On Some Topics in Lévy Insurance Risk Models

by

Tsun Yu Jeff Wong

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Actuarial Science

Waterloo, Ontario, Canada, 2019

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Dr. Alexey Kuznetsov Professor, Department of Mathematics and Statistics York University
Supervisor(s):	Dr. Bin Li Assistant Professor, Department of Statistics and Actuarial Science University of Waterloo Dr. Gordon E. Willmot Professor, Department of Statistics and Actuarial Science University of Waterloo
Internal Member:	Dr. Adam Kolkiewicz Associate Professor, Department of Statistics and Actuarial Science University of Waterloo Dr. David Landriault Professor, Department of Statistics and Actuarial Science University of Waterloo
Internal-External Member:	Dr. Qi–Ming He Professor, Department of Management Sciences University of Waterloo

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Risk management has long been the central focus within actuarial science. There are various risks a typical actuarial company would look into, solvency risk being one of them. This falls under the scope of surplus analysis. Studying of an insurer's ability to maintain an adequate surplus level in order to fulfill its future obligation would be the subject matter, which requires modeling of the underlying surplus process together with defining appropriate matrices to quantity the risk. Ultimately, it aims at accurately reflecting the solvency status to a line of business, which requires developing realistic models to predict the evolution of the underlying surplus and constructing various ruin quantities depending on the regulations or the risk appetite set internally by the company.

While there have been a vast amount of literature devoted to answering these questions in the past decades, a considerable amount of effort is devoted by different scholars in recent years to construct more accurate models to work with, and to develop a spectrum of risk quantities to serve different purposes. In the meantime, more advanced tools are also developed to assist with the analysis involved. With the same spirit, this thesis aims at making contributions in these areas.

In Chapter 3, a Parisian ruin time is analyzed under a spectrally negative Lévy model. A hybrid observation scheme is investigated, which allows a more frequent monitoring when the solvency status to a business is observed to be critical. From a practical perspective, such observation scheme provides an extra degree of realism. From a theoretical perspective, it unifies analysis to paths having either bounded or unbounded variations, a core obstacle for analysis under the context of spectrally negative Lévy model. Laplace transform to the concerned ruin time is obtained. Existing results in the literature are also retrieved to demonstrate consistency by taking appropriate limits.

In Chapter 4, the toolbox of discrete Poissonian observation is further complemented

under a spectrally negative Lévy context. By extending the classical definition of potential measures, which summarizes the law of ruin time and deficit at ruin under continuous observation, to its discrete counterpart, expressions to the Poissonian potential measures are derived. An interesting dual relation is also discovered between a Poissonian potential measure and the corresponding exit measure. This further strengthens the motivation for studying the Poissonian potential measures. To further demonstrate its usefulness, several problems are formulated and analyzed at the end of this chapter.

In Chapter 5, motivated from regulatory practices, a more conservative risk matrix is constructed by altering the traditional definition to a Parisian ruin time. As a starting point, analysis is performed using a Cramér–Lundberg model, a special case of spectrally negative Lévy model. The law of ruin time and its deficit at ruin is obtained. An interesting ordering property is also argued to justify why it is a more conservative risk measure to work with.

To ensure that the thesis flows smoothly, Chapter 1 and 2 are devoted to the background reading. Literature reviews and existing tools necessary for subsequent derivations are provided at the beginning of each chapters to ensure self–containment. A summary and concluding remarks can be found in Chapter 6.

Acknowledgements

First of all, I must take this occasion to thank God for bringing me to Waterloo and giving me the wisdom and perseverance to pursue the doctoral degree.

My gratitude goes to my supervisors, Dr. Bin Li and Professor Gordon E. Willmot, for not only their academic guidance and patience, but also their encouragements and supports in every aspects. Tolerance on my stupidity is particularly appreciated, and I am truly blessed to have them taking me as their student. No words could express my thankfulness towards all these things they have offered me throughout my study.

Meanwhile, credit goes to Dr. Ben Feng, Professor David Landriault and Dr. Ruodu Wang for all the professional advice on my career development. Furthermore, special thanks goes to my colleagues Dr. Mirabelle Huynh, Dr. Xiaobai Zhu and Dr. Kenneth Zhou for all the assistance in school.

I would also like to extend my thankfulness to my friends I met in Canada. I could not imagine how I would manage to complete my study without you all. In particular, I would like to thank Mr. Alan Pak Hay Chan, Dr. Tsz Chiu Kwok and Dr. Michael Ka Shing Ng for all the companions during my tough times. You all lessen my hardship during my research journey.

Last but not least, I must thank my mum, my dad and my sister for their unconditional love.

Dedication

In memory of my grandpa and grandma.

Table of Contents

List of Tables **xiii**

List of Figures **xiv**

1 Introduction **1**

1.1 Spectrally Negative Lévy Process . . . 1

1.2 Ruin Time . . . 3

1.2.1 Ordinary Ruin . . . 3

1.2.2 Parisian Ruin . . . 4

1.3 Occupation Time . . . 6

1.4 Observation Scheme . . . 7

1.5 Outline of Thesis . . . 8

2 Overview **10**

2.1 Gerber–Shiu function 11

2.2 Spectrally Negative Lévy Model 11

2.2.1 Model Construction 11

2.2.2 Path Variation and Regularity 13

2.2.3 Scale Functions 13

2.2.4 Potential and Exit Measure 15

2.2.5 Exit Formula 17

2.3 Cramér–Lundberg Model 18

2.3.1 Model Construction 18

2.3.2 Discounted Density Function 19

3 A Parisian Risk Model Under a Hybrid Observation Scheme 21

3.1 Introduction 21

3.2 A Hybrid Observation Scheme and the Associated Time of Ruin 23

3.3 Preliminaries 25

3.4 Main Result 26

3.5 Limiting Results 27

3.5.1 Obtaining Laplace transform of classical Parisian ruin 29

3.5.2 Retrieving classical Parisian ruin probability 29

3.6 Example 30

3.6.1 Brownian Motion 31

3.6.2 Compound Poisson Process with Exponential Claims 33

3.7 Appendix 35

3.7.1 Proof of Lemma 10 35

3.7.2 Proof of Theorem 11 35

3.7.3 Proof of Corollary 12 38

3.7.4 Proof of Lemma 13 39

3.7.5 Proof of Proposition 14 40

3.7.6 Proof of Proposition 15 42

4 Poissonian Potential Measures for Spectrally Negative Lévy Risk Models 44

4.1 Introduction 44

4.2 Preliminaries 46

4.3 Main Results 47

4.3.1 Poissonian Scale Functions 48

4.3.2 Poissonian Potential Measures 51

4.4 Interplay Between Poissonian Potential Measures and Exit Measures 54

4.5 Application – Occupation Time under Hybrid Observation Scheme 56

4.5.1 Notations 56

4.5.2 Main Results 58

4.6 Application – Parisian Ruin with Poissonian Observations 60

4.6.1 Notations 60

4.6.2 Main Results 61

4.7 Appendix 63

4.7.1 Proof of Theorem 20 63

4.7.2 Proof of Proposition 21 75

4.7.3 Proof of Corollary 23 77

4.7.4 Proof of Theorem 24 79

4.7.5 Proof of Theorem 25 81

4.7.6 Proof of Theorem 26 82

4.7.7 Proof of (4.44) 84

5 Modified Parisian Ruin Time and its Risk Management Implication 86

5.1 Introduction 86

5.2 Construction of the Modified Parisian Ruin Time 88

5.3 Preliminaries 90

5.4 Main Result 92

5.4.1 Evaluation of Gerber–Shiu Function 92

5.4.2 An Ordering Property 94

5.5 Example 95

5.6 Appendix 102

5.6.1 Proof of Theorem 28 102

5.6.2 Proof of Corollary 29 104

6 Conclusion and Future Works **105**

References **108**

List of Tables

3.1 Ruin probability for Brownian motion model with different λ. 32

3.2 Ruin probability for Brownian motion model with different b. 32

3.3 Ruin probability for Cramér–Lundberg model with different λ. 34

3.4 Ruin probability for Cramér–Lundberg model with different b. 34

5.1 Different Parisian ruin probabilities when $u = 0$. 100

5.2 Different Parisian ruin probabilities when $u = 50$. 101

List of Figures

3.1 Illustration of Parisian ruin under hybrid observation scheme. 24

4.1 Illustration of occupation time under random observation. 57

4.2 Illustration of Parisian ruin time with an exponential grace period under Poissonian observation. 61

5.1 Illustration of modified Parisian ruin under continuous observation. 90

5.2 Illustration of ordering to different ruin times. 94

Chapter 1

Introduction

Surplus analysis of an insurance company has long been a central focus in actuarial risk theory. In particular, the credibility for an insurance company so as to fulfill its future obligation would be of a crucial concern both from policyholder's and company's perspective. As a result, construction and analysis of different risk metrics pertained to the surplus process of a business has been the subject matter.

In what follows, denote $X = \{X_t\}_{t\geq 0}$ to be a surplus process defined on a complete probability space $\left(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ describing the surplus level of a business. Here, t represents the time. In order that the analysis is made possible, assumptions on the surplus process together with definitions of different risk quantities are necessary.

1.1 Spectrally Negative Lévy Process

The theoretical foundation of actuarial risk theory traces back to the time where Cramér [1955] introduces the Cramér–Lundberg model such that surplus process is modeled by a compound Poisson risk process. Since then, a lot of analysis has been performed under this framework. It is not until early 2000s that scholars attempt to generalize the setting by working with spectrally negative Lévy processes, a special class of Lévy processes.

Definition 1. A process $X = \{X_t\}_{t\geq 0}$ is said to be a Lévy process if it has the following properties:

(i) (Cádlág path) The paths of X are $\mathbb{P}$–almost surely right continuous with left limits.

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) (Stationary increment) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) (Independent increment) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u\}_{u\leq s}$.

A spectrally negative Lévy process is a special case of a Lévy process in a sense that it allows no positive jumps. As a direct consequence of the above definition, spectrally negative Lévy processes permit the modeling on sample paths having both Brownian components (corresponding to random noise in the surplus process) and downward jumps (corresponding to claim losses), which can usually be observed in actuarial empirical data.

It is worthwhile to mention that spectrally negative Lévy process indeed contains a rich class of processes that are extensively studied in the classical surplus analysis. The compound Poisson process, Brownian motion and the superposition of both are some examples. The diverse spectrum covered by spectrally negative Lévy process makes it a prevalent model to work with until these days.

Throughout the thesis, unless otherwise specified, X is assumed to evolve as a spectrally negative Lévy process. Its law and expectation when initial surplus equals $x > 0$ is denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$ respectively. Note that X under $\mathbb{P}_x$ has the same law as $x + X$ under $\mathbb{P}_0$, a property called spacial homogeneity. For brevity, $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$.

1.2 Ruin Time

Whenever a business is deemed incapable of repaying its future debt, it would be natural to consider discontinuing the business to protect the insurance company from incurring a severe loss. Ruin time is thus understood as the moment that a business is terminated. Depending on differences arose from the interpretations of debt payment capability, together with discrepancies in risk appetite among companies, ruin time takes a variety of definitions. Below is two examples that are often encountered in the literature.

1.2.1 Ordinary Ruin

It is natural to demand the surplus of a business to stay above certain threshold so as to remain solvent. This leads to the classical concept of default being the first passage time the surplus process downcrosses a threshold such that it indicates a credit hazard. A legitimate choice of threshold could be the minimal liquidity adequacy set within the company or the statutory capital requirement. Due to spacial homogeneity, the threshold is often taken to be zero so as to unify mathematical analysis.

Note that exit time constructed this way is called ordinary ruin time, as opposed to the construction of Parisian ruin time to be discussed in the following subsection.

With respect to the classical Cramér–Lundberg setting, a huge amount of literature is contributed to the study of ordinary ruin in the past few decades, particularly after the introduction of the so–called Gerber–Shiu function by Gerber and Shiu [1998], which is essentially the expected discounted penalty due at ruin with amount of penalty possibly depending on other ruin quantities, the most conventional choice being the time of ruin, the surplus prior to ruin and the deficit at ruin. A precise definition to Gerber–Shiu function is deferred to

Section 2.1. Lin and Willmot [1999] and Lin and Willmot [2000] then systematically analysis this function by introducing the idea of defective renewal equation. Keeping in mind of the possible interpretation to the Gerber–Shiu function as a Laplace transform, Dickson and Willmot [2005] successfully obtain the (defective) density of time to ruin by adopting an analytic Laplace transform inversion, which is further extended by Landriault and Willmot [2009] to derive the joint (defective) density of ruin quantities pertained to ordinary ruin. These works lay a technical foundation to many problem and model variants in surplus analysis, just to name a few examples, the barrier strategy dividend problems (e.g. Lin and Pavlova [2006]) and Sparre Andersen risk models (e.g. Gerber and Shiu [2005]).

When it comes to the spectrally negative Lévy setting, Yang and Zhang [2001] first adopted theories from Lévy processes in risk theory. Since then, surplus model of this kind attracts more attention. Analogue to the Gerber–Shiu analysis in the classical model, Biffis and Kyprianou [2010] successfully evaluated the expected discounted joint densities to ruin quantities pertained to ordinary ruin.

1.2.2 Parisian Ruin

The concept of Parisian ruin is first motivated by Parisian options in the finance literature (e.g. Chesney et al. [1997] and Schröder [2003]), where an option is knocked in or knocked out once the stock price has stayed above or below a threshold continuously for a certain amount of time. Under the context of surplus analysis, this translates to a delay in declaration of ruin. This is achieved by granting a grace period for the business to recover from the negative surplus (assuming a zero threshold) once it downcrosses the threshold. Contrary to ordinary ruin time, Parisian ruin time is hence taken as the first time the surplus process stays below the threshold continuously over the entire grace period. Such concept of ruin time can be seen more appropriate for the following reasons.

According to Gerber [1990], the probability of ordinary ruin is usually very small. Even a negative surplus is recorded, the company can still survive in long run (due to the positive loading) and, as commented by Egídio dos Reis [1993], may actually recover within a short time depending on the amount of deficit. The classical definition of ruin is thus too prudent to be used in quantifying risk in a sense that it trades off the potential profitability of a business, should it recovers quickly, for an over–conservative protection. In this regard, the definition of Parisian ruin strives for a better balance between solvency and profitability aspects.

On the other hand, it is worthwhile to mention that Parisian ruin is indeed a more consistent representation of bankruptcy and liquidation as defined by Chapter 11 US Bankruptcy Code in corporate finance. In essence, instead of immediate liquidation, a business may still remain operational should it be unable to continue fulfilling its obligation due to mild fiscal situation, during which it is given a chance for debt restructuring. A more detailed elaboration in this regard can be found in Li et al. [2014].

Parisian ruin time is first proposed by Dassios and Wu [2008] under the context of the classical compound Poisson model. After that, investigations on a compound binomial and renewal risk setting are respectively performed by Czarna et al. [2014] and Wong and Cheung [2015], while that on a spectrally negative Lévy setting are performed by Czarna and Palmowski [2011], Loeffen et al. [2013], Landriault et al. [2014] and Czarna [2016].

In effect of the Parisian concept applied to defining ruin, alternative applications are evolved within the risk theory literature. One particular examples is the Parisian implementation delay on barrier strategy dividend problems. A more detailed problem formulation together with the corresponding analyses can be found in, for example, Dassios and Wu [2009], Czarna and Palmowski [2014] and Cheung and Wong [2017].

1.3 Occupation Time

Another risk quantities that draws insight to risk management is the occupation time. It is defined as the total duration of the surplus process in a certain interval of interest, the most notable choice being the negative half–plane, which is intuitively useful in examining the health of a business. Further elaborated by Egídio dos Reis [1993], expected time spent with negative surplus can be viewed as the expected recovery time of a business, which can be used to infer whether a business may recover from ruin within a short time. The total duration in the negative half–plane is sometime called the total time spent in red. Alternatively, as pointed out by Landriault and Shi [2015], another viable choice would be an open interval indicating a low surplus level, which provides an intuition on how long a business would be exposed to liquidity stress.

Problems related to occupation time has been studied by Egídio dos Reis [1993], Dickson and Egídio dos Reis [1996] and Zhang and Wu [2002], where emphasis is put on some special classes of spectrally negative Lévy processes. Landriault et al. [2011] and Loeffen et al. [2014] then successfully attempts this problem under the spectrally negative Lévy framework, followed by Renaud [2014] and Kyprianou et al. [2014] who work on a refracted spectrally negative Lévy processes. The case for a diffusion process and a Markov additive process is also studied in Li and Zhou [2013] and Landriault and Shi [2015] respectively.

In fact, occupation time is known to be useful in characterizing some advanced derivatives in finance (e.g. Linetsky [1999] and Fusai [2000]). Besides, there is also a high resemblance between problems related to occupation time and bankruptcy in an Omega risk model. Interested readers are directed to Gerber et al. [2012] for the construction of an Omega risk process, and also discussions in Landriault et al. [2011] and Li and Zhou [2013].

1.4 Observation Scheme

Observation scheme refers to how a business is intercepted as time goes. While this has little direct implication on the risk quantification, it is more of an implicit assumption embedded in the model and, accordingly, the ruin quantities defined in Section 1.2 and 1.3.

Under most generic settings, a business is typically assumed to be continuously inspected at all time. From a practical standpoint, this can rarely be achieved on an ongoing basis due to high cost. A discrete–time observation is thus considered more reasonable such that the business is inspected only at discrete time points. However, as pointed out by Albrecher et al. [2011], ruin quantities under discrete–time model usually do not have explicit expressions. Suggested by Albrecher and Ivanovs [2016], a possible solution is to assume a Poisson observation structure, meaning observation is done at the arrival epochs of an independent Poisson process. This serves as a bridge between continuous–time and discrete–time observation while preserving the tractability of the solutions. Such observation structure is demonstrated to be useful in Albrecher and Ivanovs [2013] and Albrecher et al. [2016]. As a generalization to the Poissonian observer, Albrecher et al. [2011] and Albrecher et al. [2013] assumes time arrival between observations to be Erlang(n) distributed. The case for a deterministic periodic observation can then be seen as a limiting case of an Erlang(n) inter–arrival. This is achieved by fixing its mean and letting n goes to infinity such that its variance vanishes, a technique known as Erlangization. More works along this line can be found in Choi and Cheung [2014], Landriault et al. [2014] and Zhang and Cheung [2018].

The importance of a Poissonian observation (i.e. an Erlang(1) discrete–time observation) is particularly stressed. By the memoryless property to an exponential random variable, the ordinary ruin under the discrete observation model is equivalent to that having an exponential Parisian delay under the continuous observation model. This implies solving one problem in one of the models would provide insight over the related problem in another. Such dual relationship has been spotted and utilized by different authors. Interested readers are direct to Cheung and Wong [2017] and Baurdoux et al. [2016] for a more detailed elaboration in this regard.

Under the Lévy context, Poissonian observation also exhibits a connection to potential density of a spectrally negative Lévy process. Technical details in this aspect can be found in Bertoin [1997] and Kyprianou [2014], while a summary of results can be found in Albrecher et al. [2016].

Keeping in mind the concept of Parisian ruin and occupation time, it is indeed realistic to impose a mixed observation scheme with observation frequency depending on the surplus level of the business. Such an idea can be found in Avanzi et al. [2013], Choi and Cheung [2014] and Cheung and Wong [2017].

1.5 Outline of Thesis

The rest of the proposal is organized as follows. In Chapter 2, an overview on theories and results regarding spectrally negative Lévy processes will be summarized. Compound Poisson processes, a special case of which, will also be discussed. In Chapter 3, a model with a hybrid observation scheme will be considered under the spectrally negative Lévy model. The main focus will be on explaining the incentive behind such idea, together with the study of ruin quantity under such observation scheme. As a follow up, Chapter 4 is devoted to the development of tools that is relevant for analysis under the context of Poissonian observation. Examples are also included to demonstrate how these tools may be used for various purposes. Chapter 5 introduces a new concept to Parisian ruin. Motivations, together with the construction of the ruin quantity, will be covered. Analysis would be performed under the context of Cramér–Lundberg model. Finally, Chapter 6 serves as a concluding chapter to the thesis. Discussion of potential avenues for future research is provided.

As a final remark of this chapter, it is aware that the contents, while highly related, have a somewhat different emphasis such that the overview in Chapter 2 is clearly very limited. For better understanding, at the beginning of Chapter 3 to 5, necessary preliminaries which are relevant to the specific content will be provided.

Chapter 2

Overview

This chapter is devoted to a review of fundamental quantities and models that are commonly encountered in the surplus analysis literature.

To better assist with the presentation, we shall define the following two exit times beforehand, namely the continuously observed ruin time and the Poissonian observed ruin time. Denote the continuously observed ruin time of X for level $a \in \mathbb{R}$ as

$$\tau_a^{+(-)} = \inf \left\{ t \geq 0 : X_t > (<)\, a \right\},$$

with the convention that $\inf \emptyset = \infty$. Meanwhile, let T_i be the arrival times of an independent Poisson process of rate $\lambda > 0$. Poissonian observed ruin time of X for level $a \in \mathbb{R}$ can then be written as

$$T_a^{\lambda,+(-)} = \inf \left\{ T_i : X_{T_i} > (<)\, a \right\}.$$

Note that ruin time observed at Poisson arrival time can be seen as a generalization of continuously observed ruin time by realizing $T_a^{\lambda,+} \downarrow \tau_a^+$ and $T_a^{\lambda,-} \downarrow \tau_a^-$, both in an almost sure sense as the observation rate λ tends to infinity. This may be used to retrieve the classical exit identities.

2.1 Gerber–Shiu function

The study of Gerber–Shiu function first introduced by Gerber and Shiu [1998] has been the pivotal subject matter among the wide spectrum of models used within the literature. Denote $X_{\tau_0^- -}$ the left limit of X_t at $t = \tau_0^-$.The Gerber–Shiu function generally takes the form

$$\mathbb{E}_x \left[e^{-\delta \tau_0^-} w \left(X_{\tau_0^- -}, \left| X_{\tau_0^-} \right| \right) 1_{\left\{ \tau_0^- < \infty \right\}} \right], \quad x \geq 0,$$

where $\delta \geq 0$ represents the discount factor and w represents the penalty function satisfying some mild integrability condition. From a practical perspective, it can be interpreted as the expected discounted penalty due at ruin with penalty depending on the ruin time. From a theoretical perspective, with a proper choice of w, it can be viewed as the Laplace transform to $\left(\tau_0^-, X_{\tau_0^- -}, \left| X_{\tau_0^-} \right| \right)$ such that successfully evaluating the Gerber–Shiu function implies a full characterization to the law of the risk quantities involved due to one–one correspondence between a Laplace transform and a distribution.

Depending on the models and distributional assumptions involved, different techniques are involved in evaluating the Gerber–Shiu function. Due to the scope of this thesis, detailed discussions are omitted here. Interested readers are directed to references contained in Subsection 1.2.1

2.2 Spectrally Negative Lévy Model

2.2.1 Model Construction

There exists a representation via the Laplace exponent, which is a function $\psi : [0, \infty) \to \mathbb{R}$, such that every surplus process X can be uniquely characterized.

Theorem 1. *((1.1) of Kuznetsov et al. [2012]) For any spectrally negative Lévy processes $X = \{X_t\}_{t\geq 0}$, the Laplace exponent $\psi(\theta)$ satisfying*

$$\psi(\theta) = \frac{1}{t}\ln \mathbb{E}\left[e^{\theta X_t}\right], \quad \theta \geq 0$$

exists and is given by

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{0}\left(e^{\theta z} - 1 - \theta z 1_{(-1,0)}(z)\right)\Pi(\mathrm{d}z),$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Π is a σ–finite measure on $(-\infty, 0)$, referred to as the Lévy measure, such that

$$\int_{-\infty}^{0}\left(1 \wedge z^2\right)\Pi(\mathrm{d}z) < \infty.$$

Heuristically speaking, the Lévy measure above contains information regarding frequency and size of jumps.

Laplace exponent is also useful in expressing the net profit condition (also known as the positive loading condition). To avoid triviality, it is legitimate to consider only the case that the surplus process does not have a monotone sample path and that it drifts to infinity in long run. Consequently, unless otherwise stated, it is assumed that $|X|$ is not a subordinator and X satisfies the net profit condition given by (e.g. Kyprianou [2014])

$$\psi'(0+) = \mathbb{E}[X_1] > 0. \tag{2.1}$$

To be used later, for any given $q \geq 0$, write

$$\psi_q(\theta) = \psi(\theta) - q. \tag{2.2}$$

According to Kyprianou [2014], it is known that $\psi(0) = 0$ and $\lim_{\theta\to\infty}\psi(\theta) = \infty$. Together with the fact that it is strictly convex, the equation $\psi_q(\theta) = 0$ is know to have at least one positive solution. This allows the definition of the right inverse

$$\Phi_q = \sup\{\theta \geq 0 : \psi_q(\theta) = 0\}$$

for each $q \geq 0$. Due to the net profit condition given in (2.1), $\Phi := \Phi_0 = 0$.

2.2.2 Path Variation and Regularity

Before the introduction of the scale functions in Subsection 2.2.3, it is important to understand some additional properties to spectrally negative Lévy process which are closely linked to the properties of scale functions. Above all, it is crucial to identify whether the process X has path of bounded or unbounded variation.

Lemma 2. ((2.1) of Kuznetsov et al. [2012]) X has path of bounded variation if and only if

$$\sigma = 0 \text{ and } \int_{-1}^{0} |z| \Pi (\mathrm{d}z) < \infty.$$

The significance for X having a bounded variation is that it can be decomposed into the form $X_t = \delta t - S_t$, where $\delta = \gamma - \int_{-1}^{0} |z| \Pi (\mathrm{d}z)$ is interpreted as the drift, and $\{S_t\}_{t \geq 0}$ is a pure jump subordinator.

Path variation also impacts whether X takes an almost surely positive amount of time before visiting the open lower half line, a concept that is tied with regularity.

Lemma 3. (Page 232 of Kyprianou [2014]) 0 is regular for $(0, \infty)$ irrespective of path variation, whereas 0 is regular for $(-\infty, 0)$ if and only if X has unbounded variation.

Loosely speaking, this indicates that X takes an almost surely positive amount of time before visiting the open lower half line unless it has paths of unbounded variation.

2.2.3 Scale Functions

There are two families of function that frequently appear in identities concerning exit times. They are usually called scale functions of the first kind and the second kind in the literature.

Definition 2. For $q \geq 0$, the first scale function $W^{(q)}(x)$ is a non–negative, strictly increasing and continuous function on $[0, \infty)$ with $W^{(q)}(x) = 0$ for $x < 0$, characterized by the transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi_q(\theta)} \tag{2.3}$$

with $\theta > \Phi_q$. For brevity, $W(x) = W^{(0)}(x)$.

The asymptotic behaviors of $W^{(q)}(x)$ and $W^{(q)'}(x)$ at zero and infinity can be found in Kuznetsov et al. [2012] and are reproduced below without proof. Note that the limit at zero takes different values depending on the path variation of the underlying process.

Lemma 4. (Lemma 3.1 and 3.2 of Kuznetsov et al. [2012]) For $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} 0 & \text{for } X \text{ has unbounded variation,} \\ \dfrac{1}{\delta} & \text{for } X \text{ has bounded variation,} \end{cases}$$

and

$$W^{(q)'}(0+) = \begin{cases} \dfrac{2}{\sigma^2} & \text{where } \sigma \neq 0 \text{ or } \Pi(-\infty, 0) = \infty, \\ \dfrac{\Pi(-\infty, 0) + q}{\delta^2} & \text{where } \sigma = 0 \text{ and } \Pi(-\infty, 0) < \infty, \end{cases}$$

where the first case is understood as $+\infty$ when $\sigma = 0$.

Lemma 5. (Lemma 3.3 of Kuznetsov et al. [2012]) For $q \geq 0$,

$$\lim_{x \to \infty} e^{-\Phi_q x} W^{(q)}(x) = \frac{1}{\psi'(\Phi_q)},$$

where the right hand side is understood in the limiting sense when $q = 0$, i.e.

$$\lim_{x \to \infty} W(x) = \frac{1}{\psi'(0+)}. \tag{2.4}$$

The second scale function is defined with respect to the first scale function as follows.

Definition 3. For $q \geq 0$, the second scale function $Z^{(q)}(x, \theta)$ is defined by

$$Z^{(q)}(x, \theta) = \begin{cases} e^{\theta x}\left(1 - \psi_q(\theta) \int_0^x e^{-\theta y} W^{(q)}(y)\,\mathrm{d}y\right), & x \geq 0, \\ e^{\theta x}, & x < 0 \end{cases} \tag{2.5}$$

with $\theta \geq 0$. For brevity, $Z^{(0)}(x, \theta) = Z(x, \theta)$ and $Z^{(q)}(x, 0) = Z^{(q)}(x)$.

Note that for $\theta > \Phi_q$, by (2.3), $Z^{(q)}(x,\theta)$ can be expressed as

$$Z^{(q)}(x,\theta) = \psi_q(\theta) \int_0^\infty e^{-\theta y} W^{(q)}(x+y)\,\mathrm{d}y, \quad x \geq 0. \tag{2.6}$$

The following limiting results to scale functions are also found to be important. They are quoted here without proof.

Lemma 6. ((6), (34) of Albrecher et al. [2016] and Exercise 8.5 of Kyprianou [2014]) It holds that

$$\lim_{x\to\infty} \frac{Z(x,\theta)}{W(x)} = \frac{\psi(\theta)}{\theta - \Phi} \quad \text{for } \theta > \Phi, \tag{2.7}$$

$$\lim_{x\to 0} \int_0^x \frac{e^{-\theta y} W(y)}{W(x)} \mathrm{d}y = 0 \quad \text{for } \theta \geq 0, \tag{2.8}$$

and

$$\lim_{a\to\infty} \frac{W^{(q)}(a-x)}{W^{(q)}(a)} = e^{-\Phi_q x}. \tag{2.9}$$

As commented by Loeffen et al. [2013], closed–form expressions to scale functions may not be known explicitly under all model settings. Numerical inversion on (2.3) may become handy in obtaining an approximation to $W^{(q)}(x)$, and hence $Z^{(q)}(x,\theta)$ due to (2.5). A possible numerical method to compute the scale function could be found in Surya [2008].

It should be emphasized that scale functions are indeed robust candidates in describing a variety of fluctuation identities concerning exit problems. Despite the semi–explicitness in nature, potential measures and exit formulae (to be introduced subsequently) can both be expressed nicely via the introduction of these functions. They thereby form an important and natural family within the spectrally negative Lévy context.

2.2.4 Potential and Exit Measure

Keeping in mind the motivation of surplus analysis mentioned in Chapter 1, studying of overshoot distribution at ruin time has been the subject matter. Among the multitude of tools used in the literature, the one that is most intimately linked to it is the formula first attributed to Bertoin [1996]. Classically referred to as the q–potential measures killed on continuous exiting (where $q \geq 0$), or simply the (q–)potential measures, they shed light on a vast variety of analysis pertained to spectrally negative Lévy processes. Just to name a few examples, see Alili and Kyprianou [2005] and Avram et al. [2004] for the applications in option pricing. Also see Kluppelberg et al. [2004] in the context of ruin probability and Avram et al. [2004] in the optimal control.

Potential measures for a spectrally negative Lévy process are defined as follows.

Definition 4. For $q \geq 0$, the q–potential measures of a spectrally negative Lévy process killed on continuous exiting are defined by

$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y\right)\mathrm{d}t = \theta^{(q)}\left(y-x\right)\mathrm{d}y, \quad x, y \in \mathbb{R}, \tag{2.10}$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^+\right)\mathrm{d}t = r_+^{(q)}\left(x,y\right)\mathrm{d}y, \quad x, y \leq 0, \tag{2.11}$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^-\right)\mathrm{d}t = r_-^{(q)}\left(x,y\right)\mathrm{d}y, \quad x, y \geq 0, \tag{2.12}$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^- \wedge \tau_a^+\right)\mathrm{d}t = u^{(q)}\left(x,y;a\right)\mathrm{d}y, \quad x, y \in [0,a]\,. \tag{2.13}$$

The functions $\theta^{(q)}\left(y\right), r_+^{(q)}\left(x,y\right), r_-^{(q)}\left(x,y\right)$ and $u^{(q)}\left(x,y;a\right)$ are called potential densities. An exhaust summary to expressions of the potential densities can be found in Theorem 2.7 of Kuznetsov et al. [2012] while interested readers are referred to Kyprianou [2014] for a detailed proof to these results.

Exit measure, on the other hand, contain information on the distribution of ruin time and overshoot via the evaluation of Gerber–Shiu like functions. The following summarizes the results to these exit measures.

Lemma 7. ((2.34) of Kuznetsov et al. [2012]) Suppose $\tau = \tau_0^- \wedge \tau_a^+$. For $q \geq 0$, the exit measures are given by

$$\mathbb{E}_x\left[e^{-q\tau}1_{\left\{\tau<\infty, X_\tau\in \mathrm{d}y, X_{\tau^-}\in \mathrm{d}z\right\}}\right] = \int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}z, t<\tau\right)\Pi\left(\mathrm{d}y - z\right),$$

where $x \in [0,a]$, $y \in (-\infty, 0)$ and $z \in [0,a)$.

It can be seen that exit measure can be expressed in terms of the potential measures defined previously. This further enhances the motivation for defining and studying the potential measures.

2.2.5 Exit Formula

While probabilistic behavior to ruin time and overshoot may be successfully captured by exit measures, these may also be summarized using Laplace transforms. Expressions to these Laplace transforms are generally called exit formulae. Depending on the exit barriers considered, there are generally two types of exit quantities.

Definition 5. For $q, \theta \geq 0$ and $0 \leq x \leq a$, under the context of continuously observe ruin time, the following Laplace transforms are referred to as the one–sided and two–sided exit formulae.

one–sided exit formulae	two–sided exit formulae
$\mathbb{E}_x\left[e^{-q\tau_0^- + \theta X_{\tau_0^-}}1_{\left\{\tau_0^-<\infty\right\}}\right]$	$\mathbb{E}_x\left[e^{-q\tau_0^- + \theta X_{\tau_0^-}}1_{\left\{\tau_0^-<\tau_a^+\right\}}\right]$
$\mathbb{E}_x\left[e^{-q\tau_a^+}1_{\left\{\tau_a^+<\infty\right\}}\right]$	$\mathbb{E}_x\left[e^{-q\tau_a^+}1_{\left\{\tau_a^+<\tau_0^-\right\}}\right]$

Definition 6. For $q, \theta \geq 0$ and $0 \leq x \leq a$, under the context of Poissonian observed ruin time, the following Laplace transforms are referred to as the one–sided and two–sided exit formulae.

one–sided exit formulae	two–sided exit formulae
$\mathbb{E}_x\left[e^{-qT_0^- + \theta X_{T_0^-}} 1_{\left\{T_0^- < \infty\right\}}\right]$	$\mathbb{E}_x\left[e^{-qT_0^- + \theta X_{T_0^-}} 1_{\left\{T_0^- < \tau_a^+\right\}}\right]$
$\mathbb{E}_x\left[e^{-qT_a^+ + \theta\left(X_{T_a^+} - a\right)} 1_{\left\{T_a^+ < \infty\right\}}\right]$	$\mathbb{E}_x\left[e^{-qT_a^+ + \theta\left(X_{T_a^+} - a\right)} 1_{\left\{T_a^+ < \tau_0^-\right\}}\right]$
	$\mathbb{E}_x\left[e^{-q\tau_0^- + + \theta X_{\tau_0^-}} 1_{\left\{\tau_0^- < T_a^+\right\}}\right]$
	$\mathbb{E}_x\left[e^{-qT_0^- + \theta X_{T_0^-}} 1_{\left\{T_0^- < T_a^+\right\}}\right]$
	$\mathbb{E}_x\left[e^{-qT_a^+ + \theta\left(X_{T_a^+} - a\right)} 1_{\left\{T_x^+ < T_0^-\right\}}\right]$

Expressions to these exit quantities can be found in various literature. As an example, results pertained to the continuously observed ruin time can be found in Kyprianou [2014], whereas results pertained to the Poissonian observed ruin time can be found in Albrecher et al. [2016]. Readers are also encouraged to look into the references therein for various applications of these Laplace transforms.

It is worthwhile to point out that the one–sided exit formulae are indeed special cases of the two–sided exit formulae. Exploiting the spatial homogeneity property to a spectrally negative Lévy process, one can easily retrieve the one–sided exit formulae by taking the limit that $a \uparrow \infty$.

2.3 Cramér–Lundberg Model

2.3.1 Model Construction

Following the discussion in Subsection 2.2.1, suppose the Laplace exponent $\psi(\theta)$ takes the following more explicit form as

$$\psi(\theta) = c\theta + \lambda \int_{\mathbb{R}_+} \left(1 - e^{\theta x}\right) F(\mathrm{d}x)$$

with $c, \eta > 0$, where $\mathbb{R}_+$ and $F(\cdot)$ denotes the set of positive real numbers and the law of a random variable respectively, then the surplus process X reduces to the classical compound Poisson process such that it can be written as

$$X_t = ct - \sum_{i=1}^{N(t)} Y_i.$$

Here, c is interpreted as the constant rate of income per unit time, $\{Y_i\}_{i=1}^{\infty}$ denotes a sequence of jumps that keeps track of the $i-$th claim size, and $\{N_t\}_{t\geq 0}$ represents a Poisson counting process with intensity λ. The net profit condition in (2.1) thereby reduces to

$$c > \lambda \mathbb{E}\left[Y_i\right],$$

which has the heuristic interpretation that the rate of cash inflow should be greater than that of expected cash outflow.

2.3.2 Discounted Density Function

Analogous to potential and exit measures, under the context of compound Poisson process, discounted densities are frequently used to describe the law of ruin time and overshoot. They serve as one of the indispensable tools among many others available for analysis under the Cramér–Lundberg model.

Discounted joint density function of $\left(X_{\tau_0^- -}, \left|X_{\tau_0^-}\right|\right)$ are defined as follows.

Definition 7. For $u, \delta \geq 0$, the discounted density function $h_\delta(x, y|u)$ satisfies

$$\mathbb{E}_u\left[e^{-\delta\tau_0^-} w\left(X_{\tau_0^- -}, \left|X_{\tau_0^-}\right|\right) 1_{\left\{\tau_0^- < \infty\right\}}\right] = \int_0^\infty \int_0^\infty w(x, y)\, h_\delta(x, y|u)\, \mathrm{d}x\mathrm{d}y.$$

Denote $h_\delta(x|u)$ and $h_\delta(y|u)$ the marginal defective density function of $X_{\tau_0^- -}$ and $\left|X_{\tau_0^-}\right|$

respectively. With the above definition, they can be calculated by

$$h_\delta(x|u) = \int_0^\infty h_\delta(x, y|u)\, \mathrm{d}y,$$
$$h_\delta(y|u) = \int_0^\infty h_\delta(x, y|u)\, \mathrm{d}x.$$

There have been many papers devoted in studying this quantity. Detailed discussions, proofs and expressions to these discounted density functions can be found in Tsai [2001] and references therein.

Chapter 3

A Parisian Risk Model Under a Hybrid Observation Scheme

3.1 Introduction

This chapter is devoted to analyze Parisian ruin time under a hybrid observation scheme.

To study Parisian ruin under the context of spectrally negative Lévy process which may have an unbounded variation, in the spirit of excursion theory, researchers mainly adopt an approximation approach (e.g. Dassios and Wu [2010], Czarna and Palmowski [2011], Landriault et al. [2011] and Loeffen et al. [2013]) to overcome the difficulty caused by the infinite activity. This approximation approach essentially perturbs the sample paths of the underlying process in a spatial dimension, hence is referred as *a spatial approximation* approach. While spatial approximation successfully analyzes processes having an unbounded variation, the case for a bounded variation is tackled separately. To avoid the separate treatment of bounded and unbounded variation cases, a *temporal approximation* approach is proposed, leading to the concept of hybrid observation scheme to be further elaborated below.

Under the hybrid observation scheme, the surplus process X is first monitored discretely at Poisson arrival times with rate λ until negative surplus is observed. Then a grace period of fixed length $b > 0$ is granted to the insurer and X is observed continuously during this grace period. The insurer is considered as ruined at the end of the grace period unless the surplus recovers to a pre–specified level $a \geq 0$ within the grace period. In the latter case, the observation scheme will be switched back to the discrete scheme as soon as the surplus recovers to level a. A mathematical formulation and the associated illustrative graph of the hybrid observation scheme can be found in Section 3.2. Note that the hybrid observation scheme essentially delays the classical Parisian stopping time, hence giving rise to the temporal approximation approach.

The contribution of this work is three–fold. First, a new risk model to the actuarial risk theory is proposed. The practical meaning of the model is as follows. If the surplus is nonnegative based on the last observation, the observation scheme remains discrete, which is less onerous from the insurer point of view. Once the surplus is observed to be negative, the observation scheme is switched to the more stringent continuous scheme during the grace period, which is consistent with the potential financial seriousness of the situation. If the surplus is successfully restored to a fixed barrier a, this indicates the insurance business is healthy and observations are switched back to the discrete scheme. Second, we generalize the results of Loeffen et al. [2013] as this model reduces to the classical Parisian ruin by letting $a = 0$ and $\lambda \uparrow \infty$. Thirdly, and most interestingly from a theoretical point of view, the hybrid observation scheme lays a temporal approximation approach in handling the Parisian ruin problem. To be illustrated in Section 3.4, the case for a bounded variation and unbounded variation can be treated in a unified way.

The rest of the chapter is organized as follows. Section 3.2 is devoted to the mathematical formulation of the hybrid observation scheme as well as the associated time to ruin. Necessary results in the existing literature used for deriving the main results in Section 3.4 are reviewed in Section 3.3. Section 3.5 is devoted to retrieve different limiting quantities. Examples are illustrated in Section 3.6. Proofs to all necessary lemmas, the main result, and its corresponding corollaries and propositions are deferred to the end of this chapter.

3.2 A Hybrid Observation Scheme and the Associated Time of Ruin

Define a sequence of discrete observation times $\{\xi_n\}_{n\in\mathbb{N}}$ as follows. For ease of notation, denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. Let $\xi_0 = 0$, and for $n \in \mathbb{N}_+$,

$$\xi_n - \xi_{n-1} = \begin{cases} e_n^\lambda, & \text{if } X_{\xi_{n-1}} \geq 0, \\ \tau_a^+ \circ \theta_{\xi_{n-1}} + e_n^\lambda, & \text{if } X_{\xi_{n-1}} < 0, \end{cases} \tag{3.1}$$

where $\left\{e_n^\lambda\right\}_{n\in\mathbb{N}_+}$ is a sequence of i.i.d. exponential random variable with mean $1/\lambda > 0$, the constant $a \geq 0$ is called the recovery barrier, and θ is the Markov shift operator such that $X_t \circ \theta_s = X_{s+t}$, leading to $\tau_a^+ \circ \theta_{\xi_{n-1}} = \inf\{t \geq \xi_{n-1} : X_t > a\}$. The discrete observation scheme (3.1) indicates that the time increments between consecutive observations first follow the i.i.d. exponential distribution until the surplus is observed to be negative. It will be restored once the surplus recovers to the level a.

Denote

$$T_0^{\lambda,-} = \inf\{\xi_n : X_{\xi_n} < 0,\ n \in \mathbb{N}\}$$

as the first time the surplus is observed below level 0 under the observation scheme $\{\xi_n\}_{n\in\mathbb{N}}$. Clearly, $T_0^{\lambda,-}$ is identical to the ruin time observed at Poisson arrival times.

The ruin time under a hybrid observation scheme with recover barrier $a \geq 0$ and grace

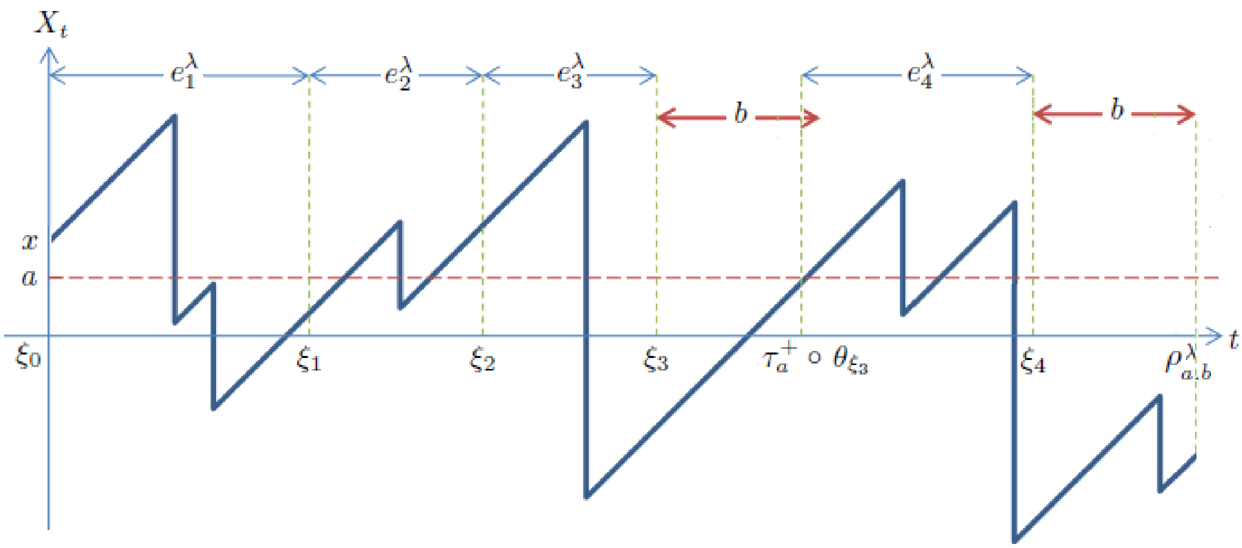

Figure 3.1: Illustration of Parisian ruin under hybrid observation scheme.

period $b > 0$ is then defined as

$$\rho_{a,b}^{\lambda} = \inf\left\{t \in \left(\xi_n, \tau_a^+ \circ \theta_{\xi_n}\right) : X_{\xi_n} < 0 \text{ and } t - \xi_n \geq b \text{ for } n \in \mathbb{N}\right\}.$$

Under the hybrid observation scheme, the surplus process X is first monitored discretely at Poisson arrival times with rate λ until the surplus is observed to be negative. Then a grace period of length b will be granted to the insurer and the surplus process will be observed continuously during this grace period. The insurer is considered ruined at the end of the grace period unless the surplus recovers to level a within the grace period. In the latter case, the observation scheme will be switched back to the discrete scheme as soon as the surplus recovers to level a. Note that as $a = 0$ and $\lambda \uparrow \infty$, the time of ruin reduces to the Parisian ruin time studied in Loeffen et al. [2013]. Figure 3.1 illustrates the Parisian ruin time under the hybrid observation scheme for a particular sample path.

3.3 Preliminaries

The following exit identity with continuous and Poisson observations would be used in deriving the main results in Section 3.4. The corresponding proofs can be found in Kyprianou [2014] and Albrecher et al. [2016] respectively.

Lemma 8. For $0 \le u \le x$ and $s, \theta \ge 0$, we have

$$\mathbb{E}_u \left[e^{-s\tau_x^+} 1_{\left\{\tau_x^+ < \infty\right\}} \right] = e^{\Phi_s (u-x)}, \tag{3.2}$$

and

$$\mathbb{E}_u \left[e^{-sT_0^{\lambda,-} + \theta X_{T_0^{\lambda,-}}}, T_0^{\lambda,-} < \infty \right] = \frac{\lambda}{\lambda - \psi_s(\theta)} \left[Z^{(s)}(u,\theta) - Z^{(s)}(u, \Phi_{\lambda+s}) \frac{\psi_s(\theta)(\Phi_{s+\lambda} - \Phi_s)}{\lambda(\theta - \Phi_s)} \right]. \tag{3.3}$$

As a special case, when $s = \theta = 0$, it is easy to see that equation (3.3) reduces to the ruin probability observed at Poisson arrival times given by

$$\mathbb{P}_u \left(T_0^{\lambda,-} < \infty \right) = 1 - \psi'(0+) \frac{\Phi_\lambda}{\lambda} Z(u, \Phi_\lambda), \quad u \ge 0, \tag{3.4}$$

which was first obtained by Landriault et al. [2011]. Further taking the limit that the observation rate goes to infinity gives the continuously observed ruin probability

$$\mathbb{P}_u \left(\tau_0^- < \infty \right) = 1 - \psi'(0+) W(u), \quad u \ge 0. \tag{3.5}$$

Meanwhile, the following lemma summarizes a few identities involving the scale function and the law of X. They can all be found in Loeffen et al. [2013], and the proof mainly relies on the Kendall's identity of spectrally negative Lévy process (e.g. Corollary VII.3 of Bertoin [1996]), i.e.,

$$r\mathbb{P}\left(\tau_z^+ \in \mathrm{d}r\right)\mathrm{d}z = z\mathbb{P}\left(X_r \in \mathrm{d}z\right)\mathrm{d}r.$$

Due to the similarity, proofs to the following lemma are left to the readers.

Lemma 9. For $u, a \geq 0$ and $\theta, s > 0$, we have

$$\int_0^\infty e^{-(\theta+r)b} \int_a^\infty e^{k(z-a)} \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z) \,\mathrm{d}b = \frac{e^{-\Phi_{\theta+r} y}}{\Phi_{\theta+r} - k} \quad \text{for } \Phi_{\theta+r} > k, \tag{3.6}$$

$$\int_0^\infty W^{(s)}(z) \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z) = e^{sb}, \tag{3.7}$$

and

$$\int_0^\infty e^{-(\theta+s)b} \int_a^\infty \left[W^{(s)}(u+z-a) - W^{(s)}(u)\right] \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z) \mathrm{d}b = \int_0^\infty \frac{e^{-\Phi_{\theta+s}(a+y)}}{\Phi_{\theta+s}} W^{(s)\prime}(u+y) \,\mathrm{d}y. \tag{3.8}$$

The following lemma will also be used in the derivation of the main results.

Lemma 10. For $\theta, a, q \geq 0$, we have

$$\int_0^\infty e^{-\theta b} \mathbb{E}\left[e^{-q\tau_a^+}, \tau_a^+ < b\right] \mathrm{d}b = \frac{1}{\theta} e^{-\Phi_{q+\theta} a}. \tag{3.9}$$

3.4 Main Result

In this section, we aim to first obtain the distribution of the ruin quantity $\rho_{a,b}^\lambda$ via the calculation of Laplace transform.

Theorem 11. *For $u, a \geq 0$ and $\lambda, b > 0$, we have*

$$\begin{aligned} &\mathbb{E}_u\left[e^{-s\rho_{a,b}^\lambda}, \rho_{a,b}^\lambda < \infty\right] \\ &= e^{-sb}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-} < \infty\right] - \int_0^b \left(\frac{s\lambda}{s+\lambda} e^{-s(b-k)} + \frac{\lambda^2}{s+\lambda} e^{\lambda(b-k)}\right) g_{u,a,\lambda}^{(s)}(k) \,\mathrm{d}k \\ &\quad + \mathbb{E}_a\left[e^{-s\rho_{a,b}^\lambda}, \rho_{a,b}^\lambda < \infty\right] \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(s)}(k) \,\mathrm{d}k, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &g_{u,a,\lambda}^{(s)}(k) \\ &= e^{-sk} \int_a^\infty \left[\frac{(\Phi_{\lambda+s} - \Phi_s) Z^{(s)}(u, \Phi_{\lambda+s})}{\lambda} e^{\Phi_s(z-a)} - W^{(s)}(u+z-a)\right] \frac{z}{k} \mathbb{P}(X_k \in \mathrm{d}z) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\mathbb{E}_a\left[e^{-s\rho_{a,b}^{\lambda}}, \rho_{a,b}^{\lambda}<\infty\right]\\&=\frac{e^{-sb}\mathbb{E}_a\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]-\int_0^b\left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)}+\frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right)g_{a,a,\lambda}^{(s)}\left(k\right)\mathrm{d}k}{1-\int_0^b\lambda e^{\lambda(b-k)}g_{a,a,\lambda}^{(s)}\left(k\right)\mathrm{d}k}.\end{aligned}\tag{3.12}$$

As a direct consequence of Theorem 11, the ruin probability $\mathbb{P}_u\left(\rho_{a,b}^{\lambda}<\infty\right)$ can be obtained without extra effort. The result is summarized in the following corollary.

Corollary 12. For $u, a \geq 0$ and $\lambda, b > 0$, we have

$$\mathbb{P}_u\left(\rho_{a,b}^{\lambda}<\infty\right)=\mathbb{P}_u\left(T_0^{\lambda,-}<\infty\right)-\mathbb{P}_a\left(\rho_{a,b}^{\lambda}=\infty\right)\int_0^b\lambda e^{\lambda(b-k)}g_{u,a,\lambda}^{(0)}\left(k\right)\mathrm{d}k,\tag{3.13}$$

where

$$g_{u,a,\lambda}^{(0)}\left(k\right)=\int_a^{\infty}\left[\frac{\Phi_\lambda}{\lambda}Z\left(u,\Phi_\lambda\right)-W\left(u+z-a\right)\right]\frac{z}{k}\mathbb{P}(X_k\in\mathrm{d}z)$$

and

$$\mathbb{P}_a\left(\rho_{a,b}^{\lambda}=\infty\right)=\frac{\mathbb{P}_a\left(T_0^{\lambda,-}=\infty\right)}{1-\int_0^b\lambda e^{\lambda(b-k)}g_{a,a,\lambda}^{(0)}\left(k\right)\mathrm{d}k}.\tag{3.14}$$

Remark 1. While the structure of the expression in Corollary 12 may seem different from that in Theorem 1 of Loeffen et al. [2013], to be demonstrated in the following section, the above findings agree with the classical results by taking appropriate limits.

3.5 Limiting Results

This section is devoted in demonstrating an alternative approach to deriving the classical Parisian ruin results by taking appropriate limits to Theorem 11. To be illustrated below, proofs in this section heavily utilize the Initial Value Theorem of Laplace transform, see, for example, Theorem 3.8.1 of Debnath and Bhatta [2015], and a more general proof can be found in Theorem 2.2.10 of Mejlbro [2010].

Keeping in mind the issue of unbounded variation, it is worthwhile to remark that the parameter a, which characterizes the effect of spatial approximation, is indeed irrelevant under the current context due to the introduction of the parameter λ, which characterizes the effect of temporal approximation. This is supported by the fact that, under the context of paths with unbounded variation, (3.20) is valid even without the effect of spatial approximation; contrasting to (10) in Loeffen et al. Loeffen et al. [2013] which degenerates to a trivial equation when the effect of spatial approximation is removed.

In what follows, it is assumed that $a = 0$ so that we concentrate our focus on the quantity $\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}}, \rho_{0,b}^{\lambda} < \infty\right] = \mathbb{E}_u\left[e^{-s\rho_{a,b}^{\lambda}}, \rho_{a,b}^{\lambda} < \infty\right]\Big|_{a=0}$.

To be used later, the following lemma turns out to be handy in the later computations.

Lemma 13. For $u, s \geq 0$ and $\lambda, b > 0$, we have

$$\begin{aligned}
&\int_0^b \lambda e^{\lambda(b-k)} g_{u,0,\lambda}^{(s)}(k)\,\mathrm{d}k \\
&= \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \\
&\quad \times \left[W^{(s)}(u+z) - (\Phi_{\lambda+s} - \Phi_s)\, e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(u+y)\,\mathrm{d}y\right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&1 - \int_0^b \lambda e^{\lambda(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k \\
&= \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \left[(\Phi_{\lambda+s} - \Phi_s)\, e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(y)\,\mathrm{d}y\right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k.
\end{aligned} \tag{3.16}$$

With this lemma in hand, we are ready to evaluate the following limiting quantities by taking appropriate limits to λ and s.

3.5.1 Obtaining Laplace transform of classical Parisian ruin

It is commented in Loeffen et al. Loeffen et al. [2013] that Laplace transform of the classical Parisian ruin time can be derived using the spatial approximation argument presented in their paper. As an alternative approach, we demonstrate how the same quantity can be derived using the temporal approximation argument. With a bit of abusing the notation, denote $\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\infty}}, \rho_{0,b}^{\infty}<\infty\right] = \lim_{\lambda\uparrow\infty}\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}}, \rho_{0,b}^{\lambda}<\infty\right]$.

Proposition 14. For $u, s \geq 0$ and $b > 0$, we have

$$\begin{aligned}&\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\infty}}, \rho_{0,b}^{\infty}<\infty\right]\\&= e^{-sb}\mathbb{E}_u\left[e^{-s\tau_0^-}, \tau_0^-<\infty\right] - \int_0^b se^{-s(b-k)}g_{u,0,\infty}^{(s)}(k)\,\mathrm{d}k + \left(1-\mathbb{E}\left[e^{-s\rho_{0,b}^{\infty}}, \rho_{0,b}^{\infty}<\infty\right]\right)g_{u,0,\infty}^{(s)}(b),\end{aligned} \tag{3.17}$$

where

$$\mathbb{E}\left[e^{-s\rho_{0,b}^{\infty}}, \rho_{0,b}^{\infty}<\infty\right] = 1 - \frac{\dfrac{s}{\Phi_s} + s\int_0^b\int_0^\infty e^{\Phi_s z}\dfrac{z}{k}\mathbb{P}(X_k\in\mathrm{d}z)\,\mathrm{d}k}{\int_0^\infty e^{\Phi_s z}\dfrac{z}{b}\mathbb{P}(X_b\in\mathrm{d}z)} \tag{3.18}$$

and

$$g_{u,0,\infty}^{(s)}(k) = e^{-sk}\int_0^\infty\left[W^{(s)}(u)e^{\Phi_s z} - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in\mathrm{d}z). \tag{3.19}$$

3.5.2 Retrieving classical Parisian ruin probability

To recover the expression for classical Parisian ruin probability, we take the limit that $s\downarrow 0$ and $\lambda\uparrow\infty$. The following proposition demonstrates the Laplace transform under the hybrid observation scheme $\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}}, \rho_{0,b}^{\lambda}<\infty\right]$ indeed reduces to the formula of classical Parisian ruin probability obtained by Loeffen et al. Loeffen et al. [2013].

Proposition 15. For $u \geq 0$ and $b > 0$, we have

$$\begin{aligned}&\lim_{s\downarrow 0}\lim_{\lambda\uparrow\infty}\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}}, \rho_{0,b}^{\lambda} < \infty\right] = \lim_{\lambda\uparrow\infty}\lim_{s\downarrow 0}\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}}, \rho_{0,b}^{\lambda} < \infty\right] \\ &= 1 - \psi'(0+)\frac{\int_0^\infty W(u+z)\, z\mathbb{P}(X_b \in \mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_b \in \mathrm{d}z)}.\end{aligned}$$

3.6 Example

According to the expression (3.11), the function $g_{u,a,\lambda}^{(s)}(k)$ plays an important role in the calculation of the Laplace transform, which involves the scale functions and the distribution of X_s. Unfortunately, as commented by Loeffen et al. [2013], the scale functions and the law of X possess explicit expressions only for a few cases, such as Brownian motion and Cramér–Lundberg model with exponential claims. For these examples, it is possible to express results explicitly using the formulas of the scale functions and law of X provided in Loeffen et al. [2013].

In this section, we will provide some numerical examples for the Parisian ruin probability under the hybrid observation scheme $\mathbb{P}_u\left(\rho_{0,b}^{\lambda} < \infty\right)$. We will study the Brownian motion model and the Cramér–Lundberg model with exponential claims. For simplicity, we assume $a = 0$ in this section.

3.6.1 Brownian Motion

Let $X_t = \mu t + \sigma B_t$, where $\mu, \sigma > 0$ and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. From Loeffen et al. [2013] with some simple algebra,

$$\begin{aligned}
\psi(\theta) &= \mu\theta + \frac{1}{2}\sigma^2\theta^2, \\
\Phi_\lambda &= \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}, \\
W(x) &= \frac{1}{\mu}\left(1 - e^{-\frac{2\mu x}{\sigma^2}}\right), \\
Z(x,\theta) &= \frac{1}{\mu}\psi(\theta)\left(\frac{1}{\theta} - \frac{e^{-\frac{2\mu x}{\sigma^2}}}{\theta + 2\mu\sigma^{-2}}\right) \quad \text{for } \theta > \Phi.
\end{aligned}$$

It follows from (3.11) that

$$g^{(0)}_{u,0,\lambda}(s) = \frac{1}{\mu s} e^{-2\mu\sigma^{-2}u}\left[\int_0^\infty e^{-2\mu\sigma^{-2}z} z\mathbb{P}(X_s \in \mathrm{d}z) - \frac{\Phi_\lambda}{\Phi_\lambda + 2\mu\sigma^{-2}}\int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z)\right].$$

Note that

$$\mathbb{P}(X_s \in \mathrm{d}z) = \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(z-\mu s)^2}{2\sigma^2 s}},$$

we have

$$\begin{aligned}
\int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z) &= \frac{\sigma\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{\mu^2 s}{2\sigma^2}} + \mu s\mathcal{N}\left(\mu\sigma^{-1}\sqrt{s}\right), \\
\int_0^\infty e^{-2\mu\sigma^{-2}z} z\mathbb{P}(X_s \in \mathrm{d}z) &= \int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z) - \mu s,
\end{aligned}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of a standard normal random variable. Thus,

$$g^{(0)}_{u,0,\lambda}(s) = \frac{1}{\mu s} e^{-2\mu\sigma^{-2}u}\left[\frac{2\mu\sigma^{-2}}{\Phi_\lambda + 2\mu\sigma^{-2}}\left(\frac{\sigma\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{\mu^2 s}{2\sigma^2}} + \mu s\mathcal{N}\left(\mu\sigma^{-1}\sqrt{s}\right)\right) - \mu s\right].$$

By Theorem 12,

$$\mathbb{P}_u\left(\rho^\lambda_{0,b} < \infty\right) = \frac{\Phi_\lambda e^{-2\mu\sigma^{-2}u}}{\Phi_\lambda + 2\mu\sigma^{-2}} - \frac{2\mu\sigma^{-2}}{\Phi_\lambda + 2\mu\sigma^{-2}} \frac{\int_0^b \lambda e^{\lambda(b-s)} g^{(0)}_{u,0,\lambda}(s)\,\mathrm{d}s}{1 - \int_0^b \lambda e^{\lambda(b-s)} g^{(0)}_{0,0,\lambda}(s)\mathrm{d}s}.$$

u	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$	$\lambda = \infty$
1	1.8639×10^{-1}	2.1541×10^{-1}	2.3596×10^{-1}	2.4905×10^{-1}	2.6546×10^{-1}
5	8.3750×10^{-2}	9.6789×10^{-2}	1.0602×10^{-1}	1.1191×10^{-1}	1.1928×10^{-1}
10	3.0810×10^{-2}	3.5607×10^{-2}	3.9003×10^{-2}	4.1168×10^{-2}	4.3881×10^{-2}
20	4.1697×10^{-3}	4.8188×10^{-3}	5.2785×10^{-3}	5.5715×10^{-3}	5.9386×10^{-3}
30	5.6430×10^{-4}	6.5216×10^{-4}	7.1437×10^{-4}	7.5402×10^{-4}	8.0371×10^{-4}

Table 3.1: Ruin probability for Brownian motion model with different λ.

u	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$
1	1.7086×10^{-1}	1.3977×10^{-1}	1.1668×10^{-1}	9.8831×10^{-2}	8.4657×10^{-2}
5	7.6770×10^{-2}	6.2802×10^{-2}	5.2426×10^{-2}	4.4408×10^{-2}	3.8039×10^{-2}
10	2.8242×10^{-2}	2.3104×10^{-2}	1.9286×10^{-2}	1.6337×10^{-2}	1.3994×10^{-2}
20	3.8222×10^{-3}	3.1267×10^{-3}	2.6101×10^{-3}	2.2109×10^{-3}	1.8939×10^{-3}
30	5.1727×10^{-4}	4.2316×10^{-4}	3.5324×10^{-4}	2.9922×10^{-4}	2.5631×10^{-4}

Table 3.2: Ruin probability for Brownian motion model with different b.

Tables 3.1 and 3.2 show the effect of model parameters on the ruin probability of the Brownian motion model. In table 3.1, we fix $\mu = 1$, $\sigma = \sqrt{10}$, and $b = 2$. It is seen that the ruin probability increases in λ as the surplus process is observed more frequently, thereby increasing the likelihood of detecting a negative surplus. In table 3.2, we fix $\mu = 1$, $\sigma = \sqrt{10}$, and $\lambda = 1$. It is seen that the ruin probability decreases in b because a negative surplus is more likely to be recovered given a longer grace period.

3.6.2 Compound Poisson Process with Exponential Claims

Let $X_t = ct - \sum_{i=1}^{N_t} Y_i$, where $\{N_t\}_{t\geq 0}$ is a Poisson process with rate η, and Y_i are i.i.d. exponential random variables with mean $1/\alpha$, which are independent of the Poisson process. It is implicitly assumed that $c > \eta\alpha^{-1}$. From Loeffen et al. [2013] with some simple algebra,

$$\begin{aligned}
\psi(\theta) &= c\theta - \eta + \frac{\eta\alpha}{\theta+\alpha}, \\
\Phi_\lambda &= \frac{-(c\alpha-\eta-\lambda)+\sqrt{(c\alpha-\eta-\lambda)^2+4c\lambda}}{2c}, \\
W(x) &= \left(c-\frac{\eta}{\alpha}\right)^{-1}\left(1-\frac{\eta}{c\alpha}e^{\left(\frac{\eta}{c}-\alpha\right)x}\right), \\
Z(x,\theta) &= \psi(\theta)\left(c-\frac{\eta}{\alpha}\right)^{-1}\left[\frac{1}{\theta}-\frac{\eta}{c\alpha}e^{\left(\frac{\eta}{c}-\alpha\right)u}\left(\theta+\alpha-\frac{\alpha}{c}\right)^{-1}\right].
\end{aligned}$$

From (3.11),

$$g^{(0)}_{u,0,\lambda}(s) = \frac{\eta}{c(c\alpha-\eta)s}e^{(\eta c^{-1}-\alpha)u}\left[\int_0^\infty e^{(\eta c^{-1}-\alpha)z} z\mathbb{P}(X_s\in \mathrm{d}z) - \frac{\Phi_\lambda}{\Phi_\lambda+\alpha-\eta c^{-1}}\int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z)\right].$$

Note that

$$\mathbb{P}(X_s\in\mathrm{d}z) = e^{-\eta r}\left[\delta_0(\mathrm{d}z) + e^{-\alpha z}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!\,(m+1)!}z^m\mathrm{d}z\right],$$

where $\delta_0(\mathrm{d}z)$ is the Dirac mass at 0. With some calculations, one can show that

$$\int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z) = e^{-\eta s}\left[cs+\sum_{m=0}^{\infty}\frac{(\eta s)^{m+1}}{m!(m+1)!}\left(cs\Gamma(m+1,\alpha cs)-\frac{1}{\alpha}\Gamma(m+2,\alpha cs)\right)\right],$$

and

$$\frac{\eta}{c\alpha}\int_0^\infty e^{(\eta c^{-1}-\alpha)z}z\mathbb{P}(X_s\in\mathrm{d}z) = \int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z) - \left(c-\frac{\eta}{\alpha}\right)s,$$

where $\Gamma(n,x) := \int_0^x e^{-t}t^{n-1}\mathrm{d}t$ for $n\in\mathbb{N}$, $x\geq 0$ is the incomplete Gamma function. By Theorem 12,

$$\mathbb{P}_u\left(\rho^\lambda_{0,b}<\infty\right) = \frac{\eta}{c\alpha}e^{(\eta c^{-1}-\alpha)u}\frac{\Phi_\lambda}{\Phi_\lambda+\alpha-\eta c^{-1}} - \frac{\alpha-\eta c^{-1}}{\Phi_\lambda+\alpha-\eta c^{-1}}\frac{\int_0^b \lambda e^{\lambda(b-s)}g^{(0)}_{u,0,\lambda}(s)\mathrm{d}s}{1-\int_0^b\lambda e^{\lambda(b-s)}g^{(0)}_{0,0,\lambda}(s)\mathrm{d}s}.$$

Tables 3.3 and 3.4 show the effect of model parameters on the ruin probability of the Cramér–Lundberg model with exponential jumps. In table 3.3 we fix $c=6$, $\eta=5$, $\alpha=1$,

u	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$	$\lambda = \infty$
1	2.0507×10^{-1}	2.3546×10^{-1}	2.5681×10^{-1}	2.7035×10^{-1}	2.8723×10^{-1}
5	1.0529×10^{-1}	1.2089×10^{-1}	1.3185×10^{-1}	1.3880×10^{-1}	1.4747×10^{-1}
10	4.5757×10^{-2}	5.2539×10^{-2}	5.7303×10^{-2}	6.0323×10^{-2}	6.4090×10^{-2}
20	8.6424×10^{-3}	9.9232×10^{-3}	1.0823×10^{-2}	1.1396×10^{-2}	1.2105×10^{-2}
30	1.6323×10^{-3}	1.8743×10^{-3}	2.0442×10^{-3}	2.1520×10^{-3}	2.2864×10^{-3}

Table 3.3: Ruin probability for Cramér–Lundberg model with different λ.

u	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$
1	1.8900×10^{-1}	1.5620×10^{-1}	1.3159×10^{-1}	1.1241×10^{-1}	9.7054×10^{-2}
5	9.7035×10^{-2}	8.0194×10^{-2}	6.7561×10^{-2}	5.7713×10^{-2}	4.9829×10^{-2}
10	4.2171×10^{-2}	3.4852×10^{-2}	2.9362×10^{-2}	2.5082×10^{-2}	2.1656×10^{-2}
20	7.9651×10^{-3}	6.5827×10^{-3}	5.5457×10^{-3}	4.7374×10^{-3}	4.0902×10^{-3}
30	1.5044×10^{-3}	1.2433×10^{-3}	1.0475×10^{-3}	8.9478×10^{-4}	7.7255×10^{-4}

Table 3.4: Ruin probability for Cramér–Lundberg model with different b.

$b = 2$, and in table 3.4 we fix $c = 6$, $\eta = 5$, $\alpha = 1$, $\lambda = 1$. Similar to the Brownian motion model, it is seen that the ruin probability increases in λ and decreases in b.

3.7 Appendix

3.7.1 Proof of Lemma 10

By Tonelli Theorem and (3.2),

$$\begin{aligned}
&\int_0^\infty e^{-\theta b}\mathbb{E}\left[e^{-q\tau_a^+}, \tau_a^+ < b\right]\mathrm{d}b \\
&= \int_0^\infty e^{-\theta b}\int_0^b e^{-qy} f_{\tau_a^+}(y)\,\mathrm{d}y\mathrm{d}b \\
&= \int_0^\infty \int_y^\infty e^{-\theta b} e^{-qy} f_{\tau_a^+}(y)\,\mathrm{d}b\mathrm{d}y \\
&= \int_0^\infty e^{-qy} f_{\tau_a^+}(y)\,\frac{1}{\theta}e^{-\theta y}\mathrm{d}y \\
&= \frac{1}{\theta}\mathbb{E}\left[e^{-(q+\theta)\tau_a^+}\right] \\
&= \frac{1}{\theta}e^{-\Phi_{q+\theta}a}.
\end{aligned}$$

3.7.2 Proof of Theorem 11

By conditioning on $T_0^{\lambda,-}$, the first time that the surplus is observed to be negative, and using the strong Markov property and spatial homogeneity, we first have

$$\begin{aligned}
&\mathbb{E}_u\left[e^{-s\rho_{a,b}^\lambda}, \rho_{a,b}^\lambda < \infty\right] \\
&= \int_{-\infty}^0 \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty\right] \\
&\quad \times \left(e^{-sb}\mathbb{P}_x\left(\tau_a^+ > b\right) + \mathbb{E}_x\left[e^{-s\tau_a^+}, \tau_a^+ < b\right]\mathbb{E}_a\left[e^{-s\rho_{a,b}^\lambda}, \rho_{a,b}^\lambda < \infty\right]\right) \\
&= e^{-sb}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty\right]\mathbb{P}\left(\tau_{a+x}^+ > b\right) \\
&\quad + \mathbb{E}_a\left[e^{-s\rho_{a,b}^\lambda}, \rho_{a,b}^\lambda < \infty\right]\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty\right]\mathbb{E}\left[e^{-s\tau_{a+x}^+}, \tau_{a+x}^+ < b\right].
\end{aligned} \tag{3.20}$$

It remains to obtain an expression for the term

$$e^{-sb}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{P}\left(\tau_{a+x}^+>b\right)$$

and

$$\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{E}\left[e^{-s\tau_{a+x}^+}, \tau_{a+x}^+<b\right].$$

We handle them term by term. By (3.9), (3.3) and (2.5), together with Tonelli Theorem and Lemma 2.6, for $\theta>0$, taking the Laplace transform of the first term with respect to b gives

$$\begin{aligned}
&\int_0^\infty e^{-\theta b}e^{-sb}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{P}\left(\tau_{a+x}^+>b\right)\mathrm{d}b\\
&=\int_0^\infty e^{-(\theta+s)b}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\left(1-\mathbb{P}\left(\tau_{a+x}^+<b\right)\right)\mathrm{d}b\\
&=\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\frac{1}{\theta+s}\left(1-e^{-\Phi_{\theta+s}(a+x)}\right)\\
&=\frac{1}{\theta+s}\left(\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]-e^{-\Phi_{\theta+s}a}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}+\Phi_{\theta+s}X_{T_0^{\lambda,-}}}, T_0^{\lambda,-}<\infty\right]\right)\\
&=\frac{1}{\theta+s}\left(\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]-e^{-\Phi_{\theta+s}a}\frac{\lambda}{\lambda-\theta}\left[Z^{(s)}\left(u,\Phi_{\theta+s}\right)-Z^{(s)}\left(u,\Phi_{\lambda+s}\right)\frac{\theta\left(\Phi_{\lambda+s}-\Phi_s\right)}{\lambda\left(\Phi_{\theta+s}-\Phi_s\right)}\right]\right)\\
&=\frac{1}{\theta+s}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]\\
&\quad-\frac{\theta}{\theta+s}e^{-\Phi_{\theta+s}a}\frac{\lambda}{\lambda-\theta}\left[\int_0^\infty e^{-\Phi_{\theta+s}y}W^{(s)}\left(u+y\right)\mathrm{d}y-\frac{\Phi_{\lambda+s}-\Phi_s}{\lambda\left(\Phi_{\theta+s}-\Phi_s\right)}Z^{(s)}\left(u,\Phi_{\lambda+s}\right)\right]\\
&=\frac{1}{\theta+s}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]\\
&\quad-\frac{\theta}{\theta+s}e^{-\Phi_{\theta+s}a}\frac{\lambda}{\lambda-\theta}\left[\frac{1}{\Phi_{\theta+s}}W^{(s)}\left(u\right)+\frac{1}{\Phi_{\theta+s}}\int_0^\infty e^{-\Phi_{\theta+s}y}W^{(s)\prime}\left(u+y\right)\mathrm{d}y-\frac{\Phi_{\lambda+s}-\Phi_s}{\lambda\left(\Phi_{\theta+s}-\Phi_s\right)}Z^{(s)}\left(u,\Phi_{\lambda+s}\right)\right]\\
&=\frac{1}{\theta+s}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right]-\frac{\theta}{\theta+s}\frac{\lambda}{\theta-\lambda}\\
&\quad\times\left[-\frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}}W^{(s)}\left(u\right)-\int_0^\infty\frac{e^{-\Phi_{\theta+s}(a+y)}}{\Phi_{\theta+s}}W^{(s)\prime}\left(u+y\right)\mathrm{d}y+\frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}-\Phi_s}\frac{\left(\Phi_{\lambda+s}-\Phi_s\right)Z^{(s)}\left(u,\Phi_{\lambda+s}\right)}{\lambda}\right].
\end{aligned}\tag{3.21}$$

Note that

$$\frac{\theta}{\theta+s}\frac{\lambda}{\theta-\lambda}=\int_0^\infty e^{-\theta b}\left(\frac{s\lambda}{s+\lambda}e^{-sb}+\frac{\lambda^2}{s+\lambda}e^{\lambda b}\right)\mathrm{d}b,\quad \theta>\lambda,$$

and recall from (3.6) and (3.8) that

$$\frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}}=\int_0^\infty e^{-\theta b}\left(e^{-sb}\int_a^\infty\frac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)\right)\mathrm{d}b,\tag{3.22}$$

$$\int_0^\infty \frac{e^{-\Phi_{\theta+s}(a+y)}}{\Phi_{\theta+s}} W^{(s)\prime}(u+y)\,\mathrm{d}y = \int_0^\infty e^{-\theta b}\left(e^{-sb}\int_a^\infty \left[W^{(s)}(u+z-a) - W^{(s)}(u)\right]\frac{z}{b}\mathbb{P}(X_b\in \mathrm{d}z)\right)\mathrm{d}b \tag{3.23}$$

and

$$\frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}-\Phi_s} = \int_0^\infty e^{-\theta b}\left(e^{-sb}\int_a^\infty e^{\Phi_s(z-a)}\frac{z}{b}\mathbb{P}(X_b\in \mathrm{d}z)\right)\mathrm{d}b. \tag{3.24}$$

By (3.22)–(3.24), the Laplace inversion of (3.21) gives

$$\begin{aligned}
&e^{-sb}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{P}\left(\tau_{a+x}^+ > b\right)\\
&= e^{-sb}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right] - \int_0^b \left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)} + \frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right) g_{u,a,\lambda}^{(s)}(k)\,\mathrm{d}k,
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
&g_{u,a,\lambda}^{(s)}(k)\\
&= -W^{(s)}(u)\,e^{-sk}\int_a^\infty \frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z) - e^{-sk}\int_a^\infty \left[W^{(s)}(u+z-a) - W^{(s)}(u)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\\
&\quad + \frac{(\Phi_{\lambda+s}-\Phi_s)\,Z^{(s)}(u,\Phi_{\lambda+s})}{\lambda}e^{-sk}\int_a^\infty e^{\Phi_s(z-a)}\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\\
&= e^{-sk}\int_a^\infty \left[\frac{(\Phi_{\lambda+s}-\Phi_s)\,Z^{(s)}(u,\Phi_{\lambda+s})}{\lambda}e^{\Phi_s(z-a)} - W^{(s)}(u+z-a)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z).
\end{aligned}$$

Similarly, for $\theta>0$, taking the Laplace transform of the second term with respect to b gives

$$\begin{aligned}
&\int_0^\infty e^{-\theta b}\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{E}\left[e^{-s\tau_{a+x}^+}, \tau_{a+x}^+ < b\right]\mathrm{d}b\\
&= \int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\frac{1}{\theta}e^{-\Phi_{\theta+s}(a+x)}\\
&= \frac{1}{\theta}e^{-\Phi_{\theta+s}a}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}+\Phi_{\theta+s}X_{T_0^{\lambda,-}}}, T_0^{\lambda,-}<\infty\right]\\
&= \frac{1}{\theta}e^{-\Phi_{\theta+s}a}\frac{\lambda}{\lambda-\theta}\left[Z^{(s)}(u,\Phi_{\theta+s}) - Z^{(s)}(u,\Phi_{\lambda+s})\frac{\theta(\Phi_{\lambda+s}-\Phi_s)}{\lambda(\Phi_{\theta+s}-\Phi_s)}\right]\\
&= \frac{\lambda}{\theta-\lambda}\left[-\frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}}W^{(s)}(u) - \int_0^\infty \frac{e^{-\Phi_{\theta+s}(a+y)}}{\Phi_{\theta+s}}W^{(s)\prime}(u+y)\,\mathrm{d}y + \frac{e^{-\Phi_{\theta+s}a}}{\Phi_{\theta+s}-\Phi_s}\frac{(\Phi_{\lambda+s}-\Phi_s)\,Z^{(s)}(u,\Phi_{\lambda+s})}{\lambda}\right].
\end{aligned}$$

Inversion gives

$$\begin{aligned}
&\int_0^\infty \mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, -X_{T_0^{\lambda,-}}\in \mathrm{d}x, T_0^{\lambda,-}<\infty\right]\mathbb{E}\left[e^{-s\tau_{a+x}^+}, \tau_{a+x}^+ < b\right]\\
&= \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(s)}(k)\,\mathrm{d}k.
\end{aligned} \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.20) gives

$$\begin{aligned}
&\mathbb{E}_u\left[e^{-s\rho_{a,b}^{\lambda}}, \rho_{a,b}^{\lambda} < \infty\right] \\
&= e^{-sb}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-} < \infty\right] - \int_0^b \left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)} + \frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right) g_{u,a,\lambda}^{(s)}(k)\,\mathrm{d}k \\
&\quad + \mathbb{E}_a\left[e^{-s\tau_{a,b}^{\lambda}}, \tau_{a,b}^{\lambda} < \infty\right] \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(s)}(k)\,\mathrm{d}k. \qquad (3.27)
\end{aligned}$$

By letting $u = a$ in (3.27), one obtains

$$\begin{aligned}
&\mathbb{E}_a\left[e^{-s\rho_{a,b}^{\lambda}}, \rho_{a,b}^{\lambda} < \infty\right] \\
&= \frac{e^{-sb}\mathbb{E}_a\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-} < \infty\right] - \int_0^b \left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)} + \frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right) g_{a,a,\lambda}^{(s)}(k)\,\mathrm{d}k}{1 - \int_0^b \lambda e^{\lambda(b-k)} g_{a,a,\lambda}^{(s)}(k)\,\mathrm{d}k}.
\end{aligned}$$

This completes the proof.

3.7.3 Proof of Corollary 12

Evaluation of $g_{u,a,\lambda}^{(0)}(k)$ is trivial. The result follows by putting $s = 0$ into (3.12) and (3.10) respectively such that (3.12) gives

$$\begin{aligned}
&\mathbb{P}_a\left(\rho_{a,b}^{\lambda} < \infty\right) \\
&= \frac{\mathbb{P}_a\left(T_0^{\lambda,-} < \infty\right) - \int_0^b \lambda e^{\lambda(b-k)} g_{a,a,\lambda}^{(0)}(k)\,\mathrm{d}k}{1 - \int_0^b \lambda e^{\lambda(b-k)} g_{a,a,\lambda}^{(0)}(k)\,\mathrm{d}k} \\
&= 1 - \frac{\mathbb{P}_a\left(T_0^{\lambda,-} = \infty\right)}{1 - \int_0^b \lambda e^{\lambda(b-k)} g_{a,a,\lambda}^{(0)}(k)\,\mathrm{d}k},
\end{aligned}$$

whereas (3.10) gives

$$\begin{aligned}
&\mathbb{P}_u\left(\rho_{a,b}^{\lambda} < \infty\right) \\
&= \mathbb{P}_u\left(T_0^{\lambda,-} < \infty\right) - \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(0)}(k)\,\mathrm{d}k + \mathbb{P}_a\left(\rho_{a,b}^{\lambda} < \infty\right) \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(0)}(k)\,\mathrm{d}k \\
&= \mathbb{P}_u\left(T_0^{\lambda,-} < \infty\right) - \mathbb{P}_a\left(\rho_{a,b}^{\lambda} = \infty\right) \int_0^b \lambda e^{\lambda(b-k)} g_{u,a,\lambda}^{(0)}(k)\,\mathrm{d}k
\end{aligned}$$

3.7.4 Proof of Lemma 13

It follows from (3.11) and (2.5) that

$$
\begin{aligned}
&\int_0^b \lambda e^{\lambda(b-k)} g^{(s)}_{u,0,\lambda}(k)\,\mathrm{d}b \\
&= \lambda e^{-\lambda b}\int_0^b e^{-(s+\lambda)k}\int_0^\infty \left[\frac{(\Phi_{\lambda+s}-\Phi_s)\,Z^{(s)}(u,\Phi_{\lambda+s})}{\lambda}e^{\Phi_s z} - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k \\
&= \lambda e^{-\lambda b}\int_0^b\int_0^\infty e^{-(s+\lambda)k}\left[(\Phi_{\lambda+s}-\Phi_s)\,e^{\Phi_s z}\int_0^\infty e^{-\Phi_{\lambda+s}y}W^{(s)}(u+y)\,\mathrm{d}y - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k.
\end{aligned}
$$

Note that by (3.6), we have

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-(s+\lambda)k}\left[(\Phi_{\lambda+s}-\Phi_s)\,e^{\Phi_s z}\int_0^\infty e^{-\Phi_{\lambda+s}y}W^{(s)}(u+y)\,\mathrm{d}y\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k \\
&= \int_0^\infty e^{-\Phi_{\lambda+s}y}W^{(s)}(u+y)\,\mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-(s+\lambda)k}W^{(s)}(u+z)\,\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k \\
&= \int_0^\infty\int_0^\infty e^{-(s+\lambda)k}W^{(s)}(u+z)\,\mathbb{P}\left(T_z^+\in \mathrm{d}k\right)\mathrm{d}z \\
&= \int_0^\infty e^{-\Phi_{\lambda+s}z}W^{(s)}(u+z)\,\mathrm{d}z.
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
&\int_0^b \lambda e^{\lambda(b-k)} g^{(s)}_{u,0,\lambda}(k)\,\mathrm{d}b \\
&= \lambda e^{-\lambda b}\int_b^\infty\int_0^\infty e^{-(s+\lambda)k}\left[W^{(s)}(u+z) - (\Phi_{\lambda+s}-\Phi_s)\,e^{\Phi_s z}\int_0^\infty e^{-\Phi_{\lambda+s}y}W^{(s)}(u+y)\,\mathrm{d}y\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k \\
&= \int_0^\infty\int_0^\infty \lambda e^{-\lambda k - s(k+b)} \\
&\quad\times\left[W^{(s)}(u+z) - (\Phi_{\lambda+s}-\Phi_s)\,e^{\Phi_s z}\int_0^\infty e^{-\Phi_{\lambda+s}y}W^{(s)}(u+y)\,\mathrm{d}y\right]\frac{z}{k+b}\mathbb{P}(X_{k+b}\in \mathrm{d}z)\,\mathrm{d}k,
\end{aligned}
$$

i.e. (3.15). Substitute $u=0$ into the above expression together with (3.7) gives

$$\begin{aligned}
&\int_0^b \lambda e^{\lambda(b-k)} g^{(s)}_{0,0,\lambda}(k)\,\mathrm{d}b \\
&= \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \left[W^{(s)}(z) - (\Phi_{\lambda+s} - \Phi_s) e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(y)\,\mathrm{d}y \right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k \\
&= 1 - \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \left[(\Phi_{\lambda+s} - \Phi_s) e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(y)\,\mathrm{d}y \right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k.
\end{aligned}$$

Rearranging gives (3.16).

3.7.5 Proof of Proposition 14

Applying Initial Value Theorem to (3.15) and (3.16) respectively gives

$$\begin{aligned}
&\lim_{\lambda\uparrow\infty} \int_0^b \lambda e^{\lambda(b-k)} g^{(s)}_{u,0,\lambda}(k)\,\mathrm{d}k \\
&= \lim_{\lambda\uparrow\infty} \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \\
&\quad \times \left[W^{(s)}(u+z) - (\Phi_{\lambda+s} - \Phi_s) e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(u+y)\,\mathrm{d}y \right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k \\
&= e^{-sb} \int_0^\infty \left[W^{(s)}(u+z) - e^{\Phi_s z} W^{(s)}(u) \right] \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z) \\
&= -g^{(s)}_{u,0,\infty}(b) \qquad (3.28)
\end{aligned}$$

and

$$\begin{aligned}
&1 - \lim_{\lambda\uparrow\infty} \int_0^b \lambda e^{\lambda(b-k)} g^{(s)}_{0,0,\lambda}(k)\,\mathrm{d}b \\
&= \lim_{\lambda\uparrow\infty} \int_0^\infty \int_0^\infty \lambda e^{-\lambda k - s(k+b)} \left[(\Phi_{\lambda+s} - \Phi_s) e^{\Phi_s z} \int_0^\infty e^{-\Phi_{\lambda+s} y} W^{(s)}(y)\,\mathrm{d}y \right] \frac{z}{k+b} \mathbb{P}(X_{k+b} \in \mathrm{d}z)\,\mathrm{d}k \\
&= \int_0^\infty e^{-sb} e^{\Phi_s z} W^{(s)}(0) \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z). \qquad (3.29)
\end{aligned}$$

Besides, applying Initial Value Theorem to (3.11) gives

$$
\begin{aligned}
&\lim_{\lambda\uparrow\infty} g_{u,0,\lambda}^{(s)}(k) \\
&= \lim_{\lambda\uparrow\infty} e^{-sk}\int_0^\infty \left[\frac{(\Phi_{\lambda+s}-\Phi_s)\, Z^{(s)}(u,\Phi_{\lambda+s})}{\lambda} e^{\Phi_s z} - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z) \\
&= \lim_{\lambda\uparrow\infty} e^{-sk}\int_0^\infty \left[(\Phi_{\lambda+s}-\Phi_s)\, e^{\Phi_s z}\int_0^\infty e^{-\Phi_{\lambda+s}y} W^{(s)}(u+y)\,\mathrm{d}y - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z) \\
&= e^{-sk}\int_0^\infty \left[W^{(s)}(u)\, e^{\Phi_s z} - W^{(s)}(u+z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)
\end{aligned}
$$

such that

$$
\begin{aligned}
&\lim_{\lambda\uparrow\infty}\int_0^b s e^{-s(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k \\
&= s e^{-sb}\int_0^b\int_0^\infty \left[W^{(s)}(0)\, e^{\Phi_s z} - W^{(s)}(z)\right]\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k \\
&= s e^{-sb}\left[\int_0^b\int_0^\infty W^{(s)}(0)\, e^{\Phi_s z}\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k - \int_0^b e^{sk}\mathrm{d}k\right] \\
&= s e^{-sb}\int_0^b\int_0^\infty W^{(s)}(0)\, e^{\Phi_s z}\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k + \left(1-e^{-sb}\right)
\end{aligned}
\tag{3.30}
$$

due to (3.7). Now, from (3.12), taking the limit together with (3.5), (3.28)–(3.30) gives

$$
\begin{aligned}
&\mathbb{E}\left[e^{-s\rho_{0,b}^\infty}, \rho_{0,b}^\infty<\infty\right] \\
&= \lim_{\lambda\uparrow\infty}\frac{e^{-sb}\mathbb{E}\left[e^{-sT_0^{\lambda,-}}, T_0^{\lambda,-}<\infty\right] - \int_0^b\left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)} + \frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right) g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k}{1-\int_0^b \lambda e^{\lambda(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k} \\
&= \frac{e^{-sb}\mathbb{E}\left[e^{-s\tau_0^-}, \tau_0^-<\infty\right] - \int_0^b s e^{-s(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k - \lim_{\lambda\uparrow\infty}\int_0^b \lambda e^{\lambda(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k}{1-\lim_{\lambda\uparrow\infty}\int_0^b \lambda e^{\lambda(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k} \\
&= 1 + \frac{e^{-sb}\mathbb{E}\left[e^{-s\tau_0^-}, \tau_0^-<\infty\right] - \int_0^b s e^{-s(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k - 1}{1-\lim_{\lambda\uparrow\infty}\int_0^b \lambda e^{\lambda(b-k)} g_{0,0,\lambda}^{(s)}(k)\,\mathrm{d}k} \\
&= 1 + \frac{e^{-sb}\left(1-\frac{s}{\Phi_s}W^{(s)}(0)\right) - s e^{-sb}\int_0^b\int_0^\infty W^{(s)}(0)\, e^{\Phi_s z}\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k + \left(1-e^{-sb}\right) - 1}{\int_0^\infty e^{-sb} e^{\Phi_s z} W^{(s)}(0)\frac{z}{b}\mathbb{P}(X_b\in \mathrm{d}z)} \\
&= 1 - \frac{\frac{s}{\Phi_s} + s\int_0^b\int_0^\infty e^{\Phi_s z}\frac{z}{k}\mathbb{P}(X_k\in \mathrm{d}z)\,\mathrm{d}k}{\int_0^\infty e^{\Phi_s z}\frac{z}{b}\mathbb{P}(X_b\in \mathrm{d}z)}.
\end{aligned}
$$

Similarly, from (3.10), taking the limit together with (3.5) and (3.28) gives

$$\begin{aligned}
&\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\infty}},\rho_{0,b}^{\infty}<\infty\right]\\
&=\lim_{\lambda\uparrow\infty}\left\{e^{-sb}\mathbb{E}_u\left[e^{-sT_0^{\lambda,-}},T_0^{\lambda,-}<\infty\right]-\int_0^b\left(\frac{s\lambda}{s+\lambda}e^{-s(b-k)}+\frac{\lambda^2}{s+\lambda}e^{\lambda(b-k)}\right)g_{u,0,\lambda}^{(s)}(k)\,\mathrm{d}k\right.\\
&\quad\left.+\mathbb{E}\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]\int_0^b\lambda e^{\lambda(b-k)}g_{u,0,\lambda}^{(s)}(k)\,\mathrm{d}k\right\}\\
&=e^{-sb}\mathbb{E}_u\left[e^{-s\tau_0^-},\tau_0^-<\infty\right]-\int_0^b se^{-s(b-k)}g_{u,0,\lambda}^{(s)}(k)\,\mathrm{d}k+\left(1-\mathbb{E}\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]\right)g_{u,0,\infty}^{(s)}(b).
\end{aligned}$$

3.7.6 Proof of Proposition 15

We first evaluate $\lim_{s\downarrow0}\lim_{\lambda\uparrow\infty}\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]$. It follows from (3.18) and Initial Value Theorem that

$$\begin{aligned}
&\lim_{s\downarrow0}\lim_{\lambda\uparrow\infty}\mathbb{E}\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]\\
&=1-\lim_{s\downarrow0}\frac{\dfrac{s}{\Phi_s}+s\int_0^b\int_0^\infty e^{\Phi_s z}\dfrac{z}{k}\mathbb{P}\left(X_k\in\mathrm{d}z\right)\mathrm{d}k}{\int_0^\infty e^{\Phi_s z}\dfrac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)}\\
&=1-\frac{\psi'(0+)}{\int_0^\infty\dfrac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)}.
\end{aligned}$$

Substituting into (3.17) together with (3.19), (3.7), (3.5) and Initial Value Theorem gives

$$\begin{aligned}
&\lim_{s\downarrow0}\lim_{\lambda\uparrow\infty}\mathbb{E}_u\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]\\
&=\mathbb{P}_u\left(\tau_0^-<\infty\right)+\left(1-\lim_{s\downarrow0}\lim_{\lambda\uparrow\infty}\mathbb{E}\left[e^{-s\rho_{0,b}^{\lambda}},\rho_{0,b}^{\lambda}<\infty\right]\right)g_{u,0,\infty}^{(0)}(b)\\
&=1-\psi'(0+)W(u)+\left(\frac{\psi'(0+)}{\int_0^\infty e^{\Phi_s z}\dfrac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)}\right)\left(\int_0^\infty\left[W(u)-W(u+z)\right]\frac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)\right)\\
&=1-\psi'(0+)\frac{\int_0^\infty W(u+z)\,z\mathbb{P}\left(X_b\in\mathrm{d}z\right)}{\int_0^\infty z\mathbb{P}\left(X_b\in\mathrm{d}z\right)}.
\end{aligned}$$

Next, we evaluate $\lim_{\lambda\uparrow\infty}\lim_{s\downarrow 0}\mathbb{E}_u\left[e^{-s\rho^{\lambda}_{0,b}},\rho^{\lambda}_{0,b}<\infty\right]$. It follows from (3.12), (2.5), (3.16) and Initial Value Theorem that

$$
\begin{aligned}
&\lim_{\lambda\uparrow\infty}\lim_{s\downarrow 0}\mathbb{E}\left[e^{-s\rho^{\lambda}_{0,b}},\rho^{\lambda}_{0,b}<\infty\right]\\
&=1-\lim_{\lambda\uparrow\infty}\frac{\mathbb{P}\left(T_0^{\lambda,-}=\infty\right)}{1-\int_0^b\lambda e^{\lambda(b-k)}g^{(0)}_{0,0,\lambda}\left(k\right)\mathrm{d}k}\\
&=1-\psi'(0+)\lim_{\lambda\uparrow\infty}\frac{\Phi_\lambda\int_0^\infty e^{-\Phi_\lambda y}W\left(y\right)\mathrm{d}y}{\int_0^\infty\int_0^\infty\lambda e^{-\lambda k}\left[\Phi_\lambda\int_0^\infty e^{-\Phi_\lambda y}W\left(y\right)\mathrm{d}y\right]\dfrac{z}{k+b}\mathbb{P}\left(X_{k+b}\in\mathrm{d}z\right)\mathrm{d}k}\\
&=1-\frac{\psi'(0+)}{\int_0^\infty\dfrac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)}.
\end{aligned}
$$

Substituting into (3.10) together with (3.5), (3.15) and Initial Value Theorem gives

$$
\begin{aligned}
&\lim_{\lambda\uparrow\infty}\lim_{s\downarrow 0}\mathbb{E}_u\left[e^{-s\rho^{\lambda}_{0,b}},\rho^{\lambda}_{0,b}<\infty\right]\\
&=\mathbb{P}_u\left(\tau_0^-<\infty\right)-\lim_{\lambda\uparrow\infty}\int_0^b\lambda e^{\lambda(b-k)}g^{(0)}_{u,0,\lambda}\left(k\right)\mathrm{d}k\left(1-\lim_{\lambda\uparrow\infty}\lim_{s\downarrow 0}\mathbb{E}\left[e^{-s\rho^{\lambda}_{0,b}},\rho^{\lambda}_{0,b}<\infty\right]\right)\\
&=1-\psi'(0+)W\left(u\right)-\int_0^\infty\left[W\left(u\right)-W\left(u+z\right)\right]\frac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)\left(\frac{\psi'(0+)}{\int_0^\infty\dfrac{z}{b}\mathbb{P}\left(X_b\in\mathrm{d}z\right)}\right)\\
&=1-\psi'(0+)\frac{\int_0^\infty W\left(u+z\right)z\mathbb{P}\left(X_b\in\mathrm{d}z\right)}{\int_0^\infty z\mathbb{P}\left(X_b\in\mathrm{d}z\right)}.
\end{aligned}
$$

Chapter 4

Poissonian Potential Measures for Spectrally Negative Lévy Risk Models

4.1 Introduction

This chapter serves as a continuation of the idea proposed in Chapter 3.

In Chapter 3, the main contribution is on the introduction of hybrid observation scheme. Via the study of the Gerber–Shiu type function, we successfully demonstrate and justify the importance of hybrid observation from both a theoretical and practical perspective. Compliments to Albrecher et al. [2016] who established a complete set of exit identities under Poissonian observation for spectrally negative Lévy process, such study is made viable. In this chapter, we further supplement their work on exit problems for a spectrally negative Lévy process observed according to an independent Poisson process by extending the analysis to its corresponding q–potential measures.

Indeed, we shall not reiterate the importance on Poissonian observation given the throughout discussions in Section 1.3 and 3.1. While the classical q–potential potential measures are fundamentally defined in connection with continuous observations, when it concerns the analysis under Poisson observations, they have reached its limited capacity. This naturally leads us to the idea of extending the notion of *q–potential measures killed on continuous exiting* to *q–potential measures killed on Poisson exiting*, or interchangeably referred to as the *Poissonian (q–)potential measures* in the sequel. As we shall see, Poissonian q–potential measures will be shown to play a fundamental role in the study of Poissonian exit measures, and as such we also revisit some exit results given in Albrecher et al. [2016] and Baurdoux et al. [2016] in the process.

The contribution of this chapter is two–fold. On one hand, to be introduced in Subsection 4.3.1, we explore a new class of scale functions, called the *Poissonian scale functions*, which will allow to state the Poissonian potential and exit measures in the same form as their analogues in the continuous-time observation scheme framework. On the other hand, we derive explicitly the Poissonian potential measures. As demonstrated in Subsection 4.3.2, they can all be expressed compactly in terms of the existing or new scale functions.

The rest of the chapter is organized as follows. Section 4.2 is used to review necessary results and introduce new intermediate functions used for deriving and expressing the main results of Poissonian potential measures in Section 4.3. Section 4.4 aims at exploring an interesting interplay between Poissonian potential measures and Poissonian exit measures, thereby reinforcing the motivation in studying the Poissonian potential measures. As an application of the Poissonian potential measures, two problems that are pertained to surplus analysis are considered in Section 4.5 and 4.6. All technical proofs are postponed to the last section.

4.2 Preliminaries

On top of the exit identities introduced in Section 3.3, the following exit identities and potential measures under continuous observation would be used thorough the remaining chapter. The corresponding proofs can be found in Kyprianou [2014] and Albrecher et al. [2016] respectively.

Lemma 16. For $q \geq 0$ and $0 \leq x \leq a$,

$$\mathbb{E}_x \left[e^{-q\tau_a^+} 1_{\left\{\tau_a^+ < \tau_0^-\right\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{4.1}$$

and

$$\mathbb{E}_x \left[e^{-q\tau_0^- + \theta X_{\tau_0^-}} 1_{\left\{\tau_0^- < \tau_a^+\right\}} \right] = Z^{(q)}(x,\theta) - \frac{W^{(q)}(x)}{W^{(q)}(a)} Z^{(q)}(a,\theta). \tag{4.2}$$

Due to (2.7), the following identity

$$\mathbb{E}_x \left[e^{-q\tau_0^- + \theta X_{\tau_0^-}} 1_{\left\{\tau_0^- < \infty\right\}} \right] = Z^{(q)}(x,\theta) - \frac{\psi_q(\theta)}{\theta - \Phi_q} W^{(q)}(x), \quad x \geq 0. \tag{4.3}$$

is immediate.

Besides the above exit identities, the following identities from Loeffen et al. [2014] are also recalled, which will be heavily relied upon in the later analysis.

Lemma 17. For any $p, q, x \geq 0$ and $p \neq q$, we have

$$\int_0^x W^{(p)}(x-y) W^{(q)}(y) \, \mathrm{d}y = \frac{W^{(p)}(x) - W^{(q)}(x)}{p-q}, \tag{4.4}$$

and

$$\int_0^x W^{(p)}(x-y) Z^{(q)}(y,\theta) \, \mathrm{d}y = \frac{Z^{(p)}(x,\theta) - Z^{(q)}(x,\theta)}{p-q}. \tag{4.5}$$

Last but not least, we quote the following results pertained to potential measures. A thorough derivation and discussion can be found in Kyprianou [2014].

Lemma 18. For $q \geq 0$ and $a > 0$, the q–potential densities $\theta^{(q)}, r_+^{(q)}, r_-^{(q)}$ and $u^{(q)}$ are given by

$$\theta^{(q)}(y) = \Phi_q' e^{-\Phi_q y} - W^{(q)}(-y), \quad y \in \mathbb{R}, \tag{4.6}$$

$$r_+^{(q)}(x,y) = e^{\Phi_q x} W^{(q)}(-y) - W^{(q)}(x-y), \quad x, y \leq 0, \tag{4.7}$$

$$r_-^{(q)}(x,y) = e^{-\Phi_q y} W^{(q)}(x) - W^{(q)}(x-y), \quad x, y \geq 0, \tag{4.8}$$

$$u^{(q)}(x,y;a) = \frac{W^{(q)}(a-y)}{W^{(q)}(a)} W^{(q)}(x) - W^{(q)}(x-y), \quad x, y \in [0,a]. \tag{4.9}$$

4.3 Main Results

In this section, we first motivate the construction of Poissonian scale functions, followed by presenting the expressions to Poissonian potential measures.

To assist with the presentation of results, the following auxiliary function is constructed, and it will be used throughout this chapter. For $q \geq 0$, $\lambda > 0$ and $x, y \in \mathbb{R}$, let

$$A^{(q,\lambda)}(x,y) = W^{(q)}(x+y) + \lambda \int_0^y W^{(q)}(x+y-z) W^{(q+\lambda)}(z) \, \mathrm{d}z, \tag{4.10}$$

which can also be rewritten as

$$A^{(q,\lambda)}(x,y) = W^{(q+\lambda)}(x+y) - \lambda \int_0^x W^{(q)}(z) W^{(q+\lambda)}(x+y-z) \, \mathrm{d}z, \tag{4.11}$$

with the help of (4.4). Note that $A^{(q,\lambda)}(x,y)$ is actually the same as $g(q,\lambda,x,y)$ defined in Baurdoux et al. [2016], and $\mathcal{W}_x^{(q,\lambda)}(x+y)$ defined in Loeffen et al. [2014]. Moreover, it is seen from (4.10) and (4.11) that

$$A^{(q,\lambda)}(x,y) = W^{(q)}(x+y), \quad y \leq 0, \tag{4.12}$$

and

$$A^{(q,\lambda)}(x,y) = W^{(q+\lambda)}(x+y), \quad x \leq 0. \tag{4.13}$$

4.3.1 Poissonian Scale Functions

As seen in Section 4.2, continuous exit identities can be written neatly in terms of the classical scale functions such that heuristic structures to these identities are expressed in a transparent way. When it comes to the Poissonian exit identities that is derived in Albrecher et al. [2016], the structures are gone. To better formulate the results under Poisson observations, we propose the following new class of scale functions, called the Poissonian scale functions. They are all analogue to the classical scale functions introduced in Subsection 2.2.3.

Definition 8. For $q \geq 0$ and $\lambda > 0$, the Poissonian scale functions are defined as

$$W^{(q,\lambda)}(x) = (\Phi_{\lambda+q} - \Phi_q) \int_0^\infty e^{-\Phi_{\lambda+q} y} W^{(q)}(x+y)\,\mathrm{d}y, \quad x \in \mathbb{R} \tag{4.14}$$

and

$$Z^{(q,\lambda)}(x) = 1 + q \frac{\Phi_{\lambda+q}}{\Phi_{\lambda+q} - \Phi_q} \frac{\lambda}{\lambda+q} \int_0^x W^{(q,\lambda)}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{4.15}$$

Using (2.6), an alternative representation to $W^{(q,\lambda)}$ is

$$W^{(q,\lambda)}(x) = \frac{\Phi_{\lambda+q} - \Phi_q}{\lambda} Z^{(q)}(x, \Phi_{\lambda+q}). \tag{4.16}$$

Example 1. Consider the case when X_t is a Brownian motion with drift and compound Poisson jumps

$$X_t = \sigma B_t + \mu t - \sum_i^{N_t} \xi_i,$$

where ξ_i are i.i.d. random variables which are exponentially distributed with mean $1/\rho$ and N_t is an independent Poisson process with intensity a. The Laplace exponent of the process is

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \mu\theta - \frac{a\theta}{\rho+\theta}, \quad \theta \geq 0.$$

According to Kuznetsov et al. [2012], it is known that the cubic equation $\psi(\theta) = q$ has exactly three real roots $\{-\zeta_2, -\zeta_1, \Phi_q\}$. It is well known that a cubic equation exhibits explicit expressions to the roots. From (2.3), the scale function $W^{(q)}(x)$ takes the form

$$W^{(q)}(x)=\begin{cases}\dfrac{e^{\Phi_q x}}{\psi'(\Phi_q)}+\dfrac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)}+\dfrac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)}, & x\geq 0,\\ 0, & x<0.\end{cases}$$

Now, for $x\geq 0$,

$$\begin{aligned}&\int_0^\infty e^{-\Phi_{\lambda+q}y}W^{(q)}(x+y)\,\mathrm{d}y\\ &=\int_0^\infty e^{-\Phi_{\lambda+q}y}\left[\frac{e^{\Phi_q(x+y)}}{\psi'(\Phi_q)}+\frac{e^{-\zeta_1(x+y)}}{\psi'(-\zeta_1)}+\frac{e^{-\zeta_2(x+y)}}{\psi'(-\zeta_2)}\right]\mathrm{d}y\\ &=\frac{1}{\Phi_{\lambda+q}-\Phi_q}\frac{e^{\Phi_q x}}{\psi'(\Phi_q)}+\frac{1}{\Phi_{\lambda+q}+\zeta_1}\frac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)}+\frac{1}{\Phi_{\lambda+q}+\zeta_2}\frac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)},\end{aligned}$$

whereas for $x<0$,

$$\begin{aligned}&\int_0^\infty e^{-\Phi_{\lambda+q}y}W^{(q)}(x+y)\,\mathrm{d}y\\ &=\int_{-x}^\infty e^{-\Phi_{\lambda+q}y}\left[\frac{e^{\Phi_q(x+y)}}{\psi'(\Phi_q)}+\frac{e^{-\zeta_1(x+y)}}{\psi'(-\zeta_1)}+\frac{e^{-\zeta_2(x+y)}}{\psi'(-\zeta_2)}\right]\mathrm{d}y\\ &=\frac{1}{\Phi_{\lambda+q}-\Phi_q}\frac{e^{\Phi_{\lambda+q} x}}{\psi'(\Phi_q)}+\frac{1}{\Phi_{\lambda+q}+\zeta_1}\frac{e^{\Phi_{\lambda+q} x}}{\psi'(-\zeta_1)}+\frac{1}{\Phi_{\lambda+q}+\zeta_2}\frac{e^{\Phi_{\lambda+q} x}}{\psi'(-\zeta_2)}.\end{aligned}$$

Thus, from (4.14),

$$W^{(q,\lambda)}(x)=\begin{cases}\dfrac{e^{\Phi_q x}}{\psi'(\Phi_q)}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_1}\dfrac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_2}\dfrac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)}, & x\geq 0,\\ e^{\Phi_{\lambda+q}x}\left[\dfrac{1}{\psi'(\Phi_q)}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_1}\dfrac{1}{\psi'(-\zeta_1)}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_2}\dfrac{1}{\psi'(-\zeta_2)}\right], & x<0.\end{cases}$$

Meanwhile, for $x\geq 0$,

$$\begin{aligned}&\int_0^x W^{(q,\lambda)}(y)\,\mathrm{d}y\\ &=\int_0^x \frac{e^{\Phi_q y}}{\psi'(\Phi_q)}+\frac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_1}\frac{e^{-\zeta_1 y}}{\psi'(-\zeta_1)}+\frac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_2}\frac{e^{-\zeta_2 y}}{\psi'(-\zeta_2)}\mathrm{d}y\\ &=\frac{e^{\Phi_q x}-1}{\psi'(\Phi_q)\Phi_q}+\frac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_1}\frac{1-e^{-\zeta_1 x}}{\psi'(-\zeta_1)\zeta_1}+\frac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_2}\frac{1-e^{-\zeta_2 x}}{\psi'(-\zeta_2)\zeta_2}.\end{aligned}$$

Hence, from (4.15),

$$Z^{(q,\lambda)}(x)=\begin{cases}1+q\dfrac{\Phi_{\lambda+q}}{\Phi_{\lambda+q}-\Phi_q}\dfrac{\lambda}{\lambda+q}\\ \qquad\times\left[\dfrac{e^{\Phi_q x}-1}{\psi'(\Phi_q)\,\Phi_q}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_1}\dfrac{1-e^{-\zeta_1 x}}{\psi'(-\zeta_1)\,\zeta_1}+\dfrac{\Phi_{\lambda+q}-\Phi_q}{\Phi_{\lambda+q}+\zeta_2}\dfrac{1-e^{-\zeta_2 x}}{\psi'(-\zeta_2)\,\zeta_2}\right], & x\geq 0,\\ 1, & x<0.\end{cases}$$

By virtue of Initial Value Theorem, it can be seen that

$$\lim_{\lambda\to\infty}W^{(q,\lambda)}(x)=W^{(q)}(x). \tag{4.17}$$

Meanwhile, let $\varepsilon>0$ be a constant. For large enough λ, one has

$$\begin{aligned}W^{(q,\lambda)}(x)&=(\Phi_{q+\lambda}-\Phi_q)\int_0^\varepsilon e^{-\Phi_{q+\lambda}y}W^{(q)}(x+y)\mathrm{d}y+(\Phi_{q+\lambda}-\Phi_q)\int_\varepsilon^\infty e^{-\Phi_{q+\lambda}y}W^{(q)}(x+y)\mathrm{d}y\\ &\leq\Phi_{q+\lambda}\int_0^\varepsilon e^{-\Phi_{q+\lambda}y}\mathrm{d}yW^{(q)}(x+\varepsilon)+\Phi_{q+\lambda}\int_\varepsilon^\infty e^{-\left(\Phi_{q+\lambda}-\Phi_{q+\varepsilon}\right)y}e^{-\Phi_{q+\varepsilon}y}W^{(q)}(x+y)\mathrm{d}y\\ &\leq W^{(q)}(x+\varepsilon)+\Phi_{q+\lambda}e^{-\left(\Phi_{q+\lambda}-\Phi_{q+\varepsilon}\right)\varepsilon}\int_\varepsilon^\infty e^{-\Phi_{q+\varepsilon}y}W^{(q)}(x+y)\mathrm{d}y\\ &\leq W^{(q)}(x+\varepsilon)+\int_\varepsilon^\infty e^{-\Phi_{q+\varepsilon}y}W^{(q)}(x+y)\mathrm{d}y\\ &\leq W^{(q)}(x+\varepsilon)+\frac{1}{\varepsilon}Z^{(q)}\left(x,\Phi_{q+\varepsilon}\right),\end{aligned}$$

where the last line holds by (2.6). Thus, by Dominated Convergence Theorem,

$$\lim_{\lambda\to\infty}Z^{(q,\lambda)}(x)=Z^{(q)}(x).$$

To conclude, the Poissonian scale functions converge to the classical scale functions as $\lambda\to\infty$. Hence, (4.14) and (4.15) can be viewed as the Poissonian analogue to the classical scale functions.

In the following lemma, we make use of the Poissonian scale functions to rephrase two Poissonian identities in Albrecher and Ivanovs [2016] in a form which is consistent with their continuous analogues (4.1) and (4.2), respectively.

Lemma 19. For $q \geq 0$, and $x \in [0, a]$,

$$\mathbb{E}_x \left[e^{-q\tau_a^+} 1_{\{\tau_a^+ < T_0^{-,\lambda}\}} \right] = \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)}, \tag{4.18}$$

and

$$\mathbb{E}_x \left[e^{-qT_0^{-,\lambda}} 1_{\{T_0^{-,\lambda} < \tau_a^+\}} \right] = Z^{(q,\lambda)}(x) - \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} Z^{(q,\lambda)}(a). \tag{4.19}$$

Remark 2. Given their importance in the subsequent analysis, we limit the review of Albrecher et al. [2016] to the above two exit identities only. We note that (4.18) was first proved by Albrecher and Ivanovs [2013]. For both identities, a spatial approximation argument is used to handle spectrally negative Lévy processes with unbounded variation paths. Alternatively, simple conditioning arguments (coupled with the classical potential measure results) can be called upon to derive these results in a more direct manner. As an illustrative example, we consider $\mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right)$. The other cases can be similarly handled.

By conditioning on the first observation time T_1 (which has the same distribution as an independent exponential random variable e_λ with mean $1/\lambda$) and then using (2.11), we have

$$\begin{aligned}
\mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right) &= \mathbb{P}\left(\tau_a^+ < e_\lambda\right) + \int_0^a \mathbb{P}\left(X_{e_\lambda} \in \mathrm{d}y, e_\lambda < \tau_a^+\right) \mathbb{P}_y\left(\tau_a^+ < T_0^{-,\lambda}\right) \\
&= \mathbb{P}\left(\tau_a^+ < e_\lambda\right) + \int_0^a \lambda r_+^{(\lambda)}(-a, x-a)\, \mathbb{P}_x\left(\tau_a^+ < T_0^{-,\lambda}\right) \mathrm{d}y. \qquad (4.20)
\end{aligned}$$

For $x \in [0, a]$, by conditioning on τ_0^- and using (3.2), one finds that

$$\begin{aligned}
\mathbb{P}_x\left(\tau_a^+ < T_0^{-,\lambda}\right) &= \mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) + \int_{-\infty}^0 \mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- < \tau_a^+\right) \mathbb{P}_y\left(\tau_0^+ < e_\lambda\right) \mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right) \\
&= \mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) + \mathbb{E}_x\left[e^{\Phi_\lambda X_{\tau_0^-}} 1_{\{\tau_0^- < \tau_a^+\}}\right] \mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right). \qquad (4.21)
\end{aligned}$$

Substituting (4.21) into (4.20) yields the desired renewal equation for $\mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right)$.

4.3.2 Poissonian Potential Measures

With the Poissonian scale functions, we are now ready to obtain an expression to the Poissonian potential measures.

Definition 9. For $q \geq 0$ and $a > 0$, the Poissonian potential measures are defined as follows.

$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{+,\lambda} \wedge \tau_a^+\right)\mathrm{d}t = r_+^{(q,\lambda)}(x,y;a)\,\mathrm{d}y, \quad x,y \leq a,$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{-,\lambda} \wedge \tau_{-a}^-\right)\mathrm{d}t = r_-^{(q,\lambda)}(x,y;-a)\,\mathrm{d}y, \quad x,y \geq -a,$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{+,\lambda}\right)\mathrm{d}t = r_+^{(q,\lambda)}(x,y)\,\mathrm{d}y, \quad x,y \in \mathbb{R},$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{-,\lambda}\right)\mathrm{d}t = r_-^{(q,\lambda)}(x,y)\,\mathrm{d}y, \quad x,y \in \mathbb{R},$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{-,\lambda} \wedge \tau_a^+\right)\mathrm{d}t = u_{d:c}^{(q,\lambda)}(x,y;a)\,\mathrm{d}y, \quad x,y \leq a,$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^- \wedge T_a^{+,\lambda}\right)\mathrm{d}t = u_{c:d}^{(q,\lambda)}(x,y;a)\,\mathrm{d}y, \ x \in [0,a], y \geq 0,$$
$$\int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{-,\lambda} \wedge T_a^{+,\lambda}\right)\mathrm{d}t = u_{d:d}^{(q,\lambda)}(x,y;a)\,\mathrm{d}y, \quad x \in [0,a], y \in \mathbb{R}.$$

Among all of these Poissonian potential measures, $r_-^{(q,\lambda)}(x,y;-a)\,\mathrm{d}y$ and $r_+^{(q,\lambda)}(x,y;a)\,\mathrm{d}y$ are the two pivotal quantities as the derivation of explicit expressions for all the other potential measures heavily relies on them. The Poissonian potential densities $r_+^{(q,\lambda)}$, $r_-^{(q,\lambda)}$, and the triplet $(u_{d:c}^{(q,\lambda)}, u_{c:d}^{(q,\lambda)}, u_{d:d}^{(q,\lambda)})$ are the Poissonian analogues to the classical potential densities $r_+^{(q)}$, $r_-^{(q)}$ and $u^{(q)}$, respectively. Note that the subscripts c and d are used to characterize the type of exit whether it is under continuous–time or discrete–time (Poissonian) observations, respectively.

The following theorem summarizes the main results on Poissonian potential measures for spectrally negative Lévy processes.

Theorem 20. *For $q \geq 0$ and $a > 0$, the Poissonian q–potential densities are given by*

$$r_{+}^{(q,\lambda)}(x,y;a) = \frac{A^{(q,\lambda)}(-y,a)}{Z^{(q+\lambda)}(a,\Phi_q)} Z^{(q+\lambda)}(x,\Phi_q) - A^{(q,\lambda)}(-y,x), \quad x,y \leq a, \tag{4.22}$$

$$r_{-}^{(q,\lambda)}(x,y;-a) = \frac{A^{(q,\lambda)}(x,a)}{Z^{(q+\lambda)}(a,\Phi_q)} Z^{(q+\lambda)}(-y,\Phi_q) - A^{(q,\lambda)}(x,-y), \quad x,y \geq -a, \tag{4.23}$$

$$r_{+}^{(q,\lambda)}(x,y) = W^{(q,\lambda)}(-y) Z^{(q+\lambda)}(x,\Phi_q) - A^{(q,\lambda)}(-y,x), \quad x,y \in \mathbb{R}, \tag{4.24}$$

$$r_{-}^{(q,\lambda)}(x,y) = W^{(q,\lambda)}(x) Z^{(q+\lambda)}(-y,\Phi_q) - A^{(q,\lambda)}(x,-y), \quad x,y \in \mathbb{R}, \tag{4.25}$$

$$u_{d:c}^{(q,\lambda)}(x,y;a) = \frac{A^{(q,\lambda)}(a,-y)}{W^{(q,\lambda)}(a)} W^{(q,\lambda)}(x) - A^{(q,\lambda)}(x,-y), \quad x,y \leq a, \tag{4.26}$$

$$u_{c:d}^{(q,\lambda)}(x,y;a) = \frac{W^{(q,\lambda)}(a-y)}{W^{(q,\lambda)}(a)} W^{(q)}(x) - W^{(q)}(x-y), \quad x \in [0,a], y \geq 0, \tag{4.27}$$

$$u_{d:d}^{(q,\lambda)}(x,y;a) = \frac{\int_a^\infty e^{-\Phi_{q+\lambda} z} A^{(q,\lambda)}(z,-y)\,\mathrm{d}z}{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z)\,\mathrm{d}z} W^{(q,\lambda)}(x) - A^{(q,\lambda)}(x,-y), \quad x \in [0,a], y \in \mathbb{R}. \tag{4.28}$$

Remark 3. It is remarked that the above theorem holds true even without the net profit condition stated in Subsection 2.2.1.

The following corollary confirms the convergence of Poissonian potential measures to the classical potential measures when the observation intensity rate λ goes to infinity.

Proposition 21. For $q \geq 0$ and $a > 0$, we have

$$\lim_{\lambda\to\infty} r_{+}^{(q,\lambda)}(x,y) = r_{+}^{(q)}(x,y), \quad \text{for } x,y \leq 0, \tag{4.29}$$

$$\lim_{\lambda\to\infty} r_{-}^{(q,\lambda)}(x,y) = r_{-}^{(q)}(x,y), \quad \text{for } x,y \geq 0, \tag{4.30}$$

$$\lim_{\lambda\to\infty} u_{d:c}^{(q,\lambda)}(x,y;a) = u^{(q)}(x,y;a), \quad \text{for } x,y \in [0,a], \tag{4.31}$$

$$\lim_{\lambda\to\infty} u_{c:d}^{(q,\lambda)}(x,y;a) = u^{(q)}(x,y;a), \quad \text{for } x,y \in [0,a], \tag{4.32}$$

$$\lim_{\lambda\to\infty} u_{d:d}^{(q,\lambda)}(x,y;a) = u^{(q)}(x,y;a), \quad \text{for } x,y \in [0,a]. \tag{4.33}$$

4.4 Interplay Between Poissonian Potential Measures and Exit Measures

As mentioned in Subsection 2.2.4, potential measures are known to play a fundamental role in the exit problems of spectrally negative Lévy processes under the continuous time observation scheme. In particular, as demonstrated in Lemma 2.11, continuous exit measures can be expressed using its classical potential measure counterparts. When it comes to the discrete Poissonian observation scheme, similar relations between exit measures and potential measures may also be found. To do so, recall that the probability an observation is made in the infinitesimal time period $\mathrm{d}t$ is $\lambda \mathrm{d}t$. With this observation, we arrive at the relationship

$$\mathbb{P}_x\left(T_0^{-,\lambda} \in \mathrm{d}t\right) = \lambda \mathbb{P}_x\left(T_0^{-,\lambda} > t, X_t < 0\right)\mathrm{d}t$$

for $x \geq 0$ due to independence between X and the observation scheme. In the same spirit, we deduce that for $x \geq 0$ and $y \leq 0$,

$$\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{T_0^{-,\lambda}<\infty, X_{T_0^{-,\lambda}}\in \mathrm{d}y\}}\right] = \lambda\int_0^\infty e^{-qt}\mathbb{P}_x\left(t < T_0^{-,\lambda}, X_t \in \mathrm{d}y\right)\mathrm{d}t = \lambda r_-^{(q,\lambda)}(x,y)\mathrm{d}y.$$

Such duality further stresses the importance of Poissonian potential measures since Poissonian exit measures are just a direct consequence of its Poissonian potential measures counterpart.

Using the same argument, we immediately have the following corollary on Poissonian exit measures.

Corollary 22. For $q \geq 0$ and $a > 0$,

$$
\begin{aligned}
\mathbb{E}_x\left[e^{-qT_0^{+,\lambda}} 1_{\{X_{T_0^{+,\lambda}} \in \mathrm{d}y, T_0^{+,\lambda} < \tau_a^+\}}\right] &= \lambda r_+^{(q,\lambda)}(x, y; a)\,\mathrm{d}y, \quad x \leq a, y \in [0, a]\,, \\
\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{X_{T_0^{-,\lambda}} \in \mathrm{d}y, T_0^{-,\lambda} < \tau_{-a}^-\}}\right] &= \lambda r_-^{(q,\lambda)}(x, y; -a)\,\mathrm{d}y, \quad x \geq -a, y \in [0, a]\,, \\
\mathbb{E}_x\left[e^{-qT_0^{+,\lambda}} 1_{\{X_{T_0^{+,\lambda}} \in \mathrm{d}y, T_0^{+,\lambda} < \infty\}}\right] &= \lambda r_+^{(q,\lambda)}(x, y)\,\mathrm{d}y, \quad x \leq 0, y \geq 0, \\
\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{X_{T_0^{-,\lambda}} \in \mathrm{d}y, T_0^{-,\lambda} < \infty\}}\right] &= \lambda r_-^{(q,\lambda)}(x, y)\,\mathrm{d}y, \quad x \geq 0, y \leq 0, \qquad (4.34) \\
\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{X_{T_0^{-,\lambda}} \in \mathrm{d}y, T_0^{-,\lambda} < \tau_a^+\}}\right] &= \lambda u_{d:c}^{(q,\lambda)}(x, y; a)\,\mathrm{d}y, \quad x \in [0, a]\,, y \leq 0, \qquad (4.35) \\
\mathbb{E}_x\left[e^{-qT_a^{+,\lambda}} 1_{\{X_{T_a^{+,\lambda}} \in \mathrm{d}y, T_a^{+,\lambda} < \tau_0^-\}}\right] &= \lambda u_{c:d}^{(q,\lambda)}(x, y; a)\,\mathrm{d}y, \quad x \in [0, a]\,, y \geq a, \\
\mathbb{E}_x\left[e^{-qT_a^{+,\lambda}} 1_{\{X_{T_a^{+,\lambda}} \in \mathrm{d}y, T_a^{+,\lambda} < T_0^{-,\lambda}\}}\right] &= \lambda u_{d:d}^{(q,\lambda)}(x, y; a)\,\mathrm{d}y, \quad x \in [0, a]\,, y \geq a, \\
\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{X_{T_0^{-,\lambda}} \in \mathrm{d}y, T_0^{-,\lambda} < T_a^{+,\lambda}\}}\right] &= \lambda u_{d:d}^{(q,\lambda)}(x, y; a)\,\mathrm{d}y, \quad x \in [0, a]\,, y \leq 0. \qquad (4.36)
\end{aligned}
$$

Corollary 22 generalizes Theorems 3.1 and 3.2 of Albrecher et al. [2016] in which the joint Laplace transforms of the Poissonian exit times and the overshoots/undershoots are given.

We also recall, in the following Corollary, another Poissonian exit measure which was first found in Baurdoux et al. [2016]. Notice that the Poissonian exit measures (4.34), (4.35) and (4.37) are actually identical to (1.12), (1.11), and (1.8), respectively, in Baurdoux et al. [2016]. This is not surprising as the Parisian ruin time τ_q in Baurdoux et al. [2016] is well known to have the same distribution as $T_0^{-,q}$ (defined in our paper). However, we point out that Baurdoux et al. [2016] also relies on the spatial approximation argument to deal with the case of unbounded variation paths, while the present derivation relies more closely on the strength of the Poisson discretization technique to derive these results.

Corollary 23. For $q \geq 0$, $a, b > 0$, $x \in [a, -b]$ and $y \in [-a, 0]\,$, we have

$$\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}} 1_{\{X_{T_0^{-,\lambda}} \in \mathrm{d}y, T_0^{-,\lambda} < \tau_{-a}^- \wedge \tau_b^+\}}\right] = \lambda\left(\frac{A^{(q,\lambda)}(x, a)}{A^{(q,\lambda)}(b, a)} A^{(q,\lambda)}(b, -y) - A^{(q,\lambda)}(x, -y)\right)\mathrm{d}y. \qquad (4.37)$$

The proof is postponed to the the last section for which the key step consists of proving the following interesting identity:

$$\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]=\frac{A^{(q,\lambda)}(x,a)}{A^{(q,\lambda)}(b,a)},\quad x\in[-a,b].$$

4.5 Application – Occupation Time under Hybrid Observation Scheme

One of the applications of the Poissonian potential measures is to study the occupation time within a certain open interval (a,b), where $a,b\in\mathbb{R}$ and $b>a$, under a hybrid observation scheme that is introduced in Chapter 3. The ultimate goal is to obtain the Laplace transform of occupation time under such observation scheme. This generalizes its continuously–observed analogue in Landriault et al. [2011] and Loeffen et al. [2014].

4.5.1 Notations

Define a sequence of discrete observation times $\{\xi_n\}_{n\in\mathbb{N}}$ as follows. For ease of notation, denote $\mathbb{N}=\{0,1,2,\ldots\}$ and $\mathbb{N}_+=\{1,2,3,\ldots\}$. Let $\xi_0=0$, and for $n\in\mathbb{N}_+$,

$$\xi_1=e_1^\lambda,$$

$$\xi_n-\xi_{n-1}=\begin{cases}e_n^\lambda, & \text{if } X_{\xi_{n-1}}\notin(a,b)\\ \left(\tau_a^-\wedge\tau_b^+\right)\circ\theta_{\xi_{n-1}}+e_n^\lambda, & \text{if } X_{\xi_{n-1}}\in(a,b),\end{cases},\quad n=2,3,4\ldots, \tag{4.38}$$

where $\left\{e_n^\lambda\right\}_{n\in\mathbb{N}_+}$ is a sequence of i.i.d. exponential random variable with mean $1/\lambda>0$, θ is the Markov shift operator such that $X_t\circ\theta_s=X_{s+t}$ and $(h\wedge k)=\min(h,k)$.

We then define the Laplace transform of occupation time the surplus process spent in

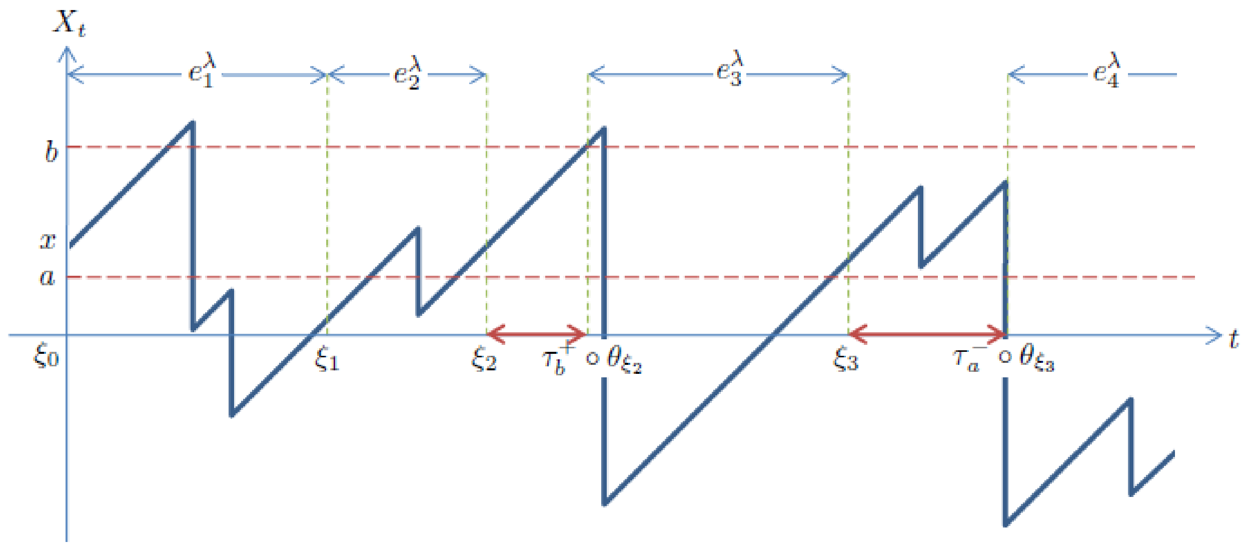

Figure 4.1: Illustration of occupation time under random observation.

(a, b) under such hybrid observation by

$$O_x^{(\lambda,q)}(a, b) = \mathbb{E}_x \left[e^{-q \int_B \mathrm{d}t} \right],$$

for $q \geq 0$. Here, the set B contains all time segments X spent in (a, b), i.e.

$$B = \bigcup_{i \in C} \left(\xi_i, \left(\tau_a^- \wedge \tau_b^+ \right) \circ \theta_{\xi_i} \right), \tag{4.39}$$

with the set C constructed as

$$C = \{ n \in \mathbb{N}_+ : X_{\xi_n} \in (a, b) \}.$$

Similar to the hybrid observation scheme in Chapter 3, the surplus process X is first monitored discretely at Poisson arrival times with rate λ until the surplus is observed to be in (a, b). Hereafter, the surplus process will be observed continuously until it leaves (a, b) such that the observation scheme switches back to the discrete fashion. Total time elapsed during the continuous observation will be contributed towards the occupation time. Figure 4.1 illustrates the occupation time under the observation scheme described by (4.38) for a particular sample path.

As a limiting case, suppose $a = -\infty$. The barrier a is understood to be unreachable such that $\left(\tau_a^- \wedge \tau_b^+\right) = \tau_b^+$ in (4.38) and (4.39).

4.5.2 Main Results

In the following derivations of the main results, the probabilities $\mathbb{P}_x\left(X_{T_b^{\lambda,-}} \in \mathrm{d}z, T_b^{\lambda,-} < \infty\right)$ (for $x \geq b, z \leq b$) and $\mathbb{P}_x\left(X_{T_a^{\lambda,+}} \in \mathrm{d}z, T_a^{\lambda,+} < \infty\right)$ (for $x \leq a, z \geq a$) are crucial. Due to spatial homogeneity property of spectrally negative Lévy process, it is immediate, from Corollary 22, that

$$\mathbb{P}_x\left(X_{T_b^{\lambda,-}} \in \mathrm{d}z, T_b^{\lambda,-} < \infty\right) = \lambda r_-^{(0,\lambda)}(x-b, z-b)\,\mathrm{d}z := f_\lambda^-(\mathrm{d}z; x, b)$$

and

$$\mathbb{P}_x\left(X_{T_a^{\lambda,+}} \in \mathrm{d}z, T_a^{\lambda,+} < \infty\right) = \lambda r_+^{(0,\lambda)}(x-a, z-a)\,\mathrm{d}z := f_\lambda^+(\mathrm{d}z; x, a)\,.$$

Occupation Time in Finite Intervals

The Laplace transform of occupation time in a finite interval $O_x^{(\lambda,q)}(a,b)$ is first considered.

Theorem 24. *For $x \in \mathbb{R}$,*

$$O_x^{(\lambda,q)}(a,b)$$
$$= \begin{cases} O_a^{(\lambda,q)}(a,b)\,, \;\; x \in (-\infty, a)\,, \\ \alpha_{x,b}^{(\lambda,q)}(a,b)\,O_b^{(\lambda,q)}(a,b) + \alpha_{x,a}^{(\lambda,q)}(a,b)\,O_a^{(\lambda,q)}(a,b) + \int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-} = \infty\right) f_\lambda^+(\mathrm{d}z; x, a)\,, \quad x \in [a,b)\,, \\ \beta_{x,b}^{(\lambda,q)}(a,b)\,O_b^{(\lambda,q)}(a,b) + \beta_{x,a}^{(\lambda,q)}(a,b)\,O_a^{(\lambda,q)}(a,b) + \mathbb{P}_x\left(T_b^{\lambda,-} = \infty\right), \;\; x \in [b,\infty)\,, \end{cases}$$

where the auxiliary functions $\alpha_{x,b}^{(\lambda,q)}(a,b)$, $\alpha_{x,a}^{(\lambda,q)}(a,b)$, $\beta_{x,b}^{(\lambda,q)}(a,b)$, $\beta_{x,a}^{(\lambda,q)}(a,b)$ are given by (4.95)–(4.96), and the boundary values $O_a^{(\lambda,q)}(a,b)$, $O_b^{(\lambda,q)}(a,b)$ are solved in (4.97)–(4.98).

Note that all the auxiliary functions are expressed in terms of either known exiting identities, laws or exit measures. Although the solution form does not look promising, they are indeed well characterized.

Occupation Time in Unbounded Intervals

As a special case, suppose one is interested in the Laplace transform of occupation time in an unbounded interval such that, without loss of generality, $a = -\infty$ and $b = 0$. It is found that a much tractable expression to the Laplace transform of occupation time $O_x^{(\lambda,q)}(-\infty, 0)$ can be obtained as follows.

Theorem 25. *For $x \in \mathbb{R}$,*

$$O_x^{(\lambda,q)}(-\infty, 0) = \frac{\psi'(0+)\,\Phi_\lambda \Phi_q}{\Phi_\lambda - \Phi_q} \left[\frac{Z(x, \Phi_q)}{q} - \frac{Z(x, \Phi_\lambda)}{\lambda} \right].$$

As a consistency check, note that

$$\lim_{\lambda \uparrow \infty} \frac{\Phi_\lambda}{\Phi_\lambda - \Phi_q} = 1,$$

and by Initial Value Theorem,

$$\lim_{\lambda \uparrow \infty} \Phi_\lambda \frac{Z(x, \Phi_\lambda)}{\lambda} = \lim_{\lambda \uparrow \infty} \int_0^\infty \Phi_\lambda e^{-\Phi_\lambda z} W(x+z)\,\mathrm{d}z = W(x)$$

since $\Phi_\lambda \uparrow \infty$ as $\lambda \uparrow \infty$. As a result,

$$\begin{aligned}
&\lim_{\lambda \uparrow \infty} O_x^{(\lambda,q)}(-\infty, 0) \\
&= \psi'(0+)\,\Phi_q \left[\frac{Z(x, \Phi_q)}{q} \lim_{\lambda \uparrow \infty} \frac{\Phi_\lambda}{\Phi_\lambda - \Phi_q} - \lim_{\lambda \uparrow \infty} \frac{\Phi_\lambda}{\Phi_\lambda - \Phi_q} \frac{Z(x, \Phi_\lambda)}{\lambda} \right] \\
&= \psi'(0+)\,\Phi_q \int_0^\infty e^{-\Phi_q z} W(x+z)\,\mathrm{d}z,
\end{aligned}$$

recovering Corollary 1 of Landriault et al. [2011].

4.6 Application – Parisian Ruin with Poissonian Observations

As another application of the Poissonian potential measures, we consider a generalization of the Parisian risk model in which the underlying spectrally negative Lévy process X is subject to a Poissonian observation scheme with intensity rate $\lambda > 0$. Our objective is to derive a Gerber–Shiu type density at the Poissonian Parisian ruin time which will generalize its continuously–observed analogue in Baurdoux et al. [2016].

4.6.1 Notations

Under a Poissonian observation scheme, an excursion of X below level 0 starts whenever the spectrally negative Lévy process X is observed below level 0 and ends whenever the spectrally negative Lévy process X is subsequently observed above level 0. Recall $\{T_n\}_{n\in\mathbb{N}}$ is the sequence of observation times which are the arrival epochs of an independent Poisson process with rate $\lambda > 0$. For $n \in \mathbb{N}$, we denote ξ_n the starting time of the n-th excursion below level 0, i.e.,

$$
\begin{aligned}
\xi_1 &= \inf\{T_i : X_{T_i} < 0\} \\
\xi_n &= \inf\left\{T_i : X_{T_i} < 0, X_{T_{i-1}} \geq 0 \text{ and } T_i > \xi_{n-1}\right\}, \text{ for } n \geq 2.
\end{aligned}
$$

Let θ be the Markov shift operator acting as $X_t \circ \theta_s = X_{t+s}$ for $s, t \geq 0$. The ending time of n-th excursion below level 0 is then given by $T_0^{+,\lambda} \circ \theta_{\xi_n}$. The excursion is deemed to have caused ruin if the length of the excursion exceeds an independent excursion–specific exponential time with mean $1/q$. Thus, the Parisian ruin time under the Poissonian observation is defined as

$$
T^{\lambda,q} = \inf\left\{\xi_n + e_n^q : T_0^{+,\lambda} \circ \theta_{\xi_n} - \xi_n > e_n^q\right\},
$$

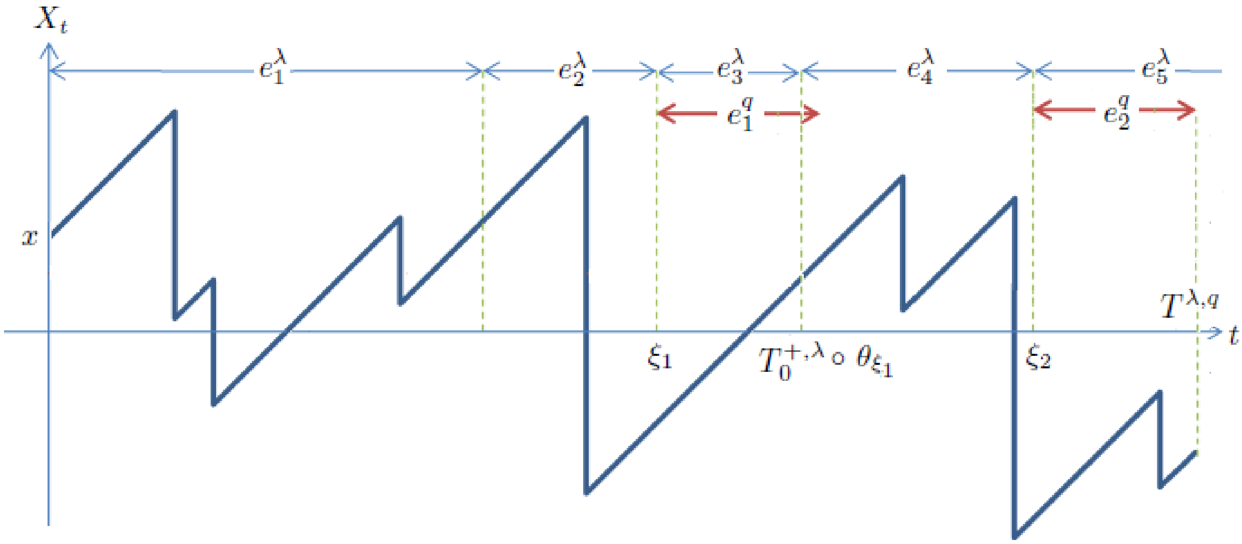

Figure 4.2: Illustration of Parisian ruin time with an exponential grace period under Poissonian observation.

where e_n^q is an independent exponential clock with mean $1/q$ generated at time ξ_n for the n-th excursion below level 0. Figure 4.2 illustrates such ruin time for a particular sample path.

Our objective is to derive an explicit expression for the following Gerber–Shiu type density at the Parisian time:

$$\mathbb{E}_x\left[e^{-sT^{\lambda,q}} 1_{\left\{X_{T^{\lambda,q}} \in \mathrm{d}y, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\right\}}\right], \quad x \in [-a, b], y \in \mathbb{R}, \tag{4.40}$$

where $a, b > 0$.

4.6.2 Main Results

For ease of notation, we define two auxiliary functions. For $x \in [a, -b]$ and $y \in \mathbb{R}$,

$$H_{a,b}^{(s,q,\lambda)}(x, y) = \int_0^a v^{(s,\lambda)}(x, -w; b) A^{(s+q,\lambda)}(a-w, y-a)\, \mathrm{d}w, \tag{4.41}$$

and

$$Z_{a,b}^{(s,q,\lambda)}(x) = \int_0^a \upsilon^{(s,\lambda)}(x, -w; b) W^{(s+q,\lambda)}(a-w)\,\mathrm{d}w, \tag{4.42}$$

where

$$\upsilon^{(s,\lambda)}(x, w; b) = \begin{cases} \delta_x(w), & x \in [-a, 0)\,, \\ \lambda u_{d:d}^{(s,\lambda)}(x, w; b), & x \in [0, b]\,, \end{cases} \tag{4.43}$$

and $\delta_x(\cdot)$ is the Dirac delta function centered at x.

Theorem 26. *For $x \in [-a, b]$ and $y \in \mathbb{R}$, the Gerber–Shiu density (4.40) is given by*

$$\frac{1}{q\mathrm{d}y}\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,\, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]$$
$$= \frac{\theta^{(s+q+\lambda)}(y) + A^{(s+q)}(a, -a-y) - \lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)H_{a,b}^{(s,q,\lambda)}(z,-y)\mathrm{d}z}{W^{(s+q,\lambda)}(a) - \lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)Z_{a,b}^{(s,q,\lambda)}(z)\mathrm{d}z} Z_{a,b}^{(s,q,\lambda)}(x) - H_{a,b}^{(s,q,\lambda)}(x,-y).$$

Remark 4. One expects the Gerber–Shiu density in Theorem 26 to reduce to the Gerber-Shiu density in Theorem 1.2 of Baurdoux et al. [2016] (or equivalently (4.37)) when the observation intensity rate λ goes to ∞. This result can be proven (see Appendix) when the spectrally negative Lévy process X has bounded variation paths, namely for $x \in [-a, b]$ and $y \in [-a, 0]$,

$$\lim_{\lambda\to\infty}\frac{1}{q\mathrm{d}y}\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,\, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] = \frac{A^{(s,q)}(x,a)}{A^{(s,q)}(b,a)}A^{(s,q)}(b,-y) - A^{(s,q)}(x,-y). \tag{4.44}$$

Unfortunately, there are non–trivial difficulties that arise in the case where the spectrally negative Lévy process X has unbounded variation paths, which are related to the evaluation of the integrals $\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)H_{a,b}^{(s,q,\lambda)}(z,-y)\mathrm{d}z$ and $\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)Z_{a,b}^{(s,q,\lambda)}(z)\mathrm{d}z$ (unless $a = \infty$). To complete this step, a non-trivial study of the two functions $H_{a,b}^{(s,q,\lambda)}(x,y)$ and $Z_{a,b}^{(s,q,\lambda)}(x)$ is necessary.

4.7 Appendix

In the rest of the section, to ease the notational burdensome, denote e_q and e_λ as two exponential random variables with mean $1/q$ and $1/\lambda$ respectively that is pertained to the Parisian delay and discrete Poissonian observation. We remark that e_q, e_λ and the underlying process X are mutually independent.

4.7.1 Proof of Theorem 20

Proof of (4.22)

For $x, y \leq a$, let

$$R_+^{(q,\lambda)}(x, \mathrm{d}y; a) = \int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{+,\lambda} \wedge \tau_a^+\right)\mathrm{d}t = \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{+,\lambda} \wedge \tau_a^+\right).$$

We consider separately the cases where $x < 0$ and $x \in [0, a]$.

For $x < 0$, conditioning on whether e_q or τ_0^+ happens first, one deduces that

$$\begin{aligned} R_+^{(q,\lambda)}(x, \mathrm{d}y; a) &= \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^+\right) + \mathbb{P}_x\left(\tau_0^+ < e_q\right) R_+^{(q,\lambda)}(0, \mathrm{d}y; a) \\ &= r_+^{(q)}(x, y)\,\mathrm{d}y + e^{\Phi_q x} R_+^{(q,\lambda)}(0, \mathrm{d}y; a), \end{aligned} \tag{4.45}$$

where the last line holds due to (2.11) and (3.2).

For $x \in [0, a]$, comparing e_q, τ_a^+, and the first Poissonian observation time e_λ, it follows that

$$\begin{aligned} R_+^{(q,\lambda)}(x, \mathrm{d}y; a) &= \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < e_\lambda \wedge \tau_a^+\right) + \int_{-\infty}^0 \mathbb{P}_x\left(X_{e_\lambda} \in \mathrm{d}z, e_\lambda < e_q \wedge \tau_a^+\right) R_+^{(q,\lambda)}(z, \mathrm{d}y; a) \\ &= r_+^{(q+\lambda)}(x-a, y-a)\mathrm{d}y + \int_{-\infty}^0 \lambda r_+^{(q+\lambda)}(x-a, z-a)\mathrm{d}z R_+^{(q,\lambda)}(z, \mathrm{d}y; a). \end{aligned} \tag{4.46}$$

Substituting (4.45) with $x=z$ into (4.46) and using (4.7) yield

$$\begin{aligned}
&R_{+}^{(q,\lambda)}(x,\mathrm{d}y;a)\\
&=r_{+}^{(q+\lambda)}(x-a,y-a)\,\mathrm{d}y+\lambda\int_{-\infty}^{0}r_{+}^{(q+\lambda)}(x-a,z-a)\,r_{+}^{(q)}(z,y)\,\mathrm{d}z\mathrm{d}y\\
&\quad+\lambda\int_{-\infty}^{0}r_{+}^{(q+\lambda)}(x-a,z-a)\,e^{\Phi_q z}\mathrm{d}zR_{+}^{(q,\lambda)}(0,\mathrm{d}y;a)\\
&=r_{+}^{(q+\lambda)}(x-a,y-a)\,\mathrm{d}y+\lambda\int_{-\infty}^{0}r_{+}^{(q+\lambda)}(x-a,z-a)\left(e^{\Phi_q z}W^{(q)}(-y)-W^{(q)}(z-y)\right)\mathrm{d}z\mathrm{d}y\\
&\quad+\lambda\int_{-\infty}^{0}r_{+}^{(q+\lambda)}(x-a,z-a)\,e^{\Phi_q z}\mathrm{d}zR_{+}^{(q,\lambda)}(0,\mathrm{d}y;a). \qquad (4.47)
\end{aligned}$$

Letting $x=0$ in (4.47), we solve for $R_{+}^{(q,\lambda)}(0,\mathrm{d}y;a)$ and obtain

$$\begin{aligned}
&R_{+}^{(q,\lambda)}(0,\mathrm{d}y;a)/\mathrm{d}y\\
&=\frac{r_{+}^{(q+\lambda)}(-a,y-a)+W^{(q)}(-y)-\lambda\int_{-\infty}^{0}r_{+}^{(q+\lambda)}(-a,z-a)W^{(q)}(z-y)\mathrm{d}z}{1-\lambda\int_{-\infty}^{0}e^{\Phi_q z}r_{+}^{(q+\lambda)}(-a,z-a)\mathrm{d}z}-W^{(q)}(-y).
\end{aligned} \qquad (4.48)$$

In what follows, we focus on specifying the two types of integrals in (4.47) and (4.48). On one hand, for $x\le a$,

$$\begin{aligned}
&\lambda\int_{-\infty}^{0}e^{\Phi_q z}r_{+}^{(q+\lambda)}(x-a,z-a)\,\mathrm{d}z\\
&=\int_{-\infty}^{a}e^{\Phi_q z}\mathbb{P}_x\left(X_{e'_\lambda}\in\mathrm{d}z,e_\lambda<e_q\wedge\tau_a^{+}\right)\mathrm{d}z-\lambda\int_{0}^{a}e^{\Phi_q z}r_{+}^{(q+\lambda)}(x-a,z-a)\,\mathrm{d}z\\
&=\int_{-\infty}^{0}e^{\Phi_q(z+a)}\mathbb{P}_{x-a}\left(X_{e_\lambda}\in\mathrm{d}z,e_\lambda<e_q\wedge\tau_0^{+}\right)\mathrm{d}z-\lambda\int_{0}^{a}e^{\Phi_q z}r_{+}^{(q+\lambda)}(x-a,z-a)\,\mathrm{d}z\\
&=e^{\Phi_q a}\mathbb{E}_{x-a}\left[e^{-qe_\lambda+\Phi_q X_{e_\lambda}}1_{\{e_\lambda<\tau_0^{+}\}}\right]-\lambda\int_{0}^{a}e^{\Phi_q z}r_{+}^{(q+\lambda)}(x-a,z-a)\,\mathrm{d}z.
\end{aligned}$$

Furthermore, using (30) of Albrecher et al. [2016], (4.7) and (2.5), one finds that

$$\begin{aligned}
&\lambda\int_{-\infty}^{0}e^{\Phi_q z}r_{+}^{(q+\lambda)}(x-a,z-a)\mathrm{d}z\\
&=e^{\Phi_q a}\left(e^{\Phi_q(x-a)}-e^{\Phi_{q+\lambda}(x-a)}\right)-e^{\Phi_{q+\lambda}(x-a)}\lambda\int_{0}^{a}e^{\Phi_q(a-z)}W^{(q+\lambda)}(z)\,\mathrm{d}z+\lambda\int_{0}^{x}e^{\Phi_q(x-z)}W^{(q+\lambda)}(z)\,\mathrm{d}z\\
&=Z^{(q+\lambda)}(x,\Phi_q)-e^{\Phi_{q+\lambda}(x-a)}Z^{(q+\lambda)}(a,\Phi_q). \qquad (4.49)
\end{aligned}$$

On the other hand, for $x \le a$ and $y < 0$,

$$
\begin{aligned}
&\lambda \int_{-\infty}^{0} r_{+}^{(q+\lambda)}(x-a, z-a) W^{(q)}(z-y) \mathrm{d}z \\
&= \lambda \int_{-\infty}^{0} \left[e^{\Phi_{q+\lambda}(x-a)} W^{(q+\lambda)}(a-z) - W^{(q+\lambda)}(x-z) \right] W^{(q)}(z-y) \mathrm{d}z \\
&= e^{\Phi_{q+\lambda}(x-a)} \lambda \int_{0}^{\infty} W^{(q+\lambda)}(a+z) W^{(q)}(-y-z) \mathrm{d}z - \lambda \int_{0}^{\infty} W^{(q+\lambda)}(x+z) W^{(q)}(-y-z) \mathrm{d}z \\
&= e^{\Phi_{q+\lambda}(x-a)} \lambda \int_{a}^{a-y} W^{(q+\lambda)}(z) W^{(q)}(a-y-z) \mathrm{d}z - \lambda \int_{x}^{x-y} W^{(q+\lambda)}(z) W^{(q)}(x-y-z) \mathrm{d}z \\
&= e^{\Phi_{q+\lambda}(x-a)} \lambda \left[\int_{0}^{a-y} W^{(q)}(a-y-z) W^{(q+\lambda)}(z) \mathrm{d}z - \int_{0}^{a} W^{(q)}\left(a-y-z\right) W^{(q+\lambda)}\left(z\right) \mathrm{d}z \right] \\
&\quad - \lambda \left[\int_{0}^{x-y} W^{(q)}(x-y-z) W^{(q+\lambda)}(z) \mathrm{d}z - \int_{0}^{x} W^{(q)}(x-y-z) W^{(q+\lambda)}(z) \mathrm{d}z \right] \\
&= e^{\Phi_{q+\lambda}(x-a)} \left[W^{(q+\lambda)}(a-y) - A^{(q,\lambda)}(-y,a) \right] - \left[W^{(q+\lambda)}(x-y) - A^{(q,\lambda)}(-y,x) \right], \qquad (4.50)
\end{aligned}
$$

where the last step is due to (4.4) and (4.10). Note that it is easily seen from (4.13) that the equality (4.50) also holds for $y \ge 0$.

Substituting (4.49) and (4.50) with $x = 0$ into (4.48), and using (4.12), it is relatively easy to show that

$$
R_{+}^{(q,\lambda)}\left(0, \mathrm{d}y; a\right) / \mathrm{d}y = \frac{A^{(q,\lambda)}\left(-y, a\right)}{Z^{(q+\lambda)}\left(a, \Phi_{q}\right)} - W^{(q)}\left(-y\right). \qquad (4.51)
$$

Lastly, substituting (4.7) and (4.51) into (4.45) yields, for $x < 0$,

$$
R_{+}^{(q,\lambda)}\left(x, \mathrm{d}y; a\right) / \mathrm{d}y = \frac{A^{(q,\lambda)}\left(-y, a\right)}{Z^{(q+\lambda)}\left(a, \Phi_{q}\right)} e^{\Phi_{q} x} - W^{(q)}\left(x-y\right).
$$

Also, substituting (4.7), (4.49), (4.50), and (4.51) into (4.47) yields, for $x \in [0, a]$,

$$
R_{+}^{(q,\lambda)}\left(x, \mathrm{d}y; a\right) / \mathrm{d}y = \frac{A^{(q,\lambda)}\left(-y, a\right)}{Z^{(q+\lambda)}\left(a, \Phi_{q}\right)} Z^{(q+\lambda)}\left(x, \Phi_{q}\right) - A^{(q,\lambda)}\left(-y, x\right).
$$

We complete the proof by unifying the above two expressions for $x \le a$.

Proof of (4.23)

For $x, y \geq -a$, let

$$R_{-}^{(q,\lambda)}(x, \mathrm{d}y; -a) = \int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < T_0^{-,\lambda} \wedge \tau_{-a}^{-}\right)\mathrm{d}t = \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^{-}\right).$$

We consider separately the cases where $y \in [-a, 0)$ and $y \geq 0$.

For $y \in [-a, 0)$, we shall have that $\tau_0^- < e_q \wedge T_0^{-,\lambda}$ almost surely. Subsequently, at level $X_{\tau_0^-}$, we know that the random time $\tau_0^+ \wedge e_q$ should occur prior to the next observation time e_λ. Therefore,

$$\begin{aligned}
R_{-}^{(q,\lambda)}(x, \mathrm{d}y; -a) &= \frac{1}{q}\int_{[-a,0]}\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}z, \tau_0^- < e_q\right)\mathbb{P}_z\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^{-}\right) \\
&= \int_{[-a,0]}\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}z, \tau_0^- < e_q\right)\mathbb{P}_z\left(\tau_0^+ < e_q \wedge e_\lambda \wedge \tau_{-a}^-\right)R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a) \\
&\quad + \frac{1}{q}\int_{[-a,0]}\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}z, \tau_0^- < e_q\right)\mathbb{P}_z\left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^+ \wedge e_\lambda \wedge \tau_{-a}^-\right).
\end{aligned}$$

Subsequently, using (4.1) and (2.13) leads to

$$\begin{aligned}
&R_{-}^{(q,\lambda)}(x, \mathrm{d}y; -a) \\
&= \int_{[-a,0]}\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}z, \tau_0^- < e_q\right)\left[\frac{W^{(q+\lambda)}(z+a)}{W^{(q+\lambda)}(a)}R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a) + u^{(q+\lambda)}(z+a, y+a; a)\,\mathrm{d}y\right] \\
&= \mathbb{E}_x\left[e^{-q\tau_0^-}\frac{W^{(q+\lambda)}\left(X_{\tau_0^-}+a\right)}{W^{(q+\lambda)}(a)}1_{\{\tau_0^- < \infty\}}\right]R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a) \\
&\quad + \mathbb{E}_x\left[e^{-q\tau_0^-}u^{(q+\lambda)}\left(X_{\tau_0^-}+a, y+a; a\right)1_{\{\tau_0^- < \infty\}}\right]\mathrm{d}y,
\end{aligned} \tag{4.52}$$

where the last line holds due to the fact that $W^{(q+\lambda)}(x) = 0$ for any $x < 0$.

For $y \geq 0$, conditioning on whether τ_0^- occurs before e_q (or not) leads to

$$\begin{aligned}
R_{-}^{(q,\lambda)}(x, \mathrm{d}y; -a) &= \frac{1}{q}\int_{[-a,0]}\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}z, \tau_0^- < e_q\right)\mathbb{P}_z\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^{-}\right) \\
&\quad + \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^-\right).
\end{aligned} \tag{4.53}$$

Since $z \leq 0$ and $y \geq 0$, by (4.1), we have

$$\begin{aligned}\frac{1}{q}\mathbb{P}_z\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^-\right) &= \mathbb{P}_z\left(\tau_0^+ < e_q \wedge e_\lambda \wedge \tau_{-a}^-\right) R_-^{(q,\lambda)}(0, \mathrm{d}y; -a) \\ &= \frac{W^{(q+\lambda)}(z+a)}{W^{(q+\lambda)}(a)} R_-^{(q,\lambda)}(0, \mathrm{d}y; -a). \end{aligned} \tag{4.54}$$

Substituting (4.54) into (4.53) and using (2.12) give

$$R_-^{(q,\lambda)}(x, \mathrm{d}y; -a) = \mathbb{E}_x\left[e^{-q\tau_0^-}\frac{W^{(q+\lambda)}\left(X_{\tau_0^-}+a\right)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}}\right] R_-^{(q,\lambda)}(0, \mathrm{d}y; -a) + r_-^{(q)}(x, y)\mathrm{d}y. \tag{4.55}$$

We further note that (4.52) and (4.55) can be expressed in a unified manner as follows: for $x, y \geq -a$,

$$\begin{aligned} &R_-^{(q,\lambda)}(x, \mathrm{d}y; -a)/\mathrm{d}y \\ &= \mathbb{E}_x\left[e^{-q\tau_0^-}\frac{W^{(q+\lambda)}\left(X_{\tau_0^-}+a\right)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}}\right] R_-^{(q,\lambda)}(0, \mathrm{d}y; -a)/\mathrm{d}y \\ &\quad + \mathbb{E}_x\left[e^{-q\tau_0^-} u^{(q+\lambda)}\left(X_{\tau_0^-}+a, y+a; a\right) 1_{\{\tau_0^- < \infty\}}\right] 1_{\{-a \leq y < 0\}} + r_-^{(q)}(x, y)\, 1_{\{y \geq 0\}}. \end{aligned} \tag{4.56}$$

To solve for $R_-^{(q,\lambda)}(0, \mathrm{d}y; -a)$, we condition on whether e_q arrives prior to the next observation time e_λ. Using (2.10), we have

$$\begin{aligned} R_-^{(q,\lambda)}(0, \mathrm{d}y; -a) &= \frac{1}{q}\mathbb{P}\left(X_{e_q} \in \mathrm{d}y, e_q < e_\lambda \wedge \tau_{-a}^-\right) + \int_0^\infty \mathbb{P}\left(X_{e_\lambda'} \in \mathrm{d}z, e_\lambda < e_q \wedge \tau_{-a}^-\right) R_-^{(q,\lambda)}(z, \mathrm{d}y; -a) \\ &= r_-^{(q+\lambda)}(a, y+a)\,\mathrm{d}y + \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a)\, R_-^{(q,\lambda)}(z, \mathrm{d}y; -a)\,\mathrm{d}z. \end{aligned} \tag{4.57}$$

Substituting (4.56) with $x = z$ into (4.57), we then solve for $R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a)$ and obtain

$$\begin{aligned}
&R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a)/\mathrm{d}y \\
&= \frac{r_{-}^{(q+\lambda)}(a, y+a) + \lambda \int_0^\infty r_{-}^{(q+\lambda)}(a, z+a)\mathbb{E}_z\left[e^{-q\tau_0^-} u^{(q+\lambda)}(X_{\tau_0^-} + a, y+a; a)1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z}{1 - \lambda \int_0^\infty r_{-}^{(q+\lambda)}(a, z+a)\mathbb{E}_z\left[e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-}+a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z} \\
&= \frac{e^{-\Phi_{q+\lambda} y} W^{(q+\lambda)}(a) - \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi_{q+\lambda} z}\mathbb{E}_z\left[e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} - y) 1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z}{e^{\Phi_{q+\lambda} a} - \lambda \int_0^\infty e^{-\Phi_{q+\lambda} z}\mathbb{E}_z\left[e^{-q\tau_0^-} W^{(q+\lambda)}\left(X_{\tau_0^-} + a\right) 1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z}
\end{aligned} \tag{4.58}$$

for $-a \le y < 0$, and

$$\begin{aligned}
&R_{-}^{(q,\lambda)}(0, \mathrm{d}y; -a)/\mathrm{d}y \\
&= \frac{r_{-}^{(q+\lambda)}(a, y+a) + \lambda \int_0^\infty r_{-}^{(q+\lambda)}(a, z+a) r_{-}^{(q)}(z, y) \, \mathrm{d}z}{1 - \lambda \int_0^\infty r_{-}^{(q+\lambda)}(a, z+a)\mathbb{E}_z\left[e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-}+a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z} \\
&= \frac{e^{-\Phi_{q+\lambda} y} W^{(q+\lambda)}(a) + \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi_{q+\lambda} z} r_{-}^{(q)}(z, y) \mathrm{d}z}{e^{\Phi_{q+\lambda} a} - \lambda \int_0^\infty e^{-\Phi_{q+\lambda} z}\mathbb{E}_z\left[e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} + a) 1_{\{\tau_0^- < \infty\}}\right] \mathrm{d}z}
\end{aligned} \tag{4.59}$$

for $y \ge 0$, thanks to (4.8) and (4.9).

Next, we focus on simplifying (4.58) and (4.59). By the spatial homogeneity of X, for any $z > 0$ and $y < 0$,

$$\begin{aligned}
\mathbb{E}_z\left[e^{-q\tau_0^-} W^{(q+\lambda)}\left(X_{\tau_0^-} - y\right) 1_{\{\tau_0^- < \infty\}}\right] &= \mathbb{E}_{z-y}\left[e^{-q\tau_{-y}^-} W^{(q+\lambda)}\left(X_{\tau_{-y}^-}\right) 1_{\left\{\tau_{-y}^- < \infty\right\}}\right] \\
&= \lim_{b \to \infty} \mathbb{E}_{z-y}\left[e^{-q\tau_{-y}^-} W^{(q+\lambda)}\left(X_{\tau_{-y}^-}\right) 1_{\left\{\tau_{-y}^- < \tau_b^+\right\}}\right].
\end{aligned} \tag{4.60}$$

Thanks to Lemma 2.2 of Loeffen et al. [2014], we deduce that

$$
\begin{aligned}
&\mathbb{E}_{z-y}\left[e^{-q\tau_{-y}^{-}}W^{(q+\lambda)}\left(X_{\tau_{-y}^{-}}\right)1_{\left\{\tau_{-y}^{-}<\tau_{b}^{+}\right\}}\right]\\
&=W^{(q+\lambda)}(z-y)-\lambda\int_{-y}^{z-y}W^{(q)}(z-y-x)W^{(q+\lambda)}(x)\mathrm{d}x\\
&\quad-\frac{W^{(q)}(z)}{W^{(q)}(b+y)}\left(W^{(q+\lambda)}(b)-\lambda\int_{-y}^{b}W^{(q)}(b-x)W^{(q+\lambda)}(x)\mathrm{d}x\right)\\
&=W^{(q+\lambda)}(z-y)-\lambda\int_{0}^{z-y}W^{(q)}(z-y-x)W^{(q+\lambda)}(x)\mathrm{d}x+\lambda\int_{0}^{-y}W^{(q)}(z-y-x)W^{(q+\lambda)}(x)\mathrm{d}x\\
&\quad-\frac{W^{(q)}(z)}{W^{(q)}(b+y)}\left(W^{(q+\lambda)}(b)-\lambda\int_{0}^{b}W^{(q)}(b-x)W^{(q+\lambda)}(x)\mathrm{d}x+\lambda\int_{0}^{-y}W^{(q)}(b-x)W^{(q+\lambda)}(x)\mathrm{d}x\right)\\
&=W^{(q)}(z-y)+\lambda\int_{0}^{-y}W^{(q)}(z-y-x)W^{(q+\lambda)}(x)\mathrm{d}x\\
&\quad-\frac{W^{(q)}(z)}{W^{(q)}(b+y)}\left(W^{(q)}(b)+\lambda\int_{0}^{-y}W^{(q)}(b-x)W^{(q+\lambda)}(x)\mathrm{d}x\right),
\end{aligned}
\tag{4.61}
$$

where the last step is due to (4.4). Taking the limit $b\to\infty$ in (4.61) and using (2.9) and (2.5), Eq. (4.60) becomes

$$
\mathbb{E}_{z}\left[e^{-q\tau_{0}^{-}}W^{(q+\lambda)}\left(X_{\tau_{0}^{-}}-y\right)1_{\{\tau_{0}^{-}<\infty\}}\right]=A^{(q,\lambda)}\left(z,-y\right)-W^{(q)}\left(z\right)Z^{(q+\lambda)}\left(-y,\Phi_{q}\right),\tag{4.62}
$$

for any $z>0$ and $y<0$. Substituting (4.62) into (4.58) and (4.59) yields

$$
\begin{aligned}
&\frac{1}{\mathrm{d}y}R_{-}^{(q,\lambda)}(0,\mathrm{d}y;-a)+W^{(q+\lambda)}(-y)\\
&=\frac{e^{-\Phi_{q+\lambda}y}W^{(q+\lambda)}(a)-\lambda W^{(q+\lambda)}(a)\int_{0}^{\infty}e^{-\Phi_{q+\lambda}z}\left[A^{(q,\lambda)}\left(z,-y\right)-W^{(q)}\left(z\right)Z^{(q+\lambda)}\left(-y,\Phi_{q}\right)\right]\mathrm{d}z}{e^{\Phi_{q+\lambda}a}-\lambda\int_{0}^{\infty}e^{-\Phi_{q+\lambda}z}\left[A^{(q,\lambda)}\left(z,a\right)-W^{(q)}\left(z\right)Z^{(q+\lambda)}\left(a,\Phi_{q}\right)\right]}
\end{aligned}
\tag{4.63}
$$

for $-a\le y<0$, and

$$
\begin{aligned}
&\frac{1}{\mathrm{d}y}R_{-}^{(q,\lambda)}(0,\mathrm{d}y;-a)+W^{(q+\lambda)}(-y)\\
&=\frac{e^{-\Phi_{q+\lambda}y}W^{(q+\lambda)}(a)+\lambda W^{(q+\lambda)}(a)\int_{0}^{\infty}e^{-\Phi_{q+\lambda}z}r_{-}^{(q)}(z,y)\mathrm{d}z}{e^{\Phi_{q+\lambda}a}-\lambda\int_{0}^{\infty}e^{-\Phi_{q+\lambda}z}\left[A^{(q,\lambda)}\left(z,a\right)-W^{(q)}\left(z\right)Z^{(q+\lambda)}\left(a,\Phi_{q}\right)\right]}
\end{aligned}
\tag{4.64}
$$

for $y\ge0$. Note that by (4.10), (2.3), (2.6) and (4.8), we have

$$
\int_{0}^{\infty}e^{-\Phi_{q+\lambda}z}A^{(q,\lambda)}\left(z,-y\right)\mathrm{d}z=\frac{e^{-\Phi_{q+\lambda}y}}{\lambda},\quad y<0,
$$

and

$$\int_0^\infty e^{-\Phi_{q+\lambda}z} r_-^{(q)}(z,y)\,\mathrm{d}z = \frac{e^{-\Phi_q y} - e^{-\Phi_{q+\lambda}y}}{\lambda}, \quad y \geq 0.$$

Thus, (4.63) and (4.64) is further reduced to

$$R_-^{(q,\lambda)}(0,\mathrm{d}y;-a)/\mathrm{d}y = \frac{W^{(q+\lambda)}(a)}{Z^{(q+\lambda)}(a,\Phi_q)} Z^{(q+\lambda)}(-y,\Phi_q) - W^{(q+\lambda)}(-y). \tag{4.65}$$

Finally, substituting (4.65) into (4.56) and using (4.8), (4.9) and (4.62) yields (4.23).

Proof of (4.24) and (4.25)

By (2.5) and (4.11), we have

$$\begin{aligned}\frac{A^{(q,\lambda)}(x,a)}{Z^{(q+\lambda)}(a,\Phi_q)} &= \frac{W^{(q+\lambda)}(x+a) - \lambda\int_0^x W^{(q)}(z)\,W^{(q+\lambda)}(x+a-z)\,\mathrm{d}z}{Z^{(q+\lambda)}(a,\Phi_q)} \\ &= \frac{\frac{W^{(q+\lambda)}(x+a)}{W^{(q+\lambda)}(a)} - \lambda\int_0^x W^{(q)}(z)\,\frac{W^{(q+\lambda)}(x+a-z)}{W^{(q+\lambda)}(a)}\mathrm{d}z}{\frac{Z^{(q+\lambda)}(a,\Phi_q)}{W^{(q+\lambda)}(a)}}.\end{aligned}$$

It follows from (2.9), (2.7) and (4.16) that

$$\lim_{a\to\infty} \frac{A^{(q,\lambda)}(x,a)}{Z^{(q+\lambda)}(a,\Phi_q)} = \frac{e^{\Phi_{q+\lambda}x} - \lambda\int_0^x W^{(q)}(z)\,e^{\Phi_{q+\lambda}(x-z)}\mathrm{d}z}{\frac{\lambda}{\Phi_{q+\lambda}-\Phi_q}} = W^{(q,\lambda)}(x).$$

Therefore, it is straightforward to see from (4.22) and (4.23), that

$$\begin{aligned} r_+^{(q,\lambda)}(x,y) &= \lim_{a\to\infty} r_+^{(q,\lambda)}(x,y;a) \\ &= \lim_{a\to\infty} \frac{A^{(q,\lambda)}(-y,a)}{Z^{(q+\lambda)}(a,\Phi_q)} Z^{(q+\lambda)}(x,\Phi_q) - A^{(q,\lambda)}(-y,x) \\ &= W^{(q,\lambda)}(-y)\,Z^{(q+\lambda)}(x,\Phi_q) - A^{(q,\lambda)}(-y,x), \end{aligned}$$

and

$$\begin{aligned} r_-^{(q,\lambda)}(x,y) &= \lim_{a\to\infty} r_-^{(q,\lambda)}(x,y;-a) \\ &= \lim_{a\to\infty} \frac{A^{(q,\lambda)}(x,a)}{Z^{(q+\lambda)}(a,\Phi_q)} Z^{(q+\lambda)}(-y,\Phi_q) - A^{(q,\lambda)}(x,-y) \\ &= W^{(q,\lambda)}(x)\,Z^{(q+\lambda)}(-y,\Phi_q) - A^{(q,\lambda)}(x,-y). \end{aligned}$$

Proof of (4.26)

For $x, y \leq a$, due to the fact that $\{t < \tau_a^+ \wedge T_0^{-,\lambda}\} = \{t < T_0^{-,\lambda}\} \backslash \{\tau_a^+ \leq t < T_0^{-,\lambda}\}$, it is immediate from (4.18) that

$$u_{d:c}^{(q,\lambda)}(x, y; a) = r_-^{(q,\lambda)}(x, y) - \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} r_-^{(q,\lambda)}(a, y). \tag{4.66}$$

Substituting (4.25) into (4.66) yields (4.26).

Proof of (4.27)

For $x \in [0, a]$ and $y \geq 0$, let

$$U_{c:d}^{(q,\lambda)}(x, \mathrm{d}y; a) = \int_0^\infty e^{-qt} \mathbb{P}_x \left(X_t \in \mathrm{d}y, t < \tau_0^- \wedge T_a^{+,\lambda} \right) \mathrm{d}t = \frac{1}{q} \mathbb{P}_x \left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^- \wedge T_a^{+,\lambda} \right).$$

Conditioning on whether or not τ_a^+ occurs prior to e_q and using (4.1) lead to

$$\begin{aligned}
&U_{c:d}^{(q,\lambda)}(x, \mathrm{d}y; a) \\
&= \frac{1}{q} \left\{ \mathbb{P}_x \left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^- \wedge \tau_a^+ \right) + \mathbb{P}_x \left(\tau_a^+ < e_q \wedge \tau_0^- \right) \mathbb{P}_a \left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^- \wedge T_a^{+,\lambda} \right) \right\} \\
&= u^{(q)}(x, y; a) \, \mathrm{d}y + \frac{W^{(q)}(x)}{W^{(q)}(a)} U_{c:d}^{(q,\lambda)}(a, \mathrm{d}y; a),
\end{aligned} \tag{4.67}$$

where we have extended the definition of $u^{(q)}$ to $u^{(q)}(x, y; a) = 0$ for $x \in [0, a]$ and $y > a$.

To solve for $U_{c:d}^{(q,\lambda)}(a, \mathrm{d}y; a)$, we condition on whether e_q occurs prior to the next observation time e_λ and arrive at

$$\begin{aligned}
U_{c:d}^{(q,\lambda)}(a, \mathrm{d}y; a) &= \frac{1}{q} \mathbb{P}_a \left(X_{e_q} \in \mathrm{d}y, e_q < \tau_0^- \wedge e_\lambda \right) + \int_0^a \mathbb{P}_a \left(X_{e_\lambda} \in \mathrm{d}x, e_\lambda < \tau_0^- \wedge e_q \right) U_{c:d}^{(q,\lambda)}(x, \mathrm{d}y; a) \\
&= r_-^{(q+\lambda)}(a, y) \, \mathrm{d}y + \lambda \int_0^a r_-^{(q+\lambda)}(a, x) \, U_{c:d}^{(q,\lambda)}(x, \mathrm{d}y; a) \, \mathrm{d}x.
\end{aligned} \tag{4.68}$$

Substituting (4.67) into (4.68) gives

$$U_{c:d}^{(q,\lambda)}(a, \mathrm{d}y; a) = \frac{\lambda \int_0^a r_-^{(q+\lambda)}(a, x) \, u^{(q)}(x, y; a) \, \mathrm{d}x + r_-^{(q+\lambda)}(a, y)}{1 - \frac{\lambda}{W^{(q)}(a)} \int_0^a r_-^{(q+\lambda)}(a, x) \, W^{(q)}(x) \, \mathrm{d}x} \mathrm{d}y. \tag{4.69}$$

Next we simplify (4.69) by evaluating the two integral terms therein. By using (4.8), (2.5) and (4.4), we have

$$\begin{aligned}
&\int_0^a r_-^{(q+\lambda)}(a,x)\,W^{(q)}(x-y)\,\mathrm{d}x \\
&= W^{(q+\lambda)}(a)\int_0^a e^{-\Phi_{q+\lambda}x}W^{(q)}(x-y)\,\mathrm{d}x - \int_0^a W^{(q+\lambda)}(a-x)\,W^{(q)}(x-y)\,\mathrm{d}x \\
&= W^{(q+\lambda)}(a)\int_0^{a-y} e^{-\Phi_{q+\lambda}(z+y)}W^{(q)}(z)\,\mathrm{d}z - \int_0^{a-y} W^{(q+\lambda)}(a-y-z)\,W^{(q)}(z)\,\mathrm{d}x \\
&= \frac{1}{\lambda}W^{(q+\lambda)}(a)\,e^{-\Phi_{q+\lambda}y}\left[1-e^{-\Phi_{q+\lambda}(a-y)}Z^{(q)}(a-y,\Phi_{q+\lambda})\right] \\
&\quad -\frac{1}{\lambda}\left[W^{(q+\lambda)}(a-y)-W^{(q)}(a-y)\right].
\end{aligned} \tag{4.70}$$

As for the other integral, using (4.9), (4.70) and (4.8) followed by simple algebraic manipulations, one finds that

$$\begin{aligned}
&\lambda\int_0^a r_-^{(q+\lambda)}(a,x)\,u^{(q)}(x,y;a)\,\mathrm{d}x \\
&= \lambda\int_0^a r_-^{(q+\lambda)}(a,x)\left[\frac{W^{(q)}(x)\,W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y)\right]\mathrm{d}x \\
&= \frac{W^{(q)}(a-y)}{W^{(q)}(a)}\left\{W^{(q+\lambda)}(a)\left(1-e^{-\Phi_{q+\lambda}a}Z^{(q)}(a,\Phi_{q+\lambda})\right)-\left(W^{(q+\lambda)}(a)-W^{(q)}(a)\right)\right\} \\
&\quad -\left\{W^{(q+\lambda)}(a)\,e^{-\Phi_{q+\lambda}y}\left[1-e^{-\Phi_{q+\lambda}(a-y)}Z^{(q)}(a-y,\Phi_{q+\lambda})\right]-\left[W^{(q+\lambda)}(a-y)-W^{(q)}(a-y)\right]\right\} \\
&= e^{-\Phi_{q+\lambda}a}W^{(q+\lambda)}(a)\left[Z^{(q)}(a-y,\Phi_{q+\lambda})-\frac{Z^{(q)}(a,\Phi_{q+\lambda})}{W^{(q)}(a)}W^{(q)}(a-y)\right]-r_-^{(q+\lambda)}(a,y).
\end{aligned} \tag{4.71}$$

With the aid of (4.16), substituting (4.70) with $y=0$ and (4.71) into (4.69) yields

$$U_{c:d}^{(q,\lambda)}(a,\mathrm{d}y;a) = \left\{\frac{W^{(q,\lambda)}(a-y)}{W^{(q,\lambda)}(a)}W^{(q)}(a)-W^{(q)}(a-y)\right\}\mathrm{d}y. \tag{4.72}$$

Finally, with the help of (4.9), (4.27) follows by substituting (4.72) into (4.67).

Proof of (4.28)

For $x\le a$ and $y\in\mathbb{R}$, let

$$U_{d:d}^{(q,\lambda)}(x,\mathrm{d}y;a) = \int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t\in\mathrm{d}y, t<T_0^{-,\lambda}\wedge T_a^{+,\lambda}\right) = \frac{1}{q}\mathbb{P}_x\left(X_{e_q}\in\mathrm{d}y, e_q<T_0^{-,\lambda}\wedge T_a^{+,\lambda}\right).$$

Conditioning on whether τ_a^+ occurs before e_q leads to

$$
\begin{aligned}
U_{d:d}^{(q,\lambda)}(x, \mathrm{d}y; a) &= \frac{1}{q}\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge \tau_a^+\right) \\
&\quad + \frac{1}{q}\mathbb{P}_x\left(\tau_a^+ < e_q \wedge T_0^{-,\lambda}\right)\mathbb{P}_a\left(X_{e_q} \in \mathrm{d}y, e_q < T_0^{-,\lambda} \wedge T_a^{+,\lambda}\right) \\
&= u_{d:c}^{(q,\lambda)}(x, y; a)\,\mathrm{d}y + \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} U_{d:d}^{(q,\lambda)}(a, \mathrm{d}y; a), \qquad (4.73)
\end{aligned}
$$

where the last step is due to (4.18) and the definition of $u_{d:c}^{(q,\lambda)}$ was extended to $u_{d:c}^{(q,\lambda)}(x, y; a) = 0$ for $y > a$ and $x \leq a$.

To solve for $U_{d:d}^{(q,\lambda)}(a, \mathrm{d}y; a)$, we consider whether e_q occurs before the next observation time e_λ and obtain

$$
\begin{aligned}
U_{d:d}^{(q,\lambda)}(a, \mathrm{d}y; a) &= \frac{1}{q}\mathbb{P}_a\left(X_{e_q} \in \mathrm{d}y, e_q < e_\lambda\right) + \frac{1}{q}\int_0^a \mathbb{P}_a\left(X_{e_\lambda} \in \mathrm{d}x, e_\lambda < e_q\right) q U_{d:d}^{(q,\lambda)}(x, \mathrm{d}y; a) \\
&= \theta^{(q+\lambda)}(y-a)\,\mathrm{d}y + \lambda \int_0^a \theta^{(q+\lambda)}(x-a)\, U_{d:d}^{(q,\lambda)}(x, \mathrm{d}y; a)\,\mathrm{d}x. \qquad (4.74)
\end{aligned}
$$

Substituting (4.73) into (4.74) and using (4.26) give

$$
\begin{aligned}
&U_{d:d}^{(q,\lambda)}(a, \mathrm{d}y; a) \\
&= \frac{\theta^{(q+\lambda)}(y-a) + \lambda \int_0^a \theta^{(q+\lambda)}(x-a)\, u_{d:c}^{(q,\lambda)}(x, y; a)\,\mathrm{d}x}{1 - \frac{\lambda}{W^{(q,\lambda)}(a)}\int_0^a \theta^{(q+\lambda)}(x-a)\, W^{(q,\lambda)}(x)\,\mathrm{d}x}\mathrm{d}y \\
&= \frac{\theta^{(q+\lambda)}(y-a) + A^{(q,\lambda)}(a, -y) - \lambda \int_0^a \theta^{(q+\lambda)}(x-a)\, A^{(q,\lambda)}(x, -y)\,\mathrm{d}x}{1 - \frac{\lambda}{W^{(q,\lambda)}(a)}\int_0^a \theta^{(q+\lambda)}(x-a)\, W^{(q,\lambda)}(x)\,\mathrm{d}x} - A^{(q,\lambda)}(a, -y).
\end{aligned}
\qquad (4.75)
$$

Next, we simplify the expression of $U_{d:d}^{(q,\lambda)}(a, \mathrm{d}y; a)$ in (4.75). Using (4.10), one obtains

$$
\begin{aligned}
\int_0^a W^{(q+\lambda)}(a-x)\, W^{(q)}(x-y)\,\mathrm{d}x &= \int_0^a W^{(q)}(-y+a-x)\, W^{(q+\lambda)}(x)\,\mathrm{d}x \\
&= \frac{A^{(q,\lambda)}(-y, a) - W^{(q)}(a-y)}{\lambda},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lambda \int_0^a W^{(q+\lambda)}(a-x) A^{(q,\lambda)}(x,-y)\,\mathrm{d}x \\
&= A^{(q,\lambda)}(-y,a) - W^{(q)}(a-y) + \lambda^2 \int_0^a \int_0^{-y} W^{(q)}(x-y-z) W^{(q+\lambda)}(z) W^{(q+\lambda)}(a-x)\,\mathrm{d}z\mathrm{d}x \\
&= \lambda \int_0^{-y} W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z,a)\,\mathrm{d}z + A^{(q,\lambda)}(-y,a) - A^{(q,\lambda)}(a,-y). \qquad (4.76)
\end{aligned}
$$

By (4.5), it can be seen that

$$
\int_0^a W^{(q+\lambda)}(a-x) W^{(q,\lambda)}(x)\,\mathrm{d}x = \frac{\Phi_{q+\lambda}-\Phi_q}{\lambda^2} e^{\Phi_{q+\lambda}a} - \frac{W^{(q,\lambda)}(a)}{\lambda}. \qquad (4.77)
$$

Invoking (4.6), (4.76), (4.77) and also (4.11) for the term $A^{(q,\lambda)}(-y,a)$, one can rewrite (4.75) as

$$
\begin{aligned}
&u_{d:d}^{(q,\lambda)}(a,y;a) \\
&= \frac{\Phi'_{q+\lambda}\left[e^{-\Phi_{q+\lambda}y} - \lambda \int_0^a e^{-\Phi_{q+\lambda}x} A^{(q,\lambda)}(x,-y)\,\mathrm{d}x\right]}{\frac{\Phi_{q+\lambda}-\Phi_q}{\lambda} - \lambda \Phi'_{q+\lambda} \int_0^a e^{-\Phi_{q+\lambda}x} W^{(q,\lambda)}(x)\,\mathrm{d}x} W^{(q,\lambda)}(a) - A^{(q,\lambda)}(a,-y) \\
&\quad + \frac{\lambda e^{-\Phi_{q+\lambda}a} \int_0^{-y}\left[W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z,a) - W^{(q+\lambda)}(a-y-z) W^{(q)}(z)\right]\mathrm{d}z}{\frac{\Phi_{q+\lambda}-\Phi_q}{\lambda} - \lambda \Phi'_{q+\lambda} \int_0^a e^{-\Phi_{q+\lambda}x} W^{(q,\lambda)}(x)\,\mathrm{d}x} W^{(q,\lambda)}(a).
\end{aligned}
\qquad (4.78)
$$

Furthermore, by (4.10), (2.6), (4.5), (4.16) and ((2.5)), it can be shown that

$$
\int_0^\infty e^{-\Phi_{q+\lambda}x} A^{(q,\lambda)}(x,-y)\,\mathrm{d}x = \frac{e^{-\Phi_{q+\lambda}y}}{\lambda},
$$

and

$$
\int_0^\infty e^{-\Phi_{q+\lambda}x} W^{(q,\lambda)}(x)\,\mathrm{d}x = \frac{\Phi_{q+\lambda}-\Phi_q}{\lambda^2 \Phi'_{q+\lambda}}.
$$

Using the above two relations, (4.78) can be rewritten as

$$
\begin{aligned}
u_{d:d}^{(q,\lambda)}(a,y;a) &= \frac{\int_a^\infty e^{-\Phi_{q+\lambda}x} A^{(q,\lambda)}(x,-y)\,\mathrm{d}x}{\int_a^\infty e^{-\Phi_{q+\lambda}x} W^{(q,\lambda)}(x)\,\mathrm{d}x} W^{(q,\lambda)}(a) - A^{(q,\lambda)}(a,-y) \\
&\quad + \frac{\int_0^{-y}\left[W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z,a) - W^{(q+\lambda)}(a-y-z) W^{(q)}(z)\right]\mathrm{d}z}{\Phi'_{q+\lambda} \int_a^\infty e^{\Phi_{q+\lambda}(a-x)} W^{(q,\lambda)}(x)\,\mathrm{d}x} W^{(q,\lambda)}(a).
\end{aligned}
\qquad (4.79)
$$

Substituting (4.79) into (4.73) leads to

$$u_{d:d}^{(q,\lambda)}(x,y;a)=\frac{\int_a^\infty e^{-\Phi_{q+\lambda}z}A^{(q,\lambda)}(z,-y)\,\mathrm{d}z}{\int_a^\infty e^{-\Phi_{q+\lambda}z}W^{(q,\lambda)}(z)\,\mathrm{d}z}W^{(q,\lambda)}(x)-A^{(q,\lambda)}(x,-y)$$
$$+\frac{\int_0^{-y}\left[W^{(q+\lambda)}(-y-z)A^{(q,\lambda)}(z,a)-W^{(q+\lambda)}(a-y-z)W^{(q)}(z)\right]\mathrm{d}z}{\Phi'_{q+\lambda}\int_a^\infty e^{\Phi_{q+\lambda}(a-z)}W^{(q,\lambda)}(z)\,\mathrm{d}z}W^{(q,\lambda)}(x).$$

In light of (4.28), it remains to show that, for any $y\in\mathbb{R}$ and $a>0$,

$$\int_0^{-y}W^{(q+\lambda)}(-y-z)A^{(q,\lambda)}(z,a)\,\mathrm{d}z=\int_0^{-y}W^{(q+\lambda)}(a-y-z)W^{(q)}(z)\,\mathrm{d}z. \tag{4.80}$$

It suffices to prove (4.80) for the case when $y<0$ because (4.80) clearly holds for $y\geq 0$. For large enough $s>0$, it follows that from (4.10), (2.3) and (2.6) that

$$\begin{aligned}
&\int_0^\infty e^{-sx}\int_0^x W^{(q+\lambda)}(x-z)A^{(q,\lambda)}(z,a)\,\mathrm{d}z\mathrm{d}x\\
&=\int_0^\infty e^{-sx}W^{(q+\lambda)}(x)\,\mathrm{d}x\int_0^\infty e^{-sz}A^{(q,\lambda)}(z,a)\,\mathrm{d}z\\
&=\frac{Z^{(q+\lambda)}(a,s)}{\psi_{q+\lambda}(s)}\frac{1}{\psi_q(s)}\\
&=\int_0^\infty e^{-sx}W^{(q+\lambda)}(a+x)\,\mathrm{d}x\cdot\int_0^\infty e^{-sz}W^{(q)}(z)\mathrm{d}z\\
&=\int_0^\infty e^{-sx}\int_0^x W^{(q+\lambda)}(a+x-z)W^{(q)}(z)\,\mathrm{d}z\mathrm{d}x.
\end{aligned}\tag{4.81}$$

Taking Laplace inversion to (4.81) yields, for $x\geq 0$,

$$\int_0^x W^{(q+\lambda)}(x-z)A^{(q,\lambda)}(z,a)\,\mathrm{d}z=\int_0^x W^{(q+\lambda)}(a+x-z)W^{(q)}(z)\,\mathrm{d}z. \tag{4.82}$$

This completes the proof of (4.80) by letting $x=-y>0$ in (4.82).

4.7.2 Proof of Proposition 21

Relations (4.29) and (4.32) are immediate from (4.17). In addition, relations (4.30) and (4.31) are direct consequences of (4.12), (4.17), and the fact that $Z^{(q)}(x,\theta)=e^{\theta x}$ for $x\leq 0$. We are only left to prove (4.33).

For $x, y \in [0, a]$, by (4.28) and (4.10),

$$u_{d:d}^{(q,\lambda)}(x, y; a) = \frac{\int_a^\infty e^{-\Phi_{q+\lambda}z} W^{(q)}(z-y)\,\mathrm{d}z}{\int_a^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z)\,\mathrm{d}z} W^{(q,\lambda)}(x) - W^{(q)}(x-y).$$

Note that by (4.14), it follows that

$$\frac{\int_a^\infty e^{-\Phi_{q+\lambda}z} W^{(q)}(z-y)\,\mathrm{d}z}{\int_a^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z)\,\mathrm{d}z} = \frac{W^{(q,\lambda)}(a-y)}{\left(\Phi_{q+\lambda} - \Phi_q\right) \int_0^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z}. \tag{4.83}$$

From (4.17) and (4.83), it remains to show that

$$\lim_{\lambda\to\infty} \left(\Phi_{q+\lambda} - \Phi_q\right) \int_0^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z = W^{(q)}(a). \tag{4.84}$$

For any fixed $\varepsilon > 0$, by (4.14), we have

$$\begin{aligned}
&\left(\Phi_{q+\lambda} - \Phi_q\right) \int_\varepsilon^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z \\
&= \left(\Phi_{q+\lambda} - \Phi_q\right)^2 \int_\varepsilon^\infty \int_0^\infty e^{-\Phi_{q+\lambda}(z+y)} W^{(q)}(z+y+a)\,\mathrm{d}y\mathrm{d}z \\
&= \left(\Phi_{q+\lambda} - \Phi_q\right)^2 \int_\varepsilon^\infty \int_z^\infty e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a)\,\mathrm{d}x\mathrm{d}z \\
&= \left(\Phi_{q+\lambda} - \Phi_q\right)^2 \int_\varepsilon^\infty (x-\varepsilon) e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a)\mathrm{d}x.
\end{aligned} \tag{4.85}$$

Observe that for any fixed $x \geq \varepsilon$, the function $\beta \mapsto \beta^2 e^{-\beta x}$ is monotone decreasing in β for any $\beta \geq \frac{2}{\varepsilon}$. By (2.2), we deduce that for any $x \geq \varepsilon$, the function $\lambda \mapsto \Phi_{q+\lambda}^2 e^{-\Phi_{q+\lambda}x}$ is monotone decreasing in λ for any $\lambda \geq \psi(\frac{2}{\varepsilon}) - q$. By the monotone convergence theorem and (4.85), we deduce that

$$\begin{aligned}
0 &\leq \limsup_{\lambda\to\infty} \left(\Phi_{q+\lambda} - \Phi_q\right) \int_\varepsilon^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z \\
&= \limsup_{\lambda\to\infty} \left(\Phi_{q+\lambda} - \Phi_q\right)^2 \int_\varepsilon^\infty (x-\varepsilon) e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a)\mathrm{d}x \\
&\leq \int_\varepsilon^\infty (x-\varepsilon) W^{(q)}(x+a) \limsup_{\lambda\to\infty} \Phi_{q+\lambda}^2 e^{-\Phi_{q+\lambda}x}\mathrm{d}x \\
&= 0.
\end{aligned} \tag{4.86}$$

Combining the fact that $\left(\Phi_{q+\lambda} - \Phi_q\right) \int_\varepsilon^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z$ is nonnegative with (4.86), one arrives at

$$\lim_{\lambda\to\infty} \left(\Phi_{q+\lambda} - \Phi_q\right) \int_\varepsilon^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a)\,\mathrm{d}z = 0.$$

On the other hand, thanks to the monotonicity of $W^{(q,\lambda)}$, we have

$$(\Phi_{q+\lambda}-\Phi_q)\int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z}W^{(q,\lambda)}(z+a)\mathrm{d}z \geq \frac{(\Phi_{q+\lambda}-\Phi_q)(1-e^{-\Phi_{q+\lambda}\varepsilon})}{\Phi_{q+\lambda}}W^{(q,\lambda)}(a),$$
$$(\Phi_{q+\lambda}-\Phi_q)\int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z}W^{(q,\lambda)}(z+a)\mathrm{d}z \leq \frac{(\Phi_{q+\lambda}-\Phi_q)(1-e^{-\Phi_{q+\lambda}\varepsilon})}{\Phi_{q+\lambda}}W^{(q,\lambda)}(a+\varepsilon).$$

It follows that

$$\liminf_{\lambda\to\infty}(\Phi_{q+\lambda}-\Phi_q)\int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z}W^{(q,\lambda)}(z+a)\mathrm{d}z \geq W^{(q)}(a), \tag{4.87}$$
$$\limsup_{\lambda\to\infty}(\Phi_{q+\lambda}-\Phi_q)\int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z}W^{(q,\lambda)}(z+a)\mathrm{d}z \leq W^{(q)}(a+\varepsilon). \tag{4.88}$$

From the arbitrariness of ε, we conclude from (4.86)–(4.88) that (4.84) holds.

4.7.3 Proof of Corollary 23

For $x\in[-a,b]$ and $y\,[-a,0]$, we have

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}}1_{\{X_{T_0^{-,\lambda}}\in \mathrm{d}y, T_0^{-,\lambda}<\tau_{-a}^-\wedge\tau_b^+\}}\right]\\
&=\mathbb{E}_x\left[e^{-qT_0^{-,\lambda}}1_{\{X_{T_0^{-,\lambda}}\in \mathrm{d}y, T_0^{-,\lambda}<\tau_{-a}^-\}}\right]-\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]\mathbb{E}_b\left[e^{-qT_0^{-,\lambda}}1_{\{X_{T_0^{-,\lambda}}\in \mathrm{d}y, T_0^{-,\lambda}<\tau_{-a}^-\}}\right]\\
&=\lambda r_-^{(q,\lambda)}(x,y;-a)\,\mathrm{d}y-\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]\lambda r_-^{(q,\lambda)}(b,y;-a)\,\mathrm{d}y.
\end{aligned}\tag{4.89}$$

In what follows, we focus on characterizing $\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]$.

Conditioning on whether τ_b^+ or τ_0^- occurs first, by (4.1) and (4.61), it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]\\
&=\int_{-a}^0\mathbb{E}_x\left[e^{-q\tau_0^-}1_{\left\{X_{\tau_0^-}\in\mathrm{d}z,\tau_0^-<\tau_b^+\right\}}\right]\mathbb{E}_z\left[e^{-q\tau_0^+}1_{\left\{\tau_0^+<e_\lambda\wedge\tau_{-a}^-\right\}}\right]\mathbb{E}\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]\\
&\quad+\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<\tau_0^-\right\}}\right]\\
&=\frac{W^{(q)}(x)}{W^{(q)}(b)}+\frac{1}{W^{(q+\lambda)}(a)}\mathbb{E}_x\left[e^{-q\tau_0^-}W^{(q+\lambda)}\left(a+X_{\tau_0^-}\right)1_{\left\{\tau_0^-<\tau_b^+\right\}}\right]\mathbb{E}\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right]\\
&=\frac{W^{(q)}(x)}{W^{(q)}(b)}+\left(\frac{A^{(q,\lambda)}(x,a)}{W^{(q+\lambda)}(a)}-\frac{W^{(q)}(x)\,A^{(q,\lambda)}(b,a)}{W^{(q)}(b)\,W^{(q+\lambda)}(a)}\right)\mathbb{E}\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<T_0^{-,\lambda}\wedge\tau_{-a}^-\right\}}\right].
\end{aligned}\tag{4.90}$$

Note that (4.90) holds for $x \in [-a, b]$. To evaluate $\mathbb{E}\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right]$, we condition on whether e_λ or τ_b^+ occurs first and obtain

$$
\begin{aligned}
&\mathbb{E}\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] \\
&= \mathbb{E}\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < e'_\lambda \wedge \tau_{-a}^-\right\}}\right] + \int_0^b \mathbb{E}\left[e^{-qe_\lambda} 1_{\left\{X_{e_\lambda} \in \mathrm{d}z, e_\lambda < \tau_{-a}^- \wedge \tau_b^+\right\}}\right] \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] \\
&= \frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \lambda \int_0^b u^{(q+\lambda)}(a, z+a; b+a)\, \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] \mathrm{d}z. \qquad (4.91)
\end{aligned}
$$

Substituting (4.90) with $x = z$ into (4.91) and using (4.9), we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] \\
&= \frac{\frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \lambda \int_0^b u^{(q+\lambda)}(a, z+a; a+b) \frac{W^{(q)}(z)}{W^{(q)}(b)} \mathrm{d}z}{1 - \lambda \int_0^b u^{(q+\lambda)}(a, z+a; a+b) \left[\frac{A^{(q,\lambda)}(z,a)}{W^{(q+\lambda)}(a)} - \frac{W^{(q)}(z) A^{(q,\lambda)}(b,a)}{W^{(q)}(b) W^{(q+\lambda)}(a)}\right] \mathrm{d}z} \\
&= \frac{\frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \frac{\lambda W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b) W^{(q)}(b)} \int_0^b W^{(q+\lambda)}(b-z) W^{(q)}(z) \mathrm{d}z}{1 - \frac{\lambda}{W^{(q+\lambda)}(a+b)} \int_0^b W^{(q+\lambda)}(b-z) \left(A^{(q,\lambda)}(z,a) - \frac{W^{(q)}(z) A^{(q,\lambda)}(b,a)}{W^{(q)}(b)}\right) \mathrm{d}z}. \qquad (4.92)
\end{aligned}
$$

From (4.82) and (4.4), one easily finds that

$$
\int_0^b W^{(q+\lambda)}(b-z) A^{(q,\lambda)}(z,a)\, \mathrm{d}z = \int_0^b W^{(q+\lambda)}(a+b-z) W^{(q)}(z)\, \mathrm{d}z,
$$

and

$$
\int_0^b W^{(q+\lambda)}(b-z) W^{(q)}(z)\, \mathrm{d}z = \frac{W^{(q+\lambda)}(b) - W^{(q)}(b)}{\lambda}.
$$

Further substituting the above two equalities into (4.92), and using (4.11) lead to

$$
\mathbb{E}\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] = \frac{W^{(q+\lambda)}(a)}{A^{(q,\lambda)}(b,a)}.
$$

Hence, (4.90) reduces to

$$
\mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\right\}}\right] = \frac{A^{(q,\lambda)}(x,a)}{A^{(q,\lambda)}(b,a)}. \qquad (4.93)
$$

Lastly, by substituting (4.93) into (4.89) and using (4.23), the proof is complete.

4.7.4 Proof of Theorem 24

To begin with, due to the positive loading condition, for $x < a$,

$$O_x^{(\lambda,q)}(a,b) = \mathbb{E}_x\left[1_{\left\{\tau_a^+ < \infty\right\}}\right] O_a^{(\lambda,q)}(a,b) = O_a^{(\lambda,q)}(a,b). \tag{4.94}$$

Using the standard conditioning argument, together with the strong Markov property and (4.94), we have, for $x \geq b$,

$$\begin{aligned}
&O_x^{(\lambda,q)}(a,b) \\
&= \mathbb{E}_x\left[\int_a^b 1_{\left\{X_{T_b^{\lambda,-}} \in \mathrm{d}z, T_b^{\lambda,-} < \infty\right\}} \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < \tau_a^-\right\}} O_b^{(\lambda,q)}(a,b) + e^{-q\tau_a^-} 1_{\left\{\tau_a^- < \tau_b^+\right\}} O_a^{(\lambda,q)}(a,b)\right]\right] \\
&\quad + \mathbb{E}_x\left[\int_{-\infty}^a 1_{\left\{X_{T_b^{\lambda,-}} \in \mathrm{d}z, T_b^{\lambda,-} < \infty\right\}} O_z^{(\lambda,q)}(a,b)\right] + \mathbb{P}_x\left(T_b^{\lambda,-} = \infty\right) \\
&= O_b^{(\lambda,q)}(a,b) \int_a^b \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < \tau_a^-\right\}}\right] f_\lambda^-(\mathrm{d}z; x, b) + \mathbb{P}_x\left(T_b^{\lambda,-} = \infty\right) \\
&\quad + O_a^{(\lambda,q)}(a,b)\left\{\int_a^b \mathbb{E}_z\left[e^{-q\tau_a^-} 1_{\left\{\tau_a^- < \tau_b^+\right\}}\right] f_\lambda^-(\mathrm{d}z; x, b) + \int_{-\infty}^a f_\lambda^-(\mathrm{d}z; x, b)\right\} \\
&:= O_b^{(\lambda,q)}(a,b)\, \beta_{x,b}^{(\lambda,q)}(a,b) + O_a^{(\lambda,q)}(a,b)\, \beta_{x,a}^{(\lambda,q)}(a,b) + \mathbb{P}_x\left(T_b^{\lambda,-} = \infty\right)
\end{aligned} \tag{4.95}$$

Similarly, for $a \leq x < b$,

$$\begin{aligned}
&O_x^{(\lambda,q)}(a,b) \\
&= \mathbb{E}_x\left[\int_a^b 1_{\left\{X_{T_a^{\lambda,+}} \in \mathrm{d}z, T_a^{\lambda,+} < \infty\right\}} \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < \tau_a^-\right\}} O_b^{(\lambda,q)}(a,b) + e^{-q\tau_a^-} 1_{\left\{\tau_a^- < \tau_b^+\right\}} O_a^{(\lambda,q)}\right]\right] \\
&\quad + \mathbb{E}_x\left[\int_b^\infty 1_{\left\{X_{T_a^{\lambda,+}} \in \mathrm{d}z, T_a^{\lambda,+} < \infty\right\}} O_z^{(\lambda,q)}(a,b)\right] + \mathbb{P}_x\left(T_a^{\lambda,+} = \infty\right) \\
&= O_b^{(\lambda,q)}(a,b) \int_a^b \mathbb{E}_z\left[e^{-q\tau_b^+} 1_{\left\{\tau_b^+ < \tau_a^-\right\}}\right] f_\lambda^+(\mathrm{d}z; x, a) \\
&\quad + O_a^{(\lambda,q)}(a,b) \int_a^b \mathbb{E}_z\left[e^{-q\tau_a^-} 1_{\left\{\tau_a^- < \tau_b^+\right\}}\right] f_\lambda^+(\mathrm{d}z; x, a) + \int_b^\infty O_z^{(\lambda,q)}(a,b)\, f_\lambda^+(\mathrm{d}z; x, a)
\end{aligned}$$

by virtue of positive loading condition. The integral in the last line can further be simplified by back substituting (4.95), leading to

$$\begin{aligned}
&\int_b^\infty O_z^{(\lambda,q)}(a,b)\, f_\lambda^+(\mathrm{d}z;x,a)\\
&= O_b^{(\lambda,q)}(a,b)\int_b^\infty \int_a^b \mathbb{E}_y\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<\tau_a^-\right\}}\right] f_\lambda^-(\mathrm{d}y;z,b)\, f_\lambda^+(\mathrm{d}z;x,a)\\
&\quad + O_a^{(\lambda,q)}(a,b)\int_b^\infty \left\{\int_a^b \mathbb{E}_y\left[e^{-q\tau_a^-}1_{\left\{\tau_a^-<\tau_b^+\right\}}\right] f_\lambda^-(\mathrm{d}y;z,b) + \int_{-\infty}^a f_\lambda^-(\mathrm{d}y;z,b)\right\} f_\lambda^+(\mathrm{d}z;x,a)\\
&\quad + \int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-}=\infty\right) f_\lambda^+(\mathrm{d}z;x,a)
\end{aligned}$$

Hence, for $a \le x < b$,

$$\begin{aligned}
&O_x^{(\lambda,q)}(a,b)\\
&= O_b^{(\lambda,q)}(a,b)\left\{\int_a^b \mathbb{E}_z\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<\tau_a^-\right\}}\right] f_\lambda^+(\mathrm{d}z;x,a)\right.\\
&\qquad\qquad \left. + \int_b^\infty \int_a^b \mathbb{E}_y\left[e^{-q\tau_b^+}1_{\left\{\tau_b^+<\tau_a^-\right\}}\right] f_\lambda^-(\mathrm{d}y;z,b)\, f_\lambda^+(\mathrm{d}z;x,a)\right\}\\
&\quad + O_a^{(\lambda,q)}(a,b)\left\{\int_a^b \mathbb{E}_z\left[e^{-q\tau_a^-}1_{\left\{\tau_a^-<\tau_b^+\right\}}\right] f_\lambda^+(\mathrm{d}z;x,a)\right.\\
&\qquad\qquad + \int_b^\infty \int_a^b \mathbb{E}_y\left[e^{-q\tau_a^-}1_{\left\{\tau_a^-<\tau_b^+\right\}}\right] f_\lambda^-(\mathrm{d}y;z,b)\, f_\lambda^+(\mathrm{d}z;x,a)\\
&\qquad\qquad \left. + \int_b^\infty \int_{-\infty}^a f_\lambda^-(\mathrm{d}y;z,b)\, f_\lambda^+(\mathrm{d}z;x,a)\right\}\\
&\quad + \int_b^\infty \mathbb{P}_z\left(T_b^-=\infty\right) f_\lambda^+(\mathrm{d}z;x,a)\\
&:= O_b^{(\lambda,q)}(a,b)\,\alpha_{x,b}^{(\lambda,q)}(a,b) + O_a^{(\lambda,q)}(a,b)\,\alpha_{x,a}^{(\lambda,q)}(a,b) + \int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-}=\infty\right) f_\lambda^+(\mathrm{d}z;x,a)\,.
\end{aligned} \tag{4.96}$$

From (4.95) and (4.96), it remains to obtain an expression for $O_a^{(\lambda,q)}(a,b)$ and $O_b^{(\lambda,q)}(a,b)$. Letting $x=b$ and $x=a$ in (4.95) and (4.96) respectively gives

$$O_b^{(\lambda,q)}(a,b) = \beta_{b,b}^{(\lambda,q)}(a,b)\, O_b^{(\lambda,q)}(a,b) + \beta_{b,a}^{(\lambda,q)}(a,b)\, O_a^{(\lambda,q)}(a,b) + \mathbb{P}_b\left(T_b^-=\infty\right)$$

and

$$O_a^{(\lambda,q)}(a,b) = \alpha_{a,b}^{(\lambda,q)}(a,b)\, O_b^{(\lambda,q)}(a,b) + \alpha_{a,a}^{(\lambda,q)}(a,b)\, O_a^{(\lambda,q)}(a,b) + \int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-}=\infty\right) f_\lambda^+(\mathrm{d}z;a,a)\,.$$

With the auxiliary functions defined in (4.95) and (4.96), solving the system of equations gives

$$\begin{aligned}
&\left(1-\alpha_{a,a}^{(\lambda,q)}(a,b)-\frac{\alpha_{a,b}^{(\lambda,q)}(a,b)\,\beta_{b,a}^{(\lambda,q)}(a,b)}{1-\beta_{b,b}^{(\lambda,q)}(a,b)}\right)O_a^{(\lambda,q)}(a,b)\\
&=\frac{\alpha_{a,b}^{(\lambda,q)}(a,b)}{1-\beta_{b,b}^{(\lambda,q)}(a,b)}\mathbb{P}_b\left(T_b^{\lambda,-}=\infty\right)+\int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-}=\infty\right)f_\lambda^+(\mathrm{d}z;a,a) \qquad (4.97)
\end{aligned}$$

and

$$\begin{aligned}
&\left(1-\beta_{b,b}^{(\lambda,q)}(a,b)-\frac{\alpha_{a,b}^{(\lambda,q)}(a,b)\,\beta_{b,a}^{(\lambda,q)}(a,b)}{1-\alpha_{a,a}^{(\lambda,q)}(a,b)}\right)O_b^{(\lambda,q)}(a,b)\\
&=\frac{\beta_{b,a}^{(\lambda,q)}(a,b)}{1-\alpha_{a,a}^{(\lambda,q)}(a,b)}\int_b^\infty \mathbb{P}_z\left(T_b^{\lambda,-}=\infty\right)f_\lambda^+(\mathrm{d}z;a,a)+\mathbb{P}_b\left(T_b^{\lambda,+}=\infty\right). \qquad (4.98)
\end{aligned}$$

Therefore, the result follows.

4.7.5 Proof of Theorem 25

Condition on the first Poissonian observation in red, using the strong Markov property together with (3.2), we have

$$\begin{aligned}
&O_x^{(\lambda,q)}(-\infty,0)\\
&=\mathbb{E}_x\left[\int_{-\infty}^0 1_{\left\{X_{T_0^{\lambda,-}}\in\mathrm{d}z,T_0^{\lambda,-}<\infty\right\}}\mathbb{E}_z\left[e^{-q\tau_0^+}1_{\left\{\tau_0^+<\infty\right\}}\right]O_0^{(\lambda,q)}(-\infty,0)\right]+\mathbb{P}_x\left(T_0^{\lambda,-}=\infty\right)\\
&=O_0^{(\lambda,q)}(-\infty,0)\,\mathbb{E}_x\left[\int_{-\infty}^0 1_{\left\{X_{T_0^{\lambda,-}}\in\mathrm{d}z,T_0^{\lambda,-}<\infty\right\}}e^{\Phi_q z}\right]+\mathbb{P}_x\left(T_0^{\lambda,-}=\infty\right)\\
&=O_0^{(\lambda,q)}(-\infty,0)\,\mathbb{E}_x\left[e^{\Phi_q X_{T_0^{\lambda,-}}}1_{\left\{T_0^{\lambda,-}<\infty\right\}}\right]+\mathbb{P}_x\left(T_0^{\lambda-}=\infty\right). \qquad (4.99)
\end{aligned}$$

(4.99) allows us to obtain an expression for $O_0^{(\lambda,q)}(-\infty,0)$, which can be done by letting $x=0$ such that

$$\begin{aligned}
O_0^{(\lambda,q)}(-\infty,0) &= \frac{\mathbb{P}_0\left(T_0^{\lambda,-}=\infty\right)}{1-\mathbb{E}_0\left[e^{\Phi(q)X_{T_0^{\lambda,-}}}1_{\left\{T_0^{\lambda,-}<\infty\right\}}\right]} \\
&= \frac{\psi'(0+)\dfrac{\Phi_\lambda}{\lambda}Z(0,\Phi_\lambda)}{1-\dfrac{\lambda}{\lambda-q}\left(Z(0,\Phi_q)-Z(0,\Phi_\lambda)\dfrac{q\Phi_\lambda}{\lambda\Phi_q}\right)} \\
&= \frac{\psi'(0+)\dfrac{\Phi_\lambda}{\lambda}}{1-\dfrac{\lambda\Phi_q-q\Phi_\lambda}{(\lambda-q)\Phi_q}} \\
&= \frac{\psi'(0+)\Phi_\lambda\Phi_q}{\Phi_\lambda-\Phi_q}\left(\frac{1}{q}-\frac{1}{\lambda}\right).
\end{aligned}$$

by using (3.3) and (3.4). Plugging in the above expression back into (4.99) and by (3.3) and (3.4) again arrives at

$$\begin{aligned}
O_x^{(\lambda,q)}(-\infty,0) &= \frac{\psi'(0+)\Phi_\lambda\Phi_q(\lambda-q)}{\lambda q(\Phi_\lambda-\Phi_q)}\frac{\lambda}{\lambda-q}\left[Z(x,\Phi_q)-Z(x,\Phi_\lambda)\frac{q\Phi_\lambda}{\lambda\Phi_q}\right] \\
&\quad+\frac{\psi'(0+)\Phi_\lambda}{\lambda}Z(x,\Phi_\lambda) \\
&= \frac{\psi'(0+)\Phi_\lambda\Phi_q}{\Phi_\lambda-\Phi_q}\left[\frac{Z(x,\Phi_q)}{q}-\frac{Z(x,\Phi_\lambda)}{\lambda}\right].
\end{aligned}$$

4.7.6 Proof of Theorem 26

For $x\in[-a,0)$ and $y\in\mathbb{R}$, we separately consider the contributions to (4.40) by the following two possible events: $\left\{e_q<T_{-a}^{-,\lambda}\wedge\tau_0^+\right\}$ and $\left\{\tau_0^+<T_{-a}^{-,\lambda}\wedge e_q\right\}$. It follows that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&=\mathbb{E}_x\left[e^{-se_q}1_{\{X_{e_q}\in\mathrm{d}y,e_q<T_{-a}^{-,\lambda}\wedge\tau_0^+\}}\right]+\mathbb{E}_x\left[e^{-s\tau_0^+}1_{\{\tau_0^+<e_q\wedge T_{-a}^{-,\lambda}\}}\right]\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\}}\right] \\
&=qu_{d:c}^{(s+q,\lambda)}(x+a,y+a;a)\,\mathrm{d}y1_{\{y<0\}}+\frac{W^{(s+q,\lambda)}(x+a)}{W^{(s+q,\lambda)}(a)}\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right],
\end{aligned} \tag{4.100}$$

where 0^- means the surplus is at level 0 and the Parisian clock is on. For $x \in [0,b]$ and $y \in \mathbb{R}$, we shall have $T_0^{-,\lambda} \leq T^{\lambda,q}$ almost surely. Hence, by conditioning on $T_0^{-,\lambda}$, one finds that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&= \int_{-a}^{0} \mathbb{E}_x\left[e^{-sT_0^{-,\lambda}}1_{\left\{X_{T_0^{-,\lambda}}\in \mathrm{d}w, T_0^{-,\lambda}<T_b^{+,\lambda}\right\}}\right] \mathbb{E}_w\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&= \lambda \int_{-a}^{0} u_{d:d}^{(s,\lambda)}(x,w;b)\, \mathbb{E}_w\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \mathrm{d}w.
\end{aligned} \tag{4.101}$$

Substituting (4.100) into (4.101) leads to

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&= q\lambda \int_{-a}^{0} u_{d:d}^{(s,\lambda)}(x,w;b)\, u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a)\, \mathrm{d}w \mathrm{d}y 1_{\{y<0\}} \\
&\quad + \lambda \int_{-a}^{0} u_{d:d}^{(s,\lambda)}(x,w;b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} \mathrm{d}w \mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right].
\end{aligned} \tag{4.102}$$

With the help of (4.43), (4.100) and (4.102) can be expressed in a unified way as follows: for any $x \in [-a,b]$ and $y \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&= q \int_{-a}^{0} \upsilon^{(s,\lambda)}(x,w;b) u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a)\, \mathrm{d}w \mathrm{d}y 1_{\{y<0\}} \\
&\quad + \int_{-a}^{0} \upsilon^{(s,\lambda)}(x,w;b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} \mathrm{d}w \mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right].
\end{aligned} \tag{4.103}$$

Next, we focus on characterizing $\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in \mathrm{d}y, T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]$. Conditioning on

whether e_λ or e_q occurs first and using (4.103), one obtains

$$\begin{aligned}
&\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]\\
&=\mathbb{E}_{0^-}\left[e^{-se_q}1_{\left\{X_{e_q}\in\mathrm{d}y,e_q<e_\lambda\right\}}\right]+\int_{-a}^{b}\mathbb{E}_{0^-}\left[e^{-se_\lambda}1_{\left\{X_{e_\lambda}\in\mathrm{d}z,e_\lambda<e_q\right\}}\right]\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]\\
&=q\theta^{(s+q+\lambda)}(y)\mathrm{d}y+\lambda q\mathrm{d}y\int_{-a}^{b}\int_{-a}^{0}\theta^{(s+q+\lambda)}(x)\upsilon^{(s,\lambda)}(x,w;b)u_{d:c}^{(s+q,\lambda)}(w+a,y+a;a)\mathrm{d}w\mathrm{d}z\\
&\quad+\lambda\int_{-a}^{b}\int_{-a}^{0}\theta^{(s+q+\lambda)}(z)\upsilon^{(s,\lambda)}(z,w;b)\frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)}\mathrm{d}w\mathrm{d}z\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\}}\right].
\end{aligned}\tag{4.104}$$

Thus, it is direct from (4.104) that

$$\begin{aligned}
&\frac{1}{q\mathrm{d}y}\mathbb{E}_{0^-}\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]\\
&=\frac{\theta^{(s+q+\lambda)}(y)+\lambda\int_{-a}^{b}\int_{-a}^{0}\theta^{(s+q+\lambda)}(z)\,\upsilon^{(s,\lambda)}(z,w;b)u_{d:c}^{(s+q,\lambda)}(w+a,y+a;a)\,\mathrm{d}w\mathrm{d}z}{1-\lambda\int_{-a}^{b}\int_{-a}^{0}\theta^{(s+q+\lambda)}(z)\,\upsilon^{(s,\lambda)}(z,w;b)\frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)}\mathrm{d}w\mathrm{d}z}\\
&=\frac{\theta^{(s+q+\lambda)}(y)+A^{(s+q,\lambda)}(a,-a-y)-\lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)H_{a,b}^{(s,q,\lambda)}(z,-y)\mathrm{d}z}{1-\lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)\frac{Z_{a,b}^{(s,q,\lambda)}(z)}{W^{(q+s,\lambda)}(a)}\mathrm{d}z}-A^{(s+q,\lambda)}(a,-a-y),
\end{aligned}\tag{4.105}$$

where the last step is due to the definitions of $u_{d:c}$, $H_{a,b}^{(s,q,\lambda)}$, and $Z_{a,b}^{(s,q,\lambda)}$, in (4.26), (4.41), and (4.42), respectively. Finally, the substitution of (4.105) into (4.102) completes the proof.

4.7.7 Proof of (4.44)

By (4.12), we have

$$\begin{aligned}
&\frac{1}{q\mathrm{d}y}\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right]\\
&=\frac{\theta^{(s+q+\lambda)}(y)+W^{(s+q)}(-y)-\lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)H_{a,b}^{(s,q,\lambda)}(z,-y)\mathrm{d}z}{W^{(s+q,\lambda)}(a)-\lambda\int_{-a}^{b}\theta^{(s+q+\lambda)}(z)Z_{a,b}^{(s,q,\lambda)}(z)\mathrm{d}z}Z_{a,b}^{(s,q,\lambda)}(x)-H_{a,b}^{(s,q,\lambda)}(x,-y),
\end{aligned}\tag{4.106}$$

where $H_{a,b}^{(s,q,\lambda)}(x,-y) = \int_0^a \upsilon^{(s,\lambda)}(x,-w;b)W^{(s+q)}(-y-w)\mathrm{d}w$. By (4.36), we know that $\upsilon^{(s,\lambda)}(x,-w;b)\mathrm{d}w$ converges to $\mathbb{E}_x\left[e^{-q\tau_0^-}1_{\left\{-X_{\tau_0^-}\in \mathrm{d}w,\tau_0^-<\tau_b^+\right\}}\right]$ as $\lambda\to\infty$. It follows that

$$\begin{aligned}
\lim_{\lambda\to\infty} H_{a,b}^{(s,q,\lambda)}(x,-y) &= \int_0^a \mathbb{E}_x\left[e^{-q\tau_0^-}1_{\left\{-X_{\tau_0^-}\in \mathrm{d}w,\tau_0^-<\tau_b^+\right\}}\right] W^{(s+q)}(-y-w)\mathrm{d}w \\
&= \mathbb{E}_x\left[e^{-q\tau_0^-}W^{(s+q)}(-y+X_{\tau_0^-})1_{\{\tau_0^-<\tau_b^+\}}\right] \\
&= A^{(s,q)}(x,-y) - \frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,-y),
\end{aligned} \tag{4.107}$$

where the last line is due to Lemma 2.2 ofLoeffen et al. [2014]. By the same argument, we have

$$\lim_{\lambda\to\infty} Z_{a,b}^{(s,q,\lambda)}(x) = \lim_{\lambda\to\infty} H_{a,b}^{(s,q,\lambda)}(x,a) = A^{(s,q)}(x,a) - \frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,a). \tag{4.108}$$

From (2.10), we deduce that $\lambda\theta^{(\lambda)}(z)$ converges to $\delta_0(z)$ when $\lambda\to\infty$. With the application of (4.107), (4.108) and (4.13), the limit of (4.106) is given by

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{q\mathrm{d}y}\mathbb{E}_x\left[e^{-sT^{\lambda,q}}1_{\left\{X_{T^{\lambda,q}}\in\mathrm{d}y,T^{\lambda,q}<T_{-a}^{-,\lambda}\wedge T_b^{+,\lambda}\right\}}\right] \\
&= \frac{W^{(s+q)}(-y)-A^{(s,q)}(0,-y)+\frac{W^{(s)}(0+)}{W^{(s)}(b)}A^{(s,q)}(b,-y)}{W^{(s+q)}(a)-A^{(s,q)}(0,a)+\frac{W^{(s)}(0+)}{W^{(s)}(b)}A^{(s,q)}(b,a)}\left(A^{(s,q)}(x,a)-\frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,a)\right) \\
&\quad - A^{(s,q)}(x,-y)+\frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,-y) \\
&= \frac{A^{(s,q)}(b,-y)}{A^{(s,q)}(b,a)}\left(A^{(s,q)}(x,a)-\frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,a)\right) - A^{(s,q)}(x,-y)+\frac{W^{(s)}(x)}{W^{(s)}(b)}A^{(s,q)}(b,-y) \\
&= \frac{A^{(s,q)}(b,-y)}{A^{(s,q)}(b,a)}A^{(s,q)}(x,a) - A^{(s,q)}(x,-y),
\end{aligned}$$

where we have used the fact that $W^{(s)}(0+)\neq 0$ when X has bounded variation paths.

Chapter 5

Modified Parisian Ruin Time and its Risk Management Implication

5.1 Introduction

This chapter is devoted to introduce a new risk quantity.

To begin with, we review the construction of Parisian ruin time under a deterministic grace setting. Recall that under the Parisian setting, a fixed time horizon with length $d > 0$ is granted once the surplus level is observed to downcross the zero barrier. As long as it recovers the negative surplus within the time horizon, the negative excursion is disregarded and a normal business (with zero starting surplus at the end of grace period) is resumed; else, ruin is declared at the end of the time horizon Under the context of a continuous observation, the business is inspected infinitely during the grace period; while as far as a Poissonian observation is concerned, it is only intercepted at discrete time points modeled by a Poisson process.

Regardless of the observation mode, the business is understood to be monitored once a negative surplus is observed until the business recovers or ruin is declared, whichever comes first. By saying so, information in relation to solvency are assumed to be accessible at a granular level along the time dimension within the grace period. While this might be feasible from a company perspective, this is rarely the case when it comes to the regulator point of view. In general, regulators usually require financial reports on a regular basis such that snapshots to the aggregated financial status can be acquired exclusively at only specific time points, say at month ends or quarter ends, depending on the regulations specified by the ordinance. Under the context of Parisian ruin, this translates to the interpretation that the surplus level at the end of the grace period alone is known and used for determining whether the business is in a good shape.

Motivated from the discussions above, this chapter aims at incorporating such idea to modify the construction of Parisian ruin so that the aforementioned feature could be captured in the risk quantity. Specifically, under the continuous observation setting, the Parisian clock is again initiated right away when a negative surplus is observed. However, we do not assume any inspections before the clock rings. Instead, aggregated financial information as reflected from the surplus level at the end of the grace period is used solely to judge whether the business is healthy. If a positive surplus is recorded, then a normal business (with a possibly non–zero positive starting surplus at the end of grace period) is resumed. Otherwise, ruin occurs at the end of the grace period. A precise construction to such modified idea of Parisian ruin time can be found in Section 5.2.

While the modified Parisian ruin is inspired to align with the general regulatory practice, to be argued in Section 5.4, the newly defined Parisian ruin always leads to an upper bound for the classical Parisian ruin probability. From a risk management perspective, it implies the modified Parisian ruin is a more conservative risk quantity to work with. This observation is indeed consistent with the intuition as reflect by the role of regulators. Keeping in mind that their primary stakeholder is the public audience, they are generally prudent as far as solvency is concerned. Hence, the modified Parisian ruin could potentially be one of the risk quantities to consider shall one wish to go conservative.

As far as a fixed grace is concerned, along the same line as discussed in Section 1.4 in handling analysis with deterministic periodic observation, we adopt an Erlangization technique to approximate the deterministic time horizon.

The contribution of this work is two–fold. On one side, we establish a new risk quantity which embed the industry practice and intuition from a regulator point of view. On the other side, to be shown in 5.4, we obtain an expression to the Gerber–Shiu function under the context of Cramér–Lundberg model specified in Section 2.3.

The rest of the chapter is organized as follows. Section 5.2 is devoted to the mathematical formulation of the modified Parisian ruin time. Via the formulation, an ordering property to several commonly encountered risk quantities is observed and summarized in this section. Any preliminary results in the literature pertained to the Cramér–Lundberg model and Erlangization technique can be found in Section 5.3. Section 5.4 performs all the necessary analysis to identify the law of the concerned ruin time, while a discussion on comparing the law of traditional and modified Parisian ruin time can be found. Examples are illustrated in Section 5.5, while the last section contains all the proofs to results in this chapter.

5.2 Construction of the Modified Parisian Ruin Time

Let $\{T_k\}_{k\in\mathbb{N}}$ be a i.i.d. sequence of random times having the same distribution as T. Here, T_k refers to the k–th grace period accompanied by the k–th regulatory check, and T is assumed to follow an Erlang(n) distribution with density

$$f_T(t) = \frac{\gamma^n t^{n-1} e^{-\gamma t}}{(n-1)!}, \quad t > 0,$$

where $n \in \mathbb{N}$ and $\gamma > 0$. Observe that by fixing the mean $\mathbb{E}[T] = n/\gamma = d > 0$, T converges in distribution to a point mass at d as $n \to \infty$. This leads to the idea of approximating a fixed grace period by choosing a sufficiently large n, a method called Erlangization.

Construct a sequence of stopping times $\left\{\tau_{0,k}^{-}\right\}_{k\in\mathbb{N}}$ as follows. For $k \in \mathbb{N}$,

$$\tau_{0,k}^{-} = \begin{cases} \inf\{t \geq 0 : X_t < 0\} & , k = 1 \\ \inf\left\{t \geq \tau_{0,k-1}^{-} + T_{k-1} : X_t < 0\right\} & , k = 2, 3, 4\ldots \end{cases}$$

In other words, $\tau_{0,k}^{-}$ marks the starting time of the Parisian clock in relation to the k–th regulatory check.

Denote

$$k^* = \min\left\{k \geq 1 : X_{\tau_{0,k}^{-}+T_k} < 0\right\}.$$

Here, k^* keeps track of the first regulatory check such that a negative surplus is observed at the end of the grace period. The modified Parisian ruin time is therefore defined as

$$\zeta_d^n = \tau_{0,k^*}^{-} + T_{k^*}.$$

Figure 5.1 illustrates the modified Parisian ruin time under a continuous observation for a particular sample path.

In order to analyze the law pertained to the modified Parisian ruin time, we study the Gerber–Shiu function for which the penalty depends only on the deficit at ruin, i.e.,

$$\phi_{d,\delta}^n(u) = \mathbb{E}_u\left[e^{-\delta\zeta_d^n} w\left(\left|X_{\zeta_d^n}\right|\right) 1_{\left\{\zeta_d^n<\infty\right\}}\right] = \int_0^\infty w(y)\, h_{d,\delta}^n(y|u)\,\mathrm{d}y, \quad u \geq 0. \tag{5.1}$$

Here, $h_{d,\delta}^n(y|u)$ (for $y > 0$) refers to the discounted density of the deficit observed at the modified Parisian ruin time under an Erlang(n) Parisian clock, i.e., $\left|X_{\zeta_d^n}\right|$.

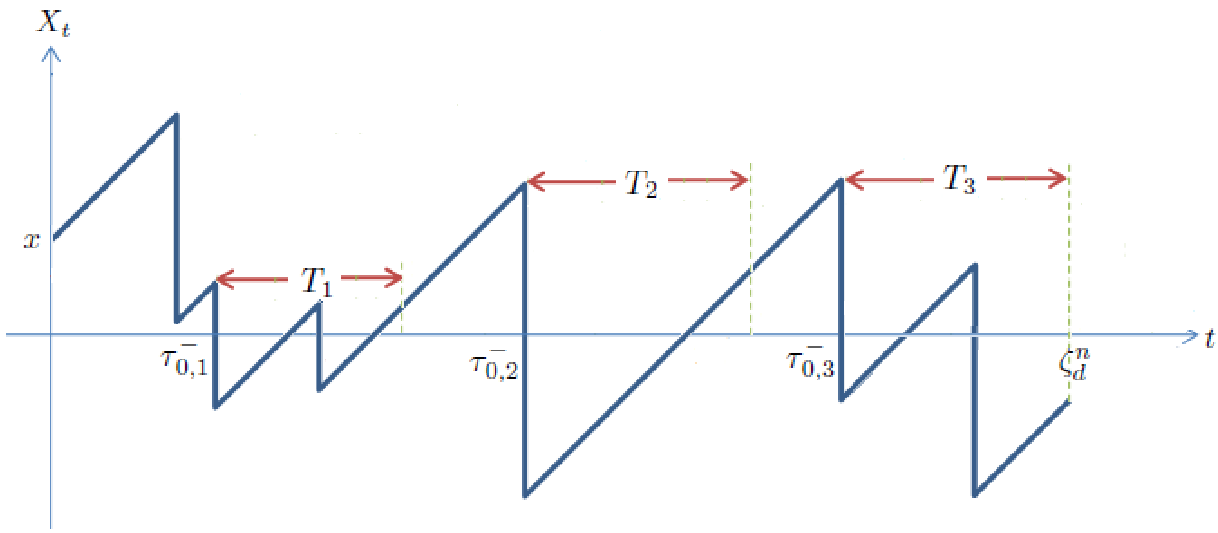

Figure 5.1: Illustration of modified Parisian ruin under continuous observation.

5.3 Preliminaries

For every $k \in \mathbb{N}$, the surplus process downcrosses the zero level at time $\tau_{0,k}^-$, and a deficit of magnitude $\left|X_{\tau_{0,k}^-}\right|$ is recorded. In order to tell whether ruin occurs at time $\tau_{0,k}^- + T_k$, we need to know the difference between the surplus level at $t = \tau_{0,k}^-$ and $t = \tau_{0,k}^- + T_k$. By virtue of spatial homogeneity, together with the construction of T_k specified in Section 5.2, the change $X_{\tau_{0,k}^-} - X_{\tau_{0,k}^- + T_k}$ has identical distribution as $\sum_{i=1}^{N(T)} Y_i - cT$. On the other hand, due to the discount factor embedded in the Gerber–Shiu function, we have to keep track of the time T as well.

From Section 3.2 of Albrecher et al. [2013], one has

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \int_{-\infty}^{\infty} e^{-sy} g_\delta(y)\,\mathrm{d}y, \tag{5.2}$$

where $g_\delta(y)$ (for $y \in \mathbb{R}$) denotes the discounted density of the increment $\sum_{i=1}^{N(T)} Y_i - cT$. To assist with the representation to $g_\delta(y)$, write

$$g_\delta(y) = g_{\delta,-}(-y)\,1_{\{y<0\}} + g_{\delta,+}(y)\,1_{\{y>0\}}$$

so that $g_{\delta,-}$ and $g_{\delta,+}$ respectively represent the case where $\sum_{i=1}^{N(T)} Y_i - cT$ is negative and positive.

It is known that there exhibit compact expressions to $g_{\delta,-}(y)$ and $g_{\delta,+}(y)$. For completeness, results pertaining to these two quantities are summarized and quoted in the following lemma without proof. Readers are directed to Albrecher et al. [2013] for more details. The structure of $g_{\delta,-}(y)$ is particularly stressed nevertheless since it would become handy in Section 5.4 when deriving for an expression to the Gerber–Shiu function.

Lemma 27. Suppose the i.i.d. claim size random variables $\{Y_i\}_{i\in\mathbb{N}}$ are having a common density given by $f_Y(\cdot)$ with a Laplace transform $\tilde{f}_Y(\cdot)$. Denote $\rho_\gamma > 0$ the unique positive root solving the following Lundberg–type equation (in x)

$$cx - (\lambda + \gamma + \delta) + \lambda \tilde{f}_Y(x) = 0. \tag{5.3}$$

Construct a proper density $f_L(\cdot)$ by

$$f_L(y) = \frac{\mathcal{T}_{\rho_\gamma} f_Y(y)}{\mathcal{T}_0 \mathcal{T}_{\rho_\gamma} f_Y(0)}, \quad y > 0, \tag{5.4}$$

where $\mathcal{T}_s$ is the so–called Dickson–Hipp operator defined as

$$\mathcal{T}_s f(y) = \int_y^\infty e^{-s(z-y)} f(z)\,\mathrm{d}z = \int_0^\infty e^{-sz} f(z+y)\,\mathrm{d}z, \quad y \geq 0$$

for any complex number s with $Re(s) \geq 0$. Then, $g_{\delta,-}(y)$ and $g_{\delta,+}(y)$ respectively admits the following representation

$$g_{\delta,-}(y) = \sum_{j=1}^{n} B_j^* \frac{y^{j-1} e^{-\rho_\gamma y}}{(j-1)!}, \quad y > 0, \tag{5.5}$$

$$g_{\delta,+}(y) = \left(\frac{\gamma}{c}\right)^n \sum_{j=1}^{\infty} \binom{n+j-1}{n-1} \phi^j \mathcal{T}_{\rho_\gamma}^n f_L^{*j}(y), \quad y > 0, \tag{5.6}$$

with

$$B_j^* = \left(\frac{\gamma}{c}\right)^n \sum_{k=1}^{\infty} \binom{n+k-1}{n-1} \phi^k \mathcal{T}_{\rho_\gamma}^{n-j+1} f_L^{*k}(0), \quad j = 1, 2, \ldots, n-1, \tag{5.7}$$

and

$$B_n^* = \left(\frac{\gamma}{c}\right)^n \left[1 - \phi \tilde{f}_L(\rho_\gamma)\right]^{-n}, \tag{5.8}$$

where

$$\phi = 1 - \frac{\gamma + \delta}{c\rho_\gamma}. \tag{5.9}$$

Here, f^{*k} denotes the k–fold convolution of f with itself, and $\mathcal{T}_s^k f(y) = \underbrace{\mathcal{T}_s \cdots \mathcal{T}_s}_{k} f(y)$.

5.4 Main Result

With all the preliminary results in hand, we are ready to look closely the distribution of ruin quantities pertained to the modified Parisian ruin time.

5.4.1 Evaluation of Gerber–Shiu Function

To assist with the presentation, the following intermediate functions and quantities are defined. Recall from Subsection 2.3.2 that $h_\delta(y|u)$ represents the discounted density function of $\left|X_{\tau_0^-}\right|$. Let

$$\chi_{\delta,w}(u) = \int_0^\infty w(z)\,\tau_\delta(z|u)\,\mathrm{d}z, \quad u \geq 0, \tag{5.10}$$

where

$$\tau_\delta(z|u) = \int_0^z g_{\delta,+}(z-y)\,h_\delta(y|u)\,\mathrm{d}y + \int_z^\infty g_{\delta,-}(y-z)\,h_\delta(y|u)\mathrm{d}y, \quad z \geq 0. \tag{5.11}$$

Also, for $i = 1, 2, \ldots, n$, let

$$\Delta_{\delta,i}(z) = \int_0^\infty \frac{u^{i-1}e^{-\rho_\gamma u}}{(i-1)!}\tau_\delta(z|u)\mathrm{d}u, \quad z \geq 0 \tag{5.12}$$

and

$$\varphi_{\delta,i}(u) = \int_0^\infty \frac{y^{i-1}e^{-\rho_\gamma y}}{(i-1)!}h_\delta(y|u)\,\mathrm{d}y, \quad u \geq 0. \tag{5.13}$$

With these in hand, we are ready to evaluate the Gerber–Shiu function.

Theorem 28. *For $u \geq 0$ and $\delta > 0$, the Gerber–Shiu function $\phi_{d,\delta}^{n}(u)$ can be expressed as*

$$\phi_{d,\delta}^{n}(u) = \chi_{\delta,w}(u) + \sum_{j=1}^{n} \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^{*} \varphi_{\delta,k}(u) \right) \eta_{\delta,j}, \tag{5.14}$$

where the constants $\{\eta_{\delta,i}\}_{i=1,\ldots,n}$ satisfy the system of linear equations

$$\eta_{\delta,i} = \int_{0}^{\infty} w(z)\Delta_{\delta,i}(z)\,\mathrm{d}z + \sum_{j=1}^{n} \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^{*} \int_{0}^{\infty} \frac{u^{i-1}e^{-\rho_{\gamma}u}}{(i-1)!} \varphi_{\delta,k}(u)\,\mathrm{d}u \right) \eta_{\delta,j}, \quad i = 1, 2, \ldots, n. \tag{5.15}$$

The above theorem completely characterizes $\phi_{d,\delta}^{n}(u)$, and hence the law of $\left(\zeta_{d}^{n}, \left|X_{\zeta_{d}^{n}}\right|\right)$ is known to full generality.

Note that the system of linear equations in (5.15) can in fact be rephrased more compactly. Define the n–dimensional column vectors $\boldsymbol{\eta}_{\delta}$ and $\boldsymbol{\Delta}_{\delta}(z)$ with the i–th element being $\eta_{\delta,i}$ and $\Delta_{\delta,i}(z)$ respectively, and the n–dimensional row vector $\boldsymbol{\sigma}_{\delta}(u)$ with the j–th element being $\sum_{k=1}^{n-j+1} B_{j+k-1}^{*} \varphi_{\delta,k}(u)\,\mathrm{d}u$. Define also the n–dimensional square matrix $\boldsymbol{\Gamma}_{\delta}$ with the (i,j)–th element being $\sum_{k=1}^{n-j+1} B_{j+k-1}^{*} \int_{0}^{\infty} \frac{u^{i-1}e^{-\rho_{\gamma}u}}{(i-1)!} \varphi_{\delta,k}(u)\,\mathrm{d}u$. Then, (5.15) can be written neatly as

$$\boldsymbol{\eta}_{\delta} = \int_{0}^{\infty} w(z)\boldsymbol{\Delta}_{\delta}(z)\,\mathrm{d}z + \boldsymbol{\Gamma}_{\delta}\boldsymbol{\eta}_{\delta}$$

such that, upon solving,

$$\boldsymbol{\eta}_{\delta} = \left(\boldsymbol{I} - \boldsymbol{\Gamma}_{\delta}\right)^{-1} \int_{0}^{\infty} w(z)\boldsymbol{\Delta}_{\delta}(z)\,\mathrm{d}z, \tag{5.16}$$

where $\boldsymbol{I}$ represents an identity matrix of size $n \times n$. Here, the invertibility of matrix $\boldsymbol{I} - \boldsymbol{\Gamma}_{\delta}$ is assumed. The defined vectors and matrix are also useful in expressing the discounted density of deficit as reflect from the following corollary.

Corollary 29. For $u \geq 0$ and $\delta > 0$, the discounted density of deficit $h_{d,\delta}^{n}(y|u)$ can be expressed as

$$h_{d,\delta}^{n}(y|u) = \tau_{\delta}(y|u) + \boldsymbol{\sigma}_{\delta}(u)\left(\boldsymbol{I} - \boldsymbol{\Gamma}_{\delta}\right)^{-1} \boldsymbol{\Delta}_{\delta}(y), \quad y > 0.$$

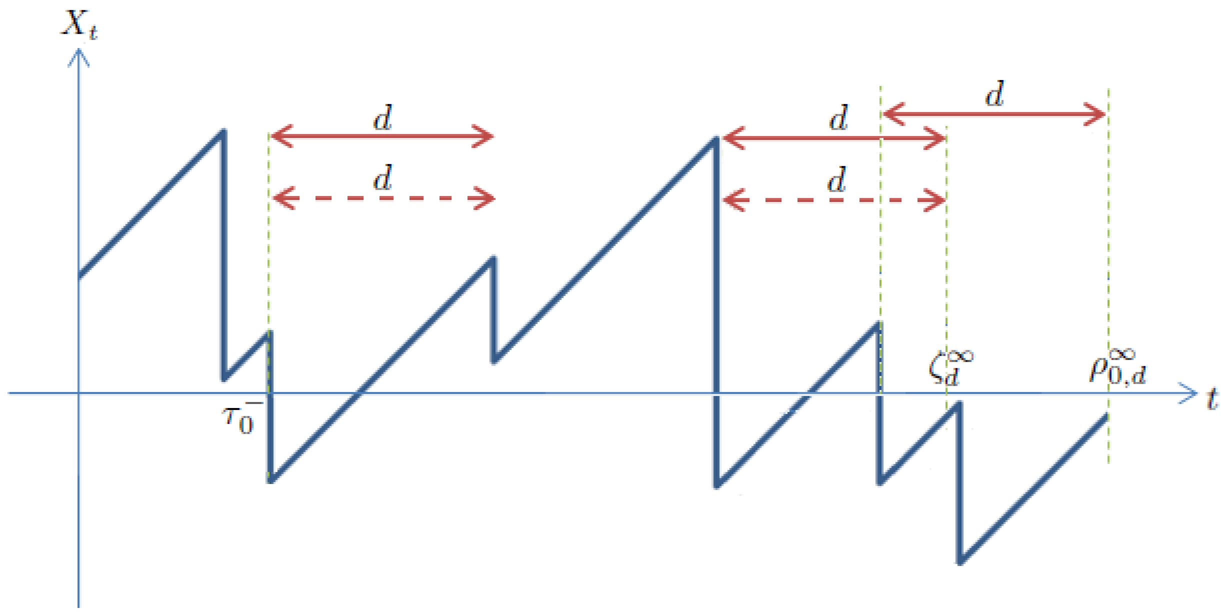

Figure 5.2: Illustration of ordering to different ruin times.

5.4.2 An Ordering Property

Recall that the essence of Erlangization is to approximate a fix time horizon by taking the limit that n goes to infinity for the Erlang(n) random variable while fixing its mean. With a bit of abusing the notation, denote ζ_d^∞ the modified Parisian ruin time with fixed grace period of length d. Definition to the traditional Parisian ruin time $\rho_{0,d}^\infty$ in Section 3.2 is also evoked.

As mentioned in Section 5.1, the modified Parisian ruin probability always bounds the traditional Parisian ruin probability from above. This can be seen by using a sample path argument. In particular, for every sample path contributing to ruin under the traditional Parisian setting, ruin must also occur under the modified Parisian setting. Along the same line, for every sample path contributing to ruin under either the traditional or modified Parisian setting, ruin must also occur under the classical setting. Figure 5.2 illustrates a particular sample path demonstrating such an idea.

Therefore, we have the following proposition.

Proposition 30. The following inequality

$$\tau_0^- \leq \zeta_d^\infty \leq \rho_{0,d}^\infty$$

holds true almost surely.

5.5 Example

In this section, a detailed work example will be demonstrated to illustrate the computability of Theorem 28. Parallel to the work by Landriault et al. [2014], we focus on the calculation of the probability of modified Parisian ultimate ruin (i.e., $\delta = 0$ and $w(\cdot) \equiv 1$ in (5.1)) with a fixed grace period by Erlangization technique, assuming an exponentially distributed claim size with mean $1/\nu$ (i.e., $f_Y(y) = \nu e^{-\nu y}$, $y > 0$).

Realize that the functions $g_{0,-}(\cdot)$, $g_{0,+}(\cdot)$ and $h_0(\cdot|u)$ are pivotal due to (5.10)–(5.13). Hence, we first obtain an expression to these functions one by one.

To begin with, note that, by simple algebra , $f_L(y)$ in (5.4) reduces to

$$f_L(y) = \nu e^{-\nu y}, \quad y > 0$$

such that one has

$$\begin{aligned}
\mathcal{T}_s^n f_L^{*k}(0) &= \int_0^\infty e^{-sz} \frac{z^{n-1}}{(n-1)!} \frac{\nu^k}{(k-1)!} z^{k-1} e^{-\nu z} \mathrm{d}z \\
&= \frac{\nu^k}{(n-1)!\,(k-1)!} \frac{(n+k-2)!}{(s+\nu)^{n+k-1}} \\
&= \begin{pmatrix} n+k-2 \\ n-1 \end{pmatrix} \frac{\nu^k}{(s+\nu)^{n+k-1}}.
\end{aligned}$$

Therefore, (5.7) reduces to

$$\begin{aligned}&B_j^* \\ &= \left(\frac{\gamma}{c}\right)^n \sum_{k=1}^{\infty} \begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix} \begin{pmatrix} n-j+k-1 \\ n-j \end{pmatrix} \left(1-\frac{\gamma+\delta}{c\rho_\gamma}\right)^k \frac{\nu^k}{(\rho_\gamma+\nu)^{n-j+k}}, \quad j=1,2,\ldots,n-1,\end{aligned}$$

and (5.8) simplifies to

$$B_n^* = \left(\frac{\gamma}{c}\right)^n \left[1-\left(1-\frac{\gamma+\delta}{c\rho_\gamma}\right)\left(\frac{\nu}{\nu+\rho_\gamma}\right)\right]^{-n}.$$

By (5.5), $g_{0,-}(\cdot)$ is fully characterized.

Remark 5. It is remarked that (4.2) of Albrecher et al. [2013] gives another characterization to B_j^*, which leads to

$$\begin{aligned}&B_j^* \\ &= (-1)^{n-j}\left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!} \frac{\mathrm{d}^{n-j}}{\mathrm{d}s^{n-j}} \left(\frac{\nu+s}{\kappa+s}\right)^n \bigg|_{s=\rho_\gamma} \\ &= (-1)^{n-j}\left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!} \sum_{k=0}^{n-j} \begin{pmatrix} n-j \\ k \end{pmatrix} \left[\frac{\mathrm{d}^{n-j-k}}{\mathrm{d}s^{n-j-k}}(\nu+s)^n\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}s^k}(\kappa+s)^{-n}\right]\bigg|_{s=\rho_\gamma} \\ &= (-1)^{n-j}\left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!} \sum_{k=0}^{n-j} \begin{pmatrix} n-j \\ k \end{pmatrix} \left[\frac{n!}{(j-k)!}(\nu+s)^{j+k}\right]\left[(-1)^k \frac{(n+k-1)!}{(n-1)!}(\kappa+s)^{-(n+k)}\right]\bigg|_{s=\rho_\gamma} \\ &= (-1)^{n-j}\left(\frac{\gamma}{c}\right)^n \sum_{k=0}^{n-j} (-1)^k \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \begin{pmatrix} n \\ j-k \end{pmatrix} (\nu+\rho_\gamma)^{j+k}(\kappa+\rho_\gamma)^{-(n+k)}, \quad j=1,2,\ldots,n,\end{aligned} \tag{5.17}$$

where

$$\kappa = \nu(1-\phi). \tag{5.18}$$

The second equality is due to Leibniz rule. While these two characterizations look different from each other, it can be demonstrated numerically that they are indeed consistent with each other.

Next, from (4.3) of Albrecher et al. [2013], $g_{0,+}(\cdot)$ is given by

$$g_{0,+}(y) = \sum_{j=1}^{n} B_j \frac{y^{j-1} e^{-\kappa y}}{(j-1)!}, \quad y > 0,$$

where B_j exhibits the following expression

$$B_j = \left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!} \frac{\mathrm{d}^{n-j}}{\mathrm{d}s^{n-j}} \left(\frac{\nu + s}{\rho_\gamma - s}\right)^n \Bigg|_{s=-\kappa}, \quad j = 1, 2, \ldots, n$$

such that, similar to (5.17), simplification gives

$$B_j = \left(\frac{\gamma}{c}\right)^n \sum_{k=0}^{n-j} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \begin{pmatrix} n \\ j+k \end{pmatrix} (\nu - \kappa)^{j+k} (\kappa + \rho_\gamma)^{-(n+k)}, \quad j = 1, 2, \ldots, n, \tag{5.19}$$

Lastly, to obtain $h_0(\cdot|u)$, we simply realize, by memoryless property to an exponential random variable, that the magnitude of deficit (given that classical ruin occurs) is again exponentially distributed with the same parameter as the claim size. Thus, it is direct that

$$h_0(y|u) = \psi_0(u) \nu e^{-\nu y} = \frac{\lambda}{c} e^{-\left(c - \frac{\lambda}{\nu}\right)\frac{\nu}{c} u - \nu y}, \quad y > 0,$$

where

$$\psi_0(u) = \frac{\lambda}{c\nu} e^{-\left(c - \frac{\lambda}{\nu}\right)\frac{\nu}{c} u}$$

is the well–known result for the classical ultimate ruin probability.

With these in hand, we are ready to compute (5.10)–(5.13). From (5.11),

$$\begin{aligned}
\tau_0(z|u) &= \int_0^z \left[\sum_{j=1}^{n} B_j \frac{(z-y)^{j-1} e^{-\kappa(z-y)}}{(j-1)!}\right] \left[\psi_0(u) \nu e^{-\nu y}\right] \mathrm{d}y \\
&\quad + \int_z^\infty \left[\sum_{j=1}^{n} B_j^* \frac{(y-z)^{j-1} e^{-\rho_\gamma(y-z)}}{(j-1)!}\right] \left[\psi_0(u) \nu e^{-\nu y}\right] \mathrm{d}y \\
&= \int_0^z \left[\sum_{j=1}^{n} B_j \frac{y^{j-1} e^{-\kappa y}}{(j-1)!}\right] \left[\psi_0(u) \nu e^{-\nu(z-y)}\right] \mathrm{d}y \\
&\quad + \int_0^\infty \left[\sum_{j=1}^{n} B_j^* \frac{y^{j-1} e^{-\rho_\gamma y}}{(j-1)!}\right] \left[\psi_0(u) \nu e^{-\nu(y+z)}\right] \mathrm{d}y, \quad z \geq 0,
\end{aligned}$$

such that (5.10) is given by

$$\begin{aligned}
\chi_{0,w}(u) &= \int_0^\infty \int_0^z \left[\sum_{j=1}^n B_j \psi_0(u)\, \nu \frac{y^{j-1} e^{-(\kappa-\nu)y-\nu z}}{(j-1)!}\right] \mathrm{d}y\mathrm{d}z \\
&\quad + \int_0^\infty \int_0^\infty \left[\sum_{j=1}^n B_j^* \psi_0(u)\, \nu \frac{y^{j-1} e^{-(\rho_\gamma+\nu)y-\nu z}}{(j-1)!}\right] \mathrm{d}y\mathrm{d}z \\
&= \int_0^\infty \left[\sum_{j=1}^n B_j \psi_0(u) \frac{y^{j-1} e^{-\kappa y}}{(j-1)!}\right] \mathrm{d}y \\
&\quad + \int_0^\infty \left[\sum_{j=1}^n B_j^* \psi_0(u) \frac{y^{j-1} e^{-(\rho_\gamma+\nu)y}}{(j-1)!}\right] \mathrm{d}y \\
&= \sum_{j=1}^n B_j \psi_0(u)\, \kappa^{-j} + \sum_{j=1}^n B_j^* \psi_0(u) (\rho_\gamma+\nu)^{-j}, \quad u \geq 0
\end{aligned}$$

while, for $i = 1, 2, \ldots, n$, (5.13) is given by

$$\begin{aligned}
\varphi_{0,i}(u) &= \int_0^\infty \frac{y^{i-1} e^{-\rho_\gamma y}}{(i-1)!} \psi_0(u)\, \nu e^{-\nu y} \mathrm{d}y \\
&= \psi_0(u)\, \nu (\rho_\gamma+\nu)^{-i}, \quad u \geq 0.
\end{aligned}$$

As a last step, the constants $\{\eta_{0,i}\}_{i=1,\ldots,n}$ in (5.15) are yet to be found in order to fully characterize (5.1). Note that

$$\begin{aligned}
\int_0^\infty \frac{u^{i-1} e^{-\rho_\delta u}}{(i-1)!} \psi_0(u)\, \mathrm{d}u &= \int_0^\infty \frac{u^{i-1} e^{-\rho_\delta u}}{(i-1)!} \left[\frac{\lambda}{c\nu} e^{-\left(c-\frac{\lambda}{\nu}\right)\frac{\nu}{c}u}\right] \mathrm{d}u \\
&= \frac{\lambda}{c\nu}\left[\rho_\delta + \left(c - \frac{\lambda}{\nu}\right)\frac{\nu}{c}\right]^{-i}.
\end{aligned}$$

Hence, (5.15) simplifies to

$$\begin{aligned}
\eta_{0,i} &= \int_0^\infty \frac{u^{i-1} e^{-\rho_\gamma u}}{(i-1)!} \left[\sum_{j=1}^n B_j \psi_0(u)\, \kappa^{-j} + \sum_{j=1}^n B_j^* \psi_0(u) (\rho_\gamma+\nu)^{-j}\right] \mathrm{d}u \\
&\quad + \sum_{j=1}^n \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^* \int_0^\infty \frac{u^{i-1} e^{-\rho_\gamma u}}{(i-1)!} \psi_0(u)\, \nu (\rho_\gamma+\nu)^{-k}\, \mathrm{d}u\right) \eta_{0,j} \\
&= \frac{\lambda}{c\nu}\left[\rho_\gamma + \left(c - \frac{\lambda}{\nu}\right)\frac{\nu}{c}\right]^{-i} \sum_{j=1}^n \left[B_j \kappa^{-j} + B_j^* (\rho_\gamma+\nu)^{-j}\right] \\
&\quad + \frac{\lambda}{c}\left[\rho_\gamma + \left(c - \frac{\lambda}{\nu}\right)\frac{\nu}{c}\right]^{-i} \sum_{j=1}^n B_{j+k-1}^* \eta_{0,j} \sum_{k=1}^{n-j+1} (\rho_\gamma+\nu)^{-k}, \quad i = 1, 2, \ldots, n. \qquad (5.20)
\end{aligned}$$

Finally, with (5.20) in hand, the required probability is given by

$$
\begin{aligned}
\mathbb{P}_u\left(\zeta_d^n<\infty\right) &= \sum_{j=1}^{n} B_j \psi_0(u)\,\kappa^{-j} + \sum_{j=1}^{n} B_j^* \psi_0(u)\left(\rho_\gamma+\nu\right)^{-j} \\
&\quad + \sum_{j=1}^{n}\left(\sum_{k=1}^{n-j+1} B_{j+k-1}^* \nu\left(\rho_\delta+\nu\right)^{-k}\left[\frac{\lambda}{c\nu} e^{-\left(c-\frac{\lambda}{\nu}\right)\frac{\nu}{c}u}\right]\right)\eta_{0,j} \\
&= \frac{\lambda}{c\nu} e^{-\left(c-\frac{\lambda}{\nu}\right)\frac{\nu}{c}u}\left\{\sum_{j=1}^{n}\left[B_j\kappa^{-j} + B_j^*\left(\rho_\gamma+\nu\right)^{-j}\right]\right. \\
&\qquad\qquad \left. \nu\sum_{j=1}^{n} B_{j+k-1}^*\eta_{0,j}\sum_{k=1}^{n-j+1}\left(\rho_\delta+\nu\right)^{-k}\right\}.
\end{aligned}
\tag{5.21}
$$

The following procedure summarizes the steps in evaluating the modified Parisian ruin probability given in (5.21).

- Step 1: calculate ρ_γ, κ and ϕ from (5.3), (5.9) and (5.18).
- Step 2: calculate $\left\{B_j^*\right\}_{j=1,\ldots,n}$ and $\left\{B_j\right\}_{j=1,\ldots,n}$ from (5.17) and (5.19).
- Step 3: solve the system of linear equations in (5.20) for $\left\{\eta_{0,i}\right\}_{i=1,\ldots,n}$.
- Step 4: substitute the constants in the above steps into (5.21).

To facilitate the comparison of results, parameters are chosen to be consistent with that in Landriault et al. [2014]. In another words, claims are assumed to arrive at a rate $\lambda = 1/3$ and claim sizes are exponentially distributed with parameter $\nu = 1/9$. Premiums are collected at a rate of $c = 4$ per unit time. Tables 5.1 and 5.2 summarize the calculated modified Parisian ruin probability (i.e. $\mathbb{P}_.\left(\zeta_d^n<\infty\right)$) for an initial surplus of $u = 0$ and $u = 50$ respectively with different grace periods. Results for the traditional Parisian ruin probability (i.e. $\mathbb{P}_.\left(\rho_{0,d}^n<\infty\right)$) in Landriault et al. [2014] are also reproduced here for easy comparison. As a side note, $\mathbb{P}_0\left(\tau_0^-<\infty\right) = 0.7500$ and $\mathbb{P}_{50}\left(\tau_0^-<\infty\right) = 0.1870$.

The following phenomenon are observed across the tables.

n	$d = 1$		$d = 2$		$d = 5$		$d = 10$	
	traditional	modified	traditional	modified	traditional	modified	traditional	modified
1	0.6886	0.6886	0.6478	0.6478	0.5676	0.5676	0.4867	0.4867
5	0.6767	0.6786	0.6195	0.6275	0.5020	0.5322	0.3879	0.4423
10	0.6748	0.6770	0.6144	0.6241	0.4910	0.5273	0.3737	0.4370
15	0.6741	0.6764	0.6126	0.6229	0.4873	0.5257	0.3690	0.4353
20	0.6737	0.6761	0.6117	0.6223	0.4854	0.5250	0.3667	0.4344
25	0.6735	0.6759	0.6112	0.6219	0.4842	0.5245	0.3653	0.4339
30	0.6733	0.6758	0.6108	0.6217	0.4835	0.5242	0.3644	0.4336
35	0.6732	0.6757	0.6105	0.6215	0.4829	0.5240	0.3637	0.4333
40	0.6732	0.6756	0.6103	0.6214	0.4825	0.5238	0.3633	0.4331
45	0.6731	0.6755	0.6102	0.6213	0.4822	0.5237	0.3629	0.4330
50	0.6731	0.6755	0.6100	0.6212	0.4820	0.5236	0.3626	0.4329
∞	0.6726	0.6751	0.6089	0.6205	0.4797	0.5227	0.3598	0.4319

Table 5.1: Different Parisian ruin probabilities when $u = 0$.

n	$d=1$		$d=2$		$d=5$		$d=10$	
	traditional	modified	traditional	modified	traditional	modified	traditional	modified
1	0.1717	0.1717	0.1615	0.1615	0.1415	0.1415	0.1213	0.1213
5	0.1687	0.1692	0.1545	0.1565	0.1252	0.1327	0.0967	0.1103
10	0.1683	0.1688	0.1532	0.1556	0.1224	0.1315	0.0932	0.1090
15	0.1681	0.1687	0.1528	0.1553	0.1215	0.1311	0.0920	0.1085
20	0.1680	0.1686	0.1525	0.1552	0.1210	0.1309	0.0914	0.1083
25	0.1679	0.1685	0.1524	0.1551	0.1207	0.1308	0.0911	0.1082
30	0.1679	0.1685	0.1523	0.1550	0.1206	0.1307	0.0909	0.1081
35	0.1679	0.1685	0.1522	0.1550	0.1204	0.1306	0.0907	0.1081
40	0.1679	0.1685	0.1522	0.1549	0.1203	0.1306	0.0906	0.1080
45	0.1679	0.1684	0.1521	0.1549	0.1202	0.1306	0.0905	0.1080
50	0.1678	0.1684	0.1521	0.1549	0.1202	0.1305	0.0904	0.1079
∞	0.1677	0.1683	0.1518	0.1547	0.1196	0.1303	0.0897	0.1077

Table 5.2: Different Parisian ruin probabilities when $u = 50$.

- When $n = 1$, the two Parisian ruin probabilities are the same. This is a direct consequence of the strong Markov property.

- The inequality $\mathbb{P}_u\left(\rho_{0,d}^{\infty} < \infty\right) \leq \mathbb{P}_u\left(\zeta_d^{\infty} < \infty\right) \leq \mathbb{P}_u\left(\tau_0^- < \infty\right)$ holds for all the demonstrated values to u and d. This is consistent with the result stated in Proposition 30.

- The modified Parisian ruin probabilities is decreasing in d. This is in line with the intuition that the longer the grace period, the more likely the business can survive the regulatory check and thereby lowering the probability of ruin.

- The difference between the two Parisian ruin probabilities is increasing in d, which translates to the interpretation that the longer the grace period, the more conservative the modified Parisian ruin time when compared to the traditional Parisian ruin time. This makes sense because according to the traditional definition of Parisian ruin, the business is said to be recovered as long as the surplus climbs back to a positive level within the grace period. Yet, with the modified definition of Parisian ruin, the business has to be consistently at a positive level through the grace period so as to survive the regulatory check in the end. The longer the grace period, the more difficult for the business to sustain a momentum for positive surplus.

5.6 Appendix

5.6.1 Proof of Theorem 28

By conditioning on $\tau_{0,1}^-$, the first time a negative surplus is observed such that a regulatory check is called upon, and revoking the definition of discounted density function introduced in Subsection 2.3.2, we arrive at

$$
\begin{aligned}
\phi_{d,\delta}^{n}(u) = \int_0^\infty \bigg[& \int_0^\infty w(y+z)\, g_{\delta,+}(z)\,\mathrm{d}z + \int_0^y w(y-z)\, g_{\delta,-}(z)\,\mathrm{d}z \\
& + \int_y^\infty \phi_{d,\delta}^{n}(z-y)\, g_{\delta,-}(z)\,\mathrm{d}z \bigg] h_\delta(y|u)\,\mathrm{d}y. \qquad (5.22)
\end{aligned}
$$

To simplify the above equation, we further study the integrals one by one. Using (5.11), the first two double integrals in (5.22) can be expressed as

$$
\begin{aligned}
& \int_0^\infty \left[\int_0^\infty w(y+z)\, g_{\delta,+}(z)\,\mathrm{d}z + \int_0^y w(y-z)\, g_{\delta,-}(z)\,\mathrm{d}z \right] h_\delta(y|u)\,\mathrm{d}y \\
&= \int_0^\infty \left[\int_y^\infty w(z)\, g_{\delta,+}(z-y)\,\mathrm{d}z \right] h_\delta(y|u)\,\mathrm{d}y + \int_0^\infty \left[\int_0^y w(z)\, g_{\delta,-}(y-z)\,\mathrm{d}z \right] h_\delta(y|u)\,\mathrm{d}y \\
&= \int_0^\infty w(z) \left[\int_0^z g_{\delta,+}(z-y)\, h_\delta(y|u)\,\mathrm{d}y + \int_z^\infty g_{\delta,-}(y-z)\, h_\delta(y|u)\,\mathrm{d}y \right] \mathrm{d}z \\
&= \int_0^\infty w(z)\, \tau_\delta(z|u)\,\mathrm{d}z. \qquad (5.23)
\end{aligned}
$$

Meanwhile, with the help of (5.5), the third integral inside the square bracket of (5.22) can be expressed as

$$
\begin{aligned}
& \int_y^\infty \phi_{d,\delta}^{n}(z-y)\, g_{\delta,-}(z)\,\mathrm{d}z \\
&= \sum_{j=1}^{n} \frac{B_j^*}{(j-1)!} \int_0^\infty \phi_{d,\delta}^{n}(z)\,(y+z)^{j-1}\, e^{-\rho_\gamma(y+z)}\mathrm{d}z \\
&= \sum_{j=1}^{n} \frac{B_j^*}{(j-1)!} \sum_{k=1}^{j} \begin{pmatrix} j-1 \\ k-1 \end{pmatrix} \left(\int_0^\infty z^{j-k} e^{-\rho_\gamma z} \phi_{d,\delta}^{n}(z)\,\mathrm{d}z \right) y^{k-1} e^{-\rho_\gamma y} \\
&= \sum_{k=1}^{n} \frac{y^{k-1} e^{-\rho_\gamma y}}{(k-1)!} \sum_{j=k}^{n} B_j^* \int_0^\infty \frac{z^{j-k} e^{-\rho_\gamma z}}{(j-k)!} \phi_{d,\delta}^{n}(z)\,\mathrm{d}z \\
&= \sum_{k=1}^{n} \frac{y^{k-1} e^{-\rho_\gamma y}}{(k-1)!} \sum_{j=1}^{n-k+1} B_{j+k-1}^* \int_0^\infty \frac{z^{j-1} e^{-\rho_\gamma z}}{(j-1)!} \phi_{d,\delta}^{n}(z)\,\mathrm{d}z \\
&= \sum_{j=1}^{n} \left(\int_0^\infty \frac{z^{j-1} e^{-\rho_\gamma z}}{(j-1)!} \phi_{d,\delta}^{n}(z)\,\mathrm{d}z \right) \sum_{k=1}^{n-j+1} B_{j+k-1}^* \frac{y^{k-1} e^{-\rho_\gamma y}}{(k-1)!},
\end{aligned}
$$

and therefore, with the help of (5.13), the last double integral in (5.22) can be expressed as

$$\begin{aligned}
&\int_0^\infty \left[\int_y^\infty \phi_{d,\delta}^n (z-y)\, g_{\delta,-}(z)\, \mathrm{d}z\right] h_\delta(y|u)\, \mathrm{d}y \\
&= \sum_{j=1}^n \left(\int_0^\infty \frac{z^{j-1}e^{-\rho_\gamma z}}{(j-1)!}\phi_{d,\delta}^n(z)\, \mathrm{d}z\right) \sum_{k=1}^{n-j+1} B_{j+k-1}^* \int_0^\infty \frac{y^{k-1}e^{-\rho_\gamma y}}{(k-1)!} h_\delta(y|u)\, \mathrm{d}y \\
&= \sum_{j=1}^n \eta_{\delta,j} \sum_{k=1}^{n-j+1} B_{j+k-1}^* \varphi_{\delta,k}(u), \qquad (5.24)
\end{aligned}$$

where

$$\eta_{\delta,j} = \int_0^\infty \frac{z^{j-1}e^{-\rho_\gamma z}}{(j-1)!}\phi_{d,\delta}^n(z)\, \mathrm{d}z, \quad j = 1,2,\ldots,n. \qquad (5.25)$$

Substituting (5.23) and (5.24) into (5.22) gives

$$\phi_{d,\delta}^n(u) = \chi_{\delta,w}(u) + \sum_{j=1}^n \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^* \varphi_{\delta,k}(u)\right)\eta_{\delta,j} \qquad (5.26)$$

due to (5.10).

It remains to show that $\{\eta_{\delta,j}\}_{j=1,\ldots,n}$ in (5.25) satisfies the system of linear equations specified in (5.15). To do so, it is instructive to note that the expression (5.26) still contains the quantity $\{\eta_{\delta,j}\}_{j=1,\ldots,n}$ defined via (5.25). Hence, back substituting (5.26) into (5.25) gives

$$\begin{aligned}
\eta_{\delta,i} &= \int_0^\infty \frac{u^{i-1}e^{-\rho_\gamma u}}{(i-1)!}\chi_{\delta,w}(u)\, \mathrm{d}u + \sum_{j=1}^n \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^* \int_0^\infty \frac{u^{i-1}e^{-\rho_\gamma u}}{(i-1)!}\varphi_{\delta,k}(u)\, \mathrm{d}u\right)\eta_{\delta,j} \\
&= \int_0^\infty w(z)\Delta_{\delta,i}(z)\, \mathrm{d}z + \sum_{j=1}^n \left(\sum_{k=1}^{n-j+1} B_{j+k-1}^* \int_0^\infty \frac{u^{i-1}e^{-\rho_\gamma u}}{(i-1)!}\varphi_{\delta,k}(u)\, \mathrm{d}u\right)\eta_{\delta,j}, \quad i = 1,2,\ldots,n
\end{aligned}$$

due to (5.12).

5.6.2 Proof of Corollary 29

Applying (5.23) and (5.16) to express (5.26) gives

$$\phi_{d,\delta}^n(u) = \int_0^\infty w(z)\, \tau_\delta(z|u)\, \mathrm{d}z + \boldsymbol{\sigma}_\delta(u)\left(\boldsymbol{I} - \boldsymbol{\Gamma}_\delta\right)^{-1}\int_0^\infty w(z)\, \boldsymbol{\Delta}_\delta(z)\, \mathrm{d}z, \quad u \geq 0.$$

Hence, the result follows directly by comparing the above expression with (5.14).

Chapter 6

Conclusion and Future Works

To summarize, Chapter 3 has provided a new methodology in analyzing ruin quantities under the spectrally negative Lévy context by introducing the hybrid observation scheme. The advantage of such method has been especially emphasized. The main contribution lies in illustrating how such observation scheme unifies the analysis for both bounded and unbounded variation case. It has also been demonstrated that Parisian ruin probability analyzed this way is indeed consistent with existing literature. As a byproduct, Laplace transform to the Parisian ruin time is also derived.

In Chapter 4, the idea of Poissonian observation inferred from the hybrid observation scheme is further leveraged under the spectrally negative Lévy setting by developing more tools for surplus analysis via defining the Poissonian potential measures, a natural extension to the classical potential measures. The main contribution lies in finding an explicit expressions to these Poissonian potential measures. Meanwhile, interesting relations between Poissonian potential measure and Poissonian exit measures are also observed, which further highlight the importance and usefulness of Poissonian potential measures.

Chapter 5 is devoted to the introduction of a modified Parisian ruin concept motivated from the regulatory practice. The merits of such modified Parisian ruin, comparing to the traditional Parisian ruin concept, is discussed with respect to the risk management context. As an initial attempt, analysis are performed under a Cramér–Lundberg setting. The main contribution lies in obtaining the Gerber–Shiu type function and the discounted density to ruin quantities pertained to the modified Parisian ruin time. An ordering property to different ruin probabilities is also developed, and an interpretation to such result is provided.

The above works can indeed be extended in several directions. With the limited time over the study period, they could only be left as future works. While some are closely related to the previous chapters, few of them are less related, though still share similarities with the questions studied. In what follows, some potential future works are proposed.

In Chapter 5, a Cramér–Lundberg model is considered. A possible extension would be to analysis the same ruin quantity under the spectrally negative Lévy context. While the analysis heavily relies on (5.2) under the Cramér–Lundberg setting, it is noted that such quantity is not known under the spectrally negative Lévy setting. To make the analysis possible, an expression for the quantity $\mathbb{E}\left[e^{-\delta T-sX_T}\right]$ (where T represents an Erlang(n) random variable), or more generally speaking, the joint law of (T, X_T) is necessary. We remark the concept of potential measure introduced in Subsection 2.2.4, a potential starting point for this problem could be to obtain an expression for $\mathbb{P}_x\left(X_T \in \mathrm{d}y\right)$, which could be seen to be a pivotal quantity for analysis in relation to Erlangization under the spectrally negative Lévy context.

Parallel to the idea of modified Parisian ruin introduced in Chapter 5, a more generalized setting of the modified Parisian ruin could be considered. In particular, a multistage regulatory checking scheme could be imposed. In brief, it means there are several evenly spaced check points spread along the grace period. Depending on the risk appetite of the company or the requirements set by the law, recovery (and hence, ruin) may take different definitions. As an example, ruin could be defined as the first instant that the business fails an inspection at one of the checkings in the grace period. Alternatively, ruin may take the definition that the business fails all the inspections in the grace period. Analyzing the law of the risk quantities pertained to such definitions of ruin would be an interesting research direction as they might provide more insights in risk management. However, it appears that this could be a challenging task, particularly on the study of the second ruin quantity since this requires keeping track of the retrospective records.

Reference

H. Albrecher and J. Ivanovs. A risk model with an observer in a markov environment. *Risks*, 1(3):148–161, 2013.

H. Albrecher and J. Ivanovs. Strikingly simple identities relating exit problems for lévy processes under continuous and poisson observations. *Stochastic Processes and their Applications*, 2016. doi: http://dx.doi.org/10.1016/j.spa.2016.06.021.

H. Albrecher, E. C. K. Cheung, and S. Thonhauser. Randomized observation periods for the compound poisson risk model: dividends. *Astin Bulletin*, 41(2):645–672, 2011.

H. Albrecher, E. C. K. Cheung, and S. Thonhauser. Randomized observation periods for the compound poisson risk model: the discounted penalty function. *Scandinavian Actuarial Journal*, 2013(6):424–452, 2013.

H. Albrecher, J. Ivanovs, and X. Zhou. Exit identities for lévy processes observed at poisson arrival times. *Bernoulli*, 22(3):1364–1382, 2016.

L. Alili and A. E. Kyprianou. Some remarks on first passage of lévy processes, the american put and pasting principles. *The Annals of Applied Probability*, 15(3):2062–2080, 2005.

B. Avanzi, E. C. K. Cheung, B. Wong, and J.-K. Woo. On a periodic dividend barrier strategy in the dual model with continuous monitoring of solvency. *Insurance: Mathematics and Economics*, 52(1):98–113, 2013.

F. Avram, A. E. Kyprianou, and M. R. Pistorius. Exit problems for spectrally negative lévy processes and applications to (canadized) russian options. *The Annals of Applied Probability*, 14(1):215–238, 2004.

E. J. Baurdoux, J. C. Pardo, J. L. Pérez, and J. F. Renaud. Gerber-shiu distribution at parisian ruin for lévy insurance risk processes. *Journal of Applied probability*, 53(2):572–584, 2016.

J. Bertoin. Lévy processes, vol. 121 of. *Cambridge Tracts in Mathematics*, 1996.

J. Bertoin. Exponential decay and ergodicity of completely asymmetric lévy processes in a finite interval. *The Annals of Applied Probability*, 7(1):156–169, 1997.

E. Biffis and A. E. Kyprianou. A note on scale functions and the time value of ruin for lévy insurance risk processes. *Insurance: Mathematics and Economics*, 46(1):85–91, 2010.

M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and parisian barrier options. *Advances in Applied Probability*, 29(1):165–184, 1997.

E. C. K. Cheung and J. T. Y. Wong. On the dual risk model with parisian implementation delays in dividend payments. *European Journal of Operational Research*, 257(1):159–173, 2017.

M. C. H. Choi and E. C. K. Cheung. On the expected discounted dividends in the cramér–lundberg risk model with more frequent ruin monitoring than dividend decisions. *Insurance: Mathematics and Economics*, 59:121–132, 2014.

H. Cramér. *Collective Risk Theory.* Jubilee volume of Forsakringsbolaget Skandia, Stockholm, 1955.

I. Czarna. Parisian ruin probability with a lower ultimate bankrupt barrier. *Scandinavian Actuarial Journal*, 2016(4):319–337, 2016.

I. Czarna and Z. Palmowski. Ruin probability with parisian delay for a spectrally negative lévy risk process. *Journal of Applied Probability*, 48(4):984–1002, 2011.

I. Czarna and Z. Palmowski. Dividend problem with parisian delay for a spectrally negative lévy risk process. *Journal of Optimization Theory and Applications*, 161(1):239–256, 2014.

I. Czarna, Z. Palmowski, and P. Światek. Binomial discrete time ruin probability with parisian delay. *arXiv preprint arXiv:1403.7761*, 2014.

A. Dassios and S. Wu. Parisian ruin with exponential claims. 2008.

A. Dassios and S. Wu. On barrier strategy dividends with parisian implementation delay for classical surplus processes. *Insurance: Mathematics and Economics*, 45(2):195–202, 2009.

A. Dassios and S. Wu. Perturbed brownian motion and its application to parisian option pricing. *Finance and Stochastics*, 14(3):473–494, 2010.

L. Debnath and D. Bhatta. *Integral transforms and their applications*. CRC press, 2015.

D. C. M. Dickson and A. D. Egídio dos Reis. On the distribution of the duration of negative surplus. *Scandinavian Actuarial Journal*, 1996(2):148–164, 1996.

D. C. M. Dickson and G. E. Willmot. The density of the time to ruin in the classical poisson risk model. *Astin Bulletin*, 35(01):45–60, 2005.

A. D. Egídio dos Reis. How long is the surplus below zero? *Insurance: Mathematics and Economics*, 12(1):23–38, 1993.

G. Fusai. Corridor options and arc-sine law. *Annals of Applied Probability*, 10(2):634–663, 2000.

H. U. Gerber. When does the surplus reach a given target? *Insurance: Mathematics and Economics*, 9(2):115–119, 1990.

H. U. Gerber and E. S. W. Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–72, 1998.

H. U. Gerber and E. S. W. Shiu. The time value of ruin in a sparre andersen model. *North American Actuarial Journal*, 9(2):49–69, 2005.

H. U. Gerber, E. S. Shiu, and H. Yang. The omega model: from bankruptcy to occupation times in the red. *European Actuarial Journal*, 2(2):259–272, 2012.

C. Kluppelberg, A. E. Kyprianou, and R. A. Maller. Ruin probabilities and overshoots for general lévy insurance risk processes. *The Annals of Applied Probability*, 14(4):1766–1801, 2004.

A. Kuznetsov, A. E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative lévy processe. pages 97–186. Springer Berlin Heidelberg, 2012.

A. E. Kyprianou. *Fluctuations of Lévy Processes with Application: Introductory Lectures*. Springer Science and Business Media, 2014.

A. E. Kyprianou, J. C. Pardo, and J. L. Pérez. Occupation times of refracted lévy processes. *Journal of Theoretical Probability*, 27(4):1292–1315, 2014.

D. Landriault and T. Shi. Occupation times in the map risk model. *Insurance: Mathematics and Economics*, 60:75–82, 2015.

D. Landriault and G. E. Willmot. On the joint distributions of the time to ruin, the surplus prior to ruin, and the deficit at ruin in the classical risk model. *North American Actuarial Journal*, 13(2):252–270, 2009.

D. Landriault, J.-F. Renaud, and X. Zhou. Occupation times of spectrally negative lévy processes with applications. *Stochastic processes and their applications*, 121(11):2629–2641, 2011.

D. Landriault, J.-F. Renaud, and X. Zhou. An insurance risk model with parisian implementation delays. *Methodology and Computing in Applied Probability*, 16(3):583–607, 2014.

B. Li and X. Zhou. The joint laplace transforms for diffusion occupation times. *Advances in Applied Probability*, 45(04):1049–1067, 2013.

B. Li, Q. Tang, L. Wang, and X. Zhou. Liquidation risk in the presence of chapters 7 and 11 of the us bankruptcy code. *Journal of Financial Engineering*, 1(03):1450023, 2014.

S. Li and J. Garrido. The Gerber–Shiu function in a Sparre Andersen risk process perturbed by diffusion. *Scandinavian Actuarial Journal*, 2005(3):161–186, 2005.

X. S. Lin and K. P. Pavlova. The compound poisson risk model with a threshold dividend strategy. *Insurance: mathematics and Economics*, 38(1):57–80, 2006.

X. S. Lin and G. E. Willmot. Analysis of a defective renewal equation arising in ruin theory. *Insurance: Mathematics and Economics*, 25(1):63–84, 1999.

X. S. Lin and G. E. Willmot. The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. *Insurance: Mathematics and Economics*, 27(1):19–44, 2000.

V. Linetsky. Step options. *Mathematical Finance*, 9(1):55–96, 1999.

R. Loeffen. On obtaining simple identities for overshoots of spectrally negative lévy processes. *arXiv preprint arXiv:1410.5341*, 2014.

R. Loeffen, I. Czarna, and Z. Palmowski. Parisian ruin probability for spectrally negative lévy processes. *Bernoulli*, 19(2):599–609, 2013.

R. L. Loeffen, J.-F. Renaud, and X. Zhou. Occupation times of intervals until first passage times for spectrally negative lévy processes. *Stochastic Processes and their Applications*, 124(3):1408–1435, 2014.

L. Mejlbro. *The Laplace Transformation I – General Theory. Complex Functions Theory a–4*. Bookboon, 2010.

J.-F. Renaud. On the time spent in the red by a refracted lévy risk process. *Journal of Applied Probability*, 51(4):1171–1188, 2014.

M. Schröder. Brownian excursions and parisian barrier options: a note. *Journal of Applied Probability*, 40(4):855–864, 2003.

B. A. Surya. Evaluating scale functions of spectrally negative lévy processes. *Journal of Applied Probability*, 45(1):135–149, 2008.

C. C. L. Tsai. On the discounted distribution functions of the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics*, 28(3):401–419, 2001.

J. T. Y. Wong and E. C. K. Cheung. On the time value of parisian ruin in (dual) renewal risk processes with exponential jumps. *Insurance: Mathematics and Economics*, 65:280–290, 2015.

H. Yang and L. Zhang. Spectrally negative lévy processes with applications in risk theory. *Advances in Applied Probability*, 33(1):281–291, 2001.

C. Zhang and R. Wu. Total duration of negative surplus for the compound poisson process that is perturbed by diffusion. *Journal of applied probability*, 39(3):517–532, 2002.

Z. Zhang and E. C. K. Cheung. A note on a lévy insurance risk model under periodic dividend decisions. *Journal of Industrial Management Optimization*, 14(1):35–63, 2018.