Sequences of Trees and Higher-Order Renormalization Group Equations

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

We gratefully acknowledge that the work of Section 4 is the intellectual property of Loïc Foissy. The content therein has been expressed in the present author's own words for the purpose of exposition.

Abstract

In 1998, Connes and Kreimer introduced a combinatorial Hopf algebra $\mathcal{H}_{CK}$ on the vector space of forests of rooted trees that precisely explains the phenomenon of renormalization in quantum field theory. This Hopf algebra has been of great interest since its inception, as it connects the disciplines of algebra, combinatorics, and physics, providing interesting questions in each.

In this thesis we introduce the notion of *higher-order renormalization group equations*, which generalize the usual renormalization group equation of quantum field theory, and further define a corresponding notion of *order* on certain sequences of trees constituting elements of the completion of $\mathcal{H}_{CK}$. We also give an explication of a result, due to Foissy, that characterizes which sequences of linear combinations of trees with one generator in each degree generate Hopf subalgebras of $\mathcal{H}_{CK}$. We conclude with some results towards classifying these sequences by their order (when such an order is admitted), and discuss the place of the Connes-Moscovici Hopf subalgebra in the context of this new framework.

Acknowledgements

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Thanks also to Joshua Ellis for the creation and maintenance of *tikz-feynman* [16], which we utilize throughout for the illustration of Feynman diagrams.

Finally, an infinity of thanks to Jason Bell and David Wagner for agreeing to read this thesis and for providing insightful feedback, and $\aleph_2$-many thanks to my thesis adviser Karen Yeats, for training me in the ancient art of mathematics, and for being the best possible mentor in every possible way. I hope that the body of my life's work will be commensurate with your investment in it.

Dedication

To my mom and dad.
For everything.

Table of Contents

List of Tables

List of Figures

1 Introduction

1.1 Overview

Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes,' he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wavefunction.' I said to him, 'You're crazy.' But he wasn't. [44] (Originally in [42]).

$\sim$ F.J. Dyson

In 1998, Connes and Kreimer [9] developed a combinatorial Hopf algebra on the vector space of forests of rooted trees that has become influential in the world of quantum field theory. The idea for the Hopf algebra was born out of the realization that the collection of Feynman diagrams representing various particle interactions posses a Hopf algebraic structure, with respect to which the antipode explains the process of renormalization [9, 10, 11]. This interpretation of renormalization as the antipode of a Hopf algebra has spurred further progress in both the development of theory and computational techniques in quantum field theory [7, 36, 15]. Quite a few surveys on the Connes-Kreimer Hopf algebra may be found, for example [36, 30, 15], while the object continues to be of interest physically, algebraically, and combinatorially to present day.

What is renormalization in quantum field theory? While the full presentation of renormalization is beyond the scope of this text, we will discuss the process in slightly more detail later on (in Section 2.3), and will start by giving an intuition for the idea here. For a particle physicist performing experiments, one of the main objectives is to predict the results of an experiment run in a particle accelerator. The idealized particle accelerator experiment can be pictured in an overly-simplified schematic as follows:

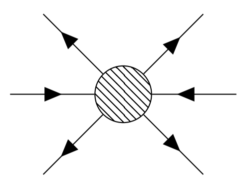


Figure 1: A diagrammatic view of a particle accelerator experiment.

Here we have particles being sent into an accelerator, interactions happening (or not happening), and then particles coming out (scattering). The probability that the particles coming out will be of a certain form knowing the particles going in is given by a quantity known as the **scattering amplitude**, and this is the quantity we wish to calculate in order to make predictions. The idea here is that we do not know what happens when the particles are in the accelerator (depicted as the shaded blob in the figure)—anything may happen with a certain probability. Hence, as stated in the words of F.J. Dyson above, we consider the sum over all possibilities ("histories"), with each possible event weighted by its own amplitude. Each of these individual histories is represented by its own schematic—a **Feynman diagram**—a graph theoretical object with vertices representing particle interactions and edges representing propagating particles (see Section 2.3.1) [49]. Hence the scattering amplitude that we ultimately desire can be expanded as a weighted sum over Feynman diagrams, with the weight of each diagram representing the contribution that that specific history makes to the overall amplitude. This expansion of the scattering amplitude into smaller calculations is known as the scattering amplitude's **perturbative expansion** [49].

A major problem arising with the above idea, however, is that even after expanding the scattering amplitude calculation in this way, we still find that the contributing amplitudes corresponding to the individual Feynman diagrams in the sum are (most often) divergent. This leaves the expressions for these integrals without meaning (as well as their sum). To solve this problem, physicists have developed a procedure by which meaningful quantities may still be extracted from the expressions, a procedure known as **renormalization**. There are in fact many different ways renormalization is carried out (known as **renormalization schemes**), though in this thesis we will think of renormalization as being performed using the **BPHZ subtraction scheme**[1]. One can think of the BPHZ scheme intuitively as follows: for Feynman diagrams with the simplest divergence structure, the corresponding integrals representing the amplitude may be convergent or may be made to converge by subtracting off a single term from the integrand. However, larger Feynman diagrams may contain multiple copies of these smaller, divergent Feynman diagrams as subgraphs, and in fact these smaller graphs may be further nested inside of each other as subgraphs. Consequently the corresponding Feynman integral will be divergent, but it will not be possible to make the integral convergent by a single subtraction—the divergent factors will be nested in the same way that the subgraphs are nested within a graph. This will call for a recursive subtraction procedure, the BPHZ scheme. In this way, by saying above that the Connes-Kreimer Hopf algebra models renormalization, we mean that the rooted tree structures of elements in the Hopf algebra model the subdivergence structure of Feynman diagrams, and that applying the Hopf algebra's antipode corresponds to performing the BPHZ subtraction scheme. Further details pertaining to subtraction schemes and fully worked-out examples may be found in [49, 40]. Examples of Feynman diagrams and how to obtain rooted trees from their subdivergence structure are given in Sections 2.3.1 and 2.3.2.

For us, the three most important objects arising from renormalization will be the notion of **Green's functions** (see Section 2.3.3), in addition to the **renormalization group** and resulting **renormalization group equation** (see Section 2.3.4). Colloquially speaking, one can think of the renormalization group as the mathematical tool enabling physicists to change their frame of reference with respect to particle interactions, thereby "zooming in" or "zooming out" to obtain more meaningful results. In order to implement the BPHZ scheme, one first needs to make a choice as to a constant known as the **reference scale** of the renormalization process. Elements of the renormalization group then represent transformations from one reference scale in the renormalization scheme to another, and can be thought of as coming from evaluations of Hopf algebra characters from the Connes-Kreimer Hopf algebra to the Hopf algebra of polynomials in some variable L over the underlying field of the Hopf algebra [32]. In other words, changes in the value of L, the **kinematic variable**, will cause changes in the Hopf character, and hence changes in the reference scale being used in the renormalization scheme. We will see in Section 2.3.4 that the renormalization group equation encodes the group law of the renormalization group, and philosophically speaking explains how different reference scales within the family of BPHZ prescription relate to one another; see Appendix A.4 of [32].

One may realize from the above description that this means the renormalization group is in fact related to the Lie group of convolution from the Connes-Kreimer Hopf algebra over a field $\mathbb{K}$ to $\mathbb{K}[L]$; the explanation of this fact is the content of Section 2.2. What's more, the renormalization group equation in this setting becomes a simple algebraic statement to the effect of [32]:

$$\phi_{L_1} * \phi_{L_2} = \phi_{L_1+L_2} \tag{1}$$

[1] BPHZ is the concatenation of the initials of Bogoliubov, Parasiuk, Hepp, and Zimmerman, who developed the scheme [49].

with ϕ_L an element of the renormalization group, $*$ the convolution product, and L_1 and L_2 representing two values of L. This Hopf-algebraic equation translates into the following analytic equation (see Appendix A.5. of [32]):

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma(x)\right)G(x, L) = 0 \tag{2}$$

Here, L is the same kinematic variable as before, x is a **coupling constant** (see Section 2.3), G is a **Green's function** in x and L, and with $\beta(x)$ and $\gamma(x)$ series in the given theory; see Section 2.3 for a brief overview and [27] for a more thorough treatment.

Now to the discrete mathematician, a differential equation of the form (2) probably seems out of place. Nevertheless, it turns out equation (2) is very significant combinatorially, as it translates (in the setting of the Connes-Kreimer Hopf algebra of trees) to a statement in terms of formal power series: namely, we can think of x as a formal counting variable, $\beta(x)$ and $\gamma(x)$ as two formal power series, and $G(x, L)$ as a generating function weighted by linear combinations of forests homogeneous of a given size, and obtained by applying an element of the renormalization group—known as **Feynman rules**—to a particular element of the completion of the Connes-Kreimer Hopf algebra of trees. Said another way, $G(x, L)$ is a generating function obtained by applying Feynman rules to certain kinds of series of linear combinations of forests, such that the elements of the series generate a Hopf subalgebra of the Connes-Kreimer Hopf algebra. Moreover, throughout the course of this thesis we will restrict our attention somewhat further and assume that

1. the linear combinations only consist of trees (and not forests), and

2. there is exactly one generator in each degree.

This leads naturally to the following question: Is there a characterization of what sequences of linear combinations of trees with one generator in each degree generate a Hopf subalgebra of $\mathcal{H}_{CK}$? In Section 3.2 we demonstrate the nuances and main hurdles of this problem, and in Section 4 we give an explication of the problem's solution, due to Foissy [24]. While we will give a more formal definition in Section 4, we will refer to the set of all sequences of linear combinations of trees satisfying 1. and 2. above, and also generating Hopf subalgebras of $\mathcal{H}_{CK}$, as $\mathcal{S}eq$.

We remark, however, that not all sequences in $\mathcal{S}eq$ satisfy an equation of the form (2), possibly the most famous example of which is the usual sequence of generators of the Connes-Moscovici Hopf subalgebra of $\mathcal{H}_{CK}$ (see Section 5 of [20] and also Section 3.1 of the present text). This leads to the introduction of a generalization of equation (2), which we call a **higher-order renormalization group equation**. Namely, we consider equations of the form:

$$\left(\frac{\partial}{\partial L} + \overline{\beta}(x, \frac{\partial}{\partial x})\right)G(x, L) = 0 \tag{3}$$

where now $\overline{\beta}$ is polynomial in the partial derivative $\frac{\partial}{\partial x}$. If this polynomial is of degree n in $\frac{\partial}{\partial x}$, we say that the generalized renormalization group equation is of **order** n. By allowing for higher-derivatives of the variable x, sequences of trees that may not have satisfied (2) may be able to satisfy (3) (including, in particular, the Connes-Moscovici Hopf subalgebra), at least with certain choices of Feynman rules. This generalization is discussed in further detail in Section 3, and ultimately leads to the main focus of this project:

Q1: Is it possible to characterize the elements of $\mathcal{S}eq$ which satisfy (3) for arbitrary Feynman rules by the order of the generalized renormalization group equation which they satisfy?

We will call such sequences k**th-order sequences** if the Green's functions they give rise to satisfy a renormalization group equation of order k . A restatement of this problem and accompanying discussion can be found in Section 3. Moreover, while we are only interested in elements of $\mathcal{S}eq$ that satisfy an equation of the form (3) of some order, for completeness we will also give examples of sequences which do not admit an order in Section 4.2.

Finally, a followup to Q1 above is the following:

Q2: Is it possible to characterize the elements of $\mathcal{S}eq$ which satisfy (2) for at least one (nontrivial) choice of Feynman rules by the order of the generalized renormalization group equation which they satisfy?

We present a near-complete solution to Q1 in Sections 5.1 and 5.2, and report on partial progress made on Q2 in Section 5.3.

1.2 Organization

While we have already given some indication of the structure of this document in the preceding discussion, we will now state a more consolidated overview of what is to follow:

We will begin in Section 1.3 with a brief overview of the pertinent literature and key results in the field. From here, we will commence the mathematical narrative in Section 2, attempting to cover all relevant background material. This section is further broken down into three parts: an introduction to Hopf algebras in general and Hopf algebras over the vector spaces of forests of rooted trees specifically (Section 2.1); a discussion about the relation of our Hopf algebras to Lie theory (Section 2.2); and finally a formal treatment of renormalization and the renormalization group with accompanying examples (Section 2.3). We continue the story with further background material in Section 3 that is less common and more specific to our problem. In particular, this will include the development of higher-order renormalization group equations (Section 3.1), and a formal statement of the main objectives of this work (Section 3.2). In Section 4, we present an overview of the solution, due to Foissy, to the question about which sequences of linear combinations of trees are in $\mathcal{S}eq$. We then move on to the main question, Q1, and present a near-complete solution in Sections 5.1 and 5.2. We also present partial progress on question Q2 in Section 5.3 and then discuss the Connes-Moscovici Hopf subalgebra and how it fits into this framework. Finally, we conclude in Section 6 with a short recapitulation of topics discussed, a presentation of some interesting open problems (Section 6.1), and some final remarks.

Appendices follow after the conclusion of the thesis. Appendix A presents our implementation of the Connes-Kreimer Hopf algebra as a SageMath class, while Appendix B contains miscellaneous code, also in SageMath, that was helpful throughout the project. Meanwhile, Appendix C provides a list of all known elements of $\mathcal{S}eq$ that have an order, by their order (up to scaling).

1.3 Related Work

The literature surrounding the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ is vast, and in this section we give an overview of some of the most related works.

The inaugural paper relating Hopf algebras to renormalization in quantum field theory is "On the Hopf algebra structure of perturbative quantum field theories" [33] due to Kreimer in 1998. Further development of $\mathcal{H}_{CK}$, together with its connections to geometry, were then developed by Connes and Kreimer in [9],[10], and [11].

In addition to these foundational works, numerous surveys of various lengths have been written. A few of these include [15, 36, 18]. While this thesis is intended to be self-contained, the interested reader is directed to these sources for an excellent overview of the material at hand. We also recommend Hoffman's paper "Combinatorics of Rooted Trees and Hopf Algebras" [30] for a similarly-excellent introduction to all aspects of the algebraic side of this thesis. Indeed, while the connections to physics are left mostly unexplained, and while the paper fulfills a much greater role than that of an overview, Hoffman's construction of the Connes-Kreimer and Grossman-Larson Hopf algebras from the ground up is very organized and insightful.

On the more abstract side of this topic, we recommend [13] for a thorough introduction to the theory of Hopf algebras. Much of the background material we include in Section 1.1 can ultimately be found there. We also cite throughout from [4] and for information related to more general combinatorial Hopf algebras [26].

The central question in this work pertains to sequences of generators of Hopf subalgebras of $\mathcal{H}_{CK}$ and to their corresponding Hopf subalgebras, as discussed above, and much of the prior work in this specified domain has been pioneered by Foissy. Indeed, in [20, 22], Foissy describes a family of Hopf subalgebras coming from combinatorial Dyson-Schwinger equations which, as we will see later on, constitute a substantial portion of the strong 1st order sequences (Section 5.2). The fact that the generators of the Connes-Moscovici Hopf subalgebra (first described as a Hopf algebra in [8] and then as a Hopf subalgebra of $\mathcal{H}_{CK}$ in [9]) did not fit into this framework was part of the motivation for pursuing a more general framework as we are doing here (see the remark at the end of Section 5 in [20]). Moreover in Section 4.2 of [22], Foissy gives a description of a sequence Y generating a Hopf algebra whose dual is isomorphic to the third FdB Lie algebra (also defined in that paper)[2]. The results we present in Theorem 5.5 characterizing the strong 0th order elements of $\mathcal{S}eq$ are in fact a generalization of Foissy's sequence Y. In particular, Y arises from Theorem 5.5 in the case when $n = 2$ and $b = 1$. Likewise the dual of the corolla Lie algebra in Section 4.1 of [22] is of course the sequence of corollas, obtained from Theorem 5.5 in the case where $n = b = 1$.

It is also possible for Hopf subalgebras generated by a sequence of nonzero linear combinations of trees to exist that do not have an associated order. Developed for other reasons, one such family of sequences with this property was also considered by Foissy in [19], in which he develops the notion of Com-PreLie bialgebras (a commutative bialgebra that contains an additional prelie product that is compatible with the product and coproduct of the bialgebra in a prescribed way) [19]. In this paper he gives combinatorial interpretations of various kinds of Com-PreLie bialgebras and explains that these structures are in one sense more general than Hopf algebras, in that the Connes-Kreimer Hopf algebra may be obtained by a specific quotient. We will give an example of a sequence arising from this context and not admitting an order in Section 4.2.

Finally, we remark that the most-specific background material, for example material relating the algebraic and analytic renormalization group equations (equations (59) and (60)), can be found in the lecture notes from Kreimer's class at Humboldt University in the winter term of 2012/13, and scribed by Lutz Klaczynski [32]. The general perspective we take on combinatorial quantum field theory may be found in this source, and especially in [49, 51, 50]

[2] As explained in [20], the name FdB Lie algebra stems from the fact that—under the correct assumptions—these Lie algebras are isomorphic to the Faá di Bruno Lie algebra

2 Background

In this section, we will review some of the background material needed to understand the problem setup and its solution. This review will begin with an introduction to Hopf algebras in their most general form, followed by the construction of two instances of Hopf algebras built on the vector space of forests of rooted trees (the Connes-Kreimer Hopf algebra and the Grossman-Larson Hopf algebra) that turn out to be dual to one another. We will then introduce the central idea from quantum field theory that will influence the problem setup—that of renormalization—and will conclude with an explanation of what renormalization looks like algebraically on our Hopf algebras of rooted trees.

2.1 Hopf Algebras on Trees

2.1.1 Overview of Hopf Algebras

The study of Hopf algebras can be traced back to the respective works of Pierre Cartier and Armand Borel, the latter of whom first used the term **Hopf algebra** in honor of Heinz Hopf [1]. The history of the development of the Hopf algebra is intricate and interesting, however we will not be discussing it here. The reader is directed to [1] for a thorough overview of this history.

We begin now with definitions. All of the material covered in this section can be found in a standard textbook on Hopf algebras, for example [13]. Given the specific topic of this work, we also recommend [4] and [26].

Given a field $\mathbb{K}$ of characteristic 0, an **algebra over** $\mathbb{K}$ is a triple $(\mathcal{A}, m, u)$, where $\mathcal{A}$ is a vector space over $\mathbb{K}$ and $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $u : \mathbb{K} \to \mathcal{A}$ are linear maps that satisfy the following two commutative diagrams:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{Id \otimes m} & \mathcal{A} \otimes \mathcal{A} \\ \big\downarrow{\scriptstyle m \otimes Id} & & \big\downarrow{\scriptstyle m} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\quad m \quad} & \mathcal{A} \end{array}$$

Figure 2: The associative property of $\mathcal{A}$

$$\begin{array}{ccccc} \mathbb{K} \otimes \mathcal{A} & \xleftarrow{1 \otimes a \leftarrow\!\shortmid a} & \mathcal{A} & \xrightarrow{a \mapsto a \otimes 1} & \mathcal{A} \otimes \mathbb{K} \\ \big\downarrow{\scriptstyle u \otimes Id} & & \big\downarrow{\scriptstyle Id} & & \big\downarrow{\scriptstyle Id \otimes u} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\ m\ } & \mathcal{A} & \xleftarrow{\ m\ } & \mathcal{A} \otimes \mathcal{A} \end{array}$$

Figure 3: The unital property of $\mathcal{A}$

where Id is the identity map. The map m is called the **multiplication** in $\mathcal{A}$ and the map u is called the **unit map** of $\mathcal{A}$. Figure 2 says that m is associative, as if we follow an element $a \otimes b \otimes c \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ through the diagram in the two possible directions from the top left to the bottom right of the schematic and write $a \cdot b := m(a, b)$, we obtain:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

The commutative diagram of Figure 3 is the **unital property** of the algebra, and can similarly be written in condensed form:

$$u(1) \cdot a = a \cdot u(1) = a$$

If one is not used to commutative diagrams, it is at first not clear why we would formalize the familiar associative and unital properties to the formal language of tensor products and maps. However, the usefulness of this construction becomes very clear in the following definition.

A **coalgebra** is a triple $(\mathcal{C}, \Delta, \epsilon)$, where $\mathcal{C}$ is a vector space over $\mathbb{K}$, and $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\epsilon : \mathcal{C} \to \mathbb{K}$ are linear maps that satisfy the following two commutative diagrams:

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \xleftarrow{Id \otimes \Delta} & \mathcal{C} \otimes \mathcal{C} \\ \uparrow{\scriptstyle \Delta \otimes Id} & & \uparrow{\scriptstyle \Delta} \\ \mathcal{C} \otimes \mathcal{C} & \xleftarrow{\quad \Delta \quad} & \mathcal{C} \end{array}$$

Figure 4: The coassociative property of $\mathcal{C}$

$$\begin{array}{ccccc} \mathbb{K} \otimes \mathcal{C} & \xrightarrow{k \otimes c \mapsto ck} & \mathcal{C} & \xleftarrow{ck c \otimes k} & \mathcal{C} \otimes \mathbb{K} \\ \uparrow{\scriptstyle \epsilon \otimes Id} & & \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id \otimes \epsilon} \\ \mathcal{C} \otimes \mathcal{C} & \xleftarrow{\quad \Delta \quad} & \mathcal{C} & \xrightarrow{\quad \Delta \quad} & \mathcal{C} \otimes \mathcal{C} \end{array}$$

Figure 5: The counital property of $\mathcal{C}$

In other words, a coalgebra over $\mathbb{K}$ is colloquially speaking like an algebra over $\mathbb{K}$ but with the arrows in the respective commutative diagrams reversed. As may be guessed, we refer to the map Δ as the **coproduct** (or **comultiplication**) of $\mathcal{C}$ and the map ϵ as the **counit** of $\mathcal{C}$.

Remark. Note that by the universal property of tensor products, the statement that m and Δ are linear maps is the same as defining $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ that are **bilinear** maps. A map $f : A \times B \to C$ is bilinear if for every $a_0 \in A$ $f(a_0, b) : \{a_0\} \times B \to C$ is linear and for every $b_0 \in B$ $f(a, b_0) : A \times \{b_0\} \to C$ is linear [2].

Examples of algebras and coalgebras are everywhere in the mathematical world. Some examples follow.

Example 2.1 (Group Algebras and Coalgebras)**.** Let G be a group with binary operation $\star$. Then there is a canonical way to make G into an algebra over $\mathbb{K}$. Namely, take $V = Span_{\mathbb{K}}G$, the linear span of elements of G with coefficients in $\mathbb{K}$, m defined as the group product $\star$ extended linearly over $\mathbb{K}$, and u given by $u(1_{\mathbb{K}}) = 1_G$. Then (V, m, u) is an algebra [13]. We can similarly define a coalgebra structure with the same V by defining $\Delta(g) = g \otimes g$ for every $g \in G$ (again extending linearly over $\mathbb{K}$) and $\epsilon(g) = \delta_{g,1_G}$, where δ is the Kronecker delta function. Then (V, Δ, ϵ) is a coalgebra. Because of the way group coalgebras relate two algebraic structures (coalgebras and groups), any element a of an arbitrary Hopf algebra with the property that $\Delta(a) = a \otimes a$ is called a **group-like element**. As such, we reintroduce them in Section 2.2 and provide a more thorough treatment of their properties.

Definition 2.2. In any coalgebra $\mathcal{C}$, we define the **set of group-like elements** as:

$$Grp(\mathcal{C}) := \{a \in \mathcal{C} : \Delta(a) = a \otimes a\} \tag{4}$$

With this new appreciation for algebras and coalgebras, we may think about how to combine the structures together in a compatible way. This combined structure is referred to in the literature as a **bialgebra**. Constructing a bialgebra is not quite as simple as just pairing an algebra structure and a coalgebra structure over the same underlying vector space, however; as alluded to above, we also need the structures to be "compatible" with one another. To explain more rigorously what we mean by compatible, we first need to introduce the notion of an algebra homomorphism.

Definition 2.3 (Algebra Homomorphism)**.** Let $\mathcal{A}$ and $\mathcal{A}'$ be arbitrary algebras over $\mathbb{K}$, and $m_{\mathcal{A}}$ and $m_{\mathcal{A}'}$ be the multiplications of $\mathcal{A}$ and $\mathcal{A}'$ respectively. Then $f : \mathcal{A} \to \mathcal{A}'$ is said to be an **algebra homomorphism** if the following diagrams involving f commute:

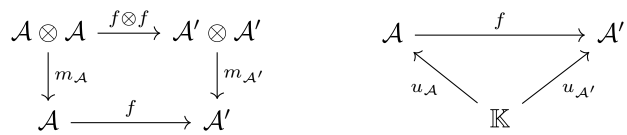


Figure 6: An algebra homomorphism

As before, if one is not used to working with tensor products, it might not at first be clear that this commutative diagram is exactly what we would expect of a homomorphism of algebras: it says that applying f and then $m_{\mathcal{A}'}$ is the same as applying $m_{\mathcal{A}}$ and then f. In other words, f preserves the algebraic structure between $\mathcal{A}$ and $\mathcal{A}'$.

An analogous notion is likely anticipated at this point: that of a coalgebra homomorphism.

Definition 2.4 (Coalgebra Homomorphism)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be arbitrary coalgebras, and $\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{C}'}$ be the comultiplications of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then $g : \mathcal{C}' \to \mathcal{C}$ is said to be a **coalgebra homomorphism** if the following commutative diagram involving g commutes:

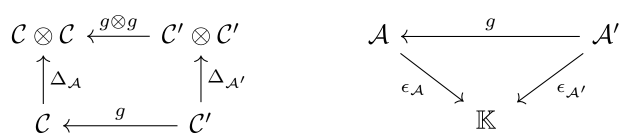


Figure 7: A coalgebra homomorphism

Once again we see that a coalgebra homomorphism preserves coalgebra structure, as expected.

With all of these definitions established, we can now define a bialgebra. A **bialgebra** over $\mathbb{K}$ is a quintuplet $(B, m, u, \Delta, \epsilon)$ such that (B, m, u) is an algebra over $\mathbb{K}$, (B, Δ, ϵ) is a coalgebra over $\mathbb{K}$, and either:

1. Δ and ϵ are algebra homomorphisms, or

2. m and u are coalgebra homomorphisms

It is important to note that one need to check only that 1. or 2. are satisfied and not both, as they are equivalent statements. To see why this is so, one merely translates the statements into commutative diagrams and observes that the resulting diagrams are the same. For example:

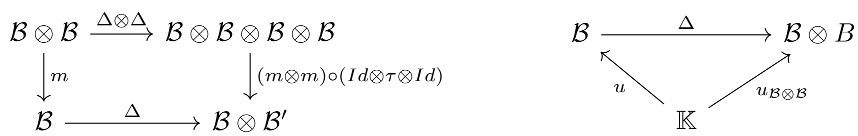


Figure 8: Δ as an algebra homomorphism

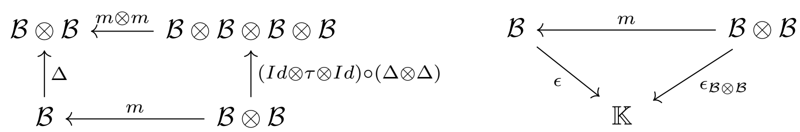


Figure 9: m as a coalgebra homomorphism

where τ is the "twist map" [2, 49]; that is: $\tau(a \otimes b) = b \otimes a$. The only nontrivial observation in constructing the diagrams is that $(m \otimes m) \circ (Id \otimes \tau \otimes Id)$ is the canonical way to define multiplication in a tensor of two algebras, while $(Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)$ is the canonical way of defining the coproduct of the tensor of two coalgebras. This multiplication and comultiplication respectively represent $m_{A'}$ and $\Delta_{\mathcal{C}'}$ in Figures 6 and 7 above. Following the two possible paths in the left diagram of each figure, we obtain the same equation:

$$\Delta \circ m = (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta) \tag{5}$$

Similarly, the map $u_{\mathcal{B}\otimes\mathcal{B}}$ is given canonically by $1_{\mathbb{K}} \mapsto 1_{\mathcal{B}} \otimes 1_{\mathcal{B}}$, so the right diagram in Figure 8 gives that:

$$\Delta(1_{\mathcal{B}}) = 1_{\mathcal{B}} \otimes 1_{\mathcal{B}} \tag{6}$$

whereas the right diagram in Figure 9 gives:

$$\epsilon(m(b_1 \otimes b_2)) = \begin{cases} 1 & \text{if } b_1 = b_2 = 1_{\mathcal{B}} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

both of which are equivalent.

An analogous argument can be made for u and ϵ as coalgebra and algebra homomorphisms, respectively. For a further exhibition, see [13, 26, 49].

Finally, given all this setup we can finish with the definition of a Hopf algebra. A **Hopf algebra** over the field $\mathbb{K}$ is a 6-tuple $(H, m, u, \Delta, \epsilon, S)$, where $(\mathcal{H}, m, u, \Delta, \epsilon)$ is a bialgebra, and S is the **antipode** map, satisfying the following commutative diagram:

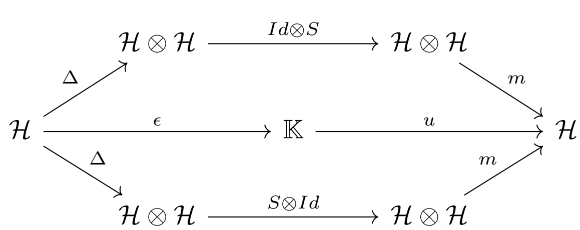


Figure 10: The antipodal property of $\mathcal{H}$.

What is this commutative diagram telling us about the relationship between S, m, and Δ? It turns out the relationship becomes much more apparent with the introduction of a new operation, called the convolution product. This operation will be very important for us in the course of this thesis.

Definition 2.5 (Convolution Product)**.** Let $Hom(\mathcal{C}, \mathcal{A})$ be the set of algebra homomorphisms from an arbitrary coalgebra $\mathcal{C}$ to an arbitrary algebra $\mathcal{A}$. Then for $f, g \in Hom(\mathcal{C}, \mathcal{A})$, the **convolution product**, $*$ of f and g is defined by:

$$f * g = m_{\mathcal{A}} \circ (f \otimes g) \circ \Delta_{\mathcal{C}} \tag{8}$$

A classic result in the theory of coalgebras is that $(Hom(\mathcal{C}, \mathcal{A}), *, u_{\mathcal{A}} \circ \epsilon_{\mathcal{C}})$ is itself an algebra [2, 13]. Of course for a given Hopf algebra $\mathcal{H}$, $\mathcal{H}$ is in particular an algebra and a coalgebra. This means that $* : \mathcal{H} \to \mathcal{H}$ is also defined, and in particular it makes $(End(\mathcal{H}), *, u \circ \epsilon)$ into an algebra! This algebra will be another object of fundamental importance to us throughout this thesis. We see its first application in the following proposition which explains the relationship of S, m, and Δ given in Figure 10:

Proposition 2.6. For any Hopf algebra $\mathcal{H}$ with antipode S, S is the inverse of $Id \in (End(\mathcal{H}), *, u \circ \epsilon)$. That is:

$$S * Id = Id * S = u \circ \epsilon \tag{9}$$

Equation (9) is obtained immediately from reading off the three possible paths in Figure 10.

Note that, as is standard in algebra, we often simply refer to a Hopf algebra by the vector space $\mathcal{H}$ and not by the full 6-tuple $(\mathcal{H}, m, u, \Delta, \epsilon, S)$. The same remark applies to algebras, coalgebras, and bialgebras throughout this thesis.

Some examples of Hopf algebras follow.

Example 2.1 (Continued). Given a group $(G, \star)$ and the vector space $V = Span_{\mathbb{K}} G$, we saw that $(V, \star, u)$ and (V, Δ, ϵ) constitute an algebra and a coalgebra, respectively. Is there a way to combine these to obtain a Hopf algebra? The answer is yes, and in a unique way. One merely checks that $(V, \star, u, \Delta, \epsilon)$ is a bialgebra. Then the unique antipode is given by $S(g) = g^{-1}$ and then extended linearly, where by g^{-1} we mean the group inverse of g in G.

Example 2.7 (Universal enveloping algebra of a Lie algebra)**.** Another example is one of primary importance, both for us and more generally in the theory of Hopf algebras. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We will now give an overview of the standard construction of $\mathcal{U}(\mathfrak{g})$; the reader is also referred to [13]. Let $[\cdot, \cdot]$ be the bracket of $\mathfrak{g}$. Intuitively, the point of $\mathcal{U}(\mathfrak{g})$ is to construct the largest possible associative algebra such that $[a, b] = a \otimes b - b \otimes a$, for a and b in the algebra. We do this as follows.

First, begin by constructing the tensor algebra of $\mathfrak{g}$, denoted $T(\mathfrak{g})$. This is a free algebra with all possible tensors products of elements of $\mathfrak{g}$. In other words:

$$T(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus ... \tag{10}$$

where $\oplus$ represents the direct sum of the pieces $(\mathfrak{g} \otimes ... \otimes \mathfrak{g})$ as vector spaces [13]. Then the universal enveloping algebra of $\mathfrak{g}$ is simply obtained by modding out by the ideal generated by all elements of the form $a \otimes b - b \otimes a - [a, b]$ for $a, b \in \mathfrak{g}$:

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \langle a \otimes b - b \otimes a - [a, b] : a, b \in \mathfrak{g} \rangle$$

where the brackets $\langle\,\rangle$ above mean “ideal generated by” as is customary.

Now $\mathcal{U}(\mathfrak{g})$ has an algebra structure, and there is also canonically the structure of a Hopf algebra: For any $a \in \mathfrak{g}$, define $\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$, $\epsilon(a) = 0$, and $S(a) = -a$. These maps are then extended uniquely to all elements $b \in \mathcal{U}(\mathfrak{g})$.

While hopefully serving as a good example of all the preceding definitions of Hopf algebras, $U(\mathfrak{g})$ is more than an example alone: it is both the primary connection between Lie algebras and Hopf algebras, and also the primary player in the famous Milnor-Moore Theorem, which is useful in determining which subalgebras of a Hopf algebra are Hopf. Hence we will see the construction $\mathcal{U}(\mathfrak{g})$ come up many times in this work, particularly in Section 4. For completeness, we include this theorem here [35]:

Theorem 2.8 (Milnor-Moore, 1965)**.** *Let $\mathcal{H}$ be a connected, graded, cocommutative Hopf algebra of finite type (that is, each graded piece of $\mathcal{H}$ is finite-dimensional as a vector space). Then:*

$$\mathcal{H} \simeq \mathcal{U}(Prim(\mathcal{H})) \tag{11}$$

That is, $\mathcal{H}$ is isomorphic to the universal enveloping algebra of its primitive elements [3]*.*

These examples inspire the inclusion of a theorem that answers a question the reader may now have; namely, how many ways are there to turn a bialgebra in a Hopf algebra?

Theorem 2.9. *If a bialgebra $\mathcal{B}$ admits a Hopf algebra structure, then the antipode S is unique.*

In other words, if a bialgebra can be turned into a Hopf algebra, it can only be done in one way [13].

Proposition 2.10. Let $\mathcal{A}$ be a graded, connected bialgebra. Let $x \in \mathcal{A}$ be arbitrary, and set $\Delta(x) := \sum_i x_{i,1} \otimes x_{i,2}$. Then $\mathcal{A}$ admits a Hopf algebra structure, with its unique antipode (from Theorem 2.9) determined recursively as follows:

$$S(x) = -x - \sum_i S(x_{i,1})x_{i,2} \tag{12}$$

Now we see one of the reasons Hopf algebras built over vector spaces of combinatorial objects are so special: in Proposition 2.10, the only requirements on the bialgebra $\mathcal{A}$ were that it be graded and connected. Now consider forming a bialgebra over some vector space of combinatorial objects. We typically obtain a grading of the vector space via the notion of size on the combinatorial objects (think length of permutations, number of vertices or number of edges in graphs, size of a partition, etc). Moreover, we typically get that the vector space is connected: this comes from the fact that combinatorial objects tend to have only one notion of an “empty element” (think the empty graph, the word of length 0, the null set, etc.). Hence, most bialgebras we could construct out of a combinatorial class satisfy the hypotheses of Proposition 2.10, and so we know one form of their antipode immediately via the recursive definition. It is exactly this recursive antipode structure that connects the Connes-Kreimer Hopf algebra of rooted trees to renormalization in quantum field theory; see [9, 40].

Before we move on, we provide one more example that ties all of these notions together.

Example 2.11 (Hopf Algebra of Polynomials in a Single Variable)**.** Consider the structure of $\mathbb{K}[L]$ of polynomials in a single variable L. $\mathbb{K}[L]$ is already an algebra with multiplication defined as regular polynomial multiplication, and the unit u given by $u(1_{\mathbb{K}}) = 1_{\mathbb{K}}$. To define the coalgebra structure, we set:

$$\Delta(L) := L \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes L \tag{13}$$

[3]We will define primitive elements formally in Definition 2.12.

We remark that in any coalgebra, an element that satisfies equation (13) is said to be a **primitive element**. Now in this case, defining Δ on L in this way determines what Δ does on every element of $\mathbb{K}[L]$. Indeed, since Δ is extended linearly over $\mathbb{K}[L]$, we only need to consider what Δ does on monomials L^n for some n. But since we also want Δ to be an algebra homomorphism since we are trying to build a Hopf algebra, we get that:

$$\begin{aligned}\Delta(L^2) &= \Delta(L)\Delta(L)\\ &= (L \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes L)(L \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes L)\\ &= L^2 \otimes 1_{\mathbb{K}} + 2L \otimes L + 1_{\mathbb{K}} \otimes L^2\end{aligned}$$

and:

$$\begin{aligned}\Delta(L^3) &= \Delta(L)\Delta(L^2)\\ &= (L \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes L)(L^2 \otimes 1_{\mathbb{K}} + 2L \otimes L + 1_{\mathbb{K}} \otimes L^2)\\ &= L^3 \otimes 1_{\mathbb{K}} + 3L^2 \otimes L + +3L \otimes L^2 + 1_{\mathbb{K}} \otimes L^3\end{aligned}$$

An easy inductive argument shows that for any n:

$$\Delta(L^n) = \sum_{j=0}^{n} \binom{n}{j} L^j \otimes L^{n-j}$$

Finally, what does the antipode S look like in $\mathbb{K}[L]$? Since $\mathbb{K}[L]$ is graded and connected, we already know of one (recursive) form for S, due to Proposition 2.10, though we can also be more explicit. Namely, since S is an **antiautomorphism** (see Proposition 2 in Section 4 of [49]), we have that $S(ab) = S(b)S(a)$ for all $a, b \in \mathbb{K}[L]$, meaning that $S(1_{\mathbb{K}}) = 1_{\mathbb{K}}$. Moreover, we have that $S(p) = -p$ for any primitive element p (see [50]). Hence $S(L) = -L$. Using these two properties together, we get that:

$$\begin{aligned}S(L^2) &= S(L)S(L) = L^2\\ S(L^3) &= S(L^2)S(L) = -L^3\end{aligned}$$

and in general for any n:

$$S(L^n) = (-1)^n L^n \tag{14}$$

and so S is completely determined over $\mathbb{K}[L]$.

The notion of primitive elements that came up in the course of the example will be central to the rest of this work. In general, we define:

Definition 2.12. For any coalgebra $\mathcal{C}$, the **set of primitive elements** of $\mathcal{C}$ is:

$$Prim(\mathcal{C}) := \{p \in \mathcal{C} : \Delta(p) = p \otimes 1_{\mathcal{C}} + 1_{\mathcal{C}} \otimes p\} \tag{15}$$

There is an interesting connection between the set of group-like elements given by Definition 4 and the set of primitive elements we have just defined, which was already foreshadowed by Theorem 2.8. Namely, for a graded and connected Hopf algebra $\mathcal{H}$ (after taking its completion, if necessary), $Prim(\mathcal{H})$ is a Lie algebra and $Grp(\mathcal{H})$ is its associated Lie group [40]. Hence:

Proposition 2.13. For a graded and connected Hopf algebra $\mathcal{H}$ and exp and log as formal power series, let $x \in \mathcal{H}$ be a primitive element and let $y \in \mathcal{H}$ be a group-like element. Then if $\exp(x)$ and $\log(y)$ are contained in $\mathcal{H}$, $\exp(x)$ is group-like and $\log(y)$ is primitive.

For a two-line proof of the proposition, see Appendix B.1 of [32].

2.1.2 The Connes-Kreimer Hopf Algebra

In this subsection, we will introduce the object of central interest for the rest of our work: the Connes-Kreimer Hopf algebra of rooted trees. We aim to give ample examples as a means of exposition, which should lend itself to casual reading.

To begin, define $\mathcal{T}$ to be the set of all (isomorphism classes of) rooted trees, and let $\mathcal{F}$ be the set of all disjoint unions of these objects (so $\mathcal{F}$ is the set of all forests of rooted trees). By a rooted tree, we mean a simple, connected, acyclic graph with a distinguished vertex known as the root, and by forests of rooted trees, we mean disjoint unions of elements of $\mathcal{T}$. We denote the empty forest by the symbol $\mathbb{1}$.

Though the trees in $\mathcal{T}$ are unlabelled, we will sometimes have the need to talk about operations on them in a way that deals with individual vertices. Whenever we require this, one can take an arbitrary labelling, perform the necessary operations on the vertices, and then forget the labelling immediately thereafter. For this reason we will often not make reference to any particular labelling, but by convention will simply refer to the vertices as though they have labels. One instance of this we rely on frequently will be to access subtrees: for a tree $t \in \mathcal{T}$ given one of these arbitrary labellings, define t_v to be the subtree of t rooted at vertex v.

Let's implement the structure of a combinatorial Hopf algebra as described in the last sections.

Definition 2.14. Take $\mathcal{H}_{CK} = (V, m, u, \Delta, \epsilon, S)$, where:

- $V = Span_{\mathbb{K}}\mathcal{F}$ is linear combinations of forests over $\mathbb{K}$.

- $m : \mathcal{H}_{CK} \otimes \mathcal{H}_{CK} \to \mathcal{H}_{CK}$ maps two forests to their disjoint union.
- $u : \mathbb{K} \to \mathcal{H}_{CK}$ sends $1_{\mathbb{K}}$ to the empty forest.
- Let $t \in \mathcal{T}$. Then $\Delta : \mathcal{H}_{CK} \to \mathcal{H}_{CK} \otimes \mathcal{H}_{CK}$ is given by

$$\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ C \text{ an antichain}}} \left(\bigcup_{v \in C} t_v \right) \otimes \left(t \setminus \bigcup_{v \in C} t_v \right) \tag{16}$$

- $\epsilon : \mathcal{H}_{CK} \to \mathbb{K}$ sends the empty forest to $1_{\mathbb{K}}$ and all other forests to 0.
- Let $t \in \mathcal{T}$. Then the antipode $S : \mathcal{H}_{CK} \to \mathcal{H}_{CK}$ is given recursively by:

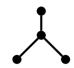
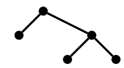
$$S(t) = -t - \sum_{\substack{\varnothing \subsetneq C \subseteq V(t) \\ C \text{ a non-root antichain}}} S\left(\bigcup_{v \in C} t_v \right) \left(t \setminus \bigcup_{v \in C} t_v \right) \tag{17}$$

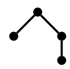
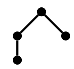
  where in the formulas for Δ and S we think of the rooted trees as posets. Moreover, for clarity we also remark that in the corner case of v being the root vertex in t, $t \setminus t = \mathbb{1}$.

  Note that we have only defined Δ and S on $\mathcal{T}$, but as $\mathcal{T}$ is a basis (via m) for $\mathcal{F}$, Δ and S are extended multiplicatively as algebra homomorphisms to all forests $F \in \mathcal{H}_{CK}$. Then $\mathcal{H}_{CK}$ is the **Connes-Kreimer Hopf algebra of rooted trees**.

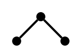
Remark. Although we have only presented one formula for the antipode above, there are in fact three distinct forms for calculating S in $\mathcal{H}_{CK}$. See [17], in which the authors present these three formulae and show that the equivalence of the three is due to the equivalence of three different renormalization schemes in quantum field theory.

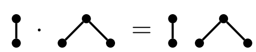
These formulae are best understood via examples.

Example 2.15. Consider some trees •, , , and . When we draw these trees, for consistency we will adopt the tendency of computer scientists and draw the root vertex at the very top, and the branches coming down from the root. We refer to vertices below a given vertex v in the poset structure of the tree as **children** of v, and conversely we refer to the vertices for which v is a child as the **parents** of v. As usual, vertices with no children will be called **leaves** of the tree.

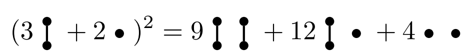
It is important to note that–while we will consistently draw trees in such a way that respects the poset structure of the tree–the combinatorial objects we are working with do not come with a planar embedding. In other words, the tree  is for us the same as the tree . In the literature, this is sometimes referred to as the **commutative Connes-Kreimer Hopf algebra**, whereas a noncommutative version works with rooted trees together with their planar embedding (and in which forests become ordered lists of rooted trees, as opposed to their unordered counterparts). Some interesting results on this noncommutative version of Connes-Kreimer Hopf algebra, together with some results that compare it to the commutative version, can be found in [20, 31] for example. In this thesis, we will only work with the commutative version of the Hopf algebra unless stated explicitly otherwise, and hence there will be no ambiguity when we simply refer to the Connes-Kreimer Hopf algebra of rooted trees.

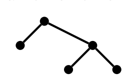
Moreover, when referring to the size of a tree t, we will often suppress the graph theory notation and write $|t|$ for $|V(t)|$.

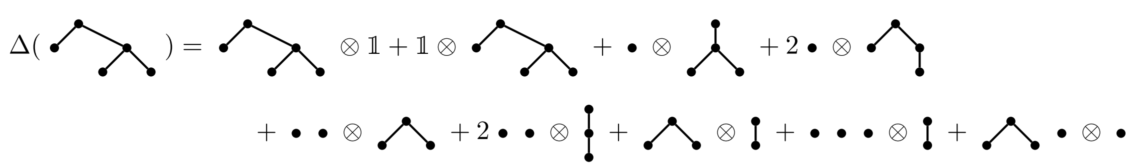
Let us now see what arithmetic looks like in $\mathcal{H}_{CK}$. Multiplication is very simple. If we use $\cdot$ to represent the map m, then the multiplication of trees  and , for example, is just:

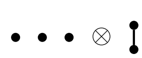
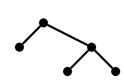
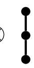
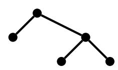


Moreover, linear combinations of trees follow exactly the principles of multiplying polynomials. In fact, the structure of $\mathcal{H}_{CK}$ as an algebra is exactly $\mathbb{K}[\mathcal{F}]$. For example:

 (18)

Less familiar, perhaps, are computations involving the coproduct and the the antipode, so we will demonstrate these now. First, consider applying the coproduct to the tree . The calculation for this is:



Where, for example, the term  is coming from the fact that  has three leaves (constituting an antichain). When we prune the three leaves (obtaining the left hand side of the tensor product, $\bullet\ \bullet\ \bullet$), we are left with a tree on two vertices, , which becomes the right hand side of the tensor product. In addition, this sum appears with coefficient 1, because there was only one way to do this: namely to prune all three leaves at once. Alternatively, note that the term $\bullet\ \bullet \otimes$  appears with a coefficient 2. This is due to the fact that there are two distinct ways to remove the forest $\bullet\ \bullet$ from  and be left with exactly .

2.1.3 Operators on $\mathcal{H}_{CK}$

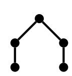
Various operators can be defined on the vector space $\mathcal{F}$, and in particular these become operators on $\mathcal{H}_{CK}$. For the purposes of this thesis, there are seven that will be of fundamental importance. We start with the notion of tree factorial:

Definition 2.16. Let $t \in \mathcal{T}$. Then define the **tree factorial**, $t!$, by:

$$t! = \prod_{v \in t} |t_v| \tag{19}$$

where $|t_v|$ denotes the number of vertices of the subtree rooted at v.

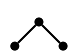
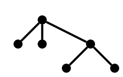
As an example, we see that:


$$! = 5 \cdot 2 \cdot 1 \cdot 2 \cdot 1$$
$$= 20$$

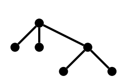
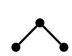
We define the rest of the operators we wish to discuss in pairs:

Definition 2.17 (Grafting Operators)**.** Let $\mathcal{H}_n$ denote the nth graded piece of $\mathcal{H}_{CK}$, and define $B^+ : \mathcal{H}_n \to \mathcal{H}_{n+1}$ to be the operator that assigns to a forest of rooted trees the tree obtained by grafting a new, common vertex onto the root of each tree in the forest and extended to linear combinations of forests as an algebra homomorphism. Let $B^- : \mathbb{K}\mathcal{T}_{n+1} \to \mathcal{H}_n$ denote the inverse of B^+ when it is restricted to $\mathbb{K}\mathcal{T}_{n+1}$.

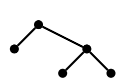
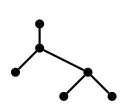
Example 2.18. To illustrate Definition 2.17, we have that:

$B^+($  $\bullet\ \bullet\ ) =$  (20)

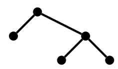
and conversely that:

$B^-($  $) =$  $\bullet\ \bullet$ (21)

In particular, note that B^+ acts on the space of trees by adding on linear segments:

$B^+($  $) =$ 

and also that every rooted tree can be expressed as an appropriate number of applications of B^+ and m recursively, with the base case being B^+ applied to the empty forest. For example:

 $= B^+(B^+(\mathbb{1})B^+(B^+(\mathbb{1})B^+(\mathbb{1})))$

We will typically write B^{+n} to mean the application of B^+ n times:

 $= B^{+4}(\mathbb{1})$

Analogous to the way B^+ and B^- grow and shrink trees from the root upwards, respectively, we can define operators that grow and shrink trees from the leaves downwards. To stay consistent with other works in the literature, we follow the notational setup of Hoffman in [30].

First, let us set up the partial order $\preceq$ on the vector space of rooted trees $\mathbb{K}[\mathcal{T}]$ as described in [30]. Namely, we say that $t_1 \preceq t_2$ if any number of non-root vertices and edges can be removed from t_2 to obtain t_1. This means in particular that $|V(t_1)| \leq |V(t_2)|$. In this poset, then, the covering relations are exactly:

$$t_1 \lhd t_2 \iff t_1 \text{ can be obtained from } t_2 \text{ by removing a leaf}$$

Hence we can define the following [30]:

Definition 2.19 (Growing and Pruning Operators)**.** Define:

$$n(t_1, t_2) = \text{the number of vertices of } t_1 \text{ to which a new child can be added to obtain } t_2$$

and

$$m(t_1, t_2) = \text{the number of leaves of } t_2 \text{ that can be removed to obtain } t_1$$

Then for any tree t define $N : \mathcal{H}_n \to \mathcal{H}_{n+1}$ by:

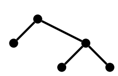
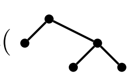
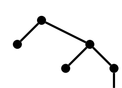
$$N(t) = \sum_{t \lhd t_2} n(t, t_2) t_2 \tag{22}$$

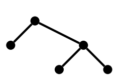
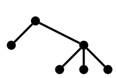
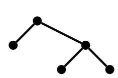
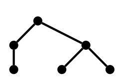
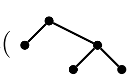
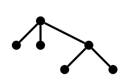
and define $P : \mathcal{H}_{n+1} \to \mathcal{H}_n$ by:

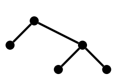
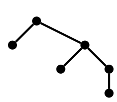
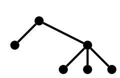
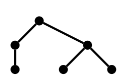
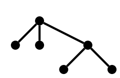
$$P(t) = \sum_{t_2 \lhd t} m(t_2, t) t_2 \tag{23}$$

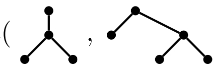
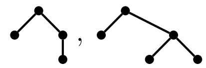
where N and P are both extended as algebra homomorphisms to all linear combinations of forests. We refer to N as **growing operator** (or **natural growth operator**) and to P as the **pruning operator**.

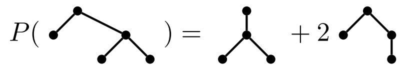
These definitions are well suited for exposition by example.

Example 2.20. As before, consider the tree  . Then since $n($  ,  $) = 2$ and

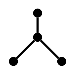
$n($  ,  $) = n($  ,  $) = n($  ,  $) = 1$, we find that:

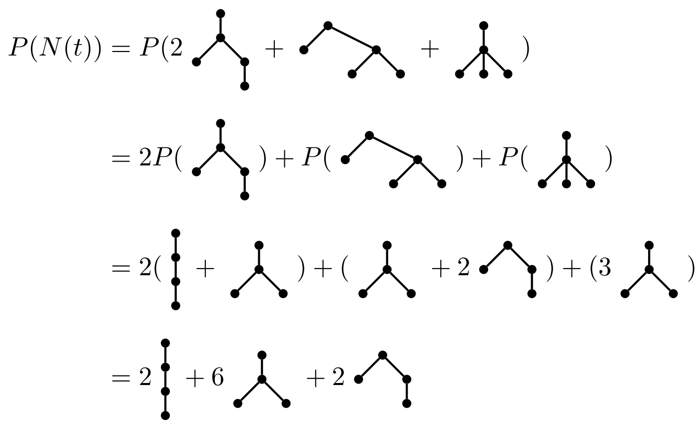
$N($  $) = 2$  $+$  $+$  $+$ 

and since $m($  $) = 1$ and $m($  $) = 2$, it follows that:

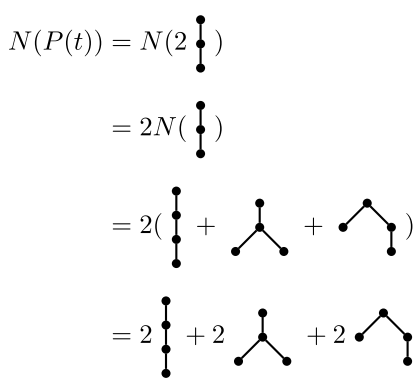


Now one would likely guess that the operators N and P are inverses to one another in the same way that B^+ and B^- are inverses, but it turns out that the situation is more subtle, as illustrated by the following example.

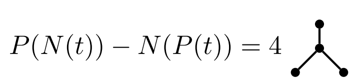
Example 2.21. Let $t =$  . Then we can compute P(N(t)) as follows:



On the other hand, we can computer N(P(t)):



Note that $P(N(t)) \neq N(P(t))$. Nevertheless, if we take their difference we find that:



which is exactly $|t|t$. It turns out that this is not just a special case, but will hold for all forests of any size (and in fact for all elements of $\mathcal{H}_{CK}$, including linear combinations of forests).

Proposition 2.22 (Proposition 2.2 of [30])**.** Let $D : \mathbb{K}\mathcal{T} \to \mathbb{K}\mathcal{T}$ be given by $D = PN - NP$ and extend D to $\mathcal{H}_{CK}$ as an algebra homomorphism. Then for any $t \in \mathcal{T}$:

$$D(t) = |t|t \tag{24}$$

Proof. See Proposition 2.2 of [30] for a proof. □

Proposition 2.22 tells us that D is a differential operator on the space of trees, and in fact that the poset $(\mathcal{T}, \preceq)$ described above is a differential poset with respect to P and N [30] (see also [47] for more information on differential posets).

Now that these definitions are in place, a natural question has probably arisen in the reader's mind: How do these operators interact with the defining maps of $\mathcal{H}_{CK}$ itself, for example the coproduct and the antipode? We now take the time to discuss this.

To begin, note that N and P are defined explicitly as algebra homomorphisms, so the way they interact with m is clear. Conversely, note that B^+ and B^- are not algebra homomorphisms, namely because $B^+(t_1t_2\cdots t_n) \neq B^+(t_1)B^+(t_2)\cdots B^+(t_n)$, as the latter is a disconnected forest while the former is a connected tree. Moreover, $B^-(t_1t_2\cdots t_n)$ is left undefined. Nevertheless, B^+ and B^- are still linear maps.

The way these operators interact with the coproduct is less trivial. For proofs of the following, see Section 4.4 of [49] for 1 and Proposition 3.5 of [30] for 2 (see also Proposition 6 of [9]).

Proposition 2.23. Let $F \in \mathcal{H}_{CK}$. Then:

1. $\Delta(B^+(F)) = B^+(F) \otimes \mathbb{1} + (Id \otimes B^+) \circ \Delta(F)$
2. $\Delta(N(F)) = (N \otimes Id + Id \otimes N + M_{\bullet} \otimes D) \circ \Delta(F)$

where $M_{\bullet}(h) = \bullet\, h$ for any $h \in \mathcal{H}_{CK}$, as in [30][4].

We point out that item 1 in the proposition is a combinatorial identity: it says that if we take the coproduct of an element $B^+(F)$, then we either prune off the whole tree (resulting in the left term) or we simply have the coproduct of F with an extra root attached to the tree on the right side of each term in the tensor product (this is essentially the proof of 1). From a different perspective, the statement 1 is saying that the map B^+ is a 1-cocyle in Hochschild cohomology; see [49, 9] and Section 6 of [15].

Remark. While one can write down the result of composing N and B^+ with the antipode of $\mathcal{H}_{CK}$, no nice identities exist such as in Proposition 2.23 that the author knows of.

Finally, we should note that the operators defined in this section not only interact with the maps of $\mathcal{H}_{CK}$, but can also interact with one another. This is the substance of Proposition 3.2 in [30], and is the following:

Proposition 2.24. Let t be a monomial of $\mathcal{H}_{CK}$. Then:

1. $B^+(P(t)) = P(B^+(t)))$

[4]We use this notation so that the equation consists strictly of operators.

2. $B^+(N(t)) = N(B^+(t)) - B^+(\bullet\, t)$

Our final pair of operators mimic the familiar notions of exp and log from other parts of mathematics, and we denote them ω and ω^{-1}, respectively. However, we postpone their definition until Section 2.2, as they will make more sense in that context.

Before we move on, we define two special families of trees in $\mathcal{H}_{CK}$ that we will often make reference to throughout this work. The names for these families of trees are standard in related literature; see [18, 49, 21, 20]:

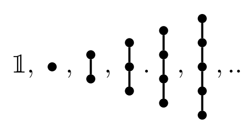
Definition 2.25. The **ladder on** i **vertices**, l_i, will be defined recursively as:

$$l_0 = \mathbb{1}$$
$$l_i = B^+(l_{i-1}) \qquad \text{for all } i \geq 1$$

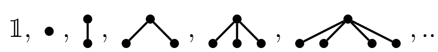
Moreover, we define the **corolla on** i **vertices**, c_i, recursively by:

$$c_0 = \mathbb{1}$$
$$c_1 = \bullet$$
$$c_i = B^+(B^-(c_{i-1})\,\bullet\,) \qquad \text{for all } i \geq 2$$

The first few ladders begin:



and the first few corollas begin:



As a further example of Definition 2.16, it is easy to compute that $l_i! = i!$ and $c_i! = i$ for all i.

2.1.4 The Grossman-Larson Hopf Algebra

In this subsection, we present an alternative way to construct a Hopf algebra on the vector space of trees which will turn out to be isomorphic to the Hopf algebra of the graded dual of $\mathcal{H}_{CK}$ [30].

To begin, we will speak often in this section (and in Section 4 to come) of the notion of **grafting** trees. We start by making a definition:

Definition 2.26. Let $t_1, t_2 \in \mathcal{T}$, and consider a vertex $v \in t_1$. Then we define the **grafting** of t_2 onto the vertex of v of t_1 to be the tree t obtained by making the root of t_2 a child of v in t_1. This operaton is represented schematically in Figure 11.

This operation will figure prominently in the following construction:

Definition 2.27. Take $\mathcal{H}_{GL} = (V, m', u', \Delta', \epsilon', S')$ where:

- $V = Span_{\mathbb{K}}\mathcal{F}$ is linear combinations of forests over $\mathbb{K}$, as before.

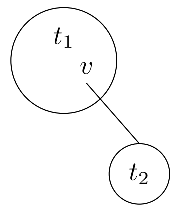


Figure 11: Grafting t_2 onto t_1 at vertex v.

- $m' : \mathcal{H}_{GL} \otimes \mathcal{H}_{GL} \to \mathcal{H}_{GL}$ is given by:

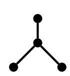
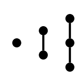
$$m'(t_1, t_2) = \sum_{v \in V(t_1)} t_2 \text{ grafted onto } t_1 \text{ at vertex } v \tag{25}$$

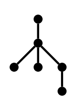
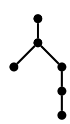
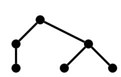
- $u'(1) = \mathbb{1}$, mapping $1_{\mathbb{K}}$ to the empty forest.
- $\Delta'(t_1 t_2 \cdots t_n) = \sum_{I \subseteq \{1,2,\ldots,n\}} \prod_{i \in I} t_i \otimes \prod_{j \notin I} t_j$
- $\epsilon'(F) = \delta_{F,\mathbb{1}}$, sending the empty forest to $1_{\mathbb{K}}$ and all other forests to 0.
- Since all of the previous operations keep $\mathcal{H}_{GL}$ graded and connected, we get a recursive formula for the antipode for free. If we write the reduced coproduct[5] $\widetilde{\Delta}(F) = \sum_i F_{i,1} \otimes F_{i,2}$ for $F \in \mathcal{H}_{GL}$, then the antipode $S : \mathcal{H}_{GL} \to \mathcal{H}_{GL}$ is given recursively by:

$$S(F) = -F - \sum_i S(F_{i,1}) F_{i,2} \tag{26}$$

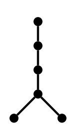
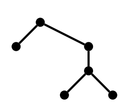
where above we use $t_1 t_2 \cdots t_n$ to represent an arbitrary forest, and where as before m is extended as an algebra homomorphism over $\mathcal{F}$.

Once again, these definitions are well-suited for a demonstration:

Example 2.28. Let $F_1 =$  , $F_2 =$  , and $F_3 =$  . Applying the definitions, we see that:

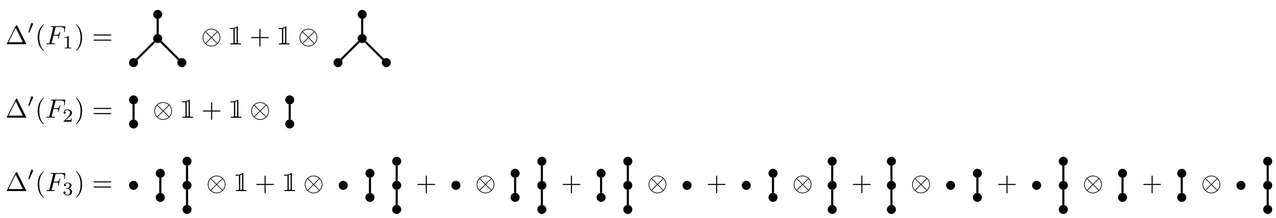
$m'(F_1 \otimes F_2) =$  $+ 2$  $+$ 

and

$m'(F_2 \otimes F_1) =$  $+$ 

[5] By reduced coproduct, we mean the coproduct minus its primitive part. That is: $\widetilde{\Delta}(a) = \Delta(a) - [a \otimes \mathbb{1} + \mathbb{1} \otimes a]$.

We also look at some examples involving Δ':



Note that—colloquially speaking—the coproduct Δ' of a forest F is simply the sum of all possible ways to partition the trees of F into two parts (separated by the tensor symbol) without making any cuts. The example also illustrates some patterns of arithmetic in $\mathcal{H}_{GL}$ that end up holding in full generality:

(i) $Prim(\mathcal{H}_{GL})$, the space of primitive elements of $\mathcal{H}_{GL}$, is exactly $\mathcal{T}$.

(ii) $\mathcal{H}_{GL}$ is cocommutative, but not in general commutative.

The following result was first credited to Panaite [39], though due to an error in the proof, the first complete result is due to Hoffman [30].

Theorem 2.29 (Panaite (2000), Hoffman (2003))**.** *The graded dual of $\mathcal{H}_{CK}$, denoted $\mathcal{H}_{CK}^*$, is isomorphic to $\mathcal{H}_{GL}$.*

The proof of the theorem is nontrivial, as the isomorphism claimed does not rely on the standard inner product one would think to use; rather, one must define an inner product on the vector space of rooted trees as follows. For any rooted trees $t_1, t_2 \in \mathcal{T}$, define $(\cdot,\cdot) : \mathcal{T} \to \mathbb{K}$ by [30]:

$$(t_1, t_2) = |Sym(t_1)|\delta_{t_1,t_2} \tag{27}$$

for δ the Kronecker delta function and $Sym(t_1)$ the symmetry group of t_1 . Defining $(\cdot,\cdot)$ in this way (and extending it to $\mathcal{H}_{CK}$) makes the pruning and growing operators P and N as defined in the last section adjoint to one another, a fact which is ultimately what delivers the proof of the theorem. Hoffman then gives the bijection $\chi : \mathcal{H}_{GL} \to \mathcal{H}_{CK}^*$ defined by:

$$\langle \chi(t), u\rangle = (B_-(t), u) = (t, B_+(u)) \tag{28}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product[6] , t is a tree in $\mathcal{H}_{GL}$, and u is a monomial in $\mathcal{H}_{CK}$. See Proposition 4.4 of [30] for the proof of this result.

2.2 Lie Theory

It is well-known that the theory of Hopf algebras is intimately connected with the theory of Lie groups and Lie algebras. In fact, we already saw one such connection in Example 2.7, and another in Theorem 2.8. In this section we will explore further connections between $\mathcal{H}_{CK}$ and Lie theory, and in particular will show that there multiple Lie groups (and their associated Lie algebras) that are related to $\mathcal{H}_{CK}$.

[6] That is, $\langle a, b\rangle := \delta_{a,b}$.

2.2.1 The Lie Algebra and Lie Group of Convolution

In Section 2.1.1, we noted that $\mathcal{A}_{\mathcal{H}} = (Hom(\mathcal{H}, \mathcal{A}), *, u \circ \epsilon)$ is an algebra, where $Hom(\mathcal{H}, \mathcal{A})$ is the set of linear maps from $\mathcal{H}$ to $\mathcal{A}$, $*$ is the convolution product, and u and ϵ are the unit and counit in $\mathcal{A}$ and $\mathcal{H}$ respectively. There exist various special subsets of $\mathcal{A}_{\mathcal{H}}$ which we now consider.

First, consider the subset of $\mathcal{A}_{\mathcal{H}}$ consisting only of the linear maps $\phi \in Hom(\mathcal{H}, \mathcal{A})$ that map the unit of $\mathcal{H}$ to the unit of $\mathcal{A}$. That is:

$$G^{\mathcal{H}}_{\mathcal{A}} := \{\phi \in Hom(\mathcal{H}, \mathcal{A}) : \phi(\mathbb{1}) = 1_{\mathcal{A}}\} \tag{29}$$

Then $G^{\mathcal{H}}_{\mathcal{A}}$ has the structure of a Lie group! In the literature, $G^{\mathcal{H}}_{\mathcal{A}}$ is referred to as the **Lie group of convolution** [40, 32].

Next, consider the subset of $\mathcal{A}_{\mathcal{H}}$ consisting only of the linear maps $\phi \in Hom(\mathcal{H}, \mathcal{A})$ that send the unit of $\mathcal{H}$ to 0 in $\mathcal{A}$. That is:

$$\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}} := \{\phi \in Hom(\mathcal{H}, \mathcal{A}) : \phi(\mathbb{1}) = 0_{\mathcal{A}}\} \tag{30}$$

As betrayed by the choice of notation, the reader may guess that $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$ is a Lie algebra. This is in fact the case, and in particular the Lie bracket is given by:

$$[f, g] = f * g - g * f \tag{31}$$

for $f, g \in \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$. We refer to $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$ as the **Lie algebra of convolution**.

The reader may now also be expecting to see the relationship between $G^{\mathcal{H}}_{\mathcal{A}}$ and $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$. We reveal this relationship in the following definition and upcoming proposition:

Definition 2.30. Let f be an element of $G^{\mathcal{H}}_{\mathcal{A}}$ and let g be an element of $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$. Then we can define the maps:

$$\exp_* : G^{\mathcal{H}}_{\mathcal{A}} \to \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}} \qquad \exp_*(g) := \sum_{n=0}^{\infty} \frac{g^{*n}}{n!} \tag{32}$$

$$\log * : \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}} \to G^{\mathcal{H}}_{\mathcal{A}} \qquad \log_*(f) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - u \circ \epsilon)^{*n}}{n} \tag{33}$$

where by f^{*n} (respectively g^{*n}) we mean the convolution product of f (respectively g) with itself n times.

While $\exp_*$ and $\log_*$ are merely formal definitions of maps, the names $\exp_*$ and $\log_*$ are justified by the similarity in the function definitions for classical exp and log as defined as formal power series. Moreover, in the case that $\mathcal{H}$ is connected (as it will be for all examples we consider throughout this work, since we will only be considering combinatorial Hopf algebras), the calculations of $\exp_*$ and $\log_*$ become finite for each $h \in \mathcal{H}$, hence there is no concern about convergence [40] (to understand why this is the case, see the discussion in Example 2.36 in the next subsection). In the setting of this text, then, the maps will also provide the correspondence between $G^{\mathcal{H}}_{\mathcal{A}}$ and $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$ as the usual exp and log do in other areas of mathematics (see Section 2.1 of [40]):

Proposition 2.31. For any $f \in G^{\mathcal{H}}_{\mathcal{A}}$ and any $g \in \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$:

$$\exp_*(\log_*(f)) = f \tag{34}$$

$$\log_*(\exp_*(g)) = g \tag{35}$$

and so $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$ is the Lie algebra of $G_{\mathcal{H}}$.

What is more, $\exp_*$ and $\log_*$ possess many of the other familiar properties of classical exp and log under certain conditions [40]:

Proposition 2.32. If $\mathcal{H}$ is cocommutative, then:

(i) $\exp_*(f+g) = \exp_*(f)\exp_*(g)$

(ii) $\log_*(f*g) = \log_*(f) + \log_*(g)$

Proof. The result follows immediately from expanding out the definitions of $\exp^*$ and $\log^*$ as formal power series. ☐

In particular, $\exp_*(0+0) = \exp_*(0)\exp_*(0)$, hence $\exp_*(0) = 1_{\mathcal{A}}$, and similarly $\log_*(1_{\mathcal{A}}) = 0$.

With this in mind, we remark that the bijection these maps provide between $G^{\mathcal{H}}_{\mathcal{A}}$ and $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$ as mentioned in Proposition 2.31 is very easy to verify, for the image of $\exp_*$ on the elements of $G^{\mathcal{H}}_{\mathcal{A}}$ is:

$$\begin{aligned}\exp_*(G^{\mathcal{H}}_{\mathcal{A}}) &= \{\exp_*(\phi) \in Hom(\mathcal{H},\mathcal{A}) : \exp_*(\phi(\mathbb{1})) = 1_{\mathcal{A}}\}\\ &= \{\psi \in Hom(\mathcal{H},\mathcal{A}) : \psi(\mathbb{1}) = 0\}\\ &= \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}\end{aligned}$$

And conversely:

$$\begin{aligned}\log_*(\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}) &= \{\log_*(\phi) \in Hom(\mathcal{H},\mathcal{A}) : \log_*(\phi(\mathbb{1})) = 0\}\\ &= \{\psi \in Hom(\mathcal{H},\mathcal{A}) : \psi(\mathbb{1}) = 1_{\mathcal{A}}\}\\ &= G^{\mathcal{H}}_{\mathcal{A}}\end{aligned}$$

Next, consider the subset of $G^{\mathcal{H}}_{\mathcal{A}}$ consisting only of the linear maps $\phi \in Hom(\mathcal{H},\mathcal{A})$ that are also algebra homomorphisms from $\mathcal{H}$ to $\mathcal{A}$, and such that $\mathcal{A}$ is a commutative algebra. Using the notation of [40], this is the subset:

$$\widetilde{G}^{\mathcal{H}}_{\mathcal{A}} := \{\phi \in Hom(\mathcal{H},\mathcal{A}) : \phi \circ m_{\mathcal{H}} = m_{\mathcal{A}} \circ (\phi \otimes \phi)\} \tag{36}$$

It turns out that this is a Lie subgroup of $G^{\mathcal{H}}_{\mathcal{A}}$, and one of particular importance to the physics community; we will see later that if we specify $\mathcal{H} = \mathcal{H}_{CK}$ and $\mathcal{A} = \mathbb{K}[L]$, then the elements of $G^{\mathcal{H}_{CK}}_{\mathcal{A}}$ that are also coalgebra homomorphisms are Feynman rules [32, 40]! We will refer to $G^{\mathcal{H}}_{\mathcal{A}}$ as the **Lie group of characters** of $\mathcal{H}$.

Since $\widetilde{G}^{\mathcal{H}}_{\mathcal{A}}$ is a Lie subgroup of $G^{\mathcal{H}}_{\mathcal{A}}$, it follows from the correspondence theorem of Lie groups and Lie algebras that the same map $\widetilde{\exp_*} = \exp_*|_{\widetilde{G}^{\mathcal{H}}_{\mathcal{A}}}$ applied to $\widetilde{G}^{\mathcal{H}}_{\mathcal{A}}$ should yield a Lie subalgebra of $\mathfrak{g}^{\mathcal{H}}_{\mathcal{A}}$, with the same bracket as before, just restricted [29]. What do the elements of this algebra look like? The answer to this question is the following proposition (this is Proposition A.3.2 of [32]):

Proposition 2.33. The characters in $\widetilde{G}^{\mathcal{H}}_{\mathcal{A}}$ are generated by the linear space of **infinitesimal characters** of $\mathcal{H}$. This is the set:

$$\widetilde{\mathfrak{g}}^{\mathcal{H}}_{\mathcal{A}} = \{\sigma \in \mathfrak{g}^{\mathcal{H}}_{\mathcal{A}} | \sigma(xy) = \sigma(x)\epsilon(y) + \epsilon(x)\sigma(y), \text{for all } x, y \in \mathcal{H}\}$$

which is a Lie algebra under the bracket $[\sigma_1, \sigma_2]_* := \sigma_1 * \sigma_2 - \sigma_2 * \sigma_1$.

An immediate corollary of Proposition 2.33 is the following:

Corollary 2.34. For every character $\phi \in \widetilde{G}^{\mathcal{H}}_{\mathcal{A}}$ there exists an infinitesimal character σ such that $\phi = \exp_*(\sigma)$.

Example 2.35 (Hopf algebra of polynomials in a single variable)**.** Let us look at an example of the above definitions, taking as our Hopf algebra $\mathcal{H} = \mathbb{K}[L]$ of polynomials in a single variable discussed in Example 2.11, and taking our algebra to be $\mathcal{A} = \mathbb{K}$, the underlying field. What does the Lie group $\widetilde{G}^{\mathbb{K}[L]}_{\mathbb{K}}$ look like? Suppose that $f \in \mathbb{K}[L]$, so that $f = k_n L^n + ... + k_1 L + k_0$, and suppose $\phi \in \widetilde{G}^{\mathbb{K}[L]}_{\mathbb{K}}$ is a character. Then by the property of algebra homomorphisms we have that:

$$\begin{aligned}\phi(f) &= \phi(k_n L^n + ... + k_1 L + k_0)\\ &= \phi(k_n L^n) + ... + \phi(k_1 L) + \phi(k_0)\\ &= k_n \phi(L^n) + ... + k_1\phi(L) + k_0\phi(1_{\mathbb{K}})\\ &= k_n \phi(L)^n + ... + k_1\phi(L) + k_0\phi(1_{\mathbb{K}})\end{aligned}$$

Hence the character ϕ is completely determined by the value in $\mathbb{K}$ chosen for L, hence it is just an evaluation of f. In other words, we have just discovered that, as sets:

$$\widetilde{G}^{\mathbb{K}[L]}_{\mathbb{K}} = \{ev_a : a \in \mathbb{K}\} \tag{37}$$

where ev_a is the evaluation function $ev_a(f) := f(a)$. So to understand the full structure of $\widetilde{G}^{\mathbb{K}[L]}_{\mathbb{K}}$ we only need to understand what the operation $*$ looks like. Let us apply the convolution of two characters to a monomial in $\mathbb{K}[L]$: for $a, b \in \mathbb{K}$ we have:

$$\begin{aligned}(ev_a * ev_b)(L^n) &= [(ev_a * ev_b)(L)]^n\\ &= [m_{\mathbb{K}} \circ (ev_a \otimes ev_b) \circ \Delta_{\mathbb{K}[L]}(L)]^n\\ &= [m_{\mathbb{K}} \circ (ev_a \otimes ev_b) \circ (L \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes L)]^n\\ &= [m_{\mathbb{K}} \circ (a \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes b)]^n\\ &= [a + b]^n\\ &= ev_{a+b}[L]^n\end{aligned}$$

And hence the group law of $\widetilde{G}^{\mathbb{K}[L]}_{\mathbb{K}}$ is just:

$$ev_a * ev_b = ev_{a+b} \tag{38}$$

giving that the neutral element of the group is e_0 and the inverse of any ev_a is just ev_{-a} [32].

Now let us look at the corresponding Lie algebra $\widetilde{\mathfrak{g}}^{\mathbb{K}[L]}_{\mathbb{K}}$. We claim that for $\sigma_a = log_*(ev_a) \in \widetilde{\mathfrak{g}}^{\mathbb{K}[L]}_{\mathbb{K}}$:

$$\sigma_a = a\partial_0 \tag{39}$$

where we use ∂_0 as shorthand for the function defined by $\partial_0(f) := \frac{\partial}{\partial L} f(L)|_{L=0}$. We will prove this claim by following the proof strategy discussed in [32]. To begin, we know that σ_a is a homomorphism, so we only have to show $\exp_*(\sigma_a)(L^n) = ev_a(L^n)$ for some monomial L^n.

First, we claim that ∂_0^{*m}, the mth convolution power of ∂_0, simply takes the mth derivative of L^n and set the remaining monomial equal to 0. We can prove this by induction on m. For $m = 1$, the result follows trivially from the definition of ∂_0. Now suppose that the result holds true for m, and consider the operator $\partial_0^{*(m+1)}$. We calculate that:

$$\begin{aligned}
\partial_0^{(m+1)}(L^n) &= \partial_0 * \partial_0^m(L^n) \\
&= m_{\mathbb{K}} \circ (\partial_0 \otimes \partial_0^m) \circ \Delta_{\mathbb{K}[L]}(L^n) \\
&= m_{\mathbb{K}} \circ (\partial_0 \otimes \partial_0^m)\left(\sum_{j=0}^{n} \binom{n}{j} L^j \otimes L^{n-j}\right) \\
&= m_{\mathbb{K}} \circ \left(\sum_{j=0}^{n} \binom{n}{j} jL^{j-1}|_{L=0} \otimes \partial_0^m L^{n-j}\right)
\end{aligned}$$

The only nonzero term in the sum above will be when $j = 1$, hence we get that:

$$\begin{aligned}
&= m_{\mathbb{K}} \circ \left(\binom{n}{1}(1) \otimes \partial_0^m L^{n-1}|_{L=0}\right) \\
&= m_{\mathbb{K}} \circ \left(\binom{n}{1}(1) \otimes (n-1)(n-2)\cdots(n-1-m+1)L^{n-1-m}|_{L=0}\right) \\
&= (n)(n-1)\cdots(n-m)L^{n-m-1}|_{L=0} \\
&= \frac{d^{m+1}}{dL^{m+1}} L^n|_{L=0}
\end{aligned}$$

as claimed.

Now it is a simple matter to verify that the elements σ_a are in fact the elements of the Lie algebra $\widetilde{\mathfrak{g}}_{\mathbb{K}}^{\mathbb{K}[L]}$ by calculating $exp_*(\sigma_a)$. Doing this, we find that (for a monomial $L^k \in \mathbb{K}[L]$):

$$\begin{aligned}
exp_*(\sigma_a)(L^k) &= \left(\sum_{n=0}^{\infty} \frac{1}{n!}\sigma_a^{*n}\right) L^k \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!}(a\partial_0)^{*n}\right) L^k \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!}a^n(\partial_0)^{*n}\right) L^k \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}a^n(\partial_0)^{*n} L^k
\end{aligned}$$

Now from the previous claim we know that every term in the above sum is equal to 0 except for the term in which $n = k$. So the above is equal to:

$$\begin{aligned}
&= k!a^k \partial_0^{*k} L^k \\
&= k!a^k k!\,(1) \\
&= a^k \\
&= ev_a(L^k)
\end{aligned}$$

as claimed. Hence:

$$\widetilde{\mathfrak{g}}_{\mathbb{K}}^{\mathbb{K}[L]} = \{\sigma_a : a \in \mathbb{K}\} \tag{40}$$

2.2.2 Further Operators on $\mathcal{H}_{CK}$

A particularly interesting special case of the maps $\exp_*$ and $\log_*$ occur when we set $\mathcal{H} = \mathcal{H}_{CK}, \mathcal{A} = \mathcal{H}_{CK}$, and define the following map:

$$\zeta(F) = \begin{cases} F & \text{if } F \text{ is a nontrivial tree, and} \\ 0 & \text{if } F \text{ is a forest (including } F = \mathbb{1}) \end{cases} \tag{41}$$

which we extend as an infinitesimal character. The reason for extending in this way is that we want ζ to be an element of $\overline{\mathfrak{g}}_{\mathcal{H}_{CK}}^{\mathcal{H}_{CK}}$. Then the map

$$\omega := exp_*(\zeta) \tag{42}$$

will be an element of $\overline{G}_{\mathcal{H}_{CK}}^{\mathcal{H}_{CK}}$. Essentially, ω is now the exponential function of rooted graph objects; that is, in analogy to exp taking the exponential generating function of some combinatorial class and returning the exponential generating function of forests of these objects, so ω takes a rooted tree T and returns all forests of rooted trees obtained from T, with weights given by the number of ways this forest can be obtained over the factorial of (one more than) the number of cut edges. In other words, for $t \in \mathcal{T}$ we have:

$$\omega(t) = \sum_{c \subseteq E(t)} \frac{1}{(n_c + 1)!} P^c(t) R^c(t) \tag{43}$$

where n_c denotes the number of edges in the cut c. We have only written this expression for ω on trees, but note that since it is $\exp_*$ applied to an infinitesimal character, we know to extend ω as an algebra homomorphism, again by Corollary 2.34.

Remark. We comment that this entire construction could also have been performed the other way around: namely, instead of defining $\exp_*$ and then defining ω, it would have been possible to define ω and then derive the definition of $\exp_*$. This is the approach of [41] (though we point out that in that context ω is denoted by $\exp_*$). The approach we have taken can be read about in more detail in both [40] and [32].

Example 2.36. Let us see an example of ω. Fix a tree $t = \overset{\bullet}{\underset{\bullet}{\bullet}}$. Then:

$$\Delta(t) = \overset{\bullet}{\underset{\bullet}{\bullet}} \otimes \mathbb{1} + \mathbb{1} \otimes \overset{\bullet}{\underset{\bullet}{\bullet}} + \overset{\bullet}{\bullet} \otimes \bullet + \bullet \otimes \overset{\bullet}{\bullet}$$

and:

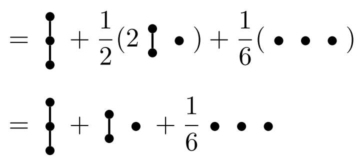
$$\Delta^2(t) = (Id \otimes \Delta) \circ \Delta(t) = \overset{\bullet}{\underset{\bullet}{\bullet}} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \overset{\bullet}{\underset{\bullet}{\bullet}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \overset{\bullet}{\underset{\bullet}{\bullet}} + \overset{\bullet}{\bullet} \otimes \bullet \otimes \mathbb{1} + \bullet \otimes \overset{\bullet}{\bullet} \otimes \mathbb{1}$$
$$+ \overset{\bullet}{\bullet} \otimes \mathbb{1} \otimes \bullet + \mathbb{1} + \overset{\bullet}{\bullet} \otimes \bullet + \bullet \otimes \bullet \otimes \bullet + \bullet \otimes \mathbb{1} \otimes \overset{\bullet}{\bullet} + \mathbb{1} \otimes \bullet \otimes \overset{\bullet}{\bullet}$$

Now note that when we apply the map ζ to each term in each tensor product, every term having a $\mathbb{1}$ in its tensor product will go to 0. Indeed, ζ is an infinitesimal character, meaning it is $\log_*$ of some algebra homomorphism ϕ (by Corollary 2.34), and hence $\log_*(\phi(\mathbb{1})) = \log_*(u \circ \epsilon) = 0$. Moreover, by the $(n-1)$st power of Δ (for n the number of vertices in F), all terms will contain a $\mathbb{1}$. Hence we can conclude that computations of ω are *locally finite*: that is, ω applied to any forest will always be a finite computation [40].

Finally, observe that every term in the expansion of $\omega(t)$ with a forest in the tensor product will also go to 0, again since ζ is infinitesimal (though there are no such terms in this example).

To finish the example, then, we have that:

$$\begin{aligned}\omega(t) &= \exp_*(\zeta)(t)\\ &= \zeta(t) + \frac{1}{2}(\zeta * \zeta)(t) + \frac{1}{6}(\zeta * \zeta * \zeta)(t) + 0\end{aligned}$$



Finally, we remark that the compositional inverse of ω also has interesting properties. Most notably:

Lemma 2.37. *Let l_i be the ladder with i vertices. The $\omega^{-1}(l_i)$ is a primitive element of $\mathcal{H}_{CK}$.*

Proof Sketch: The proof follows quickly from simple definition pushing. Solving recursively for ω^{-1} from equation (43), we find that on the space of ladders the following formula holds for ω^{-1}:

$$\omega^{-1}(l_i) = \sum_{c \subseteq E(l_i)} \frac{(-1)^{n_c+1}}{(n_c+1)} P^c(l_i) R^c(l_i) \tag{44}$$

But then this is just the ith coefficient in the series $\log\left(\sum_{n=0}^{\infty} l_i\right)$. From [21], we know that this is P_i, the ith basis element for the space of primitive elements of the Hopf algebra of ladders. □

Throughout the rest of this text, P_i will be reserved for the primitive element $\omega^{-1}(l_i)$ as it is used here. We will see these elements return later on in an important way (see Section 5.1).

Remark. We make a brief aside to mention an insightful alternative way to construct $\mathcal{H}_{CK}$. Rather than constructing the Hopf algebra out of combinatorial operations over the vector space of forests of rooted trees, it is noted on page 41 of [9] that another way to think about $\mathcal{H}_{CK}$ is as the algebra of coordinates over the Lie group $G_{\mathbb{K}}^{\mathcal{H}_{CK}}$; in this way we are constructing the Lie group first and viewing $\mathcal{H}_{CK}$ as being produced from it. This is also the perspective taken in [7] in order to apply techniques of differential geometry to speed up computations of Feynman integrals.

2.2.3 Prelie Algebras

So far we have discussed ways of obtaining various Lie algebras and groups from $\mathcal{H}_{CK}$. To conclude this subsection we present the definition of a related algebraic object [34]:

Definition 2.38. A **left prelie algebra** is an algebra $A_L = (V, \triangleright, u)$ where the product $\triangleright$ satisfies the following relation for all $a, b, c \in A_L$:

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c) \tag{45}$$

Additionally, a **right prelie algebra** is an algebra $A_R = (V, \triangleleft, u)$ such that all $a, b, c \in A_R$ satisfy:

$$(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b) \tag{46}$$

We call equation (45) the **left prelie relation** and equation (46) the **right prelie relation**).

It is valid, however, to reference both (45) and (46) as simply the **prelie relation** (as we will do later), as every left prelie algebra $A_L = (V, \triangleright, u)$ is also a right prelie algebra $A_R = (V, \triangleleft, u)$ by defining $a \triangleleft b := b \triangleright a$ for all $a, b \in A_L$ [34].

While prelie algebras are likely less well-known than their Lie algebra counterparts, the "Lie" in prelie heralds that there exists a relationship between the two objects. Namely, every left prelie algebra A_L may be turned into a Lie algebra by imposing the bracket $[a, b] := a \triangleright b - b \triangleright a$ for all $a, b \in A_L$. In the same way, a right prelie algebra A_R may be turned into a Lie algebra with the bracket $[a, b] := a \triangleleft b - b \triangleleft a$ for all $a, b \in A_R$. This claim may be verified easily by writing out the standard Jacobi identity for Lie algebras and using the prelie relations above.

The biggest reason prelie algebras are of interest to us is the following:

Proposition 2.39. The set of primitive elements of the Grossman-Larson Hopf algebra, $Prim(\mathcal{H}_{GL})$, is a left prelie algebra.

Finally, in addition to discussing the enveloping algebra of a Lie algebra as we did in Section 2, a key theorem for us later on will be a description of the universal enveloping algebra of a prelie algebra, due to Oudom and Guin (this is Proposition 2.7 of [38]):

Theorem 2.40. *Let $(L, \circ)$ be a prelie algebra, and let $S(L)$ be the symmetric algebra of L with the usual shuffle coproduct Δ. Further set $\Delta(C) = \sum_i C_{i,1} \otimes C_{i,2}$. Then there is a unique way to extend $\circ$ to $S(L)$ such that:*

i $A \circ 1 = A$

ii $T \circ BX = (T \circ B) \circ X - T \circ (B \circ X)$

iii $AB \circ C = (A \circ \sum_i C_{i,1})(B \circ \sum_i C_{i,2})$

where A, B, C are in $S(L)$ and X, T are in L.

Remark. As an interesting historical aside, the underlying notion of prelie algebras on trees was first considered by Cayley [5], before the notion of Lie algebras was even established [34].[7]

[7]Thanks to Nick Olson-Harris for first bringing this to the author's attention.

2.3 Renormalization

Up until now, we have made the claim several times that this work is motivated by the physics of quantum field theory, but so far have not given any credence to this claim. In this subsection, we will give an overview of the physics which is the motivation for this work and also for the original construction of the Connes-Kreimer Hopf algebra. This motivation is the concept of **renormalization** in quantum field theory.

The use of renormalization is widespread in physics and not isolated to quantum field theory alone. In fact, its first use was in classical particle physics in the 1950's, and similar ideas of changing the scale of a physical system can be traced back as far as ancient Greece with attempts at solving the Delian problem.

In its greatest generality, the "pipeline" of quantum field theory is as follows: physicists at large particle accelerators perform experiments involving particle interactions, which we model with graphs (in the graph theoretical sense), called **Feynman diagrams**. We then use the data encoded by these graphs to write down corresponding Feynman integrals, which are determined up to a choice of **Feynman rules** (see Section 2.3 for a formal introduction of Feynman rules). Now these integrals are the objects that physicists actually care about (we typically think of the Feynman graph as just "standing in" for the integral), however the integrals are very often divergent (especially in the cases that physicists wish to study) [49]. Hence to get around this issue, we must change the scope at which we are viewing the objects; we do this by recursively subtracting out the integrands of divergent integrals coming from proper subgraphs of our original Feynman diagram (we call these subgraphs *subdivergences*) in a prescribed way, called a **renormalization prescription**. The new, **renormalized** Feynman integral is now convergent, and gives information from which physicists are able to make predictions.

We remark that this process has been highly refined, and that quantum field theory has been dubbed by some "the most precisely tested theory in the history of science"[8] [37]. It was not until the work of Connes and Kreimer, however, that this process of renormalization was given the insightful mathematical paradigm that we have for it today: that is, renormalization ultimately works because there exists a Hopf algebra structure on the set of Feynman diagrams, whose antipode corresponds to renormalization!

The aim of this subsection is to give some more details of this process. First, we give a short overview of combinatorial quantum field theories and their associated Feynman diagrams. Following this, we will give renormalization a rigorous definition in the context of Hopf algebras (though not in the context of any involved physics), and will demonstrate the definitions with a worked-out example. Finally, we conclude with a discussion of the renormalization group and the renormalization group equation, from which our central questions emanate.

2.3.1 An Aside on Feynman Diagrams

In graph theory, a labelled graph object G is usually defined in terms of two sets—a vertex set $V(G)$ which is a subset of the natural numbers, or some other labelling set, and an edge set $E(G)$ which consists of (unordered) pairs of elements of $V(G)$. However, it is possible to start with a different paradigm and build similar objects from sets of half edges (and pairs of half edges) instead. This definition is as follows (see page 35 of [49] and also [51]):

Definition 2.41. A graph G is a set of half edges along with:

[8] In predicting the anomalous magnetic moment of the electron in quantum electrodynamics, experiments and theoretical predictions agree up to 14 significant digits. [37]

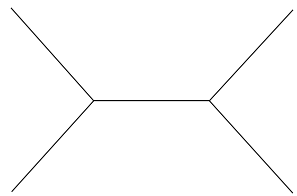


Figure 12: An unlabelled graph with half edges

- A set $V(G)$ of disjoint subsets of half edges known as **vertices** which partition the set of half edges, and
- A set $E(G)$ of disjoint pairs of half edges known as **internal edges**

The only difference between this way of defining graphs and the usual way of defining graphs is that graphs constructed from half edges will come with a set of **external edges**, which are half edges that are left unpaired (and hence do not appear in graphs in the usual sense), and **internal edges**, which are pairs of half edges constituting the edges we usually think of as belonging to a graph. One will observe that—for a given fixed number of vertices as a ground set—there are actually more graphs constructed in terms of half edges than there are in the normal graph theoretic model, since we can append different numbers of half edges to what is otherwise the same graph. However, we tend to think of the main part of the graph as that defined by the internal edges, with the external edges being extra information relevant to the underlying physics.

We also remark that in the same way that unlabelled graphs are normally defined from labelled graphs, we define unlabelled graphs with half edges to be the equivalence classes of labelled graphs having half edges with respect to the equivalence relation of graph isomorphism. See Figure 12. All graphs with half edges, including Feynman diagrams, are drawn with the *tikz-feynman* package created by Joshua Ellis [16].

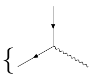
We will also include here the notion of a combinatorial physical theory, as found in [49] pp.37-38. With this definition, we are able to work with a version of quantum field theory abstracted to a combinatorial setting.

Definition 2.42 (Yeats)**.** A **combinatorial physical theory** is a set of half edge types along with:

1. a set of pairs of not necessarily distinct half edge types defining the permissible edge types,
2. a set of multisets of half edge types defining the permissible vertex types,
3. an integer associated to each edge type and each vertex type, known as a **power counting weight**, and
4. a nonnegative integer representing the dimension of spacetime.

Let us look at an example of a combinatorial physical theory. For brevity, we will only include one: that of quantum electrodynamics (QED); see [49] for a discussion of many others.

To define QED as a combinatorial physical theory, we will run through each of the items of 2.42. In QED, the set of half edges we consider is $S = \{\longrightarrow, \rightarrowtail, \rightsquigarrow\}$; that is, S consists of a left fermion half edge, a right fermion half edge, and a photon half edge. Then:

1. The possible half-edge pairings to form internal edges is ⟶ with ➤— and 〰 with 〰. Hence the set of possible internal edges is {——▶——, 〰〰〰}.
2. The only possible multiset of half edges giving an allowed vertex is when all three half edge types meet together. Therefore the set of possible vertices is {}.
3. The power counting weights associated to each internal edge type and each vertex type are:

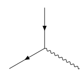
Edge/Vertex Type	Power Counting Weight
——▶——	1
〰〰〰	2
	0

Table 1: QED power counting weights

4. Finally, the dimension of spacetime is $D = 4$.

And that is all. With the four-point "recipe" given above, we have an abstract version of quantum electrodynamics relying solely on combinatorics. Graphs in the theory can be any combination of allowed vertices with allowed internal edge types (with parallel edges also allowed). For example, we can use the rules above to construct a C_4 and a K_4 with the following vertex pairings:

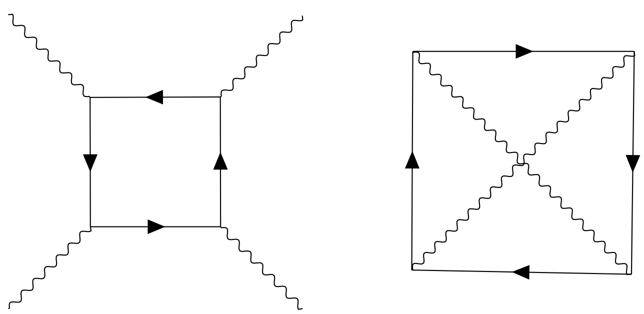


Figure 13: A C_4 and a K_4 in QED.

As we remarked above, the external edges of graphs built from half edges are thought of as extra information pertaining to the graph. Note in particular that the two graphs in Figure 13 have different multisets of external edges. This ends up being a very important feature to graphs in a combinatorial physical theory; we refer to the multiset of external edges of a graph in such a theory as the **external leg structure** of the graph. To this end, the C_4 in 13 has external leg structure {〰, 〰, 〰, 〰}, while the K_4 has external leg structure {}.

To continue, we require the following notion from graph theory:

Definition 2.43. The **loop number** (or **first Betti number**, **cyclomatic number**) of a graph is the dimension of the cycle space of the graph.

An equivalent way to calculate the loop number is to simply count the number of edges of a graph not in any one of its spanning trees [49]. For more on cycle space, see Section 1.9 of [14]. For more on the loop number of a graph and its importance in quantum field theory, see Section 5.5 of [49]

One may be wondering at this point what kind of freedom there is in coming up with a combinatorial physical theory? That is, should it be possible to fill out the items in Definition 2.42 in an arbitrary way? At this point, the answer is yes, that should in fact be possible. However, many combinatorial physical theories one may envision end up being unrealistic in the world of applied physics, and this phenomenon is captured in part by the following notion (see [49], p.39):

Definition 2.44. For a Feynman graph G in a combinatorial physical theory T, let $w(a)$ be the power counting weight of a, where a is an internal edge of a vertex of G and let D be the dimension of spacetime. The **superficial degree of divergence** is the quantity:

$$D\ell - \sum_{e \in E(G)} w(e) - \sum_{v \in V(G)} w(v) \tag{47}$$

where ℓ is the loop number of the graph.

The following translation of renormalizablity into this combinatorial language is due to Yeats [49, 53]:

Definition 2.45. A combinatorial physical theory T is said to be **renormalizable** if the superficial degree of divergence of every graph in the theory depends only on the external leg structure of each graph.

Hence the physical theories one may construct arbitrarily are likely not to be renormalizable, and as such will not be useful theories in the realm of physics. In fact, one can even view this as one reason there is not yet a useful theory of quantum gravity: if one constructs a combinatorial physical theory according to Definition 2.42, using the graviton as a particle and the correct internal edge structures, one will find that the resulting theory is not renormalizable in the sense of Definition 2.45 [50].

In the next subsection, we will briefly explain the connection between Feynman graphs and the rooted trees which are of central interest to us, and will demonstrate what renormalization looks like in the context of $\mathcal{H}_{CK}$.

2.3.2 From Feynman Diagrams to Trees

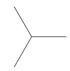
As discussed in the introduction, to each Feynman graph in a combinatorial physical theory we associate an integral (sometimes called a **Feynman integral**) which gives this particular graph's contribution to the total scattering amplitude. However, these integrals very often diverge [49, 9, 18]. One of the great advantages of the setup in the previous section is that—given a graph in a combinatorial physical theory—it is possible to tell whether or not the Feynman integral associated to the graph in the theory will diverge or not based solely on combinatorial methods[9]:

Definition 2.46 (pp. 12 and 42 of [49])**.** A graph G in a given combinatorial physical theory T is said to be **divergent** if its associated Feynman integral is divergent, and **convergent** if the integral is convergent. Furthermore, if G contains a bridgeless, proper (not necessarily connected) subgraph H whose associated integral is divergent, we say that H is a **subdivergence** of G.

[9]There are also some additional physical assumptions to make; namely, in what follows we use **divergence** to mean the **ultraviolet divergence** of the associated integrals. A discussion to this end goes beyond the scope of this work, but see Section 5.2 of [49] for the complete details.

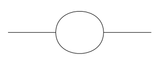
Proposition 2.47. Let G be a graph in a combinatorial physical theory T, and let $sdd(G)$ be the superficial degree of divergence of G. If $sdd(G) \geq 0$, then G is divergent. Otherwise, G is convergent.[10]

The proof of this proposition is only a matter of translating the usual method of **power counting** in physics into the combinatorial framework of the previous section; see [49]. Let us consider an example of some divergent graphs.

Example 2.48. Consider the combinatorial physical theory of unlabelled 3-regular graphs. That is, we have one half edge type ( ), one internal edge type from the pairing of the one half edge type with itself ( ), one vertex formed by the pairing of three half edges () , with the power counting weights of edges equal to 2, the power counting weight of vertices equal to 0, and the dimension of spacetime equal to 6 (this makes the theory renormalizable) [49]. This theory is known in the physics community as ϕ^3-theory.

Consider the graph G in equation (48):

$$G = \quad (48)$$

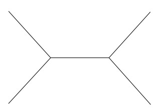


As G belongs to ϕ^3-theory, we can calculate its superficial degree of divergence according to equation (47). G has one loop, so $\ell = 1$. Moreover, in ϕ^3-theory we have that $D = 6$ and $w(e) = 2$ and $w(v) = 0$ for every $e \in E(G)$ and $v \in V(G)$. So we calculate:

$$\begin{aligned} sdd(G) &= (6)(1) - [2+2] - [0+0] \\ &= 2 \end{aligned}$$

hence G is divergent according to Proposition 2.47. On the other hand, consider the graph H:

$$H = \quad (49)$$

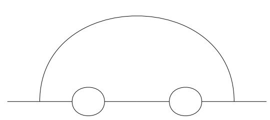


Then H is not divergent, as we calculate:

$$\begin{aligned} sdd(H) &= (6)(0) - 2 \\ &= -2 \end{aligned}$$

As one final example, consider the graph K having the form:

$$K = \quad (50)$$



[10]The situation where $sdd(G) = 0$ leads to the corresponding integral being divergent by a factor of a logarithm, so in this case we that that G is **logarithmically divergent**[49, 50].

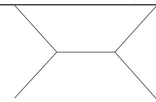
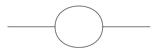
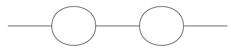
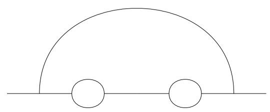
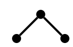
Graph in ϕ^3	Element in $\mathcal{H}_{CK}$
	$\mathbb{1}$
	
	
	

Table 2: Some graphs in ϕ^3 and their corresponding subdivergence structure.

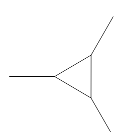
Here $\ell = 3$ and $|E(K)| = 8$, so we find that:

$$\begin{aligned} sdd(K) &= (6)(3) - (8)(2) \\ &= 2 \end{aligned}$$

and so K is also divergent.

We can now finally lend some intuition to the remarks made in Sections 1.1 and 1.1 that $\mathcal{H}_{CK}$ models renormalization according to the BPHZ renormalization scheme. If we wanted to renormalize by subtracting off divergent factors in Feynman integrals, it would be possible to renormalize the integral associated to G with a single subtraction, signalled by the fact that G contains no divergent subgraphs. However this is not the case with K. Using Proposition 2.47, we found that K is divergent in ϕ^3-theory, but the fact that is has two copies of G as subdivergences means that a single subtraction will no longer suffice; the two copies of G nested inside of K correspond to the fact that—in the Feynman integral corresponding to K—there are divergent factors (of the kind making the integral of G divergent) that are nested inside of a third. Hence the forests of rooted tree structures that make up the elements of $\mathcal{H}_{CK}$ are simply modelling the nested subdivergence structure of Feynman graphs in a given theory; see Table 2.

We may ask ourselves if the subdivergences of every graph in a combinatorial physical theory have the structure of a forest of rooted trees? That is to say, what if a graph G in a given theory contains subdivergences that are not nested in a clean way as they are in in Table 2, but rather share at least one vertex or internal edge? In fact this is possible, and the graph in Figure 14 is one such example [49]. Indeed, one can compute that the superficial degree of the graph

$L =$ 

is positive, and hence L is divergent. Yet the graph in Figure 14 has two copies L that share two vertices (the top and bottom corners of the square) and an edge (the edge splitting the square into two triangles). These kinds of subdivergences are called **overlapping subdivergences** and the renormalization procedure in

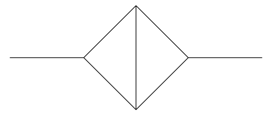


Figure 14: A graph with overlapping subdivergences

this situation calls for a separate treatment. Ultimately, however, this separate treatment is still modelled by the Hopf structure of $\mathcal{H}_{CK}$; instead of mapping the subdivergence structure to a forest of rooted trees, one takes an appropriate linear combination of forests [49, 9]. For the sake of brevity we will not discuss this case of overlapping subdivergences here, but the interested reader is directed to see the appendix of [9]. There, the authors explain the full procedure by which one resolves overlapping subdivergences, and also prove that $\mathcal{H}_{CK}$ (possibly with decorations on trees) is the only Hopf algebra needed to model this procedure [9].

2.3.3 Green's Functions

At the very center of our research problem is the notion of a Green's function. Indeed, it is these objects from which our particular interest in sequences of trees emanates. In this section, we will define Green's functions in a way that might at first seem nonstandard, and will proceed to motivate the definition by summarizing the underlying physics. As a result, this section's main goal will be to form the link between our current problem and the quantum field theory that motivates it.

We start with some definitions:

In the study of differential equations, the **method of Green's functions** is a procedure for transforming a given differential equation having a particular form into one that is simpler to solve. The solution of the modified equation, called a **Green's function** is then related to the original, desired solution via applying function convolution in a prescribed way.

More specifically:

Definition 2.49. [48] Let L be a linear differential operator satisfying:

$$Ly(t) = g(t)$$

Then a **Green's function** $G(s,t)$ is any solution of the corresponding differential equation:

$$LG(s,t) = \delta(t-s)$$

where δ is the Dirac delta function.

In applied fields of physics and engineering, Green's functions are often used as a method by which to solve ordinary and partial differential equations.

While the previous definition will still hold true, for us a Green's function will mean something even more specific:

Definition 2.50. Let $(t_n)_{n\geq 1}$ be a sequence with $t_n \in \mathbb{K}\mathcal{T}_n$ that generates a Hopf subalgebra of $\mathcal{H}_{CK}$ and such that $t_n \neq 0$ for all n, and define $X = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ to be the corresponding series of these elements. Further define Feynman rules $\phi \in \mathcal{G} = \widetilde{G}^{\mathcal{H}_{CK}}_{\mathbb{K}[L]}$. Then a **Green's function** $G(x, L)$ is:

$$G(x, L) := \phi(X) \tag{51}$$

The former definition is clear in the sense that the Green's functions we will be working with are going to be solutions to the renormalization group equation (60). However the latter definition is in need of some more rationale.

Let us begin with a very broad scope of quantum field theory. Recall the quote from F.J. Dyson included at the beginning of this work (in Section 1.1). For convenience, we include it again here:

> Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes,' he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wavefunction.' I said to him, 'You're crazy.' But he wasn't. ([44], originally quoted from [42])

This notion of "sum over histories" is precisely what Green's functions will be encapsulating for us. A sum over an infinite sequence of trees after applying Feynman rules is really representing an infinite sum of Feynman graphs after applying Feynman rules, which still further is representing an infinite sum of possible particle interactions, with Feynman rules giving the corresponding amplitude that particular history contributes to the whole amplitude.

We will now try to make this statement precise.

Unlike quantum mechanics in which we are interested in the movement of individual particles through time and space, in quantum field theory we are interested in taking into account large quantities (often of an infinite number) of particles all at once. To this end, instead of considering the kinds of models for particles arising in the quantum mechanical setting, we consider a field ϕ (not in the abstract algebra sense, but in the vector field sense) and model particles as disturbances in the field [50]. We mention the field may be a vector field, but we also remark that it may be some other type of field as well; we simply use **field** to mean a function returning some value for every point in spacetime (that is, $\mathbb{R}^n$). If the values returned are vectors, then the field is a vector field, while if the values returned are scalars, then the field is a scalar field. Other types of fields include tensor fields, spinor fields, and supersymmetric fields [46].

Hence if particles are represented by changes in the field (call it ϕ), then we need some simple way of modeling how the field changes. This is accomplished mathematically by something called a **Lagrangian density**, which generalizes the notion of a **Lagrangian** from calculus [46]. In the calculus setting, a Lagrangian is (in its simplest sense) an equation packaging up all of the components of the method of Lagrange multipliers in optimization. For example, if we are trying to optimize a two-variable function $f(x, y)$ over the constraint equation $g(x, y) = c$, then the Lagrangian is given by the equation:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c) \tag{52}$$

for λ a constant. (One can see that the equation $\nabla L = 0$ then gives the system of equations sought in the method of Lagrange multipliers, as we would expect).

Analogously, a Lagrangian density[11] is the generalization of equation (52) to describe the dynamics of a whole field—that is, to represent how the field changes with changes in both space and time. To this end, (using common notation) the Lagrangian density $\mathcal{L}$ will be a function $\mathcal{L}(\Phi(\vec{x}), \partial_\mu \Phi(\vec{x}))$ where $\vec{x} = (x_\mu)$, $\mu = (0, 1, 2, ..., D-1)$, Φ represents a field parameterized by the time-space coordinates $x_0, x_1, ..., x_{D-1}$, and D is the dimension of spacetime [46]. Here, we are using **index notation** as is standard practice in physics. Namely, we mean that μ varies over all values in the tuple above, where $\partial_\mu := \frac{\partial}{\partial x_\mu}$, and where repeated indices represent summation ranging over the index set [12]. For more on index notation, see [12].

[11] By abuse of terminology, this is often just referred to in the literature as a **Lagrangian**, too. However, as we are explaining its relation to the Lagrangian in the classical sense, we will stick with referring to it as a Lagrangian density.

The relationship between L and $\mathcal{L}$ is that L is the integral of $\mathcal{L}$ over all spatial coordinates (but excluding time). Namely:

$$L = \int \mathcal{L} dx_1 dx_2 ... dx_{D-1} \tag{53}$$

(In this setting, it is customary to have x_0 representing the time coordinate) [46].

The central idea here is that integrating the Lagrangian (in the classical mechanical setting) with respect to position and the Lagrangian density (in the field theory setting) over all spacetime yields a quantity known as the **action** (denoted by S)[12], which yields many of the most important properties of the field. For example, in the classical field theory setting, S will encode both the field laws (think laws of electromagnetism or laws of motion) and the symmetries giving rise to the field's laws of conservation [46]! In our setting of quantum field theory, the action determines the transition amplitude between quantum states [46]. The formula for the action looks like this[13]:

$$S = \int d^D x \mathcal{L}(\Phi(\vec{x}), \partial_\mu \Phi(\vec{x})) \tag{54}$$

where by d^D we mean $dx_0 dx_1 ... dx_{D-1}$. Now how does this determine the aforementioned transition amplitude, which is the key piece of information we desire? The formula for the amplitude is as follows [50]:

$$A = \int D\phi \exp(iS) \tag{55}$$

where by $D\phi$ we mean we are integrating over all possible values of the field ϕ [50]. This is very much ill-defined, but as discussed in [50], setting up the calculations in this way give a good heuristic for what is to follow.

If we add a term $J\phi$ in addition to the terms of the Lagrangian density to allow for the creation and annihilation of particles, the equation (55) becomes:

$$Z[J] = \int D\phi \exp(iS + \int J\phi) \tag{56}$$

the quantity $Z[J]$ is called the **generating functional** of the field theory; one can think of this almost like a generating function of functionals (where by functionals, we mean a function that returns another function). In another words, we have broken up the computation for the desired transition amplitude into infinitely many sub-computations, where the nth term (that is, the nth functional in the generating functional) is called the **n-point correlation function**, $\langle \phi(x_1)\phi(x_2)...\phi(x_n) \rangle$, obtained by taking n partial derivatives of $Z[J]$ [25]:

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n) \rangle = \frac{h^n}{(2\pi)^n Z[J]} \frac{\partial^n Z[J]}{\partial J(x_1)...\partial J(x_n)}|_{J=0}$$

where h is the Plank constant, $h = 6.62607004 \times 10^{-34} J \cdot s$. Performing the indicated operations on $Z[J]$ yields [25, 50]:

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n) \rangle = \frac{\int D\phi \, \phi(x_1)\phi(x_2)...\phi(x_n) \exp(iS)}{\int D\phi \, \exp(iS)} \tag{57}$$

[12] We realize that this notation is confusing in light of the fact that in the rest of this thesis we use S to mean the antipode of a Hopf algebra. However, we wish to avoid deviating too much from the standard notation of the physics literature, and believe this will not be too much of an inconvenience, as the use of S in this way will be isolated to this section alone.

[13] Note that we are following the convention of physics as used in our sources, wherein the differential elements d^D corresponding to an integral appear on the left side of the integrand rather than on the right.

Finally, this expression is a Green's function in the analytic sense of Definition 2.49. However, it also has a combinatorial description; namely, we can apply a theorem due to Wick (proved in a different setting) that informs us that the n-point correlation functions $\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle$ may be computed as certain sums over 2-point correlation functions $\langle \phi(x_i)\phi(x_j)\rangle$ [50, 54]. Using this, together with a specific protocol developed by the physics community (particularly Feynman) known as **Feynman rules**, we can assemble Feynman graphs from the terms of $\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle$; namely the sum over the 2-point correlation functions given by Wick's Theorem give the edges of the Feynman diagram. The end result will be that $\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle$ is an expansion in Feynman graphs of the theory having n external legs. The methods referenced and briefly summarized in Section 2.3.2 then take us from Feynman graphs to rooted trees, finally yielding our Definition 2.50.

We have left out many details for the sake of brevity, but will refer the reader to [25, 44, 40] for immediately accessible examples of extracting the combinatorics of $\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle$. We further recommend Chapter 1 of [54] for an excellent and accessible development of this whole topic. Namely, see Section 1.2 of [54] for a development of the path integral formulation of quantum field theory from scratch, and Sections 1.1 through 1.7 for how Feynman diagrams emerge from this perspective.

Remark. When they were first developed by Feynman, the notion Feynman rules was used to mean the rules of encoding a path integral measuring an amplitude as a Feynman diagram in order to organize the complicated computation. Nowadays, however, this notion almost universally refers to the inverse of this map that takes a Feynman diagram to its associated integral, as we have described it elsewhere in this document [32].

2.3.4 The Renormalization Group and General Tree Feynman Rules

We now have all the background knowledge we need in order to present the central mathematical tool working behind the scenes throughout this work and related works. Our running examples concerning the Hopf algebra of polynomials $\mathbb{K}[L]$ might have seemed arbitrary up until now, but now we see that they are actually an integral part of the story:

Definition 2.51. As in the last section, let $\mathcal{G} = \widetilde{G}^{\mathcal{H}_{CK}}_{\mathbb{K}[L]}$, the Lie group of characters of $\mathcal{H}_{CK}$ with target algebra the Hopf algebra of polynomials in a single indeterminate L (See Section 2.2 and Example 2.11). The indeterminate L is called a **kinematic variable**, and elements $\phi \in \mathcal{G}$ that are also coalgebra homomorphisms are called **Feynman rules**[14].

In other words, Feynman rules are the elements of the Lie group of characters from Section 2.2 when $\mathcal{A}$ is the Hopf algebra $\mathbb{K}[L]$ and when the algebra homomorphism are additionally coalgebra homomorphisms.

We mentioned in the introduction to the previous section that the process of renormalization demonstrated therein is dependant on the choice of Feynman rules, and hope that this statement has now been made clear: renormalization is the recursive subtraction of the integrands of Feynman integrals according to the terms obtained from the antipode S of $\mathcal{H}_{CK}$ applied to the rooted tree-structure of subdivergences of a Feynman graph in a given theory. However the integrands themselves are obtained by applying a map $\phi : \mathcal{H}_{CK} \to \mathbb{K}[L]$, where we now know that ϕ must be an element of the character group $\widetilde{G}^{\mathcal{H}_{CK}}_{\mathbb{K}[L]}$.

But now there is something really special about this situation: not only do we have the ability to change the value of Feynman integrals by taking different $\phi \in \mathcal{G}$, we also have the ability to vary L over different value of $\mathbb{K}$, since L is just an indeterminate! These changes in the kinematic variable L describe

[14] In the literature, Feynman rules are sometimes defined merely as algebra homomorphisms, but then a distinction is made between Feynman rules in full generality versus Feynman rules that show up in physics. See Section 3.1 of [40]

the result of changing the energy scale at which we choose to observe the underlying physical processes, and ultimately are a factor that help determine what the value of the final integral will be [32]. If we let ϕ_L denote the map from $\mathcal{H}_{CK} \times \mathbb{K} \to \mathbb{K}$ obtained by evaluating $\phi \in \mathcal{G}$ at L (in other words, for all $F \in \mathcal{H}_{CK}$, we have $\phi_L(F) = \phi(F)(L)$), then:

Definition 2.52. The **renormalization group** $\mathfrak{RG}$ is:

$$\mathfrak{RG} = \{\phi_L | L \in \mathbb{K}, \phi \in \mathcal{G}\} \tag{58}$$

where the group operation is convolution.

What kind of structure does $\mathfrak{RG}$ have? The answer to this question lies in the following lemma:

Lemma 2.53. *Let $\phi \in \mathcal{G}$ be valid Feynman rules (i.e. a Hopf algebra homomorphism). Then $\phi = \exp_*(L\sigma)$ for $\sigma : \mathcal{H}_{CK} \to \mathbb{K}$ an infinitesimal character.*

Proof. We will closely follow the proof given in [32]. To begin, let $\phi \in \mathcal{G}$, but rather than considering $\log_*(\phi)$ to obtain the desired result, consider taking an evaluation of $\log_*(\phi)$ at some value $a \in \mathbb{K}$. If we do this we can calculate:

$$\begin{aligned}
ev_a(\log_*(\phi)) &= ev_a\left(\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\phi^{*i}}{i}\right)\\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}ev_a(\phi^{*i})\\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}ev_a(m^i_{\mathbb{K}[L]} \circ (\underbrace{\phi \otimes ... \otimes \phi}_{n \text{ copies}}) \circ \Delta^{i-1}_{\mathcal{H}_{CK}}) \qquad \text{(Since } ev_a \text{ is an algebra homomorphism.)}\\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(m^i_{\mathbb{K}} \circ (\underbrace{ev_a \circ \phi \otimes ... \otimes ev_a \circ \phi}_{n \text{ copies}}) \circ \Delta^{i-1}_{\mathcal{H}_{CK}}) \qquad \text{(Since } ev_a \text{ is an algebra homomorphism.)}\\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(m^i_{\mathbb{K}} \circ (\underbrace{ev_a \otimes ... \otimes ev_a}_{n \text{ copies}}) \circ (\underbrace{\phi \otimes ... \otimes \phi}_{n \text{ copies}})\Delta^{i-1}_{\mathcal{H}_{CK}})\\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}m^i_{\mathbb{K}} \circ (\underbrace{ev_a \otimes ... \otimes ev_a}_{n \text{ copies}}) \circ \Delta^{i-1}_{\mathbb{K}[L]}(\phi) \qquad \text{(Since } \phi \text{ is a coalgebra homomorphism.)}\\
&= \left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}m^i_{\mathbb{K}} \circ (ev_a \otimes ... \otimes ev_a) \circ \Delta^{i-1}_{\mathbb{K}[L]}\right)(\phi)\\
&= \log_*(ev_a) \circ (\phi)\\
&= \sigma_a \circ \phi \qquad \text{(By Example 2.35.)}\\
&= a\partial_0\phi\\
&= ev_a(L\partial_0\phi)
\end{aligned}$$

(Note that we have used the convolution sign $*$ in two different ways as in [32], though we have made these ways explicit by the intermediate steps).

Hence we have that for any a, $ev_a(\log_*(\phi)) = ev_a(L\partial_0\phi)$. Thus $\log_*(\phi) = L\partial_0\phi$, and as $\sigma = \partial_0\phi$ is an infinitesimal character from $\mathcal{H}_{CK} \to \mathbb{K}$, we have $\log_*(\phi) = L\sigma$ and hence $\phi = \exp_*(L\sigma)$, as claimed. □

We can use the fact that the elements $\phi \in \mathcal{G}$ are Hopf algebra homomorphisms and the above lemma to show that the ϕ_L obey a special property, known as the **renormalization group equation** (c.f. Section A.4 of [32]):

Lemma 2.54 (The Renormalization Group Equation, Algebraic Version)**.**

$$\phi_{L_1} * \phi_{L_2} = \phi_{L_1+L_2} \tag{59}$$

The analytic version of this equation is:

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma(x)\right)G(x, L) = 0 \tag{60}$$

We will extensively use equation (60) in the rest of this thesis, particularly in Sections 3.1 and 3.2, when we motivate the questions at the center of this work. Nevertheless, we will not prove here the relationship between equations (59) and (60), as this has already been done to fantastic detail in Appendix A.5 in [32]. The proof is a carried out using computations and manipulations of formal power series. Hence we will use the result without proof, and direct the reader to [32] for the particular details.

Now in place of $\mathcal{H}_{CK}$ in the proof of Lemma 2.53 above, we could have worked instead with an arbitrary (graded, connected) Hopf algebra. However, as our interests lie with $\mathcal{H}_{CK}$, we can obtain an even more explicit form for what Feynman rules $\phi \in \mathcal{G}$ look like in terms of trees and forests. In what follows, we use / to represent edge contraction in the usual graph theoretical sense:

Theorem 2.55 (Yeats)**.** *Let $\phi \in \mathcal{G}$ be our Feynman rules such that $\phi = \exp_*(L\sigma)$, and let F be a forest of rooted trees. Then:*

$$\phi(L)(F) = \sum_{S \subseteq E(F)} \left(\prod_{t \in (F \setminus S)} \sigma(t)\right) \frac{L^{|F/(F\setminus S)|}}{(F/(F \setminus S))!} \tag{61}$$

where $E(F)$ is the edge set of F, $F \setminus S$ is the forest whose vertices are those of F simply with the edges in S removed, and $(F/(F \setminus S))!$ means the tree factorial of the forest $F/(F \setminus S)$.

Proof. This result follows from an application of Lemma 2.53 and some additional observations. Indeed, we already have that $\phi = \exp_*(L\sigma)$ for σ an infinitesimal character, so we just need to expand this expression out and interpret what it means combinatorially. We have:

$$\begin{aligned}
\phi(L)(F) &= \Big(\exp_*(L\sigma)\Big)(F) \\
&= \sum_{n=0}^{\infty} \frac{(L\sigma)^{*n}(F)}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(L)^n}{n!}\sigma^{*n}(F) \\
&= \sum_{n=0}^{\infty} \frac{(L)^n}{n!} m^n \circ (\underbrace{\sigma \otimes ... \otimes \sigma}_{n \text{ times}}) \circ \Delta^{n-1}(F)
\end{aligned}$$

where we are using the convention that $\sigma^{*0} = id, \sigma^{1*} = \sigma$, and hence implicitly $\Delta^{-1*} := id$. Now let us consider how the terms of $\Delta^n(F)$ interact with $\underbrace{\sigma \otimes ... \otimes \sigma}_{n \text{ times}}$ for fixed n. Since σ is an infinitesimal character, $\sigma(\mathbb{1}) = 0$ (since in particular it is an element of the convolution algebra) and σ also vanishes on nontrivial products (by virtue of the defining equation $\sigma(ab) = \sigma(a)\epsilon(b) + \epsilon(a)\sigma(b)$). Hence the only terms that will be nonzero in the composition $(\underbrace{\sigma \otimes ... \otimes \sigma}_{n \text{ times}}) \circ \Delta^n(F)$ are those corresponding to the terms of $\Delta^n(F)$ in which every element in the tensor product is a tree; this will happen precisely when a subset S of edges of cardinality $|S|= n - m$ has been removed, where we use m to mean the number of trees in the forest F. Hence so far we have that the nth term of the series $\phi(L)(F)$ will be:

$$\frac{L^n}{n!} \sum_{\substack{S \subseteq E(F) \\ |S|=n-m}} \Big(\prod_{t \in (F \setminus S)} \sigma(t) \Big) N_{F \setminus S}$$

for some integer $N_{F\setminus S}$ counting how many times the term $\prod_{t \in (F\setminus S)} \sigma(t)$ appears in $\phi(L)(F)$. We claim that $N_{F\setminus S}$ counts the number of increasing labellings of $F/(F \setminus S)$—that is, the number of increasing labellings of the forest F' obtained from F by contracting the edges not in the set S. Indeed, this follows immediately from the definition of the coproduct in terms of admissible cuts, and interpreting what these cuts mean as we apply the coproduct iteratively (see Section 2.1.2 for more on admissible cuts).

Finally, all that remains is to recognize that if $|S|= n - m$, then the forest $F/(F \setminus S)$ has $|S|$ edges. Hence $n = |S|+m$ is the number of vertices in the forest $F/(F \setminus S)$, giving us:

$$\phi(L)(F) = \sum_{S \subseteq E(F)} \frac{L^{|F/(F\setminus S)|}}{|F/(F \setminus S)|!} \Big(\prod_{t \in (F \setminus S)} \sigma(t) \Big) N_{F \setminus S}$$

Moreover, the number of increasing labellings of a forest H is equal to $\frac{|H|!}{H!}$, where $H!$ is the tree factorial, as discussed in Section 2.1.3 [49]. Consequently $N_{F\setminus S} = \frac{|F/(F\setminus S)|!}{(F/(F\setminus S))!}$, and so:

$$\begin{aligned} \phi(L)(F) &= \sum_{S \subseteq E(F)} \frac{L^{|F/(F\setminus S)|}}{|F/(F \setminus S)|!} \Big(\prod_{t \in (F \setminus S)} \sigma(t) \Big) \frac{|F/(F \setminus S)|!}{(F/(F \setminus S))!} \\ &= \sum_{S \subseteq E(F)} \Big(\prod_{t \in (F \setminus S)} \sigma(t) \Big) \frac{L^{|F/(F\setminus S)|}}{(F/(F \setminus S))!} \end{aligned}$$

as desired. □

Remark. The result above is from [52], which are some notes written during a visit of Karen Yeats and Dirk Kreimer to Spencer Bloch in Chicago in the fall of 2014. There, the result was proved inductively using Lemma 2.54, though it also had a slightly different form. Namely, if we define a map $\theta : \mathcal{H}_{CK} \to \mathbb{K}[L]$ that agrees with σ on trees, but extends as an algebra homomorphism to the rest of $\mathcal{H}_{CK}$, then we can rewrite equation (61) as:

$$\phi(L)(F) = \sum_{S \subseteq E(F)} \theta(F \setminus S) \frac{L^{|F/(F\setminus S)|}}{(F/(F \setminus S))!} \tag{62}$$

Rosen [41] showed us the connection that (62) is simply a form of the exponential map, and proved that such a map exists on arbitrary (graded, connected) Hopf algebras, which he then showed gives the result above. In particular, this explains why σ—an infinitesimal character—is more natural than θ—an algebra homomorphism. The approach we have taken here using the convolution exponential is based chiefly on the content of Appendix A.4 of [32].

Let us look at an example.

Example 2.56. Consider the tree $T = \overset{\bullet}{\overset{\bullet}{\bullet\;\bullet}}$. We use the regular formula for $\exp_*$ and see how it gives the same result as that obtained from the formula in the theorem statement. We only need to calculate the iterated reduced coproduct as the primitive part will always vanish after applying σ. We have that:

$$
\begin{aligned}
\widetilde{\Delta}(T) &= 2\bullet\otimes\substack{\bullet\\ \bullet\\ \bullet} + \bullet\;\bullet\otimes\substack{\bullet\\ \bullet} + \overset{\bullet}{\bullet\;\bullet}\otimes\bullet\\
\widetilde{\Delta}^2(T) &= 2\bullet\otimes\bullet\otimes\substack{\bullet\\ \bullet} + 2\bullet\otimes\substack{\bullet\\ \bullet}\otimes\bullet + \bullet\;\bullet\otimes\bullet\otimes\bullet\\
\widetilde{\Delta}^3(T) &= 2\bullet\otimes\bullet\otimes\bullet\otimes\bullet
\end{aligned}
$$

Hence we use the formula of $\exp_*$ to calculate:

$$
\begin{aligned}
\phi(L)(T) = \sigma(\mathbb{1}) &+ L\sigma(\overset{\bullet}{\overset{\bullet}{\bullet\;\bullet}}) + \frac{1}{2}L^2[2\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet\\ \bullet}) + \sigma(\bullet\;\bullet)\sigma(\substack{\bullet\\ \bullet}) + \sigma(\overset{\bullet}{\bullet\;\bullet})\sigma(\bullet)]\\
&+ \frac{1}{6}L^3[2\sigma(\bullet)\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet}) + 2\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet})\sigma(\bullet) + \sigma(\bullet\;\bullet)\sigma(\bullet)\sigma(\bullet)]\\
&+ \frac{1}{24}L^4[2\sigma(\bullet)\sigma(\bullet)\sigma(\bullet)\sigma(\bullet)]\\
= \sigma(\mathbb{1}) &+ L\sigma(\overset{\bullet}{\overset{\bullet}{\bullet\;\bullet}}) + \frac{1}{2}L^2[2\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet\\ \bullet}) + \sigma(\overset{\bullet}{\bullet\;\bullet})\sigma(\bullet)]\\
&+ \frac{1}{6}L^3[2\sigma(\bullet)\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet}) + 2\sigma(\bullet)\sigma(\substack{\bullet\\ \bullet})\sigma(\bullet)]\\
&+ \frac{1}{24}L^4[2\sigma(\bullet)\sigma(\bullet)\sigma(\bullet)\sigma(\bullet)]
\end{aligned}
$$

Which agrees with Theorem 2.55.

3 Problem Setup

This section will be devoted to presenting details of the central problem. We will first start by relating more background material. However, unlike the material from Section 2, this will be information more specific to the problem at hand, some of which is unpublished elsewhere. The contents of Section 3.1 are due to Spencer Bloch, Dirk Kreimer, and Karen Yeats, and arose out of discussions at the University of Chicago in the fall of 2014. The author is indebted to these scholars for allowing the inclusion of their work in the present thesis for the purpose of telling a complete mathematical story.

3.1 Higher Order Renormalization Group Equations

We saw in Section 2.3.3 that the Green's function $G(x,L)$ in the renormalization group equation (equation (60)) is in fact a series whose terms consist of Feynman rules $\phi \in \mathcal{G}$ applied to a series $X = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ whose terms are nonzero linear combination of trees. What does it mean practically speaking for $G(x,L)$ to satisfy (60)? In this section, we will answer this question and will find that doing so leads to a natural generalization of the renormalization group equation itself.

To begin let $\phi \in \mathcal{G}$ be our Feynman rules and let $X = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ be our series of linear combinations of trees. Further define $\beta(x)$ and $\gamma(x)$ to be two formal power series in x:

$$\beta(x) := \sum_{n=1}^{\infty} (-\beta_n) x^{n+1}$$

$$\gamma(x) := \sum_{n=0}^{\infty} \gamma_n x^n$$

Note that we have chosen a nonstandard way to index the coefficients in $\beta(x)$, as ultimately the coefficients are arbitrary and this convention will simplify the indexing in the final result.

We will denote by $Q_n(L)$ the polynomial in L obtained by applying ϕ to the term t_n (the natural choice of symbol would be P_n for "polynomial," but we already have P representing the pruning operator; see Section 2.1.3). This will simplify our notation, as instead of defining our Green's function as:

$$G(x,L) = 1 + \sum_{n=1}^{\infty} \phi(t_n)(L) x^n$$

we may simply write the Green's function as:

$$G(x,L) = 1 + \sum_{n=1}^{\infty} Q_n(L) x^n \tag{63}$$

As mentioned previously, x is the **coupling constant**, but for our purposes it is also the formal counting variable as it normally is in the method of generating functions.

Now the objective is to compute what each term of the renormalization group equation (60) looks like in terms of formal power series. It will turn out that the most useful way to do this is to break the equation up into two pieces: those that contain partial derivatives with respect to L on one side, and then everything else on the other side:

$$\frac{\partial}{\partial L} G(x,L) = \left(\gamma(x) - \beta(x) \frac{\partial}{\partial x} \right) G(x,L) \tag{64}$$

We will compute what each side of the equation looks like in terms of formal power series separately, and then compare the coefficients on each side. This will give us a condition describing when $G(x, L)$ satisfies (60).

Let's compute the left-hand side first. Note that the left-hand side of (64) is just:

$$\frac{\partial}{\partial L} G(x, L) = \sum_{n=1}^{\infty} \frac{\partial}{\partial L} Q_n(L) x^n \tag{65}$$

by definition of $G(x, L)$ from (63). Hence the only unknown piece of information are the factors $\frac{\partial}{\partial L} Q_n(L)$, which we now compute.

To do this, we will use what we referred to earlier as the algebraic version of the renormalization group equation (equation (59)). This is simply the property that ϕ is a Hopf algebra homomorphism. However, if we write the coproduct of t_n out explicitly, we can write equation (59) out in a more explicit form, too. Namely, start by writing the coproduct of t_n for n arbitrary as:

$$\Delta(t_n) = t_n \otimes \mathbb{1} + \mathbb{1} \otimes t_n + \sum_{i=1}^{n-1} \tau_{n,n-i} \otimes t_i$$

which we will write as:

$$\Delta(t_n) = \sum_{i=0}^{n} \tau_{n,n-i} \otimes t_i$$

using the convention that $t_0 = \tau_{n,0} = \mathbb{1}$ and $\tau_{n,n} = t_n$.

Now since the sequence $(t_n)_{n \geq 1}$ generates a Hopf subalgebra of $\mathcal{H}_{CK}$ by hypothesis, we know that the $\tau_{n,n-i}$ are polynomials in the t_j for $j < n$. Hence equation (59) applied to this situation becomes:

$$\begin{aligned}
\phi_{L_1+L_2}(t_n) &= (\phi_{L_1} * \phi_{L_2})(t_n) \\
\implies Q_n(L_1 + L_2) &= m \circ (\phi_{L_1} \otimes \phi_{L_2}) \circ \Delta(t_n) \\
&= m \circ (\phi_{L_1} \otimes \phi_{L_2})(\sum_{i=0}^{n} \tau_{n,n-i} \otimes t_i) \\
&= m(\sum_{i=0}^{n} \phi(\tau_{n,n-i})(L_1) \otimes \phi(t_i)(L_2)) \\
&= m(\sum_{i=0}^{n} \phi(\tau_{n,n-i})(L_1) \otimes Q_i(L_2)) \\
&= \sum_{i=0}^{n} \phi(\tau_{n,n-i})(L_1) Q_i(L_2)
\end{aligned}$$

where above, we are forced again to use the confusing notation $\phi(\tau_{n,n-i})(L_1)$. Since $\phi(\tau_{n,n-i})$ returns a function in L, we use $\phi(\tau_{n,n-i})(L_1)$ to mean substituting L with L_1 (where L_1 is still a variable).

Now ultimately what we want is the derivative of these terms with respect to L, in order to calculate $\frac{\partial G}{\partial L}$, so we calculate this derivative using standard undergraduate calculus. The only "trick" we will use is the fact that $\frac{\partial}{\partial x} f(x + y) = \frac{\partial}{\partial y} f(x + y)$ for any function f. So instead of differentiating with respect to L, we differentiate with respect to L_1 to get that:

$$\frac{\partial}{\partial L_1} Q_n(L_1 + L_2) = \sum_{i=0}^{n} \left(\frac{\partial}{\partial L_1} \phi(\tau_{n,n-i})(L_1) Q_i(L_2) \right)$$

Now if we make the substitution $L_1 = 0$ and use the identity regarding functions of sums of variables discussed above, this is equal to:

$$\begin{aligned} \frac{\partial}{\partial L_2} Q_n(L_1 + L_2)|_{L_1=0} &= \sum_{i=0}^{n} \frac{\partial}{\partial L_1} \phi(\tau_{n,n-i})(L_1)|_{L_1=0} Q_i(L_2) \\ \implies \frac{\partial}{\partial L_2} Q_n(L_2) &= \sum_{i=0}^{n} \frac{\partial}{\partial L_1} \phi(\tau_{n,n-i})(L_1)|_{L_1=0} Q_i(L_2) \end{aligned}$$

The factor $\frac{\partial}{\partial L_1} \phi(\tau_{n,n-i})(L_1)|_{L_1=0}$ will simply be some element of the underlying field $\mathbb{K}$, so we can represent it by some constant indexed by n and i. Let's call it $c_{n,i}$:

$$c_{n,i} := \frac{\partial}{\partial L_1} \phi(\tau_{n,n-i})(L_1)|_{L_1=0} \tag{66}$$

Then the above equation for $\frac{\partial}{\partial L_2} Q_n(L_2)$ is just:

$$\frac{\partial}{\partial L_2} Q_n(L_2) = \sum_{i=0}^{n} c_{n,i} Q_i(L_2)$$

And finally, there is no longer any need to continue using L_2 now, seeing as we have removed all appearances of L_1. Hence we set $L_2 = L$:

$$\frac{\partial}{\partial L} Q_n(L) = \sum_{i=0}^{n} c_{n.i} Q_i(L))$$

Since we finally have an expression for $\frac{\partial}{\partial L} Q_n(L)$, we can conclude that the left-hand side of (64) is just:

$$\frac{\partial}{\partial L} G(x, L) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_{n,i} Q_i(L) \right) x^n \tag{67}$$

where we have substituted the values of $\frac{\partial}{\partial L} Q_n(L)$ we have just found into (65).

Now all that remains is to compute the right hand side of (64), which we will do one term at a time.

Firstly, we have by our definition of $\beta(x)$ that:

$$\begin{aligned} -\beta(x) \frac{\partial}{\partial x} G(x, L) &= \left(\sum_{n=1}^{\infty} \beta_n x^{n+1} \right) \cdot \frac{\partial}{\partial x} \left(1 + \sum_{n=0}^{\infty} Q_n(L) x^n \right) \\ &= \left(\sum_{n=0}^{\infty} \beta_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} n Q_n(L) x^{n-1} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} i \beta_{n-i} Q_i(L) \right) x^n \end{aligned}$$

For the other term of the right-hand side of equation (64), we have:

$$\begin{aligned}\gamma(x)G(x,L) &= \left(\sum_{n=0}^{\infty}\gamma_n x^n\right)\cdot\left(1+\sum_{n=0}^{\infty}Q_n(L)x^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\gamma_{n-i}Q_i(L)\right)x^n\end{aligned}$$

where we have transitioned to the notational convention that $Q_0(L) = 1$ (Note that we know that $Q_0(L) = 1$ from the previous section, since ϕ is an algebra homomorphism). Finally, then, the renormalization group equation (64) translates as a statement about formal power series in the following way:

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}c_{n,i}Q_i(L))\right)x^n = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\gamma_{n-i}Q_i(L)\right)x^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}i\beta_{n-i}Q_i(L)\right)x^n \tag{68}$$

By comparing coefficients of x^n on both sides, we recover the following identity for any fixed $n \geq 1$:

$$\sum_{i=0}^{n}c_{n,i}Q_i(L)) = \sum_{i=0}^{n}\gamma_{n-i}Q_i(L) + \sum_{i=1}^{n}i\beta_{n-i}Q_i(L) \tag{69}$$

Now we would further like to extract coefficients of the $Q_i(L)$ for each i, but this will only be a valid operation if the $Q_i(L)$ are linearly independent. Thankfully this is the case: since ϕ are nonzero Feynman rules, at least one term of $Q_i(L)$ will be nonzero for each i. Moreover by Theorem 2.55, $Q_i(L)$ will have distinct leading terms for each i, giving that the $Q_i(L)$ are linearly independent. Hence if we fix $i = 0$ and vary n we obtain the defining relationship between the γ_i and $c_{n,k}$. Namely:

$$\gamma_n = c_{n,0} \qquad \text{for all } n \geq 1$$

If we instead fix $n \geq 1$ and vary i, we obtain the following relationship for all $i \leq n$:

$$c_{n,i} = c_{n-i,0} + i\beta_{n-i} \tag{70}$$

At long last, we can use this defining relationship to recursively solve for the $c_{n,i}$ to impose some conditions on which sequences $G(x, L)$ satisfy (64). This recursive procedure produces the following array:

$$\begin{array}{ccccccccc} &&&& c_{1,0} &&&& \\ &&& c_{2,1} && c_{2,0} &&& \\ && c_{3,2} && c_{3,1} && c_{3,0} && \\ & c_{4,3} && c_{4,2} && c_{4,1} && c_{4,0} & \\ c_{5,4} && c_{5,3} && c_{5,2} && c_{5,1} && c_{5,0} \\ &&&& \vdots &&&& \end{array}$$

where using the relationship (70) we obtain:

$$\begin{array}{ccccccccc} &&&& c_{1,0} &&&& \\ &&& \beta_1 + c_{1,0} && c_{2,0} &&& \\ && 2\beta_1 + c_{1,0} && \beta_2 + c_{2,0} && c_{3,0} && \\ & 3\beta_1 + c_{1,0} && 2\beta_2 + c_{2,0} && \beta_3 + c_{3,0} && c_{4,0} & \\ 4\beta_1 + c_{1,0} && 3\beta_2 + c_{2,0} && 2\beta_3 + c_{3,0} && \beta_4 + c_{4,0} && c_{5,0} \\ &&&& \vdots &&&& \end{array}$$

and where $c_{n,i} = 0$ for all pairs of n and i where $i \geq n$.

Hence, we have found a necessary condition for $G(x, L)$ to satisfy a renormalization group equation! That is, the array of numbers $c_{n,i}$ must be equal to first-order polynomials in i for fixed $n - i$. That is, for all $n \geq 1$ we need to have:

$$c_{n,n-i} = ai + b \tag{71}$$

Now at first this might seem like a lot of work for little reward, as so far we only know of the $c_{n,i}$ from their abstract definition as coefficients in the generating function $\frac{\partial}{\partial L}G(x, L)$. However, let us return to equation (66) and try to understand if they have any combinatorial meaning. The result in this direction will be the following:

Proposition 3.1. Let $\lambda_{n,n-i}$ denote the coefficient of $t_{n-i} \otimes t_i$ in $\Delta(t_n)$. Then $c_{n,i} = \lambda_{n,n-i}\phi(t_{n-i})$.

Proof. First, we have that $\tau_{n,n-i}$ will be a polynomial in variables $t_1, ..., t_{n-i}$. But by the grading of $\mathcal{H}_{CK}$ and the fact that Δ preserves grading, we know that this polynomial will also be homogeneous of degree $n - i$ in the t_k.

By definition of Feynman rules, ϕ is a Hopf algebra homomorphism from $\mathcal{H}_{CK}$ to $\mathbb{K}[L]$, meaning that it is enough to define ϕ on the basis $(t_n)_{n\geq 1}$, since our sequence generates a Hopf subalgebra of $\mathcal{H}_{CK}$ by hypothesis. The result $\phi(\tau_{n,n-i})$ will be a polynomial in L over $\mathbb{K}$, where every t_k in the polynomial $\tau_{n,n-i}$ contributes one factor of L by Theorem 2.55. But by definition, $c_{n,i}$ is formed by taking the derivative with respect to L and then setting $L = 0$. Hence every term in the original polynomial having a factor of L^2 or any higher power of L vanishes and we are left with $\phi(\lambda_{n,n-i}t_{n,n-i})$ from which the desired result follows. □

We remark that we have actually seen these coefficients $\lambda_{i,j}$ before—they are the $m(t_1, t_2)$ from equation (2.19) in Definition 2.19 (generalized now so that t_i is a linear combination of trees instead of a single tree as it was in Section 2.1.3)

Hence we now have a reformulation of the test for whether or not a Green's function $G(x, L)$ satisfies a renormalization group equation. We believe this statement is important enough to reiterate as a corollary:

Corollary 3.2. Let $s = (t_n)_{n\geq 1}$ be a sequence of nonzero linear combinations of trees that generates a Hopf subalgebra of $\mathcal{H}_{CK}$, and let $X_s = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ be the corresponding series. Let $\phi \in \mathcal{G}$ be our Feynman rules. Then the Green's function $G(x, L) = \phi(X)$ satisfies a renormalization group equation only if $c_{n,n-i} = ai + b$ for n fixed, i varying, and $a, b \in \mathbb{K}$.

These sequences of linear polynomials in i are precisely the leftward diagonals depicted in the arrays of the previous discussion.

We can take the abstraction to combinatorics one step further by making one more observation: now that we have a combinatorial expression for the $c_{n,i}$, we can rewrite the array containing them, setting $c_{n,i} = \lambda_{n,n-i}\phi(t_{n-i})$ once again for arbitrary $\phi \in \mathcal{G}$. This leads to:

$$\begin{array}{ccccccccc}
 & & & & \lambda_{1,1}\phi(t_1) & & & & \\
 & & & \lambda_{2,1}\phi(t_1) & & \lambda_{1,2}\phi(t_2) & & & \\
 & & \lambda_{3,1}\phi(t_1) & & \lambda_{2,2}\phi(t_2) & & \lambda_{1,3}\phi(t_3) & & \\
 & \lambda_{4,1}\phi(t_1) & & \lambda_{3,2}\phi(t_2) & & \lambda_{2,3}\phi(t_3) & & \lambda_{1,4}\phi(t_4) & \\
\lambda_{5,1}\phi(t_1) & & \lambda_{4,2}\phi(t_2) & & \lambda_{3,3}\phi(t_3) & & \lambda_{2,4}\phi(t_4) & & \lambda_{1,5}\phi(t_5) \\
 & & & & \vdots & & & &
\end{array}$$

Figure 15: An array of coefficients.

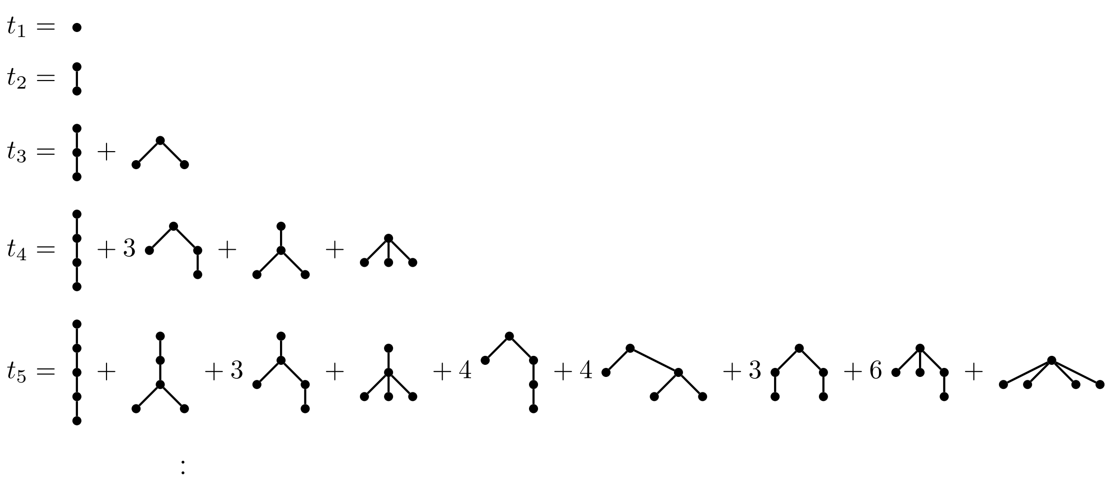
Notice that every term along the leftward-pointing diagonals (the ones that must be first-order polynomials in i) contains a constant factor of $\phi(t_j)$ for some j. In other words, a Green's function $G(x, L)$ can satisfy a renormalization group equation only when the left diagonals depicted above (the sequences $(\lambda_{i,j_0})_{i\geq 1}$) are linear. This is independent of ϕ, unless $\phi(t_j) = 0$ for some j, but this would only make it easier for $G(x, L)$ to satisfy a renormalization group equation by making faster-growing diagonals vanish.

Unfortunately, not every sequence of nonzero linear combinations of trees generating a Hopf subalgebra of $\mathcal{H}_{CK}$ satisfies a renormalization group equation. This is the essence of the following lemma:

Lemma 3.3. *Let $s = (t_n)_{n\geq 1}$ be the standard sequence of generators for the Connes-Moscovici Hopf subalgebra of $\mathcal{H}_{CK}$, and let X_s be the corresponding series: $X_s = \mathbb{1} + \sum_{n=1}^{\infty} t_n$. Then the Green's function $G(x, L) = \phi(X)$ does not satisfy a renormalization group equation for any (nonzero) choice of Feynman rules ϕ.*

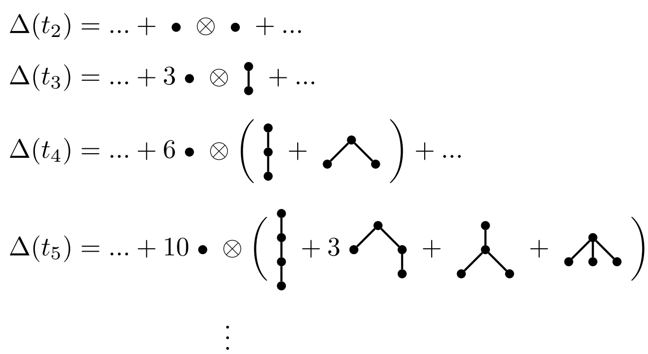
We remark that the Feynman rules $\phi(t_n) = 0$ for all $n \geq 1$ should always lead to a Green's function which will satisfy a renormalization group equation. As these rules are not very useful, however, we will not consider them in any future deliberations (unless otherwise mentioned).

Proof Sketch of Lemma 3.3. In light of the discussion above, we only need to check that—for fixed and arbitrary j—$\lambda_{i,j} \neq ai + b$ for any $a, b \in \mathbb{K}$. We can do this by direct computation. The sequence of generators described begins:



Let us compute the left-most diagonal of the array of $\lambda_{i,j}$—these will be the coefficients of $\bullet \otimes t_{n-1}$ in

$\Delta(t_n)$. We get that:



and a pattern has emerged. Indeed, it is not hard to show that the coefficients we seek form the sequence $(\lambda_{i,1})_{i\geq 1} = (\binom{i+1}{2})_{i\geq 1}$, hence as a polynomial in i we have $\lambda_{i,1} = \frac{1}{2}i^2 + \frac{1}{2}i$. Note that this is a polynomial in i of order 2. More generally, we find that (for j fixed arbitrarily) $(\lambda_{i,j})_{i\geq 1} = (\binom{i+j}{j+1})_{i\geq 1}$, which for all j forms a sequence of i of order at least 2. Hence there is no choice of (nonzero) Feynman rules ϕ that will make the array of $c_{n,i}$'s have linear left diagonals, and consequently $G(x, L) = \phi(X)$ will not satisfy a renormalization group equation for any choice of Feynman rules ϕ. □

In some sense, the results of Lemma 3.3 might seem rather dissatisfying. Indeed, the Connes-Moscovici Hopf subalgebra is one of the most natural combinatorial examples appearing the literature (in fact, one can define the sequence in terms of the *natural growth* operator of Section 2.1.3). Moreover, the array of coefficients $\lambda_{i,j}$ arising in relation to computations with a Green's function we would form from the sequence also seem nice; while they are second (and greater) order polynomials in i, the sequence of binomial coefficients is ubiquitous in combinatorics.

Hence, rather than restricting our focus to the Green's functions which satisfy the renormalization group equation in the standard sense, we can think of these inconsistencies as motivation for a generalized form of equation (60), which is the central definition of this text:

Definition 3.4 (Generalized Renormalization Group Equations)**.** For a Green's function $G(x, L)$, define a **generalized renormalization group equation** by:

$$\frac{\partial G}{\partial L} = \overline{\beta}(x, \frac{\partial}{\partial x})G \tag{72}$$

where $\overline{\beta}$ is polynomial in its second argument. If the polynomial $\overline{\beta}$ is of degree n in $\frac{\partial}{\partial x}$, we say that the generalized renormalization group equation is **of order** n.

We remark that the standard definition of the renormalization group equation can be seen as a generalized renormalization group equation of order 1. Indeed, we simply take $\overline{\beta}(x, \frac{\partial}{\partial x}) = \gamma(x) + \beta(x)\frac{\partial}{\partial x}$. We also note that this definition is informed by quantum field theory, as the case of $\beta \equiv 0$ is already a known special case in physics. Such an equation is said to have a **pure scaling solution** and is usually integrable [27]. In the language we are introducing with Definition 3.4 these are 0th-order renormalization group equations. We will give special attention to these in Sections 5.1 and 5.3.

We can ask ourselves the same question about (72) that we did at the beginning of this section regarding the usual renormalization group equation: namely, what does it mean for a Green's function to satisfy

(72)? Are we able to come up with some necessary conditions? Let us approach the question using the same method as before; we will turn equation (72) into a statement about formal power series, and then compare coefficients across the equals sign.

To fix notation again, let $\phi \in \mathcal{G}$ be our Feynman rules, $(t_n)_{n\geq 1}$ be a sequence of linear combinations of trees generating a Hopf subalgebra of $\mathcal{H}_{CK}$, and let $X = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ be the corresponding series. Then $G(x, L) = \phi(X)$ as before. Now $\overline{\beta}$ is allowed to be more general than β was before, so we will set it up in the following way:

$$\overline{\beta}(x, \frac{\partial}{\partial x}) := \sum_{j=0}^{m} \beta^{(j)}(x) \frac{\partial^j}{\partial x^j} \tag{73}$$

with each $\beta^{(i)}(x)$ a formal power series in x:

$$\beta^{(j)}(x) := \sum_{k=0}^{\infty} \beta_k^{(j)} x^k \tag{74}$$

with $\beta_k^{(j)} \in \mathbb{K}$.

At the risk of being repetitive, we remark that with this notational setup, the usual renormalization group equation is recovered by setting $\beta^{(0)}(x) = \gamma(x)$, $\beta^{(1)}(x) = \beta(x)$, and $\beta^{(k)}(x) \equiv 0$ for $k \geq 2$.

Now we proceed. Since our Green's function $G(x, L)$ looks exactly as it did before, the left-hand side of (72) is also precisely the same. We write it again for convenience:

$$\frac{\partial}{\partial L} G(x, L) = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} c_{n,i} Q_i(L)) \Big) x^n \tag{67}$$

So all that remains is to find what the right-hand side of equation (72) looks like. Let us calculate:

$$\begin{aligned}
\overline{\beta} G &= \Big(\sum_{j=0}^{m} \beta^{(j)}(x) \frac{\partial^j}{\partial x^j} \Big) \Big(\sum_{n=0}^{\infty} Q_n(L) x^n \Big) \\
&= \beta^{(0)} \Big(\sum_{n=0}^{\infty} Q_n(L) x^n \Big) + \beta^{(1)} \frac{\partial}{\partial x} \Big(\sum_{n=0}^{\infty} Q_n(L) x^n \Big) + ... + \beta^{(m)} \frac{\partial^m}{\partial x^m} \Big(\sum_{n=0}^{\infty} Q_n(L) x^n \Big) \\
&= \beta^{(0)} \Big(\sum_{n=0}^{\infty} Q_n(L) x^n \Big) + \beta^{(1)} \Big(\sum_{n=0}^{\infty} n Q_n(L) x^{n-1} \Big) + ... + \beta^{(m)} \Big(\sum_{n=0}^{\infty} n(n-1) \cdots (n-m+1) Q_n(L) x^{n-m} \Big) \\
&= \sum_{j=0}^{m} \sum_{n=0}^{\infty} \beta^{(j)} n(n-1) \cdots (n-j+1) Q_n(L) x^{n-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{m} \beta^{(j)} n(n-1) \cdots (n-j+1) Q_n(L) x^{n-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{m} \beta^{(j)} \frac{n!}{(n-j)!} Q_n(L) x^{n-j}
\end{aligned}$$

But now the $\beta^{(j)}$ are just power series, with coefficients as defined in (74), so we need to substitute these in to be able to accurately compare powers of x to the left-hand side of the equation. When we do this we get:

$$\overline{\beta}G = \sum_{n=0}^{\infty}\sum_{j=0}^{m}\left(\sum_{k=0}^{\infty}\beta_k^{(j)}x^k\right)\frac{n!}{(n-j)!}Q_n(L)x^{n-j}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{m}\sum_{k=0}^{\infty}\beta_k^{(j)}\frac{n!}{(n-j)!}Q_n(L)x^{n-j+k}$$

Now we want the sum to be indexed by powers of x, so we make a substitution on the indices: $t = n-j+k$. This yields:

$$= \sum_{t=0}^{\infty}\sum_{j=0}^{m}\sum_{n-j+k=t}\beta_k^{(j)}\frac{n!}{(n-j)!}Q_n(L)x^{t}$$

Finally we can simplify the indices of the third summation as well. If $n-j+k=t$, and j is already fixed by the second summation, it follows that n and k are partitioning $t+j$ and hence we can rewrite this as:

$$= \sum_{t=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{t+j}\beta_{t+j-n}^{(j)}\frac{n!}{(n-j)!}Q_n(L)x^{t}$$

and we are finished. We can now relate the left- and right-hand sides of equation (72) to obtain a form of the generalized renormalization group equation solely in terms of formal power series:

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}c_{n,i}Q_i(L))\right)x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{m}\sum_{i=0}^{n+j}\beta_{t+j-i}^{(j)}\frac{i!}{(i-j)!}Q_i(L)x^n \tag{75}$$

(Note that we have changed the letters of some of the indices to avoid confusion). By comparing the coefficients of powers of x on each side, we conclude that the following identity holds for all $n \geq 0$:

$$\sum_{i=0}^{n}c_{n,i}Q_i(L) = \sum_{j=0}^{m}\sum_{i=0}^{t+j}\beta_{n+j-i}^{(j)}\frac{i!}{(i-j)!}Q_i(L) \tag{76}$$

and using as before that the $Q_i(L)$ are linearly independent, we can compare their coefficients across the equals sign as well to get that:

$$c_{n,i} = \sum_{j=0}^{m}\beta_{n+j-i}^{(j)}\frac{i!}{(i-j)!}$$
$$= \beta_{n-i}^{(0)} + i\beta_{n-i+1}^{(1)} + ... + i(i-1)\cdots(i-m+1)\beta_{n-i+m}^{(m)}$$

In other words, when n is fixed we have that $c_{n,i}$ is a polynomial in i of degree m. Hence we have found a necessary condition for which $G(x,L)$ satisfy an mth-order generalized renormalization group equation, as we desired.

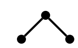
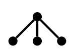
Corollary 3.5. A Green's function $G(x,L)$ satisfies an mth order generalized renormalization group equation only if—for fixed n—$c_{n,n-i}$ is a polynomial of degree m in i.

3.2 Statement of the Main Problem

As in the last section, let $s = (t_n)_{n\geq 1}$ be a sequence of nonzero linear combinations of trees, and let $X_s = \mathbb{1} + \sum_{n=1}^{\infty} t_n$ be the corresponding sequence. We can always use the elements of s to generate a subalgebra of $\mathcal{H}_{CK}$, however it is not always the case that this subalgebra will also be Hopf.

For example, we know that the sequence of ladders from Definition 2.25 $l_0 = \mathbb{1}, l_1 = \bullet, l_2 =$ $, l_3 =$  $, l_4 =$ $, \ldots$ will generate a Hopf subalgebra of $\mathcal{H}_{CK}$. To see this, let $\mathcal{A}_l$ be the algebra generated by this sequence. Then we only need to check that $\Delta(l_n) \in \mathcal{A}_l \otimes \mathcal{A}_l$. But this is true, since:

$$\Delta_{l_n} = \sum_{i=0}^{n} l_i \otimes l_{n-i}$$

Moreover, it is also easy to check that the sequence of corollas $r_0 = \mathbb{1}, r_1 = \bullet, r_2 =$  $, r_3 =$  $, r_4 =$  $, \ldots$ generates a Hopf subalgebra as well, since:

$$\Delta(r_n) = \sum_{i=0}^{n} \binom{n}{i} \bullet^{\,n-i} \otimes r_i \in \mathcal{A}_r \otimes \mathcal{A}_r$$

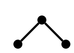
where we are letting $\mathcal{A}_r$ be the algebra generated by $(r_n)_{n\geq 0}$.

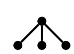
Now by Proposition 1.4.2 of [13] and the definition of a Hopf subalgebra, $\mathcal{A}_l + \mathcal{A}_r := \{a + b | a \in \mathcal{A}_l, b \in \mathcal{A}_r\}$ is also a Hopf subalgebra of $\mathcal{H}_{CK}$. However the algebra $\mathcal{A}_t$ generated by:

$$t_0 = l_0 + r_0 = 2$$

$$t_1 = l_1 + r_1 = 2\bullet$$

$t_2 = l_2 + r_2 = 2$ 

$t_3 = l_3 + r_3 =$  $+$ 

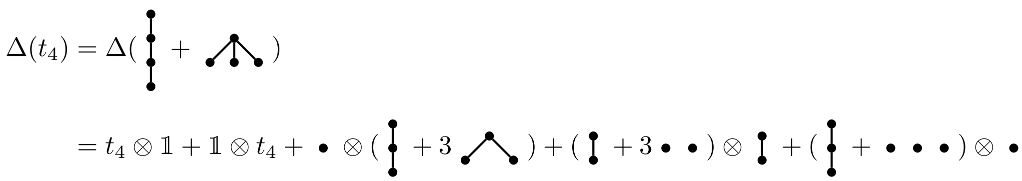
$t_4 = l_4 + r_4 =$  $+$ 

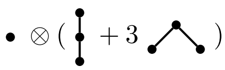
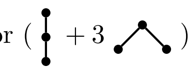
$$\vdots$$

$$t_n = l_n + r_n$$

$$\vdots$$

is not Hopf, since:



However  $\notin \mathcal{A}_t \otimes \mathcal{A}_t$, since for  to be in $\mathcal{A}_t$ we would need to be able to write it as some linear combination of products of basis elements, but this simply cannot be done. In other words, despite the fact that $\mathcal{A}_l + \mathcal{A}_r$ is a Hopf subalgebra, it is not the same algebra as that generated by $(l_n + r_n)_{n\geq 1}$, which is not Hopf.

Hence if the subalgebra generated by a sequence s as defined above is also a Hopf subalgebra, then we will refer to s as a **Hopf sequence** (or we will say that the sequence is **Hopf**). The following definition packages up all of this information in a nice way (this notation was first used by [24]):

Definition 3.6. Define $\mathcal{S}eq$ to be the set of all sequences $s = (t_n)_{n\geq 1}$ such that:

1. t_n is a nonzero linear combinations of trees, with $t_n \in \mathbb{K}[\mathcal{T}_n]$,
2. the elements of $(t_n)n \geq 1$ generates a Hopf subalgebra of $\mathcal{H}_{CK}$, and
3. $t_1 = \bullet$

Note that we include this last stipulation to avoid redundancy. Indeed, as discussed in [24], if $X' = \sum_{n=1}^{\infty} t'_n$ generates a Hopf subalgebra and $t'_1 = c\bullet$ for c a nonzero element of $\mathbb{K}$, then $X = \bullet + \sum_{n=2}^{\infty} t'_n$ is also Hopf. In other words, if we let $\mathcal{S}eq'$ be the set of all sequences $(t_n)_{n\geq 1}$ that generate a Hopf subalgebra without any condition on t_1, then there is a bijection from $\mathbb{K} \setminus 0 \times \mathcal{S}eq \to \mathcal{S}eq'$ mapping $(c, (t_n)_{n\geq 1})$ to $(ct_1, t_2, t_3, ...)$ [24].

Now suppose that $s = (t_n)_{n\geq 1}$ is in $\mathcal{S}eq$. Then we can define Feynman rules ϕ as discussed in Sections 2.3.4 and 3.1 to turn X_s into a Green's function to see if it is in fact the solution of a generalized renormalization group equation. If the series X_s satisfies a generalized renormalization group equation of order k for some choice of ϕ, then we say that s is a **kth order sequence** (or just **kth order**) with respect to ϕ (sometimes we will not make mention to ϕ when it is understood from context).

This idea of being able to choose Feynman rules leads to the following two very important notions:

Definition 3.7. If a sequence $s \in \mathcal{S}eq$ satisfies a kth order renormalization group equation for any choice of Feynman rules and for $\beta_1^{(k)} \neq 0$, then we say s is a **strong kth order sequence**.

Definition 3.8. If a sequence $s \in \mathcal{S}eq$ satisfies a kth order renormalization group equation but is not strong, we say that it is a **weak kth order sequence**. This is possible in two different ways. Either

- $G(x, L) = \phi(X_s)$ satisfies a kth-order renormalization group equation for some (but not all) choices of Feynman rules, and/or
- $G(x, L) = \phi(X_s)$ satisfies a kth-order renormalization group equation such that $\beta_1^{(k)} = 0$.

We will see examples of both of these definitions in later chapters.

Now that everything has been set up, we can restate the central goals of this work, which we already alluded to in Section 1.1:

Q1: Is it possible to characterize all strong elements of $\mathcal{S}eq$ according to their order?

Q2: Is it possible to characterize all weak elements of $\mathcal{S}eq$ according to their order?

Note that in one sense we already know that the answer to Q2 is no, this is not possible, as a given element s of $\mathcal{S}eq$ may satisfy generalized renormalization group equations of differing orders by making changes to the Feynman rules ϕ and by considering different $\overline{\beta}$-functions. Nevertheless, we will see in Section 5.3 that the question can be solved in certain settings, and in fact leads to interesting combinatorial answers.

On the other hand, sequences $s \in \mathcal{S}eq$ that lead to strong kth order sequences have a unique order by which to be classified, as the conditions imposed on them mean that their order can be read off of the left-diagonal in the array depicted in Figure 15; this order is given by the order of the sequence $(\lambda_{i,1})_{i\geq 1}$. In Sections 5.1 and 5.3, we present a near-complete solution to this problem.

Before we present these solutions, we will first take a brief detour in the next section by presenting a result due to Foissy characterizing which s belong in $\mathcal{S}eq$. Doing so will aid in the solutions to Q1 and Q2 in Section 5.

4 Bijection to Prelie Algebra of Formal Series

This section reports on the new results of Foissy [24], and we will closely follow the presentation therein.

4.1 Main Result

To begin, let $\mathcal{S}eq$ be the set of sequences of linear combinations of trees as defined in Definition 3.6. The following is the main result of this section:

Theorem 4.1 (Foissy, 2018)**.** *Let Λ be the set of all doubly-indexed sequences $(\lambda_{i,j})_{i,j\geq 1}$ that satisfy the following two properties:*

1. *(Non degeneracy) For every $n \geq 2$, there exist $i, j \geq 1$ such that $i + j = n$ and $\lambda_{i,j} \neq 0$.*
2. *(Prelie) For every $i, j, k \geq 1$, the following identity holds true:*

$$\lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} = \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} \tag{77}$$

Then there is a bijection $\Theta : \mathcal{S}eq \to \Lambda$.

To illustrate, the theorem states that every sequence we desire in $\mathcal{S}eq$ has a corresponding array[15]:

$$\begin{array}{ccccccc} & & & \lambda_{1,1} & & & \\ & & \lambda_{2,1} & & \lambda_{1,2} & & \\ & \lambda_{3,1} & & \lambda_{2,2} & & \lambda_{1,3} & \\ \lambda_{4,1} & & \lambda_{3,2} & & \lambda_{2,3} & & \lambda_{1,4} \\ & & & \vdots & & & \end{array}$$

Figure 16: A generic element of Λ.

in which the elements of the array satisfy the prelie relation (77) and in which every row has at least one nonzero entry. As in [24], we will first show the existence of the desired map Θ, and then after the proof of the theorem we will describe the map's explicit form.

Before we prove the main result we require the following lemma, which explains that the elements in the Λ-arrays above are the structure coefficients of a prelie algebra of formal variables:

Lemma 4.2. *Let $(\lambda_{i,j})_{i,j\geq 1} \in \Lambda$, and define a prelie structure on the space $V = Vect(X_i, i \geq 1)$ of formal variables X_i by:*

$$X_i \triangleright X_j = \lambda_{i,j} X_{i+j} \tag{78}$$

Then $(V, \triangleright)$ is graded, with X_i homogeneous of degree i for any i, and is generated by X_1.

Proof. First, observe that the prelie relation in $(V, \triangleright)$ is equivalent to (77). To see this, note that the standard prelie relation in any prelie algebra has the form (see Definition 2.38) :

$$(a * b) * c - a * (b * c) = (a * c) * b - a * (c * b)$$

[15] As we can always write down the doubly-indexed sequence $(\lambda_{i,j})_{i,j\geq 1}$ in this way, we will often refer to an element of Λ simply as a Λ-*array*. Moreover, if we are computing the Λ-array for a specific sequence of trees $\sigma = (t_n)_{n\geq 1} \in \mathcal{S}eq$, we will most often use the convention of writing Λ_σ to refer to the element $\Theta(\sigma) \in \Lambda$

for elements a, b, and c. Hence in $(V, \triangleright)$, this looks like:

$$(X_i \triangleright X_j) \triangleright X_k - X_i \triangleright (X_j \triangleright X_k) = (X_i \triangleright X_k) \triangleright X_j - X_i \triangleright (X_k \triangleright X_j)$$

Then applying (78), the above equation becomes:

$$(\lambda_{i,j} X_{i+j}) \triangleright X_k - X_i \triangleright (\lambda_{j,k} X_{j+k}) = (\lambda_{i,k} X_{i+k}) \triangleright X_j - X_i \triangleright (\lambda_{k,j} X_{j+k})$$
$$\Longrightarrow \lambda_{i,j}\lambda_{i+j,k} X_{i+j+k} - \lambda_{j,k}\lambda_{i,j+k} X_{i+j+k} = \lambda_{i,k}\lambda_{i+k,j} X_{i+j+k} - \lambda_{k,j}\lambda_{i,j+k} X_{i+j+k}$$

which is equivalent to (77).

Now we can prove by induction that $(V, \triangleright)$ is generated by X_1. Let V' be the subalgebra of V generated by X_1. Hence as a base case, $X_1 \in V'$ trivially. Now assume that X_i is in V' for all integer values from 1 up to k, for some k, and consider X_{k+1}. By the non degeneracy condition of the $\lambda_{i,j}$'s, there exist $i, j \geq 1$ such that $i + j = k + 1$ with $\lambda_{i,j} \neq 0$. Hence $X_i \triangleright X_j = \lambda_{i,j} X_{k+1}$, which implies that $X_{k+1} = \frac{1}{\lambda_{i,j}} X_i \triangleright X_j$ (since $\lambda_{i,j} \neq 0$), hence $X_{k+1} \in V'$ as well. Thus, $V = V'$. □

We now proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1: We will start by defining the main algebraic objects:

- Let $(t_n)_{n\geq 1}$ be an element of $\mathcal{S}eq$, and define A to be the algebra they generate.
- Define the prelie algebra $\mathfrak{g} = (Prim(\mathcal{H}_{GL}), \triangleright)$, where $\triangleright$ is the prelie product inherited from $\mathcal{H}_{GL}$ by restricting $m_{\mathcal{H}_{GL}}$ to trees (since $Prim(\mathcal{H}_{GL})$ consists only of trees. See 2.1.4). In particular, if we use $*$ to denote the product in $\mathcal{H}_{GL}$, then we may define $\triangleright$ as $x \triangleright y = x * y - xy$, for $x, y \in \mathcal{H}_{GL}$ (that is, we obtain $x \triangleright y$ simply by subtracting off the term of $x * y$ corresponding to the forest xy).
- Define $I := A^{\perp}$, the orthogonal complement of A with respect to the inner product defined in Section 2.1.4 (see equation (27)).
- Let A^* be the graded dual of A.

To prove this result, we will construct Θ by first finding a nice description of A^*, and then building Θ via a comparison between the elements of a basis for A^* with our chosen basis for A (the sequence $(t_n)_{n\geq 1}$). Hence the strategy of the proof is to show the existence of the following claimed bijection:

Claim 1:

$$A^* \leftrightarrow \mathcal{U}(\mathfrak{g}/(\mathfrak{g} \cap I)) \tag{79}$$

where we use $\leftrightarrow$ to mean bijective correspondance and as before we use $\mathcal{U}$ to mean the universal enveloping algebra of $\mathfrak{g}/(\mathfrak{g} \cap I)$. Proving this claim will tell us what A^* looks like, and will consequently allow us to define the map Θ by taking $(t_n)_{n\geq 1}$ as a basis for A and comparing it to its dual basis, which will define the values $\lambda_{i,j}$.

Proof of Claim 1: To begin, note that I is a biideal of $\mathcal{H}_{GL}$ since A is a graded Hopf subalgebra of $\mathcal{H}_{CK}$. Moreover, we get that $A^* \simeq \mathcal{H}_{GL}/I$ by virtue of the fact that I is the orthogonal complement of A. Next, recall from Section 2.1.4 that $\mathcal{H}_{GL}$ is the enveloping algebra of $\mathfrak{g}$; this comes from the Milnor-Moore Theorem (Theorem 2.8) and the fact that $\mathcal{H}_{GL}$ is connected, graded, and cocommutative. Since we identified A^* with $\mathcal{H}_{GL}/I$, it then follows that A^* is the enveloping algebra of $\mathfrak{g}/(\mathfrak{g} \cap I)$ as claimed.

The only thing left to do is prove that $\mathfrak{g} \cap I$ is in fact a prelie ideal[16] of $\mathfrak{g}$. We were able to quotient $\mathfrak{g}$ by $\mathfrak{g} \cap I$ since we knew $\mathfrak{g} \cap I$ was an algebra ideal, but if we show that it is a prelie ideal as well then $\mathfrak{g}/(\mathfrak{g} \cap I)$ will inherit a prelie product. To do this, let $x \in \mathfrak{g}$ and $y \in I$. Membership in I is determined by the orthogonality with respect to our inner product, hence we want to show that for any k and any $t_{a_1} t_{a_2} t_{a_3} \cdots t_{a_k}$ (with $a_i \geq 1$ arbitrary):

$$\langle x \triangleright y, t_{a_1} t_{a_2} t_{a_3} \cdots t_{a_k} \rangle = 0$$

By definition of the prelie product $\triangleright$, $x \triangleright y$ is a linear combination of trees (since x and y are trees—see section 2.1.4), and for $k \geq 2$, $t_{a_1} t_{a_2} t_{a_3} \cdots t_{a_k}$ is a forest, hence $\langle x \triangleright y, t_{a_1} t_{a_2} t_{a_3} \cdots t_{a_k} \rangle = 0$. If $k = 1$, then we have:

$$\begin{aligned} \langle x \triangleright y, t_{a_1} t_{a_2} t_{a_3} \cdots t_{a_k} \rangle &= \langle x \triangleright y, t_{a_1} \rangle + \langle xy, t_{a_1} \rangle \qquad \text{(Since } \langle xy, t_{a_1} \rangle = 0\text{)} \\ &= \langle x \triangleright y + xy, t_{a_1} \rangle \qquad \text{(By bilinearity)} \\ &= \langle x * y, t_{a_1} \rangle \qquad \text{(By definition of } \triangleright\text{)} \\ &= \langle x \otimes y, \Delta(t_{a_1}) \rangle \end{aligned}$$

where in the second to last line we again use $*$ to mean the multiplication $m_{\mathcal{H}_{GL}}$. Now since $\Delta(t_{a_1}) \in A \otimes A$ $\langle x \otimes y, \Delta(t_{a_1}) \rangle = 0$, since $y \in I$, the orthogonal complement of A. Hence $x \triangleright y \in \mathfrak{g} \cap I$ and $\mathfrak{g} \cap I$ is in fact a prelie ideal of $\mathfrak{g}$. This validates our claim that $\mathfrak{g}/(\mathfrak{g} \cap I)$ is a valid quotient, and it inherits a prelie structure as well (the operation of which is canonical, and will also be denoted by $\triangleright$). $\square$

Now that we have proven the claim, we may continue by comparing the basis elements of A to the basis elements of A^*. We will compute the canonical basis of A^*, which we denote $(e_k)_{k \geq 1}$. These are obtained via the relation:

$$\langle e_k, t_l \rangle = \delta_{k,l} \tag{80}$$

where $\delta_{k,l}$ is the Kronecker delta function.

This means that, as $t_1 = \bullet$, $e_1 = \bullet + I$. Moreover, as $\mathfrak{g}$ is freely generated by $\bullet$ (see [6]), it follows that $\mathfrak{g}/(\mathfrak{g} \cap I)$ is generated by e_1. Hence e_k is homogeneous of degree k for any $k \geq 1$, and since the prelie product $\triangleright$ is homogeneous as well, we get that for any $i, j \geq 1$, there exists an element $\lambda_{i,j} \in \mathbb{K}$ such that:

$$e_i \triangleright e_j = \lambda_{i,j} e_{i+j}$$

Claim 2: *The $\lambda_{i,j}$ satisfy the prelie relation in the theorem statement.*

Proof of Claim 2: This follows immediately from the calculation showed in the proof of Lemma 4.2. $\square$

Claim 3: *The $\lambda_{i,j}$ satisfy the non degeneracy condition of the theorem statement.*

Proof of Claim 3: We assume towards a contradiction that there exists $n \geq 2$ such that for every i, j with $i + j = n$, $\lambda_{i,j} = 0$. Define the prelie algebra $\mathfrak{g}' = Vect(e_k, k \neq n)$. It then follows that for any indices i and j such that $i + j = n$, $e_i \triangleright e_j = \lambda_{i,j} e_{i+j} = 0 \cdot e_{i+j} \in (0)$. Otherwise if $i + j \neq n$, $e_i \triangleright e_j = \lambda_{i,j} e_{i+j} \in \mathfrak{g}'$, since $\lambda_{i,j}$ is not 0 in this case. Consequently, $\mathfrak{g}'$ is a strict prelie subalgebra of $\mathfrak{g}/(\mathfrak{g} \cap I)$, yet it still contains e_1 (since we stipulated $n \geq 2$ above). But this is impossible, as we have already noted that e_1 generates $\mathfrak{g}/(\mathfrak{g} \cap I)$ so that $\mathfrak{g}'$ must not be strict, but rather all of $\mathfrak{g}/(\mathfrak{g} \cap I)$. Therefore we conclude that for every $n \geq 2$, there exist i, j with $i + j = n$ such that $\lambda_{i,j} \neq 0$, as desired. $\square$

[16]By *prelie ideal*, we mean exactly what the reader likely expects: That for a prelie subalgebra I of a prelie algebra L, for every $a \in I$ and $b \in L$, $a \triangleright b \in I$. In this case I is a prelie ideal of L.

This shows the existence of the map $\Theta : \mathcal{S}eq \to \Lambda$. Now all that we need to do is show that Θ is a bijection.

Claim 4: *The map Θ is injective.*

Proof of Claim 4: In both the proof of this claim and the next, the key is to use that $\mathfrak{g}$ is the unique prelie algebra freely generated by $\bullet$. To begin, assume that $\Theta((t_n)_{n\geq 1}) = (\lambda_{i,j})_{i,j\geq 1} = \Theta((t'_n)_{n\geq 1})$. We desire to show that $(t_n)_{n\geq 1} = (t'_n)_{n\geq 1}$.

To fix notation, let $(V, \triangleright)$ be the prelie algebra of Lemma 4.2, and let I and I' be the orthogonal complements of A and A' (the algebras generated by$(t_n)_{n\geq 1}$ and $(t'_n)_{n\geq 1}$ respectively). Further, define maps $\phi : \mathfrak{g}/(\mathfrak{g} \cap I) \to V$ sending e_i to X_i, and $\phi' : \mathfrak{g}/(\mathfrak{g} \cap I') \to V$ sending e'_i to X_i. These maps extend to Hopf algebra morphisms $\Phi : \mathcal{H}_{GL}/I \to \mathcal{U}(V)$ and $\Phi' : \mathcal{H}_{GL}/I' \to \mathcal{U}(V')$, by virtue of properties of the universal enveloping algebra. Finally, we can dualize the maps to get *injective* morphisms $\Phi^* : \mathcal{U}(V)^* \to \mathcal{H}_{CK}$ sending X_i^* to t_i and $\Phi'^* : \mathcal{U}(V)^* \to \mathcal{H}_{CK}$ sending X_i^* to t'_i.

But as discussed above, $\mathfrak{g}$ is the unique prelie algebra freely generated by $\bullet$, consequently there is a unique map ψ that sends $\bullet$ to X_1, as all other values of ψ are then determined. Moreover there is a unique canonical (and surjective) map from $\mathfrak{g}$ to $\mathfrak{g}/(\mathfrak{g}\cap I)$ (respectively from $\mathfrak{g}$ to $\mathfrak{g}/(\mathfrak{g}\cap I')$) by property of quotients, which we will call π (respectively π')[17]. This means that the following diagrams must commute:

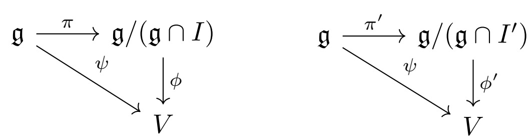


Figure 17: The maps described in the proof of claim 4.

Hence $\psi = \phi \circ \pi$ and $\psi = \phi' \circ \pi'$ and so $\phi \circ \pi = \phi' \circ \pi'$. However the map ψ is unique, hence it must be that $I = I'$ and $\phi = \phi'$, and this further implies by construction of Φ, Φ' as canonical extensions, and Φ^*, Φ'^* as dual maps that $\Phi = \Phi'$ and $\Phi^* = \Phi'^*$. Hence we also get that $(t_n)_{n\geq 1} = (t'_n)_{n\geq 1}$, and so Θ is injective as claimed. $\square$

Claim 5: *The map Θ is surjective.*

Proof of Claim 5: The proof of this claim makes use of similar maps as in the proof of Claim 4. As before, define $(V, \triangleright)$ to be the prelie algebra on formal variables as defined in Lemma 4.2. To show surjectivity, we take $(\lambda_{i,j})_{i,j\geq 1} \in \Lambda$ arbitrarily and want to show that there exists a sequence $(t_n)_{n\geq 1} \in \mathcal{S}eq$ such that $\Theta((t_n)_{n\geq 1}) = (\lambda_{i,j})_{i,j\geq 1}$. Again, since V is generated freely by X_1, there must be a unique, surjective prelie morphism $\phi : \mathfrak{g} \to V$ such that $\phi(\bullet) = X_1$. As before, we can extend this to a Hopf algebra morphism $\Phi : \mathcal{H}_{GL} \to \mathcal{U}(V)$ that preserves surjectivity, and then take the dual of this map to get $\Phi^* : \mathcal{U}(V)^* \to \mathcal{H}_{CK}$, which must be injective (by properties of duality). Now for each $n \geq 1$ set $t_n = \Phi^*(X_n^*)$. This shows the existence of $(t_n)_{n\geq 1}$ such that $\Theta((t_n)_{n\geq 1}) = (\lambda_{i,j})_{i,j\geq 1}$. We know further that $(t_{n\geq 1})$ is in fact an element of $\mathcal{S}eq$, as the fact that $\phi(\bullet + I) = X_1$ ensures that $t_1 = \bullet$, and Φ^* being injective ensures that the kernel of Φ^* is trivial, and hence that t_n is a nonzero linear combination of trees. $\square$

We have shown that $\Theta : \mathcal{S}eq \to \Lambda$ exists and is a bijection. This completes the proof of the theorem. $\square$

[17] These canonical maps are given by $\pi(a) = a + I$ and $\pi'(b) = b + I'$ [43].

Now the details of the proof of Theorem 4.1 might leave the reader without a clear sense of how to apply Θ in practical applications (as well as how to apply Θ^{-1}, which we have not yet described). However, it is straightforward to describe how to do this, which we do now.

Given $(t_n)_{n\geq 1} \in \mathcal{S}eq$, the coproduct of t_n for a given n will look like:

$$\Delta(t_n) = t_n \otimes \mathbb{1} + \mathbb{1} \otimes t_n + \left(\sum_{i=1}^{n-1} [\lambda_{i,n-i} t_{n-i} + P_{i,n-i}] \otimes t_i \right)$$

where $P_{i,n-i}$ is a polynomial in $t_1, ..., t_{n-i-1}$ homogeneous of degree $(n-i)$. In other words, in the coproduct of t_n, there will be exactly $n-1$ terms (possibly with coefficient 0) with a tree tensor another tree, and the $\lambda_{i,j}$ are found as the coefficients of such. We will see an example in the next section, and further examples in Appendix C.

Now how do we apply $\Theta^{-1} : \Lambda \to \mathcal{S}eq$? Since Θ^{-1} is actually the map Φ^* (see proof of Claim 5 in Theorem 4.1), we need to understand the structure of the universal enveloping algebra of a prelie algebra. Thankfully, this is the contents of Theorem 2.40 (Proposition 2.7 of [38]). If we denote once again by $(V, \triangleright)$ the prelie algebra described in Lemma 4.2, then the aforementioned theorem tells us how $\triangleright$ extends to the Hopf algebra $S(V)$; namely, for $X_1 \in V$ and $X_{i_1}...X_{i_k} \in S(V)$, the recurrences in the theorem give that $X_1 \triangleright X_{i_1}...X_{i_k} \in V$ (not $S(V)$). Hence there must exist a scalar $\lambda(i_1, ..., i_k)$ depending only on $i_1, ..., i_k$ such that:

$$X_1 \triangleright X_{i_1}...X_{i_k} = \lambda(i_1, ..., i_k) X_{1+i_1+...+i_k}$$

where we are also using that the prelie product $\triangleright$ is homogeneous (even after being extended to $S(V)$. Note also that this means that the coefficients $\lambda(i_1, ..., i_k)$ are symmetric in their arguments). While the use of the letter λ both for the coefficients $\lambda_{i,j}$ and for the coefficients $\lambda(i_1, ..., i_k)$ might at first seem confusing, this choice is intentional in order to stress their fundamental connection: namely, $\lambda_{i,j}$ are structure coefficients in V, and $\lambda(i_1, ..., i_k)$ are structure coefficients in $S(V)$.

Moreover, we can use Theorem 2.40 to compute the coefficients $\lambda(i_1, ..., i_k)$ by induction on k, and get that [24]:

$$\lambda(i_1, ..., i_k) = \begin{cases} 1 & \text{if } k = 0, \\ \lambda_{1,i_1} & \text{if } k = 1, \\ \lambda(i_1, ..., i_{k-1}) \lambda_{1+i_1+...+i_{k-1}, i_k} - \sum_{j=1}^{k-1} \lambda(i_1, ..., i_j + i_k, ..., i_{k-1}) \lambda_{i_j, i_k} & \text{otherwise.} \end{cases} \tag{81}$$

Finally, we again rely on the map $\phi : \mathfrak{g} \to V$ that is the unique prelie morphism such that $\phi(\bullet) = X_1$.

Lemma 4.3. *Let $t \in \mathcal{T}$, and set $\phi(t) = \mu(t) X_{|t|}$. Then:*

$$\mu(t) = \prod_{s \in V(t)} \lambda(|\alpha_1^{(s)}|, \cdots, |\alpha_{k(s)}^{(s)}|) \tag{82}$$

where $V(t)$ denotes the vertex set of t, $k(s)$ is the number of subtrees of t_s, and the $\alpha_i^{(s)}$ are the subtrees of t_s.

Proof. Suppose that $t = B^+(\alpha_1 \cdots \alpha_k)$. By definition of $\triangleright$, this means that $t = \bullet \triangleright \alpha_1 ... \alpha_k$. Hence:

$$\begin{aligned} \phi(t) &= \phi(\bullet \triangleright \alpha_1 ... \alpha_k) \\ &= X_1 \triangleright \phi(\alpha_1) \cdots \phi(\alpha_k) \\ &= \mu(\alpha_1) \cdots \mu(\alpha_k) \lambda(|\alpha_1|, ..., |\alpha_k|) X_{|t|} \end{aligned}$$

This proves the result by recursing on the $\mu(\alpha_i)$. $\square$

Hence, putting everything together:

$$t_n = \sum_{t \in \mathcal{T}_n} \frac{\mu(t)}{|Sym(t)|} t \tag{83}$$

where the factor of $\frac{1}{|Sym(t)|}$ is coming from the fact that the $\lambda(i_1, ..., i_k)$ are symmetric with respect to their arguments.

4.2 Examples

Our intention now is to tie together the material of Section 3 with that of the previous subsection via some examples. To start, let us demonstrate the method of computing $\Theta((t_n)_{n\geq 1})$ using the procedure described in the last section.

Example 4.4. We will give an example of the bijection by applying Θ to the sequence of generators of the Hopf subalgebra of binary trees. Denote this sequence by $(x_n)_{n\geq 1}$, and let $X = \mathbb{1} + \sum_{n=1}^{\infty} x_n$. Then one way we can describe the series X is by the functional equation:

$$X = \mathbb{1} + B^+(X^2) \tag{84}$$

where, due to the graded nature of $\mathcal{H}_{CK}$, we can leave out the instance of any formal counting variables. We also remark that this is an instance of a series coming from a combinatorial Dyson-Schwinger equation. These are written about in detail in [20]. One can also find the family of these sequences in Table 5 in Appendix C.

The functional equation (84) determines the sequence $(x_n)_{n\geq 1}$ recursively as follows:

$$\begin{aligned} x_1 &= \bullet \\ x_k &= 2x_{k-1} + B^+\left(\sum_{i=1}^{k-2} x_i x_{k-i}\right) \qquad \text{for } k \geq 2 \end{aligned}$$

We call the sequence $(x_n)_{n\geq 1}$ the sequence of **binary rooted trees** as each x_n will be a linear combination of trees having the structure of a binary tree (in the classical sense of every vertex in the tree having either $0, 1$, or 2 children), and the coefficient of each tree T in the sum x_n will be the number of planar embeddings of T that distinguish between left and right children for every vertex. To demonstrate, the first few x_n look like:

Figure 19: Planar embeddings of a tree that distinguish between left and right children at each vertex.

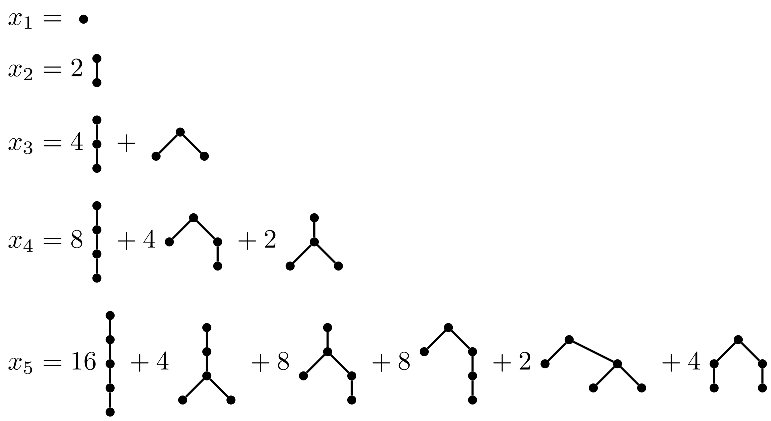
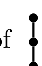
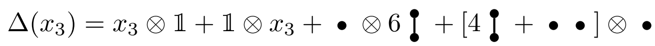


Figure 18: The first few elements in the sequence of rooted binary trees.

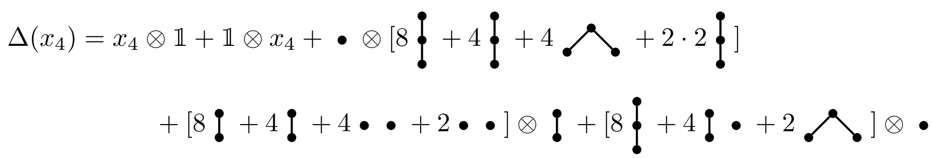
Hence the tree has coefficient 4, for example, since there are 4 different planar embeddings of  with left- and right-children distinguished. These are depicted in Figure 19.

Let us compute the coproduct of each x_n in order to find what $\Theta((x_n)_{n\geq 1})$ looks like in Λ, keeping track of the $\lambda_{i,j}$ with red text and a box. We have:

$$\Delta(x_1) = x_1 \otimes \mathbb{1} + \mathbb{1} \otimes x_1$$
$$\Delta(x_2) = x_2 \otimes \mathbb{1} + +\mathbb{1} \otimes x_2 + \boxed{2} x_1 \otimes x_1$$



$$= x_3 \otimes \mathbb{1} + \mathbb{1} \otimes x_3 + \boxed{3} x_1 \otimes x_2 + [\boxed{2} x_2 + x_1^2] \otimes x_1$$



$$= x_4 \otimes \mathbb{1} + \mathbb{1} \otimes x_4 + \boxed{4} x_1 \otimes x_3 + [\boxed{3} x_2 + 3x_1^2] \otimes x_2 + [\boxed{2} x_2 + 2x_2x_1] \otimes x_1$$
$$\vdots$$

Hence the beginning of the corresponding Λ-array will look like:

$$\begin{array}{ccccc} & & 2 & & \\ & 3 & & 2 & \\ 4 & & 3 & & 2 \\ & & \vdots & & \end{array}$$

Indeed, it is not hard to show that for the sequence of binary trees, $\lambda_{i,j} = i+1$. We will defer proof of this fact, as it is actually a special case of Foissy's parameterization of sequences coming from Dyson-Schwinger equations, as mentioned above (with the specialization $a = b = 1$). See Appendix C.

Now suppose we were in the opposite scenario, having $(\lambda_{i,j})_{i,j\geq 1} \in \Lambda$ given by $\lambda_{i,j} = i+1$, and wanted to compute the corresponding element of $\mathcal{S}eq$. Then we simply apply equations (83) and (81) algorithmically, splitting up the computation according to the size of the trees under consideration. For trees of size 1:

$$\begin{aligned} x_1' &= \sum_{t \in \mathcal{T}_1} \frac{\mu(t)}{|Sym(t)|} t \\ &= \mu(\bullet)\bullet \\ &= \bullet \end{aligned}$$

For trees of size 2:

$$x'_2 = \sum_{t \in \mathcal{T}_2} \frac{\mu(t)}{|Sym(t)|} t$$

$= \mu($  $)$ 

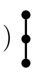
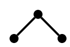
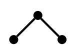
$= \lambda(1)$ 

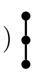
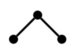
$= \lambda_{1,1}$ 

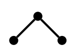
$= 2$ 

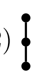
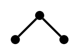
For trees of size 3:

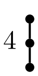
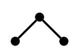
$$x'_3 = \sum_{t \in \mathcal{T}_3} \frac{\mu(t)}{|Sym(t)|} t$$

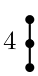
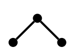
$= \mu($  $)$  $+ \frac{\mu(\ \ )}{2}$ (numerator: $\mu($  $)$) 

$= \lambda(2)\lambda(1)$  $+ \frac{\lambda(1,1)}{2}$ 

$= \lambda_{1,2}\lambda_{1,1}$  $+ \frac{\lambda(1)\lambda_{2,1} - \lambda(2)\lambda_{1,1}}{2}$ 

$= (2)(2)$  $+ \frac{\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{1,1}}{2}$ 

$= 4$  $+ \frac{(2)(3) - (2)(2)}{2}$ 

$= 4$  $+$ 

We could continue in this way to compute all x'_n, and indeed would find that $x'_n = x_n$ for all n.

Remark. We remark that the doubly-indexed sequences of elements $\lambda_{i,j}$ appearing in this section are exactly those that appeared in Section 3.1. It was not clear in Section 3.1 that the $\lambda_{i,j}$ as presented in that section had the same prelie relationship that the $\lambda_{i,j}$ here do, but the fact that the two doubly-indexed sequences are the same is clear from the fact that they are precisely the same combinatorially: both are the coefficients of the tree-tensor-tree terms in the sequence $(\Delta(t_n))_{n\geq 1}$ for $(t_n)_{n\geq 1} \in \mathcal{S}eq$.

5 Main Results

5.1 0th Order Strong

In this section, we present the full characterization of elements of $\mathcal{S}eq$ that are strong 0th-order sequences (see Definition 3.7). We build up to this result with a few lemmata.

Lemma 5.1. *Let $\omega = (\lambda_{i,j})_{i,j\geq 1}$ be an element of Λ and $s_1 = (\lambda_{i,1})_{i\geq 1}$ and $s_2 = (\lambda_{1,i})_{i\geq 1}$ such that $\lambda_{i,1} \neq 0$ for all i. Then for every $i, j \geq 1$, $\lambda_{i,j}$ can be expressed in terms of the elements of s_1 and s_2.*

Pictorially, this means that the following infinite array of $\lambda_{i,j}$'s is completely filled in after we know the array is prelie and the values on the outer two edges (and such that the left edge does not contain any zeroes):

$$\begin{array}{ccccccc} & & & \lambda_{1,1} & & & \\ & & \lambda_{2,1} & & \lambda_{1,2} & & \\ & \lambda_{3,1} & & \lambda_{2,2} & & \lambda_{1,3} & \\ \lambda_{4,1} & & \lambda_{3,2} & & \lambda_{2,3} & & \lambda_{1,4} \\ & & & \vdots & & & \end{array}$$

Proof. Let $\omega \in \Lambda$ be given as in the statement of the lemma. We will use induction on the first index of the $\lambda_{i,j}$'s to show that $(\lambda_{i,j})_{i,j\geq 1}$ is completely determined by s_1 and s_2.

To begin, recall the prelie axiom (equation (77)) for elements $(\lambda_{i,j})_{i,j\geq 1} \in \Lambda$. For convenience, we write it again here. It says that, for all $i, j, k \geq 1$:

$$\lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} = \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,k+j} \tag{77}$$

As a base case, let $k \geq 1$ be arbitrary and consider the element $\lambda_{2,k}$ in $(\lambda_{i,j})_{i,j\geq 1}$. If we set $i = j = 1$ in the prelie relation above, then we obtain:

$$\begin{aligned} &\lambda_{1,1}\lambda_{2,k} - \lambda_{1,k}\lambda_{1,1+k} = \lambda_{1,k}\lambda_{1+k,1} - \lambda_{k,1}\lambda_{1,k+1} \\ \implies &\lambda_{2,k} = \frac{\lambda_{1,k}\lambda_{1+k,1} - \lambda_{k,1}\lambda_{1,k+1} + \lambda_{1,k}\lambda_{1,1+k}}{\lambda_{1,1}} \end{aligned}$$

In particular, note that $\lambda_{1,k}, \lambda_{1,k+1}, \lambda_{1,1} \in (\lambda_{1,i})_{i\geq 1}$ and $\lambda_{k,1}, \lambda_{1+k,1} \in (\lambda_{i,1})_{i\geq 1}$, hence $\lambda_{2,k}$ is determined by $(\lambda_{1,i})_{i\geq 1}$ and $(\lambda_{i,1})_{i\geq 1}$ for all $k \geq 1$.

Now suppose that $\lambda_{i,k}$ can be expressed in terms of $(\lambda_{1,i})_{i\geq 1}$ and $(\lambda_{i,1})_{i\geq 1}$ for all positive integers $1, 2, ..., i$ and for all $k \geq 1$. Let $l \geq 1$ be arbitrary and consider the element $\lambda_{i+1,l}$. In the prelie relation above, set $i = i, j = 1, k = l$. Then we obtain:

$$\begin{aligned} &\lambda_{i,1}\lambda_{i+1,l} - \lambda_{1,l}\lambda_{i,l+1} = \lambda_{i,l}\lambda_{i+l,1} - \lambda_{l,1}\lambda_{i,l+1} \\ \implies &\lambda_{i+1,l} = \frac{\lambda_{i,l}\lambda_{i+l,1} - \lambda_{l,1}\lambda_{i,l+1} + \lambda_{1,l}\lambda_{i,l+1}}{\lambda_{i,1}} \end{aligned}$$

In particular, note that $\lambda_{1,l} \in (\lambda_{1,i})_{i\geq 1}$ and $\lambda_{l,1}, \lambda_{i,1}, \lambda_{i+l,1} \in (\lambda_{i,1})_{i\geq 1}$. Moreover, $\lambda_{i,l}, \lambda_{i,l+1}$ can be expressed in terms of $(\lambda_{1,i})_{i\geq 1}$ and $(\lambda_{i,1})_{i\geq 1}$ by the inductive hypothesis. Hence $\lambda_{i+1,l}$ can be expressed solely in terms of $(\lambda_{1,i})_{i\geq 1}$ and $(\lambda_{i,1})_{i\geq 1}$.

This proves the desired result.

□

We get for free the following consequence:

Corollary 5.2. Every strong 0th order sequence in *Seq* is completely determined by the single sequence $(\lambda_{1,i})_{i\geq 1}$.

Proof. By the non-degeneracy condition of Theorem 4.1, every element of Λ has $\lambda_{1,1} = k \neq 0$. Moreover, every strong 0th order sequence has that $(\lambda_{i,j_0})_{i\geq 0}$ is a constant sequence. Consequently $\lambda_{i,1} = k$ for all i and so by the lemma, $(\lambda_{1,i})_{i\geq 1}$ is all that is left to describe the interior of the Λ-array. ☐

Since Lemma 5.1 says that each $\lambda_{i,j}$ in a prelie array can be expressed in terms of the outer two edges s_1 and s_2 when s_1 contains no zeros, perhaps the next most natural question is what this expression looks like. This is the contents of the following:

Lemma 5.3. *For a prelie array $\omega = (\lambda_{i,j})_{i,j\geq 1}$, with s_1 and s_2 the outer sequences as in Lemma 5.1 (and such that s_1 contains no zeros), put $s_1 = (y_n)_{n\in\mathbb{N}}$ and $s_2 = (x_n)_{n\in\mathbb{N}}$. Then for all $i, j \geq 1$:*

$$\lambda_{i,j} = \frac{\sum_{n=j}^{i+j-1} \binom{i-1}{n-j} x_n \prod_{m=n+1}^{i+j-1} y_m \prod_{l=j}^{n-1} (x_l - y_l)}{\prod_{p=2}^{i-1} y_p} \tag{85}$$

Proof. The proof proceeds by a simple induction on the index i of the $\lambda_{i,j}$'s. To begin, recall the rearranged prelie relation of the Lemma 5.1. It gives that for all $i, j, k \geq 1$:

$$\lambda_{i+1,j} = \frac{\lambda_{i,j}\lambda_{i+j,1} - \lambda_{j,1}\lambda_{i,j+1} + \lambda_{1,j}\lambda_{i,j+1}}{\lambda_{i,1}} \tag{86}$$

Then for $i = 1$ and j arbitrary, we have:

$$\begin{aligned}
\lambda_{2,j} &= \frac{\lambda_{1,j}\lambda_{1+j,1} - \lambda_{j,1}\lambda_{1,j+1} + \lambda_{1,j}\lambda_{1,j+1}}{\lambda_{1,1}} \\
&= x_j y_{j+1} - y_j x_{j+1} + x_j y_{j+1} \\
&= x_{j+1}(x_j - y_j) + x_j y_{j+1}
\end{aligned}$$

which is equivalent to equation (85) for $i = 2$ and j arbitrary.

Now we induct: Suppose the statement is true for all positive integer values of the first index from 1 up to some arbitrary i. Consider the case of $i + 1$:

$$\begin{aligned}
\lambda_{i+1,j} &= \frac{\lambda_{i,j}\lambda_{i+j,1} - \lambda_{j,1}\lambda_{i,j+1} + \lambda_{1,j}\lambda_{i,j+1}}{\lambda_{i,1}} \\
&= \frac{1}{y_i}[y_{i+j}\lambda_{i,j} - y_j\lambda_{i,j+1} + x_j\lambda_{i,j+1}] \\
&= \frac{1}{y_i}[y_{i+j}\lambda_{i,j} + (x_j - y_j)\lambda_{i,j+1}] \\
&= \frac{1}{y_i}\left[\frac{y_{i+j}\sum_{n=j}^{i+j-1}\binom{i-1}{n-j} x_n \prod_{m=n+1}^{i+j-1} y_m \prod_{l=j+1}^{n-1}(x_l - y_l)}{\prod_{p=2}^{i-1} y_p}\right] \\
&\quad + \frac{1}{y_i}\left[\frac{(x_j - y_j)\sum_{n=j+1}^{i+j}\binom{i-1}{n-j-1} x_n \prod_{m=n+1}^{i+j} y_m \prod_{l=j+1}^{n-1}(x_l - y_l)}{\prod_{p=2}^{i-1} y_p}\right]
\end{aligned}$$

Now we can gather all the denominators and release $n = j$ from the first summation and $n = i + j$ from the second summation in order to write the rest with just a single sigma:

$$= \frac{1}{\prod_{p=2}^{i} y_p} \left[\left[y_{i+j} \binom{i-1}{0} x_j \prod_{m=j+1}^{i+j-1} y_m \right] + \left[(x_j - y_j) \binom{i-1}{i-1} x_{i+j} \prod_{l=j}^{i+j-1} (x_l - y_l) \right] \right.$$
$$\left. + \sum_{n=j+1}^{i+j-1} \left[y_{i+j} \binom{i-1}{n-j} x_n \prod_{m=n+1}^{i+j-1} y_m \prod_{l=j+1}^{n-1} (x_l - y_l) + (x_j - y_j) \binom{i-1}{n-j-1} x_n \prod_{m=n+1}^{i+j-1} y_m \prod_{l=j+1}^{n-1} (x_l - y_l) \right] \right]$$

Factoring, we get:

$$= \frac{1}{\prod_{p=2}^{i} y_p} \left[\left[x_j \prod_{m=j+1}^{i+j} y_m \right] + \left[x_{i+j} \prod_{l=j}^{i+j-1} (x_l - y_l) \right] \right.$$
$$\left. + \sum_{n=j+1}^{i+j-1} x_n \prod_{m=n+1}^{i+j} y_m \prod_{l=j+1}^{n-1} (x_l - y_l) \left[\binom{i-1}{n-j} + \binom{i-1}{n-j-1} \right] \right]$$

Which due to the binomial identity $\binom{i-1}{n-j} + \binom{i-1}{n-j-1} = \binom{i}{n-j}$ gives us:

$$\lambda_{i,j} = \frac{\sum_{n=j}^{i+j-1} \binom{i-1}{n-j} x_n \prod_{m=n+1}^{i+j-1} y_m \prod_{l=j}^{n-1} (x_l - y_l)}{\prod_{p=2}^{i-1} y_p}$$

which is equation 85 □

Corollary 5.4. Under the same conditions as Lemma 5.1:

(i) All zeroeth order elements of Seq have Λ-sequences such that:

$$\lambda_{i,j} = \sum_{k=j}^{i+j-1} \binom{i-1}{k-j} x_k \sum_{I \subseteq \{j, j+1, \ldots, k-1\}} (-1)^{|I|-i} x_I \tag{87}$$

With these lemmas in place, we can now present the central theorem of this subsection:

Theorem 5.5 (Classification of Strong 0th Order Sequences)**.** *Let $b \in \mathbb{K}$. Then the only elements of Seq satisfying a 0th order generalized renormalization group equation for every choice of θ in the general tree Feynman rules are of the form:*

$$X = B^{+}(\exp([\sum_{i=1}^{n-1} P_i] + bP_n)) \tag{88}$$

or

$$X = B^{+}(\exp(\sum_{i=1}^{\infty} P_i)) \tag{89}$$

where the P_i are the basis elements for the Hopf algebra of ladders, as defined in Lemma 2.37 (see also [21]).

Proof. To fix notation, set $s = (x_n)_{n\geq 0}$ and let $X = \sum_{n=0}^{\infty} x_n$ be the corresponding series. Further define A_s to be the algebra generated by the elements in s.

The proof will be broken up into three parts. First, we will show that the algebra A_s is a Hopf subalgebra of $\mathcal{H}_{CK}$. We will then show that the corresponding Λ-array, which we denote Λ_s, has the form

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad \vdots \\
1 \quad 1 \quad \vdots \quad 1 \\
1 \quad 1 \quad \vdots \quad 1 \quad b \\
1 \quad 1 \quad \vdots \quad 1 \quad b \quad 0 \\
1 \quad 1 \quad \vdots \quad 1 \quad b \quad 0 \quad 0 \\
1 \quad 1 \quad \vdots \quad 1 \quad b \quad 0 \quad 0 \quad 0
\end{array}$$

(where b is the nth element of the sequence $(\lambda_{1,i})_{i\geq 1}$), and consequently satisfies a 0th order generalized renormalization group equation for all choices of θ in the generalized tree Feynman rules (equation (62)). Finally, we will conclude by showing that all other choices of $(\lambda_{1,i})_{i\geq 1}$ result in a Λ-array that has non-constant diagonals, thereby demonstrating that equations (88) and (89) give all 0th order strong sequences.

To begin, we show that $\Delta(X) \subseteq A_s \otimes A_s$, which is sufficient to show that A_s is Hopf since the x_n generate A_s. Let n be fixed and let $Z = [\sum_{i=1}^{n-1} P_i] + bP_n$. Then:

$$\begin{aligned}
\Delta(X) &= \Delta(B^+(\exp(Z))) \\
&= (B^+(\exp(Z))) \otimes \mathbb{1} + (id \otimes B^+)(\Delta(\exp(Z))) && \text{(By the Hochschild cocycle property of } B^+\text{)} \\
&= X \otimes \mathbb{1} + (id \otimes B^+)(\exp(Z) \otimes \exp(Z)) && (\exp(Z) \text{ is group-like since } Z \text{ is primitive}) \\
&= X \otimes \mathbb{1} + \exp(Z) \otimes B^+(\exp(Z)) \\
&= X \otimes \mathbb{1} + \exp(Z) \otimes X \\
&= X \otimes \mathbb{1} + \exp([\sum_{i=1}^{n-1} P_i] + bP_n) \otimes X \\
&= X \otimes \mathbb{1} + (\sum_{m=0}^{\infty} \frac{1}{m!}(P_1 + P_2 + ... + bP_n)^m) \otimes X
\end{aligned}$$

What does the graded piece of $\Delta(X)$ of degree ℓ look like? In analogy with coefficient extraction when working with power series, we will denote this term by $[\ell]\Delta(X)$:

$$[\ell]\Delta(X) = x_\ell \otimes \mathbb{1} + \sum_{i=1}^{\ell} [i](\sum_{m=0}^{\infty} \frac{1}{m!}(P_1 + P_2 + ... + bP_n)^m) \otimes x_{\ell-i} \tag{90}$$

Hence the problem really boils down to extracting the ith piece of the power series $\sum_{m=0}^{\infty} \frac{1}{m!}(P_1 + P_2 + ... + bP_n)^m$ (notice that this is just $B^-(X)$). In the next two steps we expand the expression using the multinomial theorem and then apply coefficient extraction:

$$[i](\sum_{m=0}^{\infty}\frac{1}{m!}(P_1+P_2+...+bP_n)^m) = [i]\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k_1+k_2+...+k_n=m}\binom{m}{k_1,k_2,...,k_n}\prod_{t=1}^{m}b^{k_n}P_t^{k_t}$$
$$= \sum_{j=1}^{i}\frac{1}{j!}\sum_{\substack{k_1+k_2+...+k_n=j\\ k_1+2k_2+...nk_n=i}}\binom{j}{k_1,k_2,...,k_n}\prod_{t=1}^{m}b^{k_n}P_t^{k_t}$$

But using the condition $k_1+k_2+...+k_n=j$, we can combine the two sums under one sigma, so the previous equation is equal to:

$$= \sum_{k_1+2k_2+...nk_n=i}\frac{1}{(k_1+k_2+...+k_n)!}\binom{k_1+k_2+...+k_n}{k_1,k_2,...,k_n}\prod_{t=1}^{m}b^{k_n}P_t^{k_t}$$
$$= \sum_{k_1+2k_2+...nk_n=i}\frac{1}{(k_1+k_2+...+k_n)!}\frac{(k_1+k_2+...+k_n)!}{k_1!\,k_2!\cdots k_n!}\prod_{t=1}^{m}b^{k_n}P_t^{k_t}$$
$$= \sum_{k_1+2k_2+...nk_n=i}\frac{1}{k_1!\,k_2!\cdots k_n!}P_1^{k_1}P_2^{k_2}\cdots(bP_n)^{k_n}$$

What this means is that the terms of X are exactly:

$$[i]X = B^+(\sum_{k_1+2k_2+...nk_n=i-1}\frac{1}{k_1!\,k_2!\cdots k_n!}P_1^{k_1}P_2^{k_2}\cdots(bP_n)^{k_n}) \tag{91}$$

and the desired quantity from equation (90) is:

$$[\ell]\Delta(X) = x_\ell\otimes\mathbb{1}+\sum_{i=1}^{\ell}\left(\sum_{k_1+2k_2+...nk_n=i-1}\frac{1}{k_1!\,k_2!\cdots k_n!}P_1^{k_1}P_2^{k_2}\cdots b^{k_n}P_n^{k_n}\right)\otimes x_{\ell-i} \tag{92}$$

We claim that equation (92) implies that A_s is Hopf. To see why, recall the definition of our map $\omega=\exp_*(\zeta)$ from equation (43) in Section 2.2.2. In that section we saw that $P_i=\omega^{-1}(l_i)$ for l_i the ladder on i vertices (Definition 2.25). Note that for i at most n, the right hand side of equation 91 is exactly $\omega(P_i)$ (or $\omega(bP_n)$) , which is $\omega(\omega^{-1}(l_i))=l_i$ (respectively $\omega(\omega^{-1}(bl_n))=bl_n$), so the first $n+1$ terms of X agree with the sequence of ladders. (The reason it is $n+1$ and not n is because we are applying B^+). Hence we know that $\Delta(X)\subseteq A_s\otimes A_s$ for the first $n+1$ terms of X. But then for j greater than $n+1$, equation (90) tells us that the left side of every tensor product in $\widetilde{\Delta}(x_j)$ is just a product of $l_1,l_2,...,l_n,bl_{n+1}$. Since we already knew that the right side of every tensor product was an element of X, it follows that $\Delta(X)\subseteq A_s\otimes A_s$ as desired. Examples of the first few sequences X of this form follow the proof.

Next we want to show that Λ_s is of the form claimed. This is a straightforward computation. Firstly, for any $\ell\geq 1$, $\lambda_{\ell,1}$ is the coefficient of the term in equation (90) when $i=1$. But this term is just $P_1\otimes x_{\ell-1}$, hence $\lambda_{\ell,1}=1$ for all ℓ. On the other hand, by the analysis in the previous paragraph we know that for $\ell>n+1$, x_ℓ is B^+ applied to a specific sum of products of the $l_1,l_2...,bl_{n+1}$. Since $\lambda_{1,\ell}$ is the coefficient of $B^-(x_\ell)$, this means that $\lambda_{1,\ell}=0$ for all $\ell>n+1$ whereas $\lambda_{1,\ell}=1$ for $\ell<n$ and $\lambda_{1,n}=b$ (since the first $n+1$ terms are the first $n+1$ ladders). Finally, since A_s is Hopf by the first part of the proof and

we know the two outer edges of the array, it follows that the inside of the array is filled in according to Lemma 5.3. It is easy to verify that this is of the form claimed.

Finally, we show that the only 0th-order strong elements are of the form described in the theorem statement. Suppose we have some other 0th order element Λ_Y of Λ, so $\lambda_{i,j} \in \Lambda_Y$ are such that $\lambda_{i,1} = 1$ for all i and $(\lambda_{1,i}) = (1, 1, 1, ...1, a_n, a_{n+1}, a_{n+2}, ...)$ for $(n-1)$ 1's followed by some a_n different from 1 and a_m any element of $\mathbb{K}$, for $m > n$. Observe that this setup accounts for all Λ-arrays not of the form described above.

Now since this new array is an element of Λ and is 0th order, it follows that its members must satisfy the form given in Corollary 5.4 item (i). Let us look at the sequence $\lambda_{i,n}$ for n fixed and i varying over $\mathbb{N}$. Using the corollary, the elements of this sequence start:

$$\begin{aligned}
\lambda_{1,n} &= a_n \\
\lambda_{2,n} &= \sum_{k=n}^{n+1} \binom{1}{k-n} a_k \sum_{I \subseteq \{n-1,n,\dots,k-1\}} (-1)^{|I|-i} a_I \\
&= \binom{1}{0} a_n \cdot (1) + \binom{1}{1} a_{n+1}(-1 + a_n) \\
&= a_n a_{n+1} + a_n - a_{n+1} \\
\lambda_{3,n} &= a_n a_{n+1} a_{n+2} + 2a_n a_{n+1} - a_n a_{n+2} - a_{n+1} a_{n+2} + a_n - 2a_{n+1} + a_{n+2} \\
&\vdots
\end{aligned}$$

Now for Λ_Y to satisfy a 0th order generalized renormalization group equation for all choices of θ, we need the sequence $(\lambda_{i,n-1})_{i \geq 1}$ to be constant. In other words, we need $\lambda_{2,n} = a_n$, so:

$$a_n = a_n a_{n+1} + a_n - a_{n+1}$$

which implies that $a_n a_{n+1} = a_{n+1}$. Since $a_n \neq 1$, it follows that $a_{n+1} = 0$. Additionally, we need $\lambda_{3,n} = a_n$, which gives:

$$\begin{aligned}
& a_n = a_n a_{n+1} a_{n+2} + 2a_n a_{n+1} - a_n a_{n+2} - a_{n+1} a_{n+2} + a_n - 2a_{n+1} + a_{n+2} \\
\implies & a_n = -a_n a_{n+2} + a_n + a_{n+2}
\end{aligned}$$

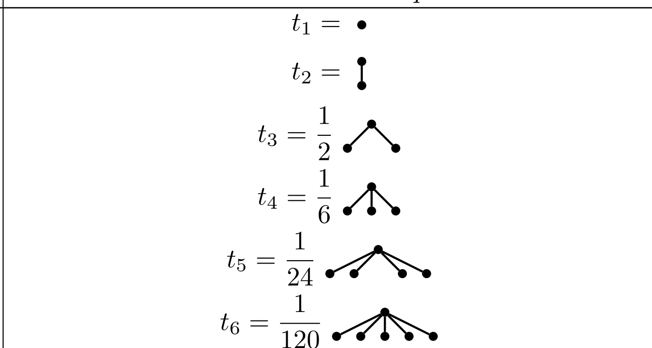
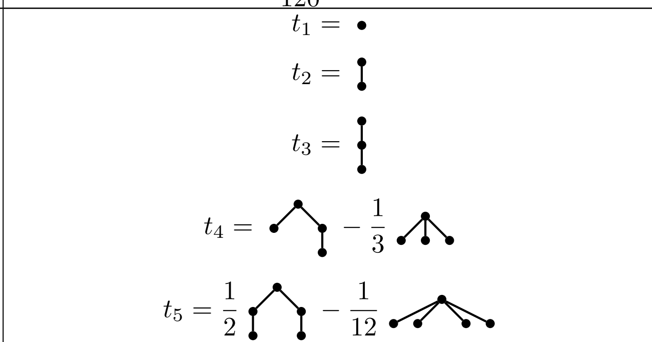
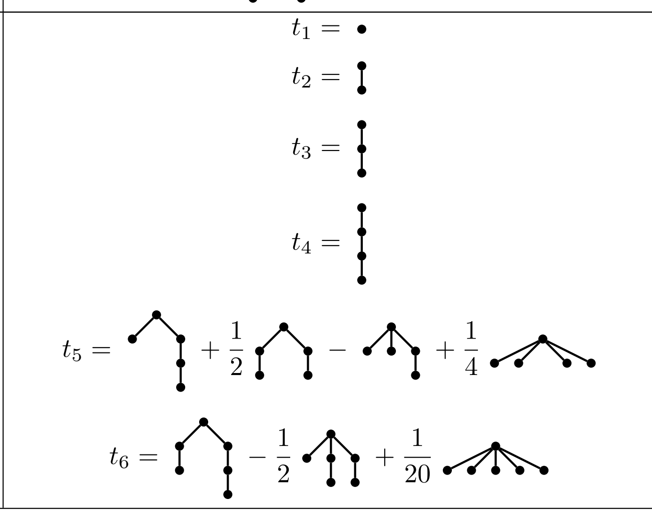
which as before implies that $a_{n+2} = 0$. By induction on i, we find that:

$$\begin{aligned}
\lambda_{i,n} &= \sum_{k=n}^{i+n-1} \binom{i-1}{k-n} a_k \sum_{I \subseteq \{j,j+1,\dots,k-1\}} (-1)^{|I|-i} a_I \\
\implies a_n &= a_n a_{n+i} + a_n - a_{n+i}
\end{aligned}$$

so $a_{n+i} = 0$ for all $i \geq 1$.

This completes the proof. □

Example 5.6. We conclude this subsection with some examples of the first few sequences of the form described in Theorem 5.5.

Λ-array	Element of Seq	Generated by:
1 1 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0		$B^+(\exp(P_1))$
1 1 1 1 1 0 1 1 0 0 1 1 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 0 1 1 0 0 0 0 0 0		$B^+(\exp(P_1 + P_2))$
1 1 1 1 1 1 1 1 1 0 1 1 1 0 0 1 1 1 0 0 0 1 1 1 0 0 0 0 1 1 1 0 0 0 0 0		$B^+(\exp(P_1 + P_2 + P_3))$

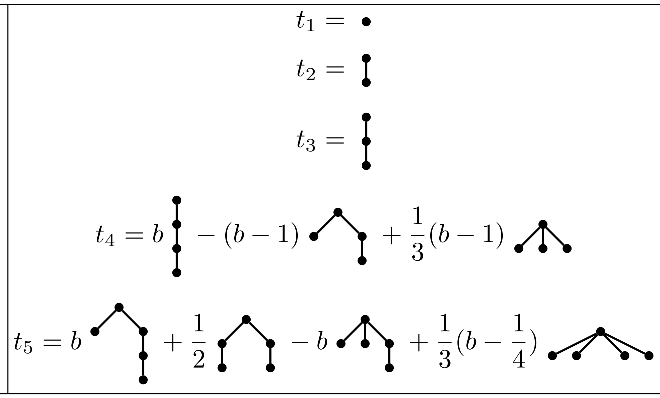
$\begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 1\quad b \\ 1\quad 1\quad b\quad 0 \\ 1\quad 1\quad b\quad 0\quad 0 \\ 1\quad 1\quad b\quad 0\quad 0\quad 0 \\ 1\quad 1\quad b\quad 0\quad 0\quad 0\quad 0 \\ 1\quad 1\quad b\quad 0\quad 0\quad 0\quad 0\quad 0 \end{array}$		$B^+(\exp(P_1 + P_2 + bP_3))$

Table 3: A few sequences X from Theorem 5.5.

5.2 kth Order Strong

In light of the classification of strong 0th order elements of *Seq* given in the last section, a natural question arises: Namely, is there a similar classification for strong kth order sequences for arbitrary k? This question is the central theme of the present section. Then in Section 5.3 we discuss the alternative next question, being: Given the results of Section 5.1, is there a similar classification for all *weak* 0th order sequences?

To begin, we start with first-order sequences and a few lemmas pertaining to them:

Lemma 5.7. *Let a and b be arbitrary elements of $\mathbb{K}$ such that $a \neq -nb$ for any positive integer $n \in \mathbb{N}$, and let $d \in \mathbb{K}$ be any element. Consider the first-order element of Seq generated by $\lambda_{i,1} = ai + b$ for $i \geq 1$ and $\lambda_{1,j} = d$ for $j \geq 2$. Then d can take on exactly four values: $d \in \{0, -a, b, a + b\}$*

Hence the lemma claims that there are only four families of Hopf subalgebra for any first-order sequence with right-diagonal of its Λ-array constant from the second index on. The general Λ-array described in the lemma has the following form:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & a+b & & & & & & & \\
 & & & & & & 2a+b & & d & & & & & & \\
 & & & & & 3a+b & & \bullet & & d & & & & & \\
 & & & & 4a+b & & \bullet & & \bullet & & d & & & & \\
 & & & 5a+b & & \bullet & & \bullet & & \bullet & & d & & & \\
 & & 6a+b & & \bullet & & \bullet & & \bullet & & \bullet & & d & & \\
 & 7a+b & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & d & \\
8a+b & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & d
\end{array}$$

Figure 20: A Λ-array.

where the • symbol is simply standing in to reduce the visual clutter of the array (recall that—under the assumption that this sequence is prelie—these elements are determined by Lemma 5.3). Moreover, the four families of sequences have Λ-arrays corresponding to one of the four families of Λ-arrays depicted in Figure 21, which we denote by $A(a, b), B(a, b), C(a, b)$, and $D(a, b)$.

$$
\begin{array}{llllllll}
a+b \\
2a+b & 0 \\
3a+b & 0 & 0 \\
4a+b & 0 & 0 & 0 \\
5a+b & 0 & 0 & 0 & 0 \\
6a+b & 0 & 0 & 0 & 0 & 0 \\
7a+b & 0 & 0 & 0 & 0 & 0 & 0 \\
8a+b & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

(a) $A(a,b)$—Case for $d=0$.

$$
\begin{array}{llllllll}
a+b \\
2a+b & a+b \\
3a+b & 2a+b & a+b \\
4a+b & 3a+b & 2a+b & a+b \\
5a+b & 4a+b & 3a+b & 2a+b & a+b \\
6a+b & 5a+b & 4a+b & 3a+b & 2a+b & a+b \\
7a+b & 6a+b & 5a+b & 4a+b & 3a+b & 2a+b & a+b \\
8a+b & 7a+b & 6a+b & 5a+b & 4a+b & 3a+b & 2a+b & a+b
\end{array}
$$

(b) $B(a,b)$—Case for $d=a+b$.

$$
\begin{array}{llllllll}
a+b \\
2a+b & b \\
3a+b & b & b \\
4a+b & b & b & b \\
5a+b & b & b & b & b \\
6a+b & b & b & b & b & b \\
7a+b & b & b & b & b & b & b \\
8a+b & b & b & b & b & b & b & b
\end{array}
$$

(c) $C(a,b)$—Case for $d=b$.

$$
\begin{array}{llllllll}
a+b \\
2a+b & -a \\
3a+b & 0 & -a \\
4a+b & 0 & 0 & -a \\
5a+b & 0 & 0 & 0 & -a \\
6a+b & 0 & 0 & 0 & 0 & -a \\
7a+b & 0 & 0 & 0 & 0 & 0 & -a \\
8a+b & 0 & 0 & 0 & 0 & 0 & 0 & -a
\end{array}
$$

(d) $D(a,b)$—Case for $d=-a$.

Figure 21: The Λ-arrays of four families of sequences

.

Remark. 1. The cases $A(a,b)$ and $B(a,b)$ constitute precisely the family of sequences coming from Dyson-Schwinger type equations and characterized by Foissy [20, 24]. On the other hand, we believe that this is the first instance of $C(a,b)$ and $D(a,b)$ appearing in the literature. See Figure 22 and Figure 23 to see what the elements look like on the level of trees.

2. We remark that various choices of a and b degenerate either to 0th order sequences or to other familiar first order sequences. For example, $D(0,b)$ is the sequence of 0th order corollas (where term i is scaled by b^i), and $C(0,b)$ is the 0th order sequence of ladders, similarly scaled. Moreover, $C(a,0) = A(a,0)$.

We now provide a proof of the result.

Proof of Lemma 5.7. This proof will be broken up into two sections. First, we show that $C(a,b)$ and $D(a,b)$ are elements of Λ for any a and b with $a \neq -nb$ (We already know $A(a,b)$ and $B(a,b)$ are in Λ. See [20] and Appendix C), then we will show that $A(a,b), B(a,b), C(a,b)$, and $D(a,b)$ are the only valid prelie arrays under our hypotheses.

First, consider $C(a,b)$. Given the array depicted in Figure 21, there are only four cases to consider:
Case 1 ($j \neq 1, k \neq 1$): This case is trivial, since equation (77) becomes:

$$(b)(b) - (b)(b) = (b)(b) - (b)(b)$$

Case 2 ($j = 1, k \neq 1$): We start with the left hand side of equation (77) and manipulate it to find:

$$\begin{aligned}
\lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} &= (ai+b)(b) - (b)(b) \\
&= (ai+b)(b) - (b)(b) + akb - akb \\
&= (ai+ak+b)(b) - (ak+b)(b) \\
&= (a[i+k]+b)(b) - (ak+b)(b) \\
&= \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,k+j}
\end{aligned}$$

Case 3 ($j \neq 1, k = 1$): As discussed in Section 4.1, the prelie relation is symmetric in j and k, so this case is the same as Case 2.
Case 4 ($j = 1, k = 1$): By symmetry of j and k again, this will also be true (though in a different way as the previous case). We can verify the computation this first time as follows:

$$\begin{aligned}
\lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} &= (ai+b)(a[i+j]+b) - (aj+b)(b) \\
&= (ai+b)(a[i+1]+b) - (a+b)(b) \\
&= (ai+b)(a[i+k]+b) - (ak+b)(b) \\
&= \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,k+j}
\end{aligned}$$

where the middle two equalities follow from the fact that $j = k = 1$.

As these four cases cover all possible values of $\lambda_{i,j}$, it follows that $C(a,b)$ is a valid prelie array. Moreover, it satisfies the non-degeneracy condition of Theorem 4.1 only when $a \neq -b$, which one can verify easily from Figure 20.

Now consider $D(a,b)$. Proving that this is a valid prelie array requires the inspection of eight cases total.

Case 1 ($i \neq 1, j \neq 1, k \neq 1$): As before, this case is trivial, since $\lambda_{m,n}$ with m and n not equal to 1 will be never be along the outer two diagonals of the array, which are the only entries that are nonzero. Hence equation (77) in this case looks like:

$$(0)(0) - (0)(0) = (0)(0) - (0)(0)$$

the ultimate triviality.
Case 2 ($i = 1, j \neq 1, k \neq 1$): In this situation, equation (77) becomes:

$$\begin{aligned} & -a(0) - (0)(-a) = -a(0) - (0)(-a) \\ \iff \ & 0 = 0 \end{aligned}$$

Case 3 ($i = 1, j = 1, k \neq 1$): Starting with the righthand side, we have:

$$\begin{aligned} \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,k+j} &= (-a)(a[i+k]+b) - (ak+b)(-a) \\ &= -a^2[i+k] - ab + a^2k + ab \\ &= -a^2i - a^2k + a^2k \\ &= -a^2i \\ &= -a^2 \\ &= -a^2 + (a+b)(0) \\ &= \lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} \end{aligned}$$

Case 4 ($i = 1, j \neq 1, k = 1$): This is the same as Case 3.
Case 5 ($i = 1, j = 1, k = 1$): This is true by the symmetry of j and k, as discussed above.
Case 6 ($i \neq 1, j = 1, k \neq 1$): In this case, the prelie relation becomes:

$$\begin{aligned} (ai+b)(0) - (-a)(0) &= (0)(a[i+k]+b) - (ak+b)(0) \\ \iff \ 0 &= 0 \end{aligned}$$

Case 7 ($i \neq 1, j \neq 1, k = 1$): This is the same as Case 6.
Case 8 ($i \neq 1, j = 1, k = 1$): This is true by the symmetry of j and k, as before.

Hence we conclude that $C(a, b)$ and $D(a, b)$ are valid elements of Λ for all $a, b \in \mathbb{K}$ such that $a \neq -nb$.

All that remains is to show that no other value of d will suffice for all a and b with $a \neq -nb$. For a given Λ-array ξ, define:

$$PL_\xi(i,j,k) = \lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} - \lambda_{i,k}\lambda_{i+k,j} + \lambda_{k,j}\lambda_{i,k+j} \tag{93}$$

Note that this is just the prelie relation of equation (77) after all terms have been moved to one side. Since we are assuming ξ must be prelie, the inner terms of the array are determined by the outer diagonals according to Lemma 5.1. We use the computer algebra system SageMath to compute the first few values of $PL(i,j,k)$ when given the array in Figure 20:

$$\begin{aligned}
PL(1,2,1) &= 0\\
PL(1,3,1) &= 0\\
PL(1,3,2) &= \frac{(a+b-d)(a+d)(b-d)d^2}{(3\,a+b)(2\,a+b)(a+b)}\\
PL(2,2,1) &= 0\\
PL(2,3,1) &= 0\\
PL(2,3,2) &= \frac{(6\,a+b+d)(a+b-d)(a+d)(b-d)d^2}{(4\,a+b)(3\,a+b)(2\,a+b)(a+b)}\\
&\vdots
\end{aligned}$$

In particular, the denominators are nonzero by our condition that $a \neq -nb$ and any value of d other than the ones specified will leave the third value nonzero for some choices of a and b. This completes the proof. □

One may be wondering what the sequences resulting from $A(a,b), B(a,b), C(a,b,)$, and $D(a,b)$ look like in terms of trees. As mentioned above, $A(a,b)$ is precisely the sequence of first-order corollas, and $B(a,b)$ is the sequence coming from Foissy's Dyson-Schwinger type equations, depicted in the top row of Table 5 in Appendix C. We can compute what $C(a,b)$ and $D(a,b)$ look like as well:

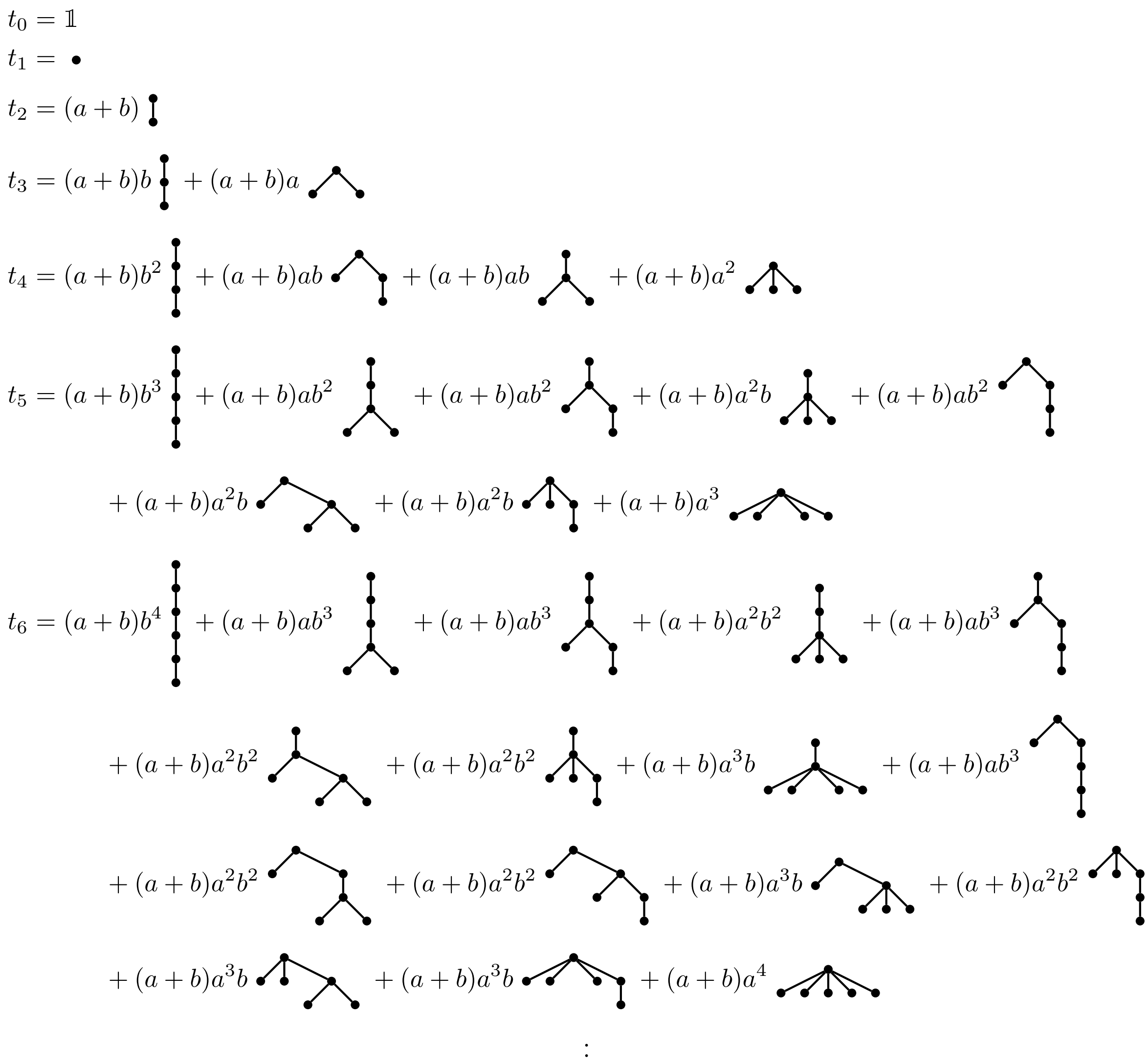


Figure 22: The element of *Seq* corresponding to $C(a,b)$.

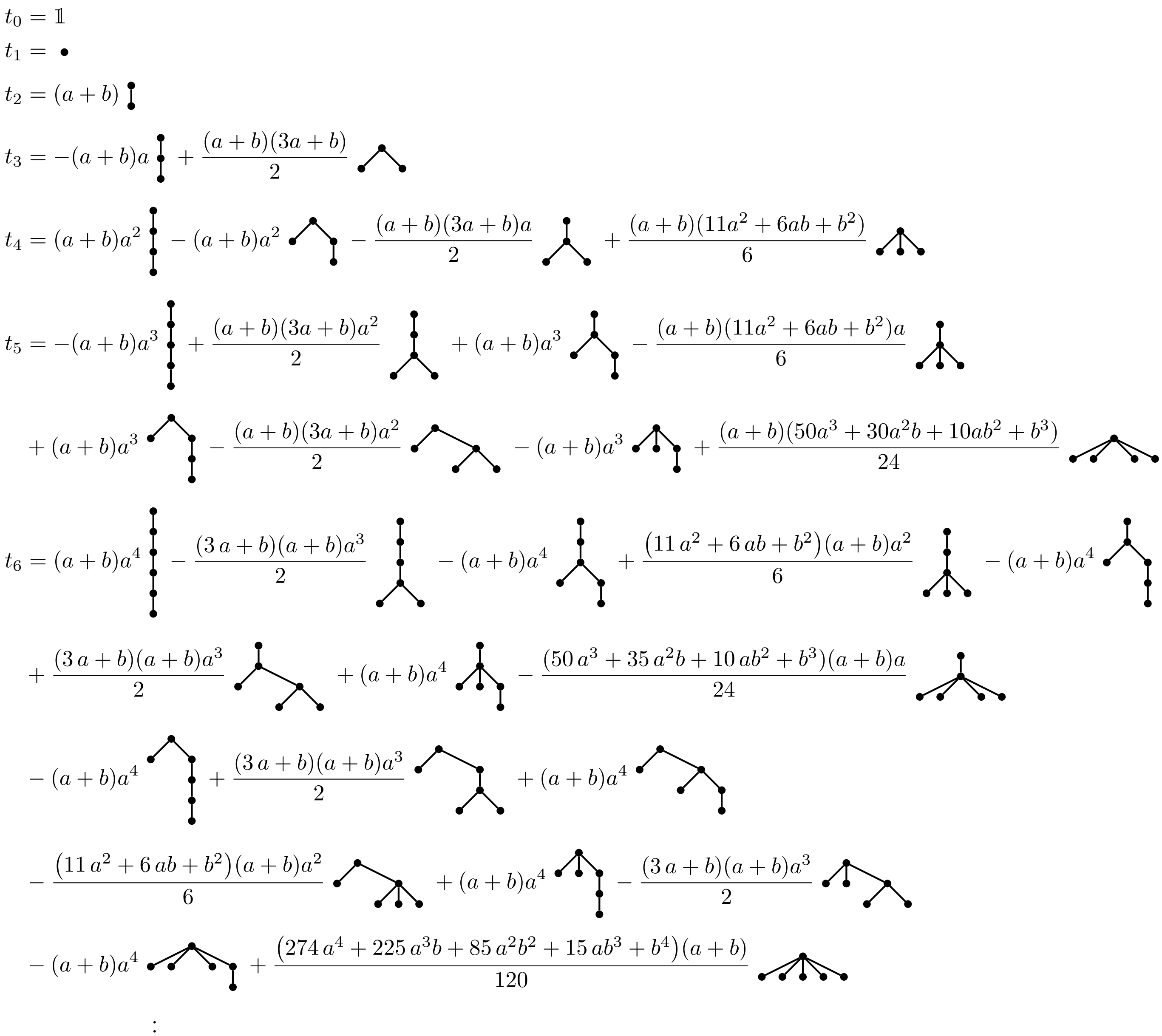


Figure 23: The element of *Seq* corresponding to $D(a, b)$.

Based on the data of Figure 22 and Figure 23, some ideas arise as to how we may describe their structure combinatorially. We begin with a definition:

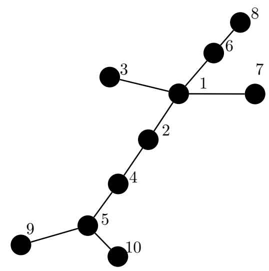


Figure 24: A caterpillar tree

Definition 5.8. A **caterpillar rooted tree** is a rooted tree such that each vertex has at most one subtree of size ≥ 2.

The name caterpillar rooted tree is in analogy with the caterpillar trees from graph theory. We emphasize that in this case the order of words in our terminology matters a lot: these trees are rooted trees first, and then are also caterpillar trees, but not every rooted caterpillar tree is a caterpillar rooted tree. For example, the graph in Figure 24 is a labelled caterpillar tree, but if we root it at vertex 2, the result will not be a caterpillar rooted tree. This distinction ultimately stems from the fact that a path graph is only a ladder when it is rooted at one of its two endpoint vertices. Hence another way to define the set of caterpillar rooted trees is to say that they are exactly the set of rooted caterpillar trees when the root is chosen to be a vertex v such that v only has at most one non-leaf neighbor.

With this distinction in mind, we present:

Lemma 5.9. *Let $X = \sum_{i=0}^{\infty} t_i$ be the sum of all terms in the element of Seq corresponding to $C(a,b)$. Then if T is a tree appearing as a term in X, T is a caterpillar rooted tree. Moreover, the coefficient of T is $(a+b)a^{m-1}b^{n-1}$ where m is the depth of T and n is the number of leaves of T.*

Proof. First, we will show that every tree in X is a caterpillar rooted tree. To do this, consider $\lambda(m,n)$ for $m,n \geq 2$. According to equation (81), this value is equal to:

$$\begin{aligned}\lambda(m,n) &= \lambda(m)\lambda_{1+m,n} - \lambda(m+n)\lambda_{m,n} \\ &= \lambda_{1,m}\lambda_{1+m,n} - \lambda_{1,m+n}\lambda_{m,n} \\ &= (b)(b) - (b)(b) \\ &= 0\end{aligned}$$

where the second equality comes from the second case of equation (81) and the second equality comes from Figure 21 and the fact that m and n are both at least 2.

Now the coefficient of a tree in X is given by equation (83):

$$\text{coef}(T) = \frac{\mu(T)}{|Sym(T)|} = \frac{1}{|Sym(T)|} \prod_{s \in V(T)} \lambda(t_1^{(s)}, ..., t_{k(s)}^{(s)})$$

Now let $l_1, ..., l_k \geq 2$ be arbitrary and consider $\lambda(m, n, l_1, ..., l_k)$. For $k = 1$ we have:

$$\begin{aligned}
\lambda_{m,n,l_1} &= \lambda(m,n)\lambda_{1+m+n,l_1} - \lambda_{m+l_1,n}\lambda_{m,l_1} - \lambda(m, n+l_1)\lambda_{n,l_1} \\
&= (0)\lambda_{1+m+n,l_1} - (0)\lambda_{m,l_1} - (0)\lambda_{n,l_1} \\
&= 0
\end{aligned}$$

where now the second equality comes from the fact that $m, n \geq 2$.

We proceed by induction on k: suppose that $\lambda(m, n, l_1, ..., l_{k-1}) = 0$, and consider $\lambda(m, n, l_1, ..., l_{k-1}, l_k)$. We get that:

$$\lambda(m, n, l_1, ..., l_{k-1}, l_k) = \lambda(m, n, l_1, ..., l_{k-1})\lambda_{1+m+n+l_1+...+l_{k-1},l_k} - \sum_{j=1}^{k+1} \lambda(n, m, ..., l_j + l_k, ...l_{k-1})\lambda_{l_j,l_k}$$

By the inductive hypothesis, $\lambda(m, n, l_1, ..., l_{k-1})$ is 0 and each factor $\lambda(n, m, ..., l_j + l_k, ...l_{k-1})$ in the sum above is also 0. Hence the entire sum is 0. This computation, together with equation (81), implies that every tree in X is a caterpillar rooted tree, since if T has a vertex with more than one subtree of size at least 2, it will have a factor of 0 in equation (81).

Now we show that every tree T in X has the coefficient claimed. The proof of this will follow from three cases:

Case 1: $T = B^+(T')$ for some tree T'. In this case, we have that:

$$\begin{aligned}
\text{coef}(T) &= \frac{1}{|Sym(T)|}\lambda(|T'|) \prod_{s \in V(T')} \lambda(t_1^{(s)}, ..., t_{k(s)}^{(s)}) \\
&= \lambda(|T'|)\text{coef}(T') \\
&= \lambda_{1,|T'|}\text{coef}(T') \\
&= b \cdot \text{coef}(T')
\end{aligned}$$

where the second equality comes from the fact that $Sym(B^+(T)) = Sym(T)$.

Case 2: Now consider the situation in which $T = B^+(\bullet^k T')$, for some tree T' of size $n \geq 2$. Then:

$$\begin{aligned}
\text{coef}(T) &= \frac{1}{|Sym(T)|}\lambda(\underbrace{1, 1, ..., 1}_{k}, n) \prod_{s \in V(T')} \lambda(t_1^{(s)}, ..., t_{k(s)}^{(s)}) \\
&= \frac{1}{k!}\lambda(\underbrace{1, 1, ..., 1}_{k}, n)\text{coef}(T')
\end{aligned}$$

Hence we only need to find the value of $\lambda(\underbrace{1, 1, ..., 1}_{k}, n)$, which we claim is equal to $k!\, a^k b$. We will prove this claim by induction on k. The base case is trivial, since when $k = 0$ we recover Case 1. So suppose that $\lambda(\underbrace{1, 1, ..., 1}_{k-1}, n) = k!\, a^{k-1} b$ and consider the value of $\lambda(\underbrace{1, 1, ..., 1}_{k}, n)$. Then:

$$\begin{aligned}
\lambda(\underbrace{1, 1, ..., 1}_{k}, n) &= \lambda(n, \underbrace{1, 1, ..., 1}_{k}) \\
&= \lambda(n, \underbrace{1, 1, ..., 1}_{k-1})\lambda_{n+k,1} - \lambda(n+1, \underbrace{1, 1, ..., 1}_{k-1})\lambda_{n,1} - \sum_{j=2}^{k-1} \lambda(n, 1..., 1, 2, 1, ...1)\lambda_{1,1}
\end{aligned}$$

where the 2 in the sum above is in the jth index of the tuple. By symmetry of the arguments of $\lambda(i_1, i_2, ..., i_m)$, the above line is equal to:

$$= \lambda(n, \underbrace{1, 1, ..., 1}_{k-1})\lambda_{n+k,1} - \lambda(n+1, \underbrace{1, 1, ..., 1}_{k-1})\lambda_{n,1} - (k-1)\lambda(n, 2, \underbrace{1, ...1}_{k-1})\lambda_{1,1}$$

But by the analysis in the first part of this proof, this last term is just 0, since there are two indices of size at least 2 in the argument of the λ. We can then apply the inductive hypothesis in each term that remains:

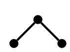
$$\begin{aligned}
&= (k-1)!\, a^{k-1}b\lambda_{n+k,1} - (k-1)!\, a^{k-1}b\lambda_{n,1} \\
&= (k-1)!\, a^{k-1}b[(n+k)a + b] - (k-1)!\, a^{k-1}b[an + b] \\
&= (k-1)!\, a^{k-1}b[\cancel{an} + ak + \cancel{b} - \cancel{an} - \cancel{b}] \\
&= k!\, a^k b
\end{aligned}$$

as claimed.

Finally, then, the coefficient of the tree T above is just:

$$\begin{aligned}
T &= \frac{1}{k!}\lambda(\underbrace{1, 1, ..., 1}_{k}, n)\mathrm{coef}(T') \\
&= \frac{1}{k!}k!\, a^k b\mathrm{coef}(T') \\
&= a^k b\mathrm{coef}(T')
\end{aligned}$$

Case 3: Finally, we consider the case in which $T = B^+(\bullet^{k+1})$ for some positive integer k. We claim that $\mathrm{coef}(T) = (a+b)a^k$. We handle this case with induction. For the base case we have that coef( ) $= a + b$ and also that:

coef() $= \frac{1}{2}\lambda(1,1)$

$$\begin{aligned}
&= \frac{1}{2}[\lambda(1)\lambda_{2,1} - \lambda(2)\lambda_{1,1}] \\
&= \frac{1}{2}[\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{1,1}] \\
&= \frac{1}{2}[(a+b)(2a+b) - (b)(a+b)] \\
&= \frac{1}{2}(a+b)(2a) \\
&= (a+b)(a)
\end{aligned}$$

as claimed. Now we can induct on k. We assume the hypothesis is true up to k and consider the case of

$T = B^+(\bullet^{k+1})$

$$\begin{aligned}
\text{coef}(T) &= \frac{1}{|Sym(T)|}\lambda(\underbrace{1,1,...,1}_{k+1}) \\
&= \frac{1}{(k+1)!}\lambda(\underbrace{1,1,...,1}_{k+1}) \\
&= \frac{1}{(k+1)!}[\lambda(\underbrace{1,1,...,1}_{k})\lambda_{k+1,1} - \sum_{j=1}^{k}\lambda(1,...,1+1,...,1)\lambda_{1,1}]
\end{aligned}$$

But by symmetry of the arguments of $\lambda(i_1, ..., i_k)$, the previous line is equivalent to:

$$= \frac{1}{(k+1)!}[\lambda(\underbrace{1,1,...,1}_{k})\lambda_{k+1,1} - k\lambda(\underbrace{1,1,...,1}_{k-1},2)\lambda_{1,1})]$$

Now by Case 2 we have that $\lambda(\underbrace{1,1,...,1}_{k-1},2) = (k-1)!\,a^{k-1}b$ and by the inductive hypothesis, we have that $\text{coef}(B^+(\bullet^k)) = (a+b)a^{k-1}$, which implies that $\lambda(\underbrace{1,1,...,1}_{k}) = k!\,(a+b)a^{k-1}$. Making these substitutions, we find:

$$\begin{aligned}
&= \frac{1}{(k+1)!}[k!\,(a+b)a^{k-1}\lambda_{k+1,1} - k(k-1)!\,a^{k-1}b\lambda_{1,1})] \\
&= \frac{1}{(k+1)!}[k!\,(a+b)a^{k-1}[a(k+1)+b] - k(k-1)!\,a^{k-1}b(a+b))] \\
&= \frac{1}{(k+1)!}[k!\,(a+b)a^{k-1}a(k+1) + \cancel{k!\,(a+b)a^{k-1}b} - \cancel{k!\,a^{k-1}b(a+b)})] \\
&= (a+b)a^k
\end{aligned}$$

exactly as desired.

The three cases show that the coefficient of t_2 is $(a+b)$ and that every larger tree is constructed from this by multiplying by b for every application of B^+ and by multiplying by a^k whenever adding in k more leaves. Hence the coefficients of the trees is exactly as claimed. This completes the proof. □

As discussed in Section 2.1.2, every tree can be constructed uniquely as applications of B^+ on the empty forest. Hence the procedure for finding the coefficients in the preceding proof gives us for free a generating function for the sequence of trees corresponding to $C(a, b)$. This is the content in the following:

Corollary 5.10. Let $X = \sum_{i=0}^{\infty} t_i$ be the sum of all terms in the element of *Seq* corresponding to $C(a, b)$. Then:

$$X = B^+\left(\frac{1+bX}{1-a\bullet}\right) \tag{94}$$

where $\bullet$ is the single-vertex tree.

5.2.1 Second Order and Higher

Given the large number of examples of strong first-order sequences given in the last subsection, the results of this section may come as a surprise. The main result of this section is the following:

Theorem 5.11 (Classification of Strong ℓth Order Sequences)**.** *For $\ell \geq 2$, the only family of strong ℓth order sequences is the family of scaled corollas.*

Corollary 5.12. The only family of strong second-order sequence is the sequences of scaled corollas, with prelie array:

$$\begin{array}{ccccccccc} & & & & a+b+c & & & & \\ & & & 4a+2b+c & & 0 & & & \\ & & 9a+3b+c & & 0 & & 0 & & \\ & 16a+4b+c & & 0 & & 0 & & 0 & \\ 25a+5b+c & & 0 & & 0 & & 0 & & 0 \\ & & & & \vdots & & & & \end{array}$$

for $a, b, c \in \mathbb{K}$ and a nonzero.

While Corollary 5.12 is an immediate consequence of the theorem and would not normally need separate consideration, we include it here as we believe it (together with its proof) is a concrete exhibition of the proof method developed in the proof of Theorem 5.11, which may otherwise seem slightly abstract.

We begin with the proof of Corollary 5.12, which we present via a sequence of lemmas.

Lemma 5.13. *Let Λ_X be the Λ-array of a second-order strong sequence, and let $\widetilde{\Lambda_X}$ be the Λ-array obtained from Λ_X by removing the left diagonal (that is, $\widetilde{\Lambda_X} = \Lambda_X \setminus \{\lambda_{k,1} : k \geq 1\}$). Then $\widetilde{\Lambda_X}$ must be a strong first-order sequence.*

Proof. To begin, consider an arbitrary strong second-order sequence with prelie array given by $\lambda_{i,j} = f_j(i)$:

$$\begin{array}{ccccccccc} & & & & f_1(1) & & & & \\ & & & f_1(2) & & f_2(1) & & & \\ & & f_1(3) & & f_2(2) & & f_3(1) & & \\ & f_1(4) & & f_2(3) & & f_3(2) & & f_4(1) & \\ f_1(5) & & f_2(4) & & f_3(3) & & f_4(2) & & f_5(1) \\ & & & & \vdots & & & & \end{array}$$

where we define:

$$f_j(i) = a_{j,1}i^2 + a_{j,2}i + a_{j,3}$$

With this setup, the statement of the lemma can be reworded as: $a_{1,1} \neq 0 \implies a_{2,1} = a_{3,1} = ... = a_{k,1} = ... = 0$.

Let $k \geq 2$ be fixed and arbitrary, and consider the four prelie relations $PL(1,1,k) = 0, PL(2,1,k) = 0, PL(3,1,k) = 0$, and $PL(4,1,k) = 0$ as follows:

$$\begin{aligned} (a_{1,1}k^2 + a_{1,2}k + a_{1,3})(a_{k+1,1} + a_{k+1,2} + a_{k+1,3}) + (a_{1,1} + a_{1,2} + a_{1,3})(4\,a_{k,1} + 2\,a_{k,2} + a_{k,3}) \\ - \Big(a_{1,1}(k+1)^2 + a_{1,2}(k+1) + a_{1,3}\Big)(a_{k,1} + a_{k,2} + a_{k,3}) \\ - (a_{k+1,1} + a_{k+1,2} + a_{k+1,3})(a_{k,1} + a_{k,2} + a_{k,3}) = 0 \quad (95) \end{aligned}$$

$$\left(a_{1,1}k^2 + a_{1,2}k + a_{1,3}\right)(4\,a_{k+1,1} + 2\,a_{k+1,2} + a_{k+1,3}) + (4\,a_{1,1} + 2\,a_{1,2} + a_{1,3})(9\,a_{k,1} + 3\,a_{k,2} + a_{k,3}) - \left(a_{1,1}(k+2)^2 + a_{1,2}(k+2) + a_{1,3}\right)(4\,a_{k,1} + 2\,a_{k,2} + a_{k,3}) - (4\,a_{k+1,1} + 2\,a_{k+1,2} + a_{k+1,3})(a_{k,1} + a_{k,2} + a_{k,3}) = 0 \quad (96)$$

$$\left(a_{1,1}k^2 + a_{1,2}k + a_{1,3}\right)(9\,a_{k+1,1} + 3\,a_{k+1,2} + a_{k+1,3}) + (9\,a_{1,1} + 3\,a_{1,2} + a_{1,3})(16\,a_{k,1} + 4\,a_{k,2} + a_{k,3}) - \left(a_{1,1}(k+3)^2 + a_{1,2}(k+3) + a_{1,3}\right)(9\,a_{k,1} + 3\,a_{k,2} + a_{k,3}) - (9\,a_{k+1,1} + 3\,a_{k+1,2} + a_{k+1,3})(a_{k,1} + a_{k,2} + a_{k,3}) = 0 \quad (97)$$

$$\left(a_{1,1}k^2 + a_{1,2}k + a_{1,3}\right)(16\,a_{k+1,1} + 4\,a_{k+1,2} + a_{k+1,3}) + (16\,a_{1,1} + 4\,a_{1,2} + a_{1,3})(25\,a_{k,1} + 5\,a_{k,2} + a_{k,3}) - \left(a_{1,1}(k+4)^2 + a_{1,2}(k+4) + a_{1,3}\right)(16\,a_{k,1} + 4\,a_{k,2} + a_{k,3}) - (16\,a_{k+1,1} + 4\,a_{k+1,2} + a_{k+1,3})(a_{k,1} + a_{k,2} + a_{k,3}) = 0 \quad (98)$$

The goal now is to isolate the highest degree terms (with respect to index i when thought of as a variable). We will do this by leveraging the fact that—as these equations are degree-2 polynomials in terms of an index i, we can use successive differences of the equations to eventually isolate the highest degree term (with respect to the index i). Taking equation (96) minus equation (95) and expanding, we get:

$$3\,a_{1,1}k^2a_{k+1,1}+a_{1,1}k^2a_{k+1,2}-3\,a_{1,1}k^2a_{k,1}-a_{1,1}k^2a_{k,2}+3\,a_{1,2}ka_{k+1,1}+a_{1,2}ka_{k+1,2}-14\,a_{1,1}ka_{k,1}-3\,a_{1,2}ka_{k,1} -6\,a_{1,1}ka_{k,2}-a_{1,2}ka_{k,2}-2\,a_{1,1}ka_{k,3}+3\,a_{1,3}a_{k+1,1}+a_{1,3}a_{k+1,2}+17\,a_{1,1}a_{k,1}+7\,a_{1,2}a_{k,1}+2\,a_{1,3}a_{k,1}-3\,a_{k+1,1}a_{k,1} - a_{k+1,2}a_{k,1} + 3\,a_{1,1}a_{k,2} + a_{1,2}a_{k,2} - 3\,a_{k+1,1}a_{k,2} - a_{k+1,2}a_{k,2} - 3\,a_{k+1,1}a_{k,3} - a_{k+1,2}a_{k,3} = 0 \quad (99)$$

Taking equation (97) minus equation (96):

$$5\,a_{1,1}k^2a_{k+1,1}+a_{1,1}k^2a_{k+1,2}-5\,a_{1,1}k^2a_{k,1}-a_{1,1}k^2a_{k,2}+5\,a_{1,2}ka_{k+1,1}+a_{1,2}ka_{k+1,2}-38\,a_{1,1}ka_{k,1}-5\,a_{1,2}ka_{k,1} -10\,a_{1,1}ka_{k,2}-a_{1,2}ka_{k,2}-2\,a_{1,1}ka_{k,3}+5\,a_{1,3}a_{k+1,1}+a_{1,3}a_{k+1,2}+43\,a_{1,1}a_{k,1}+11\,a_{1,2}a_{k,1}+2\,a_{1,3}a_{k,1}-5\,a_{k+1,1}a_{k,1} - a_{k+1,2}a_{k,1} + 5\,a_{1,1}a_{k,2} + a_{1,2}a_{k,2} - 5\,a_{k+1,1}a_{k,2} - a_{k+1,2}a_{k,2} - 5\,a_{k+1,1}a_{k,3} - a_{k+1,2}a_{k,3} = 0 \quad (100)$$

Taking equation (98) minus equation (97):

$$7\,a_{1,1}k^2a_{k+1,1}+a_{1,1}k^2a_{k+1,2}-7\,a_{1,1}k^2a_{k,1}-a_{1,1}k^2a_{k,2}+7\,a_{1,2}ka_{k+1,1}+a_{1,2}ka_{k+1,2}-74\,a_{1,1}ka_{k,1}-7\,a_{1,2}ka_{k,1} -14\,a_{1,1}ka_{k,2}-a_{1,2}ka_{k,2}-2\,a_{1,1}ka_{k,3}+7\,a_{1,3}a_{k+1,1}+a_{1,3}a_{k+1,2}+81\,a_{1,1}a_{k,1}+15\,a_{1,2}a_{k,1}+2\,a_{1,3}a_{k,1}-7\,a_{k+1,1}a_{k,1} - a_{k+1,2}a_{k,1} + 7\,a_{1,1}a_{k,2} + a_{1,2}a_{k,2} - 7\,a_{k+1,1}a_{k,2} - a_{k+1,2}a_{k,2} - 7\,a_{k+1,1}a_{k,3} - a_{k+1,2}a_{k,3} = 0 \quad (101)$$

Now taking equation (100) minus equation (99):

$$2\,a_{1,1}k^2a_{k+1,1}-2\,a_{1,1}k^2a_{k,1}+2\,a_{1,2}ka_{k+1,1}-24\,a_{1,1}ka_{k,1}-2\,a_{1,2}ka_{k,1}-4\,a_{1,1}ka_{k,2}+2\,a_{1,3}a_{k+1,1}+26\,a_{1,1}a_{k,1} + 4\,a_{1,2}a_{k,1} - 2\,a_{k+1,1}a_{k,1} + 2\,a_{1,1}a_{k,2} - 2\,a_{k+1,1}a_{k,2} - 2\,a_{k+1,1}a_{k,3} = 0 \quad (102)$$

and equation (101) minus equation (100):

$$2\,a_{1,1}k^2a_{k+1,1}-2\,a_{1,1}k^2a_{k,1}+2\,a_{1,2}ka_{k+1,1}-36\,a_{1,1}ka_{k,1}-2\,a_{1,2}ka_{k,1}-4\,a_{1,1}ka_{k,2}+2\,a_{1,3}a_{k+1,1}+38\,a_{1,1}a_{k,1} + 4\,a_{1,2}a_{k,1} - 2\,a_{k+1,1}a_{k,1} + 2\,a_{1,1}a_{k,2} - 2\,a_{k+1,1}a_{k,2} - 2\,a_{k+1,1}a_{k,3} = 0 \quad (103)$$

Finally, we take the difference of equation (103) and equation (102) we get:

$$12a_{1,1}a_{k,1}(1-k)=0 \tag{104}$$

Hence if $k=1$ the statement is true and $a_{1,1}$ is left as a free variable. We cannot have $a_{1,1}=0$, because then the sequence was actually a first-order sequence. Hence it must be that $a_{k,1}=0$. Since k was arbitrary, it follows that $a_{k,1}=0$ for all $k\geq 2$, proving the desired result. □

Lemma 5.14. *Let Λ_X be the Λ-array of a second-order strong sequence, and let $\widetilde{\Lambda_X}$ be the Λ-array obtained from Λ_X by removing the left diagonal (that is, $\widetilde{\Lambda_X}=\Lambda_X\setminus\{\lambda_{k,1}:k\geq 1\}$). Then $\widetilde{\Lambda_X}$ must consist of only constant leftward-diagonals (i.e. each sequence $(\lambda_{i,j})_{i\geq 1}$ with j fixed, must be constant).*

Proof. We can take the exact same approach as the proof of Lemma 5.13. Since Λ_X is a strong second-order array, it follows from the previous lemma that $f_j=a_{j,2}i+a_{j,3}$ for all $j\geq 2$. As before, we let $k\geq 2$ be arbitrary. This time is suffices to take only three equations $PL(1,1,k), PL(2,1,k)$ and $PL(3,1,k)$, which appear as follows:

$$\begin{aligned}&\big(a_{1,1}k^2+a_{1,2}k+a_{1,3}\big)(a_{k+1,2}+a_{k+1,3})+(a_{1,1}+a_{1,2}+a_{1,3})(2\,a_{k,2}+a_{k,3})\\&\quad-\Big(a_{1,1}(k+1)^2+a_{1,2}(k+1)+a_{1,3}\Big)(a_{k,2}+a_{k,3})-(a_{k+1,2}+a_{k+1,3})(a_{k,2}+a_{k,3})=0\end{aligned} \tag{105}$$

$$\begin{aligned}&\big(a_{1,1}k^2+a_{1,2}k+a_{1,3}\big)(2\,a_{k+1,2}+a_{k+1,3})+(4\,a_{1,1}+2\,a_{1,2}+a_{1,3})(3\,a_{k,2}+a_{k,3})\\&\quad-\Big(a_{1,1}(k+2)^2+a_{1,2}(k+2)+a_{1,3}\Big)(2\,a_{k,2}+a_{k,3})-(2\,a_{k+1,2}+a_{k+1,3})(a_{k,2}+a_{k,3})=0\end{aligned} \tag{106}$$

$$\begin{aligned}&\big(a_{1,1}k^2+a_{1,2}k+a_{1,3}\big)(3\,a_{k+1,2}+a_{k+1,3})+(9\,a_{1,1}+3\,a_{1,2}+a_{1,3})(4\,a_{k,2}+a_{k,3})\\&\quad-\Big(a_{1,1}(k+3)^2+a_{1,2}(k+3)+a_{1,3}\Big)(3\,a_{k,2}+a_{k,3})-(3\,a_{k+1,2}+a_{k+1,3})(a_{k,2}+a_{k,3})=0\end{aligned} \tag{107}$$

Subtracting equation (105) from equation (106) and expanding:

$$\begin{aligned}&a_{1,1}k^2a_{k+1,2}-a_{1,1}k^2a_{k,2}+a_{1,2}ka_{k+1,2}-6\,a_{1,1}ka_{k,2}-a_{1,2}ka_{k,2}-2\,a_{1,1}ka_{k,3}+a_{1,3}a_{k+1,2}+3\,a_{1,1}a_{k,2}\\&\quad+a_{1,2}a_{k,2}-a_{k+1,2}a_{k,2}-a_{k+1,2}a_{k,3}=0\end{aligned} \tag{108}$$

Similarly for equation (106) and equation (107):

$$\begin{aligned}&a_{1,1}k^2a_{k+1,2}-a_{1,1}k^2a_{k,2}+a_{1,2}ka_{k+1,2}-10\,a_{1,1}ka_{k,2}-a_{1,2}ka_{k,2}-2\,a_{1,1}ka_{k,3}+a_{1,3}a_{k+1,2}+5\,a_{1,1}a_{k,2}\\&\quad+a_{1,2}a_{k,2}-a_{k+1,2}a_{k,2}-a_{k+1,2}a_{k,3}=0\end{aligned} \tag{109}$$

Finally, we take equation (109) minus equation (108) and obtain:

$$-2\,a_{1,1}(2\,k-1)a_{k,2}=0 \tag{110}$$

As before, we get that $a_{1,1}\neq 0$ for Λ_X to be second order, and $(2\,k-1)\neq 0$ since k must be a positive integer. Hence it follows that $a_{k,2}=0$ for all $k\geq 2$ as claimed. □

Proof of Corollary 5.12: Given a strong second order prelie array Λ_X, Lemma 5.13 and Lemma 5.14 imply that $\lambda_{i,j} = f_j(i)$ where $f_1(i) = a_{1,1}i^2 + a_{1,3}$ and for $j \geq 2$, $f_j(i) = a_{j,3}$. Given $k \geq 2$ arbitrary, we examine equations $PL(1,1,k)$ and $PL(2,1,k)$:

$$\big(a_{1,1}k^2 + a_{1,2}k + a_{1,3}\big)a_{k+1,3} - \Big(a_{1,1}(k+1)^2 + a_{1,2}(k+1) + a_{1,3}\Big)a_{k,3} + (a_{1,1} + a_{1,2} + a_{1,3})a_{k,3} - a_{k+1,3}a_{k,3} = 0 \tag{111}$$

$$\big(a_{1,1}k^2 + a_{1,2}k + a_{1,3}\big)a_{k+1,3} - \Big(a_{1,1}(k+2)^2 + a_{1,2}(k+2) + a_{1,3}\Big)a_{k,3} + (4\,a_{1,1} + 2\,a_{1,2} + a_{1,3})a_{k,3} - a_{k+1,3}a_{k,3} = 0 \tag{112}$$

The difference of these two equations yields:

$$-2\,a_{1,1}ka_{k,3} = 0 \tag{113}$$

giving that $a_{k,3} = 0$ for all $k \geq 2$. Hence the only strong second order array is given by

$$\lambda_{i,j} = \begin{cases} a_{1,1}i^2 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

exactly as desired. □

Remark. We remark that the need to take $k \geq 2$ in the above proof is due to the fact that the prelie relations are symmetric in their last two arguments. Indeed, as we are working with prelie relations of the form $PL(i,1,k)$ for some arbitrary k and varying i, we need $k \geq 2$ to avoid the relations $PL(i,1,1)$, which are tautologies.

Proof of Theorem 5.11. We begin with the same kind of setup we had in the proof of Lemma 5.13; as mentioned previously, this proof is in fact the exact same proof, only generalized to account for more arbitrary order.

Let $\ell \geq 2$ be given, and consider the prelie array Λ_X given by $\lambda_{i,j} = f_j(i)$ where we define

$$f_j(i) = a_{j,1}i^\ell + a_{j,2}i^{\ell-1} + ... + a_{j,\ell}i + a_{j,\ell+1}$$

As before, consider an arbitrary (and fixed) $k \geq 2$, and form the sequence of relations $\mathcal{C} = \{PL(i,1,k) : i \in \mathbb{N}\}$. Now we can ask ourselves what an arbitrary member of this set looks like? We simply evaluate equation (77) at the appropriate indices and find that for arbitrary i:

$$PL(i,1,k): \qquad f_j(i)f_k(i+j) - f_k(j)f_{j+k}(i) - f_k(i)f_j(i+k) + f_j(k)f_{k+j}(i) = 0 \tag{114}$$

We remark that the collection $\mathcal{C}$ has infinitely many relations, but only $3(k+1)$ variables since k is fixed. Ultimately it is this over-saturation of equations that will enable us to prove the result.

Evaluating the f_j in (114) above with the indicated arguments yields:

$$\begin{aligned}(a_{1,1}i^\ell + a_{1,2}i^{\ell-1} + ... + a_{1,\ell}i + a_{1,\ell+1})(a_{k,1}(i+1)^\ell + a_{k,2}(i+1)^{\ell-1} + ... + a_{k,\ell}(i+1) + a_{k,\ell+1})\\ - (a_{k,1} + a_{k,2} + ... + a_{k,\ell} + a_{k,\ell+1})(a_{k+1,1}i^\ell + a_{k+1,2}i^{\ell-1} + ... + a_{k_o+1,\ell}i + a_{k+1,\ell+1})\\ - (a_{k,1}i^\ell + a_{k,2}i^{\ell-1} + ... + a_{k,\ell}i + a_{k,\ell+1})(a_{1,1}(k+i)^\ell + a_{1,2}(k+i)^{\ell-1} + ... + a_{1,\ell}(k+i) + a_{k,\ell+1})\\ + (a_{1,1}k^\ell + a_{1,2}k^{\ell-1} + ... + a_{1,\ell}k + a_{1,\ell+1})(a_{k+1,1}i^\ell + a_{k+1,2}i^{\ell-1} + ... + a_{k+1,\ell}i + a_{k+1,\ell+1}) = 0\end{aligned} \tag{115}$$

Now the expression on the left hand side of (115) is a sum of products of polynomials in the variable i, and as a consequence is also a polynomial in i. Let us examine what the highest-degree term of (115) looks like. The highest power of i appearing in (115) will be 2ℓ, with the first term contributing a term of $a_{1,1}a_{k,1}i^{2\ell}$, and the third term contributing a term of $-a_{1,1}a_{k,1}i^{2\ell}$, so taken together the coefficient of $i^{2\ell}$ is zero; the second and fourth terms will only contribute to the highest power of the polynomial when $2\ell = \ell$—that is, when $\ell = 0$ and Λ_X is 0th-order.

Hence we look instead at the next highest power of i, which is $2\ell - 1$. From the first term of (115), we obtain a factor of $(a_{1,1}a_{k,1}\ell + a_{1,1}a_{k,2} + a_{1,2}a_{k,1})i^{2\ell-1}$. From the third term of (115) we obtain a factor $-(a_{1,1}a_{k,1}k\ell + a_{1,2}a_{k,1} + a_{1,1}a_{k,2})i^{2\ell-1}$. Note that the second and fourth term of (115) will contribute to the highest power of i only when $2\ell - 1 = \ell$—that is, when $\ell = 1$ and the sequence is consequently first order! Hence since $\ell \geq 2$ we have that the coefficient of the highest power of i is equal to:

$$(a_{1,1}a_{k,1}\ell + a_{1,1}a_{k,2} + a_{1,2}a_{k,1})i^{2\ell-1} - (a_{1,1}a_{k,1}k\ell + a_{1,2}a_{k,1} + a_{1,1}a_{k,2})i^{2\ell-1}$$

which simplifies to:

$$a_{1,1}a_{k,1}\ell(1-k) \tag{116}$$

Now the power of this setup comes down to the fact that i is an index ranging over the positive integers. This means that—by definition—the sequence of relations $\mathcal{C}$ is in fact a $(2\ell-1)$th-order sequence in the variable i. But in turn, this means that taking the $(2\ell-1)$st consecutive differences of the sequence will be constant, and hence that taking the (2ℓ)th consecutive differences of $\mathcal{C}$ will be equal to 0. Now taking the first difference will cause the i^0 (that is, constant) terms to cancel, the second difference will cause the i terms to cancel, and so on. Hence by taking the (2ℓ)th consecutive differences, every term up through the terms containing $i^{2\ell-1}$ will cancel. This means that we are left with the equation:

$$Na_{1,1}a_{k,1}\ell(1-k) = 0 \tag{117}$$

for N a nonzero element of $\mathbb{K}$[18]. If $a_{1,1} = 0$, it follows that the sequence Λ_X is not actually ℓth order. Moreover, we know that $\ell \geq 2$ by hypothesis, and $k \neq 1$ (as otherwise the prelie relations are a tautology). Hence the only solution is that $a_{k,1} = 0$ for all $k \geq 2$.

We now perform induction on the second index of the $a_{k,m}$, taking as our base case the analysis above wherein $m = 1$. Suppose that $a_{k,t} = 0$ for all t from 1 up to $m-1$, and consider the case of $t = m$. (We are making the assumption that $m \leq \ell + 1$, as otherwise $a_{k,m} = 0$ already). Evaluating the f_j in (114) with the correct values as we did before yields:

$$\begin{aligned}(a_{1,1}i^{\ell} + a_{1,2}i^{\ell-1} + ... + a_{1,\ell}i + a_{1,\ell+1})(a_{k,m}(i+1)^{\ell-m+1} + a_{k,m+1}(i+1)^{\ell-m} + ... + a_{k,\ell}(i+1) + a_{k,\ell+1})\\ - (a_{k,m} + a_{k,m+1} + ... + a_{k,\ell} + a_{k,\ell+1})(a_{k+1,m}i^{\ell-m+1} + a_{k+1,m+1}i^{\ell-m} + ... + a_{k+1,\ell}i + a_{k+1,\ell+1})\\ - (a_{k,m}i^{\ell-m+1} + a_{k,m+1}i^{\ell-m} + ... + a_{k,\ell}i + a_{k,\ell+1})(a_{1,1}(k+i)^{\ell} + a_{1,2}(k+i)^{\ell-1} + ... + a_{1,\ell}(k+i) + a_{k,\ell+1})\\ + (a_{1,1}k^{\ell} + a_{1,2}k^{\ell-1} + ... + a_{1,\ell}k + a_{1,\ell+1})(a_{k+1,m}i^{\ell-m+1} + a_{k+1,m+1}i^{\ell-m} + ... + a_{k+1,\ell}i + a_{k+1,\ell+1}) = 0\end{aligned} \tag{118}$$

Now this time, the highest power of i appearing is $2\ell - m + 1$. However, we find that the first term contributes a term of $a_{1,1}a_{k,m}i^{2\ell-m+1}$ and the third term contributes $-a_{1,1}a_{k,m}i^{2\ell-m+1}$. Exactly as before, we get that the second and fourth terms contribute to the highest power of i only when $2\ell - m + 1 = \ell - m + 1$; that is, exactly when $\ell = 0$ and Λ_X is a 0th-order array.

[18] More precisely, $N = \sum_{j=0}^{\ell} \binom{\ell}{j}(i-j)^{\ell}$. A standard combinatorial exercise can be used to show that this is equal to $\ell!$.

Hence we look at the second highest power of i, namely $i^{2\ell-m}$. From the first term of (118) we get a contribution of $(a_{1,1}a_{k,m}(\ell-m+1)+a_{1,1}a_{k,m+1}+a_{1,2}a_{k,m})i^{2\ell-m}$ where we get $(\ell-m+1)$ from the binomial theorem. From the third term of (118), we get a contribution of $-(a_{1,1}a_{k,m}k\ell+a_{1,2}a_{k,m}+a_{1,1}a_{k,m+1})i^{2\ell-m}$. We note once again that the second and fourth terms of (118) contribute only when $2\ell-m=\ell-m+1$; that is, when $\ell=1$. Hence the $i^{2\ell-m}$ term of (118) has a coefficient of

$$(a_{1,1}a_{k,m}(\ell-m+1)+a_{1,1}a_{k,m+1}+a_{1,2}a_{k,m})-(a_{1,1}a_{k,m}k\ell+a_{1,2}a_{k,m}+a_{1,1}a_{k,m+1})$$

which simplifies to:

$$a_{1,1}a_{k,m}(\ell-m+1-k\ell)$$

We take the $(2\ell-m+1)$th consecutive differences of the sequence $\mathcal{C}$ to get an equation:

$$Na_{1,1}a_{k,m}(\ell-m+1-k\ell)=0 \tag{119}$$

for N a nonzero constant; see the footnote included in the argument for the base case above. As before, $a_{1,1}\neq 0$, as otherwise Λ_X is not of order ℓ. Now suppose that $\ell-m+1-k\ell=0$. This would mean in particular that $\ell=m-1+k\ell$, but since $k\geq 2$, and $\ell\geq 2$, we must then have that $m<0$. But this violates the inductive hypothesis! Hence we achieve that $\ell-m+1-k\ell\neq 0$, and so the only possibility in equation (119) is that $a_{k,m}=0$. Since $k\geq 2$ was chosen arbitrarily, it follows that $a_{k,m}=0$ for all $k\geq 2$.

This completes the proof. □

5.3 0th Order Weak

In Section 5.2, we talked about a generalization of the problem solved in Section 5.1 by looking at higher-order strong sequences. In this section, we present some results on the generalization of the problem in the other direction: that is, considering 0th order sequences that are weak.

5.3.1 $\mathcal{S}eq_{\{0,1\}}$

The first family of sequences we consider in this direction are those arising from 0th-order weak Λ-arrays such that $\lambda_{1,j}\in\{0,1\}$ for all $j\geq 1$. We denote this family by $\Lambda_{\{0,1\}}$. Put another way, $\Lambda_{\{0,1\}}$ is defined to be the set of all $0th$-order Λ-arrays where the right diagonal is a sequence of only 0's and 1's. We denote by $\mathcal{S}eq_{\{0,1\}}$ the subset of $\mathcal{S}eq$ corresponding to the Λ-arrays in $\Lambda_{\{0,1\}}$

Theorem 5.15 (Foissy, 2018 [24]). *There are only four families of arrays in $\Lambda_{\{0,1\}}$:*

- *For any $i,j\geq 1, \lambda_{i,j}=1$.*
- *Case $A(m)$: there exists $m\geq 2$ such that* $\lambda_{i,j}=\begin{cases}1 & \text{if } j\leq m\\ 0 & \text{otherwise.}\end{cases}$
- *Case $B(m)$: there exists $m\geq 2$ such that* $\lambda_{i,j}=\begin{cases}1-i & \text{if } m|j\\ 1 & \text{otherwise.}\end{cases}$
- *Case $C(m)$: there exists $m\geq 2$ such that* $\lambda_{i,j}-=\begin{cases}1-i & \text{if } j=m\\ 1 & \text{otherwise.}\end{cases}$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

(a) The Λ-array corresponding to ladders.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 0 \\
1 \quad 1 \quad 0 \quad 0 \\
1 \quad 1 \quad 0 \quad 0 \quad 0 \\
1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}$$

(b) $A(3)$.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 0 \\
1 \quad 1 \quad -1 \quad 1 \\
1 \quad 1 \quad -2 \quad 1 \quad 1 \\
1 \quad 1 \quad -3 \quad 1 \quad 1 \quad 0 \\
1 \quad 1 \quad -4 \quad 1 \quad 1 \quad -1 \quad 1 \\
1 \quad 1 \quad -5 \quad 1 \quad 1 \quad -2 \quad 1 \quad 1
\end{array}$$

(c) $B(3)$.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 0 \\
1 \quad 1 \quad -1 \quad 1 \\
1 \quad 1 \quad -2 \quad 1 \quad 1 \\
1 \quad 1 \quad -3 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad -4 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad -5 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

(d) $C(3)$.

Figure 25: The four cases of Theorem 5.15 when $m = 3$.

We will not provide a proof of this statement here, but it may be found in [24].

Note that the case of the all-1's array and case $A(m)$ taken together are a subset of those sequences considered in Theorem 5.5. Hence the only two additional sequences in $\Lambda_{\{0,1\}}$ that arise by allowing for weak sequences are $B(m)$ and $C(m)$.

5.4 Comments on kth Order Weak

As discussed previously, not a lot is known about sequences that are kth-order weak. In this section, we give a brief overview of what is known, first by discussing the notion of scaled sequences, and then by exhibiting a new second-order weak family with some nice properties.

5.4.1 Scaled Sequences

Lemma 5.16. *Let $s = (t_n)_{n\geq 1}$ be an element of $\mathcal{S}eq$, and let $k = (k_n)_{n\geq 1}$ be a sequence with $k_1 = 1_{\mathbb{K}}$ and $k_n \in \mathbb{K} \setminus \{0\}$ for all n. Then $s' := (t'_n)_{n\geq 1} = (k_n t_n)_{n\geq 1}$ is an element of $\mathcal{S}eq$.*

Proof. Let A_s be the algebra generated by s and $A_{s'}$ be the algebra generated by s'. To prove the result, we only need to show that $A_{s'}$ is Hopf, given that A_s is Hopf. For n fixed but arbitrary, set

$$\Delta(t_n) = \sum_{i=0}^{n} Q_{n,i} \otimes t_{n-i}$$

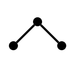
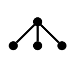
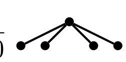
where as before we use $Q_{n,i}$ to be the polynomials in $t_1, ...t_i$ in the pruned part of the coproduct, and where we set $t_0 = Q_{n,0} = \mathbb{1}$ and $Q_{n,n} = t_n$. With this notation in place, we can calculate:

$$\begin{aligned}\Delta(t'_n) &= \Delta(k_n t_n)\\ &= k_n \Delta(t_n)\\ &= k_n \left(\sum_{i=0}^{n} Q_{n,i} \otimes t_{n-i} \right)\end{aligned}$$

But substituting $t_{n-i} = \frac{1}{k_{n-i}} t'_{n-i}$ for each t_{n-i} in the sum (including those appearing in the polynomials $Q_{n,i}$) gives us that $\Delta(t'_n) \subseteq A_{s'} \otimes A_{s'}$, since we can pull all coefficients out of the tensor product. By virtue of the t'_n being a basis for $A_{s'}$, the desired result follows. $\square$

Lemma 5.16 indicates that the notion of order on sequences is flexible; namely, by the correct choice of $(k_i)_{i\geq 1}$, we may transform a sequence that does not have an order into one that does, or we may transform a sequence that has an order into a sequence with a different order. We refer to this process as **scaling** a sequence.

Example 5.17 (Scaled Corollas)**.** Consider the sequence of corollas defined in Definition 2.25. We define a new sequence as $c'_n = \frac{1}{n!} c_n$. Hence the sequence begins:

$\mathbb{1}$, , $\frac{1}{2}$ , $\frac{1}{6}$ , $\frac{1}{24}$ , $\frac{1}{120}$ , ...

Whereas the original sequence $(c_n)_{n\geq 1}$ is 1st-order, one may easily verify that this new sequence c'_n is 0th-order.

While we may obtain a sequence of any order from an arbitrary element s of $\mathcal{S}eq$, the properties of s are not in general inherited by the scaled sequences. In particular:

Lemma 5.18. *Let $s = (t_n)_{n\geq 1}$ be an element of $\mathcal{S}eq$ that is ℓth order strong, and let $s' = (t'_n)_{n\geq 1} = (k_n t_n)_{n\geq 1}$ be a sequence obtained from scaling s to an order $\ell' \neq \ell$. Then if $\lambda_{i,1}$ and $\lambda_{1,i}$ are nonzero for all i, s' is not strong.*

Proof. We begin with the same setup as in the last section. Since the sequence s is ℓth order strong, we can represent the corresponding Λ-array by $\lambda_{i,j} := f_j(i) = a_{j,\ell}i^\ell + a_{j,\ell-1}i^{\ell-1} + ... + a_{j,1}i + a_{j,0}$. Note that scaling the sequence s to $s' = (k_n t_n)_{n\geq 1}$ scales $(\lambda_{i,1})_{i\geq 1}$ to $(\frac{k_i}{k_{i-1}}\lambda_{i,1})_{i\geq 1}$ and $(\lambda_{1,i})_{i\geq 1}$ to $(\frac{k_i}{k_{i-1}}\lambda_{1,i})_{i\geq 1}$. Since the statement of the Lemma indicates that these two diagonals do not contain any zeros, we can then use Lemma 5.3 to compute the inside of the scaled array. In particular, the same algebraic manipulations from the proof of Lemma 5.1 give the condition:

$$\lambda_{i+1,l} = \frac{\lambda_{i,l}\lambda_{i+l,1} - \lambda_{l,1}\lambda_{i,l+1} + \lambda_{1,l}\lambda_{i,l+1}}{\lambda_{i,1}}$$

which translates into our notational setup as:

$$f_\ell(i+1) = \frac{f_\ell(i)f_1(i+\ell) - f_1(\ell)f_{\ell+1}(i) + f_\ell(1)f_{\ell+1}(i)}{f_1(i)} \tag{120}$$

Let us consider the scaled array in the case that $\ell = 2$. If we let the polynomials of the scaled array be represented by $\widetilde{f_i}$, then equation (120) for the scaled array is:

$$\widetilde{f_2}(i+1) = \frac{\widetilde{f_2}(i)\frac{k_{i+2}}{k_{i+1}}f_1(i+2) - \frac{k_2}{k_1}f_1(2)\widetilde{f_3}(i) + k_1 f_2(1)\widetilde{f_3}(i)}{\frac{k_i}{k_{i-1}}f_1(i)} \tag{121}$$

For the sequence $(k_n)_{n\geq 1}$ to change the order of s, the factors $\frac{k_i}{k_{i-1}}$ must be at least linear in i (or i^{-1}), but then from equation (121) it follows that $\widetilde{f_2}(i)$ is scaled by a factor of at least i^2 (respectively i^{-2}). Hence s' is not a strong sequence. ☐

5.4.2 A Family of Second-Order Weak Sequences

We start with the following:

Lemma 5.19. *Let $\xi = (\lambda_{i,j})_{i,j\geq 1}$ be given by*

$$\lambda_{i,j} = \begin{cases} \frac{i\prod_{t=1}^{i-1}[(j+i-t)a+b]}{\prod_{s=1}^{i-2}[(i-s)a+b]} & \text{if } i \neq 1 \\ a+b & \text{if } i = 1 \end{cases} \tag{122}$$

such that $b \neq -ka$ for any $k \in \mathbb{N}$. Then ξ is an element of Λ.

Proof. To be a member of Λ, ξ must satisfy the nondegeneracy condition and prelie condition of Theorem 4.1. However, the nondegeneracy condition follows immediately from the definition of Λ, so we only focus on showing that the prelie property (77) holds for all values of $i, j, k \geq 1$. There are only eight distinct cases to consider, one case for each possibility of i, j, k being equal to 1 and not being equal to 1. However

$$
\begin{array}{ccccccccccc}
 & & & & & a+b & & & & & \\
 & & & & 4a+2b & & a+b & & & & \\
 & & & 9a+3b & & 6a+2b & & a+b & & & \\
 & & 16a+4b & & \frac{3(4a+b)(3a+b)}{(2a+b)} & & 8a+2b & & a+b & & \\
 & 25a+5b & & \frac{4(5a+b)(4a+b)}{2a+b} & & \frac{3(5a+b)(4a+b)}{2a+b} & & 10a+2b & & a+b & \\
36a+6b & & \frac{5(6a+b)(5a+b)}{2a+b} & & \frac{4(6a+b)(5a+b)(4a+b)}{(3a+b)(2a+b)} & & \frac{3(6a+b)(5a+b)}{(2a+b)} & & 12a+2b & & a+b \\
 & & & & & \vdots & & & & &
\end{array}
$$

Figure 26: The array described in Lemma 5.19.

we can cut the casework down somewhat further, first by realizing the cases $j = k = 1$ for any i is a tautology since equation (77) is symmetric in j and k, and hence the relation is already satisfied. For the same reason, we can assume without loss of generality that $j < k$. The remaining cases under these assumptions are as follows:

Case 1: $i = 1, j = 1, k \geq 2$. We have:

$$
\begin{aligned}
\lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k} &= \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} \\
\iff (a+b)\lambda_{2,k} - (a+b)(a+b) &= (a+b)\lambda_{k+1,1} - \lambda_{k,1}(a+b) \\
\iff \lambda_{2,k} - (a+b) &= \lambda_{k+1,1} - \lambda_{k,1} \\
\iff (2)[(k+1)a+b] - (a+b) &= \frac{(k+1)[(k+1)a+b]\cdots[2a+b]}{[ka+b]\cdots[2a+b]} - \frac{(k)[ka+b]\cdots[2a+b]}{[(k-1)a+b]\cdots[2a+b]} \\
\iff (2k+2)a + 2b - a - b &= (k+1)[(k+1)a+b] - (k)[ka+b] \\
\iff (2k+1)a + b &= (k^2+2k+1)a + (k+1)b - k^2a - kb \\
\iff (2k+1)a + b &= (2k+1)a + b
\end{aligned}
$$

Case 2: $i = 1, j \geq 2, k \geq 2$. We start with the right-hand side of equation (77):

$$
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} = (a+b)\left(\frac{(k+1)[(j+k)a+b]\cdots[(j+1)a+b]}{[ka+b]\cdots[2a+b]}\right) - \left(\frac{(k)[(j+k-1)a+b]\cdots[(j+1)a+b]}{[(k-1)a+b]\cdots[2a+b]}\right)(a+b)
$$

Since j is strictly less than k, some factors in the numerator and denominator of each fraction cancel to yield:[19]

$$
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} = (a+b)\left(\frac{(k+1)[(j+k)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right) - \left(\frac{(k)[(j+k-1)a+b]\cdots[ka+b]}{[ja+b]\cdots[2a+b]}\right)(a+b)
$$

[19]Note that in the corner case that factors do not cancel in second term, then $j+1=k$, and hence the new form is still valid.

Next, we take out all common factors the two terms share in common:

$$\begin{aligned}
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} &= (a+b)\left(\frac{[(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right)((k+1)[(j+k)a+b]-(k)[ka+b]) \\
&= (a+b)\left(\frac{[(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right)((k^2+jk+j+k)a+(k+1)b-k^2a-kb) \\
&= (a+b)\left(\frac{[(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right)((jk+j+k)a+b) \\
&= (a+b)\left(\frac{[(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right)((j^2+jk+j+k)a+b-j^2a+jb-jb) \\
&= (a+b)\left(\frac{[(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right)((j+1)[(j+k)a+b]-(j)[ja+b])
\end{aligned}$$

Finally, we distribute across the difference once again to obtain:

$$\begin{aligned}
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} &= (a+b)\left(\frac{(j+1)[(j+k)a+b][(j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]}\right) \\
&\qquad - \left(\frac{(j)\cancel{[ja+b]}[(j+k-1)a+b]\cdots[(k+1)a+b]}{\cancel{[ja+b]}\cdots[2a+b]}\right)(a+b) \\
&= \lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k}
\end{aligned}$$

Case 3: $i \geq 2, j = 1, k \geq 2$**.** This case proceed in a similar fashion to the previous case. Starting with the right-hand side:

$$\begin{aligned}
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} &= \left(\frac{(i)[(i+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)\left(\frac{(i+k)[(i+k)a+b]\cdots[2a+b]}{[(i+k-1)a+b]\cdots[2a+b]}\right) \\
&\qquad - \left(\frac{(k)[(k)a+b]\cdots[2a+b]}{[(k-1)a+b]\cdots[2a+b]}\right)\left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
&= \left(\frac{(i)[(i+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)(i+k)[(i+k)a+b] \\
&\qquad - (k)[ka+b]\left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)([(k+1)a+b](i+k)-(k)[ka+b]) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)((k^2+k+ik+i)a+(i+k)b-k^2a-kb) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)((k+ik+i)a+ib) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)((k+ik+i)a+ib+a+b-a-b) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)((k+ik+i)a+ib+a+b)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - (a+b)\left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
&= \left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)(i+1)(a(k+1)+b)\frac{ia+b}{ia+b} \\
&\qquad - (a+b)\left(\frac{(i)[(i+k)a+b]\cdots[(k+2)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
&= \lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k}
\end{aligned}
$$

Case 4: $i \geq 2, j \geq 2, k \geq 2$**.** The verification is just another computation:

$$
\begin{aligned}
\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} =& \left(\frac{[i][(i+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)\left(\frac{[i+k][(i+j+k-1)a+b]\cdots[(j+1)a+b]}{[(i+k-1)a+b]\cdots[2a+b]}\right) \\
&- \left(\frac{[k][(k+j-1)a+b]\cdots[(j+1)a+b]}{[(k-1)a+b]\cdots[2a+b]}\right)\left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
=& \left(\frac{[i][(i+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)\left(\frac{[i+k][(i+j+k-1)a+b]\cdots\cancel{[(j+1)a+b]}}{\cancel{[(i+k-1)a+b]}\cdots[2a+b]}\right) \\
&- \left(\frac{[k][(k+j-1)a+b]\cdots\cancel{[(j+1)a+b]}}{\cancel{[(k-1)a+b]}\cdots[2a+b]}\right)\left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right) \\
=& \left(\frac{[i][(i+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)\left(\frac{[i+k][(i+j+k-1)a+b]\cdots[(i+k)a+b]}{[ja+b]\cdots[2a+b]}\right) \\
&- \left(\frac{[k][(k+j-1)a+b]\cdots[ka+b]}{[ja+b]\cdots[2a+b]}\right)\left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]}\right)
\end{aligned}
$$

As before, we factor out as many common factors between terms as we can to obtain:

$$
\begin{aligned}
\lambda_{i,k}\lambda_{i+k,j}-\lambda_{k,j}\lambda_{i,j+k} =& \left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b][(j+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b][ja+b]\cdots[2a+b]}\right) \\
&\cdot\left([i+k][(j+k)a+b]-[k][ka+b]\right)
\end{aligned}
$$

Now the rightmost factor can be algebraically manipulated as follows:

$$
\begin{aligned}
[i+k][(j+k)a+b]-[k][ka+b] &= (ij+ik+jk+k^2)a+(i+k)b-k^2a-kb \\
&= (ij+ik+jk)a+ib \\
&= (ij+ik+jk)a+ib-j^2a+j^2a-jb+jb \\
&= (ij+ik+jk+j^2)a+(i+j)b-(j)(ja+b) \\
&= (i+j)(j+k)a+(i+j)b-(j)(ja+b) \\
&= (i+j)[(j+k)a+b]-(j)[ja+b]
\end{aligned}
$$

Hence:

$$
\lambda_{i,k}\lambda_{i+k,j}-\lambda_{k,j}\lambda_{i,j+k} = \left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b][(j+k-1)a+b]\cdots[(k+1)a+b]}{[(i-1)a+b]\cdots[2a+b][ja+b]\cdots[2a+b]}\right)
$$

$$\cdot \Big((i+j)[(j+k)a+b] - (j)[ja+b] \Big)$$

And finally redistributing:

$$\lambda_{i,k}\lambda_{i+k,j} - \lambda_{k,j}\lambda_{i,j+k} = \left(\frac{[i]}{[(i-1)a+b]\cdots[2a+b]} \right) \left(\frac{[i+j][(i+j+k-1)a+b]\cdots[(k+1)a+b]}{[ja+b]\cdots[2a+b]} \right)$$

$$- \left(\frac{[j][(j+k-1)a+b]\cdots[(k+1)a+b]\cdot\cancel{[ja+b]}}{\cancel{[ja+b]}[(j-1)a+b]\cdots[2a+b]} \right) \left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]} \right)$$

$$= \left(\frac{[i][(j+i-1)a+b]\cdots[(j+1)a+b]}{[(i-1)a+b]\cdots[2a+b]} \right) \left(\frac{[i+j][(j+i+k-1)a+b]\cdots[(k+1)a+b]}{[(i+j-1)a+b]\cdots[2a+b]} \right)$$

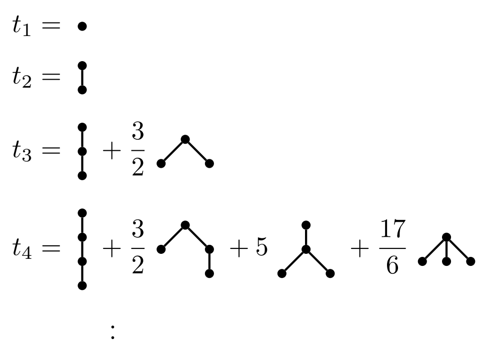
$$- \left(\frac{[j][(j+k-1)a+b]\cdots[(k+1)a+b]}{[(j-1)a+b]\cdots[2a+b]} \right) \left(\frac{[i][(i+j+k-1)a+b]\cdots[(j+k+1)a+b]}{[(i-1)a+b]\cdots[2a+b]} \right)$$

$$= \lambda_{i,j}\lambda_{i+j,k} - \lambda_{j,k}\lambda_{i,j+k}$$

This calculation finishes the proof. □

Remark. We take a moment to note some of the more salient features of the family of sequences introduced in Lemma 5.19. Firstly, note that the sequence of generators of the Connes-Moscovici subalgebra belong to this family, and can be recovered by setting $a = b = \frac{1}{2}$. Secondly, we remark that this family provides the only known examples of second-order sequences that are not scaled versions of zeroth- or first-order sequences.

Besides the sequence of generators of the Connes-Moscovici subalgebra, another sequence of the family that appears to be combinatorial in nature can be obtained by setting $a = 1$ and $b = 0$. The first few elements of this sequence begin:



While we do not yet have a complete combinatorial description of the sequence, it is interesting to note that its Λ-array is given by $\lambda_{i,j} = i\binom{i+j-1}{i-1}$ and that the coefficient sequences describing the corollas appear to be `A177208` (numerators) and `A177209` (denominators) in OEIS [45].

6 Conclusion

6.1 Open Problems

Here we present some tantalizing questions for further research. In the following list, some questions may be as in-depth as whole research problems, while others may be completely intractable, while still others may be simple exercises appropriate for graduate or even undergraduate students in combinatorics. We make no attempt to classify these questions according to their potential or difficulty, hence this classification is left to the discernment of the reader.

1. In [28], Grossman and Larson introduce the Hopf algebra on rooted trees which ends up being isomorphic to the graded dual of $\mathcal{H}_{CK}$, as discussed in Section 2.1.4. However, the Hopf algebra $\mathcal{H}_{GL}$ is just one of a few Hopf algebras Grossman and Larson introduce in [28]. In particular, another Hopf algebra they introduce is the family of rooted heap-ordered trees $\mathcal{HOT}$. Is it known if this family has any relation to $\mathcal{H}_{CK}$? Is it possible it constitutes a Hopf subalgebra?

2. In this work, we have given a complete characterization of strong 0th-order sequences and strong ℓth order sequences for $\ell \geq 2$ (Sections 5.1 and 5.2). What is the complete characterization of strong first-order sequences? We conjecture that those we have described in this thesis constitute all possible strong first-order sequences.

3. Do there exist weak kth-order sequences for any $k \geq 3$ other than scaled 0th-, 1st-, and 2nd-order sequences? If so, what is their underlying combinatorial structure? If not, then why not? What does this say about the underlying physical systems?

4. In Section 5.3, we discuss a new family of Hopf subalgebras of $\mathcal{H}_{CK}$ whose generators are an application of the exponential map to a series of ladder primitives, $X = \sum_{i=0}^{n} l_i$. However, Foissy (See section 7.6.3 of [23]) gives some primitive elements not related to the primitive elements coming from ladders. Are there any nice Hopf subalgebras coming from exponentiating series involving these elements? Moreover, are there any nice Hopf subalgebras arising from exponentiating series of k-primitive elements (in either the sense of [7] or [3]) for some k? Primitive elements correspond to 0-primitive elements in both sources, and as the elements we get in $\mathcal{S}eq$ from exponentiating primitive elements are 0th-order, is it possible there is any connection between one of the notions of k-primitiveness and our notion of kth order sequences?

6.2 Final Remarks

In this thesis, we have explored various properties of sequences of linear combinations of trees in the setting of the Connes-Kreimer Hopf algebra, and have categorized the majority of sequences that are strong. The fact that the only strong sequences of order ℓ for $\ell \geq 2$ are scaled corollas (Theorem 5.11) emphasizes the importance of the sequence of corollas, as well as the importance of first- and zeroth-order sequences, all of which have been singled out previously for their importance from a purely-physics perspective. Future investigation into the possible orders of sequences having a combinatorial description (and without scaling) is a very interesting question for further research, as mentioned in the previous section.

We have also introduced in this work a new family of second-order sequences relating to the sequence of generators of the Connes-Moscovici subalgebra. We hope these sequences will be of interest to others in the combinatorics community in the future.

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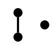
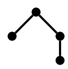
Appendices

The following appendices contain information which is integral to (but tangential to) the work done throughout this text. In Appendix A, we present our SageMath implementation of a class modelling the Connes-Kreimer Hopf algebra of forests of rooted trees. Since $\mathbb{K}[\mathcal{F}_n] \simeq \mathbb{K}[\mathcal{T}_{n+1}]$ via B^+, forests are represented as rooted trees with an extra parent vertex. Hence the forest  will be modeled in the code as . The underlying class for our implementation is that of OrderedRootedTrees which is already built in to Sage.

In Appendix B, we then present code which does not belong in the class definition of the Connes-Kreimer Hopf algebra, but which was still helpful throughout this work for various computational purposes. Examples of applications of this code are in computing the maps $\overline{S}$ and S^* on trees, and in computing elements of $\mathcal{S}eq$ from their corresponding Λ-arrays using the bijection described in Section 4.

The last appendix is a dictionary of elements of $\mathcal{S}eq$, sorted according to their order. One will find familiar examples such as those sequences coming from Dyson-Schwinger equations, in addition to the new sequences presented earlier in this text.

A SageMath Implementation of the Connes-Kreimer Hopf Algebra of Rooted Trees

```
#This file will model the Connes Kreimer Hopf algebra of rooted trees by
#implementing it as an instance of the HopfAlgebrasWithBasis class already
#in extant. Implementing it this way should make it easier in the future
#to implement Michael Hoffman's bijection to the Grossman Larson Hopf
#algebra. C.f. the Grossman Larson Hopf algebra in SageMath authored by
#Fr\'ed\'eric Chapoton.

#Author: @wtdugan
#Contact: wtdugan@uwaterloo.ca

import copy

class Hck(CombinatorialFreeModule):
    def __init__(self):
        CombinatorialFreeModule.__init__(self, QQ, RootedTrees(), category =
            HopfAlgebrasWithBasis(QQ))

    def _repr_(self):
        return "The Connes Kreimer Hopf Algebra of rooted trees over rational 
            field"

    @cached_method
    def one_basis(self):
        return(RootedTree([]))

    def product_on_basis(self, h1, h2):
        x = copy.deepcopy(h1)
        y = copy.deepcopy(h2)
        for i in y:
            x = x.graft_on_root(i)
        return(self.basis()[x])

    @cached_method
    def algebra_generators(self):
        return(Family(RootedTrees()))

    #def  coproduct_on_basis(self, h):
    #  h = self.monomial(h)
    #  return( tensor([h,h]))

    def coproduct_on_basis(self, h):
```

```
    h = h.canonical_labelling()
    P = LabelledOrderedTree(h).to_poset() #< Make a poset to find
        antichains.
    antichains_list = [[get_tv(h,j) for j in i] for i in list(P.antichains
        ())[: 1]] #< Form antichains list. Get rid of antichain
        corresponding to top vertex (as this vertex is just a vestige due
        to our model of the Hopf algebra as tres instead of forests).
    final_sum = 0  #< Initialize sum for coproduct.
    for i in antichains_list:   #< Loop over antichains.
        R = copy.deepcopy(h)
        P = self.one()
        for j in i:
            R = recursive_remove(R, j)     #< Component containing root.
            P = P*self.monomial(RootedTree([j]))        #Pruned part. Need
                to add a new fake vertex.
        final_sum = final_sum + tensor([P, self.monomial(RootedTree(R))])
    return(final_sum)

def counit_on_basis(self, h):
    return self.base_ring().one()

def antipode_on_basis(self,h):
    if(h == RootedTree([])):
        return(self.one())
    elif(h == RootedTree([[]])):
        return( self.monomial(h))
    #Else if a tree:
    elif(len(h) == 1):
        h = h.canonical_labelling()
        antipode_list = construct_antipode_list(h)
        final_sum = sum(prod((( 1)^len(j))*self.monomial(RootedTree([i]))
            for i in j) for j in antipode_list)   #< Put a list around
            each i to add back in fake vertex for each tree.
        return(final_sum)        #< Note that len(j) in the previous line
            is exactly (n_c + 1) in antipode formula.
    #Else it must be a forest with more than one component:
    else:
        list_of_trees_in_forest = [self.monomial(RootedTree([i])) for i in
            h]
        return(prod(j.antipode() for j in list_of_trees_in_forest))     #<
             Add an extra fake vertex to each root with [] wrapper.



##########################################################################
```

```
# Helper Functions:
##########################################################################

#Takes a rooted tree T and removes a subtree a:

def recursive_remove(T, a):
    with T.clone() as W:
        if a in W:
            W.remove(a)
            return(W)
        elif len(W) == 0:
            return(W)
        else:
            root_label = W.label()   #<  Obtains label of root so that
                recursively rebuilt tree maintains the same labelling as the old
                 tree.
            subtree_list = [recursive_remove(i, a) for i in T]
            U = LabelledRootedTree([], label = root_label)  #<  Creates a new
                root on which to graft altered subtrees.
            for j in subtree_list:
                U = U.graft_on_root(j)
            return(U)


#Define a helper function to get t_v  tree rooted at vertex v in t:
def get_tv(t,v):
    for i in t.subtrees():
        if i.label() == v:
            return(i)


#Takes a rooted tree and constructs all possible cuts.
def construct_antipode_list(T):
    #Start with an empty list:
    my_list = []
    for i in Subsets(range(3,T.node_number() + 1)):
        my_sublist = []
        J = copy.deepcopy(T)
        for j in reversed(i):
            W = get_tv(J,j)
            my_sublist.append(W)
            J = recursive_remove(J,W)
        my_sublist.append(J[0])          #<  Take first index of J to get rid
             of fake root.
        my_list.append(my_sublist)
    return(my_list)
```

B Other Code

```
#Return first few values of $\log(X)$:
def sequence_log(X):
    One = H(RootedTree([]))
    return( (X    One) + (1/2)*(X    One)^2    (1/3)*(X One)^3 + (1/4)*(X One)
        ^4    (1/5)*(X One)^5 )

#Return first few values of $\exp(X)$:
def sequence_exp(X):
    One = H(RootedTree([]))
    return(One + X + (1/2)*X^2 + (1/6)*X^3 + (1/24)*X^4 + (1/120)*X^5 +
        (1/720)*X^6 + (1/5040)*X^7)

#Return the component homogeneous of degree $n$:
def homogeneous_component(U, n):
    tree_list = [(i.coefficients()[0], list(i.monomials()[0])[0][0]) for i
        in U.terms()]

    return(sum(i[0]*H(RootedTree(i[1])) for i in tree_list if RootedTree(i
        [1]).node_number() == n + 1))

def B_minus(X):
    tree_list = [(i.coefficients()[0], list(i.monomials()[0])[0][0]) for i
        in X.terms() if i.monomials()[0] != H(RootedTree([]))]
    Y = sum(i[0]*H(RootedTree(i[1][0])) for i in tree_list)   #<  i[1][0]
        takes the second element of the tuple and then removes the root
        vertex of the tree (accessed as a list).
    return(Y)

def B_plus(X):
    tree_list = [(i.coefficients()[0], list(i.monomials()[0])[0][0]) for i
        in X.terms() if i.monomials()[0] != H(RootedTree([]))]
    Y = sum(i[0]*H(RootedTree([i[1]])) for i in tree_list)   #<  list(i[1])
         takes the second element of the tuple and then adds a new root
        vertex of the tree (by wrapping a new list around the old one).
    return(Y)

##########################################################################
Code related to the computation of $\overline{S}$ and $S^*$:
##########################################################################

#Write size() function. Takes t, an element of Hck. TODO: add this to the
    Hck.element class.
```

```
def element_size(t):
    h = list(t)[0][0]  #< Extracts underlying rooted tree object.
    return(h.node_number()   1)   #< Subtract one to account for fake
        vertex.

#Function to extract underlying rooted tree. TODO: add to Hck.element class.
def get_element(t):
    return(list(t)[0][0])

#Takes a forest t (as an element of Hck) and outputs $\oveline{S}$ of t.
    Only works on a single tree or forest (not a linear combination yet):
def S_log_monomial(t):
    if(element_size(t) == 0):
        return(t)
    elif(element_size(t) == 1):
        return( t)
    elif(len(get_element(t)) == 1):
        h = get_element(t)
        h = h.canonical_labelling()
        antipode_list = construct_antipode_list(h)
        final_sum = sum(prod((( 1)^len(j))*H(RootedTree([i])) for i in j)
            *(1/len(j)) for j in antipode_list)   #< Put a list around each
             i to add back in fake vertex for each tree.
        return(final_sum)       #< Note that len(j) in the previous line
            is exactly (n_c + 1) in antipode formula.
    #Else t must be a forest with more than one component:
    else:
        h = get_element(t)
        list_of_trees_in_forest = [H(RootedTree([i])) for i in h]
        return(prod(S_log(j) for j in list_of_trees_in_forest))    #< Add
            an extra fake vertex to each root with [] wrapper.

#Takes a forest t (as an element of Hck) and outputs $S^*$ of t. Only works
    on a single tree or forest (not a linear combination yet):
def S_exp_monomial(t):
    if(element_size(t) == 0):
        return(t)
    elif(element_size(t) == 1):
        return( t)
    elif(len(get_element(t)) == 1):
        h = get_element(t)
        h = h.canonical_labelling()
        antipode_list = construct_antipode_list(h)
        final_sum = sum(prod((( 1)^len(j))*H(RootedTree([i])) for i in j)
            *(1/factorial(len(j))) for j in antipode_list)   #< Put a list
            around each i to add back in fake vertex for each tree.
```

```
        return(final_sum)          #<  Note that len(j) in the previous line
            is exactly (n_c + 1) in antipode formula.
    #Else t must be a forest with more than one component:
    else:
        h = get_element(t)
        list_of_trees_in_forest = [H(RootedTree([i])) for i in h]
        return(prod(S_log(j) for j in list_of_trees_in_forest))      #<  Add
            an extra fake vertex to each root with [] wrapper.

S_log = H.module_morphism(lambda i: S_log_monomial(H.monomial(i)), codomain
    = H)
S_exp = H.module_morphism(lambda i: S_exp_monomial(H.monomial(i)), codomain
    = H)
```

C Examples of Sequences of Trees

Here we present a dictionary of the known elements of $\mathcal{S}eq$ that admit an order, sorted according to their order. We first present those that are strong sequences, and then those that are weak. In the case of strong sequences, we will indicate at each order whether the sequences presented constitute all possible strong sequences of that order (in which case we will say "complete"), or whether it is not known if there exist any more strong sequences of that order (in which case we will say "not complete").

C.1 Strong Sequences

Λ-array	Element of $\mathcal{S}eq$	Generated by:
1 1 1 1 1 ⋮ 1 1 ⋮ 1 1 1 ⋮ 1 b 1 1 ⋮ 1 b 0 1 1 ⋮ 1 b 0 0 1 1 ⋮ 1 b 0 0 0	$t_1 =$ • $t_2 =$  ⋮ $t_{n-1} = l_{n-1}$ $t_n = bl_n$ ⋮	$X = B^+(\exp([\sum_{i=1}^{n-1} P_i)] + bP_n))$
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	$t_1 =$ • $t_2 =$  $t_3 =$  $t_4 =$  ⋮	$X = B^+(\exp\left(\sum_{i=1}^{\infty} P_i\right))$ or: $X =$ • $+ B^+(X)$

Table 4: Strong 0th order sequences (complete).

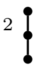
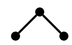
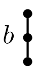
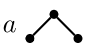
Λ-array	Element of Seq	Generated by:
$a+b$ $2a+b \quad a+b$ $3a+b \quad 2a+b \quad a+b$ $4a+b \quad 3a+b \quad 2a+b \quad a+b$	$t_1 = \bullet$ $t_2 = (a+b)$  $t_3 = (a+b)^2$  $+ \frac{(a+b)a}{2}$  $\vdots$	$X = B^+((1+bX)^{\frac{a+b}{b}}), a, b \in \mathbb{K}$ and $b \neq 0$ $X = B^+(\exp(aX)), a, b \in \mathbb{K}$ and $b = 0$
$a+b$ $2a+b \quad b$ $3a+b \quad b \quad b$ $4a+b \quad b \quad b \quad b$	$t_1 = t_1 = \bullet$ $t_2 = (a+b)$  $t_3 = (a+b)b$  $+ (a+b)a$ 	$X = B^+(\frac{1+bX}{1-a\,\bullet})$

Table 5: Strong 1st order sequences (not complete).

Λ-array	Element of Seq	Generated by:
$f(1)$ $f(2) \quad 0$ $f(3) \quad 0 \quad 0$ $f(4) \quad 0 \quad 0 \quad 0$ with $f(i) = i^k a_k + i^{k-1} a_{k-1} + ... + a_0$	Scaled corollas	

Table 6: Strong kst order sequences, for all $k \geq 2$ (complete).

C.2 Some Known Weak Sequences

Name	Λ-array
$B(n)$	1 $1 \quad \vdots$ $1 \quad \vdots \quad 1$ $1 \quad \vdots \quad 1 \quad 0$ $1 \quad \vdots \quad 1 \quad -1 \quad 1$ $1 \quad \vdots \quad 1 \quad -2 \quad 1 \quad \vdots$ $1 \quad \vdots \quad 1 \quad -3 \quad 1 \quad \vdots \quad 1$ $1 \quad \vdots \quad 1 \quad -4 \quad 1 \quad \vdots \quad 1 \quad 0$ $1 \quad \vdots \quad 1 \quad -5 \quad 1 \quad \vdots \quad 1 \quad -1 \quad 1$ $1 \quad \vdots \quad 1 \quad -5 \quad 1 \quad \vdots \quad 1 \quad -1 \quad 1 \quad \vdots$
$C(n)$	1 $1 \quad \vdots$ $1 \quad \vdots \quad 1$ $1 \quad \vdots \quad 1 \quad 0$ $1 \quad \vdots \quad 1 \quad -1 \quad 1$ $1 \quad \vdots \quad 1 \quad -2 \quad 1 \quad 1$ $1 \quad \vdots \quad 1 \quad -3 \quad 1 \quad 1 \quad 1$ $1 \quad \vdots \quad 1 \quad -4 \quad 1 \quad 1 \quad 1 \quad 1$ $1 \quad \vdots \quad 1 \quad -4 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$ $1 \quad \vdots \quad 1 \quad -4 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$

Table 7: The Λ-arrays of some weak 0th order sequences.

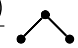
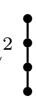
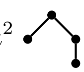
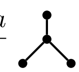
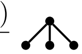
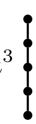
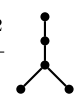
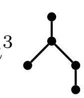
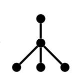
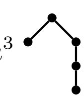
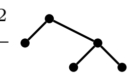
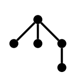
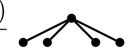
Λ-array:	$\begin{array}{cccccccc} & & & & & & & a+b \\ & & & & & & 2a+b & -a \\ & & & & & 3a+b & 0 & -a \\ & & & & 4a+b & 0 & 0 & -a \\ & & & 5a+b & 0 & 0 & 0 & -a \\ & & 6a+b & 0 & 0 & 0 & 0 & -a \\ & 7a+b & 0 & 0 & 0 & 0 & 0 & -a \\ 8a+b & 0 & 0 & 0 & 0 & 0 & 0 & -a \\ & & & \vdots & & & & \end{array}$ with $a, b \in \mathbb{K}, a \neq 0$
Sequence begins:	$t_1 =$  $t_2 = (a+b)$  $t_3 = -(a+b)a$  $+ \frac{(a+b)(3a+b)}{2}$  $t_4 = (a+b)a^2$  $- (a+b)a^2$  $- \frac{(a+b)(3a+b)a}{2}$  $+ \frac{(a+b)(11a^2+6ab+b^2)}{6}$  $t_5 = -(a+b)a^3$  $+ \frac{(a+b)(3a+b)a^2}{2}$  $+ (a+b)a^3$  $- \frac{(a+b)(11a^2+6ab+b^2)a}{6}$  $+ (a+b)a^3$  $- \frac{(a+b)(3a+b)a^2}{2}$  $- (a+b)a^3$  $+ \frac{(a+b)(50a^3+30a^2b+10ab^2+b^3)}{24}$ 

Table 8: A family of weak 1st order sequences.

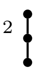
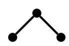
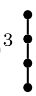
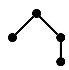
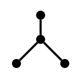
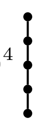
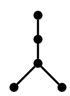
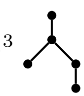
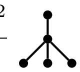
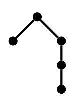
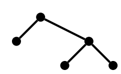
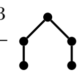
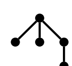
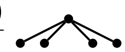
Λ-array:	$a+b$ $4a+2b \quad a+b$ $9a+3b \quad 6a+2b \quad a+b$ $16a+4b \quad \frac{3(4a+b)(3a+b)}{(2a+b)} \quad 8a+2b \quad a+b$ $25a+5b \quad \frac{4(5a+b)(4a+b)}{2a+b} \quad \frac{3(5a+b)(4a+b)}{2a+b} \quad 10a+2b \quad a+b$ $36a+6b \quad \frac{5(6a+b)(5a+b)}{2a+b} \quad \frac{4(6a+b)(5a+b)(4a+b)}{(3a+b)(2a+b)} \quad \frac{3(6a+b)(5a+b)}{(2a+b)} \quad 12a+2b \quad a+b$ $\vdots$ with $a, b \in \mathbb{K}, b \neq -ka$ for any $k \in \mathbb{N}$
Sequence begins:	$t_1 =$  $t_2 = (a+b)$  $t_3 = (a+b)^2$  $+ \frac{(3a+b)(a+b)}{2}$  $t_4 = (a+b)^3$  $+ (5a+b)(a+b)^2$  $+ \frac{(3a+b)(a+b)^2}{2}$  $+ \frac{(17a^2+6ab+b^2)(a+b)}{6}$  $t_5 = (a+b)^4$  $+ \frac{(3a+b)(a+b)^3}{2}$  $+ (5a+b)(a+b)^3$  $+ \frac{(17a^2+6ab+b^2)(a+b)^2}{6}$  $+ (7a+b)(a+b)^3$  $+ \frac{(7a+b)(3a+b)(a+b)^2}{2}$  $+ \frac{(8a+b)(3a+b)(a+b)^3}{2(2*a+b)}$  $+ \frac{(80a^3+53a^2*b+10ab^2+b^3)(a+b)^2}{2(2*a+b)}$  $+ \frac{(304a^4+201a^3*b+55a^2*b^2+15ab^3+b^4)(a+b)}{24(2*a+b)}$ 

Table 9: A family of weak 2nd order sequences[20].

[20]The Λ-array corresponding to the Connes-Moscovici subalgebra is recovered by choosing $a = b = \frac{1}{2}$.