Correlation and Communication via a Quantum Field

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 2 and parts of Chapter 1 have been adapted from Ref. [1]. I performed all of the calculations and made the majority of the contributions to the writing of this work.

Chapter 3 and parts of Chapter 1 have been adapted from Ref. [2]. I performed a majority of the calculations in this work, with the exception of some of the calculations in Sec. 3.2, which were performed by my co-author Robert Jonsson. I also made the majority of the contributions to the writing.

Chapter 4 has not been published at the time of writing this thesis. I performed all of the calculations and wrote all of the text for this chapter.

Abstract

We study the ability of qubit detectors to i) extract correlations from, and ii) transmit quantum information through, a quantum field.

We start by perturbatively studying the harvesting of correlations from thermal and squeezed coherent field states. We find that an increase in field temperature is detrimental to entanglement harvesting, but beneficial to mutual information harvesting. We also show that entanglement harvesting is independent of the field's coherent amplitude — which we relate to fundamental results regarding the entanglement structure of coherent field states — but strongly dependent on the field's squeezing amplitude. We conclude by analyzing the practical feasibility of entangling qubits using squeezed field states.

We then go on to study, non-perturbatively, the entanglement extraction by targets A and B from a quantum source S. After proving a general no-go theorem which applies for any A, B and S, we apply this theorem to the entanglement harvesting setup to prove that a wide class of i) degenerate, or ii) point-in-time coupled, detectors cannot harvest entanglement from any field state. We also discuss the role of communication in the process of entanglement extraction, and we end the chapter by presenting the simplest successful example of a non-perturbative entanglement harvesting protocol.

We conclude by studying the ability of flat spacetime observers Alice and Bob to transmit quantum information through a quantum field. We construct a perfect, field-mediated quantum channel, each use of which allows Alice to transmit a full qubit of information to Bob. This construction provides us with an understanding of how quantum information propagates through a relativistic field, which we find to be consistent with our understanding of the strong Huygens principle. Lastly, we analyze the possibility of simultaneously broadcasting a quantum message through a quantum field to multiple receivers, and discover severe fundamental limitations to such a setup.

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Chapter 1

Introduction

1.1 Harvesting entanglement from quantum fields

A critical distinction between classical and quantum systems is the existence of entanglement in the latter. Mathematically, a bipartite quantum system described by a state $\hat{\rho}$ in the Hilbert space $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$ is said to be *separable* with respect to the partition A-B if the density matrix $\hat{\rho}$ can be expressed as

$$\hat{\rho} = \sum_i p_i \, \hat{\rho}_i^{\text{A}} \otimes \hat{\rho}_i^{\text{B}}, \tag{1.1}$$

where p_i are positive numbers that sum up to 1 (i.e. probabilities), while $\hat{\rho}_i^{\text{A}}$ and $\hat{\rho}_i^{\text{B}}$ are density matrices on $\mathcal{H}_{\text{A}}$ and $\mathcal{H}_{\text{B}}$, respectively. A state on $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$ that is not separable is said to be *entangled*.

To gain some intuition on the distinction between entangled and separable states, it is beneficial to consider the simplest type of separable state, a product of two pure states: $|\psi\rangle_{\text{A}} \otimes |\psi\rangle_{\text{B}}$. This state simply tells us that when we consider system A on its own (i.e. if we are ignorant of system B/we trace out system B), its state is given by $|\psi\rangle_{\text{A}}$, and similarly for system B. Since the state of A is a pure state following the partial trace of system B, this tells us that our ignorance of B in no way induces a lack of knowledge of the state of A. In other words, A and B are completely uncorrelated.

While the product state $|\psi\rangle_{\text{A}} \otimes |\psi\rangle_{\text{B}}$ is the simplest example of a separable state, it is important to note that not every separable state (1.1) can be written in this form by a change of basis. However, if two parties, Alice and Bob, start with a joint quantum state of the form $|\psi\rangle_{\text{A}} \otimes |\psi\rangle_{\text{B}}$, then it *is* possible for them to transform it into any generic separable state of the form (1.1) simply by performing *local operations* and employing a *classical communication* channel (LOCC) [3]. Since the initial state $|\psi\rangle_{\text{A}} \otimes |\psi\rangle_{\text{B}}$ contains no correlations between A and B, and since we expect a classical communication channel to only be able to classically correlate the two systems, this tells us that a generic separable state (1.1) contains no quantum correlations. Meanwhile, a non-separable (i.e. entangled) state cannot be obtained from $|\psi\rangle_{\text{A}} \otimes |\psi\rangle_{\text{B}}$ via LOCC, and hence in this sense we say that an entangled state contains non-classical (i.e. quantum) correlations.[1]

Besides being an interesting mathematical feature unique to quantum systems, entanglement is also an extremely important physical resource that can be used for quantum information processing purposes, and which is in large part responsible for the significant advantages that quantum computers have over their classical counterparts. For instance, a shared entangled state between Alice and Bob is necessary for implementing a *quantum teleportation* protocol, which allows Alice to use a classical communication channel to transmit a quantum state to Bob [5], as well as a *superdense coding* protocol [6], in which Alice can transmit two classical bits of information by only physically sending one quantum bit. Both of these protocols are critical, for example, in the study of quantum cryptography in general, and quantum key distribution (QKD) in particular, the development of which is imperative to the security of our communication channels in the not-too-distant future [7].

With this understanding of the significance of entanglement for quantum information tasks, an important question arises: How can we entangle two quantum systems, A and B? As we already discussed, entanglement is a form of non-local quantum correlations between A and B. One way to introduce such non-local correlations into an initially separable state of A-B is to apply a global unitary onto the joint Hilbert space of $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$. While unitaries $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ acting on $\mathcal{H}_{\text{A}}$ or $\mathcal{H}_{\text{A}}$ will never succeed in entangling A and B, a unitary $\hat{U}_{\text{AB}}$ on the product space $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$ will in general be entangling. However, from a practical perspective, if A and B are quantum systems separated by a large distance, then implementing such a global unitary could be very difficult. For instance, if an experimenter wants to entangle a qubit in Europe with a qubit in China for use in a QKD experiment, it is inconceivable that the experimenter could have direct control of both qubits simultaneously.

In this case, where we would like to generate an entangled bipartite system with the two halves separated by a large distance, there are in general two ways to proceed. The first method, which is more direct, simply amounts to creating the bipartite entangled system

[1] There also exist states which contain non-classical correlations that are not entangled. For more detail on quantum correlations that are not necessarily in the form of entanglement see, e.g. Ref. [4].

locally (e.g. in a single lab) and then sending the two halves to their respective final destinations. In fact, this is the method of entanglement generation commonly employed in QKD experiments, in which the currently held record for the largest separation between two entangled qubits is over one thousand kilometres [8]. Transmitting qubits over such large distances requires being able to transmit quantum information through free space, which will be the subject of Chapter 4, and will be introduced in the next section.

There is however, a second method of entangling distant quantum systems A and B that does not require a transmission of quantum information over large distances. In this approach, A and B become entangled with one another by locally coupling to a spatially extended third quantum system F. Clearly however, the success of this method relies upon the existence of a spatially extended quantum system F, which contains preexisting entanglement between F_A, the part of F located near A, and F_B, the part of F located near B, such that A and B can extract this preexisting entanglement onto themselves. The simplest example of such an entanglement extraction protocol is if A and B are qubits that each have access to one half of a Bell pair (a maximally entangled qubit pair). Then, A can swap its quantum state with its half of the Bell pair, and B can do the same, resulting in a maximally entangled state between A and B.

Of course, the natural objection one might have to this second method of entangling A and B is the necessity of preexisting entanglement in F. After all, if our objective is to entangle spatially separated systems A and B, it seems to be a cheap trick to assume that there already exists entanglement between equally separated systems F_A and F_B. However, the presence of quantum fields precisely allows for the employment of such a cheap trick. Namely, an arbitrary state of a quantum field in general contains entanglement between two regions of spacetime, even if they are spacelike separated. Hence, two spacetime observers, Alice and Bob, that are located in these regions can become entangled with one another simply by interacting locally with the quantum field. We say that Alice and Bob *harvest entanglement* from the quantum field.

Before we review the extensive literature on entanglement harvesting, it is beneficial to take a moment to gain some intuition about why two regions of a quantum field are generally entangled. While we will discuss quantum fields much more technically in a latter section, for the present purpose it is most advantageous to simply think of a quantum field as a lattice of locally coupled quantum harmonic oscillators distributed throughout all of space, and existing for all time. Because the oscillators are locally coupled, their global ground state (i.e. the lowest energy state of the entire collection of oscillators; i.e. the global vacuum state) is not simply a tensor product of the free oscillator ground states, but rather is entangled with respect to the local modes. In other words, if the field is in its ground state and we take two oscillators of the lattice at a given time t, the reduced state of the two oscillators will be entangled. The same is also true for excited states.

Let us also note that, while we have motivated the concept of entanglement in quantum field theory from a practical perspective — i.e. by considering the useful quantum information tasks that can be achieved with the entanglement extracted from a quantum field — this is also an important and widely studied concept from more fundamental perspectives. For example, in condensed matter physics, the dynamics of a system near a quantum phase transition can be better understood by studying the entanglement entropy of the conformal field theory state describing the system [9, 10]. On the other hand, while no fully satisfactory resolution to the black hole information loss paradox exists as of yet, the possible entanglement between internal and external field degrees of freedom has been used in various proposed solutions [11–16].

The entanglement in quantum field states, particularly vacuum and thermal states, is also a very important concept in testing the AdS-CFT correspondence [17]. Namely, the correspondence allows, via the famous Ryu-Takayanagi conjecture [18], the computation of the entanglement entropy of certain conformal field states from the perspective of the gravitational theory that is holographically dual to the boundary CFT. These predictions have been shown to be in agreement with direct quantum field theoretic computations (see, e.g. Ref. [19] for a review), thus providing an encouraging positive test for the validity of the AdS-CFT conjecture. In fact, it has been proposed, initially by Mark Van Raamsdonk, that this connection between gravity and entanglement goes even deeper, and that the spacetime dynamics of the bulk theory can be directly understood by studying the entanglement of the boundary theory [20]. It is hoped that these developments can perhaps provide some insights towards a better understanding of quantum gravity.

Let us now turn our attention back to the study of entanglement harvesting from quantum field states, which, along with giving us a novel means of producing entangled pairs of qubits, which could be useful for quantum information purposes, may ultimately also provide us with more fundamental insights into the role of entanglement in quantum field theory.

The pioneering works on entanglement harvesting were by Valentini [21], and later Reznik *et. al.* [22, 23], where it was shown that it is possible for particle detectors (e.g. qubits) A and B to become entangled through local interactions with the field vacuum, *even if the detectors are spacelike separated*.[2] Since spacelike separated detectors cannot communicate with one another, this provided a simple operational proof of the fundamental fact that the field vacuum contains entanglement with respect to local modes.

[2]We will discuss in detail the model used to describe particle detectors interacting with the quantum field in Sec. 1.4

Following these initial works, the entanglement harvesting protocol has been studied in much further detail. For instance it has been shown that it is possible (albeit more difficult) to harvest entanglement from thermal states in a $(1+1)$-dimensional cavity [24], from coherent scalar field states in free space [2], as well as from the electromagnetic field vacuum using fully featured hydrogen-like atoms [25]. The sensitivities of the protocol to the properties [26] and trajectories [27] of the detectors, boundary conditions of the field [28,29], nature of the detector-field couplings [30], as well as the geometry [31–34] and topology [35] of the background spacetime have also been investigated.

Besides their fundamental significance, the above studies are important in determining the optimal conditions for an experimental realization of an entanglement harvesting protocol. On a positive note, it has been suggested that such a protocol may be within reach using current atomic and superconducting setups [36–38], and in principle could provide a constant supply of Bell pairs which could later be used for quantum information purposes in *entanglement farming* protocols [39]. However, many aspects, in particular with respect to potential implementations, still need to be explored; for instance one important question is the energetic cost of entanglement harvesting, which could be particularly high in a low number of spatial dimensions, as recently addressed in Ref. [40].

With this motivation in mind, in Chapters 2 and 3 of this thesis we will discuss several novel results which will further improve our understanding of the optimal detector and field properties required for entanglement harvesting. These results have been published in Refs. [1,2]. First, in Chapter 2, we will discuss entanglement harvesting from thermal and squeezed coherent field states. While, to our knowledge, this is the first study of squeezed state entanglement harvesting, we note that our study of thermal state harvesting differs in several crucial regards to the previous work in [24]. In [24] it was shown that for a pair of pointlike oscillator detectors interacting with a massless field in a one-dimensional cavity, the amount of entanglement extracted decays rapidly with the temperature. In contrast, i) we consider spatially smeared qubit detectors interacting with a field of any mass in a spacetime of any dimensionality, rather than pointlike oscillator detectors interacting with a massless field in (1+1)-dimensions, ii) we look at the continuum free space case rather than considering a cavity, and hence we are not forced to introduce any UV cutoffs to handle numerical sums, and iii) we directly compute the evolved detectors' density matrix from the field's one and two-point functions, rather than using the significantly different formalism of Gaussian quantum mechanics (see, e.g. [41]).

Despite these differences between our approach and that in [24], we will find that, for thermal states, our results are in qualitative agreement with the general conclusions of [24], i.e. that temperature is detrimental to entanglement harvesting. However, since we obtain analytical expressions for entanglement measures, rather than being restricted to numerical calculations, we are able to provide an explicit proof that the amount of entanglement that (qubit) detectors can harvest from the field rapidly decays with its temperature. In particular, we will show that the optimal thermal state for harvesting entanglement from the field is the vacuum. On the other hand, we will see that this is not the case for the harvesting of mutual information, which is a measure of the total (quantum and classical) correlations of the detector pair. In fact we will see that for high field temperatures T (while still in the perturbative regime) the mutual information harvested by the detectors *increases* proportionally with T.

We will then consider the case of squeezed coherent states [42], where, to our knowledge, no previous literature exists. We will first prove that the statement "entanglement harvesting is independent of the field's coherent amplitude" is true not only for non-squeezed coherent states, as was shown in [43], but also for arbitrarily squeezed coherent states. On the other hand we will show that, unlike the coherent amplitude, the choice of field's squeezing amplitude $\zeta(\boldsymbol{k})$ does in fact affect the ability of UDW detectors to become entangled, and moreover the Fourier transform of $\zeta(\boldsymbol{k})$ directly gives the locations in space near which entanglement harvesting is optimal. Perhaps surprisingly, we will also find that for highly and uniformly squeezed field states, the amount of entanglement that the detectors can harvest is independent of their spatial separation, and is often much higher than the amount obtainable from the vacuum. We will also analyze whether this advantage carries over to more experimentally attainable field configurations where states are squeezed across a narrow frequency range of field modes.

Whereas much of the previous literature focused on perturbative analyses of entanglement harvesting, the interaction between particle detectors and relativistic fields can be analyzed non-perturbatively in certain particular setups. For example, significant work has been done to develop tools that allow for the non-perturbative study of harmonic oscillator detectors in diverse contexts (see, e.g. [44–46]). On the other hand, for finite-dimensional detectors, non-perturbative time-evolution can be computed when the detector's Hamiltonian is completely degenerate (i.e. all detector states have identical energies [47–49]), or when the detector interacts with the field at a finite number of discrete instants in time (i.e via a finite sum of Dirac-δ couplings) [50, 51]. In particular, using these approaches, the following no-go entanglement harvesting theorems were proved: i) Perturbatively, it is not possible to harvest spacelike vacuum entanglement with zero-gap detectors [52], and ii) non-perturbatively it is not possible to harvest any kind of entanglement (timelike, lightlike, or spacelike) from a coherent field state using single δ-coupled detectors [2].

In Chapter 3, by making use of the above non-perturbative approaches to detector-field interactions, we will prove a general non-perturbative result that applies to any scenario of two target quantum systems A and B attempting to extract entanglement from a third quantum system F. We will first present this result in the form of an entanglement extraction *no-go theorem*, which states that for certain types of interaction unitaries describing the coupling of A and B to F, it is impossible for A and B to become entangled. Then, by applying this general theorem to the particular setup where A and B are detectors and F is a quantum field, we will immediately be able prove that the results i) and ii) stated in the above paragraph, which have only been shown to hold for very particular field states, in fact happen to be valid for *any* field state.

Finally, with this understanding of which detector-field interactions *cannot* harvest entanglement, we will construct the simplest possible coupling which *can*. Thus we will provide, to our knowledge, the first non-perturbative study of an entanglement harvesting protocol. Perhaps surprisingly, and in stark contrast with previous perturbative studies, we find that for detector-field couplings in the non-perturbative regime, an increase in coupling strength leads to a *decrease* in the amount of harvested entanglement. We will provide a physical explanation for this seemingly unintuitive phenomenon.

1.2 Communication via quantum fields

In the previous section we have discussed how a quantum field can be used as a resource of entanglement for a pair of observers Alice and Bob who want to become entangled themselves. In particular, we emphasized one of the most striking results in the study of entanglement harvesting: it is possible for Alice and Bob to become entangled with one another even if they are in spacelike separation. While this result may appear to indicate that Alice can send a superluminal signal to Bob, thus violating one of the fundamental postulates of special relativity, we stress that this is not the case. Indeed, while Alice and Bob can become entangled while in spacelike separation, there is no way for them to use this newly formed entanglement to send information between each other.

As we would expect however, the situation becomes drastically different if Bob is in the causal future of Alice. Then, special relativity does not preclude Alice from sending a signal to Bob, and we would expect that just as a quantum field can be used as a tool from which Alice and Bob can extract entanglement, it can also be used as a medium through which information can be transmitted between the two observers. Indeed, this information can in general be of two different forms: classical information and quantum information.

We are most familiar, from everyday experience, with the free-space transmission of classical information through the electromagnetic field, such as, e.g., when we make a phone call to the other side of the world. There has also been significant theoretical work in trying to understand the fundamental mechanism by which classical information is encoded in, transmitted through, and decoded from a quantum field (such as the electromagnetic field).

In the study of these fundamental classical communication protocols, the setup is quite similar to the entanglement harvesting setup discussed in the previous section. Namely, one considers two observers, Alice and Bob, who are allowed to interact with a quantum field by coupling a quantum detector (e.g. a qubit) to it. For concreteness, let us suppose that Alice would like to send a classical message to a Bob, who is in (or on) her future light cone. The simple communication protocol that has been studied in the literature is the following [53]: Alice chooses to encode the classical bit "1" in the field by coupling her detector to the field, or the bit "0" by not coupling. Then, in an attempt to receive Alice's message, Bob couples his detector (say a qubit) to the field. Following this coupling, Bob measures the quantum state of his detector in the energy eigenbasis. If he measures his qubit detector to be in the ground state, he records the bit "0", and if he measures the excited state he records "1". The efficiency of this classical communication protocol is then measured by computing its classical channel capacity, i.e. the number of bits per use of the channel that Alice can send to Bob.

With this setup in mind, an interesting question arises: Where in spacetime can Bob be located in order for the capacity of his classical communication channel with Alice to be non-zero? This question has been considered for various different spacetimes leading to some very interesting results (see Refs. [53–56]), which we will now summarize.

In all of the aforementioned literature on classical communication through a quantum field, the quantum field under consideration is taken to be massless. This is, of course, the most relevant case for our practical purposes, whether it be from the perspective of wireless telecommunication or from the hope of observing the early Universe through the electromagnetic field. Naively, since we are aware that massless field quanta propagate at the speed of light, we might expect that the classical channel capacity between Alice and Bob is non-zero only when the two observers are in lightlike separation. However, as was shown in [53], this intuition turns out to be incorrect for most spacetimes. Indeed, it was found that if Alice and Bob's detectors are initialized to coherent superpositions of ground and excited eigenstates, a timelike signaling protocol, i.e. one in which Bob is located strictly *inside* Alice's lightcone, can be established in most spacetimes. Furthermore, this protocol allows for the possibility of broadcasting a message to an arbitrary number of timelike receivers, with the energy cost of transmitting the message being paid for by the receivers themselves. Because of this, this protocol received the name *quantum collect calling*.

Despite the fact that slower-than-light communication through a massless field may at first seem physically unintuitive, from a fundamental perspective it should actually not come as too much of a surprise that it is indeed possible. The reason for this is that in considering the feasibility of information transmission through a quantum field, a necessary (but not sufficient) condition for communication is that the expectation value of the field commutator (i.e. the classical radiation Green's function) between the spacetime events of sending and receiving the message, does not vanish [57–59]. And while in (3+1)-dimensional flat spacetime it is indeed the case that a massless field's radiation Green's function only has support for lightlike separated events — this is known as the Strong Huygens principle [60] — in general spacetimes this is not the case. For instance, it is well known that the strong Huygens is violated in $(1+1)$- and $(2n+1)$-dimensional flat spacetimes, as well as in general curved spacetimes [53, 55, 60–66].

Following the initial work on quantum collect calling [53], there have been several papers studying this phenomenon in more detail for curved spacetimes, in particular Friedmann-Robertson-Walker (FRW) expanding cosmologies [54–56]. This set of spacetimes is physically relevant since it can provide a good model of our Universe on large scales, and thus we can imagine the quantum collect calling setup as us (the observer Bob) attempting to receive signals from some emitter Alice in our timelike past. Interestingly, it was shown that in an FRW universe sourced by pressureless matter, which is a polynomially expanding cosmology in the comoving time parameter, the communication channel capacity is independent of the spatial separation between Alice and Bob [54, 55]. Meanwhile, and perhaps even more surprisingly, in a cosmological constant dominated FRW cosmology, which is exponentially expanding, the channel capacity is independent of the observers' separations in time, and furthermore, actually *increases* with the rate of cosmic expansion. In other words, it would be easier for us to detect timelike signals from an event further in our past compared to a more recent one, and it would be easier to detect this signal in a more rapidly exponentially expanding universe than a slower expanding one.

The works that we have discussed so far have all focused on the transmission of *classical* information through quantum fields. However, as we mentioned in the previous section, over the last two decades there have been several experiments which — in the context of establishing entangled pairs between distant receivers for the purposes of quantum key distribution — have successfully transmitted *quantum* information with high fidelity through the electromagnetic field [8,67–72]. However, with the notable exception of Ref. [73], which we discuss below, there has been a lack of research on the fundamental mechanism by which observers can encode, transmit, and decode quantum information via a quantum field.

Indeed we expect there to be significant differences in the transmission of quantum information compared to the transmission of classical information. For instance, while it was shown in [53] that classical information can be broadcast to multiple identical receivers, we might suspect from the no-cloning theorem [74] that such a phenomenon is not possible with quantum information. This was proven to be the case, at least for identical receivers, in Ref. [73]. In this paper the authors also constructed the first example of a field-mediated quantum channel, and they showed how it can be used to transmit a qubit of information from Alice to Bob with arbitrarily low signal loss. However, a major limitation of this work is that it is particularized to (1+1)-dimensional Minkowski space, in which case there are only two directions that a signal can propagate, and is thus a significant simplification to the more relevant (3+1)-dimensional case.

In Chapter 4 of this thesis, we will generalize the work in Ref. [73] by constructing the simplest possible, field-mediated, perfect quantum channel between two observers Alice and Bob living in a flat spacetime of any dimension. We will make use of couplings between the observers and the field which are pointlike in time, and as such allow for a non-perturbative approach to the problem. In fact we will find that a non-perturbative (i.e. strong) coupling between detectors and field is necessary in order to achieve a maximal quantum channel capacity, i.e. in order to construct a perfect quantum channel.

Following the mathematical construction of our perfect quantum channel, we will explore what its physical implications are. In particular, given the location in spacetime at which Alice imprints her quantum message into the field, our construction will be able to tell us precisely where in spacetime Bob needs to be located in order to recover Alice's message. While, as discussed above, for a Bob looking to receive classical information it is enough to be located *anywhere* on Alice's lightcone (or even inside the lightcone in certain spacetimes), the situation is not nearly as straightforward in the case where Bob would like to receive *quantum* information.

In fact, if Bob would like to receive quantum information from Alice, we will show that he needs to be spatially extended in such a way that he covers a large fraction of the spacetime region into which Alice's message has propagated. Intuitively, the reason for this is that, unlike with the classical information, if there remains a large portion of spacetime in which the field still contains Alice's message (i.e. a region in which Bob has not extracted the message out of the field), then by a no-cloning type of intuition Bob cannot also have knowledge of Alice's state.

To provide a concrete example of this phenomenon, we first consider the case of (3+1)-dimensional Minkowski space, where Alice's message propagates at the speed of light. We then find that Bob needs to be covering at least half of Alice's lightcone in order to receive her quantum message. On the other hand, in the strong Huygens violating (2+1)-dimensional flat spacetime, we will see that Bob not only needs to be covering Alice's lightcone, but that he also needs to be covering a significant portion of its interior as well. In other words, the subluminal propagation of information in this spacetime actually makes it more difficult for Bob to recover Alice's message.

Finally, to conclude Chapter 4, we will attempt to overcome the limitations of broadcasting quantum information to multiple detectors discussed in [73] by considering the case of non-identical detectors, which is not studied in this previous work. However, we will find numerical evidence which suggests that it is not possible to broadcast any amount of quantum information, no matter how small, to a pair of disjoint observers, even if they happen to be non-identical. Nevertheless, this numerical study will allow us to clearly observe the relationship between the amount of information that Bob can receive from Alice, and the amount of space that he needs to cover in order to do so.

1.3 The Klein-Gordon field in flat spacetime

As discussed in the previous two sections, the major theme of this thesis will be the interaction of particle detectors with a quantum field. In particular we will focus on the Klein-Gordon field, which offers us simplicity while maintaining most of the qualitative features of more complicated fields, such as the electromagnetic field. To that end, in this section we will introduce the Klein-Gordon field and derive its mode expansion in terms of creation and annihilation operators, an expression that we will be using throughout this thesis. The derivations in this section are partially inspired by the lecture notes [75].

We begin by defining the Klein-Gordon action S_{KG} for a classical field $\phi(\boldsymbol{x}, t)$ of mass m in $(d+1)$-dimensional Minkowski spacetime. The action reads

$$\begin{aligned} S_{\text{KG}}[\phi] &= \int \mathrm{d}^d\boldsymbol{x}\, \mathcal{L}(\boldsymbol{x}, t) \\ &:= \frac{1}{2} \int \mathrm{d}^d\boldsymbol{x}\, \left(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + m^2\phi^2\right), \end{aligned} \tag{1.2}$$

where $\mathcal{L}(\boldsymbol{x}, t)$ is the Lagrangian of the theory and $\eta_{\mu\nu}$ is the Minkowski metric in the mostly positive signature. Furthermore, we use the notation $\phi_{,\nu} := \partial_\nu\phi$. The equation of motion for the field is then the Euler-Lagrange equation of the action, which is simply the wave equation,

$$\left(\Box + m^2\right)\phi(\boldsymbol{x}, t) = 0, \tag{1.3}$$

where $\Box := -\eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator in $(d+1)$-dimensions. The advantage of starting with the action for the theory is that it allows us to straightforwardly define the conjugate momentum field $\pi(\boldsymbol{x},t)$ by functionally differentiating the Lagrangian $L[\phi] := \int \mathrm{d}^d\boldsymbol{x}\,\mathcal{L}(\boldsymbol{x},t)$ with respect to $\dot{\phi} := \phi_{,0}$. Thus we obtain

$$\begin{aligned}\pi(\boldsymbol{x},t) &:= \frac{\delta L[\phi]}{\delta\dot{\phi}} \\ &= \dot{\phi}(\boldsymbol{x},t).\end{aligned} \tag{1.4}$$

We would now like to upgrade the classical, number-valued fields ϕ and π to quantum, operator valued fields $\hat{\phi}$ and $\hat{\pi}$. First, let us note that the Klein-Gordon equation (1.3) does not couple the real and imaginary parts of ϕ. Therefore, for convenience, we will assume that ϕ is real-valued. Hence, the corresponding condition for the quantum fields is that they are self-adjoint,

$$\hat{\phi}^\dagger(\boldsymbol{x},t) = \hat{\phi}(\boldsymbol{x},t), \tag{1.5}$$

$$\hat{\pi}^\dagger(\boldsymbol{x},t) = \hat{\pi}(\boldsymbol{x},t), \tag{1.6}$$

since then the spectral theorem ensures that all expectation values (i.e. possible outcomes of experiments involving the quantum field) are real.

As it stands, $\hat{\phi}$ and $\hat{\pi}$ are simply operator valued classical fields. To promote them to quantum fields we must impose non-trivial commutation relations between them. These *canonical commutation relations* read

$$[\hat{\phi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{x}',t)] = \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.7}$$

$$[\hat{\phi}(\boldsymbol{x},t), \hat{\phi}(\boldsymbol{x}',t)] = 0, \tag{1.8}$$

$$[\hat{\pi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{x}',t)] = 0, \tag{1.9}$$

and they ensure the quantum nature of the theory.

The equations of motion, (1.3) and (1.4), the self-adjointness conditions, (1.5) and (1.6), and the canonical commutation relations, (1.7)-(1.9), fully define the quantum field theory. We will now use these three defining components of the field theory to derive a different form for the operator valued fields $\hat{\phi}$ and $\hat{\pi}$, which is essentially a decomposition of these fields into their Fourier modes.

To that end, we define the spatial Fourier transform of the field $\hat{\phi}(\boldsymbol{x},t)$ as

$$\hat{\phi}_{\boldsymbol{k}}(t) := \frac{1}{\sqrt{(2\pi)^d}} \int \mathrm{d}^d\boldsymbol{x}\,\hat{\phi}(\boldsymbol{x},t)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.10}$$

and hence the inverse Fourier transform reads

$$\hat{\phi}(\boldsymbol{x},t) = \frac{1}{\sqrt{(2\pi)^d}} \int \mathrm{d}^d\boldsymbol{k}\, \hat{\phi}_{\boldsymbol{k}}(t) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.11}$$

We will use this convention for the Fourier transform throughout the thesis. Note that the self-adjointness condition $\hat{\phi}^\dagger = \hat{\phi}$ implies that $\hat{\phi}^\dagger_{\boldsymbol{k}} = \hat{\phi}_{-\boldsymbol{k}}$. Substituting the expression Eq. (1.11) for the field $\hat{\phi}(\boldsymbol{x},t)$ into the equation of motion, Eq. (1.3), we obtain

$$\frac{1}{(2\pi)^d} \int \mathrm{d}^d\boldsymbol{k} \left(\ddot{\hat{\phi}}_{\boldsymbol{k}}(t) + \omega_{\boldsymbol{k}}^2 \hat{\phi}_{\boldsymbol{k}}(t) \right) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{1.12}$$

where we have defined $\omega_{\boldsymbol{k}} := \sqrt{|\boldsymbol{k}|^2 + m^2}$. Multiplying the above expression by $e^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}}$ and making use of the integral representation of a delta function, i.e. $\delta(\boldsymbol{k} - \boldsymbol{k}') = (2\pi)^{-d} \int \mathrm{d}^d\boldsymbol{x}\, e^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}$, results in the expression

$$\ddot{\hat{\phi}}_{\boldsymbol{k}}(t) + \omega_{\boldsymbol{k}}^2 \hat{\phi}_{\boldsymbol{k}}(t) = 0 \quad \text{for all } \boldsymbol{k}. \tag{1.13}$$

Hence for all momenta $\boldsymbol{k}$, the Fourier mode $\hat{\phi}_{\boldsymbol{k}}(t)$ obeys the equation of motion for a simple harmonic oscillator, which can readily be solved to give

$$\hat{\phi}_{\boldsymbol{k}}(t) = \frac{\hat{a}_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t} + \frac{\hat{b}_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{\mathrm{i}\omega_{\boldsymbol{k}} t}. \tag{1.14}$$

Here, for later convenience, we have written the prefactors $1/\sqrt{2\omega_{\boldsymbol{k}}}$ in front of both terms. Meanwhile $\hat{a}_{\boldsymbol{k}}$ and $\hat{b}_{\boldsymbol{k}}$ are operator valued integration constants that can be fixed if the initial conditions for $\hat{\phi}$ and $\hat{\pi}$ are given for some time $t = t_0$. Note however that $\hat{a}_{\boldsymbol{k}}$ and $\hat{b}_{\boldsymbol{k}}$ are not independent: the requirement that $\hat{\phi}^\dagger_{\boldsymbol{k}} = \hat{\phi}_{-\boldsymbol{k}}$ implies that $\hat{b}_{\boldsymbol{k}} = \hat{a}^\dagger_{-\boldsymbol{k}}$, and hence $\hat{\phi}_{\boldsymbol{k}}(t)$ reads

$$\hat{\phi}_{\boldsymbol{k}}(t) = \frac{\hat{a}_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t} + \frac{\hat{a}^\dagger_{-\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{\mathrm{i}\omega_{\boldsymbol{k}} t}. \tag{1.15}$$

Substituting this into Eq. (1.11), and redefining the integration variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term of the integrand, results in the expression

$$\hat{\phi}(\boldsymbol{x},t) = \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2(2\pi)^d \omega_{\boldsymbol{k}}}} \left(\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}(\omega_{\boldsymbol{k}} t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{h.c.} \right), \tag{1.16}$$

which is the familiar expansion of the scalar field into its plane wave modes. Differentiating with respect to time then gives the expression for the momentum field:

$$\hat{\pi}(\boldsymbol{x},t) = \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2(2\pi)^d |\boldsymbol{k}|}} \left(\mathrm{i}|\boldsymbol{k}| \hat{a}^\dagger_{\boldsymbol{k}} e^{\mathrm{i}(|\boldsymbol{k}|t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{h.c.} \right). \tag{1.17}$$

Having obtained the plane wave expansions of $\hat{\phi}$ and $\hat{\pi}$, all that is left before we can make use of these expressions is to determine the commutation relations for the operators $\hat{a}_{\boldsymbol{k}}$ and their adjoints. The first step towards this is to use the commutation relations (1.7)-(1.9) for the fields $\hat{\phi}(\boldsymbol{x},t)$ and $\hat{\pi}(\boldsymbol{x},t) := \dot{\hat{\phi}}(\boldsymbol{x},t)$ to obtain the commutation relations for the modes $\hat{\phi}_{\boldsymbol{k}}(t)$ and $\dot{\hat{\phi}}_{\boldsymbol{k}}(t)$. To that end, from the Fourier transform (1.10) we obtain

$$\begin{aligned}\left[\hat{\phi}_{\boldsymbol{k}}(t), \dot{\hat{\phi}}_{\boldsymbol{k}'}(t)\right] &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{x}' \left[\hat{\phi}(\boldsymbol{x},t), \dot{\hat{\phi}}(\boldsymbol{x}',t)\right] e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} e^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}'} \\ &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{x}\, e^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} \\ &= \mathrm{i}\delta(\boldsymbol{k}+\boldsymbol{k}'), \end{aligned} \tag{1.18}$$

where for the second equality we used the commutation relation in Eq. (1.7), and for the last equality we used the integral representation of the delta function. Using analogous calculations we also find

$$\left[\hat{\phi}_{\boldsymbol{k}}(t), \hat{\phi}_{\boldsymbol{k}'}(t)\right] = \left[\dot{\hat{\phi}}_{\boldsymbol{k}}(t), \dot{\hat{\phi}}_{\boldsymbol{k}'}(t)\right] = 0. \tag{1.19}$$

Next, by inverting the expression (1.15) for $\hat{\phi}_{\boldsymbol{k}}(t)$, we can straightforwardly express the operators $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^\dagger_{\boldsymbol{k}}$ as

$$\hat{a}_{\boldsymbol{k}} = \frac{\omega_{\boldsymbol{k}}}{2}\left(\hat{\phi}_{\boldsymbol{k}}(t) - \frac{1}{\mathrm{i}\omega_{\boldsymbol{k}}}\dot{\hat{\phi}}_{\boldsymbol{k}}(t)\right), \tag{1.20}$$

$$\hat{a}^\dagger_{\boldsymbol{k}} = \frac{\omega_{\boldsymbol{k}}}{2}\left(\hat{\phi}_{-\boldsymbol{k}}(t) + \frac{1}{\mathrm{i}\omega_{\boldsymbol{k}}}\dot{\hat{\phi}}_{-\boldsymbol{k}}(t)\right). \tag{1.21}$$

Finally, using the commutators (1.18) and (1.19), we obtain the canonical commutation relations for the operators $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^\dagger_{\boldsymbol{k}}$:

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] = \delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{1.22}$$

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}\right] = 0, \tag{1.23}$$

$$\left[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] = 0. \tag{1.24}$$

$$\tag{1.25}$$

We see that these are the commutation relations satisfied by the creation and annihilation operators of a family of decoupled harmonic oscillators labeled by the continuous parameter $\boldsymbol{k}$. We will hence from here on refer to $\hat{a}^\dagger_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}$ as the creation and annihilation operators for the field mode $\boldsymbol{k}$, respectively. In analogy with the ground state of a harmonic oscillator, we define the ground or vacuum state of the field, denoted $|0\rangle$, by the condition $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$ for all momenta $\boldsymbol{k} \in \mathbb{R}^d$.

1.4 Coupling detectors to quantum fields: the Unruh-DeWitt model

Having introduced in the previous section the Klein-Gordon field, let us now review the Unruh-DeWitt (UDW) formalism [76], which is commonly used to study the interaction of a quantum field with first quantized systems. While this model is a simplification of the interaction between an atom and the full electromagnetic field, it has nevertheless been shown to produce the same qualitative predictions as the latter, at least in situations where the angular momentum exchange between light and matter can be ignored [25, 77]. Throughout this thesis we will make use of this model to study the interactions of spacetime observers, Alice and Bob, with the quantum field.

In the UDW formalism we allow Alice and Bob to each locally couple a two-level quantum system (which we refer to as a detector) to a scalar quantum field $\hat{\phi}(\boldsymbol{x}, t)$. We take the free Hamiltonian of qubit $\nu \in \{\mathrm{A}, \mathrm{B}\}$ to be $\hat{H}_\nu = \Omega_\nu \sigma_z$, with Ω_ν the energy gap and σ_z the Pauli z-operator. We denote the excited and ground states of $\hat{H}_\nu$ as $|\pm_z\rangle$, with eigenvalues $\pm\Omega_\nu$, respectively[3]. Meanwhile, the field $\hat{\phi}(\boldsymbol{x}, t)$ and $\hat{\pi}(\boldsymbol{x}, t)$ are given in terms of the creation and annihilation operators $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^\dagger$ via their plane wave expansions (1.16) and (1.17).

We can describe the interaction between a detector ν and the field by specifying a local interaction Hamiltonian, $\hat{H}_{\mathrm{I},\nu}(t)$. Working in the interaction picture of time evolution, we will consider interaction Hamiltonians of the form

$$\hat{H}_{\mathrm{I},\nu}(t) = \lambda \chi(t) \hat{m}_\nu(t) \otimes \hat{\Phi}(t). \tag{1.26}$$

Here λ is a coupling strength, $\chi(t)$ is an explicitly time-dependent switching function, and $\hat{m}_\nu(t)$ and $\hat{\Phi}(t)$ are qubit and field observables which contain an implicit time dependence coming from the fact that we are working in the interaction picture. For instance if the qubit couples through its σ_x observable, then $\hat{m}_\nu(t)$ is referred to as the monopole-moment operator, and reads

$$\hat{m}_\nu(t) = |+_z\rangle\langle -_z| e^{\mathrm{i}\Omega_\nu t} + |-_z\rangle\langle +_z| e^{-\mathrm{i}\Omega_\nu t}. \tag{1.27}$$

On the other hand, in order to ensure that the coupling between the observer ν and the field is physical, we must ensure that the field observable $\hat{\Phi}(t)$ entering the interaction Hamiltonian (1.26) is an observable that is local in spacetime to the region where observer

[3] Throughout this thesis we will use the notation $|\pm_s\rangle$ for the eigenstates of σ_s, $s \in \{x, y, z\}$, with eigenvalues of ± 1. We will alternatively sometimes denote $|+_z\rangle$ as $|e\rangle$ and $|-_z\rangle$ as $|g\rangle$, when we want to emphasize that these are the ground and excited states of the free detector Hamiltonian.

ν is located. The two simplest examples of such local observables are the smeared ϕ and π observables, defined as

$$\hat{\phi}[F](t) := \int \mathrm{d}^d\boldsymbol{x}\, F(\boldsymbol{x})\hat{\phi}(\boldsymbol{x},t), \tag{1.28}$$

$$\hat{\pi}[F](t) := \int \mathrm{d}^d\boldsymbol{x}\, F(\boldsymbol{x})\hat{\pi}(\boldsymbol{x},t), \tag{1.29}$$

where the smearing function $F(x)$ has strong support in the region of space near the observer ν. Physically, the smearing function characterizes the shape of the detector coupling to the field (if we are modeling the interaction of an atom with the electromagnetic field then $F(x)$ is related to the wavefunction of the atom). Mathematically, it is necessary to smear the field observables that enter the interaction Hamiltonian in order to avoid divergences that occur with non-smeared observables. More complicated (i.e. non-linear) local field observables $\hat{\Phi}(t)$ can also be considered, although these often lead to further divergences that cannot be removed by simply smearing the observable in space (see, e.g. [30]).

Following a specification of a qubit-field interaction Hamiltonian $\hat{H}_{\mathrm{I},\nu}(t)$ as in Eq. (1.26), we can formally write down the time-evolution unitary $\hat{U}$ generated by this Hamiltonian as

$$\hat{U} = \mathcal{T}\exp\left[-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\,\hat{H}_{\mathrm{I},\nu}(t)\right], \tag{1.30}$$

where the $\mathcal{T}$ denotes the time-ordering operation. For general detector switching functions $\chi(t)$, the need for time-ordering makes an explicit evaluation of time-evolved states impossible, instead allowing only for a perturbative approach to the problem. Of course, such a perturbative approach can only be taken when the coupling λ between qubit and field is small with respect to the other scales of the problem. In this case the detector-field interaction unitary $\hat{U}$ can be expanded in powers of λ, with the first few terms given by

$$\hat{U} = \mathbb{1} - \mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\hat{H}_{\mathrm{I},\nu}(t) - \int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\,\hat{H}_{\mathrm{I},\nu}(t)\hat{H}_{\mathrm{I},\nu}(t') + \mathcal{O}(\lambda^3). \tag{1.31}$$

We will make use of such a perturbative calculation in Chapter 2 to compute the amount of entanglement harvested by two detectors from thermal and squeezed states of the quantum field.

If instead a strong-coupling result between detectors and a field is sought after, as will be the case in Chapters 3 and 4, there are couple of ways to bypass the problems arising from the time-ordering present in the interaction unitary (1.30). In the more direct approach, we note that if we simply take the detector switching function to be $\chi(t) = \sum_{i=1}^{n} \delta(t - t_i)$ with $t_i \leq t_{i+1}$, i.e. we require that the detector only interacts with the field at discrete instant in time, then we can rewrite the time evolution unitary in Eq. (1.30) as $\hat{U} = \hat{U}_n \hat{U}_{n-1} \ldots \hat{U}_1$, where $\hat{U}_i$ is defined as

$$\hat{U}_i = \exp\left[-\mathrm{i}\lambda \hat{m}_\nu(t_i) \otimes \hat{\Phi}(t_i)\right]. \tag{1.32}$$

The derivation of this result is shown in Appendix A. Notice that the time-ordering operation $\mathcal{T}$ appearing in Eq. (1.30) has served its purpose by ensuring the unitaries $\hat{U}_i$ act in order of increasing time, and thereafter $\mathcal{T}$ no longer appears in the expression for $\hat{U}$. Therefore an exact analytical expression for the time evolved state of the detector-field system can be obtained.

Another special case which allows for a non-perturbative study of detector-field interactions can be understood if we write the unitary $\hat{U}$ in Eq. (1.30) using the Magnus expansion [78],

$$\hat{U} = \exp\left(\sum_{n=1}^{\infty} \hat{\Omega}_n\right), \tag{1.33}$$

where the lowest order terms read

$$\hat{\Omega}_1 = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t\, \hat{H}_{\mathrm{I}}(t), \tag{1.34}$$

$$\hat{\Omega}_2 = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\, [\hat{H}_{\mathrm{I}}(t), \hat{H}_{\mathrm{I}}(t')], \tag{1.35}$$

and the higher-order terms, which are obtained recursively, contain commutators of the commutators of the interaction Hamiltonian $\hat{H}_{\mathrm{I}}(t)$ at increasing orders.

However, as studied in detail in [47–49, 52], in the special case of a degenerate Unruh-DeWitt detector ν (i.e. if $\Omega_\nu = 0$) these higher order terms all vanish[4]. To see this, first note that a degenerate detector has a free Hamiltonian which is proportional to the identity, and hence $\hat{m}_\nu(t)$, the interaction picture detector observable appearing in $\hat{H}_{\mathrm{I},\nu}(t)$, has no time-dependence. Hence we write $\hat{m}_\nu(t) = \hat{m}_\nu$ and obtain

$$\hat{H}_{\mathrm{I},\nu}(t) = \lambda\chi(t)\hat{m}_\nu \otimes \hat{\Phi}(t). \tag{1.36}$$

Because the commutator of the field with itself is proportional to the identity, we have that $[\hat{H}_{\mathrm{I},\nu}(t), \hat{H}_{\mathrm{I},\nu}(t')] \propto \hat{m}_\nu \otimes \mathbb{1}_\phi$, and hence all higher order commutators of $\hat{H}_{\mathrm{I},\nu}$ with itself

[4]The energy gap Ω_ν of detector ν should not be confused with the Magnus expansion terms $\hat{\Omega}_i$.

at different times vanish. Hence $\hat{\Omega}_k$ is identically zero for all $k > 2$ and we can write the interaction unitary $\hat{U}$ in Eq. (1.33) as

$$\hat{U} = \exp\left(\hat{\Omega}_1 + \hat{\Omega}_2\right). \tag{1.37}$$

Thus, as in the case of delta coupled detectors, degenerate detectors allow us to write an explicit closed-form expression for the detector-field interaction unitary which is free of the time-ordering operation, and hence allows us to proceed with a non-perturbative calculation of evolved detector-field states.

Chapter 2

Harvesting correlations from thermal and squeezed field states

2.1 Correlation harvesting setup

In this chapter we will study the *harvesting* of correlations (in particular entanglement and mutual information) by a pair of UDW detectors from thermal and squeezed states of a quantum field. We begin by reviewing the general correlation harvesting setup that can be found in extensive literature [2, 21–35, 39, 43, 79–82].

To that end, let us suppose that the the tripartite system consisting of Alice, the field and Bob is initially in the separable state $\hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\phi} \otimes \hat{\rho}_{0,\text{B}}$. We allow this system to interact according to the interaction picture interaction Hamiltonian, $\hat{H}_{\text{I}}(t) = \hat{H}_{\text{I,A}}(t) + \hat{H}_{\text{I,B}}(t)$, where $\hat{H}_{\text{I},\nu}(t)$ is of the general form given by Eq. (1.26). For concreteness we will take

$$\hat{H}_{\text{I},\nu}(t) = \lambda_\nu \chi_\nu(t) \hat{m}_\nu(t) \int \mathrm{d}^d \boldsymbol{x}\, F_\nu(\boldsymbol{x} - \boldsymbol{x}_\nu) \hat{\phi}(\boldsymbol{x}, t). \tag{2.1}$$

As in Eq. (1.26), here λ_ν is the coupling strength of detector ν to the field, $\chi_\nu(t)$ is the time-dependent *switching function* which models the duration of the interaction and how the detector ν is turned on and off, and the $\hat{m}_\nu(t)$ are monopole moment operators given by Eq. (1.27). Notice that we suppress the identity operator $\mathbb{1}_\text{B}$ acting on subsystem B in $\hat{H}_{\text{I,A}}$, and similarly for $\hat{H}_{\text{I,B}}$.

Since we would like to determine how correlated the detectors are following their interactions with the field, we first need to compute the time-evolved two-detector state $\hat{\rho}_{\text{AB}}$.

We obtain

$$\hat{\rho}_{\text{AB}} := \text{Tr}_\phi \left[\hat{U} \left(\hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\phi} \otimes \hat{\rho}_{0,\text{B}} \right) \hat{U}^\dagger \right], \tag{2.2}$$

where, as in Eq. (1.30) the time-evolution unitary $\hat{U}$ is formally given by

$$\hat{U} = \mathcal{T} \exp \left[-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{H}_{\text{I}}(t) \right]. \tag{2.3}$$

By assuming that the detector-field coupling constants λ_ν — which have units of $L^{(d-3)/2}$ in $(d+1)$-dimensional spacetime — are small compared to other scales with the same units in the setup, we can expand $\hat{U}$ in a Dyson series in powers of λ_ν,

$$\hat{U} = \mathbb{1} \underbrace{-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t \hat{H}_{\text{I}}(t)}_{\hat{U}^{(1)}} \underbrace{- \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \hat{H}_{\text{I}}(t) \hat{H}_{\text{I}}(t')}_{\hat{U}^{(2)}} + \mathcal{O}(\lambda_\nu^3). \tag{2.4}$$

Substituting this series expansion into Eq. (2.2) allows us to perturbatively express the time-evolved two-detector state $\hat{\rho}_{\text{AB}}$ as

$$\hat{\rho}_{\text{AB}} = \hat{\rho}_{\text{AB}}^{(0)} + \hat{\rho}_{\text{AB}}^{(1)} + \hat{\rho}_{\text{AB}}^{(2)} + \mathcal{O}(\lambda_\nu^3), \tag{2.5}$$

where

$$\hat{\rho}_{\text{AB}}^{(0)} := \hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\phi} \otimes \hat{\rho}_{0,\text{B}}, \tag{2.6}$$

$$\hat{\rho}_{\text{AB}}^{(1)} := \text{Tr}_\phi \left(\hat{U}^{(1)} \hat{\rho}_{\text{AB}}^{(0)} + \hat{\rho}_{\text{AB}}^{(0)} \hat{U}^{(1)\dagger} \right), \tag{2.7}$$

$$\hat{\rho}_{\text{AB}}^{(2)} := \text{Tr}_\phi \left(\hat{U}^{(2)} \hat{\rho}_{\text{AB}}^{(0)} + \hat{U}^{(1)} \hat{\rho}_{\text{AB}}^{(0)} \hat{U}^{(1)\dagger} + \hat{\rho}_{\text{AB}}^{(0)} \hat{U}^{(2)\dagger} \right). \tag{2.8}$$

By using the definitions of $\hat{U}^{(1)}$ and $\hat{U}^{(2)}$ in Eq. (2.4) and the expression for $\hat{H}_{\text{I}}(t)$ given by Eq. (2.1), it is straightforward to show that $\hat{\rho}_{\text{AB}}^{(1)}$ and $\hat{\rho}_{\text{AB}}^{(2)}$ take the forms

$$\hat{\rho}_{\text{AB}}^{(1)} = \mathrm{i} \sum_{\nu \in \{\text{A,B}\}} \lambda_\nu \int_{-\infty}^{\infty} \mathrm{d}t \chi_\nu(t) [\hat{\rho}_{\text{AB}}^{(0)}, \hat{m}_\nu(t)] V(\boldsymbol{x}_\nu, t), \tag{2.9}$$

$$\begin{aligned} \hat{\rho}_{\text{AB}}^{(2)} = \sum_{\nu,\eta \in \{\text{A,B}\}} \lambda_\nu \lambda_\eta \Bigg[& \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \chi_\nu(t') \chi_\eta(t) \hat{m}_\nu(t') \hat{\rho}_{\text{AB}}^{(0)} \hat{m}_\eta(t) W(\boldsymbol{x}_\eta, t, \boldsymbol{x}_\nu, t') \\ & - \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \chi_\nu(t) \chi_\eta(t') \hat{m}_\nu(t) \hat{m}_\eta(t') \hat{\rho}_{\text{AB}}^{(0)} W(\boldsymbol{x}_\nu, t, \boldsymbol{x}_\eta, t') \\ & - \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \chi_\nu(t) \chi_\eta(t') \hat{\rho}_{\text{AB}}^{(0)} \hat{m}_\eta(t') \hat{m}_\nu(t) W(\boldsymbol{x}_\eta, t', \boldsymbol{x}_\nu, t) \Bigg]. \end{aligned} \tag{2.10}$$

Here, $V(\boldsymbol{x}_\nu, t)$ and $W(\boldsymbol{x}_\eta, t, \boldsymbol{x}_\nu, t')$ are given by

$$V(\boldsymbol{x}_\nu, t) := \int \mathrm{d}^d\boldsymbol{x}\, F_\nu(\boldsymbol{x} - \boldsymbol{x}_\nu) v(\boldsymbol{x}, t), \tag{2.11}$$

$$W(\boldsymbol{x}_\eta, t, \boldsymbol{x}_\nu, t') := \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{x'}\, F_\eta(\boldsymbol{x} - \boldsymbol{x}_\eta) F_\nu(\boldsymbol{x'} - \boldsymbol{x}_\nu) w(\boldsymbol{x}, t, \boldsymbol{x'}, t'), \tag{2.12}$$

while the one- and two-point correlation functions, $v(\boldsymbol{x}, t)$ and $w(\boldsymbol{x}, t, \boldsymbol{x}', t')$, of the field in the state $\hat{\rho}_\phi$, are defined as

$$v(\boldsymbol{x}, t) := \mathrm{Tr}_\phi \left[\hat{\phi}(\boldsymbol{x}, t)\hat{\rho}_\phi\right], \tag{2.13}$$

$$w(\boldsymbol{x}, t, \boldsymbol{x'}, t') := \mathrm{Tr}_\phi \left[\hat{\phi}(\boldsymbol{x}, t)\hat{\phi}(\boldsymbol{x'}, t')\hat{\rho}_\phi\right]. \tag{2.14}$$

After computing the evolved two-detector state $\hat{\rho}_{\text{AB}}$ using Eq. (2.5), we can use it to compute the amount of correlations present between the detectors A and B following their interactions with the field. We will focus on two types of correlations: entanglement and mutual information.

More precisely, we will quantify the entanglement that the detectors A and B harvest from the field by computing the negativity $\mathcal{N}$, which, for a state $\hat{\rho}_{\text{AB}}$ on the Hilbert space $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$, is defined as [9]

$$\mathcal{N}\left[\hat{\rho}_{\text{AB}}\right] := \sum_i \max\left(0, -E^{\mathbf{t}_{\text{A}}}_{\text{AB},i}\right), \tag{2.15}$$

where the $E^{\mathbf{t}_{\text{A}}}_{\text{AB},i}$ are the eigenvalues of the partially transposed matrix $\hat{\rho}^{\mathbf{t}_{\text{A}}}_{\text{AB}}$. It is well known that the negativity of a two-qubit system is an entanglement monotone that vanishes if and only if the two-qubit state is separable [83, 84]. Hence the negativity is often used as a measure of entanglement in harvesting scenarios, and it is the measure that we will use.

It is also possible for Alice and Bob to be classically correlated via their interactions with the field. We will quantify the total amount of correlations (quantum and classical) between them by computing the mutual information, I, which is defined as

$$I[\hat{\rho}_{\text{AB}}] := S[\hat{\rho}_{\text{A}}] + S[\hat{\rho}_{\text{B}}] - S[\hat{\rho}_{\text{AB}}], \tag{2.16}$$

where $S[\hat{\rho}] := -\mathrm{Tr}(\hat{\rho}\log\hat{\rho})$ is the von Neumann entropy of the state $\hat{\rho}$, while $\hat{\rho}_{\text{A}} := \mathrm{Tr}_{\text{B}}(\hat{\rho}_{\text{AB}})$ and $\hat{\rho}_{\text{B}} := \mathrm{Tr}_{\text{A}}(\hat{\rho}_{\text{AB}})$ are the reduced states of detectors A and B following the detector-field interactions. In particular, if entanglement is zero and the mutual information is not, the correlations have to be either classical correlations or quantum discord [85, 86].

2.2 Thermal field state

Let us suppose now that the two Unruh-DeWitt detectors are initially in their ground states, $\hat{\rho}_\nu = |g_\nu\rangle\langle g_\nu|$, and that the field is in a thermal state $\hat{\rho}_{\beta}$ of inverse temperature β. It will be sufficient for our purposes to formally define $\hat{\rho}_{\beta}$ as a Gibbs state in the usual way. Namely we write

$$\hat{\rho}_{\beta} := \frac{\exp(-\beta \hat{H}_\phi)}{Z}, \tag{2.17}$$

where $Z := \mathrm{Tr}[\exp(-\beta \hat{H}_\phi)]$ is the partition function of the free field. Here $\hat{H}_\phi$ is the Schrödinger picture free field Hamiltonian, which, after subtracting off an infinite zero-point energy (which does not affect any observable dynamics), takes the form

$$\hat{H}_\phi = \int \mathrm{d}^d\boldsymbol{k}\, \omega_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}. \tag{2.18}$$

We would like to emphasize that, strictly speaking, the Gibbs definition of $\hat{\rho}_{\beta}$ in Eq. (2.17) is not well defined when $\hat{H}_\phi$ is the Hamiltonian of a field in free space, since then $\hat{H}_\phi$ is an operator acting on a Hilbert space of uncountably many dimensions, and certain technical issues arise in with performing its exponentiation and trace. We could proceed rigorously by instead considering our field to be in a large box of length L, such that its Hilbert space is of countable dimension, and then in the end taking the limit $L \to \infty$. Alternatively we could formalize our treatment by making use of the Kubo-Martin-Schwinger (KMS) definition of a thermal state, which is rigorously defined even for continuous variable systems [87, 88]. In this case the definition of $\hat{\rho}_{\beta}$ would correspond to a KMS state of KMS parameter β with respect to the time t proper to both detectors. However we will shortly see that, for our limited purposes, these more rigorous definitions of $\hat{\rho}_{\beta}$ are unnecessary in the sense that formal calculations using the Gibbs definition in Eq. (2.17) yield the same results. This can be checked by comparing the results we will obtain with, e.g., the calculations in Ref. [89].

To see this concretely, from the definition (2.17) of $\hat{\rho}_{\beta}$ and the canonical commutation relations (CCRs) in Eqs. (1.22)-(1.24), we can straightforwardly calculate the one- and two-point correlation functions defined in (2.13) and (2.14). Because the field is composed of a linear superposition of $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^\dagger_{\boldsymbol{k}}$ operators, we first perform the following useful calculation:

$$
\begin{aligned}
\mathrm{Tr}_\phi\left(\hat{\rho}_\beta \hat{a}_{\boldsymbol{k}}\right) &= \frac{1}{Z}\,\mathrm{Tr}_\phi\left(e^{-\beta \hat{H}_\phi}\hat{a}_{\boldsymbol{k}}\right) \\
&= \frac{1}{Z}\,\mathrm{Tr}_\phi\left(e^{-\beta \hat{H}_\phi}\hat{a}_{\boldsymbol{k}} e^{\beta \hat{H}_\phi} e^{-\beta \hat{H}_\phi}\right) \\
&= \frac{e^{\beta\omega_{\boldsymbol{k}}}}{Z}\,\mathrm{Tr}_\phi\left(\hat{a}_{\boldsymbol{k}} e^{-\beta \hat{H}_\phi}\right) \\
&= e^{\beta\omega_{\boldsymbol{k}}}\,\mathrm{Tr}_\phi\left(\hat{a}_{\boldsymbol{k}}\hat{\rho}_\beta\right) \\
&= e^{\beta\omega_{\boldsymbol{k}}}\,\mathrm{Tr}_\phi\left(\hat{\rho}_\beta\hat{a}_{\boldsymbol{k}}\right)
\end{aligned}
\tag{2.19}
$$

where in the third line we made use of the identity $e^{-\beta \hat{H}_\phi}\hat{a}_{\boldsymbol{k}} e^{\beta \hat{H}_\phi} = e^{\beta\omega_{\boldsymbol{k}}}\hat{a}_{\boldsymbol{k}}$, which can be easily proved using the Zassenhaus formula and the CCRs. Then, comparing the first and last lines of Eq. (2.19), we conclude that $\mathrm{Tr}_\phi\left(\hat{\rho}_\beta\hat{a}_{\boldsymbol{k}}\right) = 0$. Similarly $\mathrm{Tr}_\phi(\hat{\rho}_\beta\hat{a}^\dagger_{\boldsymbol{k}}) = 0$, and therefore the one-point function $v(\boldsymbol{x}, t) = 0$. Then, from Eqs. (2.9) and (2.11), we conclude that the first order contribution $\hat{\rho}^{(1)}_{\mathrm{AB}}$ to $\hat{\rho}_{\mathrm{AB}}$ is identically zero for a thermal field state.

To calculate the two-point function $w(\boldsymbol{x}, t, \boldsymbol{x}', t')$ we first compute:

$$
\begin{aligned}
\mathrm{Tr}_\phi\left(\hat{\rho}_\beta \hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k'}}\right) &= \frac{1}{Z}\,\mathrm{Tr}_\phi\left(e^{-\beta \hat{H}_\phi}\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k'}}\right) \\
&= \frac{1}{Z}\,\mathrm{Tr}_\phi\left(e^{-\beta \hat{H}_\phi}\hat{a}_{\boldsymbol{k}} e^{\beta \hat{H}_\phi} e^{-\beta \hat{H}_\phi}\hat{a}^\dagger_{\boldsymbol{k'}}\right) \\
&= \frac{e^{\beta\omega_{\boldsymbol{k}}}}{Z}\,\mathrm{Tr}_\phi\left(\hat{a}_{\boldsymbol{k}} e^{-\beta \hat{H}_\phi}\hat{a}^\dagger_{\boldsymbol{k'}}\right) \\
&= e^{\beta\omega_{\boldsymbol{k}}}\,\mathrm{Tr}_\phi\left(\hat{\rho}_\beta\hat{a}^\dagger_{\boldsymbol{k'}}\hat{a}_{\boldsymbol{k}}\right) \\
&= e^{\beta\omega_{\boldsymbol{k}}}\left[\mathrm{Tr}_\phi\left(\hat{\rho}_\beta\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k'}}\right) + \delta(\boldsymbol{k}-\boldsymbol{k}')\right],
\end{aligned}
\tag{2.20}
$$

where in the last step we again made use of the CCRs. Comparing the first and last lines of this expression gives the result

$$
\mathrm{Tr}(\hat{\rho}_\beta\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k'}}) = \frac{e^{\beta\omega_{\boldsymbol{k}}}}{e^{\beta\omega_{\boldsymbol{k}}}-1}\delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{2.21}
$$

Similarly we obtain the identities

$$\mathrm{Tr}(\hat{\rho}_\beta \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k'}}) = \frac{1}{e^{\beta\omega_{\boldsymbol{k}}} - 1}\delta^3(\boldsymbol{k} - \boldsymbol{k'}), \tag{2.22}$$

$$\mathrm{Tr}(\hat{\rho}_\beta \hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k'}}) = 0, \tag{2.23}$$

$$\mathrm{Tr}(\hat{\rho}_\beta \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k'}}) = 0. \tag{2.24}$$

Notice that, as alluded to above, the calculations in Eqs. (2.19) and (2.20) would turn out the same if we rigorously considered the field in a box and then took the $L \to \infty$ limit in the end. In particular the only difference would be that the CCRs contain a Kronecker delta, which in the limit of free space becomes a Dirac delta, thus recovering our results in a more rigorous fashion. Furthermore, our final expressions in Eqs. (2.21)-(2.24) are equal to those obtained using the KMS definition of $\hat{\rho}_\beta$ (see equation 14.3 in [89]). Hence our formal use of the Gibbs definition of $\hat{\rho}_\beta$ in Eq. (2.17) is justified.

We can now use the identities in Eqs. (2.21)-(2.24) to write the two-point function of the field, defined by $w(\boldsymbol{x}, t, \boldsymbol{x'}, t') := \mathrm{Tr}[\hat{\rho}_\beta \hat{\phi}(\boldsymbol{x}, t)\hat{\phi}(\boldsymbol{x'}, t')]$, as

$$w(\boldsymbol{x}, t, \boldsymbol{x'}, t') = w^{\mathrm{vac}}(\boldsymbol{x}, t, \boldsymbol{x'}, t') + w^{\mathrm{th}}_\beta(\boldsymbol{x}, t, \boldsymbol{x'}, t'). \tag{2.25}$$

Here $w^{\mathrm{vac}}(\boldsymbol{x}, t, \boldsymbol{x'}, t')$ and $w^{\mathrm{th}}_\beta(\boldsymbol{x}, t, \boldsymbol{x'}, t')$ are the vacuum (β-independent) two-point function and the thermal (β-dependent) contribution, respectively, and are explicitly given by

$$w^{\mathrm{vac}}(\boldsymbol{x}, t, \boldsymbol{x'}, t') = \int \frac{\mathrm{d}^d\boldsymbol{k}}{2(2\pi)^n\omega_{\boldsymbol{k}}} e^{-\mathrm{i}\omega_{\boldsymbol{k}}(t-t')} e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}, \tag{2.26}$$

$$w^{\mathrm{th}}_\beta(\boldsymbol{x}, t, \boldsymbol{x'}, t') = \int \frac{\mathrm{d}^d\boldsymbol{k}\, \left[e^{\mathrm{i}\omega_{\boldsymbol{k}}(t-t')} e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})} + \mathrm{c.c}\right]}{2(2\pi)^n\omega_{\boldsymbol{k}}\left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)}. \tag{2.27}$$

Before we proceed to use the two-point function to calculate the time-evolved two-detector density matrix $\hat{\rho}_{\mathrm{AB}}$, it should be noted that in the literature one often finds a very different looking expression for the two-point function of a thermal field state. For instance, in [90], the thermal two-point function for a massless field in $(3+1)$-dimensions is shown to be

$$\begin{aligned} w(\boldsymbol{x}, t, \boldsymbol{x'}, t') =& \frac{1}{8\pi\Delta x\beta}\left[\coth\left(\frac{\pi(\Delta x + \Delta t)}{\beta}\right) + \coth\left(\frac{\pi(\Delta x - \Delta t)}{\beta}\right)\right] \\ &+ \frac{\mathrm{i}}{8\pi\Delta x}\left[\delta^{(3)}(\Delta x + \Delta t) - \delta^{(3)}(\Delta x - \Delta t)\right], \end{aligned} \tag{2.28}$$

where $\Delta x := |\boldsymbol{x} - \boldsymbol{x}'|$ and $\Delta t := t - t'$. The advantage of this expression over the one in Eq. (2.25) is that there are no integrals over momentum space that have to be evaluated. The disadvantage is that it is restrictive to the massless $(3+1)$-dimensional case. Furthermore the method used in [90] to obtain Eq. (2.28) is much less direct than the method we employed in obtaining Eq. (2.25). In any case, as a consistency check in Appendix B we show that the expression in Eq. (2.28) is indeed a specific case of Eq. (2.25) when $m = 0$ and $n = 3$.

We now come back to our main objective: use the two-point function $w(\boldsymbol{x}, t, \boldsymbol{x}', t')$ in Eq. (2.25) to compute the density matrix $\hat{\rho}_{\text{AB}}$ in (2.2). Substituting (2.25) into (2.10) we obtain, to second order in the coupling strength λ,

$$\hat{\rho}_{\text{AB}} = \begin{pmatrix} 1 - \mathcal{L}_{\text{AA}}(\beta) - \mathcal{L}_{\text{BB}}(\beta) & 0 & 0 & \mathcal{M}^*(\beta) \\ 0 & \mathcal{L}_{\text{BB}}(\beta) & \mathcal{L}^*_{\text{AB}}(\beta) & 0 \\ 0 & \mathcal{L}_{\text{AB}}(\beta) & \mathcal{L}_{\text{AA}}(\beta) & 0 \\ \mathcal{M}(\beta) & 0 & 0 & 0 \end{pmatrix}, \tag{2.29}$$

where we work in the basis $\{|g_{\text{A}}\rangle|g_{\text{B}}\rangle, |g_{\text{A}}\rangle|e_{\text{B}}\rangle, |e_{\text{A}}\rangle|g_{\text{B}}\rangle, |e_{\text{A}}\rangle|e_{\text{B}}\rangle\}$. The terms $\mathcal{L}_{\nu\eta}(\beta)$ and $\mathcal{M}(\beta)$ are defined to be

$$\begin{aligned} \mathcal{L}_{\nu\eta}(\beta) = \mathcal{L}^{\text{vac}}_{\nu\eta} + 2\pi\lambda_\nu\lambda_\eta \int & \frac{\mathrm{d}^d\boldsymbol{k}\, \tilde{F}^*_\nu(\boldsymbol{k})\tilde{F}_\eta(\boldsymbol{k}) e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_\eta - \boldsymbol{x}_\nu)}}{2\omega_{\boldsymbol{k}}\left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)} \\ & \times \Big[\tilde{\chi}^*_\nu(\omega_{\boldsymbol{k}} - \Omega_\nu)\tilde{\chi}_\eta(\omega_{\boldsymbol{k}} - \Omega_\eta) + \tilde{\chi}_\nu(\omega_{\boldsymbol{k}} + \Omega_\nu)\tilde{\chi}^*_\eta(\omega_{\boldsymbol{k}} + \Omega_\eta)\Big], \end{aligned} \tag{2.30}$$

$$\begin{aligned} \mathcal{M}(\beta) = \mathcal{M}^{\text{vac}} - 2\pi\lambda_{\text{A}}\lambda_{\text{B}} \int & \frac{\mathrm{d}^d\boldsymbol{k}\, \tilde{F}_{\text{A}}(\boldsymbol{k})\tilde{F}^*_{\text{B}}(\boldsymbol{k}) e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\text{A}} - \boldsymbol{x}_{\text{B}})}}{2\omega_{\boldsymbol{k}}\left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)} \\ & \times \Big[\tilde{\chi}^*_{\text{A}}(\omega_{\boldsymbol{k}} - \Omega_{\text{A}})\tilde{\chi}_{\text{B}}(\omega_{\boldsymbol{k}} + \Omega_{\text{B}}) + \tilde{\chi}_{\text{A}}(\omega_{\boldsymbol{k}} + \Omega_{\text{A}})\tilde{\chi}^*_{\text{B}}(\omega_{\boldsymbol{k}} - \Omega_{\text{B}})\Big]. \end{aligned} \tag{2.31}$$

Here we define the Fourier transform $\tilde{g} : \mathbb{R}^d \to \mathbb{C}$ of a function $g : \mathbb{R}^d \to \mathbb{R}$ as

$$\tilde{g}(\boldsymbol{k}) := \frac{1}{\sqrt{(2\pi)^d}} \int \mathrm{d}^d\boldsymbol{x}\, g(\boldsymbol{x}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.32}$$

and as always we use the superscript "vac" to denote quantities that do not depend on the inverse temperature β, i.e. those terms which arise from the "vacuum" part w^{vac} of the two-point function. The vacuum terms $\mathcal{L}^{\text{vac}}_{\nu\eta}$ and $\mathcal{M}^{\text{vac}}$ are explicitly given by

$$\mathcal{L}^{\text{vac}}_{\nu\eta} = 2\pi\lambda_\nu\lambda_\eta \int \frac{\mathrm{d}^d\boldsymbol{k}}{2\omega_{\boldsymbol{k}}} \tilde{F}^*_\nu(\boldsymbol{k})\tilde{F}_\eta(\boldsymbol{k})e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_\nu-\boldsymbol{x}_\eta)}\tilde{\chi}_\nu(\omega_{\boldsymbol{k}}+\Omega_\nu)\tilde{\chi}^*_\eta(\omega_{\boldsymbol{k}}+\Omega_\eta), \tag{2.33}$$

$$\begin{aligned}\mathcal{M}^{\text{vac}} = -\lambda_{\text{A}}\lambda_{\text{B}} \int \frac{\mathrm{d}^d\boldsymbol{k}}{2\omega_{\boldsymbol{k}}} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' e^{-\mathrm{i}\omega_{\boldsymbol{k}}(t-t')} \\ \times \Big[\tilde{F}_{\text{A}}(\boldsymbol{k})\tilde{F}^*_{\text{B}}(\boldsymbol{k})e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\text{A}}-\boldsymbol{x}_{\text{B}})}\tilde{\chi}_{\text{A}}(t)\tilde{\chi}_{\text{B}}(t')e^{\mathrm{i}(\Omega_{\text{A}}t+\Omega_{\text{B}}t')} + (\text{A}\leftrightarrow\text{B})\Big].\end{aligned} \tag{2.34}$$

2.2.1 Harvesting entanglement

Having computed the time-evolved density matrix $\hat{\rho}_{\text{AB}}$ of the Unruh-DeWitt detector pair, we can now compute the negativity of this state and thus quantify the amount of entanglement the detectors harvest from the thermal field state. Using the expression (2.29) for $\hat{\rho}_{\text{AB}}$, we find that in the same computational basis, to $\mathcal{O}(\lambda^2)$ the partially transposed matrix $\hat{\rho}^{\mathbf{t}_{\text{A}}}_{\text{AB}}$ takes the form

$$\hat{\rho}^{\mathbf{t}_{\text{A}}}_{\text{AB}} = \begin{pmatrix} 1-\mathcal{L}_{\text{AA}}(\beta)-\mathcal{L}_{\text{BB}}(\beta) & 0 & 0 & \mathcal{L}^*_{\text{AB}}(\beta) \\ 0 & \mathcal{L}_{\text{BB}}(\beta) & \mathcal{M}^*(\beta) & 0 \\ 0 & \mathcal{M}(\beta) & \mathcal{L}_{\text{AA}}(\beta) & 0 \\ \mathcal{L}_{\text{AB}}(\beta) & 0 & 0 & 0 \end{pmatrix}. \tag{2.35}$$

As discussed in [26], at $\mathcal{O}(\lambda^2)$ a matrix of this form has only one potentially negative eigenvalue:

$$E^{\mathbf{t}_{\text{A}}}_{\text{AB},1} = \frac{1}{2}\Big(\mathcal{L}_{\text{AA}}(\beta)+\mathcal{L}_{\text{BB}}(\beta) - \sqrt{(\mathcal{L}_{\text{AA}}(\beta)-\mathcal{L}_{\text{BB}}(\beta))^2+4|\mathcal{M}(\beta)|^2}\Big).$$

Hence we find that the negativity $\mathcal{N}$, defined in Eq. (2.15), can be written as

$$\mathcal{N}[\hat{\rho}_{\text{AB}}] = \max\left(0, -E^{\mathbf{t}_{\text{A}}}_{\text{AB},1}\right). \tag{2.36}$$

Now suppose that the detectors A and B are identical. That is, they have the same shapes $F(\boldsymbol{x}) = F_\nu(\boldsymbol{x})$, the same proper energy gaps $\Omega = \Omega_\nu$, the same coupling constants $\lambda = \lambda_\nu$, and the same switching profiles $\chi(t-t_\nu) = \chi_\nu(t)$. Note that we are still allowing for the detectors to couple to the field at potentially different spacetime locations $(t_{\text{A}}, \boldsymbol{x}_{\text{A}})$ and $(t_{\text{B}}, \boldsymbol{x}_{\text{B}})$. However, since the local terms $\mathcal{L}_{\nu\nu}$ are translationally invariant, we find that $\mathcal{L}_{\text{AA}}(\beta) = \mathcal{L}_{\text{BB}}(\beta)$, and the negativity can be written more simply as

$$\mathcal{N} = \max\left[0, |\mathcal{M}(\beta)| - \mathcal{L}_{\nu\nu}(\beta)\right]. \tag{2.37}$$

As acknowledged in [26], this form for the negativity makes evident the competition between the non-local term $|\mathcal{M}(\beta)|$, which increases the negativity, and the local term $\mathcal{L}_{\nu\nu}(\beta)$, which decreases it. We note however, that although this interpretation of Eq. (2.37) is pleasantly consistent with the intuition that entanglement is a non-local phenomenon, it should not be taken too literally. For instance, in [2, 43] it was shown that a detector pair interacting with a coherent field state extracts the exact same amount of entanglement as it would from a vacuum state, despite the fact that inherently local terms of $\mathcal{O}(\lambda)$ appear in $\hat{\rho}_{\text{AB}}$ for the former but not the latter case.

Having obtained an expression in (2.37) for the negativity $\mathcal{N}$ of two identical Unruh-DeWitt detectors following their interactions with a thermal field state, we would now like to determine the temperature dependence of $\mathcal{N}$. In other words, we want to answer the question, "What is the optimal field temperature for Unruh-DeWitt detectors to harvest entanglement?"

To answer this question, let us first particularize the terms $\mathcal{L}_{\nu\eta}(\beta)$ and $\mathcal{M}(\beta)$ in Eqs. (2.30) and (2.31) for identical detectors. We obtain

$$\begin{aligned}\mathcal{L}_{\nu\eta}(\beta) = \mathcal{L}_{\nu\eta}^{\text{vac}} + \pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, |\tilde{F}(\boldsymbol{k})|^2}{\omega_{\boldsymbol{k}} \left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)} e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_\eta - \boldsymbol{x}_\nu)} \\ \times \Big(|\tilde{\chi}(\omega_{\boldsymbol{k}} - \Omega)|^2 e^{\mathrm{i}(\omega_{\boldsymbol{k}}-\Omega)t_\eta} e^{-\mathrm{i}(\omega_{\boldsymbol{k}}-\Omega)t_\nu} + |\tilde{\chi}(\omega_{\boldsymbol{k}} + \Omega)|^2 e^{-\mathrm{i}(\omega_{\boldsymbol{k}}+\Omega)t_\eta} e^{\mathrm{i}(\omega_{\boldsymbol{k}}+\Omega)t_\nu}\Big),\end{aligned} \tag{2.38}$$

$$\begin{aligned}\mathcal{M}(\beta) = \mathcal{M}^{\text{vac}} - 2\pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, |\tilde{F}(\boldsymbol{k})|^2 e^{\mathrm{i}\Omega(t_{\text{A}}+t_{\text{B}})} e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\text{A}} - \boldsymbol{x}_{\text{B}})}}{\omega_{\boldsymbol{k}} \left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)} \\ \times \tilde{\chi}^*(\omega_{\boldsymbol{k}} - \Omega)\tilde{\chi}(\omega_{\boldsymbol{k}} + \Omega)\cos[\omega_{\boldsymbol{k}}(t_{\text{A}} - t_{\text{B}})].\end{aligned} \tag{2.39}$$

Now, let us consider two temperatures, $\beta_1^{-1} < \beta_2^{-1}$. Then, defining

$$h(\boldsymbol{k}) := \frac{1}{e^{\beta_2\omega_{\boldsymbol{k}}} - 1} - \frac{1}{e^{\beta_1\omega_{\boldsymbol{k}}} - 1}, \tag{2.40}$$

which is strictly greater than zero, we can rewrite $\mathcal{L}_{\nu\nu}(\beta)$ and $\mathcal{M}(\beta)$ to read

$$\mathcal{L}_{\nu\nu}(\beta_2) = \mathcal{L}_{\nu\nu}(\beta_1) + \pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k})|\tilde{F}(\boldsymbol{k})|^2}{\omega_{\boldsymbol{k}}} \Big(|\tilde{\chi}(\omega_{\boldsymbol{k}} - \Omega)|^2 + |\tilde{\chi}(\omega_{\boldsymbol{k}} + \Omega)|^2\Big), \tag{2.41}$$

$$\begin{aligned}\mathcal{M}(\beta_2) = \mathcal{M}(\beta_1) - 2\pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k})|\tilde{F}(\boldsymbol{k})|^2}{\omega_{\boldsymbol{k}}} e^{\mathrm{i}\Omega(t_{\text{A}}+t_{\text{B}})} e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\text{A}} - \boldsymbol{x}_{\text{B}})} \cos[\omega_{\boldsymbol{k}}(t_{\text{A}} - t_{\text{B}})] \\ \times \tilde{\chi}^*(\omega_{\boldsymbol{k}} - \Omega)\tilde{\chi}(\omega_{\boldsymbol{k}} + \Omega).\end{aligned} \tag{2.42}$$

Taking the magnitude of the latter expression, by the triangle inequality we obtain

$$\begin{aligned}|\mathcal{M}(\beta_2)| &\leq |\mathcal{M}(\beta_1)| + 2\pi\lambda^2 \left| \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k})|\tilde{F}(\boldsymbol{k})|^2}{\omega_{\boldsymbol{k}}} e^{\mathrm{i}\Omega(t_{\text{A}}+t_{\text{B}})} e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\text{A}}-\boldsymbol{x}_{\text{B}})} \cos[\omega_{\boldsymbol{k}}(t_{\text{A}}-t_{\text{B}})] \right. \\ &\qquad \left. \times\, \tilde{\chi}^*(\omega_{\boldsymbol{k}}-\Omega)\tilde{\chi}(\omega_{\boldsymbol{k}}+\Omega) \right| \\ &\leq |\mathcal{M}(\beta_1)| + 2\pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k})|\tilde{F}(\boldsymbol{k})|^2}{\omega_{\boldsymbol{k}}} |\tilde{\chi}^*(\omega_{\boldsymbol{k}}-\Omega)||\tilde{\chi}(\omega_{\boldsymbol{k}}+\Omega)|,\end{aligned} \tag{2.43}$$

Finally, combining Eqs. (2.41) and (2.43) we find

$$\begin{aligned}|\mathcal{M}(\beta_2)| - \mathcal{L}_{\nu\nu}(\beta_2) &\leq |\mathcal{M}(\beta_1)| - \mathcal{L}_{\nu\nu}(\beta_1) - \pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k}) D(\boldsymbol{k})}{\omega_{\boldsymbol{k}}} \\ &\leq |\mathcal{M}(\beta_1)| - \mathcal{L}_{\nu\nu}(\beta_1),\end{aligned} \tag{2.44}$$

where $D(\boldsymbol{k}) := |\tilde{F}(\boldsymbol{k})|^2(|\tilde{\chi}(\omega_{\boldsymbol{k}}-\Omega)|-|\tilde{\chi}(\omega_{\boldsymbol{k}}+\Omega)|)^2$ is a non-negative function characterized by the switching, smearing, and energy gap of the detectors. Hence, using the definition (2.37) of the negativity, Eq. (2.44) proves our first result: the amount of entanglement that two identical UDW detectors can harvest from a thermal field state decreases with the temperature β^{-1}. This is true regardless of the dimensionality of spacetime, the mass of the field, and the properties (spatial smearing, temporal switching, energy gap) of the detectors.

In fact, we can obtain a somewhat stronger statement about the negativity of a pair of detectors interacting with a thermal field state. First, notice from Eq. (2.40) that for given values of β_1 and $\boldsymbol{k}$, the value of the function $h(\boldsymbol{k})$ can be increased arbitrarily by choosing a small enough value of β_2. Therefore, from Eq. (2.44), as long as $D(\boldsymbol{k})$ is not identically equal to zero, we find that the value of $|\mathcal{M}(\beta_2)|-\mathcal{L}_{\nu\nu}(\beta_2)$ can be made negative by taking a large enough temperature β_2^{-1}. Hence, not only does the amount of entanglement harvested by a UDW detector pair decrease monotonically with the temperature, but also by increasing the temperature of the field to a high enough value we can always (as long as $D(\boldsymbol{k})$ is not identically zero) ensure that the thermal noise prevents the detectors from becoming entangled at all. This is true regardless the mass of the field, spacetime dimensionality and the detector properties.

Knowing that the negativity $\mathcal{N}$ of a detector pair decreases with the temperature of the field, we can ask what is the rate of this decrease. We can straightforwardly obtain a bound on $\mathrm{d}\mathcal{N}/\,\mathrm{d}\beta$ from Eq. (2.44). First, writing $E^{\mathbf{t}_{\text{A}}}_{\text{AB},1}(\beta) = \mathcal{L}_{\nu\nu}(\beta) - |\mathcal{M}(\beta)|$ for identical detectors, the second line of Eq. (2.44) can be expressed as

$$E^{\mathbf{t}_{\rm A}}_{\rm AB,1}(\beta_1) - E^{\mathbf{t}_{\rm A}}_{\rm AB,1}(\beta_2) \leq -\pi\lambda^2 \int \frac{\mathrm{d}^d\boldsymbol{k}\, h(\boldsymbol{k}) D(\boldsymbol{k})}{\omega_{\boldsymbol{k}}}. \tag{2.45}$$

Dividing both sides of this expression by $\beta_1 - \beta_2$, taking the limit $\beta_1 \to \beta_2$, and using the fact that

$$\lim_{\beta_1 \to \beta_2} \frac{h(\boldsymbol{k})}{\beta_1 - \beta_2} = -\frac{\mathrm{d}}{\mathrm{d}\beta_1}\left(\frac{1}{e^{\beta_1 \omega_{\boldsymbol{k}}} - 1}\right) = \frac{\omega_{\boldsymbol{k}} e^{\beta_1 \omega_{\boldsymbol{k}}}}{\left(e^{\beta_1 \omega_{\boldsymbol{k}}} - 1\right)^2}, \tag{2.46}$$

we find the rate of change of the eigenvalue $E^{\mathbf{t}_{\rm A}}_{\rm AB,1}(\beta)$ with respect to the inverse temperature β to be bounded from below according to

$$\frac{\mathrm{d}}{\mathrm{d}\beta} E^{\mathbf{t}_{\rm A}}_{\rm AB,1}(\beta) \leq -\pi\lambda^2 \int \mathrm{d}^d\boldsymbol{k}\, D(\boldsymbol{k}) \frac{e^{\beta\omega_{\boldsymbol{k}}}}{\left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)^2}. \tag{2.47}$$

Therefore in regions where the negativity $\mathcal{N}(\beta)$ is non-zero, we have that

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}\beta} \geq \pi\lambda^2 \int \mathrm{d}^d\boldsymbol{k}\, D(\boldsymbol{k}) \frac{e^{\beta\omega_{\boldsymbol{k}}}}{\left(e^{\beta\omega_{\boldsymbol{k}}} - 1\right)^2}. \tag{2.48}$$

This puts a lower bound on how fast $\mathcal{N}$ must grow with the inverse temperature β, in regions where $\mathcal{N}$ is non-zero. Of course if $\mathcal{N}$ is zero, then increasing β will only result in $\mathcal{N}$ remaining zero.

Having proven the general result that temperature is always detrimental to entanglement harvesting (at least for identical detectors), let us now consider some particular parameters for the detectors A and B, so that we may explicitly see the manifestation of this phenomenon. To that end, let us suppose that the two detectors are located in $(3+1)$ dimensional spacetime, that they have Gaussian spatial profiles of width σ,

$$F(\boldsymbol{x}) = \frac{1}{(\sqrt{\pi}\sigma)^3} e^{-\frac{|\boldsymbol{x}|^2}{\sigma^2}}, \tag{2.49}$$

and that their temporal switching functions are also Gaussians (of width τ),

$$\chi(t) = e^{-\frac{t^2}{\tau^2}}. \tag{2.50}$$

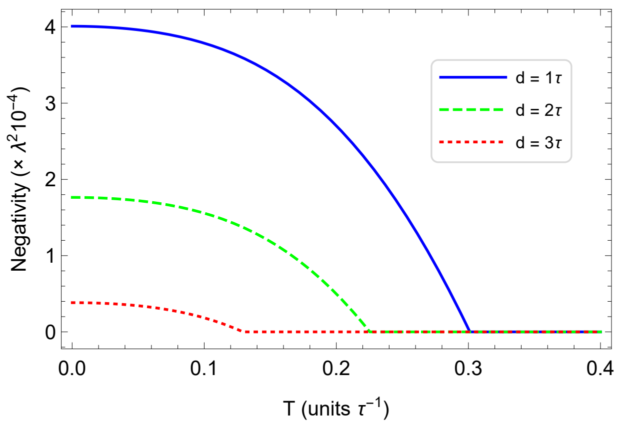

Figure 2.1: Negativity of identical detectors as a function of field temperature, for different spatial separations d of their centers of mass. The detectors are coupled to the field at the same time according to a Gaussian switching function of width τ, their spatial profiles are Gaussians of width $\sigma = \tau$, and their energy gap is $\Omega = 3/\tau$.

Then it is straightforward to show that the terms $\mathcal{M}^{\text{th}}$, $\mathcal{L}^{\text{th}}_{\text{AB}}$ and $L^{\text{th}}_{\nu\nu}$, which make up the thermal contributions to the density matrix $\hat{\rho}_{\text{AB}}$, evaluate to

$$\mathcal{M}^{\text{th}} = -\frac{\bar{\lambda}^2 e^{-\frac{1}{2}\bar{\Omega}^2} e^{\mathrm{i}\bar{\Omega}\bar{\Delta}^+}}{4\pi\bar{d}} \int_0^\infty \mathrm{d}\bar{k} \frac{e^{-\frac{1}{2}\bar{k}^2(1+\bar{\sigma}^2)}}{e^{\bar{\beta}\bar{k}}-1} \sin(\bar{d}\bar{k}) \cos(\bar{\Delta}^-\bar{k}), \tag{2.51}$$

$$\mathcal{L}^{\text{th}}_{\text{AB}} = \frac{\bar{\lambda}^2 e^{-\frac{1}{2}\bar{\Omega}^2} e^{-\mathrm{i}\bar{\Omega}\bar{\Delta}^-}}{2\pi\bar{d}} \int_0^\infty \mathrm{d}\bar{k} \frac{e^{-\frac{1}{2}\bar{k}^2(1+\bar{\sigma}^2)}}{e^{\bar{\beta}\bar{k}}-1} \sin(\bar{d}\bar{k}) \cosh[(\bar{\Omega}+\mathrm{i}\bar{\Delta}^-)\bar{k}], \tag{2.52}$$

$$\mathcal{L}^{\text{th}}_{\nu\nu} = \frac{\bar{\lambda}^2 e^{-\frac{1}{2}\bar{\Omega}^2}}{2\pi} \int_0^\infty \mathrm{d}\bar{k} \frac{\bar{k} e^{-\frac{1}{2}\bar{k}^2(1+\bar{\sigma}^2)}}{e^{\bar{\beta}\bar{k}}-1} \cosh(\bar{\Omega}\bar{k}). \tag{2.53}$$

Here, every quantity with a bar is a dimensionless expression of the scales of the problem in units of τ (e.g. $\bar{\Omega} := \Omega\tau$, $\bar{\beta} := \beta/\tau$), and we have defined $\bar{d} := |\boldsymbol{x}_{\text{A}} - \boldsymbol{x}_{\text{B}}|/\tau$ and $\bar{\Delta}^{\pm} := (t_{\text{B}} \pm t_{\text{A}})/\tau$. Meanwhile the terms $\mathcal{M}^{\text{vac}}$ and $\mathcal{L}^{\text{vac}}_{\nu\eta}$, which give the vacuum (β independent) contributions to $\hat{\rho}_{\text{AB}}$, can be found in equations 29-31 in [26].

Assuming these detector spatial profiles and switching functions, in Fig. 2.1 we show the dependence of the negativity of the detector pair on the temperature $T = \beta^{-1}$ of the field. We see that, in accordance with our general discussion above, the negativity is a monotonically decreasing function of T, and that it is identically zero after a certain finite temperature. These findings are qualitatively the same as what was found in [24], namely that harmonic oscillator detectors in a (1+1)D cavity harvest less entanglement as the field

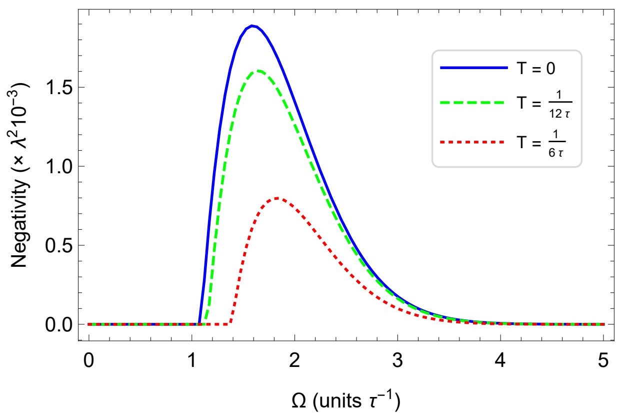

Figure 2.2: Negativity of identical detectors as a function of their energy gap, for different field temperatures T. The detectors are coupled to the field at the same time according to a Gaussian switching function of width τ, and they have Gaussian spatial profiles of width $\sigma = \tau$, the centers of which are separated in space by $d = 2\tau$.

temperature increases. This is, of course, all in agreement with our intuition that "thermal noise" is detrimental to the detectors obtaining non-local correlations. We will soon see however, that this seemingly reasonable intuition does not apply when we quantify the correlations using the mutual information rather than the negativity. In particular we will show that the mutual information between the detector pair can increase with the field temperature.

To conclude this section, let us briefly investigate how the negativity of the detectors varies with their energy gap Ω. These results are summarized in Fig. 2.2. Notice that, for a given field temperature T, the detectors cannot become entangled if their energy gap is below some finite value $\Omega_{\min}(T)$. We also notice that $\Omega_{\min}(T)$ is a monotonically increasing function of temperature. This tells us that if we have a way to control the energy gap of the detectors, then by measuring the amount of entanglement that this detector pair harvests from the field we have, in principle, a quantum thermometer capable of measuring the field temperature.

2.2.2 Harvesting mutual information

Having shown that the amount of entanglement harvested by two Unruh-DeWitt detectors decreases with the temperature of the field with which they interact, we can ask what happens to other types of correlations. As mentioned above, the mutual information $I[\hat{\rho}_{\text{AB}}]$, defined in Eq. (2.16), quantifies the total correlations (quantum and classical) present between the two detectors. Using the time-evolved density matrix $\hat{\rho}_{\text{AB}}$ in Eq. (2.29) for the two detectors, we find that $I[\hat{\rho}_{\text{AB}}]$ takes the form

$$I[\hat{\rho}_{\text{AB}}] = \mathcal{L}_+ \log(\mathcal{L}_+) + \mathcal{L}_- \log(\mathcal{L}_-) - \mathcal{L}_{\text{AA}} \log(\mathcal{L}_{\text{AA}}) - \mathcal{L}_{\text{BB}} \log(\mathcal{L}_{\text{BB}}) + \mathcal{O}(\lambda^4),$$

where $\mathcal{L}_\pm$ are defined as

$$\mathcal{L}_\pm = \frac{1}{2}\left(\mathcal{L}_{\text{AA}} + \mathcal{L}_{\text{BB}} \pm \sqrt{(\mathcal{L}_{\text{AA}} - \mathcal{L}_{\text{BB}})^2 + 4|\mathcal{L}_{\text{AB}}|^2}\right). \tag{2.54}$$

Although the general dependence of $I[\hat{\rho}_{\text{AB}}]$ on the temperature β^{-1} is highly non-trivial, from Eq. (2.54) it is straightforward to derive the asymptotic behaviour as $\beta^{-1} \to \infty$. Defining $\mathscr{L}_\pm := \beta\mathcal{L}_\pm$ and $\mathscr{L}_{\nu\eta} := \beta\mathcal{L}_{\nu\eta}$, we notice from Eq. (2.30) that $\mathscr{L}_\pm$ and $\mathscr{L}_{\nu\eta}$ are independent of β in the limit $\beta^{-1} \to \infty$. Then from Eq. (2.54) it is easy to show that in the $\beta^{-1} \to \infty$ limit the mutual information goes as

$$I[\hat{\rho}_{\text{AB}}] \sim \frac{1}{\beta}\big(\mathscr{L}_+ \log \mathscr{L}_+ + \mathscr{L}_- \log \mathscr{L}_- - \mathscr{L}_{\text{AA}} \log \mathscr{L}_{\text{AA}} - \mathscr{L}_{\text{BB}} \log \mathscr{L}_{\text{BB}}\big). \tag{2.55}$$

Combining this with the fact that the mutual information is always non-negative, we conclude that in the large temperature limit (of course with a coupling constant small enough so that we are still within the perturbative regime) the total correlations that the detectors harvest from the field grow proportionally to the temperature β^{-1}.

To see explicitly the dependence of $I[\hat{\rho}_{\text{AB}}]$ on the temperature, let us once again particularize to the case of identical detectors with Gaussian spatial smearings (2.49) and Gaussian switching functions (2.50). These results are plotted in Fig. 2.3. We see that for low $T = \beta^{-1}$ the mutual information approaches a constant finite value, which corresponds to the correlations that the detectors would obtain if they interacted with the field vacuum. For intermediate field temperatures, we find that the mutual information has a non-trivial dependence on T, and in fact, unlike the negativity, $I[\hat{\rho}_{\text{AB}}]$ does not always increase with T. However, as we showed for the case of arbitrary detectors above, in the asymptotic limit $T \to \infty$ the mutual information is proportional to T. It should be emphasized that in a full, non-perturbative calculation, this upwards trend of $I[\hat{\rho}_{\text{AB}}]$ with temperature would not continue indefinitely for the simple reason that for a two qubit system the mutual information is bounded from above by $2\log 2$. Nevertheless it is interesting that, at least in the perturbative regime (i.e. if for a given temperature we consider a small enough coupling strength), the amount of entanglement harvested from the field by an Unruh-DeWitt detector pair is hindered by high field temperatures, whereas the total correlations in fact grow with T.

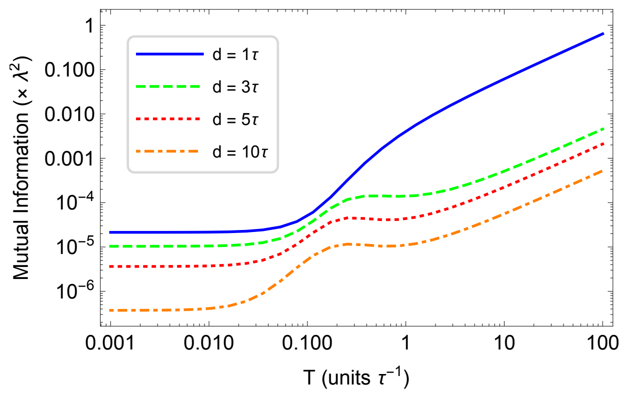

Figure 2.3: Mutual information of identical detectors as a function of field temperature, for different spatial separations d of their centers of mass. The detectors are coupled to the field at the same time according to a Gaussian switching function of width τ, their spatial profiles are Gaussians of width $\sigma = \tau$, and their energy gap is $\Omega = 3/\tau$.

2.3 Squeezed coherent field state

Again let us suppose that each Unruh-DeWitt detector is in its ground state, and that now the field is in an arbitrary, multimode, squeezed coherent state. The physical relevance of squeezed coherent states is that they are the most general set of states that saturate the Heisenberg uncertainty principle. The most general multimode squeezed coherent state is given by $|\alpha(\boldsymbol{k}), \zeta(\boldsymbol{k}, \boldsymbol{k}')\rangle := \hat{D}_\alpha \hat{S}_\zeta |0\rangle$, where the *displacement operator* $\hat{D}_\alpha$ and the *squeezing operator* $\hat{S}_\zeta$ are unitary operators defined by [42]

$$\hat{D}_\alpha := \exp\left[\int \mathrm{d}^d\boldsymbol{k}\, \left(\alpha(\boldsymbol{k})\hat{a}^\dagger_{\boldsymbol{k}} - \text{H.c.}\right)\right], \tag{2.56}$$

$$\hat{S}_\zeta := \exp\left[\frac{1}{2}\int \mathrm{d}^d\boldsymbol{k} \int \mathrm{d}^d\boldsymbol{k}'\, \left(\zeta^*(\boldsymbol{k}, \boldsymbol{k}')\hat{a}_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}'} - \text{H.c.}\right)\right]. \tag{2.57}$$

We call the complex valued distributions $\alpha(\boldsymbol{k})$ and $\zeta(\boldsymbol{k}, \boldsymbol{k}')$ the *coherent amplitude* and *squeezing amplitude* of the state $|\alpha(\boldsymbol{k}), \zeta(\boldsymbol{k}, \boldsymbol{k}')\rangle$, respectively. Through the integrals in the definitions of $\hat{D}_\alpha$ and $\hat{S}_\zeta$, these distributions generalize the familiar notion of a squeezed coherent state of a single harmonic oscillator to the case where we have an uncountably infinite number of field mode oscillators that can be pairwise two-mode squeezed with each other.

In order to calculate the one and two-point functions of the field in a squeezed coherent state, we will make use of the intertwining identities governing the action of $\hat{D}_\alpha$ and $\hat{S}_\zeta$

on the creation and annihilation operators. Namely, by using the canonical commutation relations and the Baker-Campbell-Hausdorff lemma it is straightforward to show that

$$\hat{D}_{\alpha}^{\dagger}\hat{a}_{\boldsymbol{k}}\hat{D}_{\alpha} = \hat{a}_{\boldsymbol{k}} + \alpha(\boldsymbol{k})\mathbb{1}. \tag{2.58}$$

On the other hand, we are not aware of a similarly convenient closed-form expression for $\hat{S}_{\zeta}^{\dagger}\hat{a}_{\boldsymbol{k}}\hat{S}_{\zeta}$ in the case of an arbitrary, continuous, multimode squeezing. However, since $\hat{S}_{\zeta}$ is the exponential of terms quadratic in $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^{\dagger}$, by expanding out the exponentials in $\hat{S}_{\zeta}^{\dagger}\hat{a}_{\boldsymbol{k}}\hat{S}_{\zeta}$ it is not difficult to prove that this expression takes the form of a linear superposition of $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^{\dagger}$ operators, i.e.

$$\hat{S}_{\zeta}^{\dagger}\hat{a}_{\boldsymbol{k}}\hat{S}_{\zeta} = \int \mathrm{d}^{d}\boldsymbol{k}' \left[K_1(\boldsymbol{k},\boldsymbol{k}')\hat{a}_{\boldsymbol{k}'} + K_2(\boldsymbol{k},\boldsymbol{k}')\hat{a}_{\boldsymbol{k}'}^{\dagger}\right], \tag{2.59}$$

for some bi-distributions K_1 and K_2. In particular this implies that

$$\begin{aligned}\langle\alpha(\boldsymbol{k}),\zeta(\boldsymbol{k},\boldsymbol{k}')|\hat{a}_{\boldsymbol{k}''}|\alpha(\boldsymbol{k}),\zeta(\boldsymbol{k},\boldsymbol{k}')\rangle &= \langle 0|\hat{S}_{\zeta}^{\dagger}[\hat{a}_{\boldsymbol{k}''} + \alpha(\boldsymbol{k}'')]\hat{S}_{\zeta}|0\rangle \\ &= \alpha(\boldsymbol{k}''),\end{aligned} \tag{2.60}$$

and hence, using the mode expansion (1.16) of the field operator, the one-point function (2.13) of the field in the state $|\alpha(\boldsymbol{k}),\zeta(\boldsymbol{k},\boldsymbol{k}')\rangle$ is

$$v(\boldsymbol{x},t) = \int \frac{\mathrm{d}^{d}\boldsymbol{k}}{\sqrt{2(2\pi)^{n}\omega_{\boldsymbol{k}}}}\left(\alpha(\boldsymbol{k})e^{-\mathrm{i}(\omega_{\boldsymbol{k}}t-\boldsymbol{k}\cdot\boldsymbol{x})} + \text{c.c}\right). \tag{2.61}$$

Thus we see that the one-point function is independent of the squeezing amplitude $\zeta(\boldsymbol{k},\boldsymbol{k}')$. Similarly we can show that the two-point function (2.14) in the state $|\alpha(\boldsymbol{k}),\zeta(\boldsymbol{k},\boldsymbol{k}')\rangle$ is of the form

$$w(\boldsymbol{x},t,\boldsymbol{x}',t') = w^{\text{ind}}(\boldsymbol{x},t,\boldsymbol{x}',t') + w^{\text{coh}}(\boldsymbol{x},t,\boldsymbol{x}',t'), \tag{2.62}$$

where w^{ind} is independent of the coherent amplitude $\alpha(\boldsymbol{k})$, while w^{coh} is given by a product of one-point functions,

$$w^{\text{coh}}(\boldsymbol{x},t,\boldsymbol{x}',t') = v(\boldsymbol{x},t)v(\boldsymbol{x}',t'), \tag{2.63}$$

and vanishes if $\alpha(\boldsymbol{k}) = 0$ for all $\boldsymbol{k}$.

Thus we have shown that the $\alpha(\boldsymbol{k})$-dependent contribution w^{coh} to the two-point function is the product of two one-point functions. In [2] it was shown that when this is the case, then the $\alpha(\boldsymbol{k})$-dependent contributions of $\hat{\rho}_{\text{AB}}$ arising from the one-point function exactly cancel the contributions from the two-point function, so that the eigenvalues of $\hat{\rho}_{\text{AB}}$ and

$\hat{\rho}_{\text{AB}}^{\mathbf{t}_\text{A}}$ — and therefore the negativity $\mathcal{N}[\hat{\rho}_{\text{AB}}]$ as well — are completely independent of $\alpha(\boldsymbol{k})$. This result was used in [2] to prove that the entanglement harvested by an Unruh-DeWitt detector pair is independent of the coherent amplitude of a (non-squeezed) coherent state. Since this is a general consequence of the special relationship between the $\alpha(\boldsymbol{k})$-dependent parts of the one and two-point functions, we conclude that this result is true even in the presence of squeezing. Namely, to $\mathcal{O}(\lambda^2)$, the negativity of a detector pair interacting with a general squeezed coherent state $|\alpha(\boldsymbol{k}), \zeta(\boldsymbol{k}, \boldsymbol{k}')\rangle$ is independent of the coherent amplitude distribution $\alpha(\boldsymbol{k})$. In other words, entanglement harvesting from a squeezed coherent state is insensitive to the coherent amplitude.

It is interesting to compare this result — i.e. that entanglement harvesting is insensitive to the coherent amplitude of the field — to the fact that the entanglement entropy between two regions of a spatial slice of a free field is also insensitive to the field's coherent amplitude. This later result is briefly mentioned (without proof) in Ref. [91], where the authors' interest in entanglement entropies arises out of their interest in the Ads-CFT conjecture [17]. While we are confident that a proof of such a fundamental result — i.e. that entanglement entropies are independent of the field's coherent amplitude — already exists in the literature (although we could not find it), for the sake of completeness we will now present our own proof of this result.

To that end, we note that the displacement operator $\hat{D}_\alpha$ appearing in the definition $|\alpha(\boldsymbol{k}), \zeta(\boldsymbol{k}, \boldsymbol{k}')\rangle := \hat{D}_\alpha \hat{S}_\zeta |0\rangle$ for a general squeezed coherent state can be alternatively written as

$$\hat{D}_\alpha = \exp\left[\mathrm{i}\int \mathrm{d}^d\boldsymbol{x}\, \Big(F_\phi(\boldsymbol{x})\hat{\phi}(\boldsymbol{x},0) + F_\pi(\boldsymbol{x})\hat{\pi}(\boldsymbol{x},0)\Big)\right], \tag{2.64}$$

where the smearing functions $F_\phi(\boldsymbol{x})$ and $F_\pi(\boldsymbol{x})$ are defined in terms of their Fourier transforms $\tilde{F}_\phi(\boldsymbol{k})$ and $\tilde{F}_\pi(\boldsymbol{k})$ as

$$\tilde{F}_\phi(\boldsymbol{k}) := \mathrm{i}\sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}\left[\alpha^*(-\boldsymbol{k}) - \alpha(\boldsymbol{k})\right], \tag{2.65}$$

$$\tilde{F}_\pi(\boldsymbol{k}) := \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left[\alpha^*(-\boldsymbol{k}) + \alpha(\boldsymbol{k})\right], \tag{2.66}$$

and where $\alpha(\boldsymbol{k})$ is the coherent amplitude distribution of $\hat{D}_\alpha$. This result can be straightforwardly proven by substituting Eqs. (2.65) and (2.66) into Eq. (2.64) and verifying that the defining expression for $\hat{D}_\alpha$, given by (2.56), is recovered. Furthermore, note that the Fourier transforms $\tilde{F}_\phi(\boldsymbol{k})$ and $\tilde{F}_\pi(\boldsymbol{k})$ were defined such that $\tilde{F}_\phi(-\boldsymbol{k}) = \tilde{F}_\phi^*(\boldsymbol{k})$ and $\tilde{F}_\pi(-\boldsymbol{k}) = \tilde{F}_\pi^*(\boldsymbol{k})$, and therefore the smearing functions $F_\phi(\boldsymbol{x})$ and $F_\pi(\boldsymbol{x})$ are real.

Next, notice from Eq. (2.64) that $\hat{D}_\alpha = \exp[\mathrm{i}\int \mathrm{d}^d O(\boldsymbol{x})]$, where

$$\hat{O}(\boldsymbol{x}) := F_\phi(\boldsymbol{x})\hat{\phi}(\boldsymbol{x},0) + F_\pi(\boldsymbol{x})\hat{\pi}(\boldsymbol{x},0), \tag{2.67}$$

are field observables on the $t = 0$ spatial hypersurface, and hence commute with one another, i.e. $[\hat{O}(\boldsymbol{x}), \hat{O}(\boldsymbol{x'})] = 0$ for all $\boldsymbol{x}, \boldsymbol{x'} \in \mathbb{R}^d$. Suppose now that we are given two disjoint spatial regions $\mathcal{A}$ and $\mathcal{B}$ of the $t = 0$ time slice of the (flat) manifold in which our free field lives. Also, let $\mathcal{C}$ be the complement of $\mathcal{A} \cup \mathcal{B}$ on the $t = 0$ time slice. Then, because of the above commutation property, we can express $\hat{D}_\alpha$ as a product of three commuting unitaries, namely

$$\hat{D}_\alpha = \exp\left[\mathrm{i}\int_{\mathcal{A}} \mathrm{d}^d O(\boldsymbol{x})\right] \exp\left[\mathrm{i}\int_{\mathcal{B}} \mathrm{d}^d O(\boldsymbol{x})\right] \exp\left[\mathrm{i}\int_{\mathcal{C}} \mathrm{d}^d O(\boldsymbol{x})\right]. \tag{2.68}$$

Hence we see that the displacement operator $\hat{D}_\alpha$ can be implemented by applying local unitaries to the regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and hence it cannot change the entanglement structure (as quantified by the entanglement entropy) between, for example, $\mathcal{A}$ and $\mathcal{B}$. In other words, we have proven that a free field's entanglement entropy between disjoint spatial regions $\mathcal{A}$ and $\mathcal{B}$ is independent of the field's coherent amplitude.

Hence we can now easily see the relationship between the entanglement entropy of a quantum field state and the amount of entanglement which a detector pair can harvest from that field state. Namely, since the entanglement entropy between two spacelike separated spatial regions $\mathcal{A}$ and $\mathcal{B}$ is independent of the field's coherent amplitude $\alpha(\boldsymbol{k})$, it is completely unsurprising, and indeed expected, that the amount of entanglement harvested by detectors locally coupling to these regions is also insensitive to $\alpha(\boldsymbol{k})$. Of course, as we have shown, entanglement harvesting is independent of $\alpha(\boldsymbol{k})$ even if the detectors are not in spacelike separation; however the connection with the entanglement entropy can only be made in the spacelike case, since entanglement entropies are typically computed between spatial regions located on some spacelike hypersurface.

Having observed the direct connection between entanglement entropy and entanglement harvesting, let us come back to our main problem, which is to study entanglement harvesting from squeezed coherent states.

Since we have discovered that entanglement harvesting is independent of the coherent amplitude $\alpha(\boldsymbol{k})$ of the field, we can, without loss of generality, restrict our attention only to squeezed vacuum states (i.e. we can set $\alpha(\boldsymbol{k})$ to be identically zero). Additionally, for mathematical simplicity—i.e. in order to obtain an explicit expression for $\hat{S}_\zeta^\dagger \hat{a}_{\boldsymbol{k}} \hat{S}_\zeta$ in Eq. (2.59)—from here on we will consider only squeezed coherent states in which the squeezing is not "mixed" between modes, i.e. such that the squeezing amplitude is of the form $\zeta(\boldsymbol{k},\boldsymbol{k}')=\zeta(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}')$. In this case we find that $\hat{S}_\zeta$ simplifies to

$$\hat{S}_\zeta = \exp\left[\frac{1}{2}\int \mathrm{d}^d\boldsymbol{k}\ \left(\zeta^*(\boldsymbol{k})\hat{a}_{\boldsymbol{k}}^2 - \text{H.c.}\right)\right], \tag{2.69}$$

and that $\hat{S}_\zeta^\dagger \hat{a}_{\boldsymbol{k}}\hat{S}_\zeta$ can be conveniently expressed as

$$\hat{S}_\zeta^\dagger \hat{a}_{\boldsymbol{k}}\hat{S}_\zeta = \cosh[r(\boldsymbol{k})]\hat{a}_{\boldsymbol{k}} - e^{\mathrm{i}\theta(\boldsymbol{k})}\sinh[r(\boldsymbol{k})]\hat{a}_{\boldsymbol{k}}^\dagger, \tag{2.70}$$

where we have written $\zeta(\boldsymbol{k})=r(\boldsymbol{k})e^{\mathrm{i}\theta(\boldsymbol{k})}$ in polar form. The two-point function (2.14) of the state $\hat{S}_\zeta|0\rangle$, with $\hat{S}_\zeta$ in the above form, can be written as

$$w(\boldsymbol{x},t,\boldsymbol{x}',t') = w^{\text{vac}}(\boldsymbol{x},t,\boldsymbol{x}',t') + w^{\text{sq}}(\boldsymbol{x},t,\boldsymbol{x}',t'), \tag{2.71}$$

where w^{vac} is the vacuum two-point function given in Eq. (2.26), while w^{sq} is the contribution that depends on $\zeta(\boldsymbol{k})$ and vanishes if $\zeta(\boldsymbol{k})=0$ for all $\boldsymbol{k}$. Explicitly $w^{\text{sq}}(\boldsymbol{x},t,\boldsymbol{x}',t')$ is given by

$$\begin{aligned} w^{\text{sq}}(\boldsymbol{x},t,\boldsymbol{x}',t') = \int \frac{\mathrm{d}^d\boldsymbol{k}}{2(2\pi)^n\omega_{\boldsymbol{k}}}\sinh[r(\boldsymbol{k})]\Big(&- e^{\mathrm{i}\theta(\boldsymbol{k})}\cosh[r(\boldsymbol{k})]e^{-\mathrm{i}\omega_{\boldsymbol{k}}(t+t')}e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{x}')} \\ &+ \sinh[r(\boldsymbol{k})]e^{-\mathrm{i}\omega_{\boldsymbol{k}}(t-t')}e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\Big) + \text{c.c}, \end{aligned} \tag{2.72}$$

Notice that, unlike Eq. (2.25) for a thermal field state, the two-point function for a squeezed coherent state is not invariant with respect to spacetime translations. As we will see, a physical consequence of this is that the negativity harvested by a pair of UDW detectors from a squeezed coherent state depends not only on the spacetime interval between the detectors, but also on where in the spacetime they are centered.

With the expression (2.71) for the two-point function of a squeezed vacuum field state, and with the vanishing one-point function (2.61), we can proceed to calculate the evolved state $\hat{\rho}_{\text{AB}}$ of the two UDW detectors following their interactions with this field. From (2.10), to second order in the coupling strength λ, we obtain

$$\hat{\rho}_{\text{AB}} = \begin{pmatrix} 1-\mathcal{L}_{\text{AA}}[\zeta]-\mathcal{L}_{\text{BB}}[\zeta] & 0 & 0 & \mathcal{M}^*[\zeta] \\ 0 & \mathcal{L}_{\text{BB}}[\zeta] & \mathcal{L}_{\text{AB}}^*[\zeta] & 0 \\ 0 & \mathcal{L}_{\text{AB}}[\zeta] & \mathcal{L}_{\text{AA}}[\zeta] & 0 \\ \mathcal{M}[\zeta] & 0 & 0 & 0 \end{pmatrix}, \tag{2.73}$$

where we work in the basis $\{|g_{\rm A}\rangle|g_{\rm B}\rangle, |g_{\rm A}\rangle|e_{\rm B}\rangle, |e_{\rm A}\rangle|g_{\rm B}\rangle, |e_{\rm A}\rangle|e_{\rm B}\rangle\}$. The matrix terms $\mathcal{L}_{\nu\eta}[\zeta]$ and $\mathcal{M}[\zeta]$ are now functionals of the squeezing distribution $\zeta(\boldsymbol{k})$, and they take the forms

$$\mathcal{L}_{\nu\eta}[\zeta] = \mathcal{L}_{\nu\eta}^{\rm vac} + \mathcal{L}_{\nu\eta}^{\rm sq}[\zeta], \tag{2.74}$$

$$\mathcal{M}[\zeta] = \mathcal{M}^{\rm vac} + \mathcal{M}^{\rm sq}[\zeta]. \tag{2.75}$$

As before, the vacuum terms $\mathcal{L}_{\nu\eta}^{\rm vac}$ and $\mathcal{M}^{\rm vac}$ are given by Eqs. (2.33) and (2.34), while the $\zeta(\boldsymbol{k})$ dependent terms read

$$\begin{aligned}\mathcal{L}_{\nu\eta}^{\rm sq}[\zeta] = \pi\lambda_\nu\lambda_\eta \int \frac{{\rm d}^d\boldsymbol{k}}{\omega_{\boldsymbol{k}}} & \\ \times \Big(& \sinh^2[r(\boldsymbol{k})]\bar{F}_\nu(\boldsymbol{k})\bar{F}_\eta^*(\boldsymbol{k})\bar{\chi}_\nu^*(\omega_{\boldsymbol{k}}-\Omega_\nu)\bar{\chi}_\eta(\omega_{\boldsymbol{k}}-\Omega_\eta)e^{{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_\nu-\boldsymbol{x}_\eta)} \\ & + \sinh^2[r(\boldsymbol{k})]\bar{F}_\nu^*(\boldsymbol{k})\bar{F}_\eta(\boldsymbol{k})\bar{\chi}_\nu(\omega_{\boldsymbol{k}}+\Omega_\nu)\bar{\chi}_\eta^*(\omega_{\boldsymbol{k}}+\Omega_\eta)e^{-{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_\nu-\boldsymbol{x}_\eta)} \\ & - e^{-{\rm i}\theta(\boldsymbol{k})}\sinh[r(\boldsymbol{k})]\cosh[r(\boldsymbol{k})]\bar{F}_\nu^*(\boldsymbol{k})\bar{F}_\eta^*(\boldsymbol{k})\bar{\chi}_\nu(\omega_{\boldsymbol{k}}+\Omega_\nu)\bar{\chi}_\eta(\omega_{\boldsymbol{k}}-\Omega_\eta)e^{-{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_\nu+\boldsymbol{x}_\eta)} \\ & - e^{{\rm i}\theta(\boldsymbol{k})}\sinh[r(\boldsymbol{k})]\cosh[r(\boldsymbol{k})]\bar{F}_\nu(\boldsymbol{k})\bar{F}_\eta(\boldsymbol{k})\bar{\chi}_\nu^*(\omega_{\boldsymbol{k}}-\Omega_\nu)\bar{\chi}_\eta^*(\omega_{\boldsymbol{k}}+\Omega_\eta)e^{{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_\nu+\boldsymbol{x}_\eta)}\Big),\end{aligned} \tag{2.76}$$

$$\begin{aligned}\mathcal{M}^{\rm sq}[\zeta] = 2\pi\lambda_{\rm A}\lambda_{\rm B} \int \frac{{\rm d}^d\boldsymbol{k}}{\omega_{\boldsymbol{k}}} & \\ \times \Big(& e^{-{\rm i}\theta(\boldsymbol{k})}\sinh[r(\boldsymbol{k})]\cosh[r(\boldsymbol{k})]\bar{F}_{\rm A}^*(\boldsymbol{k})\bar{F}_{\rm B}^*(\boldsymbol{k})\bar{\chi}_{\rm A}(\omega_{\boldsymbol{k}}+\Omega_{\rm A})\bar{\chi}_{\rm B}(\omega_{\boldsymbol{k}}+\Omega_{\rm B})e^{-{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\rm A}+\boldsymbol{x}_{\rm B})} \\ & + e^{{\rm i}\theta(\boldsymbol{k})}\sinh[r(\boldsymbol{k})]\cosh[r(\boldsymbol{k})]\bar{F}_{\rm A}(\boldsymbol{k})\bar{F}_{\rm B}(\boldsymbol{k})\bar{\chi}_{\rm A}^*(\omega_{\boldsymbol{k}}-\Omega_{\rm A})\bar{\chi}_{\rm B}^*(\omega_{\boldsymbol{k}}-\Omega_{\rm B})e^{{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\rm A}+\boldsymbol{x}_{\rm B})} \\ & - \sinh^2[r(\boldsymbol{k})]\bar{F}_{\rm A}^*(\boldsymbol{k})\bar{F}_{\rm B}(\boldsymbol{k})\bar{\chi}_{\rm A}(\omega_{\boldsymbol{k}}+\Omega_{\rm A})\bar{\chi}_{\rm B}^*(\omega_{\boldsymbol{k}}-\Omega_{\rm B})e^{-{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\rm A}-\boldsymbol{x}_{\rm B})} \\ & - \sinh^2[r(\boldsymbol{k})]\bar{F}_{\rm A}(\boldsymbol{k})\bar{F}_{\rm B}^*(\boldsymbol{k})\bar{\chi}_{\rm A}^*(\omega_{\boldsymbol{k}}-\Omega_{\rm A})\bar{\chi}_{\rm B}(\omega_{\boldsymbol{k}}+\Omega_{\rm B})e^{{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_{\rm A}-\boldsymbol{x}_{\rm B})}\Big).\end{aligned} \tag{2.77}$$

2.3.1 Harvesting entanglement

In order to study the dependence of field squeezing on the ability of detectors to harvest entanglement, let us once again particularize to the case of a massless field in $(3+1)$-dimensions and identical UDW detectors with Gaussian spatial profiles of width σ, given by Eq. (2.49), and Gaussian temporal switching functions of width τ, as in Eq. (2.50).

Then the matrix elements $\mathcal{L}^{\text{sq}}_{\nu\eta}[\zeta]$ and $\mathcal{M}^{\text{sq}}[\zeta]$ given by Eqs. (2.76) and (2.77) become

$$\mathcal{L}^{\text{sq}}_{\nu\eta}[\zeta] = \frac{\bar{\lambda}^2 e^{-\frac{1}{2}\bar{\Omega}^2}}{16\pi^2} \int \frac{\mathrm{d}^3\bar{\boldsymbol{k}}}{|\bar{\boldsymbol{k}}|} e^{-\frac{1}{2}|\bar{\boldsymbol{k}}|^2(1+\bar{\sigma}^2)} \Big(\sinh^2[r(\boldsymbol{k})] e^{|\bar{\boldsymbol{k}}|\bar{\Omega}} e^{-\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_\nu - \bar{t}_\eta)} e^{\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_\nu - \bar{\boldsymbol{x}}_\eta)}$$
$$+ \sinh^2[r(\boldsymbol{k})] e^{-|\bar{\boldsymbol{k}}|\bar{\Omega}} e^{\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_\nu - \bar{t}_\eta)} e^{-\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_\nu - \bar{\boldsymbol{x}}_\eta)}$$
$$- e^{-\mathrm{i}\theta(\boldsymbol{k})} \sinh[r(\boldsymbol{k})] \cosh[r(\boldsymbol{k})] e^{\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_\nu + \bar{t}_\eta)} e^{-\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_\nu + \bar{\boldsymbol{x}}_\eta)}$$
$$- e^{\mathrm{i}\theta(\boldsymbol{k})} \sinh[r(\boldsymbol{k})] \cosh[r(\boldsymbol{k})] e^{-\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_\nu + \bar{t}_\eta)} e^{\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_\nu + \bar{\boldsymbol{x}}_\eta)} \Big), \tag{2.78}$$

$$\mathcal{M}^{\text{sq}}[\zeta] = -\frac{\bar{\lambda}^2 e^{-\frac{1}{2}\bar{\Omega}^2}}{16\pi^2} \int \frac{\mathrm{d}^3\bar{\boldsymbol{k}}}{|\bar{\boldsymbol{k}}|} e^{-\frac{1}{2}|\bar{\boldsymbol{k}}|^2(1+\bar{\sigma}^2)} \Big(\sinh^2[r(\boldsymbol{k})] e^{-\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_{\text{A}} - \bar{t}_{\text{B}})} e^{\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_{\text{A}} - \bar{\boldsymbol{x}}_{\text{B}})}$$
$$+ \sinh^2[r(\boldsymbol{k})] e^{\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_{\text{A}} - \bar{t}_{\text{B}})} e^{-\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_{\text{A}} - \bar{\boldsymbol{x}}_{\text{B}})}$$
$$- e^{-\mathrm{i}\theta(\boldsymbol{k})} \sinh[r(\boldsymbol{k})] \cosh[r(\boldsymbol{k})] e^{-|\bar{\boldsymbol{k}}|\bar{\Omega}} e^{\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_{\text{A}} + \bar{t}_{\text{B}})} e^{-\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_{\text{A}} + \bar{\boldsymbol{x}}_{\text{B}})}$$
$$- e^{\mathrm{i}\theta(\boldsymbol{k})} \sinh[r(\boldsymbol{k})] \cosh[r(\boldsymbol{k})] e^{|\bar{\boldsymbol{k}}|\bar{\Omega}} e^{-\mathrm{i}|\bar{\boldsymbol{k}}|(\bar{t}_{\text{A}} + \bar{t}_{\text{B}})} e^{\mathrm{i}\bar{\boldsymbol{k}}\cdot(\bar{\boldsymbol{x}}_{\text{A}} + \bar{\boldsymbol{x}}_{\text{B}})} \Big), \tag{2.79}$$

where, as before, we denote by a bar any quantity referred to the scale τ (e.g., $\bar{\Omega} = \Omega\tau$, $\bar{\sigma} = \sigma/\tau$, etc.). With these explicit expressions for the matrix elements of $\hat{\rho}_{\text{AB}}$ at hand, we can now readily compute the negativity $\mathcal{N} = \max\big(0, |\mathcal{M}[\zeta]| - \mathcal{L}_{\nu\nu}[\zeta]\big)$, and thus quantify the amount of entanglement that the two detectors harvest from the field.

Uniform squeezing

Let us begin by considering the simplest possible type of squeezing: that in which all field modes are squeezed equally. To that end we take $\zeta(\boldsymbol{k}) = r$, where we also assume that r is real and positive. (We will shortly see what the effect is of r having a complex phase.)

In Fig. 2.4, for different values of r, we plot the negativity of the detectors following their interactions with the field as a function of their joint center of mass. We see that—as we anticipated already from the two-point function—a squeezed field state is in general not translationally invariant, and as such the entanglement harvesting ability of a pair of detectors from such a state is not translationally invariant either. In particular we find that if the detectors' center of mass is near the spatial origin of the coordinate system, then the detectors can harvest more entanglement from a uniformly squeezed field state than from the vacuum. On the other hand if the detectors are far enough away from the origin, then, regardless of the amount of squeezing, they are unable to extract entanglement. The proximity to the origin that is necessary for squeezing to be beneficial for entanglement harvesting is dictated by the amount of squeezing r: for a highly squeezed field state the

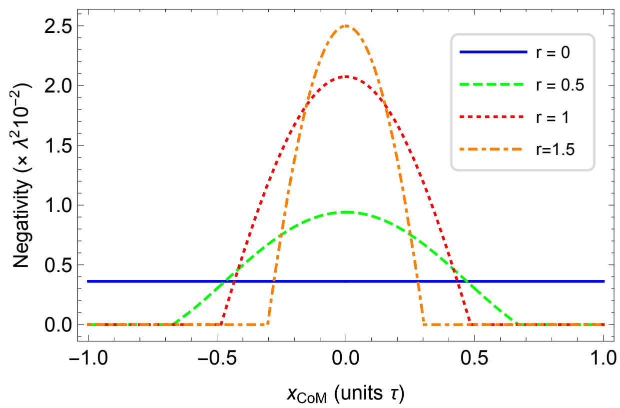

Figure 2.4: Negativity of identical detectors as a function of their center of mass position, for different values of the squeezing parameter $r = |\zeta(\boldsymbol{k})|$. Here the squeezing is uniform across all field modes. The detectors are coupled to the field through Gaussian switching functions of width τ centered at $t = 0$, and their energy gaps are $\Omega = \tau^{-1}$. The detectors are centered at $(x_{\text{COM}} \pm \tau, 0, 0)$ and have Gaussian spatial profiles of width $\sigma = \tau$.

detectors can harvest a lot more entanglement, but they have to be highly centered near the origin; for a less squeezed state the improvement in harvesting is not as noticeable, but the detectors do not need to be so precisely centered.

Let us now attempt to better understand the non-translation-invariance of squeezed field states in general, and in particular the consequences of this for entanglement harvesting from these states. Concretely, with regards to the plots in Fig. 2.4, it is natural to ask why is the spatial origin of our chosen coordinate system the preferred location of UDW detectors that hope to harvest entanglement? First, let us note once again that, as can be seen in Fig 2.4, in the absence of squeezing the translation-invariance of entanglement harvesting is restored. Therefore, the picking out of a preferred point in space near which entanglement harvesting is maximized (in this case the origin of the coordinate system) must be a direct consequence of the squeezing amplitude $\zeta(\boldsymbol{k})$ that we choose for the field. In fact, we notice that the Fourier transform of the uniform amplitude $\zeta(\boldsymbol{k}) = r$ is proportional to $\delta(\boldsymbol{x})$, and therefore the origin $\boldsymbol{x} = 0$ is clearly a special point in this case. As we will now show, this relationship between the Fourier transform of the squeezing amplitude and the preferred location of detectors trying to harvest entanglement is valid in general.

To that end, let us consider an arbitrary squeezing amplitude $\zeta(\boldsymbol{k})$. With this choice of squeezing, there will be some preferred points in space near which it is easier for detectors to harvest entanglement, and others near which it is more difficult. Suppose now that we change the squeezing by a local phase $\zeta(\boldsymbol{k}) \to \zeta'(\boldsymbol{k}) = e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_0}\zeta(\boldsymbol{k})$. How do the positions of the preferred points change?

To answer this question, let us recall from Eq. (2.2) that the state $\hat{\rho}_{\text{AB}}$ of the two detectors following their interactions with a squeezed field state with amplitude ζ' is given by

$$\hat{\rho}_{\text{AB}} = \text{Tr}_\phi \left[\hat{U}' \left(\hat{\rho}_{\text{A}} \otimes \hat{\rho}_{\text{B}} \otimes \hat{S}^\dagger_{\zeta'} |0\rangle\langle 0| \hat{S}_{\zeta'} \right) \hat{U}'^\dagger \right], \tag{2.80}$$

where $\hat{U}'$ is the time-evolution unitary

$$\hat{U}' = \mathcal{T} \exp \Big[-\mathrm{i} \int \mathrm{d}t \sum_\nu \lambda_\nu \chi_\nu(t) \hat{\mu}_\nu(t) \int \mathrm{d}^d\boldsymbol{x}\, F_\nu(\boldsymbol{x} - \boldsymbol{x}_\nu) \hat{\phi}(\boldsymbol{x}, t) \Big]. \tag{2.81}$$

Now let us define the field *momentum operator* to be $\hat{\boldsymbol{P}} := \int \mathrm{d}^3\boldsymbol{k}\, \boldsymbol{k} \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}$. Then, using the identity

$$e^{\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0} \hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0} = \hat{a}_{\boldsymbol{k}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_0}, \tag{2.82}$$

we find that we can write $\hat{S}_{\zeta'} = e^{-\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0/2} \hat{S}_\zeta e^{\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0/2}$. Making use of the cyclicity of the partial trace with respect to the subsystem being traced over, we find that $\hat{\rho}_{\text{AB}}$ can be expressed as

$$\hat{\rho}_{\text{AB}} = \text{Tr}_\phi \left[\hat{U} \left(\hat{\rho}_{\text{A}} \otimes \hat{\rho}_{\text{B}} \otimes \hat{S}^\dagger_{\zeta} |0\rangle\langle 0| \hat{S}_{\zeta} \right) \hat{U}^\dagger \right], \tag{2.83}$$

where $\hat{U} := e^{\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0/2} \hat{U}' e^{-\mathrm{i}\hat{\boldsymbol{P}}\cdot\boldsymbol{x}_0/2}$. Using (2.82) we readily obtain

$$\hat{U} = \mathcal{T} \exp \Big[-\mathrm{i} \int \mathrm{d}t \sum_\nu \lambda_\nu \chi_\nu(t) \hat{\mu}_\nu(t) \int \mathrm{d}^d\boldsymbol{x}\, F_\nu \left(\boldsymbol{x} - \boldsymbol{x}_\nu - \frac{\boldsymbol{x}_0}{2} \right) \hat{\phi}(\boldsymbol{x}, t) \Big]. \tag{2.84}$$

Hence changing the field's squeezing amplitude by a local phase $\zeta \to e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_0}\zeta$ is equivalent to shifting the detectors in space by an amount $\boldsymbol{x}_0/2$. In other words, a local phase change of the squeezing amplitude effects a translation of the points in space near which it is easier for the detectors to harvest entanglement. However, such a local phase change of ζ also effects a translation of its Fourier transform: namely $\bar{\zeta}(\boldsymbol{x}) \to \bar{\zeta}(\boldsymbol{x} - \boldsymbol{x}_0)$. Note that the discrepancy by a factor of 2 between the amount that the preferred points are translated ($\boldsymbol{x}_0/2$) and the amount that the Fourier transform $\bar{\zeta}$ is shifted by ($\boldsymbol{x}_0$) can be removed by choosing a different convention for the exponent in the definition (2.32) of a Fourier transform. Therefore we conclude that (up to a potential re-scaling) the Fourier transform of the field's squeezing amplitude ζ directly tells us where in space the UDW detectors should be centered if they want to harvest more entanglement from the squeezed field

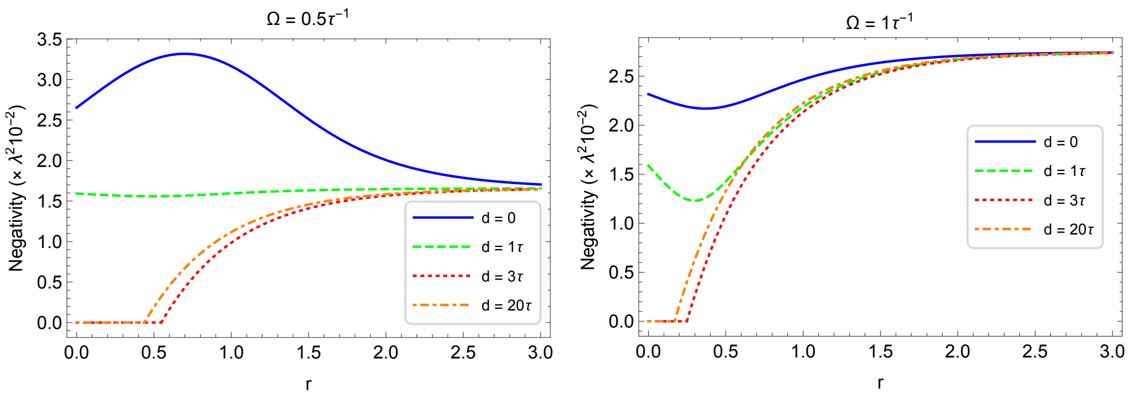

Figure 2.5: Negativity of identical detectors as a function of the squeezing parameter $r = |\zeta(\boldsymbol{k})|$, for different values of their spatial separation d and energy gaps Ω. Here the squeezing is uniform across all field modes. The detectors are coupled to the field through Gaussian switching functions of width τ centered at $t = 0$; they are centered at $(\pm d/2, 0, 0)$ and have Gaussian spatial profiles of width $\sigma = \tau$.

state. These preferred locations are commensurate with where the fluctuations of the field amplitude, and the stress energy density, are localized in space.

Having understood the dependence of the detectors' center of mass on their ability to harvest entanglement from a squeezed field state, and having related this to the local phase of the squeezing amplitude, let us now turn to the question of how the *magnitude* of the squeezing amplitude affects the detector's abilities to harvest entanglement.

In Fig. 2.5 we plot the negativity of a UDW detector pair as a function of $\zeta(\boldsymbol{k}) = r$, which we once again assume to be uniform across all field modes. We notice several interesting features from these plots.

Interestingly, high squeezing can remove the dependence of entanglement harvesting on the distance between the detectors. Indeed, we find that while at low squeezing amplitude the amount of entanglement that the detectors can harvest depends on their spatial separation $d := |\boldsymbol{x}_{\text{A}} - \boldsymbol{x}_{\text{B}}|$, at high squeezing this is not the case. In other words, in the limit of large uniform squeezing of the field, a detector pair separated by a large spatial distance will harvest the same amount of entanglement as if they were at the same location in space. A similar effect of removal of the distance scale in a setup where vacuum entanglement is relevant was seen in [92] where quantum energy teleportation could be made independent of separation between sender and receiver if one uses squeezed field states.

Furthermore, from Fig. 2.5, we find that the amount of entanglement that the detectors harvest is also independent of the squeezing parameter $\zeta(\boldsymbol{k}) = r$ in the limit as $r \to \infty$. Hence although squeezing the field modes often increases the amount of harvestable entanglement from that allowed by the field vacuum, this trend of increasing negativity does not continue indefinitely, but rather plateaus to a constant asymptotic value at large r.

Bandlimited squeezing

To an experimentalist looking to make an entanglement harvesting measurement in the lab, perhaps the most interesting results of the previous section are that i) the amount of entanglement harvested by a pair of UDW detectors from a highly (uniformly) squeezed field state is *independent* of the spatial separation of the detectors, and ii) if the detectors are centered near the "preferred" locations in space (as determined by the Fourier transform of the squeezing function $\zeta(\boldsymbol{k})$), then the amount of entanglement that they harvest could be much higher than in the case of a vacuum field state.

However such an experimentalist would be quick to note that there is an obvious difficulty with attempting to translate the theoretical results of the previous section into an actual experiment in the lab. Namely, in the previous section we assumed the field to be uniformly squeezed across all field modes, while squeezed states in experimental quantum optics [93] and superconducting setups [94] are generally bandlimited to a very narrow range of field modes. We expect that in this case, where only a narrow frequency range of modes are squeezed, the field state will behave more similarly to the vacuum state, in which case squeezing might not give much of an advantage in terms of entanglement harvesting. The key question is then: what range of field modes must be squeezed in order to produce a significant entanglement harvesting advantage over the vacuum state?

To answer this question, let us now assume that only the field modes *near* some momentum $\boldsymbol{k}$ are uniformly squeezed, while all other modes are in their vacuum states. More precisely, we set

$$\zeta(\boldsymbol{k}') = \begin{cases} r & \text{if } |k'_i - k_i| < \frac{\epsilon}{2} \text{ for } i \in \{x, y, z\} \\ 0 & \text{otherwise} \end{cases}, \tag{2.85}$$

where $\boldsymbol{k}' = (k'_x, k'_y, k'_z)$, $\boldsymbol{k} = (k_x, k_y, k_z)$, and ϵ parametrizes the bandwidth of the squeezing. With this choice of squeezing amplitude, and assuming again that the spatial and temporal profiles of the detectors are Gaussians given by Eqs. (2.49) and (2.50), the matrix elements $\mathcal{L}^{\text{sq}}_{\nu\eta}[\zeta]$ and $\mathcal{M}^{\text{sq}}[\zeta]$ of the evolved two detector density matrix $\hat{\rho}_{\text{AB}}$ are again given by the expressions in Eqs. (2.78) and (2.79), except that now the limits of momentum space integration are such that $|k'_i - k_i| < \epsilon/2$. With the use of these expressions we can compute the negativity $\mathcal{N} = \max\big(0, |\mathcal{M}[\zeta]| - \mathcal{L}_{\nu\nu}[\zeta]\big)$, and thus observe how the amount of entanglement that the detectors can harvest depends on the bandwidth ϵ of the field's squeezing amplitude.

However before showing plots of $\mathcal{N}$ versus ϵ, since we are in this section trying to upgrade our theoretical findings to the realm of what is experimentally feasible, it is important that we also discuss what values of squeezing amplitude r we can expect to obtain in our bandlimited frequency range. As far as we are aware, the highest experimentally attained squeezed state of the electromagnetic field resulted in a squeezed quadrature noise reduction of 15 dB below the vacuum level [95]. Using the conversion formula [96]

$$\Delta\text{Noise (in dB)} = 10\log_{10}\left(2\langle\Delta\hat{X}^2\rangle\right), \tag{2.86}$$

between the reduction in noise of the squeezed quadrature $\hat{X}$ and the variance $\langle\Delta\hat{X}^2\rangle := \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$ of that quadrature in the squeezed state $|\zeta(\boldsymbol{k})\rangle$, as well as the expression

$$\langle\Delta\hat{X}^2\rangle = \frac{1}{2}e^{-2r}, \tag{2.87}$$

between $\langle\Delta\hat{X}^2\rangle$ and r, we find that

$$\Delta\text{Noise (in dB)} = -20\log_{10}(e)r. \tag{2.88}$$

Hence a noise reduction of 15 dB corresponds to a squeezing amplitude of $r \approx 1.7$. To be on the safe side with respect to experimental feasibility, we will for the below discussion set $r = 1$ (corresponding to a noise reduction of ~ 8.7 dB).

In Fig. 2.6 we plot the dependence of the negativity that two UDW detectors can harvest from the field, as a function of the bandwidth ϵ of field modes that are squeezed (we assume the squeezed modes to be centered around some wavevector $\boldsymbol{k}$). In the first plot of this figure, we suppose that the detectors are near enough in space such that they are able to harvest entanglement from the field vacuum ($\epsilon = 0$). Perhaps unintuitively, we find that as we start squeezing around the mode $\boldsymbol{k}$ (i.e. we increase ϵ), the negativity of the detectors initially begins to decrease. That is, for a small bandwidth ϵ of field squeezing, regardless of the mode $\boldsymbol{k}$ around which the squeezing is being performed, the amount of entanglement that the detectors can harvest from the field is actually less than what they could harvest from the vacuum. Eventually however, as the bandwidth is increased further, the amount of entanglement that the detectors can harvest from the field becomes higher than in the vacuum case.

Meanwhile, detectors with a large spatial separation (second plot in Fig. 2.6) are unable to harvest entanglement from the vacuum ($\epsilon = 0$), as was already shown in Ref. [26]. In this

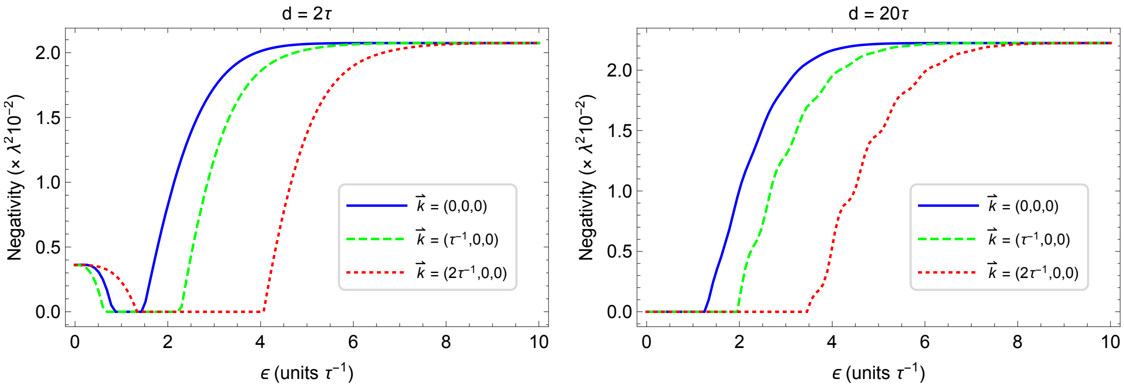

Figure 2.6: Negativity of identical detectors as a function of the bandwidth ϵ of modes squeezed, centered around a mode $\boldsymbol{k}$. The squeezing inside the bandlimited range is of uniform amplitude $r = 1$, and outside is zero. The detectors are coupled to the field through Gaussian switching functions of width τ centered at $t = 0$, they are centered at $(\pm d/2, 0, 0)$ and have Gaussian spatial profiles of width $\sigma = \tau$, and their energy gaps are $\Omega = \tau^{-1}$.

case increasing the squeezing bandwidth allows the detectors to harvest some entanglement, but this only occurs for ϵ larger than some critical value ϵ_c. Hence, regardless of separation, the ability of a pair of UDW detectors to harvest more entanglement from a squeezed field state than from the vacuum is dependent on whether a large enough frequency interval of field modes is squeezed, i.e. if the bandwidth ϵ is larger than some critical value ϵ_c.

We notice from the plots in Fig. 2.6 that the critical bandwidth ϵ_c necessary to achieve an improvement in entanglement harvesting over the vacuum is at least of the order $|\boldsymbol{k}|$, where $\boldsymbol{k}$ is the wavevector of the mode around which we squeeze. Hence for instance if we wanted to use a 300 THz squeezed laser source to entangle a pair of atomic detectors, we would need to squeeze all the modes up to 600 THz with wavevectors pointing in the direction of the laser, as well a wide range of field modes pointing in other directions. As far as we are aware, current experimental setups featuring squeezed electromagnetic field states do not squeeze such large bandwidths of field modes. Hence, in order to make use of the benefits of squeezed field states with respect to entanglement harvesting, it may be necessary to increase the experimentally achievable squeezing bandwidth. Alternatively, it might still be possible to obtain high levels of harvestable entanglement with narrowly bandlimited squeezed states, but for which the squeezing amplitude $\zeta(\boldsymbol{k})$ is non-uniform in the bandlimited range. This remains to be investigated in future work.

2.4 Conclusions

In this chapter we studied the ability of a pair of Unruh-DeWitt particle detectors to harvest quantum and classical correlations from thermal and squeezed states of a scalar field with which they interact. Let us now summarize our results:

We started by proving that the amount of entanglement that a pair of identical detectors (with arbitrary spatial profiles and time-dependent switching functions) can harvest from a thermal state of the field decreases monotonically with temperature. Additionally, we obtained a lower bound on this rate of decrease, and hence showed that for temperatures higher than a certain threshold the detectors are unable to harvest any entanglement from the field. With these findings we also extended the main results in [24], where it was numerically shown (using the very different formalism of Gaussian quantum mechanics) that temperature is detrimental to entanglement harvesting by harmonic oscillator detectors from a massless field in 1+1 dimensional spacetime. Indeed, we proved that this is also the case for qubit detectors of arbitrary shape and switching interacting with a field of any mass in any dimensionality of spacetime.

On the other hand, we found that unlike the negativity, the mutual information — which is a measure of the total (quantum and classical) correlations — that the detectors harvest from the field actually increases linearly with the field temperature (again extending the numerical findings of [24] to qubit detectors). Hence, while thermal noise hinders the ability of UDW detectors to harvest entanglement, it is beneficial in the harvesting of non-entanglement correlations.

Moving on to squeezed field states, we showed that, at least to leading perturbative order, the amount of entanglement that a UDW detector pair can harvest from a squeezed coherent state is independent of its coherent amplitude. This greatly generalizes the result of Ref. [43], which considered only unsqueezed coherent states, to hold for all general squeezed coherent states.

Moreover, this finding is fundamentally related to the fact — which we prove — that the entanglement entropy of a free field between disjoint spatial regions on some time slice is independent of the coherent amplitude of the field. Therefore, this concretely illustrates that entanglement harvesting is not merely an interesting protocol with potential practical implications, but that it also may be used as an operational tool to better understand the fundamental entanglement structure of quantum fields, and hence perhaps to better understand certain fundamental ideas which have deep relationships with the entanglement structure of spacetime, such as the Ads-CFT conjecture [17, 18] or the black hole information loss paradox [11–16].

We also showed that, unlike the coherent amplitude, the field's squeezing amplitude $\zeta(\boldsymbol{k})$ *does* affect the amount of entanglement that the detectors can harvest from the field. In particular, we found that the amount of entanglement that detectors centered at a spatial point $\boldsymbol{x}_0$ can harvest is directly related to the amplitude of the Fourier transform of $\zeta(\boldsymbol{k})$ evaluated at $\boldsymbol{x}_0$. Hence, contrary to vacuum [26], coherent [43], and thermal states, harvesting entanglement from general squeezed states is generally *not* a translationally invariant process.

However, and perhaps surprisingly, we found that for detectors centered at a particular location $\boldsymbol{x}_0$, the amount of entanglement harvested from a highly and uniformly squeezed state is independent of the spatial separation of the detectors. Moreover, this amount of entanglement is often much larger than detectors at the same separation would be able to harvest from the vacuum, raising the idea of the possibility of using squeezed states to experimentally test entanglement harvesting. This result is commensurate with the finding that squeezed states can remove the distance decay of protocols that rely on field entanglement such as quantum energy teleportation [92].

Finally, we have also studied how entanglement harvesting is modified when we allow for squeezing only in a finite frequency bandwidth of field modes. We found that if we restrict the modes of the field that are squeezed to a narrow bandwidth (namely, when the bandwidth is below the order of the frequency being squeezed), then squeezing states give no noticeable advantage over vacuum entanglement harvesting, at least for uniform squeezing. It remains to be seen whether a more general squeezing amplitude (e.g. with continuously varying magnitude and phase) can provide the necessary advantages in entanglement harvesting that we have found here for uniform squeezing, while at the same time being implementable in a lab setting. This is an important direction for future research, since such a squeezed field state could overcome the main experimental limitation of entanglement harvesting: the fast decay with detector separation.

Chapter 3

Non-perturbative entanglement extraction

In this chapter we study, in a non-perturbative setting, the ability of a pair of quantum systems A and B, which we call the targets, to extract entanglement from a source, S. Sec. 3.1 will focus on general results that can be applied to any quantum systems A, B and S. The main result of this section will be an entanglement extraction *no-go theorem*, which will provide a general condition on the interactions between the targets and the source which is necessary for the targets to become entangled. We will then, in Sec. 3.3, study the consequences of this general result to the setting of *entanglement harvesting*, where the targets are qubit detectors and the source is a quantum field.

3.1 Entanglement extraction in a general setting

The general entanglement extraction setup considers two parties, A and B, who would like to entangle their local quantum systems (the targets) by extracting correlations that are originally contained in a third quantum system (the source, S). For instance, the source might be a spatially extended quantum system such as a quantum many-body system or a quantum field. A and B couple to separate parts of S, the latter being in a state that contains entanglement between spatially separated degrees of freedom. Examples of such states include the ground states of interacting lattice theories or the vacuum state of a quantum field.

The total Hilbert space of the two targets and the source is the tensor product

$$\mathcal{H} = \mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{S}} \otimes \mathcal{H}_{\text{B}}, \tag{3.1}$$

of the Hilbert space of the source $\mathcal{H}_{\text{S}}$, and of the two targets $\mathcal{H}_{\text{A}}$ and $\mathcal{H}_{\text{B}}$. We will assume that the target Hilbert spaces are finite-dimensional, while we allow $\mathcal{H}_{\text{S}}$ to be of any (finite or infinite) dimension. Additionally, we suppose that initially, before any interactions take place, the three subsystems start out in a product state

$$\hat{\rho}_0 = \hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\text{S}} \otimes \hat{\rho}_{0,\text{B}}, \tag{3.2}$$

and the free dynamics of the system are generated by the sum of the three free Hamiltonians

$$\hat{H}_0 = \hat{H}_{\text{A}} + \hat{H}_{\text{S}} + \hat{H}_{\text{B}}, \tag{3.3}$$

where $\hat{H}_{\text{S}}$ is shorthand for $\mathbb{1}_{\text{A}} \otimes \hat{H}_{\text{S}} \otimes \mathbb{1}_{\text{B}}$, etc.

We suppose that Alice and Bob each only have access to a limited part of the source. For example, if the source is a relativistic quantum field, Alice and Bob would each only have access to the field in the region of spacetime which they occupy. Their goal is to swap entanglement that is present in the state $\hat{\rho}_{0,\text{S}}$ between their respective regions of access, onto their local target systems. To achieve this the two targets A and B locally couple to the source through time-dependent interaction Hamiltonians $\hat{H}_{\text{I,A}}(t)$ and $\hat{H}_{\text{I,B}}(t)$, such that the total interaction Hamiltonian is $\hat{H}_{\text{I}}(t) := \hat{H}_{\text{I,A}}(t) + \hat{H}_{\text{I,B}}(t)$. Note that no direct interaction Hamiltonian between the two targets is allowed, and that we do not assume any classical communication between the parties.

3.2 Entanglement extraction no-go theorem

Following a specification of the interaction Hamiltonians $\hat{H}_{\text{I},\nu}(t)$, we can, as introduced in Sec. 1.4 and used extensively in Chapter 2, write down the interaction unitary $\hat{U}$ which evolves the initial state of the system in time. Suppose now that the interaction unitary $\hat{U}$ can be factored as

$$\hat{U} = \hat{U}_{\text{BS}} \hat{U}_{\text{AS}}, \tag{3.4}$$

where $\hat{U}_{\text{BS}}$ is a unitary acting on the Hilbert space of B and S, and similarly for $\hat{U}_{\text{AS}}$. In this case, we can state a powerful claim regarding the ability of the targets A and B to become entangled via their interactions with the source. We start with a definition:

Definition 1. *A unitary* $\hat{U}_{\text{XY}}$ *acting on the Hilbert space* $\mathcal{H}_{\text{X}} \otimes \mathcal{H}_{\text{Y}}$ *is said to be* simply-generated *if it is the exponential of a Schmidt rank 1 operator, i.e. if*

$$\hat{U} = \exp\left(-\mathrm{i}\hat{X} \otimes \hat{Y}\right), \tag{3.5}$$

with $\hat{X}$ *and* $\hat{Y}$ *observables on* $\mathcal{H}_{\text{X}}$ *and* $\mathcal{H}_{\text{Y}}$*, respectively.*

We can now state the following theorem, which is the central result of this chapter:

Theorem 1. *Suppose that the initial state* $\hat{\rho}_0 = \hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\text{S}} \otimes \hat{\rho}_{0,\text{B}}$ *of the tripartite system A-S-B is evolved according to* $\hat{U} = \hat{U}_{\text{BS}}\hat{U}_{\text{AS}}$*, where* $\hat{U}_{\text{BS}}$ *is a simply-generated unitary. Then, the evolved state* $\hat{\rho}_{\text{AB}}$ *of the targets is necessarily separable.*

To prove this claim, let us begin with a standard definition from the quantum information literature.

Definition 2. *A quantum channel* ξ *taking states in* $\mathcal{H}_{\text{Y}}$ *to states in* $\mathcal{H}_{\text{Z}}$ *is said to be* entanglement breaking *if for any state* $\hat{\rho}_{\text{XY}}$ *on the product space* $\mathcal{H}_{\text{X}} \otimes \mathcal{H}_{\text{Y}}$ *the state* $(\mathbb{1} \otimes \xi)(\hat{\rho}_{\text{XY}})$ *in* $\mathcal{H}_{\text{X}} \otimes \mathcal{H}_{\text{Z}}$ *is separable.*

Physically, entanglement breaking channels are characterized by the property that when they receive only a part of a larger system as input, which may be entangled with other degrees of freedom, then the output of the channel is always in a separable state with the rest of the larger system. That is, any entanglement between the input and the environment is broken and the output is not entangled with the environment [97].

Let us proceed now to write down the time-evolved state $\hat{\rho}_{\text{AB}}$ of the targets A and B following their interactions with the source S. This state is obtained by acting on the initial state of the system in Eq. (3.2) with the interaction unitary $\hat{U}$ in Eq. (3.4), and then tracing out the source. Because of our assumption that $\hat{U}$ can be written as $\hat{U} = \hat{U}_{\text{BS}}\hat{U}_{\text{AS}}$, we can first evolve the system by $\hat{U}_{\text{AS}}$ to obtain the state $\hat{\rho}_{\text{AS}} \otimes \hat{\rho}_{0,\text{B}}$, where $\hat{\rho}_{\text{AS}} := \hat{U}_{\text{AS}}(\hat{\rho}_{0,\text{A}} \otimes \hat{\rho}_{0,\text{S}})\hat{U}_{\text{AS}}^\dagger$ is some state of the bipartite system A-S. Note that, in general, the state $\hat{\rho}_{\text{AS}}$ may contain entanglement between A and S.

The remaining steps to obtaining the time-evolved state $\hat{\rho}_{\text{AB}}$ are to i) evolve the state $\hat{\rho}_{\text{AS}} \otimes \hat{\rho}_{0,\text{B}}$ according to the unitary $\hat{U}_{\text{BS}}$, and ii) trace out the source degrees of freedom. It is useful to combine these two steps in one, and write the final result for $\hat{\rho}_{\text{AB}}$ in terms of a quantum channel ξ. Namely, we can write $\hat{\rho}_{\text{AB}}$ as

$$\hat{\rho}_{\text{AB}} = (\mathbb{1}_{\text{A}} \otimes \xi)\,(\hat{\rho}_{\text{AS}}), \tag{3.6}$$

where the channel ξ, which acts on states $\hat{\rho}_{\mathrm{S}}$ in $\mathcal{H}_{\mathrm{S}}$ and produces states $\hat{\rho}_{\mathrm{B}}$ in $\mathcal{H}_{\mathrm{B}}$, is defined by

$$\xi : \hat{\rho}_{\mathrm{S}} \mapsto \hat{\rho}_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{S}}\left[\hat{U}_{\mathrm{BS}}\left(\hat{\rho}_{\mathrm{S}} \otimes \hat{\rho}_{0,\mathrm{B}}\right)\hat{U}_{\mathrm{BS}}^{\dagger}\right]. \tag{3.7}$$

Note that the fixed state $\hat{\rho}_{0,\mathrm{B}}$, which is the initial state of target B, is part of the definition of the channel ξ. We claim that $\hat{\rho}_{\mathrm{AB}}$ in Eq. (3.6) is separable because the channel ξ is entanglement breaking.

Hence, all that remains in proving the theorem is to show that ξ, as defined by Eq. (3.7), is an entanglement breaking channel. To see that this is indeed the case, let us first note that since $\hat{U}_{\mathrm{BS}}$ is a simply-generated unitary, by definition it can be written in the form

$$\hat{U}_{\mathrm{BS}} = \exp\left(-\mathrm{i}\,\hat{m} \otimes \hat{X}\right), \tag{3.8}$$

where $\hat{m}$ is an observable of the system B, and $\hat{X}$ is a source system observable. Since we know that $\hat{X}$ is self-adjoint, the spectral theorem tells us that it can be diagonalized. In particular, if $\hat{X}$ is also a compact (and hence bounded) operator, its spectrum is discrete and we can write $\hat{X} = \sum_k x_k \left|x_k\right\rangle\!\left\langle x_k\right|$ with $x_k \in \mathbb{R}$. In this case, we can use this decomposition to recast $\hat{U}_{\mathrm{BS}}$ in the form of a controlled unitary, namely

$$\hat{U}_{\mathrm{BS}} = \sum_k \exp\left(-\mathrm{i}x_k\hat{m}\right) \otimes \left|x_k\right\rangle\!\left\langle x_k\right|, \tag{3.9}$$

Note that the more general case of a non-bounded $\hat{X}$ — which is indeed the case if $\hat{X}$ is a smeared field operator acting on the Hilbert space of a quantum field — can be treated in an analogous but more mathematically demanding fashion. This fully general case is considered in detail in Appendix C, while in this section we will assume that $\hat{X}$ is discrete and thus that the expression (3.9) is valid.

Writing $\hat{U}_{\mathrm{BS}}$ in the form of Eq. (3.9) allows us to understand the action of $\hat{U}_{\mathrm{BS}}$ as acting with the unitary $\exp\left(-\mathrm{i}x_k\hat{m}\right)$ on the target system B, conditional on the source S being in the state $\left|x_k\right\rangle$. In this sense, it can even be understood as a measurement of observable $\hat{X}$ on the source carried out by the target system. Then, from Eq. (3.7), we see that the output $\hat{\rho}_{\mathrm{B}}$ to the channel ξ acting on a state $\hat{\rho}_{\mathrm{S}}$ is

$$\hat{\rho}_{\mathrm{B}} = \sum_k \left\langle x_k\right|\hat{\rho}_{\mathrm{S}}\left|x_k\right\rangle \exp\left(-\mathrm{i}x_k\hat{m}\right)\hat{\rho}_{0,\mathrm{B}}\exp\left(\mathrm{i}x_k\hat{m}\right). \tag{3.10}$$

As shown in Refs. [97, 98], since the output of the channel ξ is of this form, the channel ξ itself must be entanglement breaking. Hence, by definition of entanglement breaking channel, this proves that the time-evolved density matrix $\hat{\rho}_{\text{AB}}$ of the two targets, as given in Eq. (3.6), is a separable state.

In fact, we see that $\hat{\rho}_{\text{AB}}$ is separable by a direct computation, without resorting to the results of Refs. [97,98]. Namely, substituting $\hat{U}_{\text{BS}}$ in the form (3.9) into the expression (3.6) for $\hat{\rho}_{\text{AB}}$, we obtain

$$\hat{\rho}_{\text{AB}} = \sum_{k,k'} \text{Tr}_{\text{S}} \left[|x_k\rangle\langle x_k|\hat{\rho}_{\text{AS}}|x_{k'}\rangle\langle x_{k'}| \right] \otimes e^{-\text{i}x_k\hat{m}}\hat{\rho}_{0,\text{B}}e^{-\text{i}x_{k'}\hat{m}}, \tag{3.11}$$

where we suppress writing the identity operator on system A. Then, using the cyclicity of the partial trace with respect to the system being traced over, we can cycle the $|x_k\rangle$ to appear to the right of the $\langle x_{k'}|$, thus producing a Kronecker delta, $\delta_{k,k'}$. Hence the above expression for $\hat{\rho}_{\text{AB}}$ becomes

$$\hat{\rho}_{\text{AB}} = \sum_{k} \langle x_k|\hat{\rho}_{\text{AS}}|x_k\rangle \otimes e^{-\text{i}x_k\hat{m}}\hat{\rho}_{0,\text{B}}e^{-\text{i}x_k\hat{m}}. \tag{3.12}$$

We now make the following definitions

$$p_k := \text{Tr}_{\text{A}} \left[\langle x_k|\hat{\rho}_{\text{AS}}|x_k\rangle \right], \tag{3.13}$$

$$\hat{\rho}_{\text{A}}^{(k)} := \frac{1}{p_k}\langle x_k|\hat{\rho}_{\text{AS}}|x_k\rangle, \tag{3.14}$$

$$\hat{\rho}_{\text{B}}^{(k)} := e^{-\text{i}x_k\hat{m}}\hat{\rho}_{0,\text{B}}e^{-\text{i}x_k\hat{m}}, \tag{3.15}$$

so that the state $\hat{\rho}_{\text{AB}}$ can be written as

$$\hat{\rho}_{\text{AB}} = \sum_{k} p_k\, \hat{\rho}_{\text{A}}^{(k)} \otimes \hat{\rho}_{\text{B}}^{(k)}. \tag{3.16}$$

Furthermore, from their definitions we clearly see that $\hat{\rho}_{\text{A}}$ and $\hat{\rho}_{\text{B}}$ are density matrices on the Hilbert spaces $\mathcal{H}_{\text{A}}$ and $\mathcal{H}_{\text{B}}$, respectively. Additionally, since $\hat{\rho}_{\text{AS}}$ is a density matrix and hence positive definite and self-adjoint, we find that p_k must be a non-negative real number. Lastly, $\sum_k p_k = \text{Tr}(\hat{\rho}_{\text{AS}}) = 1$, and hence we find that $\hat{\rho}_{\text{AB}}$ in Eq. (3.16) is manifestly separable by the definition of separable state given in Eq. (1.1).

3.2.1 Circumventing the no-go theorem

As we have seen in the previous section, if the targets A and B couple to S through a unitary of the form $\hat{U} = \hat{U}_{\text{BS}}\hat{U}_{\text{AS}}$, and $\hat{U}_{\text{BS}}$ is a simple-generated unitary, then A and B cannot extract entanglement from S. Note that this is true regardless of the form of $\hat{U}_{\text{AS}}$; in particular, it is true if $\hat{U}_{\text{AS}}$ is also a simple-generated unitary, in which case the coupling $\hat{U}$ of the targets to the source is in a sense the simplest possible coupling.

Having seen that the simplest possible unitary $\hat{U}$ (comprised of a product of two simple-generated unitaries) which couples A and B to S is not sufficient to entangle the two targets, the natural question to ask is whether it is possible to entangle the targets by combining more than two simple-generated interactions? In the following we show that it is indeed possible to get the two targets entangled by coupling one of them once and the second one twice to the source through simple-generated interactions, under certain conditions.

To that end, let us suppose that target A couples to S twice, according to the simple-generated unitaries $\hat{U}_{\text{A}_1}$ and $\hat{U}_{\text{A}_2}$ given by

$$\hat{U}_{\text{A}_1} = \exp\left(-\mathrm{i}\hat{m}_{\text{A}_1} \otimes \hat{X}_{\text{A}_1}\right), \tag{3.17}$$

$$\hat{U}_{\text{A}_2} = \exp\left(-\mathrm{i}\hat{m}_{\text{A}_2} \otimes \hat{X}_{\text{A}_2}\right), \tag{3.18}$$

and that target B couples to S once through the simple-generated unitary $\hat{U}_{\text{B}_1}$ defined as

$$\hat{U}_{\text{B}_1} = \exp\left(-\mathrm{i}\hat{m}_{\text{B}_1} \otimes \hat{X}_{\text{B}_1}\right). \tag{3.19}$$

Then it follows from the previous section that if the interaction $\hat{U}_{\text{B}_1}$ takes place last the targets A and B always end up in a separable space. Hence, A and B can only get entangled if the coupling between B and S, given by the unitary $\hat{U}_{\text{B}_1}$, takes place before at least one of A's couplings.

When target B is coupled to the source first followed by the two couplings of A, the product of the two couplings $\hat{U}_{\text{A}_1}\hat{U}_{\text{A}_2}$ must not yield an entanglement breaking channel from the field to target A, otherwise A and B would once again end up in a separable state. In order for this not to occur it is necessary that the two observables $\hat{X}_{\text{A}_1}$ and $\hat{X}_{\text{A}_2}$ do not commute.

To see this, suppose instead that $[\hat{X}_{\text{A}_1}, \hat{X}_{\text{A}_2}] = 0$. Then, the two observables $\hat{X}_{\text{A}_1}$ and $\hat{X}_{\text{A}_2}$ can be simultaneously diagonalized as

$$\hat{X}_{\text{A}_1} = \sum_k a_k^{(1)} \left|x_{\text{A},k}\right\rangle\!\left\langle x_{\text{A},k}\right|, \tag{3.20}$$

$$\hat{X}_{\text{A}_2} = \sum_k a_k^{(2)} \left|x_{\text{A},k}\right\rangle\!\left\langle x_{\text{A},k}\right|. \tag{3.21}$$

Therefore the product $\hat{U}_{\text{A}_1}\hat{U}_{\text{A}_2}$ of the unitaries governing the interactions between A and S can be expressed as

$$\hat{U}_{\text{A}_1}\hat{U}_{\text{A}_2} = \sum_k \exp\Big(-\mathrm{i}a_k^{(1)}\hat{m}_{\text{A}_1}\Big)\exp\Big(-\mathrm{i}a_k^{(2)}\hat{m}_{\text{A}_2}\Big)\otimes\big|x_{\text{A},k}\big\rangle\big\langle x_{\text{A},k}\big|, \tag{3.22}$$

which again has the form of a controlled unitary gate (performing a unitary on the target system conditional on the source system's state) and, therefore, gives rise to an entanglement breaking channel from the source to target A.

This observation, together with the fact that it is necessary to have $[\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_n}] \neq 0$ in order to obtain $[\hat{U}_{\text{B}_1}, \hat{U}_{\text{A}_n}] \neq 0$, leads to the conclusion that if more than one of the three commutators $[\hat{X}_{\text{A}_1}, \hat{X}_{\text{A}_2}]$, $[\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_1}]$ and $[\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_2}]$ vanish, then, regardless of the order in which they interact with the source S, the targets A and B always end up in a separable state. This is simply because if two of these commutators vanish then it is always possible to rearrange the product of unitaries $\hat{U}_{\text{B}_1}\hat{U}_{\text{A}_1}\hat{U}_{\text{A}_2}$ (or $\hat{U}_{\text{A}_1}\hat{U}_{\text{B}_1}\hat{U}_{\text{A}_2}$) such that it ends with an entanglement breaking coupling from the system to the corresponding target.

To demonstrate that entangling two targets via three simple-generated interactions *is* possible if one satisfies the above described necessary condition, we can construct simple toy models where both targets as well the source are modelled by single qubits. In this case we can use the CNOT-gate between two qubits as a simple-generated interaction between target and source. To explicitly see that the CNOT-gate is simple-generated, note that the unitary which implements it can be written as

$$\hat{U}_{\text{CNOT}} = \exp\Big[-\mathrm{i}\pi\,\big(2\,|-_z\rangle\langle -_z| + |+_z\rangle\langle +_z|\big)\otimes\big(2\,|+_x\rangle\langle +_x| + 3\,|-_x\rangle\langle -_x|\big)\Big]. \tag{3.23}$$

Here, we will allow either the target or the source to play the role of the control system in the CNOT. Fig. 3.1 shows examples of circuits that achieve entanglement between the target qubits through an interaction with a single source qubit. In each of the three cases a different commutator $[\hat{U}_{\text{B}_1}, \hat{U}_{\text{A}_1}]$, $[\hat{U}_{\text{B}_1}, \hat{U}_{\text{A}_2}]$, or $[\hat{U}_{\text{A}_1}, \hat{U}_{\text{A}_2}]$ vanishes.

Arguably however, the toy models of Fig. 3.1 do not technically represent entanglement extraction from the source, since the very notion of entanglement extraction from a single qubit onto two qubits does not make sense. Rather, the toy models are showing a mechanism of entangling the targets through communication via the source.

In fact, the finding above that at most one pair out of $\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_1}, \hat{X}_{\text{A}_2}$ may commute for entanglement extraction to be possible, implies that three simple-generated interactions can only entangle the target systems if they could alternatively be used to implement a communication channel from A to B or vice versa. This is because if a commutator of the

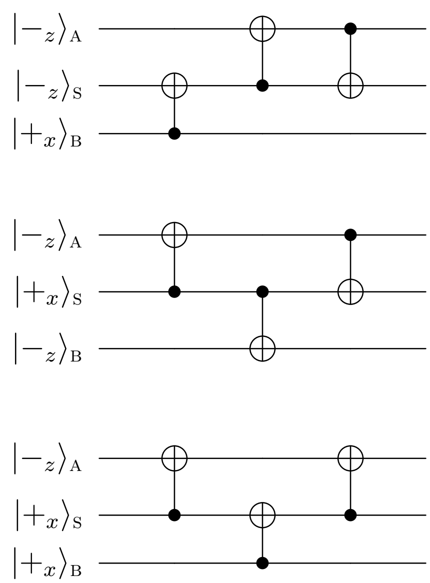

Figure 3.1: Qubit toy models for protocols that entangle the target systems A and B using three CNOT gates with the source S, which are examples of simple-generated interactions. From the top to the bottom the commutators $[\hat{U}_{B_1}, \hat{U}_{A_2}]$, $[\hat{U}_{B_1}, \hat{U}_{A_1}]$ and $[\hat{U}_{A_1}, \hat{U}_{A_2}]$ vanish respectively. The first two examples yield the final state $\frac{1}{\sqrt{2}}\left(|+_z\rangle_A|+_z\rangle_B + |-_z\rangle_A|-_z\rangle_B\right)|-_z\rangle_S$ whereas the last one yields $\frac{1}{\sqrt{2}}\left(|+_z\rangle_A|+_z\rangle_B + |-_z\rangle_A|-_z\rangle_B\right)|+_x\rangle_S$.

form $[\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_1}]$ is non-vanishing, then the corresponding pair of interactions could also be used to send information from target A to target B [99, 100].

There is another observation which suggests that entangling two target systems with three simple-generated interactions really corresponds to correlating them through communication rather than extracting entanglement from the source. This is the fact that it is not possible to genuinely extract entanglement from a source consisting of a pair of qubits with only three simple-generated interactions.

To see this, we assume that the source is given by a pair of qubits in some entangled state. Let B be the target system that couples only once, and hence only interacts with one of the source qubits. Then, in order for A and B to have any chance of extracting pre-existing entanglement from S, the target A needs to use its two interactions to couple to each of the two source qubits once, since otherwise the pre-existing entanglement between the source qubits would not be of any significance to the protocol.

Now, operators that act on only one source qubit commute with operators that act on the other source qubit. This implies that the interaction of B with one source qubit commutes with the interaction of A with the other source qubit. However, both interactions of A with the source also commute with each other because they act on different source qubits. This means that two out of the three possible pairings of observables generating the interactions, $\hat{X}_{\text{B}_1}, \hat{X}_{\text{A}_1}, \hat{X}_{\text{A}_2}$, commute. Thus by the argument above the targets end up in a separable state.

The only possible way to get the two targets to become entangled is to use all three couplings to interact with only one of the source's qubits. Clearly, such a protocol does not access the pre-existing entanglement in the source at all. In fact, entanglement between the accessed source qubit and the other source qubit impedes, rather than facilitates, the entanglement of the two target systems.

In summary, it is possible to achieve entanglement between two target systems with three simple-generated couplings. However, in these scenarios the couplings need to be such that they could also be used to send information from one of the targets to the other (not necessarily in both directions). In other words, the source system needs to play the role of a communication medium which serves to correlate the two targets. Genuine extraction of pre-existing entanglement from the source system, e.g., by spacelike separated observers, seems to require at least four simple-generated couplings. A toy model example of this is shown in Fig. 3.2.

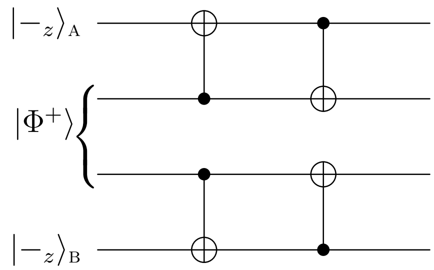

Figure 3.2: Qubit toy model demonstrating that four simple-generated interactions (here CNOT gates) can extract pre-existing entanglement from a source, which is modeled as a pair of qubits in the maximally entangled state $|\Phi^+\rangle_{\text{S}} = \frac{1}{\sqrt{2}}\big(|+_z\rangle|+_z\rangle + |-_z\rangle|-_z\rangle\big)$. The circuits swaps the entanglement onto the targets. The final state of the four qubit system reads $\frac{1}{\sqrt{2}}\big(|+_z\rangle_{\text{A}}|+_z\rangle_{\text{B}} + |-_z\rangle_{\text{A}}|-_z\rangle_{\text{B}}\big)\,|-_z\rangle|-_z\rangle$.

3.3 Applications to entanglement harvesting

A frequently studied physical system to which we will now apply our general results is the entanglement harvesting setup, studied in detail in Chapter 2, in which two qubits (the targets) attempt to become entangled by interacting with a quantum field (the source). This will allow us, in Sec. 3.3.1, to generalize previous no-go entanglement harvesting results, as well as provide a unified explanation for why they hold. Then in Sec. 3.3.2 we will non-perturbatively explore the simplest coupling scenario between qubits and field which allows for the qubits to harvest field entanglement. In particular, and in contrast to perturbative results, we will show that the amount of extracted entanglement decreases above a certain optimal value for the coupling strength.

As in Chapter 2, we consider a free massless scalar field $\hat{\phi}(\boldsymbol{x}, t)$ in $(d+1)$-dimensional flat spacetime, with mode expansion given by Eq. (1.16), along with two qubit detectors A and B which interact with the field. We suppose that the initial state of the system starts out in the separable state

$$\hat{\rho} = \hat{\rho}_{\text{A}} \otimes \hat{\rho}_{\phi} \otimes \hat{\rho}_{\text{B}}, \tag{3.24}$$

in the Hilbert space $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\phi} \otimes \mathcal{H}_{\text{B}}$. As before, we will assume that the detectors are at rest at positions $\boldsymbol{x}_\nu$ (in the $(\boldsymbol{x}, t)$ coordinate system in which we performed the field quantization), and we let $F_\nu(\boldsymbol{x})$ be real-valued distributions (with dimensions of L^{-d}) describing the detectors' spatial profiles. We allow detector ν to interact with the quantum field through the interaction Hamiltonian (in the interaction picture)

$$\hat{H}_{\mathrm{I},\nu}(t) = \lambda_\nu \chi_\nu(t) \hat{m}_\nu(t) \otimes \int \mathrm{d}^d\boldsymbol{x}\, F_\nu(\boldsymbol{x} - \boldsymbol{x}_\nu) \hat{\phi}(\boldsymbol{x}, t). \tag{3.25}$$

Once again λ_ν is a coupling strength (dimension $L^{(d-3)/2}$), $\chi_\nu(t)$ is a dimensionless switching function that describes how the detector is turned on and off, and $\hat{m}_\nu$ is the monopole moment of detector ν, given by Eq. (1.27).

The result that follows in Sec. 3.3.1 can straightforwardly be extended to detectors with arbitrary trajectories, but in this case care must be taken to specify each detector's parameters (energy gap, switching function, smearing function) in the detector's own rest frame, and then perform appropriate coordinate transformations in order to get the interaction Hamiltonian in the lab frame $(\boldsymbol{x}, t)$ [101]. In order to avoid going into these details and obscuring our main objective, we will consider only stationary detectors.

3.3.1 Null result for entanglement harvesting

Suppose now that the full interaction Hamiltonian $\hat{H}_{\mathrm{I}}(t) := \hat{H}_{\mathrm{I,A}}(t) + \hat{H}_{\mathrm{I,B}}(t)$ between the detectors and the field, with $\hat{H}_{\mathrm{I},\nu}$ defined in Eq. (3.25), is such that the time-evolution unitary of the system is of the form $\hat{U} = \hat{U}_{\mathrm{B}\phi}\hat{U}_{\mathrm{A}\phi}$ where $\hat{U}_{\nu\phi}$ is a unitary on the Hilbert space $\mathcal{H}_\nu \otimes \mathcal{H}_\phi$. Note that this form for $\hat{U}$ is achieved, for instance, if detectors A and B are spacelike separated during the times of their interactions with the field, or, alternatively, if detector A is finished coupling to the field before detector B couples, i.e. if $\operatorname{supp}\chi_{\mathrm{A}}(t) \subseteq (-\infty, \tilde{t}\,]$ and $\operatorname{supp}\chi_{\mathrm{B}}(t) \subseteq [\tilde{t}, \infty)$ for some $\tilde{t} \in \mathbb{R}$. Then, we know from our general no-go theorem in Sec. 3.2 that Alice and Bob are unable to extract harvest entanglement from the field if $\hat{U}_{\mathrm{B}\phi}$ is of the simple-generated form (3.5).

We will now show examples of two classes of Unruh-DeWitt interactions for which $\hat{U}_{\mathrm{B}\phi}$ is precisely of the simple-generated form. These two classes of interactions are i) the case of degenerate detectors [47–49, 52], and ii) the case of detectors that couple to the field at one instant in time (i.e., through a Dirac-δ switching function) [2]. Importantly, both i) and ii) are prevalent interactions considered in the literature, due to their physical significance as well as the fact that they allow for non-perturbative studies of detector-field interactions [2, 49], something that is difficult to achieve in other regimes.

Degenerate detector

Recall from Sec. 1.4 that the unitary $\hat{U}_{\mathrm{B}\phi}$ can be expressed in the Magnus form (1.33) as

$$\hat{U}_{\mathrm{B}\phi} = \exp\left(\sum_{n=1}^{\infty} \hat{\Omega}_n\right), \tag{3.26}$$

where the lowest order terms read

$$\hat{\Omega}_1 = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t\, \hat{H}_{\mathrm{I}}(t), \tag{3.27}$$

$$\hat{\Omega}_2 = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\, [\hat{H}_{\mathrm{I}}(t), \hat{H}_{\mathrm{I}}(t')], \tag{3.28}$$

and the higher-order terms $\hat{\Omega}_k$ with $k > 2$ all vanish if detector B is degenerate. A nice feature of this special case is that the unitary $\hat{U}_{\mathrm{B}\phi}$ can be written in closed-form using only the terms $\hat{\Omega}_1$ and $\hat{\Omega}_2$, and can thus allow for a non-perturbative analysis of the interaction between detector B and the field. Explicitly, $\hat{U}_{\mathrm{B}\phi}$ is given by

$$\hat{U}_{\mathrm{B}\phi} = \exp\left(-\mathrm{i}\hat{m}_{\mathrm{B}} \otimes \hat{X}\right), \tag{3.29}$$

where $\hat{m}_{\mathrm{B}} = \hat{m}_{\mathrm{B}}(t)$ is time independent due to a lack of free evolution by the degenerate detector B, and the field observable $\hat{X}$ is defined by

$$\hat{X} := \int_{-\infty}^{\infty} \mathrm{d}t\, \lambda_{\mathrm{B}}\chi_{\mathrm{B}}(t)\hat{\Phi}(t) + \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\, \lambda_{\mathrm{B}}^2\chi_{\mathrm{B}}(t)\chi_{\mathrm{B}}(t')[\hat{\Phi}(t), \hat{\Phi}(t')], \tag{3.30}$$

where $\hat{\Phi}(t) := \int \mathrm{d}^n\boldsymbol{x}\, F_{\mathrm{B}}(\boldsymbol{x} - \boldsymbol{x}_{\mathrm{B}})\hat{\phi}(\boldsymbol{x}, t)$ is the smeared field operator. Hence, by writing it in the Magnus form, we find that the unitary $\hat{U}_{\mathrm{B}\phi}$ corresponding to a degenerate detector is manifestly simple-generated.

We therefore arrive at the following conclusion: Suppose that two UDW detectors A and B interact with the field such that i) they are spacelike separated, or ii) detector A interacts with the field strictly before detector B. Then, if detector B is degenerate, the detectors cannot harvest any entanglement from the field. This is a generalization of the perturbative result found in [52], where it was shown that identical, degenerate detectors that satisfy the condition i) or ii), cannot harvest entanglement from the field vacuum. Here, just by investigating the commutator structure of the detector-field interaction Hamiltonian (i.e. without any lengthy calculations), we have shown that this is indeed true in the non-perturbative regime, for non-identical detectors, and for any field state.

Delta-coupled detector

Let us now consider another special case which allows for a non-perturbative analysis of the detector-field interaction. Namely, let us suppose that the switching function for detector B (which may now be non-degenerate) is a delta function,

$$\chi_{\mathrm{B}}(t) = \eta_{\mathrm{B}}\delta(t - t_{\mathrm{B}}). \tag{3.31}$$

Here η_{B} has dimensions of L and it characterizes the strength of detector B's coupling to the field. Since this is the same as the role played by the coupling strength λ_{B}, we will from here on combine the two by redefining $\lambda_{\mathrm{B}}\eta_{\mathrm{B}} \to \lambda_{\mathrm{B}}$. Hence we are now particularizing our discussion to interactions where detector B interacts with the field at only one instant in time, t_{B}, but with an infinite intensity, such that the total energy exchanged between detector and field is still finite. Such interactions, which we will refer to as δ-couplings, can be viewed physically as idealized limits of highly intense interactions occurring over short time intervals (see [2] for a more detailed discussion).

Assuming a switching function of the form (3.31), by substituting the interaction Hamiltonian $\hat{H}_{\mathrm{I,B}}(t)$ into the Dyson expansion (1.31), we find that the associated interaction unitary $\hat{U}$ is given by

$$\hat{U}_{\mathrm{B}\phi} = \exp\left(-\mathrm{i}\hat{m}_{\mathrm{B}} \otimes \hat{X}\right), \tag{3.32}$$

where now $\hat{m}_{\mathrm{B}} := \hat{m}(t_{\mathrm{B}})$ and $\hat{X}_{\mathrm{B}}$ is defined as

$$\hat{X}_{\mathrm{B}} := \lambda_{\mathrm{B}} \int \mathrm{d}^n\boldsymbol{x}\, F_{\mathrm{B}}(\boldsymbol{x} - \boldsymbol{x}_{\mathrm{B}})\hat{\phi}(\boldsymbol{x}, t_{\mathrm{B}}). \tag{3.33}$$

Hence, as in the case of a degenerate detector B, a point-in-time coupling also leads to a simple-generated interaction unitary $\hat{U}_{\mathrm{BS}}$ between the detector and the source S.

Applying our general no-go theorem for entanglement extraction, we therefore arrive at the following conclusion: Suppose that two UDW detectors A and B interact with the field such that i) they are spacelike separated, or ii) detector A interacts with the field strictly before detector B. Then, if detector B interacts with the field at only one instant in time (i.e. through a δ-function, then the detectors cannot harvest any entanglement from the field. This is a generalization of the result obtained in [2], where it was shown that detectors that each couple to the field once cannot harvest entanglement from any coherent state of the field. Here, just by investigating the commutator structure of the detector-field interaction Hamiltonian (i.e. without any lengthy calculations), we have shown that this is indeed true for a much more general class of coupling setups, and for any (not necessarily coherent) state of the field.

3.3.2 Simplest possible setup for entanglement harvesting

The case of Alice and Bob each δ-coupling once to the field has already been studied in Ref. [2]. The results of [2] are a particular example of the general result that we discussed in the previous section: two detectors that each δ-couple to the field once cannot become entangled with one another. In this section we will show the simplest example where the detectors *can* become entangled: Alice (A) coupling twice and Bob (B) once. The three possible coupling schemes for this to occur are AAB (A first coupling twice, then B once), ABA, and BAA. From our discussion in Sec. 3.3.1 we know that the first of these schemes (AAB coupling) is incapable of harvesting entanglement, while the harvesting abilities of detectors in the remaining two coupling setups (ABA and BAA) are constrained by the commutator structure of the observables generating the interaction unitaries. We will now explore these constraints in more detail.

For simplicity, let us work in (3 + 1)-dimensions and suppose the detectors and field are each in their free ground states, so that the initial state of the system, $|\psi_0\rangle$, reads

$$|\psi_0\rangle = |g_{\text{A}}\rangle \otimes |g_{\text{B}}\rangle \otimes |0\rangle \in \mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}} \otimes \mathcal{H}_{\phi}. \tag{3.34}$$

Furthermore we suppose the detectors are stationary in the inertial frame in which we performed the field quantization, and that their centers of mass are located at $\boldsymbol{x}_{\text{A}} = \boldsymbol{x}_{\text{B}} = 0$. We allow the detectors and field to interact according to the Hamiltonian

$$\hat{H}_{\text{I}}(t) = \hat{H}^{(1)}_{\text{I,A}}(t) + \hat{H}^{(2)}_{\text{I,A}}(t) + \hat{H}^{(1)}_{\text{I,B}}(t), \tag{3.35}$$

where the $\hat{H}^{(i)}_{\text{I},\nu}(t)$ is defined as

$$\hat{H}^{(i)}_{\text{I},\nu}(t) = \lambda_\nu \delta(t - t_{\nu i}) \hat{m}_\nu(t) \otimes \int \mathrm{d}^3\boldsymbol{x}\, F_\nu(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x}, t). \tag{3.36}$$

We will take $\lambda_{\text{A}} = \lambda_{\text{B}}/2 = \lambda$ so that detector A (which couples twice to the field) and detector B (which couples once) interact with the field with the same overall "total strength". The time-evolution unitary $\hat{U}$ generated by the interaction Hamiltonian (3.35) is given by

$$\hat{U} = \begin{cases} \hat{U}_{\text{B}_1}\hat{U}_{\text{A}_2}\hat{U}_{\text{A}_1} & \text{if } t_{\text{A}_1} \leq t_{\text{A}_2} \leq t_{\text{B}_1}, \\ \hat{U}_{\text{A}_2}\hat{U}_{\text{B}_1}\hat{U}_{\text{A}_1} & \text{if } t_{\text{A}_1} \leq t_{\text{B}_1} \leq t_{\text{A}_2}, \\ \hat{U}_{\text{A}_2}\hat{U}_{\text{A}_1}\hat{U}_{\text{B}_1} & \text{if } t_{\text{B}_1} \leq t_{\text{A}_1} \leq t_{\text{A}_2}, \end{cases} \tag{3.37}$$

where $\hat{U}_{\nu i}$ is the unitary generated by $\hat{H}^{(i)}_{\text{I},\nu}(t)$.

We will set the detector smearing function $F_\nu(\boldsymbol{x})$ to be

$$F_\nu(\boldsymbol{x}) = \frac{3}{4\pi\sigma^3}\Theta\left(1 - \frac{|\boldsymbol{x}|}{\sigma}\right), \tag{3.38}$$

where Θ is the Heaviside theta function, σ is the spatial width of the detector, and the prefactor $3/4\pi$ is chosen so that $\int \mathrm{d}^3\boldsymbol{x}\, F(\boldsymbol{x}) = 1$. Notice that the support of F_ν is the sphere of radius σ centered at $\boldsymbol{x} = 0$. Hence if detectors A and B interact with the field through unitaries $\hat{U}_\text{A}$ and $\hat{U}_\text{B}$ at times t_A and t_B, then the detectors are fully timelike separated during their interactions if and only if $|t_\text{A} - t_\text{B}| > 2\sigma$. Note also that in (3+1)D flat spacetime $[\hat{\phi}(x), \hat{\phi}(x')] \neq 0$ if and only if x and x' are null separated. Therefore for our choice of detector smearing, if $|t_\text{A} - t_\text{B}| > 2\sigma$ then $\hat{U}_\text{A}$ and $\hat{U}_\text{B}$ necessarily commute, and by the results of the previous section they cannot harvest entanglement. We will now show to what extent the detectors *can* get entangled when they *are* able to signal to each other (i.e. when they are not completely timelike nor spacelike separated).

The time-evolved state of the two detectors after their interactions with the field, denoted $\hat{\rho}_\text{AB}$, is obtained by applying the unitary $\hat{U}$ in Eq. (3.37) to the initial state $|\psi_0\rangle$ in Eq. (3.34) and tracing out the field, i.e.

$$\hat{\rho}_\text{AB} = \text{Tr}_\phi\left(\hat{U}|\psi_0\rangle\langle\psi_0|\hat{U}^\dagger\right). \tag{3.39}$$

Evaluating this expression is straightforward but rather tedious, and so we relegate the calculation to Appendix D. We find that, in the basis $\{|g_\text{A}\rangle|g_\text{B}\rangle, |g_\text{A}\rangle|e_\text{B}\rangle, |e_\text{A}\rangle|g_\text{B}\rangle, |e_\text{A}\rangle|e_\text{B}\rangle\}$, the density matrix $\hat{\rho}_\text{AB}$ reads

$$\hat{\rho}_\text{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{3.40}$$

where the entries ρ_{ij} are dependent on the choice of unitary in Eq. (3.37), and are given in Appendix D.

As in Chapter 2, we will quantify the entanglement of $\hat{\rho}_\text{AB}$ using the negativity $\mathcal{N}$, which is an entanglement monotone that vanishes only for separable states [83, 84]. Recall that the negativity is defined as $\mathcal{N} := -\sum_i \min(E_i, 0)$, where E_i are the eigenvalues of the partial transpose of $\hat{\rho}_\text{AB}$ (with respect to either system A or B). From Eq. (3.40), we find the E_i to be

$$E_1 = \frac{1}{2}\left(\rho_{22} + \rho_{33} + \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}\right), \tag{3.41}$$

$$E_2 = \frac{1}{2}\left(\rho_{22} + \rho_{33} - \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}\right), \tag{3.42}$$

$$E_3 = \frac{1}{2}\left(\rho_{11} + \rho_{44} + \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2}\right), \tag{3.43}$$

$$E_4 = \frac{1}{2}\left(\rho_{11} + \rho_{44} - \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2}\right). \tag{3.44}$$

The eigenvalues E_i of the partial transpose of $\hat{\rho}_{\text{AB}}$, and hence the negativity $\mathcal{N}$, are functions of the following parameters: the times t_{A_1}, t_{A_2}, and t_{B_1} at which the detectors couple to the field, the strength λ with which the detectors couple to the field, as well as the energy gaps Ω_{A} and Ω_{B} of the detectors. We investigate each of these dependencies below. For simplicity we will set the units of length to be σ, which is half the spatial width of a detector. Hence the units of energy and λ (in 3+1 dimensions) are σ^{-1}.

First suppose that $t_{\text{A}_1} \leq t_{\text{A}_2} \leq t_{\text{B}_1}$. With these constraints, we are not able to find any parameter values which give a non-zero negativity. This is, of course, expected from our result in Sec. 3.3.1: the single coupling between detector B and the field is entanglement breaking, and hence if B couples last, regardless of the way A couples the final state of A and B will be separable.

What happens if we constrain the coupling times by $t_{\text{B}_1} \leq t_{\text{A}_1} \leq t_{\text{A}_2}$? In this case, we do find parameter values for which the negativity is non-vanishing, as is shown in Fig. 3.3. In this plot we see that the negativity is a periodic function of the energy gap Ω_{A} of detector A, with period $\Omega_0 = 2\pi/(t_{\text{A}_2} - t_{\text{A}_1})$. This is due to the fact that detector A evolves freely for a time interval $t_{\text{A}_2} - t_{\text{A}_1}$ between its two couplings with the field. Adding a multiple of Ω_0 to the detector's free frequency Ω_{A} will not alter the phase it picks up during its free evolution.

Notice also from Fig. 3.3 that if the phase difference $\Omega_{\text{A}}(t_{\text{A}_2} - t_{\text{A}_1})$ is a multiple of 2π (i.e. $\Omega_{\text{A}} \in 4\pi\mathbb{Z}$ for the solid curve, and $\Omega_{\text{A}} \in 2\pi\mathbb{Z}$ for the dashed curve), then the detectors cannot harvest entanglement. This comes about because such a phase difference ensures that Alice's two couplings to the field are through the same detector observable, and hence they result in a unitary that is the exponential of a Schmidt rank 1 operator, which, as we have shown, results in an entanglement breaking channel. Reassuringly, we also find the negativity to be independent of Ω_{B}. This is as expected: since detector B interacts with

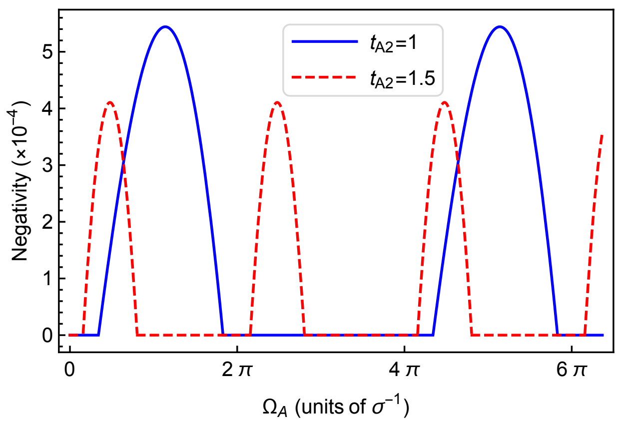

Figure 3.3: Negativity $\mathcal{N}$ of a two qubit system as a function of the energy gap $\Omega_{\rm A}$ of qubit A. Here the coupling scheme is BAA, $t_{\rm B_1} = 0$, $t_{\rm A_1} = 0.5$, $\lambda = 0.1$, and recall that σ is the spatial width of the detectors. The plot is the same for all values of $\Omega_{\rm B}$ since detector B only couples once. We plot the results for two values of $t_{\rm A_2}$. Notice that $\mathcal{N}$ is periodic in $\Omega_{\rm A}$ with period $2\pi/(t_{\rm A_2} - t_{\rm A_1})$.

the field at only one instant in time, any observable phenomenon (like the negativity) is independent of its free evolution, and thus its frequency $\Omega_{\rm B}$.

These findings allow us to strongly weigh in on the discussion presented in Ref. [2], where the authors found that two detectors that each δ-couple to the field cannot extract any entanglement. Two possible physical explanations were suggested: i) that the sudden δ-couplings induced too much local noise, which is known to have adverse effects on the amount of harvestable entanglement [23,26], or ii) that the lack of harvestable entanglement was a result of each detector not experiencing any non-trivial free dynamics, due to the fact that it only couples to the field at one instant in time. The second explanation is nicely complemented by the perturbative result that degenerate detectors, which also experience a lack of free dynamics, *cannot* harvest entanglement from the vacuum at leading order [52]. We now see that this intuition in ii) seems to be correct. Namely, we have shown that it is indeed possible to harvest entanglement by δ-coupling to the field (therefore the noisy nature of δ-couplings cannot be a critical constraint), but it is necessary for at least one of the detectors to couple more than once to the field (and hence experience non-trivial free evolution).

Let us now explore the dependence of the negativity on the coupling strength λ of the detectors to the field. In Fig. 3.4, we notice that in the weak-coupling regime $\lambda \ll 1$ (in units of σ^{-1}), $\mathcal{N}$ scales as λ^2. This is a familiar result from perturbative studies [26, 43],

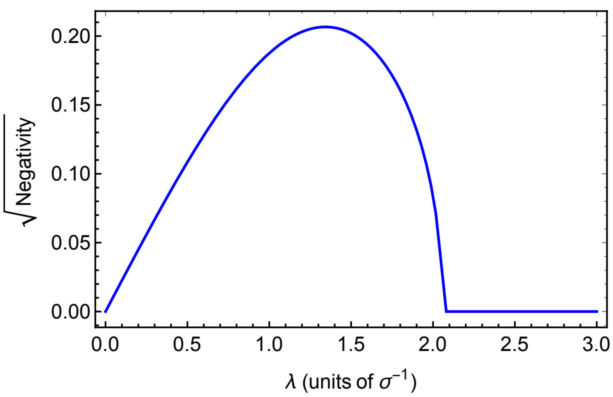

Figure 3.4: Square root of negativity $\mathcal{N}$ as a function of the strength λ with which detectors couple to the field. Here the coupling scheme is BAA, $t_{\text{B}_1} = 0$, $t_{\text{A}_1} = 0.5$, $t_{\text{A}_2} = 1$, $\Omega_{\text{A}} = 3$, and recall that σ is the spatial width of the detectors. The plot is the same for all values of Ω_{B} since detector B only couples once. As expected, $\mathcal{N} \sim \lambda^2$ for $\lambda \ll 1$. Interestingly the dependence is drastically different in the non-perturbative ($\lambda \gtrsim 1$) regime.

where it has been shown that the leading order contribution to $\mathcal{N}$ is of $\mathcal{O}(\lambda^2)$. Notice however, that this trend does not continue into the non-perturbative ($\lambda \gtrsim 1$) regime. In fact, remarkably, $\mathcal{N}$ reaches a maximum and then rapidly drops to zero at a finite value of λ, remaining zero thereafter. That is, in the strong coupling regime, *increasing the coupling strength seems to be detrimental to entanglement harvesting*, at least for delta-couplings. It is possible that this phenomenon is due to the "noisy" nature of δ-couplings becoming significant in this regime, but more work needs to be done to confirm this.

To conclude this section, let us consider how the times at which the detectors couple to the field affect whether they can become entangled. Concretely, let us again consider the BAA coupling scheme, where we set $t_{\text{B}_1} = 0$. From Fig. 3.5, we see that there is only a finite region in the $t_{\text{A}_1} - t_{\text{A}_2}$ plane in which the detectors, by appropriately tuning the energy gap Ω_{A}, could become entangled. This can be contrasted with the result in [26], where it was shown that (spacelike separated) detectors with Gaussian switching profiles can always harvest entanglement by increasing their energy gaps, regardless of separation distance.

Our result for δ-coupled detectors can be understood by our result in Sec. 3.2.1: in order for two detectors δ-coupling to the field three times in total to become entangled, the values of t_{A_1} and t_{A_2} (with t_{B_1} fixed) must be such that at least two of the three unitary commutators $[\hat{U}_{\text{B}1}, \hat{U}_{\text{A}1}]$, $[\hat{U}_{\text{B}1}, \hat{U}_{\text{A}2}]$, and $[\hat{U}_{\text{A}1}, \hat{U}_{\text{A}2}]$ are non-vanishing. (Recall that in 3+1D flat spacetime, unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ at times t_{A} and t_{B} commute iff the detectors at these

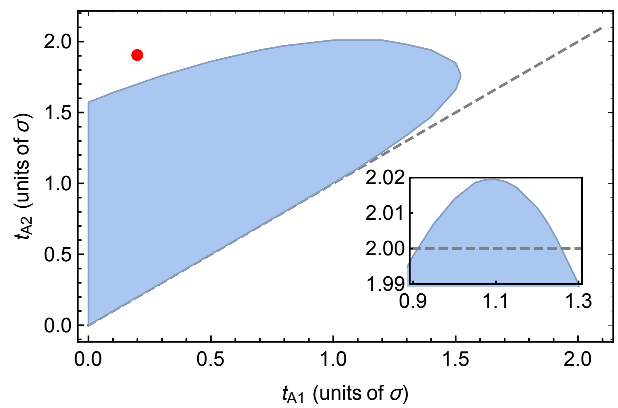

Figure 3.5: The shaded region indicates values of t_{A_1} and t_{A_2} for which detectors A and B can become entangled ($\mathcal{N} > 0$) with an appropriate choice of Ω_A. Note that in the entire shaded region $[\hat{U}_{B_1}, \hat{U}_{A_1}]$ and $[\hat{U}_{A_1}, \hat{U}_{A_2}]$ are non-zero, while the inset shows that $\mathcal{N} > 0$ is possible even if $[\hat{U}_{B_1}, \hat{U}_{A_2}] = 0$, i.e., if $t_{A_2} > 2$. The point at $(t_{A_1}, t_{A_2}) = (0.2, 1.9)$ shows that $\mathcal{N}$ could be zero even if none of the three commutators vanish. Here the coupling scheme is BAA, $t_{B_1} = 0$, $\lambda \ll 1$ such that $N \sim \lambda^2$, and Ω_A is arbitrary. The dashed line in the main plot shows $t_{A_1} = t_{A_2}$.

times are not in null contact.) Indeed, the shaded region in Fig. 3.5 corresponds to values of t_{A_1} and t_{A_2} that satisfy this property. We notice however that this commutator condition on entanglement extraction is necessary but not sufficient: there exist values of t_{A_1} and t_{A_2} (for example $t_{A_1} = 0.2$, $t_{A_2} = 1.9$) for which at least two unitary commutators are non-vanishing, yet for which entanglement harvesting is not possible.

3.4 Conclusions

In this chapter we have investigated the reasons why, in the analysis of entanglement harvesting from quantum fields, there were known regimes where entanglement harvesting was not possible. Prompted by these no-go results, we have studied the more general problem of entangling a bipartite separable system through bi-local interactions with a bipartite entangled source.

Concretely, we have considered the general setup of a separable target system A-B interacting locally with an entangled source S. Assuming knowledge of the Hamiltonian governing the time evolution of the system, we addressed the pertinent question: under what conditions does the target system A-B become entangled following the interaction?

For a general class of interaction Hamiltonians $\hat{H}_{\text{I}}(t)$ that are frequently considered in the literature, we found a necessary condition that $\hat{H}_{\text{I}}(t)$ must obey in order for A and B to be able to extract entanglement from the source. Namely, we showed that if the time-evolution unitary $\hat{U}$ generated by $\hat{H}_{\text{I}}(t)$ is of the form $\hat{U} = \hat{U}_{\text{BS}}\hat{U}_{\text{AS}}$ (i.e target A interacts with the source before target B), and $\hat{U}_{\text{BS}}$ is the exponential of a Schmidt rank 1 operator, then A and B *cannot* become entangled via their interactions with the source.

With this result we have generalized all previously known no-go theorems for entanglement harvesting [2, 52]. The significance of this result arises from the fact that Hamiltonians satisfying the above conditions are commonly used in non-perturbative studies of first quantized systems interacting with quantum fields [2, 49]. Hence the criterion stated above can be used to prove non-perturbative results for these systems.

For instance, our general result generalizes one of the main results of Ref. [52]. There it was shown that, to leading order in perturbation theory, identical and degenerate UDW detectors with non-overlapping switching functions cannot harvest any entanglement from the field vacuum in a flat spacetime of any dimensionality. In Ref. [52] it is also shown that for degenerate detectors with overlapping switchings, and spherically symmetric smearings, entanglement harvesting is only possible in timelike separation. Our result extends this claim to the non-perturbative regime, for not necessarily identical detectors of any shape, and for any arbitrary field state.

Similarly, we were able to generalize the main result from Ref. [2], where it was shown that two UDW detectors (not necessarily degenerate), each interacting with the quantum field through a single Dirac-δ switching function, cannot harvest any entanglement from a coherent field state. Namely, using our general no-go theorem for entanglement extraction we found that this is the case for *any* arbitrary field state.

It should be stressed that there is an important advantage to the method we used here to achieve these generalizations of previous entanglement harvesting results. Namely, the conclusions followed from a direct inspection of the system's Hamiltonian without the need to first explicitly evaluate the final state of the detectors, which, as can be seen in Appendix D, is often a very tedious task.

Finally, having seen that two δ-couplings are not enough to entangle a pair of UDW detectors, we showed the simplest example of a coupling scheme in which the detectors *do* become entangled through δ-interactions. For detectors that are able to communicate, i.e. are non-spacelike separated, three δ-couplings are sufficient (two for detector A, one for detector B), while for spacelike separated detectors four δ-couplings are required (two per detector).

For the case of three couplings, because our analysis was non-perturbative, we could answer the question, “What happens to entanglement harvesting in the strong coupling regime?”. We found, first of all, that when the coupling strength λ between the detectors and the field is small (compared to other scales with the same dimensions), then the amount of entanglement harvested by the detectors grows as λ^2. This was expected from previous perturbative studies [26]. However, as λ exits the perturbative regime this trend reverses itself: the extracted entanglement begins to decrease, and for λ larger than some finite critical value the detectors are not able to extract any entanglement from the field at all. We conjecture that this is due to the “noisy” nature of the sharp and intense δ-couplings, which may manifest itself only in the non-perturbative, strong coupling regime.

The results of this chapter give rise to new questions in the context of entanglement extraction, in general, and entanglement harvesting from quantum fields, in particular. For instance, our results reveal that it may be necessary to distinguish between the genuine extraction of pre-existing entanglement from a source, and the generation of entanglement between the targets through communication-assisted correlation via the source. This was illustrated by the qubit toy models which demonstrated how two target systems can become entangled by three simple generated interaction unitaries. There, pre-existing entanglement in the source system was not required, and would in fact be a hindrance to achieving entanglement between the targets. Furthermore we showed that, in full generality, when no communication between A and B is possible, i.e., when their couplings to the source commute with each other, then at least two simple generated interactions per target are necessary to achieve genuine entanglement extraction from the source.

It is particularly interesting to apply these considerations to entanglement harvesting from relativistic fields, where the ability to communicate between the targets is determined by their separation being spacelike, null or timelike. Here, our earlier discussion implies that δ-coupled detectors that are spacelike separated need at least four interactions to extract entanglement from the field. Protocols that only use three δ-couplings in total, meanwhile, can only succeed in extracting entanglement if the detectors are located such that they can communicate via the field. All these factors together suggest that the triple δ-coupling protocols, while entangling the targets through detector-field coupling, may not be an example of genuine harvesting of entanglement from the field. Instead, one would need to use at least four δ-coupling to truly harvest pre-existing entanglement from the field's degrees of freedom onto the target detectors.

As a final remark, another direction in which these results could be extended is to investigate how close to a simple generated interaction a target-source interaction can be in order for it to allow for entanglement extraction. It is likely that there is a larger class of interactions, containing the simple generated interactions, for which entanglement extraction still is not possible. A concept that may be useful to achieve this generalization is the class of entanglement-annihilating channels [102–104].[1] In particular, the entanglement breaking channels that we considered in this article are a strict subset of the 2-locally entanglement-annihilating channels [102]. However, bipartite (or more generally k-partite) entanglement extraction is impossible already if the source-target interaction yields a 2-locally (or generally k-locally) entanglement-annihilating quantum channel from the initial state of the source to the final individual partial states of the targets.

[1] As defined in [102], a channel χ mapping states on $\mathcal{H}_\text{A} \otimes \mathcal{H}_\text{B}$ to states on $\mathcal{H}_\text{A} \otimes \mathcal{H}_\text{B}$ is called *entanglement-annihilating* if $\chi(\hat{\rho}_\text{AB})$ is separable on $\mathcal{H}_\text{A} \otimes \mathcal{H}_\text{B}$ for any input state $\hat{\rho}_\text{AB}$. Note the subtle difference between entanglement-annihilating and entanglement breaking channels, the latter being defined in Definition 2.

Chapter 4

Quantum information transmission through quantum fields

While Chapters 2 and 3 studied the ability of a pair of UDW detectors to extract correlations from a quantum field, in the present chapter we study the ability of a pair of UDW detectors to *communicate* through a quantum field. As reviewed in Sec. 1.2, there has been significant work done on the classical communication ability of two UDW detectors. To complement this existing literature, we will here instead focus on purely quantum communication, which, besides the notable work done in Ref. [73], has to our knowledge not been previously studied from a fundamental light-matter interaction perspective. Furthermore, while Ref. [73] focused on the ability of a UDW detector pair to transmit quantum information in $(1+1)$-dimensions — which is the simplest possible case owing to the fact that with one spatial dimension there are only two directions in which quantum information can propagate — we will here generalize these results to an arbitrary number of spatial dimensions.

In particular, our results in this chapter will be presented in a very constructive manner. We will begin by constructing what is, in a certain sense that will be made clear later, the "simplest possible" perfect quantum channel from an observer Alice to an observer Bob in $(d{+}1)$-dimensional flat spacetime, which the observers themselves can physically implement by coupling their detectors to local field observables. In performing this construction we will naturally be led to answering the following pertinent question: if Alice couples locally to the field at time t_{A} and Bob couples to the field at time $t_{\text{B}} > t_{\text{A}}$, where in space does Bob need to be located in order to fully recover the quantum information that Alice encoded in the field?

We will find that the answer to this question has both interesting similarities as well as differences with the case of classical information transmission.

On the one hand we will find that, as is well-known for classical information, quantum information can also travel slower than light through a massless field in certain spacetimes, thus illustrating the consequences of strong Huygens principle violations [60].

On the other hand however, we will find that while sending classical information is relatively straightforward from the perspective of the observers Alice and Bob — in the sense that Bob can be located anywhere on Alice's light cone (or even inside the light cone in strong Huygens violating spacetimes) and still recover her message — we will find that the same is not true when we consider quantum information transmission. Namely, we will find in this latter case that in order for Bob to optimally recover Alice's message at a time t_{B}, he needs to be spatially delocalized *everywhere in space* where Alice's message has propagated to at this time. In particular, if Alice broadcasts her message isotropically via a massless field, then Bob needs to be smeared to cover the entirety of Alice's future light cone, as well as the interior of the light cone in strong Huygens violating spacetimes, in order to recover her quantum message. This analysis will therefore highlight, from a novel fundamental perspective, the important particular challenges that must be overcome in order to transmit quantum information through a quantum field.

4.1 Setup

As we have done throughout this thesis, we will in this chapter continue to model Alice and Bob's detectors as two-level quantum systems, which couple to the quantum field through interaction Hamiltonians of the familiar form (1.26), namely

$$\hat{H}_{\text{I},\nu}(t) = \lambda\chi(t)\hat{m}(t) \otimes \hat{\Phi}(t), \tag{4.1}$$

where $\nu \in \{\text{A}, \text{B}\}$ labels which detector we are considering. As always, λ is a coupling strength, $\chi(t)$ is an explicitly time-dependent switching function, and $\hat{m}(t)$ and $\hat{\Phi}(t)$ are qubit and field observables which contain an implicit time dependence coming from the fact that we are working in the interaction picture. In particular, in order to ensure that the coupling between the observer ν and the field is physical, we must ensure that the field observable $\hat{\Phi}(t)$ entering the interaction Hamiltonian is an observable that is local in spacetime to the region where the observer is located. To that end, for an observer coupling to the field at time t, we will allow the field observables $\hat{\Phi}(t)$ to be of the general form

$$\hat{\Phi}(t) = \hat{\phi}[F_1](t) + \hat{\pi}[F_2](t), \tag{4.2}$$

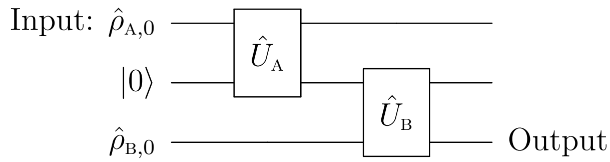

Figure 4.1: Quantum channel Ξ from Alice to Bob via a quantum field which starts in its ground state.

where for any local field operator $\hat{O}(\boldsymbol{x}, t)$ we defined the smeared operator $\hat{O}[F](t)$ as

$$\hat{O}[F](t) := \int \mathrm{d}^d\boldsymbol{x}\, F(\boldsymbol{x})\hat{O}(\boldsymbol{x}, t). \tag{4.3}$$

In order for $\hat{\Phi}(t)$ in Eq. (4.2) to indeed be a local observable for the observer in question, the smearing functions F_1 and F_2 need to have support in the region of space at time t where the observer is located. Note however that we do not require the observer to couple with the exact same smearing to the $\hat{\phi}$ and $\hat{\pi}$ fields; from a physical perspective of the full light-matter interaction this is analogous to an extended observer coupling his spatial profile differently to the electric and magnetic fields, a feat which is certainly possible.

Having motivated the physically reasonable interactions which we will allow between the observers Alice and Bob and the quantum field, let us now concretely formulate the main question that will guide us throughout this chapter.

First, let us suppose that the field starts in its vacuum state $|0\rangle$ and that qubit B is initially in some predefined state $\hat{\rho}_{\text{B},0}$. We define a quantum channel Ξ from Alice to Bob as a map which takes as input a state $\hat{\rho}_{\text{A},0}$ on Alice's Hilbert space $\mathcal{H}_{\text{A}}$ and outputs a state $\Xi[\hat{\rho}_{\text{A},0}]$ on Bob's Hilbert space $\mathcal{H}_{\text{B}}$. Concretely, we write this channel as

$$\Xi[\hat{\rho}_{\text{A},0}] := \mathrm{Tr}_{\text{A}\phi}\left[\hat{U}_{\text{B}}\hat{U}_{\text{A}}\left(\hat{\rho}_{\text{A},0}|0\rangle\langle 0|\hat{\rho}_{\text{B},0}\right)\hat{U}_{\text{A}}^{\dagger}\hat{U}_{\text{B}}^{\dagger}\right], \tag{4.4}$$

where $\hat{U}_\nu$ is a unitary between qubit ν and the field. Note that we are requiring qubit A to interact with the field prior to qubit B. A circuit diagram of the channel Ξ is shown in Fig. 4.1

The problem which we are interested in can now be formulated as follows: Suppose that qubit A has access to the field in some region of spacetime centered at $(\boldsymbol{x}_{\text{A}}, t_{\text{A}})$ and that qubit B couples to the field at some later time $t_{\text{B}} > t_{\text{A}}$. We ask:

1. Can we construct local unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ such that the channel Ξ is able to transmit quantum information from A to B?

2. Is it possible for Ξ to transmit quantum information perfectly? If so, where in space does qubit B have to be located?

Answering these two questions is the main aim of this chapter. However, before we can proceed with this, we must clarify what is meant by a channel being able to transmit quantum information, both perfectly and imperfectly. This is done in the following section.

4.1.1 Quantum channel capacity and coherent information

Suppose that we have a channel Ξ mapping the states of some Hilbert space $\mathcal{H}_{\text{A}}$ to the states of Hilbert space $\mathcal{H}_{\text{B}}$. We will later focus on the channel Ξ described in the previous subsection, but here we keep Ξ completely general. The ability of Ξ to transmit quantum information is quantified by its *quantum channel capacity*, denoted $Q(\Xi)$ which, analogous to its classical counterpart, is defined as the number of qubits that can be transmitted from sender A to receiver B per use of the channel (see, e.g., Ref [105] for a review of the quantum channel capacity).

An important result in quantum information theory states that the quantum channel capacity $Q(\Xi)$ of a quantum channel Ξ is bounded from below by the *maximal coherent information* of the channel, denoted $I_{\max}(\Xi)$ [106, 107]. Namely

$$Q(\Xi) \geq I_{\max}(\Xi) := \max_{\hat{\rho}_{\text{A},0}} I_c(\hat{\rho}_{\text{A},0}, \Xi), \tag{4.5}$$

where $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ is the *coherent information* of the channel Ξ and an input state $\hat{\rho}_{\text{A},0}$.

There are a few equivalent ways to define the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$. We define it as follows: Let $\hat{\rho}_{\text{A},0}$ be a state on the input Hilbert space $\mathcal{H}_{\text{A}}$. Then it is always possible to define a "purifying" Hilbert space $\mathcal{H}_{\text{C}}$ and a pure state $|\psi\rangle$ on the product space $\mathcal{H}_{\text{C}} \otimes \mathcal{H}_{\text{A}}$ such that $\hat{\rho}_{\text{A},0} = \text{Tr}_{\text{C}} |\psi\rangle\langle\psi|$. In other words, any generally mixed state $\hat{\rho}_{\text{A},0}$ can be thought of as a pure state of a larger system, where our lack of knowledge of the subsystem C results in the mixed nature of the partial state of A.

Next, we define the state $\hat{\rho}_{\text{CB}}$ to be the output of state $|\psi\rangle$ when we do nothing to subsystem C and act with channel Ξ on subsystem A. Namely

$$\hat{\rho}_{\text{CB}} := (\mathbb{1}_{\text{C}} \otimes \Xi) (|\psi\rangle\langle\psi|), \tag{4.6}$$

where $\mathbb{1}_{\mathrm{C}}$ is the identity operator on $\mathcal{H}_{\mathrm{C}}$. Finally, the coherent information $I_c(\hat{\rho}_{\mathrm{A},0}, \Xi)$ is defined as [105]

$$I_c(\hat{\rho}_{\mathrm{A},0}, \Xi) := S(\hat{\rho}_{\mathrm{B}}) - S(\hat{\rho}_{\mathrm{CB}}), \tag{4.7}$$

where $\hat{\rho}_{\mathrm{B}} := \Xi(\hat{\rho}_{\mathrm{A},0})$ is the output of the channel Ξ and $S(\hat{\rho}) := -\operatorname{Tr}\hat{\rho}\log_2\hat{\rho}$ is the von Neumann entropy of the state $\hat{\rho}$.

Although this definition of the coherent information is somewhat involved, it offers a very intuitive physical interpretation of its meaning. To see this, first note that before we put it through the channel Ξ, the system A was initially only entangled with the purifying system C. The coherent information $I_c(\hat{\rho}_{\mathrm{A},0}, \Xi)$ then quantifies how much of that entanglement between A and C is transferred to B and C. We can see this directly from Eq. (4.7): the first term $S(\hat{\rho}_{\mathrm{B}})$, being the entropy of system B, quantifies how entangled B is with the rest of the universe (i.e. with C as well as any additional "channel environment" implicit in the definition of the channel Ξ), while the second term, $S(\hat{\rho}_{\mathrm{CB}})$, is a measure of the entanglement of B and C with the channel environment. Hence the difference $I_c(\hat{\rho}_{\mathrm{A},0}, \Xi) = S(\hat{\rho}_{\mathrm{B}}) - S(\hat{\rho}_{\mathrm{CB}})$, analogous to the classical mutual information, quantifies the amount of correlations (in this case in the form of entanglement) between B and C. In particular, as we prove in Appendix E, $I_c(\hat{\rho}_{\mathrm{A},0}, \Xi) > 0$ only if $\hat{\rho}_{\mathrm{CB}}$ is an entangled state on $\mathcal{H}_{\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}$.

While the coherent information for a given channel Ξ is relatively easy to compute by means of Eq. (4.7), the same cannot be said for the quantum channel capacity $Q(\Xi)$. In particular it is possible to express the quantum channel capacity in terms of the maximal coherent information as [105]

$$Q(\Xi) = \lim_{n\to\infty} \frac{1}{n} I_{\max}(\Xi^{\otimes n}). \tag{4.8}$$

While this formula gives the intuitive interpretation of the quantum capacity as being the maximal coherent information of many copies of the channel being allowed to work in parallel, it is generally not possible to evaluate this expression to obtain a value for $Q(\Xi)$. For this reason, in what follows we will quantify the ability of our quantum field mediated channel Ξ to transmit quantum information from Alice to Bob by computing its coherent information. Although this is only a lower bound on the full channel capacity $Q(\Xi)$, we will show that we can construct the channel Ξ so that it has a coherent information arbitrarily close to 1 with respect to the maximally mixed input state $\hat{\rho}_{\mathrm{A},0} = \frac{1}{2}\mathbb{1}_{\mathrm{A}}$. Since our channel takes as input a single qubit state, we also know that $Q(\Xi) \leq 1$, i.e. the number of qubits that can be sent per use of the channel cannot be larger than 1. Hence we will show how to construct a channel transmitting qubits through a quantum field with a quantum channel capacity that is arbitrarily close to its maximal value; i.e. we will construct a perfect quantum channel.

4.2 Constructing a perfect quantum channel

Let us now proceed to construct a quantum channel Ξ of the form in Eq. (4.4), which allows a sender Alice to transmit a qubit of information through a quantum field to some future receiver Bob. To that end, let us suppose that Alice is located in a region of space characterized by the smearing function $F_{\text{A}}(\boldsymbol{x})$, and that she wishes to encode her message into the field around some time t_{A}. Then we would like to answer the following questions: where in space should Bob be located at time $t_{\text{B}} > t_{\text{A}}$, and what should the unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ be, in order for Bob to recover the entirety of Alice's message?

Before we proceed to answering these questions, let us remind ourselves that our ultimate goal is not to simply construct a perfect quantum channel between two spacetime observers — indeed high capacity free space channels have already been realized experimentally over distances as large as 1000+ km, typically in the context of establishing entangled pairs between distant receivers for the purposes of implementing quantum key distribution protocols [8, 67–72]. Rather our goal is to understand, from a fundamental relativistic quantum information perspective, exactly how quantum information is encoded into, propagated through, and decoded out of, a quantum field. To that end, let us attempt to construct the channel Ξ to be as simple as possible, so that we may try to understand its essential features without being distracted by the unessential ones.

With this additional requirement of simplicity for our channel Ξ, let us attempt to generate the time-evolution unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ defining the channel out of the simplest possible type of interaction Hamiltonians: those which couple the qubits A and B to the field only at discrete instants of time. Conveniently, as discussed in Sec. 1.4, by constructing our channel out of these simple couplings, we have the additional advantage that our analysis will be fully non-perturbative.

In fact, the ability to study our problem non-perturbatively is not merely a nice convenience that arises out of using simple-generated interaction unitaries. More crucially, as we will now prove, if we want the quantum channel Ξ from Alice to Bob to be a perfect quantum channel (i.e. to have a maximum possible quantum channel capacity $Q(\Xi) = 1$), then it is *necessary* that the channel is constructed out of non-perturbative couplings between Alice and Bob's qubits and the field.

The proof of this claim is rather trivial. Let us suppose that the coupling between qubits and field is quantified by some coupling strength λ, and let us consider the quantum channel capacity $Q(\Xi)$ as a power series in λ. Clearly if $\lambda = 0$, the qubits A and B do not couple to the field, and hence we would have $Q(\Xi) = 0$. Hence we can write

$$Q(\Xi) = 0 + \mathcal{O}(\lambda), \tag{4.9}$$

and thus we see that if the coupling λ is weak — i.e. $\lambda \ll 1$ in units set by the other scales in the problem — then $Q(\Xi)$ would, at best, only differ by a small amount (i.e. an amount much less than one) from zero. Hence if we want to have $Q(\Xi) \approx 1$, we must consider couplings λ in the non-perturbative regime.

4.2.1 Simple-generated couplings

The simplest possible unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ coupling the qubits A and B to the field are of the form

$$\hat{U}_\nu = \exp\left(\mathrm{i}\lambda_\nu \hat{m}_\nu \otimes \hat{\Phi}_\nu\right), \tag{4.10}$$

where $\nu \in \{\text{A}, \text{B}\}$ and we abbreviate $\hat{m}_\nu := \hat{m}(t_\nu)$ for the qubit observables and $\hat{\Phi}_\nu := \hat{\Phi}(t_\nu)$ for the field observables. As in Chapter 3, we will call unitaries of this form "simple-generated" or "rank-1 unitaries" because they are the exponential of a simple rank-1 tensor product of qubit-field observables. We claim that if either $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ are of the simple-generated form, then the channel Ξ in Eq. (4.4) is not able to transmit quantum information.

To prove this claim, let us first decompose the channel Ξ from A to B in Eq. (4.4) to read $\Xi = \Xi_{\text{B}} \circ \Xi_{\text{A}}$, where Ξ_{A} is a channel from A to the field defined by

$$\Xi_{\text{A}}(\hat{\rho}_{\text{A}}) := \mathrm{Tr}_{\text{A}}\left[\hat{U}_{\text{A}}\left(\hat{\rho}_{\text{A}} \otimes |0\rangle\langle 0|\right)\hat{U}_{\text{A}}^\dagger\right], \tag{4.11}$$

while Ξ_{B} is a channel from the field to B defined by

$$\Xi_{\text{B}}(\hat{\rho}_\phi) := \mathrm{Tr}_\phi\left[\hat{U}_{\text{B}}\left(\hat{\rho}_\phi \otimes \hat{\rho}_{\text{B},0}\right)\hat{U}_{\text{B}}^\dagger\right]. \tag{4.12}$$

Next, we recall the important result that we proved in Chapter 3, which states that unitaries of the simple-generated form necessarily give rise to *entanglement breaking channels*.[1] Thus we find that if $\hat{U}_\nu$ is simply-generated, then Ξ_ν is an entanglement breaking channel.

[1] Recall that a channel σ from states on $\mathcal{H}_{\text{A}}$ to states on $\mathcal{H}_{\text{B}}$ is said to be *entanglement breaking* if for any state $\hat{\rho}_{\text{AC}}$ on $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{C}}$ the state $(\sigma \otimes \mathbb{1}_{\text{C}})(\hat{\rho}_{\text{AC}})$ on $\mathcal{H}_{\text{B}} \otimes \mathcal{H}_{\text{C}}$ is separable.

However, we previously noted that the coherent information in Eq. (4.7) is greater than zero only if entanglement can be transferred through the channel. Hence we find that if either $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ are of the simple-generated form then the maximal coherent information $I_c(\Xi)$ of the channel Ξ is zero.

Notice however, that this does not yet prove that the quantum capacity $Q(\Xi)$ of the channel Ξ is itself zero, since $I_c(\Xi)$ is only a lower bound on $Q(\Xi)$. Let us now show that it is indeed the case that $Q(\Xi) = 0$ if either $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ are simply-generated unitaries.

The proof of this stronger claim starts with the expression Eq. (4.8) for the quantum channel capacity of Ξ in terms of the maximal coherent information of $\Xi^{\otimes n}$ for large n. Using the fact that $\Xi^{\otimes n} = \Xi_{\text{B}}^{\otimes n} \circ \Xi_{\text{A}}^{\otimes n}$ we note that if either of the $\Xi_{\nu}^{\otimes n}$ are entanglement breaking then $\Xi^{\otimes n}$ will have a maximal coherent information of zero (as discussed above), and hence $Q(\Xi) = 0$. To that end, let us start by proving that $\Xi_{\text{A}}^{\otimes n}$ is entanglement breaking.

To prove that $\Xi_{\text{A}}^{\otimes n}$ is entanglement breaking, we need to show that for any Hilbert space $\mathcal{H}_{\text{C}}$, the state $(\mathbb{1}_{\text{C}} \otimes \Xi_{\text{A}}^{\otimes n})(\hat{\rho}_{\text{C},\text{A}^n})$ is separable on the product space $\mathcal{H}_{\text{C}} \otimes \mathcal{H}_{\phi}^{\otimes n}$. We thus compute

$$(\mathbb{1}_{\text{C}} \otimes \Xi_{\text{A}}^{\otimes n})(\hat{\rho}_{\text{C},\text{A}^n}) = \text{Tr}_{\text{A}^n}\left[\hat{U}_{\text{A}_1} \dots \hat{U}_{\text{A}_n}\left(\hat{\rho}_{\text{C},\text{A}^n} \otimes \left(|0\rangle\langle 0|\right)^{\otimes n}\right)\hat{U}_{\text{A}_n}^{\dagger} \dots \hat{U}_{\text{A}_1}^{\dagger}\right]. \tag{4.13}$$

To proceed we note by the spectral theorem that we can express any qubit observable $\hat{m}_{\text{A}}$ as

$$\hat{m}_{\text{A}} = \sum_{s \in \{\pm\}} s\hat{P}_s, \tag{4.14}$$

where $\hat{P}_{\pm} := |\pm_z\rangle\langle\pm_z|$ are projectors onto the $\pm$ eigenstates of σ_z. With this decomposition of $\hat{m}_{\text{A}}$, we can write $\hat{U}_{\text{A}}$ from Eq. (4.10) as

$$\hat{U}_{\text{A}} = \sum_{s \in \{\pm\}} \hat{P}_s \otimes \hat{U}_s, \tag{4.15}$$

where $\hat{U}_s := \exp(\text{i}s\lambda\hat{\Phi}_{\text{A}})$. Note that, when written in this way, it is manifest that a simple-generated unitary $\hat{U}_{\text{A}}$ can be viewed a controlled unitary, where the state $|s_z\rangle$ of the qubit controls the unitary operation $\hat{U}_s$ performed on the field.[2] Inserting Eq. (4.15) into

[2]This can be contrasted to what was done in Chapter 3, where we applied the spectral theorem to the *field* observable, thus viewing the unitary to be a controlled unitary from the field to the qubit, rather than vice-versa.

Eq. (4.13) we obtain

$$(\mathbb{1}_{\mathrm{C}}\otimes\Xi_{\mathrm{A}}^{\otimes n})(\hat{\rho}_{\mathrm{C,A}^n})=\sum_{s_i,s_i'}\mathrm{Tr}_{\mathrm{A}^n}\left[\hat{P}_{s_1}\dots\hat{P}_{s_n}\left(\hat{\rho}_{\mathrm{C,A}^n}\hat{U}_{s_1}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_1'}\dots\hat{U}_{s_n}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_n'}\right)\hat{P}_{s_n'}\dots\hat{P}_{s_1'}\right],\tag{4.16}$$

where the sub-index i runs from 1 to n. Using the cyclicity of the partial trace with respect to the system being traced over, and the fact that $\hat{P}_{s_i}\hat{P}_{s_i'}=\hat{P}_{s_i}\delta_{s_i s_i'}$ since $\hat{P}_{s_i}$ are projectors, Eq. (4.16) simplifies to

$$(\mathbb{1}_{\mathrm{C}}\otimes\Xi_{\mathrm{A}}^{\otimes n})(\hat{\rho}_{\mathrm{C,A}^n})=\sum_{s_i}\mathrm{Tr}_{\mathrm{A}^n}\left[\hat{P}_{s_1}\dots\hat{P}_{s_n}\left(\hat{\rho}_{\mathrm{C,A}^n}\hat{U}_{s_1}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_1}\dots\hat{U}_{s_n}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_n}\right)\right].\tag{4.17}$$

Finally, defining

$$p(s_1,\dots,s_n):=\mathrm{Tr}\left[\hat{P}_{s_1}\dots\hat{P}_{s_n}\hat{\rho}_{\mathrm{C,A}^n}\right],\tag{4.18}$$

$$\hat{\rho}_{\mathrm{C}}(s_1,\dots,s_n):=\frac{1}{p_{s_i}}\mathrm{Tr}_{\mathrm{A}^n}\left[\hat{P}_{s_1}\dots\hat{P}_{s_n}\hat{\rho}_{\mathrm{C,A}^n}\right],\tag{4.19}$$

$$\hat{\rho}_{\phi^n}(s_1,\dots,s_n):=\hat{U}_{s_1}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_1}\dots\hat{U}_{s_n}|0\rangle\langle 0|\hat{U}^{\dagger}_{s_n},\tag{4.20}$$

where $\hat{\rho}_{\mathrm{C}}(s_1,\dots,s_n)$ is a density matrix on $\mathcal{H}_{\mathrm{C}}$, $\hat{\rho}_{\phi^n}(s_1,\dots,s_n)$ is a density matrix on $\mathcal{H}_{\phi}^{\otimes n}$, and $p(s_1,\dots,s_n)\geq 0$ with $\sum_{s_i}p(s_1,\dots,s_n)=1$, we can write Eq. (4.17) as

$$(\mathbb{1}_{\mathrm{C}}\otimes\Xi_{\mathrm{A}}^{\otimes n})(\hat{\rho}_{\mathrm{C,A}^n})=\sum_{s_i}p(s_1,\dots,s_n)\,\hat{\rho}_{\mathrm{C}}(s_1,\dots,s_n)\otimes\hat{\rho}_{\phi^n}(s_1,\dots,s_n),\tag{4.21}$$

which is a manifestly separable state on $\mathcal{H}_{\mathrm{C}}\otimes\mathcal{H}_{\phi}^{\otimes n}$. Hence the n-qubit channel $\Xi_{\mathrm{A}}^{\otimes n}$ is entanglement breaking for all integers $n>0$.

In an analogous fashion, we can prove that the channel Ξ_{B} from the field to B is entanglement breaking. The only subtlety with this proof compared to the one we just presented for Ξ_{A}, is that, because we are now considering a channel from the field to a qubit, rather than vice-versa, we have to perform a spectral decomposition of the field observable, rather than the qubit observable. To perform this decomposition rigorously is not trivial since the field observables acts on an uncountably infinite dimensional Hilbert space. Nevertheless, as is shown in full detail in Appendix C, since the field observables are self-adjoint operators it is possible to apply the spectral theorem to them, and hence obtain such a spectral decomposition. In this manner, we can show that a simple-generated unitary between a qubit and a field can not only be written as a controlled unitary from the qubit to the field, as in Eq. (3.9), but also as a controlled unitary from the field to the qubit (see also footnote 2). In this way, the same argument that was used above to show that Ξ_{A} is an entanglement breaking channel can also be used to arrive at the same conclusion for Ξ_{B}.

In conclusion we have shown that if either of the unitaries $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ used to define the channel Ξ are of the simple-generated form, then the quantum channel capacity of Ξ is necessarily zero. In other words, simple-generated couplings between Alice and Bob's qubits to the field are too simple for the purposes of transmitting quantum information through the field. Thus in order to achieve quantum information transmission, we will need to consider more complicated couplings.

4.2.2 Encoding a qubit into a field

In our attempt to construct a channel Ξ that allows a spacetime emitter A to send a qubit through a quantum field to a receiver B, we have come to the important conclusion that such a channel is not possible if either of the observers couple to the field through simple-generated unitaries $\hat{U}_\nu = \exp(\mathrm{i}\lambda_\nu \hat{m}_\nu \otimes \hat{\Phi}_\nu)$. The natural way to proceed with constructing Ξ is to consider the next simplest types of interaction unitaries $\hat{U}_\nu$, composed of two rank-1 unitaries performed one after the other, i.e.

$$\hat{U}_\nu = \exp\left(\mathrm{i}\lambda_{\nu 2}\hat{m}_{\nu 2} \otimes \hat{\Phi}_{\nu 2}\right) \exp\left(\mathrm{i}\lambda_{\nu 1}\hat{m}_{\nu 2} \otimes \hat{\Phi}_{\nu 1}\right), \tag{4.22}$$

where the $\lambda_{\nu i}$ are coupling constants, the $\hat{m}_{\nu i}$ are qubit observables and the $\hat{\Phi}_{\nu i}$ are field observables. As we will now show, we can indeed find unitaries of this form which ensure that the quantum capacity of the channel Ξ is not only non-zero, but is in fact arbitrarily close to its theoretically maximal value of 1.

To understand this construction of the unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$, it will be instructive to first consider a simple example of a quantum channel that we know has perfect quantum capacity. To that end, let us consider the setup in which Alice and Bob would like to transmit a qubit of information by encoding and decoding their message into and out of a third qubit, F, rather than into and out of the quantum field. In this case, we know that the channel shown in Fig. 4.2, which simply swaps qubit A with F, and then F with B, is clearly able to perfectly transfer qubit A to qubit B.

The key question thus becomes: Is it possible to construct a channel analogous to the one in Fig. 4.2 if we take the intermediary system to be a quantum field $\hat{\phi}$ rather than a single qubit F? Indeed, our expectation is that it should be possible, if only from the

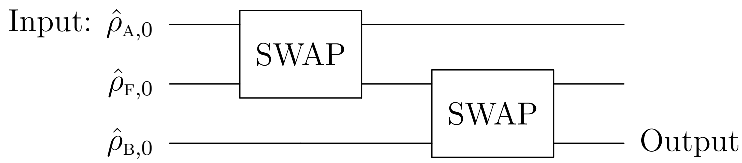

Figure 4.2: Perfect quantum channel from Alice to Bob via a third qubit, F.

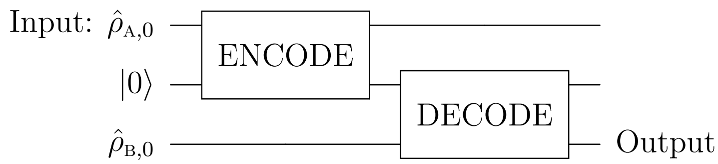

Figure 4.3: Perfect quantum channel from Alice to Bob via a quantum field, $\hat{\phi}$.

perspective that we could exclusively couple the qubits to a two-dimensional subspace of the field's Hilbert space, which is effectively equivalent to coupling to a third qubit F. However there is one important distinction between this field mediated channel and the qubit mediated channel in Fig. 4.2. Namely, while in the latter case it makes sense to construct the channel out of SWAP gates, it does not make sense to talk about a SWAP gate between a qubit and a field, since their Hilbert spaces have different dimensions and are thus not isomorphic as vector spaces. To belabour this point we will call the gate which encodes a qubit into a field an ENCODE gate, rather than a SWAP gate: indeed it should be possible to encode the two-dimensional Hilbert space of the qubit in the infinite dimensional Hilbert space of the field. Thus the field-mediated quantum channel which we are trying to construct is shown in Fig. 4.3.

Our question therefore becomes the following: How can an observer Alice, coupling to the field at time $t_{\rm A}$ and with a spatial extent $F_{\rm A}(\boldsymbol{x})$, encode the state of her qubit into the field?

Let us suppose that Alice's qubit is in some arbitrary pure state $c_1|+_z\rangle + c_2|-_z\rangle$ and that the field is initially in the vacuum state $|0\rangle$. Consider the qubit-field unitary $\hat{U}_{\rm A}$ given by

$$\hat{U}_{\rm A} = \exp\left(\mathrm{i}\sigma_x\hat{\pi}_{\rm A}\right)\exp\left(\mathrm{i}\sigma_z\hat{\phi}_{\rm A}\right), \tag{4.23}$$

where $\hat{\phi}_{\text{A}}$ and $\hat{\pi}_{\text{A}}$ are field observables defined as

$$\hat{\phi}_{\text{A}} := \lambda_\phi \int \mathrm{d}^d\boldsymbol{x}\, F_{\text{A}}(\boldsymbol{x})\hat{\phi}(\boldsymbol{x}, t_{\text{A}}), \tag{4.24}$$

$$\hat{\pi}_{\text{A}} := \lambda_\pi \int \mathrm{d}^d\boldsymbol{x}\, F_{\text{A}}(\boldsymbol{x})\hat{\pi}(\boldsymbol{x}, t_{\text{A}}). \tag{4.25}$$

Note that the coupling constants have dimensions of $[\lambda_\phi] = L^{\frac{d-1}{2}}$ and $[\lambda_\pi] = L^{\frac{d+1}{2}}$. Also note that the unitary $\hat{U}_{\text{A}}$ is generated by the interaction Hamiltonian

$$\hat{H}_{\text{I,A}}(t) = \lambda_\phi \delta(t - t_{\text{A}}^-)\hat{m}_{\text{A}}^z(t) \otimes \int \mathrm{d}^d\boldsymbol{x}\, F_{\text{A}}(\boldsymbol{x})\hat{\phi}(\boldsymbol{x}, t) + \lambda_\pi \delta(t - t_{\text{A}}^+)\hat{m}_{\text{A}}^x(t) \otimes \int \mathrm{d}^d\boldsymbol{x}\, F_{\text{A}}(\boldsymbol{x})\hat{\pi}(\boldsymbol{x}, t), \tag{4.26}$$

where $\hat{m}_{\text{A}}^z(t)$ is the σ_z operator in the interaction picture (so $\hat{m}_{\text{A}}^z(t) = \sigma_z$ for all t since σ_z is proportional to the detector's free Hamiltonian), $\hat{m}_{\text{A}}^x(t)$ is the σ_x operator in the interaction picture (i.e. it is the monopole moment operator from Eq. (1.27)), and the times $t_{\text{A}}^\pm \approx t_{\text{A}}$ are such that t_{A}^- is just slightly less than t_{A}^+.[3]

We now claim that the unitary $\hat{U}_{\text{A}}$ in Eq. (4.23) effectively encodes the state of the qubit in the state of the field, as long as the following two conditions are satisfied:

$$\left(\lambda_\phi \int \mathrm{d}^d\boldsymbol{k}\, \big|\tilde{F}_{\text{A}}(\boldsymbol{k})\big|^2\right)^2 \gg \frac{1}{2}\int \mathrm{d}^d\boldsymbol{k}\, \omega_{\boldsymbol{k}} \big|\tilde{F}_{\text{A}}(\boldsymbol{k})\big|^2, \tag{4.27}$$

$$\gamma_{\text{A}} := \lambda_\phi \lambda_\pi \int \mathrm{d}^d\boldsymbol{k}\, \big|\tilde{F}_{\text{A}}(\boldsymbol{k})\big|^2 \equiv \frac{\pi}{4} \mod 2\pi. \tag{4.28}$$

Recall that $\tilde{F}_{\text{A}}(\boldsymbol{k})$ is the d-dimensional Fourier transform of the function $F_{\text{A}}(\boldsymbol{x})$, as defined in Eq. (2.32).

This claim is straightforwardly proven by direct calculation. Acting on the initial state $\big(c_1|+_z\rangle + c_2|-_z\rangle\big)\,|0\rangle$ with the rightmost exponential in $\hat{U}_{\text{A}}$ results in the state

$$c_1|+_z\rangle|+\alpha_{\text{A}}\rangle + c_2|-_z\rangle|-\alpha_{\text{A}}\rangle, \tag{4.29}$$

where $|\pm\alpha_{\text{A}}\rangle$ are coherent field states defined by[4]

$$|\pm\alpha_{\text{A}}\rangle := \exp\big(\pm \mathrm{i}\hat{\phi}_{\text{A}}\big)|0\rangle. \tag{4.30}$$

[3]Physically this amounts to saying that $t_{\text{A}}^+ = t_{\text{A}}^- + |\epsilon|$ where $|\epsilon| \ll \Omega^{-1}$, with Ω being the free frequency of the detector.

[4]For a comprehensive overview of coherent states of a scalar field, see Refs. [2,43]

It can be shown that the magnitude of the overlap between the two coherent states $|+\alpha_{\text{A}}\rangle$ and $|-\alpha_{\text{A}}\rangle$ is (see Appendix A of [2])

$$\big|\langle+\alpha_{\text{A}}|-\alpha_{\text{A}}\rangle\big| = \exp\left[-\left(\lambda_\phi\right)^2\int\frac{\mathrm{d}^d\boldsymbol{k}}{2\omega_{\boldsymbol{k}}}\big|\tilde{F}_{\text{A}}(\boldsymbol{k})\big|^2\right]. \tag{4.31}$$

Hence we see that if $\lambda_\phi \gg 1$ in units of the characteristic length scale set by $F_{\text{A}}(\boldsymbol{x})$, then the field states $|+\alpha_{\text{A}}\rangle$ and $|-\alpha_{\text{A}}\rangle$ are almost orthogonal, and if $|c_1| = |c_2| = 1/\sqrt{2}$ the state in Eq. (4.30) is almost maximally entangled. In other words, a stronger coupling between the qubit and the field results in a more correlated (i.e. entangled) state of the two systems.

The final step in evaluating the action of the unitary $\hat{U}_{\text{A}}$ on the initial state of the qubit-field system $\big(c_1|+_z\rangle + c_2|-_z\rangle\big)\,|0\rangle$ is to apply the unitary $\exp(\mathrm{i}\sigma_x\hat{\pi}_{\text{A}})$ to the entangled state $c_1|+_z\rangle|+\alpha_{\text{A}}\rangle + c_2|-_z\rangle|-\alpha_{\text{A}}\rangle$. To perform this calculation, let us first apply the field observable $\hat{\pi}_{\text{A}}$ to the coherent state $|\pm\alpha_{\text{A}}\rangle$. To that end, using the Baker-Campbell-Hausdorff lemma we can straightforwardly prove the identity

$$\exp\big(\pm\,\mathrm{i}\hat{\pi}_{\text{A}}\big)\hat{a}_{\boldsymbol{k}}\exp\big(\mp\,\mathrm{i}\hat{\pi}_{\text{A}}\big) = \hat{a}_{\boldsymbol{k}} + \alpha_{\text{A}}(\boldsymbol{k})\mathbb{1}, \tag{4.32}$$

where the *coherent amplitude* $\alpha_\nu(\boldsymbol{k})$ is defined as

$$\alpha_\nu(\boldsymbol{k}) = \frac{\lambda_\nu^\phi}{\sqrt{2\omega_{\boldsymbol{k}}}}\tilde{F}_\nu^*(\boldsymbol{k})e^{\mathrm{i}\omega_{\boldsymbol{k}}t_\nu}, \tag{4.33}$$

and hence we find that

$$\hat{\pi}_{\text{A}}|\pm\alpha_{\text{A}}\rangle = \pm\gamma_{\text{A}}|\pm\alpha_{\text{A}}\rangle + \exp\big(\pm\,\mathrm{i}\hat{\pi}_{\text{A}}\big)\hat{\pi}_{\text{A}}|0\rangle, \tag{4.34}$$

where γ_{A} is defined in Eq. (4.28). Hence we see from Eq. (4.34) that $\hat{\pi}_{\text{A}}|\pm\alpha_{\text{A}}\rangle$ is the sum of two terms, and in particular we find that if $\gamma_{\text{A}}^2 \gg \langle 0|\hat{\pi}_{\text{A}}^2|0\rangle$ then

$$\hat{\pi}_{\text{A}}|\pm\alpha_{\text{A}}\rangle \approx \pm\gamma_{\text{A}}|\pm\alpha_{\text{A}}\rangle. \tag{4.35}$$

In other words, if $\gamma_{\text{A}}^2 \gg \langle 0|\hat{\pi}_{\text{A}}^2|0\rangle$ (which is exactly equivalent to the condition (4.27) which we are assuming to hold), then the field coherent states $|\pm\alpha_{\text{A}}\rangle$ are eigenvalues of the observable $\hat{\pi}_{\text{A}}$ with eigenvalues $\pm\gamma_{\text{A}}$. If additionally condition (4.28) is satisfied, then we find

$$\begin{aligned}\hat{U}_{\text{A}}\left(c_1|+_z\rangle + c_2|-_z\rangle\right)|0\rangle &= \exp\left(\mathrm{i}\sigma_x\hat{\pi}_{\text{A}}\right)\left(c_1|+_z\rangle|+\alpha_{\text{A}}\rangle + c_2|-_z\rangle|-\alpha_{\text{A}}\rangle\right)\\ &\approx c_1\exp\left(+\mathrm{i}\frac{\pi}{4}\sigma_x\right)|+_z\rangle|+\alpha_{\text{A}}\rangle + c_2\exp\left(-\mathrm{i}\frac{\pi}{4}\sigma_x\right)|-_z\rangle|-\alpha_{\text{A}}\rangle\\ &= |+_y\rangle\left(c_1|+\alpha_{\text{A}}\rangle - \mathrm{i}c_2|-\alpha_{\text{A}}\rangle\right).\end{aligned} \tag{4.36}$$

Note that in the second line we have used the identities $\exp(+\mathrm{i}\frac{\pi}{4}\sigma_x)|+_z\rangle = |+_y\rangle$ and $\exp(-\mathrm{i}\frac{\pi}{4}\sigma_x)|+_z\rangle = -\mathrm{i}|-_y\rangle$, which simply state that we can perform Bloch sphere rotations of the eigenstates of σ_z into the positive eigenstate $|+_y\rangle$ of σ_y by applying rotation unitaries generated by σ_x. Hence the unitary $\hat{U}_{\text{A}}$ has succeeded in encoding the orthogonal qubit superposition $c_1|+_z\rangle + c_2|-_z\rangle$ into a orthogonal superposition of field coherent states, $c_1|+\alpha_{\text{A}}\rangle - \mathrm{i}c_2|-\alpha_{\text{A}}\rangle$. The local phase of $-\mathrm{i}$ appearing in the second term does not affect the orthogonality of the first and second terms.

The results of this section can be summarized as follows. An observer Alice coupling locally to a quantum field at a time t_{A} can effectively encode her qubit into the field by implementing the unitary $\hat{U}_{\text{A}} = \exp\left(\mathrm{i}\sigma_x\hat{\pi}_{\text{A}}\right)\exp\left(\mathrm{i}\sigma_z\hat{\phi}_{\text{A}}\right)$, as long as the conditions (4.27) and (4.28) are satisfied. For instance, if Alice's qubit starts in the equally weighted superposition $\frac{1}{\sqrt{2}}(|+_z\rangle + |-_z\rangle)$, then, as long as (4.27) is satisfied, the rightmost exponential in $\hat{U}_{\text{A}}$ will maximally entangle Alice's qubit with the field. Following this, and assuming that (4.28) is satisfied, the leftmost exponential in $\hat{U}_{\text{A}}$ will then use the state of the field to perform a controlled rotation in the Bloch sphere of the qubit, thus leaving the field in an equally weighted, orthogonal superposition of coherent states. In other words, the unitary $\hat{U}_{\text{A}}$ succeeds, through local operations, in encoding Alice's qubit into the field.

4.2.3 Decoding a qubit out of a field

Having understood how Alice can ENCODE her qubit of information into the field, the final step in constructing the field-mediated quantum channel from Alice to Bob, as depicted in Fig. 4.3, is to construct the DECODE gate that allows Bob to recover Alice's message from the field. The most straightforward way to proceed is to note that the DECODE gate should simply be the inverse of the ENCODE gate. Thus, since we know the unitary $\hat{U}_{\text{A}} = \exp\left(\mathrm{i}\sigma_x\hat{\pi}_{\text{A}}\right)\exp\left(\mathrm{i}\sigma_z\hat{\phi}_{\text{A}}\right)$ implementing the encode gate, we also know that the inverse unitary $\hat{U}_{\text{A}}^{-1} = \hat{U}_{\text{A}}^{\dagger} = \exp\left(-\mathrm{i}\sigma_z\hat{\phi}_{\text{A}}\right)\exp\left(-\mathrm{i}\sigma_x\hat{\pi}_{\text{A}}\right)$ will implement the DECODE gate. We can now simply set the unitary $\hat{U}_{\text{B}}$ in Fig. 4.1, which acts on detector B and the field, to be the unitary $\hat{U}_{\text{A}}^{\dagger}$ with the understanding that the qubit observables σ_x and σ_z now act on the Hilbert space $\mathcal{H}_{\text{B}}$ rather than $\mathcal{H}_{\text{A}}$.

Note however that there is a problem with this construction of the decoding unitary $\hat{U}_{\text{B}}$. Namely, while we have modified the qubit observables in $\hat{U}_{\text{B}}$ from the ones in $\hat{U}_{\text{A}}^{\dagger}$ so that now they act on $\mathcal{H}_{\text{B}}$ rather than $\mathcal{H}_{\text{A}}$, the field observables $\hat{\phi}_{\text{A}}$ and $\hat{\pi}_{\text{A}}$ appearing in $\hat{U}_{\text{B}}$ are still defined at the time t_{A} (c.f. Eqs. (4.24) and (4.25)). But in order for Bob to implement $\hat{U}_{\text{B}}$ at a later time t_{B}, he needs to couple his qubit to field observables defined at the time t_{B}, not at t_{A}.

We will now solve this problem by proving a mathematical result which expresses the field observables $\hat{\phi}_{\rm A}$ and $\hat{\pi}_{\rm A}$ as observables at time $t_{\rm B}$. Fundamentally, this result arises due to the fact that, as discussed in Sec. 1.3, the field $\hat{\phi}(\boldsymbol{x},t)$ is by definition a solution to the wave equation, which, being a hyperbolic PDE, has a well defined initial value formulation that allows solutions at time $t_{\rm A}$ to be propagated to solutions at time $t_{\rm B}$. More concretely:

Theorem 2. *Let $\hat{\phi}(\boldsymbol{x},t)$ be a free field in any spacetime dimension with mode expansion given by Eq. (1.16). Let $\hat{\pi}(\boldsymbol{x},t)$ be the conjugate momentum field, and let $F(\boldsymbol{x})$ be any smearing function. Then*

$$\hat{\phi}[F](t_{\rm A}) = \hat{\phi}[F_2](t_{\rm B}) + \hat{\pi}[F_1](t_{\rm B}), \tag{4.37}$$

$$\hat{\pi}[F](t_{\rm A}) = \hat{\phi}[F_3](t_{\rm B}) + \hat{\pi}[F_2](t_{\rm B}), \tag{4.38}$$

where the $F_i(\boldsymbol{x})$ are related to $F(\boldsymbol{x})$ via their Fourier transforms as

$$\tilde{F}_1(\boldsymbol{k}) = \tilde{F}(\boldsymbol{k})\,\mathrm{sinc}(\Delta\omega_{\boldsymbol{k}})(-\Delta), \tag{4.39}$$

$$\tilde{F}_2(\boldsymbol{k}) = \tilde{F}(\boldsymbol{k})\cos(\Delta\omega_{\boldsymbol{k}}), \tag{4.40}$$

$$\tilde{F}_3(\boldsymbol{k}) = \tilde{F}(\boldsymbol{k})\sin(\Delta\omega_{\boldsymbol{k}})\omega_{\boldsymbol{k}}, \tag{4.41}$$

and where $\Delta := t_{\rm B} - t_{\rm A}$.

Proof. We will prove Eq. (4.37), while Eq. (4.38) is proven analogously. Starting from the mode expansion for $\hat{\phi}(\boldsymbol{x},t)$ given in Eq. (1.16) we get:

$$\begin{aligned}\hat{\phi}[F](t_{\rm A}) &= \int \mathrm{d}^d\boldsymbol{x}\, F(\boldsymbol{x}) \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2(2\pi)^d\omega_{\boldsymbol{k}}}} \left(\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}(\omega_{\boldsymbol{k}} t_{\rm A} - \boldsymbol{k}\cdot\boldsymbol{x})} + \mathrm{h.c.}\right) \\ &= \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\tilde{F}(\boldsymbol{k})\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t_{\rm A}} + \mathrm{h.c.}\right),\end{aligned} \tag{4.42}$$

with $\tilde{F}(\boldsymbol{k})$ the Fourier transform of $F(\boldsymbol{x})$ as defined in Eq. (2.32). Then, using $\Delta := t_{\rm B} - t_{\rm A}$ we obtain

$$\begin{aligned}\hat{\phi}[F](t_{\rm A}) &= \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\tilde{F}(\boldsymbol{k})\hat{a}_{\boldsymbol{k}} e^{\mathrm{i}\omega_{\boldsymbol{k}}\Delta} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t_{\rm B}} + \mathrm{h.c.}\right) \\ &= \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\tilde{F}(\boldsymbol{k}) \left[\cos(\omega_{\boldsymbol{k}}\Delta) + \mathrm{i}\sin(\omega_{\boldsymbol{k}}\Delta)\right] \hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t_{\rm B}} + \mathrm{h.c.}\right).\end{aligned} \tag{4.43}$$

By introducing the Fourier transforms $\tilde{F}_1(\boldsymbol{k})$ and $\tilde{F}_2(\boldsymbol{k})$ as defined in Eqs. (4.39) and (4.40), we can write this expression as

$$
\begin{aligned}
\hat{\phi}[F](t_{\text{A}}) &= \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\tilde{F}_2(\boldsymbol{k})\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t_{\text{B}}} + \text{h.c.}\right) + \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(-\mathrm{i}\omega_{\boldsymbol{k}}\tilde{F}_1(\boldsymbol{k})\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}\omega_{\boldsymbol{k}} t_{\text{B}}} + \text{h.c.}\right) \\
&= \int \mathrm{d}^d\boldsymbol{x}\, F_2(\boldsymbol{x}) \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2(2\pi)^d\omega_{\boldsymbol{k}}}} \left(\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}(\omega_{\boldsymbol{k}} t_{\text{B}} - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{h.c.}\right) \\
&\quad + \int \mathrm{d}^d\boldsymbol{x}\, F_1(\boldsymbol{x}) \int \frac{\mathrm{d}^d\boldsymbol{k}}{\sqrt{2(2\pi)^d\omega_{\boldsymbol{k}}}} \left(-\mathrm{i}\omega_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}(\omega_{\boldsymbol{k}} t_{\text{B}} - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{h.c.}\right) \\
&= \hat{\phi}[F_2](t_{\text{B}}) + \hat{\pi}[F_1](t_{\text{B}}),
\end{aligned} \tag{4.44}
$$

which proves Eq. (4.37). □

With this mathematical result at hand, we can now write the unitary $\hat{U}_{\text{B}}$ in Fig. 4.1 — which decodes Alice's qubit out of the field and onto Bob's detector — in terms of field observables at the time t_{B}. Namely, the theorem allows us to write the field observables $\hat{\phi}_{\text{A}}$ and $\hat{\pi}_{\text{A}}$ defined in Eqs. (4.24) and (4.25) as

$$
\begin{aligned}
\hat{\phi}_{\text{A}} &= \lambda_\phi \hat{\phi}[F_{\text{B}2}](t_{\text{B}}) + \lambda_\phi \hat{\pi}[F_{\text{B}1}](t_{\text{B}}), \\
\hat{\pi}_{\text{A}} &= \lambda_\pi \hat{\phi}[F_{\text{B}3}](t_{\text{B}}) + \lambda_\pi \hat{\pi}[F_{\text{B}2}](t_{\text{B}}),
\end{aligned} \tag{4.45}
$$

where Bob's smearing functions $F_{\text{B}i}(\boldsymbol{x})$ are defined in terms of Alice's smearing F_{A} through their Fourier transforms,

$$\tilde{F}_{\text{B}1}(\boldsymbol{k}) = \tilde{F}_{\text{A}}(\boldsymbol{k})\,\text{sinc}(\Delta\omega_{\boldsymbol{k}})(-\Delta), \tag{4.46}$$

$$\tilde{F}_{\text{B}2}(\boldsymbol{k}) = \tilde{F}_{\text{A}}(\boldsymbol{k})\cos(\Delta\omega_{\boldsymbol{k}}), \tag{4.47}$$

$$\tilde{F}_{\text{B}3}(\boldsymbol{k}) = \tilde{F}_{\text{A}}(\boldsymbol{k})\sin(\Delta\omega_{\boldsymbol{k}})\omega_{\boldsymbol{k}}. \tag{4.48}$$

Hence the unitary $\hat{U}_{\text{B}}$, defined by

$$\hat{U}_{\text{B}} = \exp\big(-\mathrm{i}\sigma_z\hat{\phi}_{\text{A}}\big)\exp\big(-\mathrm{i}\sigma_x\hat{\pi}_{\text{A}}\big), \tag{4.49}$$

can now be alternatively defined in terms of field observables at time t_{B}, namely

$$\hat{U}_{\text{B}} = \exp\left[-\mathrm{i}\lambda_\phi\sigma_z\left(\hat{\phi}[F_{\text{B}2}](t_{\text{B}}) + \hat{\pi}[F_{\text{B}1}](t_{\text{B}})\right)\right]\exp\left[-\mathrm{i}\lambda_\pi\sigma_x\left(\hat{\phi}[F_{\text{B}3}](t_{\text{B}}) + \hat{\pi}[F_{\text{B}2}](t_{\text{B}})\right)\right]. \tag{4.50}$$

In summary, we have succeeded in constructing the quantum channel shown in Fig. 4.1, which allows Alice to perfectly transmit a qubit through a quantum field to Bob. The quantum channel consists of two steps:

1. First, at time $t = t_{\text{A}}$, Alice encodes her qubit state in a spatial region of the field characterized by $F_{\text{A}}(\boldsymbol{x})$ by implementing the unitary $\hat{U}_{\text{A}}$ given in Eq. (4.23).

2. Then, at a later time $t = t_{\text{B}}$, Bob decodes the qubit from the field by coupling with the unitary $\hat{U}_{\text{B}}$ given in Eq. (4.50). In order for Bob to be able to implement this unitary, his detector must be smeared in a spatial region that contains the supports of the functions $F_{\text{B1}}(\boldsymbol{x})$, $F_{\text{B2}}(\boldsymbol{x})$, and $F_{\text{B3}}(\boldsymbol{x})$ defined by Eqs. (4.46)-(4.48).

Additionally, in order for the channel to succeed, the conditions (4.27) and (4.28) on the coupling strengths λ_ϕ and λ_π must be satisfied. Physically, Eq. (4.27) is a strong-coupling condition which ensures that Alice's qubit first gets maximally entangled with orthogonal coherent field states, while Eq. (4.28) is a fine-tuning condition which ensures that Alice's qubit is then rotated by the right amount in the Bloch sphere so that it gets completely unentangled from the field. Together, these conditions ensure that the encoding gate (and hence the decoding gate, which is just the inverse encoding gate) are implemented successfully. In particular we note that, as was discussed above, a strong (i.e. non-perturbative) coupling of detectors to the field is necessary in order for the field-mediated quantum channel from Alice to Bob to have maximal quantum channel capacity.

Despite our successes so far however, there still remain two pertinent issues that must be addressed before one can be fully satisfied with our construction of a perfect, field-mediated quantum channel from Alice to Bob. First, it should be verified, without the use of any approximations (such as the one in Eq. (4.35)), that our supposedly perfect quantum channel Ξ indeed has a maximal quantum channel capacity of $\mathcal{Q}(\Xi) = 1$. And second, the smearing function $F_{\text{B}i}(\boldsymbol{x})$ are defined in terms of their Fourier transforms, and hence it is presently not clear where in space Bob needs to be located in order to receive Alice's quantum message. We will successively address these two remaining issues in the following two sections, Sec. 4.3 and Sec. 4.4.

4.3 Numerical test of the perfect quantum channel

Let us verify that the channel Ξ which we constructed in the previous section — shown in Fig. 4.1 with $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ given by Eqs. (4.23) and (4.50) — can indeed perfectly transmit quantum information from Alice to Bob. For convenience we will assume that Bob's initial state is $|+_y\rangle$, and that Alice's initial state (i.e. the input to the channel), is the maximally mixed state, $\hat{\rho}_{\text{A},0} = \frac{1}{2}\mathbb{1}$. As discussed in Sec. 4.1, we will compute a lower bound on the quantum channel capacity $Q(\Xi)$ by computing the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ of the channel Ξ and the input state $\hat{\rho}_{\text{A},0}$.

Recall that to compute $I_c(\hat{\rho}_{\text{A},0}, \Xi)$, we must first purify the input to the channel, i.e. the maximally mixed state $\hat{\rho}_{\text{A},0}$. To that end, we suppose that the initial state of Alice is entangled with some third qubit C, and that the joint state of C and Alice is given by the maximally entangled pure state $|\psi_{\text{CA}}\rangle = \frac{1}{\sqrt{2}}(|-_z\rangle|+_z\rangle + |+_z\rangle|-_z\rangle)_{\text{CA}}$. Indeed, we can verify that this is a purification by noting that if we trace out system C the resulting state of Alice is $\hat{\rho}_{\text{A},0}$.

Next, in order to compute $I_c(\hat{\rho}_{\text{A},0}, \Xi)$, we must compute the state

$$\hat{\rho}_{\text{CB}} := (\mathbb{1}_{\text{C}} \otimes \Xi)\left(|\psi_{\text{CA}}\rangle\langle\psi_{\text{CA}}|\right), \tag{4.51}$$

on $\mathcal{H}_{\text{C}} \otimes \mathcal{H}_{\text{B}}$. Following this we can easily determine the coherent information through Eq. (4.7), i.e. as $I_c(\hat{\rho}_{\text{A},0}, \Xi) := S(\hat{\rho}_{\text{B}}) - S(\hat{\rho}_{\text{CB}})$. To that end, writing Ξ in terms of the unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$, we obtain

$$\hat{\rho}_{\text{CB}} = \text{Tr}_{\text{A}\phi}\left[\hat{U}_{\text{B}}^\dagger \hat{U}_{\text{A}}^\dagger \left(|\psi_{\text{CA}}\rangle\langle\psi_{\text{CA}}| \otimes |0\rangle\langle 0| \otimes |+_y\rangle\langle +_y|\right) \hat{U}_{\text{A}}^\dagger \hat{U}_{\text{B}}^\dagger\right]. \tag{4.52}$$

The simplest way to proceed to compute this density matrix is to decompose the unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ into products of controlled unitaries from the qubits A and B onto the field. Namely, for $\hat{U}_{\text{A}}$ we write

$$\begin{aligned}\hat{U}_{\text{A}} &= \exp\left(\text{i}\sigma_x \hat{\pi}_{\text{A}}\right) \exp\left(\text{i}\sigma_z \hat{\phi}_{\text{A}}\right) \\ &= \sum_{x,z\in\{\pm\}} \hat{P}_x \hat{P}_z \otimes e^{\text{i}x\hat{\pi}_{\text{A}}} e^{\text{i}z\hat{\phi}_{\text{A}}},\end{aligned} \tag{4.53}$$

where $\hat{P}_x$ and $\hat{P}_z$ are the projectors onto the eigenstates of σ_x and σ_z (note that to simplify notation we are using the dummy summation index x or z on the $\hat{P}$ to denote what operator the projector is associated with). Written in this form we see that the action of $\hat{U}_{\text{A}}$ is to unitarily evolve the field state with a unitary that is dependent on the outcome of a σ_z measurement of the qubit A, and then to do the same thing for a σ_x measurement. In other words $\hat{U}_{\text{A}}$ is a product of two controlled unitaries, from A to the field.

We can perform the same kind of decomposition for the unitary $\hat{U}_{\text{B}}$ by starting with the expression Eq. (4.50). However, it is more convenient to write $\hat{U}_{\text{B}}$ in the way it was initially defined, i.e. as $\hat{U}_{\text{A}}^{-1} = \hat{U}_{\text{A}}^\dagger$ with the understanding that the qubit observables are now observables on $\mathcal{H}_{\text{B}}$ rather than $\mathcal{H}_{\text{A}}$. Hence, from Eq. (4.53) we directly obtain

$$\hat{U}_{\text{B}} = \sum_{x,z\in\{\pm\}} \hat{P}_z \hat{P}_x \otimes e^{-\text{i}z\hat{\phi}_{\text{A}}} e^{-\text{i}x\hat{\pi}_{\text{A}}}, \tag{4.54}$$

where the projectors $\hat{P}_x$ and $\hat{P}_z$ are associated with the Pauli operators σ_x and σ_z on $\mathcal{H}_{\mathrm{B}}$.

Substituting Eqs. (4.53) and (4.54) for $\hat{U}_{\mathrm{A}}$ and $\hat{U}_{\mathrm{B}}$ into Eq. (4.52) for $\hat{\rho}_{\mathrm{CB}}$, and writing $|\psi_{\mathrm{CA}}\rangle = \frac{1}{\sqrt{2}}\sum_j |{-j_z}\rangle|j_z\rangle$ with $j \in \{\pm\}$, we get

$$\hat{\rho}_{\mathrm{CB}} = \frac{1}{2}\sum_{j,k,x_i,z_i} \langle 0|e^{-\mathrm{i}z_1\hat{\phi}_{\mathrm{A}}}e^{-\mathrm{i}x_1\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}x_2\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}z_2\hat{\phi}_{\mathrm{A}}}e^{-\mathrm{i}z_3\hat{\phi}_{\mathrm{A}}}e^{-\mathrm{i}x_3\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}x_4\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}z_4\hat{\phi}_{\mathrm{A}}}|0\rangle \\ \times \langle k_z|\hat{P}_{z_1}\hat{P}_{x_1}\hat{P}_{x_4}\hat{P}_{z_4}|j_z\rangle_{\mathrm{A}}|{-j_z}\rangle_{\mathrm{C}}\langle -k_z| \otimes \hat{P}_{z_3}\hat{P}_{x_3}|+_y\rangle_{\mathrm{B}}\langle +_y|\hat{P}_{x_2}\hat{P}_{z_2}, \tag{4.55}$$

where x_i stands for x_1, x_2, x_3, x_4, and similarly for z_i, and where all of the summation variables run over the set $\{+1, -1\}$, such that there are 2^{10} terms in the entire sum. This expression can straightforwardly be evaluated by a computer as long as we can first simplify the field expectation value $\langle 0|\dots|0\rangle$. In order to do so, let us first redefine the summation indices by $x_1 \mapsto -x_1, z_1 \mapsto -z_1, x_3 \mapsto -x_3$ and $z_3 \mapsto -z_3$, such that the expression for $\hat{\rho}_{\mathrm{CB}}$ reads

$$\hat{\rho}_{\mathrm{CB}} = \frac{1}{2}\sum_{j,k,x_i,z_i} \langle 0|e^{\mathrm{i}z_1\hat{\phi}_{\mathrm{A}}}e^{\mathrm{i}x_1\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}x_2\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}z_2\hat{\phi}_{\mathrm{A}}}e^{\mathrm{i}z_3\hat{\phi}_{\mathrm{A}}}e^{\mathrm{i}x_3\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}x_4\hat{\pi}_{\mathrm{A}}}e^{\mathrm{i}z_4\hat{\phi}_{\mathrm{A}}}|0\rangle \\ \times \langle k_z|\hat{P}_{-z_1}\hat{P}_{-x_1}\hat{P}_{x_4}\hat{P}_{z_4}|j_z\rangle_{\mathrm{A}}|{-j_z}\rangle_{\mathrm{C}}\langle -k_z| \otimes \hat{P}_{-z_3}\hat{P}_{-x_3}|+_y\rangle_{\mathrm{B}}\langle +_y|\hat{P}_{x_2}\hat{P}_{z_2}. \tag{4.56}$$

Using the Baker-Campbell-Hausdorff formula we can write [108]

$$e^{\mathrm{i}z_i\hat{\phi}_{\mathrm{A}}}e^{\mathrm{i}x_i\hat{\pi}_{\mathrm{A}}} = e^{x_i z_i C}e^{\mathrm{i}\hat{O}_i}, \tag{4.57}$$

where $\hat{O}_i := x_i\hat{\pi}_{\mathrm{A}} + z_i\hat{\phi}_{\mathrm{A}}$, and C is defined by

$$\begin{aligned} C &:= -\frac{1}{2}\langle[\hat{\phi}_{\mathrm{A}}, \hat{\pi}_{\mathrm{A}}]\rangle \\ &= -\frac{\mathrm{i}\lambda_\phi\lambda_\pi}{2}\int \mathrm{d}^d\boldsymbol{k}\,|\tilde{F}_{\mathrm{A}}(\boldsymbol{k})|^2. \end{aligned} \tag{4.58}$$

Here the expectation value in the first line can be taken with respect to any field state, since the field commutator is a c-number. Then, $\hat{\rho}_{\mathrm{CB}}$ can be written as

$$\hat{\rho}_{\mathrm{CB}} = \frac{1}{2}\sum_{j,k,x_i,z_i} e^{x_1z_1C}e^{-x_2z_2C}e^{x_3z_3C}e^{-x_4z_4C}\langle 0|e^{\mathrm{i}\hat{O}_1}e^{\mathrm{i}\hat{O}_2}e^{\mathrm{i}\hat{O}_3}e^{\mathrm{i}\hat{O}_4}|0\rangle \\ \times \langle k_z|\hat{P}_{-z_1}\hat{P}_{-x_1}\hat{P}_{x_4}\hat{P}_{z_4}|j_z\rangle_{\mathrm{A}}|{-j_z}\rangle_{\mathrm{C}}\langle -k_z| \otimes \hat{P}_{-z_3}\hat{P}_{-x_3}|+_y\rangle_{\mathrm{B}}\langle +_y|\hat{P}_{x_2}\hat{P}_{z_2}. \tag{4.59}$$

Next, let us make use of the identity

$$\langle 0|\prod_{l=1}^{n} e^{\mathrm{i}\hat{O}_j}|0\rangle = \prod_{l<m}^{n} e^{-W_{lm}} \prod_{l=1}^{n} e^{-\frac{1}{2}W_{ll}}, \tag{4.60}$$

where $W_{lm} := \langle 0|\hat{O}_l\hat{O}_m|0\rangle$. This identity holds for any operators $\hat{O}_j$ which are linear in the field creation and annihilation operators, and it can straightforwardly be proven using Wick's theorem [109]. Then $\hat{\rho}_{\mathrm{CB}}$ becomes

$$\begin{aligned}\hat{\rho}_{\mathrm{CB}} = \frac{1}{2}\sum_{j,k,x_i,z_i} & e^{x_1 z_1 C_1} e^{-x_2 z_2 C_2} e^{x_3 z_3 C_3} e^{-x_4 z_4 C_4} \prod_{l<m}^{4} e^{-W_{lm}} \prod_{l=1}^{4} e^{-\frac{1}{2}W_{ll}} \\ & \times \langle k_z|\hat{P}_{-z_1}\hat{P}_{-x_1}\hat{P}_{x_4}\hat{P}_{z_4}|j_z\rangle_{\mathrm{A}}|-j_z\rangle_{\mathrm{C}}\langle -k_z| \otimes \hat{P}_{-z_3}\hat{P}_{-x_3}|+_y\rangle_{\mathrm{B}}\langle +_y|\hat{P}_{x_2}\hat{P}_{z_2},\end{aligned} \tag{4.61}$$

and W_{lm} evaluates to

$$W_{lm} = \int \frac{\mathrm{d}^d\boldsymbol{k}}{2|\boldsymbol{k}|}|\tilde{F}_{\mathrm{A}}(\boldsymbol{k})|^2 \left(z_l\lambda_\phi - \mathrm{i}|\boldsymbol{k}|x_l\lambda_\pi\right)\left(z_m\lambda_\phi - \mathrm{i}|\boldsymbol{k}|x_m\lambda_\pi\right). \tag{4.62}$$

Hence we now see that if we specify the coupling constants λ_ϕ and λ_π, as well as the smearing function $\hat{F}_{\mathrm{A}}(\boldsymbol{x})$, we can straightforwardly compute C and W_{lm} (at least numerically), and hence obtain a (numerical) result for the density matrix $\hat{\rho}_{\mathrm{CB}}$.

4.3.1 Gaussian detector smearing

In order to numerically compute the coherent information of our quantum channel, let us now particularize our discussion to (3+1)-dimensions, and let us set the smearing function of Alice's detector to be a Gaussian of width σ, i.e.

$$F_{\mathrm{A}}(\boldsymbol{x}) = \frac{1}{(\sqrt{\pi}\sigma)^3}\exp\left(-\frac{|\boldsymbol{x}|^2}{\sigma^2}\right), \tag{4.63}$$

which has a Fourier transform that is given by

$$\tilde{F}_{\mathrm{A}}(\boldsymbol{k}) = \frac{1}{\sqrt{(2\pi)^3}}\exp\left(-\frac{1}{4}|\boldsymbol{k}|^2\sigma^2\right). \tag{4.64}$$

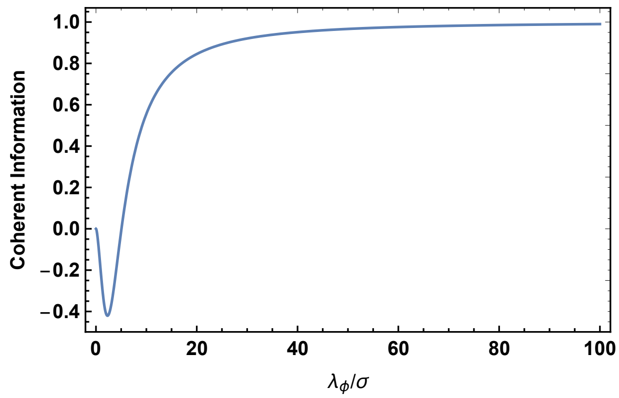

Figure 4.4: Plot of the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ of the maximally mixed state $\hat{\rho}_{\text{A},0} = \frac{1}{2}\mathbb{1}$ and the channel Ξ versus the ratio of the coupling strength λ_ϕ to the size of Alice's detector, σ. Notice that for $\lambda_\phi \gg \sigma$, the coherent information approaches its maximum possible value of 1, thus confirming that in this limit Ξ is a perfect quantum channel.

Then, the conditions (4.27) and (4.28) on the coupling strengths λ_ϕ and λ_π, which we require in order to have a perfect quantum channel, simplify to

$$\lambda_\phi \gg \sigma, \text{ and} \tag{4.65}$$

$$\frac{\lambda_\phi \lambda_\pi}{\sqrt{(2\pi)^3}\sigma^3} \equiv \frac{\pi}{4} \mod 2\pi. \tag{4.66}$$

In particular, recalling that the strong-coupling condition (4.27) is a requirement in order for Alice's qubit to become maximally entangled with the field, we find that this is only possible if the coupling strength λ_ϕ of the detector is much larger than its size. Finally, using Eqs. (4.58) and (4.62), we can also readily compute C and W_{lm}, obtaining

$$C = \frac{-\mathrm{i}\lambda_\phi \lambda_\pi}{2\sqrt{(2\pi)^3}\sigma^3}, \tag{4.67}$$

$$W_{lm} = \frac{4x_l x_m \lambda_\pi^2 + 2z_l z_m \sigma^2 \lambda_\phi^2 + \mathrm{i}\sqrt{2\pi}\sigma\lambda_\phi\lambda_\pi \left(x_m z_l - x_l z_m\right)}{8\pi^2\sigma^4}. \tag{4.68}$$

We now have all of the necessary components to compute $\hat{\rho}_{\text{CB}}$ via Eq. (4.61), and hence to compute the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ of the channel Ξ and the input state $\hat{\rho}_{\text{A},0}$.

In Fig. 4.4 we plot $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ versus λ_ϕ/σ, and we find, as expected that for $\lambda_\phi/\sigma \to \infty$, the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ approaches its maximum value of 1. Additionally, since we know that the quantum channel capacity $Q(\Xi)$ is lower bounded by $I_c(\hat{\rho}_{\text{A},0}, \Xi)$, and since we also know that $Q(\Xi) \leq 1$ (i.e. a single use of the channel can transmit at most one qubit), we thus conclude that in the limit $\lambda_\phi/\sigma \to \infty$, the quantum channel capacity $Q(\Xi)$ approaches its maximum value of 1. In other words, we have numerically verified, without the use of any approximations, that the field-mediated quantum channel from Alice to Bob is indeed a perfect quantum channel if the conditions (4.27) and (4.28) are satisfied.

4.4 Where does the quantum information propagate?

While we have mathematically verified that the quantum channel Ξ from Alice to Bob is a perfect quantum channel, we still have some work to do in order to understand the physics of quantum information propagation through a relativistic quantum field. In particular, we have yet to discuss where in space Bob needs to be located at time t_B in order to receive Alice's message, which she encoded in the field at an earlier time t_A. Let us now attempt to better understand this issue.

Recall from Theorem 2 that if Alice couples to the field at time t_A with a spatial smearing $F_\text{A}(\boldsymbol{x})$, then in order for Bob to perfectly recover Alice's message at a time t_B he needs to be able to couple his detector to the field $\hat{\phi}$ and the conjugate field $\hat{\pi}$ with three different smearing functions $F_{\text{B}i}(\boldsymbol{x})$, which are related to $F_\text{A}(\boldsymbol{x})$ via their Fourier transforms,

$$\tilde{F}_{\text{B}1}(\boldsymbol{k}) = \tilde{F}_\text{A}(\boldsymbol{k})\,\text{sinc}(\Delta\omega_{\boldsymbol{k}})(-\Delta), \tag{4.69}$$

$$\tilde{F}_{\text{B}2}(\boldsymbol{k}) = \tilde{F}_\text{A}(\boldsymbol{k})\cos(\Delta\omega_{\boldsymbol{k}}), \tag{4.70}$$

$$\tilde{F}_{\text{B}3}(\boldsymbol{k}) = \tilde{F}_\text{A}(\boldsymbol{k})\sin(\Delta\omega_{\boldsymbol{k}})\omega_{\boldsymbol{k}}, \tag{4.71}$$

where $\Delta := t_\text{B} - t_\text{A}$. Also recall that this result is valid in a flat spacetime of any dimension, and for any field mass. However, because the inverse Fourier transform is different in different spacetime dimensions, we expect the coordinate space functions $F_{\text{B}i}(\boldsymbol{x})$ to have significantly different forms in different spacetimes. To see that this is indeed the case, let us now consider the $(3+1)$ and $(2+1)$ dimensional cases, both with a massless field. We will find that quantum information propagates very differently through the relativistic field in these two spacetimes.

4.4.1 (3+1)-dimensions

In $(3+1)$-dimensions we are fortunate enough that we can obtain very simple and intuitive expressions for Bob's smearing functions $F_{\text{B}i}(\boldsymbol{x})$, which are related to Alice's smearing function $F_{\rm A}(\boldsymbol{x})$ via their Fourier transforms via Eqs. (4.69)-(4.71).

To obtain these expressions for $F_{{\rm B}i}(\boldsymbol{x})$, let us first recall the d-dimensional convolution theorem, which states that for two functions $f, g \in L^1(\mathbb{R}^d)$,

$$\mathcal{F}^{-1}\left[\mathcal{F}[f]\mathcal{F}[g]\right](\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d}}\,(f*g)\,(\boldsymbol{x}) \\ := \frac{1}{\sqrt{(2\pi)^d}}\int \mathrm{d}^d\boldsymbol{x}'\, f(\boldsymbol{x}')g(\boldsymbol{x}-\boldsymbol{x}'), \tag{4.72}$$

where $\mathcal{F}$ denotes the d-dimensional Fourier transform defined by Eq. (2.32), $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and $(f*g)(\boldsymbol{x})$ is the convolution product between $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, which is defined in the second line.

Applying the convolution theorem to Eqs. (4.69)-(4.71), we obtain

$$F_{\rm B1}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d}}\int \mathrm{d}^d\boldsymbol{x}'\, F_{\rm A}(\boldsymbol{x}')\mathcal{F}^{-1}[-\Delta\,\mathrm{sinc}(\Delta|\boldsymbol{k}|)](\boldsymbol{x}-\boldsymbol{x}'), \tag{4.73}$$

$$F_{\rm B2}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d}}\int \mathrm{d}^d\boldsymbol{x}'\, F_{\rm A}(\boldsymbol{x}')\mathcal{F}^{-1}[\cos(\Delta|\boldsymbol{k}|)](\boldsymbol{x}-\boldsymbol{x}'), \tag{4.74}$$

$$F_{\rm B3}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d}}\int \mathrm{d}^d\boldsymbol{x}'\, F_{\rm A}(\boldsymbol{x}')\mathcal{F}^{-1}[|\boldsymbol{k}|\sin(\Delta|\boldsymbol{k}|)](\boldsymbol{x}-\boldsymbol{x}'), \tag{4.75}$$

where we note that $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ since we are setting the field mass m equal to zero. Note that the above expressions are valid for any spatial dimension d of the flat spacetime. In order to proceed to calculate $F_{{\rm B}i}(\boldsymbol{x})$ via Eqs. (4.73)-(4.75), we must compute the inverse Fourier transforms of the functions $-\Delta\,\mathrm{sinc}(\Delta|\boldsymbol{k}|), \cos(\Delta|\boldsymbol{k}|)$, and $|\boldsymbol{k}|\sin(\Delta|\boldsymbol{k}|)$.

Let us now particularize to $d = 3$, in which case obtaining explicit (distributional) expressions for these inverse Fourier transforms is possible. Namely we find

$$\mathcal{F}^{-1}_{d=3}[-\Delta\,\mathrm{sinc}(\Delta|\boldsymbol{k}|)](\boldsymbol{x}) = -\sqrt{(2\pi)^3}\frac{\delta(r-\Delta)}{4\pi r}, \tag{4.76}$$

$$\mathcal{F}^{-1}_{d=3}[\cos(\Delta|\boldsymbol{k}|)](\boldsymbol{x}) = -\sqrt{(2\pi)^3}\frac{\delta'(r-\Delta)}{4\pi r}, \tag{4.77}$$

$$\mathcal{F}^{-1}_{d=3}[|\boldsymbol{k}|\sin(\Delta|\boldsymbol{k}|)](\boldsymbol{x}) = -\sqrt{(2\pi)^3}\frac{\delta''(r-\Delta)}{4\pi r}, \tag{4.78}$$

with $r := |\boldsymbol{x}|$ and where we are explicitly indicating that these are 3-dimensional inverse Fourier transforms. Here, $\delta'(\boldsymbol{x})$ and $\delta''(\boldsymbol{x})$ denote the first and second derivatives of the delta function. It is easiest to verify these results by taking the Fourier transform of the right-hand sides and checking that we get the expected answer. For example, let us verify Eq. (4.77) in this way. We find

$$\begin{aligned}\mathcal{F}_{d=3}\left[-\sqrt{(2\pi)^3}\frac{\delta'(r-\Delta)}{4\pi r}\right](\boldsymbol{k}) &= -\int \mathrm{d}^3\boldsymbol{x}\,\frac{\delta'(r-\Delta)}{4\pi r}e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -4\pi\int_0^\infty \mathrm{d}r\, r^2\frac{\delta'(r-\Delta)}{4\pi r}\frac{\sin(r|\boldsymbol{k}|)}{r|\boldsymbol{k}|}, \\ &= -\frac{1}{|\boldsymbol{k}|}\int_0^\infty \mathrm{d}r\,\delta'(r-\Delta)\sin(r|\boldsymbol{k}|),\end{aligned} \tag{4.79}$$

where in the first line we used the definition (2.32) of the Fourier transform, and in the second line we performed the angular integrals. Finally, by performing an integration by parts we can transfer the derivative from the Dirac delta function onto the $\sin(r|\boldsymbol{k}|)$. Then, as long as $0 < r < \infty$, we can discard the boundary terms since $\delta(r-\Delta)$ vanishes at the boundaries $r = 0$ and $r = \infty$. Hence we obtain

$$\begin{aligned}\mathcal{F}_{d=3}\left[-\sqrt{(2\pi)^3}\frac{\delta'(r-\Delta)}{4\pi r}\right](\boldsymbol{k}) &= \frac{1}{|\boldsymbol{k}|}\int_0^\infty \mathrm{d}r\,\delta(r-\Delta)\cos(r|\boldsymbol{k}|)|\boldsymbol{k}|, \\ &= \cos(\Delta|\boldsymbol{k}|),\end{aligned} \tag{4.80}$$

which proves Eq. (4.77). Eqs. (4.76) and (4.78) can be proven analogously.

Substituting Eqs. (4.76)-(4.78) into Eqs. (4.73)-(4.75), we find that in (3+1)-dimensions Bob's smearing functions $F_{\mathrm{B}i}(\boldsymbol{x})$ are given in terms of Alice's smearing $F_{\mathrm{A}}(\boldsymbol{x})$ as

$$F_{\mathrm{B}1}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\mathrm{A}}(\boldsymbol{x}')\frac{\delta(|\boldsymbol{x}-\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{4.81}$$

$$F_{\mathrm{B}2}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\mathrm{A}}(\boldsymbol{x}')\frac{\delta'(|\boldsymbol{x}-\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{4.82}$$

$$F_{\mathrm{B}3}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\mathrm{A}}(\boldsymbol{x}')\frac{\delta''(|\boldsymbol{x}-\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{4.83}$$

Hence, since the δ, δ' and δ'' above only have support if $|\boldsymbol{x}-\boldsymbol{x}'| = \Delta$, we find that in $(3+1)$-dimensions Bob's smearing functions $F_{\mathrm{B}i}(\boldsymbol{x})$ on the time-slice $t = t_{\mathrm{B}} = t_{\mathrm{A}} + \Delta$ only have support if they are in lightlike separation from Alice's smearing $F_{\mathrm{A}}(\boldsymbol{x})$ on the time-slice $t = t_{\mathrm{A}}$. Therefore, in order for Bob to fully receive Alice's quantum message through our field-mediated quantum channel in $(3+1)$-dimensions, he needs to be able to couple his detector on the entirety of Alice's lightcone. In other words, quantum information in $(3+1)$-dimensions propagates through a massless field precisely at the speed of light.

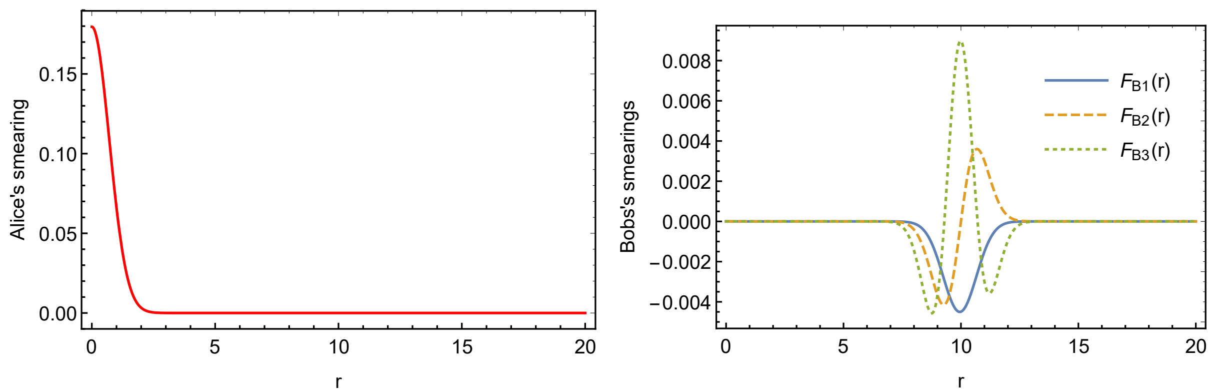

Figure 4.5: (3 + 1)-dimensions. **Left**: Location at which Alice couples to the massless field at time t_A, given by Eq. (4.84), with $\sigma = 1$ and where $r = |\boldsymbol{x}|$ is measured in units of σ. **Right**: Bob's smearing functions, which dictate where in space Bob needs to couple to the field at time $t_B = t_A + \Delta$ in order to receive Alice's message (we set $\Delta = 10$).

To conclude this section, let us illustrate the above result by considering a particular smearing $F_A(\boldsymbol{x})$ for Alice, namely the Gaussian function considered in Sec. 4.3.1,

$$F_A(\boldsymbol{x}) = \frac{1}{(\sqrt{\pi}\sigma)^3} \exp\left(-\frac{|\boldsymbol{x}|^2}{\sigma^2}\right). \tag{4.84}$$

We plot $F_A(\boldsymbol{x})$ and the resulting smearing functions $F_{Bi}(\boldsymbol{x})$ for Bob's detector in Fig. 4.5. Notice that, as expected, at time $t_B = t_A + \Delta$ Bob needs to couple to the field only near $|\boldsymbol{x}| = \Delta$ (i.e. on Alice's lightcone) in order to be able to fully recover her quantum message.

The main result of this section — i.e. that Bob needs to be lightlike separated from Alice in (3+1)-dimensions in order to receive her quantum message — is fundamentally related to the strong Huygens principle, which we recall from Sec. 1.2 holds in (3+1)-dimensional flat spacetime. Namely, recall that the strong Huygens principle states that the massless field commutator (and hence the radiation Green's function) only has support between lightlike separated events [60], and hence communication between observers via this quantum field is only possible if they are in null separation. While this has been previously studied in great detail for *classical* communication protocols (see, e.g., Refs. [53–56]), the work presented here is the first time that, to our knowledge, the effects of the strong Huygens principle have been studied in the context of *quantum* communication.

4.4.2 (2+1)-dimensions

Let us now attempt to repeat the analysis of the previous section, but this time in $(2+1)$-dimensional Minkowski space. We expect to find significant differences to the $(3+1)$D case, due to the violations of the strong Huygens principle that occur in the former but not the latter spacetime.

Recall that the key expressions directly relating Bob's smearing functions $F_{\mathrm{B}i}(\boldsymbol{x})$ in terms of Alice's smearing function $F_{\mathrm{A}}(\boldsymbol{x})$ are Eqs. (4.73)-(4.75). As in the (3+1)-dimensional case, in order to gain insight into the propagation of quantum information from these equations, we must first compute the Fourier transforms of the functions $-\Delta\,\mathrm{sinc}(\Delta|\boldsymbol{k}|)$, $\cos(\Delta|\boldsymbol{k}|)$, and $|\boldsymbol{k}|\sin(\Delta|\boldsymbol{k}|)$. Unfortunately however, we are only aware of a closed form expression for the first of these Fourier transforms, which reads

$$\mathcal{F}^{-1}_{d=2}[-\Delta\,\mathrm{sinc}(\Delta|\boldsymbol{k}|)](\boldsymbol{x}) = \begin{cases} \frac{1}{\Delta\sqrt{\Delta^2-r^2}} & r<\Delta, \\ 0 & r\geq\Delta, \end{cases} \tag{4.85}$$

where once again $r := |\boldsymbol{x}|$. Nevertheless, from this equation alone we can see an interesting feature of the propagation of quantum information in $(2+1)$-dimensions. Namely, unlike the 3D Fourier transforms given by Eqs. (4.76)-(4.78), which only had support for $r=\Delta$, the 2D Fourier transform of $\mathrm{sinc}(\Delta|\boldsymbol{k}|)$ has support inside the light cone, i.e. for $r<\Delta$. Hence, after inserting this Fourier transfrom into Eq. (4.73), we find that in $(2+1)$D the first of Bob's smearing functions, $F_{\mathrm{B}1}(\boldsymbol{x})$, is given by

$$F_{\mathrm{B}1}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{B_\Delta(\boldsymbol{x})} \mathrm{d}^2\boldsymbol{x}'\, \frac{F_{\mathrm{A}}(\boldsymbol{x}')}{\Delta\sqrt{\Delta^2-|\boldsymbol{x}-\boldsymbol{x}'|^2}}, \tag{4.86}$$

where $B_\Delta(\boldsymbol{x})$ is the ball of radius Δ centered at $\boldsymbol{x}$. Thus we see that $F_{\mathrm{B}1}(\boldsymbol{x})$ has support even if $|\boldsymbol{x}-\boldsymbol{x}'|<\Delta$, and hence we conclude that if Bob wants to receive quantum information from Alice in $(2+1)$-dimensions, then he needs to have access not only to Alice's lightcone, but also to the interior of the lightcone. In other words, quantum information in $(2+1)$-dimensions propagates slower than light via a massless field. This is in agreement with the violations of the strong Huygens principle that occur in $(2+1)$D Minkowski spacetime.

While we have come to this conclusion just by focusing on the smearing function $F_{\mathrm{B}1}(\boldsymbol{x})$ — since it is the only one out of the $F_{\mathrm{B}i}(\boldsymbol{x})$ for which we could obtain an integral expression of the form (4.86) with a closed-form integrand — let us, for the sake of completeness, now verify numerically that the smearing functions $F_{\mathrm{B}2}(\boldsymbol{x})$ and $F_{\mathrm{B}3}(\boldsymbol{x})$ also have support inside of the light cone, i.e. for $r<\Delta$.

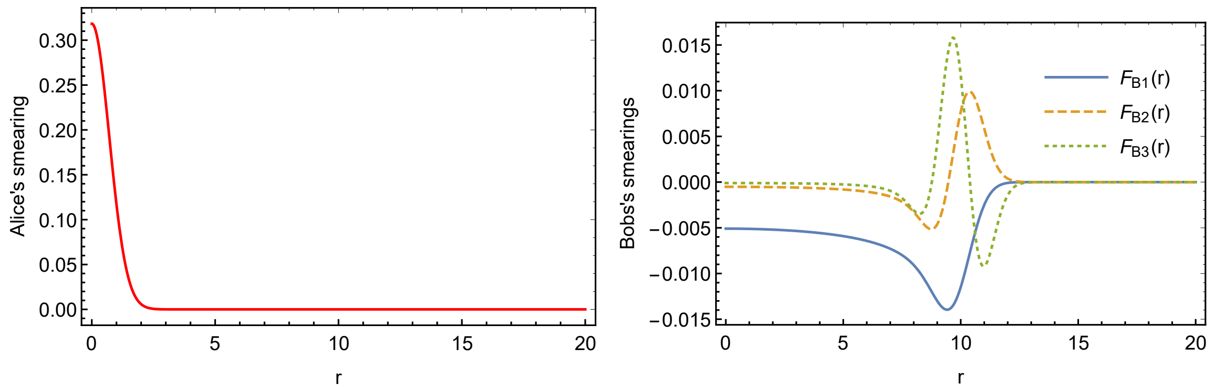

Figure 4.6: $(2+1)$-dimensions. **Left**: Location at which Alice couples to the massless field at time t_{A}, given by Eq. (4.87), with $\sigma = 1$ and where $r = |\boldsymbol{x}|$ is measured in units of σ. **Right**: Bob's smearing functions, which dictate where in space Bob needs to couple to the field at time $t_{\text{B}} = t_{\text{A}} + \Delta$ in order to receive Alice's message (we set $\Delta = 10$). Note that all three of Bob's smearing functions have support inside of the light cone, i.e. they are only polynomially, rather than exponentially, suppressed for $r \ll \Delta$.

To that end, analogous to the 3D case, let us suppose that Alice's smearing function $F_{\text{A}}(\boldsymbol{x})$ is given by the Gaussian

$$F_{\text{A}}(\boldsymbol{x}) = \frac{1}{(\sqrt{\pi}\sigma)^2} \exp\left(-\frac{|\boldsymbol{x}|^2}{\sigma^2}\right). \tag{4.87}$$

Then, in Fig. 4.6 we indeed find that all three that Bob's smearing functions $F_{\text{B}i}(\boldsymbol{x})$ have support for $r < \Delta$, and hence, as already stated above, we conclude that in order to recover Alice's quantum message in $(2+1)$-dimensional flat spacetime, Bob must couple to the massless field inside of Alice's future light cone.

4.5 Broadcasting quantum information

In the previous section we have obtained a better understanding of how quantum information propagates through a quantum field by answering the question: Where in space does Bob need to be located if he wants to receive the quantum message that Alice broadcast through the quantum field? Indeed, we found that the answer depends on the spacetime in which Alice and Bob are located. For instance, in $(3+1)$D Minkowski spacetime, Bob needs to be smeared across Alice's entire light cone, while in $(2+1)$D flat spacetime he also needs to cover the interior of the light cone. In particular, note that in both spacetimes Alice's message is broadcast isotropically in all spatial directions, which is, of course, simply a consequence of the fact that Alice's coupling to the field was fully isotropic.

Let us now consider the relevant case of $(3+1)$D Minkowski spacetime. Then, although we are fortunate that in $(3+1)$D Bob does not need to cover the interior of Alice's light cone in order to receive her full signal, from a practical perspective it is still very restrictive to require that Bob covers the entire lightcone itself (i.e. without the interior), as we found is required in order for him to receive the entirety of Alice's quantum message.

A natural question then arises: Is Alice able to transmit quantum information to Bob if he only covers a part of her light cone? This question is relevant, for instance, if Alice wants to broadcast her information to multiple disjoint receivers, each located in a different spatial direction relative to Alice.[5] In fact, this question was partially answered in Ref. [73], where the authors showed that, in a flat spacetime of any dimension, it is not possible for Alice to send any amount of quantum information to multiple *identical* Bobs.[6] In this section we will attempt to circumvent this result by considering *non-identical* Bobs.

More concretely, let us consider the setup shown in Fig. 4.7, in which two Bobs are trying to recover the message which Alice broadcast into the field. Both Bobs are spherically symmetric, with Bob B_1 covering the region of space given by $r < r_0$, and Bob B_2 covering the region $r > r_0$. We consider this setup both for its computational simplicity (owing to the fact that spherical symmetry is preserved), as well as the fact that the Bobs in this setup are not identical, thus allowing us to potentially overcome the limitations imposed upon identical Bobs [73], as discussed above. Despite the simplicity of the setup however, it will nevertheless provide us with interesting insights into the broadcasting of quantum information through a relativistic quantum field.

To proceed, let us start by setting the initial state for both Bobs B_1 and B_2 to be $|+_y\rangle$, and Alice's initial state to be the maximally mixed state, $\hat{\rho}_{\text{A},0} = \frac{1}{2}\mathbb{1}$. We set the smearing

[5] If Alice instead wanted to send her quantum message to a single Bob localized in some specified solid angle $\Omega < 4\pi$ relative to her, it would be much more prudent for her to change the way in which she couples to the field, so that it is not isotropic, but rather so that she only couples to those field modes with wavevectors pointing in Bob's direction. In this way the quantum information that Alice encodes in the field would only travel towards Bob and not in all directions, and Bob would be able to recover the full quantum message, rather than only a fraction. We leave the study of such non-isotropic couplings for a future work.

[6] Of course, from the no-cloning theorem [74] it is clear that Alice cannot perfectly send quantum information to multiple identical Bobs, since this would amount to her quantum state being cloned. The importance of the result in [73] is that it showed this to be true for *any* amount of quantum information, no matter how small.

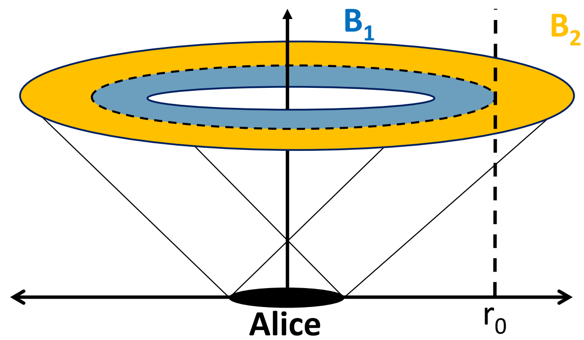

Figure 4.7: (3+1)-dimensional quantum information broadcasting setup considered in this section: Alice attempts to send quantum information to two spherically symmetric Bobs, B_1 and B_2, separated by the radius $r = r_0$.

$F_{\text{A}}(\boldsymbol{x})$ of Alice's detector to be a Gaussian of width σ, given by Eq. (4.84). Then, as we saw in Sec. 4.3, if Bob wants to recover the entirety of Alice's message, he needs to be able to couple to the field (and its conjugate momentum) via three different smearing functions, $F_{\text{B}i}(\boldsymbol{x})$, given by Eqs. (4.81)-(4.83). Namely,

$$F_{\text{B1}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}, \tag{4.88}$$

$$F_{\text{B2}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta'(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}, \tag{4.89}$$

$$F_{\text{B3}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta''(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}. \tag{4.90}$$

This is the ideal case however, where Bob has access to the entirety of Alice's lightcone. We now want to consider the less-than-ideal case of two Bobs, B_1 and B_2, that only have access to spatial regions $r < r_0$ and $r > r_0$, respectively. Hence, let us set the smearing functions for Bob B_1 to be

$$F^{(1)}_{\text{B1}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(r_0-|\boldsymbol{x}'|), \tag{4.91}$$

$$F^{(1)}_{\text{B2}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta'(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(r_0-|\boldsymbol{x}'|), \tag{4.92}$$

$$F^{(1)}_{\text{B3}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta''(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(r_0-|\boldsymbol{x}'|), \tag{4.93}$$

where the subscript (1) indicates Bob B_1, and the Θ functions ensure that these smearings are only non-zero in the ball $r < r_0$ centered on Alice. Similarly, for Bob B_2 we set the smearings to be

$$F^{(2)}_{\text{B1}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(|\boldsymbol{x}'|-r_0), \tag{4.94}$$

$$F^{(2)}_{\text{B2}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta'(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(|\boldsymbol{x}'|-r_0), \tag{4.95}$$

$$F^{(2)}_{\text{B3}}(\boldsymbol{x}) = -\int \mathrm{d}^d\boldsymbol{x}'\, F_{\text{A}}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta''(|\boldsymbol{x}'|-\Delta)}{4\pi|\boldsymbol{x}'|}\Theta(|\boldsymbol{x}'|-r_0), \tag{4.96}$$

which only have support in the spatial region $r > r_0$. Additionally, we will keep the condition $\frac{\lambda_\phi\lambda_\pi}{\sqrt{(2\pi)^3\sigma^3}} = \frac{\pi}{4}$ relating the coupling constants λ_ϕ and λ_π to the size of the detector σ, which we recall was necessary in order for Alice to be able to perfectly transmit her quantum message to (a single) Bob.

Having specified the initial quantum states as well as the smearing functions of Alice and both Bobs, we can now proceed to numerically compute the density matrices $\hat{\rho}_{\text{CB}_1}$ and $\hat{\rho}_{\text{CB}_2}$ associated with each Bob, as given by Eq. (4.61). Then, via Eq. (4.7), we can compute the coherent information associated with the channel from Alice to Bob B_1, and similarly for Bob B_2. The results are shown in Fig. 4.8 for two choices of parameters: $\lambda_\phi/\sigma = 10$ and $\lambda_\phi/\sigma = 1000$.

There are a few interesting points to note regarding Fig. 4.8. First, notice that for small enough r_0, Alice can send quantum information to Bob B_2 (but not B_1). This makes sense, since, as can be seen in Fig. 4.7, a small enough value of r_0 means that Bob B_2 has access to the entire lightcone of Alice, and thus he can recover the full quantum message (which, in $(3+1)$D propagates on the lightcone). Similarly, for large enough r_0 Alice can send quantum information to Bob B_1, but not B_2.

However, for either of the parameter ratios λ_ϕ/σ, it is not possible for Alice to simultaneously broadcast her quantum message to both Bobs, regardless of the value we take for the radius r_0 which defines the separation of B_1 and B_2. In fact we numerically verified that there is no choice of ratio λ_ϕ/σ which allows Alice to simultaneously broadcast coherent information to both Bobs. This therefore extends the no-quantum-broadcasting result proven in Ref. [73] for identical detectors to the case of spherically symmetric, non-identical detectors, and it therefore gives supporting evidence to the conjecture that it is not possible to send quantum information through a quantum field to multiple disjoint detectors, identical or not.

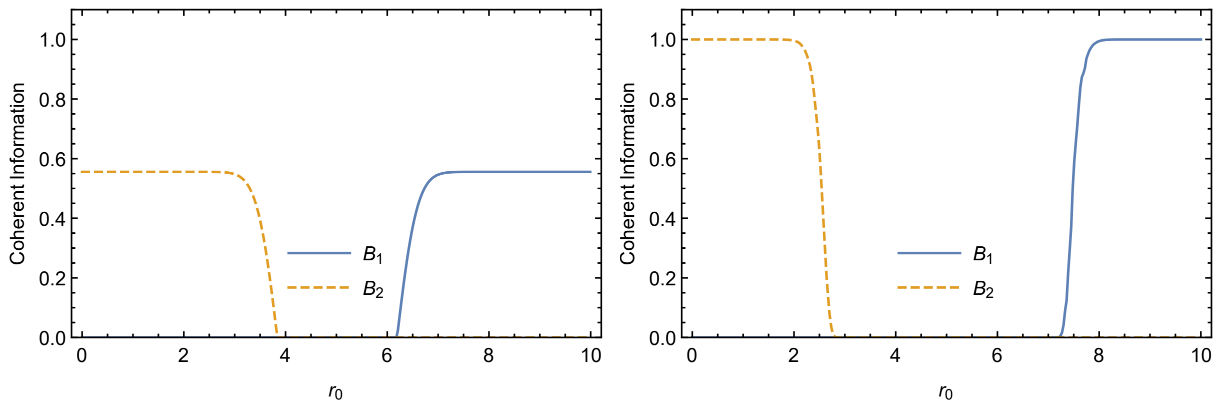

Figure 4.8: Coherent information versus r_0 (the radial separation between Bob B_1 and B_2) with $\lambda_\phi = 10$ (**left**) and $\lambda_\phi = 1000$ (**right**). We set $\sigma = 1$ for both plots. Notice that for both choices of parameters, and for any choice of r_0, it is not possible for Alice to simultaneously send coherent information to both Bobs.

Another interesting feature to note in Fig. 4.8 is the effect that increasing the ratio λ_ϕ/σ from 10 to 1000 has on the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ that Alice can transmit to the two Bobs. Namely, we see that for the smaller value of λ_ϕ/σ Alice can transmit coherent information to both Bobs for a larger range of values of r_0, but for either Bob $I_c(\hat{\rho}_{\text{A},0}, \Xi)$ never exceeds 0.6. On the other hand, if the coupling strength λ_ϕ is increased relative to σ, then there is a smaller range of r_0 values for which either Bob can receive coherent information, but in the best case scenario (large r_0 in the case of Bob B_1 and small r_0 in the case of Bob B_2) the Bobs can receive the maximum value of coherent information, $I_c(\hat{\rho}_{\text{A},0}, \Xi) = 1$. In other words, there is a "rich-get-richer, poor-get-poorer" type of trade-off associated with increasing the ratio λ_ϕ/σ: large Bobs will be able to receive more quantum information, at the expense of smaller Bobs not being able to receive any.

We can understand this trade-off on physical grounds, as follows. First of all, we know from our discussion in Sec. 4.2 that in order for Alice to perfectly transmit her quantum information to a single Bob the strong coupling condition (4.27) must be satisfied, which in $(3+1)$D is given by Eq. (4.65): $\lambda_\phi \gg \sigma$. Hence it is not surprising that if the ratio λ_ϕ/σ is increased, then a large Bob B_1 or B_2 — who would approximate the single, ideal Bob considered in the previous sections — would be able to receive more coherent information from Alice.

Furthermore, it also makes intuitive sense that a larger coupling λ_ϕ would make it more difficult for smaller, less-than-ideal Bobs to receive quantum information from Alice.

To understand this, first recall from Sec. 4.2 the physics of our perfect quantum channel from Alice to Bob. The first step to the quantum channel consists of Alice encoding her qubit into coherent states of the field, which, for larger values of λ_ϕ are increasingly more and more orthogonal to one another. Then, Bob attempts to recover the message by performing the DECODE gate between his qubit and the field, as shown schematically in Fig. 4.1. The DECODE gate, defined as the inverse to the ENCODE gate, first entangles Bob's qubit with the coherent field states, and then attempts to disentangle the field so that Alice's qubit state is coherently transmitted to Bob. However, in order for this final disentangling step to be performed successfully, Bob must have access to the entire quantum message sent out by Alice — i.e. Bob must have access to the entirety of Alice's lightcone in $(3+1)$D.

However, in this section we are manifestly considering the scenario where a less-than-ideal Bob (B_1 or B_2) does *not* have access to the entirety of Alice's lightcone, and hence in his decoding process he will *not* be able to completely disentangle his qubit from the field. Hence, following the decoding procedure the field carries partial knowledge of Bob's state, i.e. Alice's state, which she hoped to transmit to Bob. In other words, a portion of Alice's message will remain in the field, and hence, by a no-cloning type of intuition, the full message cannot get transmitted to this Bob. And since this effect of Bob remaining entangled with the field is more pronounced if the coherent field states entangled with Bob's state are more mutually orthogonal, we therefore now understand physically why a larger value of the coupling λ_ϕ, which ensures greater orthogonality between the coherent field states, requires Bob to cover a larger portion of Alice's light cone in order to receive her quantum message.

4.6 Conclusions

In this final chapter we have studied how a relativistic quantum field can be used to transmit quantum information between spacetime observers Alice and Bob, who couple to the field via Unruh-DeWitt detectors. Our analysis is applicable to flat spacetimes of any dimension, and in this sense is a generalization to Ref. [73], where a similar setup was studied in simplest case of $(1+1)$-dimensions.

We began by constructing a perfect field-mediated quantum channel from Alice to Bob, i.e. one for which the quantum channel capacity is the theoretically maximum value. The channel can be implemented by Alice first coupling to the field via a local unitary $\hat{U}_{\text{A}}$, which serves to encode Alice's qubit into the field, followed by Bob coupling to the field via a local unitary $\hat{U}_{\text{B}}$, which decodes the qubit from the field and onto Bob's detector.

The unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ defining our quantum channel are each generated by interaction Hamiltonians that couple Alice and Bob's detectors to the field only at discrete instants in time, and hence allow for a non-perturbative approach to the problem of time-evolution. Indeed, such a non-perturbative approach is necessary, since, as we showed, the field-mediated quantum channel from Alice to Bob can only be a perfect quantum channel if the observers are strongly (i.e. non-perturbatively) coupled to the field.

In particular, the unitaries $\hat{U}_{\text{A}}$ and $\hat{U}_{\text{B}}$ in our construction each take the form of a product of *two* simple-generated (i.e. rank-1 generated) unitaries. Moreover, we show that these are the simplest possible unitaries leading to a quantum channel with a non-zero quantum capacity. That is, if either $\hat{U}_{\text{A}}$ or $\hat{U}_{\text{B}}$ consists of a *single* rank-1 unitary, then the channel from Alice to Bob necessarily has zero quantum capacity. In this sense, the channel which we construct is the simplest possible field-mediated quantum channel from Alice to Bob with a non-zero quantum capacity.

Following our mathematical construction of the simplest possible perfect quantum channel, we attempted to use it to better understand how quantum information propagates through a relativistic quantum field. In particular, we asked the following question: If Alice encodes a quantum message into a quantum field at time t_{A} by coupling to the field in a spatial region characterized by the smearing function $F_{\text{A}}(\boldsymbol{x})$, then where in space does Bob have to be located at time $t_{\text{B}} > t_{\text{A}}$ in order to fully receive Alice's message?

Conveniently, the work we performed in constructing our quantum channel directly provided an answer to this question. Namely, we showed that if Bob wants to fully receive Alice's quantum message, then he must have access to the region of space containing the supports of a set of smearing functions $F_{\text{B}i}(\boldsymbol{x})$, for $i \in \{1, 2, 3\}$. These smearing functions are defined in terms of Alice's smearing $F_{\text{A}}(\boldsymbol{x})$ and the time difference $\Delta = t_{\text{B}} - t_{\text{A}}$, and they completely characterize the spacetime flow of quantum information through a Klein-Gordon field of arbitrary mass m in a flat spacetime of arbitrary dimension.

To better understand this highly general result, we then considered the particular cases of quantum information propagation through massless fields in $(2+1)$- and $(3+1)$-dimensional flat spacetimes. In $(3+1)$-dimensions we found that Bob can fully recover Alice's quantum message if he has access to her future light cone, which allowed us to conclude that in this spacetime quantum information propagates at the speed of light through the massless field. On the other hand, in the $(2+1)$-dimensional case we found that Bob additionally must have access to the full interior of Alice's lightcone in order to recover the entire message. Hence, in $(2+1)$-dimensional flat spacetime quantum information propagates subluminally through a massless field, despite the fact that the field quanta travel at the speed of light.

While this latter result may at first seem surprising, it can be simply understood by studying the validity of the strong Huygens principle, which states that the radiation Green's function of a massless field only has support for lightlike separated events. Indeed, as is well known, the strong Huygens principle does not hold in most spacetimes — including even dimensional Minkowski spaces — and in principle information can propagate slower than light in these spacetimes [60]. While this has previously been extensively investigated for *classical* information transmission [53–56], our work presented here is, to our knowledge, the first study of the effects of strong Huygens violation on *quantum* information transmission.

Having understood where in space an ideal Bob needs to be located in order to perfectly receive the quantum message that Alice sends through the field, we considered the less-than-ideal situation where Bob only covers a part of the spacetime region in which Alice's message lives. This situation is interesting from the perspective of quantum information broadcasting, a setup in which Alice hopes to simultaneously transmit at least a part of her quantum message to multiple disjoint Bobs. While the no-cloning theorem [74] precludes a perfect transmission of quantum information to multiple receivers, there appears, a priori, no reason to suspect that at least a small amount of quantum information could not be recovered by each of the Bobs.

However, as was shown in Ref. [73], it is in fact impossible for Alice to broadcast *any* amount of quantum information to multiple *identical* Bobs, a result that was proven for any spacetime dimension by noting that the quantum channel from Alice to any such Bob is anti-degradable [110]. Nevertheless this still leaves open the possibility for broadcasting quantum information to multiple, *non-identical*, disjoint Bobs, which we proceeded to study.

More concretely, we considered the case of two spherically symmetric Bobs, B_1 and B_2, covering the regions of $(3+1)$-dimensional space given by $|\boldsymbol{x}| < r_0$ and $|\boldsymbol{x}| \geq r_0$ (with r_0 some fixed radius), attempting to recover the quantum information sent out via a massless quantum field by an emitter Alice located at $\boldsymbol{x} = 0$. (The setup is depicted in Fig. 4.7.) We found that, regardless of the choice of setup parameters — such as the separation radius r_0 and the field coupling strength λ_ϕ of the detectors to the quantum field — it is not possible for Alice to simultaneously broadcast a non-zero amount of coherent information (a lower bound on the quantum information) to both Bobs. This gives support to the conjecture that it is not possible for Alice to broadcast quantum information to multiple disjoint Bobs, identical or not.

Finally, our study of quantum information broadcasting also led to an interesting result relating the coupling strength λ_ϕ of the observers to the quantum field, to the minimum size that a given Bob must be in order to receive a non-zero portion of Alice's quantum message. Namely, we showed that there is a "rich-get-richer, poor-get-poorer" type of trade-off associated with increasing the coupling strength λ_ϕ, whereby very large Bobs are able to receive more quantum information from Alice, at the cost of smaller Bobs not being able to receive any.

Physically, this trade-off arises due to the fact that an increased coupling λ_ϕ ensures that Alice's qubit is stored more coherently in the field, and hence a receiver Bob who has access to the entire portion of the field containing the qubit can better recover the qubit using his own detector. The downside of such a highly coherent encoding of Alice's qubit into the field however, is that if Bob is not able to fully access the region of the field containing the qubit, then a significant portion of Alice's message will remain in the field after Bob attempts to recover it. And since the no-cloning theorem makes it impossible for Alice's state to be simultaneously encoded in both the field and Bob's detector, we can thus understand intuitively why a spatially limited Bob would struggle to receive quantum information from Alice if λ_ϕ is large.

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APPENDICES

Appendix A

Interaction unitaries arising from delta couplings

This appendix is based on Appendix C of Ref. [2]. Let us suppose that the detector switching function entering the interaction Hamiltonian (1.26) is given by $\chi(t) = \delta(t-t_1)+\delta(t-t_2)$ with $t_1 \leq t_2$. (The more general case $\chi(t) = \sum_{i=1}^{n} \delta(t-t_i)$ follows straightforwardly by induction.) Then, the interaction unitary $\hat{U}$ given in (1.30) becomes

$$\hat{U} = \mathcal{T} \exp\left(-\mathrm{i}\left[\hat{H}_1 + \hat{H}_2\right]\right), \tag{A.1}$$

where $\hat{H}_i := \lambda \hat{m}_\nu(t_i) \otimes \hat{\Phi}(t_i)$. Expanding the exponential as a power series, one obtains

$$\hat{U} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{T}\left(-\mathrm{i}\hat{H}_1 - \mathrm{i}\hat{H}_2\right)^n. \tag{A.2}$$

Recalling that we assumed $t_1 \leq t_2$ and making use of the binomial theorem, this becomes

$$\begin{aligned} \hat{U} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \left(-\mathrm{i}\hat{H}_2\right)^m \left(-\mathrm{i}\hat{H}_1\right)^{n-m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} \left(-\mathrm{i}\hat{H}_2\right)^m \left(-\mathrm{i}\hat{H}_1\right)^{n-m}. \end{aligned} \tag{A.3}$$

Next we define the sets S_1 and S_2 to be

$$S_1 := \{(x,y) \in \mathbb{R}^2 | 0 \leq y \leq x\}, \tag{A.4}$$

$$S_2 := \{(x,y) \in \mathbb{R}^2 | 0 \leq x, 0 \leq y\}, \tag{A.5}$$

and we define the map $f : S_1 \to S_2$ by

$$f(x, y) \coloneqq (x - y, y). \tag{A.6}$$

It is straightforward to show that f is a bijection. Hence defining new summation indices k and l by $(k, l) \coloneqq f(n, m)$ allows us to write $\hat{U}$ as

$$\begin{aligned}
\hat{U}_2 &= \sum_{(n,m)\in S_1} \frac{1}{m!(n-m)!} \left(-\mathrm{i}\hat{H}_2\right)^m \left(-\mathrm{i}\hat{H}_1\right)^{n-m} \\
&= \sum_{(k,l)\in S_2} \frac{1}{l!\,k!} \left(-\mathrm{i}\hat{H}_2\right)^l \left(-\mathrm{i}\hat{H}_1\right)^k \\
&= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{l!\,k!} \left(-\mathrm{i}\hat{H}_2\right)^l \left(-\mathrm{i}\hat{H}_1\right)^k \\
&= \exp\left(-\mathrm{i}\hat{H}_2\right) \exp\left(-\mathrm{i}\hat{H}_2\right),
\end{aligned} \tag{A.7}$$

which proves the claim that $\hat{U} = \hat{U}_2\hat{U}_1$, where $\hat{U}_i$ is defined in Eq. (1.32).

Appendix B

Thermal two-point function

We will show that our expression for the thermal two-point function in Eq. (2.25) evaluates to the special case in Eq. (2.28) when $m = 0$ and $n = 3$.

Let us first evaluate the second term in Eq. (2.25), $w_\beta(\boldsymbol{x}, t, \boldsymbol{x}', t')$, which is given in Eq. (2.27). Working in polar coordinates, with $k := |\boldsymbol{k}|$, we straightforwardly obtain

$$\begin{aligned} w_\beta(\boldsymbol{x}, t, \boldsymbol{x}', t') &= \frac{1}{2\pi^2 r} \int_0^\infty \frac{\mathrm{d}k}{e^{\beta k} - 1} \sin(k\Delta x) \cos(k\Delta t) \\ &= \mathcal{P}\Bigg(- \frac{1}{4\pi^2(\Delta x^2 - \Delta t^2)} \\ &\qquad + \frac{1}{8\pi \Delta x \beta} \left[\coth\left(\frac{\pi(\Delta x + \Delta t)}{\beta}\right) + \coth\left(\frac{\pi(\Delta x - \Delta t)}{\beta}\right) \right] \Bigg), \end{aligned} \tag{B.1}$$

where $\mathcal{P}$ denotes the principal value of the integral (i.e. the expression above only has meaning as a distribution over a space of sufficiently well-behaved test functions). Interestingly, notice that the first term does not depend on the temperature.

We can similarly calculate the first term in Eq. (2.25), $w_0(\boldsymbol{x}, t, \boldsymbol{x}', t')$, which is given in

Eq. (2.26). We obtain

$$\begin{aligned}
w_0(\boldsymbol{x},t,\boldsymbol{x}',t') &= \frac{1}{8\pi^2 \mathrm{i}\Delta x}\int_0^\infty \mathrm{d}k \left(e^{-\mathrm{i}k(\Delta t-\Delta x)} - e^{-\mathrm{i}k(\Delta t+\Delta x)}\right) \\
&= \frac{1}{8\pi^2 \mathrm{i}\Delta x}\lim_{s\to\infty}\int_0^s \mathrm{d}k \left(e^{-\mathrm{i}k(\Delta t-\Delta x)} - e^{-\mathrm{i}k(\Delta t+\Delta x)}\right) \\
&= \mathcal{P}\left(\frac{1}{4\pi^2(\Delta x^2-\Delta t^2)}\right) \\
&\qquad + \lim_{s\to\infty}\frac{1}{8\pi^2\Delta x}\left[\frac{\mathrm{i}\sin\big(s(\Delta x+\Delta t)\big)}{\Delta x+\Delta t} - \frac{\mathrm{i}\sin\big(s(\Delta x-\Delta t)\big)}{\Delta x-\Delta t}\right. \\
&\qquad\qquad \left. - \frac{\cos\big(s(\Delta x+\Delta t)\big)}{\Delta x+\Delta t} - \frac{\cos\big(s(\Delta x-\Delta t)\big)}{\Delta x-\Delta t}\right]. \qquad \text{(B.2)}
\end{aligned}$$

Notice that although these limits do not converge as real functions, they do converge as distributions on test functions. Namely we have

$$\lim_{s\to\infty}\frac{\sin(sx)}{\pi x} = \delta(x), \qquad \text{(B.3)}$$

$$\lim_{s\to\infty}\frac{\cos(sx)}{\pi x} = 0 = \text{the zero distribution}. \qquad \text{(B.4)}$$

Hence Eq. (B.2) simplifies to

$$w_0(\boldsymbol{x},t,\boldsymbol{x}',t') = \mathcal{P}\left(\frac{1}{4\pi^2(\Delta x^2-\Delta t^2)}\right) + \frac{\mathrm{i}}{8\pi\Delta x}\left[\delta^{(3)}(\Delta x+\Delta t) - \delta^{(3)}(\Delta x-\Delta t)\right], \qquad \text{(B.5)}$$

where it should again be emphasized that the principal value and the delta functions only make sense as distributions. Finally, combining Eqs. (B.1) and (B.5), we find that for a massless field in $(3+1)$-dimensions our expression for the two-point function, Eq. (2.25), reduces to the distribution in Eq. (2.28), which was obtained in [90] by a completely different method.

Appendix C

No-go theorem with unbounded operators

In this appendix we will prove that our no-go theorem for entanglement extraction also holds when the source system S is a continuous variable system, and hence the observable $\hat{X}$ appearing in Eq. (3.8) is in general an unbounded, self-adjoint operator. This proof requires using the most general form of the spectral theorem, which first requires developing some preliminary functional analytic results.

To that end, let X be a set, M a σ-algebra of subsets of X, $\mathcal{H}$ a Hilbert space, and $\hat{\mu}$ a $\mathcal{B}(\mathcal{H})$-valued measure on σ. Let $|\phi\rangle \in \mathcal{H}$. Then it is straightforward to show that $\mu_\phi : M \to \mathbb{R}_+$ defined by $\mu_\phi(B) := \langle\phi|\hat{\mu}(B)|\phi\rangle$ is a positive measure on M.

Lemma 1. *Let $\hat{\rho} \in \mathcal{B}(\mathcal{H})$ be a density matrix on $\mathcal{H}$. Then $\mu_\rho : M \to \mathbb{R}_+$ defined by $\mu_\rho(B) := \mathrm{Tr}(\mu(B)\hat{\rho})$ is a positive measure on M.*

Proof. Write $\hat{\rho} = \sum_{i=1}^{\infty} \alpha_i |\phi_i\rangle\langle\phi_i|$ with $|\phi_i\rangle$ an orthonormal basis of $\mathcal{H}$, $\alpha_i \geq 0$, and

$\sum \alpha_i < +\infty$. Then

$$\begin{aligned}\mu_\rho(B) &:= \mathrm{Tr}(\mu(B)\hat{\rho}) \\ &= \sum_{j=1}^{\infty} \langle \phi_j | \hat{\mu}(B)\hat{\rho} | \phi_j \rangle \\ &= \sum_{j=1}^{\infty} \langle \phi_j | \hat{\mu}(B) \sum_{i=1}^{\infty} \alpha_i | \phi_i \rangle \langle \phi_i | \phi_j \rangle \\ &= \sum_{i=1}^{\infty} \alpha_i \langle \phi_i | \hat{\mu}(B) | \phi_i \rangle \\ &= \sum_{i=1}^{\infty} \alpha_i \mu_{\phi_i}(B). \end{aligned} \tag{C.1}$$

Hence $\mu_\rho(B) \geq 0$ for all $B \in M$ and $\mu(\varnothing) = 0$ since $\alpha_i \geq 0$ and since μ_{ϕ_i} is a positive measure for all $|\phi_i\rangle$. Also

$$\begin{aligned}\mu_\rho\left(\bigcup_{k=1}^{\infty} B_k\right) &= \sum_{i=1}^{\infty} \alpha_i \mu_{\phi_i}\left(\bigcup_{k=1}^{\infty} B_k\right) \\ &= \sum_{i=1}^{\infty} \alpha_i \sum_{k=1}^{\infty} \mu_{\phi_i}\left(B_k\right), \end{aligned} \tag{C.2}$$

where in the last step we use the fact that μ_ϕ is a measure. Since, additionally, μ_ϕ is positive, we can commute the two summations. Hence

$$\begin{aligned}\mu_\rho\left(\bigcup_{k=1}^{\infty} B_k\right) &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_i \mu_{\phi_i}\left(B_k\right) \\ &= \sum_{k=1}^{\infty} \mu_\rho\left(B_k\right). \end{aligned} \tag{C.3}$$

Hence μ_ρ is a positive measure. □

Lemma 2. *Let $f : X \to \mathbb{C}$ be a bounded, measurable function. Then* $\mathrm{Tr}(\int_X f \, \mathrm{d}\hat{\mu}\hat{\rho}) = \int_X f \, \mathrm{d}\mu_\rho$.

Proof. Write $\hat{\rho} = \sum_{i=1}^{\infty} \alpha_i |\phi_i\rangle\langle\phi_i|$ with $|\phi_i\rangle$ an orthonormal basis of $\mathcal{H}$, $\alpha_i \geq 0$, and $\sum \alpha_i < +\infty$. Then

$$\begin{aligned}
\mathrm{Tr}\left(\int_X f \, \mathrm{d}\hat{\mu}\hat{\rho}\right) &= \sum_{i=1}^{\infty} \alpha_i \langle\phi_i| \int_X f \, \mathrm{d}\hat{\mu} |\phi_i\rangle \\
&= \sum_{i=1}^{\infty} \alpha_i \int_X f \, \mathrm{d}\mu_{\phi_i} \\
&= \int_X f \, \mathrm{d}\mu_\rho, \qquad \text{(C.4)}
\end{aligned}$$

where in the second equality we used the definition of an operator-valued integral (see, e.g., Ref. [111] for details), and in the last equality we made use of Eq. (C.1). □

Suppose now that $\mathcal{H}_\mathrm{B}$ is a Hilbert space of dimension 2, and $\mathcal{H}_\mathrm{S}$ is a Hilbert space of any dimension (in particular it could be countably or even uncountably infinite dimensional). Note that the argument presented here is straightforwardly extended for any finite value of $\dim \mathcal{H}_\mathrm{B}$. Consider a unitary $\hat{U}$ on $\mathcal{H}_\mathrm{B} \otimes \mathcal{H}_\mathrm{S}$ given by

$$\hat{U} = \exp(-\mathrm{i}\hat{m} \otimes \hat{X}), \qquad \text{(C.5)}$$

where both $\hat{m}$ and $\hat{X}$ are self-adjoint operators in their respective Hilbert spaces. Since $\hat{m}$ is self-adjoint, by the spectral theorem its eigenvalues span $\mathcal{H}_\mathrm{B}$, and in this basis $\hat{m}$ can be expressed as

$$\hat{m} = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}. \qquad \text{(C.6)}$$

Since $\hat{X}$ is self-adjoint, by the most general form of the spectral theorem (see, e.g. [111]) it can be expressed as

$$\hat{X} = \int \lambda \, \mathrm{d}\hat{\mu}(\lambda), \qquad \text{(C.7)}$$

where $\hat{\mu}$ is an operator-valued measure on the Borel σ-algebra of subsets of the spectrum of $\hat{X}$. For any measurable function f we define $f(\hat{X})$ to be the operator

$$f(\hat{X}) := \int f(\lambda) \, \mathrm{d}\hat{\mu}(\lambda). \qquad \text{(C.8)}$$

Expanding $\hat{U}$ in Eq. (C.5) in a power series gives

$$\begin{aligned}\hat{U} &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \hat{m}^n \hat{X}^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \begin{pmatrix} m_{11}^n \hat{X}^n & 0 \\ 0 & m_{22}^n \hat{X}^n \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \begin{pmatrix} m_{11}^n \int \lambda^n \, \mathrm{d}\hat{\mu}(\lambda) & 0 \\ 0 & m_{22}^n \int \lambda^n \, \mathrm{d}\hat{\mu}(\lambda) \end{pmatrix},\end{aligned} \tag{C.9}$$

where in the second equality we are representing vectors in $\mathcal{H}_{\mathrm{B}}$ in the eigenbasis of $\hat{m}$, and in the third equality we have made use of Eq. (C.8). By linearity of integration this can be written as

$$\begin{aligned}\hat{U} &= \begin{pmatrix} \int \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i}m_{11}\lambda)^n \, \mathrm{d}\hat{\mu}(\lambda) & 0 \\ 0 & \int \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i}m_{22}\lambda)^n \, \mathrm{d}\hat{\mu}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} \int e^{-\mathrm{i}m_{11}\lambda} \, \mathrm{d}\hat{\mu}(\lambda) & 0 \\ 0 & \int e^{-\mathrm{i}m_{22}\lambda} \, \mathrm{d}\hat{\mu}(\lambda) \end{pmatrix}.\end{aligned} \tag{C.10}$$

Similarly, the adjoint of $\hat{U}$, denoted $\hat{U}^\dagger$, can be expressed as

$$\hat{U}^\dagger = \begin{pmatrix} \int e^{\mathrm{i}m_{11}\lambda} \, \mathrm{d}\hat{\mu}(\lambda) & 0 \\ 0 & \int e^{\mathrm{i}m_{22}\lambda} \, \mathrm{d}\hat{\mu}(\lambda) \end{pmatrix}. \tag{C.11}$$

Consider now the channel ξ which takes states (density matrices) on $\mathcal{H}_{\mathrm{S}}$ into states on $\mathcal{H}_{\mathrm{B}}$ and is given by

$$\xi(\hat{\rho}_{\mathrm{S}}) := \mathrm{Tr}_{\mathrm{S}} \left[\hat{U} \left(\hat{\rho}_{\mathrm{B}} \otimes \hat{\rho}_{\mathrm{S}} \right) \hat{U}^\dagger \right], \tag{C.12}$$

where $\hat{\rho}_{\mathrm{B}}$ is a fixed density matrix on $\mathcal{H}_{\mathrm{B}}$. We can represent $\hat{\rho}_{\mathrm{B}}$ in the eigenbasis of $\hat{m}$ as

$$\hat{\rho}_{\mathrm{B}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12}^* & b_{22} \end{pmatrix}. \tag{C.13}$$

Also in this basis, the expression for $\xi(\hat{\rho}_\mathrm{S})$ takes the form

$$\begin{aligned}
\xi(\hat{\rho}_\mathrm{S}) = \mathrm{Tr}_\mathrm{S} \Bigg[&\begin{pmatrix} \int e^{-\mathrm{i}m_{11}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) & 0 \\ 0 & \int e^{-\mathrm{i}m_{22}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) \end{pmatrix} \begin{pmatrix} b_{11}\hat{\rho}_\mathrm{S} & b_{12}\hat{\rho}_\mathrm{S} \\ b_{12}^*\hat{\rho}_\mathrm{S} & b_{22}\hat{\rho}_\mathrm{S} \end{pmatrix} \\
&\times \begin{pmatrix} \int e^{\mathrm{i}m_{11}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') & 0 \\ 0 & \int e^{\mathrm{i}m_{22}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') \end{pmatrix} \Bigg] \\
= \mathrm{Tr}_\mathrm{S} &\begin{pmatrix} \int e^{-\mathrm{i}m_{11}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{11}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{11}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') & \int e^{-\mathrm{i}m_{11}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{12}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{22}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') \\ \int e^{-\mathrm{i}m_{22}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{12}^*\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{11}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') & \int e^{-\mathrm{i}m_{22}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{22}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{22}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') \end{pmatrix} \\
= &\begin{pmatrix} \mathrm{Tr} \int e^{-\mathrm{i}m_{11}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{11}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{11}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') & \mathrm{Tr} \int e^{-\mathrm{i}m_{11}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{12}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{22}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') \\ \mathrm{Tr} \int e^{-\mathrm{i}m_{22}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{12}^*\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{11}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda') & \mathrm{Tr} \int e^{-\mathrm{i}m_{22}\lambda}\,\mathrm{d}\hat{\mu}(\lambda) b_{22}\hat{\rho}_\mathrm{S} \int e^{\mathrm{i}m_{22}\lambda'}\,\mathrm{d}\hat{\mu}(\lambda' \end{pmatrix}.
\end{aligned} \tag{C.14}$$

Note that integrals with respect to projection-valued measures have the multiplicative property [111]

$$\int f(\lambda)\,\mathrm{d}\hat{\mu}(\lambda) \int g(\lambda')\,\mathrm{d}\hat{\mu}(\lambda') = \int f(\lambda)g(\lambda)\,\mathrm{d}\hat{\mu}(\lambda). \tag{C.15}$$

Together with the cyclicity of the trace this simplifies Eq. (C.14) to read

$$\begin{aligned}
\xi(\hat{\rho}_\mathrm{S}) &= \begin{pmatrix} \mathrm{Tr} \int b_{11}\,\mathrm{d}\hat{\mu}(\lambda)\hat{\rho}_\mathrm{S} & \mathrm{Tr} \int b_{12} e^{-\mathrm{i}(m_{11}-m_{22})\lambda}\,\mathrm{d}\hat{\mu}(\lambda)\hat{\rho}_\mathrm{S} \\ \mathrm{Tr} \int b_{12}^* e^{\mathrm{i}(m_{11}-m_{22})\lambda}\,\mathrm{d}\hat{\mu}(\lambda)\hat{\rho}_\mathrm{S} & \mathrm{Tr} \int b_{22}\,\mathrm{d}\hat{\mu}(\lambda)\hat{\rho}_\mathrm{S} \end{pmatrix} \\
&= \begin{pmatrix} \int b_{11}\,\mathrm{d}\nu(\lambda) & \int b_{12} e^{-\mathrm{i}(m_{11}-m_{22})\lambda}\,\mathrm{d}\nu(\lambda) \\ \int b_{12}^* e^{\mathrm{i}(m_{11}-m_{22})\lambda}\,\mathrm{d}\nu(\lambda) & \int b_{22}\,\mathrm{d}\nu(\lambda) \end{pmatrix},
\end{aligned} \tag{C.16}$$

where in the last step we have defined the real-valued measure ν by $\nu(\cdot) := \mathrm{Tr}(\mathrm{d}\hat{\mu}(\cdot)\hat{\rho}_\mathrm{S})$ and made use of Lemma 2. Let us now define the $\mathcal{B}(\mathcal{H}_\mathrm{B})$-valued function $\hat{\rho}_\mathrm{B}(\lambda)$ so that in the eigenbasis of $\hat{m}$ it reads

$$\hat{\rho}_\mathrm{B}(\lambda) = \begin{pmatrix} b_{11} & b_{12} e^{-\mathrm{i}(m_{11}-m_{22})\lambda} \\ b_{12}^* e^{\mathrm{i}(m_{11}-m_{22})\lambda} & b_{22} \end{pmatrix}. \tag{C.17}$$

Then Eq. (C.16) can be written in a basis independent manner as

$$\xi(\hat{\rho}_\mathrm{S}) = \int \hat{\rho}_\mathrm{B}(\lambda)\nu(\lambda). \tag{C.18}$$

Finally, by Theorem 2 in [98], we see that the channel ξ is entanglement breaking.

Appendix D

Explicit calculation of simple-generated time evolution

Here we will show the procedure for calculating the expression for the density matrix $\hat{\rho}_{\text{AB}}$ for each of the three coupling setups that we consider: AAB (first Alice couples twice then Bob once), BAA, and ABA. Notice that the first two scenarios are just limiting cases of the coupling scheme AABB (up to a relabeling of A↔B). Similarly the coupling ABA is a limiting case of the four delta-coupling ABBA, where we take the two B couplings to be at the same time. We will work out the details for the AABB coupling, with the calculations for the ABBA setup performed analogously.

Let us therefore consider the case where A and B each delta-couple to the field twice, at times $t_{\text{A}_1} \leq t_{\text{A}_2} \leq t_{\text{B}_1} \leq t_{\text{B}_2}$, with coupling strengths $\lambda_{\text{A}} = \lambda_{\text{B}} = \lambda/2$. The interaction Hamiltonian is

$$\hat{H}_{\text{I}}(t) = \hat{H}^{(1)}_{\text{I,A}}(t) + \hat{H}^{(2)}_{\text{I,A}}(t) + \hat{H}^{(1)}_{\text{I,B}}(t) + \hat{H}^{(2)}_{\text{I,B}}(t), \tag{D.1}$$

with $\hat{H}^{(i)}_{\text{I},\nu}(t)$ defined in Eq. (3.36). This Hamiltonian generates the time-evolution unitary $\hat{U} = \hat{U}_{\text{B}_2}\hat{U}_{\text{B}_1}\hat{U}_{\text{A}_2}\hat{U}_{\text{A}_1}$, with the $\hat{U}_{\nu i}$ given by (see [2] for details)

$$\hat{U}_{\text{A}i} = \mathbb{1}_{\text{A}} \otimes \mathbb{1}_{\text{B}} \otimes \hat{y}^{+}_{\text{A}i} + \hat{m}_{\text{A}i} \otimes \mathbb{1}_{\text{B}} \otimes \hat{y}^{-}_{\text{A}i}, \tag{D.2}$$

$$\hat{U}_{\text{B}i} = \mathbb{1}_{\text{A}} \otimes \mathbb{1}_{\text{B}} \otimes \hat{y}^{+}_{\text{A}i} + \mathbb{1}_{\text{A}} \otimes \hat{m}_{\text{B}i} \otimes \hat{y}^{-}_{\text{A}i}, \tag{D.3}$$

where $\hat{m}_{\nu i} := \hat{m}_{\nu}(t_{\nu i})$, $\hat{y}^{+}_{\nu i} := \cosh(\hat{Y}_{\nu i})$, $\hat{y}^{-}_{\nu i} := \sinh(\hat{Y}_{\nu i})$, and we define the field observable $\hat{Y}_{\nu i} := -\mathrm{i}(\lambda/2)\int \mathrm{d}^n\boldsymbol{x}\, F_{\nu}(\boldsymbol{x})\hat{\phi}(\boldsymbol{x}, t_{\nu i})$. The unitary $\hat{U}$ evolves the initial state $|\psi_0\rangle$ given in

Eq. (3.34) into the state

$$
\begin{aligned}
\hat{U}|\psi_0\rangle =&|g_{\rm A}\rangle \otimes |g_{\rm B}\rangle \otimes \big(\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1} + e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\\
&\quad + e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1} + e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\big)|0\rangle\\
&+|g_{\rm A}\rangle \otimes |e_{\rm B}\rangle \otimes \big(e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_1}}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1} + e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_1}}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\\
&\quad + e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_2}}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1} + e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_2}}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\big)|0\rangle\\
&+|e_{\rm A}\rangle \otimes |g_{\rm B}\rangle \otimes \big(e^{\mathrm{i}\Omega_{\rm B}t_{\rm A_1}}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1} + e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_2}}\hat{y}^+_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\\
&\quad + e^{\mathrm{i}\Omega_{\rm B}t_{\rm A_1}}e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1} + e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_2}}e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}\hat{y}^-_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\big)|0\rangle\\
&+|e_{\rm A}\rangle \otimes |e_{\rm B}\rangle \otimes \big(e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_1}}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_1}}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1} + e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_2}}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_1}}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\\
&\quad + e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_1}}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_2}}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^+_{\rm A2}\hat{y}^-_{\rm A1} + e^{\mathrm{i}\Omega_{\rm A}t_{\rm A_2}}e^{\mathrm{i}\Omega_{\rm B}t_{\rm B_2}}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\hat{y}^-_{\rm A2}\hat{y}^+_{\rm A1}\big)|0\rangle. \qquad \text{(D.4)}
\end{aligned}
$$

Using this expression we can calculate the time-evolved density matrix of the two detectors as $\hat{\rho}_{\rm AB} := {\rm Tr}_\phi(\hat{U}|\psi_0\rangle\langle\psi_0|\hat{U}^\dagger)$. For example, in the basis $\{|g_{\rm A}\rangle|g_{\rm B}\rangle, |g_{\rm A}\rangle|e_{\rm B}\rangle, |e_{\rm A}\rangle|g_{\rm B}\rangle, |e_{\rm A}\rangle|e_{\rm B}\rangle\}$, the (1,1) component of $\hat{\rho}_{\rm AB}$, denoted ρ_{11}, reads

$$
\begin{aligned}
\rho_{11} =& h(++++++++) + h(++++++--)e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}+\\
&+h(++++--++)e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})} + h(++++----)e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}+\\
&+h(--++++++)e^{\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})} + h(--++++--)+\\
&+h(--++--++)e^{\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})} + h(--++----)e^{-\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}+\\
&+h(++--++++)e^{\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})} + h(++--++--)e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}+\\
&+h(++----++) + h(++------)e^{-\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}+\\
&+h(----++++)e^{\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})}e^{\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})} + h(----++--)e^{\mathrm{i}\Omega_{\rm B}(t_{\rm B_2}-t_{\rm B_1})}+\\
&+h(------++)e^{\mathrm{i}\Omega_{\rm A}(t_{\rm A_2}-t_{\rm A_1})} + h(--------). \qquad \text{(D.5)}
\end{aligned}
$$

Here $h(l_1,l_2,l_3,l_4,l_5,l_6,l_7,l_8) := \langle 0|\hat{y}^{l_1}_{\rm A_1}\hat{y}^{l_2}_{\rm A_2}\hat{y}^{l_3}_{\rm B_1}\hat{y}^{l_4}_{\rm B_2}\hat{y}^{l_5}_{\rm B_2}\hat{y}^{l_6}_{\rm B_1}\hat{y}^{l_7}_{\rm A_2}\hat{y}^{l_8}_{\rm A_1}|0\rangle$ for $l_i = \pm 1$. In order to evaluate h, it is useful write $\hat{y}^\pm_{\nu i} = [\exp(\hat{Y}_{\nu i}) \pm \exp(-\hat{Y}_{\nu i})]/2$. The expression for h then becomes

$$
h(l_1,l_2,l_3,l_4,l_5,l_6,l_7,l_8) = \frac{1}{2^8}\sum_{p_j=\pm 1}\prod_{i=1}^{8} f(l_i,p_i)K(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8), \qquad \text{(D.6)}
$$

where $K(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8) := \langle 0|e^{p_1\hat{Y}_{\rm A1}}e^{p_2\hat{Y}_{\rm A2}}e^{p_3\hat{Y}_{\rm B1}}e^{p_4\hat{Y}_{\rm B2}}e^{p_5\hat{Y}_{\rm B2}}e^{p_6\hat{Y}_{\rm B1}}e^{p_7\hat{Y}_{\rm A2}}e^{p_8\hat{Y}_{\rm A1}}|0\rangle$, and $f(l_i,p_i)$ equals -1 if $l_i = p_i = -1$ and 0 otherwise. Next we define the commutators

$\mathrm{i}\theta_\nu \mathbb{1}_\phi := [\hat{Y}_{\nu 2}, \hat{Y}_{\nu 1}]$ and $\mathrm{i}\theta_{ij} \mathbb{1}_\phi := [\hat{Y}_{\mathrm{B}i}, \hat{Y}_{\mathrm{A}j}]$, which evaluate to

$$\theta_\nu = \mathrm{i} \int \mathrm{d}^3\boldsymbol{k} \, \left(\alpha_{\mathrm{A}_1}(\boldsymbol{k}) \alpha^*_{\mathrm{A}_2}(\boldsymbol{k}) - \mathrm{c.c} \right), \tag{D.7}$$

$$\theta_{ij} = \mathrm{i} \int \mathrm{d}^3\boldsymbol{k} \, \left(\alpha_{\mathrm{A}j}(\boldsymbol{k}) \alpha^*_{\mathrm{B}i}(\boldsymbol{k}) - \mathrm{c.c} \right), \tag{D.8}$$

where $\alpha_{\nu i}(\boldsymbol{k})$ is defined by

$$\alpha_{\nu i}(\boldsymbol{k}) := -\frac{\mathrm{i}\lambda}{2\sqrt{2|\boldsymbol{k}|}} \tilde{F}^*_\nu(\boldsymbol{k}) e^{\mathrm{i}|\boldsymbol{k}| t_{\nu i}}. \tag{D.9}$$

$\tilde{F}_\nu(\boldsymbol{k})$ is the Fourier transform of the smearing function $F_\nu(\boldsymbol{x})$, given in Eq. (3.38). Calculating $\tilde{F}_\nu(\boldsymbol{k})$ we obtain

$$\begin{aligned} \tilde{F}_\nu(\boldsymbol{k}) &:= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3\boldsymbol{x} \, F_\nu(\boldsymbol{x}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin(\sigma|\boldsymbol{k}|) - \sigma|\boldsymbol{k}| \cos(\sigma|\boldsymbol{k}|)}{(\sigma|\boldsymbol{k}|)^3}. \end{aligned} \tag{D.10}$$

The expressions for θ_ν and θ_{ij} then work out to be

$$\theta_\nu = \frac{9\lambda^2}{4\pi^2} I_{\mathrm{s}}(t_{\nu 2} - t_{\nu 1}), \tag{D.11}$$

$$\theta_{ij} = \frac{9\lambda^2}{4\pi^2} I_{\mathrm{s}}(t_{\mathrm{B}_i} - t_{\mathrm{A}_j}), \tag{D.12}$$

where the function $I_{\mathrm{s}}(x)$ is given by

$$\begin{aligned} I_{\mathrm{s}}(x) &:= \int_0^\infty \mathrm{d}k \frac{(\sin(k) - k\cos k)^2}{k^5} \sin(kx) \\ &= \frac{\pi}{96} x (2 - |x|)^2 (4 + |x|) \Theta(2 - |x|). \end{aligned} \tag{D.13}$$

Using the Baker-Campbell-Hausdorff formula the expression for K becomes

$$\begin{aligned} &K(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) \\ &= \langle 0 | \hat{D}_\alpha | 0 \rangle \exp \Big[-\frac{\mathrm{i}}{2} \big((p_1 - p_8)(p_3 + p_6)\theta_{11} + (p_2 - p_7)(p_3 + p_6)\theta_{12} \\ &\qquad + (p_1 - p_8)(p_4 + p_5)\theta_{21} + (p_2 - p_7)(p_4 + p_5)\theta_{22} \\ &\qquad + (p_1 - p_8)(p_2 + p_7)\theta_{\mathrm{A}} + (p_3 - p_6)(p_4 + p_5)\theta_{\mathrm{B}} \big) \Big], \end{aligned} \tag{D.14}$$

where $\hat{D}_\alpha$ is a field *displacement operator* given by

$$\hat{D}_\alpha := \exp\left[\int \mathrm{d}^3\boldsymbol{k}\ \left(\alpha(\boldsymbol{k})a^\dagger_{\boldsymbol{k}} - \alpha(\boldsymbol{k})^* a_{\boldsymbol{k}}\right)\right], \tag{D.15}$$

and where the *coherent amplitude* α is

$$\alpha(\boldsymbol{k}) := (p_1+p_8)\alpha_{\mathrm{A}_1}(\boldsymbol{k}) + (p_2+p_7)\alpha_{\mathrm{A}_2}(\boldsymbol{k}) + (p_3+p_6)\alpha_{\mathrm{B}_1}(\boldsymbol{k}) + (p_4+p_5)\alpha_{\mathrm{B}_2}(\boldsymbol{k}). \tag{D.16}$$

Note that $\hat{D}_\alpha$ acts on the vacuum state $|0\rangle$ to create a coherent state of amplitude α, which we denote $|\alpha\rangle$. Thus the factor $\langle 0|\hat{D}_\alpha|0\rangle$ is simply the inner product between $|0\rangle$ (the coherent state of amplitude 0) and $|\alpha\rangle$. In Appendix A of [2] it is shown how to calculate the inner product of two field coherent states. The result is, as might be expected from a knowledge of coherent states of a simple harmonic oscillator,

$$\langle 0|\hat{D}_\alpha|0\rangle = \exp\left(-\frac{1}{2}\int \mathrm{d}^3\boldsymbol{k}\, |\alpha(\boldsymbol{k})|^2\right). \tag{D.17}$$

Using the definition of $\alpha(\boldsymbol{k})$ in Eq. (D.16), this simplifies to

$$\begin{aligned}\langle 0|\hat{D}_\alpha|0\rangle = \exp\Bigg[-\frac{9\lambda^2}{16\pi^2}\Bigg(&\frac{1}{4}\Big((p_1+p_8)^2+(p_2+p_7)^2+(p_3+p_6)^2+(p_4+p_5)^2\Big)\\ &+2(p_1+p_8)(p_2+p_7)I_\mathrm{c}(t_{\mathrm{A}_2}-t_{\mathrm{A}_1})+2(p_1+p_8)(p_3+p_6)I_\mathrm{c}(t_{\mathrm{B}_1}-t_{\mathrm{A}_1})\\ &+2(p_1+p_8)(p_4+p_5)I_\mathrm{c}(t_{\mathrm{B}_2}-t_{\mathrm{A}_1})+2(p_2+p_7)(p_3+p_6)I_\mathrm{c}(t_{\mathrm{B}_1}-t_{\mathrm{A}_2})\\ &+2(p_2+p_7)(p_4+p_5)I_\mathrm{c}(t_{\mathrm{B}_2}-t_{\mathrm{A}_2})+2(p_3+p_6)(p_4+p_5)I_\mathrm{c}(t_{\mathrm{B}_2}-t_{\mathrm{B}_1})\Bigg)\Bigg],\end{aligned} \tag{D.18}$$

where $I_\mathrm{c}(x)$ is defined to be

$$\begin{aligned}I_\mathrm{c}(x) &:= \int_0^\infty \mathrm{d}k\,\frac{(\sin(k)-k\cos k)^2}{k^5}\cos(kx)\\ &= \begin{cases}\frac{1}{4} & \text{if } x=0,\\ \frac{1}{12}(5-8\ln 2) & \text{if } x=\pm 2,\\ \frac{1}{96}\Big[24+4x^2-2x^2(x^2-12)\ln|x|-16|x|\ln(2+|x|)-12x^2\ln(2+|x|) & \\ \qquad +x^4\ln(2+|x|)+|x|(|x|-2)^2(4+|x|)\ln\big||x|-2\big|\Big] & \text{otherwise.}\end{cases}\end{aligned} \tag{D.19}$$

Substituting Eqs. (D.11) and (D.18) into Eq. (D.14) gives us an expression for K, which we can then substitute into Eq. (D.6) to get a concrete expression for h. Therefore we can get an expression for the matrix element ρ_{11}, which is expressed in terms of h in Eq. (D.5). The remaining elements of the two detector density matrix $\hat{\rho}_{\text{AB}}$ are calculated analogously. From the symmetries of the arguments of h in Eq. (2.40) and of K in Eq. (D.14), we can see that if h has an odd number of "$-$" arguments then it vanishes. This is the reason why half of the matrix elements of $\hat{\rho}_{\text{AB}}$ in Eq. (3.40) are zero.

Appendix E

Some technical quantum information results

Here we prove some technical results from quantum information theory, which we make use of throughout the main text.

Definition 3. *Let* $\hat{\rho}_{\mathrm{CB}}$ *be a state on* $\mathcal{H}_{\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}$*. The* conditional quantum entropy $S(C|B)_{\hat{\rho}_{\mathrm{CB}}}$ *is defined as*

$$S(C|B)_{\hat{\rho}_{\mathrm{CB}}} := S(\hat{\rho}_{\mathrm{CB}}) - S(\hat{\rho}_{\mathrm{B}}), \tag{E.1}$$

where $S(\cdot)$ *denotes the von Neumann entropy and* $\hat{\rho}_{\mathrm{B}} := \mathrm{Tr}_{\mathrm{C}}\, \hat{\rho}_{\mathrm{CB}}$*.*

Note that, with this definition, the coherent information $I_c(\hat{\rho}_{\mathrm{A},0}, \Xi)$ of a quantum channel Ξ from A to B and the input state $\hat{\rho}_{\mathrm{A},0}$, as defined by Eq. (4.7), can be written as

$$I_c(\hat{\rho}_{\mathrm{A},0}, \Xi) = -S(C|B)_{\hat{\rho}_{\mathrm{CB}}}, \tag{E.2}$$

where, recall, $\hat{\rho}_{\mathrm{CB}}$ is the output of the channel $\mathbb{1}_{\mathrm{C}} \otimes \Xi$ acting on a purification of $\hat{\rho}_{\mathrm{A},0}$.

Lemma 3. *The function taking the input* $\hat{\rho}_{\mathrm{CB}}$ *and producing the output* $S(C|B)_{\hat{\rho}_{\mathrm{CB}}}$ *is a concave function, i.e.*

$$S(C|B)_{\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2} \geq \lambda S(C|B)_{\hat{\rho}_1} + (1-\lambda) S(C|B)_{\hat{\rho}_2}, \tag{E.3}$$

for any $0 \leq \lambda \leq 1$ *and states* $\hat{\rho}_1$ *and* $\hat{\rho}_2$ *on* $\mathcal{H}_{\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}$*.*

Proof. The proof presented here is inspired by the sketch of the proof in [112]. We start by considering the state $\hat{\rho}_{\text{CBE}}$ on $\mathcal{H}_{\text{C}} \otimes \mathcal{H}_{\text{B}} \otimes \mathcal{H}_{\text{E}}$ defined by

$$\hat{\rho}_{\text{CBE}} := \lambda \hat{\rho}_1 \otimes |0\rangle\langle 0| + (1-\lambda)\hat{\rho}_2 \otimes |0\rangle\langle 0|, \tag{E.4}$$

where $\{|0\rangle, |1\rangle\}$ forms an orthonormal basis of the auxiliary qubit space $\mathcal{H}_{\text{E}}$. Then, the strong subadditivity of the von Neumann entropy reads [113]

$$S(\hat{\rho}_{\text{CBE}}) + S(\hat{\rho}_{\text{B}}) \leq S(\hat{\rho}_{\text{CB}}) + S(\hat{\rho}_{\text{BE}}), \tag{E.5}$$

where $\hat{\rho}_{\text{B}} := \text{Tr}_{\text{CE}}\, \hat{\rho}_{\text{CBE}}$, $\hat{\rho}_{\text{CB}} := \text{Tr}_{\text{E}}\, \hat{\rho}_{\text{CBE}}$ and $\hat{\rho}_{\text{BE}} := \text{Tr}_{\text{C}}\, \hat{\rho}_{\text{CBE}}$. Then, noting that $\hat{\rho}_{\text{CB}} = \lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2$ and making use of the definition (E.1) for $S(C|B)_{\hat{\rho}_{\text{CB}}}$ we find

$$S(C|B)_{\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2} \geq S(\hat{\rho}_{\text{CBE}}) - S(\hat{\rho}_{\text{BE}}). \tag{E.6}$$

Let us now evaluate $S(\hat{\rho}_{\text{CBE}})$. We obtain

$$\begin{aligned} S(\hat{\rho}_{\text{CBE}}) &:= -\,\text{Tr}\,\hat{\rho}_{\text{CBE}} \log_2 \hat{\rho}_{\text{CBE}} \\ &= -\,\text{Tr}\,\hat{\rho}_{\text{CBE}} \left(\log_2(\lambda\hat{\rho}_1) \otimes |0\rangle\langle 0| + \log_2((1-\lambda)\hat{\rho}_2) \otimes |1\rangle\langle 1|\right) \\ &= -\,\text{Tr}\left(\lambda\hat{\rho}_1 \log_2(\lambda\hat{\rho}_1) \otimes |0\rangle\langle 0| + (1-\lambda)\hat{\rho}_2 \log_2((1-\lambda)\hat{\rho}_2 \otimes |1\rangle\langle 1|\right) \\ &= -\,\text{Tr}\,\lambda\hat{\rho}_1 \log_2(\lambda\hat{\rho}_1) - \text{Tr}(1-\lambda)\hat{\rho}_2 \log_2((1-\lambda)\hat{\rho}_2) \\ &= S(\lambda\hat{\rho}_1) + S((1-\lambda)\hat{\rho}_2). \end{aligned} \tag{E.7}$$

By an analogous calculation we find

$$S(\hat{\rho}_{\text{BE}}) = S(\lambda\,\text{Tr}_{\text{C}}\,\hat{\rho}_1) + S((1-\lambda)\,\text{Tr}_{\text{C}}\,\hat{\rho}_2). \tag{E.8}$$

Then, combining Eqs. (E.6)-(E.8) we obtain

$$S(C|B)_{\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2} \geq S(\lambda\hat{\rho}_1) + S((1-\lambda)\hat{\rho}_2) - S(\lambda\,\text{Tr}_{\text{C}}\,\hat{\rho}_1) - S((1-\lambda)\,\text{Tr}_{\text{C}}\,\hat{\rho}_2). \tag{E.9}$$

Using the identity $S(\lambda\hat{\rho}) = \lambda\log_2\lambda + \lambda S(\hat{\rho})$, which is straightforwardly proven by working in the eigenbasis of $\hat{\rho}$, the above expression simplifies to

$$S(C|B)_{\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2} \geq \lambda\left[S(\hat{\rho}_1) - S(\text{Tr}_{\text{C}}\,\hat{\rho}_1)\right] + (1-\lambda)\left[S(\hat{\rho}_2) - S(\text{Tr}_{\text{C}}\,\hat{\rho}_2)\right]. \tag{E.10}$$

Finally, using the definition (E.1) for the conditional entropy $S(C|B)_{\hat{\rho}}$ we find

$$S(C|B)_{\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2} \geq \lambda S(C|B)_{\hat{\rho}_1} + (1-\lambda) S(C|B)_{\hat{\rho}_2}, \tag{E.11}$$

which completes the proof. □

We can now prove a useful result regarding the coherent information $I_c(\hat{\rho}_{\text{A},0}, \Xi)$.

Lemma 4. *Let Ξ be a quantum channel from states on $\mathcal{H}_{\text{A}}$ to states on $\mathcal{H}_{\text{B}}$, let $\hat{\rho}_{\text{A},0}$ be a state on $\mathcal{H}_{\text{A}}$, and let $\hat{\rho}_{\text{CB}}$ be the output of the channel $\mathbb{1}_{\text{C}} \otimes \Xi$ applied on the purification of $\hat{\rho}_{\text{A},0}$. Then, $I_c(\hat{\rho}_{\text{A},0}, \Xi) \leq 0$ if $\hat{\rho}_{\text{CB}}$ is separable.*

Proof. Assume $\hat{\rho}_{\text{CB}}$ is separable. Then, it is possible to find pure states $|b_i\rangle \in \mathcal{H}_{\text{B}}$ and $|c_i\rangle \in \mathcal{H}_{\text{C}}$ along with real numbers $p_i > 0$ such that

$$\hat{\rho}_{\text{CB}} = \sum_i p_i |c_i\rangle\langle c_i| \otimes |b_i\rangle\langle b_i|. \tag{E.12}$$

From Eq. (E.2) we have $I_c(\hat{\rho}_{\text{A},0}, \Xi) = -S(C|B)_{\hat{\rho}_{\text{CB}}}$ and hence from Lemma 3 we find

$$-I_c(\hat{\rho}_{\text{A},0}, \Xi) \geq \sum_i p_i S(C|B)_{|c_i b_i\rangle\langle c_i b_i|}, \tag{E.13}$$

where $|c_i b_i\rangle := |c_i\rangle \otimes |b_i\rangle$ are pure, separable states on $\mathcal{H}_{\text{C}} \otimes \mathcal{H}_{\text{B}}$. Since $S(|c_i b_i\rangle) = S(|b_i\rangle) = 0$ we see from Eq. (E.1) that $S(C|B)_{|c_i b_i\rangle\langle c_i b_i|} = 0$, and hence Eq. (E.13) reads

$$-I_c(\hat{\rho}_{\text{A},0}, \Xi) \geq 0, \tag{E.14}$$

which completes the proof. □