1}{2}y_0^2 - \frac{1}{x_0}$. Suppose we choose $\mathcal{W}(k)$ to be the centre manifold in which we define the "candidate" solution to which we compare, and we choose the upper half-plane part. The projection gives $P_c(x_0, y_0) = (x_0, 0)$. To get the associated point $(x_0, \hat{y}_0)$ on the centre manifold $\mathcal{W}(k)$, we have to solve the equation

$$\log \hat{y}_0 + \frac{1}{2}\hat{y}_0{}^2 = \frac{1}{x_0} + k, \quad \hat{y}_0 > 0.$$

To do this, we resort to an asymptotic. If $\frac{1}{x_0} + k \to -\infty$ (e.g. k is fixed and we assume $x_0 \to 0^-$) then the leading-order behaviour in $\hat{y}_0$ is dominated by the logarithm. We can then make the approximation

$$\log \hat{y}_0 \approx \frac{1}{x_0} + k,$$

so that $\hat{y}_0 \approx \exp(\frac{1}{x_0} + k)$. The associated solution, when projected onto the centre subspace, is

$$P_c v(t) = \frac{1}{\frac{1}{2}e^{\frac{2}{x_0} + 2k - 2t} + \frac{1}{x_0} - t}.$$

In Landau big theta notation, one can check directly that the asymptotic

$$P_c[u(t) - v(t)] = \Theta\left(\frac{1}{t^2}\right), \quad t \to \infty$$

is satisfied. This means that the left-hand side is asymptotically equivalent (up to multiplication by a nonzero constant) to $\frac{1}{t^2}$ as $t \to \infty$. As claimed, the components of the solutions $u(t)$ and $v(t)$ in the centre subspace converge at different rates; as $x_0 \to 0^-$, the difference in convergence rates is on the order of $\frac{1}{t^2}$

6.7 Dynamics on the centre manifold

Theorem 6.6.1 implies that in the absence of unstable directions, the centre manifold contains all of the interesting bounded-time dynamics in a neighbourhood of the relevant fixed point. This leads us to a natural question, namely: is there a nice representation of the process restricted to the centre manifold?

6.7.1 Integral equation

On the centre manifold, components of mild solutions on the centre fibre bundle are decoupled, as stated in the following lemma. The proof follows by direct calculation and the second part of Theorem 6.5.1.

Lemma 6.7.1 (Dynamics on the centre manifold: integral equation). *Let $y : \mathbb{R} \to \mathcal{RCR}$ satisfy $y(t) \in \mathcal{W}_c(t)$ with $||y|| < \delta$. Consider the projection of y onto the centre fibre bundle: $w(t) = P_c(t)y(t)$. The projection satisfies the integral equation*

$$w(t) = U(t,s)w(s) + \int_s^t U(t,\mu)P_c(\mu)\chi_0 F_{\delta,\mu}(\mu,\mathcal{C}(\mu,w(\mu)))d\mu + \sum_{s<t_i\leq t} U(t,t_i)P_c(t_i)\chi_0 G_{\delta,t_i}(i,\mathcal{C}(t_i,w(t_i)))$$

(6.11)

6.7.2 Abstract ordinary impulsive differential equation

When a solution on the centre manifold is defined by a classical solution, we can show that its projection satisfies a particular impulsive differential equation. This identification carries over to solutions that merely have enough smoothness to ensure that their right-hand derivatives exist and are elements of the space $\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. However, the following condition on the jump map is needed.

Definition 6.7.1. *A sequence of functionals $J(k,\cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the* overlap condition *if it admits a unique continuous extension to a functional $J(k,\cdot) : \mathcal{G}([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ and this extension satisfies*

$$J(k,\phi + \chi_{t_j-t_k}h) = J(k,\phi)$$

for all $\phi \in \mathcal{RCR}$ and $h \in \mathbb{R}^n$, whenever $t_j - t_k \in [-r,0)$.

The overlap condition roughly states that the jump functional does not have observable "memory" at times in the past that happen to correspond to impulse times. The extension property is satisfied for many typical jump conditions, such as those involving discrete time-varying delays or distributed delays with reasonable kernels. We make use of it in proof of the following theorem. The details are somewhat subtle, and we will spend a fair bit more time on them in Section 8.1.2.

Theorem 6.7.1 (Dynamics on the centre manifold: abstract impulsive differential equation). *Let $y \in \mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$ satisfy $y_t \in \mathcal{W}_c(t)$ with $||y|| < \delta$. Consider the projection $w(t) = P_c(t)y_t$ and define the linear operators $\mathcal{L}(t) : \mathcal{RCR}^1 \to \mathcal{RCR}$ and $\mathcal{J}(k) :*

$\mathcal{G}([-r,0],\mathbb{R}^n) \to \mathcal{G}([-r,0],\mathbb{R}^n)$ *by*

$$\mathcal{L}(t)\phi(\theta) = \begin{cases} L(t)\phi, & \theta = 0 \\ d^+\phi(\theta), & \theta < 0 \end{cases}, \qquad \mathcal{J}(k)\phi(\theta) = \begin{cases} B(k)\phi, & \theta = 0 \\ \phi(\theta^+) - \phi(\theta), & \theta < 0 \end{cases} \qquad (6.12)$$

If the jump functionals $B(k)$ and $g(k,\cdot)$ satisfy the overlap condition, then $w : \mathbb{R} \to \mathcal{RCR}^1$ satisfies, pointwise, the abstract impulsive differential equation

$$d^+w(t) = \mathcal{L}(t)w(t) + P_c(t)\chi_0 F_\delta(t, \mathcal{C}(t, w(t))), \qquad t \neq \tau_k \qquad (6.13)$$
$$\Delta w(t_k) = \mathcal{J}(k)w(t_k^-) + P_c(t_k)\chi_0 G_\delta(k, \mathcal{C}(t_k, w(t_k))), \qquad t = t_k, \qquad (6.14)$$

where $w(t_k^-)(\theta) := \lim_{\epsilon \to 0^-} w(t_k - \epsilon)(\theta)$ is the non-uniform left limit at time t_k and $\Delta w(t_k)(\theta) := w(t_k)(\theta) - w(t_k^-)(\theta)$ is the non-uniform pointwise jump at time t_k, defined for $\theta \in [-r, 0]$.

Proof. For brevity, denote $F(\mu) = F_{\delta,\mu}(\mu, \mathcal{C}(\mu, w(\mu))), \overline{F}(\mu) = \chi_0 F(\mu), \mathbf{F}(\mu) = P_c(\mu)\chi_0 F(\mu)$ and analogously for G_δ. We begin by noting that equation (6.11) allows us to write the finite difference $w_\epsilon(t) = w(t+\epsilon) - w(t)$ as

$$w_\epsilon(t) = [U(t+\epsilon, s) - U(t,s)]w(s) + (U(t+\epsilon, t) - I)\int_s^t U(t,\mu)\mathbf{F}(\mu)d\mu$$
$$+ U(t+\epsilon, t)P_c(t)\int_t^{t+\epsilon} U(t,\mu)\overline{F}(\mu)d\mu + (U(t+\epsilon, t) - I)\sum_{s < t_i \leq t} U(t,t_i)\mathbf{G}(i) \quad (6.15)$$
$$+ U(t+\epsilon, t)\sum_{t < t_i \leq t+\epsilon} U(t,t_i)\mathbf{G}(i)$$

First, we show that $d^+U(t,s)\phi = \mathcal{L}(t)U(t,s)\phi$ pointwise for $\phi \in \mathcal{RCR}$. For $\theta = 0$, we have

$$\frac{1}{\epsilon}(U(t+\epsilon, s)\phi(0) - U(t,s)\phi(0)) = \frac{1}{\epsilon}\int_t^{t+\epsilon} L(\mu)U(\mu, s)\phi d\mu,$$

which converges to $L(t)U(t,s)\phi$ as $\epsilon \to 0^+$. For $\theta < 0$ and $\epsilon > 0$ sufficiently small,

$$\frac{1}{\epsilon}(U(t+\epsilon, s)\phi(\theta) - U(t,s)\phi(\theta)) = \frac{1}{\epsilon}(\phi(t+\epsilon+\theta-s) - \phi(t+\theta-s)) \longrightarrow d^+\phi(t+\theta-s) = d^+U(t,s)\phi(\theta).$$

Therefore, $d^+U(t,s)\phi = \mathcal{L}(t)U(t,s)\phi$ pointwise, as claimed. Since $U(t,t) = I$, this also proves the pointwise convergence

$$\frac{1}{\epsilon}(U(t+\epsilon, t) - I)\phi \to \mathcal{L}(t)\phi.$$

Next, we show that

$$\frac{1}{\epsilon}U(t+\epsilon,t)P_c(t)\int_t^{t+\epsilon} U(t,\mu)\overline{F}(\mu)d\mu \to P_c(t)\overline{F}(t) = \mathbf{F}(t) \tag{6.16}$$

pointwise as $\epsilon \to 0^+$. We do this by first proving that the sequence $x_n := \frac{1}{\epsilon_n}U(t+\epsilon_n,t)P_c(t)\int_t^{t+\epsilon_n} U(t,\mu)\overline{F}(\mu)d\mu$ is pointwise Cauchy for each sequence $\epsilon_n \to 0^+$. Assuming without loss of generality that ϵ_n is strictly decreasing, we have for all $n \geq m$,

$$x_n - x_m = \left[\frac{1}{\epsilon_n}U(t+\epsilon_n,t) - \frac{1}{\epsilon_m}U(t+\epsilon_m,t)\right]P_c(t)\int_t^{t+\epsilon_n} U(t,\mu)\chi_0 F(\mu)d\mu$$
$$+ \frac{1}{\epsilon_m}U(t+\epsilon,t)\int_{t+\epsilon_m}^{t+\epsilon_n} U_c(t,\mu)P_c(\mu)\chi_0 F(\mu)d\mu$$

Both integrals can be made arbitrarily small in norm by taking $n,m \geq N$ and N large enough. Since $\frac{1}{\epsilon}U(t+\epsilon,t)$ is pointwise convergent as $\epsilon \to 0^+$, we obtain that the sequence x_n is pointwise Cauchy, and is hence pointwise convergent. Direct calculation of the limit in the pointwise sense yields (6.16). Combining all of the above results with equation (6.15) gives the pointwise equality

$$d^+w(t) = \mathcal{L}(t)U(t,s)w(s) + \mathcal{L}(t)\int_s^t U(t,\mu)\mathbf{F}(\mu)d\mu + \mathbf{F}(t) + \mathcal{L}(t)\sum_{s<t_i\leq t} U(t,t_i)\mathbf{G}(i)$$
$$= \mathcal{L}(t)w(t) + \mathbf{F}(t),$$

which is equivalent to (6.13).

To obtain the difference equation (6.14), we similarly identify $w_\epsilon(t_k)(\theta) := w(t_k)(\theta) - w(t_k-\epsilon)(\theta)$ with the decomposition

$$w_\epsilon(t_k) = [U(t_k,s) - U(t_k-\epsilon,s)]w(s) + \int_{t_k-\epsilon}^{t_k} U(t,\mu)\mathbf{F}(\mu)d\mu + \int_s^{t_k-\epsilon} [U(t_k,\mu) - U(t_k-\epsilon,\mu)]\mathbf{F}(\mu)d\mu$$
$$+ \sum_{t_k-\epsilon<t_i\leq\tau_k} U(t_k,t_i)\mathbf{G}(i) + \sum_{s<t_i\leq\tau_k-\epsilon} [U(t_k,t_i) - U(t_k-\epsilon,t_i)]\mathbf{G}(i)$$

Using Lemma 3.2.1 and Lemma 3.3.5, the above is seen to converge pointwise as $\epsilon \to 0^+$, with limit

$$\Delta w(t_k) = \tilde{\mathcal{J}}(k)U(t_k^-,s)w(s) + \tilde{\mathcal{J}}(k)\int_s^{t_k} U(t_k^-,\mu)\mathbf{F}(\mu)d\mu + \mathbf{G}(k) + \tilde{\mathcal{J}}(k)\sum_{s<t_i<t_k} U(t_k^-,t_i)\mathbf{G}(i),$$
$$\tag{6.17}$$

where $\tilde{\mathcal{J}}(k)\phi(\theta) = \chi_0(\theta)B(k)\phi + \chi_{(-r,0)}(\theta)[\phi(\theta) - \phi(\theta^-)]$, and we assume without loss of generality that $r > 0$ is large enough so that $t_k - r \neq t_j$ for all $j < k$ and all $k \in \mathbb{Z}$. Let us denote

$$U^-(t,s)\phi(\theta) = \lim_{\epsilon \to 0^+} U(t - \epsilon, s)\phi(\theta)$$

the strong left limit of the evolution family at t. This limit is well-defined pointwise, and due to the overlap condition, we have

$$\tilde{\mathcal{J}}(k)U(t_k^-, \xi)\phi = \mathcal{J}(k)U^-(t_k, \xi)\phi \tag{6.18}$$

pointwise for all $\xi < t_k$. Moreover, since

$$w(t_k^-) = U^-(t_k, s)w(s) + \int_s^{t_k} U^-(t_k, \mu)\mathbf{F}(\mu)d\mu + \sum_{s < t_i < t_k} U^-(t_k, t_i)\mathbf{G}(i), \tag{6.19}$$

we can obtain equation (6.14) by substituting (6.18) and (6.19) into (6.17). $\qquad\square$

We will not make much use of the abstract differential equation (6.13)–(6.14), and have included it mostly for the purpose of comparison with analogous results for delay differential equations. As we will see, the integral equation (6.11) will be more than sufficient.

6.7.3   A remark on coordinates and terminology

It is a slight abuse of terminology to describe (6.13)–(6.14) as an impulsive differential equation on the centre manifold. More precisely, it is the dynamical system associated to the projection onto the centre fibre bundle associated to a given solution in the centre manifold. This precise description is, however, quite verbose, and for this reason we will usually call (6.13)–(6.14) the impulsive differential equation on the centre manifold, even if this is not exactly what it is.

The evolution equation (6.13)–(6.14) is quite abstract. It is an evolution equation in the centre fiber bundle which, despite being finite-dimensional, is still rather difficult to use in practice because the fibres $\mathcal{RCR}_c(t)$ are not themselves constant in time. What is needed is an appropriate coordinate system. This would in principle allow for the derivation of an impulsive differential equation in $\mathbb{R}^p$ for $p = \dim \mathcal{RCR}_c$. We expand on precisely this idea in Section 8.1.

6.8 Reduction principle

Given a nonhyperbolic equilibrium, one may want to study the orbit structure near this equilibrium under parameter perturbation in the vector field or jump map defining the impulsive functional differential equation (5.1)–(5.2). Assuming the sufficient conditions for the existence of a centre manifold are satisfied, part 2 of Theorem 6.5.1 implies that on the centre manifold, the dynamics are completely determined by those of the component in the centre fibre bundle. Part 3 of the same theorem guarantees that the small bounded solutions are all contained on the centre manifold. Lemma 6.7.1 completely characterizes these dynamic in terms of an integral equation (6.11). As a consequence, bifurcations can be detected by analyzing this integral equation instead, and no loss of generality occurs by looking only on the centre manifold (at least for small perturbations of the parameter).

The next natural question is the following. If we detect a bifurcation on the centre manifold and the branch of solutions (or union of solutions, for example a torus) is stable when restricted to the centre manifold, are we guaranteed that this solution is stable in the infinite-dimensional system provided $\mathcal{RCR}_u$ is trivial? The answer is yes, and the following results makes this precise. They are inspired by similar results for ordinary differential equations in both finite and infinite-dimensional systems; see for instance Theorem 2.2 from Chapter 10 of Hale and Verduyn Lunel's introductory text [39] for functional differential equations, Theorem 3.22 from Chapter 2 of [40] for ordinary differential equations in Banach spaces, and the classic text of Jack Carr [14] for finite-dimensional ordinary differential equations, as well as some extensions to infinite-dimensional problems. However, we will require the vector field to be slightly more regular than previously.

Definition 6.8.1. *The functional $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ is* additive composite regulated (ACR) *if for all $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, $Y \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n \times m})$ and $z \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^m)$, the function $t \mapsto f(t, x_t + Y_t z(t))$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$.*

Remark 6.8.1. *ACR functionals are quite common in applications. For example, suppose $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ can be written in the form*

$$f(t, \phi) = F\left(t, A(t)\phi(-d(t)), \int_{-r}^{0} K(t, \theta)\phi(\theta)d\theta\right)$$

for $d \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, $A \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n \times n})$, $K : \mathbb{R} \times [-r, 0] \to \mathbb{R}^{n \times n}$ integrable in its second variable, continuous from the right in its first variable and uniformly bounded, and $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ jointly continuous from the right in its first variable and continuous in its other variables. It is clear that

$$t \mapsto A(t)[x_t(-d(t)) + Y_t(-d(t))z(t)] = A(t)[x(t - d(t)) + Y(t - d(t))z(t)]$$

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is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. As the the integral term, the function

$$t \mapsto \int_{-r}^{0} K(t, \theta)[x(t + \theta) + Y(t + \theta)z(t)]d\theta]$$

can be seen to be an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ by applying the dominated convergence theorem. From the assumptions on F, we conclude that f is ACR. The same holds true for vector fields with multiple time-varying delays and distributed delays.

Lemma 6.8.1. *Assume* $\mathcal{RCR}_u = \{0\}$. *Let* $\Phi_t = [\; \phi_t^{(1)} \;\; \cdots \;\; \phi_t^{(p)} \;]$ *be row array whose elements form a basis for* $\mathcal{RCR}_c(t)$, *the latter being p-dimensional, such that* $\Phi_t = U_c(t, 0)\Phi_0$. *Given a mild solution* $x_{(\cdot)} : I \to \mathcal{RCR}$, *write* $P_c(t)x_t = \Phi_t z(t)$ *for some* $z \in \mathbb{R}^p$, *so that*

$$x_t = \Phi_t z(t) + h(t, z(t)) + y_t^S$$

with $z \in \mathbb{R}^p$, $h(t, z) := (I - P_c(t)\mathcal{C}(t, \Phi_t z)$, *and* $y_t^S \in \mathcal{RCR}_s(t)$ *is a remainder term. Assume the matrix-valued function* $Y_c(t)$ *defined by the equation* $P_c(t)\chi_0 = \Phi_t Y_c(t)$ *is continuous from the right and possesses limits on the left. There exists positive constants* ρ, C *and* α *such that for all* $t \geq s$, *the remainder term satisfies*

$$||y_t^S|| \leq C||y_s^S - H(s, z(s))||e^{-\alpha(t-s)},$$

provided $||x_t|| \leq \rho$ *for all* $t \geq s$.

Proof. One can carefully verify that $z(t)$ and y_t^S respectively satisfy the following integral equations for all $t \geq s$:

$$z(t) = z(s) + \int_s^t Y_c(\mu)\mathcal{F}(\mu, z(\mu), y_\mu^S)d\mu + \sum_{s < t_i \leq t} U_c(t_i)\mathcal{G}(i, z(t_i), y_{t_i}^S), \qquad (6.20)$$

$$y_t^S = U(t, s)[y_s^S - h(s, z(s))] + \int_s^t U(t, \mu)P_s(\mu)\chi_0[\mathcal{F}(\mu, z(\mu), y_\mu^S) - \mathcal{F}(\mu, z(\mu), 0)]d\mu$$

$$(6.21)$$

$$+ \sum_{s < t_i \leq t} U(t, t_i)P_s(t_i)\chi_0[\mathcal{G}(i, z(t_i), y_{t_i}^S) - \mathcal{G}(i, z(t_i), 0)],$$

provided $\rho < \delta/N$, where $\mathcal{F}(t, z, y) = F_{\delta,0}(t, \Phi_t z + h(t, z) + y)$, $\mathcal{G}(k, z, y) = G_{\delta,0}(k, \Phi_{t_k} z + h(t_k, z) + y)$, and $Y_c(\mu)$ is defined by the equation $P_c(\mu)\chi_0 = \Phi_\mu Y_c(\mu)$. Because of our assumption on $Y_c(\mu)$, it follows (from the integral equation (6.20)) that z is continuous from the right and possesses limits on the left. If we remark that

$$y_t^S = (I - P_c(t))x_t = x_t - \Phi_t z(t),$$

we can use Lemma 2.1.6 to conclude that $t \mapsto ||y_t^S||$ is an element of $\mathcal{RCR}(I, \mathbb{R})$. Using spectral separation and the Lipschitz condition on the substitution operator, we can use (6.21) to get the estimate

$$||y_t^S||e^{-at} \leq Ke^{-as}||y_s^S - h(s, z(s))|| + \int_s^t KL_\delta ||y_\mu^S||e^{-a\mu}d\mu + \sum_{s < t_i \leq t} KL_\delta ||y_{t_i}^S||e^{-at_i},$$

provided $||x_t|| \leq \rho$ for ρ sufficiently small. Next, we apply the Gronwall Inequality (Lemma 2.4.1) to the function $t \mapsto ||y_t^S||e^{-at}$. After some simplifications, we get

$$||y_t^S|| \leq K(1 + KL_\delta)||y_s^S - h(s, z(s))|| \exp\left(\left(a + KL_\delta\left(1 + \frac{1}{\xi}\right)\right)(t - s)\right).$$

We can always guarantee that the exponential convergence rate is in the form $e^{-\alpha(t-s)}$ for $\alpha > 0$ by taking δ sufficiently small, since $a < 0$ and we have $L_\delta \to 0$ as $\delta \to 0$ by Corollary 6.3.0.1. The result follows. $\square$

The continuity condition on the matrix $t \mapsto Y_c(t)$ comes up in a few places in this thesis. Most noteworthy, it is used in Section 7.2 to guarantee temporal regularity properties of the centre manifold. See in particular Section 7.2.3 and the discussion of Section 7.2.7.

Theorem 6.8.1 (Local attractivity of the centre manifold). *Let the assumptions of Lemma 6.8.1 be satisfied and let f be an ACR functional. There exists a neighbourhood V of $0 \in \mathcal{RCR}$ and positive constants K_1, α_1 such that if $t \mapsto x_t$ is a mild solution satisfying $x_t \in V$ for all $t \geq s$, then there exists $u_t \in \mathcal{W}_c(t)$ with the property that*

$$||x_t - u_t|| \leq K_1 e^{-\alpha_1(t-s)}$$

for all $t \geq s$. That is, every solution that remains close to the centre manifold in forward time is exponentially attracted to a particular solution on the centre manifold. More precisely, there exists $u \in \mathcal{RCR}([s, \infty), \mathbb{R}^n)$ such that $t \mapsto \Phi_t u(t)$ satisfies the abstract integral equation (6.11) for the coordinate on the centre manifold, and we have the estimates

$$||P_c(t)x_t - \Phi_t u(t)|| \leq Ke^{-a_1(t-s)},$$
$$||P_s(t)x_t - h(t, u(t))|| \leq Ke^{-a_1(t-s)}.$$

Proof. With the same setup as in the previous proof, let $u(t; u_s)$ for $t \geq s$ denote the solution of the integral equation

$$u(t) = u_s + \int_s^t Y_c(\mu)\mathcal{F}(\mu, u(\mu), 0)d\mu + \sum_{s < t_i \leq t} Y_c(t_i)\mathcal{G}(i, u(t_i), 0),$$

for given $u_s \in \mathbb{R}^p$. Define $w(t) = z(t) - u(t; u_s)$. With $x_t = \Phi_t z(t) + h(t, z(t)) + y_t^S$, we have the following integral equations for w and y_t^S:

$$y_t^S = U(t, s)[y_s^S - h(s, w(s) + u_s)] + \int_s^t U(t, \mu) P_s(\mu) \chi_0 M_1(\mu, w(\mu) + u(\mu; u_s), y_\mu^S) d\mu$$

(6.22)

$$+ \sum_{s < t_i \leq t} U(t, t_i) P_s(t_i) \chi_0 M_2(i, w(t_i) + u(t_i; u_s), y_{t_i}^S),$$

$$w(t) = w(s) + \int_s^t Y_c(\mu) N_1(\mu, w(\mu), y_\mu^S) d\mu + \sum_{s < t_i \leq t} Y_c(t_i) N_2(i, w(t_i), y_{t_i}^S), \qquad (6.23)$$

with M_1, M_2, N_1 and N_2 defined by

$$M_1(\mu, a, b) = \mathcal{F}(\mu, a, b) - \mathcal{F}(\mu, a, 0),$$
$$M_2(i, a, b) = \mathcal{G}(i, a, b) - \mathcal{G}(i, a),$$
$$N_1(\mu, a, b) = \mathcal{F}(\mu, a + u(\mu; u_s), b) - \mathcal{F}(\mu, u(\mu; u_s), 0),$$
$$N_2(i, a, b) = \mathcal{G}(i, a + u(t_i; u_s), b) - \mathcal{G}(i, u(t_i; s), 0).$$

The idea now is to reinterpret the integral equation for w as a fixed-point equation parameterized by $y_{(\cdot)}^S$ and $u(\cdot; u_s)$. Introduce the space

$$X = \{\phi \in \mathcal{RCR}([s, \infty), \mathbb{R}^p) : ||\phi(t)||e^{a(t-s)} \leq K\}$$

equipped with the norm $||\phi|| = \sup_{t \geq s} ||\phi(t)||e^{a(t-s)}$. Define Tw by

$$(Tw)(t) = -\int_t^\infty Y_c(\mu) N_1(\mu, w(\mu), y_\mu^S) d\mu - \sum_{t < t_i < \infty} Y_c(t_i) N_2(i, w(t_i), y_{t_i}^S). \qquad (6.24)$$

If $w \in X$, then from the assumption that f is an ACR functional we can conclude that $Tw \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. So we consider the nonlinear function $T : X \to \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Notice that if w is a fixed point of T, then w satisfies the integral equation (6.23). Working backwards, it would then follow by Lemma 6.8.1 that

$$v_t := \Phi_t[w(t) + u(t; u_s)] + h(t, w(t) + u(t; u_s)) + y_t^S \qquad (6.25)$$

is a solution with the property that

$$||P_c(t)v_t - \Phi_t u(t; u_s)]|| = O(e^{-\gamma t}),$$
$$||P_s(t)v_t - h(t, u(t; u_s))|| = O(e^{-\gamma t})$$

as $t \to \infty$ (recall that if $w \in X$, then $w \to 0$ exponentially as $t \to \infty$, while $h(t, \cdot)$ is uniformly Lipschitz with respect to t). It is at this stage that we refer the reader to the proof of Theorem 2 of Carr's book [14]. The setup having been completed, the proof that T can be made a contraction on X provided δ is sufficiently small is the same as Carr's argument, and is omitted. Specifically, we have the following conclusion: for $s \in \mathbb{R}$ and any (u_s, y_s^S) is sufficiently small, $T : X \to X$ is a contraction. In particular, by making this dependence on the fixed point explicit and writing $T : (\mathbb{R} \times \mathbb{R}^p \times \mathcal{RCR}_s(s)) \times X \to X$, one can ensure that T is a uniform contraction. In the same way we proved that the centre manifold is (uniformly in t) Lipschitz continuous, one can show that the fixed point $S^*(s, u, y^S)$ of $T(s, u, y^S)$ is uniformly (with respect to s) Lipschitz continuous in $\mathbb{R}^p \times \mathcal{RCR}_s(s)$, and the Lipschitz constant can be made as small as needed by taking δ sufficiently small.

Now, for a given $\phi \in \mathcal{RCR}$, define $u_s(\phi)$ and $y_s^S(\phi)$ according to

$$P_c(s)\phi = \Phi_s u_s(\phi), \;\; y_s^S(\phi) = \phi - \Phi_s u_s(\phi) - h(s, u_s(\phi)).$$

Next, define $Q(s, \cdot, \cdot) : \mathbb{R}^p \times \mathcal{RCR}_s(s) \to \mathbb{R}^p \times \mathcal{RCR}_s(s)$ by

$$Q(s, u, \phi) = (u, \phi) + (\Phi_s S^*(s, u_s(\phi), y_s^S(\phi)), 0).$$

That is, $Q(s, \cdot, \cdot)$ is a nonlinear perturbation from the identity. If we let $\psi \in \mathcal{RCR}$, then the function $Q_\psi(s, \cdot, \cdot) : \mathbb{R}^p \times \mathcal{RCR}_s(s) \to \mathbb{R}^p \times \mathcal{RCR}_s(s)$ defined by

$$Q_\psi(s, u, \phi) = (u_s(\psi), \psi - \Phi_s u_s(\psi)) - (\Phi_s S^*(s, u_s(\phi), y_s^S(\phi)), 0)$$

satisfies the property that $Q(s, u, \phi) = (\psi_1, \psi_2)$ if and only if

$$Q_{\Phi_s\psi_1 + \psi_2}(s, u, \phi) = (u, \phi).$$

$S^*(s, \cdot, \cdot)$ is (uniformly in s) Lipschitz continuous with a Lipschitz constant that goes to zero as $\delta \to 0$. Since ψ does not factor into the nonlinear term, Q_ψ can be made a uniform (with respect to s and ψ) contraction by taking δ sufficiently small. As a consequence, every $(\psi_1, \psi_2) \in \mathbb{R}^p \times \mathcal{RCR}_s(s)$ is in the range of $Q(s, \cdot, \cdot)$ (in fact, $Q(s, \cdot, \cdot$ is a bijection).

Now, let x_t defined for $t \geq s$ be a mild solution with $||x_t||$ for $t \geq s$ sufficiently small. Write $x_s = \Phi_s x_s^c + x_s^S$ for $x_s^c \in \mathbb{R}^p$ and $x_s^S \in \mathcal{RCR}_s(s)$. Denote $(v_s^c, v_s^S) = Q^{-1}(s, \cdot, \cdot)(x_s^c, x_s^S)$. Take note that $v_s^S = x_s^S$. From the above discussion, it follows that with $u(t) = u(t; v_s^c)$, the asymptotic of the theorem is satisfied. By restricting to a sufficiently small neighbourhood of the origin, we can ignore the cutoffs on the vector field and jump map, thereby obtaining results that are applicable to mild solutions of the system without the cutoff nonlinearity. This proves the theorem. $\qquad\square$

Remark 6.8.2. *Suppose the (parameter-dependent) process $S(t, s; \epsilon) : \mathcal{RCR} \to \mathcal{RCR}$ is generated by a parameter-dependent impulsive RFDE with parameter $\epsilon \in \mathbb{R}^m$, and at $\epsilon = 0$ the equilibrium 0 is nonhyperbolic with a p-dimensional centre manifold. One then considers the spatially extended process on $\mathcal{RCR} \times \mathbb{R}^m$ defined by*

$$(\phi, \epsilon) \mapsto (S(t, s; \epsilon)\phi, \epsilon)$$

$0 \in \mathcal{RCR} \times \mathbb{R}^m$ is now nonhyperbolic with a $(p + m)$-dimensional centre fibre bundle, so that the function $(x, \epsilon) \mapsto \mathcal{C}(t, x, \epsilon)$ defines a $(p + m)$-dimensional centre manifold. The dynamics on this centre manifold are trivial in the ϵ component, while those in the x component depend for each ϵ fixed on $x \mapsto \mathcal{C}(t, x, \epsilon)$.

For small parameters $\epsilon \neq 0$, there may be small solutions in the parameter-dependent *centre manifold $\mathcal{W}_c^\epsilon$ defined by*

$$\mathcal{W}_c^\epsilon(t) = \{\mathcal{C}(t, x, \epsilon) : x \in \mathcal{RCR}_c(t)\}$$

that are locally asymptotically stable when restricted to $\mathcal{W}_c^\epsilon$. There could also be stable attractors therein. The stability condition in addition to continuity with respect to initial conditions (Theorem 5.2.1) and attractivity of the centre manifold (Theorem 6.8.1) then grants the analogous stability of such small solutions or attractors when considered in the original infinite-dimensional system (5.1)–(5.2), provided ϵ is small enough and $\mathcal{RCR}_u$ is trivial.

To summarize, when the unstable fibre bundle is trivial, the dynamics on the centre manifold completely determine all nearby dynamics. Local stability assertions associated to small solutions and attractors on the parameter-dependent centre manifold carry over to those of the original infinite-dimensional system. The parameter-dependent centre manifold contains all such small solutions and attractors.

Chapter 7

Centre manifold theory II: regularity

The content of Chapter 7.1 appears in *Smooth centre manifolds for impulsive delay differential equations* [21] by Church and Liu, while Chapter 7.2 contains material from *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [20] by the same authors. Section 6.1.4 contains a minor correction to the source material [21]; specifically, an inconsistency in the definition of the substitution operator $\tilde{R}_\delta$ and its formal derivatives $\tilde{R}_\delta^{(p)}$. Only symbolic corrections to one proof were necessary. In addition, the assumption H.3 is stronger than the condition stated in the aforementioned publication and is in fact needed in all proofs related to the smoothness of the centre manifold. The associated errata has been submitted to the publisher. Finally, by using the definition of the cutoff nonlinearity from the previous chapter, we are able to prove smoothness in state for the centre manifold without assuming periodicity on the linear part, thereby improving on the result from [21]. Only symbolic changes to proofs are needed to accomplish this.

7.1 Smoothness in state

In Section 12.3, we proved the existence of invariant centre manifolds associated to the abstract integral equation (5.3). These invariant manifolds are images of a uniformly Lipschitz continuous function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$. In this section, we will ultimately prove that this function is (fibrewise) smooth. To accomplish this, we will need to introduce an additional regularity assumption on the nonlinear parts of the vector field and jump map.

H.8 The functions c_j and sequences $\{d_j(k) : k \in \mathbb{Z}\}$ introduced in H.5 are bounded.

Note that H.8 is a purely nonautonomous property and is trivially satisfied if the vector field and jump functions are autonomous.

7.1.1 Contractions on scales of Banach spaces

The rest of this section will utilize several techniques from the theory of contraction mappings on scales of Banach spaces. In particular, many of the proofs that follow are inspired by those relating to smoothness of centre manifolds appearing in [27, 46, 86], albeit adapted somewhat so as to manage the explicitly nonautonomous and impulsive properties of the problem. The following lemma will be very helpful. It is taken from Section IX, Lemma 6.7 of [27], but also appears as Theorem 3 in [86].

Lemma 7.1.1. *Let Y_0, Y, Y_1 be Banach spaces with continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$ and let Λ be another Banach space. Consider the fixed-point equation $y = f(y, \lambda)$ for $f : Y \times \Lambda \to Y$. Suppose the following conditions hold.*

b1) *The function $g : Y_0 \times \Lambda \to Y_1$ defined by $(y_0, \lambda) \mapsto g(y_0, \lambda) = Jf(J_0y_0, \lambda)$ is of class C^1 and there exist mappings*

$$f^{(1)} : J_0Y_0 \times \Lambda \to \mathcal{L}(Y),$$
$$f_1^{(1)} : J_0Y_0 \times \Lambda \to \mathcal{L}(Y_1)$$

such that $D_1g(y_0, \lambda)\xi = Jf^{(1)}(J_0y_0, \lambda)J_0\xi$ for all $(y_0, \lambda, \xi) \in Y_0 \times \Lambda \times Y_0$ and $Jf^{(1)}(J_0y_0, \lambda)y = f_1^{(1)}(J_0y_0, \lambda)Jy$ for all $(y_0, \lambda, y) \in Y_0 \times \Lambda \times Y$.

b2) *There exists $\kappa \in [0, 1)$ such that $f(\cdot, \lambda) : Y \to Y$ is Lipschitz continuous with Lipschitz constant κ, and each of $f^{(1)}(\cdot, \lambda)$ and $f_1^{(1)}(\cdot, \lambda)$ is uniformly bounded by κ.*

b3) *Under the previous condition, the unique fixed point $\Psi : \Lambda \to Y$ satisfying the equation $\Psi(\lambda) = f(\Psi(\lambda), \lambda)$ itself satisfies $\Psi = J_0 \circ \Phi$ for some continuous $\Phi : \Lambda \to Y_0$.*

b4) *$f_0 : Y_0 \times \Lambda \to Y$ defined by $(y_0, \lambda) \mapsto f_0(y_0, \lambda) = f(J_0y_0, \lambda)$ has a continuous partial derivative*

$$D_2f : Y_0 \times \Lambda \to \mathcal{L}(\Lambda, Y)$$

b5) *The mapping $(y, \lambda) \mapsto J \circ f^{(1)}(J_0y, \lambda)$ from $Y_0 \times \Lambda$ into $\mathcal{L}(Y, Y_1)$ is continuous.*

Then, the mapping $J \circ \Psi$ is of class C^1 and $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$ for all $\lambda \in \Lambda$, where $\mathcal{A} = \mathcal{A}(\lambda)$ is the unique solution of the fixed point equation $A = f^{(1)}(\Psi(\lambda), \lambda)A + D_2f_0(\Phi(\lambda), \lambda)$.

The reason we will need this lemma is because substitution operators such as $R_\delta :$ $\mathcal{PC}^{\eta,s} \to B^{\eta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B_{t_k}^{\eta,s}(\mathbb{Z}, \mathbb{R}^n)$ defined in Corollary 6.3.0.1, though Lipschitz continuous, are generally not differentiable. The surprising result is that if one instead considers the codomain to be $B^{\zeta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B_{t_k}^{\zeta,s}(\mathbb{Z}, \mathbb{R}^n)$ for some $\zeta > \eta$, then the substitution operator becomes differentiable. Since X^η-type spaces admit continuous embeddings $J : X^{\eta_1} \hookrightarrow X^{\eta_2}$ whenever $\eta_1 \leq \eta_2$, the centre manifold itself can be considered to be embedded in any appropriate weighted Banach space with high enough exponent η. An appropriate application of Lemma 7.1.1 applied to the defining fixed-point equation (6.7) of the centre manifold will allow us to prove that a composition of the embedding operator with the fixed point is a C^1 function. An inductive argument will ultimately get us to C^m smoothness.

7.1.2 A modified fixed-point equation, substitution operator and candidate differentials

Recall the definition of the modified nonlinearities

$$F_{\delta,s}(t,x) = f(t,x)\xi\left(\frac{||P_c(s)x||}{N\delta}\right)\xi\left(\frac{||(P_s(s) + P_u(s))x||}{N\delta}\right)$$

$$G_{\delta,s}(k,x) = g(k,x_{0-})\xi\left(\frac{||P_c(s)x_{0-}||}{N\delta}\right)\xi\left(\frac{||(P_s(s) + P_u(s))x_{0-}||}{N\delta}\right).$$

Since s is fixed and $\mathcal{RCR}_c(s)$ is finite-dimensional, we may assume without loss of generality that $||\cdot||$ is smooth on $\mathcal{RCR}_c(s) \setminus \{0\}$. We introduce a symbolic modification of the fixed-point operator;

$$\mathcal{G}_\delta^{\eta,s} : \mathcal{PC}^{\eta,s} \times \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$$

defined in the same way as equation (6.7). The only difference here is that wish to make the dependence on η, s and δ explicit. We denote the associated fixed point by $\tilde{u}_{\eta,s}$, provided $\delta > 0$ is sufficiently small.

From this point on, our attention shifts to proving the smoothness of $\tilde{u}_{\eta,s} : \mathcal{RCR}_c(s) \to$ $\mathcal{PC}^{\eta,s}$ as defined by the fixed point of (6.7). We begin with some notation. Define $\mathcal{PC}^\infty = \cup_{\eta>0}\mathcal{PC}^\eta$. Let

$$V^\eta = \{u \in \mathcal{PC}^\eta : ||(I - \widehat{P}_c)u||_0 < \infty\},$$

where $\widehat{P}_c$ is the projection operator from Lemma 6.5.1. Equipped with the norm

$$||u||_{V^{\eta,s}} = ||\mathcal{P}_c u||_{\eta,s} + ||(I - \mathcal{P}_c)u||_0,$$

the space $V^{\eta,s}$ is complete, where the s-shifted definitions are as outlined at the beginning of Section 6.1 .

Let $\delta > 0$ be chosen as in Lemma 6.5.1, define

$$V_\delta^\eta = \{u \in V^\eta : ||(I - \widehat{P}_c)u||_0 < \delta\}$$

and define $V_\delta^\eta(t) \subset \mathcal{RCR}$ by $V_\delta^\eta(t) = \{u(t) : u \in V_\delta^\eta\}$. Also, define the set $V_\delta^\infty = \cup_{\eta>0}V_\delta^\eta$. Set $B^\eta = PC^\eta(\mathbb{R}, \mathbb{R}^n) \oplus B_{\tau_k}^\eta(\mathbb{Z}, \mathbb{R}^n)$ and $B^\infty = \cup_{\eta>0}B^\eta$. Finally, the bounded p-linear maps from $X_1 \times \cdots \times X_p$ to Y for Banach spaces X_i and Y will be denoted $\mathcal{L}^p(X_1 \times \cdots X_p, Y)$. We may write simply $\mathcal{L}^p$ if there is no confusion.

By construction of the modified nonlinearity $R_{\delta,s}$ and the choice of δ from Lemma 6.5.1, the functions $u \mapsto F_{\delta,s}(t, u)$ and $u \mapsto G_{\delta,s}(k, u)$ are C^m on $V_\delta^\eta(t)$ and $V_\delta^\eta(\tau_k)$ respectively, for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. We are therefore free to define

$$\tilde{F}_{\delta,s}^{(p)}u(t) = D^p\tilde{F}_{\delta,s}(t, u(t)), \qquad \tilde{G}_{\delta,s}^{(p)}u(k) = D^pG_{\delta,s}(\tau_k, u(\tau_k)),$$

for $1 \le p \le m$, where D^p denotes the pth Fréchet derivative with respect to the second variable. For each $u \in V_\delta^\infty$ we can define a p-linear map $\tilde{R}_{\delta,s}^{(p)}(u) : \mathcal{PC}^\infty \times \cdots \times \mathcal{PC}^\infty \to B^\infty$ by the equation

$$\tilde{R}_{\delta,s}^{(p)}(u)(v_1, \ldots, v_p)(t, k) = (F_{\delta,s}^{(p)}u(t)(v_1(t), \ldots, v_p(t)), G_{\delta,s}^{(p)}u(k)(v_1(t_k), \ldots, v_p(t_k))). \quad (7.1)$$

For $p = 0$, we define $\tilde{R}_{\delta,s}^{(0)} = R_{\delta,s}$.

7.1.3 Smoothness of the modified nonlinearity

In this section we elaborate on various properties of the substitution operator $R_{\delta,s}$ and its formal derivative $\tilde{R}_{\delta,s}^{(p)}$ introduced in equation (7.1). First, we will need a result that states that condition H.5 holds also for the modified nonlinearities when restricted to V_δ^∞.

Lemma 7.1.2. *For $j = 1, \ldots, m$, there exist constants $\tilde{c}_j, \tilde{d}_j, \tilde{q} > 0$ such that*

$$||D^j\tilde{F}_{\delta,s}(t, \phi) - D^j\tilde{F}_{\delta,s}(t, \psi)|| \le \tilde{c}_j||\phi - \psi||, \quad ||D^j\tilde{F}_{\delta,s}(t, \phi)|| \le \tilde{q}\tilde{c}_j \quad \phi, \psi \in V_\delta^\infty(t)$$

$$||D^j\tilde{G}_{\delta,s}(k, \phi) - D^j\tilde{G}_{\delta,s}(k, \psi)|| \le \tilde{d}_j||\phi - \psi||, \quad ||D^j\tilde{G}(k, \phi)|| \le \tilde{q}\tilde{d}_j \quad \phi, \psi \in V_{\delta,s}^\infty(\tau_k).$$

Proof. We prove only the Lipschitzian property for $D^jF_{\delta,s}$, since the boundedness and corresponding results for $D^jG_{\delta,s}$ are proven similarly. Denote

$$X(s, \phi) = \xi\left(\frac{||P_c(s)\phi||}{N\delta}\right)\xi\left(\frac{||(I - P_c(s))\phi||}{N\delta}\right).$$

When $\phi, \psi \in V_\delta^\infty(t)$, X is m-times continuously differentiable and its derivative is globally Lipschitz continuous. Moreover, the Lipschitz constant can be chosen independent of s because of the uniform boundedness (property 1) of the projection operators. Let Lip_X^k denote the Lipschitz constant for $D^k X(s, \cdot)$. Then,

$$
\begin{aligned}
D^j \tilde{F}_{\delta,s}(t, \phi) - D^j \tilde{F}_{\delta,s}(t, \psi) &= D^j \left[f(t, \phi) X(s, \phi) - f(t, \psi) X(s, \phi) \right] \\
&= \sum_{N_1, N_2 \in P_2(j)} D^{\#N_1} f(t, \phi) D^{\#N_2} X(s, \phi) - D^{\#N_1} f(t, \psi) D^{\#N_2} X(s, \psi) \\
&= \sum_{N_1, N_2 \in P_2(j)} D^{\#N_1} [f(t, \phi) - f(t, \psi)] D^{\#N_2} X(s, \phi) + D^{\#N_1} f(t, \psi) D^{\#N_2} [X(s, \phi) - X(s, \psi)],
\end{aligned}
$$

where $P_2(j)$ denotes the set of partitions of length two from the set $\{1, \ldots, j\}$ and $\#Y$ is the cardinality of Y. Restricted to the ball $B_{2\delta}(0)$, the Lipschitz constants for $D^j f(t, \cdot)$ and the boundedness estimates from H.5 then imply the estimate

$$
||D^j \tilde{F}_{\delta,s}(t, \phi) - D^j \tilde{F}_{\delta,s}(t, \psi)|| \leq \left(\sum_{N_1, N_2 \in P_2(j)} (1+q) c_{\#N_1}(t) \mathrm{Lip}_X^{\#N_2} \right) ||\phi - \psi||.
$$

As each of c_j and d_j are bounded, the Lipschitz constant admits an upper bound. Outside of $B_{2\delta}(0)$, X and all of its derivatives are identically zero. □

Lemma 7.1.3. *Let $1 \leq p \leq m$, $\mu_i > 0$ for $i = 1, \ldots, p$, $\mu = \mu_1 + \cdots + \mu_p$ and $\eta \geq \mu$. Then we have $\tilde{R}_{\delta,s}^{(p)}(u) \in \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \mathcal{PC}^{\mu_p}, B^\eta)$ for all $u \in V_\delta^\infty$, with*

$$
\begin{aligned}
||\tilde{R}_{\delta,s}^{(p)}(u)||_{\mathcal{L}^p} &\leq \sup_{t \in \mathbb{R}} ||\tilde{F}_{\delta,s}^{(p)} u(t)|| e^{-(\eta-\mu)|t|} + \sup_{k \in \mathbb{Z}} ||\tilde{G}_{\delta,s}^{(p)} u(k)|| e^{-(\eta-\mu)|\tau_k|} \\
&= ||\tilde{R}_{\delta,s}^{(p)}(u)||_{\eta-\mu}.
\end{aligned}
$$

Also, $u \mapsto \tilde{R}_{\delta,s}^{(p)}(u)$ is continuous as a mapping $\tilde{R}_{\delta,s}^{(p)} : V_\delta^\sigma \to \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \mathcal{PC}^{\mu_p}, B^\eta)$ if $\eta > \mu$, for all $\sigma > 0$.

Proof. For brevity, denote $\tilde{R}_\delta = \tilde{R}_{\delta,s}$, and similarly for $\tilde{F}$ and $\tilde{G}$. It is easy to verify that

$\tilde{R}_\delta^{(p)}(u)$ is p-linear. For boundedness,

$$
\begin{aligned}
||\tilde{R}_\delta^{(p)}(u)||_{\mathcal{L}^p} &= \sup_{\substack{t\in\mathbb{R}, k\in\mathbb{Z} \\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}_\delta^{(p)}u(t)(v_1(t),\ldots,v_p(t))||e^{-\eta|t|} + ||\tilde{G}_\delta^{(p)}u(k)(v_1(t_k),\ldots,v_p(t_k))||e^{-\eta|\tau_k|} \\
&\leq \sup_{\substack{t\in\mathbb{R} \\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}_\delta^{(p)}u(t)(v_1(t),\ldots,v_p(t))||e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z} \\ ||u||_{\vec{\mu}}=1}} ||\tilde{G}_\delta^{(p)}u(k)(w_1(k),\ldots,w_p(k))||e^{-\eta|\tau_k|} \\
&\leq \sup_{\substack{t\in\mathbb{R} \\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}_\delta^{(p)}u(t)|| \cdot \left[||v_1(t)||\cdots||v_p(t)||\right]e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z} \\ ||w||_{\vec{\mu}}=1}} ||\tilde{G}_\delta^{(p)}u(k)|| \cdot \left[||w_1(k)||\cdots||w_p(k)||\right]e^{-\eta|\tau_k|} \\
&= \sup_{t\in\mathbb{R}} ||\tilde{F}_\delta^{(p)}u(t)||e^{-(\eta-\mu)|t|} + \sup_{k\in\mathbb{Z}} ||\tilde{G}_\delta^{(p)}u(k)||e^{-(\eta-\mu)|\tau_k|},
\end{aligned}
$$

where $||v||_{\vec{\mu}=1}$ is the set of all $v = (v_1,\ldots,v_p) \in \mathcal{PC}^{\mu_1} \times \cdots \times \mathcal{PC}^{\mu_p}$ such that $||v_i||_{\mu_i} = 1$ for $i = 1,\ldots,p$. The latter term in the inequality is finite by Lemma 7.1.2 whenever $\eta \geq \mu$. In particular, the latter lemma implies that for all $\phi \in V_\delta^\infty$, one has $\sup_{t\in\mathbb{R}} ||D^j\tilde{F}_\delta(t,\phi(t))|| \leq \tilde{q}\tilde{c}_j$, and similar for $\tilde{G}_k$. This uniform boundedness can then be used to prove the continuity of $u \mapsto \tilde{R}_\delta^{(p)}(u)$ when $\eta > \mu$; the proof follows that of [Lemma 7.3 [46]] and is omitted here. $\qquad\square$

The proof of the following lemmas are essentially identical to the proofs of [Corollary 7.5, Corollary 7.6, Lemma 7.7 [46]] and are omitted here. The trivial modification stems from in our case, the nonlinearity consists of two independent components defined by F_δ and G_δ, as well as the time dependence. This latter complication is effectively resolved by Lemma 7.1.2.

Lemma 7.1.4. *Let $\eta_2 > k\eta_1 > 0$, $1 \leq p \leq k$. Then, $\tilde{R}_{\delta,s} : V_\delta^{\eta_1} \to \mathcal{L}^p(\mathcal{PC}^{\eta_1} \times \cdots \times \mathcal{PC}^{\eta_1}, B^{\eta_2})$ is C^k and $D^p\tilde{R}_{\delta,s} = \tilde{R}_{\delta,s}^{(p)}$.*

Lemma 7.1.5. *Let $1 \leq p \leq m$, $\mu_i > 0$ for $i = 1,\ldots,p$, $\mu = \mu_1 + \cdots + \mu_p$ and $\eta \geq \mu$. Then, $\tilde{R}_{\delta,s}^{(p)} : V_\delta^\sigma \to \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^{k-p} provided $\eta > \mu + (k-p)\sigma$.*

Lemma 7.1.6. *Let $1 \leq p \leq k$, $\mu_i > 0$ for $i = 1,\ldots,p$, $\mu = \mu_1 + \cdots \mu_p$ and $\eta > \mu + \sigma$ for some $\sigma > 0$. Let $X : \mathcal{RCR}_c(s) \to V_\delta^\sigma$ be C^1. Then, $\tilde{R}_{\delta,s}^{(p)} \circ X : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^1 and*

$$
D\left(\tilde{R}_{\delta,s}^{(p)} \circ X\right)(\phi)(v_1,\ldots,v_p,\psi) = \tilde{R}_{\delta,s}^{(p+1)}(X(\phi))(v_1,\ldots,v_p,X'(\phi)\psi).
$$

7.1.4 Proof of smoothness of the centre manifold and tangency to the centre fibre bundle

With our preparations complete, we can formulate and prove the statement concerning the smoothness of the centre manifold.

Theorem 7.1.1. *Let $\mathcal{J}_s^{\eta_2,\eta_1} : \mathcal{PC}^{\eta_1,s} \to \mathcal{PC}^{\eta_2,s}$ denote the (continuous) embedding operator for $\eta_1 \leq \eta_2$. Let $[\tilde{\eta}, \overline{\eta}] \subset (0, \min\{-a, b\})$ be such that $k\tilde{\eta} < \overline{\eta}$. Then, for each $p \in \{1, \ldots, m\}$ and $\eta \in (p\tilde{\eta}, \overline{\eta}]$, the mapping $J_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ is of class C^p provided $\delta > 0$ is sufficiently small.*

Proof. The proof here follows the same lines as Theorem 7.7 from Section IX of [27]. To begin, we choose $\delta > 0$ small enough so that Lemma 6.5.1 is satisfied in addition to having $\tilde{L}_\delta ||\mathcal{K}_s^\eta|| < \frac{1}{4}$ for all $\eta \in [\tilde{\eta}, \overline{\eta}]$. By Lemma 6.2.3 and Corollary 6.3.0.1, this can always be done in such a way that the inequality holds for all $s \in \mathbb{R}$.

We proceed by induction on k. For $p = 1 = k$, we let $\eta \in (\tilde{\eta}, \overline{\eta}]$ and show that Lemma 7.1.1 applies with

$$Y_0 = V_\delta^{\tilde{\eta},s}, \qquad Y = \mathcal{PC}^{\tilde{\eta},s}, \qquad Y_1 = \mathcal{PC}^{\eta,s}, \qquad \Lambda = \mathcal{RCR}_c(s)$$

$$f(u, \varphi) = \tilde{\mathcal{G}}_\delta^{\tilde{\eta},s}(u, \varphi), \quad f^{(1)}(u, \varphi) = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(u), \quad f_1^{(1)}(u, \varphi) = \mathcal{K}_s^\eta \circ \tilde{R}_{\delta,s}^{(1)}(u),$$

with embeddings $J = \mathcal{J}_s^{\eta\tilde{\eta}}$ and $J_0 : V_\delta^{\tilde{\eta},s} \hookrightarrow \mathcal{PC}^{\tilde{\eta},s}$. To check condition b1 we must first verify the C^1 smoothness of

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u, \varphi) \mapsto g(u, \varphi) = \mathcal{J}_s^{\eta\tilde{\eta}}\left(U(\cdot, s)\varphi + \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}(J_0 u)\right).$$

The embedding operator $\mathcal{J}_s^{\eta\tilde{\eta}}$ is itself C^1, as is $\varphi \mapsto U(\cdot, s)\varphi$ and $J_0 u \mapsto \tilde{R}_{\delta,s}(J_0 u)$, the latter due to Lemma 7.1.4. C^1 smoothness of g then follows by continuity of the linear embedding J_0. Verification of the equalities $D_1 g(u, \varphi)\xi = J f^{(1)}(J_0 u, \varphi) J_0 \xi$ and $J f^{(1)}(J_0 u, \varphi)\xi = f_1^{(1)}(J_0 u, \varphi) J\xi$ is straightforward. Condition b2 follows by boundedness of the embedding operators and the the small Lipschitz constant for $\tilde{\mathcal{G}}_{\delta,s}^{\tilde{\eta},s}$. For condition b3, the fixed point is $\tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\tilde{\eta},s}$, and we may factor it as $\tilde{u}_{\tilde{\eta},s} = J_0 \circ \Phi$ with $\Phi : \mathcal{RCR}_c(s) \to V_\delta^{\tilde{\eta},s}$ defined by $\Phi(\varphi) = \tilde{u}_{\tilde{\eta},s}(\varphi)$; the latter is continuous by Theorem 6.4.1 and the factorization is justified by Lemma 6.5.1. To check condition b4 we must verify that

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u, \varphi) \mapsto f_0(u, \varphi) = \tilde{\mathcal{G}}_{\delta,s}^{\tilde{\eta},s}(J_0 u, \varphi)$$

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has a continuous partial derivative in its second variable – this is clear since f_0 is linear in φ. Finally, condition b5 requires us to verify that the map $(u, \varphi) \mapsto \mathcal{J}_s^{\eta\tilde{\eta}} \circ \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(J_0 u)$ is continuous from $V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s)$ into $\mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$, but this once again follows by the continuity of the embedding operators and the smoothness of $\tilde{R}_{\delta,s}$ from Lemma 7.1.4.

The conditions of Lemma 7.1.1 are satisfied, and we conclude that $\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}$ is of class C^1 and that the derivative $D(\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}) \in \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is the unique solution $w^{(1)}$ of the equation

$$w^{(1)} = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(\tilde{u}_s^{\tilde{\eta}}(\varphi))w^{(1)} + U(\cdot, s) := F_1(w^{(1)}, \varphi). \tag{7.2}$$

The mapping $F_1 : \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for $\eta \in [\tilde{\eta}, \overline{\eta}]$ – indeed, $F_1(\cdot, \varphi)$ is Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta \cdot ||\mathcal{K}_s^\eta|| < \frac{1}{4}$; this follows from Lemma 7.1.3 and is independent of s. Thus, $\tilde{u}_s^{(1)}(\varphi) \in \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta},s}) \hookrightarrow \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ for $\eta \geq \tilde{\eta}$. Moreover, $\tilde{u}_s^{(1)} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ is continuous if $\eta \in (\tilde{\eta}, \overline{\eta}]$.

Now, let $1 \leq p \leq k$ for $k \geq 1$ and suppose that for all $q \in \{1, \ldots, p\}$ and all $\eta \in (q\tilde{\eta}, \overline{\eta}]$, the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$$

is of class C^q with $D^q(\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_s^{\tilde{\eta}}) = \mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)}$ and $\tilde{u}_{\tilde{\eta},s}^{(q)}(\varphi) \in \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{q\tilde{\eta},s})$ for each $\varphi \in \mathcal{RCR}_c(s)$, such that the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)} : \mathcal{RCR}_c(s) \to \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$$

is continuous for $\eta \in (q\tilde{\eta}, \overline{\eta}]$. Suppose additionally that $\tilde{u}_{\tilde{\eta},s}^{(q)}(\varphi)$ is the unique solution $w^{(p)}$ of an equation

$$w^{(p)} = \mathcal{K}_s^{\tilde{\eta}p} \circ \tilde{R}_{\delta,s}^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p)} + H_{\tilde{\eta}}^{(p)}(\varphi) := F_{\tilde{\eta}}^{(p)}(w^{(p)}, \varphi), \tag{7.3}$$

with $H^1 = U(\cdot, s)$ and $H_x^{(p)}(\varphi)$ for $p \geq 2$ is a finite sum of terms of the form

$$\mathcal{K}_s^{px} \circ \tilde{R}_{\delta,s}^{(q)}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}_{\tilde{\eta},s}^{(r_1)}(\varphi), \cdots, \tilde{u}_{\tilde{\eta},s}^{(r_q)}(\varphi))$$

with $2 \leq q \leq p$, $1 \leq r_i < p$ for $i = 1, \ldots, q$, such that $r_1 + \cdots + r_q = p$. Under such assumptions, the mapping $F_{\tilde{\eta}}^{(p)} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for all $\eta \in [p\tilde{\eta}, \overline{\eta}]$; see Lemma 7.1.3.

Next, choose some $\eta \in ((p+1)\tilde{\eta}, \overline{\eta}]$, $\sigma \in (\tilde{\eta}, \eta/(p+1)]$ and $\mu \in ((p+1)\sigma, \eta)$. We will verify the conditions of Lemma 7.1.1 with the spaces and functions

$$Y_0 = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma, s}), \qquad Y = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\mu, s}), \quad Y_1 = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta, s})$$

$$f(u, \varphi) = \mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))u + H_{\mu/p}^{(p)}(\varphi), \quad \Lambda = \mathcal{RCR}_c(s),$$

$$f^{(1)}(u, \varphi) = \mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi)) \in \mathcal{L}(Y),$$

$$f_1^{(1)}(u, \varphi) = \mathcal{K}_s^{\eta} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi)) \in \mathcal{L}(Y_1),$$

We begin with the verification of condition b1. We must check that

$$\mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma, s}) \times \mathcal{RCR}_c(s) \ni (u, \varphi) \mapsto \mathcal{J}^{\eta\mu} \circ \mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))u + \mathcal{J}^{\eta\mu} \circ H_{\mu/p}^{(p)}(\varphi)$$

is of class C^1, where now $\mathcal{J}^{\eta_2\eta_1} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_1, s}) \hookrightarrow \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_2, s})$. The above mapping is C^1 with respect to $u \in \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma, s})$ since it is linear in this variable. With respect to $\varphi \in \mathcal{RCR}_c(s)$, we have that $\varphi \mapsto \mathcal{J}^{\eta\mu}\mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))$ is C^1: this follows by Lemma 7.1.6 with $\mu > (p+1)\sigma$ and the C^1 smoothness of $\varphi \mapsto \mathcal{J}^{\sigma\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta}, s}(\varphi)$ with $\sigma > \tilde{\eta}$. For the C^1 smoothness of the portion $\varphi \mapsto \mathcal{J}^{\eta\mu}H_{\mu/p}^{(p)}(\varphi)$, we get differentiability from Lemma 7.1.6; we have that the derivative of $\varphi \mapsto H_{\mu/p}^{(p)}(\varphi)$ is a sum of terms of the form

$$\mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(q+1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))(\tilde{u}_{\tilde{\eta}, s}^{(r_1)}(\varphi), \ldots, \tilde{u}_{\tilde{\eta}, s}^{(r_q)}(\varphi))$$

$$+ \sum_{j=1}^{q} \mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(q)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))(\tilde{u}_{\tilde{\eta}, s}^{(r_1)}(\varphi), \ldots, \tilde{u}_{\tilde{\eta}, s}^{(r_j+1)}(\varphi), \ldots, \tilde{u}_{\tilde{\eta}, s}^{(r_q)}(\varphi)),$$

and each $\tilde{u}_{\tilde{\eta}, s}^{(j)}$ is understood as a map into $\mathcal{PC}^{j\sigma, s}$. Applying Lemma 7.1.3 with $\mu > (p+1)\sigma$ grants continuity of $DH_{\mu/p}^{(p)}(\varphi)$ and, subsequently, to $\mathcal{J}^{\eta\mu}DH_{\mu/p}^{(p)}(\varphi)$. The other embedding properties of condition b1 are easily checked. Condition b4 can be proven similarly.

The Lipschitz condition and boundedness of b2 follows by the choice of $\delta > 0$ at the beginning and the uniform contractivity of H_p described above. Condition b3 is proven by writing

$$\mathcal{J}^{\eta\mu} \circ \mathcal{K}_s^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s})(\varphi)) = \mathcal{K}_s^{\eta} \circ \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}(\varphi))$$

and applying Lemma 7.1.3 together with the C^1 smoothness of $\tilde{u}_{\tilde{\eta}, s}$ to obtain the continuity of $\varphi \mapsto \tilde{R}_{\delta, s}^{(1)}(\tilde{u}_{\tilde{\eta}, s}) \in \mathcal{L}(Y, Y_1)$. This also proves the final condition b5 of Lemma 7.1.1, and

we conclude that $\tilde{u}_{\tilde{\eta},s}^{(p)} : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is of class C^1 with $\tilde{u}_{\tilde{\eta},\mu}^{(p+1)} = D\tilde{u}_{\tilde{\eta},s}^{(p)} \in \mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,\mu})$ given by the unique solution $w^{(p+1)}$ of the equation

$$w^{(p+1)} = \mathcal{K}_s^\mu \circ \tilde{R}_{\delta,s}^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p+1)} + H_{\mu/(p+1)}^{(p+1)}(\varphi) \tag{7.4}$$

where $H_{\mu/(p+1)}^{(p+1)}(\varphi) = \mathcal{K}_s^\mu \circ \tilde{R}_{\delta,s}^{(2)}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}_{\tilde{\eta},s}^{(p)}(\varphi), \tilde{u}_{\tilde{\eta},s}^{(1)}(\varphi)) + DH_{\mu/p}^{(p)}(\varphi)$. Similar arguments to the proof of the case $k = 1$ show that the fixed point $w^{(p+1)}$ is also contained in $\mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta}(p+1),s})$, and the proof is complete. $\qquad\square$

Corollary 7.1.1.1. $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ *is C^m and tangent at the origin to the centre bundle $\mathcal{RCR}_c$. More precisely, $\mathcal{C}(t, \cdot) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ is C^k and $D\mathcal{C}(t, 0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_c(t)$.*

Proof. Let $\tilde{\eta}, \eta$ be as in the proof of Theorem 7.1.1. Define the evaluation map $ev_t : \mathcal{PC}^\eta \to \mathcal{RCR}$ by $ev_t(f) = f(t)$. Since we can decompose the centre manifold as

$$\mathcal{C}(t, \phi) = ev_t(\tilde{u}_t(\phi)) = ev_t(\mathcal{J}_t^{\eta\tilde{\eta}} \tilde{u}_t(\phi)),$$

boundedness of the linear evaluation map on the space $\mathcal{PC}^{\eta,t}$ then implies the C^k smoothness of $\mathcal{C}(t, \cdot)$. To obtain the tangent property, we remark that

$$D\mathcal{C}(t, 0)\phi = ev_t\left(D\left(\mathcal{J}_t^{\eta\tilde{\eta}} \circ \tilde{u}_t(0)\right)\phi\right) = ev_t\left(\tilde{u}_{\eta,t}^{(1)}(0)\phi\right).$$

From equation (7.2) and Theorem 6.5.1, we obtain $\tilde{u}_{\eta,t}^{(1)}(0) = U(\cdot, t)$, from which it follows that $D\mathcal{C}(t, 0)\phi = \phi$, as claimed. $\qquad\square$

As a secondary corollary, we can prove that each derivative of the centre manifold is uniformly Lipschitz continuous. The proof is similar to that of Corollary 6.4.1.1 if one takes into account the representation of the derivatives $\tilde{u}_{\tilde{\eta},s}^{(p)}$ as solutions of the fixed-point equations (7.4), whose right-hand side is a contraction with Lipschitz constant independent of s.

Corollary 7.1.1.2. *For each $p \in \{1, \ldots, k\}$, there exists a constant $L(p) > 0$ such that the centre manifold satisfies $||D^p\mathcal{C}(t, \phi) - D^p\mathcal{C}(t, \psi)|| \leq L(p)||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$.*

We readily obtain the smoothness of the centre manifold in the case where the semilinear equation is periodic. In particular, in such a situation many some of the assumptions H.1–H.8 are satisfied automatically and can be ignored.

Corollary 7.1.1.3. *Suppose the semilinear equation* (5.1)–(5.2) *satisfies the following conditions.*

P.1 *The equation is periodic with period T and c impulses per period. That is, $L(t+T) = L(t)$ and $f(t+T, \cdot) = f(t)$ for all $t \in \mathbb{R}$, and $B(k+c) = B(k)$, $g(k+c, \cdot) = g(k, \cdot)$ and $\tau_{k+c} = \tau_k + T$ for all $k \in \mathbb{Z}$.*

P.2 *Conditions H.1–H.3 and H.5–H.6 are satisfied.*

Then, the conclusions of Corollary 7.1.1.1 and Corollary 7.1.1.2 hold.

7.1.5   Periodic centre manifold

In this section we will prove that the centre manifold is itself a periodic function, provided the conditions P.1–P.2 of Corollary 7.1.1.3 are satisfied. We begin with a preparatory lemma.

Lemma 7.1.7. *Define the operator $N_s : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(s)$ by*

$$N_s(\phi) = P_c(s)S(s+T, s)\mathcal{C}(s, \phi).$$

This operator is well-defined and invertible in a neighbourhood of $0 \in \mathcal{RCR}_c(s)$. Moreover, the neighbourhood can be written $U \cap \mathcal{RCR}_c(s)$ for some open neighbourhood $U \subset \mathcal{RCR}$ of $0 \in \mathcal{RCR}$, independent of s.

Proof. To show that N_s is invertible in a neighbourhood of the origin we will use the inverse function theorem. The Fréchet derivative of N_s at 0 is given by

$$\begin{aligned}
DN_s(0)\phi &= P_c(s) \circ DS(s+T, s)(0) \circ D\mathcal{C}(s, 0)\phi \\
&= P_c(s+T) \circ U(s+T, s)\phi \\
&= U_c(s+T, s)\phi,
\end{aligned}$$

where we used Corollary 7.1.1.1 to calculate $D\mathcal{C}(s, 0)$ and Theorem 5.2.1 to calculate $DS(s+T, s)(0)$. Since $U(s+T, s)$ is an isomorphism (Theorem 4.3.1) of $\mathcal{RCR}_c(s)$ with $\mathcal{RCR}_c(s+T) = \mathcal{RCR}_c(s)$, we obtain the claimed local invertibility.

To show that the neighbourhood may be written as claimed, we notice that $DN_s(x)$ is uniformly convergent (in the variable s) as $x \to 0$ to $DN_s(0)$. Indeed, we have the estimate

$$||DN_s(x) - DN_s(0)|| \leq ||U_c(s+T, s)P_c(s)|| \cdot ||D\mathcal{C}(s, x) - D\mathcal{C}(s, 0)||,$$

and the Lipschitz property of Corollary 7.1.1.2 together with uniform boundedness of the projector $P_c(s)$ and centre monodromy operator $U_c(s+T, s)$ grants the uniform convergence as $x \to 0$. As a consequence, the implicit function may be defined on a neighbourhood that does not depend on s. $\square$

Theorem 7.1.2. *There exists $\delta > 0$ such that $\mathcal{C}(s+T, \phi) = \mathcal{C}(s, \phi)$ for all $s \in \mathbb{R}$ whenever $||\phi|| \leq \delta$.*

Proof. By Lemma 7.1.7, there exists $\delta > 0$ such that if $||\phi|| \leq \delta$, we can write $\phi = N_s(\psi)$ for some $\psi \in \mathcal{RCR}_c(s)$. By Theorem 6.5.1 and the periodicity condition P.1,

$$
\begin{aligned}
\mathcal{C}(s+T, \phi) &= \mathcal{C}(s+T, N_s(\psi)) \\
&= \mathcal{C}(s+T, P_c(s+T)S(s+T, s)\mathcal{C}(s, \psi)) \\
&= S(s+T, s)\mathcal{C}(s, \psi) \\
&= S(s, s-T)\mathcal{C}(s, \psi) \\
&= \mathcal{C}(s, P_c(s)S(s, s-T)\mathcal{C}(s, \psi)) \\
&= \mathcal{C}(s, P_c(s)S(s+T, s)\mathcal{C}(s, \psi)) \\
&= \mathcal{C}(s, N_s(\psi)) = \mathcal{C}(s, \phi),
\end{aligned}
$$

where the identity $S(s+T, s) = S(s, s-T)$ follows due to periodicity and Lemma 5.1.1. $\square$

7.2   Pointwise regularity in time

In the previous section we were concerned with the smoothness of $\phi \mapsto \mathcal{C}(t, \phi)$. To contrast, in this section we are interested in to what degree the function $t \mapsto D^k\mathcal{C}(t, \phi)$ is differentiable, for each $k = 1, \ldots, m$. Our observations in Section 3.2.2 should make it clear that we cannot expect this function to be differentiable, but we would expect $t \mapsto \mathcal{C}(t, \phi)(\theta)$ to right-continuous and regulated for each fixed θ.

Perhaps it is better to motivate our ideas on regularity in time by explaining how we will be using the centre manifold in applications. From Taylor's theorem, $\mathcal{C}(t, \phi)$ admits an expansion of the form

$$
\mathcal{C}(t, \phi) = D\mathcal{C}(t, 0)\phi + \frac{1}{2}D^2\mathcal{C}(t, 0)[\phi]^2 + \cdots + \frac{1}{m!}D^m\mathcal{C}(t, 0)[\phi]^m + O(||\phi||^{m+1}),
$$

where $[\phi]^k = [\phi, \ldots, \phi]$ with k factors, and the $O(||\phi||^{m+1})$ terms generally depend on t. By Theorem 7.1.2, under periodicity assumptions these terms will be uniformly bounded

in t for $||\phi||$ sufficiently small. This expansion can in principle be used in the dynamics equation (6.13)–(6.14) on the centre manifold or its integral version (6.11), which will permit us to classify bifurcations in impulsive RFDE. In later sections we will want to make these dynamics equations concrete – that is, to pose them in a concrete vector space such as $\mathbb{R}^p$ for some $p \in \mathbb{N}$. By analogy with ordinary and delay differential equations, this should also allow us to obtain a partial differential equation for the Taylor coefficients $D^j\mathcal{C}(t,0)$. As these coefficients are time-varying, we should suspect this PDE to contain derivatives in time as well.

In summary, we need to consider the differentiability of the function $t \mapsto D^j\mathcal{C}(t,0)$ for $j = 1, \ldots, m$. Since we know that this function will not actually be differentiable, we consider instead the differentiability of

$$t \mapsto D^j\mathcal{C}(t,0)[\phi_1, \ldots, \phi_j](\theta)$$

for each $\theta \in [-r, 0]$ and p-tuples $\phi_1, \ldots, \phi_p$. While a more realistic goal, even this is too strong a condition. It is precisely here here that we see the biggest difficulty in translating from an autonomous centre manifold formulation to a nonautonomous one based on fibre bundles: the first differential $D\mathcal{C}(t,0) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ of the centre manifold has a different domain for each t. As a consequence, we can not even define the derivative of $t \mapsto D\mathcal{C}(t,0)\phi(\theta)$, since we must have $\phi \in \mathcal{RCR}_c(t)$ for the right-hand side to be well-defined. This problem is apparent for all higher differentials.

7.2.1 A coordinate system and pointwise $PC^{1,m}$-regularity

To address the issue the centre manifold having a "time-varying domain", first note if we fix a sufficiently well-behaved coordinate system in $\mathcal{RCR}_c(t)$ – for example, let $\phi_1, \ldots, \phi_p$ be a basis for for $\mathcal{RCR}_c(0)$ and define $\phi_i(t) = U_c(t,0)\phi_i$ for $i = 1, \ldots, p$ to be a basis for $\mathcal{RCR}_c(t)$ – then the function $w(t)$ of Lemma 6.7.1 and Theorem 6.7.1 can be written as $w(t) = \Phi_t z(t)$ for $z \in \mathbb{R}^p$, where $\Phi_t = [\ \phi_1(t) \quad \cdots \quad \phi_p(t)\ ]$. This motivates us to consider instead a centre manifold in these coordinates.

Definition 7.2.1. *The function* $C : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$ *defined by*

$$C(t,z) = \mathcal{C}(t, \Phi_t z). \tag{7.5}$$

is the centre manifold in terms of the basis array Φ.

If $\mathcal{C}(t, \cdot)$ is C^m-smooth, the chain rule implies the same is true for $C(t, \cdot)$. It follows that

$$\mathcal{C}(t, w(t)) = DC(t,0)z(t) + \frac{1}{2}D^2C(t,0)[z(t), z(t)] + \cdots + \frac{1}{m!}D^mC(t,0)[z(t)]^m + O(||w(t)||^{m+1}),$$

so insofar as dynamics on the centre manifold are concerned, it is enough to study the differentiability of $t \mapsto D^jC(t,0)[z_1, \ldots, z_p](\theta)$ for p-tuples $z_1, \ldots, z_p \in \mathbb{R}^p$. Specifically, the temporal regularity we will attempt to prove is given in the following definition.

Definition 7.2.2. *A function $F : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$ is pointwise $PC^{1,m}$-regular at zero if it satisfies the following conditions.*

- *$x \mapsto F(t,x)$ is C^m in a neighbourhood of $0 \in \mathbb{R}^p$, uniformly in t;*

- *for $j = 0, \ldots, m$, $D^jF(t,0)[z_1, d, \ldots, z_j](\theta)$ is differentiable from the right with limits on the left separately with respect to t and θ, for all $z_1, \ldots, z_j \in \mathbb{R}^p$.*

With this in mind, the result we will prove is as follows.

Theorem 7.2.1. *Let $\phi_1, \ldots, \phi_p$ be a basis for $\mathcal{RCR}_c(0)$, and define*

$$\Phi_t = [\ U_c(t,0)\phi_1 \quad \cdots \quad U_c(t,0)\phi_p \ ].$$

If the centre manifold $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is (fibrewise) C^m, then the centre manifold in terms of the basis array Φ is effectively pointwise $PC^{1,m}$-regular at zero provided certain technical conditions are met (assumption H.10). Moreover, if $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, then $t \mapsto C(t,z)(\theta(t))$ is continuous from the right with limits on the left for all $z \in \mathbb{R}^p$, and $z \mapsto C(t,z)$ is Lipschitz continuous, uniformly for $t \in \mathbb{R}$.

The technical condition will be introduced in Section 7.2.3.

7.2.2 Reformulation of the fixed-point equation

Given that $C(t,z) = \mathcal{C}(t, \Phi_t z)$, we can equivalently write $C(t,z) = v_t(z)(t)$ with $v_t : \mathbb{R}^p \to \mathcal{PC}^{\eta,t}$ the unique fixed point of the equation

$$v_t(z) = \Phi_{(.)}z + \mathcal{K}_t^\eta(R_{\delta,t}(v_t(z))) \tag{7.6}$$

for each $|z|$ small enough, where $\mathcal{K}_t^\eta$ is as defined in equation (6.3) and $R_{\delta,t}$ is the substitution operator from Section 6.3. Notice also that the nonlinear operator defining the

right-hand side of the equation admits the same Lipschitz constant as original fixed-point operator $\mathcal{G}$ from equation (6.7). Up to an appropriate embedding, the jth differential $v_t^{(j)}$ satisfies for $j \geq 2$ a fixed-point equation of the form

$$v_t^{(j)} = \mathcal{K}_t^\eta \circ R_{\delta,t}^{(1)}(v_t)v_t^{(j)} + \mathcal{K}_t^\eta \circ H^{(j)}(v_t, v_t^{(1)}, \ldots, v_t^{(j-1)}), \tag{7.7}$$

with the right-hand side defining a uniform contraction in $v_t^{(j)}$. $H^{(j)}$ can be written as a finite linear combination of terms of the form

$$R_{\delta,t}^{(q)}(v_t)[v_t^{(r_1)}, \ldots, v_t^{(r_q)}],$$

for $q \in \{2, \ldots, j\}$, such that $r_1 + \cdots + r_q = j$. All of this follows from (the proof of) Theorem 7.1.1. Explicitly,

$$H^{(j)} = -R_{\delta,t}^{(1)}(v_t)v_t^{(j)} + D_z^j[R_{\delta,t}(v_t(z))],$$

and one can verify by induction on j that $H^{(j)}$ contains no term of the form $R^{(1)}(v_t)v_t^{(j)}$ and that the coefficients in the aforementioned linear combination are independent of t. To compare, for $j = 0$ and $j = 1$, we can compute directly from the definition of the fixed point and by using Corollary 7.1.1.1 and the chain rule that

$$v_t(0)(\cdot) = 0, \tag{7.8}$$

$$v_t^{(1)}(0)(\cdot) = \Phi_{(\cdot)}. \tag{7.9}$$

The assumption $Df(t,0) = Dg(k,0) = 0$ implies $R_\delta^{(1)}(0) = 0$, so the fixed-point equation (7.7) implies

$$v_t^{(j)}(0)(\mu) = \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \ldots, v_t^{(j-1)}(0)(\cdot))\right](\mu) \tag{7.10}$$

for $j \geq 2$. By definition of the basis array Φ, the following lemma is proven.

Lemma 7.2.1. *If the centre manifold is C^1, then the centre manifold in terms of the basis array Φ is pointwise $PC^{1,1}$-regular at zero. If the centre manifold is C^m, then the centre manifold in terms of the basis array Φ is pointwise $PC^{1,m}$-regular at zero provided*

$$t \mapsto \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \ldots, v_t^{(j-1)}(0)(t))\right](t)[z_1, \ldots, z_j](\theta),$$

$$\theta \mapsto \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \ldots, v_t^{(j-1)}(0)(t))\right](t)[z_1, \ldots, z_j](\theta)$$

are each, for $j = 2, \ldots, m$ differentiable from the right with limits on the left, for all $z_1, \ldots, z_j \in \mathbb{R}^p$.

7.2.3 A technical assumption on the projections $P_c(t)$ and $P_u(t)$

By definition of the bounded linear map $\mathcal{K}_t^\eta$ from (6.3), it will be necessary to differentiate (in t) integrals involving terms of the form $\mu \mapsto U(t,\mu)P_s(\mu)\chi_0$ and $\mu \mapsto U(t,\mu)P_u(\mu)\chi_0$. Generally, if we assume $\mathcal{RCR}_u(0)$ to be q-dimensional (guaranteed by Theorem 4.3.1 if the linearization is periodic), then we can fix a basis $\psi_1, \ldots, \psi_q$ for $\mathcal{RCR}_u(0)$ and construct a basis array

$$\Psi_t = [\ U_u(t,0)\psi_1 \quad \cdots \quad U_u(t,0)\psi_q\ ]$$

for $\mathcal{RCR}_u(t)$ that is formally analogous to the basis array Φ_t for the centre fibre bundle. Under spectral separation assumptions, $U_u(t,s) : \mathcal{RCR}_u(s) \to \mathcal{RCR}_u(t)$ and $U_c(t,s) : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(t)$ are topological isomorphisms, from which it follows that there exist unique $Y_c(t) \in \mathbb{R}^{p \times n}$ and $Y_u(t) \in \mathbb{R}^{q \times n}$ such that

$$\begin{aligned} P_c(t)\chi_0 &= \Phi_t Y_c(t), \\ P_u(t)\chi_0 &= \Psi_t Y_u(t). \end{aligned} \tag{7.11}$$

Even under periodicity conditions, computing the action of these projections on the functional $\chi_0 \in \mathcal{RCR}([-r,0], \mathbb{R}^{n \times n})$ is quite nontrivial and requires computing the abstract contour integral (4.5). Though this can in principle be done numerically by discretizing the monodromy operator – see Section 4.6.3 – there is little in the way of theoretical results guaranteeing, for example, that the matrix functions $t \mapsto Y_c(t)$ and $t \mapsto Y_u(t)$ are respectively elements of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p \times n})$ and $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{q \times n})$. Such a result would make the differentiation of the integrals appearing in the definition of $\mathcal{K}_t^\eta$ much more reasonable. We therefore introduce another hypothesis. We will discuss it in a bit more detail in Section 7.2.7.

H.9 There are (finite) basis arrays Φ and Ψ for $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ respectively for which the matrix functions $t \mapsto Y_c(t)$ and $t \mapsto Y_u(t)$ from equation (7.11) are continuous from the right and possess limits on the left.

7.2.4 Proof of Theorem 7.2.1

We deal first with the continuity of $t \mapsto C(t,z)(\theta(t))$ from the right and the existence of its left limits. Since $C(\cdot, z) = v_t(z)(\cdot) \in \mathcal{PC}^{\eta,t}$, it can be identified with a history function $t \mapsto c_t$ for some $c \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. But this implies $C(t,z)(\theta(t)) = c_t(\theta(t)) = c(t + \theta(t))$. The conclusion follows because $c \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, and right-continuity and limits respect composition. As for the Lipschitzian claim, it follows by

similar arguments to the proof of the original centre manifold Theorem 6.4.1 and Corollary 6.4.1.1.

Using the definition of the linear map $\mathcal{K}_t^\eta$ in (6.3) and equation (7.10), we can explicitly write $v_t^{(j)}(0)(t)$ as

$$v_t^{(j)}(0)(t) = \int_{-\infty}^{t} U(t,\mu)[I - P_c(\mu) - P_u(\mu)]\chi_0 \hat{H}_1^{(j)}(\mu)d\mu - \int_{t}^{\infty} U(t,\mu)P_u(\mu)\chi_0 \hat{H}_1^{(j)}(\mu)d\mu$$
$$+ \sum_{-\infty}^{t} U(t,t_i)[I - P_c(t_i) - P_u(t_i)]\chi_0 \hat{H}_2^{(j)}(t_i)dt_i - \sum_{t}^{\infty} U(t,t_i)P_u(t_i)\chi_0 \hat{H}_2^{(j)}(t_i)dt_i,$$

where each of $\hat{H}_1^{(j)}(\mu)$ and $\hat{H}_2^{(j)}(t_i)$ and $H^{(j)}$ are related by the equations

$$H^{(j)} = \sum_i c_i R_{\delta,t}^{(r_i)}(0)[\Phi_{(\cdot)}^{d_{i,1}}, [v_t^{(2)}(0)(t)]^{d_{i,2}}, \ldots, [v_t^{(j-1)}(0)(t)]^{d_{i,j-1}}]$$

$$\hat{H}_1^{(j)}(\mu) = \sum_i c_i D^{r_i} f(\mu,0)[\Phi_\mu^{d_{i,1}}, [v_t^{(2)}(0)(\mu)]^{d_{i,2}}, \ldots, [v_t^{(j-1)}(0)(\mu)]^{d_{i,j-1}}]$$

$$\hat{H}_2^{(j)}(t_k) = \sum_i c_i D^{r_i} g(k,0)[\Phi_{t_k}^{d_{i,1}}, [v_t^{(2)}(0)(t_k)]^{d_{i,2}}, \ldots, [v_t^{(j-1)}(0)(t_k)]^{d_{i,j-1}}].$$

The first line follows from the definition of $H^{(j)}$, while the other two come from the definition of the substitution operator. Note also that we have suppressed the inputs $z_1, \ldots, z_j$; technically, each of $\hat{H}_1^{(j)}(\mu)$ and $\hat{H}_2^{(j)}(\mu)$ are j-linear maps from $\mathbb{R}^p$ to $\mathcal{RCR}$. Using assumption H.9, we can equivalently write $v_t^{(j)}(0)(t)$ as

$$v_t^{(j)}(0)(t) = \int_{-\infty}^{t} [U(t,\mu)\chi_0 - \Phi_t Y_c(\mu) - \Psi_t Y_u(\mu)]\hat{H}_1^{(j)}(\mu)d\mu - \int_{t}^{\infty} \Psi_t Y_u(\mu)\hat{H}_1^{(j)}(\mu)d\mu$$
$$+ \sum_{-\infty}^{t} [U(t,t_i)\chi_0 - \Phi_t Y_c(t_i) - \Psi_t Y_u(t_i)]\hat{H}_2^{(j)}(t_i)dt_i - \sum_{t}^{\infty} \Psi_t Y_u(t_i)\hat{H}_2^{(j)}(t_i)dt_i.$$

$$(7.12)$$

At this stage we remark that Theorem 7.1.1 implies $v_t^{(i)}(0)(\cdot)[z_1, \ldots, z_i] \in \mathcal{PC}^\infty$ for $i = 1, \ldots, j-1$, while Φ_t is pointwise differentiable from the right by its very definition. With these details and assumption H.3, $\mu \mapsto \hat{H}_1^{(j)}(\mu)[z_1, \ldots, z_j]$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ for every tuple $z_1, \ldots, z_j \in \mathbb{R}^p$. From assumption H.9, $v_t^{(j)}(0)(t)$ is pointwise differentiable from the right if and only if the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} U(t+\epsilon,\mu)\chi_0 \hat{H}_1^{(j)}(\mu)d\mu$$

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exists pointwise. From equation (3.24) and Lemma 3.3.5, we can equivalently write the integral above in terms of the fundamental matrix solution:

$$\int_t^{t+\epsilon} U(t+\epsilon,\mu)\chi_0\hat{H}_1^{(j)}(\mu)d\mu = \int_t^{t+\epsilon} \chi_{(-\infty,t+\epsilon+\theta]}(\mu)\left(I + \int_\mu^{t+\epsilon+\theta} L(\zeta)V_\zeta(\cdot,\mu)d\zeta\right)\hat{H}_1^{(j)}(\mu)d\mu.$$

If $\theta < 0$, then the integrand vanishes when $\epsilon < -\theta$. Since $\mu \mapsto \hat{H}_1^{(j)}(\mu)$ is continuous from the right, we conclude that

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon} \int_t^{t+\epsilon} U(t,\mu)\chi_0\hat{H}_1^{(j)}(\mu)d\mu = \chi_0\hat{H}_1^{(j)}(t),$$

so that $t \mapsto v_t^{(j)}(0)(t)$ is differentiable from the right (for θ fixed), as claimed. The proof of existence of limits on the left is similar and omitted.

To get the analogous result for θ, it is worth recalling that from the fixed point formulation, $v_t^{(j)}(0)$ is a j-linear map from $\mathbb{R}^p$ to $\mathcal{PC}^{\eta,t}$. As a consequence, for all $t \in \mathbb{R}$, $\theta \in [-r,0]$ and $z_1,\ldots,z_j \in \mathbb{R}^p$ the equation

$$v_t^{(j)}(0)(t)[z_1,\ldots,z_j](\theta) = v_t^{(j)}(0)(t+\theta)[z_1,\ldots,z_j](0)$$

is satisfied. The analogous differentiability and limit results for θ therefore follow from those of t, completing the proof.

7.2.5 The hyperbolic part is pointwise $PC^{1,m}$-regular at zero

Later we will need to also consider the Taylor expansions of the *hyperbolic part* $H : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$ of the centre manifold in terms of a basis array ϕ, defined by

$$H(t,z) = (I - P_c(t))C(t,z). \tag{7.13}$$

The hyperbolic part is guaranteed to be C^m-smooth in z, since $(I - P_c(t))$ is linear. To show that it is pointwise $PC^{1,m}$-regular at zero, we notice that $H(t,z) = h_t(z)(t)$, where $h_t(z)$ can be written as

$$h_t(z) = (I - P_c(t))\Phi_{(\cdot)}z + \mathcal{K}_* \circ R_{\delta,t}(v_t(z))$$

in $\mathcal{PC}^0$. However, since $(I - P_c(t))$ is uniformly bounded, $\mathcal{K}_* = (I - P_c(t))\mathcal{K}_t^\eta$ is well-defined as a map from η-bounded inhomogeneities into $\mathcal{PC}^{\eta,t}$. Setting $z = 0$, it follows that

$$h_t(0) = 0,$$
$$h_t^{(1)}(0)(t) = 0$$
$$h_t^{(j)}(0)(t) = \mathcal{K}_* \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \ldots, v_t^{(j-1)}(0)(\cdot)).$$

On the other hand, for $z \neq 0$ we have

$$h_t(z)(t) = \mathcal{K}_* \circ R_{\delta,t}(v_t(z))(t).$$

By the same argument as in the proof of Theorem 7.2.1, we can make the following conclusion.

Corollary 7.2.1.1. *The hyperbolic part $H(t, z) = (I - P_c(t))C(t, z)$ of the centre manifold in terms of the basis array ϕ is pointwise $PC^{1,m}$-regular at zero. Moreover, if $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, then $t \mapsto H(t, z)(\theta(t))$ is continuous from the right and has limits on the left for all $z \in \mathbb{R}^p$, and $z \mapsto H(t, z)$ is Lipschitz continuous uniformly for $t \in \mathbb{R}$.*

7.2.6   Uniqueness of the Taylor coefficients

Theorem 7.2.1 guarantees that the coefficients in the Taylor expansion

$$C(t, z) = DC(t, 0)z + \frac{1}{2}D^2C(t, 0)[z, z] + \cdots + \frac{1}{m!}C^mC(t, 0)[z, \ldots, z] + O(||z||^{m+1})$$

are pointwise differentiable from the right and have limits on the left. However, the centre manifold $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ used to define the representation in terms of the basis array Φ depends non-canonically on the choice of cut-off function used to define the substitution operator $R_{\delta,t}$. However, this cutoff function does not actually factor into the coefficients $D^jC(t, 0)$. Indeed, each of $\mu \mapsto v_t^{(j)}(0)(\mu)$ is a sum of improper integrals and convergent series that depend only the lower-order terms $v_t^{(i)}(0)(\cdot)$ for $i < j$ – see equation (7.12) – and is independent of the cutoff function. By induction, we can see from (7.8)–(7.10) that, in fact, none of these lower-order terms depend on the cutoff function. The same arguments apply to the hyperbolic part. Since this is the only non-canonical element in the definition of the centre manifold (indeed, the renorming is only relevant outside of a small neighbourhood of $0 \in \mathcal{RCR}$ and so does not affect Taylor expansions), the following corollary is proven.

Corollary 7.2.1.2. *Let Φ be a basis array for $\mathcal{RCR}_c$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct centre manifolds, and let C_1 and C_2 respectively be the centre manifolds with respect to the basis array Φ. Also, let H_1 and H_2 be the respective hyperbolic parts. Then, for $j = 1, \ldots, m$, we have $D^jC_1(t, 0) = D^jC_2(t, 0)$ and $D^jH_1(t, 0) = D^jH_2(t, 0)$. That is, the Maclaurin series expansion of the centre manifold in terms of the basis array Φ is unique, as is that of the hyperbolic part.*

7.2.7 A discussion on the regularity of the matrices $t \mapsto Y_j(t)$

Hypothesis H.9 introduces a technical assumption on the matrices appearing in the decomposition (7.11). It is our goal in this section to formally demonstrate that there is reason to suspect that this hypothesis holds generally, although proving this result would likely be difficult. We will consider only $t \mapsto Y_c(t)$, since the discussion for $t \mapsto Y_u(t)$ is the same.

When the linearization "has no delayed terms" and is spectrally separated as a finite-dimensional system, $t \mapsto Y_c(t)$ is automatically continuous from the right with limits on the left. Abstractly, we would say that the functionals defining the linearization have support in the subspace $\mathcal{RCR}^0 = \{\chi_0 \xi : \xi \in \mathbb{R}^n\}$. Let us prove this claim. Let $X(t, s)$ denote the Cauchy matrix associated to the linearization

$$\dot{x} = L(t)x(t), \qquad\qquad t \neq t_k \qquad\qquad (7.14)$$
$$\Delta x = B(k)x(t^-), \qquad\qquad t = t_k. \qquad\qquad (7.15)$$

The projection $P_c(t)$ onto the associated centre fibre bundle satisfies the equation

$$X(t, s)P_c(s) = P_c(t)X(t, s)$$

for all $t \geq s$. However, since $X^{-1}(t, s)$ exists for all $t, s \in \mathbb{R}$ – see the monograph [8] for the relevant background on impulsive differential equations in finite-dimensional spaces – we have $P_c(t) = X(t, 0)P_c(0)X^{-1}(t, 0)$ for all $t \in \mathbb{R}$. Moreover, $t \mapsto X(t, 0)$ is continuous from the right and has limits on the left, from which it follows that the same is true for $P_c(t)$. Similarly, each of $P_s(t)$ and $P_u(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n \times n})$. If we write $P_c(t) = \Phi(t)Y_c(t)$ for $\Phi(t) = X(t, 0)\Phi(0)$ a matrix whose columns form a basis for $\mathcal{RCR}_c(t)$, then the observation that the columns of $\Phi(t)$ are linearly independent implies we can write

$$Y_c(t) = \Phi^+(t)P_c(t),$$

where $\Phi^+(t)$ is the left-inverse of $\Phi(t)$. Since the rank of $t \mapsto \Phi(t)$ is constant, $t \mapsto \Phi^+(t)$ is continuous from the right and has limits on the left. It follows that $t \mapsto Y_c(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p \times n})$. If (7.14)–(7.15) is now considered as an impulsive RFDE with phase space $\mathcal{RCR}([-r, 0], \mathbb{R}^n)$ for some $r > 0$, then we can write

$$U(t, s)\phi(\theta) = \begin{cases} X(t + \theta, s)\phi(0), & t + \theta \geq s \\ \phi(t + \theta - s), & t + \theta < s. \end{cases}$$

If one defines $\mathcal{P}_j(t) : \mathcal{RCR} \to \mathcal{RCR}$ by

$$\mathcal{P}_j(t)\phi(\theta) = X(t + \theta, t)P_j(t)\phi(0),$$

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one can verify directly $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated with the triple of projectors $(\mathcal{P}_s, \mathcal{P}_c, \mathcal{P}_u)$. But then,

$$\mathcal{P}_c(t)\chi_0(\theta) = X(t+\theta, t)P_c(t) = X(t+\theta, t)\Phi(t)Y_c(t) = \Phi(t+\theta)Y_c(t) = \Phi_t(\theta)Y_c(t).$$

We already know that $t \mapsto Y_c(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p \times n})$, and since this same matrix satisfies the decomposition $\mathcal{P}_c(t)\chi_0 = \Phi_t Y_c(t)$, we are done.

In the general case, the situation is far more subtle. For simplicity, consider the nonautonomous functional differential equation without impulses,

$$\dot{x} = L(t)x_t.$$

Restricted to the subspace $\mathcal{RCR}^1$, the evolution family $U(t,s)$ satisfies pointwise (in θ) the abstract ordinary differential equation

$$\frac{d}{dt}U(t,s)\phi = \mathcal{L}(t)U(t,s)\phi, \quad \mathcal{L}(t)\phi(\theta) := \chi_0(\theta)L(t)\phi + \chi_{[-r,0)}(\theta)\frac{d^+}{d\theta}\phi(\theta).$$

for all $\phi \in \mathcal{RCR}^1$. Recall that $P_c(t)$ has range in $\mathcal{RCR}^1$, so we can evaluate both sides at $\phi = P_c(s)\chi_0$. From the commutativity property $(P_c(t)U(t,s) = U(t,s)P_c(s))$ of the projectors and the identity $P_c(t)\chi_0 = \Phi_t Y_c(t)$, the equation

$$[\mathcal{L}(t)\Phi_t]Y_c(s) = \frac{d}{dt}[P_c(t)U(t,s)\chi_0]$$

holds pointwise in θ. If $t \mapsto P_c(t)$ was strongly differentiable, we could try to use the chain rule to simplify the right-hand side. This would result in the equation

$$[\mathcal{L}(t)\Phi_t]Y_c(s) = P_c'(t)U(t,s)\chi_0 + P_c(t)\mathcal{L}(t)U(t,s)\chi_0.$$

Since this equation holds for all $t \geq s$, we can set $s = t$ to get the equation

$$[\mathcal{L}(t)\Phi_t]Y_c(t) = P_c'(t)\chi_0 + P_c(t)\mathcal{L}(t)\chi_0.$$

If $\mapsto P_c(t)$ really is strongly differentiable, then we have $\frac{d}{dt}[P_c(t)\chi_0] = P_c'(t)\chi_0$. Also, we notice that $\mathcal{L}(t)\chi_0 = \chi_0 \cdot L(t)\chi_0$. Then,

$$[\mathcal{L}(t)\Phi_t]Y_c(t) = \frac{d}{dt}[\Phi_t Y_c(t)] + \Phi_t Y_c(t) \cdot L(t)\chi_0.$$

Now, suppose $t \mapsto Y_c(t)$ is differentiable. After some cancelling, this would imply

$$0 = \Phi_t Y_c'(t) + \Phi_t Y_c(t) \cdot L(t)\chi_0.$$

Recalling that the columns of Φ_t form a basis for $\mathcal{RCR}_c(t)$, this would imply that Y_c solves the non-square matrix ordinary differential equation

$$Y_c' = -Y_c \cdot L(t)\chi_0.$$

Unfortunately, the above differential equation is nonsense. If $L(t)$ has support in $[-r, 0)$ – that is, $L(t)\chi_0 = 0$ – the right-hand side is identically zero, which would imply that Y_c is constant. But since $t \mapsto e^{t\Lambda}Y_c(t)$ is periodic, this implies that the eigenvalues of Λ must be simple. This is not necessarily true, so we must conclude that $t \mapsto P_c(t)$ is not strongly differentiable. As a consequence, it is not possible in general to prove that $Y_c \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p \times n})$ using strong differentiability properties of the projector. Since $Y_c(t)$ is defined precisely by the equation $\Phi_t Y_c(t) = P_c(t)\chi_0$ and the right-hand side is defined in (4.5) by a time-varying contour integral with a singular integrand, it would seem quite difficult to prove the result generally.

The projector $t \mapsto P_c(t)$ is not even pointwise continuous. Consider the periodic case. $\mathcal{RCR}_c(t)$ is the invariant subspace of the monodromy operator V_t with the property that $||V_t\phi|| = ||\phi||$ for all $\phi \in \mathcal{RCR}_c(t)$. However, since $V_t = U(t+T, t)$, any element of $\mathcal{RCR}_c(t)$ will generally have discontinuities on the set $D_t = \{\theta \in [-r, 0] : t + \theta \in \{t_k : k \in \mathbb{Z}\}\}$. Generally, D_t is nonempty and nonconstant; the discontinuities move by translation to the left as t increases. Consequently, the discontinuities of $P_c(t)\phi$ for fixed $\phi \in \mathcal{RCR}$ are nonconstant in t, so $t \mapsto ||P_c(t)\phi||$ is generally discontinuous (from the right and left) at any $t \in \mathbb{R}$ such that D_t is nonempty. This creates further problems in proving the regularity of the matrix $Y_c(t)$.

Chapter 8

Center manifold theory III: Euclidean space representation, approximation and visualization

The content of most of this chapter appears in *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [20] by Church and Liu, although the results appearing therein are slightly less general, being only applicable to systems with discrete delays. Theorem 8.1.1 was stated without proof; we provide one here.

8.1 Euclidean space representation of the centre manifold

In this chapter we will once again be studying the semilinear equation

$$\dot{x} = L(t)x_t + f(t, x_t), \qquad\qquad t \neq t_k \qquad\qquad (8.1)$$
$$\Delta x = B(k)x_{t^-} + g(k, x_{t^-}), \qquad\qquad t = t_k. \qquad\qquad (8.2)$$

We introduce some important conditions.

C.1 The linearization is periodic with period T and c impulses per period. That is, $L(t + T) = L(t)$ for all $t \in \mathbb{R}$, $B(k + c) = B(k)$ and $t_{k+c} = t_k + T$ for all $k \in \mathbb{Z}$.

C.2 Conditions H.1–H.3, H.5–H.6 and H.9 are satisfied.

C.3 The sequences of functionals $B(k)$ and $g(k, \cdot)$ satisfy the overlap condition.

Conditions C.1 and C.2 are strong enough to guarantee that the (local) centre manifold exists, is smooth, and in terms of the basis array Φ it is pointwise $PC^{1,m}$-regular at zero. Condition C.3 will be needed to ensure the dynamics on the centre manifold are well defined. We will assume them throughout this chapter.

The centre manifold

$$C : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$$

in terms of Φ has range in $\mathcal{RCR}_c \oplus \mathcal{RCR}_s$. If $y : \mathbb{R} \to \mathcal{RCR}$ is a solution in the centre manifold – that is, $y(t) = S(t, s)y(s)$ for all $t \geq s$ with $y(t) \in \mathcal{W}_c(t)$ – then we can use property 2 of Theorem 6.5.1 and part 2 of Lemma 6.2.3 to write it as

$$y(t) = \Phi_t z(t) + H(t, z(t)) \tag{8.3}$$

where $\Phi_t z(t) = P_c(t)y(t)$ for some $z : \mathbb{R} \to \mathbb{R}^p$. By definition of the basis array Φ and the observation that any mild solution such as y defined for all time must in fact be the history function of some element of $\mathcal{RCR}^1(\mathbb{R}, \mathbb{R}^n)$ – see Lemma 5.1.1 – we must have $z \in \mathcal{RCR}^1(\mathbb{R}, \mathbb{R}^p)$. The present chapter is essentially an investigation into how the above representation of solutions in the centre manifold can help us obtain a concrete version of the dynamics on the centre manifold.

We need to introduce some extra notation. The set $M^{n \times m}(\mathbb{R}^k)$ denotes the set of $n \times m$ matrices with entries in the vector space $\mathbb{R}^k$. If $A \in M^{n \times m}(\mathbb{R}^k)$, we $A_{i,j}$ denotes the entry in its ith row and jth column. The notation $[A]_{a:b}$ denotes the $(b - a + 1) \times m$ matrix whose rows coincide with rows a through b of A.

For a j-dimensional multi-index $\xi = (\xi_1, \ldots, \xi_j)$ where $\xi_i \in \mathbb{N}$, we define $|\xi| = \sum_i \xi_i$. For $u \in \mathbb{R}^j$ and a j-dimensional multi-index ξ with $|\xi| = m$, the ξ power of u is $u^\xi = u_1^{\xi_1} \cdots u_j^{\xi_j}$. If X is a vector space and $U \in X^j$, we similarly define $U^\xi \in X^{|\xi|}$ by

$$U^\xi = (U_1, \ldots, U_1, U_2, \ldots, U_2, \ldots, U_j \ldots, U_j)$$

where the factor U_i appears ξ_i times. If $u \in X$ and $m \in \mathbb{N}$, we define $u^m \in X^m$ by $u^m = (u, \ldots, u)$.

For a vector multi-index $\xi = (\xi_1, \ldots, \xi_j)$ where each $\xi_i \in \{e'_1, \ldots, e'_k\}$ for $\{e'_i : i = 1, \ldots, k\}$ the standard ordered basis of $\mathbb{R}^{k*}$, we write $|\xi| = j$ and define $(u_1 \cdots u_j)^\xi$ for $u_i \in \mathbb{R}^k$ as follows:

$$(u_1 \cdots u_j)^\xi = (\xi_1 u_1) \cdots (\xi_j u_j).$$

For vectors in $\mathbb{R}^n$ written in component form, $(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = \sum_i u_i v_i$ denotes the standard inner product.

If $A \in \mathbb{R}^{m \times n}$ and $B \in M^{n \times k}(\mathbb{R}^\ell)$, we define the overloaded product $A * B \in M^{m \times k}(\mathbb{R}^\ell)$ by the equation

$$[A * B]_{i,j} = \sum_{u=1}^{n} A_{i,u} B_{u,j}. \tag{8.4}$$

It is readily verified that if $A \in \mathbb{R}^{m \times m}$ is invertible, then $A * B = C$ if and only if $B = A^{-1} * C$. Moreover, $*$ satisfies the Leibniz law

$$\frac{d}{dt} A(t) * B(t) = \left(\frac{d}{dt} A(t) \right) * B(t) + A(t) * \left(\frac{d}{dt} B(t) \right)$$

whenever $t \mapsto A(t)$ and $t \mapsto B(t)$ are differentiable. Clearly, when $\ell = 1$ the overloaded product reduces to the standard matrix product.

8.1.1 Definition and an approximation theorem

The first task is to replace the hyperbolic part of the centre manifold with something even more concrete. First, we recall from the Floquet Theorem 4.4.1 that we can write the basis array equivalently as

$$\Phi_t = \alpha(t) e^{tW} \Phi_0,$$

for $\alpha(t) : \mathcal{RCR}_c(0) \to \mathcal{RCR}_c(t)$ T-periodic and pointwise differentiable from the right with limits on the left, and $W \in \mathcal{L}(\mathcal{RCR}_c(0))$. Let $\Lambda \in \mathbb{C}^{p \times p}$ denote the matrix associated to W with respect to the ordered basis consisting of the columns of the array Φ_0, so that

$$e^{tW} \Phi_0 = R e^{t\Lambda}$$

for some $R \in \mathcal{L}(\mathbb{R}^p, \mathcal{RCR}_c(0))$. Then, the basis array satisfies the Floquet decomposition

$$\Phi_t = Q_t e^{t\Lambda}, \tag{8.5}$$

where $Q_t = \alpha(t)R$. By construction, we can identify $Q_t(\theta) = Q(t + \theta)$ for some $Q \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n \times p})$ that is T-periodic.

Remark 8.1.1. *We can guarantee that Λ is real by instead taking $\alpha(t) : \mathcal{RCR}_c(0) \to \mathcal{RCR}_c(t)$ to be $2T$-periodic. The reason this can be done is because we know that the space*

$\mathcal{RCR}_c(kT)$ are isomorphic for $k \in \mathbb{Z}$ via the monodromy operator V_0, by Theorem 4.2.2. This then implies that there is an invertible matrix M such that $\Phi_T = \Phi_0 M$. But then,

$$\Phi_{2T} = V_0 \Phi_0 M = \Phi_0 M^2.$$

Using (8.5), it follows that

$$\Lambda = \frac{1}{2T} \log(M^2),$$

which is guaranteed to have a real logarithm. Moreover, if $\log(M)$ is real, then $\frac{1}{T} \log(M)$ still coincides with Λ as defined above.

Let us introduce a change of variables. Starting from equation (8.3), we let $u(t) = e^{t\Lambda} z(t)$ so that $\Phi_t z(t) = Q_t u(t)$. We then define $h : \mathbb{R} \times \mathbb{R}^n \times [-r, 0] \to \mathbb{R}^n$ by the equation

$$h(t, u, \theta) = H(t, Q_t u)(\theta). \tag{8.6}$$

With this transformation completed, equation (8.3) becomes

$$y(t) = Q_t u(t) + h(t, u(t), \cdot). \tag{8.7}$$

The function $h : \mathbb{R} \times \mathbb{R}^p \times [-r, 0] \to \mathbb{R}^n$ will be referred to as the *Euclidean space representation of the centre manifold*. Introduce the left limit in the first variable

$$h(t^-, u, \theta) = \lim_{\epsilon \to 0^-} h(t + \epsilon, u, \theta).$$

We have the following approximation theorem.

Theorem 8.1.1. *The Euclidean representation $h : \mathbb{R} \times \mathbb{R}^p \times [-r, 0] \to \mathbb{R}^n$ of any centre manifold enjoys the following properties.*

1. *h admits a Taylor expansion near $u = 0$:*

$$h(t, u, \theta) = \frac{1}{2!} h_2(t, \theta) u^2 + \frac{1}{3!} h_3(t, \theta) u^3 + \cdots + \frac{1}{m!} h_m(t, \theta) u^m + O(u^{m+1}),$$

   with $h_i(t, \theta) = D_2^i h(t, 0, \theta)$, and this Taylor expansion unique and does not depend on the choice of cutoff function.

2. *$t \mapsto h_i(t, \cdot)$ is periodic for $i = 2, \ldots, m$, and each of $t \mapsto h_i(t, \theta)$ and $\theta \mapsto h_i(t, \theta)$ is differentiable from the right with limits on the left.*

3. $P_c(t)h_i(t, \cdot) = 0$ for $i = 2, \ldots, m$.

4. If $u \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and $\theta \in [-r, 0]$, then we have $\lim_{s \to 0^-} h(t + s, u(t + s), \theta) = h(t^-, u(t^-), \theta)$. Also, $t \mapsto h(t, u(t), \theta(t))$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ whenever $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$.

Proof. The Taylor expansion is a consequence of Theorem 6.4.1 and the definition of the Euclidean space representation of the centre manifold. Next, $t \mapsto \mathcal{C}(t, \phi)$ is periodic, from which it follows that the same is true of the differentials $D_2^j \mathcal{C}(t, 0)$. Since $h(t, u, \cdot) = (I - P_c(t))\mathcal{C}(t, Q_t u)$, and each of $t \mapsto Q_t$ and $t \mapsto P_c(t)$ are also periodic, the same is true for $t \mapsto h(t, \cdot, \cdot)$ and its differentials $t \mapsto D_2^j h(t, \cdot, \cdot)$. For the projection, linearity of the differential implies that $D_2^j H(t, 0) = (I - P_c(t))D_2^j C(t, 0)$, so that $P_c(t)D_2^j H(t, 0) = 0$. The same is true for h by its definition. As for the limit relation, the Fundamental Theorem of Calculus and the triangle inequality can be used to obtain

$$
\begin{aligned}
||h(t + s, &u(t + s), \theta(t + s)) - h(t^-, u(t^-), \theta(t^-))|| \\
&\leq L||u(t + s) - u(t^-)|| + ||h(t + s, u(t^-), \theta(t + s)) - h(t^-, u(t^-), \theta(t + s))|| \\
&\quad + ||h(t^-, u(t^-), \theta(t + s)) - h(t^-, u(t^-), \theta(t^-))||,
\end{aligned}
$$

where L is a Lipschitz constant for $x \mapsto D_2 h(t, x, \cdot)$ valid uniformly for all t. Note that this Lipschitz constant is guaranteed to exist by Theorem 6.4.1, the uniform boundedness of the projectors and the periodicity of $t \mapsto Q_t$. Since $u(t + s) \to u(t^-)$, the first of the two terms converges to zero as $s \to 0^-$. As for the second, since $t \mapsto H(t, z)(\theta(t))$ has limits on the left, the same is true of $t \mapsto h(t, u(t^-), \theta(t))$ for $u(t^-)$ fixed. For the third term, $\theta \mapsto h(t^-, u(t^-), \theta) \in \mathcal{RCR}$ gives the limit. This proves all assertions concerning limits from the left. Limits from the right are proven analogously. $\qquad \square$

8.1.2 Dynamics on the centre manifold in Euclidean space

Recall that the centre fibre bundle component $w(t) = P_c(t)y(t)$ satisfies the integral equation (6.11) whenever $y : \mathbb{R} \to \mathcal{RCR}$ is a mild solution with $y(t) \in \mathcal{W}_c(t)$. From (8.7), it follows that $w(t) = \Phi_t z(t)$. But also, from Lemma 5.1.1, we can formally identify $y : \mathbb{R} \to \mathcal{RCR}$ with a right-continuous regulated function $\tilde{y} : \mathbb{R} \to \mathbb{R}^n$ by way of $y(t) = \tilde{y}_t$. Substituting into the integral equation, using assumption H.9 to write $P_c(s)\chi_0 = \Phi_s Y_c(s)$ and recalling that y is a *small* solution so we can ignore the impact of the cutoff function on the nonlinearity, this gives after some simplification

$$
\Phi_t z(t) = \Phi_t z(s) + \Phi_t \int_s^t Y_c(\mu)f(\mu, y_\mu)d\mu + \Phi_t \sum_{s < t_i \leq t} Y_c(t_i)g(i, y_{t_i^-}),
$$

where we recall by Theorem 6.5.1 that $\mathcal{C}(\mu, w(\mu)) = \mathcal{C}(\mu, P_c(\mu)y(\mu)) = y(\mu)$. Since the columns of Φ_t form a basis for $\mathcal{RCR}_c(t)$, we can apply the coordinate map defined by $\phi_i(t) \mapsto e_i$ to eliminate the basis array Φ_t from each side. The result is the following integral equation in $\mathbb{R}^p$:

$$z(t) = z(s) + \int_s^t Y_c(\mu)f(\mu, y_\mu)d\mu + \sum_{s < t_i \leq t} Y_c(t_i)g(i, y_{t_i^-}). \qquad (8.8)$$

It is here that our derivation becomes a bit subtle. To motivate the next step, we prove a result concerning the overlap condition, mild solutions and regulated left-limit histories, which were introduced way back in Section 2.3.

Lemma 8.1.1. *Suppose $F(k, \cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition. If $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ defines a mild solution $t \mapsto x_t$, then $F(k, x_{t_k^-}) = F(k, x_{t_k}^-)$ for all $k \in \mathbb{Z}$.*

Proof. By Lemma 5.1.1, such a solution $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ is classical, so its discontinuities are a subset of $\{t_k : k \in \mathbb{Z}\}$. As a consequence, we can write

$$x_{t_k^-} - \sum_{t_k - t_j \in [-r, 0]} \chi_{t_k - t_j} \Delta x(t_k - t_j) = x_{t_k}^-.$$

Applying $F(k, \cdot)$ to both sides and applying the overlap condition, we obtain the claimed result. $\qquad \square$

To see why the above lemma is so helpful, let us take right-derivatives of both sides of the integral equation (8.8) and check finite differences across the jump times t_k. The result is the impulsive differential equation

$$\dot{z} = Y_c(t)f(t, y_t), \qquad\qquad\qquad t \neq t_k \qquad (8.9)$$
$$\Delta z = Y_c(t_k)g(k, y_{t_k^-}), \qquad\qquad t = t_k, \qquad (8.10)$$

where the derivative operator is understood as the right-hand derivative $\dot{x} = \frac{d^+}{dt}x(t)$. In (8.9), we can replace y_t with $\mathcal{C}(t, w(t)) = \Phi_t z(t) + h(t, z(t), \cdot)$ and get a well-behaved ordinary differential equation. In the second equation we have to take a pointwise left-limit. This is a problem, however, because the pointwise left-limit is a limit in the θ variable at a single point. In particular,

$$y_{t^-}(\theta) = \begin{cases} \Phi_t(\theta)z(t) + H(t, z(t), \theta), & \theta < 0 \\ \Phi_{t^-}(0)z(t^-) + \lim_{s \to 0^-} H(t + s, z(t + s), 0), & \theta = 0. \end{cases}$$

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Substituting this into the jump condition (8.10) would result in an implicit equation for $z(t_k)$, which makes the impulsive differential equation (8.9)–(8.10) somewhat difficult to work with. Lemma 8.1.1 solves this problem; if $g(k, \cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition, we can replace $y_{t_k^-}$ inside $g(k, \cdot)$ with the regulated left-limit $y_{t_k}^-$. In view of Theorem 8.1.1, y_t^- can be written

$$y_t^- = Q_t^- u(t^-) + h(t^-, u(t^-), \cdot). \tag{8.11}$$

Substituting (8.11) into (8.10) and completing the change of variables $u(t) = e^{t\Lambda} z(t)$ described in Section 8.1.1, the following theorem is proven.

Theorem 8.1.2. *Suppose $g(k, \cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition and conditions C.1 and C.2 are satisfied. Then, the abstract dynamics on the centre manifold described by the integral equation (6.11) in the variable $w(t) \in \mathcal{RCR}$ are equivalent under the time-periodic change of variables $w(t) = Q_t u(t)$ for $u \in \mathbb{R}^p$ to the ordinary impulsive differential equation*

$$\frac{d^+}{dt} u(t) = \Lambda u(t) + e^{t\Lambda} Y_c(t) f(t, Q_t u(t) + h(t, u(t), \cdot)), \qquad t \neq t_k \tag{8.12}$$

$$\Delta z = e^{t\Lambda} Y_c(t) g(k, Q_t^- u(t^-) + h(t^-, u(t^-), \cdot)), \qquad t = t_k. \tag{8.13}$$

Remark 8.1.2. *$t \mapsto e^{t\Lambda} Y_c(t)$ is periodic. To see this, remark that from the definition of Y_c and the periodicity of the projectors P_c,*

$$\Phi_t Y_c(t) = P_c(t)\chi_0 = P_c(t+T)\chi_0 = \Phi_{t+T} Y_c(t+T) = Q_{t+T} e^{(t+T)\Lambda} Y_c(t+T) = \Phi_t e^{T\Lambda} Y_c(t+T),$$

from which Φ_t being a basis for $\mathcal{RCR}_c(t)$ implies the equality $Y_c(t) = e^{T\Lambda} Y_c(t+T)$. As a consequence, $t \mapsto e^{t\Lambda} Y_c(t) := \tilde{Y}_c(t)$ satisfies

$$\tilde{Y}_c(t+T) = e^{(t+T)\Lambda} Y_c(t+T) = e^{(t+T)\Lambda} e^{-T\Lambda} Y_c(t) = e^{t\Lambda} Y_c(t) = \tilde{Y}_c(t),$$

so it is periodic as claimed. If the nonlinear terms are also periodic (with the same period T and $g(k+c, \cdot) = g(k, \cdot)$ for all $k \in \mathbb{Z}$) then the same is true of the impulsive differential equation (8.12)–(8.12).

Because of the reduction principle – Theorem 6.5.1 together with Theorem 6.6.1 – the ordinary impulsive differential equation (8.12)–(8.13) completely characterizes the dynamics of all small solutions. From the perspective of bifurcations, this is quite useful because we can study a concrete impulsive differential equation in $\mathbb{R}^p$ to detect the birth or destruction of periodic solutions or other invariant structures. When $\mathcal{RCR}_u(t)$ is empty, stability transitions can be analyzed. If one needs only terms of order two (eg. saddle-node bifurcation, then the center manifold does not need to be calculated. In this case, the dynamics on the center manifold are characterized by the following corollary.

Corollary 8.1.2.1. *Under the hypotheses of Theorem 8.1.2, the dynamics on the centre manifold to quadratic order are equivalent by a time-periodic change of variables to those of the impulsive differential equation*

$$\dot{u} = \Lambda u + e^{t\Lambda} Y_c(t) \left[\frac{1}{2} D^2 f(t,0)[Q_t u]^2 \right] + O(u^3), \qquad t \neq t_k \qquad (8.14)$$

$$\Delta u = e^{t\Lambda} Y_c(t) \left[\frac{1}{2} D^2 g(k,0)[Q_t^- u(t^-)]^2 \right] + O(u^3), \qquad t = t_k. \qquad (8.15)$$

For Hopf bifurcation conditions, for example, we require the reduced dynamics equations to be explicit to cubic order. Recall from Theorem 8.1.1 that we can write

$$h(t,u,\theta) = \frac{1}{2!} h_2(t,\theta) u^2 + \frac{1}{3!} h_3(t,\theta) u^3 + \cdots$$

for symmetric multilinear mappings $h_i(t,\theta) : (\mathbb{R}^p)^i \to \mathbb{R}^n$ defined by $h_i(t,\theta) = D_2^i h(t,0,\theta)$. It is then easily verified that to cubic order, the reduced dynamics are

$$\dot{u} = \Lambda u + e^{\Lambda t} Y_c(t) \left[\frac{1}{2!} D^2 f(t,0)[Q_t u]^2 + \frac{1}{3!} \left(D^3 f(t,0)[Q_t u]^3 + 3 D^2 f(t,0)[Q_t u, h_2(t,\theta)u^2] \right) \right], \quad t \neq t_k$$
$$(8.16)$$

$$\Delta u = e^{\Lambda t} Y_c(t) \left[\frac{1}{2!} D^2 g(k,0)[Q_t^- u]^2 + \frac{1}{3!} \left(D^3 g(k,0)[Q_t^- u]^3 + 3 D^2 g(k,0)[Q_t^- u, h_2(t^-,\theta)u^2] \right) \right], \quad t = t_k.$$
$$(8.17)$$

8.1.3 An impulsive evolution equation and boundary conditions

In the same way that the centre manifold associated to a nonhyperbolic equilibrium of an ordinary differential equation satisfies a nonlinear partial differential equation, the centre manifold of an impulsive RFDE satisfies a nonlinear impulsive evolution equation. This is what we prove in this section.

At this stage, we should define a pair of linear operators that are in a certain sense "generators" of the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$. They are

$$\mathcal{L}(t)\phi = \begin{cases} B(t)\phi, & \theta = 0 \\ d^+\phi(\theta), & \theta < 0 \end{cases}, \qquad \mathcal{J}(k)\phi(\theta) = \begin{cases} B(k)\phi, & \theta = 0 \\ \phi(\theta^+) - \phi(\theta), & \theta < 0. \end{cases} \qquad (8.18)$$

Also, we define $\Delta_\theta^+ : \mathcal{RCR} \to \mathcal{G}([-r,0),\mathbb{R}^n)$ by $\Delta_\theta^+ \phi(\theta) = \phi(\theta^+) - \phi(\theta)$. This operator permits a decomposition of $\mathcal{J}(k)$ into

$$\mathcal{J}(k) = \chi_0 B(k) + \chi_{[-r,0)} \Delta_\theta^+.$$

Next, we introduce yet another jump operator. $\Delta_t : \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n) \to \mathcal{G}([-r, 0], \mathbb{R}^n)$ is defined by

$$\Delta_t \phi(\theta) = \phi_t(\theta) - \lim_{s \to t^-} \phi_s(\theta).$$

We will also need the notion of the regulated left limit of an $\mathcal{RCR}$-valued function.

Definition 8.1.1. *For a function $f : \mathbb{R} \to \mathcal{RCR}$, we define the* regulated left limit $f^- : \mathbb{R} \to F([-r, 0], \mathbb{R}^n)$ *by the formal expression*

$$f^-(t)(\theta) = \lim_{s \to 0^-} f(t + s)(\theta).$$

Note that if $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, then x_t^- is an element of $\mathcal{G}([-r, 0], \mathbb{R}^n)$ and, in particular, it is continuous from the left. Also, $\Delta_t x_t = x_t - x_t^-$. The following proposition is clear, given Lemma 8.1.1.

Proposition 8.1.1. *Let $x : \mathbb{R} \to \mathbb{R}^n$ be continuous except at times t_k, where it is right-continuous and has limits on the left. Then, $\Delta_\theta^+ x_t^-(\theta) = \Delta_t x_t(\theta)$ for $\theta < 0$ and all $t \in \mathbb{R}$. If the functionals $B(k)$ and $g(k, \cdot)$ satisfy the overlap condition and x is a solution of the impulsive RFDE* (5.1)–(5.2), *then*

$$B(k)x_{t_k^-} = B(k)x_{t_k}^-, \quad g(k, x_{t_k^-}) = g(k, x_{t_k}^-). \tag{8.19}$$

Finally, if $x : \mathbb{R} \to \mathbb{R}^n$ is differentiable from the right, we define $\frac{d^+}{dt} x_t$ by the equation

$$\left[\frac{d^+}{dt} x_t \right](\theta) = \frac{d^+}{dt} x_t(\theta).$$

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a complete solution such that $t \mapsto x_t \in \mathcal{W}_t$. If assumption C.3 is satisfied, this solution satisfies the abstract evolution equation

$$\frac{d^+}{dt} x_t = \mathcal{L}(t)x_t + \chi_0 f(t, x_t), \qquad\qquad t \neq t_k \tag{8.20}$$

$$\Delta_t x_t = \mathcal{J}(k)x_t^- + \chi_0 g(k, x_t^-), \qquad\qquad t = t_k, \tag{8.21}$$

with $\mathcal{L}$ and $\mathcal{J}$ as defined in (8.18). In addition, the following boundary condition must be satisfied across the jump interfaces $t + \theta = t_k$ for all $k \in \mathbb{Z}$ and $\theta < 0$ because of Proposition 8.1.1:

$$\Delta_t x_t(\theta) = \Delta_\theta^+ x_t^-(\theta), \qquad\qquad t + \theta = t_k, \ \theta < 0. \tag{8.22}$$

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Along the lines $t + \theta = s$ for $s \notin \{t_k : k \in\}$, the condition $\Delta_t x_t(\theta) = \Delta_\theta^+ x_t^-(\theta)$ is uninformative because x is continuous at $s = t + \theta$. Note that all left-limits are now regulated left-limits because we have used equation (8.19) of Proposition 8.1.1. It is at this stage that we make the substitution (8.7) to write x_t in terms of the Euclidean space representation of the centre manifold. The following theorem characterizes the Euclidean space representation of the centre manifold in terms of an impulsive evolution equation.

Theorem 8.1.3. *For any solution u of the finite-dimensional ordinary impulsive differential equation* (8.12)–(8.13), *the Euclidean space representation of the centre manifold is a solution of the following boundary-value problem:*

$$
\begin{aligned}
Q_t(\theta)[\dot{u} - \Lambda u] + \partial_t h(t, u, \theta) + \partial_u h(t, u, \theta)\dot{u} &= \partial_\theta h(t, u, \theta), & \theta < 0, t \neq t_k \\
Q_t(\theta)\Delta u + \Delta_t h(t, u + \Delta u, \theta) + \Omega(t, h, u, \theta)\Delta u &= \Delta_\theta^+ h(t^-, u, \theta), & \theta < 0, (t = t_k \vee t + \theta = t_k)
\end{aligned}
\tag{8.23}
$$

$$
\begin{aligned}
Q_t(0)[\dot{u} - \Lambda u] + \partial_t h(t, u, 0) + \partial_u h(t, u, 0)\dot{u} &= L(t)h(t, u, \cdot) + f(t, Q_t u + h(t, u, \cdot)), & \theta = 0, t \neq t_k \\
Q_t(0)\Delta u + \Delta_t h(t, u + \Delta u, 0) + \Omega(t, h, u, 0)\Delta u &= J(k)h(t^-, u, \cdot) + g(k, Q_t u + h(t^-, u, \cdot)), & \theta = 0, t = t_k,
\end{aligned}
$$
$$\tag{8.24}$$

where we denote $u = u(t)$ when $t \neq t_k$ and $u = u(t^-)$ when $t = t_k$, $\Delta u = u(t_k) - u(t_k^-)$, we define Ω by

$$
\Omega(t, h, u, \theta) = \int_0^1 \partial_u h(t^-, u + s\Delta u, \theta)ds,
$$

and all derivatives in t and θ are the right-derivatives $\frac{\partial^+}{\partial t}$ and $\frac{\partial^+}{\partial \theta}$.

Proof. First, we remark that Q_t satisfies the following abstract impulsive differential equation:

$$
\begin{aligned}
\frac{d^+}{dt}Q_t(\theta) + Q_t\Lambda &= \chi_0 L(t)[Q_t \exp_\Lambda] + \chi_{[-r,0)}\frac{d^+}{d\theta}Q_t(\theta), & t \neq t_k \\
\Delta_t Q_t(\theta) &= \chi_0 B(k)[Q_t^- \exp_\Lambda] + \chi_{[-r,0)}\Delta_\theta^+ Q_t^-(\theta), & t = t_k,
\end{aligned}
\tag{8.25}
$$

where $\exp_\Lambda(\theta) = e^{\Lambda\theta}$. It can be derived from the equality $\Phi_t = Q_t e^{\Lambda t}$ and Proposition 8.1.1. Substituting the ansatz $x_t = Q_t u(t) + h(t, u(t), \cdot)$ into equation (8.20), we obtain when $\theta < 0$ the equality

$$
\frac{d^+}{dt}[Q_t]u + Q_t\dot{u} + \partial_t h + \partial_u h\dot{u} = \frac{d^+}{d\theta}[Q_t]u + \partial_\theta h,
$$

which is equivalent to the first equation from (8.23) if one takes into account (8.25). When $\theta = 0$, the same approach results in the first equation from (8.24).

Next, we substitute the ansatz into (8.21). If one denotes $u = u(t^-)$, when $\theta < 0$ and $t = t_k$, the result reduces to[1]

$$-Q_t \Delta u + \Delta_\theta^+ h(t^-, u, \theta) = h(t, u + \Delta u, \theta) - h(t^-, u, \theta)$$

after cancelling several duplicate terms. The above is equivalent to

$$Q_t \Delta u + \Delta_t h(t, u + \Delta u, \theta) + h(t^-, u + \Delta u, \theta) - h(t^-, u, \theta) = \Delta_\theta^+ h(t^-, u, \theta).$$

The fundamental theorem of calculus implies $\Omega(t, h, u, \theta) = h(t^-, u + \Delta u, \theta) - h(t^-, u, \theta)$, and from this we obtain the second equation of (8.23). The equation for $t + \theta = t_k$ is obtained by checking the boundary condition (8.22), while the equation for $\theta = 0$ is obtained by the same methods. $\qquad\square$

Remark 8.1.3. *Note that $h(t, \theta, u)$ must possesses discontinuities along the lines $t + \theta = t_k$ for u fixed. These discontinuities are captured by the second equation of (8.23) when $\theta < 0$, and in the second equation of (8.24) when $\theta = 0$. When $t = t_k - \theta \notin \{t_j : j < k\}$, we have $\Delta u = \Delta u(t) = 0$ in the second equation of (8.23), and the result is the constraint $h(t, u, \theta) = h(t^-, u, \theta^+)$. In particular, even though we know that $\theta \mapsto h(t, u, \theta)$ is continuous from the right, the same is* not *true of $\theta \mapsto h(t^-, u, \theta)$; the latter is continuous from the left.*

The boundary-value problem (8.23)–(8.24) is implicit in terms of the variable $\dot{u}$ and Δu. To obtain an explicit boundary-value problem for $(t, u, \theta) \mapsto h(t, u, \theta)$, one would replace every instance of $\dot{u}$ and Δu with the equations (8.12) and (8.13). The resulting equations take up a lot of space, so we do not write them out explicitly.

8.2 Approximation by Taylor expansion

Equations (8.23)–(8.24) and (8.12)–(8.13) of Theorem 8.1.3 and Theorem 8.1.2 yield a system of impulsive partial delay differential equations and boundary conditions for the Euclidean space representation of the centre manifold.

In the u coordinates, the dynamics on the centre manifold are given by (8.12)–(8.13). If one seeks to obtain the $O(||u||^k)$ dynamics on the center manifold, it is necessary to compute the terms of order $O(||u||^{k-1})$ of the center manifold h. Quadratic terms are

---

[1]Note that $Q_t^-(\theta^+) = Q_t(\theta)$ for $\theta < 0$.

needed to analyze Hopf-like bifurcations, for instance. The quadratic coefficient $h_2(t,\theta)$ of the centre manifold can be represented in the form

$$h_2(t,\theta)[u,v] = \begin{bmatrix} c_{11}^1(t,\theta)u_1v_1 + \cdots c_{1p}^1(t,\theta)u_1v_p + c_{21}^1(t,\theta)u_2v_1 + c_{22}^1(t,\theta)u_2v_2 + \cdots + c_{pp}^1(t,\theta)u_pv_p \\ \vdots \\ c_{11}^n(t,\theta)u_1v_1 + \cdots c_{1p}^n(t,\theta)u_1v_p + c_{21}^n(t,\theta)u_1v_1 + c_{22}^n(t,\theta)u_2v_2 + \cdots + c_{pp}^n(t,\theta)u_pv_p \end{bmatrix},$$

(8.26)

and similarly for the higher-order terms, where symmetrically, $c_{ij} = c_{ji}$. In terms of vector multi-indices, we can write

$$h_m(t,\theta)[u_1,\ldots,u_m] = \sum_{|\xi|=m} c_\xi(t,\theta)(u_1\cdots u_m)^\xi$$

(8.27)

for multiindex $\xi = (\xi_1,\ldots,\xi_m)$ and $\xi_i \in \{\emptyset, e_1',\ldots,e_p'\}$.

As a consequence of the above observations, one can substitute an appropriate order $O(||u||^k)$ expansion of the impulsive differential equation (8.12)–(8.13) into the evolution equation and boundary conditions (8.23)–(8.24) to obtain a $O(||u||^k)$ impulsive evolution equation for the center manifold.

8.2.1 Evolution equation and boundary conditions for quadratic terms

For the calculation of cubic order dynamics (eg. Hopf bifurcation), one needs to calculate h_2 before the dynamics on the center manifold (8.16)–(8.17) can be studied. Substituting the aforementioned equation into the evolution equation and boundary conditions (8.23)–(8.24) and keeping only the order two terms in u, we obtain the rather large equation (8.28)–(8.31), which we must unfortunately pass to LaTeX in tiny mode in order to make it fit within the page margins.

$$\tfrac{1}{2}Q_t(\theta)e^{\Lambda t}Y_c(t)D_2^2f(t,0)[Q_tu]^2 + \tfrac{1}{2}\partial_t h_2(t,\theta)u^2 + h_2(t,\theta)[\Lambda u, u] = \tfrac{1}{2}\partial_\theta h_2(t,\theta)u^2, \qquad \theta < 0, t \neq t_k \qquad (8.28)$$

$$\tfrac{1}{2}Q_t(\theta)e^{\Lambda t}Y_c(t)D^2g(k,0)[Q_t^-u]^2 + \tfrac{1}{2}\Delta_t h_2(t,\theta)u^2 = \tfrac{1}{2}\Delta_\theta^+ h_2(t^-,\theta)u^2 \qquad \theta < 0, t \in \{t_k, t_k - \theta\} \qquad (8.29)$$

$$\tfrac{1}{2}Q_t(0)e^{\Lambda t}Y_c(t)D_2^2f(t,0)[Q_tu]^2 + \tfrac{1}{2}\partial_t h_2(t,0)u^2 + h_2(t,\theta)[\Lambda u, u] = \tfrac{1}{2}L(t)h_2(t,\cdot)u^2 + \tfrac{1}{2}D_2^2f(t,0)[Q_tu]^2, \qquad \theta = 0, t \neq t_k \qquad (8.30)$$

$$Q_t(0)e^{\Lambda t}Y_c(t)\tfrac{1}{2}D^2g(k,0)[Q_t^-u]^2 + \tfrac{1}{2}\Delta_t h_2(t,0)u^2 = J(k)h_2(t^-,\cdot)u^2 + \tfrac{1}{2}D^2g(k,0)[Q_t^-u]^2, \qquad \theta = 0, t = t_k. \qquad (8.31)$$

In this equation, all partial derivatives are right-hand derivatives. Notice that upon expansion, the coefficients of each binomial $u^\xi = u_{\xi_1}u_{\xi_2}$ generates a system of coupled linear

impulsive partial differential equations for the associated coefficients c_ξ of the quadratic expansion of the center manifold. This system can be solved by a variation of the method of characteristics; this is done in Section 8.2.2.

The pattern established here continues to mth order expansions. In particular, each multinomial $u^\xi = u_1^{\xi_1} \cdots u_m^{\xi_m}$ with $\sum_m \xi_k = m$ generates a system of coupled impulsive PDEs for the u^ξ coefficient of $h_m(t, \theta)$. The order $i < m$ expansions $h_i(t, \theta)$ are generally needed to compute the order m terms, so the procedure must be done iteratively. The calculations quickly become taxing, and the use of computer algebra software is highly recommended to keep track of all of the differentials.

8.2.2   Solution by the method of characteristics

The system of impulsive partial differential equations (8.28)–(8.31) must be solved in order to obtain the quadratic order term the center manifold. A similar equation can be derived for the p-th order terms, and this equation will typically depend on the lower-order terms. For notational simplicity, we will only present the method as it applies to computing the quadratic term h_2.

First, some preparations. Given the representation (8.27) we can write

$$h_2(t, \theta)[u, u] = \sum_{\xi \in \Xi} h_2^\xi(t, \theta)(uu)^\xi,$$

for $u \in \mathbb{R}^p$, where Ξ is a set of p-dimensional multiindices in two variables that is both *permutation-free* (ie. $(e_i, e_j) \in \Xi$ implies $(e_j, e_i) \in \Xi$ if and only if $i = j$) and *complete* (ie. for every p-dimensional multiindex in two variables ζ, either $\zeta \in \Xi$ or $\zeta = (e_i, e_j)$ and $(e_j, e_i) \in \Xi$). In this setting we have

$$h_2^\xi(t, \theta) = \left\{ \begin{array}{ll} 2c_\xi, & \xi = (e_i, e_j), i \neq j \\ c_\xi, & \xi = (e_i, e_i). \end{array} \right.$$

Writing everything in terms of scalar products, there exists a $\beta \times \beta$ matrix Λ_2 with $\beta = \binom{p+1}{2}$ such that

$$h_2(t, \theta)[u, u] = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * h_2^\Xi, \tag{8.32}$$

$$h_2(t, \theta)[\Lambda u, u] = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * \Lambda_2 * h_2^\Xi, \tag{8.33}$$

where $h_2^\Xi = (h_2^{\zeta_1}, \ldots, h_2^{\zeta_\beta}) \in (\mathbb{R}^n)^\beta$ is interpreted as a $(\beta \times 1)$ array whose ith entry is $h_2^{\zeta_i}$, and $\Xi = \{\zeta_1, \ldots, \zeta_\beta\}$. As such, the matrix multiplication needs to be interpreted in an

overloaded sense as in equation (8.4). For example, with $p = 3$, $\Xi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$ and the data

$$
\Lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad
\begin{aligned}
\zeta_1 &= (e_1, e_1), & \zeta_2 &= (e_1, e_2), & \zeta_3 &= (e_1, e_3), \\
\zeta_4 &= (e_2, e_2), & \zeta_5 &= (e_2, e_3), & \zeta_6 &= (e_3, e_3).
\end{aligned}
$$

we first calculate $h_2[\Lambda u, u]$. Written in terms of the coefficients $h_2^{\zeta_i}$, it is

$$
h_2[\Lambda u, u] = h_2 \left[\begin{bmatrix} u_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right] = h_2^{11} u_1 u_2 + \frac{1}{2} h_2^{12} u_2^2 + \frac{1}{2} h_2^{13} u_2 u_3.
$$

We can then readily extract the matrix Λ_2 satisfying the expression (8.33), and we find it is

$$
\Lambda_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Next, we write

$$
Q_t(\theta) e^{\Lambda t} Y(t) D_2^2 f(t, 0)[Q_t u]^2 = [\ (uu)^{\zeta_1} \ \cdots \ (uu)^{\zeta_\beta} \ ] * \mathcal{F}(t, \theta) \tag{8.34}
$$

$$
Q_{t_k}(\theta) e^{\Lambda t_k} Y(t_k) D^2 g_k(0)[Q_{t_k^-} u]^2 = [\ (uu)^{\zeta_1} \ \cdots \ (uu)^{\zeta_\beta} \ ] * \mathcal{G}_k(\theta) \tag{8.35}
$$

$$
D_2^2 f(t, 0)[Q_t u]^2 = [\ (uu)^{\zeta_1} \ \cdots \ (uu)^{\zeta_\beta} \ ] * a(t) \tag{8.36}
$$

$$
D^2 g_k(0)[Q_{t_k^-} u]^2 = [\ (uu)^{\zeta_1} \ \cdots \ (uu)^{\zeta_\beta} \ ] * b_k, \tag{8.37}
$$

where $\mathcal{F}(t, \theta)$, $\mathcal{G}_k(\theta)$, $a(t)$, and b_k are $\beta \times 1$ arrays with entries in $\mathbb{R}^n$. Note that as $Q_t e^{\Lambda t} = \Phi_t$, equations (8.34) and (8.35) could be simplified further. Substituting equations (8.32)–(8.37) into (8.28)–(8.31) and cancelling fractions, it follows upon comparing powers

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$(uu)^{\zeta_i}$, it follows that h_2^{Ξ} must satisfy the impulsive functional differential equation

$$\mathcal{F}(t,\theta) + \partial_t h_2^{\Xi}(t,\theta) + 2\Lambda_2 * h_2^{\Xi}(t,\theta) = \partial_\theta h_2^{\Xi}(t,\theta), \qquad\qquad \theta < 0, t \neq t_k \tag{8.38}$$

$$\mathcal{G}_k(\theta) + \Delta_t h_2^{\Xi}(t,\theta) = \Delta_\theta^- h_2^{\Xi}(t,\theta), \qquad\qquad \theta < 0, t \in \{t_k, t_k - \theta\} \tag{8.39}$$

$$\mathcal{F}(t,0) + \partial_t h_2^{\Xi}(t,\theta) + 2\Lambda_2 * h_2^{\Xi}(t,0) = L(t) \odot h_2^{\Xi}(t,\cdot) + a(t), \qquad\qquad \theta = 0, t \neq t_k \tag{8.40}$$

$$\mathcal{G}_k(0) + \Delta_t h_2^{\Xi}(t,0) = J(k) \odot h_2^{\Xi}(t^-,\cdot) + b_k, \qquad\qquad \theta = 0, t = t_k, \tag{8.41}$$

and we define the overloaded operator $L(t)\odot$ by

$$L(t) \odot h_2^{\Xi}(t,\cdot) = \begin{bmatrix} L(t)h^{\zeta_1}(t,\cdot) \\ \vdots \\ L(t)h^{\zeta_\beta}(t,\cdot) \end{bmatrix},$$

and similarly for $J(k)$. Note also that all derivatives are taken from the right: $\partial_t = \frac{\partial^+}{\partial t}$ and $\partial_\theta = \frac{\partial^+}{\partial \theta}$. The following proposition characterizes the solutions of the above inhomogeneous linear impulsive system.

Proposition 8.2.1. *Every solution $z = z(t,\theta)$ of the inhomogeneous system* (8.38)–(8.41) *can be expressed in the form*

$$z(t,\theta) = e^{2\Lambda_2\theta} * \left[n(t+\theta) - \int_\theta^0 e^{-2\Lambda_2 s} * \mathcal{F}(t-s+\theta,s)ds - \sum_{\theta < t_k - t \leq 0} e^{-2\Lambda_2 t_k} * \mathcal{G}_k(t - t_k + \theta) \right], \quad (8.42)$$

where $t \mapsto n(t)$ is a solution of the inhomogeneous linear impulsive delay differential equation

$$\begin{aligned} \mathcal{F}(t,0) + \dot{n}(t) + 2\Lambda_2 * n(t) &= L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t] + m(t), \quad t \neq t_k \\ \mathcal{G}_k(0) + \Delta n(t) &= J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t-}] + n_k, \quad t = t_k \end{aligned} \tag{8.43}$$

$$m(t) = a(t) - L(t) \odot \left[\int_{(\cdot)}^0 e^{-2\Lambda_2 s} * \mathcal{F}(t-s+\cdot,s)ds + \sum_{(\cdot) < t_k - t \leq 0} e^{-2\Lambda_2 t_k} * G_k(t - t_k + \cdot) \right]$$

$$n_k = b_k - J(k) \odot \left[\int_{(\cdot)}^0 e^{-2\Lambda_2 s} * \mathcal{F}(t_k^- - s + \cdot,s)ds + \sum_{(\cdot) < t_j - t_k \leq 0} e^{-2\Lambda_2 t_j} * G_j(t_k^- - t_j + \cdot) \right]. \tag{8.44}$$

Proof Outline. Solving the equations (8.38)–(8.39) along the characteristic lines $t + \theta =$ constant, one can show that every solution has the form of (8.42) for some function n. To show that such a function n satisfies the impulsive delay differential equation (8.43)–(8.44), one substitutes the ansatz into into equations (8.40)–(8.41), taking note that $\partial_t z(t,0) = \dot{n}(t)$ and $\Delta_t z(t,0) = \Delta n(t)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Solving the inhomogeneous system (8.43) is a nontrivial matter. Moreover, there are infinitely many solutions of the form prescribed by the above proposition, since the inhomogeneous equation (8.43) can have many bounded solutions. We must recall some additional properties of the centre manifold to identify the unique solution $z = h_2^\Xi$ corresponding to the true coefficient vector of $h_2(t,\theta)$ in the expansion (8.32). We state the result in the form of a corollary. It is essentially a consequence of Theorem 8.1.1.

Corollary 8.2.0.1. *Let the centre manifold be effectively $PC^{1,2}$ at zero and let the assumptions of Theorem 6.4.1 hold. The $(\beta \times 1)$ array h_2^Ξ with $[h_2^\Xi]_i = h_2^{\zeta_i}$ in the expansion*

$$h_2(t,\theta)[u,u] = \sum_{i=1}^{\beta} h_2^{\zeta_i}(t,\theta)[uu]^{\zeta_i}$$

is the unique solution of the inhomogeneous linear impulsive PDE (8.38)–(8.41) satisfying the following constraints.

1. *Projection constraint: $P_c(t) h_2^{\zeta_i}(t,\cdot) = 0$ for all $t \in [0,T)$ and $i = 1, \ldots, \beta$.*

2. *Periodicity constraint: $t \mapsto h_2^\Xi(t,\cdot)$ is periodic.*

8.2.3 Verification of projection constraint

It is not obvious how one should verify the projection constraint in Corollary 8.2.0.1. The projection $P_c(t)$ is generally defined by the contour integral (4.5) in terms of the monodromy operator. In practice, the projection could be approximated by first discretizing the monodromy operator as in Section 4.6.3 and performing a numerical integration. There are a few details that need to be worked out here, but we will not focus on them in this thesis.

8.3 Geometry and visualization: examples

Here we consider two examples of centre manifold computation. The first (Section 8.3.1) is a finite-dimensional toy example where the centre manifolds can all be explicitly calculated. The second one (Section 8.3.2) requires Taylor expansions. Both examples contain a parameter ϵ that controls the "size" of the impulse effect and, for all ϵ small enough, the centre manifold is one-dimensional. This allows us to visualize the centre manifolds as depending on the parameter ϵ and study how the introduction of impulses affects their geometry. Both examples are intentionally simple. In particular, the linear parts contain no delays.

8.3.1 An explicit scalar example without delays

Consider the finite-dimensional impulsive system

$$\dot{x} = x^2, \qquad t \neq k, \qquad \Delta x = 0, \qquad t = k \qquad (8.45)$$
$$\dot{y} = -y, \qquad t \neq k, \qquad \Delta y = \epsilon y, \qquad t = k. \qquad (8.46)$$

This system has several useful properties. First, for all $\epsilon \in (-1, e-1)$, the unique equilibrium at the origin is nonhyperbolic with a one-dimensional centre fiber bundle $\mathcal{RCR}_c = \mathrm{span}(e_1)$ that is constant in time. Moreover, when $\epsilon = 0$ this system is a classical example of a system with infinitely-many centre manifolds, the only analytic one being the x axis. The centre manifolds are all graphs of

$$y = c \exp\left(\frac{1}{x}\right) \chi_{(-\infty,0)}(x), \qquad c \in \mathbb{R}. \qquad (8.47)$$

Our first step will be to introduce a time-dependent change of variables. Define $y = (1+\epsilon)^{\lfloor t \rfloor - t} w$. This change of variables eliminates the impulse effect from (8.45)–(8.46) entirely. The result is the autonomous system

$$\dot{x} = x^2, \qquad (8.48)$$
$$\dot{w} = (\log(1+\epsilon) - 1)w. \qquad (8.49)$$

The centre manifolds of the above autonomous system are all orbits. Explicitly solving the differential equations and rearranging shows that they can be represented in the form

$$w = c \exp\left(\frac{1}{x}(1 - \log(1+\epsilon))\right) \chi_{(-\infty,0)}(x),$$
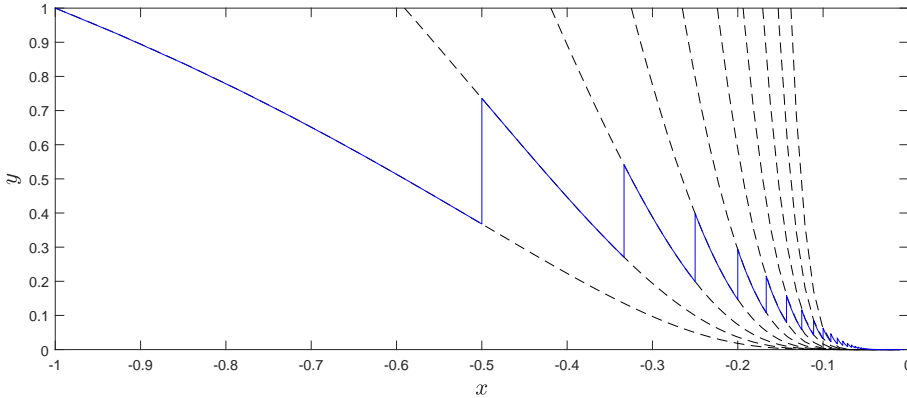
Figure 8.1: The forward orbit through $(-1, 1)$ of system (8.45)–(8.46) plotted for time $t \in [0, 200]$ and parameter $\epsilon = 1$, illustrated by the solid (blue) line. On the intervals $[k, k+1)$ for $k = 0, \ldots, 9$, the graphs of $y = c \exp\left(\frac{1}{x}\right)$ (equation (8.47)) on which the solution travels are indicated by dashed (black) lines and are plotted for $x < 0$.

where $c \in \mathbb{R}$ is a constant. Inverting the change of variables and simplifying the expression somewhat, the centre manifolds of the original system (8.45)–(8.46) can be written as the graphs of

$$y = c(1 + \epsilon)^{\lfloor t \rfloor - t} \left(\frac{e}{1 + \epsilon} \right)^{1/x} \chi_{(-\infty, 0)}(x) := h_\epsilon(t, x) \tag{8.50}$$

The solutions of (8.45)–(8.46) on the centre manifold can be understood in two ways. The most elegant description is simply that they are curves $(x(t), y(t))$ that satisfy, for a specific $c \in \mathbb{R}$, equation (8.50). A more cumbersome but perhaps more concrete interpretation is that they are curves $(x(t), y(t))$ such that for each $k \in \mathbb{Z}$, there exists $c_k \in \mathbb{R}$ such that $(x(t), y(t))$ satisfies equation (8.47) with $c = c_k$ for $t \in [k, k+1)$, and $c_{k+1} = (1 + \epsilon)c_k$. This second interpretation can be visualized with the help of Figure 8.1, while the equivalence between the two interpretations can be visualized dynamically by consulting *Supplementary Animation 1* in the UWSpace repository associated to this thesis [17], where a simulated solution is plotted as it travels along the time-varying centre manifold (8.50) for parameter $\epsilon = 1$ and $c = e/2$, while simultaneously travelling along distinct centre manifolds given by equation (8.47). The effect of varying ϵ can be seen in *Supplementary Animation 2*, a plot of the time-varying graph of $(x, \epsilon) \mapsto y = h_\epsilon(t, x)$ with $c = e/2$, plotted for $x \in [-1, 0]$ and $\epsilon \in (-1, 1]$ over five cycles, $t \in [0, 5]$.

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8.3.2 Two-dimensional example with quadratic delayed terms

We consider in this section the equation

$$
\begin{aligned}
\dot{x} &= -x + y^2, & t \neq k, & \quad \Delta x = \epsilon y(t^-), & t = 2k\pi & \quad (8.51) \\
\dot{y} &= x - x^2(t-\pi) - y^2(t-\pi), & t \neq k, & \quad \Delta y = 0, & t = 2k\pi & \quad (8.52) \\
\dot{\epsilon} &= 0, & t \neq k, & \quad \Delta \epsilon = 0, & t = 2k\pi. & \quad (8.53)
\end{aligned}
$$

Considered in isolation, the planar system (8.51)–(8.52) has, with ϵ treated as a parameter, a single zero Floquet exponent for all $\epsilon \in \mathbb{R}$. Thus, for each ϵ, the centre manifold at the origin is one-dimensional. Taking ϵ as a state variable we obtain (8.51)–(8.53), and it is for this system that we will calculate (approximate) the two-dimensional centre manifold at the origin. Taking one-dimensional slices for fixed ϵ small will produce the centre manifolds for the parameterized system (8.51)–(8.52).

The linearization of (8.51)–(8.53) at $(0,0,0)$ admits the monodromy operator V_t and associated resolvent $R(z; V_t)$

$$
V_t \xi(\theta) = \begin{bmatrix} e^{-(2\pi+\theta)} & 0 & 0 \\ 1 - e^{-(2\pi+\theta)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi(0) := V(\theta)\xi(0), \tag{8.54}
$$

$$
R(z; V_t) = z^{-1}(I_{\mathcal{RCR}} + V[I - z^{-1}V(0)]^{-1}\mathrm{ev}_0), \qquad z \neq 0, 1. \tag{8.55}
$$

One can similarly calculate a basis matrix Φ_t for the centre fiber bundle, the projection $P_c(t) : \mathcal{RCR} \to \mathcal{RCR}_c(t)$, and the matrix $Y(t)$. We find

$$
\Phi_t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P_c(t)\phi(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \phi(0), \qquad Y(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.56}
$$

It follows that we can take $Q_t = \Phi_t$ and $\Lambda = 0$ in the Floquet decomposition. The nonlinearity f of the vector field contains only the second-order term, and we have

$$
Q_t(\theta)u = \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix}, \qquad \frac{1}{2}D^2 f(0)[Q_t u]^2 = \begin{bmatrix} u_1^2 \\ -u_1^2 \\ 0 \end{bmatrix}. \tag{8.57}
$$

Similarly, the nonlinearity of the jump map g also contains only the second-order term, and we have at $t = 2k\pi$,

$$
\frac{1}{2}D^2 g(0)[Q_{2k\pi^-}u]^2 = \begin{bmatrix} u_1 u_2 \\ 0 \\ 0 \end{bmatrix}. \tag{8.58}
$$

Using (8.56), (8.57) and (8.58), we can read off the impulsive delay differential equations and boundary conditions (8.28)–(8.31) for the second-order term h_2. These are listed in 8.3.3, where the rest of the calculations are completed. In particular, the coefficients c_ξ of the quadratic-order expansion $h_2 u^2 = h_2^{11} u_1^2 + h_2^{12} u_1 u_2 + h_2^{22} u_2^2$ are computed therein. Given that $h = \frac{1}{2} h_2 u^2 + O(u^3)$, the quadratic order expansion of the centre manifold is found to be

$$h(t, u, \theta) = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_1^2 + \frac{1}{2} h_2^{12}(t, \theta) u_1 u_2 + O(u^3), \tag{8.59}$$

where the function $t \mapsto h_2^{12}(t, \theta)$ is 2π-periodic, has discontinuities along the lines $t + \theta = 2k\pi$, and is given, for $t \geq 2\pi$, by (8.67).

As ϵ is stationary in (8.51)–(8.53), we actually have $u_2 = \epsilon$. Therefore, the parameter-dependent centre manifold for the planar system (8.51)–(8.52) is obtained by replacing u_2 with ϵ and u_1 with u in (8.59), and dropping the third row, as this last row corresponds to the dynamics in ϵ. The result is

$$h_\epsilon(t, \theta, u) = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u^2 + \frac{1}{2} \epsilon \tilde{h}_2^{12}(t, \theta) u + O(u^3) := h_{\epsilon,2}(t, \theta, u) + O(u^3) \tag{8.60}$$

where $\tilde{h}_2^{12}$ denotes the first two rows of h_2^{12} (the third row is identically zero). When $\epsilon = 0$, the centre manifold is identical to the one that would be obtained by the usual adjoint-based method for autonomous delay differential equations. This can be verified by direct calculation.

Supplementary Animation 3 at the UWSpace repository [17] provides plots of the two components $(z_1, z_2) = h_{\epsilon,2}(t, \theta, u)$ of the quadratic term of the centre manifold for $u \in [-1, 1]$, $\theta \in [-2\pi, 0]$, animated over five periods: $t \in [0, 10\pi]$. The parameter $\epsilon = -\frac{1}{2}$ is chosen for illustrative purposes. In this animation it is easy to see the propagating discontinuity along the characteristics $t + \theta = 2k\pi$ predicted by Remark 8.1.3. A static portrait at the fixed moment of time $t = \pi$ is provided in Figure 8.2.

8.3.3   Detailed calculations associated to Example 8.3.2

Substituting (8.56), (8.57) and (8.58) into equations (8.28)–(8.31), we obtain

$$\partial_t h_2 u^2 - \partial_\theta h_2 u^2 = 0, \qquad\qquad \theta < 0, t \neq 2\pi k$$

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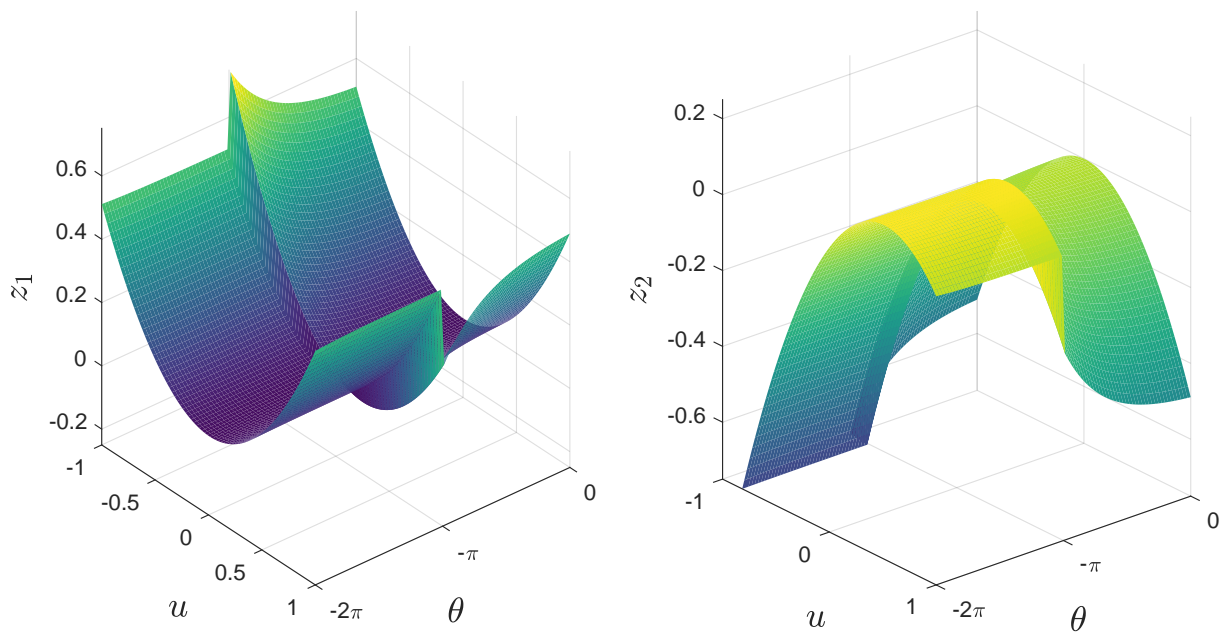
Figure 8.2: The two components $(z_1, z_2) = h_{\epsilon,2}(t, \theta, u)$ of the quadratic-order truncation of the parameter-dependent centre manifold for Example 8.3.2, plotted at the time snapshot $t = \pi$ on the grid $(u, \theta) \in [-1, 1] \times [-2\pi, 0]$, with parameter $\epsilon = -0.5$. Notice the discontinuity along the plane $\theta = -\pi$.

$$\begin{bmatrix} 0 \\ u_1 u_2 \\ 0 \end{bmatrix} + \Delta_t h_2 u^2 - \Delta_\theta^+ h_2(t^-, \theta) = 0, \qquad \theta < 0, t = 2\pi k$$

$$\partial_t h_2 u^2 - \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h_2(t,0) u^2 - \begin{bmatrix} u_1^2 \\ -u_1^2 \\ 0 \end{bmatrix} = 0, \qquad \theta = 0, t \neq 2\pi k$$

$$\begin{bmatrix} 0 \\ u_1 u_2 \\ 0 \end{bmatrix} + \Delta_t h_2 u^2 - \begin{bmatrix} u_1 u_2 \\ 0 \\ 0 \end{bmatrix} = 0, \qquad \theta = 0, t = 2\pi k$$

The second-order term $h_2(t,\theta)u^2$ is given by

$$h_2 u^2 = c_{11} u_1^2 + 2 c_{12} u_1 u_2 + c_{22} u_2^2 = h_2^{11} u_1^2 + h_2^{12} u_1 u_2 + h_2^{22} u_2^2$$

for $h_2^\zeta \in \mathbb{R}^3$, so there is a system of three 3-dimensional systems to solve. In this example, the impulsive PDEs for each of the coefficients $h_2^{\zeta_i}$ of (8.38)–(8.41) decouple because $\Lambda = 0$ implies $\Lambda_2 = 0$. With respect to the multi-index ordering $\zeta^1 = (e_1, e_1)$, $\zeta^2 = (e_1, e_2)$, $\zeta^3 = (e_2, e_2)$, we have

$$\partial_t h_2^\Xi = \partial_\theta h_2^\Xi, \qquad\qquad \theta < 0, t \neq 2\pi k \qquad (8.61)$$

$$\begin{bmatrix} 0 \\ e_2 \\ 0 \end{bmatrix} + \Delta_t h_2^\Xi = \Delta_\theta^+ h_2^\Xi(t^-, \theta), \qquad\qquad \theta < 0, t = 2\pi k \qquad (8.62)$$

$$\partial_t h_2^\Xi = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot h_2^\Xi(t,0) + \begin{bmatrix} e_1 - e_2 \\ 0 \\ 0 \end{bmatrix}, \qquad \theta = 0, t \neq 2k\pi \qquad (8.63)$$

$$\begin{bmatrix} 0 \\ e_2 \\ 0 \end{bmatrix} + \Delta_t h_2^\Xi = \begin{bmatrix} 0 \\ e_1 \\ 0 \end{bmatrix}, \qquad\qquad \theta = 0, t = 2k\pi, \qquad (8.64)$$

We will solve the equations for $h_2^{\zeta_i}$ invididually.

The u_1^2 coefficient

The partial differential equation (8.61)–(8.62) becomes the trvial transport equation:

$$\partial_t h_2^\zeta = \partial_\theta h_2^\zeta, \qquad\qquad t \neq 2k\pi \qquad (8.65)$$

$$\Delta_t h_2^\zeta = \Delta_\theta^+ h_2^\zeta(t^-, \theta), \qquad\qquad t = 2k\pi, \qquad (8.66)$$

for $\zeta = (e_1, e_1)$. Therefore, the functions $\mathcal{F}$ and $\mathcal{G}$ of Proposition 8.2.1 are both zero, and it follows that $h_2^{11}(t, \theta) = h_2^{11}(t + \theta, 0)$. The latter is determined solely by the boundary conditions (8.63)–(8.64). Namely, $t \mapsto h_2^{11}(t, 0)$ satisfies the impulsive differential equation

$$\partial_t h_2^{11}(t, 0) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h_2^{11}(t, 0) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad t \neq 2\pi k$$

$$\Delta_t h_2^{11}(t, 0) = 0, \qquad t = 2\pi k$$

It follows that $h_2^{11}(t, \theta)$ is given by

$$h_2^{11}(t, \theta) = h_2^{11}(t + \theta, 0) = \begin{bmatrix} e^{-(t+\theta)} & 0 & 0 \\ 1 - e^{-(t+\theta)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} h_2^{11}(0, 0) + \begin{bmatrix} 1 - e^{-(t+\theta)} \\ e^{-(t+\theta)} - 1 \\ 0 \end{bmatrix}.$$

Finally, we apply the constraint of Corollary 8.2.0.1 to identify the unknown constant $h_2^{11}(0, 0) = (\alpha, \beta, \gamma)$. We therefore require both $P_c(t) h_2^{11}(t, \cdot) = 0$ and $h_2^{11}(t, \theta) = h_2^{11}(t + 2\pi, \theta)$, where the period is 2π. Evaluating these two constraints and simplifying produces the systems of equations

$$\begin{bmatrix} \alpha + \beta \\ \gamma \end{bmatrix} = 0,$$

$$\begin{bmatrix} (\alpha - 1)(e^{-(t+\theta)} - e^{-(t+\theta+2\pi)}) \\ (1 - \alpha)(e^{-(t+\theta)} - e^{-(t+\theta+2\pi)}) \\ \gamma \end{bmatrix} = 0$$

It follows that $\alpha = 1$, $\beta = -1$ and $\gamma = 0$, so that the coefficient $h_2^{11}(t, \theta)$ is the constant

$$h_2^{11}(t, \theta) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The u_2^2 coefficient

With $\zeta = (e_2, e_2)$, the partial differential equation (8.61)–(8.62) for h_2^{22} similarly reduces to the transport equation (8.65)–(8.66), so that we have $h_2^{22}(t, \theta) = h_2^{22}(t + \theta, 0)$. The boundary condition (8.63)–(8.64) contains no inhomogeneous terms, and it follows that

$$h_2^{22}(t, \theta) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 1 - e^{-t} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} c_{22}(0, 0).$$

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The periodicity constraint $h_2^{22}(t+2\pi, \theta) = h_2^{22}(t, \theta)$ implies that $h_2^{22}(0,0) = (\alpha, \beta, \gamma)$ satisfies $\alpha = 0$. The projection constraint $P_c(t)h_2^{22}(t, \cdot) = 0$ then yields $\beta = 0$ and $\gamma = 0$, from which we conclude that $h_2^{22} \equiv 0$.

The $u_1 u_2$ coefficient

Contrary to the previous two coefficients, there is an inhomogeneity in the impulsive partial differential equation (8.61)–(8.62) for the final index $\zeta = (e_1, e_2)$. Specifically, in the notation of Proposition 8.2.1,

$$\mathcal{G}_k = [\, 0 \quad 1 \quad 0 \,]', \quad \mathcal{F} = 0, \quad b_k = [\, 1 \quad 0 \quad 0 \,]', \quad a = 0,$$

which means the coefficient h_2^{12} is of the form

$$h_2^{12}(t, \theta) = n(t + \theta) - \sum_{\theta < 2k\pi - t \leq 0} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

whereas n is a solution of the impulsive differential equation

$$\dot{n} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} n, \qquad\qquad t \neq 2k\pi$$

$$\Delta n = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad\qquad t = 2k\pi.$$

The general solution of the above system is given by

$$n(t) = X(t, 0) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{cases} \sum_{0 < 2k\pi \leq t} X(t, 2k\pi) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, & t > 0 \\[3em] -\sum_{t < 2k\pi \leq 0} X(t, 2k\pi) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, & t \leq 0. \end{cases}$$

$$X(t, s) = \begin{bmatrix} e^{-(t-s)} & 0 & 0 \\ 1 - e^{-(t-s)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Checking the projection condition $P_c(t)h_2^{12}(t, \cdot) = 0$, we find that $\gamma = 0$ and $\alpha + \beta = 0$. Verifying the periodicity condition $h_2^{12}(t + 2\pi, \theta) = h_2^{12}(t, \theta)$ at $\theta = 0$ and $t = 0$, we see that α must satisfy the equation $e^{-2\pi}\alpha + 1 = \alpha$, which implies

$$\alpha = \frac{1}{1 - e^{-2\pi}}, \qquad \beta = -\frac{1}{1 - e^{-2\pi}}, \qquad \gamma = 0.$$

It is not necessary to check at other arguments t and θ because Corollary 8.2.0.1 guarantees that the constants α, β, γ are uniquely specified. Therefore, the $u_1 u_2$ coefficient vector is given, for $t \geq 2\pi$, by

$$
\begin{aligned}
h_2^{12}(t, \theta) = \frac{1}{1 - e^{-2\pi}} X(t + \theta, 0) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \sum_{\theta < 2k\pi - t \leq 0} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
+ \sum_{0 < 2k\pi \leq t + \theta} X(t + \theta, 2k\pi) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\end{aligned}
\tag{8.67}
$$

For $t < 2\pi$, one may extend backwards by periodicity.

Chapter 9

Hyperbolicity

The past three chapters have been devoted to the centre manifold, $\mathcal{W}_c$. The centre manifold is essentially nonlinear version of the centre fibre bundle $\mathcal{RCR}_c$. However, there are other linear invariant fibre bundles, including the unstable fibre bundle $\mathcal{RCR}_u$ and the stable fibre bundle $\mathcal{RCR}_s$. There are also the direct sums $\mathcal{RCR}_c \oplus \mathcal{RCR}_u$ and $\mathcal{RCR}_c \oplus \mathcal{RCR}_s$ – the centre-unstable fibre bundle and the centre-stable fibre bundle. Classically, these fibre bundles have nonlinear analogues: the unstable manifold, stable manifold, centre-unstable manifold and centre-stable manifold. Under a finer notion of spectral separation – that is, an exponential N-splitting – one obtains an extended hierarchy [74] of (linear) invariant fibre bundles and associated (nonlinear) invariant manifolds. In this chapter we focus on the other two elementary invariant manifolds, namely the unstable and stable manifolds. As the proofs are very similar to the analogous ones associated to the centre manifold, many of them will be skipped over.

9.1   More bounded solutions from inhomogeneities

Let us introduce a few more spaces of exponentially weighted functions. The notation will be similar to what we have in Section 6.1. First, we define

$$PC(-s, \mathbb{R}^n) = \{f|_{(-\infty,s]} : f \in PC(\mathbb{R}, \mathbb{R}^n)\}, \quad PC(+s, \mathbb{R}^n) = \{f|_{[s,\infty)} : f \in PC(\mathbb{R}, \mathbb{R}^n)\}.$$

Next, for $s \in \mathbb{R}$, define $\mathbb{Z}_{-s} = \{k \in \mathbb{Z} : t_k \leq s\}$ and $\mathbb{Z}_{+s} = \{k \in \mathbb{Z} : t_k \geq s\}$. We introduce some exponentially weighted function spaces

$$\mathcal{PC}^{\eta,-s} = \{\phi : (-\infty, s] \to \mathcal{RCR} : \phi(t) = f_t, f \in PC(-s, \mathbb{R}^n), ||\phi||_{\eta,-s} < \infty\}$$
$$\mathcal{PC}^{\eta,+s} = \{\phi : [s, \infty) \to \mathcal{RCR} : \phi(t) = f_t, t \in PC(-(s-r), \mathbb{R}^n), ||\phi||_{\eta,+s} < \infty\}$$
$$B^{\eta,-s}(-s, \mathcal{RCR}) = \{f : (-\infty, s] \to \mathcal{RCR} : ||f||_{\eta,-s} < \infty\}$$
$$B^{\eta,+s}(+s, \mathcal{RCR}) = \{f : [s, \infty) \to \mathcal{RCR} : ||f||_{\eta,-s} < \infty\}$$
$$PC^{\eta,-s}(-s, \mathbb{R}^n) = \{f \in PC(-s, \mathbb{R}^n) : ||f||_{\eta,-s} < \infty\}$$
$$PC^{\eta,+s}(+s, \mathbb{R}^n) = \{f \in PC(+s, \mathbb{R}^n) : ||f||_{\eta,+s} < \infty\}$$
$$B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s}, \mathbb{R}^n) = \{f : \mathbb{Z}_{-s} \to \mathbb{R}^n : ||f||_{\eta,-s} < \infty\}$$
$$B^{\eta,+s}_{t_k}(\mathbb{Z}_{-s}, \mathbb{R}^n) = \{f : \mathbb{Z}_{+s} \to \mathbb{R}^n : ||f||_{\eta,+s} < \infty\},$$

with the norms

$$||f||_{\eta,-s} = \begin{cases} \sup_{t \leq s} ||f(t)|| e^{-\eta(t-s)}, & \mathrm{dom}(f) = (-\infty, s] \\ \sup_{k \in \mathbb{Z}_{-s}} ||f(k)|| e^{-\eta(t_k-s)}, & \mathrm{dom}(f) = \mathbb{Z}_{-s} \end{cases}$$

$$||f||_{\eta,+s} = \begin{cases} \sup_{t \geq s} ||f(t)|| e^{-\eta(t-s)}, & \mathrm{dom}(f) = [s, \infty) \\ \sup_{k \in \mathbb{Z}_{+s}} ||f(k)|| e^{-\eta(t_k-s)}, & \mathrm{dom}(f) = \mathbb{Z}_{+s} \end{cases}$$

Next, we introduce analogues linear operator $\mathcal{K}^\eta_s$ from Section 6.2. We have

$$\mathcal{K}^\eta_{-s} : \mathcal{PC}^{\eta,-s} \oplus B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s}, \mathbb{R}^n) \to B^{\eta,-s}(-s, \mathcal{RCR})$$
$$\mathcal{K}^\eta_{+s} : \mathcal{PC}^{\eta,+s} \oplus B^{\eta,+s}_{t_k}(\mathbb{Z}_{+s}, \mathbb{R}^n) \to B^{\eta,+s}(+s, \mathcal{RCR})$$

defined as follows:

$$\mathcal{K}^\eta_{-s}(F, G)(t) = \int_s^t U(t, \mu) P_u(\mu)[\chi_0 F(\mu)] d\mu + \int_{-\infty}^t U(t, \mu)[I - P_u(\mu)][\chi_0 F(\mu)] d\mu \quad (9.1)$$

$$+ \sum_s^t U(t, t_i) P_u(t_i)[\chi_0 G_i] dt_i + \sum_{-\infty}^t U(t, t_i)[I - P_u(t_i)][\chi_0 G_i] dt_i,$$

$$\mathcal{K}^\eta_{+s}(F, G)(t) = \int_s^t U(t, \mu) P_s(\mu)[\chi_0 F(\mu)] d\mu - \int_t^\infty U(t, \mu)[I - P_s(\mu)][\chi_0 F(\mu)] d\mu \quad (9.2)$$

$$+ \sum_s^t U(t, t_i) P_s(t_i)[\chi_0 G_i] dt_i - \sum_t^\infty U(t, t_i)[I - P_s(t_i)][\chi_0 G_i] dt_i.$$

The following result is the appropriate analogue of Lemma 6.2.3. Its proof is similar to (and easier than) that of the aforementioned result, and is omitted.

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Lemma 9.1.1. *Let H.1, H.2, H.5 and H.7 hold.*

1. *For $\eta \in (0, \min\{-a, b\})$, $\mathcal{K}^{\eta}_{-s}$ and $\mathcal{K}^{-\eta}_{+s}$ are bounded linear maps with norms that can be chosen independent of s. For any compact interval $J \subset (0, \min\{-a, b\})$, the norms are bounded uniformly for $\eta \in J$.*

2. *For $\eta \in (0, \min\{-a, b\})$, $\mathcal{K}^{\eta}_{-s}$ has range in $\mathcal{PC}^{\eta, -s}$ and $v = \mathcal{K}^{\eta}_{-s}(F, G)$ is the unique solution of (6.1) in $\mathcal{PC}^{\eta, -s}$ such that $P_s(s)v(s) = 0$.*

3. *For $-\eta \in (0, \min\{-a, b\})\mathcal{K}^{\eta}_{+s}$ has range in $\mathcal{PC}^{\eta, +s}$ and $v = \mathcal{K}^{\eta}_{+s}(F, G)$ is the unique solution of (6.1) in $\mathcal{PC}^{\eta, +s}$ such that $P_u(s)v(s) = 0$.*

9.2 Unstable manifold

Let $\eta \in (0, \min\{-a, b\})$. At this stage, we reintroduce the substitution operators

$$R : \mathcal{PC}^{\eta, -s} \to B^{\eta, -s}(-s, \mathbb{R}^n) \oplus B^{\eta, -s}_{t_k}(\mathbb{Z}_{-s}, \mathbb{R}^n),$$

defined by $R(x)(t, k) = (f(t, x(t)), g(k, x(t_k)_{0-}))$. One can then prove the following lemma.

Lemma 9.2.1. *Let H.4 and H.7 hold. The substitution operator defined above is m-times continuously differentiable. Moreover, on the ball $B_\delta(0)$ in $\mathcal{PC}^{\eta, -s}$, the substitution operator is Lipschitz continuous with Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$.*

This lemma is the reason we do not need to cut off the nonlinearity. It is a consequence of the fact that if $u \in \mathcal{PC}^{\eta, -s} \cap B_\delta(0)$ for $\eta > 0$, then $||u(t)|| \leq \delta$ for all $t \leq s$. Let $\eta \in (0, \min\{-a, b\})$ and introduce a map $\mathcal{G}_{-s} : \mathcal{PC}^{\eta, -s} \times \mathcal{RCR}_u(s) \to \mathcal{PC}^{\eta, -s}$ defined by

$$\mathcal{G}_{-s}(u, \varphi) = U(\cdot, s)\varphi + \mathcal{K}^{\eta}_{-s}(R(u)).$$

In the same way that we prove Theorem 6.4.1, one can show that if $||\varphi|| < \delta_1$ is small enough, then $\mathcal{G}_{-s}(\cdot, \varphi)$ has a unique fixed point in some ball $B_{\delta_2}(0) \cap \mathcal{PC}^{\eta, -s}$. Moreover, δ_1 and δ_2 can be chosen independent of s, and the fixed point is (uniformly in s) Lipschitz continuous with respect to φ.

Theorem 9.2.1. *Let assumptions H.1–H.7 hold. There exist δ_1 and $\delta_2 > 0$ such that for all $\varphi \in B_{\delta_1}(0) \cap \mathcal{RCR}_u(s)$, there is a unique $u^*_{-s} = u^*_{-s}(\varphi) \in B_{\delta_2}(0) \cap \mathcal{PC}^{\eta, -s}$ such that $u^*_{-s} = \mathcal{G}^{\eta}_{-s}(u^*_{-s}, \varphi)$.*

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Definition 9.2.1. *The* local unstable manifold, $\mathcal{W}_u \subset \mathbb{R} \times \mathcal{RCR}$, *is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_u(t) = \mathrm{Im}\{\mathcal{U}(t, \cdot)\}, \qquad (9.3)$$

where $\mathcal{U} : \mathcal{RCR}_u \cap B_{\delta_1}(0) \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{U}(t, \phi) = u^*_{-t}(\phi)(t)$.

Corollary 9.2.1.1. *There exists a constant $L > 0$ such that $||\mathcal{U}(t, \phi) - \mathcal{U}(t, \psi)|| \leq L ||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_u(t)$.*

The local unstable manifold is similarly negatively invariant under the nonautonomous process $S(t, s) : \mathcal{RCR} \to \mathcal{RCR}$, in the following sense. The proof is similar to the proof of part 1 of Theorem 6.5.1.

Theorem 9.2.2. *Let conditions H.1–H.7 hold. Then, for φ sufficiently small and $t \leq s$, we have $u^*_{-s}(\varphi)(t) \in \mathcal{W}_u(t)$. In particular, if $(s, \phi) \in \mathcal{W}_u$, then exists a unique mild solution $u \in \mathcal{PC}^{\eta, -s}$ of (5.1)–(5.2) with the property that $u(t) \in \mathcal{W}_u(t)$, $||u||_{\eta, -s} \leq \delta_2$, and $u(s) = \phi$.*

To prove smoothness of the unstable manifold (in the state space), we will apply the implicit function theorem to the solutions of the equation $\mathcal{F}_{-s} = 0$, with

$$\mathcal{F}_{-s}(u, \varphi) = u - \mathcal{G}_{-s}(u, \varphi).$$

Because of Lemma 9.2.1, $\mathcal{F}_s$ is m-times continuously differentiable, $\mathcal{F}_{-s}(0, 0) = 0$ and $D_u \mathcal{F}_{-s}(0, 0) = I$. One can then directly apply the implicit function theorem to guarantee the existence of an m-times continuously differentiable $\varphi \mapsto \tilde{u}^*_{-s}(\varphi)$ defined on some neighbourhood $B_\rho(0) \subset \mathcal{RCR}_u(s)$, such that $\mathcal{F}_{-s}(\tilde{u}^*_{-s}(\varphi), \varphi) = 0$. By restricting to $B_{\delta_1}(0)$, we get the equality $\tilde{u}^*_{-s} = u^*_{-s}$. Since this operation allows formal differentiation of the fixed-point equation, we immediately get $Du^*_{-s}(0) = U(\cdot, s)$. Finally, as the evaluation functional $ev_s : \mathcal{PC}^{\eta, -s} \to \mathcal{RCR}$ defined by $ev_s(f) = f(s)$ is linear and bounded, the following theorem is proven.

Theorem 9.2.3. *$\mathcal{U}(t, \cdot) : \mathcal{RCR}_u(t) \to \mathcal{RCR}$ is m times continuously differentiable, and $D\mathcal{U}(t, 0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_u(t)$.*

9.3   Stable manifold

With $-\eta \in (0, \min\{-a, b\})$, we can define the substitution operator

$$R : \mathcal{PC}^{\eta, +s} \to B^{\eta, +s}(+s, \mathbb{R}^n) \oplus B^{\eta, +s}_{t_k}(\mathbb{Z}_{+s}, \mathbb{R}^n),$$

with the same formula as previously. In the same way as before, the following lemma is applicable. It is a consequence of the fact that, if $||u||_{\eta,+s} \le \delta$ and $\eta < 0$, then $||u(t)|| \le \delta$ for all $t \ge s$.

Lemma 9.3.1. *Let H.4 and H.7 hold. The substitution operator defined above is m times continuously differentiable. Moreover, on the ball $B_\delta(0)$ in $\mathcal{PC}^{\eta,+s}$, the substitution operator is Lipschitz continuous with Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$.*

We can then proceed to define the fixed-point operator $\mathcal{G}_{+s} : \mathcal{PC}^{\eta,+s} \times \mathcal{RCR}_s(s) \to \mathcal{PC}^{\eta,+s}$ by

$$\mathcal{G}_{+s}(u,\varphi) = U(\cdot,s)\varphi + \mathcal{K}^{\eta}_{+s}(R(u)),$$

and ultimately obtain the following results. They are proven similarly to the analogous results in Section 6.4 and Section 6.5.

Theorem 9.3.1. *Let assumptions H.1–H.7 hold. There exist δ_1 and $\delta_2 > 0$ such that for all $\varphi \in B_{\delta_1}(0) \cap \mathcal{RCR}_u(s)$, there is a unique $u^*_{+s} = u^*_{+s}(\varphi) \in B_{\delta_2}(0) \cap \mathcal{PC}^{\eta,+s}$ such that $u^*_{+s} = \mathcal{G}^{\eta}_{+s}(u^*_{+s},\varphi)$.*

Definition 9.3.1. *The* local stable manifold, *$\mathcal{W}_s \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_s(t) = \mathrm{Im}\{\mathcal{T}(t,\cdot)\}, \tag{9.4}$$

where $\mathcal{T} : \mathcal{RCR}_s \cap B_{\delta_1}(0) \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{T}(t,\phi) = u^*_{+t}(\phi)(t)$.

Corollary 9.3.1.1. *There exists a constant $L > 0$ such that $||\mathcal{T}(t,\phi)-\mathcal{T}(t,\psi)|| \le S||\phi-\psi||$ for all $t \in \mathbb{R}$ and $\phi,\psi \in \mathcal{RCR}_s(t)$.*

Theorem 9.3.2. *Let conditions H.1–H.7 hold. If $(s_1,\phi) \in \mathcal{W}_s$ and ϕ is sufficiently small, then $(t, S(t,s_1)\phi) \in \mathcal{W}_s$ for all $t \ge s_1$. Additionally, for each $\gamma > 0$ there exist $\delta > 0$ and $C > 0$ such that for all $\varphi \in \mathcal{RCR}_s(s_1)$ with $||\varphi|| \le \delta$, we have the estimate*

$$||S(t,s_1)\mathcal{T}(s_1,\varphi)|| \le Ce^{(a+\gamma)(t-s_1)},$$

where we remind the reader that S is the nonlinear process introduced in Lemma 5.1.1.

Theorem 9.3.3. *$\mathcal{T}(t,\cdot) : \mathcal{RCR}_s(t) \to \mathcal{RCR}$ is m times continuously differentiable, and $D\mathcal{T}(t,0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_s(t)$.*

9.4 Linearized stability and instability

Proposition 5.3.1 states that when the unstable and centre fibre bundles are trivial, the fixed point 0 of the nonlinear equation (5.1)–(5.2) is exponentially stable. With the help of the the unstable manifold, we can prove a converse, completing our extension of the classical linearized stability/instability theorem. The following lemma will be helpful in proving the linearized instability theorem; its proof is an elementary consequence of the definition of instability, and is omitted.

Lemma 9.4.1. *Suppose there exists $\epsilon > 0$ such that for all $s \in \mathbb{R}$, there exists sequences $x_n \in \mathcal{RCR}$ and $t_n \in \mathbb{R}$ with $x_n \to 0$ and $t_n > s$ satisfying $t_n \to \infty$, such that $||S(t_n, s)x_n|| \geq \epsilon$. Then, the fixed point 0 is unstable.*

Theorem 9.4.1. *Let assumptions H.1–H.7 hold. Assume for all $\delta > 0$ sufficiently small, there exists $c(\delta) \geq 0$ satisfying $\lim_{\delta \to 0^+} c(\delta) = 0$ and such that*

$$||f(t, \phi) - f(t, \psi)|| \leq c||\phi - \psi||$$
$$||g(k, \phi) - g(k, \psi)|| \leq c||\phi - \psi||,$$

for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $\phi, \psi \in B_\delta(0)$. The fixed point $0 \in \mathcal{RCR}$ is:

- *exponentially stable if $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are both trivial;*

- *unstable if $\mathcal{RCR}_u$ is nontrivial.*

Proof. The exponential stability result is precisely Proposition 5.3.1. For the instability result, let $s \in \mathbb{R}$ be given. Let $s < t_n \to \infty$, and let $\varphi_n \in \mathcal{RCR}_u(t_n)$ be a sequence such that $||\varphi_n|| = \delta_1$. Consider the sequence $\xi_n = \mathcal{U}(t_n, \varphi_n)$. From the fixed point equation, it follows that $P_u(t_n)\xi_n = \varphi_n$ for all n. Also, we have

$$\delta_1 = ||\varphi_n|| = ||P_u(t_n)\xi_n|| \leq N||\xi_n||,$$

from which we get the lower bound $||\xi_n|| \geq \frac{1}{N}\delta_1$ for all $n \in \mathbb{N}$. From Theorem 9.2.2, there exists a mild solution $u_n \in \mathcal{PC}^{\eta, -t_n}$ satisfying $u_n(t_n) = \xi_n$. From the bound $||u_n||_{\eta, -t_n} \leq \delta_2$, we have the exponential estimate $||u_n(t)|| \leq \delta_2 e^{\eta(t-t_n)}$ for all $t \leq t_n$. In particular, we have $||u_n(s)|| \leq \delta_2 e^{\eta(s-t_n)}$. Since $t_n \to \infty$, we get $u_n(s) \to 0$ as $n \to \infty$. Applying Lemma 9.4.1 with the sequence $x_n = u_n(s)$ and $\epsilon = \frac{1}{N}\delta_1$, we get the claimed result. $\square$

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Chapter 10

Codimension-one bifurcations

The content of this chapter appears in *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [20] by Church and Liu.

This chapter contains two results. The first (Section 10.1) is a normal form calculation associated to saddle-node and transcritical bifurcations in a general periodic impulsive delay differential equation with one delay. The second (Section 10.2) considers bifurcations at Hopf points. The system to be studied will be the n-dimensional system

$$\dot{x} = A(t)x(t) + B(t)x(t - r) + q_1(t)\epsilon + f(t, x(t), x(t - r), \epsilon), \qquad t \neq k \qquad (10.1)$$

$$\Delta x = Cx(t^-) + Ex(t - r) + q_2\epsilon + g(x(t^-), x(t - r), \epsilon), \qquad t = k, \qquad (10.2)$$

where q_1 and q_2 are column vectors, and $t \mapsto f(t, \cdot, \cdot, \cdot)$, $t \mapsto A(t)$, $t \mapsto B(t)$ and $t \mapsto q_1(t)$ are periodic with period 1. We will assume that f and g are sufficiently smooth and $f(t, 0, 0, 0) = g(0, 0, 0) = 0$, so that 0 is an equilibrium point when $\epsilon = 0$. Also, we assume $Df(t, 0, 0, 0) = 0$ and $Dg(0, 0, 0) = 0$, so that f and g contain all terms of order 2 and above in $x(t)$, $x(t - r)$ and ϵ. We will also assume that the delay satisfies $r < 1$. Much of the results will depend on the properties of the linear system

$$\dot{x} = A(t)x(t) + B(t)x(t - r), \qquad t \neq k \qquad (10.3)$$

$$\Delta x = Cx(t^-) + Ex(t - r), \qquad t = k. \qquad (10.4)$$

This is precisely the linearization of (10.1)–(10.2) at the origin with parameter $\epsilon = 0$. We

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will also need the formal adjoint equation

$$\frac{d}{ds} z_1 = -z_1(s)A(s) - z_1(s+r)E(s+r), \qquad\qquad s \notin \{k, k-r\} \qquad (10.5)$$

$$\Delta z_1 = \begin{cases} -z_1(k)C(I+C)^{-1}, & s = k \\ -z_1(k)E, & s = k-r, \end{cases} \qquad (10.6)$$

and the *augmented* linear homogeneous system

$$\dot{\pi} = A(t)\pi(t) + B(t)\pi(t-r) + q_1(t)\epsilon, \qquad\qquad t \neq k \qquad (10.7)$$

$$\Delta\pi = C\pi(t^-) + E\pi(t-r) + q_2\epsilon, \qquad\qquad t = k \qquad (10.8)$$

$$\dot{\epsilon} = 0, \qquad\qquad t \neq k, \qquad (10.9)$$

$$\Delta\epsilon = 0, \qquad\qquad t = k. \qquad (10.10)$$

Finally, we denote $F(t, x_t, \epsilon) = f(t, x_t(0), x_t(-r), \epsilon)$ and $G(x_t, \epsilon) = g(x_t(0), x_t(-r), \epsilon)$.

Remark 10.0.1. *The results of this section do not depend crucially on the assumption that $r < 1$, but rather that the linear and nonlinear terms in the jump map satisfy the overlap condition and that $r \leq 1$. For example, if the jump condition (10.2) does not have delayed terms then it is no issue to allow $r = 1$. The assumption $r < 1$ is made more for ease of presentation than anything, and the methodology applied in this section of the present paper can easily be extended to accommodate $r > 1$. In particular, the generic results of Corollary 10.1.1.1 and Corollary 10.2.1.1 remain true for r arbitrary, provided the overlap condition is satisfied. They also hold if there is more than one delay and more than one impulse per period, under similar assumptions.*

10.1   Fold bifurcation

Suppose the linearized system (10.3)–(10.4) at $\epsilon = 0$ has one-dimensional centre fiber bundle: $\mathcal{RCR}_c^0(t) = \mathrm{span}\{\phi_t\}$. This means that zero is a Floquet exponent and there is a single rank 1 Floquet eigensolution ϕ_t. The superscript 0 is used to distinguish the fiber bundles of the system (10.3)–(10.4) from those of (10.7)–(10.10).

To study bifurcations at the origin for (10.1)–(10.2) at $\epsilon = 0$, we must expand the state space by taking ϵ as an additional state variable. This results in the *augmented* system

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \mathcal{L}(t)\begin{bmatrix} x_t \\ \epsilon_t \end{bmatrix} + \begin{bmatrix} f(t, x(t), x(t-r), \epsilon) \\ 0 \end{bmatrix}, \qquad\qquad t \neq k \qquad (10.11)$$

$$\Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \mathcal{J}\begin{bmatrix} x_{t^-} \\ \epsilon_{t^-} \end{bmatrix} + \begin{bmatrix} g(x(t^-), x(t-r), \epsilon) \\ 0 \end{bmatrix}, \qquad\qquad t = k, \qquad (10.12)$$

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and the linear functionals $\mathcal{L}$ and $\mathcal{J}$ are defined by

$$
\mathcal{L}(t) \left[\begin{array}{c} w \\ y \end{array}\right] = \left[\begin{array}{cc} A(t) & q_1(t) \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} w(0) \\ y(0) \end{array}\right] + \left[\begin{array}{cc} B(t) & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} w(-r) \\ y(-r) \end{array}\right],
$$

$$
\mathcal{J} \left[\begin{array}{c} w \\ y \end{array}\right] = \left[\begin{array}{cc} C & q_2 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} w(0) \\ y(0) \end{array}\right] + \left[\begin{array}{cc} E & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} w(-r) \\ y(-r) \end{array}\right].
$$

Observe that the linearization of the augmented system (10.11)–(10.12) is (10.7)–(10.10). The centre fiber bundle, in particular, has become two-dimensional, but the unstable fiber bundle remains c-dimensional.

Lemma 10.1.1. *The centre fiber bundle $\mathcal{RCR}_c(t)$ associated to the linearization of the augmented system (10.11)–(10.12) is two-dimensional. A basis matrix is*

$$
\Phi_t = \left[\begin{array}{cc} \phi_t & \pi_t \\ 0 & 1 \end{array}\right],
$$

where ϕ_t spans the centre fiber bundle of the original linearization (10.3)–(10.4) at $\epsilon = 0$, and $t \mapsto (\pi_t(0), 1)$ is a Floquet eigensolution of rank ≤ 2 with exponent zero of the augmented homogeneous system (10.7)–(10.10). Also, the unstable fiber bundle $\mathcal{RCR}_u(t)$ of the augmented system remains c-dimensional.

Proof. That $(\phi_t, 0)$ is a solution in the centre fiber bundle is clear, so $\dim \mathcal{RCR}_c(t) \geq 1$. On the other hand, any solution (x, ϵ) of the linearization (10.7)–(10.10) must satisfy $\epsilon = \text{constant}$, so in searching for other solutions in $\mathcal{RCR}_c(t)$ with such a nonzero constant, we may without loss of generality assume a solution of the form $(\pi_t, 1)$, where $t \mapsto \pi_t(0)$ solves

$$
\dot{\pi} = A(t)\pi(t) + B(t)\pi(t - r) + q_1(t), \qquad\qquad t \neq k \qquad\qquad (10.13)
$$
$$
\Delta\pi = C\pi(t^-) + E\pi(t - r) + q_2, \qquad\qquad t = k. \qquad\qquad (10.14)
$$

Observe however that for any pair of solutions π_t and ω_t of the above system, the difference $h_t = \pi_t - \omega_t$ satisfies the homogeneous equation (10.3)–(10.4) and, consequently, if $(\pi_t, 1) \in \mathcal{RCR}_c(t)$, then π_t is unique up to addition by a multiple of ϕ_t. Thus, $\dim \mathcal{RCR}_c(t) \leq 2$. Moreover, because of the inhomogeneity, any solution of the form $(\pi_t, 1)$ cannot satisfy $||\pi_t|| \to 0$ as $t \to \infty$ or $t \to -\infty$. By spectral separation, if follows that if $t \mapsto (\pi_t, 1)$ is an eigensolution, then $(\pi_t, 1) \in \mathcal{RCR}_c(t)$.

Next, we prove that $\mathcal{RCR}_u(t) = \mathcal{RCR}_u^0(t) \times \{0\}$, thereby proving that $\mathcal{RCR}_u(t)$ is c-dimensional. Let $(\omega_t, \epsilon) \in \mathcal{RCR}_u(t)$, and assume by way of contradiction that $\epsilon = 1$,

because if $\epsilon = 0$ then we must have $\omega_t \in \mathcal{RCR}_c^0(t)$, and if $\epsilon \neq 1$ we can rescale. But this means $(\omega_t, 1) \in \mathcal{RCR}_u(t)$, which is a contradiction to the above result that all such eigensolutions must be in $\mathcal{RCR}_c(t)$. Thus, $\mathcal{RCR}_u(t)$ contains only eigensolutions of the form $(\omega_t, 0)$, and so $\mathcal{RCR}_u(t) = \mathcal{RCR}_u^0(t) \times \{0\}$ is c-dimensional.

Finally, by way of contradiction, assume that dim $\mathcal{RCR}_c(t) = 1$ and $\mathcal{RCR}_c(t)$ is spanned by $(\phi_t, 0)$. Consider the element $(0, 1) \in \mathcal{RCR} = \mathcal{RCR}([-r, 0], \mathbb{R}^n) \times \mathcal{RCR}([-r, 0], \mathbb{R})$. Due to the decomposition $\mathcal{RCR} = \mathcal{RCR}_c(0) \oplus \mathcal{RCR}_s(0)$, there exists a unique $c \in \mathbb{R}$ such that $(0, 1) = c(\phi_0, 0) + (a_0, b_0)$ for some $(a_0, b_0) \in \mathcal{RCR}_s(0)$. Consequently, $b_0 = 1$, and it follows that $(a_0, 1) \in \mathcal{RCR}_s(0)$. It follows that the solution $t \mapsto (a_t, 1)$ through $(a_0, 1)$ is defined on $[0, \infty)$ and, by forward invariance of $\mathcal{RCR}_s$, we have $(a_t, 1) \in \mathcal{RCR}_s(t)$, which is a contradiction because as has been proven above, such a solution can only be in $\mathcal{RCR}_c(t)$. That π_t is a Floquet eigensolution with exponent zero and of rank ≤ 2 is a consequence the Floquet decomposition (8.5). $\square$

Lemma 10.1.2. *Let $\rho(t) \in \mathbb{R}^{n*}$ be a nontrivial periodic solution of the formally adjoint equation* (10.5)–(10.6), *normalized with respect to ϕ_t such that*

$$N(\rho, \phi) := \int_0^1 \rho(t)[\phi(t) + rB(t)\phi(t - r)]dt + \rho(0)rE\phi(-r) = 1. \tag{10.15}$$

This normalization is always attainable. Define the quantity

$$a_{01} = \int_0^1 \rho(t)q_1(t)dt + \rho(0)q_2. \tag{10.16}$$

The Floquet eigensolution $t \mapsto (\pi_t(0), 1)$ from Lemma 10.1.1 is rank 1 if and only if $a_{01} = 0$. Under the above normalization, π_t satisfies the equality

$$\pi_1 = \pi_0 + a_{01}\phi_0, \tag{10.17}$$

and the matrix Λ of the Floquet decomposition $\Phi_t = Q_t e^{\Lambda t}$ for $t \mapsto Q_t$ of period one, is

$$\Lambda = \begin{bmatrix} 0 & a_{01} \\ 0 & 0 \end{bmatrix}.$$

Proof. The proof of this result makes use of a modification and slight generalization of [[4], Theorem 1 & Lemma 5] adapted to the right-continuous solution formalism. The proof is omitted.

Proposition 10.1.1. *The inhomogeneous linear system*

$$\dot{x} = A(t)x + B(t)x(t-r) + f(t), \qquad\qquad t \neq k$$
$$\Delta x = Cx(t^-) + Ex(t-r) + g, \qquad\qquad t = k,$$

has a periodic solution if and only if

$$\int_0^1 \rho_i(s)f(s)ds + \rho_i(0)g = 0 \qquad\qquad (10.18)$$

for every nontrivial periodic solution ρ_i of the formally adjoint homogeneous system (10.5)–
(10.6). *Also, the number of linearly independent periodic solutions of the homogeneous
system* (10.3)–(10.4) *and its formal adjoint* (10.5)–(10.6) *is the same.*

Using Proposition 10.1.1, it is clear that the condition $a_{01} = 0$ is equivalent to the
periodicity of π_t. Next we show that Λ has the claimed form. To begin, we remark that
$(\pi(t), 1)$ being a Floquet eigensolution of rank ≤ 2 and exponent zero implies it can be
written in the form $\pi(t) = tv(t) + w(t)$ for periodic v and w. Substituting into (10.13)–
(10.14) and factoring, we can write

$$t(\dot{v} - Av - Bv(t-r)) + v + \dot{w} + rBv(t-r) = Aw + Bw(t-r) + q_1, \qquad t \neq k \tag{10.19}$$

$$t(\Delta v - Cv(t^-) - Ev(t-r)) + \Delta w + rEv(t-r) = Cw(t^-) + Bw(t-r) + q_2, \quad t = k. \tag{10.20}$$

Since w is periodic and hence bounded, it follows that the order t terms must vanish.
Consequently, v must in fact be a periodic solution of (10.3)–(10.4), and it follows that
$v_t = c\phi_t$ for a constant c. But this in turn means

$$\pi_1(\theta) = (1+\theta)v_1(\theta) + w_1(\theta)$$
$$= \theta v_0(\theta) + w_0(\theta) + v_0(\theta)$$
$$= \pi_0(\theta) + c\phi_0(\theta).$$

Thus, Φ_1 satisfies the decomposition

$$\Phi_1 = \Phi_0 \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} := \Phi_0 M.$$

If $a_{01} = 0$, we must chose $c = 0$, with the result being the matrix $\Lambda = \log M = 0_{2\times 2}$
as claimed. On the other hand, if $a_{01} \neq 0$, we substitute our ansatz $v(t) = c\phi(t)$ into

(10.19)–(10.20) to obtain the following inhomogeneous system for w:

$$
\begin{aligned}
\dot{w} &= Aw + Bw(t - r) + q_1 - c[\phi(t) + rB\phi(t - r)], & t \neq k \\
\Delta w &= Cw(t^-) + Ew(t - r) + q_1 - crE\phi(t - r), & t = k.
\end{aligned}
$$

Applying Proposition 10.1.1 to the above inhomogeneous system, we conclude that as w is periodic, c must satisfy the equation

$$
a_{01} - cN(\rho, \phi) = 0.
$$

It follows that $N(\rho, \phi) \neq 0$, and as it is linear in ρ, we can always attain the normalization condition $N(\rho, \phi) = 1$. Therefore, $c = a_{01}$, and we get

$$
\Lambda = \log M = \log \begin{bmatrix} 1 & a_{01} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a_{01} \\ 0 & 0 \end{bmatrix}.
$$

$\square$

Lemma 10.1.3. *Write the matrix $Y(t) \in \mathbb{R}^{2 \times (n+1)}$ associated to the projection $P_c(t) : \mathcal{RCR}([-r, 0], \mathbb{R}^{n+1}) \to \mathcal{RCR}_c(t)$ in block form*

$$
Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}
$$

for $Y_{i1} \in \mathbb{R}^{1 \times n}$ and $Y_{i2} \in \mathbb{R}^{1 \times 1}$. We have $Y_{21} = 0$ and

$$
\phi_t Y_{11}(t) = \frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t^0)^{-1} \chi_0 dz, \tag{10.21}
$$

where $V_t^0 : \mathcal{RCR} \to \mathcal{RCR}$ is the monodromy operator associated to the linear system (10.3)–(10.4) and Γ_1 is a positively-oriented contour whose interior is bounded away from zero, and enclosing $1 \in \mathbb{C}$.

Proof. By definition, we have

$$
\Phi_t \begin{bmatrix} Y_{11}(t) \\ Y_{21}(t) \end{bmatrix} = \frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t)^{-1} \mathrm{diag}(1, 1, \ldots, 1, 1, 0) \chi_0 dz, \tag{10.22}
$$

where the diagonal matrix has n ones. We start by partially solving the operator equation

$$
R(z; V_t) \mathrm{diag}(1, \ldots, 1, 0) \chi_0 = \psi \in \mathcal{RCR}([-r, 0], \mathbb{R}^{n+1}).
$$

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By linearity of the monodromy operator V_t, if we write

$$\psi = \begin{bmatrix} \psi_\pi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_\epsilon \end{bmatrix}$$

for $\psi_\pi \in \mathcal{RCR}([-r, 0], \mathbb{R}^n)$ and $\psi_\epsilon \in \mathcal{RCR}([-r, 0], \mathbb{R})$, and the operator equation is equivalent to

$$\mathrm{diag}(1, \dots, 1, 0)\chi_0 = z\psi - V_t \begin{bmatrix} \psi_\pi \\ 0 \end{bmatrix} - V_t \begin{bmatrix} 0 \\ \psi_\epsilon \end{bmatrix}.$$

Note, however, that the dynamics of the augmented system (10.7)–(10.10) are trivial in ϵ, and the left-hand side of the above equation is zero in the ϵ component. Consequently, $\psi_\epsilon \equiv 0$. Similarly, $V_t[\begin{array}{cc} \psi_\pi & 0 \end{array}]^\intercal \in \mathcal{RCR}([-r, 0], \mathbb{R}^n) \times \{0\}$, so the equation for the component ψ_π takes the form

$$\chi_0 = z\psi_\pi - V_t^0 \psi_\pi,$$

which implies ψ_π satisfies the equation $R(z; V_t^0)\chi_0 = \psi_\pi$. Taking all of the above into account together with the representation of Φ_t from Lemma 10.1.1, we see that equation (10.22) is equivalent to

$$\begin{bmatrix} \phi_t Y_{11}(t) + \pi_t Y_{21}(t) \\ Y_{21}(t) \end{bmatrix} = \frac{1}{2\pi i} \int_{\Gamma_1} \begin{bmatrix} (zI - V_t^0)^{-1}\chi_0 \\ 0 \end{bmatrix} dz,$$

which readily implies $Y_{21} = 0$ and the characterization of Y_{11} specified in equation (10.21).
$\square$

With the above three lemmas at hand we can compute the quadratic-order dynamics on the centre manifold (8.14)–(8.15). First, however, we note that because the dynamics of ϵ in (10.11)–(10.12) are trivial, we can abuse notation and write $u = (u_1, u_2) \in \mathbb{R}^2$ instead as $u = (u, \epsilon)$.

Lemma 10.1.4. *Consequently, the nontrivial dynamics on the centre manifold, to quadratic order, are given by*

$$\dot{u} = a_{01}\epsilon + \frac{1}{2}Y_{11}(t)D^2 F(t, 0)[(\phi_t u + (\pi_t - a_{01}t\phi_t)\epsilon, \epsilon)]^2, \qquad t \neq k \qquad (10.23)$$

$$\Delta u = \frac{1}{2}Y_{11}(0)D^2 G(0)[(\phi_0^- u + \pi_0^- \epsilon, \epsilon)]^2, \qquad\qquad t = k, \qquad (10.24)$$

for $|\epsilon|$ small, and all differentials are in the $\mathcal{RCR}$ variable.

Proof. First, using Lemma 10.1.1 and Lemma 10.1.2, we can calculate

$$Q_t = \begin{bmatrix} \phi_t & \pi_t - a_{01}t\phi_t \\ 0 & 1 \end{bmatrix}.$$

Next, the quadratic-order dynamics for (u, ϵ) are given by (8.14)–(8.15):

$$\frac{d}{dt} \begin{bmatrix} u \\ \epsilon \end{bmatrix} = \Lambda \begin{bmatrix} u \\ \epsilon \end{bmatrix} + \frac{1}{2} Y(t) \begin{bmatrix} D^2 F(t,0) \left[Q_t \begin{bmatrix} u \\ \epsilon \end{bmatrix} \right]^2 \\ 0 \end{bmatrix}, \qquad t \neq k,$$

$$\Delta \begin{bmatrix} u \\ \epsilon \end{bmatrix} = \frac{1}{2} Y(t) \begin{bmatrix} D^2 G(0) \left[Q_{t^-} \begin{bmatrix} u \\ \epsilon \end{bmatrix} \right]^2 \\ 0 \end{bmatrix}, \qquad t = k.$$

By periodicity of Q_t and $Y(t)$, we can replace Q_{t^-} with Q_{0^-} and $Y(t)$ with $Y(0)$ in the second equation, since the jumps occur at the integers. Also, because of the structure of $Y(t)$ supplied by Lemma 10.1.3, the above reduces to equation (10.23)–(10.24) together with the trivial equations $\dot{\epsilon} = 0$ and $\Delta \epsilon = 0$. $\qquad \square$

Theorem 10.1.1 (Fold bifurcation). *Let the centre fiber bundle associated to the linear system* (10.1)–(10.2) *be one-dimensional, with $\lambda = 0$ being the only Floquet exponent with zero real part. Let a_{01} be as stated in Lemma 10.1.2 and let π_t satisfy equation (10.17). Assume the functions $\rho(t)$ and $\phi(t)$ satisfy the normalization condition (10.15) and introduce the quantities*

$$a_{20} = \int_0^1 Y_{11}(s) D^2 F(s,0)[(\phi_s,0)]^2 ds + Y_{11}(0) D^2 G(0)[(\phi_0^-,0)]^2,$$

$$a_{11} = \int_0^1 Y_{11}(s) D^2 F(s,0)[(\phi_s,0),(\pi_s - a_{01}s\phi_s,1)] ds + Y_{11}(0) D^2 G(0)[(\phi_0^-,0),(\pi_0^-,1)].$$

The following are true.

1. *If $a_{01} \neq 0$ and $a_{20} \neq 0$, the nonlinear system (10.1)–(10.2) undergoes a fold (saddle-node) bifurcation of periodic orbits from the equilibrium 0 at parameter $\epsilon = 0$. More precisely, the iterated discrete-time dynamics on the parameter-dependent centre manifold are locally topologically equivalent near $(u, \epsilon) = 0$ to the quadratic-order truncated dynamics*

$$u \mapsto u + a_{01}\epsilon + \frac{1}{2}a_{20}u^2.$$

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2. *Suppose $a_{01} = 0$. If $a_{11} \neq 0$ and $a_{20} \neq 0$, and $x = 0$ is a solution pf* (10.1)–(10.2) *for all $|\epsilon|$ small enough, then this system undergoes a transcritical bifurcation of periodic orbits from the equilibrium 0 at parameter $\epsilon = 0$. More precisely, the iterated discrete-time dynamics on the parameter-dependent centre manifold are locally topologically equivalent near $(u, \epsilon) = 0$ to the quadratic-order truncated dynamics*

$$u \mapsto u + a_{11}\epsilon u + \frac{1}{2}a_{20}u^2.$$

3. *If $\mathcal{RCR}_u^0(t)$ is trivial, the stability of fixed points of the iterated dynamics carry over to the analogous bifurcating periodic orbits in* (10.1)–(10.2).

Proof. Starting from the quadratic-order dynamics (10.23)–(10.24) on the centre manifold, we define the stroboscopic (Poincaré) map $u \mapsto S(u, \epsilon)$ mapping the state u at time $t = 0$ to the state at time $t = 1$ for parameter ϵ. This function is smooth and following [19], it admits a Taylor expansion of the form

$$S(u, \epsilon) = u + q_{01}\epsilon + \frac{1}{2}q_{20}u^2 + q_{11}u\epsilon + \frac{1}{2}q_{02}\epsilon^2 + O(||(u, \epsilon)||^3)$$

near $(u, \epsilon) = 0$. Each of the coefficients q_{ij} is a solution of a particular initial-value problem evaluated at $t = 1$. Namely, $q_{ij} = v_{ij}(1)$ where $v_{ij}(0) = 0$ and

$$
\begin{aligned}
\dot{v}_{01} &= a_{01}, & t &\in \mathbb{R} \\
\dot{v}_{20} &= Y_{11}(t)D^2F(t,0)[(\phi_t, 0)]^2, & t &\neq k \\
\dot{v}_{11} &= \frac{1}{2}Y_{11}(t)D^2F(t,0)[(\phi_t, 0), (\pi_t - a_{01}t\phi_t, 1)], & t &\neq k \\
\Delta v_{20} &= Y_{11}(0)D^2G(0)[(\phi_0^-, 0)]^2, & t &= k \\
\Delta v_{11} &= \frac{1}{2}Y_{11}(0)D^2G(0)[(\phi_0^-, 1), (\pi_0^-, 1)], & t &= k.
\end{aligned}
$$

The differential equation for v_{02} is not shown because it will not be needed. Solving the above differential equations, it follows that the Stroboscopic map admits the Taylor expansion

$$S(u, \epsilon) = u + a_{01}\epsilon + \frac{1}{2}a_{20}u^2 + a_{11}\epsilon u + O(\epsilon^2 + ||(u, \epsilon)||^3).$$

As the dynamics on the centre manifold are periodic, the orbit structure and bifurcations can be determined by analyzing the iterated map $u \mapsto S(u, \epsilon)$. The conclusions 1–2 of the theorem now follow directly from the saddle-node bifurcation theorem for maps [92].

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The stability assertion in the presence of a trivial unstable fiber bundle follow by upper semicontinuity of the spectrum of compact operators under perturbations [51], and the continuity of the fixed points of the iterated dynamics for $|\epsilon|$ small. Alternatively, one can use Theorem 6.8.1. □

Remark 10.1.1. *The above theorem could easily be generalized to the setting where there is more than one impulse per period. The period can also be any positive real number, since one can always rescale time so that the period is unity. Note also that there is no need to calculate π_t unless the nondegeneracy condition $a_{20} \neq 0$ of the saddle-node bifurcation fails.*

The following corollary is a statement of the generic fold bifurcation theorem for the system (10.1)–(10.2). The genericity conditions are the transversality conditions $a_{01} \neq 0$ and $a_{20} \neq 0$.

Corollary 10.1.1.1 (Generic Fold Bifurcation)**.** *For any generic impulsive delay differential equation* (10.1)–(10.2) *having at $\epsilon = 0$ the equilibrium 0 with a single Floquet exponent $\lambda = 0$ and one-dimensional centre fiber bundle, there is a neighbourhood N of $0 \in \mathcal{RCR}$ and a smooth invertible change of parameters $\eta = \eta(\epsilon)$ satisfying $\eta(0) = 0$, such that for $\eta > 0$, there are exactly two periodic orbits of period 1 in N that trivialize to the equilibrium as $\eta \to 0^+$, while for $\eta < 0$ there are no periodic orbits in N.*

10.1.1 Example: fold bifurcation in a scalar system with delayed impulse

Consider the scalar equation

$$\dot{x} = \log(2)x - x^2(t - 1/2) + \epsilon\sigma(t), \qquad\qquad t \neq k \qquad\qquad (10.25)$$

$$\Delta x = -\frac{1}{\sqrt{2}}x(t - 1/2) + \epsilon, \qquad\qquad t = k, \qquad\qquad (10.26)$$

where $\sigma(t)$ is periodic with period 1. When $\epsilon = 0$, the linearization at the origin has $\phi(t) = 2^{t - \lfloor t \rfloor}$ as the unique periodic solution up to scaling. Moreover, every solution of the linearization at 0 of the above impulsive delay differential equation is eventually a solution of (ie. (10.25)–(10.26) is FD-reducible [24] to) the system

$$\dot{z} = \log(2)z, \qquad\qquad t \neq k$$

$$\Delta z = -\frac{1}{2}z, \qquad\qquad t = k.$$

Since any solution of (10.25)–(10.26) defined for all time must satisfy the above finite-dimensional system, and since it has only the Floquet exponent 0 with multiplicity 1, we conclude that the linearization of (10.25)–(10.26) has $\mathcal{RCR}_c(t) = \text{span}\{\phi_t\}$ and $\mathcal{RCR}_u(t) = \{0\}$. We are therefore in a position to apply Theorem 10.1.1.

The formal adjoint system to the linearization is

$$\dot{y} = -\log(2)y, \qquad\qquad t \neq k - 1/2$$

$$\Delta y = \frac{1}{\sqrt{2}}y(k), \qquad\qquad t = k - 1/2.$$

From this, we can calculate the nontrivial periodic solution

$$\rho(t) = \begin{cases} 2^{-t+\lfloor t \rfloor}, & 0 \leq t - \lfloor t \rfloor < \frac{1}{2} \\ 2^{1-t+\lfloor t \rfloor}, & \frac{1}{2} \leq t - \lfloor t \rfloor < 1, \end{cases}$$

by solving the equation in reverse time from $(t, y) = (1, 1)$. Next we verify the normalization condition (10.15). We have

$$N(\rho, \phi) = \int_0^1 \rho(t)\phi(t)dt + \rho(0)\frac{1}{2} \cdot \frac{-1}{\sqrt{2}}\phi(-1/2)$$

$$= \int_0^{\frac{1}{2}} 1dt + \int_{\frac{1}{2}}^1 2dt - \frac{1}{2\sqrt{2}}2^{\frac{1}{2}}$$

$$= 1,$$

so ρ is already normalized relative to ϕ. The calculation of the function $Y_{11}(t)$ is carried out in Section 10.1.2; we find that $Y_{11}(t) = \rho(t)$. We have enough information to calculate the coefficients a_{01} and a_{20}:

$$a_{01} = 1 + \int_0^{1/2} 2^{-s}\sigma(s)ds + \int_{1/2}^1 2^{1-s}\sigma(s)ds,$$

$$a_{20} = \int_0^1 -2\phi(s - 1/2)^2 ds = -\frac{3}{\log(2)}.$$

Since a_{20} is nonzero, Theorem 10.1.1 guarantees a saddle-node bifurcation occurs at $\epsilon = 0$ assuming $a_{01} \neq 0$.

For example, if we choose $\sigma(t) = \sin(2\pi t)$, then $a_{01} = 0.88881 \pm 10^{-5}$ is positive and a_{20} is negative. Reading off the discrete-time dynamics from the theorem, we predict that
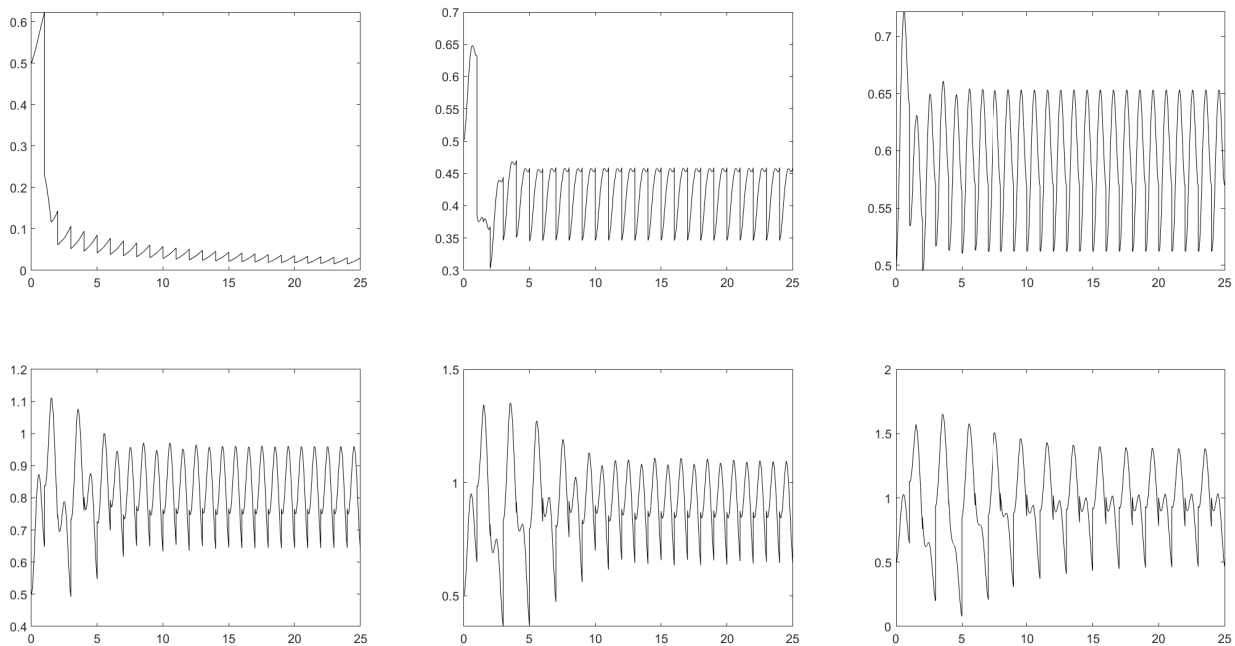
Figure 10.1: Simulations of the scalar impulsive system (10.25)–(10.26) from Section 10.1.1 for various parameters ϵ, with the forcing function $\sigma(t) = \sin(2\pi t)$, from the constant initial condition $x_0 = \frac{1}{2}$. Time t on the horizontal with $x(t)$ on the vertical axis. Top row left to right: solutions with $\epsilon = 0$, $\epsilon = 0.2$ and $\epsilon = 0.4$. Bottom row from left to right: $\epsilon = 0.8$, $\epsilon = 1$, and $\epsilon = 1.2$.

there should be a single, locally asymptotically stable periodic orbit when $\epsilon > 0$, the origin should be semistable when $\epsilon = 0$, and there should be no small periodic orbits when $\epsilon < 0$. These conclusions should all hold true provided $|\epsilon|$ is sufficiently small. Indeed, we can verify numerically that these conclusions are consistent for $0 \leq \epsilon \leq 1$, although the bifurcating periodic orbit appears to undergo a period doubling bifurcation between $\epsilon = 1$ and $\epsilon = 1.2$; see Figure 10.1. Solutions rapidly diverge in the regime $\epsilon < 0$ and we do not provide accompanying figures.

10.1.2   Calculation of the function $Y_{11}(t)$ for Example 10.1.1

First, we remark that because of the periodicity of the monodromy operator and the matrix $Y(t)$, it suffices to compute the restriction of $Y_{11}(t)$ to the interval $[0, 1)$ and extend periodically. We begin by computing the monodromy operator V_t^0 on this restriction. One

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can verify that this is given by

$$V_t^0 \xi(\theta) = \begin{cases} 2^{\theta+1}\xi(0), & t+\theta < 0, t \le \frac{1}{2} \\ 2^{\theta}\xi(0), & t+\theta \ge 0, t \le \frac{1}{2} \\ 2^{\theta}(2\xi(0) - 2^{t-\frac{1}{2}}\xi(t-1/2)), & t > \frac{1}{2} \end{cases}$$

Next, we solve the equation $(zI - V_t^0)^{-1}\chi_0 = \psi$. This is equivalent to

$$\chi_0 = z\psi - V_t^0\psi. \tag{10.27}$$

We do two cases separately: first, with $t \le \frac{1}{2}$, and then with $t > \frac{1}{2}$.

If $t \le \frac{1}{2}$, equation (10.27) evaluated at $\theta = 0$ produces the following algebraic equation for $\psi(0)$:

$$1 = z\psi(0) - \psi(0).$$

Therefore, $\psi(0) = (z-1)^{-1}$. Evaluating (10.27) at $\theta < 0$ and substituting in the constraint $\psi(0) = (z-1)^{-1}$ produces the equation

$$0 = z\psi(\theta) - \frac{1}{z-1}\begin{cases} 2^{1+\theta}, & t+\theta < 0 \\ 2^{\theta}, & t+\theta \ge 0. \end{cases}$$

Solving the above equation for $\psi(\theta)$, combining the two results and simplifying, we have determined that for $t \le \frac{1}{2}$,

$$(zI - V_t^0)^{-1}\chi_0 = \frac{2^{\theta}}{z(z-1)}\begin{cases} z, & \theta = 0 \\ 2, & t+\theta < 0 \\ 1, & t+\theta \ge 0. \end{cases} \tag{10.28}$$

Next, we consider the case $t > \frac{1}{2}$. Evaluating equation (10.27) at $\theta = 0$ and $\theta = \frac{1}{2} - t < 0$, we obtain the following pair of linear equations for the unknowns $\psi(0)$ and $\psi(1/2 - t)$:

$$1 = (z-2)\psi(0) + 2^{t-\frac{1}{2}}\psi(1/2 - t)$$
$$0 = -2^{\frac{3}{2}-t} + (z+1)\psi(1/2 - t).$$

Solving this equation, we find

$$\begin{bmatrix} \psi(0) \\ \psi(1/2 - t) \end{bmatrix} = \frac{1}{z(z-1)}\begin{bmatrix} z+1 \\ 2^{\frac{3}{2}-t} \end{bmatrix}. \tag{10.29}$$

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Next, evaluating (10.27) at $\theta < 0$ and expressing one of the terms as matrix product yields the equation

$$0 = z\psi(\theta) - 2^\theta \begin{bmatrix} 2 & -2^{t-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \psi(0) \\ \psi(1/2 - t) \end{bmatrix}.$$

Solving the equation for $\psi(\theta)$ and combining it with (10.29), we conclude that for $t > \frac{1}{2}$,

$$(zI - V_t^0)^{-1}\chi_0 = \frac{2^\theta}{z(z-1)} \begin{cases} z+1, & \theta = 0 \\ 2, & \theta < 0. \end{cases} \tag{10.30}$$

Next we calculate the Dunford integral $(2\pi i)^{-1} \int_{\Gamma_1} (zI - V_t^0)^{-1}\chi_0 dz$. Using (10.28) and (10.30) together with residue theorem, we obtain after much simplification

$$\frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t^0)^{-1}\chi_0(\theta)dz = \begin{cases} 2^{1+\theta}, & t \leq \frac{1}{2}, t + \theta < 0 \text{ or } t > \frac{1}{2} \\ 2^\theta, & t \leq \frac{1}{2}, t + \theta \geq 0. \end{cases}$$

By Lemma 10.1.3, we can calculate $Y_{11}(t)$ by multiplying the above by $1/\phi_t(\theta)$, and the result should be independent of θ. Initially, we obtain

$$\frac{1}{\phi_t(\theta)} \frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t^0)^{-1}\chi_0(\theta)dz = \begin{cases} 2^{1-t+\lfloor t+\theta \rfloor}, & t \leq \frac{1}{2}, t + \theta < 0 \text{ or } t > \frac{1}{2} \\ 2^{-t+\lfloor t+\theta \rfloor}, & t \leq \frac{1}{2}, t + \theta \geq 0. \end{cases} \tag{10.31}$$

When $t > \frac{1}{2}$, we have $1 - t + \lfloor t + \theta \rfloor = 1 - t$. Conversely, when $t \leq \frac{1}{2}$, we have

$$\begin{aligned} t + \theta < 0 & \quad \Rightarrow \quad 1 - t + \lfloor t + \theta \rfloor = -t \\ t + \theta \geq 0 & \quad \Rightarrow \quad -t + \lfloor t + \theta \rfloor = -t. \end{aligned}$$

Therefore, in both cases, we see that (10.31) can be written independent of θ, with the result being $Y_{11}(t) = \rho(t)$ on $[0, 1)$. Extending by periodicity, the claim is proven.

10.2 Cylinder bifurcation

This time we will assume (10.3)–(10.4) has a pair $\pm i\omega$ of complex conjugate Floquet exponents, and there are no other Floquet exponents with zero real part. This means the centre fibre bundle has the real basis matrix $\Phi_t^0 = [\phi_{1,t} \quad \phi_{2,t}]$, so that $\mathcal{RCR}_c^0(t) = \text{span}\{\phi_{1,t}(\theta), \phi_{2,t}(\theta)\}$. As the Floquet exponents are $\pm i\omega$, equation (8.5) implies the decomposition

$$\Phi_t^0 = Q_t^0 \exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t \right), \tag{10.32}$$

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where $t \mapsto Q_t$ is periodic with columns in $\mathcal{RCR}$. Finally, we let $\dim \mathcal{RCR}_u^0(t) = c$.

The analysis of this section is similar to the previous one, with some modifications. To motivate our first result, recall that a Neimark-Sacker bifurcation occurs in the iterated map

$$z \mapsto A(\epsilon)z + \frac{1}{2}B(\epsilon)[z,z] + \frac{1}{6}C(\epsilon)[z,z,z] + O(||z||^4) \tag{10.33}$$

for a 2×2 matrix $A(\epsilon)$ and symmetric multilinear maps $B(\epsilon)$ and $C(\epsilon)$, at $z = 0$ with parameter $\epsilon = 0$, provided the following are satisfied [54]:

- the eigenvalues $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ of $A(\epsilon)$ satisfy $\mu_i(0) = e^{\pm i\omega}$, and $e^{ik\omega} \neq 1$ for $k = 1, 2, 3, 4$;

- the crossing condition[1] $r'(0) \neq 0$ is satisfied, where

$$r'(0) = \frac{1}{2}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mu_1(\epsilon)\mu_2(\epsilon) \neq 0;$$

- the first Lyapunov coefficient [54] $d(0)$,

$$d(0) = \mathrm{Re}\left(e^{-i\omega}\frac{1}{2}\left[\langle p, C_0[q,q,\overline{q}]\rangle + 2\langle p, B_0[q,(I-A_0)^{-1}B_0[q,\overline{q}]]\rangle \right. \right. \tag{10.34}$$
$$\left. \left. + \langle p, B_0[\overline{q},(e^{2i\omega}I-A_0)^{-1}B[q,q])\rangle\right]\right),$$

satisfies $d(0) \neq 0$, where $\langle a,b\rangle = \overline{a}_1 b_1 + \overline{a}_2 b_2$ is the standard inner product on $\mathbb{C}^2$, $A_0 = A(0)$, $B_0 = B(0)$ and $C_0 = C(0)$, q satisfies $A_0 q = e^{i\omega}q$, p satisfies $A_0^\mathsf{T}p = e^{-i\omega}p$, and $\langle p, q\rangle = 1$.

Our approach in analyzing the Hopf bifurcation condition in (10.1)–(10.2) will be to first expand the state space as in equation (10.11)–(10.12) and determine the nontrivial dynamics on the parameter-dependent centre manifold near $(x_t, \epsilon) = (0,0)$. This will be a two-dimensional impulsive differential equation. The iterated dynamics of the associated stroboscopic map at parameter $\epsilon = 0$ will be compared to (10.33), while the dynamics

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[1]Precisely, this condition states the eigenvalues must cross the boundary of $|z| = 1$ in $\mathbb{C}$ transversally. This is equivalent to the modulus $|\mu_i(\epsilon)|$ being increasing or decreasing at $\epsilon = 0$ which, given that $|z| = z\overline{z}$, is equivalent to the condition we have supplied.

for $|\epsilon|$ small will provide a way to calculate $r'(0)$. This will allow us to effectively lift the Neimark-Sacker bifurcation into the nonlinear dynamics of (10.1)–(10.2).

To begin, we introduce some additional notation. The symbols D_x and D_ϵ will denote the partial derivative operators acting on functions $H : \mathcal{RCR} \times \mathbb{R} \to \mathbb{R}^n$. We then set $H_{xx} = D_x^2 H(0)$, $H_{xxx} = D_x^3 H(0)$, and $H_{x\epsilon} = D_\epsilon D_x H(0)$, where the first two are symmetric bilinear and trilinear maps on $\mathcal{RCR}$ repsectively, and the latter is a linear operator on $\mathcal{RCR}$. For $H(t, \cdot, \cdot) : \mathcal{RCR} \times \mathbb{R} \to \mathbb{R}^n$, we overload the notation and write, for example, $H_{xx}(t) = H(t, \cdot, \cdot)_{xx}$.

The following three lemmas provide the foundation of our result. They are analogues of Lemma 10.1.1, Lemma 10.1.2 and Lemma 10.1.3, and the proofs follow from the same reasoning.

Lemma 10.2.1. *The centre fiber bundle $\mathcal{RCR}_c(t)$ associated to the linearization of the parameter augmented system (10.11)–(10.12) is three-dimensional. A basis matrix is*

$$\Phi_t = \left[\begin{array}{ccc} \phi_{1,t} & \phi_{2,t} & \pi_t \\ 0 & 0 & 1 \end{array} \right],$$

where ϕ_t spans the centre fiber bundle of the original linearization (10.3)–(10.4) at $\epsilon = 0$, and $t \mapsto (\pi_t(0), 1) = (\pi(t), 1)$ is a Floquet eigensolution with exponent zero of the homogeneous impulsive delay differential equation (10.7)–(10.10). Also, the unstable fiber bundle $\mathcal{RCR}_u(t)$ of the parameter-augmented system remains c-dimensional.

Lemma 10.2.2. *The Floquet eigensolution $t \mapsto (\pi_t(0), 1)$ is rank 1, so $\pi(t) = \pi_t(0)$ is the unique periodic solution of the system (10.7)–(10.8) with $\epsilon \equiv 1$. The matrices Λ and Q_t of the Floquet decomposition $\Phi_t = Q_t e^{\Lambda t}$ are*

$$\Lambda = \left[\begin{array}{cc} \Lambda_\omega & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{array} \right], \qquad \Lambda_\omega = \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right], \qquad Q_t = \left[\begin{array}{cc} Q_t^0 & \pi_t \\ 0 & 1 \end{array} \right], \qquad (10.35)$$

where Q_t^0 is the same periodic matrix appearing in (10.32).

Lemma 10.2.3. *Write the matrix $Y(t) \in \mathbb{R}^{3\times(n+1)}$ associated to the projection $P_c(t) : \mathcal{RCR}([-r, 0], \mathbb{R}^{n+1}) \to \mathcal{RCR}_c(t)$ in block form*

$$Y = \left[\begin{array}{cc} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \\ Y_{31} & Y_{32} \end{array} \right]$$

for $Y_{i1} \in \mathbb{R}^{1 \times n}$ and $Y_{i2} \in \mathbb{R}^{1 \times 1}$. We have $Y_{31} = 0$ and

$$\phi_{1,t} Y_{11}(t) + \phi_{2,t} Y_{21}(t) = \frac{1}{2\pi i} \int_{\Gamma} (zI - V_t^0)^{-1} \chi_0 dz, \tag{10.36}$$

where $V_t^0 : \mathcal{RCR} \to \mathcal{RCR}$ is the monodromy operator associated to the linear system (10.3)–(10.4) and Γ is a positively-oriented contour whose interior is bounded away from zero, enclosing both of $e^{\pm i\omega}$ and no other eigenvalues of V_t^0.

At this stage, we should point out that because of the trivial dynamics of the parameter ϵ and the form of Φ_t, the Euclidean space representation of the three-dimensional centre manifold of the augmented system (10.11)–(10.12) takes the form

$$\overline{h}(t, (u, \epsilon), \theta) = \left[\begin{array}{c} h(t, u, \epsilon, \theta) \\ 0 \end{array} \right]$$

for $h(t, u, \epsilon, \theta) \in \mathbb{R}^n$ and $u \in \mathbb{R}^2$. Consequently, Lemma 10.2.1 though Lemma 10.2.3 together with Theorem 8.1.2 imply that the dynamics on the two-dimensional slice of the three-dimensional centre manifold at parameter ϵ are

$$\dot{u} = \Lambda_\omega u + e^{\Lambda_\omega t} \left[\begin{array}{c} Y_{11}(t) \\ Y_{21}(t) \end{array} \right] F(t, Q_t^0 u + \epsilon \tilde{\pi}_t + h(t, u, \epsilon, \cdot), \epsilon), \qquad t \neq k \tag{10.37}$$

$$\Delta u = \left[\begin{array}{c} Y_{11}(0) \\ Y_{21}(0) \end{array} \right] G(Q_{0^-}^0 u + \epsilon \tilde{\pi}_{0^-} + h(0^-, u, \epsilon, \cdot), \epsilon), \qquad t = k. \tag{10.38}$$

Next we consider $\tilde{h}(t, u, \theta) := h(t, u, 0, \theta)$, the centre manifold at parameter $\epsilon = 0$. As we will need the cubic order terms in the iterated map (10.33), it will be necessary to compute $\tilde{h}$ to quadratic order in u. As Theorem 8.1.1 implies the expansion $\tilde{h} = \frac{1}{2} \tilde{h}_2 + O(||u||^3)$, the following lemma is of use and implies the fairly striking result that, for Hopf bifurcation conditions, the projection constraint of Proposition 8.2.0.1 is not actually needed to determine the quadratic term of the centre manifold at $\epsilon = 0$.

Lemma 10.2.4. *In terms of the expansion*

$$\tilde{h}_2(t, \theta)[u, u] = h_2^{11} u_1^2 + h_2^{12} u_1 u_2 + h_2^{22} u_2^2,$$

the vector function $h_2^{\Xi} = (h_2^{11}, h_2^{12}, h_2^{22}) \in (\mathbb{R}^n)^3$ is the unique periodic solution of period one of the impulsive partial delay differential equation (8.38)–(8.41) satisfying the constraint

$P_c(t)h_2^{ij}(t,\cdot) = 0$ *for all* $t \in [0,1)$ *and unordered pairs* $i,j \in \{1,2\}$, *with the data*

$$\Lambda_2 = \begin{bmatrix} 0 & -\frac{\omega}{2} & 0 \\ \omega & 0 & -\omega \\ 0 & \frac{\omega}{2} & 0 \end{bmatrix},$$

$$\mathcal{F}(t,\theta) = \big(\phi_{1,t}(\theta)Y_{11}(t) + \phi_{2,t}(\theta)Y_{21}(t)\big) \begin{bmatrix} F_{xx}(t)[Q_{t,1}^0]^2 \\ F_{xx}(t)[Q_{t,1}^0, Q_{t,2}^0] \\ F_{xx}(t)[Q_{t,2}^0]^2 \end{bmatrix},$$

$$\mathcal{G}(\theta) = (\phi_{1,0}(\theta)Y_{11}(0) + \phi_{2,0}(\theta)Y_{21}(0)) \begin{bmatrix} G_{xx}[Q_{0-,1}^0]^2 \\ G_{xx}[Q_{0-,1}^0, Q_{0-,2}^0] \\ G_{xx}[Q_{0-,2}^0]^2 \end{bmatrix}.$$

Moreover, the inhomogeneous linear system (8.43)–(8.44) *from Proposition* 8.2.1 *has a unique periodic solution* $n(t)$ *of period* T, *so the vector* h_2^{Ξ} *of coefficients of the centre manifold at parameter* $\epsilon = 0$ *is given precisely by the right-hand side of equation* (8.42). *Also, if* $\mathcal{RCR}_u(t)$ *is trivial, the set*

$$n_t + \mathcal{RCR}_c^\dagger(t) \subset \mathcal{RCR}^3$$

is globally attracting, where $\mathcal{RCR}_c^\dagger(t)$ *is the centre fiber bundle associated to the homogeneous equation*

$$\dot{n}(t) + 2\Lambda_2 * n(t) = L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t], \qquad\qquad t \neq t_k \qquad (10.39)$$

$$\Delta n(t) = J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t^-}], \qquad\qquad t = t_k. \qquad (10.40)$$

Proof. Since $\Lambda = \Lambda_\omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, one can readily compute

$$h_2[\Lambda u, u] = h_2(t,\theta)\left[\begin{bmatrix} \omega u_2 \\ -\omega u_1 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right]$$

$$= \omega\left(h_2^{11}u_2 u_1 + \frac{1}{2}h_2^{12}u_2^2 - \frac{1}{2}h_2^{12}u_1^2 - h_2^{22}u_1 u_2\right)$$

$$= \begin{bmatrix} u_1^2 & u_2 u_1 & u_1 u_2 & u_2^2 \end{bmatrix} * \Lambda_2 * \begin{bmatrix} c_{11} \\ c_{21} \\ c_{12} \\ c_{22} \end{bmatrix}.$$

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so that Λ_2 does indeed have the claimed form. Verifying that $\mathcal{F}$ and $\mathcal{G}$ are as stated in the lemma can be done in a similar manner, taking into account Lemma 10.2.2 and Lemma 10.2.3.

Since (8.43)–(8.44) is inhomogeneous, we can write any given periodic solution in the form $n(t) = n_0(t) + M(t)y$, where $n_0(t)$ is a particular periodic solution, $M(t)$ is a matrix whose columns consist of a maximal linearly independent set (finite, due to the Floquet theory) of real periodic solutions of the homogeneous equation, and y is a real vector of appropriate dimension. To characterize $M(t)$, we write down the homogeneous equation:

$$\dot{n} + 2\Lambda_2 * n = L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t], \qquad\qquad t \neq k \qquad (10.41)$$

$$\Delta n = J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t^-}], \qquad\qquad t = k, \qquad (10.42)$$

where we remember that multiplications involving Λ_2 and its exponentials are treated as array multiplications. Introducing a change of variables $w(t) = e^{2\Lambda_2 t} * n(t)$ for $w \in (\mathbb{R}^n)^{\beta \times 1}$, we find by applying the Leibniz law that w satisfies the homogeneous equation

$$\dot{w} = L(t) \odot w_t, \qquad\qquad t \neq k \qquad (10.43)$$

$$\Delta w = J(k) \odot w_{t^-}, \qquad\qquad t = k. \qquad (10.44)$$

Thus, the dynamical system for w is merely the β-fold product of the homogeneous system (10.3)–(10.4) with itself. Recall that this system has, a priori, no nontrivial periodic solutions, and the only Floquet exponents on the imaginary axis are $\pm i\omega$. Since the eigenvalues of $2\Lambda_2$ are $\lambda = 0$ and $\lambda = \pm i2\omega$, the transformation $u \mapsto e^{2\Lambda_2 t} * u$ is uniformly bounded, so every periodic solution of (10.41)–(10.42) must be of the form $e^{-2\Lambda_2 t} * w(t)$ for a periodic solution $w(t)$ of (10.43)–(10.44). Since $\omega \neq 0$, the only periodic solution of this form is the trivial solution, thereby proving that the T-periodic solution of the inhomogeneous equation is unique. That $n_t + \mathcal{RCR}_c^\dagger(t)$ is attracting when $\mathcal{RCR}_u(t)$ is trivial follows by the uniform boundedness of the transformation and the spectral separation of $\mathcal{RCR}^3$ by the homogeneous system (10.43)–(10.43). $\qquad\square$

We cannot hope to obtain an explicit, fully general formula for the solution $c(t, \theta)$ encoding the coefficients of $\tilde{h}_2(t, \theta)$. The difficulty arises in solving the inhomogeneous impulsive delay system (8.43)–(8.44) of Proposition 8.2.1. See later Example 10.2.1.

At this stage, we will assume that one has computed the second-order term $\tilde{h}_2$ of the centre manifold at parameter $\epsilon = 0$, or some sufficiently precise numerical approximation thereof. To prove the following lemma, we could apply the method of [19], as is done in the proof of Theorem 10.1.1. However, the calculations are obviously a bit messier in this case. For the sake of transparency, we provide a self-contained proof.

Lemma 10.2.5. *The iterated dynamics defined by the Stroboscopic (Poincaré) map associated to the impulsive differential equation* (10.37)–(10.38) *on the parameter dependent centre manifold, at the fixed parameter $\epsilon = 0$, are given to cubic order by*

$$u \mapsto \Omega(1)u + \frac{1}{2!}\left(\int_0^1 \Omega(1)\tilde{Y}(s)F_{xx}(s)[\Phi_s^0 u]^2 ds + \tilde{Y}(0)G_{xx}[\Phi_{1-}^0 u]^2\right)$$

$$+ \frac{1}{3!}\left(\int_0^1 \Omega(1)\tilde{Y}(s)\left[F_{xxx}(s)[\Phi_s^0 u]^3 + 3F_{xx}(s)[\Phi_s^0 u, \tilde{h}_2(s,\cdot)[\Omega(s)u]^2] + \cdots\right.\right.$$

$$+ 3F_{xx}(s)\left[\Phi_s^0 u, \Phi_s^0 \int_0^s \tilde{Y}(t)F_{xx}(t)[\Phi_t^0 u]^2 dt\right]ds\right] + \cdots \qquad (10.45)$$

$$+ \tilde{Y}(0)\left[G_{xxx}[\Phi_{1-}^0 u]^3 + 3G_{xx}[\Phi_{1-}^0 u, \tilde{h}_2(1^-,\cdot)[\Omega(1)u]^2] + \cdots\right.$$

$$+ 3G_{xx}\left[\Phi_{1-}^0 u, \Phi_{1-}^0 \int_0^1 \tilde{Y}(t)F_{xx}(t)[\Phi_t^0 u]^2 dt\right]\right]\right) + O(||u||^4)$$

where $\Lambda_\omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $\Omega(t) = e^{\Lambda_\omega t}$, *and* $\tilde{Y}(t) = \begin{bmatrix} Y_{11}(t) \\ Y_{21}(t) \end{bmatrix}$.

Proof. The cubic-order dynamics (8.16)–(8.17) on the centre manifold at parameter $\epsilon = 0$ are

$$\dot{u} = \Lambda S + e^{\Lambda t}\tilde{Y}(t)\left[\frac{1}{2}F^2(t)[Q_t^0 u]^2 + \frac{1}{3!}(D^3 F(t)[Q_t^0 u]^3 + 3D^2 F(t)[Q_t^0 u, \tilde{h}_2(t,\cdot)u^2])\right], \quad t \neq k$$

$$\Delta u = \tilde{Y}(0)\left[\frac{1}{2!}D^2 G[Q_{k^-}u]^2 + \frac{1}{3!}(D^3 G[Q_{k^-}^0 u]^3 + 3D^2 G[Q_{k^-}^0 u, \tilde{h}_2(k^0,\cdot)u^2])\right], \qquad t = k,$$

where for brevity we write $F^3 = F_{xxx}$, $F^2 = F_{xx}$ and similarly for G. Let $t \mapsto S(t, u)$ be the unique solution of the above ordinary impulsive differential equation, defined for time $t \geq 0$ and satisfying the initial condition $S(0, u) = u$. It follows that $t \mapsto S(t, u)$ satisfies the impulsive differential equation

$$\dot{S} = \Lambda S + e^{\Lambda t}\tilde{Y}(t)\left[\frac{1}{2}F^2(t)[Q_t^0 S]^2 + \frac{1}{3!}(D^3 F(t)[Q_t^0 S]^3 + 3D^2 F(t)[Q_t^0 S, \tilde{h}_2(t,\cdot)S^2])\right], \quad t \neq k$$

$$\tag{10.46}$$

$$\Delta S = \tilde{Y}(0)\left[\frac{1}{2!}D^2 G[Q_{k^-}^0 S]^2 + \frac{1}{3!}(D^3 G[Q_{k^-}^0 S]^3 + 3D^2 G[Q_{k^-}^0 S, \tilde{h}_2(k^-,\cdot)S^2])\right], \qquad t = k.$$

$$\tag{10.47}$$

Our objective is to compute a degree three Taylor expansion of $u \mapsto S(1, u)$ near $u = 0$. The function $S : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is C^3 in its second variable and C^1 in its first variable except at times $t \in \mathbb{Z}$, where it is right-continuous [19]. Consequently, the multiple partial derivatives $t \mapsto \partial_{u^k} S := S_{u^k}$ for $k = 1, 2, 3$ themselves satisfy a set of impulsive differential equations that can be calculated by formally differentiating (10.46)–(10.47) with respect to u, keeping in mind that $S = S(t, u)$. Also, we have the initial conditions $S_u(0, 0) = I$, $S_{uu}(0, 0) = 0$ and $S_{uuu}(0, 0) = 0$. The formal differentiation process produces

$$\dot{S}_u = \Lambda S_u + e^{\Lambda t} \tilde{Y}(t) \left(D^2 F(t)[Q_t^0 S, Q_t^0 S_u] + \frac{1}{3!} \left(3 D^3 F(t)[Q_t^0 S, Q_t^0 S, Q_t^0 S_u] + \cdots \right.\right.$$
$$\left.\left. + 3 D^3 F(t)[Q_t^0 S_u, \tilde{h}_2(t, \cdot) S^2 + \tilde{h}_2(t, \cdot)[S, S_u]] \right) \right), \qquad\qquad t \neq k$$

$$\dot{S}_{uu} = \Lambda S_{uu} + e^{\Lambda t} \tilde{Y}(t) \left(D^2 F(t)[Q_t^0 S_u]^2 + D^2 F(t)[Q_t^0 S, Q_t^0 S_u] + \cdots \right.$$
$$+ \frac{1}{3!} \left(3 D^2 F(t)[Q_t^0 S, Q_t^0 S, Q_t^0 S_{uu}] + 6 D^2 F(t)[Q_t^0 S, Q_t^0 S_u, Q_t^0 S_u] + \cdots \right.$$
$$+ 3 D^2 F(t)[Q_t^0 S_{uu}, \tilde{h}_2(t, \cdot) S^2] + 12 D^2 F(t)[Q_t^0 S_u, \tilde{h}_2(t, \cdot)[S, S_u]] + \cdots$$
$$\left.\left. + 6 D^2 F(t)[Q_t^0 S, \tilde{h}_2(t, \cdot)[S_u, S_u] + \tilde{h}_2(t, \cdot)[S, S_{uu}]] \right) \right), \qquad\qquad t \neq k$$

$$\Delta S_u = \tilde{Y}(k) \left(D^2 G[Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \frac{1}{3!} \left(3 D^3 G[Q_{k^-}^0 S, Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \cdots \right.\right.$$
$$\left.\left. + 3 D^3 G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-, \cdot) S^2 + \tilde{h}_2(k^-, \cdot)[S, S_u]] \right) \right), \qquad\qquad t = k$$

$$\Delta S_{uu} = \tilde{Y}(k) \left(D^2 G[Q_{k^-}^0 S_u]^2 + D^2 G[Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \cdots \right.$$
$$+ \frac{1}{3!} \left(3 D^2 G[Q_{k^-}^0 S, Q_{k^-}^0 S, Q_{k^-}^0 S_{uu}] + 6 D^2 G[Q_{k^-}^0 S, Q_{k^-}^0 S_u, Q_{k^-}^0 S_u] + \cdots \right.$$
$$+ 3 D^2 G[Q_{k^-}^0 S_{uu}, \tilde{h}_2(k^-, \cdot) S^2] + 12 D^2 G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-, \cdot)[S, S_u]] + \cdots$$
$$\left.\left. + 6 D^2 G[Q_{k^-}^0 S, \tilde{h}_2(k^-, \cdot)[S_u, S_u] + \tilde{h}_2(k^-, \cdot)[S, S_{uu}]] \right) \right), \qquad\qquad t = k,$$

and we refrain from calculating the impulsive differential equation for S_{uuu} for now. Take note that each of $S = S(t, u)$, $S_u = D_2 S(t, u)$ and $S_{uu} = D_2^2 S(t, u)$ is evaluated at an arbitrary $u \in \mathbb{R}^2$. If one calculates the impulsive differential equation for S_{uuu} and evaluates

each of S, S_u and S_{uu} at $u = 0$, many terms cancel because $S(t,0) = 0$. The result is

$$\dot{S}_u = \Lambda S_u, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \neq k$$

$$\dot{S}_{uu} = \Lambda S_{uu} + e^{\Lambda t}\tilde{Y}(t)D^2F(t)[Q_t^0 S_u]^2, \qquad\qquad\qquad\qquad t \neq k$$

$$\dot{S}_{uuu} = \Lambda S_{uuu} + e^{\Lambda t}\tilde{Y}(t)\left[D^3F(t)[Q_t^0 S_u]^3 + 3D^2F(t)[Q_t^0 S_u, \tilde{h}_2(t,\cdot)[S_u]^2] + \cdots \qquad t \neq k \right.$$

$$\left. + 3D^2F(t)[Q_t^0 S_u, Q_t^0 S_{uu}]\right],$$

$$\Delta S_u = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = k$$

$$\Delta S_{uu} = \tilde{Y}(k)D^2G[Q_{k^-}^0 S_u]^2, \qquad\qquad\qquad\qquad\qquad\qquad\qquad t = k$$

$$\Delta S_{uuu} = \tilde{Y}(k)\left[D^3G[Q_{k^-}^0 S_u]^3 + 3D^2G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-,\cdot)[S_u]^2] + \cdots \qquad t \neq k \right.$$

$$\left. + 3D^2G[Q_{k^-}^0 S_u, Q_{k^-}^0 S_{uu}]\right]$$

It follows that $S_u(0,t) = e^{\Lambda t} = \Omega(t)$, so that $Q_t^0 S_u = \Phi_t^0$. Taking this into account, solving the above impulsive differential equations at the prescribed initial conditions and substituting into the Taylor expansion

$$S(1,u) = S_u(1,0)u + \frac{1}{2!}S_{uu}(1,0)[u,u] + \frac{1}{3!}S_{uuu}(1,0)[u,u,u] + O(||u||^4)$$

produces the right-hand side of (10.45). As $u \mapsto S(1,u)$ is precisely the stroboscopic (Poincaré) map, the lemma is proven. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 10.2.1. *If the vector field and jump map have no quadratic terms in the state x_t – that is, if $F_{xx} = G_{xx} = 0$ – one does not need to compute $\tilde{h}_2$ at all, since the evaluations of $\tilde{h}_2$ in the iterated dynamics (10.45) only appear in the action of the second differentials $D^2F(s)$ and D^2G. This assumption does not, however, preclude mixed quadratic terms of the form $\epsilon L x_t$ for L linear.*

With these preparatory lemmas in place, we are ready to state and prove our bifurcation theorem at a Hopf point.

Theorem 10.2.1 (Cylinder bifurcation). *With the notation and assumptions of Lemma 10.2.1 through Lemma 10.2.3, suppose the following nondegeneracy conditions are satisfied:*

G.1 $e^{im\omega} \neq 1$ for $m = 1, 2, 3, 4$.

G.2 $\gamma(0) \neq 0$, where

$$\gamma(0) = \frac{1}{2}\left(\text{tr}\mathcal{B} + \int_0^1 \text{tr}\mathcal{A}(s)ds\right) \qquad\qquad (10.48)$$

where $\mathcal{B} \in \mathbb{R}^{2\times 2}$ and $\mathcal{A}(t) \in \mathbb{R}^{2\times 2}$ are defined by

$$\mathcal{B} = \tilde{Y}(0)\big(G_{xx}[\pi_{0-}, Q_{0-}^0] + G_{\epsilon x}Q_{0-}^0\big)$$
$$\mathcal{A}(t) = \Omega(t)\tilde{Y}(t)\big(F_{xx}(t)[\pi_t, Q_t^0] + F_{\epsilon x}(t)Q_t^0\big)$$

G.3 The first Lyapunov coefficient $d(0)$ associated to the two dimensional discrete-time map (10.45) of Lemma 10.2.5 is nonzero.

Then, the equilibrium point at the origin of the nonlinear impulsive delay differential equation (10.1)–(10.2) undergoes a bifurcation to an invariant cylinder at the critical parameter $\epsilon = 0$. Specifically, for $|\epsilon|$ small, there is a unique periodic orbit $t \mapsto y(t, \epsilon)$ that satisfies $y_t(\cdot, \epsilon) \to 0$ as $\epsilon \to 0$, in addition to a two-dimensional parameter-dependent invariant fiber bundle $\Sigma_\epsilon \subset \mathbb{S}^1 \times \mathcal{RCR}$ that exists for $d(0)\gamma(0)\epsilon < 0$ and is periodic. The t-fiber $\Sigma_\epsilon(t)$ can be locally realized as

$$\Sigma_\epsilon(t) = Q_t^0 \sigma_\epsilon(t) + O(\epsilon),$$

where $t \mapsto \sigma_\epsilon(t) \subset \mathbb{R}^2$ is periodic with its image a curve of diameter $O(\sqrt{\epsilon})$, and continuous in the Hausdorff metric except at integer times, where it is continuous from the right. Also, if in addition $\mathcal{RCR}_u^0(t)$ is trivial, then

- *y_ϵ is asymtotically stable for $\gamma(0)\epsilon < 0$, stable for $\epsilon = 0$ and unstable for $\gamma(0)\epsilon > 0$, while $\Sigma_\epsilon(t)$ is attracting for $\gamma(0)\epsilon > 0$ provided $d(0) < 0$;*

- *y_ϵ is asymptotically stable for $\gamma(0)\epsilon < 0$ and unstable for $\gamma(0)\epsilon \geq 0$, while $\Sigma_\epsilon(t)$ is unstable for $\gamma(0)\epsilon < 0$ provided $d(0) > 0$.*

Moreover, the assertions concerning the stability and existence of the periodic orbit $y(t, \epsilon)$ for $\epsilon \neq 0$ are true regardless of the nondegeneracy condition G.3.

Proof. The persistence of the equilibrium to a periodic orbit for $|\epsilon|$ small follows by the remark that the iterated dynamics satisfy, to linear order, $u \mapsto \Omega(1)u$ at the parameter $\epsilon = 0$. As $\Omega(1)$ is invertible, the implicit function theorem guarantees the iterated dynamics posses a unique, small fixed point for $0 < |\epsilon| \ll 1$. Lifting this fixed point into the nonlinear impulsive delay differential equation, the result is a unique, small periodic orbit.

The periodic orbit $v_\epsilon(t)$ is C^1 in ϵ. Near $\epsilon = 0$, we can infer from (10.37)–(10.38) that $v_\epsilon(t) = O(\epsilon^2)$. If we perform a time- and parameter-dependent change of coordinates $y = z + v_\epsilon$, (10.37)–(10.38) becomes

$$\dot{z} = \Lambda_\omega z + e^{\Lambda_\omega t}\tilde{Y}(t)\left[F(t, Q_t^0(v_\epsilon + z) + \pi_t\epsilon, \epsilon) - F(t, Q_t^0 z + \pi_t\epsilon, \epsilon)\right], \quad t \neq k \quad (10.49)$$

$$\Delta z = e^{\Lambda_\omega k}\tilde{Y}(k)\left[G(Q_{k-}^0(v_\epsilon + z) + \pi_{k-}\epsilon, \epsilon) - G(Q_{k-}^0 v_\epsilon + \pi_{k-}\epsilon, \epsilon)\right], \qquad t = k. \quad (10.50)$$

Comparing to the iterated map (10.33), the Floquet mulipliers of the trivial equlibrium $z = 0$ of (10.49)–(10.50) are precisely the eigenvalues of $A(\epsilon)$ for the iterated map obtained via the stroboscopic (Poincaré) map for the above dynamical system. The linearization of the above system is

$$\dot{z} = \Lambda_\omega z + e^{\Lambda_\omega t}\tilde{Y}(t)\left[D_x F(t, Q_t^0 v_\epsilon + \pi_t\epsilon, \epsilon)Q_t^0\right]z, \qquad t \neq k$$

$$\Delta z = e^{\Lambda_\omega k}\tilde{Y}(k)\left[D_x G(Q_{k-}^0 v_\epsilon + \pi_{k-}\epsilon, \epsilon)Q_{k-}^0\right]z, \qquad t = k,$$

and it follows from Liouville's formula for impulsive differential equations [8] that the product of the Floquet multipliers is

$$\mu_1(\epsilon)\mu_2(\epsilon) = \det\left(I + \tilde{Y}(0)D_x G(Q_{0-}^0 v_\epsilon(0^-) + \pi_{0-}\epsilon, \epsilon)Q_{0-}^0\right)\cdots$$

$$\times \exp\left(\int_0^1 \mathrm{tr}\left[e^{\Lambda_\omega t}\tilde{Y}(t)D_x f(t, Q_t^0 v_\epsilon(t) + \pi_t\epsilon, \epsilon)Q_t^0\right]dt\right).$$

Using Jacobi's formula and the asymptotic $v_\epsilon(t) = O(\epsilon^2)$, we can readily calculate the derivative of $\mu_1(\epsilon)\mu_2(\epsilon)$ at $\epsilon = 0$. It then follows that $r'(0)$ is as given by equation (10.48).

It follows that under assumptions G.1 through G.3, the discrete-time dynamical system defined by the Stroboscopic map $S_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ of (10.49)–(10.50) undergoes a Neimark-Sacker bifurcation at parameter $\epsilon = 0$. Inverting the change of variables, the same is true of the original system (10.37)–(10.38). There is a closed curve $\sigma_\epsilon(0) \subset \mathbb{R}^2$ that exists for $d(0)\gamma(0) < 0$, is invariant under S_ϵ, and is attracting and stable for $\gamma(0)\epsilon > 0$ and unstable for $\gamma(0)\epsilon < 0$. The fixed point $v_\epsilon(0)$ of S_ϵ satisfies the stability and attraction properties of the theorem, and the same is true of $t \mapsto v_\epsilon(t)$. The stability and attraction persist when considered in the infinite-dimensional context of (10.1)–(10.2) provided $\mathcal{RCR}_u(t) = \{0\}$ because of Theorem 6.8.1.

Let $t \mapsto X_\epsilon(t; 0, w)$ denote the unique solution of (10.49)–(10.50). Because $\sigma_\epsilon(0)$ is invariant under S_ϵ, the fiber bundle

$$\sigma_\epsilon = \{(t, x(t)) : \exists w \in \sigma_\epsilon(0), x(t) = X(t; 0, w)\}$$

is invariant under the process X_ϵ and the t-fiber $\sigma_\epsilon(t)$ is periodic with period 1. The continuity of $t \mapsto X(t; 0, w)$ for $t \in [k, k+1)$ and $k \in \mathbb{Z}$ leads naturally to the continuity of fiber $t \mapsto \sigma_\epsilon(t)$. Also, $\sigma_\epsilon(0)$ has diameter [54] $O(\sqrt{\epsilon})$, and continuity implies the same of $\sigma_\epsilon(t)$. Moreover, $||\sigma_\epsilon(t)|| = O(\sqrt{\epsilon})$, and we can write $\sigma_\epsilon(t) = v_\epsilon(t) + \sigma_\epsilon^0(t)$ for another closed curve $\sigma_\epsilon^0(t)$ of diameter $O(\sqrt{\epsilon})$.

By invariance of the parameter-dependent centre manifold, it follows from the representation (8.3) of solutions on the centre manifold and the description of the matrix Q_t from (10.35) that
$$\Sigma_\epsilon(t) = Q_t^0 \sigma_\epsilon(t) + \pi_t \epsilon + \tilde{h}(t, \sigma_\epsilon(t), \epsilon, \cdot) \subset \mathcal{RCR}$$
naturally defines an invariant fiber bundle $\Sigma \subset \mathbb{R} \times \mathcal{RCR}$ of the nonlinear impulsive delay differential equation (10.1)–(10.2), with

$$\Sigma_\epsilon = \bigcup_{t \in \mathbb{R}} \{t\} \times \Sigma_\epsilon(t).$$

The fiber bundle is periodic in the sense that $\Sigma_\epsilon(t+1) = \Sigma_\epsilon(t)$, and so can be identified as being a subset of $\mathbb{S}^1 \times \mathcal{RCR}$. Since $h = O(||(u, \epsilon)||^2)$, we can write

$$\begin{aligned}
\Sigma_\epsilon(t) &= Q_t^0 \sigma_\epsilon(t) + \pi_t \epsilon + O(||(\sigma_\epsilon(t), \epsilon)||^2) \\
&= Q_t^0 \sigma_\epsilon(t) + O(\epsilon + ||(v_\epsilon(t) + \sigma_\epsilon^0(t), \epsilon)||^2) \\
&= Q_t^0 \sigma_\epsilon(t) + O(\epsilon + |\sqrt{\epsilon} + \epsilon|^2) \\
&= Q_t^0 \sigma_\epsilon(t) + O(\epsilon).
\end{aligned}$$

Finally, the attractivity properties of Σ_ϵ follow by recognizing that it is a locally attracting invariant set within the centre manifold and applying Theorem 6.8.1. $\qquad \square$

A brief discussion of the name *cylinder bifurcation* we have proposed is due. Because the system (10.1)–(10.2) is periodic, there is a natural nonautonomous forward process

$$S : \mathbb{R} \times (\mathbb{S}^1 \times \mathcal{RCR}) \to \mathbb{S}^1 \times \mathcal{RCR}, \qquad S(t, (s, \phi)) = (s + t \bmod 1, x^{(s,\phi)}(t)), \ t \geq s$$

associated to it, where $x^{(s,\phi)} : [s, s + \alpha) \to \mathcal{RCR}$ is the unique solution through the initial condition (s, ϕ) and defined on a maximal interval of existence. The process satisfies the semigroup properties $S(t, (t, \cdot)) = I_{\mathcal{RCR}}$ and

$$S(t, (s, S(s, (u, \phi)))) = S(t, (u, \phi))$$

whenever $t \geq s \geq u$. As such, $\mathbb{S}^1 \times \mathcal{RCR}$ is the state space of the nonautonomous dynamical system generated by (10.1)–(10.2); see [21] as well as the monograph [52] for related definitions.

Each of the t-fibers $\Sigma_\epsilon(t)$ of the bifurcating invariant fiber bundle are homeomorphic to the circle $\mathbb{S}^1$. The set

$$\Sigma_\epsilon = \bigcup_{t \in \mathbb{R}} \{t \bmod 1\} \times \Sigma_\epsilon(t) \subset \mathbb{S}^1 \times \mathcal{RCR}$$

naturally has the structure of a topological manifold with boundary $\{0\} \times \Sigma_\epsilon(0)$, while every interior slice $\{s\} \times \Sigma_\epsilon(s)$ for $s \neq 0$ has a neighbourhood homeomorphic to the open cylinder $(0,1) \times \mathbb{S}^1$. The nontrivial boundary is the result of $t \mapsto \sigma_\epsilon(t)$ being periodic but lacking continuity at the integers, being only right-continuous there and generically possessing a finite jump. The name *cylinder bifurcation* we propose stems from this fact. When the impulse effect is trivial, there are no discontinuities in the time evolution of $t \mapsto \sigma_\epsilon(t)$ and we obtain the classical bifurcation pattern to an invariant torus. Such torus bifurcations typically occur from periodic orbits in autonomous delay differential equations or from equilibrium points in periodically forced delay differential equations [76]. Finally, the generic cylinder bifurcation is as follows, with the genericity conditions being $\gamma(0) \neq 0$ and $d(0) \neq 0$.

Corollary 10.2.1.1 (Generic Cylinder Bifurcation). *For any generic impulsive delay differential equation* (10.1)–(10.2) *having at $\epsilon = 0$ the equilibrium 0 with a single pair of complex conjugate Floquet exponent $\lambda = \pm i\omega$ for $\omega \in (0, 2\pi)$ and two-dimensional centre fiber bundle, there is a neighbourhood N of $0 \in \mathcal{RCR}$ and a smooth invertible change of parameters $\eta = \eta(\epsilon)$ satisfying $\eta(0) = 0$ such that for $\eta > 0$, there is an invariant cylinder in $\mathbb{S}^1 \times N$ that trivializes to $\mathbb{S}^1 \times \{0\}$ (i.e. to the equilibrium) as $\eta \to 0^+$, together with a unique periodic orbit in N that persists for all $|\eta|$ sufficiently small and trivializes to the equilibrium as $\eta \to 0$.*

10.2.1 Example: impulsive perturbation from a Hopf point results in a cylinder bifurcation

We will study the effect of linear impulsive perturbations on a scalar delay differential equation at a Hopf point:

$$\dot{x} = -\frac{\pi}{2}x(t-1) + s_2(t)x^2(t-1) + s_3(t)x^3(t), \qquad t \neq kT$$

$$\Delta x = \epsilon x(t^-), \qquad t = kT,$$

where $T \geq 1$ is a fixed period and $s_2(t)$ and $s_3(t)$ are real periodic functions of period T. When $\epsilon = 0$, the impulse effect is trivial and the linearization of the associated delay

differential equation has a pair of simple complex-conjugate eigenvalues $\pm i\frac{\pi}{2}$ on the imaginary axis, and all other have negative real parts. Thus, perturbing the vector field from this configuration would generically lead to a Hopf bifurcation if s_2 and s_3 were constant. Otherwise, if they were periodic and nonconstant, we would expect a bifurcation to an invariant torus [76].

One can easily verify that $[\ \cos\frac{\pi}{2}(t+\theta)\ \ \sin\frac{\pi}{2}(t+\theta)\ ]$ is a basis matrix for $\mathcal{RCR}_c^0(t)$. Performing a rescaling of time $t \mapsto Tt$, the result is

$$\dot{x} = -\frac{T\pi}{2}x(t-1/T) + \sigma_2(t)x^2(t-1/T) + \sigma_3(t)x^3(t), \qquad t \neq k \qquad (10.51)$$

$$\Delta x = \epsilon x(t^-), \qquad\qquad\qquad\qquad\qquad\qquad\qquad t = k, \qquad (10.52)$$

where $\sigma_i(t) = Ts_i(t/T)$ is periodic with period 1. After this transformation, $\Phi_t^0 = [\ \cos(\frac{\pi}{2}T(t+\theta))\ \ \sin(\frac{\pi}{2}T(t+\theta))\ ]$ is a the new basis for $\mathcal{RCR}_c^0(t)$, $\pm i\frac{T\pi}{2}$ are the Floquet exponents on the imaginary axis, and we have the representation $\Phi_t^0 = Q_t^0 \exp(t\Lambda_{\frac{T\pi}{2}})$. Therefore, ω and Q_t^0 are

$$\omega = \frac{T\pi}{2}, \qquad Q_t^0(\theta) = Q_0(\theta) = \left[\ \cos\left(\frac{T\pi}{2}\theta\right)\ \ \sin\left(\frac{T\pi}{2}\theta\right)\ \right].$$

Using this information, we can verify the first nondegeneracy conditions G.1 of Theorem 10.2.1. We find that G.1 is equivalent to the condition

$$T \text{ and } \frac{3T}{4} \text{ are not integers.}$$

To check the nondegeneracy condition G.2, we require $\tilde{Y}(t)$ and π_t. In 10.3, we compute $\tilde{Y}(t)$ and find

$$\tilde{Y}(t) = e^{-\Lambda_\omega t}\frac{4}{4+\pi^2}\begin{bmatrix} 2 \\ \pi \end{bmatrix}.$$

The explicit calculation of π_t is a much more difficult problem: one must compute the unique periodic solution $\pi(t)$ of the linear inhomogeneous equation

$$\dot{y} = -\frac{T\pi}{2}\pi(t-1/T), \quad t \neq k \qquad\qquad\qquad (10.53)$$

$$\Delta y = 1, \qquad\qquad\qquad t = k.$$

When T is rational, the periodic solution can be computed explicitly, although the calculation can be very lengthy. Alternatively, although $\pi(t)$ is not asymptotically stable, every

solution $y(t)$ satisfies $y_t \to \pi_t + e_t$ as $t \to \infty$ for some $e_t \in \mathcal{RCR}_c(t)$, and since the latter is completely characterized, one could numerically simulate an arbitrary solution and solve an appropriate linear equation to compute $\pi(t)$ to any desired precision. Still another way is to discretize the monodromy operator and compute an approximation using a discretized variation-of-constants formula. To demonstrate one of the (in principle) more analytically tractable cases however, we will focus our attention on a specific choice of T satisfying the nondegeneracy condition G.1. One of the simplest choices is $T = \frac{3}{2}$. As demonstrated in 10.3, the periodic solution is $y(t) = e_1 \cdot S(t - \lfloor t \rfloor)$, where $S : [0, 1) \to \mathbb{R}^3$ is defined piecewise by the expression

$$S(t) = \begin{cases} e^{At}u_0, & t \in [0, 1/3) \\ e^{At}(e_3 + e^{A\frac{1}{3}}u_0), & t \in [1/3, 2/3) \\ e^{At}(e_2 + e^{A\frac{1}{3}}e_1 + e^{A\frac{2}{3}}u_0), & t \in [2/3, 1), \end{cases} \quad A = -\frac{3\pi}{4} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad (10.54)$$

$$u_0 = (I - e^A)^{-1}(e_1 + e^{\frac{1}{3}A}e_2 + e^{\frac{2}{3}A}e_3), \tag{10.55}$$

where $\{e_1, e_2, e_3\}$ is the standard ordered basis of $\mathbb{R}^3$. The expression is rather cumbersome, so we do not display it explicitly in terms of elementary functions. A plot (Figure 10.6) of this periodic solution is provided in 10.3. The nondegeneracy condition G.2, is then equivalent to

$$\gamma(0) = \frac{4}{4 + \pi^2}\left(1 - \pi \int_0^1 \sigma_2(u)e_2 \cdot S(u)du\right) \neq 0. \tag{10.56}$$

Being affine linear in the coefficient σ_2, the condition $\gamma(0) \neq 0$ is indeed generic.

At this stage, we will make a choice for the coefficient σ_2. Choosing $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, numerical integration yields

$$\gamma(0) = \frac{4}{4 + \pi^2}\left(1 - \pi \int_0^1 \frac{1}{2}\sin(2\pi u)e_2 \cdot S(u)du\right) = 0.30854 \pm 10^{-5}, \tag{10.57}$$

so the nondegeneracy condition G.2 passes and, in particular, Theorem 10.2.1 predicts the existence of a nontrivial periodic orbit $t \mapsto y(t, \epsilon)$ that is locally asymptotically stable for $\epsilon < 0$ and unstable for $\epsilon > 0$.

Next, we calculate the quadratic approximation h_2 of the centre manifold. We must solve the impulsive evolution equation (8.38)–(8.41) subject to the data and constraints from Lemma 10.2.4. The first step is to calculate the unique periodic solution $n^0(t)$ satisfying (8.43)–(8.44). The second equation (8.44) is trivial for the present example, so $n^0(t)$ satisfies the inhomogeneous linear delay differential equation

$$\mathcal{F}(t, 0) + \dot{n} + 2\Lambda_2 n(t) = -\frac{3\pi}{4}e^{-\frac{2}{3}\Lambda_2}n(t - 2/3) + m(t). \tag{10.58}$$

with the functions $\mathcal{F}(t,\theta)$ and $m(t)$ being given in 10.3.3. We devise a numerical routine in Section 10.3.2 to approximate the periodic solution. With $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, a plot of the result is provided in Figure 10.7. A numerical approximation of the coefficient vector h_2^{Ξ} is then obtained by substituting the approximation into (8.42), with the integral being computed by numerical quadrature. *Supplementary Animation 4* at the UWSpace repository [17] provides an animation of the components over two periods $t \in [0,2]$.

Visualizing the quadratic term $\frac{1}{2}\tilde{h}_2$ of the two-dimensional centre manifold is slightly more complicated because even though the two independent parameters u_1, u_2 are real, $\tilde{h}_2(t,\cdot)[u_1,u_2]$ is an element of $\mathcal{RCR}$ for each $t \in \mathbb{R}$. However, since $(u_1,u_2) \mapsto \frac{1}{2}\tilde{h}_2(t,\theta)[u_1,u_2]$ is a scalar field with a zero at the origin, we can get a sense of the geometry in a neighbourhood of this point using the Hessian matrix. To provide a coarse but faithful depiction of the geometry, we will therefore generate a plot of the eigenvalues of

$$\theta \mapsto \texttt{eig} \left[\begin{array}{cc} h_2^{11}(t,\theta) & \frac{1}{2}h_2^{12}(t,\theta) \\ \frac{1}{2}h_2^{12}(t,\theta) & h_2^{22}(t,\theta) \end{array} \right] := \texttt{eig}(\mathrm{H}(t,\theta)),$$

for $\theta \in [-1,0]$ and animated over two periods $t \in [0,2]$, where $\texttt{eig}(\mathrm{H})$ denotes the eigenvalues of H, and the matrix above is indeed the Hessian of $(u_1,u_2) \mapsto h(t,\theta)(u_1,u_2)$ at the origin. A visualization is provided by *Supplementary Animation 5* [17], and it is clear that the classification of the origin is generally nonconstant in (t,θ), as there are θ-intervals where the origin is a saddle, maximum or minimum for fixed t. These intervals themselves are nonconstant in t. Snapshots of the animation are provided in Figure 10.2.

By fixing a particular $\theta \in [-1,0]$, we can generate a contour plot of $(u_1,u_2) \mapsto \frac{1}{2}\tilde{h}_2(t,\theta)[u_1,u_2]$ and animate the result over two periods $t \in [0,2]$ to see the transitions between different topological classifications of the origin. This is done alongside a numerical computation of the Hessian determinant in *Supplementary Animation 6* [17] for $\theta = 0$ to help with identifying the classification transitions. Fixed snapshots from one period of transitions of the contour plot are provided in Figure 10.3 for the same times used in the snapshots of the Hessian eigenvalue plot (Figure 10.2).

To compute the Lyapunov coefficient $d(0)$, we will need to fix a choice of σ_3. We choose $\sigma_3(t) = -2$. For our example, the Lyapunov coefficient was calculated by first defining a MATLAB function that computes the right-hand side of (10.45) using numerical quadrature (specifically, MATLAB's built-in trapezoidal method $\texttt{trapz}$) and our previously computed approximation of $\tilde{h}_2$. Then, we used numerical differentiation (with iterated use of $\texttt{gradient}$) to calculate the associated bilinear and trilinear maps in (10.34), the normalized right- and left- eigenvectors of A_0 and, finally, the Lyapunov coefficient $d(0)$. With our choices $T = \frac{3}{2}$, $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$ and $\sigma_3(t) = -2$, our approximation of the
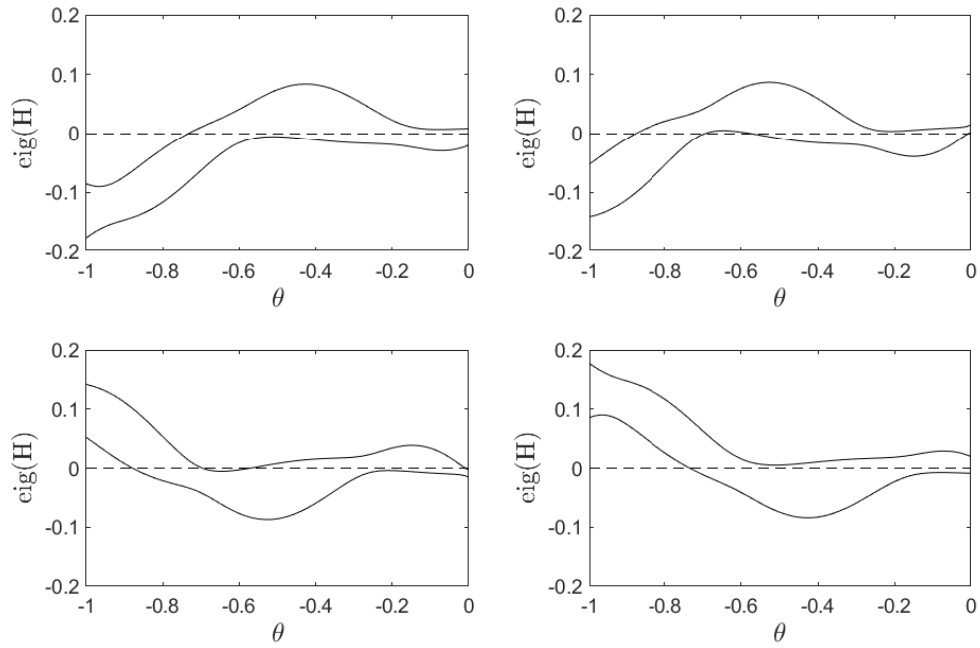
Figure 10.2: Plots of the function $\theta \mapsto \texttt{eig}(\mathrm{H}(t, \theta))$ for fixed arguments of $t \in [-1, 0]$. From top left counterclockwise, these times are $t = 0$, $t = 0, 1$, $t = 0.5$ and $t = 0.6$. Notice the varying topological classification of the origin as θ and t are varied. For each argument θ, the origin is a local minimum when both curves are above the dashed line, a local maximum when both curves are below the dashed line, and a saddle point when the dashed line separates the curves.
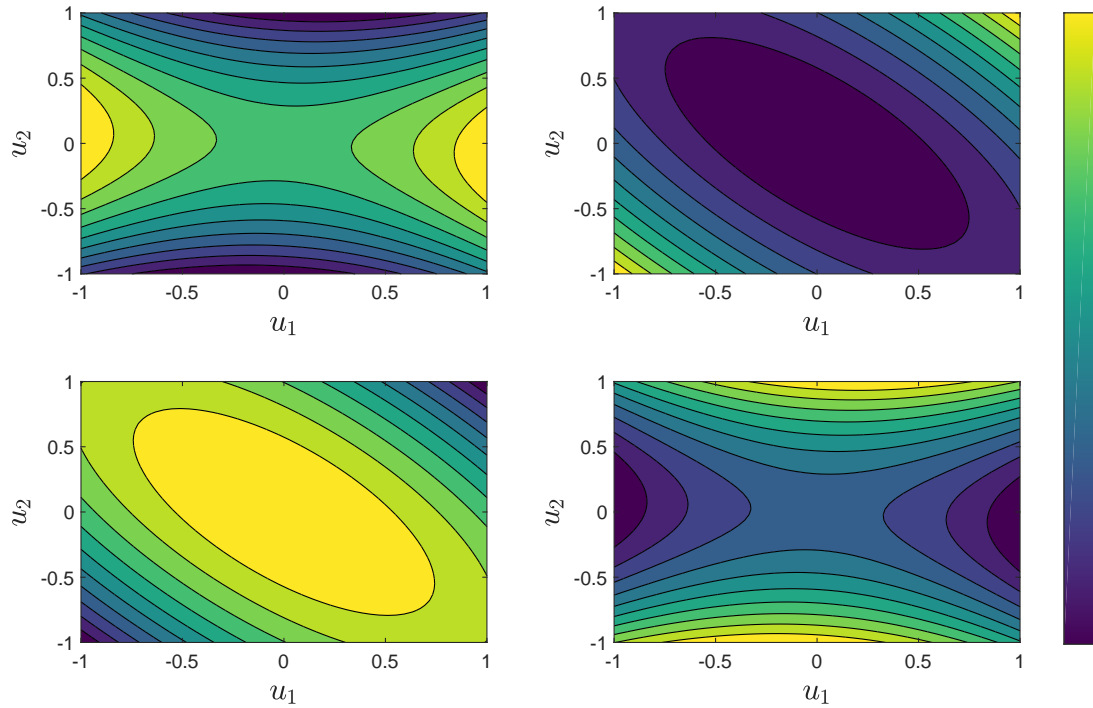
Figure 10.3: Contour plots of $(u_1, u_2) \mapsto \frac{1}{2}\tilde{h}_2(t,0)[u_1, u_2]$ for various arguments of $t \in [0, 1]$ coloured with the Viridis colourmap (displayed right) relative to each frame. Yellow (top) corresponds to more positive levels and purple (bottom) correspond to more negative levels. From top left clockwise, the plot times are $t = 0$, $t = 0.1$, $t = 0.5$ and $t = 0.6$, corresponding to the origin being a saddle point, local minimum, saddle point and local maximum, respectively. The transition times between the different topological classifications in the interval $[0, 1]$ are $t = 0.0781$, $t = 0.262$, $t = 0.576$ and $t = 0.759$.

Lyapunov coefficient is $d(0) = -0.5604$. Since $\gamma(0) > 0$ and $d(0) < 0$, Theorem 10.2.1 implies the existence of a locally attracting invariant cylinder when $\epsilon > 0$ is small.

A standard way to visualize bifurcations to invariant tori in autonomous scalar delay differential equations is to plot curves of the form $(x(t), x(t-r_1), x(t-r_2))$ for two delays r_1 and r_2 that can be chosen as desired. For periodically forced systems, the same thing can be done, or one can plot $(t \bmod T, x(t), x(t-r))$ and identify the hyperplanes $t = 0$ and $t = 1^-$ by "wrapping" the figure around a circle of fixed radius embedded in $\mathbb{R}^3$ to illustrate the torus as being a subspace of $\mathbb{S}^1 \times \mathbb{R}^2$. For impulsive delay differential equations of the form (10.1)–(10.2), one must choose the delays r_1 and r_2 to be positive integers, otherwise the curves $t \mapsto (x(t), x(t-r_1), x(t-r_2))$ will have discontinuities at times other than the integers.

For our example, we provide both with the illustrative parameter $\epsilon = \frac{1}{2}$. Figure 10.4 is a plot of the attractor in the delayed variables $x(t)$, $x(t-1)$ and $x(t-2)$, while Figure 10.5 is a plot of the curve $t \mapsto (t \bmod 1, x(t), x(t-1))$ wrapped around a cylinder of radius 7 (there is no deep significance to the choice of radius). That is, we plot the curve in the cylindrical coordinates (r, θ, z) with

$$r = |7 + x(t)|, \qquad \theta = 2\pi t, \qquad z = x(t-1).$$

In rectangular coordinates, this corresponds to a plot of $(x_{rad,1}(t), x_{rad,2}(t), x(t-1))$ with

$$\begin{bmatrix} x_{rad,1}(t) \\ x_{rad,2}(t) \end{bmatrix} = \begin{bmatrix} \cos(2\pi t)(7 + x(t)) \\ \sin(2\pi t)(7 + x(t)) \end{bmatrix}. \tag{10.59}$$

The simulation is in agreement with Theorem 10.2.1, as the cylinder is indeed present and attracting for $\epsilon > 0$.

10.3 Calculations associated to Example 10.2.1

This section is broken up into several parts. We begin with the calculations concerning the projection $P_c(t)$. Next, we calculate $\pi(t)$ and the matrices $\mathcal{A}(t)$ and $\mathcal{B}$ needed in the nondegeneracy condition $\gamma(0) \neq 0$. Following this, we calculate $\tilde{h}_2$. We conclude by checking the nondegeneracy condition $d(0) \neq 0$.
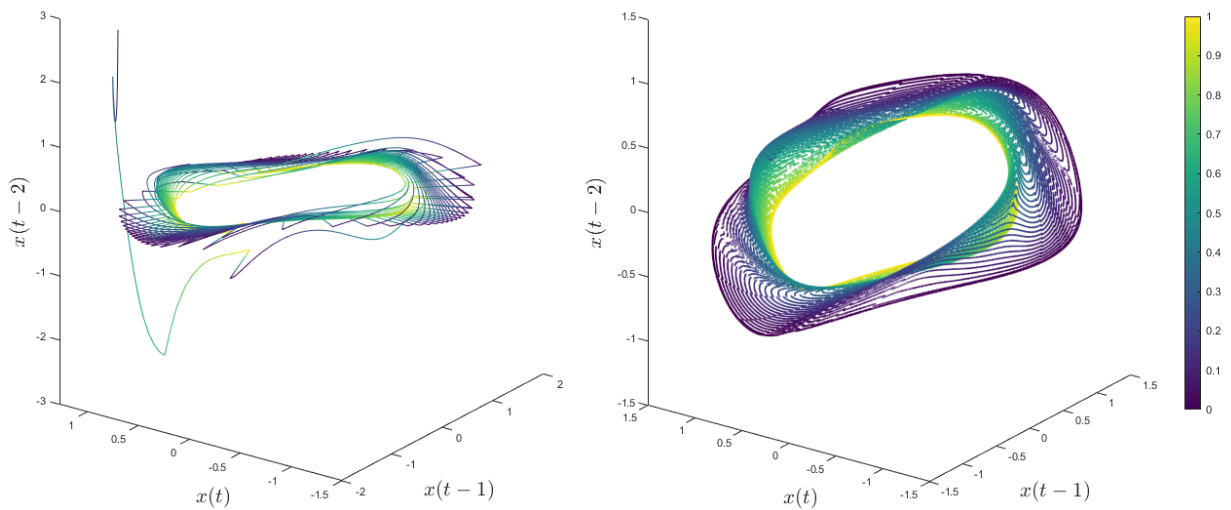
Figure 10.4: Left: trajectory through the constant initial condition $x_0 = 8$ of the system (10.52)–(10.52) from Example 10.2.1 for parameters $T = \frac{3}{2}$, $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, $\sigma_3(t) = -2$, $\epsilon = \frac{1}{2}$ in the coordinates $x(t)$, $x(t-1)$ and $x(t-2)$, plotted for time $t \in [0, 80]$. Linear interpolation between left-limits $x(k^-)$ and points $x(k)$ for integer times $k \in \mathbb{Z}$ are shown. Right: attractor to which the solution in the left pane converges. In both panes, trajectories are coloured using the Viridis colourmap (displayed right) relative to the argument t mod 1, so that purple corresponds to integer arguments of $t = k \in \mathbb{Z}$, while yellow corresponds to the left limits $t \to k^-$.
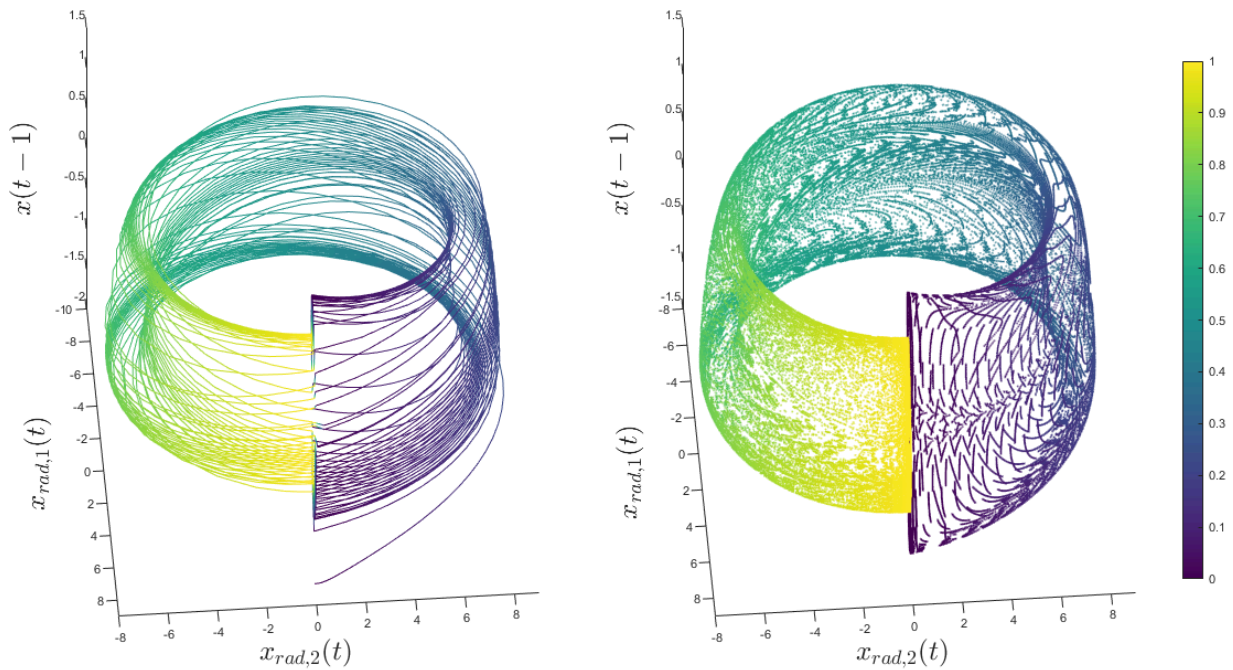
Figure 10.5: The same trajectories and attractors from Figure 10.4 in the cylindrical coordinates defined in equation (10.59). The same colourmap is also used. In these coordinates it is much easier to visualize the cylindrical topology of the attractor as well as the discontinuity along the half-plane $\{x_{rad,2}(t) = 0, x_{rad,1}(t) \geq 0\}$ corresponding to the times $t = k \in \mathbb{Z}$. If the impulse effect were replaced by a parameter-dependent continuous-time periodic linear forcing in the vector field, the structure above would generically be replaced by that of a torus, and the aforementioned half-plane discontinuity would not be present.

The projection $P_c(t)$ and matrix $\tilde{Y}(t)$

Conveniently, since $P_c(t)$ is calculated with respect to the linearization at $\epsilon = 0$ and this system is autonomous, we have $V_t^0 = V_0^0$ for all $t \in \mathbb{R}$, and the projection

$$P_c(t) = \frac{1}{2\pi i} \int_\Gamma (zI - V_0^0)^{-1} dz$$

is constant. Precisely, V_0^0 is the monodromy operator associated to the autonomous system

$$\dot{x} = -\frac{T\pi}{2} x(t - 1/T). \tag{10.60}$$

To compute the vector $Y(t)$, we remark that because $P_c(t) = P_c$ is constant and $\Phi_t^0 = Q_0 e^{\Lambda_\omega t}$ for $\omega = \frac{T\pi}{2}$, we have the representation $Y(t) = e^{-\Lambda_\omega t} Y(0)$, so it suffices to compute $Y(0)$. Since $\Phi_0 = Q_0$, $Y(0)$ satisfies the equation $P_c \chi_0 = Q_0 Y(0)$. Q_0 is precisely the basis matrix for the centre eigenspace of the infinitesimal generator associated to the semigroup of (10.60), so we can formally calculate $P_c \chi_0$ using adjoint-based methods; see [37, 39]. A basis matrix for the centre eigenspace of the infinitesimal generator of the formal adjoint of (10.60) is $\Psi(\theta) = [\cos(\frac{T\pi}{2}\theta) \quad -\sin(\frac{T\pi}{2}\theta)]^\intercal$, so applying the usual bilinear form,

$$\langle \Psi, \Phi_0 \rangle = \Psi(0)\Phi_0(0) - \frac{T\pi}{2} \int_{-1/T}^0 \Psi(\theta + 1/T)\Phi_0(\theta) d\theta$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{\pi}{2} \int_{-1}^0 \begin{bmatrix} \cos(\frac{\pi}{2}(\theta+1))\cos(\frac{\pi}{2}\theta) & \cos(\frac{\pi}{2}(\theta+1))\sin(\frac{\pi}{2}\theta) \\ -\sin(\frac{\pi}{2}(\theta+1))\cos(\frac{\pi}{2}\theta) & -\sin(\frac{\pi}{2}(\theta+1))\sin(\frac{\pi}{2}\theta) \end{bmatrix} d\theta$$

$$= \begin{bmatrix} 1/2 & \pi/4 \\ \pi/4 & -1/2 \end{bmatrix}.$$

Normalizing Ψ with respect to Φ_0, it follows that the projection $P_c \chi_0$ is given by

$$P_c \chi_0 = \Phi_0 \langle \Psi, \Phi_0 \rangle^{-1} \langle \Psi, \chi_0 \rangle$$

$$= [\cos(\frac{T\pi}{2}\theta) \quad \sin(\frac{T\pi}{2}\theta)] \begin{bmatrix} \frac{8}{4+\pi^2} & \frac{4\pi}{4+\pi^2} \\ \frac{4\pi}{4+\pi^2} & -\frac{8}{4+\pi^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \Phi_0(\theta) \frac{4}{4+\pi^2} \begin{bmatrix} 2 \\ \pi \end{bmatrix}.$$

Therefore, $\tilde{Y}(t) = e^{-\Lambda_\omega t} \frac{4}{4+\pi^2} [2 \quad \pi]^\intercal$, as claimed.
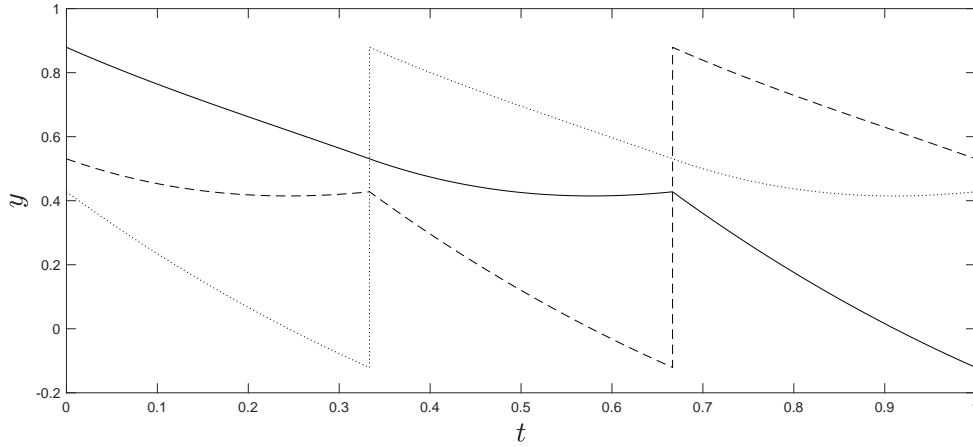
Figure 10.6: The periodic solution $\pi(t) = y_0(t)$ (solid black line) and the two shifts $y_1(t)$ and $y_2(t)$ (black dashed line and dotted lines, respectively), plotted over one period $t \in [0, 1)$.

10.3.1 Calculation of $\pi(t)$ and the matrices $\mathcal{A}(t)$ and $\mathcal{B}$

Now, we consider only the case $T = \frac{3}{2}$. To calculate $\pi(t)$ we define the shifts $y_0(t) = y(t)$, $y_1(t) = y(t - 2/3)$ and $y_2(t) = y(t - 4/3)$. Assuming y is periodic with period 1, it follows that the shifts satisfy the impulsive differential equation

$$\dot{y}_j = -\frac{3\pi}{4} y_{i+1}, \qquad\qquad t \neq k + \frac{2}{3}j,$$

$$\Delta y_j = 1, \qquad\qquad t = k + \frac{2}{3}j,$$

for $j = 0, 1, 2$ and we define $y_3 = y_0$. The sequence of impulses is periodic with period 1, and the impulse times in the interval $(0, 1]$ are $\frac{1}{3}$, $\frac{2}{3}$ and 1. If u_0 is a given initial condition at time $t = 0$, it is easy to check that the solution at time $t = 1$ is given by

$$\vec{y}(1) = e_1 + e^{\frac{1}{3}A}(e_2 + e^{\frac{1}{3}A}(e_3 + e^{\frac{1}{3}A}u_0)).$$

Imposing the periodicity constraint $\vec{y}(1) = u_0$ and solving for u_0 yields the expression (10.55), which is well-defined because 1 is not an eigenvalue of A. Since $\pi(t)$ is the unique periodic solution of (10.53), there is exactly one periodic solution of the above inhomogeneous impulsive system. Thus, its first component must coincide with $\pi(t)$, thereby proving our claim. Figure 10.6 is a plot of the periodic solution $\pi(t)$ together with the shifted components.

To check the nondegeneracy condition G.2, we calculate each of $\mathcal{B}$ and $\mathcal{A}(t)$. Since $G_{xx} = 0$, $G_{\epsilon x}\phi = \phi(0)$ and $Q_{0-}^0 = [\ 1\ \ 0\ ]$ we readily compute

$$\begin{aligned}
\mathcal{B} &= \tilde{Y}(0)(G_{xx}[\pi_{0-}, Q_{0-}^0] + G_{\epsilon x}Q_{0-}^0) \\
&= \frac{4\pi}{2}\begin{bmatrix} 2 \\ \pi \end{bmatrix}[\ 1\ \ 0\ ] \\
&= \frac{4}{4+\pi^2}\begin{bmatrix} 2 & 0 \\ \pi & 0 \end{bmatrix}.
\end{aligned}$$

On the other hand, we have $F_{xx}(t)[\phi, \psi] = 2\sigma_2(t)\phi(-1/T)\psi(-1/T)$, $F_{x\epsilon} = 0$ and $Q_t^0(-1/T) = [\ 0\ \ -1\ ]$. Since $\Omega(t)\tilde{Y}(t) = \tilde{Y}(0)$ and we have chosen $T = \frac{3}{2}$, we have

$$\begin{aligned}
\mathcal{A}(t) &= \Omega(t)\tilde{Y}(t)(F_{xx}(t)[\pi_t, Q_t^0] + F_{e\epsilon x}Q_t^0) \\
&= \frac{4}{4+\pi^2}\begin{bmatrix} 2 \\ \pi \end{bmatrix}\left(2\sigma_2(t)\pi(t-2/3)[\ 0\ \ -1\ ]\right) \\
&= -\frac{8\sigma_2(t)\pi(t-2/3)}{4+\pi^2}\begin{bmatrix} 0 & 2 \\ 0 & \pi \end{bmatrix}.
\end{aligned}$$

Taking into account that $\pi(t-2/3) = y_1(t) = e_2 \cdot S(t)$, one obtains (10.56) after substituting $\mathcal{B}$ and $\mathcal{A}(t)$ into (10.48).

10.3.2   Calculation of $n^0(t)$: a numerical routine

The routine we propose here could certainly be adapted to more general settings, and our notation will at times suggest at a more general approach. Our first step is to compute a periodic solution satisfying (10.58). To do this, we integrate the delay differential equation from the constant initial condition $0 \in \mathbb{R}^3$ until convergence is achieved to a superposition of some periodic solution $n^0(t)$ of period one, together with a linear combination of periodic solutions of period $\frac{8}{3}$ determined by the eigenvalues $\pm\frac{3\pi}{4}i$ of the homogeneous equation

$$\dot{n} + 2\Lambda_2 n(t) = -\frac{3\pi}{4}e^{-\frac{2}{3}\Lambda_2}n(t-2/3). \tag{10.61}$$

Symbolically, the solution $s(t)$ satisfying $s(0) = 0$ is simulated until the numerical convergence $s(t) \to \tilde{s}(t)$ is achieved with

$$\tilde{s}_t \in n_t^0 + \mathcal{R}\mathcal{C}\mathcal{R}_c^\dagger(t), \tag{10.62}$$

where $\mathcal{RCR}_c^\dagger(t)$ is the centre fiber bundle associated to the homogeneous equation for (10.61). That this decomposition can be realized in the limit is due to Lemma 10.2.4.

Next, we construct an approximate basis for $\mathcal{RCR}_c^\dagger(t)$. This is done by integrating the homogeneous equation associated to (10.58) from a collection $\{x_0^1, \ldots, x_0^K\}$ of arbitrary linearly independent initial conditions $x_0^i \in \mathcal{RCR}$ for $i = 1, \ldots, K$, with the integration performed until the associated solutions $s^i(t)$ numerically converge to some $\tilde{s}^i(t)$ satisfying $\tilde{s}_t^i \in \mathcal{RCR}_c^\dagger(t)$. That this convergence is attainable essentially follows be Lemma 10.2.4.

Having computed an approximate basis $\{\tilde{s}_t^1, \ldots, \tilde{s}_t^K\}$ for $\mathcal{RCR}_c^\dagger(t)$, our goal is to extract n_t^0 from $\tilde{s}_t$ in the decomposition (10.62). To this end, we define the shifts

$$v = \tilde{s}_t - \tilde{s}_{t-1}, \qquad v_i = \tilde{s}_t^i - \tilde{s}_{t-1}^i, \qquad i = 1, \ldots, K.$$

In an idealized sense, we have $v, v_i \in \mathcal{RCR}_c^\dagger(t)$, so that $v = \sum_{i=1}^K y_i v_i$ for some real constants y_i. In practice this equality is not attainable, so instead we search for a best approximation of v in the finite-dimensional subspace $W = \text{span}\{v_1, \ldots, v_K\}$, where we now interpret v and W as being in $L^2 = L^2([-1, 0], \mathbb{R}^3)$. The best approximation is $\sum_{i=1}^K y_i v_i$ with the vector $\vec{y} = (y_1, \ldots, y_K) \in \mathbb{R}^K$ being the unique solution of

$$M\vec{y} = b, \quad M_{ij} = \langle v_i, v_j \rangle, \quad b_i = \langle v, v_i \rangle, \tag{10.63}$$

where $\langle f, g \rangle = \int_{-1}^0 f(t) \cdot g(t) dt$ the standard inner product on L^2. It follows that the function

$$n^0(t) = \tilde{s}(t) - \sum_{i=1}^K y_i \tilde{s}_i(t) \tag{10.64}$$

is the best approximation to a periodic solution of (10.58), relative to the basis W, in the sense that the L^2 *periodicity error*

$$\mathsf{e}(n_t^0) = ||n_t^0 - n_{t-1}^0||_{L^2}$$

is minimized.

10.3.3 Calculation of h_2

Using equation (8.34), we find that $\mathcal{F}(t, \theta)$ and $m(t)$ are, in terms of an arbitrary $\sigma_2(t)$,

$$\mathcal{F}(t, \theta) = \frac{4}{4 + \pi^2} \left(2\cos(\frac{3\pi}{4}\theta) + \pi \sin(\frac{3\pi}{4}\theta) \right) 2\sigma_2(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.65}$$

$$m(t) = 2\sigma_2(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{3\pi}{4} e^{-\frac{4}{3}\Lambda_2} \int_{-2/3} e^{-\frac{4}{3}\Lambda_2 s} \mathcal{F}\left(t - s - \frac{2}{3}, s \right) ds, \tag{10.66}$$

$$\Lambda_2 = \frac{3\pi}{4} \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{10.67}$$

We implemented the numerical routine from Section 10.3.2 with $K = 6$ in MATLAB R2018a. All numerical integration of the delay differential equations was done using `dde23` with default error tolerances. The random initial history functions x_0^i for $i = 1, \ldots, 6$ were produced using separate calls to `randn(3,1)` at each point $t \in [-2/3, 0]$ requested by the solver. With $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, the result was the periodic solution plotted in Figure 10.7 with L^2 periodicity error $\mathtt{e}(n_t^0) < 3 \cdot 10^{-3}$.
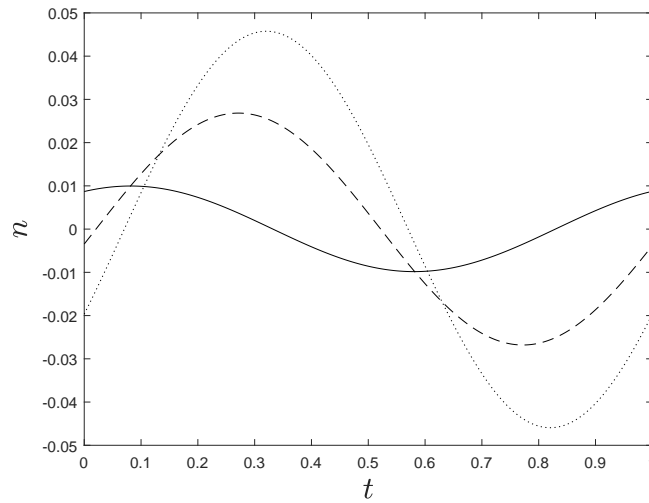
Figure 10.7: The unique periodic solution $n^0 = (n_1^0, n_2^0, n_3^0) \in \mathbb{R}^3$ of the inhomogeneous linear system (10.58) computed using the numerical routine from 10.3.2. The solid black curve is the plot of $n_1^0(t)$, while the dashed and dotted curves are those of $n_2^0(t)$ and $n_3^0(t)$ respectively.

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Chapter 11

Analysis of a SIR model with pulse vaccination and temporary immunity

The content of this chapter appears in "Analysis of a SIR model with pulse vaccination and temporary immunity: stability, bifurcation and a cylindrical attractor" by Church and Liu [22].

11.1   Background and model formulation

Pulse vaccination is a disease control policy under which at certain times, a portion of the population is vaccinated en-masse. The SIR model with pulse vaccination was first introduced by d'Onofrio in 2002 [28]. It has been argued empirically and verified analytically that pulse vaccination might be more effective than continuous vaccination in preventing epidemics that exhibit seasonality, such as measles [1, 81]. Since then, the impact on pulse vaccination has been studied in ever more complex models of disease transmission. For instance, finite infectious periods [98], saturation incidence with latent period and immune period [32], incubation period [66], force of infection by distributed delay [33], nonlinear vaccination [106], quarantine measures [72] and stochastic effects [88] have been considered.

Dynamical analysis of these pulsed vaccination models often include stability criteria for the disease-free equilibrium or periodic orbit, effectively providing a proxy for the basic reproduction number. However, due to the presence of the impulse effect, establishing the existence of an endemic periodic orbit is much more difficult. When there are no delayed terms, methods of bifurcation theory for discrete time systems have been used to

prove the existence of endemic periodic orbits from bifurcations at disease-free states; see [70, 89, 102, 106, 104] for some recent examples. In contrast, when delays are present, most analytical studies prove only permanence when $R_0 > 1$, which means that the disease persists. Numerical simulations are needed to obtain further detail, and this provides only a heuristic description of the orbit structure at a possible bifurcation point. We refer the reader to [32, 66, 72, 98] for examples.

Restricting to SIR models without pulse vaccination specifically, there are many papers that consider various forms of population dynamics and their interplay with delayed effects. Since endemic equilibrium points are often analytically available, Hopf bifurcations can often be studied analytically without the aid of numerical methods. One may consult [29, 49, 50, 62, 95] for some recent examples of this.

Our goal in this chapter is to demonstrate the power of the centre manifold theory and bifurcation theory developed in the previous chapters as they apply to a concrete infectious disease model (this is an Applied Mathematics thesis after all; there should be an application). Our starting point is the model of Kyrychko and Blyuss [56]:

$$\dot{S} = \mu - \mu S - \eta f(I(t))S(t) + \gamma I(t - \tau)e^{-\mu\tau}$$
$$\dot{I} = \eta f(I(t))S(t) - (\mu + \gamma)I(t)$$
$$\dot{R} = \gamma I(t) - \gamma I(t - \tau)e^{-\mu\tau} - \mu R(t).$$

Here, $f(I)$ is a general nonlinear incidence rate, infected individuals clear their infection at rate γ and acquire temporary immunity of length τ, η is a recruitment rate and μ is a natural death rate, with birth rate scaled accordingly so that $N(t) = S(t) + I(t) + R(t)$ approaches unity as $t \to \infty$. The incidence rate is assumed to satisfy the properties: $f(0) = 0$, $f'(0) > 0$, $f''(0) < 0$ and $\lim_{I \to \infty} f(I) = c < \infty$. Kyrychko and Blyuss [56] proved global stability of the disease-free equilibrium when $R_0 < 1$ for arbitrary nonlinear incidence satisfying the previous conditions, and considered the existence and stability of an endemic equilibrium for the particular incidence $f(I) = I/(1 + I)$. They numerically observed Hopf bifurcations at this equilibrium upon varying the immunity period τ. Soon after, Jiang and Wei [50] proved that the endemic equilibrium may indeed undergo a Hopf bifurcation, by taking η as a bifurcation parameter.

We here extend the model of Kyrychko and Blyuss to include pulse vaccination. To do this, we make the following assumptions.

1) At specific instants of time t_k for $k \in \mathbb{Z}$, any individuals that received their vaccine at time $t_k - \tau$ and are still alive lose their immunity and re-enter the susceptible cohort, at which point a fraction $v \in [0, 1)$ of the the total susceptible cohort is vaccinated.

2) Vaccinated individuals are immune to infection for a period τ (the same immunity period as having recovered from infection) and are subject to the same natural death rate μ.

3) The sequence of vaccination times is periodic with shift of τ: there exists $q > 0$ such that $t_{k+q} = t_k + \tau$ for all $k \in \mathbb{Z}$.

The interpretation of 3) is that the period of the pulse vaccination schedule is synchronized with the immunity period. This seems reasonable for seasonal flu epidemics, for example, provided most of the pulse vaccination times are clustered around the beginning of flu season.

Proposition 11.1.1. *With assumptions 1) through 3) in place, the pulse vaccination model takes the following form,*

$$\dot{S} = \mu - \mu S - \eta f(I(t))S(t) + \gamma I(t - \tau)e^{-\mu\tau}, \qquad\qquad t \neq t_k \qquad (11.1)$$
$$\dot{I} = \eta f(I(t))S(t) - (\mu + \gamma)I(t), \qquad\qquad t \neq t_k \qquad (11.2)$$
$$\Delta S = -vS(t^-) + vS(t - \tau)e^{-\mu\tau}, \qquad\qquad t = t_k, \qquad (11.3)$$

where the recovered (R) component is decoupled and has been neglected.

Proof (Derivation). Suppose V_k is the number of individuals that received a vaccine at time $t_k - \tau$. By assumption 3), $V_k e^{-\mu\tau}$ of them are still alive at time t_k. Thus, upon transferring into the susceptible cohort by assumption 1), the total number of susceptible individuals becomes $S_k := S(t_k^-) + V_k e^{-\mu\tau}$. A fraction v of these individuals are vaccinated at time t_k, so there are $(1 - v)S_k$ remaining. We can write the latter as

$$S(t_k) = (1 - v)S_k = S(t_k^-) - vS(t_k^-) + (1 - v)V_k e^{-\mu\tau}. \qquad (11.4)$$

Now, if the total number of susceptible individuals (including those that lost their immunity) at time $t_k - \tau$ is denoted S_k^-, then by 3) we have $S(t_k - \tau) = (1 - v)S_k^-$ and $V_k = vS_k^-$, which together imply $V_k = \frac{v}{1-v}S(t_k - \tau)$. Substituting into (11.4), we have that at time $t = t_k$,

$$\Delta S = S(t_k) - S(t_k^-)$$
$$= -vS(t_k^-) + (1 - v)V_k e^{-\mu\tau}$$
$$= -vS(t_k^-) + vS(t_k - \tau)e^{-\mu\tau},$$

which is precisely equation (11.3). Similarly, the number of vaccinated individuals V satisfies

$$
\begin{aligned}
\dot{V} &= -\mu V, & t &\neq t_k \\
\Delta V &= vS(t_k^-) - vS(t_k - \tau)e^{-\mu\tau}, & t &= t_k.
\end{aligned}
\tag{11.5}
$$

Note that this count of the number of vaccinated individuals is different than the one appearing in Section 11.1.1. In the latter, the component V_j is introduced to circumvent the overlap condition and it does not explicitly track deaths.   □

It is known [50] that the model of Kyrychko and Blyuss can exhibit Hopf bifurcation. Numerically, it appears as though the bifurcating periodic orbit may be globally (excluding the other two equilibria) asymptotically stable. From Section 10.2, we know that Hopf points generically result in cylinder bifurcations when impulses are present. The ramifications of this result to the present model are that, in the presence of pulse vaccination, we expect a bifurcation from an endemic periodic solution to an invariant cylinder. Verifying this hypothesis is our primary goal.

11.1.1   Vaccinated component formalism

We have indicated that it is our goal to complete a bifurcation analysis on the system (11.1)–(11.3). However, there are some technical difficulties associated with this endeavor because the overlap condition is not satisfied, since each of $t_k - \tau$ is an impulse time and equation (11.3) contains delayed terms. While the failure of the overlap condition does not complicate stability analysis, it does complicate the bifurcation analysis. To remedy this, we will at times instead consider the following modification of model (11.1)–(11.3):

$$
\begin{aligned}
\dot{x} &= \mu - \mu x - \eta f(y(t))x(t) + \gamma y(t - \tau)e^{-\mu\tau}, & t &\neq t_k & (11.6) \\
\dot{y} &= \eta f(y(t))x(t) - (\mu + \gamma)y(t), & t &\neq t_k & (11.7) \\
\dot{V}_j &= 0, & t &\neq t_k, & (11.8) \\
\Delta x &= -vx(t^-) + (1 - v)V_j(t^-)e^{-\mu\tau}, & t &= t_{j+qk} & (11.9) \\
\Delta V_j &= vx(t^-) - (1 - ve^{-\mu\tau})V_j(t^-), & t &= t_{j+qk}. & (11.10)
\end{aligned}
$$

In the above impulsive delay differential equation, j ranges from 0 to $q - 1$ where q is the period of the sequence of impulse times as defined in assumption 3). Taking note that $t_{j+qk} = t_{j+q(k-1)} + \tau$ and V is constant except at impulse times where it is continuous from

the right, we see that for $t = t_{j+qk}$,

$$
\begin{aligned}
(1-v)V_j(t^-)e^{-\mu\tau} &= (1-v)V_j(t-\tau)e^{-\mu\tau} \\
&= (1-v)[vx((t-\tau)^-) + ve^{-\mu\tau}V_j((t-\tau)^-)] \\
&= v[(1-v)x((t-\tau)^-) + (1-v)V_j((t-\tau)^-)e^{-\mu\tau}] \\
&= vx(t-\tau).
\end{aligned}
$$

Substituting the above into the jump condition for x, the result is

$$
\Delta x = -vx(t^-) + (1-v)V_j(t^-)e^{-\mu\tau} = -vx(t^-) + vx(t-\tau)e^{-\mu\tau}.
$$

This is precisely the same functional form as the jump condition (11.3) for the original model. Since the continuous-time dynamics are identical for both models, we can analyze bifurcations in (11.1)–(11.3) by equivalently studying bifurcations in the model (11.6)–(11.10) with explicit vaccinated components.

11.2 Disease-free periodic solution

In this section we will complete a thorough investigation of the local properties of disease-free states, namely their stability and bifurcations. For part of this section, the number of vaccination moments q per period will remain an arbitrary natural number. However, we will eventually specialize to the case where $q = 1$. Without loss of generality, we will assume $t_0 = 0$.

11.2.1 Existence and stability

When there is no disease – that is, on the invariant subspace $\{(S, I) : I = 0\}$ – the nontrivial dynamics are determined solely by the linear, nonhomogeneous impulsive system

$$
\begin{aligned}
\dot{z} &= -\mu z + \mu, & t &\neq t_k & (11.11) \\
\Delta z &= -vz(t^-) + vz(t-\tau)e^{-\mu\tau}, & t &= t_k. & (11.12)
\end{aligned}
$$

By the variation of constants formula (Theorem 3.3.1), every solution $z(t)$ passing through an initial condition $\phi \in \mathcal{RCR}$ at time $t = 0$ can be written

$$
z_t = U(t,0)\phi + \int_0^t U(t,s)\chi_0\mu ds,
$$

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where the integral is a weak integral and $U(t,s)$ is the evolution family associated to the homogeneous equation

$$\dot{w} = -\mu w, \qquad\qquad\qquad t \neq t_k \qquad\qquad (11.13)$$

$$\Delta w = -vw(t^-) + vw(t - \tau)e^{-\mu\tau}, \qquad\qquad t = t_k. \qquad\qquad (11.14)$$

Lemma 11.2.1. *Suppose the trivial solution of the homogeneous equation* (11.13)–(11.14) *is exponentially stable. Then, the system* (11.1)–(11.3) *has a unique disease-free periodic solution* $(\tilde{S}, 0)$, *with period* τ.

Proof. From the variation of constants formula, define the linear operator

$$V : \phi \mapsto U(\tau, 0)\phi + \int_0^\tau U(\tau, s)\chi_0\mu ds$$

on $\mathcal{RCR}$. If the trivial solution of (11.13)–(11.14) is exponentially stable, then $||U(t,s)|| \leq Ke^{-\alpha(t-s)}$ for some $\alpha > 0$ and $K \geq 1$, for all $t \geq s$. The periodicity $U(t+\tau, s+\tau) = U(t,s)$ of the evolution family, continuity and the cocycle property $U(t,s) = U(t,v)U(v,s)$ for $s \leq v \leq t$ implies that the nth iterate of V satisfies the inequality

$$||V^n\phi - V^n\psi|| \leq Ke^{-\alpha\tau n}||\phi - \psi||.$$

Consequently, $V : \mathcal{RCR} \to \mathcal{RCR}$ is an eventual contraction and has a unique fixed point, which we denote ϕ_v. From the variation of constants formula, it follows that with

$$t \mapsto \tilde{S}(t, v) = [U(t, 0)\phi_v](0) + \int_0^t [U(t, s)\chi_0\mu](0)ds,$$

$(\tilde{S}, 0)$ is the claimed disease-free periodic solution. $\qquad\qquad\qquad\qquad \square$

Lemma 11.2.2. *The trivial solution of the homogeneous equation* (11.13)–(11.14) *is exponentially stable.*

Proof. To verify the exponential stability of the trivial solution, it is enough by Corollary 4.5.1.1 for us to show that all Floquet exponents have negative real part. Let $w(t) = \phi(t)e^{\lambda t}$ be a solution of (11.13)–(11.14) with ϕ periodic. Substituting this ansatz into the dynamical system, using the periodicity condition $\phi(t) = \phi(t - \tau)$ and cancelling exponentials, we arrive at the following impulsive differential equation for ϕ:

$$\dot{\phi} + \lambda\phi = -\mu\phi, \qquad\qquad\qquad t \neq t_k \qquad\qquad (11.15)$$

$$\Delta\phi = -v\phi(t^-) + v\phi(t)e^{-(\mu+\lambda)\tau}, \qquad\qquad t = t_k. \qquad\qquad (11.16)$$

The second equation is an implicit jump condition, but we can easily rearrange it to obtain the explicit condition

$$\phi(t_k) = \frac{1-v}{1-ve^{-(\mu+\lambda)\tau}}\phi(t_k^-).$$

Calculating the solution of the above impulsive differential equation at time τ given an initial condition at time $t = 0$, one obtains

$$\phi(\tau) = e^{-(\mu+\lambda)\tau}\left(\frac{1-v}{1-ve^{-(\mu+\lambda)\tau}}\right)^q \phi(0) := D(\lambda)\phi(0).$$

ϕ is periodic provided $\phi(\tau) = \phi(0)$, so we are left with describing the location of the solutions of the transcendental equation $D(\lambda) = 1$. Defining $z = e^{-\lambda\tau}$, it follows that λ is a solution of $D(\lambda) = 1$ if and only if z is a solution of

$$0 = f(z) + g(z),$$
$$f(z) = 1,$$
$$g(z) = -ze^{-\mu\tau}\left(\frac{1-v}{1-ve^{-\mu\tau}z}\right)^q.$$

We will show that $|g(z)| < |f(z)|$ on the unit circle $|z| = 1$. We have

$$|g(z)| = e^{-\mu\tau}\left(\frac{1-v}{|1-ve^{-\mu\tau}z|}\right)^q \le e^{-\mu\tau}\left(\frac{1-v}{|1-|ve^{-\mu\tau}z||}\right)^q$$
$$= e^{-\mu\tau}\left(\frac{1-v}{1-ve^{-\mu\tau}}\right)^q \le e^{-\mu\tau} < 1 = |f(z)|,$$

as claimed. By Rouché's theorem, the equation $f(z) + g(z) = 0$ has no solutions satisfying $|z| \le 1$. Consequently, there are no Floquet exponents λ satisfying the inequality $|e^{-\lambda\tau}| \le 1$. We conclude that all Floquet exponents have negative real part and the proof is complete. $\square$

As a consequence of Lemma 11.2.1 and Lemma 11.2.2, we are guaranteed a unique disease-free periodic solution that, in the absence of infection, is globally exponentially stable.

Corollary 11.2.0.1. *The model* (11.1)–(11.3) *has a unique disease-free periodic solution* $t \mapsto (\tilde{S}(t,v),0)$ *of period* τ. *Restricted to the disease-free subspace* $D_0 = \{(S,I) : I = 0\}$, *this periodic solution is globally exponentially stable.*

Next, we will determine the stability of this periodic solution in the nonlinear system (11.1)–(11.3). Introduce the basic reproduction number

$$R_0 = \frac{\eta f'(0)}{\tau(\gamma + \mu)} \int_0^\tau \tilde{S}(t, v) dt. \tag{11.17}$$

Note that if one denotes the average of $\tilde{S}$ over the interval $[0, \tau]$ by $[\tilde{S}]$, then one can equivalently write the basic reproduction number in the more suggestive form

$$R_0 = \frac{\eta f'(0)[\tilde{S}]}{\gamma + \mu}.$$

Then, the interpretation is that R_0 is the product of the average number of susceptibles, multiplied by the small-infection (i.e. near $I = 0$) incidence rate, divided by the aggregate rate of leaving the infected class through death or clearance of the infection.

Lemma 11.2.3. $R_0 = 1$ *is an epidemiological threshold: if* $R_0 < 1$*, the disease-free periodic solution is locally asymptotically stable, while if* $R_0 > 1$ *it is unstable.*

Proof. The linearization at $(\tilde{S}, 0)$ produces the linear homogeneous impulsive system

$$\begin{aligned}
\dot{u}_1 &= -\mu u_1(t) - \eta f'(0)\tilde{S}(t, v)u_2(t) + \gamma e^{-\mu\tau}u_2(t - \tau), & t \neq t_k \\
\dot{u}_2 &= \eta f'(0)\tilde{S}(t, v)u_2(t) - (\gamma + \mu)u_2(t), & t \neq t_k \\
\Delta u_1 &= -v u_1(t^-) + v u_1(t - \tau)e^{-\mu\tau}, & t = t_k.
\end{aligned}$$

Notice that the second equation is decoupled from the first. Taking an ansatz Floquet eigensolution $u(t) = \phi(t)e^{\lambda t}$, we can examine the second component independently. Indeed, $\phi = [\ \phi_1 \ \ \phi_2\ ]^{\mathsf{T}}$ satisfies

$$\dot{\phi}_2 + \lambda\phi_2 = \eta f'(0)\tilde{S}(t, v)\phi_2 - (\gamma + \mu)\phi_2.$$

If $\phi_2 \neq 0$, then as ϕ is assumed to be periodic with period τ, the only possible Floquet exponent in this case is

$$\lambda_0 = -(\gamma + \mu) + \frac{\eta f'(0)}{\tau} \int_0^\tau \tilde{S}(t, v) dt. \tag{11.18}$$

Conversely, if $\phi_2 = 0$, then ϕ_1 and λ must satisfy (11.15)–(11.16). But it is already known that all Floquet exponents λ associated to this equation have negative real part; see Lemma 11.2.2. Consequently, the Floquet spectrum includes the special Floquet exponent λ_0 and the remainder with strictly negative real part. The equilibrium is locally asymptotically stable provided all Floquet exponents have negative real part, and is unstable if at least one has positive real part; see Theorem 9.4.1. Since λ_0 is real and the others are guaranteed to have negative real part, we obtain the conclusion of the lemma by noticing that $\lambda_0 < 0$ is equivalent to $R_0 < 1$ and that $\lambda_0 > 0$ is equivalent to $R_0 > 1$. $\qquad\square$

11.2.2 Transcritical bifurcation to endemic periodic solution

Before we can study bifurcations, we must establish the existence of a bifurcation point.

Lemma 11.2.4. *Consider the critical Floquet exponent $\lambda_0 = \lambda_0(v)$ as defined in equation (11.18). λ_0 is strictly decreasing. As consequence, if $\lambda_0(0)\lambda_0(1) \leq 0$, there is a unique $v^* \in [0,1]$ such that $\lambda_0(v^*) = 0$; that is, a critical vaccination coverage v^* at which $R_0 = 1$.*

Proof. Note that, given the explicit form of λ_0, it is enough to prove that $v \mapsto \tilde{S}(t,v)$ is decreasing for all $t \in [0,\tau]$. To accomplish this, we recall that $\tilde{S}$ is the unique periodic solution of (11.11)–(11.12). By a similar argument to the proof of Lemma 11.2.2, we can show that the jump condition can be simplified, and that $\tilde{S}$ is in fact the unique periodic solution of the impulsive differential equation without delay

$$
\begin{aligned}
\dot{z} &= \mu - \mu z, & t \neq t_k \\
\Delta z &= (\rho(v) - 1)z(t^-), & t = t_k,
\end{aligned}
$$

where $\rho(v) = (1-v)/(1 - ve^{-\mu\tau})$. If one denotes $t \mapsto z(t; z_0, v)$ the unique solution of the above impulsive differential equation for vaccination coverage v and initial condition $z(0; z_0, v) = z_0$, it is not difficult to show that $\frac{d\rho}{dv} < 0$ and, subsequently, that $\frac{\partial z}{\partial v} \leq 0$ for all $t \geq 0$. Also, one has $\frac{\partial z}{\partial z_0} > 0$ for all $t \geq 0$. Using the variation of constants formula for impulsive differential equations, routine calculations yield

$$
z(\tau; z_0, v) = e^{-\mu\tau}\rho(v)^q \left[z_0 + \sum_{i=1}^{q} \rho(v)^{1-i}(e^{\mu t_i} - e^{\mu t_{i-1}}) \right],
$$

from which we can compute the initial condition $\tilde{S}(0,v)$ of the disease-free periodic orbit, by solving the equation $z(\tau; \tilde{S}(0,v), v) = \tilde{S}(0,v)$. The result is

$$
\tilde{S}(0,v) = \frac{e^{-\mu\tau}}{1 - \rho(v)^q e^{-\mu\tau}} \sum_{k=1}^{q} \rho(v)^{q+1-i}(e^{\mu\tau_i} - e^{\mu\tau_{i-1}}),
$$

which indeed satisfies $\frac{d}{dv}\tilde{S}(0,v) < 0$. Since $\tilde{S}(t,v) = z(t; \tilde{S}(0,v), v)$, one may conclude from the chain rule that $\frac{d}{dv}\tilde{S}(t,v) < 0$ for all $t \in [0,\tau]$, so $v \mapsto \lambda_0(v)$ is decreasing. The conclusions about the critical vaccination coverage v^* follow by the intermediate value theorem. $\square$

There are several choices we can make for the bifurcation parameter. Mathematically the easiest ones to deal with are the model parameters γ, μ and η, as these appear linearly in the model and in the expression for the important Floquet exponent λ_0 in equation (11.18). Biologically, a natural choice is the vaccine coverage, v, since this is a parameter that can in principle be controlled. It is more difficult to state closed-form results for bifurcations in terms of the vaccine coverage, so for this reason we will simplify matters and assume that $q = 1$, so there is one vaccination pulse per period. That is, the sequence of impulse times is precisely $t_k = k\tau$ for $k \in \mathbb{Z}$. Then, from the previous section, we can explicitly calculate

$$\tilde{S}(t, v) = 1 - ve^{-\mu[t]_\tau}, \tag{11.19}$$

which implies that $\tilde{S}(\tau^-, v) = 1 - ve^{-\mu\tau}$ and $\tilde{S}(0, v) = 1 - v$. We can also explicitly calculate the critical vaccination coverage where $R_0 = 1$. We find

$$v^* = \frac{\mu\tau}{1 - e^{-\mu\tau}}\left(1 - \frac{\gamma + \mu}{\eta f'(0)}\right). \tag{11.20}$$

As a consequence, we have the following preliminary stability result.

Lemma 11.2.5. *If there is $q = 1$ vaccination pulse per period, the disease-free periodic solution is locally asymptotically stable provided $v > v^*$, and unstable if $v < v^*$.*

We will now pass to the equivalent system with vaccinated component (11.6)–(11.10). Define the changes of variables and parameters

$$X + \tilde{S}(\cdot, v) = x, \qquad V + \frac{v\tilde{S}(\tau^-, v)}{1 - ve^{-\mu\tau}} = V_0, \qquad Y = y, \qquad \epsilon + v^* = v.$$

The result is the following system of impulsive delay differential equations:

$$\begin{aligned}
\dot{X} &= -\mu X(t) + \eta f(Y)\big[\tilde{S}(t, v^* + \epsilon) + X\big] + \gamma Y(t - \tau)e^{-\mu\tau}, & t \neq k\tau \\
\dot{Y} &= \eta f(Y)\big[\tilde{S}(t, v^* + \epsilon) + X\big] - (\mu + \gamma)Y(t), & t \neq k\tau \\
\dot{V} &= 0, & t \neq k\tau \\
\dot{\epsilon} &= 0, & t \neq k\tau \\
\Delta X &= -(v^* + \epsilon)X(t^-) + (1 - (v^* + \epsilon))e^{-\mu\tau}V(t^-), & t = k\tau \\
\Delta Y &= 0, & t = k\tau \\
\Delta V &= (v^* + \epsilon)X(t^-) - (1 - (v^* + \epsilon)e^{-\mu\tau})V(t^-), & t = k\tau \\
\Delta\epsilon &= 0, & t = k\tau.
\end{aligned} \tag{11.21}$$

Notice that $(X, Y, V, \epsilon) = (0, 0, 0, \epsilon)$ is an equilibrium whenever $v^* + \epsilon \in [0, 1]$. The change of variables has had the effect of translating the disease-free periodic solution to the origin.

The next step is to linearize the above system at a candidate nonhyperbolic equilibrium. The origin is expected to be nonhyperbolic with a pair of Floquet exponents with zero real part, with the first zero exponent resulting from the nonhyperbolicity of $\tilde{S}$ at the critical vaccination coverage $v = v^*$, and the second zero exponent coming from the trivial dynamics equation for the parameter ϵ. The result is

$$
\begin{aligned}
\dot{u}_1 &= -\mu u_1(t) - \eta f'(0)\tilde{S}(t, v^*)u_2(t) + \gamma u_2(t - \tau)e^{-\mu\tau}, & t \neq k\tau \\
\dot{u}_2 &= \eta f'(0)\tilde{S}(t, v^*)u_2(t) - (\gamma + \mu)u_2(t), & t \neq k\tau \\
\dot{u}_3 &= 0, & t \neq k\tau \\
\dot{u}_4 &= 0, & t \neq k\tau \\
\Delta u_1 &= -v^* u_1(t^-) + (1 - v^*)e^{-\mu\tau}u_3(t^-), & t = k\tau \\
\Delta u_2 &= 0, & t = k\tau \\
\Delta u_3 &= v^* u_1(t^-) - (1 - v^* e^{-\mu\tau})u_3(t^-), & t = k\tau \\
\Delta u_4 &= 0, & t = k\tau.
\end{aligned}
\tag{11.22}
$$

Before we characterize the centre fiber bundle, we introduce a few convenience functions that will be useful both in this and subsequent sections. Define

$$
\beta(t, s; \alpha) = \exp\left(\int_s^t (-\gamma - \mu + \eta f'(0)\tilde{S}(u + \alpha, v^*))du\right).
$$

Then define the matrix $\mathbf{Z}_1(t, s; z, \alpha) \in \mathbb{C}^{2 \times 2}$ for $t \geq s$ and $z \in \mathbb{C} \setminus \{0\}$ by

$$
\mathbf{Z}_1(t, s; z, \alpha) = \begin{bmatrix} e^{-\mu(t-s)} & \int_s^t e^{-\mu(t-u)}(-\eta f'(0)\tilde{S}(u + \alpha, v^*) + \frac{1}{z}\gamma e^{-\mu\tau})\beta(u, s; \alpha)du \\ 0 & \beta(t, s; \alpha) \end{bmatrix}.
$$

Then, set $\mathbf{Z}(t, s; z, \alpha) = \text{diag}(\mathbf{Z}_1(t, s; z, \alpha), I_{2 \times 2})$. Also define the matrix $B \in \mathbb{R}^{4 \times 4}$:

$$
B = \begin{bmatrix} 1 - v^* & 0 & (1 - v^*)e^{-\mu\tau} & 0 \\ 0 & 1 & 0 & 0 \\ v^* & 0 & v^* e^{-\mu\tau} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Finally, the function β satisfies a few useful identities. They are clear from its definition:

$$
\begin{aligned}
\beta(t, s; \alpha) &= \beta(t, s, [\alpha]_\tau), \\
\beta(t, s; \alpha) &= \beta(t + \tau, s + \tau; \alpha), \\
\beta(t, s; \alpha) &= \beta(t + \alpha, s + \alpha; 0).
\end{aligned}
$$

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For convenience, we abuse notation and write $\beta(t) = \beta(t, 0; 0)$.

Since we have already determined that the dominant Floquet exponent of (11.1)–(11.3) at the disease-free periodic solution must be real – see Lemma 11.2.3 – we take the ansatz that $u(t)$ is periodic with period τ. As a consequence, $u_2(t - \tau) = u_2(t)$, and (11.22) reduces to an ordinary impulsive differential equation. If we denote $X(t, s)$ the Cauchy matrix of the resulting system, then $M = X(\tau, 0)$ is a monodromy matrix. Specifically, $M = B\mathbf{Z}(\tau, 0; 1, 0)$;

$$
M = \begin{bmatrix}
(1 - v^*)e^{-\mu\tau} & (1 - v^*)\kappa & (1 - v^*)e^{-\mu\tau} & 0 \\
0 & 1 & 0 & 0 \\
v^*e^{-\mu\tau} & v^*\kappa & v^*e^{-\mu\tau} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \kappa = e_1^\mathsf{T}\mathbf{Z}_1(\tau, 0; 1, 0)e_2.
$$

The eigenvalues are 1, 0 and $e^{-\mu\tau}$. The periodic solutions are generated by the two-dimensional generalized eigenspace associated to the eigenvalue 1. The eigenvectors are $m_1 = \begin{bmatrix} (1 - v^*)\kappa & 1 - e^{-\mu\tau} & v^*\kappa & 0 \end{bmatrix}^\mathsf{T}$ and $m_2 = e_4$. As consequence, we can completely describe the centre fiber bundle.

Lemma 11.2.6. *The centre fiber bundle, $\mathcal{RCR}_c$, associated to the nonhyperbolic equilibrium $0 \in \mathbb{R}^4$ of the system (11.21), is two-dimensional. A basis matrix Φ_t, whose columns form a basis for the t-fiber $\mathcal{RCR}_c(t)$, is periodic with period τ and is given explicitly by*

$$
\Phi_t(\theta) = \mathbf{Z}([t + \theta]_\tau, 0; 1, 0)\begin{bmatrix}
(1 - v^*)\kappa & 0 \\
1 - e^{-\mu\tau} & 0 \\
v^*\kappa & 0 \\
0 & 1
\end{bmatrix} := \begin{bmatrix} \Phi_{t,1}(\theta) & 0_{3\times 1} \\ 0 & 1 \end{bmatrix},
$$

where $\Phi_{t,1}(\theta) \in \mathbb{R}^3$.

Another ingredient necessary in the centre manifold reduction concerns the projection of χ_0 onto the centre fiber bundle. Specifically, if $P_c(t) : \mathcal{RCR} \to \mathcal{RCR}_c(t)$ denotes the spectral projection, then there exists a unique $Y(t) \in \mathbb{R}^{2\times 4}$ such that $P_c(t)\chi_0 = \Phi_t Y(t)$. It is characterized as the solution of the equation

$$
\Phi_t Y(t) = \frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t)^{-1}\chi_0 dz \tag{11.23}
$$

where V_t denotes the monodromy operator associated to the linear delay impulsive system (11.22), and Γ_1 is a simple counterclockwise-oriented closed contour in $\mathbb{C}$ such that 1 is

the only eigenvalue of V_t contained in the closure of its interior. We must compute $Y(t)$. Therefore, to proceed we solve the equation

$$zy - V_t y = \chi_0 \xi \tag{11.24}$$

for $y \in \mathcal{RCR}$, with $\xi \in \{e_1, e_2, e_3, e_4\}$. Our first task will be to obtain a representation of $V_t y$. We start by repartitioning the dynamics of (11.22) in terms of matrices. This system can equivalently written

$$
\begin{aligned}
\dot{u} &= A(t)u(t) + f(t), & t \neq k\tau \\
\Delta u &= (B - I)u(t^-), & t = k\tau, \\
A(t) &= -\mu(E_{11} + E_{22}) + \eta f'(0)\tilde{S}(t, v^*)(-E_{21} + E_{12}) - \gamma E_{22},
\end{aligned}
$$

with standard basis matrices $E_{ij} = e_i e_j^\mathsf{T} \in \mathbb{R}^{4 \times 4}$ and $f(t) = \gamma e^{-\mu\tau} E_{12} u(t - \tau)$. Note that we have treated the delayed term as a nonhomogeneous forcing. If $U_0(t, s)$ denotes the Cauchy matrix associated to the (formally) homogeneous equation (without delays), we can use the variation of constants formula to write

$$u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, r)\gamma e^{-\mu\tau} E_{12} u(r - \tau)dr.$$

Since $V_t y(\theta) = u(t + \tau + \theta; t, y)$ where $u(\cdot; t, y)$ denote the solution with initial condition $(t, y) \in \mathbb{R} \times \mathcal{RCR}$, we obtain the representation

$$V_t y(\theta) = U_0(t + \tau + \theta, t)y(0) + \int_0^{\tau+\theta} U_0(t + \tau + \theta, t + r)\gamma e^{-\mu\tau} E_{12} y(r - \tau)dr \tag{11.25}$$

for $\theta \in [-\tau, 0]$, after a few changes of variables.

Returning to equation (11.24), we notice that $zy(\theta) = V_t y(\theta)$ for $\theta < 0$. From the above representation, it follows that $\theta \mapsto V_t y(\theta)$ is differentiable except at times $\theta \in (-\tau, 0]$ where $t + \tau + \theta = k\tau$ for some $k \in \mathbb{Z}$, where it is continuous from the right. At $\theta = 0$, there is an external discontinuity because of the $\chi_0 \xi$ term in (11.24). Taking this into account, we can take derivatives in θ on both sides of $zy(\theta) = V_t y(\theta)$, and compute jumps at those times where $\theta = -[t]_\tau$. We find that $y(\theta)$ is a solution of

$$
\begin{aligned}
y' &= [A(t + \theta) + \frac{1}{z}\gamma e^{-\mu\tau} E_{12}]y, & \theta \neq -[t]_\tau \tag{11.26} \\
\Delta y &= (B - I)y(\theta^-), & \theta = -[t]_\tau. \tag{11.27}
\end{aligned}
$$

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for $\theta \in [-\tau, 0)$. Using the convenience function $\mathbf{Z}$ from earlier, we can explicitly write

$$y(\theta) = \begin{cases} \mathbf{Z}(\theta, -\tau; z, t)y(-\tau), & \theta < -[t]_\tau \\ \mathbf{Z}(\theta, -[t]_\tau; z, t)B\mathbf{Z}(-[t]_\tau, -\tau; z, t)y(-\tau), & \theta \geq -[t]_\tau \end{cases} \tag{11.28}$$

Since $y(-\tau)$ appears linearly on the right-hand side of the above, we will write it as a matrix product

$$y(\theta) = \mathbf{H}(\theta; z, t)y(-\tau) \tag{11.29}$$

Next, from (11.24) we have $zy(0) - V_t y(0) = \xi$. It is our goal to compute $y(0)$, and to facilitate this we consider two separate cases. If $[t]_\tau = 0$, then we have $V_t y(0) = BV_t y(0^-)$, as can be verified via equation (11.25). Since $V_t(\theta) = zy(\theta)$ for $\theta < 0$, it then follows that $V_t y(0) = Bzy(0^-)$. The equation $zy(0) - V_t y(0) = \xi$ is then equivalent to $zy(0) - Bzy(0^-) = \xi$. A similar argument in the case where $[t]_\tau \neq 0$ then implies that, in both cases, the end result is

$$y(0) = \frac{1}{z}\xi + \mathbf{H}(0^-; z, t)y(-\tau) \tag{11.30}$$

Our final task is to solve for $y(-\tau)$. To do this, substitute (11.30) into (11.25) and set $\theta = -\tau$. Since $V_t(-\tau) = zy(-\tau)$, the result is

$$zy(-\tau) = \frac{1}{z}\xi + \mathbf{H}(0^-; z, t)y(-\tau). \tag{11.31}$$

Lemma 11.2.7. $z \mapsto (zI - \mathbf{H}(0^-; z, t))^{-1}$ *has a pole at* $z = 1$. *In particular,* 1 *is an eigenvalue of multiplicity two for* $\mathbf{H}(0^-; z, t)$.

Proof. The spectrum (as a multiset) is found to be

$$\sigma\big(\mathbf{H}(0^-; z, t)\big) = \{e^{-\mu\tau}, \beta(0, -\tau; t), 0, 1\}.$$

The second eigenvalue in the list is, explicitly,

$$\beta(0, -\tau; t) = \exp\left(\int_{-\tau}^{0} (-\gamma - \mu + \eta f'(0)\tilde{S}(u + t, v^*))du\right),$$

which is equal to 1 because the integrand is periodic with period τ, integrates to zero on $[0, \tau]$, and t acts as a translation parameter. The result follows. $\qquad\square$

We can now calculate $y = (zI - V_t)^{-1}\chi_0$. Solving equation (11.31) and substituting the result into (11.29), the following lemma is proven.

Lemma 11.2.8. $(zI - V_t)^{-1}\chi_0$ *has the explicit form*

$$(zI - V_t)^{-1}\chi_0(\theta) = \frac{1}{z}\mathbf{H}(\theta; z, t)(zI - \mathbf{H}(0^-; z, t))^{-1}. \tag{11.32}$$

The next step is to explicitly calculate the contour integral in (11.23). The following lemma provides just enough detail for later calculations.

Lemma 11.2.9. *There exist real constants a, b such that*

$$\frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t)^{-1}\chi_0 = \mathbf{H}(\theta; 1, t)\begin{bmatrix} 0 & ab & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11.33}$$

Proof. We provide only an outline of the proof of this lemma. To begin, perform the diagonalization

$$(zI - \mathbf{H}(0^-; z, t)) = P(z)(zI - D)P(z)^{-1}$$

where $D = \mathrm{diag}(0, e^{-\mu\tau}, 1, 1)$ has the eigenvalues of $\mathbf{H}(0^-; z, t)$ on the diagonal. With this representation, both P and P^{-1} are holomorphic in a neighbourhood of $z = 1$. After lengthy calculations, one can show that

$$P = \begin{bmatrix} 1 & 1 & P_{13}(z) & 0 \\ 0 & 0 & P_{23}(z) & 0 \\ -e^{\mu[t]\tau} & \frac{v^*}{1-v^*}e^{\mu[t]\tau} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for some P_{13} and P_{23}, with $P_{23} \neq 0$ in a neighbourhood of $z = 1$. Taking into account (11.32), we have

$$\frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t)^{-1}\chi_0 = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{z}\mathbf{H}(\theta; z, t)P(z)(zI - D)^{-1}P(z)^{-1}dz$$

$$= \mathbf{H}(\theta; 1, t)P(1)\mathrm{diag}(0, 0, 1, 1)P(1)^{-1},$$

with the second line being a consequence of Cauchy's integral formula. Explicitly calculating the product $P(1)\mathrm{diag}(0, 0, 1, 1)P(1)^{-1}$, the result is the matrix on the right-hand side of (11.33) with $a = 1/P_{23}(1)$ and $b = P_{13}(1)$. $\qquad\square$

Lemma 11.2.10. *The matrix $Y(t)$ appearing in the decomposition* (11.23) *is*

$$Y(t) = \begin{bmatrix} 0 & (1 - e^{-\mu\tau})^{-1}\beta(-[t]_\tau, -\tau; t) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11.34}$$

Proof. Since the matrix $Y(t)$ appearing in (11.23) is unique and therefore independent of the argument $\theta \in [-\tau, 0]$, we can evaluate both sides of the equation at $\theta = -[t]_\tau$ to simplify the computation. Using Lemma 11.2.9 and Lemma 11.2.10, the result is the equation

$$\begin{bmatrix} (1 - v^*)\kappa & 0 \\ 1 - e^{-\mu\tau} & 0 \\ v^*\kappa & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \end{bmatrix} = \mathbf{H}(-[t]_\tau; 1, t) \begin{bmatrix} 0 & ab & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Explicitly calculating $\mathbf{H}(-[t]_\tau; 1, t)$, one immediately finds that the only nonzero entries of Y are Y_{12} and Y_{24}, the latter of which is $Y_{24} = 1$. The Y_{12} entry satisfies the equation

$$Y_{12} \begin{bmatrix} (1 - v^*)\kappa \\ 1 - e^{-\mu\tau} \\ v^*\kappa \\ 0 \end{bmatrix} = \mathbf{H}(-[t]_\tau; 1, t) \begin{bmatrix} ab \\ 1 \\ a \\ 0 \end{bmatrix}.$$

Comparing the entries in the second row, we find $Y_{12} \cdot (1 - e^{-\mu\tau}) = \beta(-[t]_\tau, -\tau; t)$, and the result follows. $\qquad\square$

Next, we can determine the quadratic-order dynamics on the centre manifold. The following is a direct consequence of Theorem 8.1.2.

Lemma 11.2.11. *The coordinate dynamics on the two-dimensional parameter-dependent centre manifold of the nonhyperbolic equilibrium $0 \in \mathbb{R}^4$ of the impulsive delay differential equation* (11.21) *are, for $||(w, \epsilon)||$ sufficiently small,*

$$\dot{w} = \eta(1 - e^{-\mu\tau})\beta(-[t]_\tau, -\tau; t)\left(g(t)w^2 + f'(0)\partial_v\tilde{S}(t, v^*)\epsilon w\right) + R(t, w, \epsilon),$$

$$\dot{\epsilon} = 0,$$

$$g(t) = \tilde{S}(t, v^*)(1 - e^{-\mu\tau})\beta(t)\left(\frac{1}{2}f''(0)(1 - e^{-\mu\tau})\beta(t)\right. \tag{11.35}$$

$$\left. + f'(0)\int_0^{[t]_\tau} e^{-\mu([t]_\tau - s)}(\gamma e^{-\mu\tau} - \eta f'(0)\tilde{S}(s, v^*))\beta(s)ds\right),$$

where $R(t, w, \epsilon)$ satisfies $R(t, 0, \epsilon) = 0$, is periodic and right-differentiable in its first argument, and is C^∞ in (w, ϵ) for fixed t.

With this lemma in place, we can state and prove our bifurcation theorem. It mostly follows from Theorem 10.1.1, although we will elaborate on some details.

Theorem 11.2.1. *For a generic set of parameters, a transcritical bifurcation occurs in the model* (11.1)–(11.3) *along the disease-free periodic solution as v crosses through the critical vaccination coverage level v^*. Specifically*

$$\ell = \int_0^\tau \beta(-[t]_\tau, -\tau; t)\tilde{S}(t, v^*)\beta(t)g(t)dt,$$

is nonzero on a generic subset of parameter space, and the following is satisfied for $|v - v^*|$ small enough and in a sufficiently small neighbourhood of $(S, I) = (\tilde{S}(t, v^*), 0)$.

- *There are at most two periodic solutions: the disease-free solution and a second solution $t \mapsto \xi(t, v)$ that is exponentially stable when $v < v^*$, unstable when $v > v^*$, and satisfies $\xi(t, v^*) = (\tilde{S}(t, v^*), 0)$.*

- *The unique periodic solution is conditionally stable when $v = v^*$ in some half-space.*

- *$\xi(\cdot, v)$ is positive (in both components) if and only if $(v - v^*)\ell > 0$.*

Proof. The time τ (Poincaré) map associated to the ordinary differential equation (11.35) is readily found to satisfy

$$w \mapsto w + \eta(1 - e^{-\mu\tau})[\ell w^2 + m\epsilon w] + h(w, \epsilon)$$
$$\epsilon \mapsto \epsilon,$$

where ℓ is as in the statement of the theorem, m is given by

$$m = \int_0^\tau \beta(-[t]_\tau, -\tau; t)f'(0)\partial_v\tilde{S}(t, v^*)dt,$$

and $h(w, \epsilon) = \int_0^\tau R(t, w, \epsilon)dt$ is a C^∞ remainder satisfying $h(0, \epsilon) = 0$ and containing all terms of order 3 and above in (w, ϵ). Note that the mixed ϵw term, m, is strictly negative because $f'(0) > 0$, $\beta > 0$, and $\partial_v\tilde{S}(t, v^*) < 0$. As for the quadratic term, the equation $\ell = 0$ is unstable with respect to perturbations in $f''(0)$, as can be verified by the functional form of $g(t)$ appearing in (11.35). Consequently, on a generic set of parameters we have $\ell \neq 0$ and $m < 0$. The first two conclusions now follow from Theorem 10.1.1.

To see that $\xi(\cdot, v)$ is positive only when $(v - v^*)\ell = \epsilon\ell > 0$, we first remark that the fixed point satisfies the estimate $w(\epsilon) = -\epsilon\frac{m}{\ell} + O(\epsilon^2)$. This follows because of the properties of the remainder term h. Also, since $\xi(t, v) \to (\tilde{S}(t, v^*), 0)$ as $v \to v^*$, it suffices to consider only the sign of the second component. This is precisely

$$
\begin{aligned}
\mathrm{sign}(\xi_2(t, v^* + \epsilon)) &= \mathrm{sign}\left(-\epsilon\frac{m}{\ell}e_2^{\mathsf{T}}\Phi_{t,1}(0)\right) \\
&= \mathrm{sign}\left(\epsilon\ell(1 - e^{-\mu\tau})\beta(t)\right) \\
&= \mathrm{sign}(\epsilon\ell),
\end{aligned}
$$

which is what was claimed. □

Remark 11.2.1. *We would typically expect $\ell < 0$ for biological reasons. Namely, $\ell < 0$ would imply that increasing the vaccination coverage to the critical level v^* drives a stable endemic (ie. positive) periodic solution toward the disease-free state. However, proving that ℓ is indeed negative on the entire parameter space for which $v^* \in [0, 1]$ seems to be rather difficult.*

11.3 Numerical bifurcation analysis

In the previous section we proved that in the event there is only one vaccination pulse per period, the disease-free periodic orbit generically undergoes a transcritical bifurcation when the vaccination coverage crosses a critical threshold. In the absence of pulse vaccination, the model (11.1)–(11.3) reduces to the SIR model of Kyrychko and Blyuss, and it is known that the endemic equilibrium can undergo Hopf bifurcation [50]. We expect that with pulse vaccination, such a Hopf bifurcation should be replaced with a cylinder bifurcation. It is therefore natural for us to track the bifurcating endemic periodic orbit and search for Hopf points. In this section, we will use the illustrative parameter choices provided in Table 11.1, and to keep results consistent with the analysis appearing in [50, 56] we will use the indence rate $f(x) = \frac{x}{1+x}$. For these parameters and incidence rates, the model of Kyrychko and Blyuss (i.e. the model without vaccination) has a disease-free equilibrium that is unstable, as well as a periodic solution that is locally asymptotically stable. It appears as though this periodic solution is the global attractor (in the positive quadrant), although this is merely conjecture.

Parameter	Numerical value/range
μ	0.5
η	50
γ	25
τ	1
v^*	0.6227
v	$[0, v^*]$

Table 11.1: Parameters used for the numerical bifurcation analysis.

11.3.1 Continuation of endemic periodic solution

As it is not possible to express the endemic periodic solution analytically, we will need to resort to numerical methods. The first step is to continue the bifurcating endemic periodic solution. To simplify matters, we will take advantage of the convenient fact that since we seek periodic solutions of period τ, it is not necessary to treat the impulsive delay differential equation (11.1)–(11.3) directly. Specifically, every instance of $S(t - \tau)$ and $I(t - \tau)$ can be replaced by $S(t)$ and $I(t)$, since we search explicitly for periodic solutions. Doing this and adjusting the jump condition accordingly, the branch $(S^*(t, v), I^*(t, v))$ of endemic periodic solutions must satisfy the following boundary-value problem:

$$\frac{d}{dt}S^* = \mu - \mu S^* - \eta f(I^*)S^* + \gamma e^{-\mu\tau}I^*, \quad \begin{aligned} S^*(0) &= \rho(v)S^*(\tau^-), \\ I^*(0) &= I^*(\tau^-), \end{aligned}$$
$$\frac{d}{dt}I^* = \eta f(I^*)S^* - (\mu + \gamma)I^*, \quad \rho(v) = (1-v)(1-ve^{-\mu\tau})^{-1}.$$

Following Theorem 11.2.1, we could use

$$S^*(t, v^* + \epsilon) = \tilde{S}(t, v^*) - \epsilon\frac{m}{\ell}e_1^\intercal\Phi_{t,1}(0)$$

$$I^*(t, v^* + \epsilon) = -\epsilon\frac{m}{\ell}e_2^\intercal\Phi_{t,1}(0) \tag{11.36}$$

as a linear-order guess for the first point on the branch, for some $|\epsilon|$ sufficiently small. Under the assumption that $\ell < 0$ – the more biologically expected case – we will do continuation for $\epsilon < 0$.

For the illustrative system parameters of Table 11.1, we solve this boundary value problem using the `bvp4c` function in MATLAB R2018a. We take the solution for perturbation parameter ϵ_n as the initial guess for perturbation parameter $\epsilon_{n+1} < \epsilon_n$, except for the first step ϵ_0 where we use the linear guess (11.36). That is, we keep things simple and
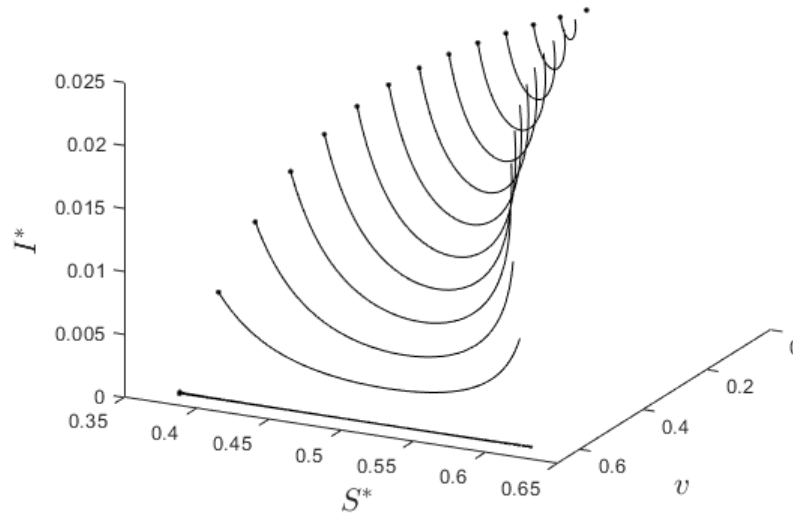
Figure 11.1: Plots of the periodic solution obtained by the continuation scheme described in Section 11.3.1 for vaccine coverage $v \in [0, v^*]$. Dots indicate the "initial" points $(S^*(0, v), I^*(0, v))$ on each periodic solution, followed by evolution along the corresponding curve at each level v with time left implicit. The periodic solution is constant in the I variable at $v = v^*$, and collapses to a fixed point at $v = 0$. To improve visibility, only fourteen vaccination coverages in the interval $[0, v^*]$ are displayed.

implement natural continuation. We do not expect any turning points along the branch (note, we expect a Hopf point, which will generically lead to an invariant cylinder on which there are no additional periodic solutions), so no difficulties are anticipated. Figure 11.1 provides a sample of the periodic solutions generated by the continuation scheme.

11.3.2 Dominant Floquet exponent and cylinder point identification

To test for additional bifurcation points, we will need to compute the dominant Floquet exponents along the periodic solution continuation. While this can in principle be done by solving an appropriate boundary-value problem, our attempts to accomplish this in MAT-
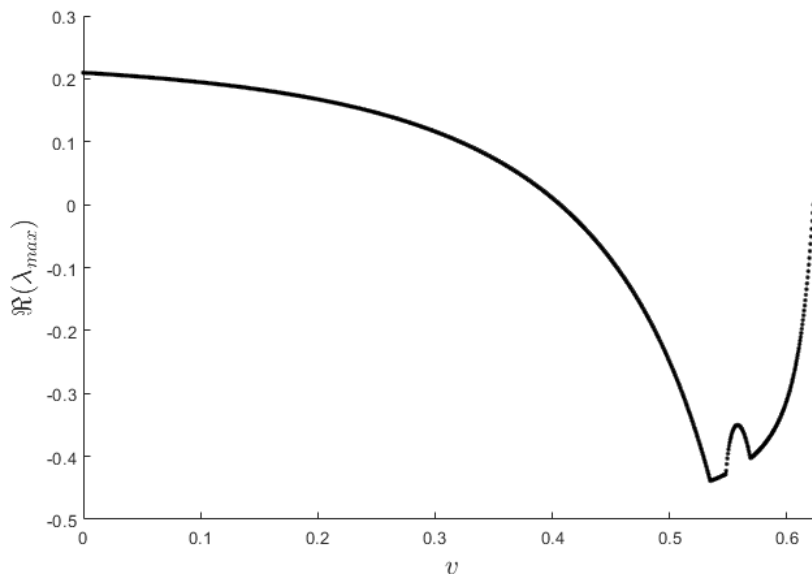
Figure 11.2: A plot of the real part of the dominant Floquet exponent of the branch $v \mapsto (S^*(\cdot, v), I^*(\cdot, v))$ of periodic solutions. The real part crosses the line $\Re(\lambda_{max}) = 0$ in the window $v \in [0.4063, 0.4068]$. All computations were done using the discretization scheme from Section 4.6.3 with $N = 200$ mesh points.

LAB using built-in boundary-value problem solvers had serious issues with convergence. Instead, we discretize the monodromy operator using the method outlined in Section 4.6.3.

The real part dominant Floquet exponent is plotted versus the vaccination coverage in Figure 11.2. Numerically windowing the crossing of the imaginary axis, we see that the real part crosses through zero for some $v \in [0.4063, 0.4068]$. The approximate Floquet spectrum (i.e. the set of Floquet exponents) for $v = 0.4068$ is plotted in Figure 11.3, where we see that, as expected, there is a pair of (approximately) imaginary Floquet exponents. To the order of discretization used (200 mesh points), the pair of Floquet exponents is simple, so we should expect (from Theorem 10.2.1) a cylinder bifurcation to at some $v_c^* \in [0.4063, 0.4068]$.

We can easily check – at least to the level of numerical accuracy achieved – two of the nondegeneracy conditions associated to the cylinder bifurcation theorem. The first condition G.1 states that we must have $e^{ki\omega} \neq 1$ for $k = 1, 2, 3, 4$, where ω is the imaginary part of the dominant Floquet exponent. Our numerical estimate (see Figure 11.3) is

$$\omega = 1.9886. \tag{11.37}$$
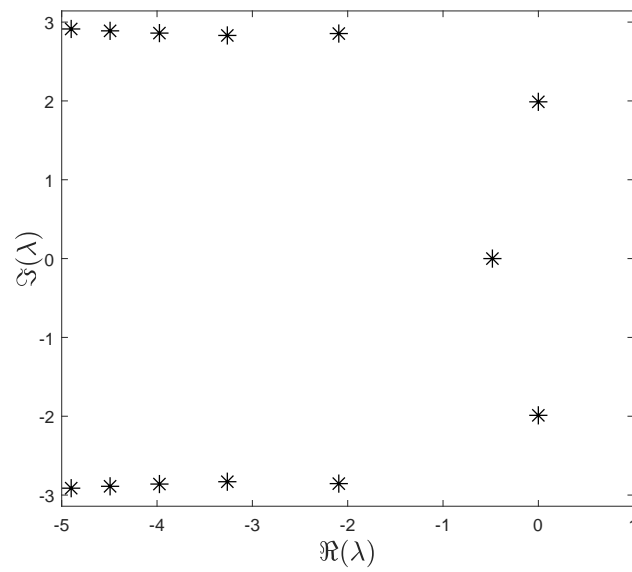
Figure 11.3: The approximate Floquet spectrum associated to the periodic solution $t \mapsto (S^*(t,v), I^*(t,v))$ for $v = 0.4063$ restricted to the strip $\{z \in \mathbb{C} : \Re(z) \in [-5, \infty)\}$. All computations were done using the discretization scheme from Section 4.6.3 with $N = 200$ mesh points.

One can then verify that $|e^{ki\omega} - 1| \geq 0.316$, and the first nondegeneracy condition is passed.

The second nondegeneracy condition G.2 pertains to the transversal crossing of the Floquet exponents across the imaginary axis. Examining the plot of real part of the dominant Floquet exponent in Figure 11.2, we see that the real part is strictly decreasing and appears smooth in the critical interval $[0.4063, 0.4068]$, from which we conclude that the second nondegeneracy condition is satisfied.

The third nondegeneracy condition requires one to calculate the quadratic term of the centre manifold at the critical parameter and, following this, compute the first Lyapunov coefficient. The benefits are somewhat limited and do not greatly aid in the exposition. We will therefore content ourselves with the first two nondegeneracy conditions, knowing that in a generic sense (i.e. up to perturbation in quadratic terms), a cylinder bifurcation does indeed occur at some $v_c^* \in [0.4063, 0.4068]$. Moreover, because the real part of the dominant Floquet exponent is decreasing at v_c^*, we obtain by Theorem 10.2.1 that the periodic solution $(S^*(t,v), I^*(t,v))$ is unstable for $v < v_c^*$ and locally asymptotically stable for $v > v_c^*$.

Remark 11.3.1. *We briefly comment that the theorem should be applied not to* (11.1)–(11.3)*, but rather to the system with vaccinated component introduced in Section 11.1.1, since the latter satisfies the overlap condition. All statements concerning the Floquet exponents carry over, however, so our conclusions remain correct.*

11.3.3 Tracking the evolving cylinder

Provided the invariant cylinder is attracting in the parameter regime where it exists, we should expect based on Theorem 10.2.1 that the cylinder is attracting for $0 \ll v < v_c^* \approx 0.4063$. In a time series, we would expect to see convergence to an oscillatory but non-periodic solution. Figure 11.4 provides such a time series for a selection of six vaccination thresholds $v \in [0, v^*]$. The solution converges to a clearly defined periodic solution for $v = 0.6$ and $v = 0.45$, while in the intermediate regime of $v \in (0, v_c^*]$ demonstrated by the second row of plots and the first plot on the final row, the dynamics are eventually oscillatory with no discernible period. A clear periodic solution is seen at $v = 0$.

The geometry of the cylindrical attractor is more clearly seen if one plots $S(t)$ and $I(t)$ together with $t \mapsto f(x_t)$ for some functional f, with $t \mapsto x_t$ the solution in the phase space $\mathcal{RCR}$. One might think the sum of vaccinated and recovered components as in Figure 11.4 to be a natural choice, but there is some transient linear dependence between these and $S(t)$ and $I(t)$ that hides some of the geometry. Instead, we plot $t \mapsto (S(t), I(t), S(t-1))$.
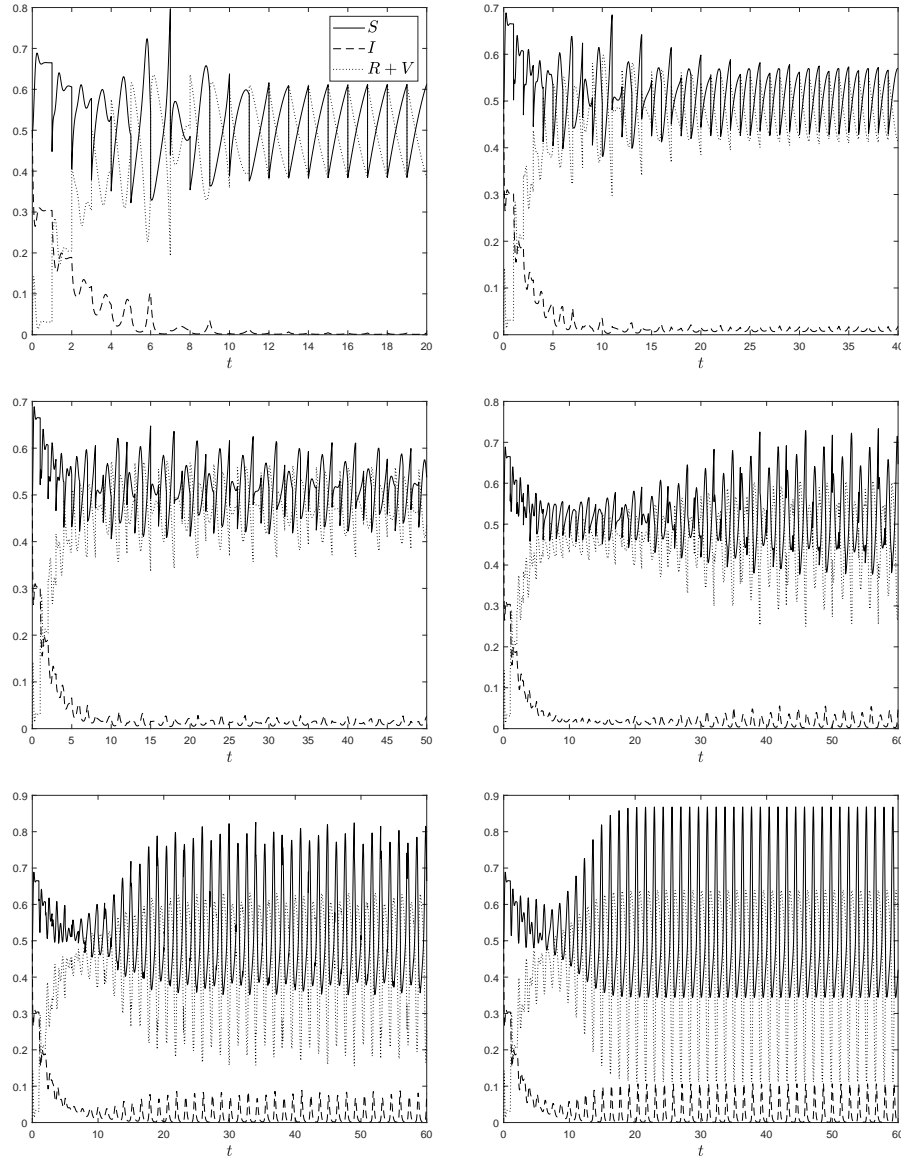
Figure 11.4: Time series from the constant initial condition $(S(0), I(0), R(0), V(0)) = (0.5, 0.5, 0, 0)$ for various vaccination coverages. Susceptible, infected and sum of recovered and vaccinated populations are plotted, with a legend inset in the first frame. The vaccinated population is governed by (11.5). (Top row: $v = 0.6$, $v = 0.45$. Middle row: $v = 0.395$, $v = 0.3$. Bottom row: $v = 0.15$, $v = 0$.)

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To illustrate the birth of the cylindrical attractor as the parameter v is varied close to the bifurcation point v_c^*, we plot both the continuation periodic solution $t \mapsto (S^*(t,v), I^*(t,v))$ and the forward time integration from the constant initial condition $(S(0), I(0)) = (0.5, 0.5)$. The delayed state $S(t-1)$ is used as a third spatial variable to aid in visualization as described above. We integrate the solution for $t \in [0, 1300]$ and plot only for $t \in [300, 1000]$. The result is provided in Figure 11.5, where we clearly see the cylindrical topology appearing at $v = 0.395$.

At $v = 0.385$ there appears to be phase locking, although the phase-locked regions still appear to lie on a cylindrical structure. To compare, as v decreases to 0.375, then to 0.35 and 0.315 – see Figure 11.6 – the radius of the cylinder becomes more variable along its length, the latter of which is contracted. The structure of the attractor bears little resemblance to the periodic solution $t \mapsto (S^*(t,v), I^*(t,v), S^*(t-1,v))$ it bifurcated from.

Further decreasing the vaccination coverage from $v = 0.25$ through to $v = 0$ shows convergence of the attractor to the periodic orbit of the Kyrychko and Blyuss model. Topologically, the cylinder contracts to a circle. This can be visualized in Figure 11.6. The complete transition of the disease-free periodic solution through the transcritical bifurcation and the cylinder bifurcation can be visualized with the help of the bifurcation diagram, Figure 11.8.
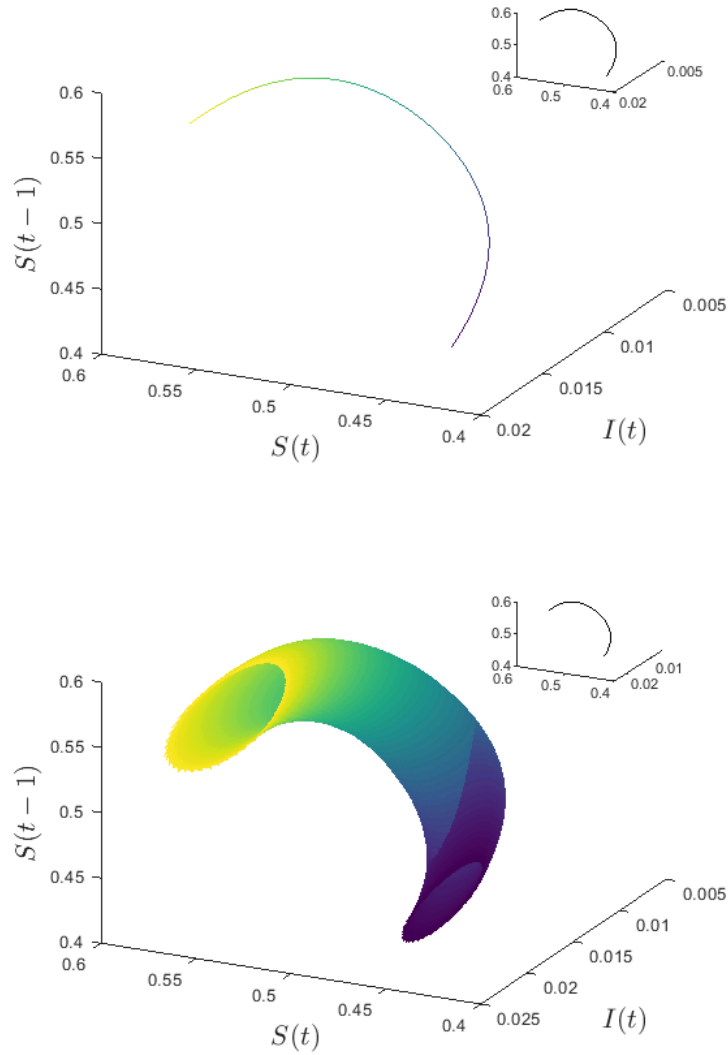
Figure 11.5: Plot of $t \mapsto (S(t), I(t), S(t-1))$ for $t \in [300, 1300]$ from a constant initial condition of $(S(0), I(0)) = (0.5, 0.5)$, for $v = 0.45$ (top) and $v = 0.395$ (bottom). Purple corresponds to arguments $t = k \in \mathbb{Z}$ and yellow to arguments $t \to k^-$. Inset: Plots of the image of $t \mapsto (S^*(t, v), I^*(t, v), S^*(t-1, v))$.
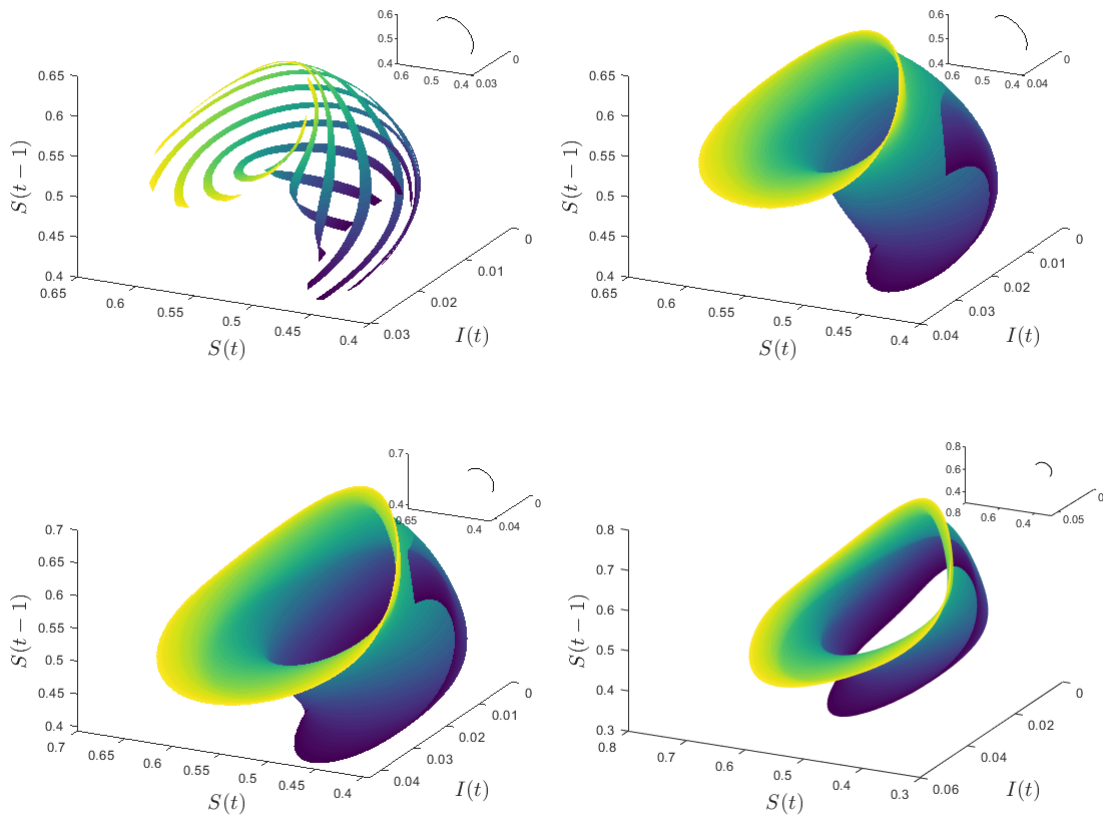
Figure 11.6: Plot of $t \mapsto (S(t), I(t), S(t-1))$ for $t \in [300, 1300]$ from a constant initial condition of $(S(0), I(0)) = (0.5, 0.5)$, for $v = 0.385$, $v = 0.375$ (top row), $v = 0.35$ and $v = 0.315$ (bottom row). Colours and insets have the same interpretation as in Figure 11.5.
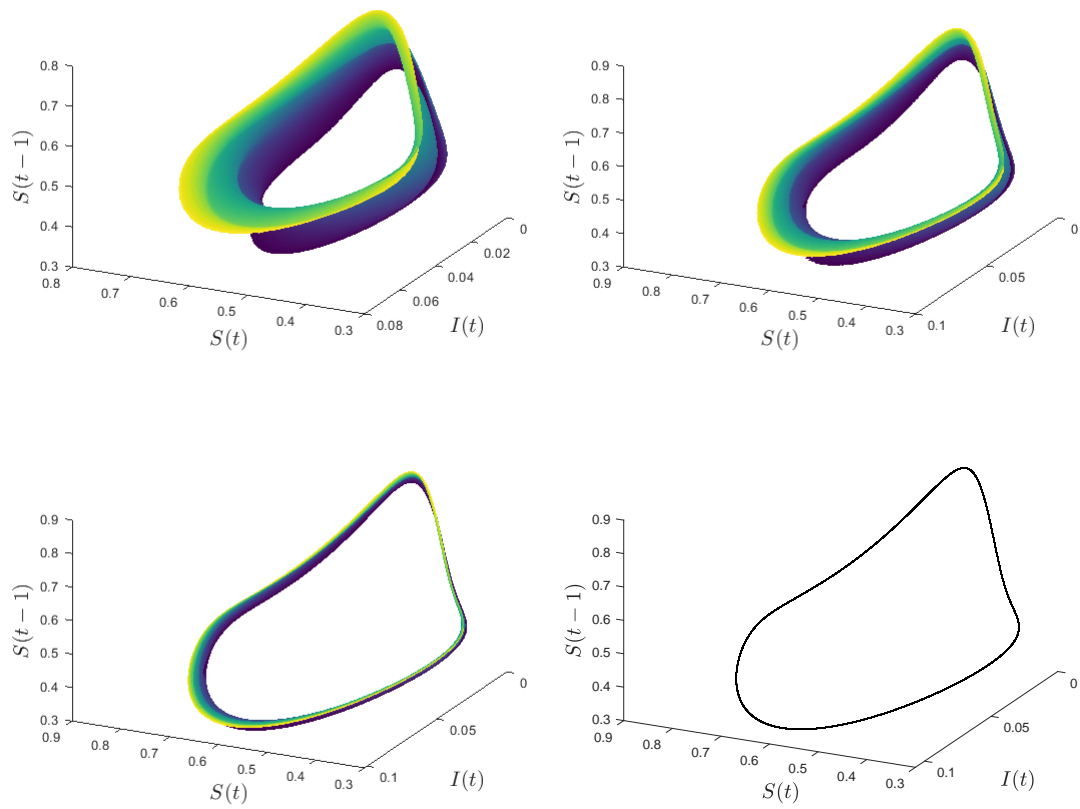
Figure 11.7: Plot of $t \mapsto (S(t), I(t), S(t-1))$ for $t \in [300, 1300]$ from a constant initial condition of $(S(0), I(0)) = (0.5, 0.5)$, for $v = 0.25$, $v = 0.15$ (top row), $v = 0.05$ and $v = 0$ (bottom row). Colours have the same interpretation as in Figure 11.5.
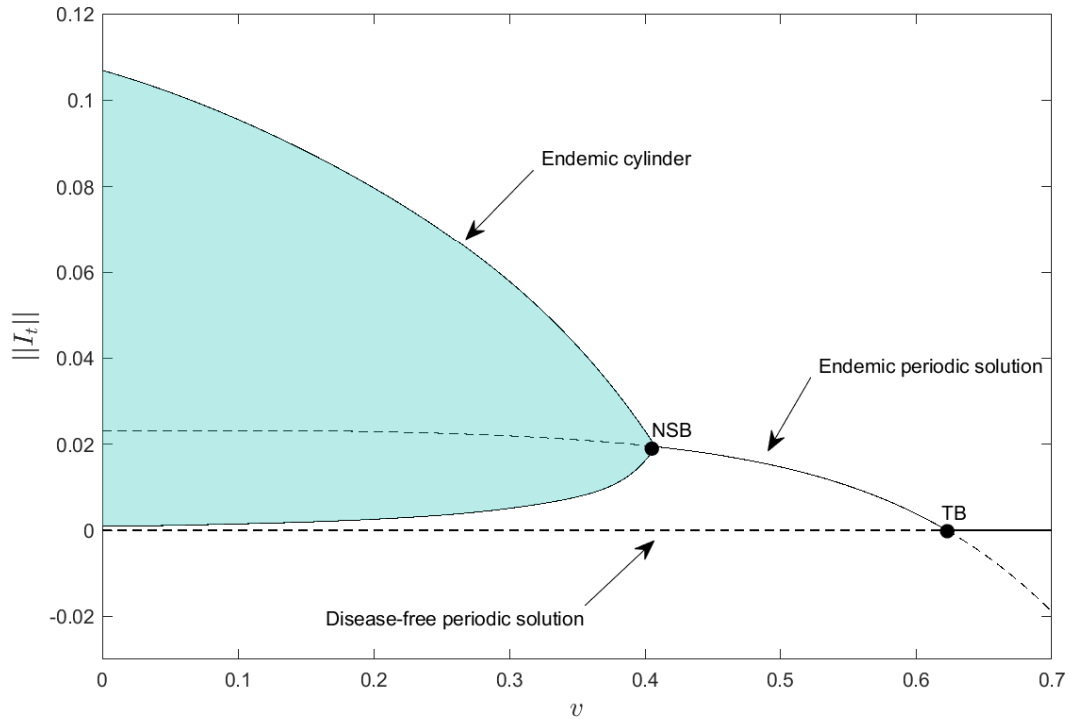
Figure 11.8: The bifurcation diagram for the SIR model with pulse vaccination and temporary immunity with the parameters from Table 11.1. TB denotes a transcritical bifurcation, while NSB denotes a Neimark-Sacker bifurcation. The latter is qualitatively a cylinder bifurcation. Dashed lines denote an unstable object, while solid lines denote an asymptotically stable object. The arcs corresponding to the endemic cylinder parameterize the minimum value (bottom arc) and maximum value (top arc) of the norm of the infected component in the phase space.

Chapter 12

Application to impulsive stabilization and bifurcation suppression

The content of this chapter appears in "Cost-effective robust stabilization and bifurcation suppression" by Kevin Church and Xinzhi Liu [23], currently in press at SIAM Journal on Control and Optimization.

12.1 Overview of impulsive stabilization methodology and current methods

Stabilization of complex networks and dynamical systems both large-scale and small play an important role in science and industry. Since the introduction of Lyapunov's direct method and its various generalizations, the technique has seen much application in the development of sufficient conditions for the stability of steady states, which can themselves be used to derive controllers guaranteeing robust stability and synchronization. The recent survey paper [82] catalogues recent developments in the stability analysis of linear time-delay systems by Lyapunov-based methods, and one may consult the references therein for background. For a short list of specifically nonlinear results, one may consult [55, 99, 100, 107].

Suppose one has an autonomous n-dimensional retarded functional differential equation depending on a parameter $\epsilon \in \mathbb{R}^p$,

$$\dot{x} = f(x_t, \epsilon), \tag{12.1}$$

and it is known that $x^* = 0$ is an equilibrium point for all $\epsilon \in N \subset \mathbb{R}^p$, a neighbourhood of the origin. The *linearization* at $x^* = 0$ is the linear system

$$\dot{y} = Df(0, \epsilon)y_t, \tag{12.2}$$

and its dynamics determine the local behaviour near $x = 0$ for the nonlinear system (12.1). The characteristic matrix is

$$\Delta(\lambda; \epsilon) = \lambda I - Df(0, \epsilon)[e^{\lambda(\cdot)}I] \tag{12.3}$$

where I the $n \times n$ identity matrix. The eigenvalues are the solutions of the generally transcendental characteristic equation

$$\det(\Delta(\lambda; \epsilon)) = 0, \tag{12.4}$$

and x^* will be (locally) exponentially stable if and only if all eigenvalues have strictly negative real part; see [39, 80] for background. A transition from stability to instability occurs when one or more eigenvalues cross the imaginary axis. In practical applications, it may be that due to some external influence or component failure, a system parameter enters a regime where stability is lost and a bifurcation occurs. The goal then shifts to *stabilization*.

One stabilization methodology that has seen a fair bit of attention in recent years is *impulsive stabilization* [42, 59, 60, 63, 65, 90, 93, 108]. These results are derived and their validity verified by means of Lyapunov functional and Lyapunov-Razumikhin methods. They are stated in terms of the existence of matrices satisfying matrix inequalities, and they typically provide global stability. However, the assumptions can be somewhat strong: global Lipschitz conditions are typically needed to guarantee convergence and finding matrices satisfying the necessary inequalities can difficult especially for large interconnected systems.

A related problem in terms of the implementation of impulsive stabilization is that, naturally, some controls may be more difficult to implement than others. In other words, there may be an explicit cost in implementing an impulsive controller. Guaranteed cost impulsive control has been considered in [61, 94, 109] among others, where the goal is to design the impulsive controller so that a running cost is minimized. To contrast, we are interested in average costs associated to impulsive controllers, where the cost may be dictated by such factors as the control gain or its structure. The latter encompasses such factors as the amount of coupling induced by the controller, the amount of diffusivity or lack thereof, or a penalty for accessing or modifying certain system states.

Simultaneously, there may be hard constraints to the types of impulsive controllers that are permitted. In pinning control, for example, the jump functionals do not induce any additional coupling between nodes – see the aforementioned references. In an input-output setting, if one only has access to system outputs, then one might want the controller to depend only on the measurements. See later an example in Section 12.2.1. We would like to incorporate such hard constraints into our stabilization methodology.

It is our goal to provide an alternative impulsive stabilization approach based on centre manifold theory. Our novel method does not require global Lipschitzian constraints and provides an algorithmic way to find an impulsive controller that achieves stabilization while simultaneously guaranteeing a prescribed *local convergence rate* and *minimizing a cost functional*. The jump functionals that leads to stabilization can be chosen from a set of admissible functionals that can be set by the control designer, thereby incorporating a wide class of *hard constraints* as described in the previous paragraph. Moreover, bifurcations are *suppressed* in the sense that any nonlinear structures such as periodic orbits that could result from parameter variation will be unstable. In other words, the dynamics near the equilibrium will be robust under parameter variation.

The idea is, at its heart, a simple geometrical construction. To linear order, each candidate jump functional will have some impact on the dynamics on the centre manifold. The goal is then to choose a jump functional in such a way that the dynamics restricted to the centre manifold become linearly stable. If the jump functional that is chosen is sufficiently small, then its inclusion will not push other eigenvalues across the imaginary axis and lead to further instabilities. In some sense, we try to choose a jump functional that stabilizes the weak instabilities due to eigenvalues on the imaginary axis, while having minimal effect on the stable modes.

With this in mind, our setup is as follows. We assume that at $\epsilon = 0$, the characteristic equation (12.4) has some number $c > 0$ of eigenvalues on the imaginary axis, while all others have strictly negative real part. This situation corresponds to one where stability of $x^* = 0$ could potentially be either gained or lost by perturbing the parameter ϵ away from zero, and a bifurcation could therefore occur. For each $\epsilon \in N$, we consider the problem of finding a *linear jump functional* B_ϵ such that, for the impulsive retarded functional differential equation

$$\dot{x} = f(x_t, \epsilon), \qquad\qquad t \notin \frac{1}{h}\mathbb{Z} \qquad\qquad (12.5)$$

$$\Delta x = B_\epsilon(t)x_{t^-}, \qquad\qquad t \in \frac{1}{h}\mathbb{Z}, \qquad\qquad (12.6)$$

the following conditions are satisfied for $|\epsilon| \leq \delta$, for some positive δ.

U.1 The equilibrium $x^* = 0$ is locally asymptotically stable with local convergence rate $O(e^{-\gamma t})$.

U.2 $B_\epsilon(t)$ is optimal in the sense that it minimizes an admissible cost function.

We will make these ideas more precise in forthcoming sections of this chapter. The frequency of impulse effect, h, is chosen beforehand. The constant $\gamma > 0$ is a chosen *rate parameter*. The functional B_ϵ is typically the action of a matrix on a vector of state observations (possibly delayed) or a sum thereof. The notion of a local convergence rate and admissible cost functions will be defined later when we formalize the problem more precisely. Condition U.1 guarantees that the equilibrium is *stabilized* and any bifurcations that could lead to a loss of stability of the equilibrium are *suppressed* in the parameter regime $|\epsilon| \leq \delta$, while specifying a worst-case convergence rate. The second condition U.2 ensures that the impulsive control (12.6) is one that minimizes an associated cost.

One might also be interested in conditions under which one can find a jump functional B_ϵ that satisfies conditions U.1 and U.2, but is *independent* of ϵ. This situation corresponds to a uniform robust cost-effective stabilization, and may be desirable when the dimension of the parameter space is very high. We will consider this problem as well.

The structure of this chapter is as follows. In Section 12.2, we precisely formulate our cost-effective impulsive stabilization and bifurcation suppression problem. The existence of solutions of the problem are considered in Section 12.3, while Section 12.4 and Section 12.4.5 are devoted to the computation of optimal solutions. The effectiveness of our stabilization method is demonstrated in Section 12.5 by way of a numerical simulation. All proofs are deferred to Section 12.7.

12.1.1 Notation

For $A \in \mathbb{R}^{a \times b}$, the notation A_{ij} denotes the entry in row i and column j.

If X is a real vector space, $Y \subset X$ is a linear subspace and $x \in X$, we denote

$$x + Y = \{x + y : y \in Y\} \subset X$$

the affine subspace spanned by Y with translation x. If W and Z are two such affine subspaces, we define for $t \in [0, 1]$ the convex combination

$$tW + (1 - t)Z = \{tw + (1 - t)z : w \in W, z \in Z\}.$$

For a vector space X, we will denote $X^k = X \times X \times \cdots \times X$ the k-fold Cartesian product of X with itself, with k factors in the product. We will sometimes abuse notation and identify elements of X^k with $1 \times k$ arrays with elements in X.

For square matrix $A \in \mathbb{R}^{b \times b}$, the symbols $\rho(A)$, $\det(A)$ and $\mathrm{tr}(A)$ will respectively denote the spectral radius, determinant and trace. We will at times suppress the parentheses and write simply ρA for the spectral radius of A.

For a linear map $L : X \to Y$ between finite-dimensional vector spaces, we denote L^+ its Moore-Penrose pseudoinverse. If L is a matrix, then L^+ is its pseudoinverse.

If $j \in \mathbb{Z}$ and $k \in \mathbb{N}^+$, we denote $[j]_k$ the remainder of j modulo k. Specifically, if we uniquely write $j = pk + r$ for some $r \in \{0, \ldots, k-1\}$ and $p \in \mathbb{Z}$, then we define $[j]_k = r$.

For a finite sequence of matrices $A_0, \ldots, A_{k-1}$ for $k \geq 1$, we define the product $\prod_{j=0}^{k-1} A_j$ by iterative composition – that is, multiplication on the left,

$$\prod_{j=0}^{k-1} A_j = A_{k-1} \cdots A_0.$$

For a function $f : X \to Y$, we will use the symbol $f(X)$ for the image of f. For an element $y \in f(X)$, we denote $f^{-1}(y)$ its preimage:

$$f^{-1}(y) = \{x \in X : f(x) = y\}.$$

When f is one-to-one, the singleton $f^{-1}(y)$ will be identified with the unique solution x of the equation $f(x) = y$.

For a complex vector $v \in \mathbb{C}^n$, we denote $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ its real and imaginary parts, respectively. That is, $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ are the unique elements of $\mathbb{R}^n$ such that $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$.

12.2   Optimal impulsive stabilization with performance target

In this section we formulate the problem outlined in Section 12.1 more precisely. We introduce the linear jump functionals and frequencies that will be considered in Section 12.2.1. The allowable cost functionals are introduced in Section 12.2.2.

12.2.1 The space $\mathcal{B}$ of discrete-delay jump functionals

We consider classes of linear jump functionals B defined by a collection of distinct discrete delays $\delta_1, \ldots, \delta_\ell \in [-d, 0]$ for some $d > 0$:

$$B\phi = A\phi(0) + \sum_{i=1}^{\ell} A_i \phi(\delta_i), \tag{12.7}$$

Note that given the frequency h of impulse effect, it will need to be assumed that the any nonzero delay $\delta_i \neq 0$ satisfies $\delta_i \notin \frac{1}{h}\mathbb{Z}$. This constraint is needed because of a technical assumption (the overlap condition) of the centre manifold theory and reduction principle for impulsive delay differential equations, of which our main results are based. From the right-hand side of (12.7), we can identify a jump functional with an element of the finite-dimensional vector space

$$\mathcal{B} = (\mathbb{R}^{n \times n})^{\ell+1}. \tag{12.8}$$

With this identification, we will abuse notation and write $B\phi$ for $B = (A, A_1, \ldots, A_\ell) \in \mathcal{B}$ and $\phi \in \mathcal{RCR}$ as the right-hand side of (12.7). Similarly, if $\Phi = [\ \phi_1\ \cdots\ \phi_c\ ] \in \mathcal{RCR}^c$, then we denote $B\Phi = [\ B\phi_1\ \cdots\ B\phi_c\ ]$.

Pinning stabilization and diagonal jump functionals

In some situations, it may not be appropriate to work with the entire space $\mathcal{B}$ of jump functionals. For example, suppose the system (12.1) has the structure of a network of N identical linearly-coupled nodes

$$\dot{x}^{(i)} = f(x_t^{(i)}) + c \sum_{j=1}^{N} a_{ij} \Gamma H(x^{(j)}(t)), \quad i = 1, \ldots, N, \quad x^{(i)} \in \mathbb{R}^n \tag{12.9}$$

with coupling strength $c > 0$, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) > 0$ an inner coupling matrix, $H : \mathbb{R}^n \to \mathbb{R}^n$ a nonlinear coupling with $H(0) = 0$, and $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ a diffusive Laplacian matrix representing the coupling configuration of the network, where any of these coupling terms may depend on parameters. In impulsive pinning stabilization and synchronization, one would choose jump functionals that act only on individual nodes – see the references [42, 65, 93] – and are functionally driven by an error system.

For simplicity, assume we wish to stabilize the trivial equilibrium $x^* = 0$, so that one does not need to consider a separate error system. An appropriate subspace of $\mathcal{B}$ in

which one could consider pinning stabilization is the set of block diagonal operators. To define these, we note that for system (12.9), we have $\mathcal{B} = (\mathbb{R}^{nN \times nN})^{\ell+1}$. Each element of $\mathbb{R}^{nN \times nN}$ can be interpreted as an $N \times N$ block matrix with $n \times n$ blocks, so that we can write an arbitrary element of $\mathcal{B}$ in the form $B = (A, A_1, \ldots, A_\ell)$ with $A_m(i,j) \in \mathbb{R}^{n \times n}$ for $m = \emptyset, 1, \ldots, \ell$ and $i, j = 1, \ldots, N$. Then, the diagonal subspace $\mathcal{B}_{\text{diag}} \subset \mathcal{B}$ is defined by

$$\mathcal{B}_{\text{diag}} = \{B = (A, A_1, \ldots, A_\ell) \in \mathcal{B} : A_m(i,j) = 0 \ \forall i \neq j, \ m = \emptyset, 1, \ldots, \ell\}. \qquad (12.10)$$

This is indeed a subspace of $\mathcal{B}$ and it contains only the linear jump functionals that induce no further coupling between different nodes.

Proportional control

Suppose one wishes to stabilize the system $\dot{x} = f(x_t, \epsilon)$ using a proportional impulsive control, with a system output y. That is, we seek jump functionals $B_\epsilon(t)$ such that $0 \in \mathbb{R}^n$ of the system

$$\dot{x} = f(x_t, \epsilon), \qquad\qquad t \notin \frac{1}{h}\mathbb{Z}$$

$$y(t) = Hx(t),$$

$$\Delta x = NB_\epsilon(t)y_{t^-}, \qquad\qquad t \in \frac{1}{h}\mathbb{Z},$$

becomes stable, where $N \in \mathbb{R}^{n \times m}$ is an input matrix, $H \in \mathbb{R}^{p \times n}$ is an output matrix and $y \in \mathbb{R}^p$ is the output. Starting with the space $\mathcal{B} = (\mathbb{R}^{(n+p) \times (n+p)})^{\ell+1}$ as introduced at the beginning of this section, our proportional control constraint on the jump functionals can be imposed by abusing notation and identifying $B_\epsilon(t)$ with an element of $\mathcal{B}$ defined in block form as

$$B_\epsilon(t) = \begin{bmatrix} 0 & NB_\epsilon(t) \\ 0 & HNB_\epsilon(t) \end{bmatrix}. \qquad (12.11)$$

We need to verify that $z = \begin{bmatrix} x & y \end{bmatrix}^\mathsf{T}$ satisfies the equation $\Delta z = B_\epsilon(t)z_{t^-}$ at times $t \in \frac{1}{h}\mathbb{Z}$.

$$\Delta z = \begin{bmatrix} \Delta x \\ H\Delta x \end{bmatrix} = \begin{bmatrix} NB_\epsilon(t)y_{t^-} \\ HNB_\epsilon(t)y_{t^-} \end{bmatrix} = \begin{bmatrix} 0 & NB_\epsilon(t) \\ 0 & HNB_\epsilon(t) \end{bmatrix} \begin{bmatrix} x_{t^-} \\ y_{t^-} \end{bmatrix} = B_\epsilon(t)z_{t^-},$$

as desired. The set of elements of $\mathcal{B}$ of the form (12.11) is indeed a proper subspace of $\mathcal{B}$, which we denote $\mathcal{B}_{\text{prop}}$. One can complete the transformation to a system of form (12.5)–(12.6) by taking the derivative of y. PID controls can be introduced in a similar way.

Cycles of jump functionals

In our impulsive functional differential equation (12.5)–(12.6), we have left the possibility for the jump function to depend on time t. That is, $B_\epsilon(t) \in \mathcal{U}$ for each t and some subspace $\mathcal{U} \subseteq \mathcal{B}$, while being generally nonconstant. As our method is based on centre manifold theory, working with infinite time horizons is difficult so we will typically take $t \mapsto B_\epsilon(t)$ to be periodic. As such, a *cycle of jump functionals* is an element of the product $\mathcal{U}^k$, for natural number $k \geq 1$ called the *period*. We can then associate a jump functional in the style of (12.7) by way of the following equivalence. If $B = (B_0, \ldots, B_{k-1}) \in \mathcal{U}^k$, we define

$$B\left(\frac{j}{h}\right)\phi = B_{[j]_k}\phi. \tag{12.12}$$

This definition is sufficient to give meaning to (12.6) since we need only define $B(t)$ at the times $t = \frac{j}{h}$ for $j \in \mathbb{Z}$.

Cycles of jump functionals in nonidentical subspaces

In some applications, it might be that certain controls (quantified by jump functionals) can only be applied intermittently due to resource limitations. As such, it is worthwhile considering the case where more generally, we have $B_\epsilon(t) \in \mathcal{U}(t)$, for $t \in \frac{1}{h}\mathbb{Z}$ and each subspace $\mathcal{U}(t)$ is generally distinct. As in Section 12.2.1 we will assume the periodicity condition

$$\mathcal{U}_{j+k} := \mathcal{U}\left(\frac{j+k}{h}\right) = \mathcal{U}\left(\frac{j}{h}\right) =: \mathcal{U}_j$$

for all $j \in \mathbb{Z}$, and associate $t \mapsto B_\epsilon(t)$ to a jump functional $B = (B_0, \ldots, B_{k-1})$ in the product space

$$\mathcal{U}^{(k)} := \mathcal{U}_0 \times \cdots \times \mathcal{U}_{k-1}.$$

The construction in the previous Section 12.2.1 corresponds to the special case where $\mathcal{U}_i = \mathcal{U}_0$ for all indices i. Regardless, it remains true that $\mathcal{U}^{(k)}$ is a linear subspace of $\mathcal{B}^k$.

12.2.2   Allowable cost functionals

An *allowable cost functional* will be a functional $\mathcal{C} : \mathcal{U} \to \mathbb{R}^+$ satisfying the following properties.

C.1 $\mathcal{C}$ is continuous, convex and positive-definite.

C.2 $\mathcal{C}$ is *radially unbounded*: if $B_n \in \mathcal{B}$ is a sequence with unbounded norm $||B_n|| \to \infty$, then $\mathcal{C}(B_n) \to \infty$.

These properties will eventually be used to show that a formalized version of the problem from Section 12.1 admits a solution satisfying the conditions U.1 and U.2. One can similarly define allowable cost functions on any linear subspace $\mathcal{U} \subseteq \mathcal{B}$ using the same definition.

Given an allowable cost functional $\mathcal{C}$, we define the cost of a cycle of jump functionals of period k in a linear subspace of $\mathcal{B}^k$ as follows. For $B = (B_0, \dots, B_{k-1}) \in \mathcal{B}^k$ we set

$$\mathcal{C}(B) = \sum_{j=0}^{k-1} \mathcal{C}(B_j). \tag{12.13}$$

We will sometimes abuse notation and write $\mathcal{C} : \mathcal{B}^k \to \mathbb{R}^+$ for the associated cost functional on the cycles of jump functionals of period k. $\mathcal{B}^k$ can be replaced with any linear subspace thereof, so this definition extends naturally to encompass cycles of jump functionals in nonidentical subspaces as in Section 12.2.1. Finally, the following definition will be useful later.

Definition 12.2.1. *An allowable cost function $\mathcal{C} : \mathcal{U} \to \mathbb{R}^+$ is* projective *if there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{C}(B) = \langle B, B \rangle$.*

Weighted matrix norms

A typical allowable cost functional can be constructed through the introduction of a weighted matrix norm $||X||_W = ||W^{\frac{1}{2}} X W^{-\frac{1}{2}}||_2$ for a symmetric positive-definite matrix W and its principal real square root $W^{\frac{1}{2}}$, and $|| \cdot ||_2$ the spectral norm. Indeed, let W_0, $W_1, \dots, W_\ell$ be positive-definite matrices, let $w_0, w_1, \dots, w_\ell \in \mathbb{R}^+$ be weight constants, and define a cost function

$$\mathcal{C}((A, A_1, \dots, A_\ell)) = w_0 ||A||_{W_0} + \sum_{i=1}^{\ell} w_i ||A_i||_{W_i} \tag{12.14}$$

The weight matrices W_i take into account limitations and costs associated to accessing and/or modifying the states of the system by impulses. The weights $w_0, w_1, \dots, w_\ell$ allow for a weighting of the individual factors that define the control (12.6) relative to each other.

12.2.3 Problem statement: parameterized and uniform

Having introduced the allowable cost functionals and the space $\mathcal{B}$ of linear jump functionals, we can more precisely formulate our problem. First, we define *local convergence rates*.

Definition 12.2.2. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying $\lim_{t\to\infty} g(t) = 0$. The asymptotic $O(g(t))$ is a* local convergence rate *of an equilibrium point x^* if there exists a neighbourhood U of x^* and a constant $K > 0$ such for all $(s, \phi) \in \mathbb{R} \times U$, the solution $t \mapsto x(t; s, \phi)$ of (12.5)–(12.6) satisfying the initial condition $x_s(\cdot; s, \phi) = \phi$ satisfies the inequality $\|x(t; s, \phi) - x^*\| \leq Kg(t - s)$ for all $t \geq s$.*

Next, we formalize the standing hypothesis that at parameter $\epsilon = 0$, our system is at a bifurcation point where stability could be either gained or lost by parameter variation.

Assumption (Spectral gap condition). *At parameter $\epsilon = 0$, the characteristic equation (12.4) has $c > 0$ eigenvalues with zero real part, and all other eigenvalues have real part less than some $\sigma < 0$. The real number σ is the* spectral gap.

Remark 12.2.1. *If the characteristic equation has a candidate bifurcation point at a parameter $\epsilon^* \neq 0$, one can perform a change of variables to shift the bifurcation point to the origin, $\epsilon = 0$. As such, no generality is lost by assuming a bifurcation point at $\epsilon = 0$.*

From this point onward, we assume the spectral gap condition. Note that because we are working with a periodic system, there is always a finite spectral gap; see part 6 of Theorem 4.2.1. With these definition at hand, the problem whose feasibility we will study and subsequently solve is the following.

Problem A. *Let $\mathcal{U}$ be a linear subspace of $\mathcal{B}^k$ for $k \geq 1$. For a given rate parameter $\gamma > 0$ and frequency h, determine whether one can, for ϵ sufficiently small, find an allowable cost functional $B_\epsilon \in \mathcal{U}$ such that the optimality condition*

$$\underset{Y \in \mathcal{Y}(\epsilon, h; \gamma)}{\arg\min} \mathcal{C}(Y) = B_\epsilon \tag{12.15}$$

is satisfied, where $\mathcal{Y}(\epsilon, h; \gamma) \subset \mathcal{U}$ is the set of all linear jump functionals for which $O(e^{-\gamma t})$ is a local convergence rate of the equilibrium point $x^* = 0$ of (12.5)–(12.6) for parameter ϵ and frequency h.

In the formulation of Problem A, we allow the optimal jump functional to depend on the parameter ϵ. However, in some settings it may be desirable to have a single jump functional provide stabilization robustly for *all* ϵ sufficiently small, or it may be computationally too expensive to generate optimal jump functionals for a large sample of parameters. Therefore, the following problem is of interest.

Problem B. *Let $\mathcal{U}$ be a linear subspace of $\mathcal{B}^k$ for some $k \geq 1$. For a given rate parameter $\gamma > 0$ and frequency h, find $\eta > 0$ and single allowable cost functional $B \in \mathcal{U}$ such that the optimality condition*

$$\underset{Y \in \mathcal{Y}^{\eta}(h;\gamma)}{\arg \min} \mathcal{C}(Y) = B \tag{12.16}$$

is satisfied, where $\mathcal{Y}^{\eta}(h;\gamma) \subset \mathcal{U}$ is the set of all linear jump functionals for which $O(e^{-\gamma t})$ is a local convergence rate of the equilibrium point $x^* = 0$ of (12.5)–(12.6) for parameters $|\epsilon| \leq \eta$ and frequency h.

12.3   Existence of an optimal solution

In this section we state our solutions to Problems A and B. The main results are provided in Section 12.3.1 and Section 12.3.2. Illustrative applications of our main results of this section as they apply to stabilization and bifurcation suppression are postponed until Section 12.5.

12.3.1   Main existence results: Problem A

A first step toward the solution of Problem A is provided by the following proposition.

Proposition 12.3.1. *Write $f(\phi, \epsilon)$ as a Taylor expansion near 0, so that*

$$f(\phi, \epsilon) = \underbrace{A_0 \phi(0) + \sum_{k=1}^{m} C_k \phi(-r_k) + \int_{-r}^{0} C(s) \phi(s) ds}_{L_0} + L(\epsilon)\phi + O(||\phi||^2),$$

for $L : \mathbb{R}^p \to \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ continuous and satisfying $L(0) = 0$ for some discrete delays $r_k \in (0, r]$ and a matrix $A_0 \in \mathbb{R}^{n \times n}$. Let $\Phi(t)$ be a real $n \times c$ matrix whose columns form a basis of the set of centre generalized eigenfunctions

$$E_0 = \bigcup_{n \in \mathbb{N}} \left\{ z(t) = \sum_{i=1}^{n} t^{i-1} e^{\lambda t} v_i, \ : \ Re(\lambda) = 0, \ \dot{z} = L_0 z_t \right\}, \tag{12.17}$$

and compute the $c \times c$ matrix Λ satisfying the identity $\frac{d}{dt}\Phi(t) = \Phi(t)\Lambda$. Introduce the transposed operator L_0^{T},

$$L_0^{\mathsf{T}} \psi = A_0 \psi(0) + \sum_{k=1}^{j} \psi(r_k) C_k + \int_{-r}^{0} \psi(-s) C(s) ds, \tag{12.18}$$

acting on $C([0, r], \mathbb{R}^{n*})$. Let $\Psi(t)$ be a real $c \times n$ matrix whose rows form a basis for the set of adjoint centre generalized eigenfunctions

$$E_0^\intercal = \bigcup_{n \in \mathbb{N}} \left\{ w(t) = \sum_{i=1}^{n} t^{i-1} e^{-\lambda t} v_i, \; : \; Re(\lambda) = 0, \; \dot{w} = -L_0^\intercal w^t \right\}, \qquad (12.19)$$

and define the invertible matrix $\Gamma \in \mathbb{R}^{c \times c}$ by the equation

$$\begin{aligned} \Gamma^{-1} = \Psi(0)\Phi(0) &- \sum_{k=1}^{j} \int_0^{r_k} \Psi(s) C_k \Phi(s - r_k) ds \quad \cdots \\ &- \int_{-r}^{0} \int_{-\theta}^{0} \Psi(s) C(s) \Phi(s + \theta) ds d\theta. \end{aligned} \qquad (12.20)$$

Introduce the centre monodromy map $\mathcal{M}_{\epsilon,h} : \mathcal{B}^k \to \mathbb{R}^{c \times c}$,

$$\mathcal{M}_{\epsilon,h}(B) = \prod_{j=0}^{k-1} (I_{c \times c} + \Gamma \Psi(0) B_j \Phi_0) \exp\left(\frac{1}{h} \left(\Lambda + \Gamma \Psi(0) L(\epsilon) \Phi_0 \right) \right), \qquad (12.21)$$

and define the set

$$\widetilde{\mathcal{Y}}(\epsilon, h; \gamma) = \{ B \in \mathcal{U} \subseteq \mathcal{B}^k : e^{\gamma/h} \rho(\mathcal{M}_\epsilon(B)) \leq 1 \}. \qquad (12.22)$$

Let an allowable cost functional $\mathcal{C}$ be given. Let h be fixed. The following are true.

1. *$\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ is closed and if the spectral gap satisfies $\gamma < \sigma$, there exists $\delta = \delta(\gamma) > 0$ such that for $|\epsilon| \leq \delta$,*

$$B_\delta(0) \cap \widetilde{\mathcal{Y}}(\epsilon, h; \gamma) = B_\delta(0) \cap \mathcal{Y}(\epsilon; \gamma). \qquad (12.23)$$

2. *For each each $\epsilon \in N$, there exists B_ϵ such that $\mathcal{C} : \widetilde{\mathcal{Y}}(\epsilon; \gamma) \to \mathbb{R}$ attains its minimum at B_ϵ, assuming the domain is nonempty.*

Remark 12.3.1. *To compute a basis for E_0, it is sufficient to compute a canonical system of Jordan chains for $\Delta(\lambda; 0)$ for all eigenvalues λ with zero real part. See Theorem 4.2 from Chapter 7 of [39] for the relevant result. To compute a basis for $E_0^\intercal$, one can use the connection between the transpose system $\dot{w} = -L_0^\intercal w^t$ and the adjoint, with the result being that a basis can be computed using a canonical system of Jordan chains for the transpose $\Delta(\lambda; 0)^\intercal$ – see Theorem 5.1 from Chapter 7 of [39] for the relevant theorem. Alternatively, one could exploit the characterization $\Psi(t) = e^{-\Lambda t} \Psi(0)$ for $\Psi(0) \in \mathbb{R}^{c \times n}$ and solve for the unknown coefficients of $\Psi(0)$ by imposing the equality $\Gamma = I$ in equation (12.20), as suggested in [13].*

Remark 12.3.2. *In the case of a Hopf bifurcation, the basis calculation is much simpler. To obtain the basis matrix for E_0, one calculates a nontrivial $v \in \mathbb{C}^n$ satisfying $\Delta(i\omega; 0)v = 0$ for the critical eigenvalue $\lambda = i\omega$. Then, the basis matrix is*

$$\Phi(t) = \left[\begin{array}{cc} Re(ve^{i\omega t}) & Im(ve^{i\omega t}) \end{array} \right].$$

For the transpose basis $E_0^\intercal$, one computes a nontrivial $w \in \mathbb{C}^n$ satisfying $w^\intercal \Delta(i\omega; 0) = 0$, and obtains the basis matrix

$$\Psi(t) = \left[\begin{array}{c} Re(w^\intercal e^{-i\omega t}) \\ Im(w^\intercal e^{-i\omega t}) \end{array} \right].$$

One must then compute Γ explicitly after.

Thus, the existence of a small solution to Problem A is equivalent to the set $\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ being nonempty. It is therefore important that we determine conditions under which that set is nonempty. Also, it would be prudent to ensure that as $\gamma \to 0$, the cost of ensuring the convergence rate $O(e^{-\gamma t})$ for a fixed pair $(\epsilon, B_\epsilon^\gamma)$ should become arbitrarily small and that the intersections (12.23) are nonempty. Our sufficient condition is the following.

Theorem 12.3.1. *Let $\mathcal{U} = \mathcal{B}^k$. $\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ is nonempty provided $\Phi(0)$ and $\Psi(0)$ are of rank c. If this is the case, for any selection $\gamma \mapsto B_\epsilon^\gamma$ of minimizing jump functionals for rate parameter γ, one has $\lim_{\gamma \to 0^+} B_\epsilon^\gamma = 0$. In particular, if $\gamma > 0$ is sufficiently small, the sets in (12.23) are nonempty.*

The subspace and rank condition of Theorem 12.3.1 allows us to quickly exclude memoryless systems and systems at fold bifurcation points. We have the following corollary.

Corollary 12.3.1.1. *If $c = 1$ or $L_0\phi = C\phi(0)$ for an $n \times n$ matrix C, then $\widetilde{\mathcal{Y}}(\epsilon; \gamma)$ is nonempty.*

We can derive a more general sufficient condition that is implied by the subspace and rank condition of Theorem 12.3.1. It is captured by the following corollary whose proof is omitted since it is similar to that of the previous theorem.

Corollary 12.3.1.2. *The subspace condition $\mathcal{U} = \mathcal{B}^k$ and the rank condition on $\Phi(0)$ and $\Psi(0)$ in Theorem 12.3.1 can be replaced with the condition $\mathcal{M}_\epsilon(\mathcal{U}) = \mathbb{R}^{c \times c}$, and the conclusions of the theorem hold.*

12.3.2   Main existence results: Problem B

As it turns out, the proofs of Proposition 12.3.1 and Theorem 12.3.1 work with minimal modifications to solve Problem B. We have the following analogues and the appropriate variant of Corollary 12.3.1.1.

Proposition 12.3.2. *With the same notation as in Proposition 12.3.1, define the set*

$$\widetilde{\mathcal{Y}}^{\eta}(h;\gamma) = \{B \in \mathcal{U} \subseteq \mathcal{B}^k : \forall |\epsilon| \leq \eta, \ e^{\gamma/h}\rho(\mathcal{M}_{\epsilon,h}(B)) \leq 1\}. \tag{12.24}$$

Let h be fixed. The following are true.

1. *$\widetilde{\mathcal{Y}}^{\eta}(h;\gamma)$ is closed and if the spectral gap satisfies $\gamma < \sigma$, there exists $\delta > 0$ such that for $\eta \leq \delta$,*

$$B_{\delta}(0) \cap \widetilde{\mathcal{Y}}^{\eta}(h;\gamma) = B_{\delta}(0) \cap \mathcal{Y}^{\eta}(h;\gamma). \tag{12.25}$$

2. *$\widetilde{\mathcal{Y}}^{\eta}(h;\gamma)$ can be written as the intersection*

$$\widetilde{\mathcal{Y}}^{\eta}(h;\gamma) = \bigcap_{|\epsilon| \leq \eta} \widetilde{\mathcal{Y}}(\epsilon;\gamma). \tag{12.26}$$

3. *There exists B such that $\mathcal{C} : \widetilde{\mathcal{Y}}^{\eta}(h;\gamma) \to \mathbb{R}$ attains its minimum at B, assuming the domain is nonempty.*

Theorem 12.3.2. *Let $\mathcal{U} = \mathcal{B}^k$. $\widetilde{\mathcal{Y}}^{\eta}(h;\gamma)$ is nonempty provided $\Phi(0)$ and $\Psi(0)$ are of rank c. If this is the case, then for any selection $\gamma \mapsto B^{\gamma}$ of minimizing jump functionals for rate parameter γ, one has $\lim_{\gamma \to 0^+} B^{\gamma} = 0$. In particular, if $\gamma > 0$ is sufficiently small, the sets in (12.25) are nonempty.*

Corollary 12.3.2.1. *If $c = 1$ or $L_0\phi = C\phi(0)$ for an $n \times n$ matrix C, then $\widetilde{\mathcal{Y}}^{\eta}(h;\gamma)$ is nonempty.*

Corollary 12.3.2.2. *The subspace condition $\mathcal{U} = \mathcal{B}^k$ and the rank condition on $\Phi(0)$ and $\Psi(0)$ in Theorem 12.3.2 can be replaced with the condition $\mathcal{M}_{\epsilon}(\mathcal{U}) = \mathbb{R}^{c \times c}$ for $|\epsilon| \leq \eta$, and the conclusions of the theorem hold.*

Remark 12.3.3. *Since $\epsilon \mapsto \mathcal{M}_{\epsilon}$ is continuous, the conclusion of the above corollary is guaranteed to hold for some $\eta > 0$, provided $\mathcal{M}_{0,h}(\mathcal{U}) = \mathbb{R}^{c \times c}$. This can be seen by vectorizing the monodromy map and recalling that rank function $X \mapsto rank(X)$ is lower semicontinuous.*

12.4 Practical computation: centre probe method

Solving the optimization problem

$$\begin{aligned} \text{minimize} \quad & \mathcal{C}_h(Y), \\ \text{subject to} \quad & Y \in \widetilde{\mathcal{Y}}(\epsilon; \gamma), \end{aligned} \tag{Y}$$

directly appears to be very difficult. The feasible set, introduced in Proposition 12.3.1, is characterized as a sublevel set of $B \mapsto \rho(\mathcal{M}_{\epsilon,h}(B))$. While $B \mapsto \mathcal{M}_{\epsilon,h}(B)$ is smooth, the spectral radius is is notably irregular and nonconvex. If one wishes to guarantee feasibility it is generally necessary to optimize in the space $\mathcal{B}^k$, whose dimension is $kn^2(\ell+1)$ and can be quite large even for small networks. It is therefore critical that we reduce the dimension of the problem before even considering performing an optimization task.

From this point onward, we will assume that a rate parameter γ, system parameter ϵ and frequency h have been chosen. We will suppress all dependence on these variables unless necessary. Also, we make the following simplifying assumption on our chosen space of cycles of jump functionals of period $k \geq 1$.

Assumption. *The subspace $\mathcal{U}^{(k)} \subseteq \mathcal{B}^k$ of cycles of jump functionals is given by k copies of a single subspace $\mathcal{U} \subseteq \mathcal{B}$. That is, $\mathcal{U}^{(k)} = \mathcal{U}^k$.*

This assumption is not strictly needed. All constructions (eg. probe space) can be appropriately generalized to allow the individual subspaces making up the product $\mathcal{U}^{(k)}$ to be distinct, and the major theorems (Theorem 12.4.1 and Theorem 12.4.2) have appropriate and similarly strong analogues. However, the notation can make the presentation difficult to follow. For this reason, we will specialize to this particular case.

To avoid ambiguity later, we define $\mathcal{M}_1 : \mathcal{U} \to \mathbb{R}^{c \times c}$ by

$$\mathcal{M}_1(B) = (I_{c \times c} + \Gamma \Psi(0) B \Phi_0) \exp\left(\frac{1}{h} \left(\Lambda + \Gamma \Psi(0) L(\epsilon) \Phi_0 \right) \right),$$

and reserve the symbol $\mathcal{M}$ for the centre monodromy operator $\mathcal{M} : \mathcal{U}^k \to \mathbb{R}^{c \times c}$. Note that this implies the factorization $\mathcal{M}(B) = \prod_{j=0}^{k-1} \mathcal{M}_1(B_j)$.

12.4.1 The probe space $\mathcal{P}$

The map $\mathcal{M}_1 : \mathcal{U} \to \mathbb{R}^{c \times c}$ is affine, and we can decompose it as

$$\mathcal{M}_1 = M_0 + Z, \;\; Z = \exp\left(\frac{1}{h} \left(\Lambda + \Gamma \Psi(0) L(\epsilon) \Phi_0 \right) \right), \;\; M_0(B) = \Gamma \Psi(0) B \Phi_0 Z,$$

where $M_0 : \mathcal{U} \to \mathbb{R}^{c \times c}$ is linear.

Definition 12.4.1. *The* probe space $\mathcal{P} \subset \mathbb{R}^{c \times c}$ *is the image of* $\mathcal{M}_1$. *That is,* $\mathcal{P} = \operatorname{im}(\mathcal{M}_1)$. *Elements* $P \in \mathcal{P}$ *are called* probe elements. *The* k-probe space *is the kth Cartesian power* $\mathcal{P}^k$, *and its elements are* k-probe elements.

The following lemmas describe the geometry of the probe space, k-probe space and preimages of probe elements under the centre monodromy operator. The proofs are omitted.

Lemma 12.4.1. *The probe space is an affine space of dimension at most* c^2. *It can written in the form* $\mathcal{P} = Z + \operatorname{im}(M_0)$.

Remark 12.4.1. *If the subspace/rank condition of Theorem 12.3.1 / Theorem 12.3.2 is satisfied, then* $\mathcal{P}$ *is a vector subspace of* $\mathbb{R}^{c \times c}$ *and it is precisely* $\mathcal{P} = \operatorname{im}(M_0)$.

Lemma 12.4.2. *The* k-probe space $\mathcal{P}^k$ *is convex.*

Lemma 12.4.3. *Let* $P \in \mathcal{P}$. *The preimage of* P *under the centre monodromy map* $\mathcal{M}_1$ *is an affine space, and can be written*

$$\mathcal{M}_1^{-1}(P) = M_0^+(P - Z) + \ker(M_0). \tag{12.27}$$

The correspondence $\mathcal{M}_1^{-1} : \mathcal{P} \rightrightarrows \mathcal{U}$ exhibits a concavity-like property that will be essential in the next section. It is summarized by the following lemma.

Lemma 12.4.4. *Let* $X, Y \in \mathcal{P}$. *For all* $t \in [0, 1]$,

$$t\mathcal{M}_1^{-1}(X) + (1 - t)\mathcal{M}_1^{-1}(Y) \subseteq \mathcal{M}_1^{-1}(tX + (1 - t)Y). \tag{12.28}$$

At this stage we should define the precise link between the k-probe space and the image of $\mathcal{M} : \mathcal{U}^k \to \mathbb{R}^{c \times c}$. Define the product function $G : (\mathbb{R}^{c \times c})^k \to \mathbb{R}^{c \times c}$ by

$$G(X_0, \ldots, X_{k-1}) = X_{k-1} \cdots X_0.$$

In terms of the product function, the image of $\mathcal{M} : \mathcal{U}^k \to \mathbb{R}^{c \times c}$ can be written equivalently as the image of $G : \mathcal{P}^k \to \mathbb{R}^{c \times c}$. From this equivalence, we obtain the following lemma.

Lemma 12.4.5. *Let* $P \in \mathcal{P}^k$. *For all* $B \in \mathcal{M}^{-1}(G(P))$, *we have* $\rho(\mathcal{M}(B)) = \rho(G(P))$.

The above lemma provides one way to pull k-probe elements back to cycles of jump functionals of period k. Namely, compute the ordered product and take the preimage under $\mathcal{M} : \mathcal{U}^k \to \mathbb{R}^{c \times c}$. However, the preimage under $\mathcal{M} : \mathcal{U}^k \to \mathbb{R}^{c \times c}$ of a given $c \times c$ matrix is not necessarily convex. To remedy this, we can alternatively define a map $\mathcal{M}_k : \mathcal{U}^k \to (\mathbb{R}^{c \times c})^k$ by

$$\mathcal{M}_k(B_0, \dots, B_{k-1}) = (\mathcal{M}_1(B_0), \dots, \mathcal{M}_1(B_{k-1})).$$

Then, an analogue of Lemma 12.4.5 is as follows, strengthened by the convexity of $\mathcal{M}_k^{-1}(P)$ for any $P \in \mathcal{P}^k$ due to the componentwise convexity afforded by Lemma 12.4.3.

Lemma 12.4.6. *Let $P \in \mathcal{P}^k$. For all $B \in \mathcal{M}_k^{-1}(P)$, we have $\rho(\mathcal{M}(B)) = \rho(G(P))$. Also, $\mathcal{M}_k^{-1}(P)$ is convex and given $P = (P_0, \dots, P_{k-1})$, it can be written*

$$\mathcal{M}_k^{-1}(P) = \{(B_0, \dots, B_{k-1}) : B_i \in \mathcal{M}_1^{-1}(P_i),\ i = 0, \dots, k-1\}.$$

12.4.2 Compatible probe cost

Broadly speaking, our idea for solving the nonlinear program (Y) is to first solve a related optimization problem in the probe space and obtain an optimal solution P^*, pull this optimal solution into the convex space $\mathcal{M}_k^{-1}(P^*)$ and find a minimizer Y_{P^*} of the cost functional. Since every element of $\mathcal{M}^{-1}(P^*)$ will be feasible provided $\rho(P^*) \leq e^{-\gamma/h}$ – see Lemma 12.4.5 – the spectral constraint does not need to be checked at this final optimization stage. The following definition allows us to define the appropriate optimization problem in $\mathcal{P}^k$.

Definition 12.4.2. *A continuous, radially unbounded convex function $\tilde{C} : \mathcal{P}^k \to \mathbb{R}^+$ is a:*

- local probe-compatible cost (LPCC) *for the nonlinear program (Y) if one of the following conditions hold:*

  1. $\tilde{C} \circ \mathcal{M}_k(B) \leq \mathcal{C}_h(B)$ for all $B \in \mathcal{U}$, with equality if

  $$B \in \underset{X \in \mathcal{M}^{-1}(\mathcal{M}(B))}{\arg\min}\ \mathcal{C}_h(X). \tag{12.29}$$

  In this case we say $\tilde{C}$ is a type 1 LPCC.

  2. $\tilde{C} \circ \mathcal{M}_k(X) \leq \tilde{C} \circ \mathcal{M}_k(Y)$ implies $\mathcal{C}_h(X) \leq \mathcal{C}_h(Y)$. In this case we say $\tilde{C}$ is a type 2 LPCC.

- global probe-compatible cost (GPCC) *if for any global optimum B^* of the nonlinear program (Y) and any other feasible solution B, one has $\tilde{C} \circ \mathcal{M}_k(B^*) \leq \tilde{C} \circ \mathcal{M}_k(B)$, with equality holding if and only if B is also a global optimum.*

- uniform probe-compatible cost (UPCC) *if it is both a LPCC and a GPCC.*

The existence of GPCC/LPCCs will be addressed in Theorem 12.4.2. Given a continuous, radially unbounded convex function $\tilde{C} : \mathcal{P}^k \to \mathbb{R}^+$, we define a new nonlinear program in the probe space, which we call the *probe program*:

$$
\begin{aligned}
\text{minimize} \quad & \tilde{C}(X), \\
\text{subject to} \quad & X \in \mathcal{P}^k, \\
& \rho \circ G(X) \leq e^{-\gamma/h}.
\end{aligned} \qquad (\mathcal{P}\tilde{C})
$$

The program ($\mathcal{P}\tilde{C}$) possesses a global optimum so long as $\widetilde{\mathcal{Y}}$ is nonempty. We also define a family of convex programs indexed by $P \in \mathcal{P}^k$ with feasible set given by the preimage $\mathcal{M}_k^{-1}(P)$. We call this the *inverse probe program*:

$$
\begin{aligned}
\text{minimize} \quad & \mathcal{C}(B), \\
\text{subject to} \quad & B \in \mathcal{M}_k^{-1}(P).
\end{aligned} \qquad (Y_P)
$$

Remark 12.4.2. *As $\mathcal{M}_k^{-1}(P)$ is an external direct sum of k affine subspaces of $\mathcal{B}$ – see Lemma 12.4.3 and Lemma 12.4.6 – the inverse probe program is actually unconstrained after an appropriate affine linear change of variables.*

Our first theorem of this section relates the programs ($\mathcal{P}\tilde{C}$) and (Y_P) to solutions of (Y) under the assumption that $\tilde{C}$ is a LPCC or a GPCC. The second theorem establishes the existence of at least one UPCC.

Theorem 12.4.1 (centre probe method). *Assume $\widetilde{\mathcal{Y}}$ is nonempty. Let $\tilde{C} : \mathcal{P}^k \to \mathbb{R}^+$ be a global (resp. local) PCC for the nonlinear program (Y). Let $P \in \mathcal{P}^k$ be a global (resp. local) optimum for the program ($\mathcal{P}\tilde{C}$). Let $B^* \in \mathcal{M}_k^{-1}(P)$ be a local optimum for the convex program (Y_P). Then, B^* is a global (resp. local) optimum for the program (Y).*

We will refer to the procedure of determining a solution of the program (Y) using the probe program in conjunction with the inverse probe program collectively as the *centre probe method* (CPM). A cartoon drawing of the method is provided in Figure 12.1.
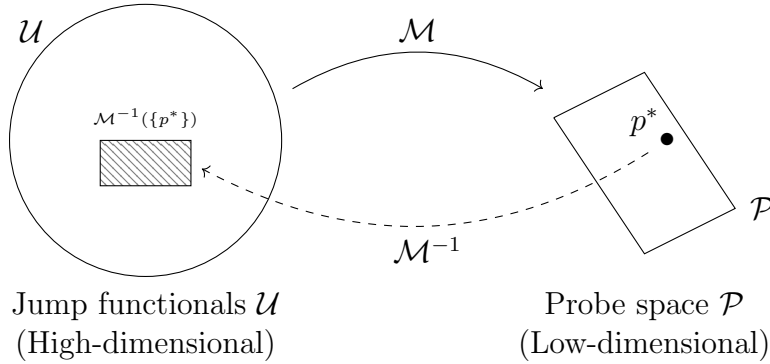
Figure 12.1: A cartoon drawing of the centre probe method. After finding an optimal probe element p^* in the probe space, the inverse probe program is run in the preimage under $\mathcal{M}$ if necessary.

Theorem 12.4.2. *Define* $\tilde{C} : \mathcal{P}^k \to \mathbb{R}^+$ *by*

$$\tilde{C}(P) = \min_{X \in \mathcal{M}_k^{-1}(P)} \mathcal{C}_h(X). \tag{12.30}$$

This function is continuous, radially unbounded, convex, and a UPCC. We will refer to it as the trivializing UPCC.

The trivializing UPCC is so named because it makes the final step of running the nonlinear program (Y_P) unnecessary. Indeed, once one has computed an optimal solution P for $(\mathcal{P}\tilde{C})$, the optimal cost for the program (Y) is precisely $\tilde{C}(P)$. In other words, running the program (Y_P) is equivalent to computing $\tilde{C}(P)$.

12.4.3 Explicit formula for trivializing UPCC

Suppose the cost functional $\mathcal{C} : \mathcal{U} \to \mathbb{R}^+$ is projective. Recall this means that $\mathcal{C}(B) = \langle B, B \rangle$ for an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{U}$ (note, on $\mathcal{U}^k$ one defines the cost additively according to (12.13)). See Section 12.4.3 for examples. Because of Lemma 12.4.3, we can calculate $\tilde{C}(P)$ for the trivializing UPCC according to

$$\tilde{C}(P) = k\mathcal{C}_h(0) + \sum_{i=0}^{k-1} \min \left\{ \langle X, X \rangle : X \in M_0^+(Z - P_i) + \ker(M_0) \right\}. \tag{12.31}$$

$\tilde{C}(P)$ and the associated minimizer in $\mathcal{M}^{-1}(P)$ in fact has an explicit solution, due the computation of $\tilde{C}(P)$ being equivalent to squared distance minimization to an affine subspace of a Hilbert space. The proof is elementary and is omitted.

Proposition 12.4.1. *Let $\{m_1, \ldots, m_b\}$ be an orthonormal basis for $\ker(M_0)$. For a given $P \in \mathcal{P}$, define $X(P) \in \mathcal{U}$ by*

$$X(P) = M_0^+(P - Z) - \sum_{j=1}^{b} \langle M_0^+(P - Z), m_j \rangle m_j. \tag{12.32}$$

If $\tilde{C}$ is the trivializing UPCC and the cost functional is projective, then for any $P = (P_0, \ldots, P_{k-1}) \in \mathcal{P}^k$, if we define $\mathbf{X}(P) = (X(P_0), \ldots, X(P_{k-1}))$, then

$$\tilde{C}(P) = \mathcal{C}(\mathbf{X}(P)).$$

Moreover, $M_0^+(P - Z)$ in equation (12.32) can be replaced by any solution Y of the linear equation $M_0 Y + Z = P$, and these are

$$Y = M_0^+(P - Z) + (I - M_0^+ M_0)w$$

for any $w \in \mathcal{U}$.

Example of projective cost functionals

A projective cost functional modeled on (12.14) and suitable for implementation (ie. vectorized) is

$$\mathcal{C}((A, A_1, \ldots, A_\ell)) = w_0 \vec{A}^\intercal W_0 \vec{A} + \sum_{i=1}^{\ell} w_i \vec{A}_i^\intercal W_i \vec{A}_i, \tag{12.33}$$

where $w_0, w_1, \ldots, w_\ell > 0$, $W_0, W_1, \ldots, W_\ell \in \mathbb{R}^{n^2 \times n^2}$ are symmetric positive-definite matrices and for $X \in \mathbb{R}^{n \times n}$, we define $\vec{X} \in \mathbb{R}^{n^2}$ to be its standard vectorization, obtained by stacking the columns of X on top of one another from left to right. The cost (12.33) is projective with the inner product

$$\langle (A, A_1, \ldots, A_\ell), (B, B_1, \ldots, B_\ell) \rangle = w_0 \vec{A}^\intercal W_0 \vec{B} + \sum_{i=1}^{\ell} w_i \vec{A}_i^\intercal \vec{B}_i.$$

12.4.4   Implementation and complexity of the CPM

Here we discuss the complexity of the CPM and implementation strategies. Given a global (respectively, local) PCC, the centre probe method can be split into three stages. Note that in any implementation, the operator $M_0 : \mathcal{B} \to \mathbb{R}^{c \times c}$ will need to be vectorized, with the result being a $c^2 \times n^2(\ell + 1)$ matrix.

1. Initialization step: computation of Z, a pseudoinverse M_0^+, a basis for $M_0(\mathcal{B})$ and a basis for $\ker(M_0)$

2. Probe program: compute a global (respectively, local) optimum $P \in \mathcal{P} = Z + M_0(\mathcal{B})$ of the program $(\mathcal{P}\tilde{C})$.

3. Inverse probe program: compute a local optimum $B^* \in \mathcal{B}$ of the program (Y_P) in the space $\mathcal{M}^{-1}(P) = M_0^+(Z - P) + \ker(M_0)$.

The initialization step requires a $c \times c$ matrix exponentiation and several matrix multiplications, as well a pseudoinverse calculation and row reductions for the basis calculations for $M_0(\mathcal{B})$ and $\ker(M_0)$. To obtain Z and M_0 we must first compute the basis matrix Φ_0 associated to the centre eigenspace E_0 and the evaluation $\Psi(0)$ of the basis matrix for $E_0^\intercal$. This task is very much related to the calculation of the zeroes of the characteristic equation (12.4); see Remark 12.3.1. One may consult [12, 30, 47, 87, 97] for relevant methods. As the initialization step needs only to be run once, however, its computational burden is minimal.

The inverse probe program is a convex program in the convex set $\mathcal{M}_k^{-1}(P) \subset \mathcal{B}^k$. Using Lemma 12.4.6 together with Lemma 12.4.3, this can be written as the Cartesian product

$$\mathcal{M}_k^{-1}(P) = \prod_{j=0}^{k-1}(M_0^+(P_j - Z) + \ker(M_0)).$$

By a change of variables, it is equivalent to an unconstrained problem in $\mathbb{R}^{kd}$ for $d = \dim(\ker(M_0))$, which in the worst case is $d = n^2(\ell + 1) - 1$. As this can be quite large, gradient descent variants are likely to perform best. In the special case where we choose the trivializing UPCC and the cost functional is projective, the inverse probe program is not necessary.

Thanks to the reduction principle inherent to the probe method, the optimization for the probe program is done in the convex set $\mathcal{P}^k \subseteq (\mathbb{R}^{c \times c})^k$, which after vectorization is

equivalent by a change of coordinates to $\mathbb{R}^{c'}$ for $c' \leq kc^2$. For a one-dimensional parameter $\epsilon \in \mathbb{R}$, the generic bifurcations have $c \leq 2$, while for two-dimensional parameters $\epsilon \in \mathbb{R}^2$ the generic bifurcations have $c \leq 4$. Moreover, arguably the most common "naturally occurring" bifurcation in delay differential equations – the Hopf bifurcation – has $c = 2$. Thus, it is typical for the ambient space of the probe program to be (relatively) low-dimensional.

Formulas for $\tilde{C}$ and any gradients can be explicitly derived for the trivializing UPCC, assuming a projective cost functional. Specifically, Proposition 12.4.1 together with equation (12.31) yields

$$\tilde{C}(P) = k\mathcal{C}_h(0) + \sum_{i=0}^{k-1} \langle X(P_i), X(P_i) \rangle,$$

with $X : \mathcal{P} \to \mathcal{B}$ being linear. We can then explicitly calculate the gradient:

$$\nabla \tilde{C}(P) = 2 \sum_{i=0}^{k-1} \left\langle X(P_i), M_0^+ - \sum_{j=1}^{b} m_j \langle m_j, M_0^+ \rangle \right\rangle, \tag{12.34}$$

with $\{m_1, \ldots, m_b\}$ the orthonormal basis for $\ker(M_0)$ computed in the initialization step. The term on the right inside the inner product can be computed once and stored in memory, with the result being that each gradient is equal in computational requirements to an evaluation of $\mathbf{X}(P)$ and the inner product calculation. A crude worst-case complexity (assuming the worst case for d as well) of this calculation is $O(kc^2 n^4)$ floating point operations assuming naive matrix multiplications.

The nonconvexity of the probe program does pose certain difficulties insofar as implementation is concerned. These all centre around the characterization of the feasible set as being the intersection of $\mathcal{P}^k$ with the sublevel set $\{X \in (\mathbb{R}^{c \times c})^k : \rho \circ G(X) \leq e^{-\gamma/h}\}$. As we recalled previously at the beginning of Section 12.4, the spectral radius $\rho : \mathbb{R}^{c \times c} \to \mathbb{R}$ is non-convex and non-smooth. However, on the set

$$V = \{X \in \mathbb{R}^{c \times c} : \text{every eigenvalue of } X \text{ is simple}\},$$

the spectral radius is continuously differentiable and locally Lipschitz continuous. V is also open and dense in $\mathbb{R}^{c \times c}$. As a consequence, except in exceptional circumstances where $\mathcal{P} \cap V$ is not dense in $\mathcal{P}$, methods based on gradient sampling [25, 67] can be used to solve the program $(\mathcal{P}\tilde{C})$ with guaranteed convergence results.

12.4.5 Extension to the uniform problem

The centre probe method of Section 12.4 can be adapted to find solutions of Problem B. Recall that the our proposed solution to Problem B derived from Proposition 12.3.2 requires us to find a constant $\eta > 0$ such that there exists a solution of the nonlinear program

$$
\begin{aligned}
\text{minimize} \quad & \mathcal{C}(Y), \\
\text{subject to} \quad & Y \in \widetilde{\mathcal{Y}}^{\eta}(h; \gamma).
\end{aligned}
\tag{Y^{η}}
$$

It is important to remark that a solution of Problem B *includes* the *robustness parameter* η. That is to say, η is not chosen at the outset. This is crucial, and in general one cannot take η as arbitrary.

Suppose for the sake of argument that one could choose $\eta > 0$ at the outset. One may recall from Proposition 12.3.2 that a solution B^{η} of (Y^{η}) can only be guaranteed to solve the nonlinear program (12.16) if $||B^{\eta}|| < \delta(\eta)$, otherwise there may be secondary bifurcations involving eigenvalues crossing the imaginary axis from the left. As $\delta(\eta)$ is generally decreasing with respect to η – see the proof of the aforementioned proposition – one can only *guarantee* stabilization with an increasingly trivial linear jump functional. However, unless the equilibrium $x^{*} = 0$ is already exponentially stable with rate $O(e^{-\gamma t})$, a trivial jump functional should not be able to provide stabilization. Therefore, generally, it is not possible to choose the robustness parameter; a controller that stabilizes the equilibrium at the parameter $\epsilon = 0$ will generally fail to stabilize the equilibrium if the parameter is taken too large.

12.4.6 Uniform CPM

Because of the smoothness of the vector field (12.1), any linear jump functional that guarantees the local convergence rate $O(e^{-(\gamma+s)t})$ at parameter $\epsilon = 0$ for some $s > 0$ will automatically guarantee the local convergence rate $O(e^{-\gamma t})$ for $|\epsilon| \leq \eta$, for some $\eta > 0$ that generally depends on s. This follows from hemicontinuity arguments; see the related proof of Proposition 12.3.1.

Theorem 12.4.3 (Uniform CPM). *Let $B \in \mathcal{U}^{k}$ be a linear jump functional produced by the CPM (local or global) at parameter $\epsilon = 0$ and convergence rate γ' satisfying $\sigma > \gamma' = \gamma + s$, for $s > 0$ a small safety parameter. There exists $\eta = \eta(s) > 0$ such that B is a feasible solution of the program (Y^{η}).*

The above theorem does not guarantee optimality of the candidate solution. However, for typical problems where the performance (linear-order convergence rate) of the candidate B deteriorates when $|\epsilon|$ becomes large, we are guaranteed that the solution is optimal for some robustness parameter.

Corollary 12.4.3.1 (Optimality). *With the notation from the previous theorem, suppose for some $\epsilon \neq 0$, $\rho(\mathcal{M}_\epsilon(B)) > e^{-\gamma/h}$. Then, there exists $\eta \in (0, |\epsilon|)$ such that B is an optimum (local or global) of the program* (Y^η).

12.5 Stabilization of a complex network near a Hopf point

In this section we consider primarily by way of example how our uniform CPM can be use to stabilize a complex network near a Hopf point. We incorporate an inhomogeneous weighting on the cost of controlling each node and address how one can minimize the number of controlled nodes while still taking advantage of the performance improvements inherent to .

The neural network we will consider in this section is a slight modification of an example considered in [65]. For $x_i \in \mathbb{R}^2$ for $i = 1, \ldots, 100$, consider the nonlinear network model

$$\dot{x}_i = -x_i(t) + \epsilon \big[B \tanh(x_i(t)) + D \tanh(x_i(t-1)) \big] + \sum_{j=1}^{N} a_{ij} x_j(t) \qquad (12.35)$$

with connection weight matrices

$$B = \begin{bmatrix} 2 & -0.11 \\ -5 & 3.2 \end{bmatrix}, \qquad D = \begin{bmatrix} -1.6 & -0.1 \\ -0.18 & -2.4 \end{bmatrix}$$

and linear coupling determined by the matrix $A = (a_{ij})_{N \times N}$, which is the negative of a graph Laplacian associated to a small world network graph on 100 nodes. We define $\tanh(y) = [\ \tanh(y_1)\ \ \tanh(y_2)\ ]^\intercal$ for $y = (y_1, y_2) \in \mathbb{R}^2$, and $\epsilon \geq 0$ is a neural activation strength parameter. It is known [65] that when decoupled, the individual nodes determined by the dynamical system

$$\dot{y} = -y + C \tanh(y(t)) + D \tanh(y(t-1))$$

exhibit chaotic dynamics with a double-scroll-like attractor, and that the origin is unstable. Thus, when $\epsilon = 1$, the diffusivity condition implies that the nonlinear network model

(12.35) is chaotic and the origin is unstable. However, when $\epsilon = 0$, the origin is globally asymptotically stable. There is therefore a bifurcation at some critical activation strength $\epsilon^* \in (0, 1)$.

Our goal is to robustly stabilize the network (12.35) in a parameter neighbourhood of ϵ^* with frequency $h = 1$ and period $k = 1$ using our uniform CPM. We will assume decoupled controls, so that we take our candidate jump functionals from the diagonal subspace $\mathcal{B}_{diag}$. To introduce some heterogeneity, we will assume that some nodes are more difficult to control than others, and we will incorporate this into the associated cost functional. Specifically, nodes with a higher degree of connectivity will be assigned a higher cost. We will also attempt to minimize the number of controlled nodes.

Our methodology is as follows. First, we recall the small-world network topology and introduce the cost functional we will be using in Section 12.5.1. Then, we encode the parameter-dependent system (12.35) as the user input to DDE-BIFTOOL and use the included `GetStability` routine to determine the critical parameter ϵ^* where the bifurcation occurs, and determine its type. We also use this routine to compute the critical eigenvalues $\lambda = \pm i\omega$ on the imaginary axis. We then implement the uniform CPM at the critical parameter ϵ^* and determine a locally cost-minimizing controller B^0 and a generating probe element P^* for a target local convergence rate $O(e^{-\frac{t}{5}})$ – that is, we take the target rate parameter to be $\gamma = \frac{1}{5}$. These steps are carried out in Section 12.5.2.

The feasible jump functional B^0 is cost-minimizing in the affine space $\mathcal{M}^{-1}(P^*)$ but might not minimize the number of controlled nodes. We provide a solution to this secondary minimization problem in Section 12.5.3.

In Section 12.5.4 we assess the performance of the jump functional derived using the uniform CPM in conjunction with the node-minimizing process of Section 12.5.3. Specifically, we compare the output of the neural network model with and without our pinning control at a range of parameters $\epsilon \in [\epsilon^*, \epsilon_2^*)$. The bifurcation is suppressed in this regime, but a secondary bifurcation point is identified at the parameter value $\epsilon = \epsilon_2^* < 1$. The implications of this bifurcation are discussed.

12.5.1   Network topology, cost functional and the trivializing UPCC

Small-world networks [91] capture a network topology involving both a high degree of clustering and short average path-lengths. They can be constructed by starting with a ring lattice on N vertices with k edges per vertex (specifically, k nearest neighbours) and "rewiring" each edge randomly with a specified probability, p.
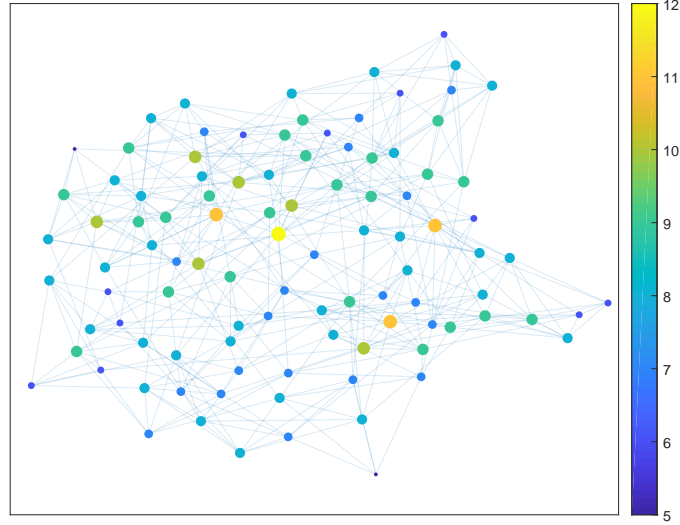
Figure 12.2: The small-world graph used in the example of Section 12.5. Nodes are coloured with intensity varying based on their degree, while the sizes indicate relative degree. Parameters were $N = 100$ nodes with initial connections to $k = 8$ nearest neighbours and rewiring probability $p = 0.3$.

We wrote a script in MATLAB R2018a to generate a Watts-Strogatz small-world graph **G** on $N = 100$ vertices with parameters $k = 8$ and $p = 0.3$. The matrix A defining the linear coupling in (12.35) is then obtained by taking the negative graph Laplacian: $A = -\texttt{laplacian}(\mathbf{G})$. The computed graph that was used in this example is displayed in Figure 12.2.

The degree of node i is precisely $|a_{ii}| \geq 1$. Based on this, if we uniquely write $B \in \mathcal{B}_{diag}$ as a tuple $B = (B_1, \ldots, B_{100})$ for $B_i \in \mathbb{R}^{2 \times 2}$, we can define the cost functional $\mathcal{C} : \mathcal{B}_{diag} \to \mathbb{R}$,

$$\mathcal{C}(B) = \frac{1}{\Delta(\mathbf{G})} \sum_{i=1}^{100} |a_{ii}| \langle B_i, B_i \rangle_F,$$

where $\langle \cdot, \cdot \rangle_F$ is the Frobenius inner product on $\mathbb{R}^{2 \times 2}$. Thus, the cost of controlling node i is linearly scaled relative to its degree, normalized with respect to the maximum degree. This cost functional is projective with inner product

$$\langle X, Y \rangle = \frac{1}{\Delta(\mathbf{G})} \sum_{i=1}^{100} |a_{ii}| \langle X_i, Y_i \rangle_F.$$

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As such, the trivializing UPCC has an explicit formula from Proposition 12.4.1:

$$\tilde{C}(P) = \left\| M_0^+(Z-P) - \sum_{j=1}^{b} \langle M_0^+(Z-P), m_j \rangle m_j \right\|^2 \tag{12.36}$$

and $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$ is the norm induced by $\langle \cdot, \cdot \rangle$.

12.5.2   Precomputation and uniform CPM

DDE-BIFTOOL [30] was used to identify parameters where bifurcations in the neural network model (12.35) occur. The tool detected a Hopf bifurcation (centre subspace of dimension $c = 2$) at parameter $\epsilon = 0.5621$. Further numerical examination revealed that the trivial equilibrium is locally stable at parameter $\epsilon \leq 0.5620$, and unstable at parameter $\epsilon \geq 0.5621$. For our purposes, we chose $\epsilon^* = 0.5621$ to be the approximate bifurcation point. The critical eigenvalues were also computed by DDE-BIFTOOL, and it was found that they are $\lambda^* = 0.0001 \pm 0.375i$. Note that the real part is positive, which is a consequence of our choosing the parameter on the unstable side of the bifurcation point. We allow ourselves to be content with this approximation.

To calculate the matrices $\Phi(t)$ and $\Psi(t)$, we numerically computed the right and left eigenvectors associated to the eigenvalue of the characteristic matrix $\Delta(\lambda^*; \epsilon^*)$ with the smallest absolute value. If ϵ^* were the true bifurcation parameter rather than a numerical approximation and λ^* was the true critical eigenvalue, we would merely compute the kernel. Then, we calculate the matrices $\Phi(t)$ and $\Psi(t)$ following Remark 12.3.2. The rank of $\Phi(0)$ and $\Psi(0)$ were both verified to be equal to 2. Following this we defined a three-dimensional cell array $U = \texttt{cell}(100, 2, 2)$ and populated it with basis elements for $\mathcal{B}_{diag}$ according to the assignment

$$U\{i, j, k\} = (0, \ldots, 0, E_{jk}, 0, \ldots, 0),$$

with the nonzero entry is in the ith position. An ordered basis $\{V_1, \ldots, V_{400}\}$ was then defined according to the rule

$$V_{4(i-1)+2(j-1)+k} = U\{i, j, k\}, \qquad i = 1, \ldots, 100, \ j, k \in \{1, 2\}.$$

Then, relative to the aforementioned ordered basis for $\mathcal{B}_{diag}$ and the standard ordered basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ for $\mathbb{R}^{2 \times 2}$, the map $M_0 : \mathcal{B}_{diag} \to \mathbb{R}^{2 \times 2}$ was vectorized as a 4×400 matrix. The result was rank 4, and it follows that our example satisfies Corollary 12.3.2.2 with $\eta = 0$ (after a change of coordinates, treating ϵ^* as zero). Following Remark 12.3.3, we are guaranteed that the uniform CPM possesses at least one feasible solution.

The probe program ($\mathcal{P}\tilde{C}$) was solved in MATLAB R2018a using the smooth constrained solver `fmincon` from the optimization toolbox. The explicit formula for the trivializing UPCC from (12.36) was used, and we supplied the solver with an explicit gradient. The solver was initialized at the infeasible matrix (vectorized)

$$P^0 = \left[\begin{array}{cc} 100 & 100 \\ 100 & 100 \end{array} \right]$$

and converged quickly to the feasible matrix (vectorized)

$$P^* = \left[\begin{array}{cc} 0.7317 & 0.2672 \\ -1.4729 & 0.3783 \end{array} \right]$$

satisfying first-order optimality. As the objective is not actually smooth, the optimality is not guaranteed (internally, MATLAB checks stationarity conditions using finite differences, and since the spectral radius map is generally non-differentiable these are not typically informative). We attempted to refine the solution using the non-smooth solver `patternsearch`, but to standard tolerances no improvement could be made. Proposition 12.4.1 provides a feasible jump functional $B^0 = X(P^*)$, but as we mentioned earlier, this candidate does not minimize the number of controlled nodes.

12.5.3 Minimizing the number of controlled nodes

As discussed in the outline, the jump functional B^0 minimizes the cost functional, but it does not necessarily control a minimal number of nodes. To address this, we recall that because of Lemma 12.4.3 and Proposition 12.4.1, the cost functional is in fact constant on an affine hyperplane. Specifically, this hyperplane is

$$\mathcal{H}(P^*) = \left\{ M_0^+(P^* - Z) + Qy - \sum_{j=1}^{b} \langle M_0^+(P^* - Z) + Qy, m_j \rangle m_j : y \in \mathcal{U} \right\},$$

$$= \left\{ B^0 + Qy - \sum_{j=1}^{b} \langle Qy, m_j \rangle m_j : y \in \mathcal{U} \right\},$$

where $Q = I - M_0^+ M_0$. Thus, if we wish to minimize the number of controlled nodes while maintaining the minimal cost, it suffices to solve the following sequence of unconstrained

nonlinear programs:

$$\text{minimize} \quad \sum_{i=1}^{100} \tanh\left(\mu||B_i^0 + \pi_i(\mathcal{Q}y)\,||_2\right), \tag{N^μ}$$
$$\text{subject to} \quad y \in \mathcal{U},$$

where we define the linear operator $\mathcal{Q} : \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{Q}y = Qy - \sum_{j=1}^{b} \langle Qy, m_j \rangle m_j,$$

and $\pi_i(B_1, \ldots, B_{100}) = B_i$ is the projection onto the ith factor. The objective function of (N^μ) precisely counts the number of nodes controlled by the candidate $B^0 + \mathcal{Q}y \in \mathcal{H}(P^*)$ in the limit $\mu \to \infty$. Consequently, if y^μ is a solution for parameter μ and $y^\mu \to y^*$, then

$$B^* = B^0 + \mathcal{Q}y^* \tag{12.37}$$

is a feasible jump functional having optimal cost that controls a minimal number of nodes.

After appropriate vectorization consistent with Section 12.5.2, the nonlinear program (N^μ) was solved using the unconstrained solver `fminunc` from MATLAB R2018a initially with parameter $\mu = 1$ and initialized at $y = 0$. Subsequently, the program was solved with increments $\mu \mapsto \mu + \frac{1}{2}$ and initialized at the previous solution. Upon reaching $\mu = 3$, the solver made step sizes smaller than 10^{-7} and we manually halted the process.

The resulting jump functional $B^* = (B_1^*, \ldots, B_{100}^*)$ was cleaned by setting $B_i^* \mapsto 0$ if $||B_i^*||_F < 10^{-4}$, where $|| \cdot ||_F$ is the Frobenius norm. The result was a jump functional that pinned fourteen nodes. Defining $||B_i^*||$ to be the absolute gain of node i, a plot of the absolute gain relative to the degree is provided in Figure 12.3. From this figure, it is clear that our algorithm prioritized the pinning of nodes that have a low degree. Indeed, all nodes with degree 5 and 6 are pinned, and the only other node to be pinned was the degree 9 node at index 61.

12.5.4 Performance of the controller and a secondary bifurcation

Plots of sample trajectories of the system without the pinning controller are given in Figure 12.4, while those with pinning are provided in Figure 12.5. Several illustrative parameters ϵ are used.
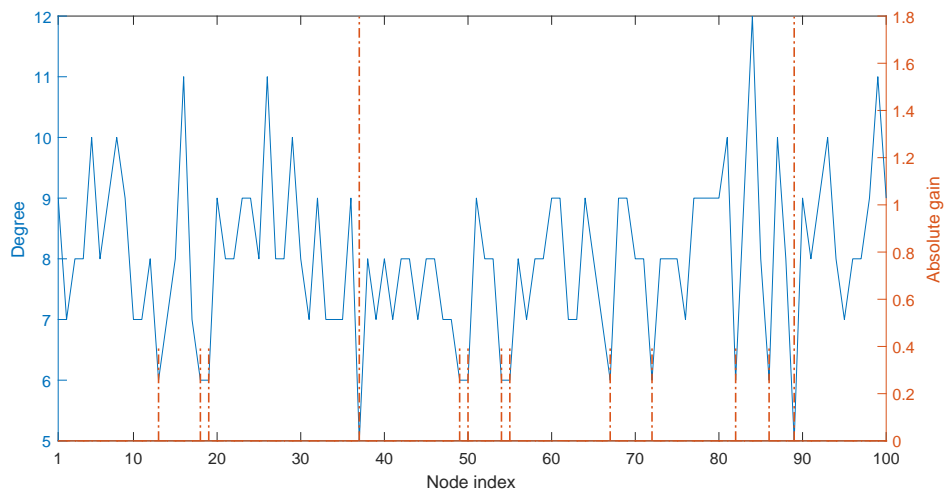
Figure 12.3: Plot of the degree (curve, solid blue) of nodes $i = 1, \ldots, 100$ versus the absolute gain (stems, dashed dot orange) of node i by the cleaned jump functional obtained by the uniform CPM for the example in Section 12.5. Fourteen nodes are controlled – the small gain applied to node 61 is not visible under the scale of the present graph, as the associated absolute gain was $||B_{61}^*|| = 0.01$. Note that more expensive nodes were assigned a lower gain control, and vice versa.

(a) $\epsilon = 0.5621$

(b) $\epsilon = 0.59$

(c) $\epsilon = 0.61$

(d) $\epsilon = 0.63$

Figure 12.4: Sample time series (time on horizontal axis) from random constant initial conditions drawn from the standard normal distribution, rescaled to the interval $[-1, 1]$, for the example from Section 12.5. The neural activation parameter ϵ for the given simulation is listed below its frame. Notice the transition from a stable periodic orbit to a stable equilibrium in the parameter interval $[0.61, 0.63]$, indicative of another bifurcation point.

(a) $\epsilon = 0.5621$ (b) $\epsilon = 0.59$

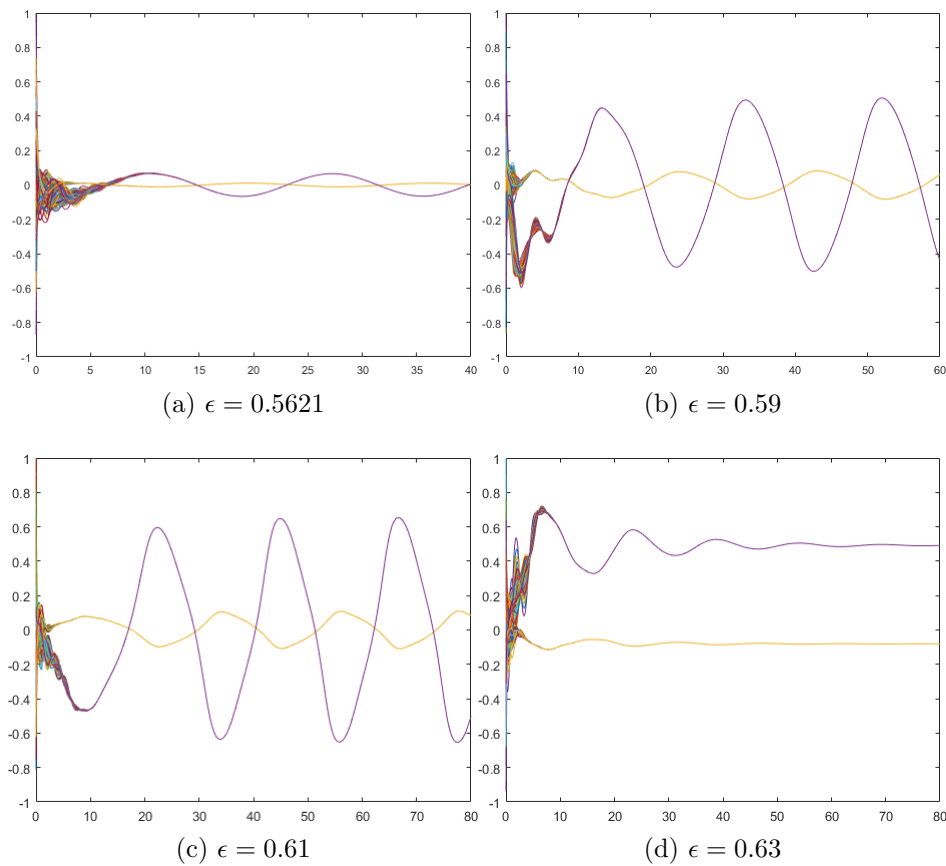(c) $\epsilon = 0.61$ (d) $\epsilon = 0.63$
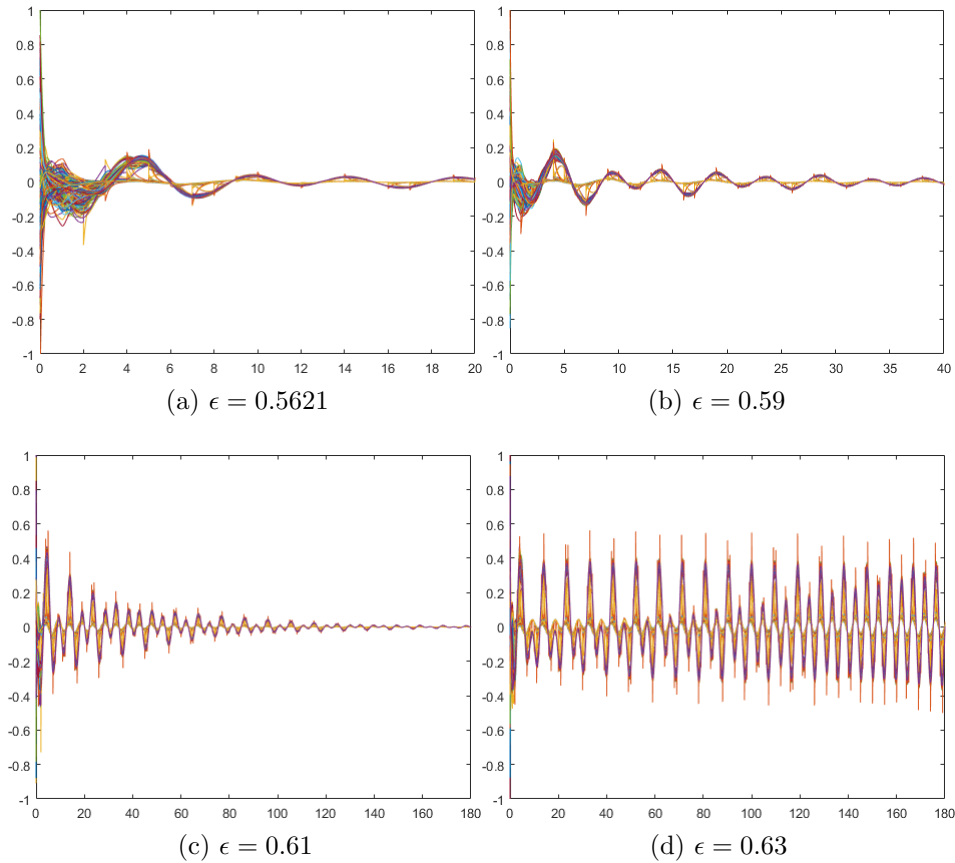
Figure 12.5: Sample time series (time on horizontal axis) from random constant initial conditions drawn from the standard normal distribution, rescaled to the interval $[-1, 1]$, with different neural activation parameters. (a) Activation parameter $\epsilon = 0.5621$; the trivial equilibrium is quickly stabilized. (b) Activation parameter $\epsilon = 0.59$; the trivial equilibrium is exponentially stabilized but with a slightly smaller rate parameter. (c) Activation parameter $\epsilon = 0.61$; the trivial equilibrium is stabilized, but the rate parameter is low. (d) Activation parameter $\epsilon = 0.63$; the control no longer stabilizes the trivial equilibrium. A secondary bifurcation has occurred.

By comparing the two figures, it should be clear that our pinning control B^* derived from the uniform CPM with subsequent node minimization succeeds in exponentially stabilizing the trivial equilibrium in the activation parameter interval $[\epsilon^*, 0.61]$, while when our controller is not present, the equilibrium is unstable. Increased volatility and decreased convergence rates follow as the parameter increases, and a secondary bifurcation occurs somewhere in the parameter interval $[0.61, 0.63]$. This is consistent with our observation that the system without the pinning control appears to undergo a bifurcation in the interval $[0.61, 0.63]$; see Figure 12.4. We suggested in the preamble to Section 12.4.5 that such secondary bifurcations of the model without impulses may be responsible for poor performance or complete failure of the impulsive controller if the parameter too far away from the bifurcation point. The presence of a secondary bifurcation in the interval $[0.61, 0.63]$ in both cases – with and without the controller – is consistent with this claim.

12.6   Strengths and weaknesses of the CPM

We have proposed a method of stabilizing a candidate equilibrium point of an autonomous delay dynamical system at or near a bifurcation point using a novel impulsive stabilization approach based on invariant manifold theory. We have also introduced cost structure; our method identifies a jump functional (impulsive controller) that minimizes a given cost functional, should certain controllers have cost associated to their implementation. The method, which we call the centre probe method (CPM), takes advantage of the dimension reduction inherent to the dynamics on the centre manifold. If the cost is projective, the CPM can be implemented with great efficiency.

Our method differs from the majority of impulsive stabilization methods in that one does not design the controller to satisfy a system of linear matrix inequalities, but rather solves a pair of optimization problems. The most computationally expensive of the two – the probe program $(\mathcal{P}\tilde{C})$ – is done in a low-dimensional space determined by the number of eigenvalues crossing the imaginary axis at the bifurcation point. The second one – the inverse probe program (Y_P) – is convex, smooth and unconstrained (in an appropriate coordinate system), and can be solved efficiently using out-of-the-box nonlinear solvers. This is the greatest strength of our method compared to those derived from Lyapunov functionals based on matrix inequalities: constraint satisfaction for the CPM is done in a low-dimensional space, whereas the matrix inequalities of the other methods are checked in the original (potentially) high-dimensional space and must also be synthesized therein.

Being an inherently local method, the CPM does not rely on Lipschitzian conditions on the vector field or any boundedness constraints. This is one of its strengths compared

to Lyapunov functional-based sufficient conditions for impulsive stabilization. The latter universally require Lipschitzian or Lipschitz-like conditions on the vector field in order to rigorously ensure stability. This being said, the locality of the method is also a weakness: it does not guarantee global stabilization.

A weakness of our method in its current state is that such secondary bifurcations are barriers to further impulsive stabilization. Suppose for example that the uniform CPM is implemented at scalar parameter $\epsilon = 0$ and a controller $B(t)$ is found, but that a secondary bifurcation occurs at the parameter $\epsilon^* \neq 0$. That is, the system

$$
\begin{aligned}
\dot{x} &= f(x_t, \epsilon), & t &\notin \frac{1}{h}\mathbb{Z} \\
\Delta x &= B(t)x_{t^-}, & t &\in \frac{1}{h}\mathbb{Z}
\end{aligned}
$$

undergoes a bifurcation at the parameter $\epsilon = \epsilon^*$. Technically, the CPM cannot be applied at this parameter because the system is no longer an autonomous delay differential equation. In principle, the CPM could be extended to this case, but the computation of basis functions in the centre fiber bundle becomes more difficult, and these are absolutely essential to the CPM. A discretization scheme such as the one described in Section 4.6.3 could be helpful in this endeavor.

Another limitation of our method is that it cannot in general be used to stabilize an unstable equilibrium point in a delay differential equation. The method only has a chance of success if there is a sufficiently nearby perturbation of the system such that the equilibrium satisfies the spectral gap condition. In the absence of such a nearby perturbation, traditional impulsive stabilization approaches based on matrix inequalities derived from Lyapunov functional-based sufficient conditions for stability are more appropriate. One possible way to extend the centre probe method to this more general setting would be to use data from the centre-unstable manifold to design the controller and iterate the process if necessary.

12.7   Proofs

This section contains proofs of most results of the chapter. Some have been omitted since they are similar to others.

Proof of Proposition 12.3.1

1. That the set $\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ is closed follows by the continuity of the spectral radius function $\rho : \mathbb{R}^{c \times c} \to \mathbb{R}^+$. To prove the equality (12.23), consider the dynamics on the c-dimensional slice of the parameter-dependent centre manifold at the point $x = 0$, for small parameter $(\epsilon, B) \in \mathbb{R}^p \times \mathcal{U}$ of the impulsive functional differential equation

$$\dot{x} = L_0 x_t + L(\epsilon) x_t + O(||x_t||^2), \qquad\qquad t \notin \frac{1}{h}\mathbb{Z} \qquad (12.38)$$

$$\Delta x = B(t) x_{t^-}, \qquad\qquad t \in \frac{1}{h}\mathbb{Z}, \qquad (12.39)$$

where we have overloaded the notation and identify B with the functional on the right-hand side of (12.7) and, if $k > 1$, using (12.12). At the parameter $(\epsilon, B) = 0$, the centre fiber bundle (equivalently, centre subspace) is determined by the homogeneous linear equation without impulses

$$\dot{y} = L_0 y_t,$$

so the claim concerning $\Phi(t)$ and $\Psi(t)$ as being basis matrices associated to the eigenspaces E_0 and $E_0^\intercal$ is true – see [39]. Moreover, we trivially have the Floquet decomposition

$$\Phi_t(\theta) = \Phi(0) e^{\Lambda \theta} e^{\Lambda t} := Q(\theta) e^{\Lambda t}$$

due to the characterization $\frac{d}{dt}\Phi(t) = \Phi(t)\Lambda$. This implies that $Q_t = \Phi_0$ is constant in t. The projection operator $P_c(t)$ is constant and we therefore have

$$P_c(t)\chi_0 = \Phi_0 \langle \Psi_0, \Phi_0 \rangle^{-1} \langle \Psi_0, \chi_0 \rangle = Q\Gamma\Psi(0) = \Phi_t e^{-\Lambda t}\Gamma\Psi(0)$$

Applying Theorem 8.1.2, the dynamics on the c-dimensional slice of the parameter-dependent centre manifold are given to linear order in $z \in \mathbb{R}^c$ by

$$\dot{z} = \Lambda z + \Gamma\Psi(0)L(\epsilon)\Phi_0 z + O(||z||^2), \qquad\qquad t \notin \frac{1}{h}\mathbb{Z} \qquad (12.40)$$

$$\Delta z = \Gamma\Psi(0)B(t)\Phi_0 z(t^-), \qquad\qquad t \in \frac{1}{h}\mathbb{Z}, \qquad (12.41)$$

for (ϵ, B) sufficiently small. The monodromy matrix associated to the equilibrium $z = 0$ is precisely $\mathcal{M}_{\epsilon,h}(B)$, and the dynamics restricted to the centre manifold therefore has $O(e^{-\gamma t})$ as a convergence rate if and only if $\rho \mathcal{M}_{\epsilon,h}(B) \leq e^{-\gamma/h}$. Since all other Floquet exponents of the linearization of (12.38)–(12.39) have strictly negative real parts and the spectrum of the monodromy operator is upper hemicontinuous with respect to perturbations in (ϵ, B) – see the perturbation theory of compact operators in the reference [51] –

the same convergence rate is achieved locally near $x = 0$ in the original nonlinear system provided $\gamma < \sigma$ and $|(\epsilon, B)|$ is sufficiently small, hence the intersection in equation (12.23). The converse follows by the same argument.

2. Since the cost $\mathcal{C} : \mathcal{U} \to \mathbb{R}$ is nonnegative, the set $X = \mathcal{C}_h(\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)) \subset \mathbb{R}$ is bounded below, so there exists a sequence $B_n \in \widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ such that $\mathcal{C}(B_n) \to \inf X$ as $n \to \infty$. The sequence B_n cannot be unbounded because $\mathcal{C}$ is radially unbounded, from which we conclude by the Bolzano-Weierstrass theorem that B_n admits a convergent subsequence having a limit $B_\epsilon \in \widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$, with the latter inclusion justified by part 1. As $\mathcal{C}$ is continuous, we conclude that $\mathcal{C}(B_\epsilon) = \inf X$.

Proof of Theorem 12.3.1

To begin, we assume $k = 1$. If there exists a linear jump functional B^* such that $\mathcal{M}_{\epsilon,h}(B^*) = e^{-\gamma/h} I$, then the nonemptiness of $\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$ will follow. Thus, it suffices to solve the equation

$$(I + \Gamma \Psi(0) B^* \Phi_0) \exp\left(\frac{1}{h}(\Lambda + \Psi(0) L(\epsilon) \Phi_0)\right) = e^{-\gamma/h} I. \qquad (12.42)$$

If $\Phi(0)$ and $\Psi(0)$ are rank c, there exists a left-inverse $\Phi^+(0)$ and right-inverse $\Psi^+(0)$ such that $\Phi^+(0)\Phi(0) = I_{n \times n}$ and $\Psi(0)\Psi^+(0) = I_{c \times c}$. Then, the jump functional $B^*_{\epsilon;\gamma}$ defined by

$$B^*_{\epsilon;\gamma}\xi = \Psi^+(0)\Gamma^{-1}(e^{-\gamma/h} - 1)\exp\left(-\frac{1}{h}(\Lambda + \Psi(0)L(\epsilon)\Phi_0)\right)\Phi^+(0)\xi(0)$$

satisfies (12.42). Since $B^*_{\epsilon;\gamma} \to 0$ as $\gamma \to 0^+$, it follows by minimality that also $\mathcal{C}_h(B^*_{\epsilon;\gamma}) \to 0$ as $\gamma \to 0^+$ for any selection $\gamma \mapsto B^*_{\epsilon;\gamma}$ of cost-minimizing jump functionals for rate parameter γ. As $\mathcal{C}$ is continuous and positive-definite, we conclude that $B^*_{\epsilon;\gamma} \to 0$.

If $k > 1$, consider the period k cycle of jump functionals $B = (B^*_{\epsilon;\gamma/k}, \ldots, B^*_{\epsilon;\gamma/k})$. Then $\mathcal{M}_{\epsilon,h}(B) = e^{-\gamma/h} I$ and the argument proceeds as before. The intersections in (12.23) are nonempty because we have just shown that by taking $\gamma \to 0$, we can construct an arbitrarily small element of $\widetilde{\mathcal{Y}}(\epsilon, h; \gamma)$.

Proof of Corollary 12.3.1.1

If $c = 1$, then $\Phi(t)$ is a $n \times 1$ column vector and, as it constitutes a basis for E_0, it cannot be identically zero. Moreover, as $\Phi(t) = \Phi(0)e^{\Lambda t}$, we cannot have $\Phi(0) = 0$. Consequently, $\Phi(0)$ has rank one. The same argument applies to $\Psi(0)$.

If $L_0\phi = C\phi(0)$ for an $n \times n$ matrix C, then $\Phi(t) = Z(t)M$ for some $n \times c$ with matrix M and $Z(t)$ is the fundamental matrix solution of the finite-dimensional linear equation $\dot{z} = Cz$ satisfying $Z(0) = I$. As each column of $\Phi(t)$ is a linear combination of the columns of $Z(t)$, this set of columns is linearly independent if and only if M has maximal rank. As $c \leq n$, it follows that M has rank c, so that the same is true of $\Phi(0)$. The same argument applies to $\Psi(0)$.

Proof of Lemma 12.4.4

Let $x \in \mathcal{M}_1^{-1}(X)$ and $y \in \mathcal{M}_1^{-1}(Y)$. Then, we have

$$
\begin{aligned}
\mathcal{M}_1(tx + (1-t)y) &= M_0(tx + (1-t)y) + Z \\
&= t(M_0(x) + Z) + (1-t)(M_0(y) + Z) \\
&= t\mathcal{M}_1(x) + (1-t)\mathcal{M}_1(y) \\
&= tX + (1-t)Y.
\end{aligned}
$$

Consequently, $tx + (1-t)y \in \mathcal{M}_1^{-1}(tX + (1-t)Y)$ for all $t \in [0,1]$, $x \in \mathcal{M}_1^{-1}(X)$ and $y \in \mathcal{M}_1^{-1}(Y)$, from which the inclusion (12.28) follows.

Proof of Theorem 12.4.1

Before we begin, we remark that by virtue of convexity, any local optimum of the program (Y_P) is automatically a global optimum. As such, we will refer to these as global optimums.

Let $\tilde{C}$ be a GPCC, let P be a global optimum for (Y) and B be a global optimum for the problem (Y_P). Suppose by way of contradiction that B is not a global optimum for (Y). By Proposition 12.3.1, there must exist a global optimum B^*, and as $\tilde{C}$ is a GPCC we have $\tilde{C} \circ \mathcal{M}_k(B^*) < \tilde{C} \circ \mathcal{M}_k(B)$. But $P = \mathcal{M}_k(B)$, and the latter inequality implies P is not optimal for the program $(\mathcal{P}\tilde{C})$, a contradiction. Therefore, B is a global optimum for (Y).

Now, let $\tilde{C}$ be a LPCC, let P be a local optimum for (Y) and B be a global optimum for the problem (Y_P). Suppose by way of contradiction that B is not a local optimum for (Y). Then, there exists a sequence $B_n \to B$ such that $\mathcal{C}(B_n) < \mathcal{C}(B)$. By continuity of $\mathcal{M}_k : \mathcal{U}^k \to (\mathbb{R}^{c \times c})^k$, we have $P_n := \mathcal{M}_k(B_n) \to \mathcal{M}_k(B) = P$, but local optimality of P implies that for n sufficiently large, $\tilde{C}(P) \leq \tilde{C}(P_n)$. Using the LPCC property (type 1 or type 2) of $\tilde{C}$, it follows that $\mathcal{C}(B) \leq \mathcal{C}(B_n)$ for n sufficiently large, which is a contradiction. The result follows.

Proof of Theorem 12.4.2

Radial unboundedness follows from radial unboundedness of $\mathcal{C} : \mathcal{U} \to \mathbb{R}^+$. For continuity, we remark that $\tilde{C}$ is the optimal value function associated to the family of convex programs (Y_P) indexed by the parameter $P \in \mathcal{P}^k$. We have $\mathcal{M}_k^{-1} : \mathcal{P}^k \rightrightarrows \mathcal{U}^k$ is closed-valued, convex-valued, and can be shown to be continuous (see the characterization provided by Lemma 12.4.3). The cost $\mathcal{C}$ is convex and continuous, and the solution set is bounded for each P due to the radial unboundedness of $\mathcal{C}$. Using Theorem 1 of [84], we obtain continuity of $\tilde{C}$. Using Lemma 12.4.4, Lemma 12.4.6 and the convexity of $\mathcal{C}$, if we use the subscript notation $P = (P_0, \ldots, P_{k-1})$ for arbitrary k-probe elements, then

$$
\begin{aligned}
\tilde{C}(tX + (1-t)Y) &= \min\{\mathcal{C}(V) : V \in \mathcal{M}_k^{-1}(tX + (1-t)Y)\} \\
&= \min\{\mathcal{C}(V_0, \ldots, V_{k-1}) : V_i \in \mathcal{M}_1^{-1}(tX_i + (1-t)Y_i)\} \\
&\leq \min\{\mathcal{C}(V_0, \ldots, V_{k-1}) : V_i \in t\mathcal{M}_1^{-1}(X_i) + (1-t)\mathcal{M}_1^{-1}(Y_i)\} \\
&= \sum_{i=0}^{k-1} \min\{\mathcal{C}(tx_i + (1-t)y_i) : x_i \in \mathcal{M}_1^{-1}(X_i), y_i \in \mathcal{M}_1^{-1}(Y_i)\} \\
&\leq \sum_{i=0}^{k-1} \min\{t\mathcal{C}(x_i) + (1-t)\mathcal{C}_h(y_i) : x_i \in \mathcal{M}_1^{-1}(X_i), y_i \in \mathcal{M}_1^{-1}(Y_i)\} \\
&\leq t\tilde{C}(X) + (1-t)\tilde{C}(Y),
\end{aligned}
$$

and we conclude that $\tilde{C}$ is convex.

Next we prove that $\tilde{C}$ is a GPCC. If B^* is a global optimum and B is feasible, then

$$
\begin{aligned}
\tilde{C} \circ \mathcal{M}_k(B^*) &= \min_{x \in \mathcal{M}_k^{-1}(\mathcal{M}_k(B^*))} \mathcal{C}(X) = \mathcal{C}(B^*) \\
&\leq \min_{x \in \mathcal{M}_k^{-1}(\mathcal{M}_k(B))} \mathcal{C}(X) = \tilde{C} \circ \mathcal{M}_k(B).
\end{aligned}
$$

B is also a global minimum if and only if $\mathcal{C}(B^*) = \mathcal{C}(B)$, and in this case the calculation above is simplified. We have

$$
\tilde{C} \circ \mathcal{M}_k(B^*) = \mathcal{C}(B^*) = \mathcal{C}(B) = \tilde{C} \circ \mathcal{M}_k(B).
$$

Our final task is to prove that $\tilde{C}$ is a LPCC. We have

$$
\tilde{C} \circ \mathcal{M}_k(B) = \min_{X \in \mathcal{M}_k^{-1}(\mathcal{M}_k(B))} \mathcal{C}(X) \leq \mathcal{C}(B)
$$

because $B \in \mathcal{M}_k^{-1}(\mathcal{M}_k(B))$. Equality holds precisely when B satisfies (12.29).

Proof of Corollary 12.4.3.1

Under the assumption of the corollary, the set $\{x \geq 0 : \forall |\epsilon| \leq x,\ \rho(\mathcal{M}_{\epsilon,h}(B)) \leq e^{-\gamma/h}\}$ is bounded above, and therefore has a supremum, η. By continuity of the spectral radius, it follows that B is a feasible solution for the program (Y^η). To see that it is optimal, we recall that due to part 2 of Proposition 12.3.2, we have the inclusion $\widetilde{\mathcal{Y}}^\eta(h;\gamma) \subseteq \widetilde{\mathcal{Y}}(0,h;\gamma)$. Since B minimizes (locally or globally) $\mathcal{C} : \widetilde{\mathcal{Y}}(0,h;\gamma) \to \mathbb{R}$ and is feasible for (Y^η), it must also minimize (locally or globally) $\mathcal{C} : \widetilde{\mathcal{Y}}^\eta(h;\gamma) \to \mathbb{R}$.

Chapter 13

Conclusion

Establishing a dynamical systems framework for impulsive RFDE from linear systems to nonlinear systems, we have demonstrated the existence and smoothness of centre manifolds at nonhyperbolic equilibrium points. In contrast to autonomous systems of ordinary differential equations or functional differential equations, these manifolds are time-varying objects – they would more accurately be referred to as invariant fibre bundles. We proved that the centre manifold is locally attracting in the absence of an unstable fibre bundle and, in particular, every small solution converges exponentially to a particular solution on the centre manifold. We obtained two abstract equations – one an integral equation, and the other an impulsive differential equation – that essentially determine the dynamics on the centre manifold. By introducing a coordinate system, we made this differential equation concrete and were able to prove that the centre manifold is sufficiently regular with respect to deviation in time that it satisfies a boundary-value problem. By taking Taylor expansions, one can compute the centre manifold to any desired level of truncation.

Using our centre manifold theory, we established analogues of the fold bifurcation and Hopf bifurcation for impulsive delay differential equations. The fold bifurcation condition generically results in a fold of periodic solutions, while the Hopf condition involves the birth of an invariant cylinder. We studied (by way of numerical methods) a particular instance of the latter bifurcation in a model of infectious disease transmission with a finite incubation period and pulse vaccination. We detected a Hopf point along a branch of nontrivial endemic periodic solutions and witnessed the birth of an attracting invariant cylinder in the extended phase space.

We applied the centre manifold theory to the synthesis of impulsive stabilization schemes. Starting from an autonomous functional differential equation at a nonhyperbolic equilib-

rium with a trivial unstable subspace, we outlined conditions under which the equilibrium can be impulsively stabilized while guaranteeing a specified convergence rate and minimizing a given user-specified cost. We then used the low-dimensional properties of the centre manifold to reduce the complexity of the associated optimization problem. We refer to the latter as the centre probe method.

Some additional results developed in this thesis include a variation-of-constants formula for linear equations in the phase space $\mathcal{RCR}$ of right-continuous regulated functions, a thorough investigation into the properties of homogeneous periodic systems including Floquet theory, a heuristic monodromy operator discretization scheme, and a result on the smoothness of solutions of impulsive RFDE with respect to initial conditions. On the topic of invariant manifolds, we proved the existence and smoothness of unique, local unstable and stable manifolds. We were also able to obtain a linearized stability/instability result.

There are several ways in which the contributions of this thesis could be extended. For example, there are a few technical conditions that we have assumed to guarantee temporal regularity (i.e. effective $PC^{1,m}$ regularity at zero) of the Euclidean space representation of the centre manifold. One of these is the condition that the matrices $t \mapsto Y_j(t)$ are continuous from the right and have limits on the left. Establishing conditions under which this assumption holds generally would improve the robustness of the centre manifold theory.

The overlap condition is needed to get a concrete Euclidean space impulsive differential equation for the coordinate dynamics on the centre manifold. This condition can be circumvented by defining additional states in the same way we did in Section 11.1.1 for the SIR model. We find this approach unsatisfying, however. We suspect that the apparent difficult occurs because the one-point left limit is used to define the jump condition in (1.2). If a regulated left-limit were used instead, we conjecture the invariant manifold theory would be more transparent and would not rely on an overlap-type condition to ensure concrete dynamics on the invariant manifolds. This might, however, cause additional complications, since the regulated left-limit operator is not a map from $\mathcal{RCR}$ into itself, and to compute x_t^- one requires data on an interval $[t - (r + \epsilon), t]$ for some finite $\epsilon > 0$.

We have established analogues of the fold and Hopf/Neimark-Sacker bifurcation for impulsive delay differential equations with a single delay. As for other codimension-one bifurcations, there is also the period-doubling bifurcation from maps that, given the discrete properties we have exploited, should have a natural generalization to periodic impulsive delay differential equations. In the same way, one could study how other generic codimension-two bifurcations of maps manifest themselves in infinite-dimensional impulsive RFDE using the centre manifold reduction.

The monodromy operator discretization scheme we proposed in Section 4.6.3 is far from

rigorous. It would be incredibly beneficial to have convergence guarantees for this scheme with respect to the Floquet multipliers and Floquet eigensolutions after an appropriate embedding into the phase space is established. The scheme could also be extended to allow for more general functional dependence such as distributed delays, multiple delays and multiple impulses per period. This would allow for the numerical analysis of bifurcations involving more complicated linear terms.

The centre probe method is clearly effective at reducing the complexity of the impulsive stabilization problem, but as it stands the applicability is limited because we require the unstable subspace to be trivial. There is a natural way to extend the method so that one designs the controller using data from the centre-unstable fibre bundle, treating the controller as a perturbation in the centre-unstable manifold. For weak instabilities, such a method could very well stabilize the equilibrium, and the same computational complexity reduction should be expected. For stronger instabilities, successive iteration of the method may be helpful, although guaranteeing convergence of such an iterative method would be difficult. This would inevitably also require efficient numerical methods to approximate the centre-unstable fibre bundle of an arbitrary homogeneous linear impulsive RFDE. This could be accomplished by improvements to our monodromy operator discretization scheme. More broadly speaking, one could envision other forms of invariance-based stabilization for infinite-dimensional systems based on the same idea of designing a controller using centre-unstable-manifold data. The methodology need not be restricted to impulsive controllers, and there is no reason to apply the method solely to delay differential equations.

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