From Gravity to Hopf Algebra Lattice Models

by

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Statement of contributions

Chapter 4 is based on the paper [1], which I co-authored with Maïté Dupuis, Florian Girelli and Wolfgang Wieland.

Section 5.2 of Chapter 5 is based on a yet to be published work [2] with Maïté Dupuis, Laurent Freidel and Florian Girelli.

The last two sections of chapter 7 is based on the paper [3], which I co-authored with Florian Girelli and Prince K. Osei.

Abstract

This thesis addresses three different problems related to quantum gravity. In the first problem we will discuss the two natural ways to encode gravity through geometric structures. One is the much acclaimed Einstein's general relativity and the other is teleparallel gravity, where torsion as opposed to curvature encodes the dynamics of gravitational degrees of freedom. We will show that the Einstein–Cartan action, the general relativity first-order formulation, can also be seen as the first-order formulation of teleparallel gravity. We then discuss how the discretization of the Einstein-Cartan action in three dimensional spacetime affects the equivalence of these two formulations.

We investigate then how one can derive the quantum group structure in loop quantum gravity with a non-zero cosmological constant Λ from the continuum action. It is well known that the cosmological constant implies a quantum group deformation of the internal gauge group at the quantum level, but it is usually introduced by hand. We show for the first time that we can derive this q-deformation from the 3d gravity action using the discretization scheme already discussed above. The key element is to find the right set of variables to discretize. The discretized symplectic form is then the one for the Heisenberg double which upon quantization generates the quantum group structure. This allows to see 3d loop quantum gravity (with topological defects) as a specific Hopf algebra lattice theory.

The last part of the thesis studies Hopf algebras lattice models, defined as representations for certain quantum groups. We propose a set of requirements for these models which should allow to generalize the construction to different types of quantum groups. We show how the Kitaev's quantum double model satisfies such requirements. We then propose a new version of the Kitaev model based on Majid's bicrossproduct quantum group. The construction of this new model is relatively natural as it relies on the use of the covariant Hopf algebra actions. We obtain an exactly solvable Hamiltonian for the model and provide a definition of the ground state in terms of a tensor network representation. Finally, we discuss how Kitaev's quantum double model and its dual version can be related to the two formulations of gravity.

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Dedication

This thesis is dedicated to my father Mr. Awudu K. Osumanu and my late mother Ms. Emefa Agidi

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 The two faces of gravity

Universality of free fall, a distinct property of gravity, ensures particles move with the same acceleration irrespective of their mass and composition. The concept of the universality of free fall gives rise to two equivalent ways to encode gravity into spacetime geometry [5]. The first is to consider a connection which is torsionless (Levi-Civita connection). This formulation is given by Einstein's *general relativity* in which the curvature of the Levi-Civita connection encodes the dynamics of gravity. Massive degrees of freedom are most naturally described in this framework and are the natural source of curvature. This is possible because the universal nature of free fall allows the geometrization of gravity through curvature. In other words, in the absence of universality, there is a breakdown in the description of gravity from general relativity's point of view.

The second and not so popular way to encode gravity through geometric structures [6] is to consider the so called Weitzenböck connection which has zero curvature. This is the *teleparallel* formulation in which torsion encodes the gravitational degrees of freedom [7, 5, 8, 9, 10]. This formulation was actually studied by Einstein in his attempt to unify general relativity with electromagnetic theory. His attempt failed when it was realized electromagnetism requires a background field to propagate whiles general relativity does not. This work however led to the formulation of teleparallel gravity, a gauge theory for the translation group. In the teleparallel formulation, spin degrees of freedom are the natural degrees of freedom to consider, they are the natural source for torsion. Its description as a gauge theory means in the absence of universality gravitational interactions still lead to a consistent theory of gravity.

The Levi-Civita and Weitzenböck connections are both metric compatible, and contain inertia degrees of freedom. The Levi-Civita connection in addition has gravitational degrees of freedom. With the above descriptions of both general relativity and teleparallel gravity, their respective underlying spacetime are the pseudo-Riemannian space and the Weitzenboc̈k spacetime. The dynamics of general relativity can be derived from the familiar Einstein – Hilbert action

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4v_g\, \overset{\circ}{\mathrm{R}}[g], \tag{1.1}$$

where $\overset{\circ}{\mathrm{R}}$ is the Ricci scalar built for the metric tensor g_{ab}, and $\mathrm{d}^4v_g = \sqrt{-\det g_{\mu\nu}}\mathrm{d}x^0 \wedge \ldots \mathrm{d}x^3$ is the canonical volume element. In the case of teleparallel gravity the dynamics is governed by the action

$$S_{//}[e;\overset{\bullet}{\omega}] = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4v_g\left(\frac{1}{4}\overset{\bullet}{\mathrm{T}}_{cab}\overset{\bullet}{\mathrm{T}}{}^{cab} - \frac{1}{2}\overset{\bullet}{\mathrm{T}}_{cab}\overset{\bullet}{\mathrm{T}}{}^{abc} - \overset{\bullet}{\mathrm{T}}{}^{b}{}_{ab}\overset{\bullet}{\mathrm{T}}{}^{ca}{}_{c}\right), \tag{1.2}$$

where $\overset{\bullet}{\mathrm{T}}{}^{a}{}_{bc}$ is the torsion tensor. The two actions are equal up to a boundary term [11, 9].

$$S_{\mathrm{EH}}[g] = S_{//}[g,\overset{\bullet}{\omega}] - \frac{1}{16\pi G}\int_{\mathcal{M}} d^4v_g\, {}^{\bullet}\nabla_a \overset{\bullet}{\mathrm{T}}{}^{ba}{}_{b}, \tag{1.3}$$

thereby underlying the fact that both general relativity and teleparallel gravity give equivalent description of gravitation. Here ${}^{\bullet}\nabla$ is the covariant derivative with respect to the Weitzenböck connection.

Faced with the difficulty to introduce non-zero spin degrees of freedom in the standard general relativity case, Weyl introduced the concept of frame field e, which led ultimately to the Sciama-Kibble-Einstein-Cartan formalism for gravity which puts on equal footing mass as source of curvature and spin as source of torsion

$$S_{\mathrm{EC}}[e,A] = \int \big\langle B[e] \wedge F[A]\big\rangle_{\mathfrak{so}(1,n-1)}, \tag{1.4}$$

where $F = \mathrm{d}A + \frac{1}{2}[A,A]$ is the curvature of the spin connection A, and $B = *(\wedge^{n-2}e)$ is the $(n-2)$-form built from the internal hodge dual of $(n-2)$ frame fields e. On the space of histories, the connection A has both non-trivial curvature and torsion. This is the first order formalism for gravity, because the action only contains first derivative of the fundamental configuration variables. If there are no spin degrees of freedom, we get as an equation of motion that A should be torsionless. Plugging this back into the action

(1.4), we get the Palatini formalism for gravity, in terms of frame fields and a torsionless connection (a second order formalism).

Interestingly, while there is the duality relation (1.3), to our knowledge, there is no similar derivation of the teleparallel action from a first order action. Indeed, having a zero-curvature connection solves the equation of motion of the frame field (when there is no massive degrees of freedom), but plugging this back into the action (1.4) leads to a zero action. Some ways to avoid a zero action is to either supplement (1.4) with a constraint implementing the zero-curvature [12], or even to add quadratic contribution in the torsion and curvature [13].

Chapters 2 and 3 of this thesis provides a review of both general relativity and teleparallel gravity respectively.

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1.2 Quantum general relativity

Recent developments in the *loop quantum gravity* framework indicate that there ought to be a more symmetric treatment between general relativity and teleparallel formulations. One of the goals of this thesis is to explore this symmetry between the two formulations. Loop quantum gravity is one of several approaches to solve the problem of *quantum gravity*, which is an attempt to reconcile the principles of quantum mechanics and general relativity. Both quantum mechanics and general relativity have profoundly changed our understanding of the nature of the universe.

In general relativity, the geometry of spacetime is a dynamical quantity and at the heart of it, is the gravitational field. This has led to the concept of background independence and general covariance – which is that there is no preferred inertial frame of reference when it comes to the description of physical phenomenon. On the other hand, quantum mechanics requires a dynamical field to be quantized, and in addition a fixed non-dynamical background spacetime for it to take place. From these two theories, the open question is, can we find a theory of gravity that takes into account quantum effects? The answer so far has been inconclusive from both the theoretical and the experimental directions for a number of reasons. One of such reasons has been the lack of consensus within the community for what features a quantum gravity theory should possess. Another is the highly non-linear nature of Einstein's equations which are not easily solvable.

Since their formulation both general relativity and quantum physics have enjoyed some level of success through our understanding of the universe and the nature of matter interactions respectively. For instance we understand the universe through cosmology and astrophysics, and the nature of matter interactions through atomic and condensed matter physics. However despite their success both theories are conceptually incomplete. For example in the context of cosmology and for that matter the FLRW models[1], they assume the initial phase of the universe began with a big bang[2], resulting in the appearance of singularities, thus suggesting a breakdown of general relativity [14, 15]. At the center of the black hole (thus a region of space from which nothing can escape), thus before one reaches the singularity, it happens that both gravitational and quantum effects have to be considered at the same time. This phenomenon occurs at a Planck length smaller [14] than $10^{-33}cm$. Matter at this stage collapses reaching an infinite energy density and the quantum behaviour of the gravitational field sets in or becomes relevant [16]. Hence the need for a quantum theory of gravity to avoid or regularize these singularities.

A number of approaches have been proposed to solve the problem of quantum gravity. These include string theory, emergent gravity, non-commutative geometry, spin foams, loop quantum gravity, etc. Our focus however will be on loop quantum gravity, which is a canonical approach to quantizing gravity and pays special attention to conceptual lessons of general relativity such as background independence.

In loop quantum gravity, the starting point is the Hamiltonian formulation [17, 16, 18] of general relativity. Hence spacetime is foliated into a family of hypersurfaces of constant time. In the initial Hamiltonian formulation of general relativity, the ADM[3] or the metric variables were used in the construction. A quantization technique by Dirac[4] was applied to the constraints (Gauss, spatial diffeomorphism and Hamiltonian) of these variables. It was quickly realized in these ADM variables, the quantization program gave no meaningful result due the extremely complicated shape of the constraints. With a nice change of variables, Ashtekar [19] was able to simplify these constraints and thus opened up the quantization program as we know it today. In the Ashtekar variables i.e., the Ashtekar connection A^i_b and the densitized triad E^a_j, general relativity takes the structure of an SU(2) gauge theory[5]. Wave functions become gauge invariant functional of the connection A^i_b

[1]The FLRW model is an exact solution of Einstein's equations of general relativity which describes a homogeneous, isotropic and expanding /contracting universe.

[2]At the big bang, spacetime curvature and matter densities become infinite.

[3]It is named after the initial inventors, Arnowitt, Deser and Misner.

[4]This quantization approach involves three steps: The first is to find a representation of the phase space variables as operators, acting on some kinematical Hilbert space $\mathcal{H}_{kin}$. This representation of the phase space variables should map Poisson brackets into commutators. Next is to define the constraints into self-adjoint operators in $\mathcal{H}_{kin}$, and finally the solutions of these constraints are characterize to define an inner product leading to a physical Hilbert space $\mathcal{H}_{phy}$.

[5] In this formulation we are restricting to the case of zero cosmological constant Λ. Later in the chapter

naturally based on a Wilson loop, which is characterized by holonomies – the configuration variables of loop quantum gravity.

Following the works of Rovelli and Smolin [20], spin network states were introduced as the building blocks of loop quantum gravity, most importantly they are the basis of the kinematical Hilbert space $\mathcal{H}_{kin}$ of the theory. The spin networks states are the quantum states of spatial quantum geometry. These are graphs Γ decorated by the irreducible unitary representations $D^{(j_l)}(h_l)$ of SU(2) holonomy h_l along each link l of Γ, and an intertwiner i on each node n of Γ

$$\psi_{(\Gamma,j_l,i_n)} = \otimes_l D^{(j_l)}(h_l) \otimes_n i_n. \tag{1.5}$$

By considering a flux[6] $X_l \in \mathfrak{su}(2)$ which is conjugate to the holonomy h_l, one finds that the Poisson brackets of these variables coincide with those of the cotangent bundle of SU(2). Thus the phase space corresponding to the spin networks state of a graph is $T^*\,\mathrm{SU}(2)$ for each link l of the graph. Taking a limit of graph refinement, the kinematical Hilbert space of loop quantum gravity is

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma} \mathcal{H}_{\Gamma,kin}. \tag{1.6}$$

The discovery of the spin networks states as basis of loop quantum gravity allowed to extract information about the quantum geometry through geometric operators [21], which are the quantization of classical expressions that measure area and volume. The spin network states diagonalize these operators resulting in the discrete nature of their eigenvalues. For instance, the area operator's $\hat{A}(\mathcal{S})$ action on a spin network Ψ_Γ is given by

$$\hat{A}(\mathcal{S})\Psi_\Gamma = 8\pi\,\gamma\,l_P^2 \sum_{p\in S\cup\Gamma} \sqrt{j_p(j_p+1)}\Psi_\Gamma, \tag{1.7}$$

from which the discreteness of the area eigenvalue is given in terms of the Planck's length l_P. Here p is the intersection point between Ψ_Γ and a two dimensional surface $\mathcal{S}$, j_p is the spin associated to p, and the factor γ is called the Barbero-Immirzi parameter. In addition to the discreteness of the area and volume operators, calculation of the black hole entropy [22] was successfully carried out in loop quantum gravity.

Three dimensional gravity

Despite the achievements of loop quantum gravity, the construction of its physical Hilbert space has not yet been realized. That is, the dynamics of the theory which should be

we will return and discuss more on loop quantum gravity with $\Lambda \neq 0$.

[6]The flux is the result of an integration of the densitzed over a surface dual to a link.

implemented by solution of the Hamiltonian constraint has not been found. This limitation, we should point out occurs only in the 4d gravity case. When it comes to 3d Euclidean gravity with $\Lambda = 0$, where general relativity can be reformulated as a topological BF theory for SU(2) [23], the dynamics is fully solved [24]. Conceptually, 3d gravity is similar to 4d gravity and as such lessons about (loop) quantum gravity can be learned from it. From now on, we will focus on 3d gravity as this thesis is based on it.

The spin network states as already discussed characterize quantum discrete geometries [20] and this results in some discrepancies between the classical continuum variables and the corresponding quantum operators. It happens that at the classical level the Poisson bracket of the fluxes is trivial, however when quantized, the quantum fluxes algebra coincide with a SU(2) algebra. Hence one should first discretize the frame field and then quantize it. This means in the loop quantum gravity scheme, there is at the same time a discretization and quantization that are performed. Disentangling these two different facet is paramount in when describing the dynamics [25]. To carry out the discretization process is to first subdivide the system by defining a cellular decomposition of the spatial manifold and then followed by truncating the degrees of freedom of the theory in each cell of the decomposition [25, 26].

In 3d gravity there are two constraints: the Gauss (or torsion) and the curvature (or flatness constraint). At the classical level there is no preference as to which is imposed first, however at the quantum level this matters. The standard loop quantum gravity framework is based on imposing the Gauss constraint first, which amounts – at least in 3d – to deal with a torsionless connection. At the quantum level, this leads to the so-called Ashtekar-Lewandowski vacuum. From this perspective, loop quantum gravity can be seen as quantization of the general relativity formalism (at least in 3d) [16]. Dittrich and Geiller suggested that there should be another interesting "vacuum" to start with, not based on the imposition of the Gauss constraint first, but instead on a zero curvature constraint. At the quantum level, this leads to the so-called BF vacuum [27, 28]. A posteriori, this could be viewed as a quantization of the teleparallel formulation since we deal with a flat connection.

A bit later, looking at the discretization picture underlying the loop quantum gravity framework at the classical framework, [29] showed that there are two natural ways to discretize Einstein-Cartan gravity action (the work is done in 3d for BF theory but it should carry similarly in the 4d case). One that essentially implements the Gauss constraint first (this is the loop gravity picture), another one that implements a zero curvature first (this is the dual loop gravity picture). This last case can be viewed as another classical derivation of the idea suggested by Dittrich and Geiller [28]. The discretization procedure started in each case from the BF action so while it seems pretty clear that the second derivation, i.e., the dual loop gravity picture, should be related to a discretization of the teleparallel formulation, it was not shown explicitly.

To complete the picture regarding possible discretization/quantization related to the teleparallel formulation we recall that in [30], the authors argue that t'Hooft discrete approach to 3d gravity can be seen as a discretization of the teleparallel formulation (still hinging on the assumption that the dual loop picture is related to the teleparallel picture). In [31], the authors presented a quantization of the dual loop gravity model (slightly different than Dittrich and Geiller's) which led to the Dijkgraaf-Witten model.

Based on the above arguments illustrating the symmetric relation between general relativity and teleparallel gravity, and also on the duality relation (1.3), we will present the results of [1] in chapter 4 showing indeed the Einstein–Cartan action, a first-order formulation of the standard general relativity theory, is also a first-order formulation of the teleparallel theory up to a boundary term. This is done first in the three-dimensional Euclidean case where the Einstein-Cartan action is simply the SU(2) BF action. We then generalize our derivation for a Lorentzian signature in any dimensions. The main idea of the derivation is to decompose the Einstein–Cartan connection into a fiducial reference connection plus a contorsion tensor. Then by choosing some specific reference connections and solving some of the equations of motion strongly, we show that, depending on the choice of reference connection, the Einstein–Cartan action is equal *on shell* to either the Palatini action of general relativity or the teleparallel action up to boundary terms.

In the second part of chapter 4, we will then discuss the different discretizations performed in [29, 32] (for three-dimensional gravity) in light of the fact that both the general relativity and teleparallel frameworks can be derived from the same first-order action (up to a boundary term). In the Hamiltonian picture, each of these frameworks can be naturally associated to a choice of polarization. The physical equivalence of the different polarizations is the natural translation of the equivalence between the general relativity and teleparallel frameworks. We will argue however that different choices of polarization at the continuum level lead to different discretized theories. More explicitly, the choice of polarization in the continuum and the discretization procedure used in [29, 32] do not commute. As a consequence, we will discuss how the dual loop gravity picture can be seen as a discretized version of the teleparallel formulation.

1.3 Loop quantum gravity and the cosmological constant

The nature of the cosmological constant Λ has long been an open question. For instance in the classical picture, it is seen as a left over of a scalar degree of freedom [33, 34] from the point of view of modify gravity. In a recent proposal [35], it was assumed to be related to torsion thereby making it a dynamical variable. In the quantum realm, and in the context of unimodular gravity, Λ is a constant of integration, hence it was suggested to accounts for lack of energy-momentum conservation [36]. Physically, its observed value of $\Lambda^{obs} \approx 10^{-52}m^{-2}$ is off by up to 120 orders[7] of magnitude [37] of the calculated energy vacuum [38]. However, in this thesis, we are not going to discuss physical phenomenon generating it but rather we will take it as a constant, which could/should possibly run.

Several works [39, 40, 41] have shown that in 3d, a non-zero cosmological constant leads to a quantum group (quasitriangular Hopf algebra) deformation of internal gauge group. As a consequence in loop quantum gravity with $\Lambda \neq 0$, not only is the gauge group deformed but also the spin network representation is deformed. The relevant parameter q which plays the role of a deformation parameter is related to the cosmological constant through [42]

$$q = \exp\left(-\frac{\hbar G\sqrt{\Lambda}}{c}\right), \tag{1.8}$$

with values in either $\mathbb{R}$ or the unit circle. $\hbar$ denotes the Planck's constant and c is the speed of light in vacuum, and is imaginary in an Euclidean spacetime.

A general feature of the q-deformation in the loop quantum gravity framework is that, it is normally introduced by hand. The justification for such a method comes from other quantization programs of 3d gravity. Programs such the combinatorial quantization of Chern-Simons theory [43, 44, 45] and the construction of the Turaev-Viro spin foam model [46].

Aside for their mathematical elegance, the instances in which quantum groups have been introduced by hand, have seen them throw more light on their physical relevance. In [47], the authors attempted to derive a quantum group structure in 3d gravity but was found in [48] to be a regularization tool for solving the Hamiltonian constraint in the quantum theory. Again in [49, 50], they were introduced to regularize expectation values of observables.

[7] This is what is known as the cosmological constant problem.

The reason why in the loop quantum gravity set up, the appearance of such quantum group structure is not visible for $\Lambda \neq 0$, has to do with the fact that Λ appears in 3d or 4d through the flatness or Hamiltonian constraint respectively, and solving these constraints on the kinematical Hilbert space has so far been elusive. One expects the quantum states should be built on deformed spin network states, however that is not the case, as the Gauss constraint does not depend on Λ.

In a number of works [51, 4, 52, 39, 40], efforts have been made to clarify this issue. Most of these works specialize on the mathematical structures of these deformed symmetries in the presence of $\Lambda \neq 0$. Using lattice gauge theory techniques, a proposed model [4] based on Heisenberg doubles of SU(2) to describe 3d gravity is constructed. The Heisenberg double as it turns out is the phase space of deformed loop quantum gravity with a negative cosmological constant. This model was then quantized [51], its kinematical Hilbert space is spanned by spin networks with a q-deformation and solution to the Hamiltonian constraint gave rise to the Turaev-Viro amplitude with q real.

In chapter 5 of this thesis, we will present a new result showing how to derive a q-deformation from the 3d gravity action with a non-zero cosmological constant. This result is based on computing the phase space of the theory through the discretization process explained above. The obtained phase space is precisely given in terms of the Heisenberg double framework. Our derivation is based on the following argument: When $\Lambda \neq 0$, in the standard Hamiltonian formulation of the action, Λ does not appear in the Gauss constraint. So we need to find a change of variable, typically a canonical transformation so that Λ appears in the Gauss constraints hence generating the deformed spin network structure. Furthermore, the constraints expressed in the new variables should be easily discretized.

Guided by the Chern-Simons formulation which put in a sense on equal footing the phase space variables, we use a different polarization guided by the different decompositions one can have of the Lorentz group. In particular the Iwasawa decomposition provides the key to find the right change of variables.

In terms of these new variables, the constraints appear to be the natural continuum candidates behind the discrete constraints used to define a discrete model of 3d gravity with $\Lambda \neq 0$ [4]. To define such new constraints, we have to deal with a lattice gauge theory based on a matched pairs of groups, which was first discussed, to the best of our knowledge, by Majid [53]. We will then discuss the relation between our derivation and the discrete model of [4].

1.4 Hopf algebra lattice models

In Landau's symmetry breaking theory [54], every material or physical phase can be classified according to the organization or order of the particles that constitute it. However, it turned out not every physical phases of matter can be explained by the symmetry breaking theory. An example is the phenomenon of fractional quantum Hall effect [55], here all the different states have the same symmetry. This is attributed to the stability of the ground state degeneracy[8] against any arbitrary local perturbations [56]. The stability of the ground state degeneracy implies the existence of universal internal orders of the internal structures of the states in the fractional quantum Hall effect. These universal internal orders are term *topological orders* and they provide an insight into the study of phases of matter. Apart from their relevance in condensed matter, topological orders have found their way into quantum information, precisely in kitaev models.

The Kitaev quantum double models [57] were originally proposed to exploit topological phases of matter for fault-tolerant quantum computation. The models are based on quantum many-body systems exhibiting topological order. Their physics is obtained from Topological quantum field theories (TQFTs), while their underlying mathematical structure is based on Hopf algebras. For a given finite group G, Kitaev constructed an 'extended' Hilbert space on a triangulated oriented surface Σ and an exactly solvable Hamiltonian[9], whose ground state or protected space is a topological invariant of the surface. It turns out that, this triangulations or graph defines a representation of the Drinfeld quantum double $D(G)$. A well known example of these models is the Kitaev toric code, which is based on the cyclic group $\mathbb{Z}_2$ [57]. See also [58] for a recent account. It was anticipated in [57] that these models could be generalized to that based on a finite-dimensional Hopf algebra H. This was achieved in [59].

The Kitaev quantum double models can be understood to describe the moduli space of flat connections on a 2d surface with defect excitations. From the point of view of quantum gravity, they are of strong interest as they are directly related to certain 3d TQFTs defined in terms of (quasitriangular) Hopf algebras. It is known that the protected space of a Kitaev model for a finite-dimensional semisimple Hopf algebra H on an oriented surface Σ is exactly the vector space that the Turaev-Viro TQFTs [60, 46] for the representation category of H assigns to Σ [61, 62, 63, 64]. The construction of these models is also closely related to BF theory with defects [65, 66, 67, 68, 69], a TQFT describing locally

[8]This degeneracy depends on the topology of space.

[9]Hamiltonians that are solvable are the ones whose eigenvalues and eigenvectors can be determined exactly.

flat connections. Other recent examples include a dual picture which was introduced in the quantum gravity setting where the excitations have been swapped [29]. Even though this was discovered independently, this result could have been guessed in light of the notion of electro-magnetic duality well known in topological quantum computing [70]. A recent paper by Meusburger show that Kitaev's model for a finite-dimensional semisimple Hopf algebra H is equivalent to the combinatorial quantization of Chern-Simons theory for the Drinfeld double $D(H)$ [71]. This emerges in a gauge theoretic framework, in which both models are viewed as Hopf algebra-valued lattice gauge theories [72].

These results have opened new perspectives on the relations between topological quantum information(TQI) and quantum gravity. Although each framework comes with its own motivation, they share similar mathematical concepts. For example, in the case of TQI cases, one deals with a (ribbon) graph decorated by Hopf algebra elements and constructs an exactly solvable Hamiltonian defined in terms of operators acting on the nodes and faces of the graph. The vacuum state of this can be interpreted from the quantum gravity perspective as the pure gravity case, whereas the excitations of the TQI Hamiltonian, used to perform quantum computations, are interpreted as particles with mass or spin depending on their location. In the case of loop quantum gravity, one has torsion excitations on the nodes, i.e. spin, whereas on the faces, one has curvature excitations, i.e. mass. The most relevant algebraic structure to deal with representations which classify particles for example and indicate their braiding, is not only the Hopf algebra H but the associated Drinfeld double $D(H)$. Once again, this structure was identified using different arguments in each of the different frameworks. In the TQI case, one deals with the Drinfeld's quantum double of finite dimensional (semisimple) Hopf algebras (e.g. built from finite groups) [57, 73, 59] whereas in the quantum gravity case one makes use of the quantum double of Hopf algebras built from Lie groups or their quantum deformation [43, 44, 74, 75, 76, 77, 78, 79, 80, 81, 42].

As described above, the Drinfeld quantum double is in a sense the common quantum group which arise in the quantum computing setting. However, from the point of view of quantum gravity, other quantum groups emerge. In particular, the bicrossproduct quantum group originally proposed by Majid [53] as a new foundation for quantum gravity. The bicrossproduct quantum groups are interpreted here as algebras of observables of quantum systems so that one can view them as functions on a quantum phase space. These bicrossproduct quantum groups are also known to be valid candidates for the combinatorial quantization of Chern-Simons theory of 3d gravity [82, 83, 84, 85].

It turns out that the bicrossproduct quantum group is physically related to the Drinfeld double, through a semi-dualization map [86]. This stems from Majid's idea of 'quantum born reciprocity', proposed for quantum gravity where one can exchange position and momentum degrees of freedom in an algebraic framework [87].

From the above considerations, while the bicrossproduct quantum group emerges in the quantum gravity framework, it is yet to be explored for the topological quantum computation models. It is therefore natural to ask whether it is possible to construct lattice models for quantum computation based on quantum groups other than the quantum doubles, in particular the bicrossproduct quantum groups.

Chapter 6 of this thesis provides the mathematical structure behind the construction of the bicrossproduct quantum group through the semi-duallization map. In chapter 7, we will then present a proposed Kitaev lattice model based on the bicrossproduct quantum group, this result is based on the work in [3].

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Chapter 2

Classical gravity theories

Before the advent of general relativity, gravity according to Newton is an attractive instantaneous force acting between two massive objects. This notion of gravity as a force was successful in explaining the motion of the planets in our solar system. Then special relativity told us that instantaneous interaction ceases to exist and time is no longer absolute. Two simultaneous events in an inertial frame are not necessarily simultaneous in another thereby changing our understanding of simultaneity. This brought the realization Newton's laws do not give the full picture in understanding the Universe. In 1915, Einstein, in his seminal paper described gravity as a curvature of spacetime, thus moving away from the notion of gravity as a force. The outcome of this was the formulation of the theory of General Relativity (GR).

Physically, GR is the discovery that spacetime and gravitational field are the same entity [17]. This can be seen from the universality of free-fall exhibited by particles. If we recall, this is when objects, regardless of their composition, experience gravitation, will have the same acceleration in a gravitational field. This a property only unique to gravity compared to the other known interactions or forces of Nature. Einstein's insights was that the gravitational field is characterized by an underlying curvature of spacetime and accordingly the gravitational interaction makes then no reference to any concept of force. Since it formulation, general relativity has passed several observational and experimental tests such as relativistic astrophysics, cosmology and the recently verified gravitational waves [88] during a merger of two black holes. Einstein's general relativity is formulated in the language of (pseudo)- Riemannian geometry. This makes differential geometry a very important subject in the formulation of the theory.

Naturally when describing gravity, there is the non-trivial parallel transport of vectors and the object responsible for this, is the connection. The connection compensate for the non-convariant nature of the ordinary derivatives of vectors transported on the manifold. If a vector parallel transported comes back rotated with respect to its initial value [89], the spacetime is said to contain *curvature*. On the other hand, if the vector is translated with respected to it initial value the spacetime is said to contain *torsion*. In general relativity, the (Levi-Civita) connection may have a non-trivial curvature and no torsion. Spinless particles follow geodesics

$$\ddot{x}^{\mu} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\sigma}\,\dot{x}^{\nu}\,\dot{x}^{\sigma} = 0, \tag{2.1}$$

here x^{μ} are the particle coordinates and $\overset{\circ}{\Gamma}$ is the Levi-Civita connection. The geodesics equation (2.1) tells us how particles move in curved spacetime. There are instances however where spinning particles are included in the GR description. In such situation, the particle's trajectory do not obey the geodesics equation but rather follow the dynamical equations of the test particle, as first determined by Papapetrou [90] and subsequently applied in [91] for the Schwarzschild field.

Suppose, one considers including in a non-trivial torsion into the description of gravitation, then no longer are we describing GR but some generalized theories of gravity instead. The Einstein-Cartan gravity theory includes torsion as an independent degrees of freedom alongside curvature [92]. Here the spins of matter fields acts as the source of torsion just as the energy-momentum tensor act as the source of curvature in general relativity. The relevant connection here is the Cartan connection and contains both curvature and torsion. If there is no spin, the two theories are equivalent, since they have the same equations of motion.

So far, we have discussed gravity theories in which dynamics are described by only curvature, and by both curvature and torsion. There is however yet another gravity theory, namely teleparallel gravity, in which a non-trivial torsion describes dynamics with the Weitzenbočk connection encoding it. It was initially studied by Einstein just after the formulation of general relativity in his bid to unify electromagnetism and general relativity. The underlying spacetime describing teleparallel gravity is globally flat, this is so because the Weitzenbočk connection has a vanishing Riemannian curvature. We will discuss a lot more about this formulation of gravity in the next chapter.

We will present general relativity in both the metric and tetrad formulation. Just as the other fundamental interactions of nature, thus electromagnetism, weak and strong forces, gravity can also be formulated as a gauge theory. In considering torsion as an independent propagating field in addition to curvature we will then study Einstein-Cartan theory as a generalization of general relativity and show why general relativity can been viewed as a gauge theory.

2.1 Metric formulation of general relativity

The metric formulation of Einstein's gravity is specified by the pair $(\mathcal{M}, g_{\mu\nu})$, a (pseudo)-Riemannian spacetime manifold. We start with the Einstein-Hilbert action for the metric $g_{\mu\nu}$ which describes the dynamics of spacetime

$$S_{\rm EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\int_{\mathcal{M}} {\rm d}^4x\sqrt{-\,{\rm g}}\,(\overset{\circ}{\rm R} - 2\Lambda), \tag{2.2}$$

where $\overset{\circ}{\rm R}$ is the Ricci scalar of the Levi-Civita connection $\overset{\circ}{\Gamma}$, Λ is the cosmological constant, G is Newton's gravitational constant, and g is the determinant of the metric $g_{\mu\nu}$. Here $\mu, \nu, ... = 0, 1, 2, 3$ are spacetime indices. The action is a scalar and contain derivatives of the metric. The variation of the action leads to the vacuum Einstein equations

$$\overset{\circ}{\rm R}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,\overset{\circ}{\rm R} + \Lambda\,g_{\mu\nu} = 0. \tag{2.3}$$

One could also include matter fields into spacetime, in that regard, such a theory can then be defined by specifying a Lagrangian $\mathcal{L}_m$ for matter fields Ψ to include their first covariant derivative. This leads to the action for matter fields

$$S_m[\Psi] = \int {\rm d}^4x\sqrt{-{\rm g}}\,\mathcal{L}_m(\Psi). \tag{2.4}$$

Its variation with respect to the inverse metric $g^{\mu\nu}$ gives the energy momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-{\rm g}}}\frac{\delta S_m[\Psi]}{\delta g^{\mu\nu}}. \tag{2.5}$$

Hence by supplementing matter fields with gravitational fields, the full gravitational action for general relativity becomes

$$S_{\rm GR} = S_{\rm EH}[g_{\mu\nu}] + S_m[\Psi], \tag{2.6}$$

and its variation leads to Einstein's equations

$$G_{\mu\nu} = 8\pi G\,T_{\mu\nu}, \tag{2.7}$$

where

$$G_{\mu\nu} = \overset{\circ}{\rm R}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,\overset{\circ}{\rm R} + \Lambda g_{\mu\nu} \tag{2.8}$$

is the Einstein tensor and c is the speed of light. It is this relation (2.7) which determines how the distribution of mass and momentum densities induces curvature on spacetime.

The Einstein equations (2.7) holds in any dimension. In the physical theory (4 dimensions), there are twenty components of the Riemann curvature tensor of which ten are the Ricci tensor comprising of coupled second order partial differential equation and the other ten are the Weyl tensor $C_{\mu\nu\rho\sigma}$ defined in terms of the Riemman curvature

$$\overset{\circ}{\mathrm{R}}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \left(g_{\mu[\rho}\overset{\circ}{\mathrm{R}}_{\sigma]\nu} - g_{\nu[\rho}\overset{\circ}{\mathrm{R}}_{\sigma]\mu} \right) - \frac{1}{3}\overset{\circ}{\mathrm{R}}\, g_{\mu[\rho}g_{\sigma]\nu}. \tag{2.9}$$

These differential equations of the Ricci tensor are difficult to solve, however imposing some symmetry principles, one is able to get exact solutions for the gravitational field for some physical situations. The Schwarzshild solution - solution of Einstein vacuum equations with spherical symmetry - is one such instance, it describes the exterior of a static spherical star or a black hole. In three dimensions, the Riemman tensor is entirely dependent on the Ricci tensor as the Weyl tensor vanishes identically:

$$\overset{\circ}{\mathrm{R}}_{\mu\nu\rho\sigma} = g_{\mu\rho}\,\overset{\circ}{\mathrm{R}}_{\nu\sigma} + g_{\nu\sigma}\,\overset{\circ}{\mathrm{R}}_{\mu\rho} - g_{\nu\rho}\,\overset{\circ}{\mathrm{R}}_{\mu\sigma} - g_{\mu\sigma}\,\overset{\circ}{\mathrm{R}}_{\nu\rho} - \frac{1}{2}\left(g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)\overset{\circ}{\mathrm{R}}. \tag{2.10}$$

By taking the trace of (2.3) the vacuum equation becomes

$$\overset{\circ}{\mathrm{R}}_{\mu\nu} = 2\Lambda g_{\mu\nu}. \tag{2.11}$$

Thus in the absence of a cosmological constant, the Riemann tensor vanishes implying every solution of the Einstein vacuum equations is $\eta_{\mu\nu}$, thus the flat metric. In that sense spacetime is locally isomorphic to the Minkowski space $(\mathcal{M}, \eta_{\mu\nu})$ and in the presence of a cosmological constant, any solution has constant curvature. An important thing to note is the dependence of the Riemann tensor (2.10) in terms of the Ricci tensor. This implies general relativity in three dimensions has no local degrees of freedom: there are no gravitational waves in the classical theory and no gravitons in the quantum theory. The theory however admits topological degrees of freedom [23].

2.2 Tetrad formulation of general relativity

In this section we will show how gravity can be written as a gauge theory through the use of the tetrads (orthonormal frames) and differential forms. The importance of this formulation is that it is similar to the gauge theories of electromagnetism, weak and strong forces and so possibly easier to quantize. Another importance of the formulation is that, it is suitable for introducing particles with spin.

We introduce the tetrad field e^I_μ defined as $e^I = e^I_\mu \mathrm{d}x^\mu$ which is related to the metric

$$g_{\mu\nu} = \eta_{IJ}\, e^I_\mu\, e^J_\nu, \tag{2.12}$$

this provides a local isomorphism between a general (reference) frame field and inertial frame characterized by the flat metric η_{IJ}. In addition, the tetrad provides an isormorphism between the tangent bundle $T(\mathcal{M}) = \bigcup_p T_p(\mathcal{M})$ of $\mathcal{M}$ and the Lorentz principal bundle $(\mathcal{M}, SO(n-1,1))$. For more on the frame field fields see appendix A.1. On the principal bundle, a spin connection ω^{IJ}_μ, a one-form with value in the Lorentz Lie algebra $\mathfrak{so}(n-1,1)$ is defined. This spin connection transforms as a connection under the Lorentz transformations in the internal space

$$\omega^I{}_{J\mu} = \Lambda^I{}_K\, \omega^K{}_{L\mu}(\Lambda^{-1})^L{}_J - (\Lambda^{-1})^I{}_L\, \partial_\mu \Lambda^L{}_J, \tag{2.13}$$

showing the spin connection possess gravitational and inertial effects according to the first and second terms respectively. This is not case in the teleparallel equivalent of gravity, where there the spin connection has only inertial effects.

A general connection $\Gamma^\rho{}_{\mu\nu}$ allows for the introduction of a covariant derivative ∇_μ (defined in appendix A.2) for tensors and in particular vectors in $T(\mathcal{M})$. Analogously for a vectors v^I carrying internal index, we can define its covariant derivative as

$$D_\mu v^I = \partial_\mu v^I + \omega^I{}_{J\mu}\, v^J. \tag{2.14}$$

Subsequently, for objects with both internal and spacetime indices, such as the frame field, their covariant derivative is defined to be

$$\mathcal{D}_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega^I{}_{J\mu}\, e^J_\nu - \Gamma^\rho{}_{\nu\mu}\, e^I_\rho. \tag{2.15}$$

Just like we required metric compatibility of the Levi-Civita connection in the metric formulation of general relativity, here we also require tetrad compatibility of the spin connection, $\mathcal{D}_\mu e^I_\nu = 0$. This gives a solution of the spin connection as a function of the tetrad and Levi-Civita connection,

$$\omega^I{}_{J\mu} = e^I_\nu \nabla_\mu e^\nu_J, \tag{2.16}$$

and this solution of the spin connection $\omega^I{}_J$ is what is known as Cartan's first structure equation

$$\mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J = 0. \tag{2.17}$$

The curvature of the spin connection is define as

$$F^{IJ}(\omega) = \mathrm{d}\omega^{IJ} + \omega^{I}{}_{K} \wedge \omega^{KJ}, \tag{2.18}$$

and componentwise this reads

$$F^{IJ}_{\mu\nu}(\omega) = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^{I}{}_{K\mu}\,\omega^{KJ}_\nu - \omega^{I}{}_{K\nu}\,\omega^{KJ}_\mu. \tag{2.19}$$

By inserting the solution of the spin connection (2.16) into (2.19), we can relate $F^{IJ}_{\mu\nu}$ and $\overset{\circ}{\mathrm{R}}_{\mu\nu\rho\sigma}$

$$\begin{aligned} F^{IJ}_{\mu\nu}(\omega(e)) \;&=\; \partial_\mu e^I_\rho\,\partial_\nu e^{\rho J} + \partial_\mu e^I_\rho\,\Gamma^\rho_{\sigma\nu}\,e^{\sigma J} + e^I_\rho\,\partial_\mu(\Gamma^\rho_{\sigma\nu})e^{\sigma J} + e^I_\rho\,\Gamma^\rho_{\sigma\nu}\,\partial_\mu e^{\sigma J} \\ &\quad + e^I_\rho\,\partial_\mu e^\rho_K\,e^K_\sigma\,\partial_\nu e^{\sigma J} + e^I_\rho\,\Gamma^\rho_{\alpha\mu}\,e^\alpha_K\,e^K_\sigma\,\partial_\nu e^{\sigma J} + e^I_\rho\,\partial_\mu e^\rho_K\,e^K_\sigma\,\Gamma^\sigma_{\alpha\nu}\,e^{\alpha J} \\ &\quad + e^I_\rho\,\Gamma^\rho_{\alpha\mu}\,e^\alpha_K\,e^K_\sigma\,\Gamma^\sigma_{\eta\nu}\,e^{\eta J} - (\mu \longleftrightarrow \nu). \end{aligned} \tag{2.20}$$

Making use of the following expressions

$$e^I_\rho\,e^\rho_K = \delta^I_K, \quad e^I_\rho\,\partial_\mu e^K_\rho = -\,\partial_\mu(e^I_\rho)e^\rho_K. \tag{2.21}$$

Finally we have

$$F^{IJ}_{\mu\nu}(\omega(e)) = 2\,e^{I\rho}\,e^{J\sigma}\left(\partial_{(\mu}\Gamma^\rho_{\sigma\nu)} + \Gamma^\rho_{\delta(\mu}\Gamma^\delta_{\sigma\nu)}\right) \equiv e^{I\rho}\,e^{J\sigma}\,\overset{\circ}{\mathrm{R}}_{\mu\nu\rho\sigma}(e), \tag{2.22}$$

where $\overset{\circ}{\mathrm{R}}_{\mu\nu\rho\sigma}(e)$ is the Riemann tensor constructed from the tetrad. This is the Cartan second structure equations, and it shows general relativity is a gauge theory whose local gauge group is the Lorentz group, and the Riemann tensor is the field-strength of the spin connection.

Furthermore, we can express the volume element in terms of the tetrad as

$$\mathrm{d}x^4\sqrt{-\mathrm{g}} = \frac{1}{24}\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge e^K \wedge e^L = \mathrm{d}x^4\,e, \tag{2.23}$$

where $e = \sqrt{-\mathrm{g}}$ is the determinant of the tetrad. Then with equations (2.22) and (2.23) we can rewrite the action (2.2) as

$$\begin{aligned} S_{\mathrm{EH}}[g_{\mu\nu}(e)] \;&=\; \int \mathrm{d}x^4\sqrt{-\det g_{\mu\nu}}\left(e^\mu_I\,e^{\nu I}\overset{\circ}{\mathrm{R}}_{\mu\rho\nu\sigma}(e)\,e^\rho_J\,e^{\sigma J} - 2\Lambda\right) \\ S_{\mathrm{EH}}[e^I_\mu] \;&=\; \int \frac{1}{2}\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge F^{KL}(\omega(e)) \;+\; \beta\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge e^K \wedge e^L, \end{aligned} \tag{2.24}$$

where $\beta = -\frac{\Lambda}{12}$. This form of the action is what is known as the second order formulation of gravity, since it contains second derivatives of the tetrad.

We can in fact work with the tetrad and spin connection considered as independent variables and the action reads

$$S_{\mathrm{EC}}[e^I_\mu, \omega^{IJ}_\mu] = \int \frac{1}{2}\,\varepsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega) \;+\; \beta\, \varepsilon_{IJKL}\, e^I \wedge e^J \wedge e^K \wedge e^L. \tag{2.25}$$

In this formulation which is often called the Einstein-Cartan first order formulation of gravity, both the massive and spinning degrees of freedom are put on equal footing. As already stated, in this theory, both curvature and torsion represent different gravitational degrees of freedom. Thus the general connection (or Cartan connection) has both curvature and torsion, and can be decomposed into the Levi-Civita connection and the contorsion K

$$\Gamma = \overset{\circ}{\Gamma} + K. \tag{2.26}$$

Just as in general relativity the energy momentum tensor is determined by curvature, the spin acts as the source of torsion in the new theory. Let us show this. First let us note that the Einstein-Cartan action takes the same form as the gravitational action S_{GR} (2.6) with the only difference being that, the Cartan connection is not the Levi-Civita connection, and the matter part of the action is not really independent of torsion. Earlier we have seen the variation of the matter action $S_m[\Psi]$ of (2.6) with respect to the metric gives the energy momentum tensor $T_{\mu\nu}$ (2.5). Now if we take the variation of $S_m[\Psi]$ with respect to the contorsion, this leads to

$$s^\rho{}_{\mu\nu} \equiv \frac{\delta S_m[\Psi]}{\delta K^{\mu\nu}{}_\rho} = \frac{c^4}{8\pi G}\left(T^\rho{}_{\mu\nu} + \delta^\rho_\mu T^\alpha{}_{\nu\alpha} - \delta^\rho_\nu T^\alpha{}_{\mu\alpha}\right), \tag{2.27}$$

where $T^\rho{}_{\mu\nu}$ is the torsion tensor and $s^\rho{}_{\mu\nu}$ is the *spin tensor*. Rewriting (2.27) in the form

$$T^\rho{}_{\mu\nu} = \frac{8\pi G}{c^4}\left(s^\rho{}_{\mu\nu} + \frac{1}{2}\,\delta^\rho_\mu s^\alpha{}_{\nu\alpha} - \frac{1}{2}\,\delta^\rho_\nu s^\alpha{}_{\mu\alpha}\right) \tag{2.28}$$

shows that the torsion is related to the spin tensor of the matter. Hence in the presence of spinless matter, torsion vanishes while on the other hand, the presence of spinning particles suggests a non-vanishing torsion with torsion acting as a non-propagating wave field.

The fields equations for (2.25) are derived as follows. Varying with respect to the spin connection and with the help of the Palatini identity $\delta_\omega F^{KL}(\omega) = \mathrm{d}_\omega \delta\omega^{KL}$ we obtain

$$\varepsilon_{IJKL}\, e^I \wedge \mathrm{d}_\omega e^J = 0. \tag{2.29}$$

Assuming invertibility of the tetrad, we obtain that the torsion

$$\mathrm{d}_\omega e^J = 0 \tag{2.30}$$

is vanishing and this is just the Cartan first structure equation (2.17), whose solution is the Levi-Civita spin connection (2.16). Variation of the action (2.25) with respect to the tetrad yields the fields equation

$$F^{KL}(\omega) = \frac{\Lambda}{3}\, e^K \wedge e^L. \tag{2.31}$$

Upon the substitution of the solution (2.16), the field equation becomes the vacuum Einstein equations $G_{\mu\nu} = 0$ in the absence of a cosmological constant, thus the torsionless condition (2.30) and (2.31) provide the same solution space for the metric formulation of general relativity. This shows the metric and the tetrad formulations of gravity coincide on shell, when there is no spin.

Chapter 3

The teleparallel formulation of general relativity

After the formulation of Einstein's general relativity, H. Weyl [93] in 1918 considered the unification of gravity and electromagnetism. Though unsuccessful in unifying the two theories, his work led to the introduction of the concept of gauge transformations, which later gave birth to gauge theories. A decade later, Einstein using the mathematical structure of teleparallelism tried to find a unified theory of gravitational and electromagnetic fields. Teleparallelism also known as distant or absolute parallelism is the distant comparison of the direction of tangent vectors at different points on the manifold [8, 94]. This is achieved by introducing the tetrad field, a field of orthonormal bases of the tangent space define at each point of the four dimensional spacetime manifold.

Einstein's use of teleparallelism was to capitalize on the sixteen components of the tetrad field against the ten independent components of the metric. According to Einstein's initial guess, these additional six components of the tetrad, should be related to those of the electromagnetic field, which also has six components. However, his unification idea did not work, since the additional degrees of freedom of the tetrad are eliminated by the six parameters of the local Lorentz invariance of the theory, i.e, $e'^{I}_{\mu} = \Lambda^{I}{}_{K} e^{K}_{\mu}$, where $\Lambda^{I}{}_{K}$ is the Lorentz transformation. The failure to unify these two theories using teleparallelism led instead to an alternative way of formulating gravity, which today is referred to as *teleparallel gravity.*

The mathematics behind distant parallelism were first developed by Cartan and Weitzenböeck [94]. In Einstein-Cartan gravity theory, the connection contains both curvature and torsion with spin as the source of torsion. However, Weitzenböeck noticed he could introduce a connection on a manifold that is compatible with the tetrad field, such that the underlying spacetime has a vanishing Riemannian curvature but a non-trivial torsion. In such situation, parallel transport is independent of the path taken, and angles are invariant under parallel transport. In developing teleparallel gravity, Einstein used the Weitzenböck connection and by so doing was able to show for a specific choice of free parameters its equivalence to general relativity. This implies both torsion and curvature provide alternative ways to describe gravitation independently. Using torsion to parametrize gravity is referred as the *teleparallel equivalent of general relativity* or TEGR for short.

In the course of developing TEGR, Einstein had several correspondence with some notable mathematicians and physicists, including Herman Müntz, with whom together they exploited various ways to find the right field equations for the new theory in 1928. Subsequently, Einstein had contact with Roland Weitzenböck, who pointed out some relevant result in deriving these field equations on the basis of a variational formulation. For more on the early historical account on the unification of electromagnetic and gravitational fields, and the formulation of teleparallel gravity from Einstein's perspective, one should see the reference [94] and references therein.

There was a hiatus in new research directions in the aftermath of Einstein's findings on teleparallel gravity for about three decades. Moller in the 1960's started to look at teleparallelism again with no unification motivation but rather as a gauge theory for gravitation. Using the findings of his work, Pellegrini and Plebanski found a Lagrangian [11] formulation for teleparallel gravity, which later was found to be equivalent to the Einstein-Hilbert action up to a divergence. At the same time, Hayashi pointed out the relationship between a gauge theory of translation and teleparallel gravity [95]. This is evident as the tetrad is related to the gauge potential of translations whose field strength is torsion. This suggest teleparallel gravity can be seen as a gauge theory of the translation group.

In as much as teleparallel gravity through torsion and general relativity through curvature describe the same degrees of freedom of gravitation, they have different physical interpretation. Gravitational interaction in TEGR is described by torsion which acts as a force implying there is no geodesics equation but rather a force equation similar to those in Maxwell's electromagnetic theory. On the contrary, in GR, curvature geometrize gravitational interaction and trajectories are determined by geodesics.

The concept of universality is so important that in its absence, the description of gravitation breaks down from the viewpoint of Einstein's GR. This means in order to attribute gravitation to curvature, the weak equivalence principle - which establishes the equality of inertial and gravitational masses - must be true. As a gauge theory of the translation group, there is then the question as to whether teleparallel gravity can describe gravitational interaction in the absence of weak equivalence principle. It turns out with or without the weak equivalence principle, teleparallel gravity is able to describe gravitational interaction, contrary to the GR case.

The validity of the weak equivalence principle in classical physics is in no doubt, however there are several issues which suggest a not well defined principle in the quantum realm. Fields in quantum physics are generally spread across all of space hence the notion of locality which is crucial for the validity of the weak equivalence principle is absent. Again expectation values of fields in quantum physics are generally peaked (thus the fields are constantly fluctuating) due to the metric's background independence [96]. With these and several reasons, one can draw an inference to suggest that teleparallel gravity which requires no equivalence principle to describe gravitational interaction is the ideal gravity theory to quantize compared to general relativity.

In what follows, we will review the concept of teleparallelism, the foundations of teleparallel gravity, as a gauge theory for the translation group. In addition, we will show the equivalence of the force equation of teleparallel gravity and the geodesic equation of general relativity.

The review on teleparallel gravity in this chapter is based on the book [11] and the following publications [5, 7, 94, 97, 98].

3.1 The mathematics of teleparallelism

In this section we will review the concepts behind what is called today teleparallel gravity. As noted in the introduction of the chapter, TEGR is a formulation whereby gravitation is described by torsion as opposed to curvature. This formulation is possible if one chooses a curvature - free connection and a metric tensor field - both defined in terms of a dynamical tetrad field - on the same spacetime manifold.

Let us introduce a spin connection valued one-form $\omega^I_{\mu J}$ on the tangent bundle $T(\mathcal{M}) = \bigcup_p T_p(\mathcal{M})$, which is compatible with the tetrad field in the sense that the covariant derivative of the tetrad vanishes i.e.,

$$\mathcal{D}_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega^I_{\mu J}\, e^J_\nu - \Gamma^\rho_{\nu\mu} e^I_\rho = 0, \tag{3.1}$$

from this we can solve to obtain the spin connection in terms of the connection $\Gamma^\rho_{\nu\mu}$ and the tetrad

$$\omega^I_{\mu J} = e^I_\nu \nabla_\mu e^\nu_J, \tag{3.2}$$

where ∇_ν denote the covariant derivative associated to $\Gamma^\rho_{\nu\mu}$. Suppose we take the spin connection $\omega^I_{\mu J}$ to represent only inertial effects, in that case we choose it to be of the form

$$\overset{\bullet}{\omega}^I_{\mu J} = \Lambda^I{}_K \partial_\mu \Lambda_J{}^K, \tag{3.3}$$

whereby in the new theory there exist no additional degrees of freedom. This choice of spin connection corresponds to the gravitational theory described by teleparallel gravity. The connection $\Gamma^\rho_{\nu\mu}$, defined in (3.2), corresponding to such $\overset{\bullet}{\omega}^I_{\mu J}$, is called the Weitzenböck connection, and for the rest of the thesis will be denoted $\overset{\bullet}{\Gamma}^\rho_{\nu\mu}$. In the class of frames in which the spin connection $\overset{\bullet}{\omega}^I_{\mu J}$ is vanishing, the Weitzenböck connection becomes

$$\overset{\bullet}{\Gamma}^\rho_{\nu\mu} = -e^I_\nu \partial_\mu e^\rho_I. \tag{3.4}$$

We note that the vanishing of the spin connection (3.3) leading to the condition that uniquely determines the Weitzenböck connection (3.4) is the absolute parallelism condition, from where the name teleparallel gravity comes from.

To be precise, let us consider a vector with components V^I at a point p in the spacetime manifold parallel transported to another vector with components V'^I at a point p$'$. These vectors are said to be parallel if their respective components expressed in terms of the tetrads are equal, thus $V^\mu = V'^\mu$, hence by parallel transporting we get the condition

$$0 = \mathrm{d}V^I = \mathrm{d}(e^I_\mu V^\mu) = V^\mu \partial_\nu e^I_\mu \mathrm{d}x^\nu + e^I_\mu \mathrm{d}V^\mu, \tag{3.5}$$

multiplying by the co-tetrad e^μ_I we get

$$\mathrm{d}V^\mu = -\Gamma^\mu{}_{\rho\nu} V^\rho \mathrm{d}x^\nu, \tag{3.6}$$

where the connection

$$\Gamma^\rho{}_{\mu\nu} = e^I_\mu \partial_\nu e^\rho_I \tag{3.7}$$

defined as before in equation (A.8) [94], is the Weitzenböck connection encoding the distant parallelism property.

The Riemann curvature tensor for the connection (3.4) vanishes identically

$$\overset{\bullet}{\mathrm{R}}^\rho{}_{\lambda\mu\nu} = \partial_\nu \overset{\bullet}{\Gamma}^\lambda{}_{\mu\rho} - \partial_\rho \overset{\bullet}{\Gamma}^\lambda{}_{\mu\nu} + \overset{\bullet}{\Gamma}^\lambda{}_{\sigma\nu} \overset{\bullet}{\Gamma}^\sigma{}_{\mu\rho} - \overset{\bullet}{\Gamma}^\lambda{}_{\sigma\rho} \overset{\bullet}{\Gamma}^\sigma{}_{\mu\nu} = 0, \tag{3.8}$$

whereas its torsion tensor defined as

$$\overset{\bullet}{\mathrm{T}}^\rho{}_{\mu\nu} = \overset{\bullet}{\Gamma}^\rho{}_{\nu\mu} - \overset{\bullet}{\Gamma}^\rho{}_{\mu\nu}, \tag{3.9}$$

does not vanish in general and as such the Weitzenböck connection does not determine the tetrad field.

The Weitzenböck connection can be written as a sum of the Levi-Civita connection and the contorsion tensor

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}, \tag{3.10}$$

we will use later this expression to show the equivalence of general relativity and TEGR.

There is however a drawback when assuming the strict vanishing of the spin connection $\overset{\bullet}{\omega}{}^{I}{}_{\mu J}$ which yield the Weitzenböck connection (3.4). In this approach, termed pure-tetrad formulation [99], the torsion tensor (3.9) is only defined in terms of the derivative of the frame field. Hence it is not a Lorentz tensor and the Lorentz invariance of the theory is broken. This means we have a theory which is not manifestly covariant with the effects adding up to surface terms leading to a theory with infrared divergences [100]. It was shown in [101] by choosing an appropriate spin connection and using the "background substraction" method of Gibbson and Hawking [102], the local Lorentz invariance of the pure-tetrad formulation of teleparallel gravity can be restored. This new spin connection introduced has to be flat or purely inertial [99]. This new formulation is then a gauge theory for the Poincaré group.

This approach where the spin connection is purely inertial, we shall pursue in chapter 4 when discussing the first order formulation of teleparallel gravity.

3.2 Foundations of teleparallel gravity

Rooted in the formulation of teleparallel gravity is the fact that the underlying field is related to the gauge potential of translation. It is this relationship which we will shortly see that makes it possible to understand teleparallel gravity as a gauge theory.

The tangent bundle provides the geometrical set up of teleparallel gravity. At each point of spacetime, there is a tangent space attached to it on which the gauge transformations take place [11]. We note the Greek alphabets $\mu, \nu, \dots = 0, 1, 2, 3$ denote spacetime indices and the middle Latin alphabets $I, J, \dots = 0, 1, 2, 3$ denote tangent space indices. This means spacetime and tangent space coordinates are denoted respectively by x^{μ} and x^{I}, and with these coordinates as functions of each other their basis (derivatives) for vector fields are related as follows

$$\partial_{\mu} = (\partial_{\mu} x^{I})\partial_{I}, \quad \partial_{I} = (\partial_{I} x^{\mu})\partial_{\mu}, \tag{3.11}$$

where $\partial_\mu x^I$ is a trivial tetrad and $\partial_I x^\mu$ is its inverse.

As a gauge theory for the translation group, teleparallel gravity has the translational gauge potential 1-form denoted B_μ, valued in the Lie algebra of the translation group as its fundamental field

$$B_\mu = B^I{}_\mu \, P_I \tag{3.12}$$

where $B^I{}_\mu$ are components of the gauge potential and $P_I = \partial_I$ is the generator of translation satisfying the commutation relation

$$[P_I, P_J] = 0. \tag{3.13}$$

Given an infinitesimal parameter $\epsilon^I(x^\mu)$, the local (point dependent) translations[1] of the tangent space coordinates x^I are

$$x'^I = x^I + \epsilon^I(x^\mu), \tag{3.14}$$

and these are the gauge transformation of teleparallel gravity. Its corresponding infinitesimal transformation is

$$\delta x^I = \epsilon^J \, P_J \, x^I. \tag{3.15}$$

Consider a general source field $\Psi \equiv \Psi(x^\mu)$ which transforms under an infinitesimal gauge translation as

$$\delta\Psi = \epsilon^I P_I \Psi. \tag{3.16}$$

The associated derivative of the field Ψ transforms

$$\delta(\partial_\mu \Psi) = \epsilon^I \partial_I(\partial_\mu \Psi) + (\partial_\mu \epsilon^I)\partial_I \Psi \tag{3.17}$$

and this is not covariant under the above transformation. However by introducing the gauge potential (3.12), we define the translational covariant derivative of Ψ to be

$$e_\mu \Psi := \partial_\mu \Psi + B^I{}_\mu P_I \Psi, \tag{3.18}$$

and this transform covariantly $\delta(e_\mu \Psi) = \epsilon^I \partial_I (e_\mu \Psi)$, provided the potential transforms as $\delta B^I{}_\mu = -\partial_\mu \epsilon^I$. Alternatively, the covariant derivative (3.18) can be rewritten as

$$e_\mu \Psi = e^I{}_\mu P_I \Psi, \tag{3.19}$$

[1]Point dependent in the sense that the different gauge transformations in the tangent space take place at different points of the spacetime manifold.

where the component defined as[2]

$$e^I{}_\mu = \partial_\mu x^I + B^I{}_\mu \equiv e_\mu x^I \tag{3.20}$$

is a non-trivial tetrad field decomposed into a trivial inertial part and a gravitational potential which encodes gravity. From equation (3.14) and the infinitesimal transformation of the gauge potential, the tetrad field is invariant under the gauge transformation (3.15). We note from the non-trivial tetrad (3.20), the tangent space coordinates are now functions on the manifold, which means we are now considering a section of the tangent bundle and this is so because of the expressions in (3.11).

A remark about the translational covariant derivative (3.18): in the second piece there is the action of the generators (derivatives) on the source field Ψ and subsequent coupling to the translational gravitational potential $B^I{}_\mu$, this shows that every source field in nature will feel gravitation in a similar manner. This gives rise to the concept of universality from the perspective of teleparallel gravity.

Computing the commutation relation of the covariant derivative (3.18)

$$[e_\mu, e_\nu]\Psi = \overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu} P_I \Psi \tag{3.21}$$

results in the field strength

$$\overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu} \equiv \partial_\mu B^I{}_\nu - \partial_\nu B^I{}_\mu = \partial_\mu e^I_\nu - \partial_\nu e^I_\mu \tag{3.22}$$

of teleparallel gravity and its assumes values in the Lie algebra of the translation group. Due to the invariance of the tetrad the field strength $\overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu}$ is also invariant under a gauge transformation. This is the pure-tetrad formulation of the theory.

Suppose we perform a local Lorentz transformation in addition to the local translation, this will become relevant when we consider the first order formulation of teleparallel gravity, then the tangent coordinates x^I, the source field Ψ and the gauge potential $B^I{}_\mu$ transform as follows

$$x^I = \Lambda^I{}_J\, x^J, \quad \Psi = U(\Lambda)\Psi, \quad B^I{}_\mu = \Lambda^I{}_J\, B^J{}_\mu, \tag{3.23}$$

where $U(\Lambda)$ is an element of the Lorentz group in the representation Λ acting on Ψ. Inferring from previous transformations, one can see the covariant derivative (3.19) becomes

$$e_\mu \Psi = U(\Lambda)\, e^I{}_\mu P_I \Psi \tag{3.24}$$

[2] We note here that we have used a different notation for the tetrad compared to what is normally used in the teleparallel community: $h^I{}_\mu$

and transforms covariantly, where

$$e^I{}_\mu = \partial_\mu x^I + \overset{\bullet}{\omega}{}^I{}_{J\mu}\, x^J + B^I{}_\mu \tag{3.25}$$

is the non-trivial tetrad and the inertial Lorentz connection $\overset{\bullet}{\omega}{}^I{}_{J\mu}$ is defined just as in equation (3.3), which is what we want in teleparallel gravity. The covariance of the Lorentz and translation covariant derivative (3.24) implies that the gauge potential has to transform as

$$\delta B^I{}_\mu = -(\partial_\mu \epsilon^I + \overset{\bullet}{\omega}{}^I{}_{J\mu}\, x^J) \tag{3.26}$$

under a gauge translation $\delta x^I = \epsilon^I$, and this leads to the invariance of the tetrad: $\delta e^I{}_\mu = 0$. Introducing a new notation for the covariant derivative (3.24), the commutation relation gives

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Psi = \overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu} P_I \Psi \tag{3.27}$$

and this gives rise to the field strength defined as the torsion

$$\overset{\bullet}{\mathrm{T}}{}^I \equiv \mathrm{d}B^I + \overset{\bullet}{\omega}{}^I{}_J \wedge B^J = \mathrm{d}e^I + \overset{\bullet}{\omega}{}^I{}_J \wedge e^J. \tag{3.28}$$

It takes value in the Lie algebra of the translational part of the Poincaré group. Due to the invariance of the tetrad, the torsion is also invariant under the translational gauge transformation.

3.3 The equivalence of the force and geodesic equations

The approach of teleparallel gravity is that gravitational interactions are described by contorsion and spinless particles follow force equations, wheres in general relativity particles follow geodesics. In the presence or not of the weak equivalence principle, we will show how the force equations of teleparallel gravity are equivalent or not to the geodesics equation of general relativity, and provide answers to why teleparallel gravity does not require universality to describe gravitational interactions.

Let us consider the motion of a spinless particle in a gravitational field $B^I{}_\mu$, then the action integral of this interaction is ($c = 1$)

$$S = \int_a^b \Big(- m_i\, \mathrm{d}\sigma - m_g\, B^I{}_\mu\, u_I\, \mathrm{d}x^\mu \Big), \tag{3.29}$$

this is similar to the action integral for a particle in an electromagnetic field. $\mathrm{d}\sigma = \big(\eta_{IJ}\, \mathrm{d}x^I \mathrm{d}x^J\big)^{1/2}$ is the Minkowski space interval in the tangent space, u^I is the particle four velocity in the tetrad frame, m_i and m_g are the respective inertial and gravitational mass of the particle. The first term in (3.29) is the action of a free particle, and the second term is coupling of the particle to the gravitational field. In the above action (3.29), if the weak equivalence principle is assumed, thus $m_i = m_g = m$. The variation of the action gives

$$e^I_\mu \frac{\mathrm{d}u_I}{\mathrm{d}s} = \overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu}\, u_I\, u^\nu \tag{3.30}$$

as the equation of motion with the teleparallel field strength $\overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu}$ acting as the gravitational force, where $\mathrm{d}s = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the Riemannian spacetime invariant interval and

$$u^\nu = \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \equiv e^\nu_I\, u^I \tag{3.31}$$

is the four-velocity. Given the contorsion is expressed in terms of the Weitzenböck torsion

$$K^\lambda{}_{\mu\rho} = \frac{1}{2}\left(\overset{\bullet}{\mathrm{T}}{}_\mu{}^\lambda{}_\rho + \overset{\bullet}{\mathrm{T}}{}_\rho{}^\lambda{}_\mu - \overset{\bullet}{\mathrm{T}}{}^\lambda{}_{\mu\rho}\right), \tag{3.32}$$

and by the antisymmetry of the torsion tensor one obtains the identity

$$\overset{\bullet}{\mathrm{T}}{}^\lambda{}_{\mu\rho}\, u_\lambda\, u^\rho = -K^\lambda{}_{\mu\rho}\, u_\lambda\, u^\rho. \tag{3.33}$$

By writing the Weitzenböck torsion in terms of the tetrad basis

$$\overset{\bullet}{\mathrm{T}}{}^I{}_{\mu\nu} = e^I_\rho\, \overset{\bullet}{\mathrm{T}}{}^\rho{}_{\mu\nu}, \tag{3.34}$$

using the identity (3.33) and considering the relation (3.10), the force equation (3.30) gives the geodesic equation of general relativity

$$\frac{\mathrm{d}u^\lambda}{\mathrm{d}s} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\, u^\mu\, u^\nu = 0. \tag{3.35}$$

Hence in the presence of WEP, the force equation (3.30) yields the geodesic equation (3.35) of general relativity. Suppose now that $m_g \neq m_i$, that is in the absence of universality (the weak equivalence principle), the force equation (3.30) reads

$$\left(\partial_\mu x^I + \frac{m_g}{m_i} B^I{}_\mu\right)\frac{\mathrm{d}u_I}{\mathrm{d}s} = \frac{m_g}{m_i}\, T^I{}_{\mu\nu}\, u_I\, u^\nu. \tag{3.36}$$

With torsion still playing the role of a force, the new force equation shows teleparallel gravity is still able to describe the motion of a particle even with a violation of the weak equivalence principle. Furthermore, despite the dependence of the force equation (3.36) on the expression m_g/m_i, neither the gauge potential $B^I{}_\mu$ nor the force $T^I{}_{\mu\nu}$ depends on it, this implies the field equations (3.36) can be solved for $B^I{}_\mu$.

Again making use of the same identity (3.33) and relation (3.10), the corresponding "geodesic" equation for (3.36) is given to be

$$\frac{\mathrm{d}u^\lambda}{\mathrm{d}s} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\, u^\mu\, u^\nu = \left(\frac{m_g - m_i}{m_g}\right) \partial_\mu x^I \,\frac{\mathrm{d}u_I}{\mathrm{d}s}. \tag{3.37}$$

Of course one should note that this is not a proper geodesic equation since the RHS of (3.37) is non-zero and that it contains the difference between the gravitational and inertial masses. The consequences of this is that the equation of motion (3.36) does not obey general relativity description of gravity where the trajectories of particles are governed by geodesics equations.

3.4 Lagrangian and field equations

We now write down the teleparallel Lagrangian – quadratic in the torsion tensor, the field strength of the theory. To begin, we note the Hodge dual operator. Consider a p-form β, on a n-dimensional manifold, i.e,

$$\beta = \frac{1}{p!}\beta_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots . \wedge \mathrm{d}x^{\mu_p},$$

then its Hodge dual is given by

$$*\,\beta = \frac{e}{(n-p)!p!}\,\varepsilon_{\mu_1 \ldots \mu_n}\,\beta^{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots . \wedge \mathrm{d}x^{\mu_n}, \tag{3.38}$$

an $(n-p)$-form. Here $e = \det(e^I_\mu) = \sqrt{-\mathrm{g}}$ is the determinant of the tetrad field and $\varepsilon_{\mu_1 \ldots \mu_n}$ is the anti-symmetric Levi-Civita symbol. The Hodge dual satisfies the following properties

$$*\,*\beta = (-1)^{p(n-p)+(n-s)/2}\beta, \qquad *^{-1} = (-1)^{p(n-p)+(n-s)/2}*, \tag{3.39}$$

with s the metric signature. Suppose now if β is a vector valued p-form taking values in some vector space $\mathcal{V}$ then its dual is a vector valued $(n-p)$-form

$$*\,\beta = \frac{e}{(n-p)!p!}\,\varepsilon_{\mu_1 \ldots \mu_n}\, J_A\, \beta^{A\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots . \wedge \mathrm{d}x^{\mu_n}, \tag{3.40}$$

where $\{J_A\}$ is a set of basis vectors in $\mathcal{V}$ and A is an internal space index.

By considering all possible index contraction of the torsion tensor in conjunction with equation (3.40) we can write its generalized dual as

$$* T^{\rho}{}_{\mu\nu} = e\,\varepsilon_{\mu\nu\lambda\sigma}\left(aT^{\rho\lambda\sigma} + bT^{\lambda\rho\sigma} + cT^{\alpha\lambda}{}_{\alpha}\,g^{\rho\sigma}\right), \tag{3.41}$$

where a, b, c are some constant coefficients to be specified. These different possibilities are due to the anti-symmetric nature of the torsion tensor in the last two indices and the presence of the tetrad field. Like in any gauge theory, the Lagrangian is obtained by contraction of the field strength, therefore by introducing the Killing form

$$\langle T, *T\rangle \tag{3.42}$$

we have the Lagrangian of teleparallel gravity to be

$$\mathcal{L}_{//}[e^I_\mu] = e\left(aT^{\rho}{}_{\nu\sigma}\,T_{\rho}{}^{\nu\sigma} + bT^{\sigma}{}_{\rho\alpha}\,T^{\rho\alpha}{}_{\sigma} + cT^{\mu}{}_{\nu\mu}\,T^{\rho\nu}{}_{\rho}\right). \tag{3.43}$$

The three different expressions in the above Lagrangian are invariant under both general coordinate transformation and rotations of the tetrad fields. The invariance of these terms were first proved by Weitzenböck [103]. In his attempt to derive field equations for the new theory, Einstein tried different combinations of the constants a, b, c to get the Lagrangian which will yield a consistent gravitational theory described by torsion, as well as to recover vacuum field equations of general relativity.

It turns out if one considers the constants $a = 1$ and $b = c = 0$, the vacuum Einstein and Maxwell equations are obtained. This however gives an inconsistent superficial theory with no physical or experimental evidence. The same inconsistency is registered if one chooses $a = b = 0$ and $c = 1$. The choice $a = 1/2$, $b = 1/4$ and $c = -1$ however yields the desired outcome, and by introducing superpotential tensor

$$S^{\rho\mu\nu} := K^{\mu\nu\rho} - g^{\rho\nu}\,T^{\sigma\mu}{}_{\sigma} + g^{\rho\mu}\,T^{\sigma\nu}{}_{\sigma} = -S^{\rho\nu\mu} \tag{3.44}$$

the dual torsion tensor reads

$$* T^{\rho}{}_{\mu\nu} = \frac{e}{2}\,\varepsilon_{\mu\nu\lambda\sigma}\,S^{\rho\lambda\sigma}. \tag{3.45}$$

The torsion scalar is given by

$$T = \frac{1}{2}\,T_{\rho\mu\nu}\,S^{\rho\mu\nu} = \left(\frac{1}{4}T^{\sigma}{}_{\rho\alpha}\,T^{\rho\alpha}{}_{\sigma} + \frac{1}{2}T^{\rho}{}_{\nu\sigma}\,T_{\rho}{}^{\nu\sigma} - T^{\mu}{}_{\nu\mu}\,T^{\rho\nu}{}_{\rho}\right), \tag{3.46}$$

where the first term corresponds to the Lagrangian of internal gauge theories. The action of TEGR in compact form reads

$$S_{//}[e^I_\mu] = \frac{1}{16\pi G}\int T e \, \mathrm{d}x^4. \tag{3.47}$$

Alternatively by making use of the identity

$$T^\mu{}_{\mu\rho} = K^\mu{}_{\rho\mu}, \tag{3.48}$$

the Lagrangian can be written in terms of the contorsion and this take the form

$$\mathcal{L}_{//} = \dot{\mathrm{K}}^{\mu\nu\rho}\,\dot{\mathrm{K}}_{\rho\nu\mu} - \dot{\mathrm{K}}^{\mu\rho}{}_{\mu}\,\dot{\mathrm{K}}^{\nu}{}_{\rho\nu}. \tag{3.49}$$

If we now include a Lagrangian $\mathcal{L}_m$ of a general matter field Ψ and varying the total Lagrangian $\mathcal{L} = \mathcal{L}_{//} + \mathcal{L}_m$ with respect to the gauge potential, we obtain the teleparallel version of the gravitational field equations

$$\partial_\sigma \left(e\, S_I{}^{\rho\sigma}\right) - e\, j_I{}^{\rho} = e\, \Theta_I{}^{\rho}, \tag{3.50}$$

where $S_I{}^{\rho\sigma} = e_I{}^{\lambda}\, S_\lambda{}^{\rho\sigma}$, $\Theta_I{}^{\rho} = e^\mu{}_I\, \Theta_\mu{}^{\rho}$ is the standard energy-momentum tensor defined by

$$\Theta_I{}^{\rho} \equiv -\frac{\delta\mathcal{L}_m}{\delta B^I{}_\rho} \equiv -\frac{\delta\mathcal{L}_m}{\delta e^I{}_\rho} = -\left(\frac{\partial\mathcal{L}_m}{\partial e^I{}_\rho} - \partial_\lambda \frac{\partial\mathcal{L}_m}{\partial_\lambda\, \partial e^I{}_\rho}\right), \tag{3.51}$$

and

$$e\, j_I{}^{\rho} \equiv -\frac{\partial\mathcal{L}}{\partial e^I{}_\rho} = e\, e_I{}^{\lambda}\, S_J{}^{\nu\rho}\, T^J{}_{\nu\lambda} - e_I{}^{\rho}\mathcal{L} \tag{3.52}$$

is the gauge current, which represents the energy and momentum of the gravitational field.

3.5 Equivalence of teleparallel gravity and general relativity

Teleparallel gravity is equivalent to GR up to some boundary term. This can be demonstrated either at the level of the action or at the level of the field equations. Here, we will illustrate the equivalence at the level of the action. To begin, let us consider the Weitzenböck connection decomposition (3.10) and plug it into (3.8), then the Riemann curvature tensor of the Weitzenböck connection in terms of the Levi-Civita connection and contorsion is given by

$$\begin{aligned}
\overset{\bullet}{\mathrm{R}}{}^{\alpha}{}_{\mu\rho\nu} &= \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu,\rho} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\rho,\nu} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu,\rho} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\rho,\nu} + (\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu})(\overset{\circ}{\Gamma}{}^{\beta}{}_{\rho\beta} + \overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\rho\beta}) \\
&\quad -(\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\beta} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\beta})(\overset{\circ}{\Gamma}{}^{\beta}{}_{\nu\rho} + \overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\nu\rho}) \\
&= \overset{\circ}{\mathrm{R}}{}^{\alpha}{}_{\mu\rho\nu} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu,\rho} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\rho,\nu} + \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu}\overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\rho\beta} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu}\overset{\circ}{\Gamma}{}^{\beta}{}_{\rho\beta} - (\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\beta}\overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\nu\rho} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\beta}\overset{\circ}{\Gamma}{}^{\beta}{}_{\nu\rho}) \\
&= \overset{\circ}{\mathrm{R}}{}^{\alpha}{}_{\mu\rho\nu} + \overset{\bullet}{\mathrm{Q}}{}^{\alpha}{}_{\mu\rho\nu},
\end{aligned} \tag{3.53}$$

where $\overset{\circ}{\mathrm{R}}{}^{\alpha}{}_{\mu\rho\nu}$ is the Riemann tensor associated to the Levi-Civita connection and

$$\begin{aligned}
\overset{\bullet}{\mathrm{Q}}{}^{\alpha}{}_{\mu\rho\nu} &= \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu,\rho} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\rho,\nu} + (\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\nu} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu})\overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\rho\beta} + \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu}(\overset{\bullet}{\Gamma}{}^{\beta}{}_{\rho\beta} - \overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\rho\beta}) \\
&\quad -(\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\beta} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\beta})\overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\nu\rho} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\beta}(\overset{\bullet}{\Gamma}{}^{\beta}{}_{\nu\rho} - \overset{\bullet}{\mathrm{K}}{}^{\beta}{}_{\nu\rho}),
\end{aligned} \tag{3.54}$$

is a two-form tensor with values in the Lie algebra of the Lorentz group and we used the relation $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\nu} - \overset{\bullet}{\mathrm{K}}{}^{\alpha}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ to get the expression (3.54). Taking the appropriate traces in equation (3.53), we have the Ricci scalar of the Weitzenböck connection expressed in terms of the Ricci scalar of the usual Riemann tensor – which is the general relativity Lagrangian – plus a contribution of the contorsion – which is essentially the teleparallel Lagrangian, plus a covariant derivative of the torsion tensor $\overset{\bullet}{\mathrm{T}}$

$$\overset{\bullet}{\mathrm{R}} = \overset{\circ}{\mathrm{R}} + \overset{\bullet}{\mathrm{Q}} - 2\nabla^{\mu}(\overset{\bullet}{\mathrm{T}}{}^{\nu}{}_{\nu\mu}), \tag{3.55}$$

with

$$\overset{\bullet}{\mathrm{Q}} = \overset{\bullet}{\mathrm{K}}{}^{\mu\nu\rho}\,\overset{\bullet}{\mathrm{K}}{}_{\rho\nu\mu} - \overset{\bullet}{\mathrm{K}}{}^{\mu\rho}{}_{\mu}\,\overset{\bullet}{\mathrm{K}}{}^{\nu}{}_{\rho\nu}, \tag{3.56}$$

which is the same expression (3.49), the Lagrangian of teleparallel gravity. Imposing the vanishing of the Riemann curvature of the Weitzenböck connection and using the relation $\nabla_\mu V^\mu = e^{-1}\partial_\mu(e\,V^\mu)$ in (3.55) to replace the covariant derivative with a total derivative of the torsion, we get

$$-\overset{\circ}{\mathrm{R}} = \overset{\bullet}{\mathrm{Q}} - \frac{2}{e}\,\partial_\mu(e\,\overset{\bullet}{\mathrm{T}}{}^{\nu\mu}{}_{\nu}), \tag{3.57}$$

where e is the determinant of the tetrad field. The second term in the above expression is the boundary term and has no consequences on the dynamics of the system. Hence the Einstein-Hilbert Lagrangian is equal to the Teleparallel Lagrangian up to to a boundary term.

Chapter 4

First-order formulation of teleparallel gravity and dual loop gravity

Following our discussion in the introductory chapter 1, many arguments suggests that the teleparallel formulation should also be present in the Einstein-Cartan formulation. In this chapter we will formalize this idea by illustrating how the GR Palatini formalism and the teleparallel formulation can be obtained in the same way from the Einstein-Cartan formulation.

We will then recall how the discretization of the Einstein-Cartan formulation in 3d leads to the two different pictures, loop gravity and the dual loop gravity [29]. We will discuss how the dual loop gravity formalism can be seen as a discretization of the teleparallel formulation.

This chapter is essentially based on the paper [1].

4.1 First order action for teleparallel gravity

We detail below how the Einstein-Cartan action (1.4) can be seen as the first-order formulation of the teleparallel action. We first focus on the three dimensional Euclidean case as a warm-up. Three-dimensional Euclidean gravity is very well understood in the Loop Quantum gravity framework [104, 105]. We then study the general D-dimensional Lorentzian case.

The key idea is that the connection $A^I{}_{Ja}$ can be written as $A^I{}_{Ja} = \omega^I{}_{Ja} + \mathcal{K}^I{}_{Ja}$, a reference connection ω plus the contorsion $\mathcal{K}$, which encodes the dynamical degrees of freedom. We will pick the two choices of metric compatible connection which do not bring any new degrees of freedom, namely the Weitzenböck connection, $\overset{\bullet}{\omega}$, and the Levi-Civita connection $\overset{\circ}{\omega}$. They have respectively no curvature or no torsion.

Solving the equations of motion for $\mathcal{K}$ will allow to re-express the Einstein-Cartan action (1.4) as the teleparallel action provided the reference connection is the Weitzenböck connection, while the other choice gives the standard GR case.

4.1.1 The BF action in three dimensions

The starting point is the Einstein-Cartan (1.4) for three-dimensional Euclidean gravity, so that the $\mathfrak{su}(2)$ Lie algebra-valued one-form $B^I{}_a$ is the cotriad $e^I{}_a$.

$$S_{\mathrm{EC}}[e,A] = -\frac{1}{8\pi G}\int_{\mathcal{M}} \big\langle e \wedge F[A]\big\rangle, \quad F[A] = \mathrm{d}A + \frac{1}{2}[A,A], \tag{4.1}$$

and where $\langle X, Y\rangle$ is the Killing form for $\mathfrak{su}(2)$. Taking into account the split $A^I{}_a = \omega^I{}_a + \mathcal{K}^I{}_a$ of the connection into an arbitrary reference connection $\omega^I{}_a$ and a displacement vector $\mathcal{K}^I{}_a$, the $SU(2)$ field strength becomes

$$\begin{aligned} F[A] &= \mathrm{d}A + \frac{1}{2}\big[A,A\big] = \mathrm{d}\omega + \frac{1}{2}\big[\omega,\omega\big] + \mathrm{d}\mathcal{K} + [\omega,\mathcal{K}] + \frac{1}{2}\big[\mathcal{K},\mathcal{K}\big] \\ &= F[\omega] + {}^{\omega}D\mathcal{K} + \frac{1}{2}\big[\mathcal{K},\mathcal{K}\big]. \end{aligned} \tag{4.2}$$

where ${}^{\omega}D = \mathrm{d} + [\omega,\cdot]$ is the exterior covaraint derivative with respect to the reference connection. At the level of the action, we thus have,

$$S_{\mathrm{EC}}[e,A] = S_{\mathrm{EC}}[e,\mathcal{K};\omega] = -\frac{1}{8\pi G}\int_{\mathcal{M}} \Big\langle e \wedge F[\omega] - \mathrm{d}(e\wedge\mathcal{K}) + {}^{\omega}T\wedge e + \frac{1}{2} e\wedge\mathcal{K}\wedge\mathcal{K}\Big\rangle \tag{4.3}$$

where ${}^{\omega}T = \mathrm{d}e + [\omega, e]$. The second term is a total exterior derivative, using Stokes's theorem it turns into a surface integral.

Let us then consider the case where $\omega = \overset{\bullet}{\omega}$, which by definition is such that $F[\overset{\bullet}{\omega}] = 0$. Hence the first term in the action (4.3) goes away and we have up to a boundary term,

$$S_{\mathrm{EC}}[e,\mathcal{K};\overset{\bullet}{\omega}] \;=\; -\frac{1}{8\pi G}\int_{\mathcal{M}} \Big\langle \overset{\bullet}{\mathrm{T}}\wedge\mathcal{K} + \frac{1}{2} e\wedge[\mathcal{K}\wedge\mathcal{K}]\Big\rangle, \tag{4.4}$$

where we denote $\overset{\bullet}{\mathrm{T}} = \overset{\bullet}{\mathcal{D}}e = \mathrm{d}e + [\overset{\bullet}{\omega}, e]$. Variations in terms of e and $\mathcal{K}$ respectively give,

$$\overset{\bullet}{\mathcal{D}}\mathcal{K} + \frac{1}{2}[\mathcal{K} \wedge \mathcal{K}] = 0 \tag{4.5}$$

$$\overset{\bullet}{\mathrm{T}} + [e \wedge \mathcal{K}] = 0. \tag{4.6}$$

Provided the frame field is invertible, we can solve the last equation of motion, and actually express the contorsion $\mathcal{K}$ in terms of the frame field and the torsion tensor $\overset{\bullet}{\mathrm{T}}{}^I{}_{ab}$ associated to the Weitzenböck connection.

$$\mathcal{K}^I{}_a = -\frac{1}{2}\,\epsilon^I{}_{JK}\left(e^{bJ}\,\overset{\bullet}{\mathrm{T}}{}^K{}_{ab} - \frac{1}{2}\,e^{cJ}\,e^{bK}\,\overset{\bullet}{\mathrm{T}}{}_{acb}\right), \tag{4.7}$$

where $\overset{\bullet}{\mathrm{T}}{}^a{}_{bc} = e_I{}^a\,\overset{\bullet}{\mathrm{T}}{}^I{}_{bc}$. We can now plug this expression back in the action (4.4). After some algebra, we recover the teleparallel action [11].

$$S_{\mathrm{EC}}[e, A] \approx -\frac{1}{16\pi G}\int d^3v_e\left(\frac{1}{4}\,\overset{\bullet}{\mathrm{T}}{}^a{}_{bc}\,\overset{\bullet}{\mathrm{T}}{}_a{}^{bc} - \frac{1}{2}\,\overset{\bullet}{\mathrm{T}}{}^c{}_{ab}\,\overset{\bullet}{\mathrm{T}}{}^{ab}{}_c - \,\overset{\bullet}{\mathrm{T}}{}^c{}_{bc}\,\overset{\bullet}{\mathrm{T}}{}^{ab}{}_a\right) =: S^{\mathrm{eucl.}}_{//}[e; {}^\bullet\omega]. \tag{4.8}$$

where $\approx$ means we went on-shell in terms of the equation of motion for $\mathcal{K}$, and $d^3v_e = \frac{1}{6}\epsilon_{IJK}e^I \wedge e^J \wedge e^K$ is the three-volume element.

The Einstein–Cartan action is therefore a first-order formulation of teleparallel gravity. As we have just shown the standard teleparallel action is recovered by choosing as reference connection ω the Weitzenböck connection $\overset{\bullet}{\omega}$ and by plugging back the equations of motion coming from the variations with respect to $\mathcal{K}$ into the Einstein–Cartan action. The equality between the two actions (4.8) is indeed valid up to a boundary term and on-shell.

A similar construction can be used to recover the Palatini formulation of GR in the 2nd order formalism. We take instead the reference connection ω to be the Levi-Civita connection $\overset{\circ}{\omega}$, which is such that $\overset{\circ}{\mathrm{T}} = De = 0$. The action (4.3) becomes then up to a boundary term

$$S_{\mathrm{EC}}[e, A] = S_{\mathrm{EC}}[e, \overset{\circ}{\omega} + \mathcal{K}] \;=\; -\frac{1}{8\pi G}\int_{\mathcal{M}}\Big\langle e \wedge F[\overset{\circ}{\omega}] + \frac{1}{2}e \wedge [\mathcal{K} \wedge \mathcal{K}]\Big\rangle. \tag{4.9}$$

Variations along $\mathcal{K}$ give $[e, \mathcal{K}] = 0$. Assuming again that e is invertible, the solution of such equation is given by $\mathcal{K} = 0$. Plugging back this solution in (4.9) allows to recover the Palatini action for 3d gravity, in the second order formalism.

$$S_{\mathrm{EC}}[e, A] \approx S[e, \overset{\circ}{\omega}] \;=\; -\frac{1}{8\pi G}\int \langle e \wedge F(\overset{\circ}{\omega})\rangle \equiv S_{\mathrm{Palatini}}[e]. \tag{4.10}$$

4.1.2 Teleparallel gravity in D dimensions from D-dimensional Einstein – Cartan action

The same construction holds in D Lorentzian spacetime dimensions. Consider the Einstein – Cartan action

$$S_{\rm EC}[A,e] = \frac{1}{16\pi G}\int_{\mathcal{M}} B_{IJ}[e]\wedge F^{IJ}[A], \tag{4.11}$$

where $F^I{}_J$ is the curvature two-form

$$F^I{}_J = \mathrm{d}A^I{}_J + A^I{}_M \wedge A^M{}_J, \tag{4.12}$$

and B_{IJ} is the bivector-valued $(D-2)$-form

$$B_{IJ} = \frac{1}{(D-2)!}\epsilon_{IJK_1\dots K_{D-2}} e^{K_1}\wedge\dots\wedge e^{K_{D-2}}. \tag{4.13}$$

To write the action in a more familiar form, we decompose the curvature two-form into its components with respect to the D-bein, namely

$$F^I{}_J = \frac{1}{2}F^I{}_{JKL}[A,e]\, e^K\wedge e^L, \tag{4.14}$$

which is possible as long as the D-bein is invertible. A short calculation gives,

$$S[A,e] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D v_e\, F^{IJ}{}_{IJ}[A,e], \tag{4.15}$$

where we introduced the D-dimensional volume element,

$$d^D v_e = \frac{1}{D!}\epsilon_{I_1\dots I_D} e^{I_1}\wedge\dots\wedge e^{I_D}. \tag{4.16}$$

Let us now explain how to recover the GR and teleparallel formulations. Consider first an arbitrary origin $\omega^I{}_J$ in the affine space of connections and parametrise any connection in terms of $\omega^I{}_J$ and a displacement vector $\mathcal{K}^I{}_J$, which is an $\mathfrak{so}(1,D-1)$-valued one form. Thus,

$$A^I{}_J = \omega^I{}_J + \mathcal{K}^I{}_J. \tag{4.17}$$

Let now ${}^\omega D$ denote the exterior covariant derivative with respect to $\omega^I{}_J$. The curvature two-form satisfies

$$F^I{}_J[A] = F^I{}_J[\omega] + {}^\omega D\mathcal{K}^I{}_J + \mathcal{K}^I{}_L\wedge\mathcal{K}^L{}_J. \tag{4.18}$$

If $\omega^I{}_{Ja}$ is the torsionless Levi-Civita spin connection $\overset{\circ}{\omega}{}^I{}_{Ja}$, the corresponding curvature two-form is nothing but the Riemann curvature tensor. In components,

$$F^I{}_{Jab}[\overset{\circ}{\omega}] = e^I{}_c e_J{}^d R^c{}_{dab}[g]. \tag{4.19}$$

In this case, the action (4.15) does indeed reduce to the usual metrical Einstein–Hilbert action on-shell where $\mathcal{K} = 0$.

If we are interested in teleparallel gravity, the relevant reference connection is the Weitzenböck connection $\overset{\bullet}{\omega}$, which has vanishing curvature. Performing a partial integration, we are then left with the following expression for the action,

$$\begin{aligned} S[A,e] + (-1)^{D-1}\int_{\partial\mathcal{M}} B_{IJ}\wedge\mathcal{K}^{IJ} = \\ = \frac{1}{16\pi G}\int_{\mathcal{M}}\Bigg[\frac{1}{(D-3)!}\epsilon_{L_1\dots L_{D-3}IJK}e^{L_1}\wedge\dots\wedge e^{L_{D-3}}\wedge\overset{\bullet}{\mathrm{T}}{}^I\wedge\mathcal{K}^{JK} + \\ + \frac{1}{(D-2)!}\epsilon_{L_1\dots L_{D-2}IJ}e^{L_1}\wedge\dots\wedge e^{L_{D-2}}\wedge\mathcal{K}^I{}_L\wedge\mathcal{K}^{LJ}\Bigg], \end{aligned} \tag{4.20}$$

where we introduced the Weitzenböck torsion,

$$\overset{\bullet}{\mathrm{T}}{}^I = \overset{\bullet}{\mathcal{D}}e^I. \tag{4.21}$$

The algebraic structure of the action (4.20) can be considerably simplified by noting that

$$\epsilon_{L_1\dots L_{d-n}I_1\dots I_n}\epsilon^{L_1\dots L_{d-n}J_1\dots J_n} = -n!(d-n)!\delta^{[J_1}_{I_1}\dots\delta^{J_n]}_{I_n}. \tag{4.22}$$

Consider then the components of the Weitzenböck torsion with respect to the D-bein,

$$\overset{\bullet}{\mathrm{T}}{}^I = \overset{\bullet}{\mathcal{D}}e^I = \frac{1}{2}\overset{\bullet}{\mathrm{T}}{}^I{}_{LM}e^L\wedge e^M. \tag{4.23}$$

This allows us to write the action (4.20) in the following compact form

$$\begin{aligned} S[A,e] + \frac{(-1)^{D-1}}{16\pi G}\int_{\partial\mathcal{M}} B_{IJ}\wedge\mathcal{K}^{IJ} = \\ = \frac{1}{16\pi G}\int_{\mathcal{M}} d^Dv_e\Big[3\,\overset{\bullet}{\mathrm{T}}{}^I{}_{MN}\mathcal{K}^{JK}{}_R\delta^{[M}_I\delta^N_J\,\delta^{R]}_K + 2\delta^{[M}_I\delta^{N]}_J\mathcal{K}^I{}_{LM}\mathcal{K}^{LJ}{}_N\Big] = \\ = \frac{1}{16\pi G}\int_{\mathcal{M}} d^Dv_e\Big[2\,\overset{\bullet}{\mathrm{T}}{}^M{}_{MJ}\mathcal{K}^{JN}{}_N + \overset{\bullet}{\mathrm{T}}{}_{IJK}\mathcal{K}^{JKI} - \mathcal{K}^I{}_{LI}\mathcal{K}^{JL}{}_J - \mathcal{K}^N{}_{[LM]}\mathcal{K}^{LM}{}_N\Big], \end{aligned} \tag{4.24}$$

where we decomposed the contortion one-form $\mathcal{K}^I{}_J$ into its components $\mathcal{K}^I{}_J = \mathcal{K}^I{}_{JM} e^M$ with respect o the D-bein e^I.

To express this action in terms of the torsion two-form alone, we have to impose strongly the torsionless condition at the level of the action. In other words, part of the equations of motion are plugged back into the action. Consider first the variation of the action with respect to the contortion one-form $\mathcal{K}^I{}_{Ja}$, which yields the torsionless condition,

$$\overset{\bullet}{\mathrm{T}}{}^I + \mathcal{K}^I{}_J \wedge e^J = 0. \tag{4.25}$$

In terms of its components, the torsionless condition (4.25) is now the same as to say

$$\mathcal{K}_{IJK} = -\mathcal{K}_{JIK} = \frac{1}{2}\big(\overset{\bullet}{\mathrm{T}}_{IJK} + \overset{\bullet}{\mathrm{T}}_{JKI} - \overset{\bullet}{\mathrm{T}}_{KIJ}\big). \tag{4.26}$$

This in turn implies

$$\mathcal{K}^N{}_{[LM]} = \frac{1}{2}\overset{\bullet}{\mathrm{T}}{}^N{}_{LM}, \tag{4.27a}$$

$$\mathcal{K}^{JN}{}_N = \overset{\bullet}{\mathrm{T}}{}_N{}^{NJ}. \tag{4.27b}$$

If we now insert (4.26, 4.27a, 4.27b) back into (4.24), we get the usual teleparallel action which is now quadratic in the components of the torsion two-form,

$$\begin{aligned} S[A,e] + \frac{(-1)^{D-1}}{16\pi G}\int_{\partial\mathcal{M}} B_{IJ}\wedge\mathcal{K}^{IJ} \approx \\ \approx \frac{1}{16\pi G}\int_{\mathcal{M}} d^D v_e\Big[\overset{\bullet}{\mathrm{T}}{}^M{}_{MJ}\overset{\bullet}{\mathrm{T}}{}_N{}^{NJ} + \frac{1}{2}\overset{\bullet}{\mathrm{T}}_{NLM}\,\overset{\bullet}{\mathrm{T}}{}^{LMN} - \frac{1}{4}\overset{\bullet}{\mathrm{T}}_{NLM}\,\overset{\bullet}{\mathrm{T}}{}^{NLM}\Big], \end{aligned} \tag{4.28}$$

where $\approx$ denotes terms that vanish provided the torsionless condition (4.25) is satisfied.

As in the three-dimensional Euclidean case, we have proved in the Lorenztian D-dimensional case that the Einstein-Cartan action, a well-known first order formulation of the standard GR formulation (Palatini action), is also a first order formulation of the teleparallel action up to a boundary term.

4.2 Relating the dual loop picture to the teleparallel formulation in 3d

We now focus on the three-dimensional Euclidean case, and restrict ourselves to a trivial topology $\mathcal{M} \sim \mathbb{R}\times\Sigma$, with the spatial manifold Σ having no boundary for simplicity. As in section 4.1.1, e, A are 1-forms with value in $\mathfrak{su}(2)$.

We will show that starting from the Einstein-Cartan action there are two natural symplectic potentials that appear, related by an integration by parts. They amount to different choices of polarization. Following our previous result, namely that the Einstein–Cartan action can be seen as the first order action of both GR and teleparallel gravity, we will argue that the different choices of polarization are naturally related to the choice of description of gravity, either the GR or teleparallel frameworks.

We will then recall how the discretization procedure described in [29, 32] gives rise to different discrete theories. Each discrete theory can then be naturally identified with the different choices of polarization in the continuum. Hence we will argue that the dual loop gravity discrete theory can naturally be seen as a discretization of the teleparallel framework

4.2.1 Pre-symplectic forms in the continuum

Standard calculations for the Einstein-Cartan action

$$S_{\mathrm{EC}}[e, A] = \frac{1}{8\pi G}\int_{\mathcal{M}} \big\langle e \wedge F[A]\big\rangle, \tag{4.29}$$

lead to the pre-symplectic potential

$$\Theta_{\mathrm{EC}} = -\frac{1}{8\pi G}\int_{\Sigma} \tilde{E}_I{}^a \delta A^I{}_a, \tag{4.30}$$

where δ is the differential in field space and $\tilde{E}_I{}^a$ denotes the densitized triad[1]

$$\tilde{E}_I{}^a = \tilde{\varepsilon}^{ba} e_{Ib}. \tag{4.31}$$

On the other hand, we now also have on field space

$$\delta A^I{}_a = \delta\big[\omega^I{}_a + \mathcal{K}^I{}_a\big] = \delta \mathcal{K}^I{}_a, \tag{4.32}$$

since ω is a reference connection, which is kept fixed on field space.

$$\Theta_{\mathrm{EC}} = -\frac{1}{8\pi G}\int_{\Sigma} \tilde{E}_I{}^a \delta \mathcal{K}^I{}_a. \tag{4.33}$$

[1] In the following, indices $a, b, c, \dots$ are two-dimensional abstract tensor indices and $\tilde{\varepsilon}^{ba}$ is the Levi-Civita skew-symmetric tensor density on the spatial slice.

Let us now choose as reference connection, the Weitzenbock connection $\overset{\bullet}{\omega}$. Then, the action (4.29) becomes (4.4)

$$S_{\rm EC}[e,\overset{\bullet}{\omega}] = \frac{1}{8\pi G}\int_{\mathcal{M}} \Big\langle \overset{\bullet}{\mathrm{T}} \wedge \mathcal{K} + \frac{1}{2} e \wedge [\mathcal{K}\wedge\mathcal{K}] \Big\rangle. \tag{4.34}$$

We refer to section 4.1.1 for more details. Direct calculations lead this time to the symplectic potential

$$\Theta_{//} = -\frac{1}{8\pi G}\int_{\Sigma} \big\langle \delta e \wedge \mathcal{K} \big\rangle = \frac{1}{8\pi G}\int_{\Sigma} \tilde{\varepsilon}^{ab}\delta e_{Ia}\,\mathcal{K}^{I}{}_{b} = \frac{1}{8\pi G}\int_{\Sigma} \tilde{\mathcal{K}}_{I}{}^{a}\delta e^{I}{}_{a}, \tag{4.35}$$

where we introduced the densitized contorsion

$$\tilde{\mathcal{K}}_{I}{}^{a} = \tilde{\varepsilon}^{ab}\mathcal{K}_{Ib}. \tag{4.36}$$

We refer to this symplectic potential as the symplectic potential for the teleparallel picture since (4.34) is the teleparallel action (4.8) on shell.

The actions (4.29) and (4.34) are related by an integration by part. The relevant connection variables for the symplectic form are actually given in terms of the contorsion tensor. We note that because we are dealing with densitized fields, the canonical map relating the two choices of polarization also implements a (Poincaré) dualization implemented by the Levi-Civita tensor density $\tilde{\varepsilon}^{ab}$.

$$(\tilde{\varepsilon}^{ba}e_{Ib}, \mathcal{K}^{I}{}_{a}) \to (\tilde{\varepsilon}^{ab}\mathcal{K}_{Ib}, e^{I}{}_{a}). \tag{4.37}$$

These two sets of variables amount to two polarization choices to describe our theory, either of the GR or teleparallel formulations. Physics does not depend on the choice of polarization. A polarization is chosen for a convenient description of the physical system at hand. This is another way to say that to discuss gravity we can equivalently work with the GR or teleparallel formulations according to the system we are looking at.

Hence from an abstract perspective, the choice of polarization does not matter at the continuum level. At the discrete level however things will be more subtle. Indeed, the discretization procedure is sensitive to the presence of the Levi-Civita tensor density $\tilde{\varepsilon}^{ba}$. Let us describe now the discretization scheme we intend to use.

4.2.2 Symplectic forms in the discrete picture

We recall the construction of [29], neglecting the possible existence of curvature/torsion defects at the vertices of the triangulation. For further details about these, see [106].

The phase space underlying the spin network quantum states can be obtained through a discretization procedure relying on two steps. We implement a discretization of the spatial manifold using a triangulation as well as a truncation of the degrees of freedom by assuming that on the faces c^* of the triangulation we have the equations of motion/constraints satisfied. The solutions of such zero torsion and zero curvature constraints are respectively given by

$$e(x) = g_c^{-1} \mathrm{d}y_c g_c, \quad A = g_c^{-1} \mathrm{d}g_c, \tag{4.38}$$

with x any point of a given face c^* of the triangulation, $g_c(x)$ the holonomy joining the reference point c to x in c^*, and y_c a Lie algebra element.

We intend to discretize the pre-symplectic potential Θ_{EC} (4.33) and not $\Theta_{//}$ (4.35), as the latter cannot be brought to an expression depending on the boundary only. Nevertheless, we will still be able to have the discrete analogue of the potential $\Theta_{//}$ (4.35) precisely because the discretized version of Θ_{EC} (4.33) will be an exact 2-form, essentially allowing for the integration by parts relating the discrete version of Θ_{EC} to what can be seen as a discrete version of $\Theta_{//}$.

Starting from Θ_{EC} (4.33), within a face c^* of the triangulation, we replace the frame field and the connection by their respective discrete expression given in (4.38)

$$\Theta_{EC} = \frac{1}{8\pi G} \int_{c^*} \big\langle e \wedge \delta A \big\rangle = \frac{1}{8\pi G} \int_{c^*} \big\langle \mathrm{d}y_c \wedge \mathrm{d}(\delta g_c g_c^{-1}) \big\rangle \,. \tag{4.39}$$

As the integrand is an exact 2-form, this integral can be evaluated on the boundary of c^* and there are two possible choices.

$$\int_{c^*} \big\langle \mathrm{d}y_c \wedge \mathrm{d}(\delta g_c g_c^{-1}) \big\rangle = - \int_{\partial c^*} \big\langle \mathrm{d}y_c \, (\delta g_c g_c^{-1}) \big\rangle = \int_{\partial c^*} \big\langle y_c \, \mathrm{d}(\delta g_c g_c^{-1}) \big\rangle. \tag{4.40}$$

Such discretization can be performed for any face, in particular for the face c'^* which shares an edge ℓ as boundary with c^*. Furthermore the fields $g_{c'}(x)$ and $y_{c'}(x)$ being evaluated on ℓ can be related to the fields $g_c(x)$ and $y_c(x)$ evaluated at the same point on ℓ .

$$g_{c'} = h_{c'c} g_c, \quad y_{c'} = h_{c'c}(y_c + x_{cc'}) h_{c'c}^{-1}. \tag{4.41}$$

These are the continuity conditions at ℓ, the common edge of the faces c^* and c'^*. Implementing these relations for each contributions c^*, c'^* for the edge $\ell = [vv']$, which is dual to the the spin network link $[cc'] = \ell^*$, we get the two different potentials, for each edge ℓ

$$\Theta^{\ell}_{\mathrm{LG}} = -\frac{1}{8\pi G} \Big\langle \left(\int_{\ell} \mathrm{d}y_c \right) \delta h_{\ell^*} h_{\ell^*}^{-1} \Big\rangle = -\frac{1}{8\pi G} \Big\langle X_{\ell} \, \delta h_{\ell^*} h_{\ell^*}^{-1} \Big\rangle, \tag{4.42}$$

$$\Theta^{\ell}_{\mathrm{LG}^*} = +\frac{1}{8\pi G} \Big\langle (g_{vc} x_{\ell^*} g_{cv}) \delta g_{\ell} g_{\ell}^{-1} \Big\rangle = +\frac{1}{8\pi G} \Big\langle X_{\ell^*} \, \delta g_{\ell} g_{\ell}^{-1} \Big\rangle, \tag{4.43}$$

where $\Theta^\ell_{\rm LG}$ refers to the loop gravity potential[2], whereas $\Theta^\ell_{\rm LG^*}$ refers to the dual loop gravity potential. Indeed, by construction, the fluxes X_ℓ satisfy the Gauss constraint when summing over the edges of a given triangle.

$$\sum_{\ell\in\partial c^*} X_\ell = 0. \tag{4.44}$$

This is the discretized version of dealing with a torsionless connection. The data $(X_\ell, g_{\ell^*}, \Theta^\ell_{LG})$ provides the classical phase space for the usual spin networks: we have holonomies decorating the dual of the triangulation, ie the spin network. This is often coined *loop gravity.*

On the other hand we also have the dual picture where the holonomies around the triangles satisfy the flatness constraint.

$$\prod_{\ell\in\partial c^*} g_\ell = 1, \tag{4.45}$$

This is the discretized version of dealing with a flat connection. The data $(g_\ell, X_{\ell^*}, \Theta^\ell_{\rm LG^*})$ provides the classical phase space for the "dual" spin networks: we have fluxes decorating the dual of the triangulation, ie the spin network. This is naturally coined *dual loop gravity.* Such discrete theory was shown to be related to t'Hooft theory [30], or the Dijkgraaf-Witten model [31].

The parallel with the previous section should now be clear. The configuration variables $\mathcal{K}^I{}_a$, $e^I{}_a$, are discretized along the link ℓ^*, whereas the momentum variables $\tilde{E}_I{}^a$, $\tilde{\mathcal{K}}_I{}^a$, are discretized along the edge ℓ.

"GR polarization"	$\to$	LQG
$\tilde{E}_I{}^a$	$\to$	X_ℓ
$A^I{}_a$	$\to$	g_{ℓ^*}
$\Theta_{\rm EC}$	$\to$	$\Theta^\ell_{\rm LG}$

"Teleparallel polarization"	$\to$	dual LQG
$\tilde{\mathcal{K}}_I{}^a$	$\to$	g_ℓ
$e^I{}_a$	$\to$	X_{ℓ^*}
$\Theta_{//}$	$\to$	$\Theta^\ell_{\rm LG^*}$

Dual loop gravity can be interpreted as the discretization of the teleparallel framework, just like loop gravity can be seen as a discretization of general relativity. The momentum variables are discretized on structures dual to the ones which the configuration variables

[2] The loop gravity symplectic potential (4.42) was already obtained in [107] but in the context of multi particles in $2+1$ gravity.

are associated to. Hence different polarizations in the continuum are associated *different* discretization pictures. Change of polarization at the continuum level and discretization do not commute.

At the end of the day, physics should still not depend on the choice of polarization. The discretization procedure should not lead to different physics. Hence this means that the two discretizations must be related by a duality map, encoding their equivalence. Such duality was conjectured in [31] and probably related to the one found in the context of the Kitaev model [70]. We will leave this for further investigations.

Chapter 5

Discretization of the 3d gravity phase space

It is well known that the phase space $T^* \mathrm{SU}(2)$ can also be described as the Heisenberg double of the Lie group SO(3) provided with a trivial cocycle. Such feature was used in [4] to discuss 3d loop quantum gravity. In the same paper, $\mathrm{SL}(2, \mathbb{C})$ seen as the Heisenberg double of SU(2) is also introduced as a phase space for the non-zero cosmological constant framework, and is then seen as a deformation of the phase space $T^* \mathrm{SU}(2)$. The construction of both phase space were carried out in the Euclidean gravity settings by making use of Poisson-Lie group formalism. Subsequently, in [51] the deformed model of [4] was quantized by applying LQG techniques. The kinematical Hilbert space of this deformed model interestingly is spanned by $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ spin networks, a q-deformation of the spin networks in LQG with $\Lambda = 0$, thus providing a glimpse on how quantum groups appear in the LQG framework.

The appearance of a quantum group at the kinematical level is a long standing issue since the cosmological constant appears only in the curvature constraints, and not in the Gauss constraint which ultimately generates the spin networks. Hence from the standard Hamiltonian picture, whether Λ is zero or not, we would always deal with standard SU(2) spin networks.

The works [51, 4] are one of several approaches in which quantum groups are usually introduced by hand to account for the presence of a non-zero cosmological constant. We show here we can derive the q-deformation naturally starting from an action through a specific process of discretization and a canonical transformation. The canonical transformation allows to "deform" the Gauss constraint so that the cosmological constant also appears there. Hence such canonical transformation will be the key to recover the notion of quantum groups in the quantum realm.

The first part of this chapter introduces 3d gravity with $\Lambda \neq 0$ in the context of the BF and Chern-Simons formulations. Their equivalence is demonstrated and we carry out their respective Hamiltonian analysis and discus the symmetries the constraints generate. We introduce the canonical map. In the new variables, we derive the phase space of 3d gravity through some discretization steps already discussed in [29]. This derivation is based on a yet to be published work [2].

5.1 3d gravity with a cosmological constant

Just as 4d gravity can be formulated as a gauge theory so is 3d gravity. We will review the first order formulation of 3d gravity. For a thorough treatment, one should see [23]. Our reason for this review is two-fold: (i) Show the equivalence between BF theory and Chern-Simons theory in the presence of a non-zero cosmological constant. (ii) for purposes of obtaining the deformed phase space of 3d gravity via discretization from the viewpoint of loop quantum gravity.

5.1.1 BF and Chern-Simons Actions: Part I

Following [78, 79], let us consider the Lie algebra $\mathfrak{g}$ generated by J_A and P_A, $A = 1, 2, 3$ with the properties $J_A^\dagger = -J_A$, $P_A^\dagger = P_A$ and satisfying the Lie brackets

$$[J_A, J_B] = \epsilon_{ABC} J^C, \quad [J_A, P_B] = \epsilon_{ABC} P^C, \quad [P_A, P_B] = -c^2 \, \Lambda \, \epsilon_{ABC} J^C. \tag{5.1}$$

J_A and P_A are generators of the 3-dimensional Lorentz group and translation group respectively. The indices are raised with the metric η_{AB}, the Minkowski or Euclidean metric, according to the choice of spacetime signature. We also have $c^2 = -1$ or $c^2 = 1$ respectively for a Euclidean or Lorentzian spacetime. Λ is the cosmological constant. With these definitions, we can get the Lie algebra of the symmetries of any 3d homogenous spacetime. For completeness, we summarize the Lie algebra $\mathfrak{g}$ for the different signatures and sign of the cosmological constant which is given in table 5.1

A convenient parametrization of $\mathfrak{g}$ is given by setting [78]

$$P_A = \theta J_A, \text{ with } \theta^2 = -c^2 \Lambda. \tag{5.2}$$

Cosm. const.(Λ)	Euclidean signature	Lorentzian signature
$\Lambda = 0$	$\mathfrak{iso}(3)$	$\mathfrak{iso}(1,2)$
$\Lambda > 0$	$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(1,3)$
$\Lambda < 0$	$\mathfrak{so}(1,3)$	$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$

Table 5.1: Local isometry Lie algebras in 3d gravity.

When dealing with $\Lambda = 0$, θ can be seen as a Grassmanian number. When $\Lambda > 0$, then $\theta = i\sqrt{\Lambda}$ is purely imaginary. When $\Lambda < 0$, then $\theta = I\sqrt{-\Lambda}$ is purely split imaginary ($I^2 = 1$). Following [78], given two real numbers a, b, we have for any type of θ,

$$\overline{(a + \theta b)} = a - \theta b. \tag{5.3}$$

To build an action, we need to introduce a pairing between the generators, i.e., an invariant bilinear form over $\mathfrak{g}$. The relevant one, relating Chern-Simons to gravity, is[1]

$$\langle J_A, P_B \rangle = \eta_{AB} = \langle P_A, J_B \rangle, \quad \langle J_A, J_B \rangle = \langle P_A, P_B \rangle = 0. \tag{5.4}$$

To proceed we consider a 3d manifold $\mathcal{M}$ with no boundary. Let us begin with the BF action and the Einstein-Hilbert action (2.2) in the case where the triad e^I_μ and the spin connection ω^{IJ}_μ are the dynamical variables and are not related. The triad e^I_μ is related to the metric through the relation $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$. Considered as Lie algebra $\mathfrak{g}$ valued 1-forms, the triad and spin connection are defined as

$$e \equiv e^I P_I \text{ with } e^I = e^I_\mu \mathrm{d}x^\mu, \qquad \omega \equiv \omega^I J_I, \text{ with } \omega^I = \frac{1}{2}\epsilon^{IJK}\omega_{JK}. \tag{5.5}$$

Their transformation properties are given by the Lorentz gauge transformations parametrized by $\psi \equiv \psi^I J_I$,

$$\omega \to \omega + \mathrm{d}_\omega \psi, \quad e \to [e, \psi], \tag{5.6}$$

and the infinitesimal "curved" translation parameterized by $\phi \equiv \phi^I P_I$,

$$\omega \to \omega - c^2\Lambda[e, \phi], \quad e \to e + \mathrm{d}_\omega \phi. \tag{5.7}$$

We recall the curvature of the spin connection (2.18) with two indices

$$F^{IJ}(\omega) = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}, \tag{5.8}$$

[1]See [79] for a discussion on the most general pairing one can consider.

however using the spin connection with one index (5.5), we can rewrite the curvature also with one index as

$$F^I[\omega] = d\omega^I + \frac{1}{2}\epsilon^I{}_{JK}\omega^J \wedge \omega^K. \tag{5.9}$$

The dynamics of 3d gravity with a cosmological constant are specified by the action

$$\begin{aligned} S_{GR}[e,\omega] &= \frac{1}{16\pi G}\int_{\mathcal{M}} (e^I \wedge F_I - c^2\frac{\Lambda}{3}\,\epsilon_{IJK}\, e^I \wedge e^J \wedge e^K) \\ &= \frac{1}{16\pi G}\int_{\mathcal{M}} (e^I \wedge F_I + \frac{1}{3}\, e_I \wedge [e\wedge e]^I), \end{aligned} \tag{5.10}$$

where in the last line we introduced the Lie bracket (5.1). We note however the non-degeneracy of the triad field which is assumed in general relativity is dropped in the present formulation.

By varying (5.10) with respect to the connection and triad, the equations of motion are

$$T^I := \mathrm{d}_\omega e^I = \mathrm{d}e^I + \epsilon^I{}_{JK}\omega^J \wedge e^K = 0, \qquad F^I = c^2\Lambda\,\epsilon^I{}_{JK} e^J \wedge e^K. \tag{5.11}$$

The equations of motion encode that the connection has no torsion and that the curvature of ω is characterized by the cosmological constant. The torsionless condition ensures that the connection is related to the triad

$$\mathrm{d}e = -\omega \wedge e. \tag{5.12}$$

We now look at the Chern-Simons action. We recall $\mathcal{M}$ as a 3d manifold, equipped with the Cartan connection $\mathcal{A}$ with value in $\mathfrak{g}$. The frame field e and spin connection ω combine into the Cartan connection defined as

$$\mathcal{A} = \omega^A J_A + e^A P_A. \tag{5.13}$$

The infinitesimal gauge transformation for $\mathcal{A}$, characterized by χ a scalar field with value in $\mathfrak{g}$, is

$$\mathcal{A} \to \mathcal{A} + \delta_\chi \mathcal{A} = \mathcal{A} + \mathrm{d}\mathcal{A} + [\mathcal{A}, \chi] = \mathcal{A} + \mathrm{d}_{\mathcal{A}}\chi. \tag{5.14}$$

In terms of these components, the infinitesimal gauge transformations (5.14) generated by $\chi = \psi \cdot J + \phi \cdot P$ are respectively for $\phi = 0$ and $\psi = 0$,

$$\omega \to \omega + \mathrm{d}_\omega \psi, \quad e \to [e,\psi], \quad \omega \to \omega - c^2\Lambda[e,\phi], \quad e \to e + \mathrm{d}_\omega \phi, \tag{5.15}$$

which coincide with those in (5.6) and (5.7).

The well-known Chern-Simons Lagrangian is now given by

$$\mathcal{L}_{CS}(\mathcal{A}) = \langle \mathcal{A} \wedge \mathrm{d}\mathcal{A} \rangle + \frac{1}{3} \langle \mathcal{A} \wedge [\mathcal{A} \wedge \mathcal{A}] \rangle. \tag{5.16}$$

The equations of motion simply state that the curvature $\mathfrak{F}$ of $\mathcal{A}$ is zero.

$$\mathfrak{F} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}] = 0. \tag{5.17}$$

By plugin the expression of the Cartan connection (5.13) into (5.17), the equations of motion become

$$\mathfrak{F} = 0 \Leftrightarrow \left| \begin{array}{lcl} T = \mathrm{d}_\omega e = \mathrm{d}e + [\omega \wedge e] & = & 0 \\ \\ F^I(\omega) - c^2 \Lambda \epsilon^I{}_{BC}\, e^B \wedge e^C & = & 0. \end{array} \right. \tag{5.18}$$

Again these equations of motion coincide with the ones in (5.11) for the BF theory. Using these components, the Chern-Simons Lagrangian becomes the BF action with a cosmological constant Λ up to a boundary term.

$$\begin{aligned} \mathcal{L}_{CS}[\omega, e] &= \omega^A \wedge \mathrm{d}e_A + e^A \wedge \mathrm{d}\omega_A + \epsilon^{ABC} \omega_A \wedge \omega_B \wedge e_C + \theta^2 \frac{1}{3} \epsilon^{ABC} e_A \wedge e_B \wedge e_C \\ &= 2 \left(e^A \wedge F_A - c^2 \frac{\Lambda}{3} \epsilon^{ABC}\, e_A \wedge e_B \wedge e_C \right) + \mathrm{d}(e_A \wedge \omega^A) \\ &= \mathcal{L}_{BF}[\omega, e]. \end{aligned} \tag{5.19}$$

We thus have seen that the equivalence of the BF theory and Chern-Simons are not only at the level of the Lagrangian but also at the level of the equations of motion and also their infinitesimal transformation coincide. Let us note that this equivalence is based on the Cartan decomposition of $\mathfrak{g}$. Also, we note that the frame field e, while considered as a tensor in the usual general relativity case is actually seen as a piece of the connection. Note however that in a sense, it is not exactly a frame field, since as part of a connection, it does transform as a connection. When $\Lambda = 0$, i.e., with $\mathfrak{g}$ being the Poincaré/Euclidian ($\mathfrak{iso}(3)/\mathfrak{iso}(1,2)$) Lie algebra, forgetting that the BF Lagrangian is the same as the Chern-Simons one, one usually says that the BF action is invariant under the translation of the frame field, $e \to e + \mathrm{d}_\omega \phi$, which is nothing else than the transformation (5.15) restricted to the case where $\mathfrak{g}$ is the Poincaré/Euclidian Lie algebra. Hence this "extra" symmetry from the gravity point of view is nothing else but the fact that the frame field can be viewed as part of a grand connection. .

5.1.2 Hamiltonian formalism and momentum maps

Let us consider now the Hamiltonian formulation of the previous actions by foliating the spacetime $\mathcal{M}$ into a a family of hypersurfaces Σ_t characterized by time t. The spacetime $\mathcal{M}$ is of the form $\mathbb{R} \times \Sigma$, where Σ represent space. Without loss of generality, we will denote coordinates now by $t = x^0, x^1, x^2$. Lower case indices are space indices, $a, b = 1, 2$. First the Chern-Simons action becomes

$$\mathcal{S}_{CS}[\mathcal{A}] = \int_{\Sigma\times\mathbb{R}} dx^2 \left(\varepsilon_{ab}\delta_{IJ}\mathcal{A}^{aI}\dot{\mathcal{A}}^{bJ} + \mathcal{A}^{tI}\mathfrak{F}_I \right), \tag{5.20}$$

with $\epsilon^{ab} \equiv \epsilon^{abt}$. We can read out the Poisson structure on the connection space

$$\{\mathcal{A}^I_a(x), \mathcal{A}^J_b(y)\} = \epsilon_{ab}\delta^{IJ}\delta^2(x-y), \quad x, y \in \Sigma, \tag{5.21}$$

and its associated symplectic form is

$$\Omega^{CS} = \int_\Sigma \langle \delta\mathcal{A} \wedge \delta\mathcal{A} \rangle, \tag{5.22}$$

where δ encodes the differential over the field space and squares to zero $\delta^2 = 0$. The Lagrange multiplier $\mathcal{A}^{tI}$ enforces the curvature constraint

$$\mathfrak{F}^I = 0, \tag{5.23}$$

which is first class. This set of constraints implement the infinitesimal gauge transformations on $\mathcal{A}$. Considering a smearing of the curvature constraint on the $\mathfrak{g}$-valued scalar χ,

$$\{\mathcal{A}^I_a, \int (\chi^L \mathfrak{F}_L)\} = -(\partial_a \chi^I(x) + \epsilon^I_{JK}\mathcal{A}^J_a\chi^K) = -\delta_\chi \mathcal{A}. \tag{5.24}$$

Let us now consider the same Hamiltonian formalism but in the gravity variables (still omitting the boundary term).

$$\mathcal{S}_{BF}[\omega, e] = \int_{\Sigma\times\mathbb{R}} d^2x \left(\epsilon^{ab}\delta_{IJ}(e^I_b\partial_t\omega^J_a) + \omega_{tI}(\epsilon^{ab}D_a e^I_b) + \frac{1}{2}e_{tI}\epsilon^{ab}\left(F^I_{ab} - c^2\,\Lambda\,\epsilon^I_{BC}\,e^B_a\,e^C_b\right)\right), \tag{5.25}$$

with $\epsilon^{ab} \equiv \epsilon^{abt}$, D_a is the restriction of the covariant derivative on the slice Σ. We therefore identify phase space variables $(\omega^I_a(x), E^b_J(x) = \epsilon^{bc}e_{cJ}(x))$, with the non-zero Poisson bracket between the pair

$$\{\omega^I_a(x), E^b_J(y)\} = \delta^b_a\delta^I_J \quad x, y \in \Sigma, \tag{5.26}$$

and with ω_a^I the configuration variable the corresponding symplectic form of (5.26) is

$$\Omega_{\text{grav}}^{LQG} = \int_\Sigma \langle \delta\omega \wedge \delta e \rangle. \tag{5.27}$$

However, one could perform an integration by part on the first term in (5.25) and obtain $\epsilon^{ab}\delta_{IJ}(\omega_a^J \partial_t e_b^I)$. Here e_b^I becomes the configuration variable with

$$\Omega_{\text{grav}}^{LGQ^*} = \int_\Sigma \langle \delta e \wedge \delta\omega \rangle, \tag{5.28}$$

the symplectic form of dual LQG.

The terms e_t^J and ω_t^J in the action (5.25) are Lagrange multipliers enforcing respectively the curvature constraint and the Gauss constraint,

$$\mathcal{F}^I = \epsilon^{ab}(F_{ab}^I - c^2\,\Lambda\,\epsilon^I_{BC}\,e_a^B\,e_b^C) = F^I - c^2\,\Lambda\,\epsilon^I_{BC}\,\epsilon^{cd}\,E_c^B E_d^C = 0 \tag{5.29}$$

$$T_J = \epsilon^{ab} D_a e_{bJ} = \partial_a E_J^a + \epsilon_{JAB}\,\omega_a^A E^{aB} = 0. \tag{5.30}$$

Smeared with appropriate functions, i.e $\mathcal{T} = \int \xi_I T^I$ and $\mathcal{F} = \int \phi_I \mathcal{F}^I$ and using the Poisson bracket (5.26) one finds the constraint algebra or commutation relations between the constraints to be

$$\{\mathcal{T}[\xi], \mathcal{T}[\xi']\} = \mathcal{T}\,[[\xi, \xi']] \tag{5.31a}$$

$$\{\mathcal{F}[\phi], \mathcal{T}[\xi]\} = \mathcal{F}\,[[\phi, \xi]] \tag{5.31b}$$

$$\{\mathcal{F}[\phi], \mathcal{F}[\phi']\} = c^2\Lambda\mathcal{T}[[\phi, \phi']]. \tag{5.31c}$$

The above commutation relations suggests both constraints are first class and by a standard computation one can show that the theory has zero dynamical degrees of freedom.

We can check again what are the the transformations that generate these constraints (5.29) and (5.30), by considering a smeared version over the field $\chi = \phi \cdot J + \psi \cdot P$, we have that[2]

$$\{e, \int (\psi \cdot T)\} = -[e, \psi] \tag{5.32}$$

$$\{\omega, \int (\psi \cdot T)\} = -\mathrm{d}_\omega \psi \tag{5.33}$$

$$\{e, \int \big(\phi_I(F^I - c^2\Lambda\epsilon^I_{BC}\epsilon^{cd}\,E_c^B\,E_d^C)\big)\} = -\mathrm{d}_\omega \phi \tag{5.34}$$

$$\{\omega, \int \big(\phi_I(F^I - 2c^2\Lambda\epsilon^I_{BC}\epsilon^{cd}\,E_c^B\,E_d^C)\big)\} = -c^2\Lambda[e, \phi]. \tag{5.35}$$

[2] We use that $\epsilon_{ab}\epsilon_{bc} = -\delta_{ac}$ so that $E_{aJ} = \epsilon_{ab}e_{bJ} \Leftrightarrow e_{cJ} = -\epsilon_{ca}E_J^a$.

As we could expect, we recover the infinitesimal transformations (5.15). We see that the zero torsion constraint is a momentum map (these are maps which encode the symmetries of phase space) implementing infinitesimal gauge transformations: the connection is infinitesimally gauge transformed (5.32), and the frame field undergoes an infinitesimal Lorentz transformation (5.33). On the other hand if (5.34) can be seen as a infinitesimal gauge transformation of the frame field also called a "translation", (5.35) has no simple geometric interpretation and this makes quantization or discretization of the theory problematic, unless $\Lambda = 0$.

If from the classical perspective this is just a fact, this is an issue from the quantum perspective. We would need to find a realization of the symmetries which does implement the above transformations. While it might be possible to do so, it is not known to us how to proceed in such formulation. So instead, in close spirit to the basic philosophy behind LQG, we want to find another set of variables where the symmetries are realized in a more suitable manner.

5.1.3 BF and Chern-Simons Actions: Part II

The new variables can be obtained by a change of basis for $\mathfrak{g}$. In the previous section we use the Cartan decomposition, we now use the Iwasawa decomposition. We define

$$\tau_A = P_A + n^B \epsilon_{AB}{}^C J_C, \quad n^A n_A = \Lambda, \tag{5.36}$$

which satisfy the $\mathfrak{an}_2$ algebra commutation relations

$$[\tau_A, \tau_B] = C_{AB}{}^D \tau_D \quad \text{with } C_{AB}{}^D = \delta^D_B n_A - \delta^D_A n_B, \tag{5.37}$$

where n_A is a vector. The canonical choice is to take $n_A = \theta \delta_{A3}$. We emphasize that the structure constant $C_{AB}{}^D$ is not cyclic as $\epsilon_{AB}{}^C$. In this new basis, $\mathfrak{g}$ is now expressed as a matched pair of Lie algebras, $\mathfrak{g} \sim \mathfrak{su} \bowtie \mathfrak{an}_2$ [53], with $\mathfrak{su}$ being $\mathfrak{su}(1,1)$ or $\mathfrak{su}(2)$ according to the signature of spacetime. Note that such decomposition does not exist for the case $\mathfrak{g} = \mathfrak{so}(4)$, which coincide with the case of Euclidian gravity with a positive cosmological constant. This case, known to be technically more tricky than the others, will be discussed again later on. At this time, we disregard it.

$$[J_A, J_B] = \epsilon_{AB}{}^C J_C, \qquad [\tau_A, \tau_B] = C_{AB}{}^D \tau_D \tag{5.38}$$

$$[J_A, \tau_B] = C_{BDA} J^D + \epsilon_{AB}{}^D \tau_D. \tag{5.39}$$

The cross commutator (5.39) is more involved since it includes an action and "back"-action and as such has components both in $\mathfrak{su}$ and $\mathfrak{an}_2$. We will note therefore the projection in the respective components as follows

$$[J_A, \tau^B]_{|\mathfrak{su}} \equiv C^{BD}{}_A\, J_D\,, \quad [J_A, \tau^B]_{|\mathfrak{an}_2} \equiv \varepsilon_{DA}{}^B\, \tau^D\,. \tag{5.40}$$

The relations (5.38) and (5.39) are the counterparts of (5.1). They are the defining relations of $\mathfrak{g}$ as a Drinfeld double of $\mathfrak{su}$ (with a non-trivial cocycle) [53]. The Killing form (5.4) in this new basis is now

$$\langle J_A, \tau_B \rangle = \delta_{AB}, \quad \langle J_A, J_B \rangle = \langle \tau_A, \tau_B \rangle = 0. \tag{5.41}$$

The change of basis in the Lie algebra $\mathfrak{g}$ can be seen as a change of variables in the fields. In particular the new connection $\mathcal{A}$ will depend both on ω and the frame field e.

We consider now the $\mathfrak{g}$ valued (Chern-Simons) connection $\mathcal{A}$ in the Iwasawa decomposition. In terms of the gravity variables, we have

$$\mathcal{A} = \omega^K J_K + e^K P_K = \omega^K J_K - n^A \epsilon_{AB}{}^K e^B J_K + n^A \epsilon_{AB}{}^K e^B J_K + e^A P_A = \tilde{\omega}^K J_K + e^K \tau_K, \tag{5.42}$$

with $\tilde{\omega}^K = \omega^K - n^A \epsilon_{AB}{}^K e^B$. Hence, the spin connection is modified to incorporate a piece of the frame field, whereas the frame field is now valued in a Lie algebra $\mathfrak{an}_2$.

In these coordinates, the $\mathfrak{g}$ valued curvature $\mathfrak{F}$ becomes,

$$\begin{aligned} \mathfrak{F}(\tilde{\omega}, e) &= \left(\mathrm{d}\tilde{\omega}^D + \frac{1}{2} \epsilon_{AB}{}^D\, \tilde{\omega}^A \wedge \tilde{\omega}^B + C_B{}^D{}_A\, \tilde{\omega}^A \wedge e^B \right) J_D \\ &\quad + \left(\mathrm{d}e^D + \epsilon_{AB}{}^D\, \tilde{\omega}^A \wedge e^B + \frac{1}{2} C_{AB}{}^D\, e^A \wedge e^B \right) \tau_D \\ &= \mathcal{F}^D\, J_D + \mathcal{T}^D\, \tau_D. \end{aligned} \tag{5.43}$$

Under the limit $\Lambda \to 0$ and $\mathfrak{an}_2 \to \mathbb{R}^3$, $\mathcal{T}$ can be seen as the analogue of the torsion, whereas $\mathcal{F}$ is the analogue of the curvature. Note however that we can still defined the direct crossproduct algebra in the case of $\Lambda = 0$ and using a Grassmanian number θ. We also emphasize that now the torsion is with value in a non-abelian Lie algebra. The direct crossproduct structure makes the usually two different objects curvature and torsion much more symmetric. There is contribution of the connection ω in the torsion (as usual) but there is also now a contribution of the frame field in the curvature. The two types of structure constant $C_{AB}{}^D$ and $\epsilon_{AB}{}^C$ appear in both $\mathcal{F}$ and $\mathcal{T}$.

The corresponding infinitesimal gauge transformations of (5.14) spanned by the scalar $\chi = X^A J_A + Y^A \tau_A$ with value in $\mathfrak{g}$ on the gauge connection $\mathcal{A}$ are

$$\delta_\chi \mathcal{A} \;=\; \mathrm{d}X^A J_A + \mathrm{d}Y^A \tau_A + [\tilde{\omega}^B J_B + e^B \tau_B \,,\, X^A J_A + Y^A \tau_A]. \tag{5.44}$$

We use the Lie algebra structure of $\mathfrak{g}$ given in (5.38) and (5.39) to get the explicit transformations for the components.

$$\begin{aligned} \delta_\chi \mathcal{A} \;=\; & \left(\mathrm{d}X^D + \epsilon_{AB}{}^D X^B \tilde{\omega}^A - C_B{}^D{}_A X^B e^A + C_A{}^D{}_B Y^A \tilde{\omega}^B\right) J_D \\ & + \left(\mathrm{d}Y^D + C_{AB}{}^D Y^B e^A - \mathcal{E}_{AB}{}^D X^A e^B + \mathcal{E}_{BA}{}^D Y^A \tilde{\omega}^B\right) \tau_D. \end{aligned} \tag{5.45}$$

In particular, we have respectively for $Y = 0$ and $X = 0$,

$$\delta_X \tilde{\omega}^D = \mathrm{d}X^D + \epsilon_{AB}{}^D X^B \tilde{\omega}^A - C_B{}^D{}_A X^B e^A \equiv D_{\mathfrak{h}} X^D, \quad \delta_X e^D = -\mathcal{E}_{AB}{}^D X^A e^B, \tag{5.46}$$

$$\delta_Y \tilde{\omega}^D = C_A{}^D{}_B Y^A \tilde{\omega}^B, \quad \delta_Y e^D = \mathrm{d}Y^D + C_{AB}{}^D Y^B e^A + \mathcal{E}_{BA}{}^D Y^A \tilde{\omega}^B \equiv D_{\mathfrak{L}} Y^D. \tag{5.47}$$

These transformations are nice in the sense that the variation of $\tilde{\omega}$ or e is still given in terms of $\tilde{\omega}$ or e, unlike in the previous splitting. We have either a "translation" or a "rotation" in each of the sectors $\mathfrak{su}$ and $\mathfrak{an}_2$.

The gravity action in terms of the $(\tilde{\omega}, e)$ variables up to a boundary term now looks

$$\begin{aligned} \mathcal{L}_{CS}(\tilde{\omega}, e) \;&=\; \tilde{\omega}^A \wedge \mathrm{d}e_A + e^A \wedge \mathrm{d}\tilde{\omega}_A + \tilde{\omega}_D \wedge \tilde{\omega}_B \wedge e_C \, \epsilon_{BC}{}^D + \tilde{\omega}_A \wedge e_B \wedge e_C \, C^{BC}{}_A \\ &=\; -\mathrm{d}(\tilde{\omega}^A \wedge e_A) + 2e^A \wedge \mathrm{d}\tilde{\omega}_A + \tilde{\omega}_D \wedge \tilde{\omega}_B \wedge e_C \, \epsilon_{BC}{}^D + \tilde{\omega}_A \wedge e_B \wedge e_C \, C^{BC}{}_A \\ &=\; 2\left(e^A \wedge \mathrm{d}\tilde{\omega}_A + \frac{1}{2}\epsilon^{ABC} \, \tilde{\omega}_A \wedge \tilde{\omega}_B \wedge e_C + \frac{1}{2} C^{BC}{}_A \, \tilde{\omega}_A \wedge e_B \wedge e_C\right). \end{aligned} \tag{5.48}$$

The equations of motion still implement that $\mathfrak{F} = 0$, that is each of the curvature sectors is flat.

$$\mathcal{F}^D \;=\; \mathrm{d}\tilde{\omega}^D + \frac{1}{2}\epsilon_{AB}{}^D \, \tilde{\omega}^A \wedge \tilde{\omega}^B + C_B{}^D{}_A \, \tilde{\omega}^A \wedge e^B = 0, \tag{5.49}$$

$$\mathcal{T}^D \;=\; \mathrm{d}e^D + \epsilon_{AB}{}^D \, \tilde{\omega}^A \wedge e^B + \frac{1}{2} C_{AB}{}^D \, e^A \wedge e^B = 0. \tag{5.50}$$

First the analogue of the torsion, that is $\mathcal{T}$ is still zero. The most important point is that now the analogue of the curvature, that is $\mathcal{F}$, is really flat, even when $\Lambda \neq 0$. Hence we have reabsorbed the homogenous curvature into a (pair of) connection which are "flat". The homogenous curvature is encoded in the match pair structure or equivalently in the change of variables $\omega \rightarrow \tilde{\omega}$.

We have used the Chern-Simons variables as a guiding tool to construct the new theory, but instead we could have proposed from scratch, from the gravity point of view, the change of variables

$$\tilde{\omega}^K = \omega^K - e^A n^B \, \epsilon_{AB}{}^K. \tag{5.51}$$

This change of variables is a symplectic transformation for the gravity phase space, since using (5.26)

$$\begin{aligned} &\{\tilde{\omega}_a^I(x), E_J^b(y)\} = \delta_a^b \, \delta_J^I \delta^2(x-y), \quad \{E_I^a(x), E_J^b(y)\} = 0, \quad x, y \in \Sigma, \\ &\{\tilde{\omega}_a^I(x), \tilde{\omega}_c^J(y)\} = \{\omega_a^I(x) - e^A(x)\, n^B \, \epsilon_{AB}{}^K, \omega_c^J(y) - e^A(y)\, n^B \, \epsilon_{AB}{}^K\} = 0. \end{aligned} \tag{5.52}$$

Proceeding as earlier, we can study the Hamiltonian picture, and recover first the new Poisson bracket (5.52) and the two curvatures as (first class constraints).

$$\tilde{\mathcal{F}}^I = \epsilon^{ab}\partial_a\tilde{\omega}_b^I + \frac{1}{2}\,\epsilon^{ab}\,\epsilon_{AB}{}^I\,\tilde{\omega}_a^A\,\tilde{\omega}_b^B + C_B{}^I{}_A\,\tilde{\omega}_a^A E^{aB} \approx 0 \tag{5.53}$$

$$\tilde{\mathcal{T}}^I = \partial_a E^{aI} + \epsilon_{KL}{}^I\,\tilde{\omega}_a^K E_{aL} + \epsilon^{ab}\frac{1}{2}C_{KL}{}^I\,E_a^K E_b^L \approx 0. \tag{5.54}$$

As one can anticipate, if we smear the constraints $\tilde{\mathcal{F}}^I$ and $\tilde{\mathcal{T}}^I$ over some scalar fields φ and ψ with value respectively in $\mathfrak{su}$ and $\mathfrak{an}_2$, these constraints generate the infinitesimal transformations (5.46) and (5.47).

$$\{\tilde{\omega}_d^M, \int \tilde{\mathcal{F}}_1^D \varphi_D\} = C^{MD}{}_A\,\tilde{\omega}_d^A\,\varphi_D \tag{5.55}$$

$$\begin{aligned} \{E_d^M, \int \tilde{\mathcal{F}}_1^D \varphi_D\} &= \epsilon^a{}_d\partial_a\varphi^M + \epsilon_d{}^b\,\epsilon^{MA}{}_B\,\varphi_A\tilde{\omega}_b^B + C^{MA}{}_B\,\varphi_A E^{aB} \\ &= D_d^{\mathfrak{su}}\varphi^M. \end{aligned} \tag{5.56}$$

We see that the curvature component in J_A generates a rotation on the connection and a translation on the frame field. In a similar way,

$$\{\tilde{\omega}_d^M, \int \tilde{\mathcal{T}}^I\psi_I\} = -\left(\partial_d\psi^M + \epsilon_K{}^{IM}\,\psi_I\,\tilde{\omega}_d^K + \epsilon_d{}^b C_L{}^{MI}\,\psi_I E_b^L\right), \tag{5.57}$$

$$\{E_d^M, \int \tilde{\mathcal{T}}^I\psi_I\} = \epsilon_{ML}{}^I\,E_{dL}\psi_I. \tag{5.58}$$

Now the curvature component in τ_A generates a rotation on the frame field and a translation on the connection. This ultimately paves the way for discretization in the homogeneously curved spacetime.

5.2 Discretization of the gravity phase space

The LQG phase space, when $\Lambda = 0$, can be recovered through a discretization procedure: first the manifold is discretized using a triangulation, then there is a truncation of the degrees of freedom by imposing the constraints on the faces of the triangulation. These two steps put together allowed to recover $T^* \mathrm{SU}(2)$ for each edge of the dual graph to the triangulation, together with the Gauss constraint for each face or triangle of the triangulation. This is the LQG kinematical phase space. We intend now to generalize this to the case when $\Lambda \neq 0$ [4].

To achieve this, we first need to recall the notion of the *Heisenberg double* [108]. Such structure arises when considering pairs of Lie group, with the same dimension, which put together can be associated with a non-degenerate Poisson structure, or equivalently a symplectic form, see Appendix B for more details.

Definition: Consider the Lie group $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$ built from the n-dimensional Lie groups $\mathcal{G}_i$, and the associated Killing form $\langle , \rangle$, for which $Lie\mathcal{G}_1$ and $Lie\mathcal{G}_2$ are dual to each other. The Heisenberg double (i.e., phase space) associated to $\mathcal{G}$ has for symplectic form [109, 110]

$$\Omega = \frac{1}{2}\Big(\langle \Delta\tilde{h} \wedge \Delta\ell \rangle + \langle \underline{\Delta} h \wedge \underline{\Delta}\tilde{\ell} \rangle\Big), \tag{5.59}$$

where we use the notation $\Delta v = \delta v v^{-1}$, $\underline{\Delta} v = v^{-1}\delta v$, and $\ell, \tilde{\ell} \in \mathcal{G}_1$, $h, \tilde{h} \in \mathcal{G}_2$ are such that

$$\ell\, h = \tilde{h}\; \tilde{\ell}. \tag{5.60}$$

In the following, we will be interested in the case where the group $\mathcal{G}$ is ISU(2) and $\mathrm{SL}(2, \mathbb{C})$ for the Heisenberg double.

5.2.1 Preliminary steps in the discretization of the gravity phase space

As stated above, the discretization scheme involves two process. First a decomposition of the spatial manifold followed by a truncation of the degrees of freedom using the constraints. Here we will give more details on the first step. We choose a triangulation Γ^* of the 2d surface Σ and Γ a 3-valent graph dual to Γ^*. The vertices of the triangulation Γ^* are denoted v, v' and the corresponding edges by $\mathfrak{e} = [vv']$, where v and v' are the respective source and target of the edge $\mathfrak{e}$. Inside each face of the triangulation denoted $[v_1 v_2 v_3]$ is a center point or node c, with the duality between the node and face denoted $c^* = [v_1 v_2 v_3]$. The centers c and c' in two different faces are connected by links $[cc'] = \mathfrak{e}^*$. The links and nodes are components of the 3-valent graph Γ. One finds out the links of Γ and edges of Γ^* satisfy $[vv'] = c^* \cap c'^*$, provided they are dual to each other.

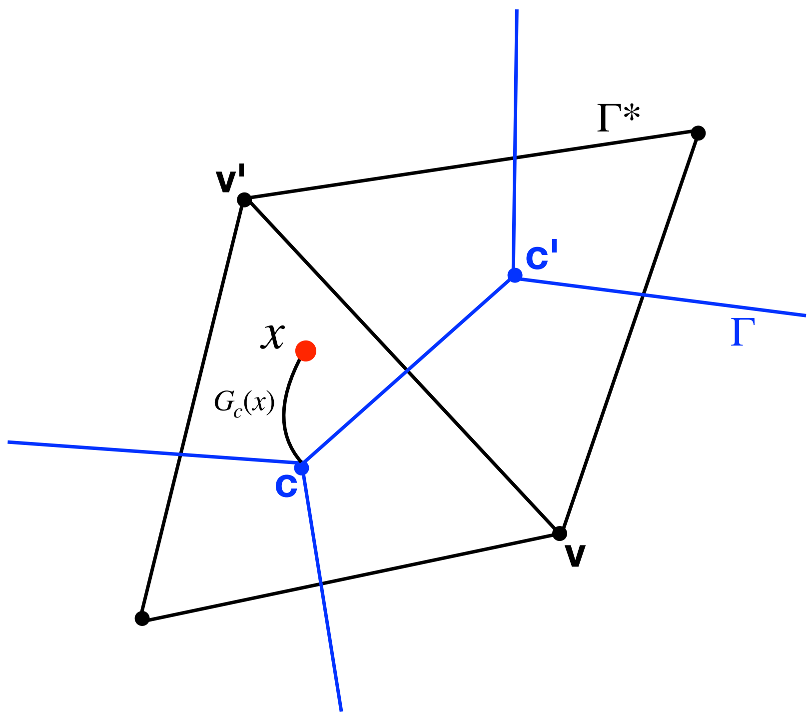


Figure 5.1: Components of the graphs Γ^*, Γ and the holonomy $G_c(x)$. The faces c^* and c'^* share the edge $\mathfrak{e} = [vv']$, dual to the link $[cc']$.

We intend to use the Chern-Simons phase space variables $(\mathcal{A}_a^I, \mathcal{A}_b^J)$ to carry out the truncation step of the discretization process. On each of the triangles or cells of the triangulation, we impose the constraints $\mathfrak{F} = 0$ which in terms of the components is simply $\mathcal{F} = 0 = \mathcal{T}$. Any curvature or torsion is localized on the vertices v of the triangulation. Since we assume that on the face of the triangles c_i^* we have a flat $\mathfrak{g}$ connection $\mathcal{A}$, we can express it in terms of its holonomy $G_{c_i}(x)$. This holonomy starts at c_i some "center" of the face and goes to $x \in c_i^*$.

$$\mathcal{A}(x) = (G_{c_i}^{-1} \mathrm{d} G_{c_i})(x). \tag{5.61}$$

Next we discretize the Chern-Simons symplectic structure on all the faces c_i^* [29],

$$\Omega^{CS} = \int_M \langle \delta\mathcal{A} \wedge \delta\mathcal{A} \rangle = \sum_i \int_{c_i^*} \langle \delta\mathcal{A} \wedge \delta\mathcal{A} \rangle. \tag{5.62}$$

The variation of the flat connection $\mathcal{A}$ (5.61) gives

$$\delta A = \delta(G_{c_i}^{-1} \mathrm{d} G_{c_i}) = G_{c_i}^{-1} \mathrm{d}\left(\delta G_{c_i} G_{c_i}^{-1}\right) G_{c_i} = G_{c_i}^{-1}(\mathrm{d}\Delta G_{c_i}) G_{c_i}, \tag{5.63}$$

where we use the notation $\Delta G := \delta G_{c_i} G_{c_i}^{-1}$ in the last equality. We get then that the symplectic form (5.62) with the help of the invariance of the Killing form reads

$$\Omega^{CS} = \sum_i \int_{c_i^*} \langle \mathrm{d}\Delta G_{c_i} \wedge \mathrm{d}\Delta G_{c_i} \rangle = \sum_i \int_{\partial c_i^*} \langle \mathrm{d}\Delta G_{c_i} \wedge \Delta G_{c_i} \rangle. \tag{5.64}$$

The Killing form of $Lie\mathcal{G}$ allows us to split the contribution into two.

$$\langle \mathrm{d}\Delta G_{c_i} \wedge \Delta G_{c_i} \rangle = \langle (\mathrm{d}\Delta G_{c_i})_{|Lie\mathcal{G}_1} \wedge (\Delta G_{c_i})_{|Lie\mathcal{G}_2} \rangle + \langle (\mathrm{d}\Delta G_{c_i})_{|Lie\mathcal{G}_2} \wedge (\Delta G_{c_i})_{|Lie\mathcal{G}_1} \rangle. \tag{5.65}$$

From the case $\Lambda = 0$, we expect that each contribution should lead to different phase spaces, one being the loop gravity one, the other the dual loop gravity one [29]. In fact it was shown that these two were dual symplectic pairs. We shall show how this extends to the case $\Lambda \neq 0$. It will be useful to define the notation $\Omega^{CS} = \Omega^{LQG} + \Omega^{LQG^*}$

$$\Omega^{LQG} \equiv \sum_i \int_{\partial c_i^*} \langle (\mathrm{d}\Delta G_{c_i})_{|Lie\mathcal{G}_2} \wedge (\Delta G_{c_i})_{|Lie\mathcal{G}_1} \rangle \tag{5.66}$$

$$\Omega^{LQG^*} \equiv \sum_i \int_{\partial c_i^*} \langle (\mathrm{d}\Delta G_{c_i})_{|Lie\mathcal{G}_1} \wedge (\Delta G_{c_i})_{|Lie\mathcal{G}_2} \rangle. \tag{5.67}$$

Let us say a bit more about the two contributions in (5.66) and (5.67) using the decomposition (5.60). First recall that the direct crossproduct structure $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$ is inherited from the direct crossproduct structure at the Lie algebra level $Lie\mathcal{G} = Lie\mathcal{G}_1 \bowtie Lie\mathcal{G}_2$. We have an action of $Lie\mathcal{G}_1$ on $Lie\mathcal{G}_2$ and conversely of $Lie\mathcal{G}_2$ on $Lie\mathcal{G}_1$. Going forward, we will specify the Lie group to be $\mathcal{G} := \mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \bowtie AN_2$ with its corresponding Lie algebra $Lie\mathcal{G} := \mathfrak{sl}(2,\mathbb{C})$ expressed in the Iwasawa decompositon as $\mathfrak{su}(2) \bowtie \mathfrak{an}_2$. The Lie brackets are given in equations (5.38) and (5.39).

5.2.2 Recovering the deformed LQG phase space

Let us consider $\mathcal{A}$ is an $\mathfrak{sl}(2,\mathbb{C})$ valued connection express in terms of the continuum variables $(\tilde{\omega}, e)$ or its associated holonomy $G_{c_i} \in SL(2,\mathbb{C})$ as

$$\mathcal{A} = \tilde{\omega}^K J_K + e^K \tau_K = G_{c_i}^{-1} \mathrm{d} G_{c_i}. \tag{5.68}$$

With the Iwasawa decomposition, we can decompose the holonomy

$$G_{c_i} = \ell_{c_i} u_{c_i} = \tilde{u}_{c_i} \tilde{\ell}_{c_i} \equiv (\ell_{c_i} \triangleright u_{c_i})(\ell_{c_i} \triangleleft u_{c_i}), \quad \ell_{c_i} \triangleright u_{c_i} = \tilde{u}_{c_i}, \quad \ell_{c_i} \triangleleft u_{c_i} = \tilde{\ell}_{c_i}, \tag{5.69}$$

with u_{c_i}, $\tilde{u}_{c_i} \in SU(2)$ and ℓ_{c_i}, $\tilde{\ell}_{c_i} \in AN_2$, then we can write down expressions relating the discretize variables and the holonomies u_{c_i}, ℓ_{c_i}, i.e.,

$$\begin{aligned}
&\mathcal{A} = G_{c_i}^{-1} \mathrm{d} G_{c_i} = \tilde{\omega}^I J_I \, + \, e^I \tau_I = \Big(u_{c_i}^{-1} \mathrm{d} u_{c_i} \, + \, [u_{c_i}^{-1} (\ell_{c_i}^{-1} \mathrm{d} \ell_{c_i}) u_{c_i}]_{|\mathfrak{su}(2)} \Big) \, + \, [u_{c_i}^{-1} (\ell_{c_i}^{-1} \mathrm{d} \ell_{c_i}) u_{c_i}]_{|\mathfrak{an}_2} \, , \\
&\tilde{\omega} = u_{c_i}^{-1} \mathrm{d} u_{c_i} \, + \, [u_{c_i}^{-1} (\ell_{c_i}^{-1} \mathrm{d} \ell_{c_i}) u_{c_i}]_{|\mathfrak{su}(2)} , \qquad e = [u_{c_i}^{-1} (\ell_{c_i}^{-1} \mathrm{d} \ell_{c_i}) u_{c_i}]_{|\mathfrak{an}_2} \, .
\end{aligned} \tag{5.70}$$

where u_{c_i} and ℓ_{c_i} are the respective SU(2) and AN_2 holonomies joining c_i to a point x in c_i^*. Due to the back-action of AN_2 on SU(2) the term $u_{c_i}^{-1}(l_{c_i}^{-1} \mathrm{d} l_{c_i}) u_{c_i}$ contributes both to $\tilde{\omega}$ and e. These relations (5.70) are the deformed version of the standard discrete picture with $\Lambda = 0$ (and $\mathbf{n} = 0$), initially obtained in [29]

$$\omega = u_{c_i}^{-1} \mathrm{d} u_{c_i}, \quad e = u_{c_i}^{-1} \mathrm{d} X \, u_{c_i}, \quad \text{with } \ell = 1 + X. \tag{5.71}$$

Going forward, we will focus only on two cells c^* and c'^* sharing the same edge $\mathfrak{e} = [vv']$.

Demanding the continuity of the connection $\mathcal{A}$ across the edge $\mathfrak{e}$, this implies that

$$G_c^{-1} \mathrm{d} G_c(x) = \mathcal{A}(x) = G_{c'}^{-1} \mathrm{d} G_{c'}(x), \quad \forall x \in \mathfrak{e} \tag{5.72}$$

this indicate the holonomies $G_c(x)$ and $G_{c'}(x)$ are evaluated at the same point $x \in \mathfrak{e}$. The continuity condition suggest there exist a constant holonomy $H_{c'c} = m_{c'c} \, h_{c'c}$ such that

$$G_{c'}(x) = H_{c'c} \, G_c(x). \tag{5.73}$$

We note that if we swap between the centers c and c', we naturally get $H_{c'c} = H_{cc'}^{-1}$.

In terms of the different components of the holonomies, we have from (5.73)

$$\begin{aligned}
G_{c'}(x) \;\; &= \;\; \ell_{c'x} u_{c'x} = (m_{c'c} h_{c'c}) \, (\ell_{cx} u_{cx}) \\
&= \;\; m_{c'c} \, (h_{c'c} \, \ell_{cx}) \, u_{cx} = m_{c'c} \, (h_{c'c} \triangleright \ell_{cx}) \, (h_{c'c} \triangleleft \ell_{cx}) \, u_{cx},
\end{aligned} \tag{5.74}$$

where $h_{c'c} \in \mathrm{SU}(2)$ and $m_{c'c} \in AN_2$ are the constant holonomies in the different sub-Lie group.

Let us focus on each contribution for each group component

$$\ell_{c'x} = m_{c'c}\,(h_{c'c} \triangleright \ell_{cx}) \quad \Leftrightarrow \quad m_{cc'}\ell_{c'x} = (h_{c'c} \triangleright \ell_{cx}), \tag{5.75}$$

$$u_{c'x} = (h_{c'c} \triangleleft \ell_{cx})\, u_{cx} \quad \Leftrightarrow \quad (h_{c'c} \triangleleft \ell_{cx}) = u_{c'x}u_{xc} \equiv h^{x}_{c'c}. \tag{5.76}$$

The relations (5.75), (5.76) provide us the meaning of the action of h on ℓ and respectively of ℓ on h.

$$(h_{c'c} \triangleright \ell_{cx}) \equiv m_{cc'}\ell_{c'x} \tag{5.77}$$

$$(h_{c'c} \triangleleft \ell_{cx}) \equiv u_{c'x}u_{xc}. \tag{5.78}$$

We also highlight the constraint which is an analog of (5.60).

$$h_{c'c}\ell_{cx} = (h_{c'c} \triangleright \ell_{cx})\,(h_{c'c} \triangleleft \ell_{cx}) \quad \Leftrightarrow \quad \ell_{cx}\,(h_{c'c} \triangleleft \ell_{cx})^{-1} = h_{cc'}\,(h_{c'c} \triangleright \ell_{cx}). \tag{5.79}$$

Before getting the statement of the main result, let us explain how we can actually anticipate it. For this we need to pick some specific point x and using the geometric interpretation of the actions together with the decomposition (5.79), we will get some constraint analog to (5.60) which will suggest the Heisenberg double structure we should consider.

Let us pick x to be either of v or v'. We have then (5.77) that becomes respectively for $x = v$ and $x = v'$,

$$m_{cc'}\ell_{c'v} = (h_{c'c} \triangleright \ell_{cv}), \quad m^{v'}_{cc'}\ell_{c'v'} = (h_{c'c} \triangleright \ell_{cv'}). \tag{5.80}$$

Putting them together and getting rid of $m_{cc'}$ we have that

$$\ell^{c'}_{vv'} \equiv \ell_{vc'}\ell_{c'v'} = (h_{c'c} \triangleright \ell_{vc})(h_{c'c} \triangleright \ell_{cv'}) = h_{c'c} \triangleright (\ell_{vc}\ell_{cv'}) \equiv (h_{c'c} \triangleright \ell^{c}_{vv'}). \tag{5.81}$$

This tells us what is the action of $h_{c'c}$ on $\ell^{c}_{vv'} = \ell_{vc}\ell_{cv'}$. We also have the geometric meaning of the action of ℓ on u, through (5.78).

$$(h_{c'c} \triangleleft \ell_{cv}) = u_{c'v}u_{vc} \equiv h^{v}_{c'c}, \quad (h_{c'c} \triangleleft \ell_{cv'}) = u_{c'v'}u_{v'c} \equiv h^{v'}_{c'c}. \tag{5.82}$$

Finally, we get the relations identifying the relevant phase space variables from (5.79), evaluated respectively at v and v'.

$$\ell_{cv}\,(h_{c'c} \triangleleft \ell_{cv})^{-1} = h_{cc'}\,(h_{c'c} \triangleright \ell_{cv}), \quad \ell_{cv'}\,(h_{c'c} \triangleleft \ell_{cv'})^{-1} = h_{cc'}\,(h_{c'c} \triangleright \ell_{cv'}). \tag{5.83}$$

Combined together by getting rid of $h_{cc'}$, we have that

$$\ell^c_{vv'}\,(h_{cc'} \triangleleft \ell_{cv'}) = (h_{cc'} \triangleleft \ell_{cv})\,(h_{c'c} \triangleright \ell^c_{vv'}) \tag{5.84}$$
$$\Leftrightarrow \ell^c_{vv'}\,\tilde{h}^{v'}_{cc'} = \tilde{h}^v_{cc'}\,\tilde{\ell}^{c'}_{vv'}, \tag{5.85}$$

where we used in the last equation the definitions we obtained in (5.81) and (5.82). Where the holonomies $\tilde{h}^v_{cc'}$ and $\tilde{h}^{v'}_{cc'}$ are located at the vertices v and v' of Γ^* respectively but they depend on the link $[cc']$ of Γ. Whiles the holonomies $\ell^c_{vv'}$ and $\tilde{\ell}^{c'}_{vv'}$ (taken to be the fluxes) sits on the nodes c and c' of Γ respectively but they depend on the edge $[vv']$ of Γ^*. The expression (5.85) provides therefore a candidate for the discretized symplectic form Ω^{LQG}. This leads to the following theorem:

Theorem 5.2.1. *Consider a triangulation Γ^* of a spatial manifold Σ and its corresponding dual Γ. Consider also the Lie group $\mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \bowtie AN_2$ such that $G = \ell u$, where $\ell \in AN_2$, $u \in \mathrm{SU}(2)$. Then the LQG symplectic structure associated to a single link $[cc']$ of Γ is*

$$\Omega^{LQG}_{cc'} = \frac{1}{2}\left(\langle \Delta\tilde{h}^v_{cc'} \wedge \Delta\ell^c_{vv'}\rangle + \langle \underline{\Delta}\tilde{h}^{v'}_{cc'} \wedge \underline{\Delta}\tilde{\ell}^{c'}_{vv'}\rangle\right), \tag{5.86}$$

and its generates the Poisson bracket of $\mathrm{SL}(2,\mathbb{C})$.

Proof. The symplectic form (5.66) associated to the LQG part is

$$\Omega^{LQG}_{cc'} \equiv \int_{[vv']\subset\partial c^*} \langle (\mathrm{d}\Delta G_c)_{|\mathfrak{an}_2} \wedge (\Delta G_c)_{|\mathfrak{su}(2)}\rangle + \int_{[vv']\subset\partial c'^*} \langle (\mathrm{d}\Delta G_{c'})_{|\mathfrak{an}_2} \wedge (\Delta G_{c'})_{|\mathfrak{su}(2)}\rangle. \tag{5.87}$$

Note that the edge $[vv']$ has a different orientation depending whether it is belonging to ∂c^* or $\partial c'^*$. Hence

$$\Omega^{LQG}_{cc'} \equiv \int_{[vv']\subset\partial c^*} \left(\langle (\mathrm{d}\Delta G_c)_{|\mathfrak{an}_2} \wedge (\Delta G_c)_{|\mathfrak{su}(2)}\rangle - \langle (\mathrm{d}\Delta G_{c'})_{|\mathfrak{an}_2} \wedge (\Delta G_{c'})_{|\mathfrak{su}(2)}\rangle\right). \tag{5.88}$$

With the decomposition $G = \ell u$, one obtains the expressions

$$\Delta G = \Delta\ell + \ell\Delta u \ell^{-1}, \qquad \mathrm{d}\Delta G = \mathrm{d}\Delta\ell + \ell(\mathrm{d}\Delta u + [\ell^{-1}\mathrm{d}\ell, \Delta u])\ell^{-1}. \tag{5.89}$$

By considering just a face c^*, it follows from equation (5.88) we have

$$\begin{aligned}
\int_{c^*} \delta e \wedge \delta\tilde{\omega} &= -\int_{\partial_{c^*}} \left(\ell_c^{-1}\mathrm{d}(\Delta\ell_c)\ell_c + [\ell_c^{-1}\mathrm{d}\ell_c, \Delta u_c] + \mathrm{d}\Delta u_c\right)_{|\mathfrak{an}} \wedge \left(\ell_c^{-1}(\Delta\ell_c)\ell_c + \Delta u_c\right)_{|\mathfrak{su}} \\
&= -\int_{\partial_{c^*}} \left(\ell_c^{-1}\mathrm{d}(\Delta\ell_c)\ell_c \wedge \Delta u_c + \frac{1}{2}\ell_c^{-1}\mathrm{d}\ell_c \wedge [\Delta u_c, \Delta u_c]\right).
\end{aligned} \tag{5.90}$$

In the process of obtaining the second equality, we used that the commutator of $\mathfrak{su}(2)$ on $\mathfrak{an}_2$ restricted to $\mathfrak{an}_2$, which is actually given by the $\mathfrak{su}_2$ structure constant. The contribution for the edge $[vv']$ from the face c'^* is then

$$\begin{aligned}\Omega^{LQG}_{cc'} &= -\int_{[vv']}\left(\ell^{-1}_{c'}\mathrm{d}(\Delta\ell_{c'})\ell_{c'}\wedge\Delta u_{c'}+\frac{1}{2}\ell^{-1}_{c'}\mathrm{d}\ell_{c'}\wedge[\Delta u_{c'},\Delta u_{c'}]\right)\\ &= -\int_{[vv']}\left(\tilde{\ell}^{-1}_{c}\mathrm{d}(\Delta\tilde{\ell}_{c})\tilde{\ell}_{c}\wedge(\Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c})\right.\\ &\qquad\left.+\frac{1}{2}\,\tilde{\ell}^{-1}_{c}\mathrm{d}\tilde{\ell}_{c}\wedge[(\Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c},\Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c})]\right),\end{aligned}\tag{5.91}$$

where to get from the first equality to the second equality we used the expression (C.4). Now we expand the different contributions of (5.91). Starting with the first term, and using the expression (C.7) to simplify

$$\begin{aligned}\int_{[vv']}\tilde{\ell}^{-1}_{c}\mathrm{d}(\Delta\tilde{\ell}_{c})\tilde{\ell}_{c}\wedge\Delta\tilde{h}_{c'c} &= \int_{[vv']}\left\{\tilde{h}_{c'c}\ell^{-1}_{c}(\mathrm{d}\Delta\ell_c)\ell_c\tilde{h}^{-1}_{c'c}-[\tilde{\ell}^{-1}_{c}\mathrm{d}\tilde{\ell}_{c},\Delta\tilde{h}_{c'c}])\right\}\wedge\Delta\tilde{h}_{c'c}\\ &= \int_{[vv']}\left\{\tilde{h}_{c'c}\ell^{-1}_{c}(\mathrm{d}\Delta\ell_c)\ell_c\tilde{h}^{-1}_{c'c}-[\tilde{h}_{c'c}(\ell^{-1}_{c}\mathrm{d}\ell_{c})\tilde{h}^{-1}_{c'c},\Delta\tilde{h}_{c'c}])\right\}\wedge\Delta\tilde{h}_{c'c}\\ &= -\int_{[vv']}\left\{\ell^{-1}_{c}(\mathrm{d}\Delta\ell_c)\ell_c\wedge\Delta\tilde{h}_{cc'}+(\ell^{-1}_{c}\mathrm{d}\ell_{c})[\Delta\tilde{h}_{cc'},\Delta\tilde{h}_{cc'}]\right\}.\end{aligned}\tag{5.92}$$

Using (C.7) again, the second term of (5.91) is simplified as follows

$$\begin{aligned}\int_{[vv']}\left(\tilde{\ell}^{-1}_{c}\mathrm{d}(\Delta\tilde{\ell}_{c})\tilde{\ell}_{c}\wedge\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c}\right) &= \int_{[vv']}\ell^{-1}_{c}(\mathrm{d}\Delta\ell_c)\ell_c\wedge\Delta u_c-\int_{[vv']}[\tilde{\ell}^{-1}_{c}\mathrm{d}\tilde{\ell}_{c},\Delta\tilde{h}_{c'c}])\wedge\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c}\\ &= \int_{[vv']}\ell^{-1}_{c}(\mathrm{d}\Delta\ell_c)\ell_c\wedge\Delta u_c+\int_{[vv']}\ell^{-1}_{c}\mathrm{d}\ell_{c}\,[\Delta\tilde{h}_{cc'},\Delta u_c].\end{aligned}\tag{5.93}$$

To get the last equation, we used that the commutator of $\mathfrak{su}(2)$ on $\mathfrak{an}_2$ restricted to $\mathfrak{an}_2$ is actually given by the $\mathfrak{su}(2)$ structure constant. The rest of the other contributions from (5.91) is simplified as follows

$$\begin{aligned}&\frac{1}{2}\int_{[vv']}\tilde{\ell}^{-1}_{c}\mathrm{d}\tilde{\ell}_{c}\wedge[(\Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c},\Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c})]\\ =\;&\frac{1}{2}\int_{[vv']}(\ell^{-1}_{c}\mathrm{d}\ell_{c})\wedge\tilde{h}^{-1}_{c'c}\left([\Delta\tilde{h}_{c'c},\Delta\tilde{h}_{c'c}]+2[\Delta\tilde{h}_{c'c},\tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c}]+\tilde{h}_{c'c}[\Delta u_c,\Delta u_c]\tilde{h}^{-1}_{c'c}\right)\tilde{h}_{c'c}\\ =\;&\frac{1}{2}\int_{[vv']}(\ell^{-1}_{c}\mathrm{d}\ell_{c})\wedge\left([\Delta\tilde{h}_{cc'},\Delta\tilde{h}_{cc'}]-2[\Delta\tilde{h}_{cc'},\Delta u_c]+[\Delta u_c,\Delta u_c]\right).\end{aligned}\tag{5.94}$$

Here we used the first expression of equation (C.5) to get from the first to the second equality. Putting the left over contributions from (5.92), (5.93) and (5.94) together we get that

$$\begin{aligned}\Omega^{LQG}_{cc'} &= \int_{[vv']} \left\{ \ell_c^{-1}(\mathrm{d}\Delta\ell_c)\ell_c \wedge \Delta\tilde{h}_{cc'} + \frac{1}{2}\,(\ell_c^{-1}\mathrm{d}\ell_c)[\Delta\tilde{h}_{cc'}, \Delta\tilde{h}_{cc'}] \right\} \\ &= \int_{[vv']} \left\{ \delta Y \wedge \Delta\tilde{h}_{cc'} + \frac{1}{2}\, Y\,[\Delta\tilde{h}_{cc'}, \Delta\tilde{h}_{cc'}] \right\}, \end{aligned} \tag{5.95}$$

where we set $Y = \ell_c^{-1}\mathrm{d}\ell_c$ to arrive at the second equation. With the help of the second expression of (C.10) we get

$$\begin{aligned}\Omega^{LQG}_{cc'} &= \int_{[vv']} \left\{ \delta Y \wedge \Delta\tilde{h}_{cc'} + \frac{1}{2}(\delta\overline{Y} \wedge \underline{\Delta}\tilde{h}_{cc'} - \delta Y \wedge \Delta\tilde{h}_{cc'}) \right\} \\ &= \frac{1}{2}\int_{[vv']} \left\{ \delta Y \wedge \Delta\tilde{h}_{cc'} + \delta\overline{Y} \wedge \underline{\Delta}\tilde{h}_{cc'} \right\} \\ &= \frac{1}{2}\int_{[vv']} \Big(\ell_c\delta Y\ell_c^{-1} \wedge (\Delta h_{cc'}) + \ell_c\delta Y\ell_c^{-1} \wedge h_{cc'}(\Delta\tilde{\ell}_c)h_{cc'}^{-1} \\ &\qquad - \tilde{\ell}_c\,\delta\overline{Y}\,\tilde{\ell}_c^{-1} \wedge h_{cc'}^{-1}(\Delta\ell_c)\,h_{cc'} + \tilde{\ell}_c\delta\overline{Y}\,\tilde{\ell}_c^{-1} \wedge \underline{\Delta}h_{cc'} \Big), \end{aligned} \tag{5.96}$$

where we again plugin equations (C.12) and (C.13) to get the last equation of (5.96). Now, using the second expressions in equations (C.8) and (C.9)

$$\begin{aligned}\Omega^{LQG}_{cc'} &= \frac{1}{2}\Big\{ [\int_{[vv']} \mathrm{d}(\Delta\ell_c)] \wedge (\Delta h_{cc'}) + \int_{[vv']} \mathrm{d}(\Delta\ell_c) \wedge h_{cc'}(\Delta\tilde{\ell}_c)h_{cc'}^{-1} \\ &\qquad - \int_{[vv']} \mathrm{d}(\Delta\tilde{\ell}_c) \wedge h_{cc'}^{-1}(\Delta\ell_c)\,h_{cc'} + [\int_{[vv']} \mathrm{d}(\Delta\tilde{\ell}_c)] \wedge \underline{\Delta}h_{cc'} \Big\} \\ &= \frac{1}{2}\Big\{ [\int_{[vv']} \mathrm{d}(\Delta\ell_c)] \wedge (\Delta h_{cc'}) + \int_{[vv']} \mathrm{d}(\Delta\ell_c) \wedge h_{cc'}(\Delta\tilde{\ell}_c)h_{cc'}^{-1} \\ &\qquad + \int_{[vv']} (\Delta\ell_c) \wedge h_{cc'}\mathrm{d}(\Delta\tilde{\ell}_c)h_{cc'}^{-1} + [\int_{[vv']} \mathrm{d}(\Delta\tilde{\ell}_c)] \wedge \underline{\Delta}h_{cc'} \Big\} \\ &= \frac{1}{2}\Big\{ [\int_{[vv']} \mathrm{d}(\Delta\ell_c)] \wedge (\Delta h_{cc'}) + \int_{[vv']} \mathrm{d}\left(\Delta\ell_c \wedge h_{cc'}\Delta\tilde{\ell}_c h_{cc'}^{-1}\right) \\ &\qquad + [\int_{[vv']} \mathrm{d}(\Delta\tilde{\ell}_c)] \wedge \underline{\Delta}h_{cc'} \Big\}. \end{aligned} \tag{5.97}$$

In the last equation we used the relation

$$\mathrm{d}\left(\Delta\ell_c \wedge\, h_{cc'}(\Delta\tilde{\ell}_c)h^{-1}_{cc'}\right) = \mathrm{d}(\Delta\ell_c)\wedge\, h_{cc'}(\Delta\tilde{\ell}_c)h^{-1}_{cc'} + \Delta\ell_c\wedge\,(h_{cc'}\mathrm{d}(\Delta\tilde{\ell}_c)h^{-1}_{cc'}). \tag{5.98}$$

Carrying out the integration in (5.97), we obtain

$$\begin{aligned}\Omega^{LQG}_{cc'} &= \frac{1}{2}\Big\{(\Delta\ell_{cv'}-\Delta\ell_{cv})\wedge(\Delta h_{cc'}) + (\Delta\tilde{\ell}_{cv'}-\Delta\tilde{\ell}_{cv})\wedge\underline{\Delta}h_{cc'} \\ &\qquad + \left(\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv'}h^{-1}_{cc'} - \Delta\ell_{cv}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'}\right)\Big\} \\ &= \frac{1}{2}\Big\{\ell_{cv}\,(\Delta\ell_{vv'})\,\ell^{-1}_{cv}\wedge(\Delta h_{cc'}) + \tilde{\ell}_{c'v}\,(\Delta\tilde{\ell}^{c'}_{vv'})\,\tilde{\ell}^{-1}_{c'v}\wedge\underline{\Delta}h_{cc'} \\ &\qquad + \left(\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv'}h^{-1}_{cc'} - \Delta\ell_{cv}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'}\right)\Big\},\end{aligned} \tag{5.99}$$

where we used in the last equation that $(\Delta\tilde{\ell}_{cv'}-\Delta\tilde{\ell}_{cv}) = \tilde{\ell}_{c'v}\,(\Delta\tilde{\ell}^{c'}_{vv'})\,\tilde{\ell}^{-1}_{c'v}$ since $\tilde{\ell}_c$ actually sits at c'. Simplifying

$$\begin{aligned}\Omega^{LQG}_{cc'} &= \frac{1}{2}\Big\{\,(\Delta\ell_{vv'})\,\wedge\ell^{-1}_{cv}(\Delta h_{cc'})\ell_{cv} + \left(\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv'}h^{-1}_{cc'} - \Delta\ell_{cv}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'}\right) \\ &\quad +\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'} - \Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'} + \tilde{\ell}_{c'v}\,(\Delta\tilde{\ell}_{vv'})\,\tilde{\ell}^{-1}_{c'v}\wedge\underline{\Delta}h_{cc'}\Big\} \\ &= \frac{1}{2}\Big\{\,(\Delta\ell_{vv'})\,\wedge\ell^{-1}_{cv}(\Delta h_{cc'})\ell_{cv} + \Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv'}h^{-1}_{cc'} + \ell_{cv}\Delta\ell_{vv'}\ell^{-1}_{cv}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'} \\ &\quad -\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'} + \tilde{\ell}_{c'v}\,(\Delta\tilde{\ell}_{vv'})\,\tilde{\ell}^{-1}_{c'v}\wedge\underline{\Delta}h_{cc'}\Big\} \\ &= \frac{1}{2}\Big\{\,\Delta\ell^{c}_{vv'})\,\wedge\Delta\tilde{h}^{v}_{cc'} + \Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv'}h^{-1}_{cc'} \\ &\quad -\Delta\ell_{cv'}\wedge h_{cc'}\Delta\tilde{\ell}_{cv}h^{-1}_{cc'} + \tilde{\ell}_{c'v}\,(\Delta\tilde{\ell}^{c'}_{vv'})\,\tilde{\ell}^{-1}_{c'v}\wedge\underline{\Delta}h_{cc'}\Big\} \\ &= \frac{1}{2}\Big\{\,(\Delta\ell^{c}_{vv'})\,\wedge\Delta\tilde{h}^{v}_{cc'} + \tilde{\ell}^{c'}_{v'v}\Delta\tilde{\ell}^{c'}_{vv'}\tilde{\ell}^{c'}_{vv'}\wedge\underline{\Delta}\tilde{h}^{v'}_{cc'}\Big\} \\ &= \frac{1}{2}\Big\{\,(\Delta\ell^{c}_{vv'})\,\wedge\Delta\tilde{h}^{v}_{cc'} + \underline{\Delta}\tilde{\ell}^{c'}_{vv'}\wedge\underline{\Delta}\tilde{h}^{v'}_{cc'}\Big\}.\end{aligned} \tag{5.100}$$

This gives exactly the expression (5.86) as expected. The above symplectic structure (5.100) is just for a single link $[cc']$ of the graph Γ, hence extending the construction to all links of the graph Γ, one gets the full phase space. $\square$

From the above symplectic structure (5.100), one can recover the symplectic potential (4.42) which is the case for $\Lambda = 0$. We now demonstrate how this can be achieved.

If $\Lambda = 0$, the basic LQG phase space $T^*\,\mathrm{SU}(2)$ is isomorphic to the Heisenberg double $\mathrm{ISU}(2) \sim \mathrm{SU}(2) \triangleright\!\!< \mathbb{R}^3$. In this case we have $\ell \in \mathbb{R}^3$, $h \in \mathrm{SU}(2)$ and $\tilde{\ell} = h^{-1}\ell h$, $\tilde{h} = h$. Since $\mathbb{R}^3$ is an abelian group, $\Delta\ell = \underline{\Delta}\ell$. We can use that[3] $\ell \sim 1 + X$, so that plugging these in (5.100), we recover the standard $T^*\,\mathrm{SU}(2)$ symplectic form.

$$\Omega = \frac{1}{2}\left(\langle \delta h h^{-1} \wedge \delta X\rangle + \langle h^{-1}\delta h \wedge \delta \tilde{X}\rangle\right). \tag{5.101}$$

In [71], the Kitaev model was formulated as a Hopf algebra gauge theory. Our construction then allows us to view 3d loop quantum gravity with topological defects as the classical version of such formulation.

5.2.3 Recovering the deformed dual LQG phase space

In the last section we guessed the deformed phase space of LQG starting from equation (5.66). To obtain the dual, we will use the contribution (5.67), this leads to the symplectic form associated to dual LQG as

$$\Omega^{LQG^*}_{cc'} \equiv \int_{[vv']\subset\partial c^*} \left(\langle (\mathrm{d}\Delta G_c)_{|\mathfrak{su}(2)} \wedge (\Delta G_c)_{|\mathfrak{an}_2}\rangle - \langle (\mathrm{d}\Delta G_{c'})_{|\mathfrak{su}(2)} \wedge (\Delta G_{c'})_{|\mathfrak{an}_2}\rangle\right), \tag{5.102}$$

where we have already taken into consideration the orientation of the edge $\mathfrak{e}$.

Let us now guessed the symplectic contribution Ω^{LQG^*}. In this case, we split the relation $G_{c'}(x) = H_{c'c}G_c(x)$ using the other Iwasawa decomposition

$$G_{c_i} = \tilde{u}_{c_i}\,\tilde{\ell}_{c_i}, \quad \tilde{u}_{c_i} \in \mathrm{SU}(2),\, \tilde{\ell}_{c_i} \in AN_2. \tag{5.103}$$

This becomes

$$\begin{aligned} G_{c'}(x) &= \tilde{u}_{c'x}\tilde{\ell}_{c'x} = (\tilde{h}_{c'c}\,\tilde{m}_{c'c})\,(\tilde{u}_{cx}\,\tilde{\ell}_{cx}) \\ &= \tilde{h}_{c'c}\,(\tilde{m}_{c'c}\,\tilde{u}_{cx})\,\tilde{\ell}_{cx} = \tilde{h}_{c'c}(\tilde{m}_{c'c} \triangleright \tilde{u}_{cx})(\tilde{m}_{c'c} \triangleleft \tilde{u}_{cx})\tilde{\ell}_{cx}, \end{aligned} \tag{5.104}$$

where $\tilde{h}_{c'c} \in \mathrm{SU}(2)$ and $\tilde{m}_{c'c} \in AN_2$ are constant holonomies. We deduce that we have

$$\tilde{u}_{c'x} = \tilde{h}_{c'c}(\tilde{m}_{c'c} \triangleright \tilde{u}_{cx}) \quad \Leftrightarrow \quad \tilde{m}_{c'c} \triangleright \tilde{u}_{cx} \equiv \tilde{h}_{cc'}\tilde{u}_{c'x} \tag{5.105}$$

$$\tilde{\ell}_{c'x} = (\tilde{m}_{c'c} \triangleleft \tilde{u}_{cx})\tilde{\ell}_{cx} \quad \Leftrightarrow \quad \tilde{m}_{c'c} \triangleleft \tilde{u}_{cx} \equiv \tilde{\ell}_{c'x}\tilde{\ell}_{xc} \equiv \tilde{\ell}^{x}_{c'c} \tag{5.106}$$

[3] We have also that $\ell_1\ell_2 = (1 + X_1)(1 + X_2) \equiv 1 + X_1 + X_2$.

which provides us the definition of the different actions m on u and u on m respectively

$$\tilde{m}_{c'c} \triangleright \tilde{u}_{cx} \equiv \tilde{h}_{cc'}\tilde{u}_{c'x} \quad (5.107)$$

$$\tilde{m}_{c'c} \triangleleft \tilde{u}_{cx} \equiv \tilde{\ell}_{c'x}\tilde{\ell}_{xc}. \quad (5.108)$$

Evaluating them at $x = v$ and $x = v'$, and putting them together we have the new analog of (5.60) as well as (5.83) and (5.85).

$$\tilde{m}_{c'c} \triangleright \tilde{h}^{c}_{vv'} = \tilde{u}_{vc'}\tilde{u}_{c'v'} = \tilde{h}^{c'}_{vv'}, \quad \tilde{m}_{c'c} \triangleleft \tilde{u}_{cv} = \tilde{m}^{v}_{c'c}, \quad \tilde{m}_{c'c} \triangleleft \tilde{u}_{cv'} = \tilde{m}^{v'}_{c'c} \quad (5.109)$$

$$\tilde{m}^{v}_{c'c}\tilde{h}^{c}_{vv'} = h^{c'}_{vv'}\tilde{m}^{v'}_{c'c} \Leftrightarrow \tilde{h}^{c}_{vv'}\,\tilde{m}^{v'}_{cc'} = \tilde{m}^{v}_{cc'}\,h^{c'}_{vv'}, \quad (5.110)$$

where the holonomies $h^{c}_{vv'}$ and $\tilde{h}^{c'}_{vv'}$ are located at the nodes c and c' of Γ respectively but they depend on the edge $[vv']$ of Γ^*. The holonomies $\tilde{m}^{v}_{cc'}$ and $\tilde{m}^{v'}_{cc'}$ sits on the vertices v and v' of Γ^* respectively but they depend on the link $[cc']$ of Γ. The expression (5.110) provides therefore a candidate for the discretized symplectic form $\Omega^{LQG^*}_{cc'}$. This leads to the following theorem:

Theorem 5.2.2. *Consider a triangulation Γ^* of a spatial manifold Σ and its corresponding dual Γ. Consider also the Lie group $\mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \bowtie AN_2$ such that $G = \tilde{u}\tilde{\ell}$, where $\tilde{\ell} \in AN_2$, $\tilde{u} \in \mathrm{SU}(2)$. Then the dual LQG symplectic structure associated to a single link $[cc']$ of Γ is*

$$\Omega^{LQG^*}_{cc'} = \frac{1}{2}\left(\langle \Delta\tilde{m}^{v}_{cc'} \wedge \Delta\tilde{h}^{c}_{vv'}\rangle + \langle \underline{\Delta}\tilde{m}^{v'}_{cc'} \wedge \underline{\Delta}h^{c'}_{vv'}\rangle\right), \quad (5.111)$$

and its generates the Poisson bracket of $\mathrm{SL}(2,\mathbb{C})$.

The proof of the above theorem is contained in Appendix (C.3).

5.3 Recovering quantum group spin network

In section 5.1.2, we recovered the Heisenberg double associated to each link of the graph Γ. This is the basic building block that was considered in [4], which led ultimately to the appearance of quantum group spin networks as well as the Turaev-Viro model.

For completeness, let us review the model [4] and how the quantum group spin networks appeared [51]. After reviewing the classical model, we will discuss how it relates to our result in section 5.2.2

For simplicity we will focus on the Euclidean case, with $\Lambda < 0$. Hence the Heisenberg double we have obtained in (5.86) is characterized by $\mathrm{SL}(2,\mathbb{C})$.

These $\mathrm{SL}(2,\mathbb{C})$ elements are associated to the dual graph Γ to build a new deformed LQG phase space. This review will help us link both the lattice gauge theory and our discrete theory. We then review the quantization of the model [4] carried out in [51]. This quantization leads to the Turaev-Viro spin foam model.

Before we proceed with the construction of the deformed lattice gauge theory, we consider some mathematical structures. Recall we considered $\mathrm{SL}(2,\mathbb{C})$ as a direct crossproduct of $\mathrm{SU}(2) \bowtie AN_2$ given by the Iwasawa decomposition

$$G = \ell u = \tilde{u}\tilde{\ell}, \quad u, \tilde{u} \in \mathrm{SU}(2), \;\; \ell, \tilde{\ell} \in AN_2. \tag{5.112}$$

If we focus on the left Iwasawa decomposition: ℓu, the different elements expressed in a matrix form are

$$u = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \quad \ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R}_+, z \in \mathbb{C} \tag{5.113}$$

and since z is complex, we consider its conjugate $\overline{z}$ such that $l = (\ell^{\dagger})^{-1}$ where

$$l = \begin{pmatrix} \lambda^{-1} & -\overline{z} \\ 0 & \lambda \end{pmatrix}. \tag{5.114}$$

The configuration variable is u with ℓ the momentum variable. In the case of the right Iwasawa decomposition: $\tilde{u}\tilde{\ell}$, the elements expressed in matrix form are

$$\tilde{u} = \begin{pmatrix} \tilde{\alpha} & -\overline{\tilde{\beta}} \\ \tilde{\beta} & \overline{\tilde{\alpha}} \end{pmatrix}, \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix}, \quad \tilde{\lambda} \in \mathbb{R}_+, \tilde{z} \in \mathbb{C}, \tag{5.115}$$

likewise we have $\tilde{l} = (\tilde{\ell}^{\dagger})^{-1}$ which takes into account the conjugate of $\tilde{z}$. The Poisson structures for both decompositions can be found in [4] and we refer the reader to the said reference for a full algebraic derivation of $\mathrm{SL}(2,\mathbb{C})$ as a phase space.

Let us now recall the model construction presented in [4].

Let us specify a graph Γ embedded in a 2d manifold Σ. Following the standard lattice gauge theory techniques, one usually decorates the edge of a graph with the elements of the phase space. For instance in LQG with $\Lambda = 0$, for each edge l, we have the phase space $T^*\,\mathrm{SU}(2)$, which contains the pair $(h_l, X_l) \in G \times \mathfrak{su}^*(2)$, where h_l is the holonomy on the edge and X_l the flux . If we consider instead $\mathrm{SL}(2,\mathbb{C})$ phase space where an element can be expressed either as ℓu or as $\tilde{u}\tilde{\ell}$, then we will need to replace an edge with a box (as illustrated in Fig 5.2) to build the lattice to implement or accommodate both decompositions. The box is seen as the fattening of an edge into a ribbon with orientations on its boundary. A ribbon graph is a 2d cell decomposition of an oriented manifold with

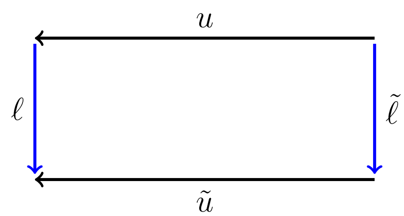


Figure 5.2: A ribbon graph oriented such that $\ell u = \tilde{u}\tilde{\ell}$ is satisfied.

the black edges belonging to the standard graph Γ and the blue edges belong to the dual graph Γ^*. The black and blue edges together are called ribbon edges, and are incident on the ribbon vertices. Due to the Iwasawa decomposition, the ribbon graph has SU(2) and AN_2 holonomies associated to the relevant sides. The SU(2) elements $u, \tilde{u}$ are on the side of the edges and the AN_2 elements $\ell, \tilde{\ell}$ lie[4] where the ribbon edges are glued to the ribbon vertices. The ribbon geometrically encodes the Iwasawa decomposition $\ell u = \tilde{u}\tilde{\ell}$.

To define the set of first class constraints on the ribbon graphs, we need to know how we can glue two ribbon edges together at a vertex. The only allowed operation of a ribbon graph is rotation in the plane. This means that the product of SU(2) elements along the black edges only contain some u and/or $\tilde{u}^{-1}$ or some u^{-1} and/or $\tilde{u}$ depending on the face orientation. Likewise the product of the AN_2 elements along the blue edges only contains ℓ and/or $\tilde{\ell}^{-1}$ or some ℓ^{-1} and/or $\tilde{\ell}$ depending on the orientation. With all these possible combinations of the products, it turns out there are only four possible ways to glue two ribbon edges at a vertex. See Fig 5.3 for the computation of one such product.

With the four possible ways of computing the product of two ribbon edges, consider the graph on the left side of Fig 5.4 and fatten it to get the corresponding ribbon graph on the right. Looking at the ribbon graph of Fig. 5.4, there exists two closed ribbon vertices at the top. Taking the products around these vertices gives $\ell_1\ell_2\tilde{\ell}_7^{-1}$ (for the left closed vertex) and $\tilde{\ell}_6^{-1}\ell_5\tilde{\ell}_1^{-1}$ (for the right closed vertex). Generalizing these products at the closed ribbon

[4]In the LQG context the variables ℓ and $\tilde{\ell}$ are the fluxes.

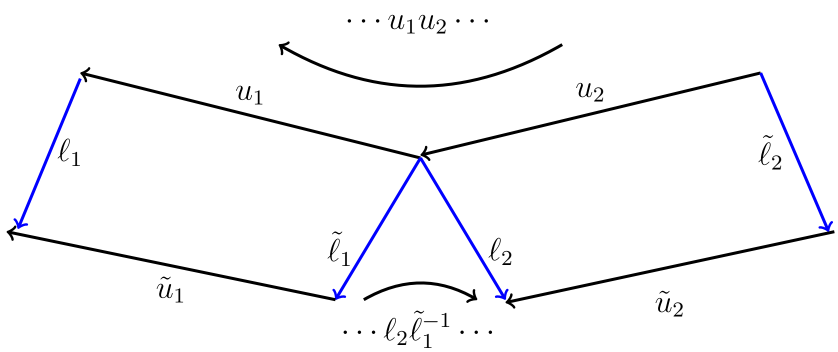


Figure 5.3: A product of two ribbon edges. In computing this product, the orientation matters but one can start with any edge. Changing the orientations of the faces will result in the other three products.

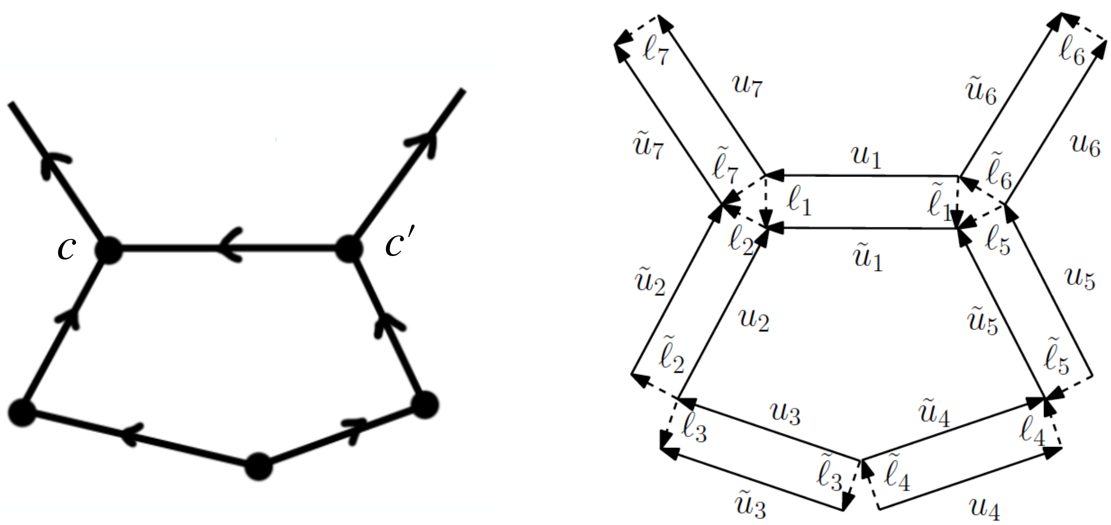


Figure 5.4: Fattening the graph on the left results in the ribbon graph on the right. The edges $1, 2, 3, 4, 5$ of the ribbon graph form a closed face with constraint $\tilde{u}_4 \tilde{u}_5 \tilde{u}_1 u_2^{-1} u_3^{-1} = 1$. There are two closed vertices with constraints $\ell_1 \ell_2 \tilde{\ell}_7^{-1} = 1$ and $\tilde{\ell}_6^{-1} \ell_5 \tilde{\ell}_1^{-1} = 1$. Diagram taken from [4].

vertices to N_v number of edges around the vertex v, gives the constraint

$$\mathfrak{G}_v \equiv \prod_{i=1}^{N_v} \mathfrak{L}_i = 1, \tag{5.116}$$

with $\mathfrak{L}_i = \ell_i$ or $\mathfrak{L}_i = \tilde{\ell}_i^{-1}$ as AN_2 elements on each edge. Back to Fig. 5.4, there exist only one closed face with SU(2) elements labels and their product gives $\tilde{u}_4\tilde{u}_5\tilde{u}_1 u_2^{-1} u_3^{-1}$. If this is generalized to N_f number of edges around the face f, it gives the constraint

$$\mathfrak{C}_f \equiv \prod_{i=1}^{N_f} \mathfrak{U}_i = 1, \tag{5.117}$$

where $\mathfrak{U}_i = u_i$ or $\mathfrak{U}_i = \tilde{u}_i^{-1}$ are SU(2) elements on the edges around the face. It was shown in [4], that $\mathfrak{G}_v$ and $\mathfrak{C}_f$ are the respective Gauss and flatness constraints and they generate the SU(2) transformations and deformed translations.

Before going into the quantization of the above lattice model, let us see how it compares with our discrete model introduce earlier on in this chapter. We recall the symplectic form (5.86) for the deformed LQG phase space

$$\Omega^{LQG}_{cc'} = \frac{1}{2}\left(\langle \Delta h^v_{cc'} \wedge \Delta \ell^c_{vv'}\rangle + \langle \underline{\Delta} h^{v'}_{cc'} \wedge \underline{\Delta} \ell^{c'}_{vv'}\rangle\right). \tag{5.118}$$

From the above expression, notice the AN_2 holonomies $\ell^c_{vv'}$ and $\ell^{c'}_{vv'}$ are both linked to the edge $[vv']$ but are situated at the centers c and c' respectively. Similarly for the SU(2) holonomies $h^v_{cc'}$ and $h^{v'}_{cc'}$, they are both associated to the link $[cc']$ and located at the vertices v and v' respectively. Putting this together, we are naturally led to the ribbon model reviewed above. The ribbon structure encodes again the different Iwasawa decompositions

$$\ell^c_{vv'}\, h^{v'}_{cc'} = h^v_{cc'}\, \ell^{c'}_{vv'} \Leftrightarrow \ell u = \tilde{u}\tilde{\ell}. \tag{5.119}$$

Fig 5.5 visualizes this concept graphically.

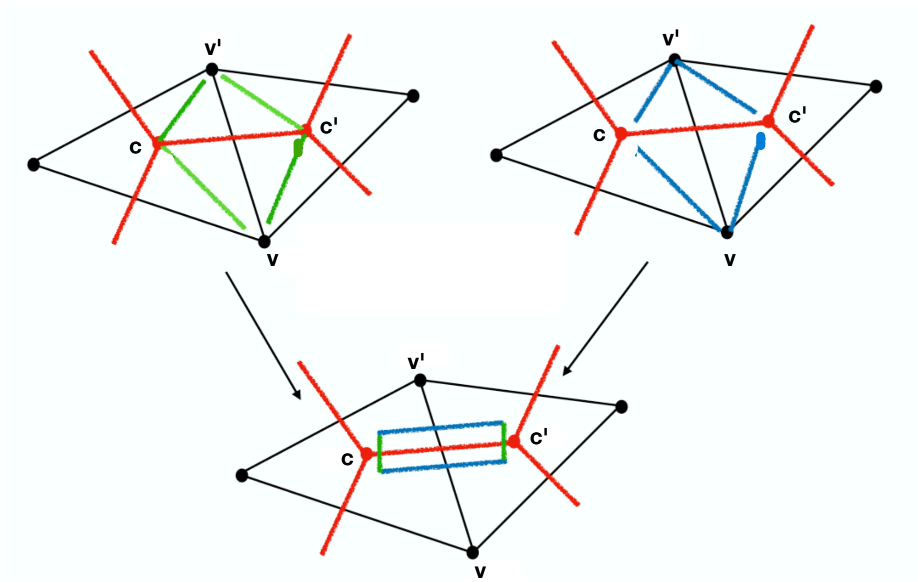


Figure 5.5: The top left diagram shows where the holonomies $\ell^{c}_{vv'}$ and $\ell^{c'}_{vv'}$ linked to the edge $[vv']$ depicted in green. The top right diagram shows where the holonomies $h^{v}_{cc'}$ and $h^{v'}_{cc'}$ associated to the link $[cc']$ indicated in blue. The bottom diagram is what we get by putting together the two top diagrams, leading then to the notion of ribbons.

Regarding the quantization of the ribbon model, we will only focus on the Gauss constraint which will generate the deformed spin networks. The Gauss constraint (5.117) is defined in terms of the momentum variables $\ell_i, \tilde{\ell}_i^{-1}$. To quantize the Gauss constraint is equivalent to quantizing the algebra of functions on momentum space, which can be viewed as the algebra[5] $k(AN_2)$ generated by the matrix elements of AN_2 given in equation (5.113). First we quantize the momentum variables ℓ and $(\ell^{\dagger})^{-1}$, and the quantization rule for the matrix elements of AN_2 are given as

$$\lambda \to K, \quad \lambda^{-1} \to K^{-1}, \quad z \to \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) J_+, \quad \overline{z} \to -\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) J_-, \tag{5.120}$$

where $K, J_\pm$ are generators and $q = \exp(\hbar\kappa)$ is a deformation parameter. Applying the above quantization rule, the quantum matrices of AN_2 (5.113) are

$$\widehat{\ell} = \begin{pmatrix} K & 0 \\ \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) J_+ & K^{-1} \end{pmatrix}, \quad (\widehat{\ell^{\dagger}})^{-1} = \begin{pmatrix} K^{-1} & -\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) J_- \\ 0 & K \end{pmatrix}. \tag{5.121}$$

[5] Here k is a field of complex numbers.

The quantization of the Poisson brackets satisfied by the matrix elements of AN_2 leads to the commutation relations of the generators $K, J_\pm$

$$KJ_+K^{-1} = q^{\frac{1}{2}} J_+, \quad KJ_-K^{-1} = q^{-\frac{1}{2}} J_-, \quad [J_+, J_-] = \frac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{5.122}$$

which are the commutation relations of the quantum group.

As algebra of functions on momentum space, $k(AN_2)$ has a coalgbera structure (see Appendix D for the definition) given by the coproduct Δ and the antipode S defined respectively

$$\Delta \ell_{ij} = \ell_{ik} \otimes \ell_{kj}, \quad S(\ell_{ij}) = \ell_{ij}^{-1}. \tag{5.123}$$

Quantization of the above coproduct leads to a total quantum angular momentum, which in matrix form is given by

$$\Delta \widehat{\ell} = \begin{pmatrix} K \otimes K & 0 \\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(J_+ \otimes K + K^{-1} \otimes J_+) & K^{-1} \otimes K^{-1} \end{pmatrix} \tag{5.124}$$

and similarly we obtain for

$$\Delta(\widehat{\ell}^\dagger)^{-1} = \begin{pmatrix} K \otimes K & -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(J_- \otimes K + K^{-1} \otimes J_-) \\ 0 & K^{-1} \otimes K^{-1} \end{pmatrix}. \tag{5.125}$$

These coproducts are then the quantum analogue of the Gauss constraint. From the above two coproducts, one can deduce the coproducts of the generators $K, J_\pm$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes k^{\pm 1}, \quad \Delta(J_\pm) = J_\pm \otimes K + K^{-1} \otimes J_\pm \tag{5.126}$$

which are those of $\mathcal{U}_\mathrm{q}(\mathrm{SU}(2))$. In assuming a quantization map of the form $\ell^{-1} \longrightarrow S(\widehat{\ell})$, one can readily obtain the antipodes of $K, J_\pm$ by solving the equation $\widehat{\ell} S(\widehat{\ell}) = 1$

$$S(K^{\pm 1}) = K^{\mp 1}, \quad S(J_\pm) = -KJ_\pm K^{-1} = -q^{\pm \frac{1}{2}} J_\pm \tag{5.127}$$

and these are the antipodes of $\mathcal{U}_\mathrm{q}(\mathfrak{su}(2))$. Equations (5.122), (5.126) and (5.127) together constitute the the quantum group structure of $\mathcal{U}_\mathrm{q}(\mathfrak{su}(2))$, which is the algebra of quantum momentum observables. Hence we see that qauntizing $k(AN_2)$ leads to $\mathcal{U}_\mathrm{q}(\mathfrak{su}(2))$.

Using these quantum observables we are going to recall how the kinematical Hilbert space is spanned by $\mathcal{U}_\mathrm{q}(\mathfrak{su}(2))$ spin networks. Let us note that there are different ways to compute the Gauss constraint depending on the orientation of the edges with respect to the vertex in consideration. For our purpose, we shall restrict to a 3-valent graph with all edges incident on the vertex v. The total Hilbert space for such a 3-valent graph is

$$\mathcal{H}_v = \bigotimes_{l=0}^{3} \mathcal{H}_{j_l}, \tag{5.128}$$

where $\mathcal{H}_{j_l}$ is an irreducible representation of $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ to each edge l. We note here by choosing q to be real, the representations of $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ are also classified by j as the representations of $\mathfrak{su}(2)$, hence the reason for j_l at the far right of (5.128). A general state of (5.128) is denoted by

$$i_{j_1j_2j_3} = \sum_{m_i} i^{j_1j_2j_3}_{m_1m_2m_3} |j_1m_1, j_2m_2, j_3m_3\rangle. \tag{5.129}$$

The Gauss law is given as $\ell_1\ell_2\ell_3 = 1$ and the corresponding quantum version is $\Delta^2\widehat{\ell} = 1 \otimes 1 \otimes 1$, which in terms of the generators translates as

$$\begin{aligned} \Delta^2(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1} \otimes K^{\pm 1} = 1, \\ \Delta^2(J_\pm) &= J_\pm \otimes K \otimes K + K^{-1} \otimes J_\pm \otimes K + K^{-1} \otimes K^{-1} \otimes J_\pm = 0. \end{aligned} \tag{5.130}$$

Looking for the states $i_{j_1j_2j_3}$ that solves the above constraints i.e., $(\Delta^2 K^{\pm 1}) i_{j_1j_2j_3} = i_{j_1j_2j_3}$ and $(\Delta^2 J_\pm) i_{j_1j_2j_3} = 0$, one obtains the 3-valent intertwiners in terms of the $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ Clebsh-Gordon coefficients

$$i_{j_1j_2j_3} = \sum_{m_i} (-1)^{j_3-m_3} q^{-\frac{m_3}{2}} C^{j_1\ j_2\ j_3}_{m_1 m_2 -m_3} |j_1m_1, j_2m_2, j_3m_3\rangle. \tag{5.131}$$

Contracting the Clebsh-Gordon coefficients together according to the combinatorics specified by the graph Γ, we recover the $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ spin network associated to the graph Γ.

This concludes the derivation of the quantum group structure starting from the continuum BF action with a cosmological constant, which was a long standing problem up to now. Note that a very similar construction should work for the different signatures and signs of Λ, except for the Euclidian case and $\Lambda > 0$. In this case the change of variables leads to a complexification of the variables and more care is required. It is in fact well known that one should recover in this case $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ with q root of unity, which is much more tricky to construct than the q real case. We leave this for further studies.

Chapter 6

Hopf algbras as symmetry and representation tools

Symmetries play an important role in physics such that knowledge of their presence simplifies certain physical or mathematical problems. Symmetries are a reflection of the invariance properties of physical objects. In classical physics symmetries are usually captured or played by Lie groups. The concept of Lie groups is viewed as a set of transformations which come with some multiplication and inverse maps satisfying certain axioms. However in quantum physics and more specifically in three dimension, symmetries are better captured by Hopf algberas or quantum groups, a generalization of the idea of Lie groups.

As generalizations of Lie groups, elements in Hopf algebras do not all have an inverse. Instead these transformations come with a weaker structure called the antipode. Hopf algebras are the mathematical structures underlying the symmetries of systems such as quantum inverse scattering, low-dimensional topology and exactly solvable lattice models (such as the Kitaev models). Certain properties possessed by Hopf algebras make them very natural for such systems. One such property is the comultiplication, which can been seen as the dual of multiplication. Suppose we denote $\mathcal{A}$ as the algebra of observables for a one particle state with momentum $\underline{P}$, then for a two particle state, the expression[1] $\Delta\underline{P} = \underline{P} \otimes 1 + 1 \otimes \underline{P}$ is the representation on tensor product and this is made possible by the coproduct. Another relevant property of Hopf algebras is duality or self-duality. Duality is a concept of having two mutually valid but distinct physical descriptions of a system. This will be a constant theme throughout this chapter and the next chapter.

[1]This could be more complicated and can be extended to a finite number of tensor products of representation.

In this chapter, relevant concepts necessary for the construction of our new lattice model in the next chapter will be reviewed: concepts such as double crossproducts, bicrossproducts, and the machinery of semi-dualization will be presented. We will mostly follow the literature in [108, 53] and for more on Hopf algebras we direct the reader to references therein.

6.1 Notation and Conventions

We follow the theory and conventions for Hopf algebras in the book [53]. Unless otherwise specified, we work over a field k of characteristic zero. A Hopf algebra or quantum group[2] H is an algebra and a coalgebra, with a linear coproduct $\Delta : H \to H \otimes H$ which is an algebra homomorphism and satisfies the coassociativity condition $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. We use Sweedler notation for the coproduct so that for all $h \in H$, $\Delta(h) = h_{(1)} \otimes h_{(2)} = h^{(1)} \otimes h^{(2)}$. There is also a counit $\epsilon : H \to k$ and an antipode $S : H \to H$ satisfying the following properties

$$\begin{aligned} (Sh_{(1)})h_{(2)} &= h_{(1)}Sh_{(2)} = \epsilon(h), \quad \forall h \in H, && (6.1)\\ S(hg) &= S(g)S(h), \quad S(1_H) = 1_H, && (6.2)\\ Sh_{(1)} \otimes Sh_{(2)} &= (Sh)_{(2)} \otimes (Sh)_{(1)}, \quad \epsilon Sh = \epsilon h. && (6.3)\end{aligned}$$

If H is finite-dimensional, then S^{-1} exist. We denote by $H^{\otimes n}$, $n \in \mathbb{N}$ the n-fold tensor product of H. The composition of n coproducts is the map $\Delta^{(n)} : H \to H^{\otimes(n+1)}$ defined by $\Delta^{(n)}(h) = h_{(1)} \otimes h_{(2)} \otimes ... \otimes h(n+1)$. This is well defined since the coproduct is coassociativity. We denote by H^* the dual Hopf algebra with dual pairing given by the non-degenerate bilinear map $\langle \, , \, \rangle$ and H^{cop}, H^{op} denote taking the opposite coproduct or opposite product in H.

6.2 Bicrossproduct Hopf algebras

In this section we briefly review the features of the bicrossproduct construction which are required in the current application and refer to the book [53] for a comprehensive discussion of these quantum groups.

[2] A comprehensive definition, representation and examples of Hopf algebras can be found in Appendix D.1.

6.2.1 Double crossproducts and semidualization

Consider a Hopf algebra H which factorizes into two sub-Hopf algebras H_1, H_2 and built on the vector space $H_1 \otimes H_2$. Factorization here implies an isomorphism of linear spaces given by the map $H_1 \otimes H_2 \to H$. This gives rise to the actions $\triangleright : H_2 \otimes H_1 \to H_1$ and $\triangleleft : H_2 \otimes H_1 \to H_2$ of each Hopf algebra on the vector space of the other and defines a double crossproduct Hopf algebra $H_1 \bowtie H_2$. The actions enter the definition of the product on $H_1 \otimes H_2$ as $(1 \otimes a).(h \otimes 1) = a_{(1)} \triangleright h_{(1)} \otimes a_{(2)} \triangleleft h_{(2)}$. The coproduct of $H_1 \bowtie H_2$ is given by the tensor coproduct. That is if the coproduct of H_1 and H_2 are $\Delta(h) = h_{(1)} \otimes h_{(2)}$ and $\Delta(a) = a_{(1)} \otimes a_{(2)}$ respectively then the coproduct of $H_1 \bowtie H_2$ is

$$\Delta(h \otimes a) = h_{(1)} \otimes a_{(1)} \otimes h_{(2)} \otimes a_{(2)}. \tag{6.4}$$

Furthermore, there is a canonical right action of $H_1 \bowtie H_2$ on the vector space H_2

$$b \triangleleft (h \otimes a) = (b \triangleleft h).a, \quad \forall b \in H_2, \quad h \otimes a \in H_1 \otimes H_2 \tag{6.5}$$

which respect the coalgebra structure of H_2. There is then a covariant left action of $H_1 \bowtie H_2$ on H_2^* as a module algebra. In this module algebra, what happens is that, H_1 acts on H_2^* by dualising the right action $\triangleleft$ on H_2

$$\langle h \triangleright \phi, a \rangle = \langle \phi, a \triangleleft h \rangle, \quad \forall \phi \in H_2^*, \quad a \in H_2, \quad h \in H_1 \tag{6.6}$$

and H_2 acts on H_2^* by the co-regular action

$$a \triangleright \phi = \langle a, \phi_{(1)} \rangle \phi_{(2)}. \tag{6.7}$$

This leads to a left covariant system $(H_1 \bowtie H_2, H_2^*)$ with $H_2^* \rtimes (H_1 \bowtie H_2)$ as its associated left cross product algebra.

The semidual of the double cross product is obtained by dualising half of the match pair data and gives a bicrossproduct Hopf algebra. More precisely, replacing H_2 with H_2^* gives a bicrossproduct Hopf algebra $H_2^* \blacktriangleright\!\!\triangleleft H_1$, which then acts covariantly on H_2 from the right as an algebra. The covariant action of the bicrossproduct gives the right covariant system $(H_2^* \blacktriangleright\!\!\triangleleft H_1, H_2)$ with $(H_2^* \blacktriangleright\!\!\triangleleft H_1) \ltimes H_2$ as its associated right cross product algebra. It is shown in [86] that the algebras $H_2^* \rtimes (H_1 \bowtie H_2)$ and $(H_2^* \blacktriangleright\!\!\triangleleft H_1) \ltimes H_2$ are the same when built in the vector space $H_2^* \otimes H_1 \otimes H_2$ but have different physical interpretation.

The explicit details are given in [53]. See also [111] for a recent account. The left action $\triangleright : H_1 \otimes H_2^* \to H_2^*$ of H_1 on H_2^* and a right coaction $\Delta_R : H_1 \to H_1 \otimes H_2^*$ of H_2^* on H_1 are defined by

$$(h \triangleright \phi)(a) := \phi(a \triangleleft h), \quad \phi \in H_2^*, \quad a \in H_2 \quad h \in H_1$$

$$h^0\langle h^1, a\rangle = a\triangleright h, \quad h\in H_1, \quad a\in H_2, \quad \Delta_R h = h^{(\bar 1)}\otimes h^{(\bar 2)}\in H_1\otimes H_2^*.$$

These define the bicrosspropduct $H_2^*{\blacktriangleright\!\!\triangleleft} H_1$ by a left cross product $H_2^*{>\!\!\triangleleft} H_1$ as an algebra and a right cross coproduct $H_2^*{\blacktriangleright\!\!<} H_1$ as coalgebra:

$$(\phi\otimes h)(\psi\otimes g) = \phi(h_{(1)}\triangleright\psi)\otimes h_{(2)}g, \qquad h\in H_1, \quad \phi,\psi\in H_2^*, \tag{6.8}$$

$$\Delta(\phi\otimes h) = (\phi_{(1)}\otimes h^{(\bar 1)}{}_{(1)})\otimes(\phi_{(2)}h^{(\bar 2)}{}_{(1)}\otimes h_{(2)}), \tag{6.9}$$

$$S(\phi\otimes h) = (1\otimes S(h^{(\bar 1)}))(S(\phi h^{(\bar 2)})\otimes 1). \tag{6.10}$$

The canonical right action of $H_2^*{\blacktriangleright\!\!\triangleleft} H_1$ on H_2 is

$$a\triangleleft(\phi\otimes h) = a_{(2)}\triangleleft h\langle\phi, a_{(1)}\rangle, \quad \forall h\in H_1, \quad a\in H_2, \quad \phi\in H_2^*. \tag{6.11}$$

The semidualisation described above is known as the B-model as noted in [86]. One could also have a different bicrossproduct model via semidualisation where we dualise H_1 to obtain $H_2{\triangleright\!\!\blacktriangleleft} H_1^*$ acting on the left on H_1 while $H_1{\bowtie} H_2$ acts on the right on H_1^*. This is the A-model. We refer to [53, 86] for more details.

6.2.2 Quantum double and Mirror bicrossproduct quantum group

A well known example of the double cross product is the Drinfeld quantum double $D(H) = H{\bowtie} H^{*\mathrm{op}}$, built on $H\otimes H^*$ as a vector space[3]. It is given via a double semidirect product by a mutual left coadjoint action of $H^{*\mathrm{op}}$ on H and a right coadjoint action of H on $H^{*\mathrm{op}}$ which are given respectively by

$$\phi\triangleright h = h_{(2)}\langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(2)}\rangle, \quad \phi\triangleleft h = \phi_{(2)}\langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(2)}, \phi_{(3)}\rangle, \quad h\in H, \ \ \phi\in H^{*\mathrm{op}}. \tag{6.12}$$

This product is given by

$$(h\otimes\phi)(g\otimes\psi) = hg_{(2)}\otimes\psi\phi_{(2)}\langle g_{(1)}, \phi_{(1)}\rangle\langle Sg_{(3)}, \phi_{(3)}\rangle, \quad h, g\in H, \ \ \phi,\psi\in H^{*\mathrm{op}}. \tag{6.13}$$

Here, $1\otimes H^{*\mathrm{op}}$ and $H\otimes 1$ appear as subalgebras but with mutual commutation relation fully determined by

$$\phi g := (1\otimes\phi)(g\otimes 1) = g_{(2)}\otimes\phi_{(2)}\langle g_{(1)}, \phi_{(1)}\rangle\langle Sg_{(3)}, \phi_{(3)}\rangle, \tag{6.14}$$

[3]Note that in [59], $\bowtie$ is refered to as bicrossproduct. However, we refer to it as a double crossproduct built from the two semidirect product $>\!\!\triangleleft$ and $\triangleright\!\!<$ put together. The bicrossproduct is ${\blacktriangleright\!\!\triangleleft}$ (or ${\triangleright\!\!\blacktriangleleft}$) and is the semidual of $\bowtie$ as explained in section 6.2.1.

where the identifications $\phi \to 1_H \otimes \phi$ and $g \to g \otimes 1_{H^{*\mathrm{op}}}$ are algebra morphisms. The coproduct is the tensor product coproduct of the individual Hopf algebras H and $H^{*\mathrm{op}}$,

$$\Delta(h \otimes \phi) = h_{(1)} \otimes \phi_{(1)} \otimes h_{(2)} \otimes \phi_{(2)}. \tag{6.15}$$

Following the general construction of double crossproduct above, $D(H)$ canonically acts on $(H^{*\mathrm{op}})^* = H^{\mathrm{cop}}$ from the left as an algebra and we have $(D(H), H^{\mathrm{cop}})$ as a left covariant system.

It is easy to see from the previous section that one can semidualise the quantum double $D(H)$ and get the mirror bicrossproduct $M(H) = H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H$. The left action of H on H^{cop} and the right coaction of H^{cop} on H are given respectively as

$$h \triangleright a = h_{(1)} a S h_{(2)}, \qquad \Delta_R h = h_{(2)} \otimes h_{(1)} S h_{(3)}. \tag{6.16}$$

The algebra is

$$(a \otimes h)(b \otimes g) = a(h_{(1)} b S h_{(2)}) \otimes h_{(3)} g, \quad h, g \in H, \quad a, b \in H^{\mathrm{cop}}. \tag{6.17}$$

Here, $H^{\mathrm{cop}} \otimes 1$ and $1 \otimes H$ appear as subalgebras but with mutual commutation relation fully determined by

$$hb := (1 \otimes h)(b \otimes 1) = (h_{(1)} b S h_{(2)}) h_{(3)}, \tag{6.18}$$

where the identification $h \to 1_{H^{\mathrm{cop}}} \otimes h$ and $b \to b \otimes 1_H$ are algebra morphisms. The coproduct and antipode are respectively

$$\Delta(a \otimes h) = a_{(2)} \otimes h_{(2)} \otimes a_{(1)} h_{(1)} S h_{(3)} \otimes h_{(4)}, \tag{6.19}$$

$$S(a \otimes h) = (1 \otimes S h_{(2)})(S(a h_{(1)} S h_{(3)}) \otimes 1). \tag{6.20}$$

The Hopf algebra $H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H$ acts covariantly on $H^{*\mathrm{op}}$ from the right according to

$$\phi \triangleleft (a \otimes h) = \langle a h_{(1)}, \phi_{(1)} \rangle \langle S h_{(2)}, \phi_{(3)} \rangle \phi_{(2)}, \tag{6.21}$$

and using (D.20) with the antipode (6.20) of $H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H$, this gives rise to covariant left action on H^*

$$(a \otimes h) \triangleright \phi = \langle S h_{(1)} S a, \phi_{(1)} \rangle \langle h_{(2)}, \phi_{(3)} \rangle \phi_{(2)}. \tag{6.22}$$

We thus have the right covariant system $(H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H, H^{*\mathrm{op}})_R$ as the left covariant system[4] $(H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H, H^*)_L$. Again, we refer to [53] for details. Extracting the covariant actions of H^{cop} on H^* and H on H^* in the covariant system $(H^{\mathrm{cop}}\blacktriangleright\!\!\triangleleft H, H^*)$, we get

$$a \triangleright \phi = \langle S a, \phi_{(1)} \rangle \phi_{(2)}, \quad h \triangleright \phi = \langle S h_{(1)}, \phi_{(1)} \rangle \langle h_{(2)}, \phi_{(3)} \rangle \phi_{(2)} \tag{6.23}$$

respectively.

[4]This is refer to as the co-Schrodinger representation.

Chapter 7

Kitaev lattice models for Hopf algebras

The requirements for 2d lattice models based on a Hopf algebra K acting on an algebra $\mathcal{A}$ are presented in this chapter. These requirements puts together Kitaev's quantum double $D(H)$ model [59] and a recently constructed lattice model [3] based on the mirror bicrossproduct $M(H)$ on an equal footing through an axiomatic construction. It turns out that in the double model one works on the graph of a 2d surface whose edges are labeled by H, while in the $M(H)$ model, its the dual graph whose edges are identified by H^*. As a consequence the Hilbert space describing any of these two models carry either the representation of the quantum double or the mirror bicrossproduct.

By construction the quantum double and the mirror bicrossproduct come with representations on H and H^* respectively, these are called respectively the Schrödinger and co-Schrödinger representations. Hence a graph with a single edge can be viewed as carrying either one of these representations depending on which model one is interested in. When working with a graph with many edges in Kitaev's double model, one introduces by hand other representations of $D(H)$ on multiple copies of H in order for the underlying graph to yield a $D(H)$-module. This is because the Schröodinger representation does not contain all the irreducible representation of $D(H)$ [59]. However, in the $M(H)$ model, the extension of the canonical covariant action (introduced in the last chapter) of the mirror bicrossproduct on the tensor product of H^* is all one requires.

The concept of duality has played an integral role in understanding the structures of mathematical and physical theories, some of which include Hopf algebras (discussed in the previous chapter), loop quantum gravity [29] and quantum many-body systems. These theories or systems have two independent valid but distinct physical descriptions and this is a phenomenon associated with systems exhibiting duality. For example in [29], it was shown for 3d gravity, a different choice of polarization resulted in a new discretization based on the triad as opposed to the connection as in standard loop quantum gravity. In this new framework which is dual to LQG, the kinematics is implemented by the flatness constraint with the dynamics described by the Gauss constraint. This duality realized in this choice of polarization manifests the electric-magnetic (EM) duality, which is the exchange of magnetic and electric charges between the two different discretization schemes. The quantum double Kitaev model is an Hamiltonian formulation of BF theory with particles, it too has a dual model [112] and as such manifest EM duality.

In the following we begin by describing the graph representation of the Hopf algebra K acting on several copies of another Hopf algebra $\mathcal{A}$. Given a Hopf algebra $\mathcal{A}$, we define an extended Hilbert space on a polytope decomposition Γ of an oriented surface Σ with each edge of Γ decorated by $\mathcal{A}$. We show that Γ defines a representation of the Hopf algebra K by obtaining local vertex and face operators which act on the Hilbert space satisfying the commutation relations in K. Using this description we review the Kitaev model base on the form of the quantum double of section 6.2.2. The dual Kitaev model is also introduced with the help of a unitary map. The new lattice model based on the mirror bicrossproduct is presented. In both lattice models we will begin with a graph consisting of an edge, then two edges and finally extend to a graph with an arbitrary number of edges. Finally we will determine a representation of the ground state of the mirror bicrossproduct model, this is achieved through a tensor network representation.

The review of the quantum double model and its dual is based on the papers [112, 59]. The contents in sections 7.3 and 7.4 are based on the paper [3].

7.1 Graph representation

There are two main components of the Kitaev lattice models. The first is a polytope decomposition Γ of a 2d compact oriented surface Σ, possibly with boundaries or equivalently, a graph with cyclic ordering of edge ends at each vertex. A polytope decomposition of an oriented surface naturally gives a cyclic ordering of edge ends at each vertex. On the other hand, a cyclic ordering of edge ends gives rise to the notion of a face and one can glue in each face to get a surface. We denote by V, E, F respectively the set of vertices v, edges e, faces p of a graph Γ. The other component of the lattice models is a finite-dimensional Hopf algebra $\mathcal{A}$.

From the data above, we define an extended Hilbert space

$$\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{A}, \tag{7.1}$$

the $|E|$-fold tensor product of the finite-dimensional Hopf algebra $\mathcal{A}$ with each copy assigned to an edge of Γ. To define the inner product on the Hilbert space, we will require a stronger condition on $\mathcal{A}$ working over $\mathbb{C}$ and that it be a Hopf $\star$-algebra.

Since the polytope decomposition is a graph embedded in a surface or a graph from which one can construct a surface, it has a Poincaré dual. The Poincaré duality suggest working on the dual graph denoted Γ^* for models in which the Hilbert space is constructed from the dual Hopf algebra or vice versa.

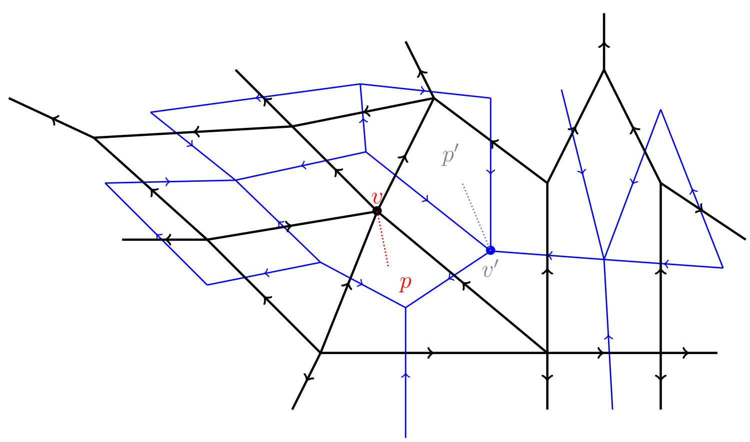


Figure 7.1: This figure shows a graph Γ and its dual Γ^*. The edges of Γ^* are in blue. A site $s = (v, p)$ of Γ is indicated in red and $s' = (v', p')$ (in grey) is a site of Γ^*.

For completeness, we describe Γ^*: In each face p of Γ is a center point or node c. The duality between the faces and the nodes is expressed as $c = p^*$. These nodes denotes the vertices of Γ^*. Given any two nodes c and c', they connect to give a link l, which is an edge of Γ^*. The oriented links connect to give rise to the notion of plaquettes f, which are the faces of Γ^*. There is a duality then between the vertices v in Γ and the faces f in Γ^* expressed as $f = v^*$. The duality exhibited between Γ and Γ^* comes from the duality between the oriented edges in both graphs. This is depicted in Figure 7.1. Throughout this chapter, we shall maintain the same notations for the components in both Γ and Γ^*.

To each edge $e \in E$, we assign a family of basic linear operators $(L^h_\pm)_e$, $(T^a_\pm)_e$ which are linear maps on the Hilbert space, respectively indexed by elements of some finite-dimensional Hopf algebras H_1 and H_2. They act on the edge in question and act only on the copy of the Hopf algebra associated to the edge. These operators are called triangle operators following the initial work by Kitaev [57] and are termed so for the following reason. Consider an edge in Γ and its corresponding dual in Γ^*. Joining the two edges at their respective vertices, one obtains four triangles. These four triangles can be seen as equivalent to thickening of the edges (in Γ and Γ^*) to give a small rectangle (in red).

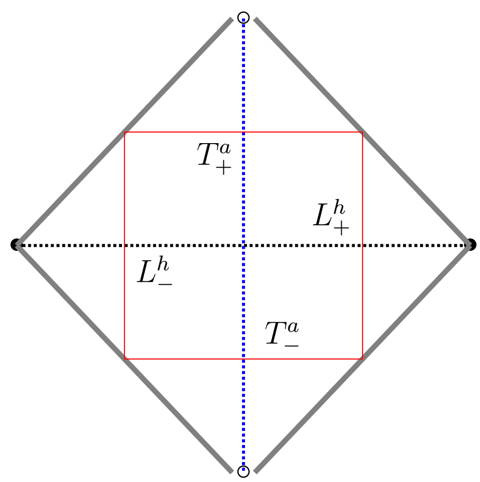


Figure 7.2: The horizontal black dotted edge lives in Γ, the vertical blue dotted edge lives in Γ^*. The boundaries of the four triangles are in gray.

Depending on how one views figure 7.2, these triangle operators especially those of the quantum double model can be related to the holonomies of our discrete theory of chapter 5. That is the holonomies are the classical analogue to Kitaev's triangle operators. The holonomies $\ell^c_{vv'}$, $\ell^{c'}_{vv'}$ (which are linked to the edge $[vv']$) of the LQG phase space put together gives the vertex operator (to be introduced shortly) of the Kitaev model. Whiles the holonomies $h^v_{cc'}$, $h^{v'}_{cc'}$ (associated to the link $[cc']$) put together give the face operator (to be introduced shortly) of the Kitaev model.

We can now define the triangle operators as follows:

Definition 7.1.1. *Let $\mathcal{A}$ be a finite-dimensional Hopf algebra which is a left H_1-module coalgebra and a left H_2-module algebra, let Γ be a graph with cyclic ordering at edge ends.*

The triangle operators for an edge $e \in E$ are linear maps

$$(L^h_{\pm})_e : \mathcal{A}^{\otimes |E|} \to \mathcal{A}^{\otimes |E|}, \quad (T^a_{\pm})_e : \mathcal{A}^{\otimes |E|} \to \mathcal{A}^{\otimes |E|},$$

where $L^h_+, T^a_+ : \mathcal{A} \to \mathcal{A}$ are the left actions

$$L^h_+(\phi) := h \triangleright \phi, \quad T^a_+(\phi) := a \triangleright \phi, \qquad h \in H_1, a \in H_2, \phi \in \mathcal{A} \tag{7.2}$$

of H_1 on $\mathcal{A}$ and H_2 on $\mathcal{A}$ respectively such that either H_1 or H_2 is dual to $\mathcal{A}$. By using the antipode to reverse orientation, the left actions L_+ and T_+ are replaced by the following left actions L_- and T_- respectively obtained via the relations[1]

$$L^h_-(\phi) = (S \circ L^h_+ \circ S^{-1})(\phi), \quad T^a_-(\phi) = (S \circ T^a_+ \circ S^{-1})(\phi). \tag{7.3}$$

Geometrically, for an edge $e \in E$ ending at vertex v, the operator $(L^h_{\pm})_e$ is $(L^h_-)_e$ if v is the source of the edge and $(L^h_+)_e$ otherwise. Similarly, $(T^a_-)_e$ (respectively $(T^a_+)_e$) for the adjacent face p on the left (the right) of the edge e. This rule is illustrated in Figure 7.3. However the expressions in (7.3) relating the triangle operator $L^h_+(T^a_+)$ with $L^h_-(T^a_-)$ does not consider all possible edge orientations on the same level [59].

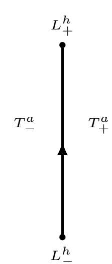


Figure 7.3: Kitaev convention for triangle operators acting on an edge.

Although $\mathcal{A}$ is associated to one edge of Γ, one is required to extend the actions of H_1 and H_2 to many edges so as to cover the full extended Hilbert space. This corresponds to extending the action of H_1 and H_2 to many copies of $\mathcal{A}$.

Next, the triangle operators are used to define vertex and face operators $A^h(v,p) = A^h_v$ and $B^a(v,p) = B^a_p$ on the extended Hilbert space. These operators are also called geometric

[1] In [71], the relation $L^h_- = S \circ L^{Sh}_+ \circ S$ is instead used, this however gives a right action due to the antipode on h.

operators. Both operators depend on a pair of vertex and face that are adjacent to each other. They require linear ordering of edges at each vertex and in each face. This is specified by a site [57] $s = (v, p)$, which consist of a face p and adjacent vertex v and represented by dotted lines as shown in Figure 7.1. To each face, to get a site, one has to choose a vertex that belongs to the face and to each vertex choose a face belonging to the vertex.

Definition 7.1.2.

1. *Let Γ a graph with cyclic ordering of edge ends at each vertex. For a given vertex v at a site, we define the vertex operator $A_v^h : \mathcal{A}^{\otimes |E|} \to \mathcal{A}^{\otimes |E|}$, for $h \in H_1$ which encodes the action of H_1 at the site by*

$$A_v^h = \prod_{e \in C(v)} L_{\pm}^{h_{(i)}}, \tag{7.4}$$

   where $\Delta^n(h) = h_{_{(1)}} \otimes h_{_{(2)}} \otimes ... \otimes h_{(n)}$ and $C(v)$ represent the set of edges connected to the vertex v. The face operator $B_p^a : \mathcal{A}^{\otimes |E|} \to \mathcal{A}^{\otimes |E|}$, for $a \in H_2$ for a give face p at a site which encodes the action of H_2 at the site is defined by

$$B_p^a = \prod_{e \in \partial p} T_{\pm}^{a_{(i)}}, \tag{7.5}$$

   where $\Delta^n(a) = a_{_{(1)}} \otimes a_{_{(2)}} \otimes ... \otimes a_{(n)}$ and ∂p is the set of edges bounded by the face p. Here, the choice of triangle operator L_+ or L_- (resp. T_+ or T_-) in A (resp. B) depends on the orientation of the edge of interest.

2. *The graph Γ defines a representation of a Hopf algebra K built from the Hopf algebras H_1 and H_2 provided the associated vertex and face operators generate the algebra in K. More explicitly Γ defines a representation if there exist an injective algebra homomorphism given by*

$$\rho : K \to End(\mathcal{A}^{\otimes |E|}), \quad ah \mapsto B_p^a \circ A_v^h, \tag{7.6}$$

   taking elements in K and sending it to the associated product of the vertex and face operators. Here, the coproducts $\Delta^n(h)$ and $\Delta^n(a)$ which enter the definition of the vertex and face operators in (7.4) and (7.5) are those of H_1 and H_2 in K.

It is important to note that as a vector space, K is *not* necessarily $H_1 \otimes H_2$ in this general construction and the coproducts of H_1 and H_2 in K may not be the tensor product one as we will see later in the example of the bicrossproduct model.

When the geometric operators do not act at the same site, they essentially commute as stated in the following Lemma.

Lemma 7.1.3. *Let Γ a graph with cyclic ordering at edge ends, and $h, g \in H_1$ and $a, b \in H_2$.*

(i) For all sites, $A_v^h \circ A_w^g = A_w^g \circ A_v^h$ provided the two vertices v and w do not coincide.

(ii) For all sites, $B_p^a \circ B_q^b = B_q^b \circ B_p^a$ if the two faces p and q do not coincide.

(iii) At disjoint sites, $A^h(v,p) \circ B^b(v',p') = B^b(v',p') \circ A^h(v,p)$.

7.1.1 Hamiltonian

Let H_1 and H_2 be finite dimensional Hopf C^*-algebras and by proposition D.4.2, both H_1 and H_2 have the notion of normalized Haar integral. We use the normalized Haar integrals to define projectors A_v^l for each vertex and B_p^k for each face. We note that by considering a finite dimensional Hopf C^*-algebras (i.e., a Hopf algebra with the property $S^2 = \text{id}$), the problem of edge orientation is resolved by identifying $\phi \mapsto S(\phi)$ if the edge is reversed, with $\phi \in \mathcal{A}$.

Lemma 7.1.4. *Let $l \in H_1$, $k \in H_2$ be Haar integrals of the finite-dimensional Hopf C^*-algebras H_1 and H_2. The vertex and face operators $A_v^l, B_p^k : \mathcal{A}^{\otimes |E|} \to \mathcal{A}^{\otimes |E|}$ form a set of commuting projectors independent of a site.*

Proof. The proof follows directly from the properties of the Haar integral outlined in appendix D.4.

$$A_v^2 = A_v^{l^2} = A_v^l, \quad B_p^2 = B_p^{k^2} = B_p^k.$$

□

These projectors commute no matter the vertex or face you pick and are independent of sites at v and at p. They depend on the structure of the polytope decomposition which is the cyclic ordering of the edge ends at each vertex but no longer on the starting point one has to make. From the projection operators, one can then define the Hamiltonian of the theory:

Definition 7.1.5. *Let Γ a graph with cyclic ordering of edge ends at each vertex, and Let $l \in H_1$, $k \in H_2$ be Haar integrals of the finite-dimensional Hopf C^*-algebras H_1 and H_2. The Hamiltonian for the Kitaev model on Γ decorated by the finite-dimensional Hopf C^*-algebra $\mathcal{A}$ is given by*

$$\mathfrak{H} = \sum_{v \in V} (\mathrm{id} - A_v) + \sum_{p \in F} (\mathrm{id} - B_p). \tag{7.7}$$

The space of ground states or protected space of the Hamiltonian (7.7) is given by the invariant subspace $\mathcal{P}_\Gamma$ of $\mathfrak{H}$:

$$\mathcal{P}_\Gamma := \{\phi \in \mathcal{H}_\Gamma : A_v(\phi) = \phi,\ B_p(\phi) = \phi,\ \forall v, p\}. \tag{7.8}$$

By requiring the operators A_v and B_p to be self-adjoint, one ensures that the Hamiltonian is self-adjoint. The protected space is also a topological invariant[2] of the oriented surface Σ. In the general construction the extended Hilbert space looks quite different depending on the graph. This is because one can have different graphs describing the same surface which can be sub divided to look different. However, the protected space depend only on the associated surface and *not* the choice of the graph.

In principle, one could also extend the theory of ribbon operators from in [57, 73] to this general framework. These operators are constructed from certain elementary opeartors associated with two type of triangles which any ribbon path can be decomposed into and are exactly how $L_\pm$ and $T_\pm$ implement the H_1 and H_2-module structure. The algebraic properties of the ribbon operators then allows one to extend topological properties such as degeneracy of the ground state sector as well as the exotic statistics of the quasiparticle excitations whose anyonic nature is revealed via braiding and fusion operations.

7.2 Kitaev quantum double model

We shall now review the Kitaev quantum double model in line with the general framework given in section 7.1. Let H be a finite-dimensional Hopf algebra. Recall from the above section that, the quantum double of H, $D(H)$ is built from H and it dual H^*. We take $\mathcal{A}$ to be H, H_1 to be H and H_2 to be H^* and give a graph representation for $K = D(H)$ [57, 59]. Here, we view the quantum double as $H \bowtie H^{*\mathrm{op}}$ as described above and *not* $H^{*\mathrm{op}} \bowtie H$ as in [59].

[2]The invariance of the protected space holds at least in the quantum double model case.

Following the general set-up in section 7.1 we assign a copy of H to each edge of Γ and define the extended Hilbert space as in (7.1). The definition of the triangle operators in the quantum double model is motivated by the following theorem.

Theorem 7.2.1. *[53] The left regular action L of a bialgebra or Hopf algebra H on itself is $L_h(g) = hg$, and makes H into an H-module coalgebra. For a finite-dimesional bialgebra or Hopf algebra H, the left coregular action R of H^* on H is $R_h(\phi) = \sum h_{(1)}\langle \phi, h_{(2)}\rangle$, and makes H into an H^*-module algebra.*

The triangle operators for the quantum double model are defined from the actions in the above theorem following Definition 7.1.1.

Definition 7.2.2. *[57, 59] Let H be a finite-dimensional Hopf algebra and Γ a graph with cyclic ordering at edge ends. For an edge $e \in E$, $h, g \in H$ and $\phi \in H^*$, the triangle operators are the linear maps $L^h_\pm, T^a_\pm : H \to H$ defined by*

$$L^h_+(g) = hg, \qquad L^h_-(g) = gSh,$$
$$T^\phi_+(g) = g_{(2)}\langle \phi, g_{(1)}\rangle, \qquad T^\phi_-(g) = g_{(1)}\langle \phi, Sg_{(2)}\rangle. \tag{7.9}$$

Note that these actions are not the canonical covariant actions (in the sense of covariance in (D.19)) obtained naturally via the constructions of the quantum double as a double cross product described in section 6.2.1. Of course as pointed out in Definition 7.1.2 of the general construction, we only seek to make H into a $D(H)$-module and not module algebra. Combining the triangle operators (7.9) of the edges at each vertex v and in each face p of Γ according to Definition 7.1.2, we define the vertex and face operators A^h_v and B^a_p at a given site (v,p) of Γ.

Definition 7.2.3. *[57, 59] Let Γ a graph with cyclic ordering of edge ends at each vertex. Let $h \in H$, $\phi \in H^*$ and e_i is the ith edge of Γ. For a given vertex v at a site, we define the vertex operator $A^h_v : H^{\otimes |E|} \to H^{\otimes |E|}$ at the site by*

$$A^h_v = (L^{h_{(1)}}_\pm)_{e_1} \otimes (L^{h_{(2)}}_\pm)_{e_2} \otimes \ldots \otimes (L^{h_{(n)}}_\pm)_{e_n} : H^{\otimes |E|} \to H^{\otimes |E|}. \tag{7.10}$$

The face operator $B^\phi_p : H^{\otimes |E|} \to H^{\otimes |E|}$ for a give face p at a site is defined by

$$B^\phi = (T^{\phi_{(1)}}_\pm)_{e_1} \otimes (T^{\phi_{(2)}}_\pm)_{e_2} \otimes \ldots \otimes (T^{\phi_{(n)}}_\pm)_{e_n} : H^{\otimes |E|} \to H^{\otimes |E|}. \tag{7.11}$$

Note that the coproducts which enter the definitions of these operators are those of H and H^* respectively since the coproduct in $D(H)$ is the tensor product one define in (6.15).

Lemma 7.2.4. *[57, 59] Let* Γ *a graph with cyclic ordering of edge ends at each vertex. For each given site* (v, p)*, the vertex operator* A_v^h *and face operator* B_p^ϕ *define a representation of* $D(H)$ *by satisfying the commutation relation* (6.13) *via*

$$B_p^\phi \circ A_v^h = \langle h_{(1)}, \phi_{(1)} \rangle \langle Sh_{(3)}, \phi_{(3)} \rangle A_v^{h_{(2)}} \circ B_p^{\phi_{(2)}}, \tag{7.12}$$

where the map

$$\rho : D(H) \to End(H), \quad \phi \otimes h \mapsto B_p^\phi \circ A_v^h, \tag{7.13}$$

is an injective algebra homomorphism.

Recall in the introductory part of the chapter, we mentioned how the quantum double has a natural representation on H called the Schrödinger representation. This is given in the following definition:

Definition 7.2.5. *The Schrödinger representation of* $D(H)$ *on* H *is defined as*

$$h \triangleright g = h_{(1)} g Sh_{(2)}, \quad \phi \triangleright g = \langle \phi, g_{(1)} \rangle g_{(2)}, \quad h, g \in H,\ \phi \in H^{*op} \tag{7.14}$$

and makes H *into a* $D(H)$*-module algebra.*

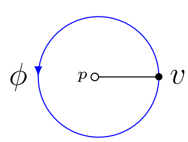


Figure 7.4: A minimal graph Γ_e as an $H \bowtie H^{*\mathrm{op}}$-module on H.

Proof. From the above representation and figure 7.4, we define the following operators

$$A^h(g) = h_{(1)} g Sh_{(2)}, \quad B^\phi(g) = \langle \phi, g_{(1)} \rangle g_{(2)}. \tag{7.15}$$

We show how the graph 7.4 with a single edge satisfy the commutation relation (7.12) and consequently yields the representation (7.14). The proof goes as follows

$$\begin{aligned}
& \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle A^{h_{(2)}} \circ B^{\phi_{(2)}}(g) \\
= & \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle A^{h_{(2)}}\left(\langle \phi_{(2)}, g_{(1)}\rangle g_{(2)}\right) \\
= & \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle\langle \phi_{(2)}, g_{(1)}\rangle h_{(2)(1)}\, g_{(2)}\, Sh_{(2)(2)} \\
= & \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(4)}, \phi_{(3)}\rangle\langle \phi_{(2)}, g_{(1)}\rangle h_{(2)}\, g_{(2)}\, Sh_{(3)} \\
= & \langle \phi, g_{(1)} Sh_{(4)} h_{(1)}\rangle h_{(2)}\, g_{(2)}\, Sh_{(3)} \\
= & \langle \phi, g_{(1)} (Sh_{(2)})_{(1)} h_{(1)(1)}\rangle h_{(2)}\, g_{(2)}\, (Sh_{(2)})_{(2)} \\
= & \langle \phi, g_{(1)}\rangle h_{(1)} g_{(2)} Sh_{(2)} = B^{\phi} A^{h}(g).
\end{aligned} \tag{7.16}$$

In the third equality, a renumbering is performed due to coassociativity in H. We use the dual pairing axiom (D.15) in the four equality to put together the evaluation maps in the third equality. Note, in using (D.15) in the fourth equality we used the fact that the opposite multiplication is the appropriate dual to swap the comultiplication in $H^{*\mathrm{op}}$. By performing another renumbering in H, using the anticoalgebra map of the antipode, the antipode and counit axioms in the fifth equality one gets the last equality.

For a graph with two edges the proof is given below. We refer to the diagram 7.7 for this computation and for $g^1, g^2 \in H$, the vertex and face operators are defined as

$$A^{h}(g^1 \otimes g^2) = L_{+}^{h_{(1)}}(g^1) \otimes L_{-}^{h_{(2)}}(g^2), \tag{7.17}$$

$$B^{\phi}(g^1 \otimes g^2) = T_{+}^{\phi_{(1)}}(g^1) \otimes T_{+}^{\phi_{(2)}}(g^2). \tag{7.18}$$

We start with the LHS of (7.12)

$$\begin{aligned}
B^{\phi} A^{h}(g^1 \otimes g^2) &= B^{\phi}\left(L_{+}^{h_{(1)}}(g^1) \otimes L_{-}^{h_{(2)}}(g^2)\right) = B^{\phi}\left(h_{(1)}\, g^1 \otimes g^2 Sh_{(2)}\right) \\
&= T_{+}^{\phi_{(1)}}\left(h_{(1)}\, g^1\right) \otimes T_{+}^{\phi_{(2)}}\left(g^2 Sh_{(2)}\right) \\
&= \langle \phi_{(1)}, h_{(1)(1)}\, g^1_{(1)}\rangle h_{(1)(2)} g^1_{(2)} \otimes \langle \phi_{(2)}, g^2_{(1)} (Sh_{(2)})_{(1)}\rangle g^2_{(2)} (Sh_{(2)})_{(2)} \\
&= \langle \phi_{(1)}, h_{(1)}\, g^1_{(1)}\rangle h_{(2)} g^1_{(2)} \otimes \langle \phi_{(2)}, g^2_{(1)} Sh_{(4)}\rangle g^2_{(2)} Sh_{(3)} \\
&= \langle \phi, g^2_{(1)} Sh_{(4)} h_{(1)} g^1_{(1)}\rangle h_{(2)}\, g^1_{(2)} \otimes g^2_{(2)} Sh_{(3)} \\
&= \langle \phi, g^2_{(1)} Sh_{(2)(2)} h_{(1)(1)} g^1_{(1)}\rangle h_{(1)(2)}\, g^1_{(2)} \otimes g^2_{(2)} Sh_{(2)(1)} \\
&= \langle \phi, g^2_{(1)} (Sh_{(2)})_{(1)} h_{(1)(1)} g^1_{(1)}\rangle h_{(1)(2)}\, g^1_{(2)} \otimes g^2_{(2)} (Sh_{(2)})_{(2)} \\
&= \langle \phi, g^2_{(1)} (Sh_{(2)} h_{(1)})_{(1)} g^1_{(1)}\rangle h_{(1)(2)}\, g^1_{(2)} \otimes g^2_{(2)} (Sh_{(2)})_{(2)} \\
&= \langle \phi, g^2_{(1)} g^1_{(1)}\rangle h_{(1)}\ g^1_{(2)} \otimes g^2_{(2)} Sh_{(2)}.
\end{aligned} \tag{7.19}$$

To get to the fifth equality, we applied the anticoalgebra map of the antipode and then perform a renumbering in H from the fourth equality. We use the dual pairing axiom (D.15) in the sixth equality and carry out a renumbering in the seventh equality. We then use the anticoalgebra map of the antipode again in the eighth equality, this leads to the application of the antipode and counit axioms in the last but one equality.

The proof of the RHS of (7.12) is

$$
\begin{aligned}
& \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle A^{h_{(2)}} \circ B^{\phi_{(2)}}(g^1 \otimes g^2) \\
= \;& \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle A^{h_{(2)}} \left(T_+^{\phi_{(2)(1)}}(g^1) \otimes T_+^{\phi_{(2)(2)}}(g^2)\right) \\
= \;& \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle \langle \phi_{(2)(1)}, g^1_{(1)}\rangle\langle \phi_{(2)(2)}, g^2_{(1)}\rangle A^{h_{(2)}} \left(g^1_{(2)} \otimes g^2_{(2)}\right) \\
= \;& \langle h_{(1)}, \phi_{(1)}\rangle\langle Sh_{(3)}, \phi_{(3)}\rangle \langle \phi_{(2)}, g^2_{(1)} g^1_{(1)}\rangle \, L_+^{h_{(2)(1)}}(g^1_{(2)}) \otimes L_-^{h_{(2)(2)}}(g^2_{(2)}) \\
= \;& \langle \phi, g^2_{(1)} g^1_{(1)} Sh_{(3)} h_{(1)}\rangle h_{(2)(1)} g^1_{(2)} \otimes g^2_{(2)} Sh_{(2)(2)} \\
= \;& \langle \phi, g^2_{(1)} g^1_{(1)} Sh_{(4)} h_{(1)}\rangle h_{(2)} g^1_{(2)} \otimes g^2_{(2)} Sh_{(3)} \\
= \;& \langle \phi, g^2_{(1)} g^1_{(1)}\rangle h_{(1)} \; g^1_{(2)} \otimes g^2_{(2)} Sh_{(2)}.
\end{aligned}
\tag{7.20}
$$

Renumbering in $H^{*\mathrm{op}}$ is performed in the third equality and the dual pairing property (D.15) is used in the fourth equality. A renumbering in H, the anticoalgebra map of the antipode, the antipode and counit axioms are all used in the fifth equality to get to the last equality. □

We refer to [59] for the proof with regards to the arbitrary graph. Since H is finite-dimensional, the existence of Haar integrals for H and H^* allows the definition of the vertex and face projectors similar to Lemma 7.1.4. These projectors also satisfy the properties in Lemma 7.1.3 and the Hamiltonian of the system is defined similar to (7.1.5). It is shown in [57, 59] that the ground state of the quantum double system is topological invariant.

In the case of a group algebra of a finite group, $H = \mathbb{C}[G]$, the distinguished basis of H are used consisting of group elements of G, that is $\{|g\rangle : g \in G\}$. Kitaev's toric code uses the cyclic group $\mathbb{Z}_2$[57]. A basis of $\mathcal{H}_\Gamma$ is given by assigning to any edge of Γ a group element g. The group elements g are interpreted as the holonomy of a connection along the edge. The projection by the operator A_v implements gauge invariance at the vertex v by averaging with respect to the Haar measure. The projection by the operator B_p implements that locally on the face p the connection flat. Indeed, integrals project to invariants and thus for the holonomy around a face, we have $(g_1 \cdot g_2 \cdot ... \cdot g_n) = 1$ which amounts to the flatness condition $(g_1 \cdot g_2 \cdot ... \cdot g_n) = e$. The ground state corresponds to the space of flat G-connections modulo gauge transformations (conjugation)[57].

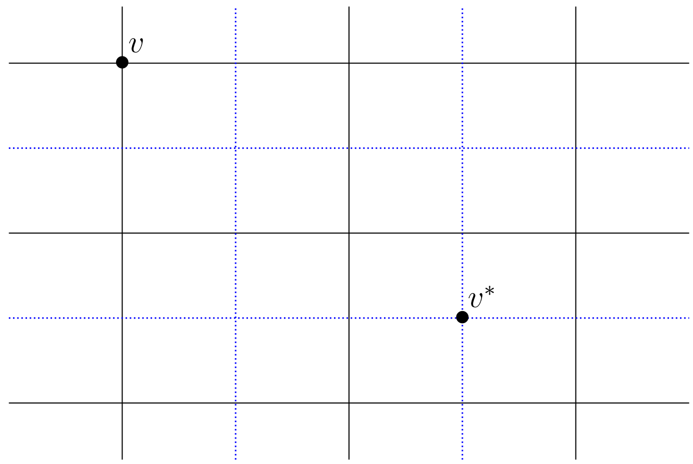


Figure 7.5: The blue dotted lines constitute the dual graph Γ^* which is characterized by the nodes p^* and plaquettes v^*. Here torsion (Gauss constraint) sits on v^* while curvature (flatness constraint) is on p^* in the $D(H^*)$ model.

Buerschaper and collaborators [112] have shown that the dual Kitaev quantum double model exist for both the finite group and Hopf algebra. This is realized in terms of EM duality. In the finite group model, electric charges sits on the vertices of the lattice and are labeled by irreducible representation of G. Magnetic charges sits on the faces and they correspond to conjugacy class of the gauge group. In gravity context, we can relate this to torsion excitation at the vertex and curvature excitation on the face. Therefore one expects in the dual model, there will be exchange of degrees of freedom with the original model.

The EM duality of the Kitaev models especially in the Hopf algebra context appears naturally from the algebraic settings. We have seen earlier how the axioms of finite dimensional Hopf algebras H are self-dual, hence in that regard H^* is finite dimensional and its structure determined by H. Note that since we will associate Hopf algebra H to graph Γ, the dualization of the Hopf algebra goes together with a dualization of the graph.

$$(\Gamma, H) \to (\Gamma^*, H^*). \tag{7.21}$$

By defining a unitary map $U : H \to H^*$ such that

$$f_a(b) = \sqrt{\dim H}\, \phi(ab), \tag{7.22}$$

where $a, b \in H$ and $\phi \in H^*$, one associate this map with the transformation of Γ into its dual Γ^*. This ensures the degree of freedom associated to an edge in Γ becomes those of an edge in Γ^*. The unitarity of the map easily follows from the above definition and using the fact that

$$\langle f^\star, h\rangle = \overline{\langle f, (Sh)^\star\rangle}. \tag{7.23}$$

Putting together the unitary maps for the individual edges of Γ, the following global map

$$U_\Gamma := \bigotimes_{e\in\Gamma} U_e \tag{7.24}$$

is obtained which sends the vertex A^h and face B^ϕ operators of the quantum double $D(H)$ on Γ into the face $\tilde{B}^h$ and vertex $\tilde{A}^\phi$ operators associated to the representations of the $D(H^*)$ model on Γ^* through the following expression

$$U_\Gamma\, A^h\, U_\Gamma^\dagger = \tilde{B}^h, \quad U_\Gamma\, B^\phi\, U_\Gamma^\dagger = \tilde{A}^\phi. \tag{7.25}$$

The proof of the duality expressed in (7.25) can be found in [112]. Ultimately, the global map U_Γ is unitarily transforming the Hilbert spaces and Hamiltonian of the $D(H)$ model into the $D(H^*)$ model. Hence the spin and mass excitations are exhibited by the face operator B and vertex operator A respectively in the latter model.

Based on the exchange of excitations between Kitaev's quantum double model and its dual, one can relate them to the two formulations of gravity.

7.3 Lattice representation for mirror bicrossproduct Hopf algebras

In this section, we will construct a lattice representation based on the mirror bicrossproduct $M(H)$ acting on H^*. We take $\mathcal{A}$ as H^*, H_1 as H and H_2 as H^{cop} in the general picture outlined in Section 7.1. We work over $\mathbb{C}$ and require that the Hopf algebras be Hopf $\star$-algebras. Our goal is to find local vertex and face operators A^h, B^a which act on $\mathcal{H}_\Gamma$ and represent both copies H, H^{cop} in the mirror product bicrossproduct $M(H) = H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft} H$ so that they satisfy the product in the bicrossproduct $M(H)$.

Lattice and Hilbert space: From the notion in the correspondence between Poincaré and Hopf algebra duality pointed out earlier, we take Γ to be the 1-skeleton of the *dual* graph of the polytope decomposition. Each edge of Γ is then decorated by elements of H^*. The extended Hilbert space is therefore given by (7.1) as

$$\mathcal{H}_\Gamma = \bigotimes_{e\in\Gamma} H^*,$$

the $|E|$-fold tensor product of H^*.

Triangle operators: It is interesting to note that for the bicrossproduct covariant system $(H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H, H^*)$ described in section 6.2.2, the canonical left action of H^{cop} on H^* is a coregular action and makes H^* an H^{cop}-module algebra by construction while the canonical left action of H on H^* is a coadjoint action and makes H^* an H-module coalgebra. We therefore choose the covariant actions (6.23) as natural candidates for defining the triangle operators. Thus although Definition 7.1.1 does not demand that the action of K on $\mathcal{A}$ be covariant, in the case of the bicrossproduct we get get this for free.

Definition 7.3.1. *Let H be a finite-dimensional Hopf algebra and Γ a graph with cyclic ordering of edge ends at each vertex. Let $h \in H$, $\phi \in H^*$ and $a \in H^{cop}$. The triangle operators for an edge $e \in E$ are linear maps*

$$(L^h_\pm)_e : H^{*\otimes|E|} \to H^{*\otimes|E|}, \quad (T^a_\pm)_e : H^{*\otimes|E|} \to H^{*\otimes|E|},$$

where $L^h_+, T^a_+ : H^* \to H^*$ are given by

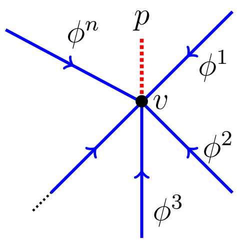
$$\begin{aligned} L^h_+(\phi) &= \langle h, S\phi_{(1)}\phi_{(3)}\rangle\phi_{(2)}, \qquad L^h_-(\phi) = \langle h, \phi_{(3)}S^{-1}\phi_{(1)}\rangle\phi_{(2)}, \\ T^a_+(\phi) &= \langle Sa, \phi_{(1)}\rangle\phi_{(2)}, \qquad\qquad T^a_-(\phi) = \langle a, \phi_{(2)}\rangle\phi_{(1)}. \end{aligned} \tag{7.26}$$

Here, the operators L_+ and T_+ are the canonical left action (6.23) of the bicrossproduct $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$ on H^*. The L_- and T_- are also left actions obtained using the relations in (7.3). The algebra of the triangle operators are given by

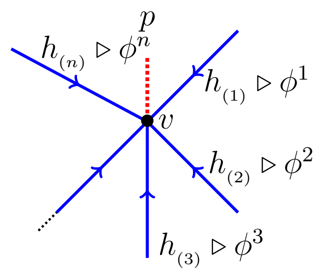
$$\begin{aligned} [L^h_+, L^g_-] &= 0, \quad [T^a_+, T^b_-] = 0, \quad \left[L^h_-, T^a_-\right] = 0 \\ L^h_- T^a_- &= T_-^{h_{(1)}aSh_{(2)}} L_-^{h_{(3)}}, \quad L^h_+ T^a_- = T_-^{h_{(1)}aSh_{(2)}} L_+^{h_{(3)}}, \\ L^h_+ T^a_+ &= T_+^{h_{(1)}aSh_{(2)}} L_+^{h_{(3)}}, \quad \text{for all } \ h, g \in H, \ a, b \in H^{\mathrm{cop}}. \end{aligned} \tag{7.27}$$

Geometric operators: We now define the vertex and face operators for the bicrossproduct model. Due to the duality between the vertex and face operators for a graph and its dual, the operators get swapped in the bicrossproduct model. Since we are working on the dual graph, the vertex operators become face operators while the face operators become vertex operators.

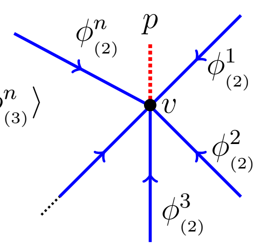
Definition 7.3.2. *Let (v,p) be the site of Γ with all edges incoming, and $h \in H$, $a \in H^{cop}$, $\phi^i \in H^*$. The vertex operator $A^h_v : H^{*\otimes|E|} \to H^{*\otimes|E|}$ which encodes the action of H in $H^{cop} \blacktriangleright\!\!\triangleleft H$ at the site by*



$\longmapsto$



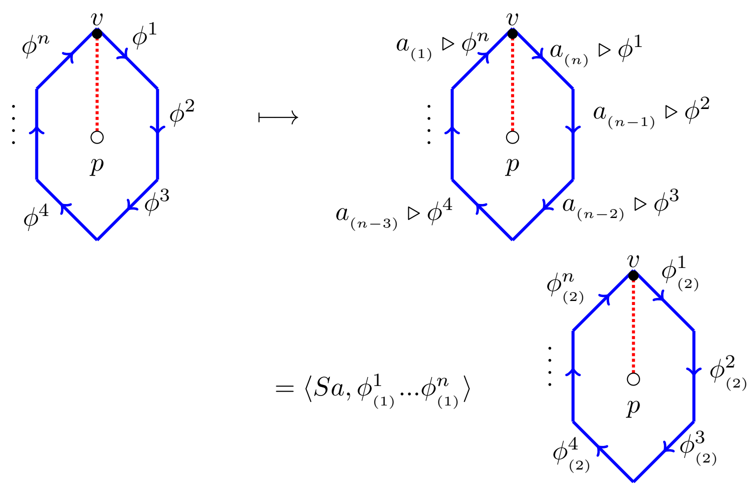
$= \langle h, (S\phi^1_{(1)})\phi^1_{(3)} .... (S\phi^n_{(1)})\phi^n_{(3)} \rangle$



or

$$A^h_v = L_+^{h_{(1)}} \otimes \cdots \otimes L_+^{h_{(n)}},$$

where $\Delta^{(n)}(h) = h_{(1)} \otimes h_{(2)} \otimes ... \otimes h_{(n+1)}$ with $\Delta(h)$ given by the coproduct of $H^{cop} \blacktriangleright\!\!\triangleleft H$ in (6.19). The antipode is applied when there is a change in orientation away from the vertex v to map the action L_+ to the action L_- as described in (7.3). The face operator $B^a_p : H^{*\otimes|E|} \to H^{*\otimes|E|}$ for the face p which encodes the action of H^{cop} in $H^{cop} \blacktriangleright\!\!\triangleleft H$ at the site is defined by



or

$$B_p^a = T_+^{a_{(n)}} \otimes \cdots \otimes T_+^{a_{(1)}}$$

where the coproduct $\Delta(a)$ *is given by* (6.19)*. The antipode is applied when there is a change in orientation away from the vertex* v *to map the action* T_+ *to the action* T_-*.*

The definition of A^h follows by assigning the coproduct of $h \in H$ along the edges in a clockwise manner taking into account the site (v, p), and then the appropriate action of h depending on the edge orientation is taking. In a like manner, the operator B^a is defined, but the edges associated to the face p are assigned the coproduct of $a \in H^{\mathrm{cop}}$ clockwise starting from the vertex v. The action of a is then taking depending on whether the edge orientation is on the left or right of the face p.

We shall now show how Γ, equipped with these operators admits a local mirror bicrossproduct $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$-representation at the sites of arbitrary graphs. We need to show that the vertex and face operators represent their respective copies of $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$ and that their commutation relations arising from common edges implement the algebra in the bicrossproduct quantum group $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$.

Theorem 7.3.3. *Let* H *be a finite-dimensional Hopf algebra with dual* H^* *and* Γ *a graph with cyclic ordering of edge ends at each vertex. Let* H^* *be assigned to each edge of* Γ*. Then each site* (v, p) *of* Γ *admits a bicrossproduct* $H^{cop} \blacktriangleright\!\!\triangleleft H$*-module structure via the vertex and face operators* $A_v^h, B_p^a : H^{* \otimes |E|} \to H^{* \otimes |E|}$ *given in Definition 7.3.2, i.e. the operators satisfy the commutation relation in the bicrossproduct quantum group*

$$A^h \circ B^a = B^{(h_{(1)} a S h_{(2)})} \circ A^{h_{(3)}}, \quad \forall h \in H, \ a \in H^{cop}. \tag{7.28}$$

Proof. Let us consider first the simplest or minimal graph Γ_e, with one vertex and one edge, as in Figure 7.6. The covariant system $(H^{\text{cop}} \blacktriangleright\!\!\triangleleft H, H^*)$ provides a representation of $H^{\text{cop}} \blacktriangleright\!\!\triangleleft H$ for Γ_e. The vertex and face operators for this graph is defined by

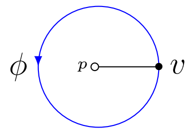


Figure 7.6: A minimal graph Γ_e as an $H^{\text{cop}} \blacktriangleright\!\!\triangleleft H$-module on H^*.

$$A^h(\phi) := L_+^h(\phi) = \langle h, (S\phi_{(1)})\phi_{(3)}\rangle \phi_{(2)}, \quad B^a(\phi) := T_-^a(\phi) = \langle a, \phi_{(2)}\rangle \phi_{(1)} \tag{7.29}$$

We proceed to show that the operators A^h and B^a defined in (7.29) satisfy (7.28). The LHS of (7.28) is computed as follows

$$\begin{aligned} A^h B^a(\phi) &= A^h\left(\langle a, \phi_{(2)}\rangle \phi_{(1)}\right) = \langle a, \phi_{(2)}\rangle A^h(\phi_{(1)}) \\ &= \langle a, \phi_{(2)}\rangle \langle h, (S\phi_{(1)(1)})\phi_{(1)(3)}\rangle \phi_{(1)(2)} \\ &= \langle a, \phi_{(4)}\rangle \langle h, (S\phi_{(1)})\phi_{(3)}\rangle \phi_{(2)} \\ &= \langle a, \phi_{(4)}\rangle \langle h_{(1)}, S\phi_{(1)}\rangle \langle h_{(2)}, \phi_{(3)}\rangle \phi_{(2)} \\ &= \langle a, \phi_{(2)(2)}\rangle \langle Sh_{(1)}, \phi_{(1)(1)}\rangle \langle h_{(2)}\phi_{(2)(1)}\rangle \phi_{(1)(2)} \\ &= \langle a, \phi_{(3)}\rangle \langle Sh_{(1)}h_{(2)}, \phi_{(1)}\rangle \phi_{(2)} \\ &= \epsilon(h)\langle a, \phi_{(3)}\rangle \epsilon(\phi_{(1)})\phi_{(2)} = \epsilon(h)\langle a, \phi_{(2)}\rangle \phi_{(1)}. \end{aligned} \tag{7.30}$$

In the third and fifth equalities we renumbered the indices as a result of coassiciativity in H^*. We applied the pairing axioms (D.15) and (D.16) in both the fourth and sixth equalities. In the last equality we used the antipode axiom in H and H^*, and the counity axiom in H^*. Computing the RHS of (7.28) we have

$$\begin{aligned}
B^{h_{(1)}aSh_{(2)}}A^{h_{(3)}}(\phi) &= B^{h_{(1)}aSh_{(2)}}(\langle h_{(3)},(S\phi_{(1)})\phi_{(3)}\rangle\phi_{(2)}) = \langle h_{(3)},(S\phi_{(1)})\phi_{(3)}\rangle T_{-}^{h_{(1)}aSh_{(2)}}(\phi_{(2)}) \\
&= \langle h_{(3)},(S\phi_{(1)})\phi_{(3)}\rangle\langle h_{(1)}aSh_{(2)},\phi_{(2)(2)}\rangle\phi_{(2)(1)} \\
&= \langle h_{(3)},(S\phi_{(1)})\phi_{(4)}\rangle\langle h_{(1)}aSh_{(2)},\phi_{(3)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}aSh_{(2)},\phi_{(3)}\rangle\langle h_{(3)},S\phi_{(1)}\rangle\langle h_{(4)},\phi_{(4)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}aSh_{(2)}h_{(4)},\phi_{(3)}\rangle\langle Sh_{(3)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}aSh_{(2)(1)}h_{(3)},\phi_{(3)}\rangle\langle Sh_{(2)(2)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a(Sh_{(2)})_{(2)}h_{(3)},\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a(Sh_{(2)})_{(2)}h_{(4)}\epsilon(h_{(3)}),\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a(Sh_{(2)})_{(2)}h_{(3)(2)}\epsilon(h_{(3)(1)}),\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)}.
\end{aligned} \tag{7.31}$$

We did some renumbering in the third equality equality due to the coassociativity in H^*. The pairing axioms (D.15) and (D.16) were used in the the fourth and fifth equalities. In the seventh equality we used the fact that S is an anticoalgebra map. In the last but one equality we inserted $h_{(3)} = h_{(4)}\epsilon(h_{(3)})$ by the counity axiom and then carry out a renumbering in the last equality in H. Simplifying further we get

$$\begin{aligned}
B^{h_{(1)}aSh_{(2)}}A^{h_{(3)}}(\phi) &= \langle h_{(1)}a(Sh_{(2)}h_{(3)})_{(2)}\epsilon(h_{(3)})_{(1)},\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a\,\epsilon(h_{(2)})_{(2)}\epsilon(h_{(3)}),\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a\,\epsilon(Sh_{(2)})_{(2)}\epsilon(h_{(3)}),\phi_{(3)}\rangle\langle (Sh_{(2)})_{(1)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a\,\epsilon(Sh_{(2)(1)})\epsilon(h_{(3)}),\phi_{(3)}\rangle\langle Sh_{(2)(2)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a\,\epsilon(h_{(2)(1)})\epsilon(h_{(3)}),\phi_{(3)}\rangle\langle Sh_{(2)(2)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)}a\,,\phi_{(3)}\rangle\langle Sh_{(2)},\phi_{(1)}\rangle\phi_{(2)} \\
&= \langle h_{(1)},\phi_{(2)(1)}\rangle\langle a,\phi_{(2)(2)}\rangle\langle Sh_{(2)},\phi_{(1)(1)}\rangle\phi_{(1)(2)} \\
&= \langle a,\phi_{(3)}\rangle\langle Sh_{(2)}h_{(1)},\phi_{(1)}\rangle\phi_{(2)} = \epsilon(h)\langle a,\phi_{(2)}\rangle\phi_{(1)}.
\end{aligned} \tag{7.32}$$

The antipode axiom $h_{(1)}Sh_{(2)} = \epsilon(h)$ is used in the first equality. From (7.30) and (7.32) we conclude the operators (7.29) form a representation of $M(H)$ on the minimal graph 7.6.

Consider next that Γ is made up of two edges connecting each other at two different vertices as shown in Figure 7.7. The associated Hilbert space is $\mathcal{H}_\Gamma = H^* \otimes H^*$. Note that the coproduct (6.19) of the bicrossproduct $H^{\text{cop}}\blacktriangleright\!\!\triangleleft H$ is not a tensor product one but rather entangled due to the presence of the coaction (6.16). From (6.19), we have

$$\Delta(a) = a_{(2)} \otimes 1 \otimes a_{(1)} \otimes 1, \quad \Delta(h) = 1 \otimes h_{(2)} \otimes h_{(1)}Sh_{(3)} \otimes h_{(4)}$$

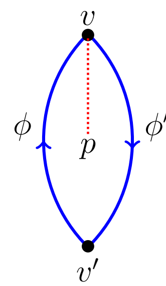


Figure 7.7: A graph representing the Hilbert space of $H^* \otimes H^*$
admitting an $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$-module.

in $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H \otimes H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$. We may sometimes use the identification $H^{\mathrm{cop}} \cong H^{\mathrm{cop}} \otimes 1$ and $H \cong 1 \otimes H$ in $H^{\mathrm{cop}} \blacktriangleright\!\!\triangleleft H$ so that we can write for example $\Delta(a) = a_{(2)} \otimes a_{(1)}$. Using the triangle operators of (7.26), the vertex and the face operators at the site (v, p) of Figure 7.7 are given by

$$
\begin{aligned}
A_v^h(\phi \otimes \phi') &= L_+^{1 \otimes h_{(2)}}(\phi) \otimes L_-^{h_{(1)} S h_{(3)} \otimes h_{(4)}}(\phi'), && (7.33)\\
B_p^a(\phi \otimes \phi') &= T_+^{a_{(2)}}(\phi) \otimes T_+^{a_{(1)}}(\phi'). && (7.34)
\end{aligned}
$$

Before we proceed, let see how we can compute the operator $L_-^{h_{(1)} S h_{(3)} \otimes h_{(4)}}(\phi)$ as it is not defined in (7.26). From the convariant action (6.22), we have

$$
\begin{aligned}
L_+^{h_{(1)} S h_{(3)} \otimes h_{(4)}}(\phi) &= (h_{(1)} S h_{(3)} \otimes h_{(4)}) \triangleright \phi = \langle S h_{(4)(1)} S(h_{(1)} S h_{(3)}), \phi_{(1)} \rangle \langle h_{(4)(2)}, \phi_{(3)} \rangle \phi_{(2)} \\
&= \langle S h_{(4)} S^2 h_{(3)} S h_{(1)}, \phi_{(1)} \rangle \langle h_{(5)}, \phi_{(3)} \rangle \phi_{(2)} \\
&= \langle S(h_{(1)} S h_{(3)} h_{(4)}), \phi_{(1)} \rangle \langle h_{(5)}, \phi_{(3)} \rangle \phi_{(2)} \\
&= \langle S h_{(1)}, \phi_{(1)} \rangle \langle h_{(3)}, \phi_{(3)} \rangle \phi_{(2)}, && (7.35)
\end{aligned}
$$

and subsequently using (7.3) we get

$$
L_-^{h_{(1)} S h_{(3)} \otimes h_{(4)}}(\phi) = \langle h_{(1)}, \phi_{(3)} \rangle \langle h_{(3)}, S^{-1} \phi_{(1)} \rangle \phi_{(2)}. \qquad (7.36)
$$

We now show that these operators satisfy (7.28). The proof follows a direct calculation, evaluating both side of the formula on arbitrary elements $\phi, \phi' \in H^*$. Evaluating the left hand side, we have

$$
\begin{aligned}
& A^h B^a(\phi\otimes\phi') \\
= \;& A^h\left(T_+^{a_{(2)}}(\phi)\otimes T_+^{a_{(1)}}(\phi')\right) = A^h\left(\langle a_{(2)}, S\phi_{(1)}\rangle\phi_{(2)}\otimes\langle a_{(1)}, S\phi'_{(1)}\rangle\phi'_{(2)}\right) \\
= \;& \langle a_{(2)}, S\phi_{(1)}\rangle\langle a_{(1)}, S\phi'_{(1)}\rangle A^h(\phi_{(2)}\otimes\phi'_{(2)}) \\
= \;& \langle a_{(2)}, S\phi_{(1)}\rangle\langle a_{(1)}, S\phi'_{(1)}\rangle L_+^{h_{(2)}}(\phi_{(2)})\otimes L_-^{h_{(1)}Sh_{(3)}\otimes h_{(4)}}(\phi'_{(2)}) \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle h_{(2)}, (S\phi_{(2)(1)})\phi_{(2)(3)}\rangle\phi_{(2)(2)}\otimes\langle h_{(1)}, \phi'_{(2)(3)}\rangle\langle h_{(3)}, S^{-1}\phi'_{(2)(1)}\rangle\phi'_{(2)(2)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle h_{(2)}, (S\phi_{(2)})\phi_{(4)}\rangle\phi_{(3)}\otimes\langle h_{(1)}, \phi'_{(4)}\rangle\langle h_{(3)}, S^{-1}\phi'_{(2)}\rangle\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle h_{(2)}, S\phi_{(2)}\rangle\langle h_{(3)}, \phi_{(4)}\rangle\phi_{(3)}\otimes\langle h_{(1)}, \phi'_{(4)}\rangle\langle h_{(4)}, S^{-1}\phi'_{(2)}\rangle\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle Sh_{(2)}, \phi_{(1)(2)}\rangle\langle h_{(3)}, \phi_{(2)(2)}\rangle\langle h_{(1)}, \phi'_{(4)}\rangle\langle h_{(4)}, S^{-1}\phi'_{(2)}\rangle\phi_{(2)(1)}\otimes\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle Sh_{(2)}h_{(3)}, \phi_{(2)}\rangle\ \langle h_{(1)}, \phi'_{(4)}\rangle\langle h_{(4)}, S^{-1}\phi'_{(2)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\epsilon(h_{(2)})\,\epsilon(\phi_{(2)})\langle h_{(1)}, \phi'_{(4)}\rangle\langle h_{(3)}, S^{-1}\phi'_{(2)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)(1)}\rangle\langle h_{(1)}, \phi'_{(2)(2)}\rangle\langle h_{(2)}, S^{-1}\phi'_{(1)(2)}\rangle\phi_{(2)}\otimes\phi'_{(2)(1)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)(1)}\rangle\langle S^{-1}h_{(2)}, \phi'_{(1)(2)}\rangle\langle h_{(1)}, \phi'_{(2)(2)}\rangle\phi_{(2)}\otimes\phi'_{(2)(1)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\langle S^{-1}h_{(2)}h_{(1)}, \phi'_{(2)}\rangle\phi_{(2)}\otimes\phi'_{(3)} \\
= \;& \langle Sa_{(2)}, \phi_{(1)}\rangle\langle Sa_{(1)}, \phi'_{(1)}\rangle\,\epsilon(h)\,\epsilon(\phi'_{(2)})\phi_{(2)}\otimes\phi'_{(3)} \\
= \;& \langle (Sa)_{(2)}, \phi'_{(1)}\rangle\langle (Sa)_{(1)}, \phi_{(1)}\rangle\,\epsilon(h)\,\phi_{(2)}\otimes\phi'_{(2)} \\
= \;& \langle Sa, \phi'_{(1)}\phi_{(1)}\rangle\,\epsilon(h)\,\phi_{(2)}\otimes\phi'_{(2)}.
\end{aligned}
\tag{7.37}
$$

We used the definitions of the triangle operators (7.26) in the first and fourth equalities. In the fifth and tenth equalities we did some renumbering due to coassociativity in H^*. The dual pairing property (D.15) is applied in the sixth equality and the antipode axiom is used in the eighth equality.

Computing the right hand side of (7.28), we have

$$
\begin{aligned}
& B^{h_{(1)}aSh_{(2)}}A^{h_{(3)}}(\phi\otimes\phi') \\
= \;& B^{h_{(1)}aSh_{(2)}}\left(L_+^{h_{(3)(2)}}(\phi)\otimes L_-^{h_{(3)(1)}Sh_{(3)(3)}\otimes h_{(3)(4)}}(\phi')\right) \\
= \;& B^{h_{(1)}aSh_{(2)}}\left(\langle h_{(3)(2)},S\phi_{(1)}\phi_{(3)}\rangle\phi_{(2)}\otimes\langle h_{(3)(1)},\phi'_{(3)}\rangle\langle h_{(3)(3)},S^{-1}\phi'_{(1)}\rangle\phi'_{(2)}\right) \\
= \;& \langle h_{(3)(2)},S\phi_{(1)}\phi_{(3)}\rangle\langle h_{(3)(1)},\phi'_{(3)}\rangle\langle h_{(3)(3)},S^{-1}\phi'_{(1)}\rangle B^{h_{(1)}aSh_{(2)}}\left(\phi_{(2)}\otimes\phi'_{(2)}\right) \\
= \;& \langle h_{(3)(2)},S\phi_{(1)}\phi_{(3)}\rangle\langle h_{(3)(1)},\phi'_{(3)}\rangle\langle h_{(3)(3)},S^{-1}\phi'_{(1)}\rangle\langle h_{(1)}aSh_{(2)},S\phi_{(2)(1)}S\phi'_{(2)(1)}\rangle\phi_{(2)(2)}\otimes\phi'_{(2)(2)} \\
= \;& \langle h_{(3)(2)},S\phi_{(1)}\phi_{(4)}\rangle\langle h_{(3)(1)},\phi'_{(4)}\rangle\langle h_{(3)(3)},S^{-1}\phi'_{(1)}\rangle\langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle h_{(4)},S\phi_{(1)}\phi_{(4)}\rangle\langle h_{(3)},\phi'_{(4)}\rangle\langle h_{(5)},S^{-1}\phi'_{(1)}\rangle\langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\langle h_{(3)},\phi'_{(4)}\rangle\langle h_{(4)},S\phi_{(1)}\rangle\langle h_{(5)},\phi_{(4)}\rangle\langle h_{(6)},S^{-1}\phi'_{(1)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\langle h_{(3)(1)},\phi'_{(4)}\rangle\langle h_{(3)(2)},S\phi_{(1)}\rangle\langle h_{(4)(1)},\phi_{(4)}\rangle\langle h_{(4)(2)},S^{-1}\phi'_{(1)}\rangle\phi_{(3)}\otimes\phi'_{(3)}
\end{aligned}
$$

The definitions of the triangle operators (7.26) are used in the third and the fifth equalities. The antipode axiom $(Sh_{(1)})h_{(2)} = h_{(1)}Sh_{(2)} = \epsilon(h)$ is used in the sixth equality. We renumber in the sixth equality due to coassociativity in H^* and H. The counity axiom $h = h_{(1)}\epsilon(h_{(2)})$ is used in the eighth and the last equalities. We used the dual pairing property (D.15) in the last but one equality. Simplifying further, we apply the counit axiom to the last equality, which then reads as follows

$$
\begin{aligned}
& B^{h_{(1)}aSh_{(2)}}A^{h_{(3)}}(\phi\otimes\phi') \\
= \;& \langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\langle h_{(3)(2)},S\phi_{(1)}\rangle\langle h_{(4)(2)},S^{-1}\phi'_{(1)}\rangle\langle h_{(3)(1)},\phi'_{(4)}\rangle\langle h_{(4)(1)},\phi_{(4)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle h_{(1)}aSh_{(2)},S\phi_{(2)}S\phi'_{(2)}\rangle\langle h_{(3)(2)},S\phi_{(1)}S^{-1}\phi'_{(1)}\rangle\langle h_{(3)(1)},\phi'_{(4)}\phi_{(4)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle h_{(2)}SaSh_{(1)},\phi'_{(2)}\phi_{(2)}\rangle\langle Sh_{(3)(2)},\phi'_{(1)}\phi_{(1)}\rangle\langle h_{(3)(1)},\phi'_{(4)}\phi_{(4)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sh_{(2)(2)},\phi'_{(1)}\phi_{(1)}\rangle\langle h_{(2)(1)}SaSh_{(1)},\phi'_{(2)}\phi_{(2)}\rangle\langle h_{(3)},\phi'_{(4)}\phi_{(4)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sh_{(2)(2)}h_{(2)(1)}SaSh_{(1)},\phi'_{(1)}\phi_{(1)}\rangle\langle h_{(3)},\phi'_{(3)}\phi_{(3)}\rangle\phi_{(2)}\otimes\phi'_{(2)} \\
= \;& \langle\epsilon h_{(2)(1)}SaSh_{(1)},\phi'_{(1)}\phi_{(1)}\rangle\langle h_{(2)(2)},\phi'_{(3)}\phi_{(3)}\rangle\phi_{(2)}\otimes\phi'_{(2)} \\
= \;& \langle SaSh_{(1)},\phi'_{(1)}\phi_{(1)}\rangle\langle\epsilon h_{(2)(1)}h_{(2)(2)},\phi'_{(3)}\phi_{(3)}\rangle\phi_{(2)}\otimes\phi'_{(2)} \\
= \;& \langle SaSh_{(1)},\phi'_{(1)}\phi_{(1)}\rangle\langle h_{(2)},\phi'_{(3)}\phi_{(3)}\rangle\phi_{(2)}\otimes\phi'_{(2)} \\
= \;& \langle Sa,\phi'_{(1)}\phi_{(1)}\rangle\langle Sh_{(1)},\phi'_{(2)}\phi_{(2)}\rangle\langle h_{(2)},\phi'_{(4)}\phi_{(4)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sa,\phi'_{(1)(1)}\phi_{(1)(1)}\rangle\langle Sh_{(1)},\phi'_{(1)(2)}\phi_{(1)(2)}\rangle\langle h_{(2)},\phi'_{(2)(2)}\phi_{(2)(2)}\rangle\phi_{(2)(1)}\otimes\phi'_{(2)(1)} \\
= \;& \langle Sa,\phi'_{(1)}\phi_{(1)}\rangle\langle Sh_{(1)}h_{(2)},\phi'_{(2)}\phi_{(2)}\rangle\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sa,\phi'_{(1)}\phi_{(1)}\rangle\,\epsilon(h)\,\epsilon(\phi'_{(2)})\,\epsilon(\phi_{(2)})\,\phi_{(3)}\otimes\phi'_{(3)} \\
= \;& \langle Sa,\phi'_{(1)}\phi_{(1)}\rangle\,\epsilon(h)\,\phi_{(2)}\otimes\phi'_{(2)}.
\end{aligned}
\tag{7.38}
$$

We apply the pairing properties (D.15) and (D.17) to get to the eighth equality. In the last but third equality, the pairing property (D.15) is used while the counit axiom is used to get to the last equality. The equivalence of (7.37) and (7.38) shows that the operators A^h and B^a define a representation of $H^{\text{cop}}\blacktriangleright\!\!\triangleleft H$ on the loop in Figure 7.7.

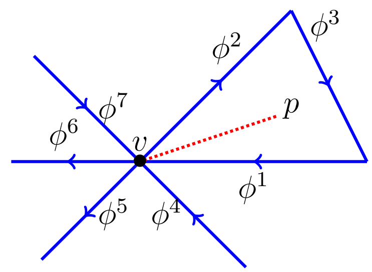


Figure 7.8: Graph representing seven copies of H^* used in the proof of Theorem 7.3.3.

The proof for an arbitrary graph follows. We use Figure 7.8 to proof Theorem 7.3.3. It is sufficient to show that (6.18) holds on this graph. Before proceeding with the proof, we note the followings: (i) First there are six edges connected to the vertex v of Figure 7.8 and this require we compute the fifth coproduct of $M(H)$

$$\begin{aligned}\Delta^5(a\otimes h) \quad = \quad & a_{(6)}\otimes h_{(6)}\otimes a_{(5)}h_{(5)}Sh_{(7)}\otimes h_{(8)}\otimes a_{(4)}h_{(4)}Sh_{(9)}\otimes h_{(10)}\otimes a_{(3)}h_{(3)}Sh_{(11)} \\ & \otimes h_{(12)}\otimes a_{(2)}h_{(2)}Sh_{(13)}\otimes h_{(14)}\otimes a_{(1)}h_{(1)}Sh_{(15)}\otimes h_{(16)}, \end{aligned} \tag{7.39}$$

from which the fifth coproducts of the sub-Hopf algebras $H^{\text{cop}}\otimes 1$ and $1\otimes H$ are obtained respectively

$$\begin{aligned}\Delta^5(a\otimes 1) \quad &= \quad a_{(6)}\otimes 1\otimes a_{(5)}\otimes 1\otimes a_{(4)}\otimes 1\otimes a_{(3)}\otimes 1\otimes a_{(2)}\otimes 1\otimes a_{(1)}\otimes 1, \\ \Delta^5(1\otimes h) \quad &= \quad 1\otimes h_{(6)}\otimes h_{(5)}Sh_{(7)}\otimes h_{(8)}\otimes h_{(4)}Sh_{(9)}\otimes h_{(10)}\otimes h_{(3)}Sh_{(11)} \\ & \quad\quad \otimes h_{(12)}\otimes h_{(2)}Sh_{(13)}\otimes h_{(14)}\otimes h_{(1)}Sh_{(15)}\otimes h_{(16)}. \end{aligned} \tag{7.40}$$

(ii) Secondly, the face and vertex operators associated with Figure 7.8 are respectively

$$\begin{aligned}B^a(\phi^1\otimes ...\otimes\phi^7) \quad &= \quad T_+^{a(3)}(\phi^1)\otimes T_+^{a(2)}(\phi^2)\otimes T_+^{a(1)}(\phi^3)\otimes\phi^4\otimes\phi^5\otimes\phi^6\otimes\phi^7 \\ A^h(\phi^1\otimes ...\otimes\phi^7) \quad &= \quad L_+^{h_{(6)}}(\phi^1)\otimes L_-^{h_{(5)}Sh_{(7)}\otimes h_{(8)}}(\phi^2)\otimes\phi^3\otimes L_+^{h_{(4)}Sh_{(9)}\otimes h_{(10)}}(\phi^4) \\ & \quad\quad \otimes L_-^{h_{(3)}Sh_{(3)(11)}\otimes h_{(12)}}(\phi^5)\otimes L_-^{h_{(2)}Sh_{(13)}\otimes h_{(14)}}(\phi^6)\otimes L_+^{h_{(1)}Sh_{(15)}\otimes h_{(16)}}(\phi^7). \end{aligned} \tag{7.41}$$

(iii) Lastly, for Figure 7.8 to yield a well defined theorem, in the definition of the face operator above, we first applied the coproduct of $a\in H^{\text{cop}}\otimes 1$ clockwise along the edges enclosing the face p and then continued clockwise with the rest of the edges connecting the vertex v. In a similar manner, we define the vertex operator accordingly but this time applying the coproduct of $h \in 1 \otimes H$ instead to the edges of the arbitrary graph 7.8.

We proceed by a direct calculation, let $h \in H$, $a \in H^{\text{cop}}$ and $\phi^i \in H^*$, where $i \in \{1, 2, .., 7\}$. Starting with the LHS of equation (7.28), we have

$$\begin{aligned}
& A^h B^a(\phi^1 \otimes ... \otimes \phi^7) \\
= \;& \langle a_{(3)}, S\phi^1_{(1)} \rangle \langle a_{(2)}, S\phi^2_{(1)} \rangle \langle a_{(1)}, S\phi^3_{(1)} \rangle A^h \left(\phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3_{(2)} \otimes \phi^4 \otimes \phi^5 \otimes \phi^6 \otimes \phi^7 \right) \\
= \;& \langle a, S\phi^1_{(1)} S\phi^2_{(1)} S\phi^3_{(1)} \rangle L_+^{h_{(6)}}(\phi^1_{(2)}) \otimes L_-^{h_{(5)}Sh_{(7)} \otimes h_{(8)}}(\phi^2_{(2)}) \otimes \phi^3_{(2)} \otimes L_+^{h_{(4)}Sh_{(9)} \otimes h_{(10)}}(\phi^4) \\
& \otimes L_-^{h_{(3)}Sh_{(11)} \otimes h_{(12)}}(\phi^5) \otimes L_-^{h_{(2)}Sh_{(13)} \otimes h_{(14)}}(\phi^6) \otimes L_+^{h_{(1)}Sh_{(15)} \otimes h_{(16)}}(\phi^7) \\
= \;& \langle a, S\phi^1_{(1)} S\phi^2_{(1)} S\phi^3_{(1)} \rangle \langle h_{(6)}, S\phi^1_{(2)(1)} \phi^1_{(2)(3)} \rangle \phi^1_{(2)(2)} \otimes \langle h_{(5)}, \phi^2_{(2)(3)} \rangle \langle h_{(7)}, S^{-1}\phi^2_{(2)(1)} \rangle \phi^2_{(2)(2)} \\
& \otimes \phi^3_{(2)} \otimes \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \phi^4_{(2)} \otimes \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \phi^5_{(2)} \\
& \otimes \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \phi^6_{(2)} \otimes \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \phi^7_{(2)} \\
= \;& \langle a, S\phi^1_{(1)} S\phi^2_{(1)} S\phi^3_{(1)} \rangle \langle h_{(6)}, S\phi^1_{(2)} \phi^1_{(4)} \rangle \phi^1_{(3)} \otimes \langle h_{(5)}, \phi^2_{(4)} \rangle \langle h_{(7)}, S^{-1}\phi^2_{(2)} \rangle \phi^2_{(3)} \\
& \otimes \phi^3_{(2)} \otimes \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \phi^4_{(2)} \otimes \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \phi^5_{(2)} \\
& \otimes \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \phi^6_{(2)} \otimes \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \phi^7_{(2)} \\
= \;& \langle a, S\phi^1_{(1)(1)} S\phi^2_{(1)(1)} S\phi^3_{(1)} \rangle \langle h_{(6)}, S\phi^1_{(1)(2)} \phi^1_{(2)(2)} \rangle \phi^1_{(2)(1)} \otimes \langle h_{(5)}, \phi^2_{(2)(2)} \rangle \langle h_{(7)}, S^{-1}\phi^2_{(1)(2)} \rangle \phi^2_{(2)(1)} \\
& \otimes \phi^3_{(2)} \otimes \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \phi^4_{(2)} \otimes \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \phi^5_{(2)} \\
& \otimes \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \phi^6_{(2)} \otimes \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \phi^7_{(2)} \\
= \;& \langle a, S\phi^1_{(1)(1)} S\phi^2_{(1)(1)} S\phi^3_{(1)} \rangle \langle Sh_{(6)(1)}, \phi^1_{(1)(2)} \rangle \langle h_{(6)(2)}, \phi^1_{(2)(2)} \rangle \phi^1_{(2)(1)} \otimes \langle h_{(5)}, \phi^2_{(2)(2)} \rangle \langle h_{(7)}, S^{-1}\phi^2_{(1)(2)} \rangle \phi^2_{(2)(1)} \\
& \otimes \phi^3_{(2)} \otimes \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \phi^4_{(2)} \otimes \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \phi^5_{(2)} \\
& \otimes \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \phi^6_{(2)} \otimes \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \phi^7_{(2)} \\
= \;& \langle a, S\phi^1_{(1)} S\phi^2_{(1)(1)} S\phi^3_{(1)} \rangle \phi^1_{(2)} \otimes \langle h_{(5)}, \phi^2_{(2)(2)} \rangle \langle S^{-1}h_{(6)}, \phi^2_{(1)(2)} \rangle \phi^2_{(2)(1)} \\
& \otimes \phi^3_{(2)} \otimes \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \phi^4_{(2)} \otimes \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \phi^5_{(2)} \\
& \otimes \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \phi^6_{(2)} \otimes \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \phi^7_{(2)} \\
= \;& \langle a, S\phi^1_{(1)} S\phi^2_{(1)} S\phi^3_{(1)} \rangle \langle Sh_{(4)}, \phi^4_{(1)} \rangle \langle h_{(9)}, \phi^4_{(3)} \rangle \langle h_{(3)}, \phi^5_{(3)} \rangle \langle h_{(11)}, S^{-1}\phi^5_{(1)} \rangle \langle h_{(2)}, \phi^6_{(3)} \rangle \langle h_{(13)}, S^{-1}\phi^6_{(1)} \rangle \\
& \langle Sh_{(1)}, \phi^7_{(1)} \rangle \langle h_{(15)}, \phi^7_{(3)} \rangle \, \phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3_{(2)} \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)}
\end{aligned} \tag{7.42}$$

We used the definiton of the triangle operators (7.26) in the first and third equalities. The fifth coproduct of H in $M(H)$ is used in the second equality to label the $L_\pm$ operators. The dual pairing property (D.15) is used in the second and the sixth equalities. A renumbering is carried out in the fourth, fifth, sixth and seventh equalities using coassociativity in H^*. The counity axiom is used on H in the last two equalities.

In computing the RHS of equation (7.28), we first consider the vertex operetor $A^{h_{(3)}}$ acting on Figure 7.8

$$\begin{aligned}
& A^{h_{(3)}}(\phi^1 \otimes ... \otimes \phi^7) \\
= \; & L_+^{h_{(3)(6)}}(\phi^1) \otimes L_-^{h_{(3)(5)}Sh_{(3)(7)}\otimes h_{(3)(8)}}(\phi^2) \otimes \phi^3 \otimes L_+^{h_{(3)(4)}Sh_{(3)(9)}\otimes h_{(3)(10)}}(\phi^4) \\
& \otimes L_-^{h_{(3)(3)}Sh_{(3)(11)}\otimes h_{(3)(12)}}(\phi^5) \otimes L_-^{h_{(3)(2)}Sh_{(3)(13)}\otimes h_{(3)(14)}}(\phi^6) \otimes L_+^{h_{(3)(1)}Sh_{(3)(15)}\otimes h_{(3)(16)}}(\phi^7) \\
= \; & \langle h_{(3)(6)}, S\phi^1_{(1)}\phi^1_{(3)}\rangle\phi^1_{(2)} \otimes \langle h_{(3)(5)}, \phi^2_{(3)}\rangle\langle h_{(3)(7)}, S^{-1}\phi^2_{(1)}\rangle\phi^2_{(2)} \otimes \phi^3 \otimes \langle Sh_{(3)(4)}, \phi^4_{(1)}\rangle\langle h_{(3)(9)}, \phi^4_{(3)}\rangle\phi^4_{(2)} \\
& \otimes \langle h_{(3)(3)}, \phi^5_{(3)}\rangle\langle h_{(3)(11)}, S^{-1}\phi^5_{(1)}\rangle\phi^5_{(2)} \otimes \langle h_{(3)(2)}, \phi^6_{(3)}\rangle\langle h_{(3)(13)}, S^{-1}\phi^6_{(1)}\rangle\phi^6_{(2)} \\
& \otimes \langle Sh_{(3)(1)}, \phi^7_{(1)}\rangle\langle h_{(3)(15)}, \phi^7_{(3)}\rangle\phi^7_{(2)} \\
= \; & \langle h_{(8)}, S\phi^1_{(1)}\phi^1_{(3)}\rangle\phi^1_{(2)} \otimes \langle h_{(7)}, \phi^2_{(3)}\rangle\langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle\phi^2_{(2)} \otimes \phi^3 \otimes \langle Sh_{(6)}, \phi^4_{(1)}\rangle\langle h_{(11)}, \phi^4_{(3)}\rangle\phi^4_{(2)} \\
& \otimes \langle h_{(5)}, \phi^5_{(3)}\rangle\langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle\phi^5_{(2)} \otimes \langle h_{(4)}, \phi^6_{(3)}\rangle\langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle\phi^6_{(2)} \otimes \langle Sh_{(3)}, \phi^7_{(1)}\rangle\langle h_{(17)}, \phi^7_{(3)}\rangle\phi^7_{(2)}
\end{aligned} \tag{7.43}$$

We used the fifth coproduct of $h_{(3)} \in H$ in $M(H)$ in the first equality to label the $L_\pm$ operators. The definitions of the $L_\pm$ in equation (7.26) are used in the third and fifth equalities. Renumbering is done in the last equality as a result of the coassociativity in

H^*. We now apply the operator $B^{h_{(1)}aSh_{(2)}}$ on (7.43) to get

$$
\begin{aligned}
& B^{h_{(1)}aSh_{(2)}} A^{h_{(3)}} (\phi^1 \otimes ... \otimes \phi^7) \\
= \; & B^{h_{(1)}aSh_{(2)}} \left(\phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3 \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)}\right) \\
& \langle h_{(8)}, S\phi^1_{(1)}\phi^1_{(3)}\rangle \langle h_{(7)}, \phi^2_{(3)}\rangle \langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle \langle Sh_{(6)}, \phi^4_{(1)}\rangle \langle h_{(11)}, \phi^4_{(3)}\rangle \langle h_{(5)}, \phi^5_{(3)}\rangle \langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle \\
& \langle h_{(4)}, \phi^6_{(3)}\rangle \langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle \langle Sh_{(3)}, \phi^7_{(1)}\rangle \langle h_{(17)}, \phi^7_{(3)}\rangle \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(2)(1)} S\phi^2_{(2)(1)} S\phi^3_{(1)}\rangle \langle h_{(8)}, S\phi^1_{(1)}\phi^1_{(3)}\rangle \langle h_{(7)}, \phi^2_{(3)}\rangle \langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle \langle Sh_{(6)}, \phi^4_{(1)}\rangle \langle h_{(11)}, \phi^4_{(3)}\rangle \\
& \langle h_{(5)}, \phi^5_{(3)}\rangle \langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle \langle h_{(4)}, \phi^6_{(3)}\rangle \langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle \langle Sh_{(3)}, \phi^7_{(1)}\rangle \langle h_{(17)}, \phi^7_{(3)}\rangle \\
& \phi^1_{(2)(2)} \otimes \phi^2_{(2)(2)} \otimes \phi^3_{(2)} \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)} \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(2)} S\phi^2_{(2)} S\phi^3_{(1)}\rangle \langle h_{(8)}, S\phi^1_{(1)}\phi^1_{(4)}\rangle \langle h_{(7)}, \phi^2_{(4)}\rangle \langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle \langle Sh_{(6)}, \phi^4_{(1)}\rangle \langle h_{(11)}, \phi^4_{(3)}\rangle \\
& \langle h_{(5)}, \phi^5_{(3)}\rangle \langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle \langle h_{(4)}, \phi^6_{(3)}\rangle \langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle \langle Sh_{(3)}, \phi^7_{(1)}\rangle \langle h_{(17)}, \phi^7_{(3)}\rangle \\
& \phi^1_{(3)} \otimes \phi^2_{(3)} \otimes \phi^3_{(2)} \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)} \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(1)(2)} S\phi^2_{(2)} S\phi^3_{(1)}\rangle \langle h_{(8)}, S\phi^1_{(1)(1)}\phi^1_{(2)(2)}\rangle \langle h_{(7)}, \phi^2_{(4)}\rangle \langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle \langle Sh_{(6)}, \phi^4_{(1)}\rangle \\
& \langle h_{(11)}, \phi^4_{(3)}\rangle \langle h_{(5)}, \phi^5_{(3)}\rangle \langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle \langle h_{(4)}, \phi^6_{(3)}\rangle \langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle \langle Sh_{(3)}, \phi^7_{(1)}\rangle \langle h_{(17)}, \phi^7_{(3)}\rangle \\
& \phi^1_{(2)(1)} \otimes \phi^2_{(3)} \otimes \phi^3_{(2)} \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)} \\
= \; & \langle h_{(1)}aSh_{(2)}, (S\phi^1_{(1)})_{(1)} S\phi^2_{(2)} S\phi^3_{(1)}\rangle \langle h_{(8)}, (S\phi^1_{(1)})_{(2)}\phi^1_{(2)(2)}\rangle \langle h_{(7)}, \phi^2_{(4)}\rangle \langle h_{(9)}, S^{-1}\phi^2_{(1)}\rangle \langle Sh_{(6)}, \phi^4_{(1)}\rangle \\
& \langle h_{(11)}, \phi^4_{(3)}\rangle \langle h_{(5)}, \phi^5_{(3)}\rangle \langle h_{(13)}, S^{-1}\phi^5_{(1)}\rangle \langle h_{(4)}, \phi^6_{(3)}\rangle \langle h_{(15)}, S^{-1}\phi^6_{(1)}\rangle \langle Sh_{(3)}, \phi^7_{(1)}\rangle \langle h_{(17)}, \phi^7_{(3)}\rangle \\
& \phi^1_{(2)(1)} \otimes \phi^2_{(3)} \otimes \phi^3_{(2)} \otimes \phi^4_{(2)} \otimes \phi^5_{(2)} \otimes \phi^6_{(2)} \otimes \phi^7_{(2)}.
\end{aligned}
\tag{7.44}
$$

In the second equality, we apply the definition of the face operator whiles in the third equality, we carry out a renumbering due to coassociativity in H^*. In moving from the fourth to the fifth equality, we used the property of the antipode as an anticoalgebra map.

Applying the antipode axiom in the last equality and a further renumbering in H^* gives

$$\begin{aligned}
& B^{h_{(1)}aSh_{(2)}}A^{h_{(3)}}(\phi^1\otimes\ldots\otimes\phi^7) \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(1)}S\phi^2_{(1)(2)}S\phi^3_{(1)}\rangle\langle h_{(7)},\phi^2_{(2)(2)}\rangle\langle h_{(8)},S^{-1}\phi^2_{(1)(1)}\rangle\langle Sh_{(6)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle \\
& \langle h_{(5)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(4)},\phi^6_{(3)}\rangle\langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle\langle Sh_{(3)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle \\
& \phi^1_{(2)}\otimes\phi^2_{(2)(1)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(1)}(S\phi^2_{(1)})_{(1)}S\phi^3_{(1)}\rangle\langle h_{(7)},\phi^2_{(2)(2)}\rangle\langle h_{(8)},(S^{-1}\phi^2_{(1)})_{(2)}\rangle\langle Sh_{(6)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle \\
& \langle h_{(5)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(4)},\phi^6_{(3)}\rangle\langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle\langle Sh_{(3)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle \\
& \phi^1_{(2)}\otimes\phi^2_{(2)(1)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle h_{(1)}aSh_{(2)}, S\phi^1_{(1)}S\phi^2_{(1)}S\phi^3_{(1)}\rangle\langle Sh_{(6)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle\langle h_{(5)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(4)},\phi^6_{(3)}\rangle \\
& \langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle\langle Sh_{(3)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle\,\phi^1_{(2)}\otimes\phi^2_{(2)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle h_{(2)}SaSh_{(1)}, \phi^3_{(1)}\phi^2_{(1)}\phi^1_{(1)}\rangle\langle Sh_{(6)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle\langle h_{(5)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(4)},\phi^6_{(3)}\rangle \\
& \langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle\langle Sh_{(3)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle\,\phi^1_{(2)}\otimes\phi^2_{(2)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle h_{(2)}, \phi^3_{(1)(1)}\phi^2_{(1)(1)}\phi^1_{(1)(1)}\rangle\langle Sa, \phi^3_{(1)(2)}\phi^2_{(1)(2)}\phi^1_{(1)(2)}\rangle\langle Sh_{(1)}, \phi^3_{(1)(3)}\phi^2_{(1)(3)}\phi^1_{(1)(3)}\rangle \\
& \langle Sh_{(6)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle\langle h_{(5)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(4)},\phi^6_{(3)}\rangle\langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle \\
& \langle Sh_{(3)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle\,\phi^1_{(2)}\otimes\phi^2_{(2)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle h_{(1)(2)}, \phi^3_{(1)(1)}\phi^2_{(1)(1)}\phi^1_{(1)(1)}\rangle\langle Sh_{(1)(1)}, \phi^3_{(2)(1)}\phi^2_{(2)(1)}\phi^1_{(2)(1)}\rangle\langle Sa, \phi^3_{(1)(2)}\phi^2_{(1)(2)}\phi^1_{(1)(2)}\rangle \\
& \langle Sh_{(5)},\phi^4_{(1)}\rangle\langle h_{(11)},\phi^4_{(3)}\rangle\langle h_{(4)},\phi^5_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(3)},\phi^6_{(3)}\rangle\langle h_{(15)},S^{-1}\phi^6_{(1)}\rangle \\
& \langle Sh_{(2)},\phi^7_{(1)}\rangle\langle h_{(17)},\phi^7_{(3)}\rangle\,\phi^1_{(2)(2)}\otimes\phi^2_{(2)(2)}\otimes\phi^3_{(2)(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle Sa, \phi^3_{(1)}\phi^2_{(1)}\phi^1_{(1)}\rangle\langle Sh_{(4)},\phi^4_{(1)}\rangle\langle h_{(9)},\phi^4_{(3)}\rangle\langle h_{(3)},\phi^5_{(3)}\rangle\langle h_{(11)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(2)},\phi^6_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^6_{(1)}\rangle \\
& \langle Sh_{(1)},\phi^7_{(1)}\rangle\langle h_{(15)},\phi^7_{(3)}\rangle\,\phi^1_{(2)}\otimes\phi^2_{(2)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)} \\
= \; & \langle a, S\phi^1_{(1)}S\phi^2_{(1)}S\phi^3_{(1)}\rangle\langle Sh_{(4)},\phi^4_{(1)}\rangle\langle h_{(9)},\phi^4_{(3)}\rangle\langle h_{(3)},\phi^5_{(3)}\rangle\langle h_{(11)},S^{-1}\phi^5_{(1)}\rangle\langle h_{(2)},\phi^6_{(3)}\rangle\langle h_{(13)},S^{-1}\phi^6_{(1)}\rangle \\
& \langle Sh_{(1)},\phi^7_{(1)}\rangle\langle h_{(15)},\phi^7_{(3)}\rangle\,\phi^1_{(2)}\otimes\phi^2_{(2)}\otimes\phi^3_{(2)}\otimes\phi^4_{(2)}\otimes\phi^5_{(2)}\otimes\phi^6_{(2)}\otimes\phi^7_{(2)}. \qquad (7.45)
\end{aligned}$$

Again we applied the property of the antipode as an anticoalgebra map from the first to the second equality. To get to the third equality, we used the pairing property (D.15) and the antipode axiom on H^*. In the fourth equality we used both the pairing property (D.17) and the fact that the antipode is an antialgebra map. The paring conditions (D.15) and (D.16) are used in the fifth equality. While a renumbering is done in the sixth equality on both H and H^*. The counit and antipode axioms are used in to get to the seventh equality from the sixth equality. Finally we see that the equations (7.42) and (7.45) are the same, and hence this proves Theorem 7.3.3. □

Hamiltonian: We are now ready to define the Hamiltonian of the mirror bicrossproduct Kitaev model. Let $H^{\rm cop}$ and H be finite-dimensional Hopf C^*-algebras and hence are endowed with normalized Haar integrals. We refer to appendix D.4 for some properties of Haar integrals.

The non-degenerate Hermitian inner product on H^* is defined by [113, 114]

$$\langle\phi|\psi\rangle_{H^*} := \langle l, \phi^\star\psi\rangle, \quad \phi,\psi\in H^*, \tag{7.46}$$

where l is the normalized Haar integral of H. The inner product (7.46) makes the triangle operators $L_\pm$ and $T_\pm$ into $\star$-representations with adjoint maps given by

$$(L^h_\pm)^\dagger = L^{h^\star}_\pm, \qquad (T^a_\pm)^\dagger = T^{a^\star}_\pm.$$

For example, we check this for T^a_- as follows:

$$\begin{aligned}
\langle\phi|T^a_-(\psi)\rangle_{H^*} &= \langle l, \phi^\star T^a_-(\psi)\rangle = \langle l, \phi^\star\langle a, \psi_{(2)}\rangle\psi_{(1)}\rangle = \langle l, \langle a_{(3)}, \psi_{(2)}\rangle\langle a_{(2)}Sa_{(1)}, \phi^*_{(2)}\rangle\phi^*_{(1)}\psi_{(1)}\rangle\\
&= \langle l, \langle a_{(3)}, \psi_{(2)}\rangle\langle a_{(2)}, \phi^*_{(2)}\rangle\langle Sa_{(1)}, \phi^*_{(3)}\rangle\phi^*_{(1)}\psi_{(1)}\rangle\\
&= \langle l, \langle a_{(2)}, \psi_{(2)}\phi^*_{(2)}\rangle\langle Sa_{(1)}, \phi^*_{(3)}\rangle\phi^*_{(1)}\psi_{(1)}\rangle\\
&= \langle l, \langle a_{(2)}, (\psi\phi^*_{(1)})_{(2)}\rangle\langle Sa_{(1)}, \phi^*_{(2)}\rangle(\phi^*_{(1)}\psi)_{(1)}\rangle\\
&= \langle l, \langle a_{(2)}, \phi^*_{(2)}\rangle\phi^*_{(1)}\psi\rangle = \langle T^{a^*}_-(\phi)|\psi\rangle.
\end{aligned}$$

Similarly for $L_\pm$ and T_+. Consequently, the operators A^h and B^a are Hermitian since they are tensor products of the $L_\pm$ and $T_\pm$ operators, i.e.,

$$(A^h(v,p))^\dagger = A^{h^\star}(v,p), \quad (B^a(v,p))^\dagger = B^{a^\star}(v,p). \tag{7.47}$$

Now since the Haar integrals commute with every other element in the Hopf algebra, we use them to define projectors $A_v := A^l_v$ for each vertex and $B_p := B^k_p$ for each face. It follows one can then state Lemma 7.1.4 for the mirror bicrossproduct model which leads to the definition of its Hamiltonian $\mathfrak{H}$ which has a similar expression to (7.7). The exception being that the geometric operators are now those of the mirror bicrossproduct lattice model. The space of ground state of the Hamiltonian $\mathfrak{H}$ is given by the invariant subspace

$$\mathcal{P}_\Gamma := \{\phi\in\mathcal{H}_\Gamma : A_v(\phi)=\phi,\ B_p(\phi)=\phi,\ \forall v,p\}. \tag{7.48}$$

7.4 Tensor network representations for mirror bicrossproduct models

Following [59], we would like to construct a tensor network representation for one of the ground states of the mirror bicrossproduct model of Section 7.3. Other eigenstates may be obtained by the application of the appropriate ribbon operator [59]. Our starting point is to provide the diagrammatic framework for the tensor network states built on Γ and decorated by H^*. The construction includes graphs whose underlying surface has boundaries. We define then the notion of tensor trace which allows to construct the tensor network states on Γ.

7.4.1 Diagrammatic scheme for tensor network states and tensor trace

To each oriented edge $e \in \Gamma$, we associate a tensor as indicated in Figure 7.9 below. Here the black dot represents the orientation of the edge inherited from the underlying

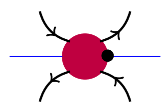


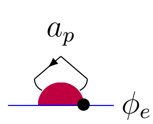
Figure 7.9:

graph (physical edge). The black arrows (virtual edges) attached to the tensor represents the indices of the tensor. The association of a tensor to each edge of Γ also amounts to placing an anti-clockwise oriented virtual loop in each face of the graph Γ. A virtual loop determines a face $p \in F$, to which we associate an element $a_p \in H^{\text{cop}}$.

The rule for contraction of the tensor network is as follows: one first splits the tensor and then contracts each pair of virtual edges separately and then glue these pieces together. This splitting process is implemented by the coproduct in H^*. An element $\phi_e \in H^*$ associated to each edge $e \in E$ can be split into two elementary parts depending on the orientation of the underlying edge according to the rule

$$((S \otimes \text{id}) \circ \Delta)(\phi_e) = S\phi_e^{(1)} \otimes \phi_e^{(2)}. \tag{7.49}$$

Thus, to the left and right adjacent face of e, we can assign $\phi_e^{(2)}$ and $S\phi_e^{(1)}$ respectively as shown in Fig. 7.10. For any $a_p \in H^{\text{cop}}$ and $\phi_e \in H^*$, the contraction of a pair of virtual edges is given by the canonical pairing as shown in (7.50)



$$:= \langle \phi_e, a_p \rangle. \tag{7.50}$$

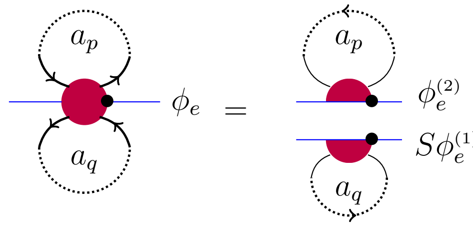
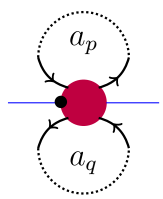


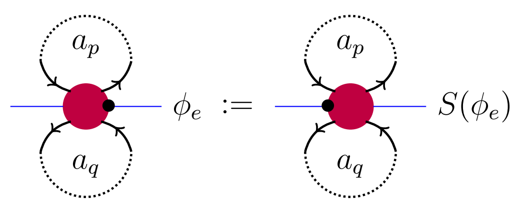
Figure 7.10: Splitting rule

To evaluate a given virtual loop p one performs a clockwise multiplication of all elements labelling the physical edges of the loop and canonically pair the result with a_p. Graphically this takes the form



$$:= \langle S\phi_e^{(1)} \cdots , a_p \rangle \langle \phi_e^{(2)} \cdots , a_q \rangle. \tag{7.51}$$

A change in the orientation of a physical edge using the antipode in H^* changes the orientation of the corresponding tensor as shown in (7.52)



(7.52)

Definition 7.4.1. *(Tensor trace) Let Γ be the dual graph corresponding to a surface without boundaries. The Hopf tensor trace associated with the graph Γ is the map $ttr_\Gamma : H^{*\otimes|E|} \otimes H^{cop\otimes|F|} \to \mathbb{C}$, defined by*

$$\bigotimes_{e\in E} \phi_e \bigotimes_{p\in F} a_p \longmapsto ttr_\Gamma(\{\phi_e\}; \{a_p\}) \tag{7.53}$$

is given in terms of diagrams and evaluated using equations (7.51) and (7.52).

Note that this Hopf tensor trace in the bicrossproduct model acts on a space dual to that of the quantum double model defined in [59], $(H \otimes H^{*\mathrm{op}})^* = H^* \otimes H^{\mathrm{cop}}$. This can be regarded as the wave function amplitude of a quantum many body-system.

We consider next the case where Γ is embedded on a surface with boundaries. The set of edges E of the graph Γ corresponding to the surface Σ may be decomposed into a disjoint union of interior edges and boundary edges while the set of faces F have no boundary faces as depicted in Figure 7.11. The inherent features of the face and the edge sets of a graph Γ embedded in a surface Σ may be classified depending on whether they are in the interior and/or on the boundary of the surface. Naturally, the surface has no boundary faces but by deforming two or more boundary edges, a new (complete) face is created. This is illustrated in Figure 7.11 below. These features makes it possible to discuss tensor networks not only for graphs embedded in a surface with boundaries but also at points between where regions of a surface with or without boundary meet.

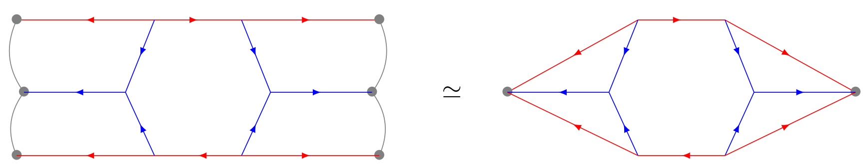


Figure 7.11: The interior and boundary faces of a graph. The edges in gray are those where the boundary of a surface Σ do not meet the edge of a graph Γ. In creating new faces, the gray vertices of the left diagram identify themselves upon deformation of boundary edges as shown on the right diagram. Boundary edges are shown in red.

Forming a set consisting of the different sections of the boundary edges and faces, a natural ordering is inherited from the orientation of the boundary of the surface by this set. This occurs once a specific section is fixed. For our discussion, the orientation of any boundary is fixed at an anticlockwise direction with regards to the interior of the surface. Taking into account boundaries of Σ, the graphical definition of the tensor networks previously given changes. Given an edge either from the set of interior or boundary edges, for any $a_p \in H^{\mathrm{cop}}$ and $\phi_e \in H^*$, the canonical pairing is given by (7.50). Different orientations of graph edges and loops are related using the relevant antipode as shown in Figure 7.12.

Let us now discuss how we can extend these diagrams to higher numbers of edges or faces. First if the face p has more edges e in its boundary, we extend the diagrams in Figure 7.12 as follows. Consider another edge e' which shares a common vertex with e. We define a glueing operation as in Figure 7.13. Here the arrows indicate the order in which

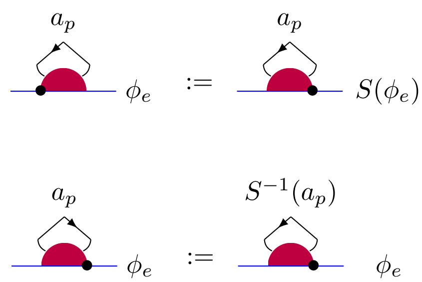


Figure 7.12: The antipode is used to change the orientation.

the coproduct of a_p is applied to the basic diagrams. The red dot indicates the origin of this coproduct. Note that if a_p is cocommutative, the red dot can be ignored in which case

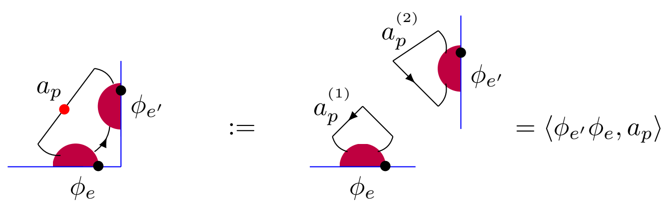
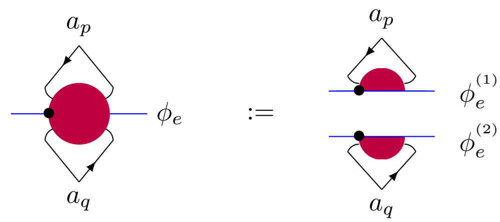


Figure 7.13: Glueing edges with the coproduct of H^{cop}

the order of the coproduct does not matter. However in the above instance a_p cannot be cocommutative since $a_p \in H^{\mathrm{cop}}$ and as such the order of the coproduct must be taken into account.

Secondly, the edge e will be in general adjacent to two faces, since there are no boundaries. So for any edge e with adjacent faces p, q, we pick $\phi_e \in H^*$ and $a_p, a_q \in H^{\mathrm{cop}}$ and define the face glueing operation as in Figure 7.54. If the faces p and q have many edges, we have to put together Figure 7.13 and Figure 7.54. If furthermore one loop is outgoing we have to also consider the antipode following Figure 7.12.



(7.54)

These tensors are then evaluated using the tensor trace which is nothing but the graphical rules we just set up. The fully contracted tensor network, which is a complex number, for a certain ground state of the bicrossproduct model on the graph Γ can be interpreted as a collection of virtual loops in the faces of Γ that have been suitably glued together to form the physical degrees of freedom.

Definition 7.4.2. *(Hopf tensor trace with boundaries) Let ∂E and ∂F be sets of boundary edges and faces respectively of Γ. The Hopf tensor trace associated with the graph Γ is the function $ttr_\Gamma : H^{*\otimes|E|} \otimes H^{cop\otimes|F|} \otimes H^{*\otimes|\partial E|} \otimes H^{cop\otimes|\partial F|} \to \mathbb{C}$,*

$$\bigotimes_{e\in E} \phi_e \bigotimes_{p\in F} a_p \bigotimes_{e\in\partial E} \phi'_e \bigotimes_{q\in\partial F} a_q \longmapsto ttr_\Gamma(\{\phi_e\};\{a_p\};\{\phi'_e\};\{a_q\}) \tag{7.55}$$

which is defined via diagrams and the evaluation rules given in Figures 7.51, 7.13 and 7.54.

7.4.2 Quantum state

We now use the tensor trace to define quantum states for the bicrossproduct model.

Definition 7.4.3. *Let $\phi_e \in H^*$ and $a_p \in H^{cop}$. Let Γ be the dual of the graph embedded in a surface Σ with no boundaries. The Hopf tensor network state on the graph Γ is given by*

$$|\Psi_\Gamma(\{\phi_e\};\{a_p\})\rangle := ttr_\Gamma(\{\phi_e^{(2)}\};\{a_p\}) \bigotimes_{e\in E_{\Sigma I}} |\phi_e^{(1)}\rangle. \tag{7.56}$$

We shall now proceed to solve the bicrossproduct model in this framework of Hopf tensor network states. We choose a particular Hopf tensor network state as a ground state of the model so that this state is topologically ordered. We consider the case of a surface without boundaries.

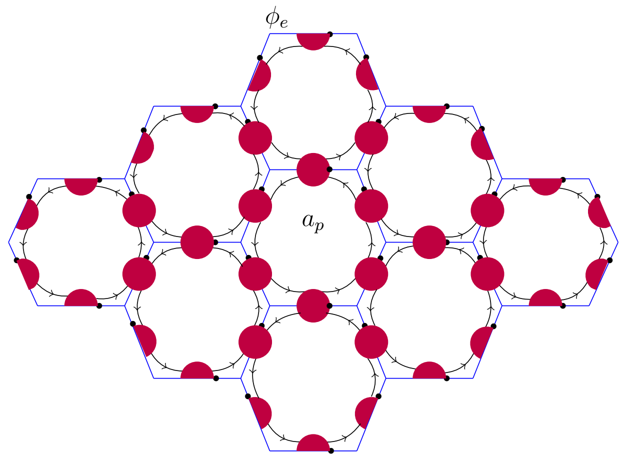


Figure 7.14: Example of a tensor network, for a hexagonal graph Γ.

Theorem 7.4.4. *(Ground state of the mirror bicrossproduct model). Let η and k the Haar integrals of H^* and H^{cop} respectively. Then a degenerate ground state of the mirror bicrossproduct $H^{cop} \blacktriangleright\!\!\triangleleft H$-model is*

$$|\Psi_\Gamma\rangle := |\Psi_\Gamma(\{\eta_e\}; \{k_p\})\rangle \tag{7.57}$$

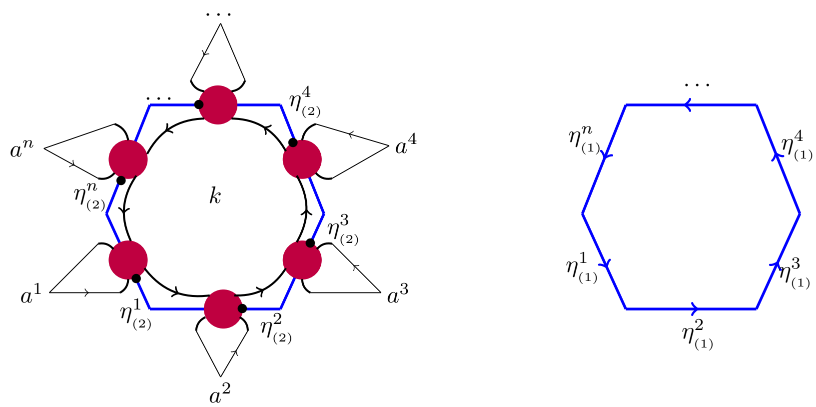
where for $\phi_e \in H^*$ and $a_p \in H^{cop}$ we have

$$|\Psi_\Gamma(\{\phi_e\}; \{a_p\})\rangle := ttr_\Gamma(\{\phi_e^{(2)}\}; \{a_p\}) \bigotimes_{e \in \Gamma} |\phi_e^{(1)}\rangle. \tag{7.58}$$

Recall that the Hamiltonian for the mirror bicrossproduct model is a sum of local commuting terms A_v and B_p. Hence it is sufficient to show that the operators A_v and B_p leave the state $|\Psi_\Gamma\rangle$ invariant individually.

Proof. Consider a face p with a boundary consisting of n edges. A face of Γ, decorated by the Haar integral k, leads to the contribution $|\Psi_\Gamma\rangle$ given by

$|\Psi_p(a^1, ..., a^n)\rangle =$



Here, we have not included the other faces specified by a^i. Note that the left diagram is the tensor trace function. The state $|\Psi_p(a^1, ..., a^n)\rangle$ written in an explicit form reads

$$\begin{aligned} |\Psi_p(a^1, ..., a^n)\rangle &= \langle \eta^1_{(2)(2)} \cdots \eta^n_{(2)(2)}, k\rangle \langle S\eta^1_{(2)(1)}, a^1\rangle \cdots \langle S\eta^n_{(2)(1)}, a^n\rangle |\eta^1_{(1)}\rangle \otimes \cdots \otimes |\eta^n_{(1)}\rangle \\ &= \langle \eta^1_{(3)} \cdots \eta^n_{(3)}, k\rangle \langle S\eta^1_{(2)}, a^1\rangle \cdots \langle S\eta^n_{(2)}, a^n\rangle |\eta^1_{(1)}\rangle \otimes \cdots \otimes |\eta^n_{(1)}\rangle \\ &= \langle \eta^1_{(3)} \cdots \eta^n_{(3)}, k\rangle \prod_j^n \langle S\eta^j_{(2)}, a^j\rangle \, |\eta^1_{(1)}\rangle \otimes \cdots \otimes |\eta^n_{(1)}\rangle, \end{aligned} \tag{7.59}$$

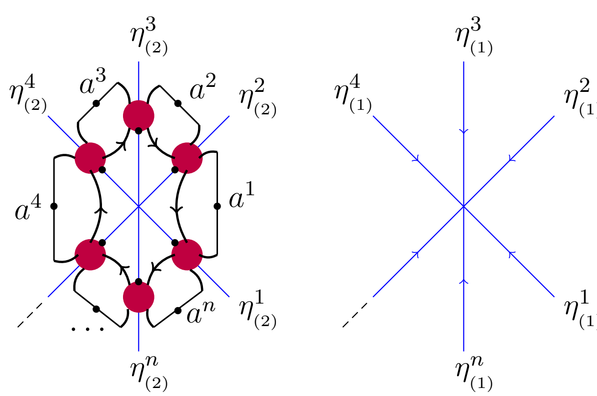
where the Haar integrals $\eta^i = \eta \in H^*$ and the expression $\eta^1_{(3)} \cdots \eta^n_{(3)}$ denote composition of the elements η. We used (7.58) to write down the contribution to the state $|\Psi_\Gamma\rangle$. To each edge on the left hand side of the above diagram, we labelled it by $\eta_{(2)}$. We split $\eta_{(2)}$ according to (7.49), then assign $\eta_{(2)(2)}$ and $S\eta_{(2)(1)}$, to the left and right adjacent faces of each edge respectively. To each of the outer nontrivial faces, we evaluate them according to Figure 7.13. The state (7.59) is invariant under the action of B_p as shown below

$$
\begin{aligned}
B_p^k|\Psi_p(a^1,.....,a^n)\rangle &= \langle \eta^1_{(3)}\cdots\eta^n_{(3)},\, k\rangle\langle \eta^1_{(1)(1)}\cdots\eta^n_{(1)(1)},\, k\rangle \prod_j^n \langle S\eta^j_{(2)},\, a^j\rangle\, |\eta^1_{(1)(2)}\rangle\otimes\cdots\otimes|\eta^n_{(1)(2)}\rangle \\
&= \langle \eta^1_{(4)}\cdots\eta^n_{(4)},\, k\rangle\langle \eta^1_{(1)}\cdots\eta^n_{(1)},\, k\rangle \prod_j^n \langle S\eta^j_{(3)},\, a^j\rangle\, |\eta^1_{(2)}\rangle\otimes\cdots\otimes|\eta^n_{(2)}\rangle \\
&= \langle \eta^1_{(3)}\cdots\eta^n_{(3)},\, k\rangle\langle \eta^1_{(4)}\cdots\eta^n_{(4)},\, k\rangle \prod_j^n \langle S\eta^j_{(2)},\, a^j\rangle\, |\eta^1_{(1)}\rangle\otimes\cdots\otimes|\eta^n_{(1)}\rangle \\
&= \langle (\eta^1_{(3)}\cdots\eta^n_{(3)})^2,\, k\rangle \prod_j^n \langle S\eta^j_{(2)},\, a^j\rangle\, |\eta^1_{(1)}\rangle\otimes\cdots\otimes|\eta^n_{(1)}\rangle \\
&= \langle \eta^1_{(3)}\cdots\eta^n_{(3)},\, k\rangle \prod_j^n \langle S\eta^j_{(2)},\, a^j\rangle\, |\eta^1_{(1)}\rangle\otimes\cdots\otimes|\eta^n_{(1)}\rangle \\
&= |\Psi_p(a^1,.....,a^n)\rangle. \qquad (7.60)
\end{aligned}
$$

In the first equality, we used the action of the face operator B_p^k of Definition 7.3.2 on the state $|\Psi_p(a^1,.....,a^n)\rangle$. We perform a renumbering on η in the second equality. The fifth equality uses the property of the Haar integral.

Next we consider a vertex v with n ingoing edges. The vertex contribution to Ψ_Γ is

$$|\Psi_v(a^1,.....,a^n)\rangle = \sum_{\eta^i}$$



where once again the left diagram is a tensor trace function. In a more explicit form the state contribution from the vertex is

$$
\begin{aligned}
|\Psi_v(a^1, ..., a^n)\rangle &= \langle S\eta^1_{_{(2)(1)}} \cdot \eta^2_{_{(2)(2)}}, \, a^1\rangle \cdots \langle S\eta^{n-1}_{_{(2)(1)}} \cdot \eta^n_{_{(2)(2)}}, \, a^{n-1}\rangle \langle S\eta^n_{_{(2)(1)}} \cdot \eta^1_{_{(2)(2)}}, \, a^n\rangle \, |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle \\
&= \langle S\eta^1_{_{(2)}} \cdot \eta^2_{_{(3)}}, \, a^1\rangle \cdots \langle S\eta^{n-1}_{_{(2)}} \cdot \eta^n_{_{(3)}}, \, a^{n-1}\rangle \langle S\eta^n_{_{(2)}} \cdot \eta^1_{_{(3)}}, \, a^n\rangle \, |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle \\
&= \langle S\eta^n_{_{(2)}} \cdot \eta^1_{_{(3)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(2)}} \cdot \eta^{j+1}_{_{(3)}}, \, a^j\rangle |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle. \qquad (7.61)
\end{aligned}
$$

This is then invariant under the action of A_v:

$$
\begin{aligned}
& A^l_v|\Psi_v(a^1, ..., a^n)\rangle \\
= & \langle S\eta^n_{_{(2)}} \cdot \eta^1_{_{(3)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(2)}} \cdot \eta^{j+1}_{_{(3)}}, \, a^j\rangle \langle (S\eta^j_{_{(1)(1)}})\eta^j_{_{(1)(3)}}, l^j\rangle \, |\eta^1_{_{(1)(2)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)(2)}}\rangle \\
= & \langle S\eta^n_{_{(4)}} \cdot \eta^1_{_{(5)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(4)}} \cdot \eta^{j+1}_{_{(5)}}, \, a^j\rangle \langle (S\eta^j_{_{(1)}})\eta^j_{_{(3)}}, l^j\rangle \, \rangle |\eta^1_{_{(2)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(2)}}\rangle \\
= & \langle S\eta^n_{_{(4)}} \cdot \eta^1_{_{(5)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(4)}} \cdot \eta^{j+1}_{_{(5)}}, \, a^j\rangle \langle (S\eta^j_{_{(3)}})\eta^j_{_{(2)}}, l^j\rangle \, \rangle |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle \\
= & \langle S\eta^n_{_{(3)}} \cdot \eta^1_{_{(4)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(3)}} \cdot \eta^{j+1}_{_{(4)}}, \, a^j\rangle \, \epsilon(\eta^j_{_{(2)}}) \, \epsilon(l^j) \, \rangle |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle \\
= & \langle S\eta^n_{_{(2)}} \cdot \eta^1_{_{(3)}}, \, a^n\rangle \prod_{j=1}^{n-1} \langle S\eta^j_{_{(2)}} \cdot \eta^{j+1}_{_{(3)}}), \, a^j\rangle \, |\eta^1_{_{(1)}}\rangle \otimes \cdots \otimes |\eta^n_{_{(1)}}\rangle = |\Psi_v(a^1, ..., a^n)\rangle. \, (7.62)
\end{aligned}
$$

We used the definition of the vertex operator of 7.3.2 in the first equality. We permute cyclicly the different components of η in the definition of the vertex operator in the second equality. The third equality uses the counit property of a Hopf algebra. While in the fourth equality we used the fact that $\epsilon(\eta^j_{_{(2)}}) = \epsilon(l^j) = 1$, to get to the fifth equality.

□

The quantum state $|\Psi_\Gamma\rangle$ is nothing but a trivial representation of the $H^{\rm cop} \bowtie H$ and the vacuum of the model and as such it has trivial topological charge everywhere.

Chapter 8

Conclusion and Discussion

In this thesis we have looked at three different topics related to quantum gravity. These topics were on: (i) the equivalence of general relativity and teleparallel gravity as first order formulations of gravity, and how the discretization of the Einstein-Cartan action in three dimensions translate into this equivalence (ii) the derivation of the quantum group structure in loop quantum gravity with a non-zero cosmological constant from the continuum action (iii) the definition of new Hopf algebra lattice models

We will review these results and point out some problems which have been left open for further investigation.

In chapter 4, we presented the result of formulating a first order teleparallel gravity. Bearing in mind that the teleparallel action is obtained from the general relativity one by an integration by part it is not so surprising to see that the first order formalism for teleparallel gravity is obtained from an integration by parts of the first order formulation of the general relativity action. The key-idea to recover this is to split the connection degrees of freedom in terms of a reference connection and the contorsion, which is slightly different than what is usually done. Then, this allows us to show that as the Einstein-Cartan action is seen as the first order formulation of the general relativity action, it is also the first order formulation of the teleparallel action.

Such result allows to justify that statements made in [29]. Namely on one hand that dual loop gravity is related to the teleparallel picture and that furthermore the loop gravity and dual loop gravity can be viewed as a change of polarization. These two polarizations are equivalent in the continuum but lead to two different discrete theories. We expect that the equivalence of choice of polarization should lead to an equivalence of discretization schemes expressed as a duality (implementing the Poincaré duality found in the continuum). This is currently investigated.

In chapter 5, we addressed a long standing question about the origin of the quantum group structure in loop quantum gravity with $\Lambda \neq 0$. Our approach was to derive the q-deformation from the 3d gravity action through the process of discretization of the spatial manifold. The presence of this quantum structure leads then to quantum homogeneously curved discrete geometries. We used the Chern-Simons formulation, an equivalent formulation of 3d gravity, to package the phase variables with the help of the Iwasawa decomposition. In this new phase space variables, we obtained nice symmetries of the new constraints and computing the sympletic form of the discretized phase space we derived the Heisenberg double phase space formulation, which upon quantization gives a quantum group structure.

Our result naturally led to the ribbon model of [4] whose quantization in [51] led to the recovering of the Turaev-Viro amplitude with a q-deformation. Our result has opened up the possibility of also deriving a quantum group structure from a 4d gravity action with $\Lambda \neq 0$. This is an ongoing investigation. Another interesting question is to see how the cosmological constant modifies the construction of the flat teleparallel formulation described. We expect such theory to exist since we have seen that at least in 3d, there is a deformed dual loop gravity theory. Also Dittrich and Geiller [115] discussed how the dual BF vacuum construction is also deformed using quantum group structures. This suggests that there must be a teleparallel formulation of gravity that is discretized along some teleparallel analogue of homogeneously curved geometries. We are currently exploring what should be this analogue.

In chapter 7 we proposed for the first time a Kitaev lattice model built not based on the Drinfeld quantum double, but instead on the (mirror) bicrossproduct quantum group. Given a graph with cyclic ordering of edge ends at each vertex, our construction of a Hilbert space for the bicrossproduct model for a Hopf algebra H is based on the extension of the canonical covariant action of the bicrossproduct quantum group $H^{\text{cop}}\blacktriangleright\!\!\triangleleft H$ on H^* to an action on $H^{*\otimes|E|}$, the $|E|$-fold tensor product of H^*, where $|E|$ is the number of edges. This action which enter the definition of the triangle operators and consequently the vertex and face operators are in general not required to be covariant as we seek a bicrossproduct module and not a module algebra. We obtain an exactly solvable Hamiltonian, whose ground state or protected space is invariant under the actions of the operators comprising the Hamiltonian. This invariance was shown by introducing a tensor network representation and identifying topologically ordered quantum states in this framework.

This new model opens up new directions to explore. From the quantum gravity perspective, the vertex and face operators are related to the Gauss constraint and the Flatness constraint, which are usually characterized in terms of symmetries by the Drinfeld double in the quantum double model. It would be interesting to determine whether the bicrossproduct case has also some geometrical meaning. The semi-duality between the quantum groups seems to indicate naively that we dualize somehow for example the Flatness constraint into another Gauss constraint, or vice versa. Investigations are currently underway to see if this argument can be made more rigorous.

As the Kitaev quantum double model is known to be equivalent to the combinatorial quantization of Chern-Simons theory based on the Drinfeld double [71]. It would be interesting to see whether this result extends to the bicrossproduct case, namely that our model can be related to the combinatorial quantization of Chern-Simons theory based on the bicrossproduct quantum group. In the case of the Drinfeld double, one required a Hopf gauge theoretic framework [72]. This provides another interesting question to address in the context of the bicrossproduct model. For this construction, one required a universal R-matrix. This is now known explicitly for the bicrossproduct quantum group due to recent work in [111] which provides an explicit expression of the R-matrix for this quantum group.

APPENDICES

Appendix A

Geometric settings

A.1 The metric tensor and frame field

Let us consider an n-dimensional spacetime manifold with no boundary denoted $\mathcal{M}$. A metric tensor $\mathbf{g}$ on $\mathcal{M}$ is a symmetric, non-degenerate rank two tensor which measures the infinitesimal squared distance and also determines the causal structure of spacetime. In coordinate basis the metric tensor can be written in terms of its components $g_{\mu\nu}(x)$ as

$$\mathbf{g} = g_{\mu\nu}(x)\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}, \tag{A.1}$$

where x^{μ} is any spacetime coordinate on $\mathcal{M}$ with $\mu = 0, 1, ..., n-1$ denoting spactime indices, and Einstein summation convention is used, i.e., summation with repeated indices is understood. Alternatively, $\mathrm{d}s^2$ is used instead of $\mathbf{g}$ to represent the metric tensor, with $\mathrm{d}s$ known as the line element.

The non-degeneracy of $\mathbf{g}$ implies it has an inverse whose components are denoted by $g^{\mu\nu}$ and such that they satisfy

$$g_{\mu\nu}\, g^{\nu\rho} = \delta^{\rho}_{\mu}. \tag{A.2}$$

The metric has the signature $(\sigma, 1, 1, ..., 1)$, where in the Euclidean settings $\sigma = +1$ and the underlying spacetime is termed Riemannian spacetime, while in the Lorenztian signature[1] $\sigma = -1$, where the underlying spacetime is known as pseudo-Riemannian spacetime. Metrics of spacetime with Lorenztian signature are said to lack positive definiteness, in the sense that the inner product of non-zero vectors can be null or negative.

[1]Depending on the convention the signature $(-1, -1, -1, ..., 1)$ is sometimes used in the case of the Lorenztian signature.

Instead of using the metric $g_{\mu\nu}$ to encode gravitational degrees of freedom, we can use a frame field $\{e_I\}$. $\{e_I\}$ and $\{e^I\}$ are the respective bases of the tangent space $T_p(\mathcal{M})$ and the co-tangent space $T_p^*(\mathcal{M})$ at every point p in $\mathcal{M}$. They can be expressed in their canonical bases ∂_μ and $\mathrm{d}x^\mu$ respectively as

$$e_I = e_I^\mu \, \partial_\mu, \quad e^I = e_\mu^I \, \mathrm{d}x^\mu, \tag{A.3}$$

where the frame field e_I^μ and its inverse e_μ^I obey the duality conditions

$$e_I^\mu \, e_\nu^I = \delta_\nu^\mu, \quad e_I^\mu \, e_\mu^J = \delta_I^J. \tag{A.4}$$

The orthonormality of the frame field implies

$$g_{\mu\nu} \, e_I^\mu \, e_J^\nu = e_{\nu I} \, e_J^\nu = \eta_{IJ}, \tag{A.5}$$

and therefore the physical metric $g_{\mu\nu}$ can be written in terms of the inverse frame field

$$g_{\mu\nu} = \eta_{IJ} \, e_\mu^I \, e_\nu^J, \tag{A.6}$$

and the flat Minkowski metric η_{IJ}, however one can use δ_{IJ} if working in the Euclidean signature. Capital Latin alphabets $A, B, ...I, J, ... = 0, 1, 2, ..., \mathrm{n} - 1$ denote tangent space indices. In general, the frame field has n^2 individual components. In three and four dimensions the frame fields are called triads and tetrads respectively. From (A.6), the frame field provides a local isomorphism between a general reference frame and an inertial one, characterized by the flat metric η_{IJ}. A local inertial frame is defined up to a Lorentz transformation

$$\tilde{e}_\mu^I = \Lambda^I{}_J \, e_\mu^J, \tag{A.7}$$

where $\Lambda^I{}_J$ is an element of the Lorentz group $SO(n-1, 1)$ and that the definition (A.6) is invariant under this transformation. The implication of this is that the "internal" index I carries a representation of the Lorentz group. The expression (A.6) clearly suggest the frame field now encodes the metric structure of spacetime, implying one can formulate a dynamical theory base on the frame field, precisely formulating gravity as a gauge theory.

A.2 Connection, Torsion and Curvature

In the previous section, we saw how $g_{\mu\nu}(x)$ describes the geometry of a manifold. Yet another way to describe the geometry of a manifold is through the introduction of an affine connection. This way of describing the non-trivial geometry of a manifold manifest itself through the non-triviality of parallel transport of vectors.

Let us consider the basis vectors $\{\partial_\mu\}$ in a local coordinate chart of the tangent space $T_p(\mathcal{M})$, then by parallelly transporting a basis vector, the connection coefficients $\Gamma^\rho{}_{\mu\nu}(x)$ on the spacetime manifold $\mathcal{M}$ are defined as

$$\nabla_{\partial_\mu}\partial_\nu := \Gamma^\rho{}_{\mu\nu}(x)\,\partial_\rho, \tag{A.8}$$

where ∇ is a covariant derivative, a tensorial generalization of the ordinary derivative. More importantly, once we know the action of ∇ on the basis vectors, $\Gamma^\rho{}_{\mu\nu}(x)$ determines the action of the covariant derivative ∇ on any vector V^μ and its co-vectors V_μ respectively as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\lambda}V^\lambda \tag{A.9}$$
$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\lambda{}_{\nu\mu}V_\lambda. \tag{A.10}$$

These definitions generalizes to any arbitrary tensor of higher rank. The connection coefficients defines the geodesic structure of the spacetime, more explicitly, the connection is an object which specifies how a vector is transported along a curve. We note that in any theory the connection can be defined independently of the metric, and as such one can define a connection on a manifold without it possessing a metric. In such instances compatibility conditions are required when physical requirements are impose on the spacetime.

The transformation property of the connection under a change of spacetime coordinates from x^μ to $\tilde{x}^\mu$ is given by

$$\tilde{\Gamma}^\kappa{}_{\mu\nu} = \frac{\partial \tilde{x}^k}{\partial x^\rho}\left(\frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} + \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\,\Gamma^\rho{}_{\alpha\beta}\right), \tag{A.11}$$

so that it is not a tensor. This means that it may vanish in one coordinate system and not in other coordinate systems - despite this, the covariant derivatives define above do transform as tensors. The non-tensorial nature of the connection is due to the non-linearity in Γ of the expression (A.11). This leads to naming them as affine connection. We have seen how the affine connection specifies the parallel transport of vectors and this specifies the geometry of the manifold but the question is how is this possible? The simple answer is through the notions of torsion and curvature, which we will now introduce.

Suppose η_1 and η_2 are tangent vectors at $p \in \mathcal{M}$, the torsion map $\mathcal{J} : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \longrightarrow T_p(\mathcal{M})$ is defined as

$$\mathcal{J}(\eta_1, \eta_2) := \nabla_{\eta_1}\eta_2 - \nabla_{\eta_2}\eta_1 - [\eta_1, \eta_2], \tag{A.12}$$

where $[\cdot,\cdot]$ is a Lie bracket. The map $\mathcal{J}(\eta_1,\eta_2)$ is the amount by which an infinitesimal parallelogram spanned by the vectors η_1 and η_2 does not close. Through the definition of the torsion map, one can define the torsion tensor

$$T(\vartheta,\eta_1,\eta_2) := \langle \vartheta, \mathcal{J}(\eta_1,\eta_2)\rangle, \tag{A.13}$$

where ϑ is a co-vector in the co-tangent space $T_p^*(\mathcal{M})$ and $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $T(\mathcal{M})$ and $T^*(\mathcal{M})$. By choosing the canonical bases $\vartheta := \mathrm{d}x^\rho$, $\eta_1 := \partial_\mu$, $\eta_2 := \partial_\nu$ and using the definition (A.8) one can write the torsion tensor in terms of the connection

$$T^\rho{}_{\mu\nu} := \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}, \tag{A.14}$$

which is twice the anti-symmetric part of the connection. From our previous discussion, we have seen the connection does not transform as a tensor under coordinate transformation however its anti-symmetric part does transform as a tensor. It then turns out that the torsion tensor (A.14) is anti-symmetric in its last two indices. The failure of a parallelogram to close when built infinitesimally in the spacetime manifold by parallel transport of vectors is what geometrically signifies the existence of the torsion tensor. In general relativity, one assumes there is no torsion, implying the connection (Christoffel symbols) is symmetric. Let us elaborate more on this: the connection which specifies the parallel transport of vectors on a spacetime manifold expresses gravitational forces and requires that around each point $p \in \mathcal{M}$, there exist a chart such that the connection vanishes.

We pick another tangent vector η_3, in addition to η_1 and η_2 as before, then the curvature map $\mathcal{R} : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \times T_p(\mathcal{M}) \longrightarrow T_p(\mathcal{M})$ is defined as

$$\mathcal{R}(\eta_1,\eta_2,\eta_3) := \left(\nabla_{\eta_1}\nabla_{\eta_2} - \nabla_{\eta_2}\nabla_{\eta_1} - \nabla_{[\eta_1,\eta_2]}\right)\eta_3. \tag{A.15}$$

This map defines the curvature tensor

$$R(\vartheta,\eta_1,\eta_2,\eta_3) := \langle \vartheta, \mathcal{R}(\eta_1,\eta_2,\eta_3)\rangle. \tag{A.16}$$

In coordinate basis, the curvature tensor is written in terms of the connection

$$R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda{}_{\mu\rho} - \partial_\rho \Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\rho} - \Gamma^\lambda{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu}. \tag{A.17}$$

The Riemann curvature tensor measures the strength of the variation of tidal forces, that is it expresses changes in gravity. Geometrically, its measures the curvature which is expressed through the non-commutativity of two parallel transports. That is the deviation of a vector from its original position when parallel transported.

The curvature tensor has $\frac{n^2(n^2-1)}{12}$ independent components which can be decomposed into the trace and the trace-free parts. Taking the trace of the curvature tensor over its first and third (or equivalently, the second and fourth) indices defines the Ricci tensor

$$R_{\mu\rho} := R^{\lambda}{}_{\mu\lambda\rho} \tag{A.18}$$

and is symmetric. By taking the trace of the Ricci tensor we obtain the Ricci scalar defined

$$R := R^{\mu}{}_{\mu}. \tag{A.19}$$

The trace-free part of $R^{\lambda}{}_{\mu\nu\rho}$ is the Weyl tensor $C^{\lambda}{}_{\mu\nu\rho}$ and is defined for dimensions $n \geq 3$, it vanishes identically for dimension three and below.

Let us point out that since the connection is not a property of the manifold, one can define different connections on the same manifold. We can then infer from this that there is nothing as the curvature or torsion of a manifold but rather curvature or torsion of a connection. For instance in general relativity it is the Levi-Civita connection which is defined in terms of the metric

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \frac{1}{2}\, g^{\rho\lambda} \left(\partial_{\nu} g_{\lambda\mu} + \partial_{\mu} g_{\lambda\nu} - \partial_{\lambda} g_{\mu\nu}\right). \tag{A.20}$$

The Levi-Civita connection is metric compatible, in the sense that

$$\nabla_{\rho} g_{\mu\nu} \equiv \partial_{\rho} g_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\rho\mu} g_{\alpha\nu} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\rho\nu} g_{\alpha\mu} = 0, \tag{A.21}$$

and torsion free

$$\mathrm{T}(\overset{\circ}{\Gamma}) := \overset{\circ}{\mathrm{T}}{}^{\rho}{}_{\mu\nu} = 0. \tag{A.22}$$

Any general connection can be decomposed into two parts,

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} \tag{A.23}$$

the Levi-Civita part $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ and the co-torsion tensor part $K^{\rho}{}_{\mu\nu}$ defined in terms of the torsion tensor

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}\left(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} + T^{\rho}{}_{\mu\nu}\right). \tag{A.24}$$

This decomposition is relevant when discussing the teleparallel formulation of gravity.

Appendix B

Classical phase space: Poisson Lie groups and Lie bialgebras

Lie groups which are compatible with Poisson structures in a certain way are called Poisson Lie groups. These are the classical structures behind some quantum groups, which in turn are seen as deformations of classical Lie group symmetries. The infinitesimal version of the Poisson Lie groups are the Lie bialgebras. In [51, 4], Poisson Lie groups were used to understand how a negative cosmological constant leads to a deformation of the classical symmetry to a quantum group symmetry in loop quantum gravity. In this chapter, we will review the concept of Poisson-Lie groups and Lie bialgebras following mainly the book [108]. With this review we will see how the classical double of a Lie algebra when integrated results in either a Drinfeld double (Poisson-Lie group) or a Heisenberg double (symplectic phase space).

B.1 Poisson Lie groups

The phase space is the most convenient way to describe classical physics and is given by the pair $(T^*(\mathcal{M}), \{\cdot,\cdot\})$, where $T^*(\mathcal{M})$ is the cotangent bundle of a smooth manifold $\mathcal{M}$ and $\{\cdot,\cdot\}$ is the Poisson bracket.

Definition B.1.1. *(**Poisson Structure**) Let $\mathcal{M}$ be a finite smooth manifold and $C^\infty(\mathcal{M})$ be the algebra of smooth functions on $\mathcal{M}$. A Poisson structure on $\mathcal{M}$ is a $\mathbb{R}$-bilinear map*

$$\{\cdot,\cdot\}_{\mathcal{M}} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M}), \tag{B.1}$$

called the Poisson bracket satisfying the following properties

$$\{f_1, f_2\}_{\mathcal{M}} = -\{f_2, f_1\}_{\mathcal{M}}, \tag{B.2}$$
$$\{f_1, \{f_2, f_3\}\}_{\mathcal{M}} + \{f_2, \{f_3, f_1\}\}_{\mathcal{M}} + \{f_3, \{f_1, f_2\}\}_{\mathcal{M}} = 0, \tag{B.3}$$
$$\{f_1 f_2, f_3\}_{\mathcal{M}} = f_1\{f_2, f_3\}_{\mathcal{M}} + \{f_1, f_3\}_{\mathcal{M}} f_2, \tag{B.4}$$

for all $f_1, f_2, f_3 \in C^\infty(\mathcal{M})$*. The subscript* $\mathcal{M}$ *on the Poisson bracket* $\{\cdot,\cdot\}_{\mathcal{M}}$ *will be omitted to avoid any confusion or when the manifold under consideration is obvious. The equation* (B.2) *is the anti-symmetric property of the Poisson bracket,* (B.3) *is the Jacobi identity and together with the anti-symmetry property* (B.2) *makes* $\{\cdot,\cdot\}$ *is a Lie bracket on* $C^\infty(\mathcal{M})$*. Property* (B.4) *is the Leibniz identity or a derivation on* $C^\infty(\mathcal{M})$*, in the sense that there exist a vector field* X_{f_1} *on* $\mathcal{M}$ *such that* $X_{f_1}(f_2) = \{f_2, f_1\}$ *for all* f_2*. Such a vector field which arises in this manner is called a Hamiltonian vector field and it generates time evolution (symmetry). Putting together* $\mathcal{M}$ *and* $\{\cdot,\cdot\}$ *as one structure gives a Poisson manifold.*

The Poisson bracket on $\mathcal{M}$ *defines what is called a Poisson bivector* $\Pi_{\mathcal{M}}$ *as*

$$\{f_1, f_2\} = \langle \Pi_M, \mathrm{d}f_1 \otimes \mathrm{d}f_2 \rangle \tag{B.5}$$

where $\Pi_{\mathcal{M}} \in \bigwedge^2 T\mathcal{M}$*,* $\mathrm{d}f_1, \mathrm{d}f_2 \in T^*(\mathcal{M})$ *and* $\langle\cdot,\cdot\rangle$ *is the natural pairing between* $T(\mathcal{M})$ *and* $T^*(\mathcal{M})$*. In coordinates* $x_i \in \mathcal{M}$*, the Poisson bracket* (B.5) *reads*

$$\{f_1, f_2\}(x) = \Pi_{ij}(x)\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j} \tag{B.6}$$

and where the Poisson bivector is

$$\Pi_x := \Pi_{ij}(x)\partial_i \otimes \partial_j, \quad \textit{with} \quad \Pi_{ij} = -\Pi_{ji}, \tag{B.7}$$

where Einstein's summation convention is assumed.

Every manifold admits the trivial Poisson structure $\{f_1, f_2\} = 0$ *for which the Poisson bivector* $\Pi_{\mathcal{M}} = 0$*. If the bivector* $\Pi_{\mathcal{M}}$ *is everywhere non-degenerate, i.e.,* $\Pi_{\mathcal{M}} \neq 0$ *for all* $f_i \in C^\infty(\mathcal{M})$*, then its called symplectic. This means it can be inverted to give everywhere a non-degenerate closed 2-form* $\omega = \Pi_{\mathcal{M}}^{-1}$*, where* $\mathrm{d}\omega = 0$ *is equivalently to the Jacobi identity* (B.3)*.*

Next we consider maps between Poisson manifolds.

Definition B.1.2. ***(Poisson Maps)*** *A smooth map $F : N \longrightarrow M$ between two Poisson manifolds $(N, \{\cdot,\cdot\})$ and $(M, \{\cdot,\cdot\})$ is called a Poisson map if it preserves the Poisson brackets of M and N*

$$\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N, \tag{B.8}$$

for all $f_1, f_2 \in C^\infty(M)$.

Given two Poisson manifolds $(N, \{\cdot,\cdot\}_N)$ and $(M, \{\cdot,\cdot\}_M)$, we can equip their product $M \times N$ with a Poisson structure

Definition B.1.3. ***(Product Poisson Structure)*** *For all $f_1, f_2 \in C^\infty(M \times N)$, $x \in M$, $y \in N$, the product Poisson structure on $M \times N$ is given by*

$$\{f_1, f_2\}_{M\times N}(x, y) = \{f_1(\cdot, y), f_2(\cdot, y)\}_{M(x)} + \{f_1(x,\cdot), f_2(x,\cdot)\}_{N(y)} \tag{B.9}$$

Definition B.1.4. ***(Poisson Lie group)*** *A Poisson Lie group is a Lie group G, equipped with a Poisson bracket $\{\cdot,\cdot\}$, such that $\{\cdot,\cdot\}$ is consistent with the group multiplication of G, $\mu : G \times G \longrightarrow G$, $\mu(g_1 g_2) = g_1 g_2$ such that is a Poisson map, where $G \times G$ is given the product Poisson structure defined above.*

B.2 Lie bialgebras

The infinitesimal version of a Lie group is the Lie algebra and hence the obvious question one will like to ask is, what are the additional structures on the Lie algebra of G for one to obtain a Poisson Lie-group.

Definition B.2.1. ***(Lie bialgebras)*** *The pair $(\mathfrak{g}, \delta_\mathfrak{g})$ is a Lie bialgebra where $\mathfrak{g}$ is a Lie algebra and $\delta_\mathfrak{g} : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a skew-symmetric, linear 1-cocycle called the cocommutator and satisfies the cocycle condition*

$$\delta_\mathfrak{g}([X, Y]) = (ad_X \otimes 1 + 1 \otimes ad_X)\, \delta_\mathfrak{g}(Y) - (ad_Y \otimes 1 + 1 \otimes ad_Y)\, \delta_\mathfrak{g}(X) \tag{B.10}$$

and defines a canonical Lie algebra structure on the dual Lie algebra $\mathfrak{g}^*$ via

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = (\mathrm{d}\{f_1, f_2\})_e = \delta^*_\mathfrak{g}\,(\xi_1 \otimes \xi_2) \tag{B.11}$$

where $\xi_i \in \mathfrak{g}^*$ and $\xi_i = (\mathrm{d}f_i)_e$. One further requires that for a Lie algebra homomorphisms $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$, we have $(\psi \otimes \psi) \circ \delta_\mathfrak{g} = \delta_\mathfrak{h} \circ \psi$.

Alternatively, we can write (B.11) *as*

$$\langle X | [\xi_1, \xi_2]_{\mathfrak{g}^*}\rangle = \langle X | \mathrm{d}\{f_1, f_2\}_e\rangle = \langle X | \delta^*_\mathfrak{g}\,(\xi_1 \otimes \xi_2)\rangle = \langle \delta(X) | \xi_1 \otimes \xi_2\rangle, \tag{B.12}$$

where $X \in \mathfrak{g}$ and $\langle\cdot|\cdot\rangle : \mathfrak{g} \times \mathfrak{g}^* \longrightarrow \mathbb{R}$ is the canonical pairing.

Definition B.2.2. ***(Classical double)*** *The classical double $(\mathfrak{d}(\mathfrak{g}), \delta_{\mathfrak{d}})$ associated to a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is the canonical Lie bialgebra defined on $\mathfrak{g} \otimes \mathfrak{g}^*$, where $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{g} \otimes \mathfrak{g}^*$, such that the inclusions $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g}^* \hookleftarrow (\mathfrak{g}^*)^{op}$ are homomorphisms of Lie bialgebras.*

We define a non-degenerate, symmetric bilinear form on $\mathfrak{g} \otimes \mathfrak{g}^*$ via

$$(X, \xi) = \langle X|\xi\rangle, \quad (X, Y) = 0 = (\xi, \eta), \quad X, Y \in \mathfrak{g}, \quad \xi, \eta \in \mathfrak{g}^* \tag{B.13}$$

and the cocommutator of $\mathfrak{d}(\mathfrak{g})$ is defined with $t \in \mathfrak{g} \otimes \mathfrak{g}^*$ via

$$\delta_{\mathfrak{d}}(u) = (ad_u \otimes id + id \otimes ad_u)(t), \tag{B.14}$$

where $u \in \mathfrak{d}(\mathfrak{g})$ and t corresponds to the identity map $\mathfrak{g} \longrightarrow \mathfrak{g}$, thus $t = X_i \otimes \xi^i$.

Consider $\{X_i\}$ and $\{\xi_i\}$ as the respective basis of $\mathfrak{g}$ and $\mathfrak{g}^*$ with the bilinear map $\langle X_i|\xi^j\rangle = \delta_i^j$. The Lie algebra structures of $\mathfrak{g}$ and $\mathfrak{g}^*$ are denoted

$$[X_i, X_j] = f_{ij}{}^k X_k, \quad [\xi^i, \xi^j] = c^{ij}{}_k \, \xi^k \tag{B.15}$$

which means the cocommutator of $\mathfrak{g}$ is $\delta_{\mathfrak{g}}(X_i) = c_i{}^{jk} X_j \otimes X_k$ and also the cocommutator of $\mathfrak{g}^*$ is given by $\delta_{\mathfrak{g}^*}(\xi^i) = f^i{}_{jk} \, \xi^j \otimes \xi^k$. $f_{ij}{}^k$ and $c^{ij}{}_k$ are the respective structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$. The Lie bracket between X_i and ξ^j can be calculated by noting that the invariance of the inner product (B.13) implies the $\mathfrak{g}$-component

$$\big([X_i, \xi^j]\big)^k = \big([X_i, \xi^j], \xi^k\big) = \big(X_i, [\xi^j, \xi^k]\big) = c^{jk}{}_i, \tag{B.16}$$

likewise for the $\mathfrak{g}^*$-component

$$\big([X_i, \xi^j]\big)_k = \big([X_i, \xi^j], X_k\big) = \big([X_k, X_i], \xi^j\big) = f_{ki}{}^j, \tag{B.17}$$

and combining (B.16) and (B.17) we obtain

$$[X_i, \xi^j] = c^{jk}{}_i X_k + f_{ki}{}^j \, \xi^k = c^{jk}{}_i X_k - f_{ik}{}^j \, \xi^k. \tag{B.18}$$

Therefore the Lie algebra structures (B.15) and (B.18) are those of the classical double $\mathfrak{d}(\mathfrak{g})$.

The inner product (B.13) is invariant under the adjoint action of $\mathfrak{g} \otimes \mathfrak{g}^*$ on itself if and only if $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra. We note that the classical double $\mathfrak{d}(\mathfrak{g})$ can be integrated to get the Lie group $\mathcal{D} = G \bowtie G^*$ which can be equipped with two different Poisson structures, characterized by the Poisson bivector $\Pi_{\pm}$. The group $\mathcal{D}$ equipped with Π_- is called the Drinfeld double of G and is a Poisson Lie group, and as such is not a symplectic symplectic space. The group $\mathcal{D}$ equipped with Π_+ is called the Heisenberg double and is a symplectic space.

Appendix C

Proof of the symplectic form

C.1 Expressions used in the LQG computations

Given the continuity equations are

$$\ell_{c'}u_{c'} = m_{c'c}h_{c'c}\,\ell_c u_c = m_{c'c}\tilde{\ell}_c\,\tilde{h}_{c'c}u_c = m_{c'c}(h_{c'c}\triangleright\ell_c)(\ell_c\triangleright h_{c'c})u_c, \tag{C.1}$$

we obtain the following

$$\begin{aligned}\ell_{c'} &= m_{c'c}\tilde{\ell}_c, \quad \tilde{\ell}_c = h_{c'c}\triangleright\ell_c = h^{-1}_{c'c}\,\ell_c\,h_{c'c} && \text{(C.2)}\\ u_{c'} &= \tilde{h}_{c'c}u_c, \qquad \tilde{h}_{c'c} = \ell_c\triangleright h_{c'c} = \ell_c^{-1}\,h_{c'c}\,\ell_c, && \text{(C.3)}\end{aligned}$$

where we note that $\tilde{\ell}$ actually sits at c'. From the first expressions in (C.2) and (C.3) we get

$$\begin{aligned}\Delta\ell_{c'} &= \Delta m_{c'c} + m_{c'c}\Delta\tilde{\ell}_c m^{-1}_{c'c}, \quad \ell^{-1}_{c'}\mathrm{d}(\Delta\ell_{c'})\ell_{c'} = \tilde{\ell}^{-1}_c\mathrm{d}(\Delta\tilde{\ell}_c)\tilde{\ell}_c, \quad \ell^{-1}_{c'}\mathrm{d}\ell_{c'} = \tilde{\ell}^{-1}_c\mathrm{d}\tilde{\ell}_c,\\ \Delta u_{c'} &= \Delta\tilde{h}_{c'c} + \tilde{h}_{c'c}\Delta u_c\tilde{h}^{-1}_{c'c} = \tilde{h}_{c'c}(\tilde{h}^{-1}_{c'c}\delta\tilde{h}_{c'c} + \Delta u_c)\tilde{h}^{-1}_{c'c} && \text{(C.4)}\end{aligned}$$

To relate the contribution for c^*, we need to expand the contribution in $\tilde{\ell}$, hence from (C.1) one gets the expression $h_{c'c}\ell_c = \tilde{\ell}_c\tilde{h}_{c'c}$ and subsequently after some algebra we get the followings

$$\begin{aligned}\tilde{\ell}^{-1}_c\mathrm{d}\tilde{\ell}_c &= \tilde{h}_{c'c}(\ell^{-1}_c\mathrm{d}\ell_c)\tilde{h}^{-1}_{c'c}, \qquad \Delta\tilde{\ell}_c = \left\{h_{c'c}\Delta\ell_c h^{-1}_{c'c} - \tilde{\ell}_c(\Delta\tilde{h}_{c'c})\tilde{\ell}^{-1}_c\right\}_{|\mathfrak{an}_2} && \text{(C.5)}\\ \mathrm{d}(\Delta\tilde{\ell}_c) &= h_{c'c}\mathrm{d}\Delta\ell_c h^{-1}_{c'c} - \tilde{\ell}_c(\mathrm{d}(\Delta\tilde{h}_{c'c}) + [\tilde{\ell}^{-1}_c\mathrm{d}\tilde{\ell}_c, \Delta\tilde{h}_{c'c}])\tilde{\ell}^{-1}_c && \text{(C.6)}\\ \tilde{\ell}^{-1}_c\mathrm{d}(\Delta\tilde{\ell}_c)\tilde{\ell}_c &= \left\{\tilde{h}_{c'c}\ell^{-1}_c(\mathrm{d}\Delta\ell_c)\ell_c\tilde{h}^{-1}_{c'c} - [\tilde{\ell}^{-1}_c\mathrm{d}\tilde{\ell}_c, \Delta\tilde{h}_{c'c}])\right\}_{|\mathfrak{an}_2}. && \text{(C.7)}\end{aligned}$$

If we consider $Y=\ell_c^{-1}\mathrm{d}\ell_c$, we obtain the following expressions

$$\delta Y=\delta(\ell_c^{-1}\mathrm{d}\ell_c)=\ell_c^{-1}\mathrm{d}(\Delta\ell_c)\ell_c,\quad \ell_c\delta Y\ell_c^{-1}=\mathrm{d}(\Delta\ell_c). \tag{C.8}$$

It follows then that

$$\begin{aligned}
\overline{Y} &= \left\{\tilde{h}_{cc'}^{-1}Y\tilde{h}_{cc'}\right\}_{|\mathfrak{an}_2}=\left\{\tilde{h}_{cc'}^{-1}(\ell_c^{-1}\mathrm{d}\ell_c)\tilde{h}_{cc'}\right\}_{|\mathfrak{an}_2}=\left\{\tilde{h}_{cc'}^{-1}\ell_c^{-1}\mathrm{d}(\ell_c\tilde{h}_{cc'})-\tilde{h}_{cc'}^{-1}\mathrm{d}\tilde{h}_{cc'}\right\}_{|\mathfrak{an}_2}=\tilde{\ell}_c^{-1}\mathrm{d}\tilde{\ell}_c\\
\delta\overline{Y} &= \left\{\Delta(\tilde{\ell}_c^{-1}\mathrm{d}\tilde{\ell}_c)-\tilde{h}_{cc'}^{-1}\mathrm{d}\Delta\tilde{h}_{cc'}\,\tilde{h}_{cc'}\right\}_{|\mathfrak{an}_2},\qquad \tilde{\ell}_c\,\delta\overline{Y}\,\tilde{\ell}_c^{-1}=\mathrm{d}(\Delta\tilde{\ell}_c) && \text{(C.9)}\\
\delta\overline{Y} &= \tilde{h}_{cc'}^{-1}(\delta Y+[Y,\Delta\tilde{h}_{cc'}])\tilde{h}_{cc'},\qquad Y\,[\Delta\tilde{h}_{cc'},\Delta\tilde{h}_{cc'}]=\delta\overline{Y}\wedge\underline{\Delta}\tilde{h}_{cc'}-\delta Y\wedge\Delta\tilde{h}_{cc'}. && \text{(C.10)}
\end{aligned}$$

From the continuity condition, we derive the expression $h_{c'c}\,\ell_c=\tilde{\ell}_c\,\tilde{h}_{c'c}$, from which we get

$$\tilde{h}_{c'c}=\tilde{\ell}_c^{-1}\,h_{c'c}\ell_c,\qquad \tilde{h}_{cc'}=\ell_c^{-1}\,h_{c'c}^{-1}\,\tilde{\ell}_c=\ell_c^{-1}\,h_{cc'}\,\tilde{\ell}_c, \tag{C.11}$$

and then

$$\begin{aligned}
\underline{\Delta}\tilde{h}_{cc'} &= \left\{-(h_{cc'}\tilde{\ell}_c)^{-1}(\Delta\ell_c)\,h_{cc'}\tilde{\ell}_c+\tilde{\ell}_c^{-1}(\underline{\Delta}h_{cc'})\tilde{\ell}_c\right\}_{|\mathfrak{su}(2)}, && \text{(C.12)}\\
\Delta\tilde{h}_{cc'} &= \left\{\ell_c^{-1}(\Delta h_{cc'})\ell_c+\ell_c^{-1}\,h_{cc'}(\Delta\tilde{\ell}_c)(\ell_c^{-1}\,h_{cc'})^{-1}\right\}_{|\mathfrak{su}(2)}. && \text{(C.13)}
\end{aligned}$$

C.2 Expressions used in the dual LQG computations

Starting from the continuity equations

$$\tilde{u}_{c'}\tilde{\ell}_{c'}=(\tilde{h}_{c'c}\,\tilde{m}_{c'c})\,(\tilde{u}_c\,\tilde{\ell}_c)=\tilde{h}_{c'c}\,(u_c\,m_{c'c})\,\tilde{\ell}_c=\tilde{h}_{c'c}(\tilde{m}_{c'c}\triangleright\tilde{u}_c)(\tilde{m}_{c'c}\triangleleft\tilde{u}_c)\tilde{\ell}_c, \tag{C.14}$$

from this we obtain

$$\begin{aligned}
\tilde{u}_{c'} &= \tilde{h}_{c'c}u_c,\qquad u_c=\tilde{m}_{c'c}\triangleright\tilde{u}_c= && \text{(C.15)}\\
\tilde{\ell}_{c'} &= m_{c'c}\tilde{\ell}_c\qquad m_{c'c}=\tilde{m}_{c'c}\triangleleft\tilde{u}_c= && \text{(C.16)}
\end{aligned}$$

we note that u sits at c'. The first expressions in (C.15) and (C.16) gives

$$\begin{aligned}
\Delta\tilde{u}_{c'} &= \Delta\tilde{h}_{c'c}+\tilde{h}_{c'c}(\Delta u_c)\tilde{h}_{c'c}^{-1},\quad \tilde{u}_{c'}^{-1}\mathrm{d}(\Delta\tilde{u}_{c'})\tilde{u}_{c'}=u_c^{-1}\mathrm{d}(\Delta u_c)u_c,\quad \tilde{u}_{c'}^{-1}\mathrm{d}\tilde{u}_{c'}=u_c^{-1}\mathrm{d}u_c\\
\Delta\tilde{\ell}_{c'} &= \Delta m_{c'c}+m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}=m_{cc'}(m_{c'c}^{-1}\delta m_{c'c}+\Delta\tilde{\ell}_c)m_{c'c}^{-1}. && \text{(C.17)}
\end{aligned}$$

In order to relate the contribution for c^*, we need to expand the contributions in u, hence from (C.14) we get the expression $\tilde{m}_{c'c}\tilde{u}_c = u_c m_{c'c}$. After some algebra we get

$$\begin{aligned} u_c^{-1}\mathrm{d}u_c &= m_{c'c}(\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c)m_{c'c}^{-1}, \quad \Delta u_c = \left\{\tilde{m}_{c'c}(\Delta\tilde{u}_c)\tilde{m}_{c'c}^{-1} - u_c(\Delta m_{c'c})u_c^{-1}\right\}_{|\mathfrak{su}(2)} && \text{(C.18)} \\ \mathrm{d}(\Delta u_c) &= \tilde{m}_{c'c}(\mathrm{d}\Delta\tilde{u}_c)\tilde{m}_{c'c}^{-1} - u_c(\mathrm{d}(\Delta m_{c'c}) + [u_c^{-1}\mathrm{d}u_c, \Delta m_{c'c}])u_c^{-1} && \text{(C.19)} \\ u_c^{-1}\mathrm{d}(\Delta u_c)u_c &= \left\{m_{c'c}\tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c m_{c'c}^{-1} - [u_c^{-1}\mathrm{d}u_c, \Delta m_{c'c}])\right\}_{|\mathfrak{su}(2)} && \text{(C.20)} \end{aligned}$$

If we consider $X = \tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c$, we obtain the following expressions

$$\delta X = \delta(\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c) = \tilde{u}_c^{-1}\mathrm{d}(\Delta\tilde{u}_c)\tilde{u}_c, \qquad \tilde{u}_c\delta X\tilde{u}_c^{-1} = \mathrm{d}(\Delta\tilde{u}_c) \tag{C.21}$$

It follows then

$$\begin{aligned} \overline{X} &= \left\{m_{cc'}^{-1}Xm_{cc'}\right\}_{|\mathfrak{su}(2)} = \left\{m_{cc'}^{-1}\tilde{u}_c^{-1}\mathrm{d}(\tilde{u}_c m_{cc'}) - m_{cc'}^{-1}\mathrm{d}m_{cc'}\right\}_{|\mathfrak{su}(2)} = u_c^{-1}\mathrm{d}u_c \\ \delta\overline{X} &= \left\{\Delta(u_c^{-1}\mathrm{d}u_c) - m_{cc'}^{-1}\mathrm{d}\Delta m_{cc'}\, m_{cc'}\right\}_{|\mathfrak{su}(2)}, \quad u_c\,\delta\overline{X}\,u_c^{-1} = \mathrm{d}(\Delta u_c) && \text{(C.22)} \\ \delta\overline{X} &= m_{cc'}^{-1}(\delta X + [X, \Delta m_{cc'}])m_{cc'}, \quad [\Delta m_{cc'}, \Delta m_{cc'}]\ X = \underline{\Delta} m_{cc'} \wedge \delta\overline{X} - \Delta m_{cc'} \wedge \delta X && \text{(C.23)} \end{aligned}$$

From the continuity condition, we derive the expression $\tilde{m}_{c'c}\tilde{u}_c = u_c m_{c'c}$, from which we get

$$m_{c'c} = u_c^{-1}\tilde{m}_{c'c}\tilde{u}_c, \qquad m_{cc'} = \tilde{u}_c^{-1}\tilde{m}_{c'c}^{-1}u_c = \tilde{u}_c^{-1}\tilde{m}_{cc'}u_c, \tag{C.24}$$

and then we get

$$\begin{aligned} \underline{\Delta} m_{cc'} &= = \left\{-(\tilde{m}_{cc'}u_c)^{-1}(\Delta\tilde{u}_c)\,\tilde{m}_{cc'}u_c + u_c^{-1}(\underline{\Delta}\tilde{m}_{cc'})u_c\right\}_{|\mathfrak{an}_2} && \text{(C.25)} \\ \Delta m_{cc'} &= \left\{\tilde{u}_c^{-1}(\Delta\tilde{m}_{cc'})\tilde{u}_c + \tilde{u}_c^{-1}\,\tilde{m}_{cc'}(\Delta u_c)(\tilde{u}_c^{-1}\,\tilde{m}_{cc'})^{-1}\right\}_{|\mathfrak{an}_2} && \text{(C.26)} \end{aligned}$$

C.3 Dual LQG phase space

Given the decomposition $G = \tilde{u}\tilde{\ell}$, we obtain the following

$$\Delta G = \Delta\tilde{u} + \tilde{u}(\Delta\tilde{\ell})\tilde{u}^{-1}, \qquad \mathrm{d}\Delta G = \mathrm{d}\Delta\tilde{u} + \tilde{u}(\mathrm{d}\Delta\tilde{\ell} + [\tilde{u}^{-1}\mathrm{d}\tilde{u}, \Delta\tilde{\ell}])\tilde{u}^{-1}. \tag{C.27}$$

Our starting point will be equation (5.102), considering just a face c^* it follows

$$\begin{aligned} \int_{c^*}\delta\tilde{\omega}\wedge\delta e &= -\int_{\partial_{c^*}}\left(\Delta\tilde{\ell}_c + \tilde{u}_c^{-1}(\Delta\tilde{u}_c)\tilde{u}_c\right)_{|\mathfrak{an}} \wedge \left(\tilde{u}_c^{-1}\mathrm{d}(\Delta\tilde{u}_c)\tilde{u}_c + [\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c, \Delta\tilde{\ell}_c] + \mathrm{d}\Delta\tilde{\ell}_c\right)_{|\mathfrak{su}} \\ &= -\int_{\partial_{c^*}}\left(\Delta\tilde{\ell}_c \wedge \tilde{u}_c^{-1}\mathrm{d}(\Delta\tilde{u}_c)\tilde{u}_c + \frac{1}{2}[\Delta\tilde{\ell}_c, \Delta\tilde{\ell}_c] \wedge \tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c\right). && \text{(C.28)} \end{aligned}$$

The commutator of $\mathfrak{su}(2)$ on $\mathfrak{an}_2$ restricted to $\mathfrak{su}(2)$ is given by the $\mathfrak{su}(2)$ structure constant, which is used to get from the first to second equality actually. The contribution for the edge $\mathfrak{e}$ from the face c'^* is then

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \int_{\partial_{c^*}} \left(\Delta\tilde{\ell}_{c'} \wedge \tilde{u}_{c'}^{-1}\mathrm{d}(\Delta\tilde{u}_{c'})u_{c'} + \frac{1}{2}[\Delta\tilde{\ell}_{c'}, \Delta\tilde{\ell}_{c'}] \wedge \tilde{u}_{c'}^{-1}\mathrm{d}\tilde{u}_{c'}\right) \\ &= \int_{\mathfrak{e}} \Bigg(\left(\Delta m_{c'c} + m_{c'c}\Delta\tilde{\ell}_c m_{c'c}^{-1}\right) \wedge u_c^{-1}\mathrm{d}(\Delta u_c)u_c \\ &\quad + \frac{1}{2}[\Delta m_{c'c} + m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}, \Delta m_{c'c} + m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}] \wedge u_c^{-1}\mathrm{d}u_c\Bigg), \end{aligned} \tag{C.29}$$

where we used the different expressions in (C.17) to arrive at the second equality. We expand the different terms of (C.29). The first term gives

$$\begin{aligned}\int_{\mathfrak{e}} \Delta m_{c'c} \wedge u_c^{-1}\mathrm{d}(\Delta u_c)u_c &= \int_{\mathfrak{e}} \Delta m_{c'c} \wedge \left\{m_{c'c}\tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c m_{c'c}^{-1} - [u_c^{-1}\mathrm{d}u_c, \Delta m_{c'c}])\right\} \\ &= -\int_{\mathfrak{e}} \Delta m_{cc'} \wedge \left\{\tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c + [\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c, \Delta m_{cc'}])\right\} \\ &= -\int_{\mathfrak{e}} \left\{\Delta m_{cc'} \wedge \tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c + [\Delta m_{cc'}, \Delta m_{cc'}]\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c\right\} \end{aligned} \tag{C.30}$$

where we used the expression (C.20) to simplify. Using again (C.20), the second term simplifies to

$$\begin{aligned}\int_{\mathfrak{e}} \left(m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1} \wedge u_c^{-1}\mathrm{d}(\Delta u_c)u_c\right) &= \int_{\mathfrak{e}} \Delta\tilde{\ell}_c \wedge \tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c - \int_{\mathfrak{e}} \Delta\tilde{\ell}_c \wedge [\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c, m_{c'c}^{-1}\delta m_{c'c}] \\ &= \int_{\mathfrak{e}} \Delta\tilde{\ell}_c \wedge \tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c + \int_{\mathfrak{e}} [\Delta m_{cc'}, \Delta\tilde{\ell}_c]\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c \end{aligned} \tag{C.31}$$

To get the last equation, we used that the commutator of $\mathfrak{su}(2)$ on $\mathfrak{an}_2$ restricted to $\mathfrak{an}_2$ is actually given by the $\mathfrak{su}(2)$ structure constant. The other contributions from (C.29) is simplified as follows

$$\begin{aligned} & \frac{1}{2}\int_{\mathfrak{e}} [\Delta m_{c'c} + m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}, \Delta m_{c'c} + m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}] \wedge u_c^{-1}\mathrm{d}u_c \\ = & \frac{1}{2}\int_{\mathfrak{e}} m_{c'c}^{-1}\left([\Delta m_{c'c}, \Delta m_{c'c}] + m_{c'c}[\Delta\tilde{\ell}_c, \Delta\tilde{\ell}_c]m_{c'c}^{-1} + 2[\Delta m_{c'c}, m_{c'c}(\Delta\tilde{\ell}_c)m_{c'c}^{-1}]\right) m_{c'c} \wedge \tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c \\ = & \frac{1}{2}\int_{\mathfrak{e}} \left([\Delta m_{cc'}, \Delta m_{cc'}] + [\Delta\tilde{\ell}_c, \Delta\tilde{\ell}_c] - 2[\Delta m_{cc'}, \Delta\tilde{\ell}_c]\right) \wedge \tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c, \end{aligned} \tag{C.32}$$

where we used the first expression of (C.18) to go from the first to second equality. The left over contributions from (C.30), (C.31) and (C.32) gives

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \int_{\mathfrak{e}} \left\{ \Delta m_{cc'} \wedge \tilde{u}_c^{-1}(\mathrm{d}\Delta\tilde{u}_c)\tilde{u}_c + \frac{1}{2}[\Delta m_{cc'}, \Delta m_{cc'}]\tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c) \right\} \\ &= \int_{\mathfrak{e}} \left\{ \Delta m_{cc'} \wedge \delta X + \frac{1}{2}[\Delta m_{cc'}, \Delta m_{cc'}]X \right\}, \end{aligned} \tag{C.33}$$

where we set $X = \tilde{u}_c^{-1}\mathrm{d}\tilde{u}_c$ to arrive at the second equation. Making use of the second expression of (C.23) we get

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \int_{\mathfrak{e}} \left\{ \Delta m_{cc'} \wedge \delta X + \frac{1}{2}\left(\underline{\Delta} m_{cc'} \wedge \delta\overline{X} - \Delta m_{cc'} \wedge \delta X \right) \right\} = \frac{1}{2}\int_{\mathfrak{e}} \{ \Delta m_{cc'} \wedge \delta X + \underline{\Delta} m_{cc'} \wedge \delta\overline{X} \} \\ &= \frac{1}{2}\int_{\mathfrak{e}} \{ \Delta\tilde{m}_{cc'} \wedge \tilde{u}_c(\delta X)\tilde{u}_c^{-1} + \tilde{m}_{cc'}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge \tilde{u}_c(\delta X)\tilde{u}_c^{-1} \\ &\qquad -\tilde{m}_{cc'}^{-1}(\Delta\tilde{u}_c)\,\tilde{m}_{cc'} \wedge u_c(\delta\overline{X})u_c^{-1} + \underline{\Delta}\tilde{m}_{cc'} \wedge u_c(\delta\overline{X})u_c^{-1} \}, \end{aligned} \tag{C.34}$$

where we again plugin equations (C.25) and (C.26). Using the second equations in (C.22) and (C.23)

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \frac{1}{2}\left\{ \int_{\mathfrak{e}} \Delta\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \int_{\mathfrak{e}} \tilde{m}_{cc'}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge \mathrm{d}(\Delta\tilde{u}_c) \right. \\ &\qquad \left. - \int_{\mathfrak{e}} \tilde{m}_{cc'}^{-1}(\Delta\tilde{u}_c)\,\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta u_c) + \int_{\mathfrak{e}} \underline{\Delta}\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta u_c) \right\} \\ &= \frac{1}{2}\left\{ \int_{\mathfrak{e}} \Delta\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \int_{\mathfrak{e}} \tilde{m}_{cc'}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge \mathrm{d}(\Delta\tilde{u}_c) \right. \\ &\qquad \left. - \int_{\mathfrak{e}} (\Delta\tilde{u}_c) \wedge \tilde{m}_{cc'}\mathrm{d}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} + \int_{\mathfrak{e}} \underline{\Delta}\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta u_c) \right\} \\ &= \frac{1}{2}\left\{ \int_{\mathfrak{e}} \Delta\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \int_{\mathfrak{e}} \underline{\Delta}\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta u_c) \right. \\ &\qquad \left. + \int_{\mathfrak{e}} \tilde{m}_{cc'}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \int_{\mathfrak{e}} \tilde{m}_{cc'}\mathrm{d}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge (\Delta\tilde{u}_c) \right\} \\ &= \frac{1}{2}\int_{\mathfrak{e}} \{ \Delta\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \mathrm{d}\left(\tilde{m}_{cc'}(\Delta u_c)\tilde{m}_{cc'}^{-1} \wedge \Delta\tilde{u}_c \right) + \underline{\Delta}\tilde{m}_{cc'} \wedge \mathrm{d}(\Delta u_c) \} \end{aligned} \tag{C.35}$$

where in the last equation we used the relation

$$\mathrm{d}\left(\tilde{m}_{cc'}(\Delta u_c)\tilde{m}_{cc'}^{-1} \wedge \Delta\tilde{u}_c \right) = \tilde{m}_{cc'}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge \mathrm{d}(\Delta\tilde{u}_c) + \tilde{m}_{cc'}\mathrm{d}(\Delta u_c)\,\tilde{m}_{cc'}^{-1} \wedge (\Delta\tilde{u}_c). \tag{C.36}$$

We carry out the integration in , which gives

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \frac{1}{2}\{\Delta\tilde{m}_{cc'}\wedge(\Delta\tilde{u}_{cv'}-\Delta\tilde{u}_{cv})+\underline{\Delta}\tilde{m}_{cc'}\wedge(\Delta u_{cv'}-\Delta u_{cv})\\ &\qquad +\left(\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}-\tilde{m}_{cc'}(\Delta u_{cv})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv}\right)\}\\ &= \frac{1}{2}\left\{\Delta\tilde{m}_{cc'}\wedge\tilde{u}_{cv}\,(\Delta\tilde{u}_{vv'})\,\tilde{u}^{-1}_{cv}+\underline{\Delta}\tilde{m}_{cc'}\wedge u_{c'v}\left(\Delta u^{c'}_{vv'}\right)u^{-1}_{c'v}\right.\\ &\qquad \left.+\left(\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}-\tilde{m}_{cc'}(\Delta u_{cv})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv}\right)\right\}\end{aligned}\tag{C.37}$$

We used in the last equation that $(\Delta\tilde{u}_{cv'}-\Delta\tilde{u}_{cv})=\tilde{u}_{c'v}\,(\Delta\tilde{u}^{c'}_{vv'})\,\tilde{u}^{-1}_{c'v}$ since $\tilde{u}_c$ actually sits at c'. Simplifying further

$$\begin{aligned}\Omega^{LQG^*}_{cc'} &= \frac{1}{2}\{\Delta\tilde{m}_{cc'}\wedge\tilde{u}_{cv}\,(\Delta\tilde{u}_{vv'})\,\tilde{u}^{-1}_{cv}+\left(\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}-\tilde{m}_{cc'}(\Delta u_{cv})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv}\right)\\ &\quad +\left(\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}-\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}\right)+\underline{\Delta}\tilde{m}_{cc'}\wedge u_{c'v}\left(\Delta u^{c'}_{vv'}\right)u^{-1}_{c'v}\Big\}\\ &= \frac{1}{2}\{\Delta\tilde{m}_{cc'}\wedge\tilde{u}_{cv}\,(\Delta\tilde{u}_{vv'})\,\tilde{u}^{-1}_{cv}+\tilde{m}_{cc'}(\Delta u_{cv})\tilde{m}^{-1}_{cc'}\wedge\tilde{u}_{cv}(\Delta\tilde{u}_{vv'})\tilde{u}^{-1}_{cv}+\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}\\ &\quad -\tilde{m}_{cc'}(\Delta u_{cv'})\tilde{m}^{-1}_{cc'}\wedge\Delta\tilde{u}_{cv'}+\underline{\Delta}\tilde{m}_{cc'}\wedge u_{c'v}\left(\Delta u^{c'}_{vv'}\right)u^{-1}_{c'v}\Big\}\\ &= \frac{1}{2}\left\{\Delta\overline{m}^{v}_{cc'}\wedge\Delta u^{c}_{vv'}-m^{-1}_{cc'}\,\Delta u_{cv'}\,m_{cc'}\wedge\,\overline{u}_{c'v}(\Delta\overline{u}^{c'}_{vv'})\overline{u}^{-1}_{c'v}+\underline{\Delta}m_{cc'}\wedge\overline{u}_{c'v}\left(\Delta\overline{u}^{c'}_{vv'}\right)\overline{u}^{-1}_{c'v}\right\}\\ &= \frac{1}{2}\left\{\Delta\overline{m}^{v}_{cc'}\wedge\Delta u^{c}_{vv'}+\underline{\Delta}\overline{m}^{v'}_{cc'}\wedge\overline{u}^{c'}_{v'v}\left(\Delta\overline{u}^{c'}_{vv'}\right)\overline{u}^{c'}_{vv'}\right\}\\ &= \frac{1}{2}\left\{\Delta\overline{m}^{v}_{cc'}\wedge\Delta u^{c}_{vv'}+\underline{\Delta}\overline{m}^{v'}_{cc'}\wedge\underline{\Delta}\overline{u}^{c'}_{vv'}\right\}.\end{aligned}\tag{C.38}$$

Appendix D

Comprehensive details on Hopf algebras

Most of the materials collected in this appendix are standard definitions and results on Hopf algebras used in this thesis and can be found in the following books [53, 108]. Where needed we will indicate appropriate references other the two books listed.

D.1 Hopf algebras

Let k be a field of characteristic zero. The 3-tuple or triple (A, μ, η) is an algebra over a field k, where A is a vector space, μ is a multiplication (or product) linear map $\mu : A \otimes A \longrightarrow A$ defined as $\mu(a \otimes b) = ab$ such that the associative property[1]

$$\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}), \tag{D.1}$$

is satisfied, and η is a unit linear map $\eta : k \longrightarrow A$ defined $\eta(\lambda) = \eta\, 1_A$ such that the unit property

$$\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta) \tag{D.2}$$

is satisfied. The above condition implies that the element $1_A \in A$ is both a left and a right unit for the multiplication map μ.

Equivalently, μ and η are defined in such a way the following diagrams commute

Given any two algebras (A, μ, η) and (A', μ', η') over k, then the linear map $f : A \longrightarrow A'$ is an algebra homomorphism if it satisfies

$$\mu' \circ (f \otimes f) = f \circ \mu, \quad f \circ \eta = \eta'. \tag{D.3}$$

[1]Explicitly the associativity of μ is $a(bc) = ab(c)$

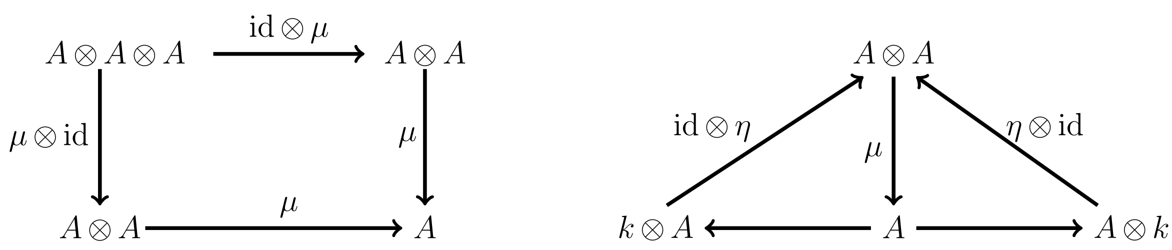


Figure D.1: Associativity and unit element expressed as commutative diagrams

The dual notion of an algebra is the triple (C, Δ, ϵ) called a coalgebra, where C is a vector space over the field k, Δ is the dual map to μ, called the comultiplication (or coproduct) map $\Delta : C \otimes C \longrightarrow C$, and $\epsilon : C \longrightarrow k$ is the counit map. Δ satisfy the coassoaciative property

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{D.4}$$

while the counit map obey the property

$$(\epsilon \otimes \mathrm{id}) \circ \Delta(c) = c = (\mathrm{id} \otimes \epsilon) \circ \Delta(c), \quad \text{for all } c \in C. \tag{D.5}$$

Equivalently, Δ and ϵ are defined in such a way the following diagrams commute

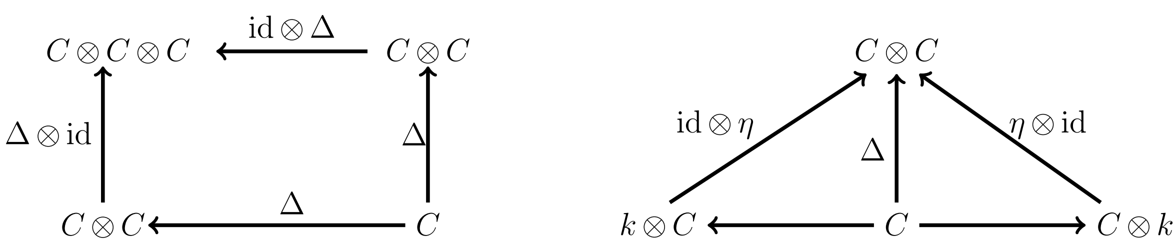


Figure D.2: Coassociativity and counit element expressed as commutative diagrams

Unlike the product which combines two elements together, the coproduct splits up an element apart. Notation wise, the coproduct of an element $c \in C$ is written as

$$\Delta(c) = \sum_{i=1}^{n} c_{i(1)} \otimes c_{i(2)}, \tag{D.6}$$

this way of writing the coproduct is known as the Sweedler notation, where $c_{i(1)}, c_{i(2)} \in C$ and the sum denote an element of $C \otimes C$. For brevity the index i is suppressed and the sum is omitted then equation (D.6) reads[2] $\Delta(c) = c_{(1)} \otimes c_{(2)}$. In Sweedler notation, the coassociativity and the counit property are respectively

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \tag{D.7}$$

$$\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)}). \tag{D.8}$$

A linear map $f : C \longrightarrow C'$ is a coalgebra homomorphism if it satisfies

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad \epsilon = \epsilon' \circ f, \tag{D.9}$$

with (C, Δ, ϵ) and (C', Δ', ϵ') as the coalgebra structures.

A bialgebra H is a vector space over k, which is both an algebra (H, μ, η) and a coalgebra (H, Δ, ϵ) with the following compatibility conditions

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1_H) = 1_k \tag{D.10}$$

for all $h, g \in H$.

A Hopf algebra H over k is then a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ equipped with a linear map $S : H \longrightarrow H$ satisfying

$$\mu(S \otimes \mathrm{id}) \circ \Delta = \mu(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{D.11}$$

S is referred to as the antipode and plays the role of an inverse. Though in most instance, S^{-1} is not presumed to exist. In this thesis we will consider H to be finite-dimensional which means S^{-1} exist. The antipode S satisfy the following properties

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \tag{D.12}$$

$$(S \otimes S) \circ \Delta h = \tau \circ \Delta \circ Sh, \quad \epsilon Sh = \epsilon h, \tag{D.13}$$

where $\tau : V \otimes W \longrightarrow W \otimes V$ is the transposition map defined as $\tau(v \otimes w) = w \otimes v$, for all $v \in V, w \in W$. The set of equations in (D.12) implies that S is an antialgebra map and the other set of equations in (D.13) has S as an anticoalgebra map. In Sweedler notation the first equation in (D.13) reads

$$(Sh)_{(1)} \otimes (Sh)_{(2)} = Sh_{(2)} \otimes Sh_{(1)}. \tag{D.14}$$

A Hopf algebra is called commutative if $\mu(h_1, h_2) = \mu(h_2, h_1)$ for every $h_1, h_2 \in H$. It is called cocommutative if $\Delta = \tau \circ \Delta$. A Hopf subalgebra of H is a subalgebra F such that

[2] In this thesis we shall also use the Sweedler notation with superscripts $\Delta(c) = c_{(1)} \otimes c_{(2)} = c^{(1)} \otimes c^{(2)}$

$\Delta(F) \subset F \otimes F$ and $S(F) \subset F$. Quantum groups are usually seen as noncocommutative Hopf algebra.

Other Hopf algebras can be constructed from H. We denote H^{op} with structure $(H, \mu^{\text{op}}, \eta, \Delta, \epsilon, S^{-1})$ as the opposite Hopf algebra, where $\mu^{\text{op}}(a \otimes b) = ba$ is the opposite product. Furthermore we denote H^{cop} as the the Hopf algebra with an opposite coproduct $\Delta^{\text{cop}} = c_{(2)} \otimes c_{(1)}$.

The axioms in a Hopf algebra are self-dual, in that, when the maps in the Hopf algebra are reversed and μ, η are interchanged with Δ, ϵ the same set of axioms are obtained. We denote by H^* the dual Hopf algebra with dual pairing given by the non-degenerate bilinear map $\langle \cdot, \cdot \rangle : H^* \times H \longrightarrow k$, defined as the evaluation $(a, b) \longmapsto \langle a, h \rangle = a(h)$, such that, for all $a, b \in H^*$ and $h, g \in H$, the following are satisfied

$$\begin{aligned} &\langle \Delta(a), g \otimes h \rangle = \langle a, gh \rangle, \quad \langle a \otimes b, \Delta(h) \rangle = \langle ab, h \rangle \\ &\langle a, 1 \rangle = \epsilon(a), \quad \langle 1, h \rangle = \epsilon(h) \end{aligned} \tag{D.15}$$

Note that for the property (D.15) we have extend the dual pairing on the tensor products by

$$\langle a \otimes b, g \otimes h \rangle = \langle a, g \rangle \langle b, h \rangle, \tag{D.16}$$

and from the properties of the dual pairing it follows that

$$\langle S(a), h \rangle = \langle a, Sh \rangle. \tag{D.17}$$

The structure of H^* as a Hopf algebra is $(H^*, \mu^*, \eta^*, \Delta^*, \epsilon^*, S^*)$, where μ^*, η^*, Δ^*, ϵ^* and S^* denote the product, unit, coproduct, counit and antipode in H^* respectively. We can also define the dual $H^{\text{op}*}$ of H^{op}, whose Hopf algebra structure is $(H^*, (\mu^{\text{op}})^*, \eta^*, \Delta^*, \epsilon^*, (S^{-1})^*)$.

We consider a few examples of Hopf algebras.

Example D.1.1. *Let G be a finite group. The group algebra denoted kG is a k-vector space with basis $\{u | u \in G\}$. An element of kG has the form $\{\sum_{u \in G} \alpha(u) u\}$, with the coefficients $\alpha(u) \in k$. For $g, h \in G$ and $\alpha \in K$, the algebra structure of kG is defined using the product in G, i.e.,*

$$\mu(g \otimes h) = gh$$

extended by linearity to kG, and the unit is the unit $1_G = e$ element of G. The coalgebra structure of kG is determined by the coproduct and the conunit

$$\Delta u = u \otimes u, \quad \epsilon(u) = 1,$$

the antipode is $S(u) = u^{-1}$.

The algebra and coalgebra property can be easily checked, and one can also check that S *satisfies the antipode axiom. It is also easily checked that* kG *is cocommutative but non-commutative in general.*

Example D.1.2. *Given a finite group* G *with identity* e*, the group function algebra denoted by* $k(G)$*, is the set of linear functions on* G *to* k*. The product and unit are defined by*

$$(\phi\psi)(u) = \phi(u)\psi(u), \quad \eta(\lambda)(u) = \lambda, \quad \phi, \psi \in k(G), u \in G.$$

The coproduct, counit and antipode are respectively

$$(\Delta\phi)(u, v) = \phi(uv), \quad \epsilon(\phi) = \phi(e), \quad (S\phi)(u) = \phi(u^{-1}).$$

$k(G)$ *is a Hopf algebra over* k *and is dual by definition to the group algebra* kG*. One can check that* $k(G)$ *is commutative but in general non-cocommutative.*

Example D.1.3. *Let* $\mathfrak{g}$ *be a Lie algebra over a field* k*. The universal enveloping algebra* $U(\mathfrak{g})$ *of* $\mathfrak{g}$ *is the quotient of the tensor algebra* $T(\mathfrak{g}) := k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})$..... *by the two-sided ideal* $I(\mathfrak{g})$ *generated by the elements* $x \otimes y - y \otimes x - [x, y]$ *for all* $x, y \in \mathfrak{g}$*. The coproduct, counit and antipode are*

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x. \tag{D.18}$$

As an algebra, $U(\mathfrak{g})$ *is noncommutative.*

D.2 Representation: Actions and Coactions

One of the many things Hopf algebras do is act on vector spaces. We will focus on left representations (for both actions and coactions) in this section. Analogous definitions of right representations also exist and these can be found in Appendix **??**.

A left action or representation of an algebra A is a pair $(\triangleright, V)$, where V is a vector space and $\triangleright : A \otimes V \to V$ is a linear map such that

$$a\triangleright v \in V, \quad (ab)\triangleright v = a\triangleright(b\triangleright v), \quad 1\triangleright v = v.$$

We say that the algebra A acts on the left of the vector space V or V is a left A-module depending on whether we want to emphasise the map $\triangleright$ or the space on which the algebra acts. If the Hopf algebra H acts on vector spaces V, W, then it also act on the tensor product $V \otimes W$ by

$$h \triangleright (v \otimes w) = h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w, \quad \forall h \in H, v \in V \text{ and } w \in W.$$

This is another way to see how the existence of the coproduct allows one to extend a Hopf algebra representation to a tensor product representation.

Apart from acting on vector spaces, Hopf algebras act on structures, such as algebras, coalgebras, bialgebras or Hopf algebras. In each case, it is natural to ask that the relevant structure be respected by the action in some way.

An algebra A is said to be an H-module *algebra* if A is a left H-module (i.e. H acts on it from the left) and this action is covariant, i.e.

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \qquad h \triangleright 1 = \epsilon(h)1, \;\; a \in A, \, h \in H. \tag{D.19}$$

We refer to the pair $(H, A)_L$ (resp. $(H, A)_R$) as a left (resp. right) *covariant system* if A is a module algebra under the left (resp. right) action of H. If H acts covariantly on A from the right then one can turn this to a left action of H on A^{op}, according to the relation

$$(H, A)_R \to (H, A^{\mathrm{op}})_L, \quad h \triangleright a = a \triangleleft S^{-1}h. \tag{D.20}$$

We will mean a left covariant system when no index L, R is specified.

A coalgebra C is a left H-module coalgebra if

$$\Delta(h \triangleright c) = h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleright c_{(2)}, \quad \epsilon(h \triangleright c) = \epsilon(h)\epsilon(c), \;\; c \in C, \, h \in H. \tag{D.21}$$

This says that $\triangleright : H \otimes C \longrightarrow C$ is a coalgebra map, where $H \otimes C$ has the tensor product coalgebra structure, i.e. $\Delta(h \triangleright c) = (\Delta h) \triangleright \Delta c$.

Some examples of module algebras and module coalgebras that arise in the lattice models we will be studying later are the following:

Example D.2.1. *(i) The left regular action of a bialgebra or Hopf algebra H on it self is $h \triangleright g = hg$, and makes H into an H-module coalgebra.*

(ii) The left co-regular of a bialgebra or Hopf algebra H on H^* is $h \triangleright \phi = \langle h, \phi_{(2)} \rangle \phi_{(1)}$ and makes H^* into an H-module algebra.

(iii) The left adjoint action of a Hopf algebra on itself is $h \triangleright g = h_{(1)} g S h_{(2)}$, and makes H into an H-module algebra.

(iv) *The left coadjoint action of a Hopf algebra H on H^* is $h \triangleright \phi = \langle h, S\phi_{(1)}\phi_{(3)}\rangle \phi_{(2)}$, and makes H^* into an H-module coalgebra.*

Proposition D.2.2. *Let H be a bialgebra or Hopf algbra, and let A be a left H-module algebra. There is a left cross product algebra $A \rtimes H$ built on $A \otimes H$ with product*

$$(a \otimes h)(b \otimes g) = a\,(h_{(1)} \triangleright b) \otimes h_{(2)} g, \quad a, b \in A, \quad h, g \in H, \tag{D.22}$$

and the unit element is $1 \otimes 1$.

Example D.2.3. *The left Heisenberg double of H is the cross product for the left co-regular action of H on H^**

$$(\phi \otimes h)(\psi \otimes g) = \langle \psi_{(2)}, h_{(1)}\rangle \psi\phi_{(1)} \otimes h_{(2)} g$$

where $\phi, \psi \in H^*$, $h, g \in H$.

Dualizing the concept of modules gives rise to the concept of comodules. A left coaction (or corepresentation or V is a left H-comodule) of a coalgebra H is a pair (β, V), where V is vector space and $\beta : V \longrightarrow H \otimes V$ is a linear map such that

$$(\mathrm{id} \otimes \beta) \circ \beta = (\Delta \otimes \mathrm{id}) \circ \beta \quad \text{and} \quad \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \beta.$$

Writing the linear map β explicitly as $\beta(v) = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}$, where the right hand side is an explicit representation of an element of $H \otimes V$, then the left comodule axiom reads

$$\sum v^{(\bar{1})} \otimes v^{(\bar{2})(\bar{1})} \otimes v^{(\bar{2})(\bar{2})} = \sum v^{(\bar{1})}{}_{(1)} \otimes v^{(\bar{1})}{}_{(2)} \otimes v^{(\bar{2})} \tag{D.23a}$$

$$\sum \epsilon(v^{(\bar{1})}) v^{(\bar{2})} = v. \tag{D.23b}$$

An algebra A is a left H-comodule algebra if it is a left comodule and

$$\beta(ab) = \beta(a)\beta(b), \quad \beta(1) = 1 \otimes 1. \tag{D.24}$$

Here $H \otimes A$ has the tensor product algebra structure and we are requiring β to be an algebra map.

Finally, a coalgebra C is a left comodule coalgebra if the left coaction satisfies

$$\sum c^{(\bar{1})} \otimes c^{(\bar{2})}{}_{(1)} \otimes c^{(\bar{2})}{}_{(2)} = \sum c_{(1)}{}^{(\bar{1})} c_{(2)}{}^{(\bar{1})} \otimes c_{(1)}{}^{(\bar{2})} \otimes c_{(2)}{}^{(\bar{2})} \tag{D.25a}$$

$$\sum c^{(\bar{1})} \epsilon(c^{(\bar{2})}) = \epsilon(c). \tag{D.25b}$$

Proposition D.2.4. *Let H be a bialgebra or Hopf algebra, and let C be a left H-comodule coalgebra. There is a left cross coproduct coalgebra $C{>\!\!\blacktriangleleft} H$ built on $C \otimes H$ with the coalgebra structure*

$$\Delta(c \otimes h) = \sum c \otimes c_{(2)}{}^{(\bar{1})} h_{(1)} \otimes c_{(2)}{}^{(\bar{2})} \otimes h_{(2)}, \quad \epsilon(c \otimes h) = \epsilon(c)\epsilon(h) \tag{D.26}$$

for $h \in H$, $c \in C$.

An important proposition, which will be become useful in the next chapter is the following.

Proposition D.2.5. *If H is a finite dimensional Hopf algebra, then a left action of H corresponds to a right coaction of H^* on the same vector space. Explicitly, if $\beta(v) = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}$ is the coaction of H^* on, then $h \triangleright v = \sum v^{(\bar{1})} \langle h, v^{(\bar{2})} \rangle$ is the corresponding action of H on. If A is a left H-module algebra, then it is a right H^*-comodule algebra. If C is a left H-module coalgebra, then it is a right H^*-comodule coalgebra.*

D.3 Star structure

If $k = \mathbb{C}$, then a Hopf algebra may sometimes be equipped with an antilinear map.

Definition D.3.1. *If $k = \mathbb{C}$. Given an antilinear map $\star : H \to H$ satisfying the condition*

$$\star^2 = id, \quad (hg)^\star = g^\star h^\star, \qquad \forall h, g \in H, \tag{D.27}$$

then it turns H into a $\star$-algebra. Hence H is a Hopf $\star$-algebra if condition (D.27) and the following are satisfied

$$\Delta h^\star = (\Delta h)^{\star \otimes \star}, \quad \epsilon(h^\star) = \overline{\epsilon(h)}, \quad (S \circ \star)^2 = id. \tag{D.28}$$

If A, H are two $\star$-Hopf algebras, they are dually paired if they are dually paired as Hopf algebras and in addition

$$\langle \phi^\star, h \rangle = \overline{\langle \phi, (Sh)^\star \rangle} \tag{D.29}$$

for all $h \in H$ and $\phi \in A$.

D.4 Haar integrals

Definition D.4.1. *Let H be a Hopf algebra H. A (normalised) Haar integral in H is an element $\ell \in H$ with $h \cdot \ell = \ell \cdot h = \epsilon(h)\ell$ for all $h \in H$ and $\epsilon(\ell) = 1$.*

Proposition D.4.2. *[72] Let H be a finite-dimensional Hopf C^*-algebra.*

1. *If $\ell \in H$ is a Haar integral, then $\ell^2 = \ell$,*

2. *If $\ell \in H$ is a Haar integral, then $\ell^* = \ell$,*

3. *If $\ell, \ell' \in H$ are Haar integrals, then $\ell = \ell'$*

4. *If $\ell \in H$ is a Haar integral, then $\Delta^{(n)}(\ell)$ is invariant under cyclic permutations i.e. $\ell \in Cocom(H)$ and $S(\ell) = \ell$.*

5. *If $\ell \in H$ is a Haar integral, then the element $e = (id \otimes S)(\Delta(\ell))$ is a separability idempotent in H, i.e. one has $\mu(e) = \ell_{(1)} S(\ell_{(2)}) = 1$, $e.e = e$ and for all $h \in H$*

$$(h \otimes 1) \cdot \Delta(\ell) = (1 \otimes Sh) \cdot \Delta(\ell) \quad \Delta(\ell)(h \otimes 1) = \Delta(\ell)(1 \otimes Sh),$$

   where μ is the linear unit map $\mu : k \to H$.

6. *If $\ell \in H$ is a Haar integral, then $\langle \alpha_{(1)}, \ell \rangle \alpha_{(2)} = \langle \alpha_{(2)}, \ell \rangle \alpha_{(1)} = \langle \ell, \alpha \rangle 1$ for all $\alpha \in H^*$.*

D.5 Bicrossproduct models

Theorem D.5.1. *Let A, H be bialgebras, let A be a left H-module algebra and let H be a right A-comodule coalgebra by maps*

$$\alpha : H \otimes A \longrightarrow A,\ \alpha(h \otimes a) = h \triangleright a, \quad \beta : H \longrightarrow H \otimes A,\ \beta(h) = \sum h^{(\bar{1})} \otimes h^{(\bar{2})}$$

obeying the compatibility conditions

$$\epsilon(h \triangleright a) = \epsilon(h)\epsilon(a), \quad \Delta(h \triangleright a) = \sum {h_{(1)}}^{(\bar{1})} \triangleright a_{(1)} \otimes {h_{(1)}}^{(\bar{2})} (h \triangleright a_{(2)}), \tag{D.30a}$$

$$\beta(1) = 1 \otimes 1, \quad \beta(gh) = \sum {g_{(1)}}^{(\bar{1})} h^{(\bar{1})} \otimes {g_{(1)}}^{(\bar{2})} (g_{(2)} \triangleright h^{(\bar{2})}), \tag{D.30b}$$

$$\sum {h_{(2)}}^{(\bar{1})} \otimes (h_{(1)} \triangleright a) {h_{(2)}}^{(\bar{2})} = \sum {h_{(1)}}^{(\bar{1})} \otimes {h_{(1)}}^{(\bar{2})} (h_{(2)} \triangleright a), \tag{D.30c}$$

for all $a, b \in A$ and $h, g \in H$. Then $A \rtimes H$ and $A \blacktriangleright\!\!< H$ form a bialgebra: the left-right bicrossproduct bialgebra associated to the compatible (co)actions and denoted by $A \blacktriangleright\!\!\triangleleft H$. If A, H are Hopf algebras then so is $A \blacktriangleright\!\!\triangleleft H$. Its antipode is

$$S(a \otimes h) = \sum (1 \otimes S h^{(\bar{1})})(S(a h^{(\bar{2})}) \otimes 1). \tag{D.31}$$

There is a very similar construction with left and right interchanged.

Theorem D.5.2. *Let A, H be bialgebras, let A be a right H-module algebra and let H be a left A-comodule coalgebra by maps*

$$\alpha : A \otimes H \longrightarrow A,\ \alpha(a \otimes h) = a \triangleleft h, \quad \beta : H \longrightarrow A \otimes H,\ \beta(h) = \sum h^{(\bar{1})} \otimes h^{(\bar{2})}$$

obeying the compatibility conditions

$$\epsilon(a \triangleleft h) = \epsilon(a)\epsilon(h), \quad \Delta(a \triangleleft h) = \sum (a_{(1)} \triangleleft h_{(1)}) {h_{(2)}}^{(\bar{1})} \otimes a_{(2)} \triangleleft {h_{(2)}}^{(\bar{2})}, \tag{D.32a}$$

$$\beta(1) = 1 \otimes 1, \quad \beta(hg) = \sum (h^{(\bar{1})} \triangleleft g_{(1)}) {g_{(2)}}^{(\bar{1})} \otimes h^{(\bar{2})} {g_{(2)}}^{(\bar{2})}, \tag{D.32b}$$

$$\sum {h_{(1)}}^{(\bar{1})} (a \triangleleft h_{(2)}) \otimes {h_{(1)}}^{(\bar{2})} = \sum (a \triangleleft h_{(1)}) {h_{(2)}}^{(\bar{1})} \otimes {h_{(2)}}^{(\bar{2})}, \tag{D.32c}$$

for all $h, g \in H$ and $a, b \in A$. Then $H \rtimes A$ and $H \blacktriangleright\!\!< A$ form a bialgebra. It is the right-left bicrossproduct bialgebra and is denoted by $H \triangleright\!\!\blacktriangleleft A$. If H, A are Hopf algebras then so is $H \triangleright\!\!\blacktriangleleft A$. Its antipode is

$$S(h \otimes a) = \sum (1 \otimes S(h^{(\bar{1})} a))(S h^{(\bar{2})} \otimes 1). \tag{D.33}$$

Appendix E

Proof of Lemma 7.1.3

(i). Suppose there exist at least one edge connecting the vertices v and w, and the orientation is from v to w, then we have

$$A^h(v,p)(\phi) = L_+^h(\phi), \quad A^g(w,p')(\phi) = L_-^g(\phi). \tag{E.1}$$

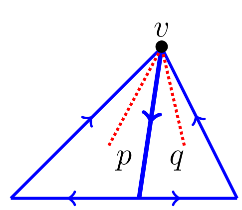
From the definition of $L_\pm$ and equation (7.27) we get $A^h(A^g(\phi)) = A^g(A^h(\phi)) = \epsilon(g)\langle h, S\phi_{(1)}\phi_{(3)}\rangle\phi_{(2)}$, and this can be generalized to any number of edges connecting v and w.

Suppose also the incident edges to v and and to w are disjoint, then we have for one incident edge

$$A^h(v)(\phi) = L_+^h(\phi), \quad A^g(w)(\phi) = L_+^g(\phi), \tag{E.2}$$

and obviously these two operators are commuting.

(ii) Consider the diagram below with faces p and q sharing a common edge. For the face p, starting at the vertex v and moving clockwise, the face operator for p reads



$$B^a(\phi^1 \otimes \phi^2 \otimes \phi^3) = T_+^{a(3)}(\phi^1) \otimes T_+^{a(2)}(\phi^2) \otimes T_+^{a(1)}(\phi^3). \tag{E.3}$$

Likewise for the face q, starting at the vertex v and moving anti-clockwise, its operator is

$$B^b(\phi^1 \otimes \phi^2 \otimes \phi^3) = T_-^{b_{(3)}}(\phi^1) \otimes T_-^{b_{(2)}}(\phi^2) \otimes T_-^{b_{(1)}}(\phi^3). \tag{E.4}$$

With the above face operators, we compute

$$\begin{aligned} B^a B^b(\phi^1 \otimes \phi^2 \otimes \phi^3) &= \langle b, \phi^1_{(2)}\phi^2_{(2)}\phi^3_{(2)}\rangle B^a\left(\phi^1_{(1)} \otimes \phi^2_{(1)} \otimes \phi^3_{(1)}\right) \\ &= \langle b, \phi^1_{(2)}\phi^2_{(2)}\phi^3_{(2)}\rangle\langle Sa, \phi^1_{(1)(1)}\phi^2_{(1)(1)}\phi^3_{(1)(1)}\rangle\phi^1_{(1)(2)} \otimes \phi^2_{(1)(2)} \otimes \phi^3_{(1)(2)} \\ &= \langle b, \phi^1_{(3)}\phi^2_{(3)}\phi^3_{(3)}\rangle\langle Sa, \phi^1_{(1)}\phi^2_{(1)}\phi^3_{(1)}\rangle\phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3_{(2)} \end{aligned}$$

and also

$$\begin{aligned} B^b B^a(\phi^1 \otimes \phi^2 \otimes \phi^3) &= \langle Sa, \phi^1_{(1)}\phi^2_{(1)}\phi^3_{(1)}\rangle B^b\left(\phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3_{(2)}\right) \\ &= \langle Sa, \phi^1_{(1)}\phi^2_{(1)}\phi^3_{(1)}\rangle\langle b, \phi^1_{(2)(2)}\phi^2_{(2)(2)}\phi^3_{(2)(2)}\rangle\phi^1_{(2)(1)} \otimes \phi^2_{(2)(1)} \otimes \phi^3_{(2)(1)} \\ &= \langle Sa, \phi^1_{(1)}\phi^2_{(1)}\phi^3_{(1)}\rangle\langle b, \phi^1_{(3)}S\phi^2_{(3)}\phi^3_{(3)}\rangle\phi^1_{(2)} \otimes \phi^2_{(2)} \otimes \phi^3_{(2)}. \end{aligned}$$

This then shows $B^aB^b(\phi^1 \otimes \phi^2 \otimes \phi^3) = B^bB^a(\phi^1 \otimes \phi^2 \otimes \phi^3)$.

(iii) Consider the Figure E.1 below with two different sites (v, p) and (v', p'). The vertex operator for the site (v, p) moving clockwise reads

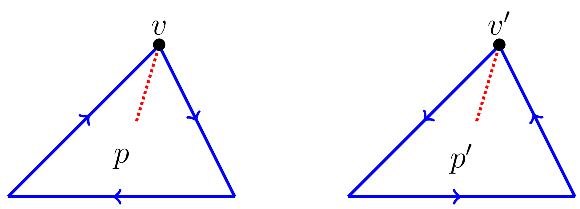


Figure E.1: A diagram depicting two graphs with disjoint sites (v, p) and (v', p').

$$A_v^h(\phi^1 \otimes \phi^2 \otimes \phi^3) = L_-^{h_{(2)}}(\phi^1) \otimes \phi^2 \otimes L_+^{h_{(1)}Sh_{(3)} \otimes h_{(4)}}(\phi^3) \tag{E.5}$$

and the face operator for the site (v', p') moving counterclockwise is

$$B_{v'}^a(\phi^1 \otimes \phi^2 \otimes \phi^3) = T_-^{a_{(3)}}(\phi^1) \otimes T_-^{a_{(2)}}(\phi^2) \otimes T_-^{a_{(1)}}(\phi^3). \tag{E.6}$$

Therefore from the definition of the triangle operators (7.26) we have

$$
\begin{aligned}
& A^h B^a(\phi^1 \otimes \phi^2 \otimes \phi^3) = \langle a, \phi^1_{(2)}\phi^2_{(2)}\phi^3_{(2)}\rangle A^h \left(\phi^1_{(1)} \otimes \phi^2_{(1)} \otimes \phi^3_{(1)}\right) \\
= \;& \langle a, \phi^1_{(2)}\phi^2_{(2)}\phi^3_{(2)}\rangle \langle h_{(2)}, \phi^1_{(1)(3)} S^{-1}\phi^1_{(1)(1)}\rangle \langle Sh_{(1)}, \phi^3_{(1)(1)}\rangle \langle h_{(3)}, \phi^3_{(1)(3)}\rangle \phi^1_{(1)(2)} \otimes \phi^2_{(1)} \otimes \phi^3_{(1)(2)} \\
= \;& \langle a, \phi^1_{(4)}\phi^2_{(2)}\phi^3_{(4)}\rangle \langle h_{(2)}, \phi^1_{(3)} S^{-1}\phi^1_{(1)}\rangle \langle Sh_{(1)}, \phi^3_{(1)}\rangle \langle h_{(3)}, \phi^3_{(3)}\rangle \phi^1_{(2)} \otimes \phi^2_{(1)} \otimes \phi^3_{(2)} \\
= \;& \langle a, \phi^1_{(3)}\phi^2_{(2)}\phi^3_{(3)}\rangle \, \epsilon(h) \, \phi^1_{(2)} \otimes \phi^2_{(1)} \otimes \phi^3_{(2)}. \qquad \text{(E.7)}
\end{aligned}
$$

Similarly, we find

$$
\begin{aligned}
& B^a A^h(\phi^1 \otimes \phi^2 \otimes \phi^3) = \langle h_{(2)}, \phi^1_{(3)} S^{-1}\phi^1_{(1)}\rangle \langle Sh_{(1)}, \phi^3_{(1)}\rangle \langle h_{(3)}, \phi^3_{(3)}\rangle B \left(\phi^1_{(2)} \otimes \phi^2 \otimes \phi^3_{(2)}\right) \\
= \;& \langle h_{(2)}, \phi^1_{(3)} S^{-1}\phi^1_{(1)}\rangle \langle Sh_{(1)}, \phi^3_{(1)}\rangle \langle h_{(3)}, \phi^3_{(3)}\rangle \langle a, \phi^1_{(2)(2)}\phi^2_{(2)}\phi^3_{(2)(2)}\rangle \phi^1_{(2)(1)} \otimes \phi^2_{(1)} \otimes \phi^3_{(2)(1)}) \\
= \;& \langle h_{(2)}, \phi^1_{(4)} S^{-1}\phi^1_{(1)}\rangle \langle Sh_{(1)}, \phi^3_{(1)}\rangle \langle h_{(3)}, \phi^3_{(4)}\rangle \langle a, \phi^1_{(3)}\phi^2_{(2)}\phi^3_{(3)}\rangle \phi^1_{(2)} \otimes \phi^2_{(1)} \otimes \phi^3_{(2)} \\
= \;& \langle a, \phi^1_{(3)}\phi^2_{(2)}\phi^3_{(3)}\rangle \, \epsilon(h) \, \phi^1_{(2)} \otimes \phi^2_{(1)} \otimes \phi^3_{(2)}. \qquad \text{(E.8)}
\end{aligned}
$$

With these operators, we have shown $A^h B^a(\phi^1 \otimes \phi^2 \otimes \phi^3) = B^a A^h(\phi^1 \otimes \phi^2 \otimes \phi^3)$. This can be generalized to any graph and easily shown that the vertex and face operators at two different sites commute. Note that in this case the orientation of the edges belonging to the different sites (v, p) and (v', p') should not coincide. This is illustrated in Figure E.1.

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