Sampled-Data Control of Invariant Systems on Exponential Lie Groups

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis examines the dynamics and control of a class of systems furnished by kinematic systems on exponential matrix Lie groups, when the plant evolves in continuous-time, but whose controller is implemented in discrete-time. This setup is called sampled-data and is ubiquitous in applied control. The class of Lie groups under consideration is motivated by our previous work concerning a similar class of kinematic systems on commutative Lie groups, whose local dynamics were found to be linear, which greatly facilitated control design. This raised the natural question of what class of systems on Lie groups, or class of Lie groups, would admit global characterizations of stability based on the linear part of their local dynamics. As we show in this thesis, the answer is—or at least includes—left- or right-invariant systems on exponential Lie groups, which are necessarily solvable, nilpotent, or commutative.

We examine the stability of a class of difference equations that arises by sampling a right- or left-invariant flow on a matrix Lie group. The map defining such a difference equation has three key properties that facilitate our analysis: 1) its Lie series expansion enjoys a type of strong convergence; 2) the origin is an equilibrium; 3) the algebraic ideals enumerated in the lower central series of the Lie algebra are dynamically invariant. We show that certain global stability properties are implied by stability of the Jacobian linearization of dynamics at the origin, in particular, global asymptotic stability. If the Lie algebra is nilpotent, then the origin enjoys semiglobal exponential stability.

We then study the synchronization of networks of identical continuous-time kinematic agents on a matrix Lie group, controlled by discrete-time controllers with constant sampling periods and directed, weighted communication graphs with a globally reachable node. We present a smooth, distributed, nonlinear discrete-time control law that achieves global synchronization on exponential matrix Lie groups, which include simply connected nilpotent Lie groups as a special case. Synchronization is generally asymptotic, but if the Lie group is nilpotent, then synchronization is achieved at an exponential rate. We first linearize the synchronization error dynamics at the identity, and show that the proposed controller achieves local exponential synchronization on any Lie group. Building on the local analysis, we show that, if the Lie group is exponential, then synchronization is global. We provide conditions for deadbeat convergence when the communication graph is unweighted and complete.

Lastly, we examine a regulator problem for a class of fully actuated continuous-time kinematic systems on Lie groups, using a discrete-time controller with constant sampling period. We present a smooth discrete-time control law that achieves global regulation on simply connected nilpotent Lie groups. We first solve the problem when both the plant state and exosystem state are available for feedback. We then present a control law for the case where the plant state and a so-called plant output are available for feedback. The class of plant outputs considered includes, for example, the quantity to be regulated. This class of output allows us to use the classical Luenberger observer to estimate the exosystem states. In the full-information case, the regulation quantity on the Lie algebra is shown to decay exponentially to zero, which implies that it tends asymptotically to the identity on the Lie group.

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Nomenclature

$A\|\mathcal{V}$	The restriction of A to the A-invariant subspace $\mathcal{V}$	20
H	The ideal hold operator	12
I	The identity matrix of appropriate dimension	19
I_n	The $n \times n$ identity matrix	19
S	The ideal sample operator	12
T	The sampling period	13
U	The neighbourhood on which the principal matrix logarithm is well defined	25
$[\cdot, \cdot]$	The Lie bracket	15
$\mathbb{F}$	The field of $\mathfrak{g}$; either $\mathbb{C}$ or $\mathbb{R}$	11
G	A Lie group	11
$\mathsf{GL}(n, \mathbb{F})$	The general linear group of $n \times n$ invertible matrices over the field $\mathbb{F}$	22
Id_S	The identity operator on the set S	19
Log	The matrix logarithm	14
$\mathbb{N}_N$	$\{1, \dots, N\}$	19
$\mathbb{R}$	The real numbers	2
$\mathbb{R}^-$	The nonpositive real numbers	19
$\mathsf{SE}(n)$	The special Euclidean group of homogeneous transformations on $\mathbb{R}^n$	4
$\mathsf{SO}(n)$	The special orthogonal group of proper rotations on $\mathbb{R}^n$	3
$\bar{A}$	The map induced in $\mathcal{X}/\mathcal{V}$ by A	28
$\bar{x}$	The coset of x in $\mathcal{X}/\mathcal{V}$	19
det	Determinant	22
exp	The matrix exponential	24
$\mathfrak{g}$	A Lie algebra, typically the Lie algebra of G	11
$\mathfrak{g}^{(i)}$	The ith entry in the lower central series of $\mathfrak{g}$	27
$\mathfrak{g}_i$	The ith entry in the derived series of $\mathfrak{g}$	27
$\mathfrak{h}$	The ideal of $\mathfrak{g}$ that satisfies Assumption 1(c)	34
$\mathbf{1}_n$	The column vector of ones	19
$\mathfrak{gl}(n, \mathbb{F})$	The Lie algebra of $n \times n$ matrices over the field $\mathbb{F}$	23

$\mathfrak{sup}(n)$	The Lie algebra of $\mathsf{SUP}(n)$	7
Ad_X	The adjoint operator of $X \in \mathsf{G}$	24
ad_x	The adjoint operator of $x \in \mathfrak{g}$	14
$\mathsf{SUP}(n)$	The Lie group of $n \times n$ unipotent matrices with unit determinant that models systems in chained form	7
$\mathsf{SU}(n)$	The special unitary group of $n \times n$ unitary matrices with unit determinant over the field of complex numbers	5
$\mathsf{U}(n)$	The group of $n \times n$ unitary matrices over the complex field	5
$\mathcal{N}_i$	The neighbour set of agent i	59
$\mathcal{X}/\mathcal{V}$	The quotient space $\mathcal{X}$ modulo $\mathcal{V}$	19
$\mathcal{X}$	A vector space; typically the state space of the plant	19
ω	A word, i.e., a nested Lie bracket	24
$\sigma(A)$	The spectrum of A	20
$\mathfrak{so}(n)$	The special orthogonal Lie algebra of real $n \times n$ skew symmetric matrices	3
$\sqcup$	Disjoint union	19
$\top$	Non-Hermitian matrix transpose	19
$\widetilde{W}$	$\{W_1, \ldots, W_r\}$	35
$\widetilde{X}$	$\{X_1, \ldots, X_n\}$	35
c_ω	The coefficient of word ω	33
$x + \mathcal{V}$	The coset of x in $\mathcal{X}/\mathcal{V}$	19

Chapter 1

Introduction

The main results of this thesis, presented in Chapter 3, give easily checkable conditions to ensure, among other properties, global asymptotic stability of the origin for a class of discrete-time dynamical systems on solvable Lie algebras with state X and exogenous signal W. We make no general assumptions on the exogenous signal W. We show that, for this class of systems, global stability properties can be determined from the linear part of the dynamics.

As we delineate in this chapter, study of this class of systems is motivated by the sampled-data control of right- (or left-) invariant systems on matrix Lie groups:

$$\dot{X}(t) = \left(A + \sum_{i=1}^{m} B_i u_i(t) \right) X(t), \tag{1.1}$$

where the state $X(t)$ evolves on the Lie group, $A, B_1, \ldots, B_m$ belong to the associated Lie algebra, and $u_1(t), \ldots, u_m(t)$ are the control signals. Our stability results in Chapter 3 for system (3.1) are applied to synchronization and regulation problems for system (1.1) in Chapters 4 and 5, respectively.

In particular, we consider the case where the Lie group is solvable or nilpotent. Our study of this class of Lie groups was initially motivated by our synchronization [73] and regulation [74] results for commutative Lie groups. In those works, it was found that, locally, the dynamics of (1.1) were linear. This raised the natural question of what class of systems on Lie groups, or class of Lie groups, would admit global characterizations of stability based on the linear part of their local dynamics. As we show in this thesis, for systems of the form (1.1), the answer is—or at least includes—exponential Lie groups, which are necessarily solvable, nilpotent, or commutative.

The mathematical object that has been come to be known as a Lie group was initially studied by Sophus Lie in the latter half of the nineteenth century. He sought to do for ordinary differential equations, what Évariste Galois had done for polynomial equations—that is, identify groups characterizing symmetries of these equations, thereby granting insight to their solubility and solutions [87]. In 1868, Wilhelm Killing published the first of a series of papers that would lay the foundation for the classification of what would become known as semisimple Lie algebras [42, §II]. In his doctoral dissertation, Élie Cartan extended Killing's discoveries on semisimple and solvable Lie algebras [42, §III.6.2]. In the early twentieth century, Hermann Weyl made significant advancements in the study of irreducible representations of Lie groups [42, §IV]. In 1900, David Hilbert published his list of *Mathematische Probleme* [45] (Mathematical Problems [46]). The fifth of these problems is commonly interpreted as *"are all locally Euclidean topological groups Lie groups? [108]"* This question was answered in the affirmative in 1952 by Deane Montgomery, Leo Zippin [79], and Andrew M. Gleason [36]. This result leads to the simple characterization of Lie groups: they are groups that are also topological manifolds.

1.1 Control Systems on Matrix Lie Groups

Control systems on Lie groups differ from classical control systems in that the state does not evolve on a vector space. Such systems are often controlled using differential geometric techniques, i.e., using coordinate charts on the Lie group to represent the system dynamics in local coordinates on $\mathbb{R}^n$. This effects artificial singularities that arise from the choice of local coordinates, rather than being intrinsic to the system's dynamics.

Example 1.1.1. (Rotating rigid body in $\mathbb{R}^3$) Rotating rigid bodies appear frequently in engineering. Examples include UAVs [110] and robotic manipulators [101]. Fix an inertial reference frame

$$\Sigma_{\text{intertial}} = \{O_{\text{inertial}}, \{g_1, g_2, g_3\}\},$$

which allows us to treat points in space as vectors in $\mathbb{R}^3$, relative to the origin O_{inertial}, with components taken relative to the basis $\{g_1, g_2, g_3\}$. To the rigid body, attach a body reference frame

$$\Sigma_{\text{body}} = \{O_{\text{body}}, \{b_1, b_2, b_3\}\},$$

that moves with the body.

If $O_{\text{inertial}} = O_{\text{body}}$, and the bases $\{g_1, g_2, g_3\}$ and $\{b_1, b_2, b_3\}$ have the same orientation, then the matrix $R \in \mathbb{R}^{3\times 3}$, whose ith column is the components of b_i expressed in the basis

$\{g_1, g_2, g_3\}$, represents the orientation of the rigid body frame with respect to the inertial frame. The matrix R is an element of the special orthogonal group of 3×3 orthogonal matrices $\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : \det(R) = 1, R^\top R = I\}$. It is easy to show that the evolution of the orientation of the rigid body $R(t)$ is modelled by

$$\dot{R} = R\Omega_b, \tag{1.2}$$

where $\Omega_b \in \mathfrak{so}(3)$, the Lie algebra of 3×3 skew-symmetric matrices. If the system is fully actuated, then Ω_b has three independent control inputs and the system can be taken to be of the form (1.1) with $A = 0$,

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These dynamics can also be expressed in the coordinate frame Σ_{inertial} as

$$\dot{R} = \Omega_g R. \tag{1.3}$$

▲

In the previous example, no local coordinates were used; the elements of $\mathsf{SO}(3)$ were represented as matrices embedded in $\mathbb{R}^{3\times 3}$. However, such matrices have only 3 independent entries. This motivates many researchers to use the minimum number of differential equations to describe the dynamics (1.2) (or (1.3)). This is done by treating $\mathsf{SO}(3)$ as a 3-dimensional manifold and with local coordinate charts. The next example illustrates how the common practice of using local coordinates introduces artificial singularities.

Example 1.1.2. (Rotating rigid body in $\mathbb{R}^3$ in local coordinates) Consider the rotating rigid body of Example 1.1.1 with dynamics (1.2). A common choice of local coordinates is roll-pitch-yaw, whose coordinate map is defined only on the open subset $U = \{R \in \mathsf{SO}(3) : -1 \text{ not in spectrum of } R\}$ on which it is invertible. Given a rotation matrix $R \in U$, the coordinate chart (U, φ) is given by $\varphi : U \to \varphi(U) \subset \mathbb{R}^3$,

$$\varphi(R) \coloneqq \begin{bmatrix} \operatorname{atan2}(R_{32}, R_{33}) \\ \operatorname{atan2}(R_{21}, R_{11}) \\ \operatorname{atan2}(-R_{31}, \cos(\operatorname{atan2}(R_{21}, R_{11}))\, R_{11} + \sin(\operatorname{atan2}(R_{21}, R_{11}))\, R_{21}) \end{bmatrix},$$

where atan2 is the 4 quadrant arctan, and with inverse,

$$\varphi^{-1}(\xi) = \begin{bmatrix} \cos\xi_2\cos\xi_3 & \cos\xi_2\sin\xi_3\sin\xi_1 - \sin\xi_2\cos\xi_1 & \cos\xi_2\sin\xi_3\cos\xi_1 + \sin\xi_2\sin\xi_1 \\ \sin\xi_2\cos\xi_3 & \sin\xi_2\sin\xi_3\sin\xi_1 + \cos\xi_2\cos\xi_1 & \sin\xi_2\sin\xi_3\cos\xi_1 - \cos\xi_2\sin\xi_1 \\ -\sin\xi_3 & \cos\xi_3\sin\xi_1 & \cos\xi_3\cos\xi_1, \end{bmatrix}.$$

The variables ξ_1, ξ_2, and ξ_3 represent rotations in $\varphi(U) \subset \mathbb{R}^3$. The dynamics (1.2) can be expressed in local coordinates by applying the chain rule:[1]

$$\dot{\xi} = \left.\frac{\partial\varphi}{\partial R}\right|_{R=\varphi^{-1}(\xi)} \left.\dot{R}\right|_{R=\varphi^{-1}(\xi)} = \begin{bmatrix} \omega_1 + \tan(\xi_3)(\omega_3\cos(\xi_1) + \omega_2\sin(\xi_1)) \\ \frac{\omega_3\cos(\xi_1)+\omega_2\sin(\xi_1)}{\cos(\xi_3)} \\ \omega_2\cos(\xi_1) - \omega_3\sin(\xi_1) \end{bmatrix}.$$

We see that $\dot{\xi}_1$ and $\dot{\xi}_2$ are unbounded as $\xi_3 \to \pm\frac{\pi}{2}$.

The vector field $\dot{\xi}$ has singularities at $\cos(\xi_3) = \cos(\xi_2)\cos(\xi_3) = \sin(\xi_2)\cos(\xi_3) = 0$ [4]. These singularities are mathematical artifacts associated with the choice of local coordinates; they are not physical properties of any rotating rigid body. ▲

The study of (1.1) was pioneered in the 1970s with the works of Jurdjevic, Sussmann [54] and Brocket [12] on controllability; the latter also addressed observability and realization theory. Brockett's observability results in [12] were extended, and necessary and sufficienct conditions were identified in [20]. We refer the reader to [93] for a more recent comprehensive treatment of control theory on Lie groups.

Many engineering systems can be modelled on Lie groups, which is advantageous because it eliminates dependence on local coordinates, thereby avoiding singularities in the dynamical model. The study of systems in this global framework allows one to identify its intrinsic properties, i.e., the properties that are invariant under smooth diffeomorphisms. Networks of coupled oscillators are common in science and engineering: biological systems such as neural networks, crickets chirping, and pacemaker cells of the heart [102], parts of the power grid [71, 97, 31], and robots moving on a plane [57]. The Kuramoto oscillator is used to model the synchronization behaviour of networks of oscillators. The Kuramoto oscillator evolves on the circle [30], which is isomorphic to the commutative Lie group $\mathsf{SO}(2)$, the group of rotations on $\mathbb{R}^2$. The motion of robots in a plane is modelled on the solvable Lie group $\mathsf{SE}(2)$ [55, 53]; motion in space, such as that of underwater vehicles [64], UAVs [62, 91], and robotic arms [101], is modelled on $\mathsf{SE}(3)$, where $\mathsf{SE}(n)$ is the special

[1]Here we are using the fact that $\mathsf{SO}(3)$ is embedded in $\mathbb{R}^{3\times 3}$ in order to apply the chain rule.

Euclidean group of homogeneous transformations on $\mathbb{R}^n$. Quantum systems evolve on the groups $\mathsf{U}(n)$ [65] of $n \times n$ unitary matrices over the complex field, and $\mathsf{SU}(n)$ [88, 3, 2], the subgroup of unitary matrices with unit determinant. Even the noise responses of some circuits evolve on Lie groups [115], specifically the solvable Lie group of invertible upper-triangular matrices. Control on the nilpotent Heisenberg group H has also been the object of much study [78, 6]. In continuous-time, certain classes of vector fields can be approximated as being on nilpotent Lie algebras [43, 104] and, more generally, solvable Lie algebras [23]. This is of interest because of the relatively simple Lie algebraic structure of nilpotent and solvable Lie algebras. In particular, left- and right-invariant systems on nilpotent matrix Lie groups have trajectories characterized by finite series of Lie brackets, as will be made apparent when we discuss the Magnus expansion in Section 1.2.

Our main results concern nilpotent and solvable Lie groups. In this section, we outline how these classes of Lie groups are relevant in engineering applications. Consider Brockett's nonholonomic integrator [14]

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= u_1 x_2 - x_1 u_2.\end{aligned} \tag{1.4}$$

This system can be used to model, for example, a special case of the dynamical constraints that furnish the equations of planar motion of a particle in a magnetic field [10, §7.5]. The nonholonomic integrator is an example of a nonlinear system that is globally controllable, but whose linearization at the origin is not useful for control design:

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= 0,\end{aligned}$$

which is not even stabilizable.

The nonholonomic integrator (1.4) can be expressed as a system on the nilpotent 3-dimensional real Heisenberg group H of matrices of the form

$$X = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

A basis for its Lie algebra $\mathfrak{h}$ is

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose nonzero Lie brackets are given by $[e_1, e_2] = e_3$. Consider the control system

$$\dot{X} = (e_1 u_1 + e_2 u_2)X, \tag{1.5}$$

where $e_1, e_2 \in \mathfrak{h}$ are the input vector fields, $X \in \mathsf{H}$ is the state, and $u_1, u_2 \in \mathbb{R}$ are the inputs. System (1.5) is of the form (1.1), with $A = 0$, $B_1 = e_1$ and $B_2 = e_2$. From (1.5) we have

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2. \end{aligned} \tag{1.6}$$

Applying the global change of coordinates [112]

$$\begin{aligned} \xi_1 &\coloneqq x_1 \\ \xi_2 &\coloneqq x_2 \\ \xi_3 &\coloneqq 2x_3 - x_1 x_2, \end{aligned}$$

we recover the nonholonomic integrator (1.4), which shows that (1.4) and (1.5) are differentially equivalent.

System (1.6) is an example of a system in so-called *chained form*, introduced by Murray [83]. In particular, system (1.6) is an example of a *one chain* system:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ \dot{x}_4 &= x_3 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1. \end{aligned} \tag{1.7}$$

Such systems are expressions of a kinematic system on a particular nilpotent Lie group

$\mathsf{SUP}(n)$ [103] of matrices of the form

$$X = \begin{bmatrix} 1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_n \\ 0 & 1 & x_1 & \frac{1}{2}x_1 & \frac{1}{6}x_1^3 & \cdots & \frac{1}{(n-2)!}x_1^{n-2} \\ \vdots & \ddots & 1 & x_1 & \frac{1}{2}x_1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & x_1 & \ddots & \frac{1}{6}x_1^3 \\ \vdots & & & \ddots & 1 & \ddots & \frac{1}{2}x_1^2 \\ \vdots & & & & \ddots & \ddots & x_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

where $x \coloneqq (x_1, \ldots, x_n) \in \mathbb{R}^n$, in exponential coordinates of the second kind, i.e.,

$$X = \exp(e_1 x_1) \cdots \exp(e_n x_n).$$

A basis for its Lie algebra $\mathfrak{sup}(n)$ is

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \ldots, \; e_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

whose nonzero Lie brackets are given by

$$[e_1, e_i] = (-1)^{i+1} e_{i+1}, \qquad i = 1, \ldots, n-1.$$

It is easy to verify that $\mathsf{SUP}(n)$ is a nilpotent Lie group (see Definition 2.3.4). We remark that $\mathsf{SUP}(3)$ is isomorphic to the Heisenberg group H, which is the Lie group of the nonholonomic integrator (1.4). This can be seen by observing that their respective Lie algebras have the same bases, up to reordering. Again consider a control system of the form (1.5), where $e_1, e_2 \in \mathfrak{sup}(n)$ and $X \in \mathsf{SUP}(n)$. Expressing these dynamics in exponential coordinates of the second kind, we obtain (1.7).

Example 1.1.3. (Car-like robot, [82]) Consider the vehicle in Figure 1.1. We can

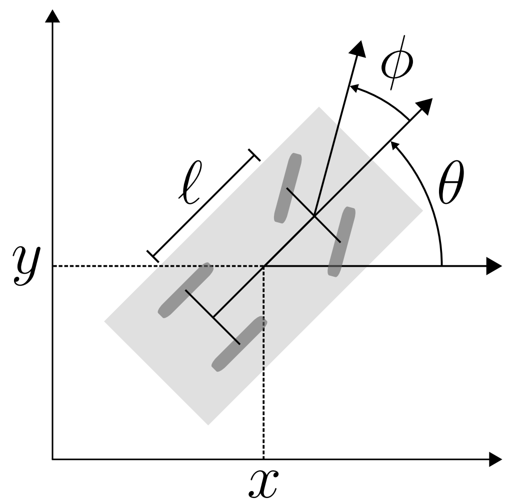


Figure 1.1: Car-like robot.

model this vehicle using the kinematic model

$$
\begin{aligned}
\dot{x} &= \cos(\theta)u_1 \\
\dot{y} &= \sin(\theta)u_1 \\
\dot{\phi} &= u_2 \\
\dot{\theta} &= \frac{1}{\ell}\tan(\phi)u_1.
\end{aligned}
\tag{1.8}
$$

Applying the coordinate and feedback transformation [112, §4] given by

$$
\begin{aligned}
x_1 &\coloneqq x \\
x_2 &\coloneqq \frac{1}{\ell}\sec(\theta)^3\tan(\phi) \\
x_3 &\coloneqq \tan(\theta) \\
x_4 &\coloneqq y \\
u_1 &= \sec(\theta)v_1 \\
u_2 &= \frac{-3}{\ell}\sin(\phi)^2\tan(\theta)\sec(\theta)v_1 + \ell\cos(\phi)^2\cos(\theta)^3 v_2,
\end{aligned}
$$

we obtain a system in chained form:

$$\begin{aligned}\dot{x}_1 &= v_1\\ \dot{x}_2 &= v_2\\ \dot{x}_3 &= x_2 v_1\\ \dot{x}_4 &= x_3 v_1,\end{aligned}$$

which can be expressed as a system on the nilpotent matrix Lie group $\mathsf{SUP}(4)$:

$$\dot{X} = X\left(\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix} v_1 + \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} v_2\right),$$

where

$$X := \begin{bmatrix}1 & x_2 & x_3 & x_4\\ 0 & 1 & x_1 & \frac{1}{2}x_1^2\\ 0&0&1&x_1\\ 0&0&0&1\end{bmatrix}.$$

▲

Finally, if a left- or right-invariant kinematic control system on a Lie group does not evolve on a nilpotent Lie group, it can always be approximated by one that does. The following example illustrates the nilpotent approximation technique of [104].

Example 1.1.4. ([104, Example 3.3]) Consider an underactuated rigid body in space, which can be modelled as a left-invariant system on the Lie group $\mathsf{SE}(3)$

$$\dot{X} = X\left(e_1u_1 + e_2u_2 + e_4u_3 + e_5u_4\right), \tag{1.9}$$

where $\{e_1, \ldots, e_6\}$ is a basis for the Lie algebra $\mathfrak{se}(3)$, whose nonvanishing Lie brackets are given by

$$\begin{aligned}&[e_1,e_2] = e_3, \quad [e_1,e_3] = -e_2, \quad [e_1,e_5] = e_6, \quad [e_1,e_6] = -e_5,\\ &[e_2,e_3] = e_1, \quad [e_2,e_4] = -e_6, \quad [e_2,e_6] = e_4,\\ &[e_3,e_4] = e_5, \quad [e_3,e_5] = -e_4.\end{aligned}$$

In exponential coordinates of the second kind, the dynamics (1.9) are

$$\dot{x} = \begin{bmatrix} \sec(x_2)\cos(x_5) & -\sec(x_2)\sin(x_5) & 0 & 0 \\ \sin(x_5) & \cos(x_5) & 0 & 0 \\ x_4\tan(x_2)\cos(x_5) - x_6\sin(x_5) & -x_4\tan(x_2)\sin(x_5) - x_6\cos(x_5) & \cos(x_5) & -\sin(x_5) \\ \cos(x_5)(x_6 - x_3\tan(x_2)) & \sin(x_5)(x_3\tan(x_2) - x_6) & \sin(x_5) & \cos(x_5) \\ -\tan(x_2)\cos(x_5) & \tan(x_2)\sin(x_5) & 0 & 0 \\ x_3\sin(x_5) - x_4\cos(x_5) & x_3\cos(x_5) + x_4\sin(x_5) & 0 & 0 \end{bmatrix} u. \tag{1.10}$$

The nilpotent approximation technique of [104] first constructs a nilpotent Lie algebra by copying the structure constants of the original Lie algebra, then judiciously setting some of them to zero, such that the resultant Lie algebra is nilpotent, while preserving a subset of the original Lie brackets. The nilpotent Lie algebra in this example has nonvanishing Lie brackets given by

$$[n_1, n_2] = n_5, \quad [n_1, n_4] = n_6, \quad [n_2, n_3] = -n_6.$$

We define the dynamics on the nilpotent Lie group associated with the foregoing nilpotent Lie algebra:

$$\dot{Y} = Y(n_1u_1 + n_2u_2 + n_3u_3 + n_4u_4),$$

which in exponential coordinates of the second kind is

$$\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= u_3 \\ \dot{y}_4 &= u_4 \\ \dot{y}_5 &= -y_2u_1 \\ \dot{y}_6 &= -y_4u_1 + y_3u_2. \end{aligned} \tag{1.11}$$

To see how the dynamics (1.11) are an approximation for (1.10), first define

$$\begin{aligned} F_k^e &\coloneqq \mathrm{span}_{\mathbb{R}}\{\text{words over the letters } e_1, e_2, e_4, e_5 \text{ of length at most } k\}, \\ F_k^n &\coloneqq \mathrm{span}_{\mathbb{R}}\{\text{words over the letters } n_1, n_2, n_3, n_4 \text{ of length at most } k\}. \end{aligned}$$

The sequences $\{F_k^e\}$ and $\{F_k^n\}$ are called the *control filtrations* of $\mathrm{Lie}_{\mathbb{R}}\{e_1, e_2, e_4, e_5\}$ and

$\mathrm{Lie}_{\mathbb{R}}\{n_1, n_2, n_3, n_4\}$, respectively. We have

$$\begin{aligned}
F_1^e &= \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_4, e_5\} \\
F_2^e &= \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_4, e_5, [e_1, e_2], [e_1, e_5], [e_2, e_4]\} = \mathfrak{se}(3) \\
F_1^n &= \mathrm{span}_{\mathbb{R}}\{n_1, n_2, n_3, n_4\} \\
F_2^n &= \mathrm{span}_{\mathbb{R}}\{n_1, n_2, n_3, n_4, [n_1, n_2], [n_1, n_4], [n_2, n_3]\} = \mathfrak{se}(3)
\end{aligned}$$

The dynamics (1.11) are an approximation for (1.10) in the sense that Y evolves on the nilpotent Lie group, whose Lie algebra is given by a subset of the structure constants of $\mathfrak{se}(3)$, and

$$\begin{aligned}
\dim F_1^e &= \dim F_1^n \\
\dim F_2^e &= \dim F_2^n.
\end{aligned}$$

More generally, the method of [104] furnishes approximations such that for all k, $\dim F_k^e = \dim F_k^n$. ▲

The foregoing examples serve to motivate the study of nilpotent Lie groups in the context of control theory. In the case of the nonholonomic integrator (1.4), the system is described as a system on a nilpotent matrix Lie group (1.5) via a coordinate transformation. In the car-like robot example, the dynamics (1.8) are put into chained form via a feedback transformation. Note that nilpotent Lie groups are a special case of solvable Lie groups.

Remark 1.1.1. A so-called **left-invariant** dynamical system on a Lie group is

$$\dot{X} = X\left(A + \sum_{i=1}^{m} B_i u_i\right), \tag{1.12}$$

where X evolves on a Lie group G, $A, B_1, \dots, B_m$ belong to the associated Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$, and the input u takes values $\mathbb{F}^m$. The name *left-invariant* indicates that the action of G on $\mathfrak{g}$ is left-invariant, i.e., if $X, Y \in \mathsf{G}$ and $\dot{X} = f(X)$, then $f(YX) = Yf(X)$. Such systems include Schrödinger's equation in $\mathsf{SU}(n)$ [3] and the rotational dynamics of rigid bodies on $\mathsf{SO}(3)$ [116].

Applying the change of coordinates $Z \coloneqq X^{-1}$ to (1.12), we obtain a **right-invariant** system of the form (1.1) [93, §3.2]. Results applying to left-invariant systems apply *mutatis mutandis* to right-invariant systems. Henceforth, we use whichever of (1.12) or (1.1) is most convenient. ♦

1.2 Sampled-Data Feedback Control of Systems on Matrix Lie Groups

This thesis is concerned with the control of plants on Lie groups in the sampled-data setting, i.e., a plant evolving in continuous-time and a controller evolving in discrete-time. The plant often has an output $Y = h(X, u)$, where $h : \mathsf{G} \times \mathbb{F}^m \to \mathsf{Y}$ and Y is the output set: a Cartesian product of Lie groups or vector spaces. In the standard feedback control system setup, there is a reference signal $R \in \mathsf{Y}$, which represents the desired value of the output. A closed-loop sampled-data setup is illustrated in Figure 1.2, where C is the controller, E is the tracking error, and S and H are ideal sample and hold operators, respectively.

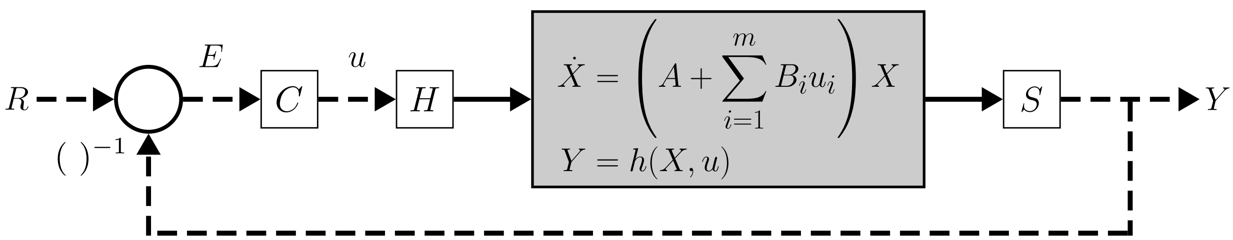


Figure 1.2: Sampled-data error feedback system on a Lie group.

The sampled-data setup is ubiquitous in applied control [38]. In this context, the plant, as seen from the controller's perspective, is the composition of the plant dynamics with the sample and hold operators, as illustrated in Figure 1.3. In the LTI case, the plant

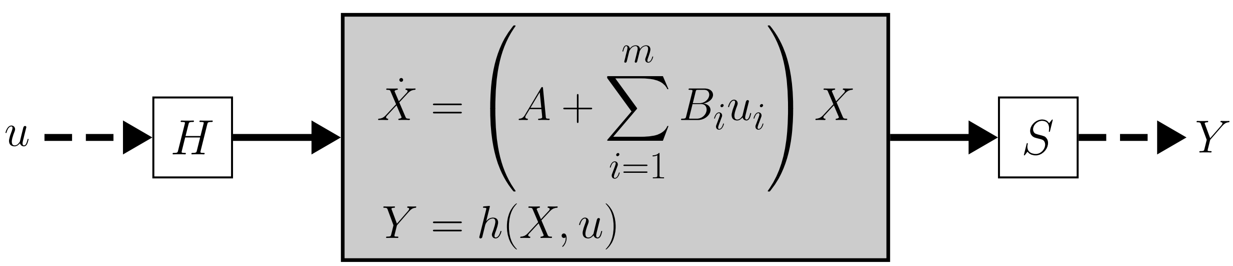


Figure 1.3: Sampled-data plant.

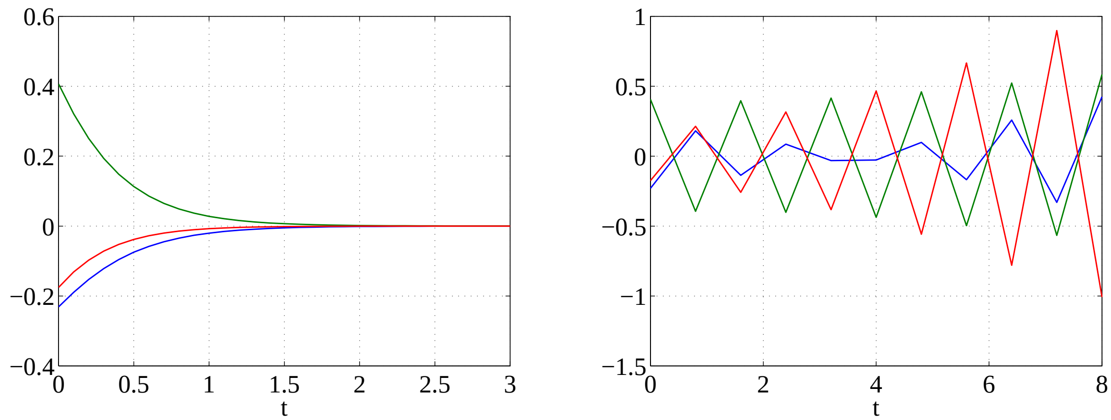
can be exactly discretized, i.e., the map $u[k] \mapsto y[k]$ has an exact closed-form expression and, if the plant is stabilizable, i.e., the restriction of the dynamics to the unstable modal subspace are controllable, and the sampling period is not pathological, then a discrete-time controller can be designed such that closed-loop stability is achieved. Such stability guarantees cannot generally be enforced for nonlinear plants, as nonlinear ODEs generally do not have closed-form solutions, necessitating the use of approximations for discrete-time design or implementation. As discussed in the next subsection, right- and left-invariant systems on matrix Lie groups are an exception. The design of a discrete-time controller for the discretized plant is called *direct design*. Since nonlinear ODEs generally do not have closed-form solutions, the plant dynamics are usually approximately discretized, for example, using series approximations of the state trajectories [77]. For a broad class of nonlinear systems, sufficient conditions for closed-loop stability of the sampled-data system were identified in [84]: 1) given a fixed sampling period, stability is achieved if the plant discretization is sufficiently accurate [84, Theorem 1]; 2) if the approximation's accuracy is a function of only the sampling period, e.g., Euler's method, then stability is achieved for sufficiently small sampling periods [84, Theorem 2]. Approximation-based direct design has two main weaknesses [84]: 1) closed-loop stability may be impossible for a given discretization method; 2) when closed-loop stability is achievable, it relies on fast sampling, which may be infeasible. For example, when using machine vision, the sampling rate may be limited by the framerate of the camera [69]. The latter issue is also the main weakness of *emulation*—solving the control problem in continuous-time, but implementing a discrete-time controller that approximates the continuous-time controller at the sampling instants [85].

Example 1.2.1. Consider a network of 3 Kuramoto oscillators of the form

$$\dot{\theta}_i = \omega_{0i} - \sum_{i=1}^{3} \sin(\theta_i - \theta_j), \qquad i \in \{1, 2, 3\}. \tag{1.13}$$

Frequency synchronization is achieved, i.e., for all i, $\dot{\theta}_i \to \omega$ as $t \to \infty$, but can be lost under sampling. We simulate this network, where each agent can only update its neighbours' phase information at the sampling instants. As seen in Figure 1.4, synchronization is achieved with sampling period $T = 0.1$, but not for $T = 0.8$. Although not surprising, this example illustrates one of the weaknesses of emulation. ▲

The limitations inherent to approximate discretization do not necessarily pose a problem for the class of kinematic systems on matrix Lie groups (1.1), which are nonlinear, yet have dynamics that admit exact closed-form solutions [32], thereby enabling direct design using discretized plant models whose states coincide with those of the continuous-time plant at the sampling instants. To our knowledge, sampled-data control of systems on Lie groups has not been explored extensively in the literature. Brockett and Willsky introduced and established controllability and observability properties for the class of *group*



(a) $T = 0.1$ (b) $T = 0.8$

Figure 1.4: Phases of the agents (1.13) under different sampling periods.

homomorphic sequential systems:

$$\begin{aligned} x[k+1] &= b(u[k])a(x[k]) \\ y[k] &= c(x[k]), \end{aligned}$$

where a, b, and c are morphisms of groups [13, 15, 114]. However, they only studied these systems in the case where the group is finite.

Consider the general class of time-invariant right-invariant kinematic systems on matrix Lie groups, whose dynamics are governed by

$$\dot{X} = A(t,u)X, \tag{1.14}$$

where $X(t) \in \mathsf{G}$, $u(t) \in \mathbb{F}^m$, $A : \mathbb{R} \times \mathbb{R}^m \to \mathfrak{g}$. For piecewise constant inputs u, the state trajectory can be solved exactly, thereby furnishing a step-invariant transform, as for linear systems. The solution $X(t)$ for (1.14) is given by the Magnus expansion [9], which provides an expression for $\mathrm{Log}(X(t)) \in \mathfrak{g}$ wherever the principal logarithm $\mathrm{Log} : \mathsf{G} \to \mathfrak{g}$ is well-defined. Given fixed $A \in \mathfrak{g}$, its adjoint operator is $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto [A, X]$, where the Lie bracket $[\cdot,\cdot]$ is the commutator $AX - XA$. Recursively define

$$
\begin{aligned}
\Omega_1(t) &\coloneqq \int_0^t A(\tau, u(\tau))\mathrm{d}\tau \\
\Omega_n(t) &\coloneqq \sum_{j=1}^{n-1} \frac{B_j}{j!} \sum_{\substack{k_1+\cdots+k_j=n-1 \\ k_1,\ldots,k_j\geq 1}} \int_0^t \mathrm{ad}_{\Omega_{k_1}(s)} \cdots \mathrm{ad}_{\Omega_{k_j}(s)} A(s, u(s))\mathrm{d}s, \quad n \geq 2 \\
\Omega(t) &\coloneqq \sum_{n=1}^{\infty} \Omega_n(t),
\end{aligned}
\tag{1.15}
$$

where the B_j are the Bernoulli numbers.[2] Then, whenever the series defining Ω, which is a linear combination of the integral of A and nested Lie brackets $\Omega_n(t)$, $n \geq 2$, converges,

$$
X(t) = \exp(\Omega(t))X(0), \tag{1.16}
$$

In the sampled-data setup, due to the hold operator H, the plant is driven by a piecewise constant input signal. When $A(t, u)$ is constant over the interval $t \in [kT, (k+1)T)$, from (1.15), a straightforward computation yields

$$
\begin{aligned}
\Omega[k+1] = \Omega[k] &+ \int_{kT}^{(k+1)T} A(\tau, u[k])\mathrm{d}\tau \\
&+ \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{B_j}{j!} \sum_{\substack{k_1+\cdots+k_j=n-1 \\ k_1,\ldots,k_j\geq 1}} \int_{kT}^{(k+1)T} \mathrm{ad}_{\Omega_{k_1}(s)} \cdots \mathrm{ad}_{\Omega_{k_j}(s)} A(s, u[k])\mathrm{d}s,
\end{aligned}
$$

which furnishes an exact discretization of (1.16):

$$
\begin{aligned}
X[k+1] = \exp\Bigg(&\Omega[k] + \int_{kT}^{(k+1)T} A(\tau, u[k])\mathrm{d}\tau \\
&+ \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{B_j}{j!} \sum_{\substack{k_1+\cdots+k_j=n-1 \\ k_1,\ldots,k_j\geq 1}} \int_{kT}^{(k+1)T} \mathrm{ad}_{\Omega_{k_1}(s)} \cdots \mathrm{ad}_{\Omega_{k_j}(s)} A(s, u[k])\mathrm{d}s\Bigg) X(0).
\end{aligned}
$$

[2] Using the convention $B_1 = \frac{1}{2}$.

If, for all $t_1, t_2 \in [kT, (k+1)T)$, $A(t_1, u(t_1))$ commutes with $A(t_2, u(t_2))$—as is the case for (1.1) under sampling, since $u(t_1) = u(t_2)$—then this expression simplifies significantly:

$$\begin{aligned} X[k+1] &= \exp\left(\Omega[k] + \int_{kT}^{(k+1)T} A(\tau, u[k])\mathrm{d}\tau\right) X(0) \\ &= \exp\left(\int_{kT}^{(k+1)T} A(\tau, u[k])\mathrm{d}\tau\right) \underbrace{\exp\left(\Omega[k]\right) X(0)}_{X[k]}. \end{aligned}$$

which yields a *step-invariant transform* on the group G:

$$X[k+1] = \exp\left(\int_0^T A(\tau, u[k])\mathrm{d}\tau\right) X[k]. \tag{1.17}$$

There also exist approximate discretization methods for such systems [32, §1.8], for example, the Euler discretization

$$X[k+1] = \left(I + T\left(A + \sum_{i=1}^m B_i u_i[k]\right)\right) X[k] \tag{1.18}$$

and the Padé approximant

$$X[k+1] = \left(I + \frac{T}{2}\left(A + \sum_{i=1}^m B_i u_i[k]\right)\right)\left(I - \frac{T}{2}\left(A + \sum_{i=1}^m B_i u_i[k]\right)\right)^{-1} X[k]. \tag{1.19}$$

The Euler discretization is attractive from the perspective of analysis because it preserves the algebraic form of (1.12), where A and B_i have been replaced by $I + TA$—which is not necessarily in $\mathfrak{g}$—and TB_i, respectively; additionally, the approximation error tends to 0 as $T \to 0$. The step-invariant transform does not preserve this structure; its right side is the product of two elements of G. The vector fields in (1.18) and (1.19) are not necessarily in $\mathfrak{g}$, consequently, X may leave G.

Exact solutions, and therefore step-invariant transforms, are not unique to right- (or left-) invariant vector fields. For example, the ODE in the variable $X \in \mathfrak{g}$,

$$\dot{X} = \mathrm{ad}_A X = XA - AX$$

has the closed-form solution

$$X(t) = e^{\mathrm{ad}_{tA}} X(0),$$

where $e^{\mathrm{ad}_{tA}} := \mathrm{Id}_{\mathfrak{g}} + \mathrm{ad}_{tA} + \frac{1}{2!}\mathrm{ad}_{tA}^2 + \frac{1}{3!}\mathrm{ad}_{tA}^3 + \cdots$, which furnishes the step-invariant transform

$$X[k+1] = e^{\mathrm{ad}_{TA}} X[k] = X[k] + T[A, X[k]] + \frac{T^2}{2!}[A,[A,X[k]]] + \frac{T^3}{3!}[A,[A,[A,X[k]]]] + \cdots .$$

Other than the Euler discretization and Padé approximant, the local expressions one obtains via the matrix logarithm of all the sampled dynamics presented in this section are examples of Lie functions, in particular, they belong to class-$\mathcal{A}$, which we define in Section 3.1 and is the main class of systems studied in this thesis.

1.3 Relevant Stability Concepts

The main results of this thesis—Theorems 3.2.3, 3.3.3, 3.3.7, and Corollaries 3.3.6 and 3.3.9—assert that, if the spectral radius of the Jacobian linearization of the dynamics is sufficiently small, then various global stability properties of the origin are implied, the weakest and strongest of which, are global attractivity and global asymptotic stability of the origin, respectively. This is a rare property. Of course, Lyapunov's Second Method can be used to establish *local* stability of an equilibrium, and it is a strong and surprising result when this method establishes global stability for a class of dynamical systems. In continuous-time, Krasovskii's Method [56, p. 183] asserts that, given dynamics $\dot{x} = f(x)$, if there exists a symmetric positive definite $P \in \mathbb{R}^{n\times n}$, which, for all $x_0 \in \mathbb{R}^n$, solves the Lyapunov equation,

$$\left.\frac{\partial f}{\partial x}\right|_{x_0} P + P\left.\frac{\partial f}{\partial x}\right|_{x_0} = -Q, \tag{1.20}$$

where Q is positive definite, then the (unique) equilibrium is globally asymptotically stable. Again in the continuous-time case, the Markus-Yamabe Conjecture [75] supposes that global attractivity of a (unique) equilibrium is implied by the Jacobian of the vector field being everywhere Hurwitz; this conjecture is true for vector fields on $\mathbb{R}^2$, but is in general false. The Conjecture asserts only that (1.20) is everywhere solvable, whereas Krasovskii's Method asserts that there exists a P that solves (1.20) at all $x_0 \in \mathbb{R}^n$. The discrete-time analog of the Conjecture—the key difference being that it supposes that the Jacobian is everywhere Schur—similarly to the continuous-time case, is true for polynomial maps on $\mathbb{R}^2$ [21, Theorem B] and in general false on $\mathbb{R}^n$, $n \geq 3$. However, it is true for triangular maps on $\mathbb{R}^n$ [21, Theorem A].

The proofs of our main stability results in Chapter 3 leverage a structure of solvable Lie algebras that results in "subsystems" that are evocative of a cascade or triangular structure. Consider a dynamical system of the form

$$\begin{aligned} x[k+1] &= f(x[k], z[k]) \\ z[k+1] &= g(z[k]), \end{aligned} \tag{1.21}$$

which can be viewed as a cascade of a subsystem with state $z[k] \in \mathbb{R}^m$ driving another subsystem with state $x[k] \in \mathbb{R}^n$.

Theorem 1.3.1 ([106, Theorem 2])**.** *Suppose that the origin of the respective state spaces is a globally asymptotically stable equilibrium of $f(x, 0)$ and $g(z)$. If all trajectories of (1.21) are bounded, then the origin in $\mathbb{R}^n \times \mathbb{R}^m$ is a globally asymptotically stable equilibrium of (1.21).*

Thus, when a system's dynamics can be decomposed into a cascade connection of subsystems, it potentially greatly facilitates the stability analysis of the overall system. However, the foregoing theorem asserts that the trajectories of the system are bounded. If this is not the case, then global asymptotic stability of both subsystems is not sufficient for stability of the cascade. The *peaking phenomenon* [107] is the observation that when the trajectories of the driving system exhibit large transients, the state of a globally asymptotically stable driven system can be forced irreversibly far away from the origin.

The notion of *input-to-state stability (ISS)*, introduced by Sontag [98], has proven to be a powerful concept in the analysis of cascaded systems.

Definition 1.3.2 ([100, Definition 2.1])**.** *Given a C^1 map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the continuous-time dynamical system*

$$\dot{x} = f(x, u)$$

*is **input-to-state stable (ISS)** if there exists a class-$\mathscr{KL}$ function β and a class-$\mathscr{K}$ function γ such that for any bounded input $u \in \mathcal{L}_\infty^m$ and any initial condition $x(0)$,*

$$\|x(t)\| \leq \beta(x(0), t) + \gamma(\|u\|_\infty)$$

for all $t \geq 0$.

Intuitively, the ISS property asserts that the state of a system tends to zero if the input tends to zero. The function β bounds the transient behaviour, and γ bounds the steady-state behaviour effected by the input. As with most concepts in nonlinear control, ISS has been studied primarily in the continuous-time setting, however, there is an analogous discrete-time formulation [52]. In cascade, the state of one subsystem serves as the input to the subsequent subsystem. Thus, one would intuitively expect that if each subsystem is ISS, and the driving system is globally asymptotically stable, then the state of the entire system will be driven to zero. This is in fact the case: the cascade connection of a globally asymptotically stable subsystem driving an ISS system is globally asymptotically stable [99, §4].

The foregoing concepts and results are important and useful in the broader context of the stability and control of cascaded systems, but we take a different approach in our stability analysis in Chapter 3. We instead take the perspective that each subsystem is a "larger piece" of the overall system. Each subsystem is a quotient system induced by a subalgebra of the overall state space. The subalgebra inducing each driven system is strictly contained in the subalgebra that induces its driving system. We recognize a form to the dynamics of each subsystem that resembles a linear system with an input, and we explicitly bound the convergence of the state trajectories.

1.4 Notation and Terminology

Given $N \in \mathbb{N}$, let $\mathbb{N}_N \coloneqq \{1, \ldots, N\}$. Given a matrix $M \in \mathbb{C}^{n\times n}$, $M^\top$ is its (non-Hermitian) transpose. Let $\mathbf{1}_n \in \mathbb{R}^n$ denote the column vector of ones. The symbol I denotes the identity matrix; to avoid ambiguity, we will sometimes use I_n to refer to the $n \times n$ identity matrix. Let $\mathbb{R}^-$ denote the set of nonpositive real numbers. Given a set S, let $\mathrm{Id}_S : S \to S$ denote the identity map on S, and $|S|$ its cardinality. The disjoint union of two sets S_1 and S_2 is denoted by $S_1 \sqcup S_2$. Given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, let $A \otimes B \in \mathbb{C}^{mp\times nq}$ denote their (generally noncommutative) Kronecker product.

Given a set $\mathcal{X}$, a map $x : \mathbb{Z} \to \mathcal{X}$ is a *discrete-time signal*. The notation $x[k]$, with brackets, in contrast to parentheses, implies that the domain of x is the integers. The notation x and x^+ will often be used as shorthand for $x[k]$ and $x[k+1]$, respectively, when the time index is clear or irrelevant. When a discrete-time signal appears in a continuous-time expression, it is to be understood as having passed through an ideal zero order hold.

All vector spaces in this thesis are finite dimensional. The symbol 0 will be used to represent the additive identity on any vector space. Many of our results hold whether the Lie algebra is either a real or complex vector space. We will denote the field of $\mathfrak{g}$ by $\mathbb{F}$, which is either $\mathbb{C}$ or $\mathbb{R}$. Given a vector space $\mathcal{X}$ with subspace $\mathcal{V} \subseteq \mathcal{X}$, $\mathcal{X}/\mathcal{V}$ denotes the quotient (or factor) space with cosets $\bar{x} := \{v \in \mathcal{X} : x - v \in \mathcal{V}\}$; we will sometimes use the notation $x + \mathcal{V}$ for this same coset. If $\mathcal{T}$ is a Cartesian product of a vector space $\mathcal{X}$ with itself n times, and a subspace $\mathcal{V} \subseteq \mathcal{X}$, we will sometimes use the notation $\mathcal{T}/\mathcal{V}$ as shorthand for $\mathcal{T}/\mathcal{V}^n = \mathcal{X}^n/\mathcal{V}^n = (\mathcal{X}/\mathcal{V})^n$. Given an endomorphism of vector spaces $A : \mathcal{X} \to \mathcal{X}$, let $\sigma(A)$ denote its spectrum, including repeated eigenvalues, and $\rho(A)$ denote its spectral radius, and $\|A\|$ denote the operator norm induced by the vector norm $\|\cdot\|$ on $\mathcal{X}$; unless stated otherwise, the choice of norm is immaterial. Given an A-invariant subspace $\mathcal{V} \subseteq \mathcal{X}$, let $A|\mathcal{V} : \mathcal{V} \to \mathcal{V}$ denote the restriction of $A : \mathcal{X} \to \mathcal{X}$ to $\mathcal{V}$. Given vector spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, with respective norms $\|\cdot\|_{\mathcal{X}_1}, \dots, \|\cdot\|_{\mathcal{X}_n}$, we define the product norm on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ by $\|(X_1, \dots, X_n)\| := \sum_{i=1}^n \|X_i\|_{\mathcal{X}_i}$.

1.5 Organization and Contributions

Chapter 2 covers the preliminaries of the mathematics used throughout this thesis. Chapter 3 presents stability results for a class of discrete-time dynamics on solvable Lie algebras. The main results are Theorem 3.2.3, which asserts conditions for semiglobal exponential stability on nilpotent Lie algebras, and Theorem 3.3.3, which asserts conditions for global attractivity on solvable Lie algebras. The former furnishes Corollary 3.2.6, which asserts simpler conditions for semiglobal exponential stability on nilpotent Lie algebras when the exogenous signal is bounded, and the latter furnishes Corollary 3.3.6, which asserts conditions for global asymptotic stability on solvable Lie algebras. Another interesting result in this chapter is Theorem 3.3.7, which asserts conditions for finite-time convergence on solvable Lie algebras. Corollaries 3.2.6 and 3.3.6 are used to solve regulation and synchronization problems in Chapters 4 and 5, respectively. Chapter 4 presents a solution to the sampled-data synchronization problem for fully actuated kinematic systems on exponential matrix Lie groups; the main result is Theorem 4.4.8. In Chapter 5, we propose a solution to a sampled-data regulator problem for a class of fully actuated kinematic systems on nilpotent matrix Lie groups. The main results are Theorems 5.3.10 and 5.3.16, which assert conditions for regulation using full information and partial information, respectively. The former is proven by invoking Theorem 3.3.7. We briefly address how the regulator problem can be solved on solvable Lie groups.

Chapter 2

Mathematical Preliminaries

In this chapter, we introduce the mathematical concepts used in this thesis. Section 2.1 formalizes the concepts of Lie groups and Lie algebras. The Lie theoretic content of this chapter is is based primarily on Brian C. Hall's algebraic treatment of matrix Lie groups [41] and Veeravalli S. Varadarajan's much more general and abstract treatment [111], as well as the comprehensive, yet accessible trilogy by Vladimir V. Gorbatsevich, Arkadij L. Onishchik, and Ernest B. Vinberg [39]. Section 2.2 discusses the matrix exponential and matrix logarithm maps, which are fundamental in the study of Lie groups. In Section 2.3, we discuss the class of Lie groups and Lie algebras we focus on in this thesis: solvable and nilpotent. In Section 2.4, we review basic concepts from linear algebra, and establish several lemmas that are used in this thesis.

2.1 Lie Groups and Lie Algebras

Definition 2.1.1 ([81, p. 145] Topological Group)**.** *If G is a group that is also a topological space, such that sets of finitely many points are closed, and the group product and group inverse operations are continuous, then G is called a **topological group**.*

Definition 2.1.2 ([111, §2.1] Lie Group)**.** *A **Lie group** is a topological group that is also an analytic manifold*

Remark 2.1.3. It is common to define a Lie group as having only a smooth structure [61, 16]. However, any such Lie group can be equipped with an analytic structure [89, §53]. If the Lie group is real, then this analytic structure is unique [111, Theorem 2.11.3]. ♦

Example 2.1.1. (The General Linear Group $\mathsf{GL}(n,\mathbb{F})$) Let $\mathsf{GL}(n,\mathbb{F})$ denote the set of invertible $n \times n$ matrices over the field $\mathbb{F}$. This collection of matrices, equipped with the operation of matrix multiplication, is a group, and further, it is a Lie group. By definition, if $A \in \mathsf{GL}(n,\mathbb{F})$, then A is invertible. Clearly, its inverse A^{-1} is also invertible, and therefore is also in $\mathsf{GL}(n,\mathbb{F})$. Using the property of determinants that $\det(AB) = \det(A)\det(B)$, it is easy to see that the product of invertible matrices is also invertible, and therefore $\mathsf{GL}(n,\mathbb{F})$ is closed under matrix multiplication. The identity matrix serves as the identity element. Therefore, $\mathsf{GL}(n,\mathbb{F})$ is a group with matrix multiplication as its group operation. That $\mathsf{GL}(n,\mathbb{F})$ is a Lie group follows from matrix inversion and multiplication being analytic in the entries of the matrices. The Lie group $\mathsf{GL}(n,\mathbb{F})$ is called the *general linear group.* ▲

The general linear group $\mathsf{GL}(n,\mathbb{F})$ is an example of a *matrix Lie group.*

Definition 2.1.4 ([41, Definition 1.4] Matrix Lie Group)**.** *A subgroup $\mathsf{G} \subseteq \mathsf{GL}(n,\mathbb{F})$ is a **matrix Lie group** if the limit of every convergent sequence in G is either in G or not in $\mathsf{GL}(n,\mathbb{F})$.*

Example 2.1.2. (Nonconvergent sequence in $\mathsf{GL}(n,\mathbb{F})$) Consider the sequence $\{X_k\} \subset \mathsf{GL}(n,\mathbb{F})$, where $X_k = \frac{1}{k}I$. For all $k \in \mathbb{N}$, $X_k \in \mathsf{GL}(n,\mathbb{F})$, but $\lim_{k\to\infty} X_k = 0 \notin \mathsf{GL}(n,\mathbb{F})$.

Despite this sequence, one can verify that the set of invertible diagonal matrices, equipped with the operation of matrix multiplication, is a matrix Lie group. This is because the limit, which is not an invertible diagonal matrix, is not in $\mathsf{GL}(n,\mathbb{F})$, by virtue of not being invertible. ▲

Definition 2.1.4 is equivalent to asserting that G be closed as a subset of $\mathsf{GL}(n,\mathbb{F})$; it does not require that G be closed as a subset of $\mathbb{F}^{n\times n}$. The Lie group G is closed in the subspace topology if and only if G is an embedded submanifold of $\mathbb{F}^{n\times n}$.

Definition 2.1.5 ([111, §2.2] Lie Algebra)**.** *A vector space $\mathfrak{g}$ over $\mathbb{F}$ is a **Lie algebra** if there exists a binary operator $[\cdot,\cdot]$ on $\mathfrak{g}$, called the **Lie bracket**, satisfying*

1. *(bilinearity) for all $x, y, z \in \mathfrak{g}$ and all $\alpha, \beta \in \mathbb{F}$,*

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$$
$$[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z];$$

2. *(skew-symmetry) for all $x, y \in \mathfrak{g}$,*

$$[x, y] = -[y, x];$$

3. *(Jacobi identity) for all $x, y, z \in \mathfrak{g}$,*

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0.$$

Example 2.1.3. (The General Linear Algebra $\mathfrak{gl}(n,\mathbb{F})$) The space of $n \times n$ matrices over the field $\mathbb{F}$ equipped with the *matrix commutator* $[A,B] := AB - BA$ is a Lie algebra, called the *general linear algebra* $\mathfrak{gl}(n,\mathbb{F})$. ▲

Example 2.1.4. (The Special Euclidean Algebra $\mathfrak{se}(2)$) The *special Euclidean algebra* has basis

$$e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose Lie bracket is the matrix commutator, yielding the nonvanishing Lie brackets $[e_1, e_3] = -e_2$ and $[e_2, e_3] = e_1$. ▲

Definition 2.1.6 ([111, §2.2, p. 49] Linear Representation)**.** *A **representation of $\mathfrak{g}$ in a vector space** V is a map $\pi : \mathfrak{g} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the vector space of endomorphims of V, such that*

1. *π is linear;*
2. *for all $x, y \in \mathfrak{g}$, $\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$.*

If a representation π is injective, then it is said to be **faithful**. When a Lie algebra is finite-dimensional, the definition of a representation is equivalent to a morphism of Lie algebras $\pi : \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{F})$, where the Lie bracket of $\mathfrak{gl}(n,\mathbb{F})$ is the **matrix commutator**

$$[x,y] = xy - yx.$$

Theorem 2.1.7 ([39, Theorem 5.3] Ado's Theorem)**.** *Any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ admits a faithful finite-dimensional linear representation over $\mathbb{F}$.*

Ado's Theorem greatly facilitates the study and use of finite-dimensional Lie algebras, as it allows us to assume, without loss of generality, that the Lie algebra of interest is a matrix Lie algebra.

Definition 2.1.8 (Lie Algebra of a Lie Group)**.** ***The Lie algebra** $\mathfrak{g}$ **of the Lie group** G is the tangent space to G at the identity element.*

Example 2.1.5. ([93, **Example 2.12**]) The Lie algebra of $\mathsf{GL}(n, \mathbb{F})$ is $\mathfrak{gl}(n, \mathbb{F})$. For arbitrary $A \in \mathfrak{gl}(n, \mathbb{F})$, define the curve $X(t) = I + tA$. Then $X(t) \in \mathsf{GL}(n, \mathbb{F})$ for t sufficiently small, $X(0) = I$, and $\dot{X}(0) = A$. Since $A \in \mathfrak{gl}(n, \mathbb{F})$ was arbitrary, the tangent space at $I \in \mathsf{GL}(n, \mathbb{F})$ is all of $\mathfrak{gl}(n, \mathbb{F})$. ▲

Definition 2.1.9 (Adjoint Operators). *Given $X \in \mathsf{G}$, its **adjoint operator** is an action of the manifold G on its tangent space $\mathfrak{g}$, equal to the differential of the conjugate operator $Y \mapsto XYX^{-1}$. If G is a matrix Lie group, then the definition reduces to*

$$\begin{aligned} \mathrm{Ad}_X : \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto XyX^{-1}. \end{aligned}$$

*Given $x \in \mathfrak{g}$, its **adjoint operator** is*

$$\begin{aligned} \mathrm{ad}_x : \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto [x, y]. \end{aligned}$$

Definition 2.1.10 ((Lie) Subalgebra of a Lie Algebra). *Given a Lie algebra $\mathfrak{g}$, a subset $\mathfrak{h} \subseteq \mathfrak{g}$ is a **subalgebra** if $\mathfrak{h}$ is a Lie algebra with the Lie bracket inherited from $\mathfrak{g}$.*

Given two Lie subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$, $[\mathfrak{h}_1, \mathfrak{h}_2] := \{[H_1, H_2] \in \mathfrak{g} : H_1 \in \mathfrak{h}_1, H_2 \in \mathfrak{h}_2\}$.

Definition 2.1.11 (Ideal of a Lie Algebra). *Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is an **ideal** if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.*

Definition 2.1.12 (Centre of a Lie Algebra). *Given a Lie algebra $\mathfrak{g}$, its **centre** $\mathfrak{z}$ is the maximal subalgebra (in terms of subspace inclusion) such that $[\mathfrak{z}, \mathfrak{g}] = 0$.*

Given a Lie group G, let $\mathfrak{g}$ be its associated Lie algebra; more generally, a Lie group and its associated Lie algebra are denoted by sans-serif majuscule letters and the corresponding Fraktur minuscule letters, respectively. A *word* $\omega \in \mathfrak{g}$ with *length* $|\omega| \in \mathbb{N}$ over the $n \in \mathbb{N}$ *letters* $X_1, \ldots, X_n \in \mathfrak{g}$ is a (nested) Lie bracket $[X_{\omega_1}, [X_{\omega_2}, [\ldots X_{\omega_{|\omega|}}] \cdots]$, where $X_{\omega_i} \in \{X_1, \ldots, X_n\}$.

2.2 The Matrix Exponential and Logarithm

Of fundamental importance in the study of Lie groups and Lie algebras is the exponential map $\exp : \mathfrak{g} \to \mathsf{G}$, which for matrix Lie groups is given by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Definition 2.2.1 ([41, Definition 3.18] Lie Algebra of a Matrix Lie Group)**.** ***The Lie algebra*** $\mathfrak{g}$ ***of the matrix Lie group*** G *is the set of all matrices* x *such that, for all* $t \in \mathbb{R}$, $\exp(tx) \in \mathsf{G}$*, equipped with the matrix commutator as the Lie bracket.*

Example 2.2.1. Based on this most recent definition, it is obvious that the Lie algebra of $\mathsf{GL}(n, \mathbb{F})$ is $\mathfrak{gl}(n, \mathbb{F})$, since the exponential of any matrix is invertible. ▲

The exponential map is invertible in a neighbourhood of the origin of $\mathfrak{g}$.

Theorem 2.2.2 ([44, Theorem 1.31])**.** *Let* $X \in \mathbb{C}^{n \times n}$ *have no eigenvalues in* $\mathbb{R}^-$*. There is a unique logarithm* $A \in \mathbb{C}^{n \times n}$ *of* X*, all of whose eigenvalues lie in the strip* $\{z : -\pi < \operatorname{Im}(z) < \pi\}$*. If* $X \in \mathbb{R}^{n \times n}$*, then* $A \in \mathbb{R}^{n \times n}$*.*

The unique matrix A from Theorem 2.2.2 is called the **principal logarithm of** X and is denoted by $\operatorname{Log}(X)$. If $\|X - I\| < 1$, then

$$\operatorname{Log}(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (X - I)^k.$$

Given a Lie group G, the principal logarithm is well-defined only on some neighbourhood of the identity $U \subseteq \mathsf{G}$, where in general $U \neq \mathsf{G}$. For example, $\operatorname{Log}(X)$ is guaranteed to be well-defined for all $X = \exp(A)$ such that $\|A\| < \operatorname{Log}(2)$. Wherever $\operatorname{Log} : \mathsf{G} \to \mathfrak{g}$ is well-defined, it is the inverse of $\exp : \mathfrak{g} \to \mathsf{G}$. Hereinafter, the symbol U always denotes the neighbourhood of the identity in G on which Log is well-defined.

The Baker-Campbell-Hausdorff (BCH) formula relates the product of two elements on the Lie group G to an analytic function of their principal logarithms. If $A, B \in \mathfrak{g}$, then the BCH formula has the series representation

$$\operatorname{Log}(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \cdots,$$

where the remaining terms are nested brackets of increasing order [41, §5.6]. The convergence of the BCH series is the subject of much study [8], but for our purposes, we will be content with the fact that there is always some nonempty neighbourhood of the origin on which the series converges.

The BCH formula can be generalized to arbitrary finite products [25, §5]. Given $A_1, \ldots, A_n \in \mathfrak{g}$,

$$\operatorname{Log}(\exp(A_1) \cdots \exp(A_n)) = \sum_{i=1}^{n} A_i + \frac{1}{2} \sum_{i<j} [A_i, A_j] + \cdots, \tag{2.1}$$

where the coefficients for third- and higher-order terms are increasingly complicated, specifically, not all words of the same length have the same coefficient. It is easy to see that the linearization of (2.1) at the origin of $\mathfrak{g}^n$ is

$$\operatorname{Log}(\exp(A_1)\cdots\exp(A_n)) \approx A_1 + \cdots + A_n. \tag{2.2}$$

We will use (2.1) and (2.2) extensively in Chapters 4 and 5.

Given a Lie group G with Lie algebra $\mathfrak{g}$, if $\exp : \mathfrak{g} \to \mathsf{G}$ is a global diffeomorphism, then G is said to be **exponential**. The following theorem characterizes when a simply connected Lie group enjoys the rather strong property of being exponential.

Theorem 2.2.3 ([39, Theorem 6.4])**.** *If G is simply connected, then the following are equivalent:*

1. $\exp : \mathfrak{g} \to \mathsf{G}$ *is a global diffeomorphism;*
2. $\exp : \mathfrak{g} \to \mathsf{G}$ *is globally injective;*
3. $\exp : \mathfrak{g} \to \mathsf{G}$ *is locally injective;*
4. *for all $x \in \mathfrak{g}$, ad_x has no nonzero imaginary eigenvalues;*
5. *for all $X \in \mathsf{G}$, Ad_X has no nonunity eigenvalues of unit modulus;*
6. *there is no ideal $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathfrak{se}(2)$.*

Remark 2.2.4. If G is exponential, then $U = \mathsf{G}$. ♦

Borrowing from the definition of complex powers of scalars [59, §III.6], we define complex powers of a matrix, which we will use in Chapter 4.

Definition 2.2.5 (Matrix Power)**.** *Suppose $X \in \mathbb{C}^{n\times n}$ has no eigenvalues in $\mathbb{R}^-$. If $\alpha \in \mathbb{C}$, then $X^\alpha := \exp(\alpha \operatorname{Log}(X))$.*

2.3 Solvability and Nilpotency

In this thesis we study dynamics on solvable Lie algebras. A Lie algebra is *solvable* if and only if its derived length (see Definition 2.3.2) is finite. The complementary classification of Lie algebras is called *semi-simple*, which is defined as those Lie algebras whose maximal solvable ideal—the *radical*—is zero. Any Lie algebra $\mathfrak{g}$ admits a *Levi decomposition*, $\mathfrak{g} = \mathfrak{l} \Subset \mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$, $\mathfrak{l}$ is a semi-simple subalgebra of $\mathfrak{g}$, and $\Subset$ means *semidirect sum*.[1] This establishes that solvable Lie algebras are of fundamental importance in Lie theory. Of particular interest in this thesis, is that any exponential Lie group is solvable [39, Theorem 6.3].

Definition 2.3.1 (Derived Series)**.** *The **derived series** of a Lie algebra* $\mathfrak{g}$ *is defined recursively by* $\mathfrak{g}_0 := \mathfrak{g}$, $\mathfrak{g}_{i+1} := [\mathfrak{g}_i, \mathfrak{g}_i]$, *for* $i \geq 0$.

A consequence of the definition of $\mathfrak{g}_i$ is that for all $i \geq 0$, $\mathfrak{g}_i \supseteq \mathfrak{g}_{i+1}$.

Definition 2.3.2 (Solvable)**.** *A Lie algebra* $\mathfrak{g}$ *is **solvable** if there exists a finite* v *such that* $\mathfrak{g}_{v+1} = 0$. *The smallest such* v *is called the **derived length** of* $\mathfrak{g}$. *A Lie group is solvable if its Lie algebra is solvable.*

If $\mathfrak{g}$ is solvable with derived length v, then for all $i \leq v$, the containment $\mathfrak{g}_i \supset \mathfrak{g}_{i+1}$ is strict.

Example 2.3.1. Consider the 6-dimensional real upper triangular algebra, whose nonvanishing Lie brackets are given by

$$[e_1, e_4] = e_4,\ [e_1, e_6] = e_6,\ [e_2, e_4] = -e_4,\ [e_2, e_5] = e_5,$$
$$[e_3, e_5] = -e_5,\ [e_3, e_6] = -e_6,\ [e_4, e_5] = e_6.$$

It follows that

$$\mathfrak{g}_2 = \mathrm{Lie}_{\mathbb{R}}\{e_4, e_5, e_6\}$$
$$\mathfrak{g}_3 = \mathrm{span}_{\mathbb{R}}\{e_6\},$$

and its derived length is 3. ▲

Definition 2.3.3 (Lower Central Series)**.** *The **lower central series** of a Lie algebra* $\mathfrak{g}$ *is defined recursively by* $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}]$, *for* $i \geq 1$.

There are two important consequences of Definition 2.3.3: the algebras of the lower central series $\mathfrak{g}^{(i)}$ are ideals, and for all $i \geq 1$, $\mathfrak{g}^{(i)} \supseteq \mathfrak{g}^{(i+1)}$.

Definition 2.3.4 (Nilpotent)**.** *A Lie algebra* $\mathfrak{g}$ *is **nilpotent** if there exists a finite* p *such that* $\mathfrak{g}^{(p+1)} = 0$. *The smallest such* p *is called the **nilindex** of* $\mathfrak{g}$. *A Lie group is nilpotent if its Lie algebra is nilpotent.*

[1] A detailed treatment of this decomposition can be found in, for example, [39, §4] or [111, §3.14].

The property that serves as the foundation of our analysis, is that if $\mathfrak{g}$ is nilpotent, then

$$\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(p)} \supset \mathfrak{g}^{(p+1)} = 0.$$

Example 2.3.2. The Heisenberg algebra $\mathfrak{h}$ has basis vectors e_1, e_2, and e_3, whose nonzero commutator relations are given by $[e_1, e_2] = e_3$. Therefore, $\mathfrak{h}^{(2)} = \mathrm{span}_{\mathbb{R}}\{e_3\}$, and $p = 2$. ▲

Lemma 2.3.5 ([22, Lemma 1.1.1])**.** *The ideals of the lower central series of a Lie algebra* $\mathfrak{g}$ *satisfy* $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{(i+j)}$.

Although Definition 2.3.2 is the formal definition of solvability, it is the structure endowed by the following theorem that will be leveraged in our analysis.

Theorem 2.3.6 ([39, p. 9, Corollary 3])**.** *A Lie algebra* $\mathfrak{g}$ *over* $\mathbb{F}$ *is solvable if and only if its* ***derived algebra*** $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Theorem 2.3.7 ([22, Theorem 1.2.1])**.** *If* G *is a simply connected nilpotent Lie group with Lie algebra* $\mathfrak{g}$*, then* $\exp : \mathfrak{g} \to \mathsf{G}$ *is an analytic diffeomorphism.*

2.4 Linear Algebraic Results

In the proofs of our main results in Chapter 3, we examine the quotient dynamics on the quotient spaces modulo the ideals of the lower central series. To that end, we require the notion of canonical projection.

Definition 2.4.1 (Canonical Projection)**.** *Let* $\mathcal{X}$ *be a vector space with subspace* $\mathcal{V} \subseteq \mathcal{X}$*. The* ***canonical projection*** *of* $\mathcal{X}$ *onto* $\mathcal{V}$ *is the unique linear map* $P : \mathcal{X} \to \mathcal{X}/\mathcal{V}$, $x \mapsto x + \mathcal{V}$.

Proposition 2.4.2 ([117, §0.7])**.** *Given a linear map* $A : \mathcal{X} \to \mathcal{X}$ *and an* A*-invariant subspace* $\mathcal{V} \subseteq \mathcal{X}$*, i.e.,* $A\mathcal{V} \subseteq \mathcal{V}$*, there exists a unique linear map* $\bar{A} : \mathcal{X}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ *such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "A"] \arrow[d, "P"'] & \mathcal{X} \arrow[d, "P"] \\ \mathcal{X}/\mathcal{V} \arrow[r, "\bar{A}"'] & \mathcal{X}/\mathcal{V} \end{tikzcd}$$

The map $\bar{A}$ in Proposition 2.4.2 is called the *map induced in $\mathcal{X}/\mathcal{V}$ by A*, or in short, the *induced map*.

Lemma 2.4.3. *Let $\mathcal{X}$ be a vector space with subspace $\mathcal{V} \subseteq \mathcal{X}$ and $P : \mathcal{X} \to \mathcal{X}/\mathcal{V}$ be the canonical projection. For any right-inverse $\imath : \mathcal{X}/\mathcal{V} \to \mathcal{X}$ of P, $(\mathrm{Id}_{\mathcal{X}} - \imath \circ P)\mathcal{X} \subseteq \mathcal{V}$.*

Proof. $P(\mathrm{Id}_{\mathcal{X}} - \imath \circ P) = P - P \circ \imath \circ P = P - P = 0$, which implies $(\mathrm{Id}_{\mathcal{X}} - \imath \circ P)\mathcal{X} \subseteq \operatorname{Ker} P$. □

Definition 2.4.4 (Quotient Norm)**.** *Given a vector space $\mathcal{X}$ with norm $\|\cdot\|$ and subspace $\mathcal{V} \subseteq \mathcal{X}$, if $x \in \mathcal{X}$, then the **quotient norm** of the coset $x + \mathcal{V}$ is*

$$\|x + \mathcal{V}\|_{\mathcal{X}/\mathcal{V}} := \inf_{v \in \mathcal{V}} \|x + v\|.$$

The following result is an obvious consequence of Definition 2.4.4. We formally state it because it is important in the proofs of our main results.

Lemma 2.4.5. *Let $\mathcal{X}$ be a normed vector space with subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. For all $x \in \mathcal{X}$, $\|x + \mathcal{V}_2\|_{\mathcal{X}/\mathcal{V}_2} \leq \|x + \mathcal{V}_1\|_{\mathcal{X}/\mathcal{V}_1} \leq \|x\|$.*

The following result is elementary, but we state and prove it for completeness, and will use it in our analysis.

Proposition 2.4.6. *Let $\mathcal{X}$ be a vector space with norm $\|\cdot\|$, and let $\mathcal{V} \subseteq \mathcal{X}$ be a subspace. If the quotient norm is used on $\mathcal{X}/\mathcal{V}$, then the canonical projection $P : \mathcal{X} \to \mathcal{X}/\mathcal{V}$ has unit norm.*

Proof. Beginning with the definition of operator norm, we have

$$\begin{aligned}
\|P\| :&= \max_{\|x\|=1} \inf_{v \in \mathcal{V}} \|x + v\| \\
&\leq \max_{\|x\|=1} \inf_{v \in \mathcal{V}} \{\|x\| + \|v\|\} \\
&= \max_{\|x\|=1} \|x\| \\
&= 1,
\end{aligned}$$

which establishes an upper bound of 1.

Consider a vector $x \in \mathcal{X}$, $x \notin \mathcal{V}$. For all $v \in \mathcal{V}$

$$\begin{aligned} & \|Px\|_{\mathcal{X}/\mathcal{V}} = \|P(x+v)\|_{\mathcal{X}/\mathcal{V}} \leq \|P\|\|x+v\| \\ \implies \quad & \|Px\|_{\mathcal{X}/\mathcal{V}} \leq \|P\| \underbrace{\inf_{v\in\mathcal{V}} \|x+v\|}_{\|Px\|_{\mathcal{X}/\mathcal{V}}} \\ \iff \quad & 1 \leq \|P\|. \end{aligned}$$

This establishes a lower bound of 1, and so $\|P\| = 1$. □

Throughout the majority of this thesis, any choice of norm is immaterial. However, in some specific circumstances, a norm of the class described in the following theorem will be used.

Theorem 2.4.7 ([28, §7]). *Given a linear map $A : \mathcal{X} \to \mathcal{X}$ and a constant $\varepsilon > 0$, there exists a vector norm $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ such that the induced operator norm satisfies $\|A\| < \rho(A) + \varepsilon$.*

Remark 2.4.8. Given a matrix Lie algebra $\mathfrak{g}$ with norm $\|\cdot\|$, there exists $\mu \in [0, 2]$, such that for all $X, Y \in \mathfrak{g}$, $\|[X, Y]\| \leq \mu\|X\|\|Y\|$.

The lower bound of 0 holds when $\mathfrak{g}$ is commutative, and the upper bound of 2 is verified by the triangle inequality and submultiplicativity of induced norms:

$$\|[X, Y]\| = \|XY - YX\| \leq \|X\|\|Y\| + \|Y\|\|X\| = 2\|X\|\|Y\|.$$

The constant μ is not necessarily either 0 or 2. For example, if $\mathfrak{g}$ is any matrix Lie algebra equipped with the Frobenius norm, then $\mu = \sqrt{2}$ [11, Theorem 2.2]. ♦

Chapter 3

Stability on Solvable Lie Algebras

In this chapter, we examine the stability of a class of difference equations that arises by sampling a right- or left-invariant flow on a matrix Lie group. The map defining such a difference equation has three key properties that facilitate our analysis: 1) its Lie series expansion enjoys a type of strong convergence; 2) the origin is an equilibrium; 3) the algebraic ideals enumerated in the lower central series of the Lie algebra are dynamically invariant. We show that certain global stability properties are implied by stability of the Jacobian linearization of dynamics at the origin, in particular, global asymptotic stability. If the Lie algebra is nilpotent, then the origin enjoys semiglobal exponential stability, as characterized in Theorem 3.2.3. In the most general case, when the Lie algebra is solvable, then the origin is globally attractive, as characterized in Theorem 3.3.3. Under certain additional hypotheses, if the linearization has spectral radius zero, then even on a solvable Lie algebra, the state converges to the origin in finite time, as characterized in Theorem 3.3.7.

3.1 The Class of Systems

Recall that the right-invariant kinematic model (1.1) admits an exact solution, thereby furnishing the step invariant transform (1.17), which is a product of elements on the Lie group G. Applying the Baker-Campbell-Hausdorff formula (2.2) to the step invariant transform (1.17), we express the sampled dynamics on the Lie algebra:

$$\begin{aligned}\mathrm{Log}(X[k+1]) = \mathrm{Log}(X[k]) + \int_0^T A(\tau, u[k])\mathrm{d}\tau + \frac{1}{2}\left[\int_0^T A(\tau, u[k])\mathrm{d}\tau, \mathrm{Log}(X[k])\right] \\ + \frac{1}{12}\left[\int_0^T A(\tau, u[k])\mathrm{d}\tau, \left[\int_0^T A(\tau, u[k])\mathrm{d}\tau, \mathrm{Log}(X[k])\right]\right] \\ + \frac{1}{12}\left[\mathrm{Log}(X[k]), \left[\mathrm{Log}(X[k]), \int_0^T A(\tau, u[k])\mathrm{d}\tau\right]\right] + \cdots,\end{aligned}$$

which is a linear combination of words of all lengths. These dynamics are a function of the state X and an exogenous signal u. In this thesis, we study a generalization of these dynamics. We allow for multiple plant states $X = (X_1, \ldots, X_n) \in \mathfrak{g}^n$, and multiple exogenous signals $W = (W_1, \ldots, W_r) \in \mathfrak{g}^r$. We study the dynamics of X, namely

$$X^+ = f(X, W), \tag{3.1}$$

where $X \in \mathcal{X} := \mathfrak{g}^n$, $n \geq 1$, $W \in \mathcal{W} := \mathfrak{g}^r$, $r \geq 0$, and $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ is a *Lie function* that belongs to *class* $\mathcal{A}$, which we define in this section. We make no general assumptions on the evolution of W. To state the assumptions we impose on (3.1), we must first introduce several key concepts.

Definition 3.1.1 (Lie Element)**.** *Let $X_1, \ldots, X_n$ be elements of a Lie algebra $\mathfrak{g}$. The elements $X_1, \ldots, X_n$ are called **Lie elements (in** $\{X_1, \ldots, X_n\}$) of degree one. The Lie brackets $[X_i, X_j]$ are Lie elements of degree two, $[X_i, [X_j, X_k]]$ Lie elements of degree three, and so forth. Any $\mathbb{F}$-linear combination of Lie elements—not necessarily finite or convergent—is also a Lie element.*

Definition 3.1.2. *A map $f : \mathfrak{g}^n \to \mathfrak{g}$ is a **Lie function** if there exists a domain $\mathcal{D} \subseteq \mathfrak{g}^n$ containing the origin, and a Lie element $E \in \mathcal{D}$ in $\{X_1, \ldots, X_n\}$, such that, for all $X \in \mathcal{D}$, $f(X) = E$; the Lie element E is called the **Lie series of** f **on the domain** $\mathcal{D}$. A product map $f_1 \times \cdots \times f_m : \mathfrak{g}^n \to \mathfrak{g}^m$ is a Lie function if each component map is a Lie function.*

We now develop a convenient and succinct expression for product Lie functions. Define the tensor product $(\mathbb{F}^n \otimes \mathfrak{g}, \otimes)$. If $f_1, \ldots, f_m$ are Lie functions, whose scalar coefficients of the word ω are respectively $c^1_\omega, \ldots, c^m_\omega \in \mathbb{F}$, where $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$, then

$$f(X_1, \ldots, X_n) := \begin{bmatrix} f_1(X_1, \ldots, X_n) \\ \vdots \\ f_m(X_1, \ldots, X_n) \end{bmatrix} = \begin{bmatrix} \sum_\omega c^1_\omega \omega \\ \vdots \\ \sum_\omega c^m_\omega \omega \end{bmatrix} = \sum_\omega \begin{bmatrix} c^1_\omega \\ \vdots \\ c^m_\omega \end{bmatrix} \otimes \omega,$$

which we write compactly as

$$f(X) = \sum_{\omega} c_\omega \otimes \omega. \tag{3.2}$$

Given $f : \mathfrak{g}^n \to \mathfrak{g}$, the following theorem can be used to test whether it is a Lie function.

Theorem 3.1.3 (Friedrichs' Theorem [68, Theorem 1])**.** *A map $f : \mathfrak{g}^n \to \mathfrak{g}$ equals a Lie element if and only if, for all $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathfrak{g}$ such that for all i, j, $[X_i, Y_j] = 0$,*

$$f(X_1 + Y_1, \ldots, X_n + Y_n) = f(X_1, \ldots, X_n) + f(Y_1, \ldots, Y_n).$$

We refine the class of functions considered in the following definition. We consider systems whose dynamical maps are Lie functions, but we also impose that they enjoy a strong form of convergence, as characterized in the following definition.

Definition 3.1.4 (Class-$\mathcal{A}$ Function)**.** *Let $\mathfrak{g}$ be a normed Lie algebra, and let $\mu > 0$ be such that, for all $X, Y \in \mathfrak{g}$, $\|[X, Y]\| \leq \mu \|X\| \|Y\|$. A Lie function $f : \mathfrak{g}^n \to \mathfrak{g}$ belongs to **class**-$\mathcal{A}$—which we write as $f \in \mathcal{A}$—if there exists a neighbourhood of the origin in $\mathfrak{g}^n$ where the Lie series of f satisfies the strong absolute convergence property*

$$\sum_{\omega} \mu^{|\omega|-1} |c_\omega| \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| < \infty. \tag{3.3}$$

A product map $f_1 \times \cdots \times f_m : \mathfrak{g}^{n_1} \times \cdots \times \mathfrak{g}^{n_m} \to \mathfrak{g}^m$ belongs to class-$\mathcal{A}$ if each component map belongs to class-$\mathcal{A}$.

Remark 3.1.5. Property (3.3), enjoyed by $f \in \mathcal{A}$, is stronger than absolute convergence, i.e., $\sum_\omega |c_\omega| \|\omega\| < \infty$, since $\|\omega\| \leq \mu^{|\omega|-1} \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\|$. ♦

Remark 3.1.6. By the Baker-Campbell-Hausdorff formula (2.2), we have that the map

$$\mathrm{Log}(\exp(X)\exp(Y))$$

belongs to class-$\mathcal{A}$. To see that (2.2) satisfies (3.3), refer to [25, Proof of Theorem 8] or [8], and the references therein. That the BCH is class-$\mathcal{A}$ precipitates out of the proofs the various characterizations of its regions of convergence. ♦

Remark 3.1.7. That $\mathrm{Log}(\exp(X)\exp(Y))$ belongs to class-$\mathcal{A}$ means that the sampled-data dynamics of a system on a matrix Lie group of the form (1.17) have local dynamics that are class-$\mathcal{A}$, which, as discussed in Chapter 1 and the beginning of this section, motivates the study of this class of systems. ♦

Proposition 3.1.8. *If the product map* (3.2) *belongs to class-$\mathcal{A}$, then*

$$\sum_{\omega} \mu^{|\omega|-1} \|c_\omega\| \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| < \infty.$$

Proof. By definition, $f \in \mathcal{A}$ implies $f_i \in \mathcal{A}$, which means that for all $i \in \{1, \dots, m\}$,

$$\sum_{\omega} \mu^{|\omega|-1} |c^i_\omega| \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| < \infty. \tag{3.4}$$

Summing (3.4) over $1 \leq j \leq m$:

$$\sum_{\omega} \mu^{|\omega|-1} \|c_\omega\|_1 \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| < \infty,$$

where $\| \cdot \|_1$ is the 1-norm. On a finite dimensional vector space, all norms are equivalent, so this summation differs from that in the proposition by at most a constant, finite factor γ, i.e., $\|c_\omega\| \leq \gamma \|c_\omega\|_1$, which implies

$$\sum_{\omega} \mu^{|\omega|-1} \|c_\omega\| \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| \leq \gamma \sum_{\omega} \mu^{|\omega|-1} \|c_\omega\|_1 \|X_{\omega_1}\| \cdots \|X_{\omega_{|\omega|}}\| < \infty.$$

□

If the Lie algebra $\mathfrak{g}$ is nilpotent, then only finitely many words are nonzero; consequently (3.5) trivially satisfies the class-$\mathcal{A}$ convergence property (3.3) globally. The function (3.2) can be written in the form (3.1) by relabeling r of the variables as $W \in \mathfrak{g}^r$ and redefining X as the remaining variables. We now impose the major structural assumption on the class of systems (3.1) under consideration.

Assumption 1. *The function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ in* (3.1) *enjoys the following properties:*

(a) f belongs to class-$\mathcal{A}$;

(b) the origin of the state-space $\mathcal{X}$ is a unique equilibrium,

$$f(X, \mathcal{W}) = 0 \iff X = 0;$$

(c) there exists an ideal $\mathfrak{h} \subseteq \mathfrak{g}$ with nilindex p, such that $\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}]$, whereof each ideal in the lower central series of $\mathfrak{h}$, $\left(\mathfrak{h}^{(i)}\right)^n \subseteq \mathfrak{g}^n$ is invariant under f, i.e.,

$$f\left(\left(\mathfrak{h}^{(i)}\right)^n, \mathcal{W}\right) \subseteq \left(\mathfrak{h}^{(i)}\right)^n.$$

Remark 3.1.9. Assumption 1(c) may seem restrictive, however, in the context of control theory, it is not unreasonable, because the control signal can be used to enforce invariance. Consider, for example, the step-invariant transform of the driftless kinematics of a fully actuated rigid body with velocity inputs on the solvable Lie group $\mathsf{SE}(2)$:

$$X[k+1] = \exp\left(T\left(\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} u_1[k] + \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} u_2[k] + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} u_3[k]\right)\right) X[k],$$

where $X \in \mathsf{SE}(2)$, $u_1, u_2, u_3 \in \mathbb{R}$, $T > 0$. The inputs u_1, u_2, u_3 can be chosen to make any subspace of $\mathfrak{se}(2)$ invariant under the local dynamics. ♦

Define the notation $\widetilde{X} := \{X_1, \ldots, X_n\}$ and $\widetilde{W} := \{W_1, \ldots, W_r\}$. Henceforth, we adopt the convention that summations over ω are restricted to words of length at least 2; words of length 1 will be written separately, in particular, under Assumption 1, the dynamics (3.1) can be written as

$$f(X, W) = AX + BW + \sum_{\omega} c_\omega \otimes \omega, \tag{3.5}$$

where $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{W} \to \mathcal{X}$ are linear maps, ω is a word with letters in $\widetilde{X} \cup \widetilde{W}$, and $c_\omega \in \mathbb{F}^n$ is the vector of coefficients of ω in the series representation of each component function f_i.

Proposition 3.1.10. *If the function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ in (3.1) is a Lie function that satisfies Assumption 1(b), then every word in the series of f has at least one letter in $\widetilde{X}$.*

Proof. By bilinearity of the Lie bracket, all words with at least one letter in $\widetilde{X}$ vanish at $X = 0$. Setting $X = 0$ in (3.5) yields

$$0 = BW + \sum_{\omega \text{ with no letters in } \widetilde{X}} c_\omega \otimes \omega, \tag{3.6}$$

which holds identically for all $W \in \mathcal{W}$. □

Therefore, without loss of generality, we can take B and the coefficients of all words ω with no letters in $\widetilde{X}$ to be zero. By Proposition 3.1.10, henceforth, systems that satisfy Assumption 1 will be written:

$$X^+ = AX + \sum_{\omega} c_\omega \otimes \omega, \tag{3.7}$$

where every word ω has at least one letter in $\widetilde{X}$.

Proposition 3.1.11. *If the function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ in (3.1) satisfies Assumptions 1(a) and 1(b), then its linearization at the origin, $(X, W) = (0, 0) \in \mathcal{X} \times \mathcal{W}$, is $f(X, W) \approx AX$.*

Proof. The Fréchet derivative of $f(X, W)$ at the origin in the direction $H := (H_X, H_W) \in \mathcal{X} \times \mathcal{W}$ is the unique linear map $Df := D_X f \times D_W f$ that satisfies

$$\lim_{H \to 0} \frac{\|f(H_X, H_W) - f(0,0) - DfH\|}{\|H\|} = 0. \tag{3.8}$$

Substituting definitions, and invoking Assumption 1(b) and Proposition 3.1.10 to set $B = 0$, the left side of (3.8) becomes

$$\lim_{H \to 0} \frac{\|(A - D_X f)H_X + \sum_\omega c_\omega \otimes \omega - D_W f H_W\|}{\|H\|},$$

where the letters of ω are $H_1, \ldots, H_n$ instead of $X_1, \ldots, X_n$ and $H_{n+1}, \ldots, H_{n+r}$ instead of $W_1, \ldots, W_r$. Suppose $D_X f = A$ and $D_W f = 0$, then

$$\lim_{H \to 0} \frac{\|f(H_X, H_W) - f(0,0) - DfH\|}{\|H\|} = \lim_{H \to 0} \frac{\|\sum_\omega c_\omega \otimes \omega\|}{\|H\|}.$$

By the result discussed in Remark 2.4.8,

$$\|\omega\| = \|[H_{\omega_1}, [\ldots, H_{\omega_{|\omega|}}] \cdots]\| \le \mu^{|\omega|-1} \|H_{\omega_1}\| \cdots \|H_{\omega_{|\omega|}}\| \le \mu^{|\omega|-1} \|H\|^{|\omega|}.$$

By the triangle inequality,

$$\left\| \sum_\omega c_\omega \otimes \omega \right\| \le \sum_\omega \|c_\omega\| \mu^{|\omega|-1} \|H\|^{|\omega|},$$

whose right side converges, by Assumption 1(a). Therefore,

$$\lim_{H \to 0} \frac{\|\sum_\omega c_\omega \otimes \omega\|}{\|H\|} \le \lim_{H \to 0} \frac{\sum_\omega \|c_\omega\| \mu^{|\omega|-1} \|H\|^{|\omega|}}{\|H\|} = 0.$$

Since any such Df is unique, the choice of $Df = A \times 0$ is the Fréchet derivative of f at the origin. Therefore, near the origin, $f(X, W) \approx AX$. □

Our main results assert that global stability properties of (3.1) under Assumption 1 can be inferred from its Jacobian linearization, as quantified in Proposition 3.1.11. The following proposition asserts that the dynamical invariance described in Assumption 1(c) can also be inferred from the Jacobian linearization. This latter result is due to strong centrality of the lower central series, i.e., the property described in Lemma 2.3.5.

Proposition 3.1.12. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. If the function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ in (3.1) is a Lie function that satisfies Assumption 1(b), then $f\left(\left(\mathfrak{h}^{(i)}\right)^n, \mathcal{W}\right) \subseteq \left(\mathfrak{h}^{(i)}\right)^n$ if and only if $\left(\mathfrak{h}^{(i)}\right)^n$ is invariant under A.*

Proof. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. Suppose $X \in \left(\mathfrak{h}^{(i)}\right)^n$. Under Assumption 1(b), by Proposition 3.1.10, every word ω has at least one letter in $\widetilde{X}$. Since $\mathfrak{h}^{(i)}$ is an ideal, every word ω belongs to $\mathfrak{h}^{(i)}$. From (3.7), we conclude $f\left(\left(\mathfrak{h}^{(i)}\right)^n, \mathcal{W}\right) \subseteq \left(\mathfrak{h}^{(i)}\right)^n$ if and only if $\left(\mathfrak{h}^{(i)}\right)^n$ is invariant under A. □

Corollary 3.1.13. *If the function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{W}$ in (3.1) is a Lie function that satisfies Assumption 1(b), then it satisfies Assumption 1(c) if and only if $\left(\mathfrak{h}^{(i)}\right)^n$ is invariant under A.*

Our next result emphasizes that A-invariant subspaces induce well-defined quotient systems associated with the nonlinear dynamics.

Proposition 3.1.14. *If the function $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ in (3.1) satisfies Assumptions 1(a) and 1(b), then, given an A-invariant ideal $\mathcal{V} \subseteq \mathcal{X}$ with canonical projection $P : \mathfrak{g} \to \mathfrak{g}/\mathcal{V}$, there exists a unique function $\bar{f} : \mathcal{X}/\mathcal{V} \times \mathcal{W}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ that satisfies Assumptions 1(a) and 1(b), and makes the following diagram commute.*

$$\begin{tikzcd}
\mathcal{X} \times \mathcal{W} \arrow[r, "f"] \arrow[d, "(I_n \otimes P) \times (I_r \otimes P)"'] & \mathcal{X} \arrow[d, "I_n \otimes P"] \\
\mathcal{X}/\mathcal{V} \times \mathcal{W}/\mathcal{V} \arrow[r, "\bar{f}"'] & \mathcal{X}/\mathcal{V}
\end{tikzcd}$$

Proof. Along the path $\mathcal{X} \times \mathcal{W} \xrightarrow{f} \mathcal{X} \xrightarrow{I_n \otimes P} \mathcal{X}/\mathcal{V}$, we have

$$(I_n \otimes P) f(X, W) = (I_n \otimes P) A X + (I_n \otimes P) \sum_{\omega} c_\omega \otimes \omega.$$

By Proposition 2.4.2, there exists a unique map $\bar{A} : \mathcal{X}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ such that $(I_n \otimes P)A = \bar{A}(I_n \otimes P)$. Using the property of tensor products that $(M_1 \otimes N_1)(M_2 \otimes N_2) = (M_1 M_2) \otimes$

(N_1N_2), the projection of the summation over ω equals $\sum_\omega c_\omega \otimes (P\omega)$. Then, since the canonical projection of an algebra onto an ideal is a morphism of algebras [67, p. 537],[1] we have

$$P\omega = P[Y_{\omega_1}, [\ldots, Y_{\omega_{|\omega|}}]\cdots] = [PY_{\omega_1}, [\ldots, PY_{\omega_{|\omega|}}]_{\mathfrak{g}/\mathcal{V}}\cdots]_{\mathfrak{g}/\mathcal{V}}, \qquad Y_{\omega_i} \in \widetilde{X} \cup \widetilde{W}.$$

The map $\bar{f} : \mathcal{X}/\mathcal{V} \times \mathcal{W}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ is then given by

$$\bar{f}(\bar{X}, \bar{W}) := \bar{A}\bar{X} + \sum_\omega c_\omega \otimes [\bar{Y}_{\omega_1}, [\ldots, \bar{Y}_{\omega_{|\omega|}}]_{\mathfrak{g}/\mathcal{V}}\cdots]_{\mathfrak{g}/\mathcal{V}},$$

where $\bar{Y}_{\omega_i} = PY_{\omega_i}$. That $\bar{f}$ satisfies Assumption 1(a) follows from Lemma 2.4.5; satisfaction of Assumption 1(b) is clear from the definition of $\bar{f}$. □

3.2 Nilpotent Lie Algebras

In this section, we present a global stability result in the case that $\mathfrak{g}$ is nilpotent, and the ideal $\mathfrak{h}$ satisfying Assumption 1(c) is $\mathfrak{g}$ itself. We devote this section to this specific case because, as will be seen, the results are much stronger than in the general case. The general case where Assumption 1(c) is satisfied by a proper ideal is addressed in Section 3.3. The stability property proved in this section is *semiglobal-exponential stability*. The following definition is the natural adaptation of a continuous-time definition, taken from [66].

Definition 3.2.1 ([66, Definition 2.7]). *Given a discrete-time dynamical system $x^+ = f(k,x)$, $x \in \mathcal{X}$, the origin of $\mathcal{X}$ is* ***semiglobally exponentially stable*** *if for all $M > 0$, there exist $\alpha \geq 0$, $\lambda < 1$ such that if $\|x[0]\| \leq M$, then for all $k \geq 0$,*

$$\|x[k]\| \leq \alpha\lambda^k \|x[0]\|.$$

It follows immediately from the definition that semiglobal exponential stability implies local exponential stability. Exponential stability differs from semiglobal exponential stability, in that α and λ do not depend on M. Our main result in the nilpotent case is that a sufficiently small spectral radius of A implies semiglobal exponential stability. Additionally, the constant λ is shown to not depend on M.

Our proof of the main result of this section makes extensive use of canonical projections of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{g}^{(i+1)}$, where $\mathfrak{g}^{(i+1)}$ is an ideal of the lower central series of $\mathfrak{g}$ (recall

[1] In [67], a proof is provided in the context of *graded* algebras, but this additional structure is not used.

Definition 2.3.3). Throughout this section, let $P_i : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}^{(i+1)}$ denote the canonical projection of $\mathfrak{g}$ onto $\mathfrak{g}^{(i+1)}$, and let $\imath_i : \mathfrak{g}/\mathfrak{g}^{(i+1)} \to \mathfrak{g}$ denote any linear injection such that $P_i \circ \imath_i = \mathrm{Id}_{\mathfrak{g}/\mathfrak{g}^{(i+1)}}$. Before proving the main result, we establish the following lemma.

Lemma 3.2.2. *Let $\mathfrak{g}$ be a Lie algebra. Given a word ω with letters $Y_1, \ldots, Y_{|\omega|} \in \mathfrak{g}$,*

$$P_i\omega = P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\ldots, \imath_{i-1} \circ P_{i-1}Y_{|\omega|}] \cdots].$$

Proof. By bilinearity of the Lie bracket and Lemma 2.4.3,

$$P_i\omega = \underbrace{P_i\,\underbrace{[\underbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1}_{\in\mathfrak{g}^{(i)}}, [Y_2, [\ldots, Y_{|\omega|}] \cdots]}_{\in\mathfrak{g}^{(i+1)}}}_{0} + P_i[\imath_{i-1} \circ P_{i-1}Y_1, [Y_2, [\ldots, Y_{|\omega|}] \cdots], \tag{3.9}$$

where membership in $\mathfrak{g}^{(i+1)}$ follows from the property of the ideals discussed in Lemma 2.3.5; the first term is zero, since $P_i\mathfrak{g}^{(i+1)} = 0$, by definition of P_i. Applying the same decomposition to the second letter yields

$$P_i\omega = P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\imath_{i-1} \circ P_{i-1}Y_2, [Y_3, [\ldots, Y_{|\omega|}] \cdots].$$

Continuing in this way completes the proof. □

Theorem 3.2.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with nilindex p, and define $\mathcal{X} := \mathfrak{g}^n$ and $\mathcal{W} := \mathfrak{g}^r$. Consider the dynamics (3.1) and suppose $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ satisfies Assumption 1, where Assumption 1(c) is satisfied with $\mathfrak{h} = \mathfrak{g}$. If there exist $\beta \geq 0$, $s \geq 1$ such that $\|W[k]\| \leq \beta s^k$, and $\rho(A) < s^{-\frac{p(p-1)}{2}}$, then the origin of $\mathcal{X}$ is semiglobally exponentially stable.*

Proof of Theorem 3.2.3. Assume that there exist $\beta \geq 0$, $s \geq 1$ such that $\|W[k]\| \leq \beta s^k$, and that $\rho(A) < s^{-\frac{p(p-1)}{2}}$; the latter implies that A is Schur, since $p, s \geq 1$. Let $M > 0$ be arbitrary and assume $\|X[0]\| \leq M$. We examine the quotient dynamics on $\mathcal{X}/\mathfrak{g}^{(i+1)}$ for all i. Since $\mathfrak{g}$ is nilpotent, the quotient algebra $\mathfrak{g}/\mathfrak{g}^{(i+1)}$ is nilpotent with nilindex i, thus for all $|\omega| > i$, $P_i\omega = 0$. By Proposition 3.1.14,

$$\bar{X}_i^+ = \bar{A}_i\bar{X}_i + \sum_{|\omega|\leq i} c_\omega \otimes (P_i\bar{\omega}_{i-1}), \tag{3.10}$$

where $\bar{\omega}_{i-1} \in \mathfrak{g}$ is the word ω with $\imath_{i-1} \circ P_{i-1}$ applied to each of its letters, per Lemma 3.2.2.

Since $A : \mathfrak{g}^n \to \mathfrak{g}^n$ is Schur, every induced map $\bar{A}_i : \left(\mathfrak{g}/\mathfrak{g}^{(i+1)}\right)^n \to \left(\mathfrak{g}/\mathfrak{g}^{(i+1)}\right)^n$ is also Schur. The quotient dynamics (3.10) have the form of a linear system with state $\bar{X}_i$ and exogenous input

$$u_i := \sum_{|\omega| \leq i} c_\omega \otimes (P_i \bar{\omega}_{i-1}), \tag{3.11}$$

which does not depend on $\bar{X}_i$. Even though quotient state $i-1$ drives quotient state i, the analysis does not exploit a serial structure; rather, each subsequent quotient system is a "larger piece" of the full dynamics. We will show that each quotient system is semiglobally exponentially stable. Our proof is by finite induction. The approach is to show that each quotient system is semiglobally exponentially stable, and, since $\mathfrak{g}^{(i)} = 0$ for $i > p$, the pth quotient system is simply the original system.

Before proceeding, we define some key values. Since A is Schur, for any $\varepsilon \in (0, 1-\rho(A))$, define $\Lambda := \rho(A) + \varepsilon$, then there exists a $\sigma \geq 0$ such that for all $k \geq 0$, $\|A^k\| \leq \sigma \Lambda^k$ [60, §5]. Define

$$\Lambda_i := \rho(\bar{A}_i) + \frac{i}{p+1}\varepsilon, \quad 1 \leq i \leq p,$$

then for all i, there exists $\sigma_i \geq 0$ such that $\|\bar{A}_i^k\| \leq \sigma_i \Lambda_i^k$. Note $\Lambda_1 < \cdots < \Lambda_p < \Lambda < 1$.

We begin with the base case, $i = 1$:

$$\bar{X}_1^+ = \bar{A}_1 \bar{X}_1,$$

which is an unforced linear time-invariant system. Consequently, $\bar{X}_1[k] = \bar{A}_1^k \bar{X}_1[0]$, so we have $\|\bar{X}_1[k]\| \leq \sigma_1 \Lambda_1^k \|\bar{X}_1[0]\| \leq \sigma_1 \Lambda^k \|\bar{X}_1[0]\|$. Let $\alpha_1 := \sigma_1$ and $\lambda_1 := \Lambda$.

By way of induction, we assert that there exists $\alpha_{i-1} > 0$ such that

$$\|\bar{X}_{i-1}[k]\| \leq \alpha_{i-1} \lambda_{i-1}^k \|\bar{X}_{i-1}[0]\|, \tag{3.12}$$

where for $1 \leq i-1 \leq p-1$, $\lambda_{i-1} := \Lambda s^{\frac{(i-1)(i-2)}{2}}$. We remark that $\frac{(i-1)(i-2)}{2}$ is the sum of all natural numbers less than $i-1$. Note also that by Lemma 2.4.5, $\|X[0]\| \leq M$ implies $\|\bar{X}_{i-1}[0]\| \leq M$.

We now prove that case $i-1$ implies case i. Fix $1 \leq j \leq n$ and choose an arbitrary word ω in the series of f_j. Denote its letters by $Y_k \in \widetilde{X} \cup \widetilde{\widetilde{W}}$, $k \in \{1, \ldots, |\omega|\}$, and the number of these letters in $\widetilde{X}$ by q. We will show that the projection of each word $P_i\omega$ converges to zero exponentially. Beginning with Lemma 3.2.2,

$$P_i\omega = P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\ldots, \imath_{i-1} \circ P_{i-1}Y_{|\omega|}] \cdots],$$

then

$$\|P_i\omega\| \le \mu^{|\omega|-1}\|\imath_{i-1}\|^{|\omega|}\prod_{j=1}^{|\omega|}\|P_{i-1}Y_j\|.$$

We have $\|P_{i-1}X_j\| \le \|(I_n \otimes P_{i-1})X\|$, and Lemma 2.4.5 implies $\|P_{i-1}W_j\| \le \|W\|$. Combining these inequalities with the induction hypothesis (3.12) yields

$$\begin{aligned}\|P_i\bar{\omega}_{i-1}[k]\| &\le \mu^{|\omega|-1}\|\imath_{i-1}\|^{|\omega|}\|\bar{X}_{i-1}[k]\|^q\|W[k]\|^{|\omega|-q}\\ &\le \mu^{|\omega|-1}\|\imath_{i-1}\|^{|\omega|}\left(\alpha_{i-1}\lambda_{i-1}^k\|\bar{X}_{i-1}[0]\|\right)^q(\beta s^k)^{|\omega|-q}\\ &= \mu^{|\omega|-1}\|\imath_{i-1}\|^{|\omega|}\alpha_{i-1}^q\beta^{|\omega|-q}(\lambda_{i-1}^q s^{|\omega|-q})^k\|\bar{X}_{i-1}[0]\|^q. \end{aligned} \tag{3.13}$$

Since $\|X[0]\| \le M$, in (3.13), we use Lemma 2.4.5 to upper bound $q-1$ of the factors of $\|\bar{X}_{i-1}[0]\|$ by M, and the single remaining factor by $\|\bar{X}_i[0]\|$:

$$\|P_i\bar{\omega}_{i-1}[k]\| \le \mu^{|\omega|-1}\|\imath_{i-1}\|^{|\omega|}\alpha_{i-1}^q\beta^{|\omega|-q}(\lambda_{i-1}^q s^{|\omega|-q})^k M^{q-1}\|\bar{X}_i[0]\|. \tag{3.14}$$

Claim 3.2.4. *There exists $\gamma_i \ge 0$ such that the norm of the exogenous input (3.11) satisfies*

$$\|u_i[k]\| \le \gamma_i(\underbrace{\lambda_{i-1}s^{i-1}}_{\lambda_i})^k\|\bar{X}_i[0]\|.$$

Proof of Claim 3.2.4. Fix the word length $\ell \ge 2$ and the number of letters in $\widetilde{X}$, $1 \le q \le \ell$. There are n^q choices of letters in $\widetilde{X}$, $r^{\ell-q}$ choices of letters in $\widetilde{W}$, and $\binom{\ell}{q}$ ways to position the letters in $\widetilde{X}$. Thus, there are $\binom{\ell}{q}n^q r^{\ell-q}$ words of length ℓ with q letters in $\widetilde{X}$. First, recall from (3.11), that $u_i := \sum_{|\omega|\le i} c_\omega \otimes (P_i\bar{\omega}_{i-1})$. Applying (3.14), we have

$$\|u_i[k]\| \le \left(\sum_{\substack{2\le\ell\le i\\1\le q\le\ell}}\max_{|\omega|=\ell}\{\|c_\omega\|\}\binom{\ell}{q}n^q r^{\ell-q}\mu^{\ell-1}\|\imath_{i-1}\|^\ell\alpha_{i-1}^q\|\bar{X}_{i-1}[0]\|^q\beta^{\ell-q}\right)\max_{\substack{2\le\ell\le i\\1\le q\le\ell}}\{\lambda_{i-1}^q s^{\ell-q}\}^k,$$

whose right side is bounded above by

$$\underbrace{\left(\sum_{\ell=2}^{i}\max_{|\omega|=\ell}\{\|c_\omega\|\}\mu^{\ell-1}\|\imath_{i-1}\|^\ell\sum_{q=1}^{\ell}\binom{\ell}{q}n^q r^{\ell-q}\alpha_{i-1}^q M^{q-1}\beta^{\ell-q}\right)}_{=:\gamma_i}\underbrace{\max_{\substack{2\le\ell\le i\\1\le q\le\ell}}\{\lambda_{i-1}^q s^{\ell-q}\}^k}_{\lambda_i}\|\bar{X}_i[0]\|.$$

Since $0 < \lambda_{i-1} < 1$ and $s \ge 1$, the maximization is solved by $\ell = i$ and $q = 1$, thus, the maximization term is equal to λ_i. $\square$

Note that even though $\bar{X}_i$ and $\bar{X}_{i-1}$ are both projections of the state X, by the induction hypothesis, the trajectory of $\bar{X}_{i-1}$ is fixed, i.e., a function of only time. Thus, despite $\bar{X}_{i-1}[k]$ partially determining $\bar{X}_i[k]$, we can view $\bar{X}_{i-1}$ in the dynamics of $\bar{X}_i$ as an exogenous signal.

By linear systems theory, we can express $\bar{X}_i[k]$ as the sum of a zero-input response $\bar{X}_i^{\mathrm{zi}}[k] = \bar{A}_i^k \bar{X}_i[0]$ and a zero-state response $\bar{X}_i^{\mathrm{zs}}[k] = \sum_{j=0}^{k-1} \bar{A}_i^j u_i[k-1-j]$. We now bound the zero-state response thus:

$$\begin{aligned}
\|\bar{X}_i^{\mathrm{zs}}[k]\| &\leq \sum_{j=0}^{k-1} \|\bar{A}_i^j\| \|u_i[k-1-j]\| \\
&\leq \sum_{j=0}^{k-1} \sigma_i \Lambda_i^j \gamma_i \lambda_i^{k-1-j} \|\bar{X}_i[0]\| \qquad \text{(by Claim 3.2.4)} \\
&\leq \sigma_i \gamma_i \lambda_i^{k-1} \|\bar{X}_i[0]\| \sum_{j=0}^{\infty} \left(\frac{\Lambda_i}{\lambda_i}\right)^j .
\end{aligned}$$

Recall that for all $1 \leq i \leq p$, $\Lambda_i < \Lambda$, and that by the induction hypothesis, $\lambda_i \geq \Lambda$. Therefore, for all $1 \leq i \leq p$, $\lambda_i > \Lambda_i$. Hence,

$$\|\bar{X}_i^{\mathrm{zs}}[k]\| \leq \frac{\sigma_i \gamma_i}{\lambda_i - \Lambda_i} \lambda_i^k \|\bar{X}_i[0]\|.$$

Applying the triangle inequality to $\bar{X}_i[k] = \bar{X}_i^{\mathrm{zi}}[k] + \bar{X}_i^{\mathrm{zs}}[k]$, we have

$$\begin{aligned}
\|\bar{X}_i[k]\| &\leq \sigma_i \Lambda_i^k \|\bar{X}_i[0]\| + \frac{\sigma_i \gamma_i}{\lambda_i - \Lambda_i} \lambda_i^k \|\bar{X}_i[0]\| \\
&\leq \underbrace{\sigma_i \left(1 + \frac{\gamma_i}{\lambda_i - \Lambda_i}\right)}_{=:\alpha_i} \lambda_i^k \|\bar{X}_i[0]\|.
\end{aligned}$$

This proves that the origin of $P_i \mathcal{X} = \mathfrak{g}^n / \left(\mathfrak{g}^{(i+1)}\right)^n$ is semiglobally exponentially stable. This concludes the induction. Recall that $P_{p+j}\mathfrak{g} = \mathfrak{g}/\mathfrak{g}^{(p+j)} = \mathfrak{g}/0 \cong \mathfrak{g}$, so step $i = p$ of the induction proves that the origin of $\mathcal{X} = \mathfrak{g}^n$ is semiglobally exponentially stable. □

Remark 3.2.5. The assertion that W is bounded by a function of the form βs^k implies that it is Z-transformable. ♦

Corollary 3.2.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ satisfy Assumption 1, where Assumption 1(c) is satisfied with $\mathfrak{h} = \mathfrak{g}$. If W is bounded, then the origin of $\mathcal{X}$ is semiglobally exponentially stable if and only if A is Schur.*

Proof. Suppose A is Schur. Since W is bounded, $\|W[k]\| \leq \beta s^k$, for $s = 1$ and some finite β. Theorem 3.2.3 implies semiglobal exponential stability. For necessity, recall that semiglobal exponential stability implies local exponential stability, which implies that the linearization is Schur. By Proposition 3.1.11, the linearization is $X^+ \approx AX$. $\square$

Remark 3.2.7. If $\mathfrak{g}$ has nilindex 1, i.e., $\mathfrak{g}$ is commutative, then the dynamics (3.1) reduce to a linear time-invariant system. This was exploited for output regulation and synchronization on commutative matrix Lie groups in [74] and [73], respectively. ◆

In the following example, we illustrate the application of Theorem 3.2.3 to control design. We will first define a simple regulator problem, then, using Theorem 3.2.3, we will show that the error dynamics are semiglobally exponentially stable. This foreshadows our treatment of the regulator problem in Chapter 5.

Example 3.2.1. Let $\mathfrak{g}$ be the 3-dimensional Heisenberg algebra, which is defined by the commutator relations

$$[h_1, h_2] = h_3, \quad [h_1, h_3] = 0, \quad [h_2, h_3] = 0.$$

The lower central series of $\mathfrak{g}$ is $\mathfrak{g} =: \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \mathfrak{g}^{(3)} = 0$, where $\mathfrak{g}^{(2)} = \mathrm{Lie}_{\mathbb{R}}\{h_3\} \cong \mathrm{Span}_{\mathbb{R}}\{h_3\}$, thus, $\mathfrak{g}$ has nilindex $p = 2$.

Consider the right-invariant dynamical system with state $X \in \mathsf{G}$

$$\dot{X} = (h_1 u_1 + h_2 u_2 + h_3 u_3)X,$$

where $u \in \mathbb{R}^3$ is the control input. Suppose this system is sampled with period $T = 1$. The step-invariant transform of this system is

$$X^+ = \exp(h_1 u_1 + h_2 u_2 + h_3 u_3)X. \tag{3.15}$$

Suppose we want X to track a reference that is given implicitly by the tracking error

$$E = \exp((h_1 + 2h_2 + 3h_3)w)X,$$

where $w \in \mathbb{R}$ is a known exogenous signal, which evolves according to

$$w^+ = 2w. \tag{3.16}$$

The goal is to choose u such that E tends to the identity in G. This is equivalent to driving $\mathrm{Log}(E) \in \mathfrak{g}$ to 0, where we express $e := \mathrm{Log}(E)$ in the basis $\{h_1, h_2, h_3\}$:

$$\mathrm{Log}(E) =: e_1 h_1 + e_2 h_2 + e_3 h_3.$$

Using (3.15) and the definition of E, we find

$$\begin{aligned}
E^+ &= \exp(2(h_1 + 2h_2 + 3h_3)w)\exp(h_1u_1 + h_2u_2 + h_3u_3)X \\
&= \exp(2(h_1 + 2h_2 + 3h_3)w)\exp(h_1u_1 + h_2u_2 + h_3u_3)\exp(-(h_1 + 2h_2 + 3h_3)w)E.
\end{aligned}$$

Using the generalized BCH (2.1), we express the error dynamics on the Lie algebra:

$$\begin{aligned}
e^+ = {} & 2(h_1 + 2h_2 + 3h_3)w + (h_1u_1 + h_2u_2 + h_3u_3) - (h_1 + 2h_2 + 3h_3)w + e \\
& + \frac{1}{2}[2(h_1 + 2h_2 + 3h_3)w, h_1u_1 + h_2u_2 + h_3u_3] \\
& + \frac{1}{2}[2(h_1 + 2h_2 + 3h_3)w, -(h_1 + 2h_2 + 3h_3)w] \\
& + \frac{1}{2}[2(h_1 + 2h_2 + 3h_3)w, e] + \frac{1}{2}[h_1u_1 + h_2u_2 + h_3u_3, -(h_1 + 2h_2 + 3h_3)w] \\
& + \frac{1}{2}[h_1u_1 + h_2u_2 + h_3u_3, e] + \frac{1}{2}[-(h_1 + 2h_2 + 3h_3)w, e] \\
= {} & (w + u_1)h_1 + (2w + u_2)h_2 + (3w + u_3)h_3 + e \\
& + \frac{1}{2}[(w + u_1)h_1 + (2w + u_2)h_2 + (3w + u_3)h_3, e] \\
& - \frac{3}{2}[h_1u_1 + h_2u_2 + h_3u_3, \underbrace{(h_1 + 2h_2 + 3h_3)w}_{=:W}].
\end{aligned}$$

The independent signal W evolves according to

$$\begin{aligned}
W^+ &= (h_1 + 2h_2 + 3h_3)w^+ \\
&= 2(h_1 + 2h_2 + 3h_3)w \\
&= 2W,
\end{aligned}$$

which yields

$$W[k] = 2^k W[0]$$
$$\|W[k]\| = 2^k \|W[0]\|.$$

Thus, setting $\beta = \|W[0]\|$ and $s = 2$, we have $\|W[k]\| \leq \beta s^k$.

To apply Theorem 3.2.3 to the dynamics of e, we must choose the control law u such that Assumption 1 is satisfied, and the linear part of (3.17) has spectral radius smaller than $s^{-1} = \frac{1}{2}$. After choosing our control law u, we will verify that each of Assumptions 1(a), 1(b), and 1(c) are satisfied. Per Proposition 3.1.10, Assumption 1(b) is satisfied only if the linear part of the dynamics does not depend on W. This observation, in part, motivates the control law

$$u = \begin{bmatrix} -0.75 & 0.25 & 0 \\ -0.25 & -0.75 & 0 \\ 0 & 0 & -0.99 \end{bmatrix} e - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} w.$$

Substituting into the dynamics of e, we obtain

$$\begin{aligned} e^+ = (0.25e_1 + 0.25e_2)h_1 + (-0.25e_1 + 0.25e_2)h_2 + (0.01e_3)h_3 \\ + \frac{1}{2}[(0.25e_1 + 0.25e_2)h_1 + (-0.25e_1 + 0.25e_2)h_2, e_1h_1 + e_2h_2] \\ - \frac{3}{2}[(0.25e_1 + 0.25e_2)h_1 + (-0.25e_1 + 0.25e_2)h_2, (h_1 + 2h_2)w]. \end{aligned} \tag{3.17}$$

The dynamics (3.17) are of the form

$$e^+ = Ae + \sum_{|\omega|=2} c_\omega \omega,$$

where in the basis $\mathfrak{g} = \mathrm{Lie}_{\mathbb{R}}\{h_1, h_2, h_3\} \cong \mathrm{Span}_{\mathbb{R}}\{h_1, h_2, h_3\}$, $A : \mathfrak{g} \to \mathfrak{g}$ has matrix representation

$$\mathrm{Mat}A = \begin{bmatrix} 0.25 & 0.25 & 0 \\ -0.25 & 0.25 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \tag{3.18}$$

We now verify that (3.17) satisfies Assumption 1. By the form of (3.17) and nilpotency of $\mathfrak{g}$, the dynamics of e are clearly class-$\mathcal{A}$, thus Assumption 1(a) is satisfied.

That $e = 0$ is an equilibrium is verified by substituting $e = 0$ into (3.17). To verify that $e = 0$ is the only equilibrium, note that by the definition of the Lie bracket on $\mathfrak{g}$, the bracket terms in (3.17) lie in $\mathrm{Span}_{\mathbb{R}}\{h_3\}$. Therefore, a point e is an equilibrium only if

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 \\ -0.25 & 0.25 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

which holds if and only if $e_1 = e_2 = 0$. If $e_1 = e_2 = 0$, then (3.17) reduces to $e[k+1] = 0.01e_3h_3$, whose only equilibrium is $e_3 = 0$. This verifies Assumption 1(b).

The block diagonal structure of (3.18) makes it clear that $\mathfrak{g}^{(2)} = \mathrm{Lie}_{\mathbb{R}}\{h_3\} \cong \mathrm{Span}_{\mathbb{R}}\{h_3\}$ is invariant. By Corollary 3.1.13, this verifies Assumption 1(c). By Theorem 3.2.3, $e = 0$

is semiglobally exponentially stable if $\rho(A) < s^{-1} = \frac{1}{2}$. The eigenvalues of (3.18) are $\{-0.25 + i0.25, -0.25 - i0.25, 0.01\}$, thus $\rho(A) = \frac{1}{2\sqrt{2}}$. Therefore, $e = 0$ is semiglobally exponentially stable. We simulate the dynamics of the tracking error using the initial conditions $e[0] = 3h_1 + 2h_2 - h_3$, $w[0] = 1$. The trajectory of e is in Figure 3.1. As can be seen, e tends to 0. ▲

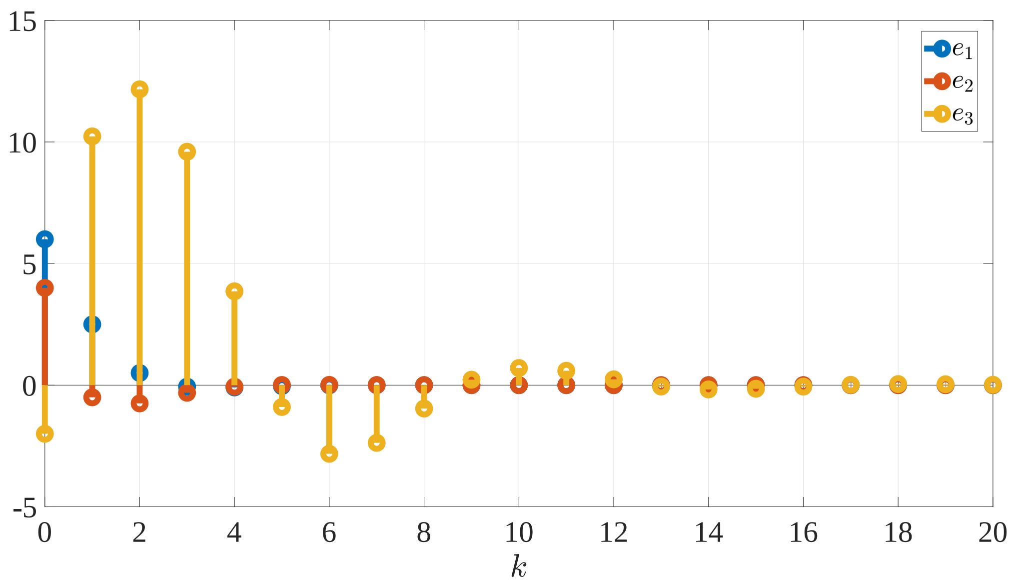


Figure 3.1: The tracking error $e \in \mathfrak{g}$ at the sampling instants.

3.3 Solvable Lie Algebras

In this section we present various global stability results in the case that $\mathfrak{g}$ is solvable, but not necessarily nilpotent. Our analysis exploits the structure endowed by Theorem 2.3.6. The proof of the main result of this section takes a similar geometric approach to that of Theorem 3.2.3, but the analysis is significantly complicated by the nontrivial quotient space $\mathfrak{K} := \mathfrak{g}/\mathfrak{h}$. The dynamics on $\mathfrak{K}$ will be treated from an analysis perspective, rather than using geometric arguments, and be shown to converge to the origin via contradiction. Throughout this section, let $P_i : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}^{(i+1)} \cong \mathfrak{K} \oplus \mathfrak{h}/\mathfrak{h}^{(i+1)}$ denote the canonical projection of $\mathfrak{g}$ onto $\mathfrak{h}^{(i+1)}$. We will require the following lemma, which is the solvable analogue of Lemma 3.2.2 in the nilpotent case.

Lemma 3.3.1. *Let $\mathfrak{g}$ be a solvable Lie algebra. Then, given a word ω with letters $Y_1, \dots, Y_{|\omega|}$,*

$$\begin{aligned} P_i\omega = {} & P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\dots, \imath_{i-1} \circ P_{i-1}Y_{|\omega|}] \cdots] \\ & + P_i[\overbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})}^{1st\ letter} Y_1, [\imath_0 \circ P_0 Y_2, [\dots, \imath_0 \circ P_0 Y_{|\omega|}] \cdots] \\ & + P_i[\imath_0 \circ P_0 Y_1, [\overbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})}^{2nd\ letter} Y_2, [\imath_0 \circ P_0 Y_3, [\dots, \imath_0 \circ P_0 Y_{|\omega|}] \cdots] + \cdots \\ & \cdots + P_i[\imath_0 \circ P_0 Y_1, [\dots, [\imath_0 \circ P_0 Y_{|\omega|-1}, \underbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})}_{|\omega|th\ letter} Y_{|\omega|}] \cdots]. \end{aligned}$$

Proof. Using $\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1} + \imath_{i-1} \circ P_{i-1} = \mathrm{Id}_{\mathfrak{g}}$ and bilinearity of the Lie bracket,

$$P_i\omega = P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [Y_2, [\dots, Y_{|\omega|}] \cdots] + P_i[\imath_{i-1} \circ P_{i-1}Y_1, [Y_2, [\dots, Y_{|\omega|}] \cdots], \quad (3.19)$$

where $Y_j \in \widetilde{X} \cup \widetilde{W}$.

We next decompose the second letter of the first term in (3.19) with respect to $\imath_0 \circ P_0$ and invoke Lemma 2.4.3:

$$\begin{aligned} P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [Y_2, [\dots, Y_{|\omega|}] \cdots] = {} & P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [\imath_0 \circ P_0 Y_2, [\dots, Y_{|\omega|}] \cdots] \\ & + P_i \underbrace{[\underbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1}_{\in \mathfrak{h}^{(i)}}, [\underbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_0 \circ P_0)Y_2}_{\in \mathfrak{h}^{(1)}}, [\dots, Y_{|\omega|}] \cdots]}_{\in \mathfrak{h}^{(i+1)}}, \end{aligned} \quad (3.20)$$

where membership in $\mathfrak{h}^{(i+1)}$ follows from Lemma 2.3.5; the second term is zero, since $P_i\mathfrak{h}^{(i+1)} = 0$. Decomposing the rest of the letters in (3.20) with respect to $\imath_0 \circ P_0$ yields

$$P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [Y_2, [\dots, Y_{|\omega|}] \cdots] = P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [\imath_0 \circ P_0 Y_2, [\dots, \imath_0 \circ P_0 Y_{|\omega|}] \cdots].$$

Now decompose the second letter of the second term in (3.19) with respect to $\imath_{i-1} \circ P_{i-1}$:

$$\begin{aligned} P_i[\imath_{i-1} \circ P_{i-1}Y_1, [Y_2, [\dots, Y_{|\omega|}] \cdots] = {} & P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\overbrace{(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_2}^{\in \mathfrak{h}^{(i)}}, [Y_3, [\dots, Y_{|\omega|}] \cdots] \\ & + P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\imath_{i-1} \circ P_{i-1}Y_2, [Y_3, [\dots, Y_{|\omega|}] \cdots]. \end{aligned}$$

We continue in a fashion similar to that following (3.19), the only noteworthy difference is the decomposition of $\imath_{i-1} \circ P_{i-1}Y_1$ with respect to $\imath_0 \circ P_0$.

Claim 3.3.2. *For all $i \geq 1$, the following diagram commutes.*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "P_{i-1}"] \arrow[drr, "P_0"'] & \mathfrak{g}/\mathfrak{h}^{(i)} \arrow[r, "\imath_{i-1}"] & \mathfrak{g} \arrow[d, "P_0"] \\ & & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

Proof of Claim 3.3.2. From the definitions of P_0, P_{i-1}, and $\imath_{i-1}$, we have $\mathfrak{g} = \operatorname{Im} \imath_{i-1} \oplus \mathfrak{h}^{(i)}$ and $\operatorname{Ker} P_0 = \mathfrak{h} \supseteq \mathfrak{h}^{(i)} = \operatorname{Ker} P_{i-1}$. Then $P_0\mathfrak{g} = P_0 \operatorname{Im} \imath_{i-1} \oplus P_0\mathfrak{h}^{(i)} = P_0 \operatorname{Im} \imath_{i-1}$. $\square$

It follows immediately from Claim 3.3.2 that $\imath_0 \circ P_0 \circ \imath_{i-1} \circ P_{i-1} = \imath_0 \circ P_0$. Thus, the decomposition process specified above yields

$$\begin{aligned} P_i\omega = P_i[\imath_{i-1} \circ P_{i-1}Y_1, [\imath_{i-1} \circ P_{i-1}Y_2, [Y_3, [\ldots, Y_{|\omega|}] \cdots] \\ + P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [\imath_0 \circ P_0Y_2, [\ldots, \imath_0 \circ P_0Y_{|\omega|}] \cdots] \\ + P_i[\imath_0 \circ P_0Y_1, [(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_2, [\imath_0 \circ P_0Y_3, [\ldots, \imath_0 \circ P_0Y_{|\omega|}] \cdots]. \end{aligned} \tag{3.21}$$

Applying this process to the rest of the letters in the first word of (3.21) completes the proof. $\square$

Theorem 3.3.3. *Let $\mathfrak{g}$ be a solvable Lie algebra, and define $\mathcal{X} := \mathfrak{g}^n$ and $\mathcal{W} := \mathfrak{g}^r$. Consider the dynamics (3.1) and suppose $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ satisfies Assumption 1. If A is Schur, and as $k \to \infty$, $W[k] \to \mathfrak{h}^r$, then there exists $\beta > 0$ such that if $\limsup_{k\to\infty} \|W[k]\| \leq \beta$, then the origin of $\mathcal{X}$ is globally attractive.*

Proof. Analogous to the proof of Theorem 3.2.3, we will examine the quotient dynamics on $\mathcal{X}/\mathfrak{h}^{(i+1)}$, where $i \geq 0$. By Proposition 3.1.14, the quotient dynamics on $\mathcal{X}/\mathfrak{h}^{(i+1)}$ are

$$\bar{X}_i^+ = \bar{A}_i\bar{X}_i + \sum_{\omega} c_\omega \otimes (P_i\omega). \tag{3.22}$$

We begin by examining the quotient dynamics on $\mathcal{X}/\mathfrak{h} = \mathfrak{K}^n$:

$$\bar{X}_0^+ = \bar{A}_0\bar{X}_0, \tag{3.23}$$

which is an unforced linear time-invariant system. That A is Schur implies $\bar{A}_0$ is Schur, so the origin of $P_0\mathcal{X} = \mathfrak{g}^n/\mathfrak{h}^n \cong \mathfrak{K}^n$ is globally exponentially stable under the quotient dynamics (3.22).

We assert the induction hypothesis that the origin of $P_{i-1}\mathcal{X} \cong \mathfrak{K}^n \oplus \left(\mathfrak{h}/\mathfrak{h}^{(i)}\right)^n$ is globally asymptotically stable. We now show that the origin of $P_i\mathcal{X} \cong \mathfrak{K}^n \oplus (\mathfrak{h}/\mathfrak{h}^{(i+1)})^n$ is globally asymptotically stable.

By Lemma 3.3.1,

$$
\begin{aligned}
P_i\omega = P_i\,&\overbrace{[\imath_{i-1} \circ P_{i-1}Y_1, [\ldots, \imath_{i-1} \circ P_{i-1}Y_{|\omega|}]\cdots]}^{\hat{\omega}_{i-1}:=} \\
&+ P_i[(\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_1, [\imath_0 \circ P_0 Y_2, [\ldots, \imath_0 \circ P_0 Y_{|\omega|}]\cdots] + \cdots \\
&\cdots + P_i[\imath_0 \circ P_0 Y_1, [\ldots, [\imath_0 \circ P_0 Y_{|\omega|-1}, (\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_{|\omega|}]\cdots].
\end{aligned}
$$

By the induction hypothesis, each term $P_{i-1}Y_j$ in $\hat{\omega}_{i-1}$ tends to zero, which implies $\hat{\omega}_{i-1} \to 0$. We now show $P_i\omega \to 0$. By the result discussed in Remark 2.4.8, Lemma 3.3.1, and that P_i is a morphism of algebras, the norm of each projected word can be bounded thus

$$
\|P_i\omega\| \leq \|\hat{\omega}_{i-1}\| + \mu^{|\omega|-1} \sum_{j=1}^{|\omega|} \left(\|P_i \circ (\mathrm{Id}_{\mathfrak{g}} - \imath_{i-1} \circ P_{i-1})Y_j\| \prod_{\ell \neq j} \|P_i \circ \imath_0 \circ P_0 Y_\ell\| \right). \tag{3.24}
$$

By submultiplicativity of operator norms and Proposition 2.4.6, we have

$$
\|P_i \circ \imath_0 \circ P_0 Y_j\| \leq \|\imath_0 \circ P_0 Y_j\| \leq \|\imath_0\| \|P_0 Y_j\|. \tag{3.25}
$$

By Proposition 2.4.6 and the triangle inequality, we have

$$
\|(P_i - P_i \circ \imath_{i-1} \circ P_{i-1})Y_j\| \leq \|P_i Y_j\| + \|\imath_{i-1}\| \|P_{i-1} Y_j\| \leq (1 + \|\imath_{i-1}\|)\|P_i Y_j\|, \tag{3.26}
$$

where the second inequality follows from Lemma 2.4.5. Recalling the notation $\widetilde{X} = \{X_1, \ldots, X_n\}$ and $\widetilde{W} = \{W_1, \ldots, W_r\}$, we partition the words into the sets $\Omega_X := \{\omega :$ every letter is in $\widetilde{X}\}$ and $\Omega_W := \{\omega :$ at least one letter is in $\widetilde{W}\}$. First consider $\omega \in \Omega_X$. Applying (3.25) and (3.26) to (3.24), we obtain

$$
\begin{aligned}
\|P_i\omega\| &\leq \|\hat{\omega}_{i-1}\| + (\mu\|\imath_0\|)^{|\omega|-1}(1 + \|\imath_{i-1}\|)\|\bar{X}_i\| \sum_{j=1}^{|\omega|} \prod_{\ell \neq j} \|P_0 Y_\ell\| \\
&\leq \|\hat{\omega}_{i-1}\| + (\mu\|\imath_0\|)^{|\omega|-1}|\omega|(1 + \|\imath_{i-1}\|)\|\bar{X}_0\|^{|\omega|-1}\|\bar{X}_i\|,
\end{aligned} \tag{3.27}
$$

where we have used $\|P_i X_j\| \leq \|(I_n \otimes P_i)X\| = \|\bar{X}_i\|$, for all $j \in \{1, \ldots, n\}$.

Now consider $\omega \in \Omega_W$ and let $1 \leq q \leq |\omega| - 1$ be the number of letters in $\widetilde{X}$. Without loss of generality, suppose $Y_1, \ldots, Y_q \in \widetilde{X}$, and $Y_{q+1}, \ldots, Y_{|\omega|} \in \widetilde{W}$. Then

$$\sum_{j=1}^{q} \|P_i Y_j\| \prod_{\ell \neq j} \|P_0 Y_\ell\| \leq (\mu \|\imath_0\|)^{|\omega|-1} (1 + \|\imath_{i-1}\|) \|\bar{X}_i\| q \|\bar{X}_0\|^{q-1} \|\bar{W}_0\|^{|\omega|-q}$$

and

$$\sum_{j=q+1}^{|\omega|} \|P_i Y_j\| \prod_{\ell \neq j} \|P_0 Y_\ell\| \leq (\mu \|\imath_0\|)^{|\omega|-1} (1 + \|\imath_{i-1}\|) \|W\| (|\omega| - q) \|\bar{X}_0\|^{q} \|\bar{W}_0\|^{|\omega|-q-1}.$$

Using the bounds $q, |\omega| - q \leq |\omega| - 1$, we have

$$\|P_i \omega\| \leq \|\hat{\omega}_{i-1}\| + (\mu \|\imath_0\|)^{|\omega|-1} (1 + \|\imath_{i-1}\|)(\|\bar{X}_i\| + \|W\|)(|\omega| - 1) \max\{\|\bar{X}_0\|, \|\bar{W}_0\|\}^{|\omega|-1}. \tag{3.28}$$

Using (3.27) and (3.28), we upper bound $\|\bar{X}_i^+\|$:

$$\begin{aligned}
\|\bar{X}_i^+\| &\leq \|\bar{A}_i\| \|\bar{X}_i\| + \sum_{\omega \in \Omega_W} \|c_\omega\| \|P_i \omega\| + \sum_{\omega \in \Omega_X} \|c_\omega\| \|P_i \omega\| \\
&\leq \|\bar{A}_i\| \|\bar{X}_i\| + \sum_{\omega \in \Omega_W \cup \Omega_X} \|c_\omega\| \|\hat{\omega}_{i-1}\| \\
&\qquad + (1 + \|\imath_{i-1}\|) \|\bar{X}_i\| \sum_{\omega \in \Omega_X} |\omega| (\mu \|\imath_0\|)^{|\omega|-1} \|c_\omega\| \|\bar{X}_0\|^{|\omega|-1} \\
&\qquad + (1 + \|\imath_{i-1}\|)(\|\bar{X}_i\| + \|W\|) \sum_{\omega \in \Omega_W} (|\omega| - 1)(\mu \|\imath_0\|)^{|\omega|-1} \|c_\omega\| \max\{\|\bar{X}_0\|, \|\bar{W}_0\|\}^{|\omega|-1} \\
&\leq \|\bar{A}_i\| \|\bar{X}_i\| + \sum_{\omega} \|c_\omega\| \|\hat{\omega}_{i-1}\| \\
&\qquad + (1 + \|\imath_{i-1}\|)(2\|\bar{X}_i\| + \|W\|) \sum_{\omega} |\omega| (\mu \|\imath_0\|)^{|\omega|-1} \|c_\omega\| \max\{\|\bar{X}_0\|, \|\bar{W}_0\|\}^{|\omega|-1}.
\end{aligned}$$

Claim 3.3.4. *There exists $\varrho > 0$ such that for all $\|\bar{X}_0\|, \|\bar{W}_0\| < \varrho$,*

$$\sum_{\omega} |\omega| (\mu \|\imath_0\|)^{|\omega|-1} \|c_\omega\| \max\{\|\bar{X}_0\|, \|\bar{W}_0\|\}^{|\omega|-1} < \infty. \tag{3.29}$$

Proof of Claim 3.3.4. Suppose f satisfies (3.3). In particular, suppose there exists $\varrho_1 \leq 1$ such that

$$\|X_1\|, \ldots, \|X_n\|, \|W_1\|, \ldots, \|W_r\| < \varrho_1.$$

On this domain, we have $\|\omega\| \leq \mu^{|\omega|-1}\varrho_1^{|\omega|}$ and

$$\sum_{\omega} \mu^{|\omega|-1}\|c_\omega\|\varrho_1^{|\omega|} < \infty.$$

We can rewrite this summation by grouping all words of the same length:

$$\sum_{\ell=2}^{\infty} \mu^{\ell-1}\left(\sum_{|\omega|=\ell} \|c_\omega\|\right)\varrho_1^{\ell},$$

which can be viewed as a series over the single index ℓ. Since this series converges, by the root test [92, Theorem 3.33], we have

$$\begin{aligned}
\limsup_{\ell\to\infty} \sqrt[\ell]{\mu^{\ell-1}\varrho_1^{\ell}\sum_{|\omega|=\ell}\|c_\omega\|} &= \varrho_1 \limsup_{\ell\to\infty} \mu^{1-\frac{1}{\ell}} \limsup_{\ell\to\infty} \sqrt[\ell]{\sum_{|\omega|=\ell}\|c_\omega\|} \\
&= \varrho_1\mu \limsup_{\ell\to\infty} \sqrt[\ell]{\sum_{|\omega|=\ell}\|c_\omega\|} \\
&\leq 1.
\end{aligned}$$

Let $0 < \varrho_2 < \frac{\varrho_1}{\|\imath_0\|}$. Applying the root test to the series

$$\sum_{\omega}(\mu\|\imath_0\|)^{|\omega|-1}|\omega|\|c_\omega\|\varrho_2^{|\omega|}, \tag{3.30}$$

we have

$$\begin{aligned}
\limsup_{\ell\to\infty} \sqrt[\ell]{\ell(\mu\|\imath_0\|)^{\ell-1}\varrho_2^{\ell}\sum_{|\omega|=\ell}\|c_\omega\|} &= \varrho_2\mu\|\imath_0\| \limsup_{\ell\to\infty}\sqrt[\ell]{\ell} \limsup_{\ell\to\infty} \sqrt[\ell]{\sum_{|\omega|=\ell}\|c_\omega\|} \\
&= \varrho_2\mu\|\imath_0\| \limsup_{\ell\to\infty} \sqrt[\ell]{\sum_{|\omega|=\ell}\|c_\omega\|} \\
&< \varrho_1\mu \limsup_{\ell\to\infty} \sqrt[\ell]{\sum_{|\omega|=\ell}\|c_\omega\|} \\
&< 1,
\end{aligned}$$

which implies that (3.30) converges. Let $\varrho \leq \varrho_2^2$, then for all $|\omega| \geq 2$, $\varrho^{|\omega|-1} < \varrho_2^{|\omega|}$. Then, by the comparison test [92, Theorem 3.25], if $\|\bar{X}_0\|, \|\bar{W}_0\| \leq \varrho$, then (3.29) converges. $\square$

First, note that the hypothesis $W[k] \to \mathfrak{h}^r$ implies $\bar{W}_0 \to 0$. Now, since (3.29) converges for $\|\bar{X}_0\|, \|\bar{W}_0\|$ sufficiently small, it follows that since $\bar{X}_0$ and $\bar{W}_0$ tend to zero as $k \to \infty$, that (3.29) tends to zero.

We divide both sides by $\|\bar{X}_i\|$ and upper bound the limiting supremum thus

$$\limsup_{k\to\infty} \frac{\|\bar{X}_i^+\|}{\|\bar{X}_i\|} \leq \|\bar{A}_i\| + \frac{\limsup_{k\to\infty} \sum_\omega \|c_\omega\| \left(\|\hat{\omega}_{i-1}\| + (1 + \|\imath_{i-1}\|)\|W\||\omega|(\mu\|\imath_0\|)^{|\omega|-1} \max\{\|\bar{X}_0\|, \|\bar{W}_0\|\}^{|\omega|-1}\right)}{\liminf_{k\to\infty} \|\bar{X}_i\|}.$$

Suppose, by way of contradiction, that $\liminf_{k\to\infty} \|\bar{X}_i\| > 0$. Since $\hat{\omega}_{i-1} \to 0$ and $\bar{W}_0 \to 0$ by hypothesis, W is bounded, and $\bar{X}_0 \to 0$, the limiting supremum on the right side is 0, so

$$\limsup_{k\to\infty} \frac{\|\bar{X}_i^+\|}{\|\bar{X}_i\|} \leq \|\bar{A}_i\|. \tag{3.31}$$

All our analysis heretofore has been independent of a specific choice of norm. However, at this point, we invoke Theorem 2.4.7 and choose the norm $\|\cdot\| : \mathfrak{g} \to \mathbb{R}$ such that for some $\varepsilon \in (0, 1 - \rho(\bar{A}_i))$, $\|\bar{A}_i\| = \rho(\bar{A}_i) + \varepsilon < 1$. By (3.31), we have $\lim_{k\to\infty} \|\bar{X}_i\| = 0$, which is a contradiction. Therefore, $\liminf_{k\to\infty} \|\bar{X}_i\| = 0$, so given any $\varepsilon > 0$, there exists a time k_ε such that $\|\bar{X}_i[k_\varepsilon]\| < \varepsilon$. By Proposition 3.1.11, A Schur and $W = 0$ implies local exponential stability of the origin, so by a standard perturbation argument, for W sufficiently small, the origin remains locally exponentially stable. Thus, there exist $\beta > 0$, $\bar{k} \geq 0$ such that if, for all $k \geq \bar{k}$, $\|W[k]\| \leq \beta$, then the origin of $\mathcal{X}$ is locally attractive. Therefore, $\bar{X}_i$ eventually enters the basin of attraction, so $\bar{X}_i \to 0$. This establishes that the origin is globally attractive. This proves the induction. □

Remark 3.3.5. Since the dynamics on $\mathcal{X}/\mathfrak{h}$ are linear, it could be argued that $[\mathfrak{g}, \mathfrak{g}]$ is the "best" possibility for $\mathfrak{h}$, since this maximizes the dimension of $\mathcal{X}/\mathfrak{h}$. However, the choice of $\mathfrak{h}$ does not change the analysis or results. ♦

Theorem 3.3.3 is somewhat weaker than Theorem 3.2.3 for the nilpotent case. Although Theorem 3.3.3 would of course apply when the Lie algebra is nilpotent, Theorem 3.2.3 is not a special case of Theorem 3.3.3. If we assert that W is bounded, rather than the less restrictive condition in Theorem 3.3.3, then we can strengthen the attractivity result of Theorem 3.3.3 to stability.

Corollary 3.3.6. *Let $\mathfrak{g}$ be a solvable Lie algebra, and define $\mathcal{X} := \mathfrak{g}^n$ and $\mathcal{W} := \mathfrak{g}^r$. Consider the dynamics (3.1) and suppose $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ satisfies Assumption 1. If A is*

Schur, and as $k \to \infty$, $W[k] \to \mathfrak{h}^r$, then there exists $\beta > 0$ such that if $\|W[k]\| \leq \beta$, then the origin of $\mathcal{X}$ is globally asymptotically stable.

Proof. The proof is the same as that of Theorem 3.3.3, where $\bar{k} = 0$ (defined near the end of the proof of Theorem 3.3.3), which implies that the origin of $\mathcal{X}$ is locally exponentially stable for all $k \geq 0$. □

The requirement that W be indeterminately small in Theorem 3.3.3 and Corollary 3.3.6 is rather restrictive. However, when the map A has spectral radius 0, W need not be bounded, and we can even relax the assumption that f belongs to class-$\mathcal{A}$.

Theorem 3.3.7. *Consider the dynamics (3.1). Let $\mathfrak{g}$ be a solvable Lie algebra and $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ be a Lie function that satisfies Assumptions 1(b) and 1(c). If $\rho(A) = 0$ and for all $k \geq 0$, $W[k] \in \mathfrak{h}^r$, then X converges to zero in finite time.*

Proof. The quotient dynamics on $\mathcal{X}/\mathfrak{h} = \mathfrak{K}^n$ are

$$\bar{X}_0^+ = \bar{A}_0 \bar{X}_0.$$

That A has spectral radius zero implies that $\bar{A}_0 : \mathfrak{K}^n \to \mathfrak{K}^n$ has spectral radius zero, which implies $\bar{A}_0^{\dim \mathfrak{K}} = 0$. Therefore, for all $k \geq \dim \mathfrak{K}$, we have $\bar{X}_0[k] = 0$.

By way of induction, we assert that for all $k \geq i \dim \mathfrak{g} - \sum_{j=1}^{i} \dim \mathfrak{h}^{(j)}$, $\bar{X}_{i-1}[k] = 0$.

Define $\hat{\omega}_{i-1}$, q, Ω_X, and Ω_W as in the proof of Theorem 3.3.3. If $\omega \in \Omega_X$, then from (3.27), for all $k \geq \dim \mathfrak{K}$, $\|P_i \omega\| \leq \|\hat{\omega}_{i-1}\|$. Since $\|\bar{W}_0\| = 0$, if $\omega \in \Omega_W$, then from (3.28),

$$\|P_i \omega\| \leq \|\hat{\omega}_{i-1}\| + (1 + \|\imath_{i-1}\|)(\mu \|\imath_0\|)^{|\omega|-1} \|W\| \|\bar{X}_0\|^{|\omega|-1},$$

which for $k \geq \dim \mathfrak{K}$, simplifies to $\|P_i \omega\| \leq \|\hat{\omega}_{i-1}\|$. Since every word ω has at least one letter in $\widetilde{X}$, the induction hypothesis implies $\hat{\omega}_{i-1} = 0$ for all $k \geq i \dim \mathfrak{g} - \sum_{j=1}^{i} \dim \mathfrak{h}^{(j)}$. Therefore, for all $k \geq i \dim \mathfrak{g} - \sum_{j=1}^{i} \dim \mathfrak{h}^{(j)}$, the quotient dynamics reduce to

$$\bar{X}_i^+ = \bar{A}_i \bar{X}_i,$$

where $\rho(\bar{A}_i) = 0$, and so $\bar{A}_i^{\dim\left(\mathfrak{g}/\mathfrak{h}^{(i+1)}\right)} = 0$, where $\dim\left(\mathfrak{g}/\mathfrak{h}^{(i+1)}\right) = \dim \mathfrak{g} - \dim \mathfrak{h}^{(i+1)}$; in particular, $\dim \mathfrak{K} = \dim \mathfrak{g} - \dim \mathfrak{h}$. Thus, for all $k \geq (i+1) \dim \mathfrak{g} - \sum_{j=1}^{i+1} \dim \mathfrak{h}^{(j)}$, $\bar{X}_i[k]$ is zero.

Since p is the nilindex of $\mathfrak{h}$, we have $P_p \mathfrak{g} = \mathfrak{g}/\mathfrak{h}^{(p+1)} = \mathfrak{g}/0 \cong \mathfrak{g}$, and so the induction terminates at $i = p$. Consequently, for all $k \geq (p+1) \dim \mathfrak{g} - \sum_{j=1}^{p} \dim \mathfrak{h}^{(j)}$, $X[k] = 0$. □

Corollary 3.3.8. *Consider the dynamics* (3.1)*. Let $\mathfrak{g}$ be a solvable Lie algebra and $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ be a Lie function that satisfies Assumptions 1(b) and 1(c). If $\rho(A) = 0$ and for all $k \geq 0$, $W[k] \in \mathfrak{h}^r$, then the origin of $\mathcal{X}$ is globally attractive.*

Proof. By Theorem 3.3.7, the state X tends to the origin for any initial conditions. □

Corollary 3.3.9. *Consider the dynamics* (3.1)*. Let $\mathfrak{g}$ be a solvable Lie algebra and $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ be a Lie function that satisfies Assumptions 1(b) and 1(c). If $\rho(A) = 0$, there exists $\beta \geq 0$ such that $\|W\| \leq \beta$, and for all $k \geq 0$, $W[k] \in \mathfrak{h}^r$, then the origin of $\mathcal{X}$ is semiglobally exponentially stable.*

Proof. By Theorem 3.3.7, $X[k]$ converges to zero in finite time. Define $\bar{k} := \arg\min_k \{X[k] = 0\}$ and let $M \geq 0$ be arbitrary. Since $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is continuous, $\|X[k]\|$ attains its maximum on the compact set $\{X[k] : 0 \leq k \leq \bar{k},\ \|W[k]\| \leq \beta,\ \|X[0]\| \leq M\}$. Choosing any $\lambda \in [0, 1)$, there exists finite $\alpha > 0$ such that $\|X[k]\| \leq \alpha\lambda^k \|X[0]\|$, where α depends on $\|X[0]\|$ and β. □

Remark 3.3.10. Theorem 3.3.7 and Corollaries 3.3.8 and 3.3.9 easily extend to the case where there exists $k_{\mathfrak{h}} \in \mathbb{Z}_{\geq 0}$ such that, for all $k \geq k_{\mathfrak{h}}$, $W[k] \in \mathfrak{h}^r$, but $W[0]$ is not necessarily in $\mathfrak{h}^r$. ♦

Example 3.3.1. Consider the 6-dimensional Lie algebra of 4×4 real upper triangular matrices, with basis $\{t_1, \ldots, t_6\}$, such that the nonvanishing Lie brackets are given by

$$[t_1, t_4] = t_4,\ [t_1, t_6] = t_6,\ [t_2, t_4] = -t_4,\ [t_2, t_5] = t_5,\ [t_3, t_5] = -t_5,\ [t_3, t_6] = -t_6,\ [t_4, t_5] = t_6.$$

The derived algebra is $\mathfrak{h} = \mathrm{Lie}_{\mathbb{R}}\{t_4, t_5, t_6\}$, which has lower central series $\mathfrak{h} =: \mathfrak{h}^{(1)} \supset \mathfrak{h}^{(2)} \supset \mathfrak{h}^{(3)} = 0$, where $\mathfrak{h}^{(2)} = \mathrm{Lie}_{\mathbb{R}}\{h_6\} \cong \mathrm{Span}_{\mathbb{R}}\{h_6\}$. We remark that the derived algebra $\mathfrak{h}$ and the Heisenberg algebra are isomorphic as Lie algebras.

We will consider a dynamical system driven by the exogenous signal $W := (W_1, W_2) \in \mathfrak{g}^2 =: \mathcal{W}$

$$\begin{aligned} W_1^+ &= 2\left(1 - k(1.1)^{-0.5k}\right)\sin(10k)W_0 \\ W_2^+ &= \left(2 - k^2(1.1)^{-2k}\right)\cos(20k)W_0, \end{aligned}$$

where $W_0 = t_4 + 7t_5 + 6t_6 \in \mathfrak{h}$. Note that W is bounded.

Consider the dynamical system with state $X := (X_1, X_2) \in \mathfrak{g}^2 =: \mathcal{X}$

$$\begin{aligned} X_1^+ &= \frac{1}{2}\exp(W_1)X_1\exp(-W_1) - \exp(X_2)X_1\exp(-X_2) + \frac{1}{2}\exp(W_2)X_2\exp(-W_2) \\ X_2^+ &= \frac{1}{2}\exp(X_2)X_1\exp(-X_2) + \frac{1}{4}\exp(X_1 + W_1)X_2\exp(-(X_1 + W_1)), \end{aligned}$$

where for all $Y \in \mathfrak{g}$, $\exp(Y)X_i\exp(-Y) \in \mathfrak{g}$ [41, Propositions 2.16, 2.17]. To see that these dynamics are indeed a Lie function, we use $\exp(Y)X_i\exp(-Y) = e^{\operatorname{ad}_Y}X_i$ [41, Proposition 3.35]:

$$\begin{aligned}
X_1^+ &= \left(\frac{1}{2}e^{\operatorname{ad}_{W_1}} - e^{\operatorname{ad}_{X_2}}\right)X_1 + \frac{1}{2}e^{\operatorname{ad}_{W_2}}X_2\\
X_2^+ &= \frac{1}{2}e^{\operatorname{ad}_{X_2}}X_1 + \frac{1}{4}e^{\operatorname{ad}_{X_1+W_1}}X_2.
\end{aligned}$$

Recall $e^{\operatorname{ad}_Y} = \operatorname{Id}_{\mathfrak{g}} + \operatorname{ad}_Y + \frac{1}{2!}\operatorname{ad}_Y^2 + \frac{1}{3!}\operatorname{ad}_Y^3 + \cdots$, yielding

$$\begin{aligned}
X_1^+ &= -\frac{1}{2}X_1 + \frac{1}{2}X_2 + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left(\left(\frac{1}{2}\operatorname{ad}_{W_1}^{\ell} - \operatorname{ad}_{X_2}^{\ell}\right)X_1 + \frac{1}{2}\operatorname{ad}_{W_2}^{\ell}X_2\right)\\
X_2^+ &= \frac{1}{2}X_1 + \frac{1}{4}X_2 + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left(\frac{1}{2}\operatorname{ad}_{X_2}^{\ell}X_1 + \frac{1}{4}\operatorname{ad}_{X_1+W_1}^{\ell}X_2\right).
\end{aligned}$$

Using the basis $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ for $\mathfrak{g}$, and letting $I_6 \in \mathbb{R}^{6\times 6}$ be the identity matrix, we can express the dynamics of X as

$$X^+ = \underbrace{\left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4}\end{bmatrix} \otimes I_6\right)}_{\operatorname{Mat}A} X + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\begin{bmatrix} \left(\frac{1}{2}\operatorname{ad}_{W_1}^{\ell} - \operatorname{ad}_{X_2}^{\ell}\right)X_1 + \frac{1}{2}\operatorname{ad}_{W_2}^{\ell}X_2 \\ \frac{1}{2}\operatorname{ad}_{X_2}^{\ell}X_1 + \frac{1}{4}\operatorname{ad}_{X_1+W_1}^{\ell}X_2 \end{bmatrix}.$$

We now verify that Assumption 1 is satisfied. For all $Y \in \mathfrak{g}$, $\|\operatorname{ad}_Y^{\ell}X_i\| \leq \mu^{\ell}\|Y\|^{\ell}\|X_i\|$, yielding

$$\|e^{\operatorname{ad}_Y}X_i\| \leq \sum_{\ell=0}^{\infty}\frac{(\mu\|Y\|)^{\ell}}{\ell!}\|X_i\| = e^{\mu\|Y\|}\|X_i\| < \infty,$$

so the dynamics of X belong to class-$\mathcal{A}$, thereby satisfying Assumption 1(a).

That $X = 0$ is an equilibrium is verified by substituting $X = 0$ into the dynamics. To verify that $X = 0$ is the only equilibrium, recall that the derived algebra is $\operatorname{Lie}_{\mathbb{R}}\{t_4, t_5, t_6\}$, so a point is an equilibrium only if

$$P_0X = \underbrace{\left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4}\end{bmatrix} \otimes I_3\right)}_{\operatorname{Mat}\bar{A}_0} P_0X,$$

where $\rho(\bar{A}_0) = \bigsqcup_{i=1}^{3} \left\{-\frac{3}{4}, \frac{1}{2}\right\}$, implying that $\bar{A}_0$ is bijective. Therefore, a point can be an equilibrium only if $P_0 X = 0$, or equivalently, $X \in \mathfrak{h}^2$. As mentioned, $\mathfrak{h}$ is isomorphic to the Heisenberg algebra, so the rest of the argument that Assumption 1(b) is satisfied is similar to that in Example 3.2.1.

It is clear from the form of Mat$A$ that $A\mathfrak{h}_i^2 \subseteq \mathfrak{h}_i^2$. By Corollary 3.1.13, this verifies Assumption 1(c).

From Mat$A$, we find $\rho(A) = \bigsqcup_{i=1}^{6} \left\{-\frac{3}{4}, \frac{1}{2}\right\}$. Thus, by Theorem 3.3.3, if the limiting supremum of W is sufficiently small, then the origin of $\mathcal{X}$ is globally attractive. By Corollary 3.3.6, if W is bounded sufficiently small, then the origin is globally asymptotically stable. We illustrate simply that for the arbitrary choice of W in this example, that $X \to 0$ as $k \to \infty$, as seen in Figure 3.2. ▲

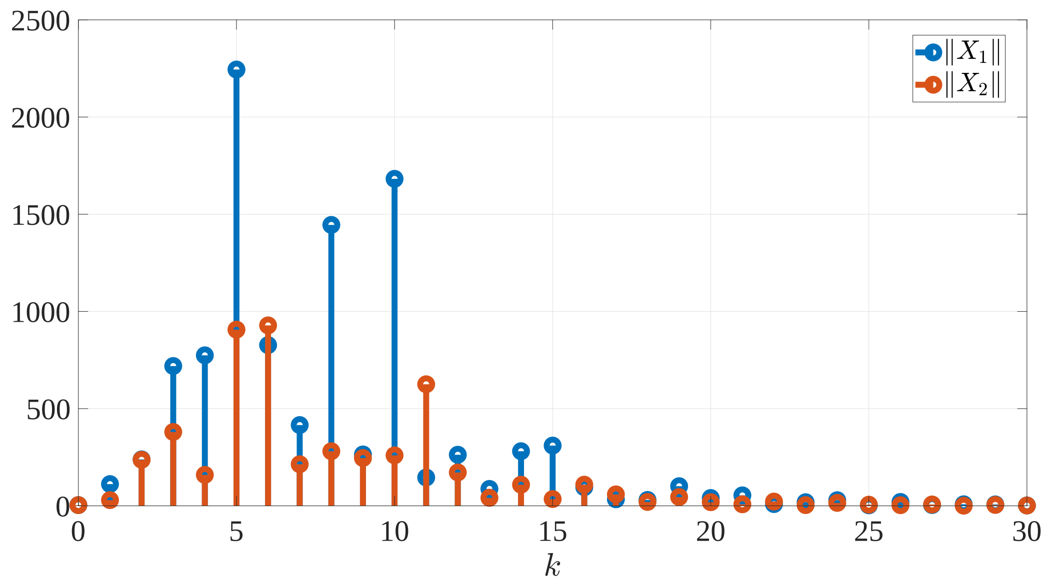


Figure 3.2: The norms of the states $X_1, X_2 \in \mathfrak{g}$.

Remark 3.3.11. In this chapter, we have studied series of words over letters in $\widetilde{X}$ and $\widetilde{W}$. In Chapter 5, we will encounter words of this form, as well as words where these letters are acted on by linear maps.

Consider a collection of linear operators $M_i : \mathfrak{g} \to \mathfrak{g}$, satisfying $M_i \mathfrak{h}^{(j)} \subseteq \mathfrak{h}^{(j)}$ for all j, where $i \in \mathcal{I} \subseteq \mathbb{N}$, such that $M := \sup_{i \in \mathbb{N}} \|M_i\| < \infty$, and $\mathfrak{h}$ is the nilpotent ideal satisfying

Assumption 1(c). If each word ω in (3.1) of length at least 2 were instead of the form

$$\omega = [M_{i_{\omega_1}} Y_{\omega_1}, [\ldots, M_{i_{\omega_{|\omega|}}} Y_{\omega_{|\omega|}}] \cdots],$$

then all our results still hold. The invariance property of the maps admits straightforward extensions of Lemmas 3.2.2 and 3.3.1. Then, in the proof of Theorem 3.2.3, the presence of these linear maps can easily be accounted for by multiplying γ_i by a factor of $\max\{M^i, 1\}$ in the bound determined in Claim 3.2.4. In the proof of Theorem 3.3.3, the linear maps merely result in a scaled estimate of the basin of attraction for the linearized dynamics, in particular, the estimate of the basin of attraction would be scaled by a factor of M^{-1}. ♦

Chapter 4

Synchronization of Homogeneous Networks on Exponential Matrix Lie Groups

In this chapter, we examine the synchronization of networks of identical continuous-time kinematic agents on a matrix Lie group, controlled by discrete-time controllers with constant sampling periods and directed, weighted communication graphs with a globally reachable node. We present a distributed discrete-time control law that achieves global synchronization on exponential matrix Lie groups. As characterized in Theorem 4.4.8, synchronization is generally asymptotic, but if the Lie group is nilpotent, then synchronization on the associated Lie algebra is achieved at an exponential rate, as characterized in Proposition 4.4.6. We first linearize the synchronization error dynamics at identity, and show that the proposed controller achieves local exponential synchronization on any Lie group, as characterized in Corollary 4.3.3. Building on the local analysis, we show that, if the Lie group is exponential, then synchronization is global. Proposition 4.4.10 provides conditions for deadbeat convergence when the communication graph is unweighted and complete.

4.1 Introduction

Synchronization, or consensus, has received a tremendous amount of attention in the literature [113, 90]. In engineering, synchronization captures problems such as satellite at-

A preliminary version of this work was presented at the 2018 American Control Conference [73].

titude alignment and vehicle formation control. Such systems are naturally modelled in a Lie group framework. Synchronization of networks on $\mathsf{SE}(3)$ was achieved using passivity in [47]; synchronization under sampling was studied for a network of Kuramoto-like oscillators in [35] and harmonic oscillators with a time-varying period in [105], and path following in [48]. The Kuramoto network model was extended from $\mathsf{SO}(2)$ to $\mathsf{SO}(n)$ in [65]. A framework for coordinated motion on Lie groups was developed in [94], where the synchronization problem that we consider is a special case of what the authors call "bi-invariant coordination". In [29], linear consensus algorithms were applied to continuous-time networks on $\mathsf{SE}(3)$ with unweighted communication graphs that are complete, ring-shaped, or spanning trees.

We present a control law that achieves global synchronization for a network of identical agents on exponential Lie groups—meaning the exponential map is a global diffeomorphism—with driftless dynamics and a directed communication graph with a globally reachable node. We use the stability results of Chapter 3 to generalize and extend the results of [73], which considered only unweighted graphs with agents on one-parameter Lie subgroups. The controller requires that each agent have access to its relative state with respect to each of its neighbours. We show that, irrespective of the Lie group, if the sampled dynamics achieve synchronization, then the agents synchronize in continuous-time as well. If the Lie group is exponential, then we show that the proposed controller achieves global asymptotic synchronization. In the special case that the Lie group is simply connected and nilpotent, and hence exponential, synchronization is achieved at an exponential rate. We also provide conditions for global deadbeat convergence on any exponential Lie group when the communication graph is unweighted and complete. These results hold locally when the Lie group is only solvable, rather than exponential, or nilpotent but not simply connected.

We use weighted, directed graphs to model communication constraints between agents. A graph $\mathcal{G}$ is a triple $(\mathcal{V}, \mathscr{E}, w)$ consisting of a finite set of vertices $\mathcal{V} = \mathbb{N}_N$, a set of edges $\mathscr{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weight function $w : \mathscr{E} \to \mathbb{R}_{\geq 0}$. If agent i has access to its relative state with respect to agent j, then $(i,j) \in \mathscr{E}$. A graph is *complete* if for all $i, j \in \mathcal{V}$, $(i,j) \in \mathscr{E}$. Define vertex i's neighbour set as $\mathcal{N}_i \coloneqq \{j \in \mathbb{N}_N : (i,j) \in \mathscr{E}\}$. A node $i \in \mathcal{V}$ is *globally reachable*, if for all $k \in \mathcal{V}$, $k \neq i$, there exists a sequence $\{j_1, \ldots, j_\ell\} \subset \mathcal{V}$, such that $(i, j_1), (j_2, j_1), \ldots, (k, j_\ell) \in \mathscr{E}$. The weight $w_{ij} \coloneqq w((i,j))$ is nonzero only if $(i,j) \in \mathscr{E}$. We assume that $\mathcal{G}$ has no self-loops. A graph is *unweighted* if, for all $i \neq j \in \mathbb{N}_N$, $w_{ij} \in \{0, 1\}$. Associated with $\mathcal{G}$ is the Laplacian $L \in \mathbb{R}^{N \times N}$, defined elementwise as

$$L_{ij} = \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{j \in \mathcal{N}_i} w_{ij}, & i = j. \end{cases}$$

4.2 Sampled-Data Synchronization Problem

We consider a network of N controlled agents, each modelled by the differential equation

$$\dot{X}_i = X_i\left(B_i u_i\right), \qquad i \in \mathbb{N}_N. \tag{4.1}$$

Here, $X_i \in \mathsf{G}$, where G is an m-dimensional subgroup of the complex general linear group $\mathsf{GL}(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries, which itself contains the real general linear group $\mathsf{GL}(n, \mathbb{R})$ as a proper subgroup. We allow for such generality in the choice of G in only Section 4.3, as it allows us to establish local results on any Lie group. Our strongest results, discussed in Section 4.4, hold when G is an *exponential Lie group*, which is a special case of a solvable Lie group. The control input to each agent is $u_i \in \mathbb{F}^m$ and $B_i : \mathbb{F}^m \to \mathfrak{g}$ are linear maps. Equation (4.1) is a kinematic model of a system evolving on a matrix Lie group G. Each agent is assumed to be fully actuated in the sense that

$$\operatorname{Im} B_i = \mathfrak{g}.$$

We are interested in the sampled-data control of this multi-agent system in which each agent's control law is implemented on an embedded computer, which we explicitly model using the setup in Figure 4.1. The blocks H and S in Figure 4.1 are, respectively, the ideal hold and sample operators. Sample and hold are, respectively, idealized models of A/D and D/A conversion. The following assumption is made throughout this chapter.

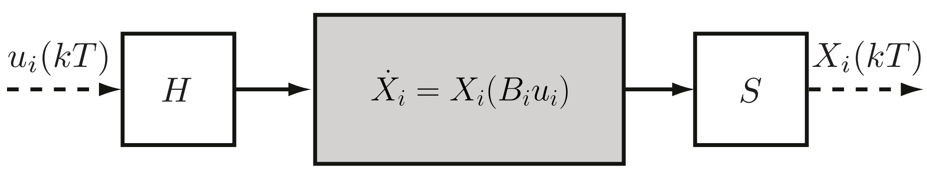


Figure 4.1: Sampled-data agent on a matrix Lie group G.

Assumption 2. *All sample and hold blocks operate at the same period $T > 0$ and the blocks are synchronized for the multi-agent system* (4.1).

Under Assumption 2, letting $X_i[k] \coloneqq X_i(kT)$ and $u_i[k] \coloneqq u_i(kT)$, the discretized dynamics of each agent are given by

$$X_i^+ = X_i \exp\left(T B_i u_i\right), \qquad i \in \mathbb{N}_N \tag{4.2}$$

which is an exact discretization of (4.1).

Given a network of N agents with kinematic dynamics (4.1) or (4.2), define the error quantities $E_{ij} \coloneqq X_i^{-1}X_j$, $i, j \in \mathbb{N}_N$. Observe that $E_{ij} = I$ if and only if, $X_i = X_j$. The error matrix E_{ij} is called left-invariant [58], since for all $X \in \mathsf{G}$, $(XX_i)^{-1}(XX_j) = X_i^{-1}X_j$.

Local Synchronization on Matrix Lie Groups : Given a network of N agents with continuous-time dynamics (4.1), sampling period $T > 0$ and a weighted communication graph $\mathcal{G} = (\mathcal{V}, \mathscr{E}, w)$ with a globally reachable node, find, if possible, distributed control laws u_i, $i \in \mathbb{N}_N$, such that for all initial errors in a neighbourhood of the identity in G^N, for all $i, j \in \mathbb{N}_N$, $E_{ij} \to I$ as $t \to \infty$. •

By a distributed control law we mean that agent i's control signal u_i depends on E_{ij} only if $(i, j) \in \mathscr{E}$. We propose the distributed feedback control law

$$u_i = \frac{1}{T} B_i^{-1} \operatorname{Log}\left(\left(\prod_{j \in \mathcal{N}_i} E_{ij}^{w_{ij}}\right)^{\frac{1}{K}}\right), \tag{4.3}$$

where $K \in \mathbb{R}$ is a gain.

The gain of exactly $1/T$ in (4.3) greatly simplifies our analysis by eliminating the sampling period T in the plant and error dynamics. However, this gain is not truly an independent parameter. The control law (4.3) simplifies to

$$u_i = \frac{1}{TK} \operatorname{Log}\left(\prod_{j \in \mathcal{N}_i} E_{ij}^{w_{ij}}\right),$$

so the gain of $1/T$ could be absorbed into the gain of $1/K$ without loss of generality, but we will not do this.

The control law (4.3) does not require agent i to know agent j's state X_j, nor its own state X_i, but instead requires knowledge of the relative state E_{ij}. The expression (4.3) is well-defined so long as the product $\prod_{j \in \mathcal{N}_i} E_{ij}$ has no eigenvalues in $\mathbb{R}^-$, as discussed in Chapter 2; this condition is always satisfied when when the relative states are sufficiently close to the identity. The control law (4.3) is motivated by exponential coordinates for Lie groups, classical consensus algorithms in $\mathbb{R}^n$, and the notion of Riemannian mean of rotations on $\mathsf{SO}(3)$, which on a one-parameter subgroup thereof can be explicitly computed as $\prod_{i=1}^{N} R_i^{\frac{1}{N}}$ [76]. When the control law (4.3) is well-defined, the closed-loop discrete-time dynamics are

$$X_i^+ = X_i \left(\prod_{j \in \mathcal{N}_i} E_{ij}^{w_{ij}}\right)^{\frac{1}{K}}, \qquad i \in \mathbb{N}_N \tag{4.4}$$

and the synchronization error dynamics are

$$\begin{aligned}E_{ij}^{+} &= \left(X_i^{+}\right)^{-1} X_j^{+} \\ &= \left(\prod_{p\in\mathcal{N}_i} E_{ip}^{w_{ip}}\right)^{-\frac{1}{K}} X_i^{-1} X_j \left(\prod_{q\in\mathcal{N}_j} E_{jq}^{w_{jq}}\right)^{\frac{1}{K}},\end{aligned}$$

which, using the definition of E_{ij}, yields

$$E_{ij}^{+} = \left(\prod_{p\in\mathcal{N}_i} E_{ip}^{w_{ip}}\right)^{-\frac{1}{K}} E_{ij} \left(\prod_{q\in\mathcal{N}_j} E_{jq}^{w_{jq}}\right)^{\frac{1}{K}}. \tag{4.5}$$

Remark 4.2.1. The order of multiplication in (4.3) need not be common to all agents or even constant. ♦

A key advantage of direct design over emulation, is that stability can be guaranteed at the sampling instants. As mentioned in Section 4.1, on $\mathsf{SE}(3)$, the relative error E_{ij} can be computed using machine vision, where the speed of sampling is limited by the frame rate of the camera, for example, 25 Hz [69]. This limits the feasibility of emulation-based design. However, direct design does not guarantee good performance between sampling instants. But in the specific case of the plant and problem discussed in this chapter, achieving synchronization at the sampling instants implies synchronization between the sampling instants.

Proposition 4.2.2. *Suppose that each agent's controller u_i is continuous in the synchronization errors,*[2] *and vanishes if synchronization is achieved at the sampling instants. If the agents synchronize at the sampling instants, then under the plant dynamics (4.1), synchronization is achieved.*

Proof. If $E_{ij}[k] \to I$, then $u_i[k], u_j[k] \to 0$. Fix $0 < \delta < T$, then

$$\begin{aligned}\lim_{k\to\infty} E_{ij}(kT+\delta) &= \lim_{k\to\infty} \exp\left(\delta B_i u_i[k]\right)^{-1} E_{ij}[k] \exp\left(\delta B_j u_j[k]\right) \\ &= \lim_{k\to\infty} \exp\left(\delta B_i u_i[k]\right)^{-1} \lim_{k\to\infty} E_{ij}[k] \lim_{k\to\infty} \exp\left(\delta B_j u_j[k]\right) \\ &= I^3 = I.\end{aligned}$$

Since δ is arbitrary, this implies that $E_{ij}(t) \to I$. □

[2] As is the case with the proposed controller (4.3).

Proposition 4.2.2 means that asymptotically stabilizing the set where $E_{ij} = I$, for all $i, j \in \mathbb{N}_N$, at the sampling instants is sufficient for solving the synchronization problem. Thus, hereinafter we can conduct all analysis in the discrete-time setting and do not rely on T being sufficiently small.

4.3 Linear Analysis on General Lie Groups

We first show that the linearization of the error dynamics (4.5) is exponentially stable. This will establish that the proposed control law achieves synchronization on any Lie group if the agents are initialized sufficiently close to one another. In the next section, leveraging the results of Chapter 3, we will use the stability of the linearization to establish global stability on exponential Lie groups.

Using (2.2), we establish conditions on the controller gain $K \in \mathbb{R}$ for local stability. First, we apply the identity that, for all $i, j \in \mathbb{N}_N$, $E_{ij} = E_{1i}^{-1} E_{1j}$ to (4.5):

$$E_{ij}^{+} = \left(\prod_{p \in \mathcal{N}_i} \left(E_{1i}^{-1} E_{1p} \right)^{w_{ip}} \right)^{-\frac{1}{K}} E_{1i}^{-1} E_{1j} \left(\prod_{q \in \mathcal{N}_j} \left(E_{1j}^{-1} E_{1q} \right)^{w_{jq}} \right)^{\frac{1}{K}}. \tag{4.6}$$

From (2.2), the linearization of the dynamics of the local error $\mathcal{E}_{ij} \coloneqq \mathrm{Log}(E_{ij})$ is

$$\begin{aligned}
\mathcal{E}_{ij}^{+} &\approx \mathcal{E}_{1j} - \mathcal{E}_{1i} - \frac{1}{K} \sum_{p \in \mathcal{N}_i} w_{ip} (\mathcal{E}_{1p} - \mathcal{E}_{1i}) + \frac{1}{K} \sum_{q \in \mathcal{N}_j} w_{jq} (\mathcal{E}_{1q} - \mathcal{E}_{1j}) \\
&= \mathcal{E}_{1j} - \mathcal{E}_{1i} - \frac{1}{K} \left(\sum_{p \in \mathcal{N}_i} w_{ip} \mathcal{E}_{1p} - \left(\sum_{p \in \mathcal{N}_i} w_{ip} \right) \mathcal{E}_{1i} \right) + \frac{1}{K} \left(\sum_{q \in \mathcal{N}_j} w_{jq} \mathcal{E}_{1q} - \left(\sum_{q \in \mathcal{N}_j} w_{jq} \right) \mathcal{E}_{1j} \right) \\
&= \mathcal{E}_{1j} - \mathcal{E}_{1i} + \frac{1}{K} ((e_1^\top L) \otimes \mathrm{Id}_{\mathfrak{g}}) \mathcal{E} - \frac{1}{K} ((e_j^\top L) \otimes \mathrm{Id}_{\mathfrak{g}}) \mathcal{E} \\
&= \mathcal{E}_{1j} - \mathcal{E}_{1i} + \frac{1}{K} ((e_1 - e_j)^\top L) \otimes \mathrm{Id}_{\mathfrak{g}}) \mathcal{E},
\end{aligned}$$

where $\mathcal{E} \coloneqq (\mathcal{E}_{11}, \ldots, \mathcal{E}_{1N}) \in \mathfrak{g}^N$. Setting $i = 1$ and "stacking" the last line over all j, we obtain

$$
\begin{aligned}
\mathcal{E}^+ &\approx \left(\left((I_N - \mathbf{1}_N e_i^\top) + \frac{1}{K}(\mathbf{1}_N e_i^\top - I_N) L \right) \otimes \mathrm{Id}_{\mathfrak{g}} \right) \mathcal{E} \\
&= \underbrace{\left(\overbrace{\left((I_N - \mathbf{1}_N e_1^\top) \left(I_N - \frac{1}{K} L \right) \right)}^{M :=} \otimes \mathrm{Id}_{\mathfrak{g}} \right)}_{=: A} \mathcal{E}.
\end{aligned} \tag{4.7}
$$

The eigenvalues of the state matrix $A = M \otimes \mathrm{Id}_{\mathfrak{g}}$ in (4.7) are the m-times-repeated eigenvalues of M [7, Chapter 12, §5]. The linear dynamics (4.7) are (exponentially) stable if and only if the matrix M is Schur. Therefore, we now establish sufficient conditions on the gain K.

Lemma 4.3.1. *The spectrum of M is*

$$(\sigma(I_N - L/K) \setminus \{1\}) \sqcup \{0\}.$$

Proof. Let $\mathcal{V} \coloneqq \mathrm{span}_{\mathbb{R}}\{\mathbf{1}_N\}$. For any Laplacian, we have $\operatorname{Ker} L \subseteq \mathcal{V}$, which implies $(I_N - L/K)|\mathcal{V} = 1$. Direct calculation verifies $\operatorname{Ker}(I_N - \mathbf{1}_N e_1^\top) = \mathcal{V}$. This means that $\mathcal{V} \subseteq \operatorname{Ker} M$. The spectrum of a map under which $\mathcal{V}$ is invariant equals the disjoint union of the spectra of its own and induced map. Thus $\sigma(M) = \sigma(M|\mathcal{V}) \sqcup \sigma(\bar{M})$, where $\sigma(M|\mathcal{V}) = 0$.

Since $\mathcal{V}$ is invariant under both $(I_N - \mathbf{1}_N e_1^\top)$ and $(I_N - L/K)$, the map induced on $\mathbb{R}^N/\mathcal{V}$ by their composition is the composition of their respective induced maps, i.e., $\bar{M} = \overline{(I_N - \mathbf{1}_N e_1^\top)} \cdot \overline{(I_N - L/K)}$. In the basis $\{\bar{e}_2, \ldots, \bar{e}_N\}$, $\overline{(I_N - \mathbf{1}_N e_1^\top)} = I_{N-1}$, thus $\bar{M}$ is similar to $\overline{(I_N - L/K)}$. Since the eigenvalue of $(I_N - L/K)$ associated with the eigenvector $\mathbf{1}_N$ is 1, $\sigma\overline{(I_N - L/K)} = \sigma(I_N - L/K) \setminus \{1\}$. □

Lemma 4.3.1 holds given the Laplacian of any graph, but under our hypothesis that $\mathcal{G}$ has a globally reachable node, L has a simple zero eigenvalue [1, Theorem 4], which, by the Spectral Mapping Theorem, implies that $(I_N - L/K)$ has a simple eigenvalue of 1. The matrix $(I_N - L/K)$ is the Perron matrix from the discrete-time linear consensus problem [86, §II.C]. Lemma 4.3.1 asserts that the spectrum of M is the spectrum of $(I_N - L/K)$, where the simple eigenvalue of 1 has been replaced with a simple eigenvalue of 0. Thus, Lemma 4.3.1 establishes a local equivalence between discrete-time linear consensus and synchronization of kinematic systems on arbitrary Lie groups using the proposed controller (4.3). Thus, the controller gain $K \in \mathbb{R}$ can be chosen according to any of the myriad well-established criteria in the discrete-time linear consensus problem, e.g., $K = N \max_{i,j}\{w_{ij}\}$.

Assumption 3. *The gain $K \in \mathbb{R}$ solves the discrete-time linear consensus problem for the directed graph $\mathcal{G}$ with Laplacian L.*

Lemma 4.3.2. *Under Assumption 3, the origin is exponentially stable under the linearized Lie algebra error dynamics (4.7).*

Proof. We have $\sigma(M \otimes \mathrm{Id}_{\mathfrak{g}}) = \bigsqcup_{i=1}^{m} \sigma(M)$. Under Assumption 3, by Lemma 4.3.1, $M \otimes \mathrm{Id}_{\mathfrak{g}}$ is Schur. $\square$

Corollary 4.3.3. *For any Lie group G with communication graph $\mathcal{G}$ containing a globally reachable node, if Assumption 3 holds, then the equilibrium $\{E_{ij} = I : i, j \in \mathbb{N}_N\}$ is locally exponentially stable.*

4.4 Synchronization on Exponential Lie Groups

In this section, we consider the case where G is exponential, and therefore that its Lie algebra $\mathfrak{g}$ is solvable. Under this hypothesis, we identify the error $E_{ij} \in \mathsf{G}$ with its logarithm, $\mathcal{E}_{ij} \in \mathfrak{g}$, and examine the network's synchronization error on $\mathfrak{g}^N$. We prove that synchronization is achieved for any initial conditions when G is exponential. To do this, we use the results of Chapter 3.

4.4.1 Synchronization on Exponential Lie Groups on Networks with a Globally Reachable Node

To apply the results of Chapter 3, we must verify that the error dynamics (4.5) satisfy Assumption 1 when expressed on $\mathfrak{g}^N$. Applying the generalized BCH to the reformulated global error dynamics (4.6), a straightforward but tedious computation verifies

$$\begin{aligned}
\mathcal{E}_{ij}^{+} = \mathcal{E}_{1j} - \mathcal{E}_{1i} + \frac{1}{K}((e_i - e_j)^\top L) \otimes \mathrm{Id}_{\mathfrak{g}})\mathcal{E} \\
- \left(1 + \frac{1}{2K}\sum_{p\in\mathcal{N}_i} w_{ip} - \frac{1}{2K}\sum_{q\in\mathcal{N}_j} w_{jq} + \frac{1}{2K^2}\sum_{p\in\mathcal{N}_i}\sum_{q\in\mathcal{N}_j} w_{ip}w_{jq}\right)[\mathcal{E}_{1i}, \mathcal{E}_{1j}] \\
- \frac{1}{2K}\left(1 - \frac{1}{K}\sum_{q\in\mathcal{N}_j} w_{jq}\right)\sum_{p\in\mathcal{N}_i} w_{ip}[\mathcal{E}_{1p}, \mathcal{E}_{1j}] \\
- \frac{1}{2K}\left(1 - \frac{1}{K}\sum_{p\in\mathcal{N}_i} w_{ip}\right)\sum_{q\in\mathcal{N}_j} w_{jq}[\mathcal{E}_{1i}, \mathcal{E}_{1q}] \\
+ \frac{1}{2K}\sum_{\substack{r,s\in\mathcal{N}_i \\ r<s}} w_{ir}w_{is}\left([\mathcal{E}_{1i}, \mathcal{E}_{1s}] + [\mathcal{E}_{1r}, \mathcal{E}_{1i}] - [\mathcal{E}_{1r}, \mathcal{E}_{1s}]\right) \\
+ \frac{1}{2K}\sum_{\substack{u,v\in\mathcal{N}_j \\ u<v}} w_{ju}w_{jv}\left(-[\mathcal{E}_{1j}, \mathcal{E}_{1v}] - [\mathcal{E}_{1u}, \mathcal{E}_{1j}] + [\mathcal{E}_{1u}, \mathcal{E}_{1v}]\right) \\
- \frac{1}{2K^2}\sum_{p\in\mathcal{N}_i}\sum_{q\in\mathcal{N}_j} w_{ip}w_{jq}[\mathcal{E}_{1p}, \mathcal{E}_{1q}] + \cdots,
\end{aligned}$$

where the first line was already found in Section 4.3. The exact form of the Lie bracket terms is not our main focus, rather, we wish to impress upon the reader that the error dynamics on $\mathfrak{g}^N$ are in the form of (3.1), which we explicitly rewrite as

$$\mathcal{E}^{+} = f(\mathcal{E}) = A\mathcal{E} + \sum_{|\omega|\geq 2} c_\omega \otimes \omega, \tag{4.8}$$

where $A : \mathfrak{g}^N \to \mathfrak{g}^N$ is defined in (4.7), the words ω are over the letters $\mathcal{E}_{11}, \ldots, \mathcal{E}_{1N}$, and the coefficients c_ω are determined by the repeated application of the generalized BCH and collecting like terms. This establishes that the error dynamics on $\mathfrak{g}^N$ are a Lie function. To verify that (4.8) satisfies Assumption 1, we begin with Assumption 1(a).

Lemma 4.4.1. *The error dynamics* (4.8) *belong to class-$\mathcal{A}$.*

Proof. The generalized BCH is class-$\mathcal{A}$ [25, Proof of Theorem 6]. Therefore,

$$\mathrm{Log}(\exp(-TB_i u_i)E_{ij}\exp(TB_j u_j))$$

converges in the class-$\mathcal{A}$ sense of Definition 3.1.4 for TB_iu_i, TB_ju_j, $\mathcal{E}_{ij}$ sufficiently small. Recalling the choice of u_i in (4.3), we see that u_i converges in the class-$\mathcal{A}$ sense if, for all $p \in \mathcal{N}_i$, $\mathcal{E}_{ip}$ is sufficiently small. $\square$

The verification of Assumption 1(b) is facilitated by first verifying that Assumption 1(c) holds, which is a corollary of a stronger result, which we present on its own, as it could be of independent interest.

Lemma 4.4.2. *Every Lie subalgebra of $\mathfrak{g}^N$ is invariant under the dynamics* (4.8).

Proof. A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ generates a unique connected Lie subgroup $\mathsf{H} \subseteq \mathsf{G}$ whose Lie algebra is $\mathfrak{h}$ [41, Theorem 5.20].

The error dynamics (4.6) are the composition of finitely many group products and powers, under which any Lie subgroup on which the latter operation is well-defined, is invariant. Consequently, if $E \in \mathsf{H}^N$, then $E^+ \in \mathsf{H}^N$. It follows immediately that $\mathcal{E} \in \mathfrak{h}^N$ implies $\mathcal{E}^+ \in h^N$. $\square$

Corollary 4.4.3. *Given any nilpotent ideal $\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}]$ with nilindex p, $\mathfrak{h}$ and $\left(\mathfrak{h}^{(i)}\right)^N$, $i \in \mathbb{N}_p$, are invariant under the dynamics* (4.8).

In Proposition 3.1.14, it is shown that, if there exists an f-invariant ideal $\mathfrak{h}^N \subset \mathfrak{g}^N$, then (4.8) admits well-defined quotient dynamics $\bar{\mathcal{E}}^+ = \bar{f}(\bar{\mathcal{E}})$ on $\mathfrak{g}^N/\mathfrak{h}^N$. In particular, given the canonical projection $P : \mathfrak{g}^N \to \mathfrak{g}^N/\mathfrak{h}^N$, the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{g}^N \arrow[r, "f"] \arrow[d, "P"'] & \mathfrak{g}^N \arrow[d, "P"] \\
\mathfrak{g}^N/\mathfrak{h}^N \arrow[r, "\bar{f}"'] & \mathfrak{g}^N/\mathfrak{h}^N
\end{tikzcd}$$

We exploit this fact to prove the following lemma.

Lemma 4.4.4. *Under Assumptions 3 and 1(c), the origin is the unique equilibrium of* (4.8).

Proof. It is clear from (4.6) that the identity is an equilibrium on the Lie group, so the origin is an equilibrium of (4.8) on the associated Lie algebra.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be any ideal that satisfies Assumption 1(c). For $i \in \{0\} \cup \mathbb{N}_p$, define the canonical projection $P_i : \mathfrak{g}^N \to \mathfrak{g}^N / \left(\mathfrak{h}^{(i+1)}\right)^N$, and let $\bar{f}_i : \mathfrak{g}^N / \left(\mathfrak{h}^{(i)}\right)^N \to \mathfrak{g}^N / \left(\mathfrak{h}^{(i)}\right)^N$ be the unique map satisfying $P_i f = \bar{f}_i \circ P_i$. Note that, since $\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}]$, the Lie bracket on $\mathfrak{g}^N/\mathfrak{h}^N$ is identically zero, so $\bar{f}_0 = \bar{A}_0$, where $\bar{A}_0 : \mathfrak{g}^N/\mathfrak{h}^N \to \mathfrak{g}^N/\mathfrak{h}^N$ is the unique linear map satisfying $P_0 A = \bar{A}_0 P_0$.

Now suppose $\mathcal{E}^\star \in \mathfrak{g}^N$ is an equilibrium. We will show that $\mathcal{E}^\star$ is necessarily the origin. The error dynamics (4.8) on $\mathfrak{g}^N/\mathfrak{h}^N$ are $\bar{\mathcal{E}}_0^+ = \bar{A}_0\bar{\mathcal{E}}_0$. Under Assumption 3, A is Schur, implying that $\bar{A}_0$ is Schur, so $\bar{\mathcal{E}}_0^\star = \bar{A}_0\bar{\mathcal{E}}_0^\star$ if and only if $\bar{\mathcal{E}}_0^\star = 0$, which holds if and only if $\mathcal{E}^\star \in \mathfrak{h}^N$.

By way of induction, suppose for $j \in \mathbb{N}_{i-1}$, $\bar{\mathcal{E}}_j^{\star+} = \bar{f}_j(\bar{\mathcal{E}}_j^\star)$ is at equilibrium only if $\bar{\mathcal{E}}_j^\star$ is zero. This means that $\mathcal{E}^\star$ is an equilibrium only if $\mathcal{E}^\star \in \left(\mathfrak{h}^{(i)}\right)^N$.

By Lemma 2.3.5, the image of the Lie bracket on $\mathfrak{h}^{(i)}$ is contained in $\mathfrak{h}^{(2i)} \subsetneq \mathfrak{h}^{(i+1)}$. Therefore, if $\mathcal{E}^\star \in \left(\mathfrak{h}^{(i)}\right)^N$, then P_i maps all Lie brackets to zero, so $\bar{\mathcal{E}}_i = \bar{A}_i\bar{\mathcal{E}}_i$, where $\bar{A}_i : \mathfrak{g}^N/\left(\mathfrak{h}^{(i+1)}\right)^N \to \mathfrak{g}^N/\left(\mathfrak{h}^{(i+1)}\right)^N$ is the unique linear map satisfying $P_i A = \bar{A}_i P_i$. Under Assumption 3, A is Schur, so $\bar{A}_i$ is Schur, so $\bar{\mathcal{E}}_i^\star = \bar{A}_i\bar{\mathcal{E}}_i^\star$ if and only if $\bar{\mathcal{E}}_i^\star$ is zero. This means that $\mathcal{E}^\star$ is an equilibrium only if $\mathcal{E}^\star \in \left(\mathfrak{h}^{(i+1)}\right)^N$. This proves the induction. Since $\mathfrak{h}^{(p+1)}$ is zero, $P_p\mathfrak{g}^N \cong \mathfrak{g}^N$. Therefore, $\mathcal{E}^\star$ is an equilibrium only if it is the origin. □

Lemma 4.4.1, Corollary 4.4.3, and Lemma 4.4.4 together establish that Assumption 1 is satisfied.

Proposition 4.4.5. *Given any nilpotent ideal $h \supseteq [\mathfrak{g}, \mathfrak{g}]$, the error dynamics (4.8) satisfy Assumption 1.*

By Theorem 2.3.6, $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$ always satisfies Assumption 1(c), but this is not necessarily the only valid choice. For example, the largest nilpotent ideal of $\mathfrak{g}$, called the *nilradical*, contains $[\mathfrak{g}, \mathfrak{g}]$ when $\mathfrak{g}$ is solvable, but the two do not necessarily coincide [39, §5.1]. This is most obvious when $\mathfrak{g}$ is nilpotent, so the nilradical is the entire Lie algebra.

Equipped with Proposition 4.4.5, our two main synchronization results follow immediately from direct application of Corollaries 3.2.6 and 3.3.6.

Proposition 4.4.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Under Assumption 3, the origin is semiglobally exponentially stable under* (4.8).

Proposition 4.4.7. *Let $\mathfrak{g}$ be a solvable Lie algebra. Under Assumption 3, the origin is globally asymptotically stable under (4.8).*

Since Propositions 4.4.6 and 4.4.7 pertain to the error dynamics on the Lie algebra $\mathfrak{g}$, their implications for synchronization on the group G are limited to the region where

$\mathrm{Log} : \mathsf{G} \to \mathfrak{g}$ is well-defined. If G is exponential, then the Lie algebra error dynamics (4.8) are a globally valid local representation of the Lie group error dynamics (4.6), which are equivalent to (4.5).

Theorem 4.4.8. *Let G be an exponential Lie group. Under Assumption 3, the identity is globally asymptotically stable under* (4.5)*.*

Note that, since Lie groups are generally not vector spaces, the exponential convergence rate on $\mathfrak{g}$, characterized in Proposition 4.4.6, does not readily translate to the group G.

Remark 4.4.9. Under the hypotheses Theorem 4.4.8, the origin is Lyapunov stable, so given a non-exponential, but nilpotent or solvable, Lie group, there exists some neighbourhood of the identity $U_1 \subseteq U$ where this theorem holds locally. ♦

4.4.2 Deadbeat Performance with Unweighted Complete Graphs

If the communication graph $\mathcal{G}$ is unweighted and complete, then the synchronization error E can be driven to identity in finitely many time-steps.

Proposition 4.4.10. *Let G be an exponential Lie group and suppose* (4.8) *satisfies Assumptions 1(b) and 1(c). If $\mathcal{G}$ is unweighted and complete, and $K = N$, then synchronization is achieved in at most $(p+1)\dim\mathfrak{g} - \sum_{j=1}^{p}\dim\mathfrak{h}^{(j)}$ time-steps.*

Proof. Since $\mathcal{G}$ is unweighted and complete, L has a simple eigenvalue of 0, and an eigenvalue of N with multiplicity $N-1$. By the Spectral Mapping Theorem, if $K = N$, then all eigenvalues of $(I_N - L/K)$ are zero. By Lemma 4.3.1, M is nilpotent as an endomorphism. Proposition 4.4.10 follows immediately from the bound determined in the proof of Theorem 3.3.7. □

We remark that this property is not robust. If K is not exactly N, then deadbeat performance will not be achieved. However, even in the case of linear systems, deadbeat performance requires all eigenvalues of the state matrix to be exactly zero, with no robustness.

4.5 Simulations

4.5.1 Network on the Heisenberg Group

To illustrate Proposition 4.4.6 and Theorem 4.4.8, we simulate a network on the simply connected 3-dimensional Heisenberg group $\mathsf{H} \subset \mathsf{GL}^+(3, \mathbb{R})$ with $N = 10$, $K = 11$, $T = 1$, with communication graph depicted in Figure 4.2.

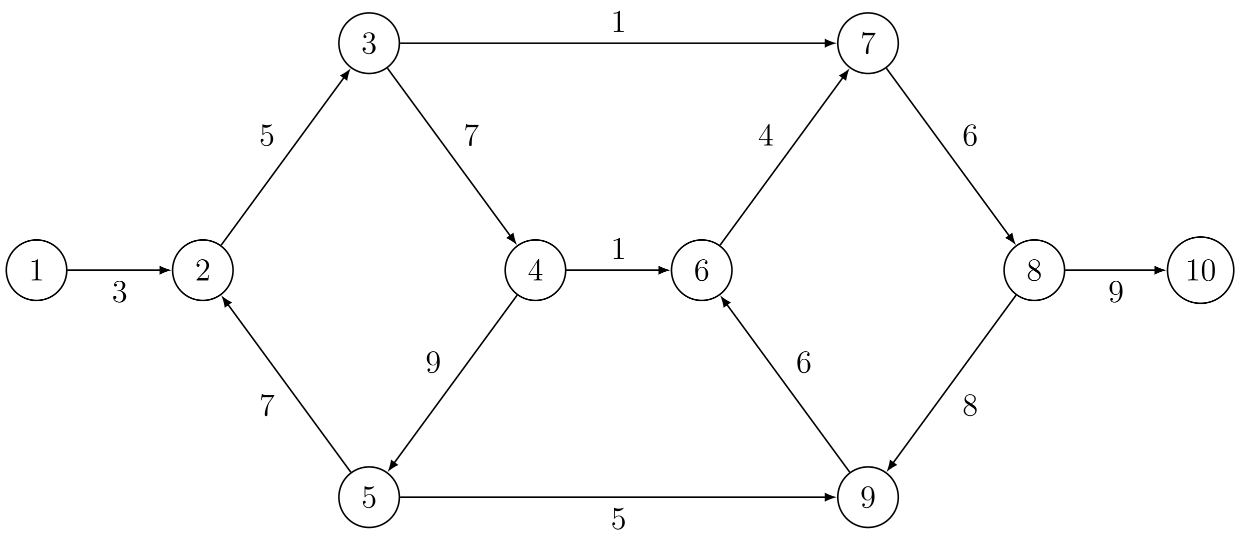


Figure 4.2: Communication graph for Heisenberg and upper-triangular networks.

The agents are initialized at

$$X_i(0) = \begin{bmatrix} 1 & \sin(i) & \sin(3i) + \frac{1}{2}\sin(i)\sin(2i) \\ 0 & 1 & \sin(2i) \\ 0 & 0 & 1 \end{bmatrix}, \quad i \in \mathbb{N}_{10}.$$

By Proposition 4.4.6, the origin of the Lie algebra under the discrete-time error dynamics is semiglobally exponentially stable. The continuous-time synchronization error is shown in Figure 4.3. As predicted by Proposition 4.2.2, the continuous-time error is driven to zero.

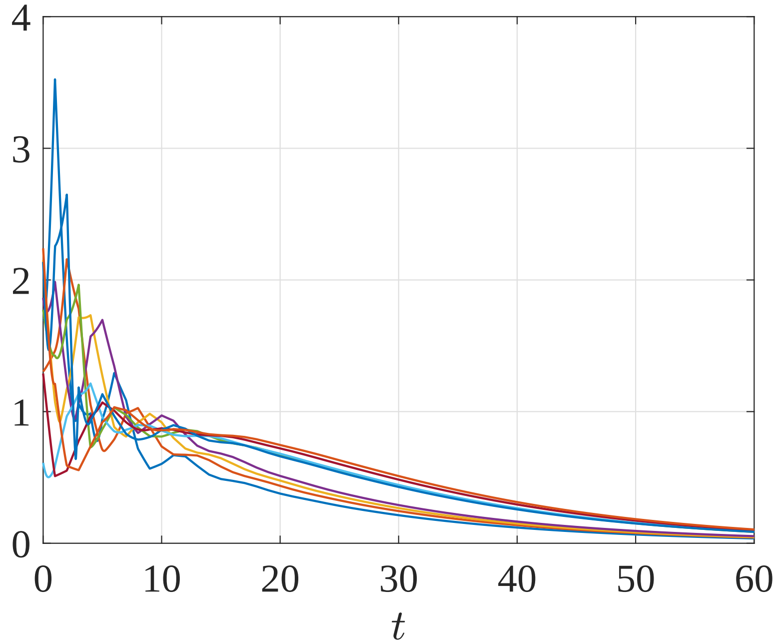


Figure 4.3: $\| \operatorname{Log} E_{1j}(t)\|$, $j \in \{2, \dots, 10\}$ for a network on H with $T = 1$, $N = 10$, $K = 11$, and graph depicted in Figure 4.2.

4.5.2 Network on the Upper Triangular Group

To illustrate Theorem 4.4.8, we simulate a network on the 6-dimensional Lie group of upper triangular matrices with positive diagonal entries $\mathsf{T}(3) \subset \mathsf{GL}^+(3, \mathbb{R})$. The graph $\mathcal{G}$ and controller gain K are the same as in Section 4.5.1. The initial conditions are given by $X_i[0] = \exp(x_i[0])$, where

$$x_i(0) = \begin{bmatrix} \sin(i) & \sin(4i) & \sin(6i) \\ 0 & \sin(2i) & \sin(5i) \\ 0 & 0 & \sin(3i) \end{bmatrix}, \quad i \in \mathbb{N}_{10}.$$

By Theorem 4.4.8, the origin of the Lie algebra under the discrete-time error dynamics is asymptotically stable. The continuous-time synchronization error is shown in Figure 4.4. As predicted by Proposition 4.2.2, the continuous-time error is driven to zero.

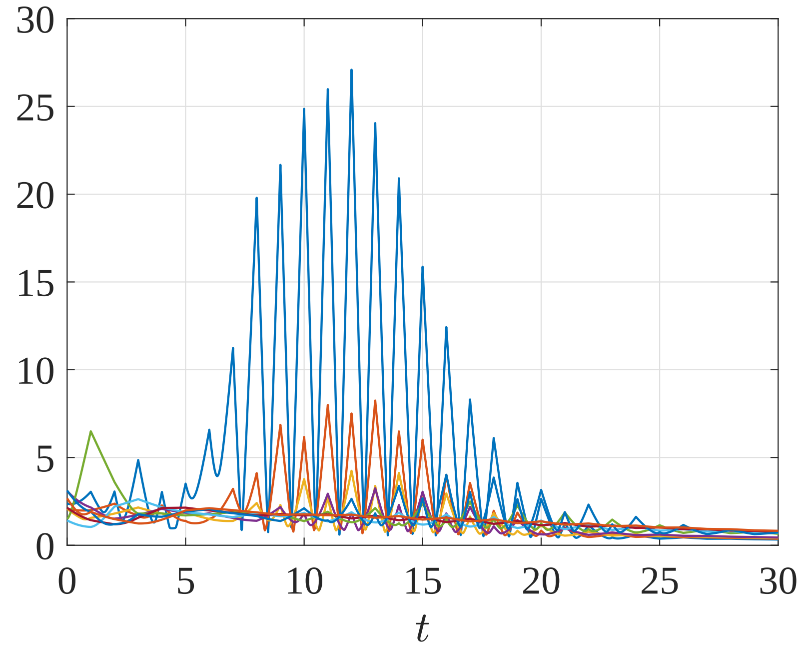


Figure 4.4: $\|\operatorname{Log} E_{1j}(t)\|$, $j \in \{2, \ldots, 10\}$ for a network on $\mathsf{T}(3)$ with $T = 1$, $N = 10$, $K = 11$, and graph depicted in Figure 4.2.

4.5.3 Deadbeat Performance on the Upper Triangular Group

To illustrate Proposition 4.4.10, we simulate a network with a complete connectivity graph on the simply connected 10-dimensional Lie group of 4×4 invertible upper triangular matrices with positive diagonal entries $\mathsf{T}(4) \subset \mathsf{GL}^{+}(4, \mathbb{R})$ with $K = N = 20$, $T = 1$. The agents are initialized at $X_i(0) = \exp(x_i(0))$, where

$$x_i = \begin{bmatrix} x_{i1} & x_{i5} & x_{i8} & x_{i10} \\ 0 & x_{i2} & x_{i6} & x_{i9} \\ 0 & 0 & x_{i3} & x_{i7} \\ 0 & 0 & 0 & x_{i4} \end{bmatrix} \in \mathfrak{g}$$

and

$$x_{ij}(0) = \sin(ij)\cos(ij), \quad i \in \mathbb{N}_{20}, \quad j \in \mathbb{N}_{10}.$$

The derived algebra $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$ has nilindex $p = 2$, $\dim \mathfrak{g} = 10$, $\dim \mathfrak{h} = 6$, $\dim \mathfrak{h}^{(2)} = 3$, $\dim \mathfrak{h}^{(3)} = 1$, so the bound on the time of convergence is $2 \cdot 10 - (6 + 3 + 1) = 10$ time-steps.

As seen in Figure 4.5, the discrete-time error on the Lie algebra is driven to zero in three time-steps, which is consistent with the upper bound of 10 time-steps asserted by Proposition 4.4.10. As predicted by Proposition 4.2.2, the continuous-time error is driven to zero.

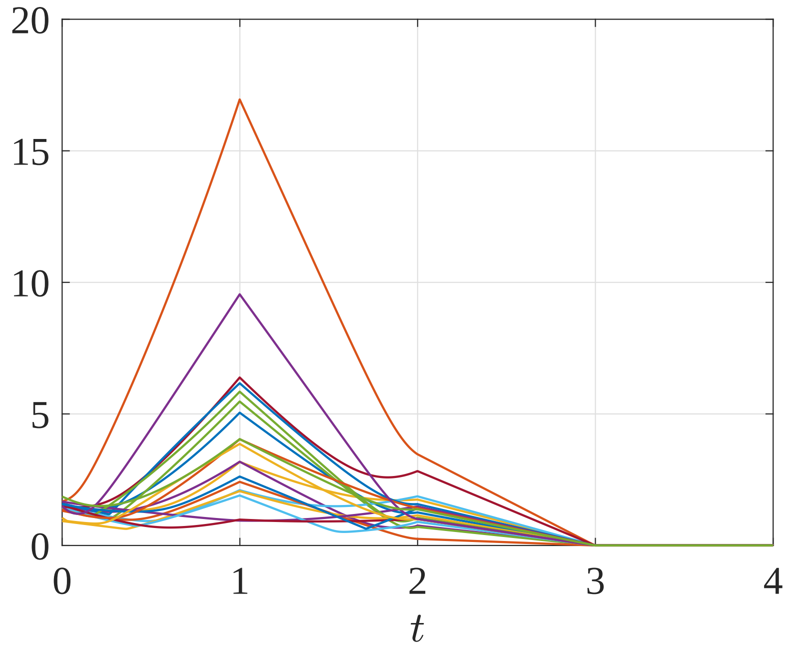


Figure 4.5: $\|\operatorname{Log} E_{1j}(t)\|$, $j \in \{2, \ldots, 20\}$ for a network on $\mathsf{T}(4)$ with $T = 1$, $K = N = 20$, and $\mathcal{G}$ unweighted and complete.

4.5.4 Network on $\mathsf{SU}(2)$

Lastly, we simulate a network on $\mathsf{SU}(2)$ to demonstrate our local synchronization result, Corollary 4.3.3, on a complex, semi-simple—and therefore, by definition, not solvable—Lie group. We simulate a network with $N = 6$, $K = 3.5$, and graph Laplacian

$$L = \begin{bmatrix} 0.5 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 \\ 0 & 0.8 & -0.2 & -0.2 & -0.2 & -0.2 \\ 0 & 0 & 0.9 & -0.3 & -0.3 & -0.3 \\ 0 & 0 & 0 & 0.8 & -0.4 & -0.4 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

The Pauli matrices constitute the canonical basis of $\mathfrak{su}(2)$:

$$\sigma_1 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

We use the Pauli matrices to generate the initial conditions:

$$x_i(0) = i \sin(i)\sigma_1 + (7 - i)\cos(i)\sigma_2 + i \sin(i^3)\sigma_3, i \in \mathbb{N}_6.$$

On the group, the agents are initialized at $X_i(0) = \exp(x_i(0))$.

By Corollary 4.3.3, the origin of $\mathfrak{su}(2)^6$ under the discrete-time error dynamics is locally exponentially stable. The continuous-time synchronization error is shown in Figure 4.6. As predicted by Proposition 4.2.2, the continuous-time error is driven to zero.

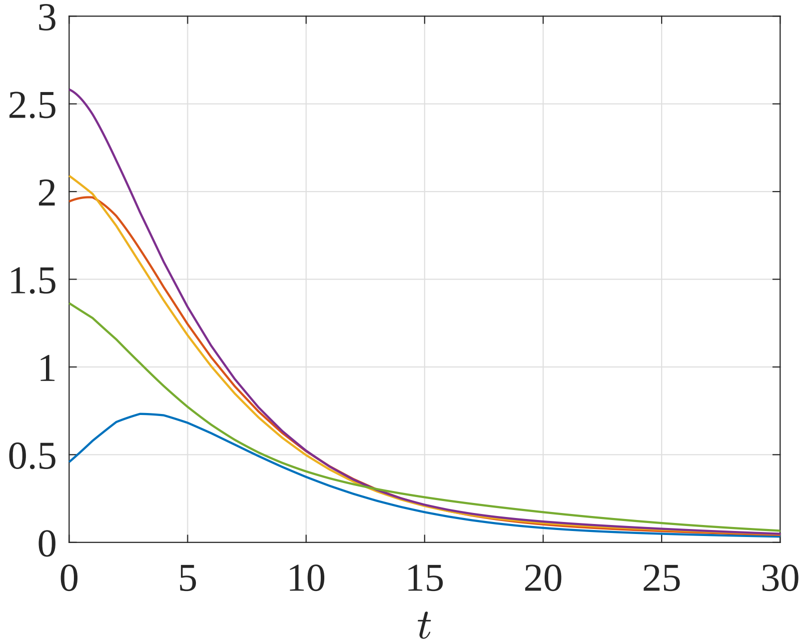


Figure 4.6: $\|\operatorname{Log} E_{1j}(t)\|$, $j \in \{2, \ldots, 6\}$ for a network on $\mathsf{SU}(2)$ with $N = 6$, $K = 3.5$, $T = 1$, and Laplacian (4.9).

Chapter 5

Regulation on Simply Connected Nilpotent Matrix Lie Groups

We examine a regulator problem for a class of fully actuated continuous-time kinematic systems on Lie groups, using a discrete-time controller. We present a discrete-time control law that achieves global regulation on simply connected nilpotent Lie groups. We first solve the problem when both the plant state and exosystem state are available for feedback. We then present a control law for the case where the plant state and a so-called plant output are available for feedback. The class of plant outputs considered includes, for example, the quantity to be regulated. This class of output allows us to use the classical Luenberger observer to estimate the exosystem states. Theorem 5.3.10 asserts that in the full-information case, the regulation quantity on the Lie algebra is shown to decay exponentially to zero, which implies that it tends asymptotically to the identity on the Lie group. In the plant output feedback case, Theorem 5.3.16 asserts that regulation is achieved. In Section 5.4, we briefly address the more general case where the Lie group is solvable, but not necessarily nilpotent.

5.1 Introduction

Tracking in the presence of disturbances is one of the central problems addressed in control theory. It is closely related to the stabilization problem, since stabilizing the origin of the error dynamics implies that the output tracks the reference signal. The tracking problem

A preliminary version of this chapter was presented at the 2017 American Control Conference [74].

for systems on Lie groups has received some attention in the continuous-time setting. The stability of driftless systems on Lie groups was studied in [80], and an almost-global controller for tracking for the class of left-invariant systems on compact Lie groups was identified in [70] using state feedback. A class of regulation problem on $\mathsf{SE}(3)$ was solved using output feedback and an observer, and a separation principle was identified, in [95]. In the literature, UAVs are a popular application. An almost-global output feedback tracking controller for systems on $\mathsf{SE}(3)$ was designed in [18]. A tracking controller that is robust to modelling errors was identified in [63]. In [37], a tracking controller was designed for a UAV carrying a load suspended by a flexible cable.

The regulator problem is central to control theory; it combines reference tracking, disturbance rejection, and stabilization. The problem was completely solved in the continuous-time linear case in the seminal papers [24, 33, 34]. These linear results were extended to nonlinear systems in [49], wherein the continuous-time nonlinear regulator equations—the celebrated Francis-Byrnes-Isidori equations—were developed. Necessary and sufficient conditions for solvability of the regulator problem for nonlinear systems were identified in [49], and for structural stability in [17] by using an internal model. The case of an uncertain exosystem model was solved in [96] by using an adaptive internal model. More recently, researchers have tried to specialize continuous-time regulator problems to classes of systems evolving on Lie groups. The output regulation problem was solved for a class of systems evolving on the special Euclidean group $\mathsf{SE}(n)$ in [95] by identifying a separation principle. In [26], an almost-global solution to the output regulation problem for a class of systems on $\mathsf{SE}(3)$ was proposed; these results were extended to local results on arbitrary Lie groups in [27].

Necessary and sufficient conditions for solving the regulator equations were identified for general nonlinear discrete-time systems in [109], by borrowing from the ideas of [49]. Although dynamics can be intrinsically discrete, such dynamics most often arise in practice through sampling. In [19], solutions to the continuous-time regulator equations were used to approximate solutions to the discrete-time regulator problem for the sampled-data system. To the author's knowledge, the literature on the sampled-data regulator problem for systems on Lie groups is sparse, currently comprising only our preliminary work [74] on commutative Lie groups, and step tracking using passivity for general Lie groups [72].

We show that, when the group is nilpotent and the plant is fully actuated, the origin of the Lie algebra can be made semiglobally exponentially stable; as a corollary, the identity of the Lie group is globally asymptotically stable. We show that, when the trajectories of the exosystem are bounded, the intersample behaviour of the closed-loop system is also bounded. Using the results of Chapter 3, with the same form of controller as [74], which solved a similar regulator problem on commutative Lie groups, we solve the regulator problem for sampled-data systems on simply connected nilpotent Lie groups, and briefly address the more general case of simply connected solvable Lie groups.

5.2 Sampled-Data Regulator Problem

We study the sampled-data control problem for the system illustrated in Figure 5.1.

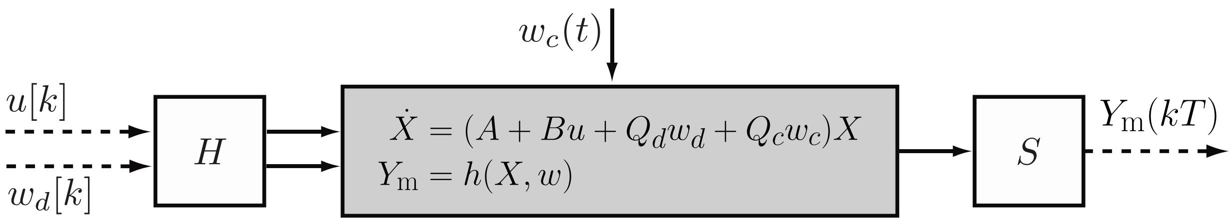


Figure 5.1: Sampled-data plant on a Lie group G.

The plant is modelled by the differential equation

$$\dot{X}(t) = (A + Bu(t) + Q_d w_d(t) + Q_c w_c(t))X(t). \tag{5.1}$$

The system has a **measured output** Y_{m}, which models the information that is available to the controller. It is convenient to model a so-called **plant output**

$$Y(t) = \exp(C + D_d w_d(t) + D_c w_c(t))X(t), \tag{5.2}$$

which models information that is always available for feedback, either through direct measurement or algebraic computation. This signal could be, for example, the quantity to be regulated.

We assume, as is typical, that the exogenous signals w_d, w_c evolve according to known dynamics, modelled as

$$\begin{aligned} w_d[k+1] &= S_d w_d[k] \\ \dot{w}_c(t) &= S_c w_c(t). \end{aligned} \tag{5.3}$$

In (5.1) and (5.2), $X \in \mathsf{G}$ where $\mathsf{G} \subseteq \mathsf{GL}(N, \mathbb{C})$ is an n-dimensional simply connected nilpotent Lie group—which is formalized in Assumption 4 below—over the complex field $\mathbb{C}$ which includes, as a special case, real Lie groups. The associated Lie algebra is $\mathfrak{g}$, which is an n-dimensional vector space over the field $\mathbb{F}$. The control input is $u \in \mathbb{F}^n$, the discrete- and continuous-time exostates are $w_d \in \mathbb{F}^{r_d}$ and $w_c \in \mathbb{F}^{r_c}$, respectively, and $S_d \in \mathbb{F}^{r_d \times r_d}$,

$S_c \in \mathbb{F}^{r_c \times r_c}$. The quantities A and C are elements of $\mathfrak{g}$, and $B : \mathbb{F}^n \to \mathfrak{g}$, $Q_d : \mathbb{F}^{r_d} \to \mathfrak{g}$, $Q_c : \mathbb{F}^{r_c} \to \mathfrak{g}$, $D_d : \mathbb{F}^{r_d} \to \mathfrak{g}$, and $D_c : \mathbb{F}^{r_c} \to \mathfrak{g}$ are linear maps.

Equation (5.1) is a kinematic model of a system evolving on a Lie group G, where the plant output (5.2) models information that is always available for feedback. The exosystem (5.3) comprises both discrete- and continuous-time subsystems. This enables modelling of, for example, physical plants that are subject to continuous-time disturbances, but are sent reference signals from a computer. In this section, we impose four standing assumptions; unless explicitly stated otherwise, these assumptions hold hereinafter.

Assumption 4. *The Lie group G is simply connected, and nilpotent with nilindex p.*

Under Assumption 4, the Lie Group G is diffeomorphic to its Lie algebra $\mathfrak{g}$, which is isomorphic to $\mathbb{F}^n$; in particular, the exponential map $\exp : \mathfrak{g} \to \mathsf{G}$ is a (global) diffeomorphism.

Assumption 5. *The spectra of S_d and S_c lie outside the open unit disc and in the closed right half plane, respectively.*

Assumption 5 incurs no loss of generality; if necessary, S_d and S_c can be redefined as their restrictions to their respective unstable modal subspaces [117, §2.3].

Assumption 6. *The plant is fully actuated:* $\operatorname{Im} B = \mathfrak{g}$.

The foregoing assumption is necessary for the linearization of the tracking error dynamics to be stabilizable, as will be seen in Section 5.3.1. The following technical assumption greatly simplifies our analysis, and guarantees well-posedness of the closed-loop dynamics, as it obviates use of the Magnus expansion in deriving the exact discretization of (5.1). See the proof of Proposition 5.2.1.

Assumption 7. *The image of Q_c is contained in the centre of $\mathfrak{g}$.*[2]

Assumption 7 can be interpreted as the continuous exostate acting as a purely linear disturbance in the plant dynamics on the Lie algebra. Under Assumption 7, letting $X[k] \coloneqq X(kT)$, $u[k] \coloneqq u(kT)$, and $w_c[k] \coloneqq w_c(kT)$, the plant (5.1) and exosystem (5.3) have exact discretizations, as we will prove at the end of this section:

$$X^+ = \exp\left(TA + TBu + TQ_d w_d + Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \, w_c\right) X,$$

[2]It would be sufficient to assume only that $\operatorname{Im} Q_c$ lies in the centralizer of $\{A\} \cup \operatorname{Im} B \cup \operatorname{Im} Q_d$, but under Assumption 6, this is equivalent to Assumption 7.

and

$$\begin{bmatrix} w_d^+ \\ w_c^+ \end{bmatrix} = \underbrace{\begin{bmatrix} S_d & 0 \\ 0 & \mathrm{e}^{TS_c} \end{bmatrix}}_{=:S} \underbrace{\begin{bmatrix} w_d \\ w_c \end{bmatrix}}_{=:w}. \tag{5.4}$$

To simplify the notation, let $r \coloneqq r_d + r_c$ and identify $\mathbb{F}^{r_d} \times \mathbb{F}^{r_c}$ with $\mathbb{F}^r$, define $Q : \mathbb{F}^r \to \mathfrak{g}$, $(w_d, w_c) \mapsto TQ_d w_d + Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau w_c$, and $D : \mathbb{F}^r \to \mathfrak{g}$, $(w_d, w_c) \mapsto D_d w_d + D_c w_c$. Equipped with this notation, we rewrite the discretized plant dynamics and the plant output in the compact form

$$\begin{aligned} X^+ &= \exp(TA + TBu + Qw)X \\ Y &= \exp(C + Dw)X. \end{aligned} \tag{5.5}$$

Proposition 5.2.1. *The dynamics* (5.5) *are the exact discretization of* (5.1)*, in the sense that for all $k \geq 0$, $X[k] = X(kT)$.*

Proof. For $t \in [kT, (k+1)T)$, let $U(t) \coloneqq (A + Bu(kT) + Q_d w_d(kT)) + Q_c w_c(t)$. Under Assumption 7, $[U(t_1), U(t_2)] = 0$ for all t_1, t_2. Thus, using (1.17), over this sampling period, the ODE (5.1) is satisfied by

$$X(t) = \exp\left((t - kT)A + (t - kT)Bu(kT) + (t - kT)Q_d w_d(kT) + Q_c \int_{kT}^t w_c(\tau)\mathrm{d}\tau\right) X(kT).$$

The rest of the proof follows from routine calculation. □

The goal of the regulator problem is to drive a regulation quantity to identity. We take the regulation quantity to be

$$Z(t) = \exp(F + Gw(t))X(t), \tag{5.6}$$

where $F \in \mathfrak{g}$ and $G : \mathbb{F}^r \to \mathfrak{g}$ is a linear map.

More generally, the problem considered in this chapter is as follows. We consider systems of the form

$$\begin{aligned} X^+ &= f(X, u, w) \\ Y_\mathrm{m} &= h(X, w) \\ Z &= g(X, w), \end{aligned}$$

where $f : \mathsf{G} \times \mathbb{F}^m \times \mathbb{F}^r \to \mathsf{G}$ is given by (5.5), $Y_\mathrm{m} : \mathsf{G} \times \mathbb{F}^n \times \mathbb{F}^r \to \mathcal{Y}$, where $\mathcal{Y}$ is some Cartesian product of G and $\mathbb{F}^r$, is the measured output, i.e., the information available to the controller, and Z is the regulation quantity given by (5.6). The objective of the sampled-data regulator problem is to find, if possible, a control law of the form

$$\begin{aligned} x_c^+ &= f_c(x_c, Y_\mathrm{m}) \\ u &= h_c(x_c, Y_\mathrm{m}), \end{aligned}$$

where x_c belongs to some *a priori* unspecified, possibly commutative Lie group,[3] such that

1. the closed-loop system is well-posed;
2. for all initial conditions, $Z[k] \to I$ as $k \to \infty$.

We impose no requirements on internal stability or the intersample behaviour. Concerning the latter, see Remark 5.3.3. Regarding the former, it follows from (5.6) that, when $w \equiv 0$, there is a unique constant $X^\star$ such that if $X = X^\star$, then $Z = I$; when $F = 0$, this constant is $X^\star = I$. Thus internal stability is trivially satisfied by any regulating control law. We consider two cases: 1) $Y_\mathrm{m} = (X[k], w[k])$; 2) $Y_\mathrm{m} = (Y[k], X[k])$. The former is called the **regulator problem with full information**, the latter the **regulator problem with plant state information**. In both cases, since there is no direct feedthrough from u to Y_m, the closed-loop system is well-posed.

5.3 The Solution

In this section, we show that the regulator problem has a solution under the standing assumptions of Section 5.2. We first solve the regulator problem with full information, i.e, $Y_\mathrm{m} = (X[k], w[k])$, which is equivalent to $Y_\mathrm{m} = (Y[k], w[k])$, since the plant state can be computed algebraically as $X[k] = Y[k] \exp(C + Dw[k])^{-1}$. We then solve the regulator problem when the exostate is not measured, i.e., $Y_\mathrm{m} = (X[k], Y[k])$. To prove our main results, we will invoke Theorems 3.2.3 and 3.3.7.

5.3.1 Regulator Problem with Full Information

We solve the regulator problem with full information in two steps: 1) make the tracking manifold $\mathcal{T} := \{(X, w) \in \mathsf{G} \times \mathbb{F}^r : Z = I\}$ positively invariant in discrete-time; 2) make

[3] For example, $\mathbb{R}^{n_c}$ as an additive group.

the tracking manifold globally attractive. The tracking manifold $\mathcal{T}$ is positively invariant if there exist $\Pi : \mathbb{F}^r \to \mathsf{G}$ and $\Psi : \mathbb{F}^r \to \mathfrak{g}$ satisfying the regulator equations:

$$\begin{aligned}\Pi(Sw) &= \exp(TA + TB\Psi(w) + Qw)\Pi(w)\\ I &= \exp(F + Gw)\Pi(w).\end{aligned}$$

Straightforward arithmetic yields the **state reference**

$$\Pi(w) = \exp(F + Gw)^{-1} \tag{5.7}$$

and the **friend**

$$\Psi(w) = \frac{1}{T}B^{-1}\left(\operatorname{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) - TA - Qw\right). \tag{5.8}$$

By construction, if the restriction of u to the tracking manifold equals the friend Ψ, then the tracking manifold is controlled-invariant under the dynamics (5.5).

Remark 5.3.1. These regulator equations are always solvable, because Z is a product of group elements on G, and B is invertible under Assumption 6; this decouples the two equations, which makes it possible for $X[k]$ to track any $\Pi(w[k])$ when properly initialized.

In particular, that B is invertible allows us to choose u such that the discrete-time plant dynamics are of the form $X^+ = UX$, where $U \in \mathsf{G}$ can be set arbitrarily. Choosing $U[k] = \Pi(Sw[k])X[k]^{-1}$ results in one-step-ahead deadbeat reference tracking for any sampling period. Technically, such a control law would solve the regulator problem, but, in practice, this control law would generally be impractical, as it would (attempt to) effect potentially very large actuations in very small time scales, e.g., rotating a 0.5-meter-long robotic arm by 60 degrees in 1 millisecond.

The class of control laws that we propose in this section allows for arbitrary finesse in the control action, and although deadbeat performance will be achievable, it is not necessary. Additionally, although we do not explore the underactuated case in this thesis, an interesting avenue of future research would be to use multirate sampling in tandem with the Lie bracket to exert control effort in a subspace of $\mathfrak{g}$ that is complementary to $\operatorname{Im} B$ [51]. ♦

Remark 5.3.2. A more general problem formulation would define Z on a simply connected nilpotent Lie group H with Lie algebra $\mathfrak{h}$:

$$Z = \exp(F + Gw)\Phi(X),$$

where $F \in \mathfrak{h}$, $\operatorname{Im} G \subseteq \mathfrak{h}$, and $\Phi : \mathsf{G} \to \mathsf{H}$ is an epimorphism[4] of groups. In this case, none of our analysis or results change in any significant way. The regulator equation (5.7) in this case becomes

$$\Phi(\Pi(w)) = \exp(F + Gw)^{-1},$$

which always has a (non-unique) solution $\Pi : \mathbb{F}^r \to \mathsf{G}$. We present a natural choice of Π, based on the local vector space structure, to guarantee internal stability.

Define $\widehat{\Pi} : \mathbb{F}^r \to \mathsf{H}$, $w \mapsto \exp(F + Gw)^{-1}$. There exists a unique linear morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ [41, Theorem 3.28] such that $\Phi = \exp \circ \phi \circ \operatorname{Log}$. Next define $\pi : \mathbb{F}^r \to \mathfrak{g}$ such that

$$\widehat{\Pi} = \exp \circ \phi \circ \pi, \tag{5.9}$$

which has at least one solution π, by bijectivity of exp, under Assumption 4, and surjectivity of ϕ. Since ϕ is a linear map between vector spaces, it has a unique Moore-Penrose pseudoinverse $\phi^\dagger : \mathfrak{h} \to \mathfrak{g}$, which is a right inverse, by surjectivity. By well-known properties of the Moore-Penrose pseudoinverse,

$$\pi := \phi^\dagger \circ \operatorname{Log} \circ \widehat{\Pi},$$

is the solution to (5.9) whose matrix representation has the smallest Frobenius norm. The state reference is then given by $\Pi = \exp \circ \pi$. In terms of the original data:

$$\Pi(w) = \exp\left((\operatorname{Log} \circ \Phi)^\dagger (F + Gw)\right)^{-1}. \tag{5.10}$$

This construction is summarized by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{F}^r \arrow[r, "\Pi"] \arrow[rr, bend left, "\widehat{\Pi}"] \arrow[dr, "\pi"'] & \mathsf{G} \arrow[r, "\Phi"] \arrow[d, "\operatorname{Log}"] & \mathsf{H} \\
& \mathfrak{g} \arrow[r, "\phi"] & \mathfrak{h} \arrow[u, "\exp"']
\end{tikzcd}$$

By construction, π is the bounded linear operator of least Frobenius norm that satisfies (5.10), therefore, it furnishes a local state reference as close to the origin as possible, as quantified by the Frobenius norm. This implies that Π furnishes a state reference on the group that is as close to the identity as possible. ♦

[4] This is without loss of generality, since if a morphism of groups Φ is not surjective, then it can be redefined as the codomain restriction $\operatorname{Im} \Phi | \Phi$.

Remark 5.3.3. Regulation at the sampling instants does not imply intersample regulation. If $u[k] = \Psi(w[k])$, then the continuous-time plant dynamics (5.1) over the sampling period $t \in [kT, (k+1)T)$ are

$$\dot{X} = \left(\frac{1}{T} \operatorname{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right) + Q_c w_c - \frac{1}{T} Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau w_c[k] \right) X[k].$$

Solving for $X(kT+\delta)$, where $\delta \in [0,T)$, and setting $X[k] = \Pi(w[k])$, we have

$$X(kT+\delta) = \exp\left(\frac{\delta}{T} \operatorname{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right) + Q_c \left(\int_0^\delta \mathrm{e}^{\tau S_c} \mathrm{d}\tau - \frac{\delta}{T} \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \right) w_c[k] \right) \Pi(w[k]). \quad (5.11)$$

which shows $X(t) \neq \Pi(w[k])$ for all $t \in [kT, (k+1)T)$. ♦

Remark 5.3.3 may seem ominous, but under the standard assumption that the exosystem (5.4) is neutrally stable, we can partially characterize the intersample behaviour.

Proposition 5.3.4. *If the trajectories of* (5.3) *are bounded and* $X[k] = \Pi(w[k])$*, then for all* $t \geq 0$*,* $X(t)$ *is bounded.*

Proof. Given a square matrix A, $\|\exp(A)\| \leq \exp(\|A\|)$. Applying this property to (5.11), we obtain

$$\|X(kT+\delta)\| \leq \exp\left(\frac{\delta}{T} \left\|\operatorname{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right)\right\| + \left\| Q_c \left(\int_0^\delta \mathrm{e}^{\tau S_c} \mathrm{d}\tau - \frac{\delta}{T} \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \right) \right\| \|w_c[k]\| \right) \|\Pi(w[k])\|.$$

Since w is bounded, $\Pi(w)$, its inverse, and $\Pi(Sw)$ are bounded. Since $\operatorname{Log} : \mathsf{G} \to \mathfrak{g}$ is continuous, the boundedness of $\Pi(Sw)\Pi(w)^{-1}$ implies that $\operatorname{Log}(\Pi(Sw)\Pi(w)^{-1})$ is bounded. Noting that w_c is bounded completes the proof. □

The next result addresses the important special case of step reference tracking and disturbance rejection.

Proposition 5.3.5. *If w_d and w_c are constant and $X[k] = \Pi(w[k])$, then Z is identity for all $t \geq kT$.*

Proof. Without loss of generality, we take $S_d = I$ and $S_c = 0$. Then (5.11) simplifies to $X(kT + \delta) = \Pi(w)$. Thus, $X(t)$ is constant. The result follows immediately from (5.6) and (5.7). □

The two foregoing Propositions furnish analogous corollaries for the intersample behaviour of Z, which follow immediately from (5.6).

We now use state feedback to make the tracking manifold $\mathcal{T}$ globally attractive. Define the **state-tracking error** $E \coloneqq X\Pi(w)^{-1}$. By definition, if $E = I$, then $(X, w) \in \mathcal{T}$. We will design a control law that stabilizes the Jacobian linearization of the tracking error dynamics on $\mathfrak{g}$; this of course implies local exponential stability of the tracking manifold $\mathcal{T}$ on any Lie group. We then invoke the results of Chapter 3 to show that such a controller achieves global regulation on nilpotent Lie groups.

We propose a controller of the form

$$u = \Gamma(X, w) + \Psi(w),$$

where Ψ is given by (5.8), and $\Gamma(X, w) \coloneqq K \operatorname{Log}(E)$, where $K : \mathfrak{g} \to \mathbb{F}^n$ is a yet-to-be-specified gain. The tracking manifold $\mathcal{T}$ is rendered invariant by the friend Ψ, and attractive by the state feedback Γ. Define $e \coloneqq \operatorname{Log}(E)$ and $\pi \coloneqq \operatorname{Log} \circ \Pi$; under Assumption 4, e is well-defined for all $E \in \mathsf{G}$. Using the proposed controller, the error dynamics on G are

$$\begin{aligned} E^+ &= X^+\Pi(w^+)^{-1} \\ &= \exp\left(TBKe + \operatorname{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) X\Pi(Sw)^{-1} \\ &= \exp\left(TBKe + \operatorname{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) E \left(\Pi(Sw)\Pi(w)^{-1}\right)^{-1}. \end{aligned} \tag{5.12}$$

We examine the tracking error dynamics (5.12) on the Lie algebra $\mathfrak{g}$. To facilitate this, we again invoke the generalized BCH (2.1). Applying the BCH to $\operatorname{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)$ in (5.8), and the generalized BCH to (5.12), performing some simplifications and rearranging, we obtain

$$\begin{aligned} e^+ = (I + TBK)e + \frac{1}{2}[TBKe, e] + \frac{1}{2}[\pi(Sw), e] + \frac{1}{2}[-\pi(w), e] \\ + \frac{1}{2}[TBKe, \pi(w)] + \frac{1}{2}[TBKe, -\pi(Sw)] + \cdots, \end{aligned}$$

which can be written in the form of (3.1) thus

$$e^+ = (I + TBK)e + \sum_\omega c_\omega \otimes \omega, \tag{5.13}$$

where the words ω are over the letters $e, BKe, \pi(Sw), \pi(w)$; the disturbance signals are $\pi(Sw)$ and $\pi(w)$. Recall Remark 3.3.11, wherein we discuss that dynamics of the form (5.13) are amenable to the results of Chapter 3. Note that the pair (I, TB) is stabilizable if and only if Assumption 6 is satisfied. To leverage the results of Chapter 3 in the proof of the main result of this section, we require the following Lemma.

Lemma 5.3.6. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the tracking error dynamics on the Lie algebra (5.13) satisfy Assumption 1.*

Proof. We verify that (5.13) satisfies each of Assumptions 1(a), 1(b), and 1(c), in order. Since $\mathfrak{g}$ is nilpotent, Assumption 1(a) holds trivially, since there are only finitely many words in (5.13).

Claim 5.3.7. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$, such that $(I + TBK)$ is Schur and every subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is BK-invariant.*

Proof of Claim 5.3.7. Fix $\alpha \in (0, 2)$ and $K = -\alpha(TB)^{-1}$. Then $BK = -\alpha T^{-1} I$ leaves any subspace invariant, and $(I + TBK) = (1 - \alpha)I$ is Schur. □

Fix $K : \mathfrak{g} \to \mathbb{F}^n$ such that $(I + TBK)$ is Schur and $BK\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^{(i)}$ for all $i \in \mathbb{N}_p$.

Claim 5.3.8. *The dynamics (5.13) satisfy Assumption 1(b).*

Proof of Claim 5.3.8. Note that $e \in \mathfrak{g}$ is a fixed point of (5.13) if and only if $E = \exp(e)$ is a fixed point of (5.12). Solving for the fixed points of (5.12),

$$\begin{aligned}
& \Pi(Sw)\Pi(w)^{-1} = \exp\left(TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) \\
\iff\quad & \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) = TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) \\
\iff\quad & TBKe = 0.
\end{aligned}$$

By the Spectral Mapping Theorem, $(I + TBK)$ is Schur only if 0 is not an eigenvalue of TBK, implying that TBK is an isomorphism. Therefore, $TBKe = 0$ if and only if $e = 0$. □

Claim 5.3.9. *The dynamics (5.13) satisfy Assumption 1(c).*

Proof of Claim 5.3.9. By our choice of K, $(I + TBK)\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^{(i)}$. By Proposition 3.1.12, $\mathfrak{g}^{(i)}$ is invariant under (5.13). $\square$

Claims 5.3.8, 5.3.9, and the initial argument verify the Lemma. $\square$

Equipped with Lemma 5.3.6, we prove the main result of this section.

Theorem 5.3.10. *If $\Pi : \mathbb{F}^r \to \mathsf{G}$ and $\Psi : \mathbb{F}^r \to \mathbb{F}^n$ are given by (5.7) and (5.8), respectively, then there exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that if*

$$u = K \operatorname{Log}(X\Pi(w)^{-1}) + \Psi(w), \tag{5.14}$$

then (5.14) solves the regulator problem with full information.

Proof. Let $K = -(TB)^{-1}$, which is the controller used in the proof of Lemma 5.3.6, with $\alpha = 1$, therefore, by Lemma 5.3.6, Assumption 1 is satisfied. In particular, $(I + TBK) = 0$. By Theorem 3.3.7, the tracking error e converges to 0 in finite time. Consequently, E converges to identity in finite time. $\square$

Remark 5.3.11. The use of the deadbeat control law $K = -(TB)^{-1}$ in the proof of Theorem 5.3.10 is merely an expeditious existence proof technique. All subspaces are I-invariant, and B is an isomorphism under Assumption 6, hence all subspaces of $\mathfrak{g}$ are controllability subspaces of the pair (I, TB) [117, §5]. Due to the ordering of the lower central series $\mathfrak{g}^{(i)} \supset \mathfrak{g}^{(i+1)}$, $i \in \mathbb{N}_p$, every ideal $\mathfrak{g}^{(i)}$ can be made simultaneously invariant under $(I + TBK)$, where the eigenvalues of each restriction $(I + TBK)|\mathfrak{g}^{(i)}$ can be placed arbitrarily. Clearly, a subspace is invariant under $(I + TBK)$ if and only if it is invariant under BK. Any such gain $K : \mathfrak{g} \to \mathbb{F}^n$ would furnish a static feedback control law (5.14) that solves the regulator problem with full information. ♦

5.3.2 Rate of Convergence

In the proof of Theorem 5.3.10, we invoked Theorem 3.3.7 to demonstrate the existence of $K : \mathfrak{g} \to \mathbb{F}^n$ such that the state-tracking error converges to zero in finite time; in the proof of Theorem 3.3.7, this time is found to be bounded above by the summation of the dimensions of $\mathfrak{g}$ and the ideals of the lower central series of $\mathfrak{h}$. In this section, we characterize the general rate of convergence. In anticipation of invoking Theorem 3.2.3, we establish the following lemmas, which assert that the growth rates of the exogenous signals are independent of the choice of norm; this is important, because per Theorem 3.2.3, this growth rate defines a sufficiently small spectral radius for stability.

Lemma 5.3.12. *Consider the discretized exosystem* (5.4)*. There exists $s \geq 1$, such that given any norm $\|\cdot\| : \mathbb{F}^r \to \mathbb{R}$ and any initial condition $w[0]$, there exists $\beta \geq 0$ such that $\|w[k]\| \leq \beta s^k$.*

We emphasize that Lemma 5.3.12 establishes a bound on the rate of growth of w independent of the norm chosen.

Proof. Let $\|\cdot\| : \mathbb{F}^r \to \mathbb{R}$ be arbitrary. Fix $\varepsilon > 0$ and let $\|\cdot\|_\varepsilon : \mathbb{F}^r \to \mathbb{R}$ be a norm such that its induced norm satisfies $\|S\|_\varepsilon = \rho(S) + \varepsilon =: s$. Since all norms are equivalent on finite-dimensional vector spaces, there exists $\alpha > 0$ such that for all $w \in \mathbb{F}^r$, $\|w\| \leq \alpha\|w\|_\varepsilon$. Since the solution to (5.4) is $w[k] = S^k w[0]$, we have $\|w[k]\|_\varepsilon \leq s^k\|w[0]\|_\varepsilon$, so

$$\|w[k]\| \leq \alpha\|w[k]\|_\varepsilon = \underbrace{(\alpha\|w[0]\|_\varepsilon)}_{=:\beta} s^k.$$

Since ε was arbitrary, s can be made arbitrarily close to $\rho(S)$, which, under Assumption 5, is at least 1. $\square$

Lemma 5.3.13. *There exists $s \geq 1$ such that, given any norms on $\mathbb{F}^r$ and $\mathfrak{g}$, and any initial condition $w[0] \in \mathbb{F}^r$, there exists $\beta \geq 0$ such that*

$$\left\| \begin{bmatrix} \pi(Sw) \\ \pi(w) \\ C + Dw \end{bmatrix} \right\| \leq \beta s^k.$$

Proof. We first bound the norm of $\pi(Sw[k])$:

$$\begin{aligned} \|\pi(Sw[k])\| &= \|F + GSw[k]\| \\ &\leq \|F\| + \|GS\|\|w[k]\|. \end{aligned}$$

Applying Lemma 5.3.12,

$$\begin{aligned} \|\pi(Sw[k])\| &\leq \|F\| + \|GS\|\beta' s^k \\ &\leq (\|F\| + \|GS\|\beta')s^k, \end{aligned}$$

where we have used that $s \geq 1$. Similarly, we establish $\|\pi(w[k])\| \leq (\|F\| + \|G\|\beta')s^k$ and $\|C + Dw[k]\| \leq (\|C\| + \|D\|)\beta' s^k$. Let

$$\beta := \max\{\|F\| + \max\{\|GS\|, \|G\|\}, \|C\| + \|D\|\}\beta'.$$

$\square$

Proposition 5.3.14. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the origin of $\mathfrak{g}$ is semiglobally exponentially stable under* (5.13).

Proof. Fix $\alpha \in \left(1 - \rho(S)^{-\frac{p(p-1)}{2}}, 1 + \rho(S)^{-\frac{p(p-1)}{2}}\right)$ and $K = -\alpha(TB)^{-1}$. Then $I + TBK = (1-\alpha)I$, whose spectral radius is $|1-\alpha| < \rho(S)^{-\frac{p(p-1)}{2}}$, and the letters ω in (5.13) reduce to $\{e, \pi(Sw), \pi(w)\}$, where the exogenous signals are $\pi(Sw)$ and $\pi(w)$. Stacking the exogenous signals into a single variable $W \in \mathfrak{g} \times \mathfrak{g}$, we apply Lemma 5.3.13. The result then follows by direct application of Theorem 3.2.3. □

Since $\mathfrak{g}$ is diffeomorphic to G, we translate Proposition 5.3.14 to the group.

Theorem 5.3.15. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the identity of G is globally asymptotically stable under the group tracking-error dynamics* (5.12).

5.3.3 Regulator Problem with Plant State Information

Four natural choices of a measured output Y_{m} are 1) $Y_{\mathrm{m}} = (X[k], w[k])$; 2) $Y_{\mathrm{m}} = (Y[k], w[k])$; 3) $Y_{\mathrm{m}} = (Y[k], X[k])$; 4) $Y_{\mathrm{m}} = Y[k]$.

The first case is that of full information studied in the previous subsection. The second case is equivalent to the first, because it allows us to algebraically compute $X[k] = \exp(C + Dw[k])^{-1}Y[k]$. The third case includes, for example, the case where the plant state X and the regulation quantity Z are measured, i.e., $Y = Z$, $F = C$, and $D = G$. The fourth case characterizes the regulator problem with output information. In this section, we treat the third case, and leave the fourth case as a topic for future research.

When $Y_{\mathrm{m}} = (Y[k], X[k])$, at each sampling instant, we can compute

$$Dw[k] = \mathrm{Log}(Y[k]X[k]^{-1}) - C. \tag{5.15}$$

We therefore propose the linear observer

$$\hat{w}^+ = S\hat{w} + L(D\hat{w} - Dw), \tag{5.16}$$

which yields the estimation error dynamics

$$e_w^+ = (S + LD)e_w. \tag{5.17}$$

Under Assumption 5, $L : \mathfrak{g} \to \mathbb{F}^r$ can be chosen such that (5.17) is stable if and only if the pair (D, S) is observable.

Theorem 5.3.16. *If the pair (D, S) is observable, then there exist $K : \mathfrak{g} \to \mathbb{F}^n$ and $L : \mathfrak{g} \to \mathbb{F}^r$ such that the control law defined by (5.7), (5.8), (5.16), and*

$$u = K \operatorname{Log}(X\Pi(\hat{w})^{-1}) + \Psi(\hat{w}) \tag{5.18}$$

solves the regulator problem with plant state information.

Proof. If (D, S) is observable, then there exists $L : \mathfrak{g} \to \mathbb{F}^r$ such that $\rho(S + LD) = 0$; fix such an L. Then for all $k \geq r$, $e_w[k] = 0$, or equivalently, $\hat{w}[k] = w[k]$. For all $k \geq r$, the control law (5.18) reduces to (5.14) from the full-information case. Since all the dynamics under consideration are polynomial in the dynamical variables, none of the trajectories can exhibit finite escape time, so for $k \leq r$, the trajectories are well-defined, and for all $k \geq r$, the tracking error dynamics are (5.12) on the group, and (5.13) on the algebra. The proof follows from applying the arguments used in the proof of Theorem 5.3.10 to establish global attractivity of the origin under the tracking error dynamics. □

Remark 5.3.17. Analogous to the discussion in Remark 5.3.2, we could instead define the plant output as $Y = \exp(C + Dw)\Phi(X)$. Equation (5.15) would then instead be $Dw[k] = \operatorname{Log}(Y[k]\Phi(X[k])^{-1}) - C$. The rest of the analysis is identical. ⧫

5.4 Solvable Lie Groups

We focus on the full-information case, but obvious extensions can be made to the plant-state-information case. When $\mathfrak{g}$ is solvable, we cannot take $\mathfrak{h}$ to be $\mathfrak{g}$ in Assumption 1(c), but instead, for example, the derived algebra or nilradical of $\mathfrak{g}$ are viable candidates. The general convergence results are also somewhat weaker (recall Corollary 3.3.6). However, as in the nilpotent case, we can again invoke Theorem 3.3.7.

Following the same analysis as in the nilpotent case, it soon becomes apparent that to satisfy the hypotheses of Theorem 3.3.7, we require $\operatorname{Im} \pi \subseteq \mathfrak{h}$, which holds for all w if and only if $F \in \mathfrak{h}$ and $\operatorname{Im} G \subseteq \mathfrak{h}$, which implies $\exp(F + Gw) \in \mathsf{H} := \exp \mathfrak{h}$. From (5.6), we see that on the quotient group G/H, Z and X are equivalent, thus regulation on this quotient group is equivalent to attractivity of the identity under the plant dynamics (5.5). The regulator problem on the nilpotent subgroup $\mathsf{H} \subset \mathsf{G}$ is solved in the way already described in the previous sections.

5.5 Simulations on the Heisenberg Group

To illustrate our results, we simulate a system on the Heisenberg group $\mathsf{G} \subset \mathsf{GL}(3, \mathbb{R})$. The Heisenberg system is a prototypical example for nonlinear control problems, and can be viewed as an approximation to some physical systems whose linearizations provide little insight, such as reorientation and locomotion systems [10]. We choose the basis for the Heisenberg algebra $\mathfrak{g}$ to be $\{g_1, g_2, g_3\}$, where

$$g_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The lower central series is $\mathfrak{g} =: \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \mathfrak{g}^{(3)} = 0$, where $\mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}] = \mathrm{Lie}_{\mathbb{R}}\{g_3\} = \mathrm{span}_{\mathbb{R}}\{g_3\}$, thus the nilindex is $p = 2$.

We present several examples with different exosystems. In every case, $\rho(S) = 1$, and the observer gain L is chosen such that $\rho(S + LD) = 0$. We first consider a system with exosystem parameters $S_d = 1$ and $S_c = 0$, which both define steps in discrete- and continuous-time, respectively; plant parameters:

$$\begin{aligned} A &= g_1 + g_2 + g_3, & B_1 &= g_1 \\ Q_{d1} &= g_1 + g_2 + g_3, & B_2 &= g_2 \\ Q_{c1} &= g_3, & B_3 &= g_3, \end{aligned}$$

where $Bu = \sum_{i=1}^{3} B_i u_i$; plant output parameters:

$$C = g_1 + 2g_2 + 3g_3, \quad D_{d1} = g_1 - g_2, \quad D_{c1} = g_2 + g_3,$$

and regulation quantity parameters:

$$\begin{aligned} F &= -3g_1 - 2g_2 - g_3, \\ G_{d1} &= -g_1 + 2g_2 - 3g_3, \quad G_{c1} = 2g_1 + g_2 - 3g_3. \end{aligned}$$

We use a sampling period of $T = 1$ and initialize with

$$\begin{aligned} X(0) &= \exp(g_1 + 2g_2 - 3g_3), \\ w_d[0] &= 1, & \hat{w}_d[0] &= 0, \\ w_c(0) &= 1, & \hat{w}_c[0] &= 0. \end{aligned}$$

We choose $K = -(1/2)I$, which yields $(I + TBK) = (1/2)I$, whose spectral radius is $1/2 \leq \rho(S)^{-\frac{p(p-1)}{2}} = 1$. By Theorem 5.3.16, this choice of K and L furnishes a control law that solves the regulator problem. As predicted by Proposition 5.3.5, since w_d and w_c are constant, $z(t) \to 0$ as $t \to \infty$, as seen in Figure 5.2.

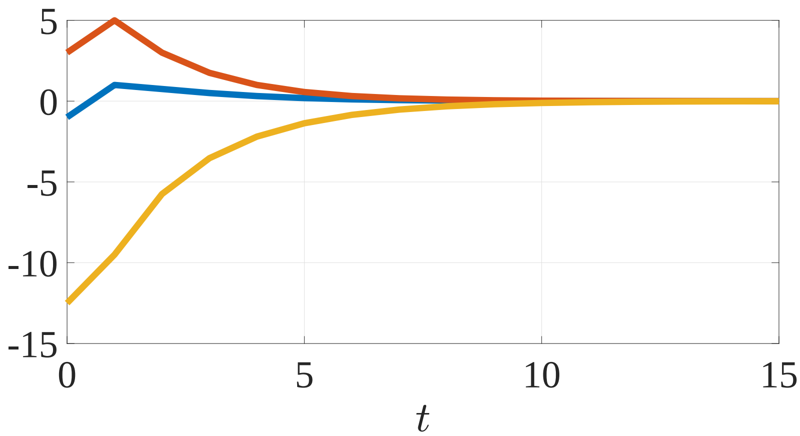


Figure 5.2: Regulation quantity $\mathrm{Log}(Z)$ for constant w.

To illustrate non-step-tracking behaviour, we redefine the exosystem dynamics as

$$S_d = \begin{bmatrix} \cos(1) & -\sin(1) \\ \cos(1) & \sin(1) \end{bmatrix}, \qquad S_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which define discrete- and continuous-time sinusoids, respectively, both with unit frequency. We extend the plant, plant output, and regulation quantity definitions with the parameters

$$\begin{aligned} Q_{d2} &= -g_1 - g_2, & Q_{c2} &= -g_3, \\ D_{d2} &= g_1 + g_2 + g_3, & D_{c2} &= g_1 + g_3 \\ G_{d2} &= g_3, & G_{c2} &= g_1 + 2g_2 + 3g_3, \end{aligned}$$

where now $Q_d w_d = \sum_{i=1}^{2} Q_{di} w_{di}$, etc.

We use the same sampling period $T = 1$ and initial condition $X(0)$, and initialize the observer states at the origin, but now initialize the exostates at

$$w_d[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w_c(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

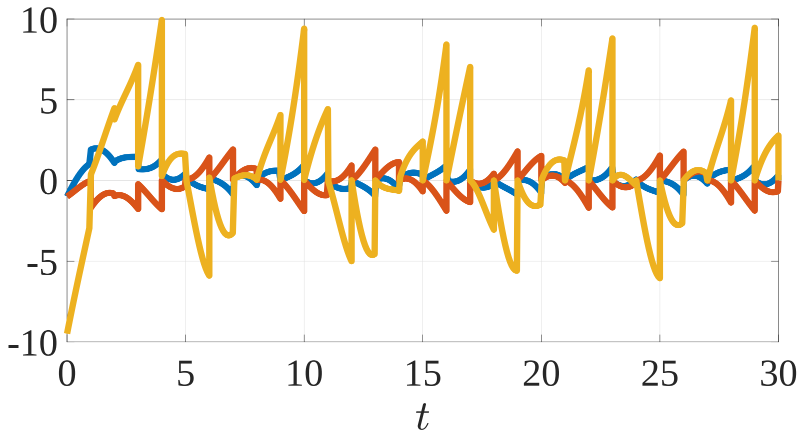


Figure 5.3: Regulation quantity Log(Z) for sinusoidal w.

We use the same tracking-error feedback gain K. Since the exostates are bounded, Proposition 5.3.4 predicts that $z(t)$ will be bounded, which is what we see in Figure 5.3.

Repeating the simulation again, but changing the discrete-time exosystem dynamics to

$$S_d = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which defines a ramp, the regulation quantity exhibits the behaviour seen in Figure 5.4. At the sampling instants, $z[k] \to 0$, however, the intersample behaviour of $z(t)$ is unbounded.

However, if we remove the continuous-time exostate w_c, or equivalently set $w_c(0) = 0$, then we make the interesting observation that $z(t)$ is bounded, as seen in Figure 5.5. From (5.11), it is not surprising that eliminating the continuous-time disturbance improves intersample behaviour, and it seems plausible that when, in addition, the growth rate of w_d is bounded, that the intersample behaviour is bounded. However, due to the nonlinearity of (5.11), it is not obvious that this is indeed always the case. We leave this as a topic for future research.

Remark 5.5.1. The nondiagonal gain

$$K = -\frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

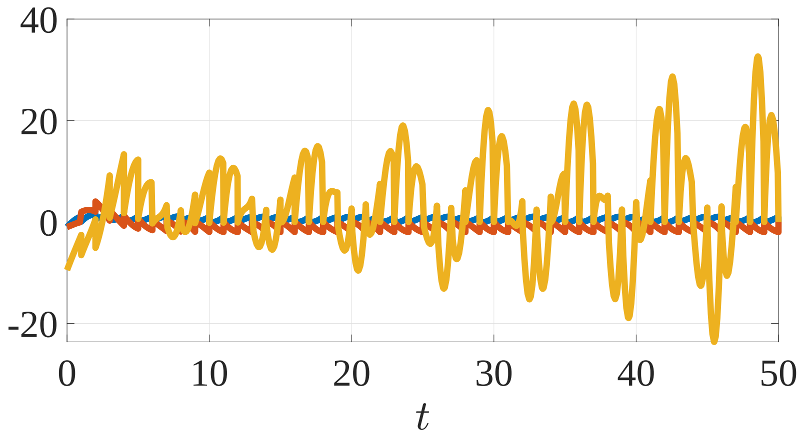


Figure 5.4: Regulation quantity $\mathrm{Log}(Z)$ for ramp w_d and sinusoidal w_c.

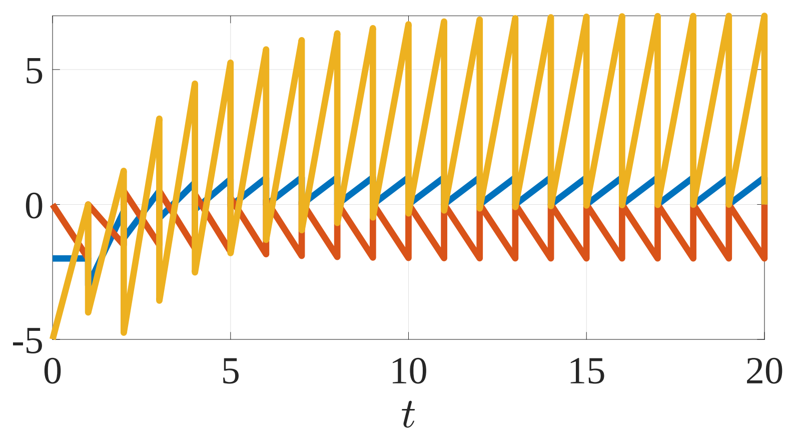


Figure 5.5: Regulation quantity $\mathrm{Log}(Z)$ for ramp w_d and $w_c \equiv 0$.

endows $(I+TBK)$ with the same spectral radius, and leaves $\mathfrak{g}^{(2)}$ invariant under $(I+TBK)$ and BK. Simulations (not shown) yield qualitatively similar performance to that seen in the respective simulations, which is consistent with Remark 3.3.11. ⧫

Chapter 6

Summary and Future Research

In this thesis, we solved synchronization and regulation problems for kinematic systems on exponential matrix Lie groups. The exponential property allows for globally valid analysis on the Lie algebra of the Lie group. We exploited this in order to abstract the synchronization and regulation problems into a stability problem on solvable Lie algebras.

In Chapter 3, we showed that for a class of systems evolving on solvable Lie algebras, global stability properties can be inferred from the linear part the dynamics. If the Lie algebra is solvable, then global asymptotic stability can be established. If the Lie algebra is nilpotent, then semiglobal exponential stability can be established.

In Chapter 4, we proposed a simple sampled-data control law for global synchronization of identical kinematic agents on exponential Lie groups, whose network connectivity graphs have a globally reachable node. On the associated Lie algebra, the error dynamics are evocative of those in linear consensus. Synchronization is asymptotic in the general case, exponentially fast if the Lie group is simply connected and nilpotent, and deadbeat synchronization is achieved with a specific choice of gain when the communication graph is connected and unweighted. For the class of systems considered, synchronization at the sampling instants implies continuous-time synchronization.

In Chapter 5, we solved a regulator problem for a class of kinematic systems on simply connected nilpotent Lie groups in two cases: 1) when the plant state and exostate are available for feedback; 2) when the plant state and a so-called plant output are available for feedback. In the latter, we used a Luenberger observer to estimate the exostates, thereby furnishing a dynamical control law. In the full-information case, we showed that the origin of the Lie algebra is semiglobally exponentially stable under the tracking error dynamics.

In the area of stability, future work should attempt to strengthen the results in the case where the Lie algebra is solvable, but not necessarily nilpotent. Although the notion of a class-$\mathcal{A}$ series includes the important class of kinematic systems on matrix Lie groups, it would be of interest to confirm whether this notion of convergence is truly distinct from more familiar notions of convergence, such as absolute or strong convergence. Given an arbitrary finite-dimensional Lie algebra, it would be interesting to explore the use of the Levi decomposition to study the quotient dynamics on the radical, and see what utility this offers for studying stability on the full Lie algebra.

In the area of regulation, the case where only the plant output Y is available for feedback should be addressed. The last simulation in Section 5.5 suggests that our conditions for bounded intersample behaviour can be refined. Another natural extension is to remove Assumption 7, and use the Magnus expansion to express the local trajectory of the plant's state and design control laws. It would also be of interest to identify conditions for robustness to noise and structural stability of the error dynamics.

In the areas of both synchronization and regulation, future work includes the treatment of underactuated, but controllable plants, i.e., $\mathrm{Lie}_{\mathbb{F}}\{\mathrm{Im}\, B\} = \mathfrak{g}$, rather than $\mathrm{Im}\, B = \mathfrak{g}$. This could perhaps be achieved using multirate sampling, and would require appeals to nonlinear tests for controllability. The problem objectives would need to be redefined, e.g, *there exists an integer* $\alpha > 1$ *such that* $Z[\alpha k] \to I$, since it would take multiple time steps to generate the directions that are missing from $\mathrm{Im}\, B$.

The methods described in this thesis should be combined with the nilpotent and solvable approximation methods of [43, 104] and [23], respectively. Although not an extension of the work in this thesis, it could be fruitful to identify conditions under which feedback transformations to chained form are preserved under sampling. Recall the the car-like robot of Example 1.1.3. Its dynamics (1.8) were put into chained form via a feedback transformation. Thus, if the control signals are constrained to update only at discrete time instants, as in the sampled-data setting, the feedback transformation will be destroyed. This is the general effect in the case of feedback linearization as well [50, 5]. However, feedback linearization can be achieved using multirate sampling [40]. An interesting avenue of future research would be feedback transformation of systems into chained form under sampling. This would render yet another large and interesting class of systems amenable to the techniques developed in this thesis.

References

[1] Agaev, R. P. and Chebotarev, P. Y. (2005). On the spectra of nonsymmetric Laplacian matrices. *Linear Algebra and its Applications*, 399(1-3):157–168. 64

[2] Albertini, F. and D'Alessandro, D. (2015). Minimum time optimal synthesis for two level quantum systems. *Journal of Mathematical Physics*, 56(1). 5

[3] Altafini, C. and Ticozzi, F. (2012). Modeling and control of quantum systems: an introduction. *IEEE Transactions on Automatic Control*, 57(8):1898–1917. 5, 11

[4] Ang, M. and Tourassis, V. (1987). Singularities of Euler and roll-pitch-yaw representations. *IEEE Transactions on Aerospace and Electronic Systems*, AES-23(3):317–324. 4

[5] Arapostathis, A., Jakubczyk, B., Lee, H.-G., Marcus, S., and Sontag, E. D. (1989). The effect of sampling on linear equivalence and feedback linearization. *Systems & Control Letters*, 13(5):373–381. 96

[6] Bartlett, C. E., Biggs, R., and Remsing, C. C. (2016). Control systems on the Heisenberg group: equivalence and classification. *Publicationes Mathematicae Debrecen*, 88(1-2):217–234. 5

[7] Bellman, R. (1960). *Introduction to Matrix Analysis*. McGraw Hill. 64

[8] Blanes, S. and Casas, F. (2004). On the convergence and optimization of the Baker-Campbell-Hausdorff formula. *Linear Algebra and its Applications*, 378(1-3):135–158. 25, 33

[9] Blanes, S., Casas, F., Oteo, J., and Ros, J. (2009). The Magnus expansion and some of its applications. *Physics Reports*, 470(5-6):151–238. 14

[10] Bloch, A. M. (2003). *Nonholonomic Mechanics and Control*, volume 24 of *Interdisciplinary Applied Mathematics*. Springer New York, New York, NY. 5, 91

[11] Böttcher, A. and Wenzel, D. (2008). The Frobenius norm and the commutator. *Linear Algebra and its Applications*, 429(8-9):1864–1885. 30

[12] Brockett, R. (1972). System theory on group manifolds and coset spaces. *SIAM Journal on Control*, 10(2):265–284. 4

[13] Brockett, R. and Willsky, A. (1972). Finite group homomorphic sequential systems. *IEEE Transactions on Automatic Control*, 17(4):483–490. 14

[14] Brockett, R. W. (1983). Asymptotic stability and feedback stabilization. *Differential Geometric Control Theory*, 27(1):181–191. 5

[15] Brockett, R. W. and Willsky, A. S. (1975). Some structural properties of automata defined on groups. In Manes, E. G., editor, *Category Theory Applied to Computation and Control*, pages 112–118, San Francisco. Springer. 14

[16] Bullo, F. and Lewis, A. D. (2005). *Geometric Control of Mechanical Systems*, volume 49 of *Texts in Applied Mathematics*. Springer New York, New York, NY. 21

[17] Byrnes, C., Priscoli, F., Isidori, A., and Kang, W. (1997). Structurally stable output regulation of nonlinear systems. *Automatica*, 33(3):369–385. 77

[18] Cabecinhas, D., Cunha, R., and Silvestre, C. (2008). Output-feedback control for almost global stabilization of fully-actuated rigid bodies. In *IEEE Conference on Decision and Control*, volume 58, pages 3583–3588, Cancun, Mexico. 77

[19] Castillo, B., Di Gennaro, S., Monaco, S., and Normand-Cyrot, D. (1997). On regulation under sampling. *IEEE Transactions on Automatic Control*, 42(6):864–868. 77

[20] Cheng, D., Dayawansa, W. P., and Martin, C. F. (1990). Observability of systems on Lie groups and coset spaces. *SIAM Journal on Control and Optimization*, 28(3):570–581. 4

[21] Cima, A., Gasull, A., and Mañosas, F. (1999). The discrete Markus–Yamabe problem. *Nonlinear Analysis: Theory, Methods & Applications*, 35(3):343–354. 17

[22] Corwin, L. and Greenleaf, F. (2004). *Representations of Nilpotent Lie Groups and Their Applications: Volume 1, Part 1, Basic Theory and Examples*. Cambridge Studies in Advanced Mathematics. Cambridge University Press. 28

[23] Crouch, P. E. (1984). Solvable approximations to control systems. *SIAM Journal on Control and Optimization*, 22(1):40–54. 5, 96

[24] Davison, E. J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control*, 21(1):25–34. 77

[25] Day, J., So, W., and Thompson, R. C. (1991). Some properties of the Campbell Baker Hausdorff series. *Linear and Multilinear Algebra*, 29(3-4):207–224. 25, 33, 66

[26] de Marco, S., Marconi, L., Hamel, T., and Mahony, R. (2016). Output regulation on the Special Euclidean Group $\mathsf{SE}(3)$. In *Conference on Decision and Control*, pages 4734–4739, Las Vegas, USA. 77

[27] de Marco, S., Marconi, L., Mahony, R., and Hamel, T. (2018). Output regulation for systems on matrix Lie-groups. *Automatica*, 87:8–16. 77

[28] Desoer, C. A. and Vidyasagar, M. (1975). *Feedback Systems: Input–Output Properties*. Academic Press. 30

[29] Dong, R. and Geng, Z. (2013). Consensus based formation control laws for systems on Lie groups. *Systems & Control Letters*, 62(2):104–111. 59

[30] Dörfler, F. and Bullo, F. (2014). Synchronization in complex networks of phase oscillators: A survey. *Automatica*, 50(6):1539–1564. 4

[31] Dörfler, F., Chertkov, M., and Bullo, F. (2013). Synchronization in complex oscillator networks and smart grids. *Proceedings of the National Academy of Sciences of the United States of America*, 110(6):2005–10. 4

[32] Elliott, D. (2009). *Bilinear Control Systems - Matrices in Action*. Springer. 13, 16

[33] Francis, B. and Wonham, W. (1976). The internal model principle of control theory. *Automatica*, 12(5):457–465. 77

[34] Francis, B. A. (1977). The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, 15(3):486–505. 77

[35] Giraldo, J., Mojica-Nava, E., and Quijano, N. (2013). Synchronization of dynamical networks with a communication infrastructure: A smart grid application. In *IEEE Conference on Decision and Control*, pages 4638–4643, Florence, Italy. 59

[36] Gleason, A. M. (1952). Groups without small subgroups. *The Annals of Mathematics*, 56(2):193. 2

[37] Goodarzi, F. A., Lee, D., and Lee, T. (2014). Geometric stabilization of a quadrotor UAV with a payload connected by flexible cable. In *Proceedings of the American Control Conference*, pages 4925–4930, Portland, OR. 77

[38] Goodwin, G. C., Agüero, J. C., Cea Garridos, M. E., Salgado, M. E., and Yuz, J. I. (2013). Sampling and sampled-data models: the interface between the continuous world and digital algorithms. *IEEE Control Systems*, 33(5):34–53. 12

[39] Gorbatsevich, V. V., Onishchik, A. L., and Vinberg, E. B. (1994). *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*. Springer-Verlag, Berlin, Heidelberg. 21, 23, 26, 27, 28, 68

[40] Grizzle, J. and Kokotovic, P. (1988). Feedback linearization of sampled-data systems. *IEEE Transactions on Automatic Control*, 33(9):857–859. 96

[41] Hall, B. C. (2015). *Lie Groups, Lie Algebras, and Representations*, volume 222 of *Graduate Texts in Mathematics*. Springer International Publishing, Cham, Switzerland, 2nd edition. 21, 22, 25, 55, 67, 83

[42] Hawkins, T. (2000). *Emergence of the Theory of Lie Groups*. Springer New York, New York, NY. 2

[43] Hermes, H. (1986). Nilpotent approximations of control systems and distributions. *SIAM Journal on Control and Optimization*, 24(4):731–736. 5, 96

[44] Higham, N. J. (2008). *Functions of Matrices*. Society for Industrial and Applied Mathematics. 25

[45] Hilbert, D. (1900). Mathematische Probleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 254–297. 2

[46] Hilbert, D. (1902). Mathematical problems. *Bulletin of the American Mathematical Society*, 8(10):437–480. 2

[47] Igarashi, Y., Hatanaka, T., Fujita, M., and Spong, M. W. (2009). Passivity-based attitude synchronization in $\mathsf{SE}(3)$. *IEEE Transactions on Control Systems Technology*, 17(5):1119–1134. 59

[48] Ihle, I.-A. F., Arcak, M., and Fossen, T. I. (2007). Passivity-based designs for synchronized path-following. *Automatica*, 43(9):1508–1518. 59

[49] Isidori, A. and Byrnes, C. I. (1990). Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2):131–140. 77

[50] Jakubczyk, B. and Sontag, E. D. (1987). The effect of sampling on feedback linearization. In *IEEE Conference on Decision and Control*, pages 1374–1379. IEEE. 96

[51] Jakubczyk, B. and Sontag, E. D. (1990). Controllability of nonlinear discrete-time systems: a Lie-algebraic approach. *SIAM Journal on Control and Optimization*, 28(1):1–33. 82

[52] Jiang, Z.-P. and Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6):857–869. 18

[53] Jin, J., Green, A., and Gans, N. (2014). A stable switched-system approach to obstacle avoidance for mobile robots in $\mathsf{SE}(2)$. In *International Conference on Intelligent Robots and Systems*, pages 1533–1539, Chicago, IL. 4

[54] Jurdjevic, V. and Sussmann, H. J. (1972). Control systems on Lie groups. *Journal of Differential Equations*, 12(2):313–329. 4

[55] Justh, E. W. and Krishnaprasad, P. S. (2004). Equilibria and steering laws for planar formations. *Systems & Control Letters*, 52(1):25–38. 4

[56] Khalil, H. K. (2002). *Nonlinear Systems*. Prentice Hall, 3 edition. 17

[57] Klein, D. J., Lee, P., Morgansen, K. A., and Javidi, T. (2008). Integration of communication and control using discrete time Kuramoto models for multivehicle coordination over broadcast networks. *IEEE Journal on Selected Areas in Communications*, 26(4):695–705. 4

[58] Lageman, C., Trumpf, J., and Mahony, R. (2010). Gradient-like observers for invariant dynamics on a lie group. *IEEE Transactions on Automatic Control*, 55(2):367 –377. 61

[59] Lang, S. (1999). *Complex Analysis*, volume 103. Springer-Verlag, New York, NY. 26

[60] LaSalle, J. P. (1986). *The Stability and Control of Discrete Processes*, volume 62 of *Applied Mathematical Sciences*. Springer, New York. 40

[61] Lee, J. M. (2003). *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer New York, New York, NY. 21

[62] Lee, T., Leok, M., and McClamroch, N. H. (2010). Geometric tracking control of a quadrotor UAV on $\mathsf{SE}(3)$. In *IEEE Conference on Decision and Control*, pages 5420–5425, Atlanta, GA. 4

[63] Lee, T., Leok, M., and McClamroch, N. H. (2013). Nonlinear robust tracking control of a quadrotor UAV on $\mathsf{SE}(3)$. *Asian Journal of Control*, 15(2):391–408. 77

[64] Leonard, N. E. (1997). Stability of a bottom-heavy underwater vehicle. *Automatica*, 33(3):331–346. 4

[65] Lohe, M. (2009). Non-Abelian Kuramoto models and synchronization. *Journal of Physics A: Mathematical and Theoretical*, 42(39):395101. 5, 59

[66] Loría, A. and Panteley, E. (2005). 2 Cascaded Nonlinear Time-Varying Systems: Analysis and Design. In *Advanced Topics in Control Systems Theory*, pages 23–64. Springer, London. 38

[67] MacLane, S. and Birkhoff, G. (1999). *Algebra*. American Mathematical Society, 3 edition. 38

[68] Magnus, W. (1953). Algebraic Aspects in the Theory of Systems of Linear Differential Equations. 33

[69] Mahboubi, Z., Kolter, Z., Wang, T., and Bower, G. (2011). Camera based localization for autonomous UAV formation flight. In *Infotech@Aerospace*, St. Louis, Missouri. American Institute of Aeronautics and Astronautics. 13, 62

[70] Maithripala, D., Berg, J., and Dayawansa, W. (2006). Almost-global tracking of simple mechanical systems on a general class of Lie groups. *IEEE Transactions on Automatic Control*, 51(2):216–225. 77

[71] Mangesius, H., Hirche, S., and Obradovic, D. (2012). Quasi-stationarity of electric power grid dynamics based on a spatially embedded Kuramoto model. In *Proceedings of the American Control Conference*, volume 2, pages 2159– 2164, Montreal, QC. 4

[72] McCarthy, P. J. and Nielsen, C. (2016). Passivity-based control of sampled-data systems on Lie groups with linear outputs. In *Symposium on Nonlinear Control Systems*, volume 49, pages 1006–1011, Monterey, CA. 77

[73] McCarthy, P. J. and Nielsen, C. (2017). Local synchronization of sampled-data systems on one-parameter Lie subgroups. In *Proceedings of the American Control Conference*, pages 3914–3919, Seattle, WA. 1, 43, 58, 59

[74] McCarthy, P. J. and Nielsen, C. (2018). A local solution to the output regulation problem for sampled-data systems on commutative matrix Lie groups. In *Proceedings of the American Control Conference*, pages 6055–6060, Milwaukee, WI. 1, 43, 76, 77

[75] Meisters, G. H. (1996). A biography of the Markus-Yamabe conjecture. In *Hong Kong Conference on Algebra & Geometry in Complex Analysis*, Hong Kong. 17

[76] Moakher, M. (2002). Means and averaging in the group of rotations. *SIAM Journal on Matrix Analysis and Applications*, 24(1):1–16. 61

[77] Monaco, S. and Normand-Cyrot, M.-D. (2007). Accessibility under multirate control for nilpotent lie algebra. In *IEEE Conference on Decision and Control*, pages 6262–6267. 13

[78] Monroy-Pérez, F. and Anzaldo-Meneses, A. (1999). Optimal control on the Heisenberg group. *Journal of Dynamical and Control Systems*, 5(4):473–499. 5

[79] Montgomery, D. and Zippin, L. (1952). Small subgroups of finite-dimensional groups. *Proceedings of the National Academy of Sciences*, 38(5):440–442. 2

[80] Morin, P. and Samson, C. (2003). Practical stabilization of driftless systems on lie groups: The transverse function approach. *IEEE Transactions on Automatic Control*, 48(9):1496–1508. 77

[81] Munkres, J. R. (2000). *Topology*. Prentice Hall, Upper Saddle River, NJ, 2 edition. 21

[82] Murray, R. and Sastry, S. (1993). Nonholonomic motion planning: steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716. 8

[83] Murray, R. M. and Sastry, S. S. (1991). Steering nonholonomic systems in chained form. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1121–1126, Brighton, England. 6

[84] Nešić, D. and Teel, A. R. (2004). A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control*, 49(7):1103–1122. 13

[85] Nešić, D., Teel, A. R., and Carnevale, D. (2009). Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, 54(3):619–624. 13

[86] Olfati-Saber, R., Fax, J. A., and Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233. 64

[87] Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics*. Springer US, New York, NY, 2 edition. 2

[88] Petersen, I. and Dong, D. (2010). Quantum control theory and applications: a survey. *IET Control Theory & Applications*, 4(12):2651–2671. 5

[89] Pontryagin, L. S. (1946). *Topological Groups*. Princeton University Press. 21

[90] Qin, J., Ma, Q., Shi, Y., and Wang, L. (2017). Recent advances in consensus of multi-agent systems: a brief survey. *IEEE Transactions on Industrial Electronics*, 64(6):4972–4983. 58

[91] Roza, A. and Maggiore, M. (2014). A class of position controllers for underactuated VTOL vehicles. *IEEE Transactions on Automatic Control*, 59(9):2580–2585. 4

[92] Rudin, W. (1976). *Principles of Mathematical Analysis*. McGraw-Hill, 3 edition. 51

[93] Sachkov, Y. (2009). Control theory on Lie groups. *Journal of Mathematical Sciences*, 156(3):381–439. 4, 11, 24

[94] Sarlette, A., Bonnabel, S., and Sepulchre, R. (2010). Coordinated motion design on Lie groups. *IEEE Transactions on Automatic Control*, 55(5):1047–1058. 59

[95] Schmidt, G. S., Ebenbauer, C., and Allgöwer, F. (2014). Output regulation for control systems on $\mathsf{SE}(n)$: a separation principle based approach. *IEEE Transactions on Automatic Control*, 59(11):3057–3062. 77

[96] Serrani, A., Isidori, A., and Marconi, L. (2001). Semi-global nonlinear output regulation with adaptive internal model. *IEEE Transactions on Automatic Control*, 46(8):1178–1194. 77

[97] Simpson-Porco, J. W., Dörfler, F., and Bullo, F. (2013). Synchronization and power sharing for droop-controlled inverters in islanded microgrids. *Automatica*, 49(9):2603–2611. 4

[98] Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443. 18

[99] Sontag, E. D. (2004). Input to State Stability: Basic Concepts and Results. In *Nonlinear and Optimal Control Theory*. Springer. 19

[100] Sontag, E. D. and Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359. 18

[101] Spong, M. W., Hutchinson, S., and Vidyasagar, M. (2006). *Robot Modeling and Control*. Wiley. 2, 4

[102] Strogatz, S. H. and Stewart, I. (1993). Coupled oscillators and biological synchronization. *Scientific American*, 269(6):102–109. 4

[103] Struemper, H. K. (1997). *Motion Control for Nonholonomic Systems on Matrix Lie*. Ph.d., University of Maryland. 7

[104] Struemper, H. K. (1998). Nilpotent approximation and nilpotentization for underactuated systems on matrix Lie groups. In *IEEE Conference on Decision and Control*, volume 4, pages 4188–4193. IEEE. 5, 9, 10, 11, 96

[105] Sun, W., Yu, X., Lü, J., and Chen, S. (2014). Synchronisation of directed coupled harmonic oscillators with sampled-data. *IET Control Theory & Applications*, 8(11):937–947. 59

[106] Sundarapandian, V. (2002). Global asymptotic stability of nonlinear cascade systems. *Applied Mathematics Letters*, 15(3):275–277. 18

[107] Sussmann, H. and Kokotovic, P. (1991). The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 36(4):424–440. 18

[108] Tao, T. (2014). *Hilbert's Fifth Problem and Related Topics*, volume 153 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island. 2

[109] Tiecheng, L., Zhaolin, W., and Junfeng, L. (1999). Output regulation of the general nonlinear discrete-time system. *Applied Mathematics and Mechanics*, 20(2):200–204. 77

[110] Valavanis, K. P. (2007). *Advances in Unmanned Aerial Vehicles*. Intelligent Systems, Control and Automation: Science and Engineering. Springer Netherlands, Dordrecht. 2

[111] Varadarajan, V. S. (1984). *Lie Groups, Lie Algebras, and Their Representations*, volume 102 of *Graduate Texts in Mathematics*. Springer, New York. 21, 22, 23, 27

[112] Wang, S. Y., Huo, W., and Xu, W. L. (1999). Order-reduced stabilization design of nonholonomic chained systems based on new canonical forms. In *IEEE Conference on Decision and Control*, volume 4, pages 3464–3469, Phoenix, AZ. 6, 8

[113] Wei Ren, Beard, R., and Atkins, E. (2005). A survey of consensus problems in multi-agent coordination. In *Proceedings of the American Control Conference*, pages 1859–1864. IEEE. 58

[114] Willsky, A. S. (1975). Invertibility of finite group homomorphic sequential systems. *Information and Control*, 27(2):126–147. 14

[115] Willsky, A. S. and Marcus, S. I. (1976). Analysis of bilinear noise models in circuits and devices. *Journal of the Franklin Institute*, 301(1-2):103–122. 5

[116] Wittenburg, J. (1977). *Dynamics of Systems of Rigid Bodies*. Leitfäden der angewandten Mathematik und Mechanik. Vieweg+Teubner Verlag, Wiesbaden. 11

[117] Wonham, W. M. (1979). *Linear Multivariable Control: a Geometric Approach*. Springer, New York. 28, 79, 87