Compressible Matrix Algebras and the Distance from Projections to Nilpotents

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 2 and 3 are based on joint work with Laurent W. Marcoux and Heydar Radjavi [7]. I am the sole author of Chapters 4 and 5, which are based on articles [6] and [5], respectively.

Abstract

In this thesis we address two problems from the fields of operator algebras and operator theory. In our first problem, we seek to obtain a description of the unital subalgebras $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ with the property that $E\mathcal{A}E$ is an algebra for all idempotents $E \in \mathbb{M}_n(\mathbb{C})$. Algebras with this property are said to be *idempotent compressible.* Likewise, we wish to determine which unital subalgebras of $\mathbb{M}_n(\mathbb{C})$ satisfy the analogous property for projections (i.e., self-adjoint idempotents). Such algebras are said to be *projection compressible.*

We begin by constructing various examples of idempotent compressible subalgebras of $\mathbb{M}_n(\mathbb{C})$ for each integer $n \geq 3$. Using a case-by-case analysis based on reduced block upper triangular forms, we prove that our list includes all unital projection compressible subalgebras of $\mathbb{M}_3(\mathbb{C})$ up to similarity and transposition. A similar examination indicates that the same phenomenon occurs in the case of unital subalgebras of $\mathbb{M}_n(\mathbb{C})$, $n \geq 4$. We therefore demonstrate that the notions of projection compressibility and idempotent compressibility coincide for unital subalgebras of $\mathbb{M}_n(\mathbb{C})$, and obtain a complete classification of the unital algebras admitting these properties up to similarity and transposition.

In our second problem, we address the question of computing the distance from a nonzero projection to the set of nilpotent operators acting on $\mathbb{C}^n$. Building on MacDonald's results in the rank-one case, we prove that the distance from a rank $n-1$ projection to the set of nilpotents in $\mathbb{M}_n(\mathbb{C})$ is $\frac{1}{2}\sec\left(\frac{\pi}{\frac{n}{n-1}+2}\right)$. For each $n \geq 2$, we construct examples of pairs (Q, T) where Q is a projection of rank $n-1$ and $T \in \mathbb{M}_n(\mathbb{C})$ is a nilpotent of minimal distance to Q. Moreover, it is shown that any two such pairs are unitarily equivalent. We end by discussing possible extensions of these results in the case of projections of intermediate ranks.

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Dedication

I dedicate this thesis to my Papa, Jack.

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Chapter 1

Introduction

The fields of operator algebras and operator theory concern the study of bounded linear transformations (operators) acting on a Hilbert space. In this thesis we will present two problems from these respective fields. Our focus will be the case in which the Hilbert space in question is complex and finite-dimensional. That is, we will be concerned with operators acting on $\mathbb{C}^n$ for some $n \in \mathbb{N}$.

To introduce the first problem, suppose that $\mathcal{A}$ is an algebra of linear operators acting on $\mathbb{C}^n$. By fixing an orthonormal basis for $\mathbb{C}^n$, we may identify $\mathcal{A}$ with a subalgebra of $\mathbb{M}_n(\mathbb{C})$, the algebra of all complex $n \times n$ matrices. Given an idempotent $E \in \mathbb{M}_n(\mathbb{C})$ (i.e., an operator satisfying the equation $E^2 = E$), each $A \in \mathcal{A}$ can be expressed as a sum

$$A = EAE + EA(I - E) + (I - E)AE + (I - E)A(I - E).$$

Accordingly, we may identify each A with a block 2×2 matrix

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

written with respect to the algebraic direct sum decomposition $\mathbb{C}^n = \operatorname{ran}(E) \dotplus \ker(E)$. Here, the notation $\dotplus$ indicates that every vector in $\mathbb{C}^n$ may be expressed uniquely as a sum of a vector from $\operatorname{ran}(E)$ and a vector in $\ker(E)$, though these summands need not be orthogonal. Under this identification, the set $E\mathcal{A}E \coloneqq \{EAE : A \in \mathcal{A}\}$ corresponds to the collection of all $(1,1)$-blocks from elements of $\mathcal{A}$.

It is clear that for any idempotent $E \in \mathbb{M}_n(\mathbb{C})$, the set $E\mathcal{A}E$ is a linear space. Moreover, dimension considerations imply that when $n \leq 2$, $E\mathcal{A}E$ is in fact, an algebra. When $n \geq 3$, however, this is often not the case, as $E\mathcal{A}E$ frequently fails to be multiplicatively closed. We are therefore interested in determining which subalgebras $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$, $n \geq 3$, have the property that with respect to every direct sum decomposition $\mathbb{C}^n = \operatorname{ran}(E) \dotplus \ker(E)$, the compression of $\mathcal{A}$ to the $(1,1)$-corner is an algebra of linear maps acting on $\operatorname{ran}(E)$. This condition will be known as the *idempotent compression property*. An algebra that admits this property will be called *idempotent compressible*. We will focus mainly on understanding this property for unital subalgebras of $\mathbb{M}_n(\mathbb{C})$.

An interesting variant on the above problem arises when considering only the *orthogonal* direct sum decompositions of $\mathbb{C}^n$. We remind the reader that an idempotent operator whose range is orthogonal to its kernel is called a *projection*, and that the projections in $\mathbb{M}_n(\mathbb{C})$ are exactly the idempotents that are self-adjoint. If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n(\mathbb{C})$ such that $P\mathcal{A}P$ is an algebra for every projection $P \in \mathbb{M}_n(\mathbb{C})$, we shall say that $\mathcal{A}$ exhibits the *projection compression property* or that $\mathcal{A}$ is *projection compressible*. As in the case of idempotents, we will focus primarily on studying this property for unital algebras.

While it is immediate from the definitions that every idempotent compressible algebra is also projection compressible, the converse is much less clear. As we will see, all of our preliminary examples indicate either the presence of the idempotent compression property or the absence of the projection compression property, thus providing evidence to the affirmative. Despite this evidence, however, our attempts at obtaining an intrinsic proof that these notions coincide have been unsuccessful. Instead, we use a systematic case-by-case analysis to investigate whether or not such an equivalence exists.

We begin in Chapter 2 by introducing the notation and basic theory surrounding idempotent and projection compressibility. Here we also develop a list of examples of idempotent compressible subalgebras of $\mathbb{M}_n(\mathbb{C})$ for every integer $n \geq 3$. In many cases, these algebras belong to a general family of idempotent compressible algebras with members in $\mathbb{M}_n(\mathbb{C})$ for each $n \geq 3$. We do, however, encounter three exceptional examples in $\mathbb{M}_3(\mathbb{C})$ that do not appear to admit analogues in higher dimensions. This chapter ends with an overview of key results from [14] and [13] concerning the structure theory for matrix algebras. These facts will be used extensively in Chapters 3 and 4.

Next, we turn our attention to assessing the completeness of our list of idempotent compressible algebras from Chapter 2. Since certain pathological examples were observed in $\mathbb{M}_3(\mathbb{C})$, we devote Chapter 3 to the classification of unital idempotent compressible algebras that exist in this setting. Using a case-by-case analysis based on block triangular forms, we show that the examples from Chapter 2 account for all unital idempotent compressible algebras in $\mathbb{M}_3(\mathbb{C})$ up to transposition and similarity. A closer examination of the unital algebras that lack the idempotent compression property reveals that in fact, no such algebra is projection compressible. From this it follows that the notions of projection compressibility and idempotent compressibility coincide for unital subalgebras of $\mathbb{M}_3(\mathbb{C})$.

In Chapter 4, we implement a similar analysis to classify the projection compressible subalgebras of $\mathbb{M}_n(\mathbb{C})$ when $n \geq 4$. There we present a key tool—Theorem 4.1.2—which greatly restricts the possible block triangular forms of such an algebra. Through a three-part examination of the remaining block triangular forms, we obtain a description of the unital projection compressible algebras that exist in this setting up to transposition and unitary equivalence. Furthermore, we observe that every such algebra is similar to one of the unital idempotent compressible algebras presented in Chapter 2. We therefore prove that for any n, a unital subalgebra of $\mathbb{M}_n(\mathbb{C})$ is projection compressible if and only if it is idempotent compressible.

To introduce our second problem, let $\mathcal{H}$ be a complex Hilbert space of (possibly infinite) dimension n. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all linear operators $T : \mathcal{H} \to \mathcal{H}$ that are bounded with respect to the operator norm

$$\|T\| \coloneqq \sup_{\|x\|=1} \|Tx\|.$$

Consider the sets

$$\begin{aligned} \mathcal{P}(\mathcal{H}) &= \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\} \setminus \{0\}, \text{ and} \\ \mathcal{N}(\mathcal{H}) &= \{N \in \mathcal{B}(\mathcal{H}) : N^j = 0 \text{ for some } j \in \mathbb{N}\} \end{aligned}$$

consisting of all non-zero projections acting on $\mathcal{H}$ and all *nilpotent* operators acting on $\mathcal{H}$, respectively. We are interested in the problem of understanding the distance between these two sets, measured in operator norm on $\mathcal{B}(\mathcal{H})$. This quantity will be denoted by δ_n:

$$\delta_n \coloneqq \operatorname{dist}(\mathcal{P}(\mathcal{H}), \mathcal{N}(\mathcal{H})) = \inf \{\|P - N\| : P \in \mathcal{P}(\mathcal{H}), N \in \mathcal{N}(\mathcal{H})\}.$$

The problem of computing δ_n is by no means new to the world of operator theory. In 1972, Hedlund [9] proved that $\delta_2 = 1/\sqrt{2}$, and that $1/4 \leq \delta_n \leq 1$ for all $n \geq 3$. This lower bound was increased to $1/2$ by Herrero [10] shortly thereafter. At this time Herrero also showed that $\delta_n = 1/2$ whenever n is infinite, thus reducing the problem to the case in which $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$, $n \geq 3$.

Various estimates on the values of δ_n were obtained in the early 1980's. One such estimate established by Salinas [19] states that

$$\frac{1}{2} \leq \delta_n \leq \frac{1}{2} + \frac{1 + \sqrt{n-1}}{2n} \quad \text{for all } n \in \mathbb{N}.$$

One may note that this upper bound approaches $1/2$ as n tends to infinity, and hence Salinas' inequality leads to an alternative proof that $\delta_{\aleph_0} = 1/2$. Herrero [11] subsequently improved upon this upper bound for large values of n by showing that

$$\frac{1}{2} \leq \delta_n \leq \frac{1}{2} + \sin\left(\frac{\pi}{\lfloor \frac{n+1}{2} \rfloor}\right) \quad \text{for } \ n \geq 2,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

For many years the bounds obtained by Salinas and Herrero remained the best known. In 1995, however, MacDonald [16] established a new upper bound that would improve upon these estimates for all values of n. In order to describe MacDonald's approach, we first make the following remarks.

(i) Any two projections in $\mathbb{M}_n(\mathbb{C})$ of equal rank are unitarily equivalent and thus of equal distance to $\mathcal{N}(\mathbb{C}^n)$. Thus, $\delta_n = \min\limits_{1 \leq r \leq n} \nu_{r,n}$, where

$$\nu_{r,n} \coloneqq \inf\left\{\|P - N\| : P \in \mathcal{P}(\mathbb{C}^n),\, rank(P) = r, N \in \mathcal{N}(\mathbb{C}^n)\right\}.$$

(ii) Straightforward estimates show that when computing $\nu_{r,n}$, one need only consider nilpotents of norm at most 2. From here, one may use the compactness of the set of projections of rank r in $\mathbb{M}_n(\mathbb{C})$ and the set of nilpotents in $\mathbb{M}_n(\mathbb{C})$ of norm at most 2 to show that $\nu_{r,n}$ is achieved by some projection-nilpotent pair. Consequently, so too is δ_n.

(iii) If $\{e_i\}_{i=1}^n$ denotes the standard basis for $\mathbb{C}^n$, then

$$\nu_{r,n} = \min\left\{\|P - N\| : P \in \mathcal{P}(\mathbb{C}^n),\, rank(P) = r, N \in \mathcal{T}_n\right\},$$

where $\mathcal{T}_n$ is the subalgebra of $\mathbb{M}_n(\mathbb{C})$ consisting of all operators that are strictly upper triangular with respect to $\{e_i\}_{i=1}^n$. This follows from Schur Triangularization.

The reduction from $\mathcal{N}(\mathbb{C}^n)$ to $\mathcal{T}_n$ described in (iii) may seem innocuous at first glance. This alternate formulation, however, allows one to make use of a theorem of Arveson [1] that describes the distance from an operator in $\mathcal{B}(\mathcal{H})$ to a nest algebra. The version of this result that we require was established by Power [18], and is presented below for the algebra $\mathcal{T}_n$. Note that for vectors $x, y \in \mathbb{C}^n$, the notation $x \otimes y^*$ will be used to denote the rank-one operator $z \mapsto \langle z, y \rangle x$ acting on $\mathbb{C}^n$.

Theorem 1.0.1 (Arveson Distance Formula)**.** *Let $\{e_i\}_{i=1}^n$ denote the standard basis for $\mathbb{C}^n$. Define $E_0 \coloneqq 0$ and $E_k \coloneqq \sum_{i=1}^k e_i \otimes e_i^*$ for each $k \in \{1, 2, \ldots, n\}$. For any $A \in \mathbb{M}_n(\mathbb{C})$,*

$$\mathrm{dist}(A, \mathcal{T}_n) = \max_{1 \leq i \leq n} \|E_{i-1}^{\perp} A E_i\|.$$

Using Arveson's formula, MacDonald successfully determined the exact value of $\nu_{1,n}$, the distance from a rank-one projection in $\mathbb{M}_n(\mathbb{C})$ to $\mathcal{N}(\mathbb{C}^n)$.

Theorem 1.0.2. [16, Theorem 1] *For every positive integer n, the distance from a rank-one projection in $\mathbb{M}_n(\mathbb{C})$ to $\mathcal{N}(\mathbb{C}^n)$ is*

$$\nu_{1,n} = \frac{1}{2} \sec\left(\frac{\pi}{n+2}\right).$$

The expression for $\nu_{1,n}$ described above provides an upper bound on δ_n that is sharper than those previously obtained by Herrero and Salinas for all $n \in \mathbb{N}$. In addition, MacDonald proved that this bound is in fact optimal when $n = 3$ [16, Corollary 4]. These results led to the formulation of the following conjecture.

Conjecture 1.0.3 (MacDonald, [16])**.** *The closest non-zero projections to $\mathcal{N}(\mathbb{C}^n)$ are of rank 1. That is,*

$$\delta_n = \nu_{1,n} = \frac{1}{2} \sec\left(\frac{\pi}{n+2}\right) \quad \text{for all } n \in \mathbb{N}.$$

Conjecture 1.0.3 has since been verified for $n = 4$ [17, Theorem 3.4], but remains open for all $n \geq 5$.

MacDonald's success in computing $\nu_{1,n}$ was largely due to the rigid structure of rank-one projections in $\mathbb{M}_n(\mathbb{C})$. Specifically, the decomposition of such a projection as a simple tensor $P = e \otimes e^*$ for some unit vector $e \in \mathbb{C}^n$ made it feasible to obtain a closed-form expression for $\|E_{i-1}^{\perp} P E_i\|$ in terms of the entries of P. With this in hand, it became possible to show that the rank-one projections of minimal distance to $\mathcal{T}_n$ are such that $\|E_{i-1}^{\perp} P E_i\| = \nu_{1,n}$ for all $i \in \{1, 2, \ldots, n\}$. An exact expression for $\nu_{1,n}$ was then derived through algebraic and combinatorial arguments.

Extending the above approach to accommodate projections of intermediate ranks appears to be a formidable task; when P is not expressible as a simple tensor $e \otimes e^*$ it becomes significantly more challenging to obtain an explicit formula for $\|E_{i-1}^{\perp} P E_i\|$. One may note, however, that the rigidity that led to success in the rank-one case can also be observed in projections of rank $n-1$. It is therefore our goal to extend MacDonald's approach to determine the exact value of $\nu_{n-1,n}$. This will be the focus of Chapter 5.

We accomplish this goal in three stages. Motivated by the analogous result for projections of rank 1, we show in §5.1 that any projection Q of rank $n-1$ that is of minimal distance to $\mathcal{T}_n$ must be such that $\|E_{i-1}^{\perp} Q E_i\| = \nu_{n-1,n}$ for all i. In §5.2, we then apply these equations to determine a list of candidates for $\nu_{n-1,n}$ via arguments adapted from [16]. Finally, we prove that exactly one such candidate satisfies a certain necessary norm inequality from [17], and therefore deduce that this value must be $\nu_{n-1,n}$.

In §5.3, we describe a construction of the pairs (Q, T) where $Q \in \mathbb{M}_n(\mathbb{C})$ is a projection of rank $n-1$, T is an element of $\mathcal{T}_n$, and $\|Q - T\| = \nu_{n-1,n}$. We prove that for each $n \in \mathbb{N}$, any two such pairs are in fact, unitarily equivalent. Lastly, in §5.4 we propose a possible formula for $\nu_{r,n}$ in the case of projections of arbitrary rank, which can be seen to closely resemble numerical estimates for $\nu_{r,n}$ when n is small. We explain how this formula and its consequences may be used to answer MacDonald's conjecture in the affirmative.

Chapter 2

Compressibility Preliminaries

§2.1 Definitions and First Results

In this section we introduce some of the preliminary results concerning algebras that admit the idempotent or projection compression properties. Our first task will be to establish the notation and terminology that will be used throughout.

Since we will only be concerned with algebras of $n \times n$ matrices over $\mathbb{C}$, we will write $\mathbb{M}_n$ in place of $\mathbb{M}_n(\mathbb{C})$ from here on.

Definition 2.1.1. Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$.

(i) We say that $\mathcal{A}$ is *idempotent compressible*, or that $\mathcal{A}$ admits the *idempotent compression property*, if $E\mathcal{A}E$ is an algebra for all idempotents $E \in \mathbb{M}_n$.

(ii) We say that $\mathcal{A}$ is *projection compressible*, or that $\mathcal{A}$ admits the *projection compression property*, if $P\mathcal{A}P$ is an algebra for all projections $P \in \mathbb{M}_n$.

It is immediate from the definitions that every idempotent compressible algebra is also projection compressible.

We begin by addressing the claim from Chapter 1 that every subalgebra of $\mathbb{M}_2$ is idempotent compressible. Before stating this result formally, let us first recall the following facts concerning matrices of rank one.

Proposition 2.1.2. *If $E \in \mathbb{M}_n$ is an operator of rank 1, then the linear space $\mathbb{C}E$ is an algebra, and $E\mathbb{M}_nE$ is contained in $\mathbb{C}E$.*

Proof. Let E be as above. As a rank-one operator, E is either nilpotent or a scalar multiple of an idempotent. Hence, $\mathbb{C}E$ is closed under multiplication. Writing $E = x \otimes y^*$ for some vectors $x, y \in \mathbb{C}^n$, we have that for any $A \in \mathbb{M}_n$,

$$EAE = (x \otimes y^*)A(x \otimes y^*) = \langle Ax, y\rangle(x \otimes y^*) = \langle Ax, y\rangle E \in \mathbb{C}E.$$

Thus, $E\mathbb{M}_n E \subseteq \mathbb{C}E$. ■

Proposition 2.1.3. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$, and let $E \in \mathbb{M}_n$ be idempotent.*

(i) $E\mathcal{A}E$ is an algebra if and only if $E\mathcal{A}E$ is multiplicatively closed.

(ii) If $\mathrm{rank}(E) = 1$, then $E\mathcal{A}E$ is an algebra.

(iii) If $n \leq 2$, then $\mathcal{A}$ is idempotent compressible.

Proof. Since $E\mathcal{A}E$ is a linear space, statement (i) is immediate. For (ii), note that if $A, B \in \mathcal{A}$, then by Proposition 2.1.2, $EBE = \lambda E$ for some $\lambda \in \mathbb{C}$. Thus,

$$(EAE)(EBE) = E(\lambda A)E \in E\mathcal{A}E.$$

It follows that $E\mathcal{A}E$ is closed under multiplication. By (i), $E\mathcal{A}E$ is an algebra. Statement (iii) is now an immediate consequence of (ii). ■

In light of Proposition 2.1.3, we will devote our attention to studying the compression properties for subalgebras of $\mathbb{M}_n$, $n \geq 3$.

We now investigate some permanence results for algebras admitting the idempotent or projection compression properties. These facts will be used extensively throughout Chapters 2-4 without mention. The first result in this vein states that if $\mathcal{A}$ admits one of the compression properties, then so too does $P\mathcal{A}P$ for every projection P.

Proposition 2.1.4. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$ that admits the idempotent (resp. projection) compression property, and let P be a projection in $\mathbb{M}_n$. When restricted to an algebra of linear maps acting on* $\mathrm{ran}(P)$*, the algebra $P\mathcal{A}P$ is idempotent (resp. projection) compressible.*

Proof. Assume that $\mathcal{A}$ is idempotent compressible. Given an idempotent E acting on $\mathrm{ran}(P)$, we have that $PE = EP = E$. Thus, $E(P\mathcal{A}P)E = E\mathcal{A}E$ is an algebra, as $\mathcal{A}$ is idempotent compressible.

An analogous argument may be used in the case that $\mathcal{A}$ is projection compressible. ■

Note that the set of idempotents in $\mathbb{M}_n$ is closed under transposition and similarity, whereas the set of projections in $\mathbb{M}_n$ is closed under transposition and unitary equivalence. This leads to our second permanence property for compressible algebras, Proposition 2.1.6. In order to simplify the statement of this result, as well as much of the exposition in the chapters to come, we first introduce the following definitions.

Definition 2.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{M}_n$. Define the *transpose* of $\mathcal{A}$ to be the set

$$\mathcal{A}^T := \left\{ A^T : A \in \mathcal{A} \right\}.$$

If $\mathcal{A}$ or $\mathcal{A}^T$ is similar to $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are *transpose similar*. If $\mathcal{A}$ or $\mathcal{A}^T$ is unitarily equivalent to $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are *transpose equivalent*.

It is easy to verify that transpose similarity and transpose equivalence are equivalence relations that generalize the notions of similarity and unitary equivalence, respectively.

The proof of the following result follows immediately from the comments preceding Definition 2.1.5.

Proposition 2.1.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathbb{M}_n$.*

(i) If $\mathcal{A}$ and $\mathcal{B}$ are transpose similar, then $\mathcal{A}$ is idempotent compressible if and only if $\mathcal{B}$ is idempotent compressible.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are transpose equivalent, then $\mathcal{A}$ is projection compressible if and only if $\mathcal{B}$ is projection compressible.

Definition 2.1.7. Given $A \in \mathbb{M}_n$, define the *anti-transpose* of A to be the matrix

$$A^{aT} := JA^TJ,$$

where $J = J^*$ is the unitary matrix whose (i, j)-entry is $\delta_{j,n-i+1}$. If $\mathcal{A}$ is a subset of $\mathbb{M}_n$, then we will define the *anti-transpose* of $\mathcal{A}$ to be the set

$$\mathcal{A}^{aT} := J\mathcal{A}^TJ = \left\{ A^{aT} : A \in \mathcal{A} \right\}.$$

While transposition has the effect of reflecting a matrix about its main diagonal, anti-transposition has the effect of reflecting a matrix about its *anti-diagonal* (i.e., the diagonal from the $(n, 1)$-entry to the $(1, n)$-entry).

Since an algebra $\mathcal{A}$ and its anti-transpose $\mathcal{A}^{aT}$ are easily seen to be transpose equivalent, we obtain the following useful consequence of Proposition 2.1.6.

Corollary 2.1.8. *If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n$, then $\mathcal{A}$ is idempotent (resp. projection) compressible if and only if $\mathcal{A}^{aT}$ is idempotent (resp. projection) compressible.*

Next we will show that if an algebra $\mathcal{A}$ admits the idempotent (resp. projection) compression property, then so too does its unitization $\mathcal{A}+\mathbb{C}I$. A counterexample following the proof of Corollary 2.1.14 demonstrates that the converse is false.

Proposition 2.1.9. *If $\mathcal{A}$ is an idempotent (resp. a projection) compressible subalgebra of $\mathbb{M}_n$, then its unitization*

$$\tilde{\mathcal{A}} \coloneqq \mathcal{A} + \mathbb{C}I$$

is idempotent (resp. projection) compressible.

Proof. Assume that $\mathcal{A}$ is idempotent (resp. projection) compressible, and let E be a idempotent (resp. projection) in $\mathbb{M}_n$. Let $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, so that $A + \alpha I$ and $B + \beta I$ define elements of $\tilde{\mathcal{A}}$. Since $EAE \cdot EBE$ belongs to $E\mathcal{A}E$, we can write $EAE \cdot EBE = ECE$ for some $C \in \mathcal{A}$. As a result,

$$\begin{aligned} E(A+\alpha I)E \cdot E(B+\beta I)E &= EAE \cdot EBE + \beta EAE + \alpha EBE + \alpha\beta E \\ &= E((C + \beta A + \alpha B) + \alpha\beta I)E \end{aligned}$$

Since $(C + \beta A + \alpha B) + \alpha\beta I$ belongs to $\widetilde{A}$, we conclude that $E\widetilde{\mathcal{A}}E$ is an algebra. ■

The following proposition describes an obvious sufficient condition for an algebra to exhibit the projection or idempotent compression property, and will be useful in building our first class of examples.

Proposition 2.1.10. *Let n be a positive integer, and let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$. If $AEB \in \mathcal{A}$ for all A, $B \in \mathcal{A}$, and all idempotents (resp. projections) $E \in \mathbb{M}_n$, then $\mathcal{A}$ is idempotent (resp. projection) compressible.*

Proof. Let E be an idempotent (resp. a projection) in $\mathbb{M}_n$. Given $A, B \in \mathcal{A}$, we have that $AEB \in \mathcal{A}$, and hence

$$(EAE)(EBE) = E(AEB)E \in E\mathcal{A}E.$$

Thus, $E\mathcal{A}E$ is an algebra by Proposition 2.1.3 (i). ■

The sufficient condition for idempotent compressibility from Proposition 2.1.10 strongly resembles the multiplicative absorption property satisfied by ideals. In particular, this result implies that any (one- or two-sided) ideal of $\mathbb{M}_n$ exhibits the idempotent compression property. It will be shown in Corollary 2.1.14 that this property also holds for the intersection of one-sided ideals, or equivalently, the intersection of a single left ideal with a single right ideal. Thus, we make following definition.

Definition 2.1.11. If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n$ given by an intersection of a left ideal and a right ideal in $\mathbb{M}_n$, then $\mathcal{A}$ is said to be an $\mathcal{LR}$-*algebra*.

It is straightforward to show that any algebra that is transpose similar to an $\mathcal{LR}$-algebra $\mathcal{A}$ is again an $\mathcal{LR}$-algebra. Indeed, if $\mathcal{A} = \mathcal{L} \cap \mathcal{R}$ for some left ideal $\mathcal{L}$ and right ideal $\mathcal{R}$ of $\mathbb{M}_n$, then $\mathcal{R}^T$ is a left ideal, $\mathcal{L}^T$ is a right ideal, and $\mathcal{A}^T = \mathcal{R}^T \cap \mathcal{L}^T$. Hence, $\mathcal{A}^T$ is also an $\mathcal{LR}$-algebra. If $\mathcal{B}$ is transpose similar to $\mathcal{A}$, then by replacing $\mathcal{A}$ with $\mathcal{A}^T$ if necessary, we may assume that

$$\mathcal{B} = S^{-1}\mathcal{A}S = \left(S^{-1}\mathcal{L}S\right) \cap \left(S^{-1}\mathcal{R}S\right)$$

for some invertible $S \in \mathbb{M}_n$. Since $S^{-1}\mathcal{L}S$ and $S^{-1}\mathcal{R}S$ are left and right ideals of $\mathbb{M}_n$, respectively, $\mathcal{B}$ is again an $\mathcal{LR}$-algebra.

It is well-known that the one-sided ideals in $\mathbb{M}_n$ can be described in terms of projections. In particular, each left ideal of $\mathbb{M}_n$ has the form $\mathbb{M}_n Q$ for some orthogonal projection Q, while each right ideal has the form $P\mathbb{M}_n$ for some orthogonal projection P. More generally, we have the following classical ring-theoretic result concerning $\mathbb{M}_n$-submodules of the $n \times p$ and $p \times n$ matrices. A proof of this result is presented in the complex case for completeness, though a more general argument applicable to matrix algebras over division rings may be found in [13, Theorem 3.3].

Theorem 2.1.12. *Let n and p be positive integers.*

(i) If $\mathcal{S} \subseteq \mathbb{M}_{n\times p}$ is a left $\mathbb{M}_n$-module, then there is a projection $Q \in \mathbb{M}_p$ such that $\mathcal{S} = \mathbb{M}_{n\times p}Q$.

(ii) If $\mathcal{S} \subseteq \mathbb{M}_{p\times n}$ is a right $\mathbb{M}_n$-module, then there is a projection $P \in \mathbb{M}_p$ such that $\mathcal{S} = P\mathbb{M}_{p\times n}$.

Proof. Observe that (ii) follows from (i), as $\mathcal{S}$ is a left $\mathbb{M}_n$-module if and only if $\mathcal{S}^T$ is a right $\mathbb{M}_n$-module. Thus, it suffices to prove (i).

Let $\mathcal{S}$ be a left $\mathbb{M}_n$-module, and consider the subspace

$$\mathcal{V} = \left(\bigcap_{S \in \mathcal{S}} \ker(S) \right)^{\perp}$$

of $\mathbb{C}_p$. Let Q denote the orthogonal projection of $\mathbb{C}^p$ onto $\mathcal{V}$, and suppose that $\dim \mathcal{V} = m$. It will be shown that $\mathcal{S} = \mathbb{M}_{n \times p} Q$.

To see this, let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$. With respect to the decomposition $\mathbb{C}^p = \mathcal{V} \oplus \mathcal{V}^{\perp}$, every $S \in \mathcal{S}$ can be expressed as a matrix of the form

$$S = \left[\begin{array}{cccc|ccc} s_{11} & s_{12} & \cdots & s_{1m} & 0 & \cdots & 0 \\ s_{21} & s_{22} & \cdots & s_{2m} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nm} & 0 & \cdots & 0 \end{array} \right]$$

for some s_{ij} in $\mathbb{C}$. For each i and j in $\{1, 2, \ldots, n\}$, let E_{ij} denote the $n \times n$ matrix unit $e_i \otimes e_j^*$. Since $\mathcal{S}$ is a left $\mathbb{M}_n$-module, the product

$$E_{ii} S = \left[\begin{array}{cccc|ccc} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ s_{i1} & s_{i2} & \cdots & s_{im} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right]$$

belongs to $\mathcal{S}$ for all i. Moreover, one may multiply by E_{ji} on the left to move the i^{th} row of this matrix to the j^{th} row. In particular, one may move any row to the first row and vice-versa.

Consider the subspace

$$\mathcal{W} = \left\{ (w_1, w_2, \ldots, w_m)^T : (s_{11}, s_{12}, \ldots, s_{1m}) = (w_1, w_2, \ldots, w_m) \text{ for some } S = (s_{ij}) \text{ in } \mathcal{S} \right\}$$

of $\mathcal{V}$. From the above remarks, it follows that each row of each $S \in \mathcal{S}$ are of the form

$$(w_1, w_2, \ldots, w_m, 0, 0, \ldots, 0)$$

for some $(w_1, w_2, \ldots, w_m)^T$ in $\mathcal{W}$. This means that if $\mathcal{W} \neq \mathcal{V}$, then $\mathcal{V}$ contains a non-zero vector x that is orthogonal to $\mathcal{W}$, and hence $Sx = 0$ for all $S \in \mathcal{S}$. This would then contradict our choice of $\mathcal{V}$, so it must be the case that $\mathcal{W} = \mathcal{V}$. As a result, $\mathcal{S} = \mathbb{M}_{n \times p} Q$. $\blacksquare$

Corollary 2.1.13. *A subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is an $\mathcal{LR}$-algebra if and only if there are projections P and Q in $\mathbb{M}_n$ such that $\mathcal{A} = P\mathbb{M}_n Q$.*

The description of $\mathcal{LR}$-algebras presented in Corollary 2.1.13 allows one to quickly deduce that these algebras admit the idempotent compression property.

Corollary 2.1.14. *Every $\mathcal{LR}$-algebra is idempotent compressible.*

Proof. Let $\mathcal{A}$ be an $\mathcal{LR}$-algebra, so $\mathcal{A} = P\mathbb{M}_n Q$ for some projections P and Q in $\mathbb{M}_n$. If E is an idempotent in $\mathbb{M}_n$, then for any $A, B \in \mathcal{A}$,

$$AEB = (PAQ)E(PBQ) = P(AQEPB)Q \in P\mathbb{M}_n Q = \mathcal{A}.$$

Thus, $\mathcal{A}$ satisfies the assumptions of Proposition 2.1.10 in the case of idempotents. We conclude that $\mathcal{A}$ is idempotent compressible. ■

As the following Proposition demonstrates, the algebra generated by a rank-one operator is an $\mathcal{LR}$-algebra. This result will be referenced at the end of Chapter 4.

Proposition 2.1.15. *If $R \in \mathbb{M}_n$ is an operator of rank 1, then $Alg(R)$—the algebra generated by R—is an $\mathcal{LR}$-algebra. Consequently, $Alg(R)$ is idempotent compressible.*

Proof. In light of the remarks following Definition 2.1.11, it suffices to prove that $Alg(R)$ is similar to an $\mathcal{LR}$-algebra.

Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$. As a rank-one operator, R is either nilpotent or a scalar multiple of an idempotent. If R is nilpotent, then R is unitarily equivalent to $e_1 \otimes e_2^*$. Consequently, $Alg(R)$ is unitarily equivalent to $\mathbb{C}e_1 \otimes e_2^*$. If instead R is a multiple of an idempotent, then R is similar to $\alpha e_1 \otimes e_1^*$ for some non-zero $\alpha \in \mathbb{C}$. Consequently, $Alg(R)$ is similar to $\mathbb{C}e_1 \otimes e_1^*$. In either case, $Alg(R)$ is an $\mathcal{LR}$-algebra. ■

The fact that $\mathcal{LR}$-algebras admit the idempotent compression property gives us a means to disprove the converse to Proposition 2.1.9. We will exhibit a subalgebra of $\mathbb{M}_3$ that is not projection compressible, but whose unitization is idempotent compressible.

Indeed, let $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{C}^3$ and for each i, let Q_i denote the orthogonal projection onto the span of $\{e_i\}$. Consider the algebra $\mathcal{A} = \mathbb{C}(Q_1 + Q_2)$. Note that the unitization of $\mathcal{A}$ is also the unitization of the $\mathcal{LR}$-algebra $\mathcal{B} := \mathbb{C}Q_3 = Q_3\mathbb{M}_3 Q_3$. By Corollary 2.1.14 and Proposition 2.1.9, $\tilde{A}$ is idempotent compressible, a fortiori, projection compressible.

To see that $\mathcal{A}$ is not projection compressible, consider the matrix

$$P = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and note that $\frac{1}{3}P$ is a projection in $\mathbb{M}_3$. We claim that $\left(\frac{1}{3}P\right)\mathcal{A}\left(\frac{1}{3}P\right)$ is *not* an algebra. Of course, since $(\frac{1}{3}P)\mathcal{A}(\frac{1}{3}P)$ is an algebra if and only if $P\mathcal{A}P$ is an algebra, it suffices to prove that $P\mathcal{A}P$ is not multiplicatively closed.

One may verify that every element $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation $b_{22}+5b_{23} = 0$. With $B = e_1 \otimes e_1^* + e_2 \otimes e_2^*$, however, we have that

$$(PBP)^2 = \begin{bmatrix} 42 & -39 & -3 \\ -39 & 42 & -3 \\ -3 & -3 & 6 \end{bmatrix}.$$

This matrix clearly does not satisfy the above equation, and hence $(PBP)^2$ does not belong to $P\mathcal{A}P$. Thus, $P\mathcal{A}P$ is not an algebra, so $\mathcal{A}$ is not projection compressible.

Remark 2.1.16. When determining whether or not a corner $E\mathcal{A}E$ is an algebra, it is often more computationally convenient to consider a multiple of the idempotent E rather than E itself. This simplification will frequently be used without mention.

§2.2 Examples of Idempotent Compressible Algebras

While $\mathcal{LR}$-algebras comprise a large collection of idempotent compressible algebras, they are not the only examples. The purpose of §2.2 is to expand our library of matrix algebras that admit the idempotent compression property.

In §2.2.1 we showcase three distinct families of idempotent compressible algebras that arise as subalgebras of $\mathbb{M}_n$ for each $n \geq 3$. In §2.2.2, we present three additional examples of idempotent compressible algebras that occur uniquely in the setting of 3×3 matrices. The algebras presented in these sections lay the groundwork for the classification of compressible algebras in Chapters 3 and 4.

§2.2.1 Subalgebras of $\mathbb{M}_n$, $n \geq 3$

This section is devoted to the exposition of three families of idempotent compressible algebras that exist in $\mathbb{M}_n$ for each $n \geq 3$. These families are described in Examples 2.2.1, 2.2.3, and 2.2.6, respectively.

Example 2.2.1. *Let $n \geq 3$ be an integer. If Q_1, Q_2, and Q_3 are projections in $\mathbb{M}_n$ which sum to I, then the algebra*

$$\begin{aligned}\mathcal{A} &\coloneqq \mathbb{C}Q_1 + (Q_1+Q_2)\mathbb{M}_n(Q_2+Q_3) \\ &= \left\{ \begin{bmatrix} \alpha I & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & 0 \end{bmatrix} : \alpha \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\}\end{aligned}$$

has the idempotent compression property. Consequently, its unitization

$$\begin{aligned}\widetilde{\mathcal{A}} &= \mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1+Q_2)\mathbb{M}_n(Q_2+Q_3) \\ &= \left\{ \begin{bmatrix} \alpha I & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & \beta I \end{bmatrix} : \alpha, \beta \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\}\end{aligned}$$

has the idempotent compression property as well.

Proof. Define $\mathcal{A}_1 \coloneqq \mathbb{C}Q_1$ and $\mathcal{A}_2 \coloneqq (Q_1+Q_2)\mathbb{M}_n(Q_2+Q_3)$, so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2$. Let E be an idempotent in $\mathbb{M}_n$. We will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for each choice of i and j.

Since $\mathcal{A}_2$ is an $\mathcal{LR}$-algebra, it is easy to see that $(E\mathcal{A}_2E)^2$ is contained in $E\mathcal{A}E$. What's more, the equation $Q_1 = (Q_1+Q_2)Q_1$ shows that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ is contained in $E\mathcal{A}_2E$, and hence in $E\mathcal{A}E$. To see that $(E\mathcal{A}_1E)^2$ is contained in $E\mathcal{A}E$, write

$$(EQ_1E)^2 = EQ_1E - E(Q_1+Q_2)Q_1E \cdot E(Q_2+Q_3)E.$$

Finally, if $T \in \mathbb{M}_n$, then the equation

$$\begin{aligned}E(Q_1+Q_2)T(Q_2+Q_3)E \cdot EQ_1E &= E(Q_1+Q_2)T(Q_2+Q_3)E \\ &\quad - E(Q_1+Q_2)T(Q_2+Q_3)E \cdot E(Q_2+Q_3)E,\end{aligned}$$

proves that $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$ is contained in $E\mathcal{A}E$. ■

Remark 2.2.2. Let $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{C}^3$. For each $i \in \{1, 2, 3\}$, let Q_i denote the orthogonal projection of $\mathbb{C}^3$ onto $\mathbb{C}e_i$. By Example 2.2.1, the algebra

$$\mathcal{A} = \mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_n(Q_2 + Q_3) = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma, x, y, z \in \mathbb{C} \right\}$$

of all 3×3 upper triangular matrices is idempotent compressible.

Example 2.2.3. *Let $n \geq 3$ be an integer. If Q_1 and Q_2 are mutually orthogonal rank-one projections in $\mathbb{M}_n$, and $Q_3 = I - Q_1 - Q_2$, then the algebra*

$$\begin{aligned} \mathcal{A} &\coloneqq \mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_1 + Q_2)\mathbb{M}_n Q_3 \\ &= \left\{ \begin{bmatrix} \alpha & 0 & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\} \end{aligned}$$

has the idempotent compression property. Consequently, its unitization

$$\begin{aligned} \widetilde{\mathcal{A}} &= \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_n Q_3 \\ &= \left\{ \begin{bmatrix} \alpha & 0 & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\} \end{aligned}$$

has the idempotent compression property as well.

Proof. Define $\mathcal{A}_1 \coloneqq \mathbb{C}Q_1$, $\mathcal{A}_2 \coloneqq \mathbb{C}Q_2$, and $\mathcal{A}_3 \coloneqq (Q_1+Q_2)\mathbb{M}_n Q_3$, so that $\mathcal{A} = \mathcal{A}_1 \dot{+} \mathcal{A}_2 \dot{+} \mathcal{A}_3$. Let E be an idempotent in $\mathbb{M}_n$. As in the previous proof, we will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_i E \cdot E\mathcal{A}_j E$ for all choices of i and j.

Note that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras, so $E\mathcal{A}E$ contains $(E\mathcal{A}_i E)^2$ for all i. Moreover, it is easy to see that $E\mathcal{A}_1 E \cdot E\mathcal{A}_3 E$ and $E\mathcal{A}_2 E \cdot E\mathcal{A}_3 E$ are contained in $E\mathcal{A}E$. From these inclusions it follows that $E\mathcal{A}_1 E \cdot E\mathcal{A}_2 E$ and $E\mathcal{A}_2 E \cdot E\mathcal{A}_1 E$ are contained in $E\mathcal{A}E$, as

$$\begin{aligned} EQ_1E \cdot EQ_2E &= EQ_1E - EQ_1E \cdot EQ_1E - E(Q_1 + Q_2)Q_1E \cdot EQ_3E, \quad \text{and} \\ EQ_2E \cdot EQ_1E &= EQ_2E - EQ_2E \cdot EQ_2E - E(Q_1 + Q_2)Q_2E \cdot EQ_3E. \end{aligned}$$

The proof will be complete upon showing that $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ and $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$ are contained in $E\mathcal{A}E$. To demonstrate that this is the case, observe that for any $T \in \mathbb{M}_n$,

$$\begin{aligned} E(Q_1 + Q_2)TQ_3E \cdot EQ_1E = EQ_1TQ_3E \cdot EQ_1E - EQ_2TQ_3E \cdot EQ_2E \\ + EQ_2T(I - Q_3E)Q_3E. \end{aligned}$$

By Proposition 2.1.2, the first two summands on the right-hand side of this equation belong to $E\mathcal{A}_1E$ and $E\mathcal{A}_2E$, respectively. Moreover, the summand belongs to $E\mathcal{A}_3E$. Consequently, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ is contained in $E\mathcal{A}E$. The inclusion $E\mathcal{A}_3E \cdot E\mathcal{A}_2E \subseteq E\mathcal{A}E$ can be deduced in a similar fashion. ■

It was fairly routine to verify that the algebras presented in Examples 2.2.1 and 2.2.3 admit the idempotent compression property. Showing that this condition holds for the algebra $\mathcal{A}$ in our next example is not so straightforward. We will first present two lemmas that describe sufficient conditions for an arbitrary corner of this algebra to be an algebra itself. It will then be shown in Example 2.2.6 that every such corner of $\mathcal{A}$ must satisfy one of these conditions. This will prove that this algebra is indeed idempotent compressible.

Lemma 2.2.4. *Let $n \geq 3$ be an integer, let $Q_1, Q_2 \in \mathbb{M}_n$ be mutually orthogonal rank-one projections, and define $Q_3 := I - Q_1 - Q_2$. Consider the subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ given by*

$$\begin{aligned} \mathcal{A} &:= \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3 \\ &= \left\{ \begin{bmatrix} \alpha & x & M_{13} \\ 0 & \alpha & M_{23} \\ 0 & 0 & 0 \end{bmatrix} : \alpha, x \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\}. \end{aligned}$$

If E is an idempotent in $\mathbb{M}_n$ and $E\mathcal{A}E$ contains EQ_2E, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be a fixed idempotent in $\mathbb{M}_n$ and suppose that $EQ_2E \in E\mathcal{A}E$. Define

$$\begin{aligned} \mathcal{A}_0 &:= \mathbb{C}Q_1 + \mathbb{C}Q_2 + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3 \\ &= \mathbb{C}Q_1 + (Q_1 + Q_2)\mathbb{M}_n(Q_2 + Q_3), \end{aligned}$$

and note that $\mathcal{A}_0$ is idempotent compressible by Example 2.2.1. It then follow that

$$\begin{aligned} E\mathcal{A}E &= \mathbb{C}E(Q_1 + Q_2)E + EQ_1\mathbb{M}_nQ_2E + E(Q_1 + Q_2)\mathbb{M}_nQ_3E \\ &= \mathbb{C}EQ_1E + \mathbb{C}EQ_2E + EQ_1\mathbb{M}_nQ_2E + E(Q_1 + Q_2)\mathbb{M}_nQ_3E \\ &= E\mathcal{A}_0E \end{aligned}$$

is an algebra. ■

Lemma 2.2.5. *Let $n \geq 3$ be an integer, let $Q_1, Q_2 \in \mathbb{M}_n$ be mutually orthogonal rank-one projections, and define $Q_3 \coloneqq I - Q_1 - Q_2$. Let $\mathcal{A}$ denote the subalgebra of $\mathbb{M}_n$ given by*

$$\begin{aligned}\mathcal{A} &\coloneqq \mathbb{C}(Q_1+Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1+Q_2)\mathbb{M}_nQ_3 \\ &= \left\{ \begin{bmatrix} \alpha & x & M_{13} \\ 0 & \alpha & M_{23} \\ 0 & 0 & 0 \end{bmatrix} : \alpha, x \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\}.\end{aligned}$$

If E is an idempotent in $\mathbb{M}_n$ such that $EQ_1 = Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent in $\mathbb{M}_n$ such that $EQ_1 = Q_1$. Define $\mathcal{A}_1 \coloneqq \mathbb{C}(Q_1+Q_2)$, $\mathcal{A}_2 \coloneqq Q_1\mathbb{M}_nQ_2$, and $\mathcal{A}_3 \coloneqq (Q_1+Q_2)\mathbb{M}_nQ_3$, so that $\mathcal{A} = \mathcal{A}_1 \dot{+} \mathcal{A}_2 \dot{+} \mathcal{A}_3$. As in the previous examples, we will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for all i and j.

Since $\mathcal{A}_2$ and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras, it is easy to see that $E\mathcal{A}E$ contains $(E\mathcal{A}_2E)^2$ and $(E\mathcal{A}_3E)^2$. Moreover, it is clear that $E\mathcal{A}_2E \cdot E\mathcal{A}_3E$ is contained in $E\mathcal{A}_3E$, and hence in $E\mathcal{A}E$. Observe that since the algebra $\mathcal{A}_0 \coloneqq \mathcal{A}_1 \dot{+} \mathcal{A}_3$ was shown to be idempotent compressible in Example 2.2.1, we have that $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$, and $(E\mathcal{A}_1E)^2$ are contained in $E\mathcal{A}_0E \subseteq E\mathcal{A}E$. Proving these inclusions directly is also straightforward.

The equation $EQ_1 = Q_1$ will now be used to obtain the remaining inclusions. We have that for all S and T in $\mathbb{M}_n$,

$$\begin{aligned} E(Q_1+Q_2)SQ_3E \cdot EQ_1TQ_2E &= 0, \\ E(Q_1+Q_2)E \cdot EQ_1TQ_2E &= EQ_1TQ_2E, \text{ and} \\ EQ_1TQ_2E \cdot E(Q_1+Q_2)E &= EQ_1(TQ_2E)Q_2E. \end{aligned}$$

The right-hand side of each expression above is easily seen to belong to $E\mathcal{A}E$. As a result, $E\mathcal{A}E$ contains $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$, $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$, and $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$, as claimed. ∎

Example 2.2.6. *Let $n \geq 3$ be a positive integer, let Q_1 and Q_2 be mutually orthogonal rank-one projections in $\mathbb{M}_n$, and define $Q_3 \coloneqq I - Q_1 - Q_2$. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_n$ given by*

$$\begin{aligned}\mathcal{A} &\coloneqq \mathbb{C}(Q_1+Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1+Q_2)\mathbb{M}_nQ_3 \\ &= \left\{ \begin{bmatrix} \alpha & x & M_{13} \\ 0 & \alpha & M_{23} \\ 0 & 0 & 0 \end{bmatrix} : \alpha, x \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\},\end{aligned}$$

then $\mathcal{A}$ is idempotent compressible. Consequently, its unitization

$$\begin{aligned}\tilde{\mathcal{A}} &= \mathbb{C}(Q_1+Q_2)+Q_1\mathbb{M}_nQ_2+(Q_1+Q_2)\mathbb{M}_nQ_3+\mathbb{C}Q_3\\ &= \left\{\begin{bmatrix}\alpha & x & M_{13}\\ 0 & \alpha & M_{23}\\ 0 & 0 & \beta I\end{bmatrix} : \alpha,\beta,x\in\mathbb{C}, M_{ij}\in Q_i\mathbb{M}_nQ_j\right\}\end{aligned}$$

is also idempotent compressible.

Proof. In light of Lemmas 2.2.4 and 2.2.5, it suffices to prove that if $r \in \{2,3,\ldots,n-1\}$ and E is an idempotent in $\mathbb{M}_n$ of rank r, then either $EQ_2E \in E\mathcal{A}E$ or $EQ_1 = Q_1$.

Fix such an integer r and idempotent E. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that $e_1 \in \operatorname{ran}(Q_1)$ and $e_2 \in \operatorname{ran}(Q_2)$, and consider the projection

$$P \coloneqq \sum_{i=1}^{r} e_i \otimes e_i^*.$$

Since $rank(P) = r$, there is an invertible matrix $S = (s_{ij})$ in $\mathbb{M}_n$ such that $E = SPS^{-1}$.

The product EQ_2E belongs $E\mathcal{A}E$ if and only if there is an $A \in \mathcal{A}$ such that

$$PS^{-1}(A-Q_2)SP = 0.$$

In showing this equality it suffices to exhibit an $A \in \mathcal{A}$ such that $(A-Q_2)SP = 0$. To this end, observe that for any $A \in \mathcal{A}$, the operator $B \coloneqq A - Q_2$ admits the following matrix representation with respect to the basis $\mathcal{B}$:

$$B = \begin{bmatrix}\alpha & w_2 & w_3 & \cdots & w_n\\ 0 & \alpha-1 & v_3 & \cdots & v_n\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

Since the last $n-2$ rows of B and the last $n-r$ columns of P are zero, the product BSP is zero whenever $(BS)_{ij} = 0$ for all $i \in \{1,2\}$ and $j \in \{1,2,\ldots,r\}$. That is, such a B exists if there is a solution to the following non-homogeneous $2r \times 2(n-1)$ system of linear equations:

$$\begin{array}{cccccccc|c}
w_2 & w_3 & \cdots & w_n & \alpha & v_3 & \cdots & v_n & \\
\end{array}$$
$$\left[\begin{array}{cccccccc|c}
s_{21} & s_{31} & \cdots & s_{n1} & s_{11} & 0 & \cdots & 0 & 0 \\
s_{22} & s_{32} & \cdots & s_{n2} & s_{12} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_{2r} & s_{3r} & \cdots & s_{nr} & s_{1r} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & s_{21} & s_{31} & \cdots & s_{n1} & s_{21} \\
0 & 0 & \cdots & 0 & s_{22} & s_{32} & \cdots & s_{n2} & s_{22} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & s_{2r} & s_{3r} & \cdots & s_{nr} & s_{2r}
\end{array}\right].$$

If the rank of the above (non-augmented) matrix is $2r$, then its columns span $\mathbb{C}^{2r}$ and a solution exists. In this case, EQ_2E belongs to $E\mathcal{A}E$, so $E\mathcal{A}E$ is an algebra by Lemma 2.2.4.

Suppose that this is not the case, so the above (non-augmented) matrix has rank $< 2r$. It is then apparent that

$$S_0 := \begin{bmatrix} s_{21} & s_{31} & \cdots & s_{n1} \\ s_{22} & s_{32} & \cdots & s_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2r} & s_{3r} & \cdots & s_{nr} \end{bmatrix}$$

has rank $< r$. From here we will demonstrate that $EQ_1 = Q_1$, or equivalently, that $PS^{-1}Q_1 = S^{-1}Q_1$.

To see that this is the case, note that if $S^{-1} = (t_{ij})$, then $t_{i1} = 0$ for all $i > r$. Indeed,

$$t_{i1} = \frac{C_{1i}}{\det(S)}$$

where C_{ij} denotes the (i, j)-cofactor of S. When $i > r$, C_{1i} is equal to $(-1)^{i+1}\det(M)$, where M is an $(n-1) \times (n-1)$ matrix of the form

$$M = \begin{bmatrix} s_{21} & s_{22} & \cdots & s_{2r} & * & \cdots & * \\ s_{31} & s_{32} & \cdots & s_{3r} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nr} & * & \cdots & * \end{bmatrix}.$$

Since the $(n-1) \times r$ matrix obtained by keeping only the first r columns of M is exactly S_0^T and $\operatorname{rank}(S_0) < r$, one has

$$\operatorname{rank}(M) < r + (n-1-r) = n-1.$$

Consequently, $t_{i1} = 0$ for all $i > r$. A straightforward computation now shows that $PS^{-1}Q_1 = S^{-1}Q_1$. ∎

§2.2.2 Exceptional Subalgebras of $\mathbb{M}_3$

In §2.2.1 we introduced various examples of unital idempotent compressible subalgebras of $\mathbb{M}_n$ for each integer $n \geq 3$. It will be shown that when $n \geq 4$, these examples are the only unital idempotent compressible subalgebras of $\mathbb{M}_n$ up to similarity and transposition. In fact, we will see that for $n \geq 4$, our examples also represent all unital *projection* compressible subalgebras of $\mathbb{M}_n$ up to similarity and transposition. Proving these results is the focus of Chapter 4.

Unfortunately, the story for unital subalgebras of $\mathbb{M}_3$ is somewhat more complicated. As we will see in this section, there exist several examples of unital idempotent compressible subalgebras of $\mathbb{M}_3$ that are not accounted for in §2.2.1. One explanation as to why these pathological examples arise is due to dimension. Just as $\mathbb{M}_2$ is simply "too small" to contain the projections required to disprove the existence of the compression properties for any of its subalgebras, certain subalgebras of $\mathbb{M}_3$ acquire the compression properties because $\mathbb{M}_3$ does not contain projections of large enough rank. Support for this explanation is given by Theorem 4.1.2, which demonstrates that in the case of $\mathbb{M}_n$, $n \geq 4$, one can very often prove that an algebra lacks the compression properties using projections of rank 3.

Example 2.2.7. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := \mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_2 + Q_3)\mathbb{M}_3 Q_3 = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & x \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma, x \in \mathbb{C} \right\},$$

then $\mathcal{A}$ is idempotent compressible.

Proof. Define $\mathcal{A}_1 := \mathbb{C}Q_1$, $\mathcal{A}_2 := \mathbb{C}Q_2$, and $\mathcal{A}_3 := (Q_2+Q_3)\mathbb{M}_3 Q_3$, so $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. Let E be an idempotent in $\mathbb{M}_3$. We will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for all i and j.

For each $i \in \{1,2,3\}$, $\mathcal{A}_i$ is an $\mathcal{LR}$-algebra; hence $(E\mathcal{A}_iE)^2 \subseteq E\mathcal{A}_iE \subseteq E\mathcal{A}E$. Moreover, since $EQ_2\mathbb{M}_3Q_3E$ is contained in $E\mathcal{A}_3E$, we have that $E\mathcal{A}_2E \cdot E\mathcal{A}_3E \subseteq E\mathcal{A}E$ as well. These inclusions, together with the identities

$$\begin{aligned} EQ_1E \cdot EQ_2E = EQ_1E - EQ_1E \cdot EQ_1E - EQ_3E \\ + EQ_2E \cdot EQ_3E + EQ_3E \cdot EQ_3E \end{aligned}$$

and

$$EQ_2E \cdot EQ_1E = EQ_2E - EQ_2E \cdot EQ_2E - EQ_2E \cdot EQ_3E,$$

demonstrate that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ and $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$ are contained in $E\mathcal{A}E$. Furthermore, if T is an arbitrary element of $\mathbb{M}_3$, then by writing

$$\begin{aligned} EQ_1E \cdot E(Q_2+Q_3)TQ_3E = E(Q_2+Q_3)TQ_3E \\ - E(Q_2+Q_3)E \cdot E(Q_2+Q_3)TQ_3E, \end{aligned}$$

it becomes apparent that $EQ_1E \cdot E(Q_2+Q_3)TQ_3E \in E\mathcal{A}E$. Consequently, $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$ is contained in $E\mathcal{A}E$.

For the final inclusions, it will be helpful to first prove that $EQ_3E \cdot EQ_2E \in E\mathcal{A}E$. Indeed, this is a consequence of the identity

$$\begin{aligned} EQ_3E \cdot EQ_2E = EQ_3E - EQ_3E \cdot EQ_3E - EQ_1E \\ + EQ_1E \cdot EQ_1E + EQ_2E \cdot EQ_1E \end{aligned}$$

and the inclusions established above. One may then apply Proposition 2.1.2 to the rank-one operator Q_3 to deduce that $EQ_3\mathbb{M}_3Q_3E \cdot EQ_2E$ is contained in $E\mathcal{A}E$. Thus, for $T \in \mathbb{M}_3$, we have that

$$E(Q_2+Q_3)TQ_3E \cdot EQ_2E = EQ_2TQ_3E \cdot EQ_2E + EQ_3TQ_3E \cdot EQ_2E$$

and

$$\begin{aligned} E(Q_2+Q_3)TQ_3E \cdot EQ_1E = E(Q_2+Q_3)TQ_3E - E(Q_2+Q_3)TQ_3E \cdot EQ_3E \\ - EQ_2TQ_3E \cdot EQ_2E - EQ_3TQ_3E \cdot EQ_2E, \end{aligned}$$

belong to $E\mathcal{A}E$. We conclude that $E\mathcal{A}E$ contains $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$ and $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$, and therefore $E\mathcal{A}E$ is an algebra. ∎

Proving the existence of the idempotent compression property for our next two examples will be somewhat more challenging. In the same spirit of the proof of Example 2.2.6, Examples 2.2.10 and 2.2.13 will each be preceded by two lemmas that highlight sufficient conditions for a corner of the algebra to be an algebra itself. We will then prove that all corners of these algebras must satisfy one of these two conditions.

Lemma 2.2.8. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1+Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2+Q_3) = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x, y \in \mathbb{C} \right\}.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_2E \in E\mathcal{A}E$, then $E\mathcal{A}E$ is an algebra.

Proof. Suppose that $E \in \mathbb{M}_3$ is an idempotent such that $EQ_2E \in E\mathcal{A}E$, and define

$$\mathcal{A}_0 \coloneqq \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2+Q_3).$$

We have that

$$\begin{aligned} E\mathcal{A}E &= \mathbb{C}E(Q_1+Q_2)E + \mathbb{C}EQ_3E + EQ_1\mathbb{M}_3(Q_2+Q_3)E \\ &= \mathbb{C}EQ_1E + \mathbb{C}EQ_2E + \mathbb{C}EQ_3E + EQ_1\mathbb{M}_3(Q_2+Q_3)E \\ &= E\mathcal{A}_0E. \end{aligned}$$

Since $\mathcal{A}_0^{aT}$ is the unital algebra from Example 2.2.3, $\mathcal{A}_0$ is idempotent compressible. Thus, $E\mathcal{A}_0E = E\mathcal{A}E$ is an algebra. ■

Lemma 2.2.9. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1+Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2+Q_3) = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x, y \in \mathbb{C} \right\}.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1 = Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent such that $EQ_1 = Q_1$. Define $\mathcal{A}_1 \coloneqq \mathbb{C}(Q_1+Q_2)$, $\mathcal{A}_2 \coloneqq \mathbb{C}Q_3$, and $\mathcal{A}_3 \coloneqq Q_1\mathbb{M}_3(Q_2+Q_3)$, so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. To show that $E\mathcal{A}E$ is an algebra, we will verify that the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ is contained in $E\mathcal{A}E$ for all i and j.

Observe that $\mathcal{A}_2$ and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras. Thus, $(E\mathcal{A}_iE)^2 \subseteq E\mathcal{A}_iE \subseteq E\mathcal{A}E$ for each $i \in \{2,3\}$. Moreover, since

$$\begin{aligned} E(Q_1+Q_2)E \cdot EQ_3E &= EQ_3E - EQ_3E \cdot EQ_3E, \\ EQ_3E \cdot E(Q_1+Q_2)E &= EQ_3E - EQ_3E \cdot EQ_3E, \quad \text{and} \\ E(Q_1+Q_2)E \cdot E(Q_1+Q_2)E &= E - 2EQ_3E + EQ_3E \cdot EQ_3E, \end{aligned}$$

it follows that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$, $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$, and $(E\mathcal{A}_1E)^2$ are all contained in $E\mathcal{A}E$.

For the remaining inclusions, note that for any $T \in \mathbb{M}_3$,

$$\begin{aligned} EQ_1T(Q_2+Q_3)E \cdot E(Q_1+Q_2)E = EQ_1T(Q_2+Q_3)E& \\ - EQ_1T(Q_2+Q_3)E \cdot EQ_3(Q_2+Q_3)E& \end{aligned}$$

and

$$EQ_1T(Q_2+Q_3)E \cdot EQ_3E = EQ_1T(Q_2+Q_3)E \cdot EQ_3(Q_2+Q_3)E.$$

Consequently, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ and $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$ are contained in $E\mathcal{A}_3E \subseteq E\mathcal{A}E$. Finally, since $EQ_1 = Q_1$ by hypothesis, we have that

$$\begin{aligned} E(Q_1+Q_2)E \cdot EQ_1T(Q_2+Q_3)E &= EQ_1T(Q_2+Q_3)E \quad \text{and} \\ EQ_3E \cdot EQ_1T(Q_2+Q_3)E &= 0. \end{aligned}$$

This implies that $E\mathcal{A}E$ contains $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$ and $E\mathcal{A}_2E \cdot E\mathcal{A}_3E$. ■

Example 2.2.10. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1+Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2+Q_3) = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x, y \in \mathbb{C} \right\},$$

then $\mathcal{A}$ is idempotent compressible.

Proof. It is obvious that $E\mathcal{A}E$ is an algebra whenever E is an idempotent of rank 1 or 3. In light of Lemmas 2.2.8 and 2.2.9, it suffices to show that for every rank-two idempotent E in $\mathbb{M}_3$, either EQ_2E belongs to $E\mathcal{A}E$ or $EQ_1 = Q_1$.

To this end, suppose that E is a rank-two idempotent in $\mathbb{M}_3$ such that EQ_2E does not belong to $E\mathcal{A}E$, and consider the projection $P \coloneqq (Q_1 + Q_2)$. By rank considerations, there is an invertible matrix $S = (s_{ij})$ with inverse $S^{-1} = (t_{ij})$ such that $E = SPS^{-1}$.

Since EQ_2E is not contained in $E\mathcal{A}E$, there is no $A \in \mathcal{A}$ that satisfies the equation

$$SPS^{-1}(A - Q_2)SPS^{-1} = 0.$$

In particular, there is no $A \in \mathcal{A}$ such that $(A - Q_2)SP = 0$. Since every $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

with respect to the decomposition $\mathbb{C}^3 = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2) \oplus \operatorname{ran}(Q_3)$, it follows that there do not exist constants $\alpha, \beta, x, y \in \mathbb{C}$ that solve the following system of linear equations:

$$\begin{cases} \alpha s_{11} + x s_{21} + y s_{31} & = & 0 \\ \alpha s_{12} + x s_{22} + y s_{32} & = & 0 \\ \alpha s_{21} & = & s_{21} \\ \alpha s_{22} & = & s_{22} \\ \qquad\qquad\qquad \beta s_{31} & = & 0 \\ \qquad\qquad\qquad \beta s_{32} & = & 0 \end{cases}.$$

Note that if the determinant of $S_0 \coloneqq \begin{bmatrix} s_{21} & s_{31} \\ s_{22} & s_{32} \end{bmatrix}$ were non-zero, then a solution to the above system could be obtained by taking $\alpha = 1$, $\beta = 0$, and x and y such that

$$x \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} + y \begin{bmatrix} s_{31} \\ s_{32} \end{bmatrix} = \begin{bmatrix} -s_{11} \\ -s_{12} \end{bmatrix}.$$

It must therefore be the case that $\det S_0 = 0$.

We end the proof by showing that $EQ_1 = Q_1$, or equivalently, that $PS^{-1}Q_1 = S^{-1}Q_1$. It is easy to see that this equation holds when $t_{31} = 0$. But if C_{ij} denotes the (i, j)-cofactor of S, then indeed,

$$t_{31} = \frac{C_{13}}{\det(S)} = \frac{\det(S_0^T)}{\det(S)} = 0.$$

■

Lemma 2.2.11. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := Q_1\mathbb{M}_3(Q_2+Q_3) + Q_2\mathbb{M}_3Q_3 + \mathbb{C}I = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y, z \in \mathbb{C} \right\}.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1E \in E\mathcal{A}E$, then $E\mathcal{A}E$ is an algebra.

Proof. Suppose that E is an idempotent such that $EQ_1E \in E\mathcal{A}E$, and define

$$\mathcal{A}_0 := \mathbb{C}Q_1 + \mathbb{C}(Q_2+Q_3) + Q_1\mathbb{M}_3(Q_2+Q_3) + Q_2\mathbb{M}_3Q_3.$$

We have that

$$\begin{aligned} E\mathcal{A}E &= EQ_1\mathbb{M}_3(Q_2+Q_3)E + EQ_2\mathbb{M}_3Q_3E + \mathbb{C}E \\ &= EQ_1\mathbb{M}_3(Q_2+Q_3)E + EQ_2\mathbb{M}_3Q_3E + \mathbb{C}EQ_1E + \mathbb{C}E(Q_2+Q_3)E \\ &= E\mathcal{A}_0E. \end{aligned}$$

Since $\mathcal{A}_0^{aT}$ is the unital algebra from Example 2.2.6, $\mathcal{A}_0$ is idempotent compressible. Thus, $E\mathcal{A}_0E = E\mathcal{A}E$ is an algebra. ∎

Lemma 2.2.12. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := Q_1\mathbb{M}_3(Q_2+Q_3) + Q_2\mathbb{M}_3Q_3 + \mathbb{C}I = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y, z \in \mathbb{C} \right\}.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1 = Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent such that $EQ_1 = Q_1$. Define $\mathcal{A}_1 := Q_1\mathbb{M}_3(Q_2+Q_3)$, $\mathcal{A}_2 := Q_2\mathbb{M}_3Q_3$ and $\mathcal{A}_3 := \mathbb{C}I$, so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. Yet again, to show that $E\mathcal{A}E$ is an algebra, we will prove that the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ is contained in $E\mathcal{A}E$ for all i and j.

Observe that $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ is clearly contained in $E\mathcal{A}E$ when $i = 3$ or $j = 3$. Moreover, it is easy to see that $(E\mathcal{A}_1E)^2$ and $(E\mathcal{A}_2E)^2$ are contained in $E\mathcal{A}E$, as $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{LR}$-algebras.

Given $T, S \in \mathbb{M}_3$, we have

$$EQ_1S(Q_2+Q_3)E \cdot EQ_2TQ_3E = EQ_1S(Q_2+Q_3)E \cdot EQ_2TQ_3(Q_2+Q_3)E,$$

so $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ is contained in $E\mathcal{A}_1E$, and hence in $E\mathcal{A}E$. Finally, we may use the fact that $EQ_1 = Q_1$ to deduce that $EQ_2SQ_3E \cdot EQ_1T(Q_2+Q_3)E = 0$, and therefore $E\mathcal{A}_2E \cdot E\mathcal{A}_1E = \{0\}$. ∎

Example 2.2.13. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq Q_1\mathbb{M}_3(Q_2+Q_3) + Q_2\mathbb{M}_3Q_3 + \mathbb{C}I = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y, z \in \mathbb{C} \right\},$$

then $\mathcal{A}$ is idempotent compressible.

Proof. It is obvious that $E\mathcal{A}E$ is an algebra whenever E is an idempotent of rank 1 or 3. In light of Lemmas 2.2.11 and 2.2.12, it suffices to show that for every rank-two idempotent E in $\mathbb{M}_3$, either EQ_1E belongs to $E\mathcal{A}E$, or $EQ_1 = Q_1$.

To this end, suppose that E is a rank-two idempotent in $\mathbb{M}_3$ such that EQ_1E does not belong to $E\mathcal{A}E$. Define $P \coloneqq (Q_1+Q_2)$, and let $S = (s_{ij})$ be an invertible matrix with inverse $S^{-1} = (t_{ij})$ satisfying $E = SPS^{-1}$.

Since EQ_1E is not contained in $E\mathcal{A}E$, then there is no $A \in \mathcal{A}$ satisfying the equation

$$SPS^{-1}(A-Q_1)SPS^{-1} = 0.$$

In particular, there is no $A \in \mathcal{A}$ such that $(A-Q_1)SP = 0$. Since every $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix}$$

with respect to the decomposition $\mathbb{C}^3 = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2) \oplus \operatorname{ran}(Q_3)$, it follows that there do not exist constants $\alpha, x, y, z \in \mathbb{C}$ that solve the following system of equations :

$$\begin{cases} \alpha s_{11} + xs_{21} + ys_{31} & & = & s_{11} \\ \alpha s_{12} + xs_{22} + ys_{32} & & = & s_{12} \\ \alpha s_{21} & + zs_{31} & = & 0 \\ \alpha s_{22} & + zs_{32} & = & 0 \\ \alpha s_{31} & & = & 0 \\ \alpha s_{32} & & = & 0 \end{cases}.$$

Observe, however, that if the determinant of $S_0 := \begin{bmatrix} s_{21} & s_{31} \\ s_{22} & s_{32} \end{bmatrix}$ were non-zero, then a solution could be obtained by taking $\alpha = z = 0$, and x and y such that

$$x \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} + y \begin{bmatrix} s_{31} \\ s_{32} \end{bmatrix} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}.$$

It must therefore be the case that $\det S_0 = 0$.

We are now prepared to show that $EQ_1 = Q_1$, or equivalently, that $PS^{-1}Q_1 = S^{-1}Q_1$. This equality is easily verified in the case that $t_{31} = 0$. We have, however, that if C_{ij} denotes the (i, j) cofactor of S, then

$$t_{31} = \frac{C_{13}}{\det(S)} = \frac{\det(S_0^T)}{\det(S)} = 0.$$

■

§2.3 Structure Theory for Matrix Algebras

In §2.2, we introduced several families of unital algebras admitting the idempotent compression property. By Proposition 2.1.6, any algebra obtained by applying a transposition or similarity to one of these algebras also enjoys the idempotent compression property. It becomes interesting to ask whether or not this list is exhaustive. That is, is every unital idempotent compressible subalgebra of $\mathbb{M}_n$ transpose similar to one of the idempotent compressible algebras from §2.2? In order to decide whether or not additional examples exist, it will be necessary to establish a systematic approach to listing the unital subalgebras of $\mathbb{M}_n$. Thus, this section will be devoted to recording a few key results concerning the structure theory for matrix algebras over $\mathbb{C}$. The primary reference for this section is [14].

Perhaps the most important result in this vein is the following theorem of Burnside [4], a simple proof of which can be found in [15]. First, recall that a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is *irreducible* if the only subspaces of $\mathbb{C}^n$ that are invariant for all $A \in \mathcal{A}$ are $\{0\}$ and $\mathbb{C}^n$ itself.

Theorem 2.3.1 (Burnside's Theorem)**.** *If $\mathcal{A}$ is an irreducible algebra of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over an algebraically closed field, then $\mathcal{A}$ is the algebra of all linear transformations on $\mathcal{V}$.*

The notion of irreducibility for a collection linear transformations is strongly related to that of transitivity. Recall that a set $\mathcal{A}$ of linear transformations from a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is *transitive* if for every non-zero $x \in \mathcal{V}$ and arbitrary $y \in \mathcal{W}$, there is an $A \in \mathcal{A}$ such that $Ax = y$. In the case that $\mathcal{A}$ is an algebra of linear transformations mapping $\mathcal{V}$ into itself, the subspace $\{Ax : A \in \mathcal{A}\}$ for any $x \in \mathcal{V}$ is invariant for $\mathcal{A}$. Thus, when $\dim \mathcal{V} \geq 2$, an algebra $\mathcal{A}$ of linear transformations on $\mathcal{V}$ is irreducible if and only if it is transitive. The notion of transitivity for subspaces of $\mathbb{M}_n$ will be important in Chapter 4.

As a consequence of Burnside's Theorem, every proper subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ can be block upper triangularized with respect to some orthonormal basis for $\mathbb{C}^n$. Since the diagonal blocks in this decomposition are themselves algebras, Burnside's Theorem may be applied to these blocks successively to obtain a maximal block upper triangularization of $\mathcal{A}$.

Definition 2.3.2. [14, Definition 9] A subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is said to have a *reduced block upper triangular form* with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 \dot{+} \mathcal{V}_2 \dot{+} \cdots \dot{+} \mathcal{V}_m$ if

(i) when expressed as a matrix, each A in $\mathcal{A}$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\ 0 & A_{22} & A_{23} & \cdots & A_{2m} \\ 0 & 0 & A_{33} & \cdots & A_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{mm} \end{bmatrix}$$

with respect to this decomposition, and

(ii) for each i, the algebra $\mathcal{A}_{ii} := \{A_{ii} : A \in \mathcal{A}\}$ is irreducible. That is, either $\mathcal{A}_{ii} = \{0\}$ and $\dim \mathcal{V}_i = 1$, or $\mathcal{A}_{ii} = \mathbb{M}_{\dim \mathcal{V}_i}$.

If $\mathcal{A}$ is a reduced block upper triangular algebra and $A \in \mathcal{A}$, define the *block-diagonal* of A to be the matrix $BD(A)$ obtained by replacing the block-'off-diagonal' entries of A with zeros. In addition, define the *block-diagonal* of $\mathcal{A}$ to be the algebra

$$BD(\mathcal{A}) = \{BD(A) : A \in \mathcal{A}\}.$$

By definition, the non-zero diagonal blocks of a reduced block upper triangular matrix algebra $\mathcal{A}$ are full matrix algebras. There may, however, exist dependencies among different diagonal blocks. That is, while it may be the case that any matrix of suitable size can be realized as a diagonal block for some element of $\mathcal{A}$, there is no guarantee that matrices for different blocks can be chosen at will simultaneously. The following result states that any dependencies that occur among the diagonal blocks of $\mathcal{A}$ can be described in terms of dimension and similarity.

Theorem 2.3.3. [14, Corollary 14] *If a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 \dotplus \mathcal{V}_2 \dotplus \cdots \dotplus \mathcal{V}_m$, then the set $\{1, 2, \ldots, m\}$ can be partitioned into disjoint sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that*

(i) If $i \in \Gamma_s$ and $\mathcal{A}_{ii} \neq \{0\}$, then there exists $G^{<i>} \in \mathcal{A}$ such that $G^{<i>}_{jj} = I_{\mathcal{V}_j}$ for all $j \in \Gamma_s$, and $G^{<i>}_{jj} = 0$ for all $j \notin \Gamma_s$.

(ii) If i and j belong to the same Γ_s, then $\dim \mathcal{V}_i = \dim \mathcal{V}_j$, and there is an invertible linear map $S_{ij} : \mathcal{V}_i \to \mathcal{V}_j$ such that

$$A_{ii} = S_{ij}^{-1} A_{jj} S_{ij} \text{ for all } A \in \mathcal{A}.$$

(iii) If i and j do not belong to the same Γ_s, then

$$\{(A_{ii}, A_{jj}) : A \in \mathcal{A}\} = \{A_{ii} : A \in \mathcal{A}\} \times \{A_{jj} : A \in \mathcal{A}\}.$$

Definition 2.3.4. Let $\mathcal{A}$ be an algebra of the form described in Theorem 2.3.3. Indices i and j are said to be *linked* if they belong to the same Γ_s, and are said to be *unlinked* otherwise.

It should be noted that if $\mathcal{A}$ is an algebra in reduced block upper triangular form and S is an invertible matrix that is block upper triangular with respect to the same decomposition as that of $\mathcal{A}$, then $S^{-1}\mathcal{A}S$ has a reduced block upper triangular form with respect to this decomposition, and indices i and j are linked in $S^{-1}\mathcal{A}S$ if and only if they are linked in $\mathcal{A}$.

The following Jordan-Hölder-type result describes the extent to which the reduced block upper triangular form of a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is unique.

Theorem 2.3.5. [14, Theorem 23] *Suppose that a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 \dotplus \mathcal{V}_2 \dotplus \cdots \dotplus \mathcal{V}_k$, as well as with respect to a decomposition $\mathbb{C}^n = \mathcal{W}_1 \dotplus \mathcal{W}_2 \dotplus \cdots \dotplus \mathcal{W}_m$. Then $k = m$ and there is a permutation π on $\{1, 2, \ldots, k\}$ such that*

(i) i is linked to j in the $\mathcal{V}$-decomposition if and only if $\pi(i)$ is linked to $\pi(j)$ in the $\mathcal{W}$-decomposition, and

(ii) for each i there is an invertible linear map $S_i : \mathcal{V}_i \to \mathcal{W}_{\pi(i)}$ such that

$$A\big|_{\mathcal{V}_i} = S_i^{-1} A\big|_{\mathcal{W}_{\pi(i)}} S_i \quad \textit{for all } A \in \mathcal{A}.$$

The theorems presented above provide insight into the structure of the block-diagonal of a reduced block upper triangular matrix algebra $\mathcal{A}$. It will now be important to develop an understanding of the blocks that are located above the block-diagonal. First, recall the following definitions.

Definition 2.3.6. The *nil radical* of a ring $\mathcal{R}$ is the unique largest ideal of $\mathcal{R}$ consisting entirely of nilpotent elements. It is denoted by $Rad(\mathcal{R})$.

It is well-known that every ring $\mathcal{R}$ admits a unique largest ideal of nilpotent elements [13, Lemma 10.25], and hence $Rad(\mathcal{R})$ is well defined. Moreover, if $\mathcal{R} = \mathcal{A}$ is a finite-dimensional algebra over a field $\mathbb{F}$, then $Rad(\mathcal{A})$ coincides with the Jacobson radical of $\mathcal{A}$ [13, Theorem 4.20]. In particular, this result holds for subalgebras $\mathcal{A}$ of $\mathbb{M}_n$.

Definition 2.3.7. A subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is said to be *semi-simple* if $Rad(\mathcal{A}) = \{0\}$.

As described in [14, Corollary 28], if $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n$, then there is a semi-simple subalgebra $\mathcal{S}$ of $\mathcal{A}$ such that $\mathcal{A}$ decomposes as an algebraic direct sum $\mathcal{A} = \mathcal{S} \dotplus Rad(\mathcal{A})$. If $\mathcal{A}$ is in reduced block upper triangular form, then $\mathcal{S}$ is block upper triangular and $Rad(\mathcal{A})$ consists of all strictly block upper triangular elements of $\mathcal{A}$ [14, Proposition 19]. Thus, the blocks above the block-diagonal are, in general, comprised of blocks from $\mathcal{S}$ and blocks from $Rad(\mathcal{A})$. In the simplest scenario $\mathcal{S}$ is equal to $BD(\mathcal{A})$.

Definition 2.3.8. Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$ that has a reduced block upper triangular form with respect to some decomposition of $\mathbb{C}^n$. The algebra $\mathcal{A}$ is said to be *unhinged* with respect to this decomposition if

$$\mathcal{A} = BD(\mathcal{A}) \dotplus Rad(\mathcal{A}).$$

The following result indicates that if $\mathcal{A}$ is an algebra in reduced block upper triangular form with respect to some decomposition of $\mathbb{C}^n$, then $\mathcal{A}$ can be unhinged with respect to this decomposition via conjugation by a block upper triangular similarity.

Theorem 2.3.9. [14, Corollary 30] *If a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to a decomposition of $\mathbb{C}^n$, then after an application of a block upper triangular similarity, $\mathcal{A}$ has an unhinged reduced block upper triangular form with respect to this decomposition.*

It is useful to note that if $\mathcal{A}$ is in reduced block upper triangular form and $BD(\mathcal{A}) = \mathbb{C}I$, then Theorem 2.3.9 implies that $\mathcal{A} = \mathbb{C}I + Rad(\mathcal{A})$. Thus, $\mathcal{A}$ is unhinged with respect to any decomposition in which it admits a reduced block upper triangular form.

It will be shown in Chapter 4 that the blocks of a unital algebra $\mathcal{A}$ in reduced block upper triangular form must exhibit a very particular structure in order for $\mathcal{A}$ to be projection compressible. Once this result is obtained, it will be important to understand the structure of $Rad(\mathcal{A})$ when $\mathcal{A}$ is an algebra of the correct form. We therefore conclude this section with a technical lemma concerning the independence of the blocks in the radical of such an algebra.

Lemma 2.3.10. *Let n be a positive integer, and let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$ in reduced block upper triangular form with respect to a decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$. Suppose that there is an index k, $1 < k < m$, that is unlinked from all indices $i \neq k$. Let Q_1, Q_2, and Q_3 denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively, and assume that $Q_1 Rad(\mathcal{A}) Q_1 = Q_3 Rad(\mathcal{A}) Q_3 = \{0\}$.*

(i) For every $R \in Rad(\mathcal{A})$, there are elements $R' = Q_1 R'$ and $R'' = R'' Q_3$ in $Rad(\mathcal{A})$ such that $R' Q_2 = Q_1 R Q_2$ and $Q_2 R'' = Q_2 R Q_3$.

(ii) If there exist projections $Q_1' \leq Q_1$ and $Q_3' \leq Q_3$ such that $Q_1 Rad(\mathcal{A}) Q_2 = Q_1' \mathbb{M}_n Q_2$, $Q_2 Rad(\mathcal{A}) Q_3 = Q_2 \mathbb{M}_n Q_3'$, and $Q_1 Rad(\mathcal{A}) Q_3 = Q_1' Rad(\mathcal{A}) Q_3'$ then

$$Rad(\mathcal{A}) = Q_1' \mathbb{M}_n Q_2 \dotplus Q_1' \mathbb{M}_n Q_3' \dotplus Q_2 \mathbb{M}_n Q_3'.$$

(iii) If $\mathcal{A}$ is unhinged with respect to this decomposition, then

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3.$$

Proof. For (i), let R belong to $Rad(\mathcal{A})$. Since $\mathcal{V}_k$ is unlinked from all other spaces $\mathcal{V}_i$, there is an element $A \in \mathcal{A}$ such that $Q_1 A Q_1 = Q_3 A Q_3 = 0$ and $Q_2 A Q_2 = Q_2$. Thus, with respect to the decomposition $\mathbb{C}^n = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2) \oplus \operatorname{ran}(Q_3)$, A and R may be expressed as

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & I & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

for some A_{ij} and R_{ij}. It is then easy to see that $R' := RA$ and $R'' := AR$ define elements of $Rad(\mathcal{A})$ that satisfy the requirements of (i).

For (ii), let M_1 and M_2 denote arbitrary elements of $Q_1'\mathbb{M}_n Q_2$ and $Q_2\mathbb{M}_n Q_3'$, respectively. By (i), there are elements S_1 and S_2 in $Q_1\mathbb{M}_n Q_3$ such that $M_1 + S_1$ and $M_2 + S_2$ belong to $Rad(\mathcal{A})$. Moreover, since $Q_1 Rad(\mathcal{A}) Q_3 = Q_1' Rad(\mathcal{A}) Q_3'$, we have that S_1 and S_2 are contained in $Q_1'\mathbb{M}_n Q_3'$.

Observe that $R \coloneqq (M_1 + S_1)(M_2 + S_2)$ belongs to $Rad(\mathcal{A})$. With respect to the decomposition of $\mathbb{C}^n$ described above, this element can be expressed as

$$R = \begin{bmatrix} 0 & M_1 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & S_2 \\ 0 & 0 & M_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & M_1 M_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But since M_1 and M_2 were arbitrary, this implies that $Q_1'\mathbb{M}_n Q_3' \subseteq Rad(\mathcal{A})$. In particular, $Rad(\mathcal{A})$ contains S_1 and S_2. It then follows that M_1 and M_2 belong to $Rad(\mathcal{A})$ as well. We conclude that $Rad(\mathcal{A})$ contains $Q_1'\mathbb{M}_n Q_2$ and $Q_2\mathbb{M}_n Q_3'$, as M_1 and M_2 were arbitrary.

Finally, let us prove (iii). Assume that $\mathcal{A}$ is unhinged with respect to $\bigoplus_{i=1}^m \mathcal{V}_i$, and let R be an element of $Rad(\mathcal{A})$. Since $\mathcal{V}_k$ is unlinked from $\mathcal{V}_i$ for all $i \neq k$, the projection Q_2 belongs to $\mathcal{A}$. It follows that the operators $RQ_2 = Q_1 R Q_2$ and $Q_2 R = Q_2 R Q_3$ belong to $Rad(\mathcal{A})$, and therefore so too does $R - Q_2 R - R Q_2 = Q_1 R Q_3$. We conclude that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3,$$

as claimed. ■

Chapter 3

Compressibility in $\mathbb{M}_3$

We now turn our attention to assessing the completeness of the list of idempotent compressible algebras established in §2.2. That is, we wish to determine whether or not there exist additional examples of unital idempotent compressible algebras up to transpose similarity.

Our findings in §2.2.2 suggest that there may exist pathological examples of such algebras in $\mathbb{M}_3$. For this reason, we devote this chapter to classifying the unital subalgebras in $\mathbb{M}_3$ that admit the idempotent compression property, and reserve the classification of such subalgebras of $\mathbb{M}_n$, $n \geq 4$, for Chapter 4.

Using the structure theory established in §2.3, we will show in §3.1 that up to transposition and similarity, the only unital idempotent compressible subalgebras of $\mathbb{M}_3$ are those constructed in §2.2. As a consequence of this analysis, we will observe that a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_3$ that lacks the idempotent compression property is necessarily transpose similar to one of the following algebras:

$$\begin{aligned}
\mathcal{B} &:= \left\{ \begin{bmatrix} \alpha & x & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\}, \\
\mathcal{C} &:= \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & x \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\}, \quad \text{or} \\
\mathcal{D} &:= \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.
\end{aligned}$$

This observation has interesting implications for *projection* compressibility $\mathbb{M}_3$. In particular, it leads to an avenue for proving that in the case of unital subalgebras of $\mathbb{M}_3$, the notions of projection compressibility and idempotent compressibility coincide. For if there were a unital projection compressible subalgebra $\mathcal{A}$ of $\mathbb{M}_3$ that did not exhibit the idempotent compression property, then $\mathcal{A}$ must be similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$. Thus, one could establish the equivalence of these notions by proving that no algebra similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ is projection compressible. This approach will be used in §3.2.

§3.1 Classification of Idempotent Compressibility

Here we begin the classification of unital idempotent compressible subalgebras of $\mathbb{M}_3$ up to transposition and similarity. By the results outlined in §2.3, we may assume that our algebras are expressed in reduced block upper triangular form with respect to an orthogonal decomposition of $\mathbb{C}^3 = \bigoplus_{i=1}^m \mathcal{V}_i$, and are unhinged with respect to this decomposition. That is, we will assume that

$$\mathcal{A} = BD(\mathcal{A}) \dotplus Rad(\mathcal{A}),$$

where $Rad(\mathcal{A})$ consists of all strictly block upper triangular elements of $\mathcal{A}$. With this in mind, the algebras in this list will be organized according to the configuration of their block-diagonal and the dimension of their radical.

If $\mathcal{A} = \mathbb{M}_3$, then $\mathcal{A}$ is clearly idempotent compressible. Furthermore, if some $\mathcal{V}_i$ has dimension 2, then Theorem 2.1.12 implies that $\mathcal{A}$ is transpose equivalent to $\mathbb{C} \oplus \mathbb{M}_2$ or

$$\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

In either case, $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, and hence is idempotent compressible.

Thus, we may assume from here on that all spaces $\mathcal{V}_i$ have dimension 1. For each i, let e_i be a unit vector in $\mathcal{V}_i$, and let Q_i denote the orthogonal projection onto $\mathcal{V}_i$.

Case I: $\dim BD(\mathcal{A}) = 3$. If $\dim BD(\mathcal{A}) = 3$, then the spaces $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are mutually unlinked. An application of Lemma 2.3.10 (iii) then shows that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3.$$

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A} = \mathcal{D}$, one of the three algebras presented at the outset of Chapter 3. It will be shown in Theorem 3.2.6 that no algebra similar to $\mathcal{D}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then there is exactly one pair of indices (i, j) such that $i < j$ and $Q_i Rad(\mathcal{A}) Q_j$ is non-zero. In this case, $\mathcal{A}$ is unitarily equivalent to

$$\mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_2 + Q_3)\mathbb{M}_3 Q_3,$$

the algebra described in Example 2.2.7. Consequently, $\mathcal{A}$ is idempotent compressible.

(iii) If $\dim Rad(\mathcal{A}) = 2$, then $Q_i Rad(\mathcal{A}) Q_j = \{0\}$ for exactly one pair of indices (i, j) with $i < j$. By considering products of elements in $Rad(\mathcal{A})$, one can show that $Q_1 Rad(\mathcal{A}) Q_3$ is non-zero whenever both $Q_1 Rad(\mathcal{A}) Q_2$ and $Q_2 Rad(\mathcal{A}) Q_3$ are non-zero. This means that either $Q_1 Rad(\mathcal{A}) Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A}) Q_3 = \{0\}$; hence $\mathcal{A}$ is transpose equivalent to

$$\mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_3 Q_3.$$

This algebra was shown to admit the idempotent compression property in Example 2.2.3. Therefore, $\mathcal{A}$ is idempotent compressible.

(iv) If $\dim Rad(\mathcal{A}) = 3$, then $\mathcal{A}$ is equal to

$$\mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_3(Q_2 + Q_3),$$

the unital algebra from Example 2.2.1. Consequently, $\mathcal{A}$ is idempotent compressible.

Case II: $\dim BD(\mathcal{A}) = 2$. If $\dim BD(\mathcal{A}) = 2$, then exactly two of the spaces $\mathcal{V}_i$ and $\mathcal{V}_j$ are linked. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $\mathcal{V}_1$ is one of the linked spaces.

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A}$ is unitarily equivalent to $\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3$, the unitization of the $\mathcal{LR}$-algebra $\mathbb{C}Q_3$. Consequently, $\mathcal{A}$ is idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then $Rad(\mathcal{A}) = \mathbb{C}R$ for some strictly upper triangular element

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $R^2 \in Rad(\mathcal{A})$, we have that $R^2 = \alpha R$ for some $\alpha \in \mathbb{C}$. From this it follows that at least one of r_{12} or r_{23} is equal to zero.

First consider the case in which $\mathcal{V}_2$ is not linked to $\mathcal{V}_1$. By Lemma 2.3.10 (iii),

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3.$$

If $r_{12} = r_{13} = 0$ or $r_{13} = r_{23} = 0$, then $\mathcal{A}$ or $\mathcal{A}^{aT}$ is equal to

$$\mathcal{A} = Q_2 \mathbb{M}_3 (Q_2 + Q_3) + \mathbb{C}I.$$

In this case, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $r_{12} = r_{23} = 0$, then $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}$, one of the three algebras described at the beginning of Chapter 3. It will be shown in Theorem 3.2.2 that no algebra similar to $\mathcal{B}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

Now consider the case in which $\mathcal{V}_1$ is linked to $\mathcal{V}_2$. Since $\mathcal{V}_3$ is therefore unlinked from $\mathcal{V}_1$ and $\mathcal{V}_2$, and one may argue as in the proof of Lemma 2.3.10 (iii) to show that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus (Q_1 + Q_2) Rad(\mathcal{A}) Q_3.$$

If $r_{12} = 0$, then $\mathcal{A}$ is unitarily equivalent to

$$(Q_2 + Q_3) \mathbb{M}_3 Q_3 + \mathbb{C}I.$$

In this case, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $r_{12} \neq 0$, then $r_{13} = r_{23} = 0$ and hence $\mathcal{A}$ is equal to $\mathcal{B}$.

(iii) Suppose now that $\dim Rad(\mathcal{A}) = 2$. If $\mathcal{V}_2$ is the unlinked space, then

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3.$$

It then follows that either $Q_1 Rad(\mathcal{A})Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A})Q_3 = \{0\}$, so $\mathcal{A}$ is transpose equivalent to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2 + Q_3).$$

This algebra was shown to admit the idempotent compression property in Example 2.2.10, and hence $\mathcal{A}$ is idempotent compressible.

Now consider the case in which $\mathcal{V}_2$ is linked to $\mathcal{V}_1$. Since $\mathcal{V}_3$ is therefore unlinked from $\mathcal{V}_1$ and $\mathcal{V}_2$, we have that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A})Q_2 \dot{+} (Q_1 + Q_2)Rad(\mathcal{A})Q_3.$$

If $Q_1 Rad(\mathcal{A})Q_2 = \{0\}$, then

$$\mathcal{A} = \mathbb{M}_3 Q_3 + \mathbb{C}I.$$

Consequently, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $Q_1 Rad(\mathcal{A})Q_2 = Q_1\mathbb{M}_3 Q_2$, then $(Q_1+Q_2)Rad(\mathcal{A})Q_3$ is 1-dimensional. Thus, there is a non-zero matrix $R \in (Q_1 + Q_2)\mathbb{M}_3 Q_3$ such that

$$Rad(\mathcal{A}) = Q_1\mathbb{M}_3 Q_2 \dot{+} \mathbb{C}R.$$

It is then easy to see that $\langle Re_3, e_2\rangle = 0$. For if not, $Rad(\mathcal{A})$ would contain an element of the form $e_2 \otimes e_3^* + te_1 \otimes e_3^*$ for some $t \in \mathbb{C}$; hence

$$e_1 \otimes e_3^* = (e_1 \otimes e_2^*)\,(e_2 \otimes e_3^* + te_1 \otimes e_3^*) \in Rad(\mathcal{A})$$

This would then imply that $Rad(\mathcal{A})$ is 3-dimensional—a contradiction.

Thus, $\langle Re_3, e_2\rangle = 0$, so $\mathcal{A}$ is equal to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2 + Q_3),$$

the idempotent compressible algebra from Example 2.2.10. In all cases, $\mathcal{A}$ is idempotent compressible.

(iv) Suppose that $\dim Rad(\mathcal{A}) = 3$. If $\mathcal{V}_2$ is the unlinked space, then $\mathcal{A}$ is equal to

$$(Q_1 + Q_2)\mathbb{M}_3(Q_2 + Q_3) + \mathbb{C}I.$$

In this case $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, and hence is idempotent compressible. If instead $\mathcal{V}_2$ is linked to $\mathcal{V}_1$, then $\mathcal{A}$ is equal to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3Q_2 + (Q_1 + Q_2)\mathbb{M}_3Q_3,$$

the unital algebra described in Example 2.2.6. Consequently, $\mathcal{A}$ is idempotent compressible.

Case III: $\dim BD(\mathcal{A}) = 1$. Suppose now that $\dim BD(\mathcal{A}) = 1$, so that all spaces $\mathcal{V}_i$ are mutually linked. That is, $BD(\mathcal{A}) = \mathbb{C}I$.

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A} = \mathbb{C}I$. Clearly $\mathcal{A}$ is idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then $Rad(\mathcal{A}) = \mathbb{C}R$ for some strictly upper triangular matrix

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

As in part (ii) of the previous case, one can show that $r_{12} = 0$ or $r_{23} = 0$, so R necessarily has rank 1. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $r_{12} = 0$. But then $\mathcal{A}$ is unitarily equivalent to

$$Q_2\mathbb{M}_3Q_3 + \mathbb{C}I,$$

the unitization of an $\mathcal{LR}$-algebra. Thus, $\mathcal{A}$ is idempotent compressible.

(iii) Suppose that $\dim Rad(\mathcal{A}) = 2$. If $Q_1 Rad(\mathcal{A})Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A})Q_3 = \{0\}$, then $\mathcal{A}$ or $\mathcal{A}^{aT}$ is equal to

$$Q_1\mathbb{M}_3(Q_2 + Q_3) + \mathbb{C}I.$$

Thus, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra.

Now consider the case in which $Rad(\mathcal{A})$ contains an element

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

with $r_{12} \neq 0$ and $r_{23} \neq 0$. When this occurs, $Rad(\mathcal{A})$ contains $\frac{1}{r_{12}r_{23}}R^2 = e_1 \otimes e_3^*$; hence

$$Rad(\mathcal{A}) = \text{span}\{e_1 \otimes e_2^* + re_2 \otimes e_3^*, e_1 \otimes e_3^*\}$$

where $r := r_{23}/r_{12}$. Consequently,

$$\mathcal{A} = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & rx \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\},$$

which is easily seen to be similar to the algebra $\mathcal{C}$ described at the outset of Chapter 3. It will be shown in Theorem 3.2.4 that no algebra similar to $\mathcal{C}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

(iv) If $\dim Rad(\mathcal{A}) = 3$, then $\mathcal{A}$ is equal to

$$Q_1\mathbb{M}_3(Q_2 + Q_3) + Q_2\mathbb{M}_3Q_3 + \mathbb{C}I,$$

the idempotent compressible algebra described in Example 2.2.13.

This concludes our classification of the unital idempotent compressible subalgebras of $\mathbb{M}_3$. Our findings are summarized in the following theorem.

Theorem 3.1.1. *Let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_3$.*

(i) $\mathcal{A}$ is idempotent compressible if and only if $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra or transpose similar to one of the following algebras:

$$\left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma, x, y, z \in \mathbb{C} \right\}, \qquad \text{(Example 2.2.1)}$$

$$\left\{ \begin{bmatrix} \alpha & 0 & x \\ 0 & \beta & y \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, x, y \in \mathbb{C} \right\}, \qquad \text{(Example 2.2.3)}$$

$$\left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, \gamma, x, y, z \in \mathbb{C} \right\}, \qquad \text{(Example 2.2.6)}$$

$$\left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & x \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma, x \in \mathbb{C} \right\}, \qquad \text{(Example 2.2.7)}$$

$$\left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x, y \in \mathbb{C} \right\}, \qquad \text{(Example 2.2.10)}$$

$$\left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}; \qquad \text{(Example 2.2.13)}$$

(ii) $\mathcal{A}$ does not admit the idempotent compression property if and only if $\mathcal{A}$ is transpose similar to one of the algebras $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$, presented at the outset of Chapter 3.

Although the algebras presented in Theorem 3.1.1(i) may appear to share little in common beyond the idempotent compression property, there do exist other interesting characterizations of this collection. For instance, aside from the unitizations of $\mathcal{LR}$-algebras, the unital idempotent compressible algebras are exactly those that are not 3-dimensional.

Corollary 3.1.2. *A unital subalgebra of $\mathcal{A}$ of $\mathbb{M}_3$ is idempotent compressible if and only if $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is not 3-dimensional.*

In addition, one may observe that the unital algebras lacking the idempotent compression property are exactly those that are generated by a matrix in which every Jordan block corresponds to a distinct eigenvalue. Such matrices are said to be *nonderogatory* [12, Definition 1.4.4].

Corollary 3.1.3. *A unital subalgebra of $\mathbb{M}_3$ is idempotent compressible if and only if it is not singly generated by an invertible nonderogatory matrix.*

As we will see in the following section, the algebras described in Theorem 3.1.1(i) also represent the complete list of unital *projection* compressible subalgebras of $\mathbb{M}_3$ up to transpose similarity.

§3.2 Equivalence of Idempotent and Projection Compressibility

Our final goal of this chapter is to show that no unital subalgebra of $\mathbb{M}_3$ can possess the projection compression property without also possessing the idempotent compression property. If such an algebra did exist, it would necessarily be transpose similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ by the analysis in §3.1. Thus, to show that the notions of projection compressibility and idempotent compressibility agree for unital subalgebras of $\mathbb{M}_3$, it suffices to prove that no algebra similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ is projection compressible. This goal will be accomplished by first characterizing the algebras similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ up to unitary equivalence.

Lemma 3.2.1. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{B} = \left\{ \begin{bmatrix} \alpha & x & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\},$$

then there are constants $s, t \in \mathbb{C}$ such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{B}_{st} := \left\{ \begin{bmatrix} \alpha & s(\alpha-\beta) & x \\ 0 & \beta & t(\alpha-\beta) \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\}.$$

Proof. If the matrices in $\mathcal{B}$ are expressed with respect to the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{C}^3$, then $\mathcal{B}$ is spanned by $\{E_{11}+E_{22}, E_{12}, E_{33}\}$, where $E_{ij} := e_i \otimes e_j^*$. Thus, if S is an invertible matrix in $\mathbb{M}_3$ such that $\mathcal{A} = S^{-1}\mathcal{B}S$, then $\mathcal{A}$ is spanned by $\{E'_{11}+E'_{22}, E'_{12}, E'_{33}\}$, where $E'_{ij} := S^{-1}E_{ij}S$.

Since E'_{12} is a rank-one nilpotent, there is a unitary $U \in \mathbb{M}_3$ and a non-zero $y_0 \in \mathbb{C}$ such that

$$U^*E'_{12}U = \begin{bmatrix} 0 & 0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $x_{ij} \in \mathbb{C}$ be such that $U^*(E'_{11}+E'_{22})U = (x_{ij})$. Using the fact that

$$(E'_{11}+E'_{22})E'_{12} = E'_{12}(E'_{11}+E'_{22}) = E'_{12},$$

one can show that $x_{21} = x_{31} = x_{32} = 0$ and $x_{11} = x_{33} = 1$. Moreover, since $U^*(E'_{11}+E'_{22})U$ is an idempotent of trace 2, it follows that $x_{22} = 0$ and $x_{13} = -x_{12}x_{23}$. Thus,

$$U^*(E'_{11}+E'_{22})U = \begin{bmatrix} 1 & x_{12} & -x_{12}x_{23} \\ 0 & 0 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, we have that

$$U^* E'_{33} U = I - U^*(E'_{11} + E'_{22})U = \begin{bmatrix} 0 & -x_{12} & x_{12}x_{23} \\ 0 & 1 & -x_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

As a result,

$$U^* \mathcal{A} U = \operatorname{span} \left\{ \begin{bmatrix} 1 & x_{12} & -x_{12}x_{23} \\ 0 & 0 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -x_{12} & x_{12}x_{23} \\ 0 & 1 & -x_{23} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \mathcal{B}_{st},$$

where $s := x_{12}$ and $t := x_{23}$. ∎

Theorem 3.2.2. *For any $s, t \in \mathbb{C}$, the algebra $\mathcal{B}_{st}$ as in Lemma 3.2.1 is not projection compressible. Consequently, no algebra similar to $\mathcal{B}$ is projection compressible.*

Proof. Consider the elements A and B of $\mathcal{B}_{st}$ given by

$$A = \begin{bmatrix} 1 & s & 0 \\ 0 & 0 & t \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will construct a matrix P that is a multiple of a projection in $\mathbb{M}_3$, and such that $(PAP)(PBP)$ does not belong to $P\mathcal{B}_{st}P$. To accomplish this goal, let k be any element of $\mathbb{R} \setminus \{0, s, t\}$, and define

$$P := \begin{bmatrix} k^2+1 & -k & -1 \\ -k & 2 & -k \\ -1 & -k & k^2+1 \end{bmatrix}.$$

Note that $\frac{1}{k^2+2}P$ is a projection in $\mathbb{M}_3$.

If $(PAP)(PBP)$ were an element of $P\mathcal{B}_{st}P$, there would exist a matrix

$$C = \begin{bmatrix} \alpha_0 & s(\alpha_0 - \beta_0) & x_0 \\ 0 & \beta_0 & t(\alpha_0 - \beta_0) \\ 0 & 0 & \alpha_0 \end{bmatrix} \in \mathcal{B}_{st}$$

such that $PAPBP - PCP = (g_{ij})$ is equal to 0. By examining the value of g_{31}, one can show that

$$x_0 = k(\alpha_0 - \beta_0 + 1)(2k - s - t) + 2(\alpha_0 + 1) + k^2\beta_0.$$

From here, direct computations reveal that

$$(k-s)g_{11} - (k-t)g_{33} = k(k^2+2)(k-s)(k-t).$$

Since $g_{11} = g_{33} = 0$, but the right-hand side of the above equation is non-zero by construction, we have reached a contradiction. Thus, there does not exist a C as above, so $P\mathcal{B}_{st}P$ is not an algebra. The final claim is now a consequence of Lemma 3.2.1. ■

Lemma 3.2.3. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{C} := \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & x \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\},$$

then there is a non-zero constant $r \in \mathbb{C}$ such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{C}_r := \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & rx \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\}.$$

Proof. Observe that $\mathcal{C}$ is spanned by $\{I, N_1, N_2\}$, where

$$N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, if $S \in \mathbb{M}_3$ is an invertible matrix such that $\mathcal{A} = S^{-1}\mathcal{C}S$, then $\mathcal{A}$ is spanned by $\{I, N_1', N_2'\}$, where $N_i' = S^{-1}N_iS$ for $i \in \{1,2\}$.

It is evident that N_1' and N_2' are nilpotent operators of rank 1 and 2, respectively, and $N_1'N_2' = N_2'N_1' = 0$. In particular, since N_1' and N_2' commute, there is a unitary $U \in \mathbb{M}_3$ such that $U^*N_1'U$ and $U^*N_2'U$ are upper triangular. If a_{ij} and b_{ij} are such that

$$U^*N_1'U = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U^*N_2'U = \begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

then rank considerations imply that neither b_{12} nor b_{23} is equal to 0. But

$$(U^*N_1'U)(U^*N_2'U) = \begin{bmatrix} 0 & 0 & a_{12}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (U^*N_2'U)(U^*N_1'U) = \begin{bmatrix} 0 & 0 & a_{23}b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so it must be that $a_{12} = a_{23} = 0$. By setting $r = b_{23}/b_{12}$, it follows that

$$U^*\mathcal{A}U = \operatorname{span}\left\{ I, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & b_{13}/b_{12} \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \right\} = \mathcal{C}_r.$$

■

Theorem 3.2.4. *For every non-zero $r \in \mathbb{C}$, the algebra $\mathcal{C}_r$ as in Lemma 3.2.3 is not projection compressible. Consequently, no algebra similar to $\mathcal{C}$ is projection compressible.*

Proof. Consider the elements $A, B \in \mathcal{C}_r$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, define the matrix

$$P := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

so $\frac{1}{3}P$ is a projection in $\mathbb{M}_3$.

We claim that $(PAP)(PBP)$ does not belong to $P\mathcal{C}_rP$. For if it did, there would exist an element

$$C = \begin{bmatrix} \alpha_0 & x_0 & y_0 \\ 0 & \alpha_0 & rx_0 \\ 0 & 0 & \alpha_0 \end{bmatrix}$$

in $\mathcal{C}_r$ such that $PAPBP - PCP = (g_{ij})$ is equal to 0. Direct computations show that

$$0 = g_{31} = 3\alpha_0 - (x_0 + 1)(r + 1) - y_0,$$

hence $y_0 = 3\alpha_0 - (x_0 + 1)(r + 1)$. From here, further calculations reveal that $g_{21} - rg_{32} = 3r$. Since $g_{21} = g_{32} = 0$ but $r \neq 0$, we have reached a contradiction. Thus, there does not exist an element $C \in \mathcal{C}_r$ as described above. This shows that $(PAP)(PBP) \notin P\mathcal{C}_rP$, so $\mathcal{C}_r$ is not projection compressible. The final claim is now immediate from Lemma 3.2.3. ■

Lemma 3.2.5. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{D} = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\},$$

then there are constants $r, s, t \in \mathbb{C}$ such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{D}_{rst} := \left\{ \begin{bmatrix} \alpha & r(\alpha-\beta) & s(\alpha-\gamma) - rt(\gamma-\beta) \\ 0 & \beta & t(\gamma-\beta) \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Proof. If $\mathcal{D}$ is written with respect to the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{C}^3$, then $\mathcal{D}$ is spanned by $\{E_{11}, E_{22}, E_{33}\}$ where $E_{jj} = e_j \otimes e_j^*$. Let S be an invertible element of $\mathbb{M}_3$ such that $\mathcal{A} = S^{-1}\mathcal{D}S$. Clearly $\mathcal{A}$ is spanned by $\{E'_{11}, E'_{22}, E'_{33}\}$ where $E'_{jj} = S^{-1}E_{jj}S$.

Observe that the matrices E'_{jj} commute, so there is a unitary $U \in \mathbb{M}_3$ such that $U^*E'_{jj}U$ is upper triangular for each $j \in \{1, 2, 3\}$. Furthermore, since each $U^*E'_{jj}U$ is an idempotent of rank 1, and

$$(U^*E'_{ii}U)(U^*E'_{jj}U) = \delta_{ij}U^*E'_{jj}U$$

for all i and j, one may re-index the matrices E'_{jj} if necessary to write

$$U^*E'_{11}U = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U^*E'_{22}U = \begin{bmatrix} 0 & y_{12} & y_{12}y_{23} \\ 0 & 1 & y_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad U^*E'_{33}U = \begin{bmatrix} 0 & 0 & z_{13} \\ 0 & 0 & z_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

for some x_{ij}, y_{ij}, and z_{ij} in $\mathbb{C}$. The fact that these matrices add to I implies that

$$y_{12} = -x_{12}, \quad y_{23} = -z_{23}, \quad \text{and} \quad z_{13} = -x_{13} - x_{12}z_{23} \ .$$

As a result,

$$U^*\mathcal{A}U = \operatorname{span}\left\{ \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -x_{12} & x_{12}z_{23} \\ 0 & 1 & -z_{23} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -x_{13} - x_{12}z_{23} \\ 0 & 0 & z_{23} \\ 0 & 0 & 1 \end{bmatrix} \right\} = \mathcal{D}_{rst},$$

where $r := x_{12}$, $s := x_{13}$, and $t := z_{23}$. ∎

Theorem 3.2.6. *For any $r, s, t \in \mathbb{C}$, the algebra $\mathcal{D}_{rst}$ as in Lemma 3.2.5 is not projection compressible. Consequently, no algebra similar to $\mathcal{D}$ is projection compressible.*

Proof. Consider the elements A and B of $\mathcal{D}_{rst}$ given by

$$A = \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -r & rt \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{bmatrix}.$$

We wish to construct a matrix P that is a multiple of a projection in $\mathbb{M}_3$, and such that $(PAP)(PBP)$ does not belong to $P\mathcal{D}_{rst}P$. To accomplish this goal, choose elements $k, m \in \mathbb{R} \setminus \{0\}$ subject to the following constraints:

$$\begin{array}{rcrcr} tk & & & \neq & 1, \\ & & rm & \neq & 1, \\ sk & + & m & \neq & -r, \quad \text{and} \\ k & - & (rt+s)m & \neq & -t. \end{array}$$

Of course, such k and m always exist. Using these values, define

$$P = \begin{bmatrix} k^2+1 & -m & -mk \\ -m & k^2+m^2 & -k \\ -mk & -k & m^2+1 \end{bmatrix}.$$

It is straightforward to check that $\frac{1}{k^2+m^2+1}P$ is a projection in $\mathbb{M}_3$.

Suppose to the contrary that $(PAP)(PBP)$ were an element of $P\mathcal{D}_{rst}P$. In this case, there is a matrix

$$C = \begin{bmatrix} \alpha_0 & r(\alpha_0 - \beta_0) & s(\alpha_0 - \gamma_0) - rt(\gamma_0 - \beta_0) \\ 0 & \beta_0 & t(\gamma_0 - \beta_0) \\ 0 & 0 & \gamma_0 \end{bmatrix} \in \mathcal{D}_{rst}$$

such that $PAPBP - PCP = (g_{ij})$ is equal to 0. We will obtain a contradiction by examining specific entries g_{ij}.

Firstly, one may check that

$$0 = g_{31} - kg_{21} = km(k^2+m^2+1)(tk-1)(\beta_0 - \gamma_0).$$

By construction, the product on the right-hand side is zero if and only if $\beta_0 = \gamma_0$. But if this is the case, then

$$0 = kg_{23} - g_{33} = \beta_0(k^2+m^2+1),$$

and therefore $\beta_0 = \gamma_0 = 0$. Direct computations then show that

$$
\begin{aligned}
&(r(k^2+m^2) - sk - m)g_{21} - (k^2 - skm - rm + 1)g_{22} \\
&\qquad\qquad\qquad = km(k^2+m^2+1)(rm-1)(sk+m+r)(k-(rt+s)m+t).
\end{aligned}
$$

Since $g_{21} = g_{22} = 0$ while the right-hand side of this equation is non-zero by construction, we obtain the required contradiction.

Thus, $(PAP)(PBP)$ does not belong to $P\mathcal{D}_{rst}P$, so $\mathcal{D}_{rst}$ is not projection compressible. The final claim now follows from Lemma 3.2.5. ■

Combining Theorems 3.2.2, 3.2.4, and 3.2.6 with Theorem 3.1.1 and its subsequent corollaries, we obtain the following classification of unital subalgebras of $\mathbb{M}_3$ that admit one, and hence both of the compression properties.

Theorem 3.2.7. *If $\mathcal{A}$ is a unital subalgebra of $\mathbb{M}_3$, then the following are equivalent:*

(i) $\mathcal{A}$ is projection compressible;

(ii) $\mathcal{A}$ is idempotent compressible;

(iii) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is not 3-dimensional;

(iv) $\mathcal{A}$ is not singly generated by an invertible nonderogatory matrix;

(v) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose similar to one of the algebras from Theorem 3.1.1(i).

Chapter 4

Compressibility in $\mathbb{M}_n$, $n \geq 4$

In Chapter 3 we obtained a complete description of the unital projection compressible subalgebras of $\mathbb{M}_3$ up to transpose similarity. In doing so, we established the surprising result that the notions of projection compressibility and idempotent compressibility coincide for unital algebras in this setting. We now consider the problem of classifying the unital projection compressible subalgebras $\mathbb{M}_n$ when $n \geq 4$.

In §4.1 we present a certain necessary condition for a unital subalgebra of $\mathbb{M}_n$, $n \geq 4$, to admit the projection compression property. This condition imposes substantial restrictions on the reduced block upper triangular form such an algebra can take. In particular, it provides a systematic approach for classifying the unital projection compressible algebras in this setting based on the block upper triangular forms that can arise. Using this strategy, our analysis may be divided into three stages which are addressed in §4.2, §4.3, and §4.4, respectively.

§4.1 A Strategy for Classification

Before initiating our classification, it will be helpful to record a list of the projection compressible subalgebras of $\mathbb{M}_n$, $n \geq 4$, that we have studied up to now. First, we have the class of $\mathcal{LR}$-algebras which were shown to exhibit the idempotent compression property in Corollary 2.1.14. Additionally, the following examples describe the three distinct families of projection compressible algebras that were encountered in §2.1. It was shown in Examples 2.2.1, 2.2.3, and 2.2.6, respectively, that each of these algebras is in fact, idempotent compressible.

Example 4.1.1. *Let $n \geq 4$ be an integer, and let Q_1, Q_2, and Q_3 be projections in $\mathbb{M}_n$ that sum to I. In what follows, all matrices are expressed with respect to the decomposition $\mathbb{C}^n = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2) \oplus \operatorname{ran}(Q_3)$.*

(i) The algebra

$$\begin{aligned}\mathcal{A} &= \mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_n(Q_2 + Q_3) \\ &= \left\{ \begin{bmatrix} \alpha I & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & \beta I \end{bmatrix} : \alpha, \beta \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\}\end{aligned}$$

is idempotent compressible.

(ii) If rank(Q_1) = rank$(Q_2) = 1$, then the algebra

$$\begin{aligned}\mathcal{A} &= \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_nQ_3 \\ &= \left\{ \begin{bmatrix} \alpha & 0 & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\}\end{aligned}$$

is idempotent compressible.

(iii) If rank(Q_1) = rank$(Q_2) = 1$, then the algebra

$$\begin{aligned}\mathcal{A} &= \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3 + \mathbb{C}Q_3 \\ &= \left\{ \begin{bmatrix} \alpha & x & M_{13} \\ 0 & \alpha & M_{23} \\ 0 & 0 & \beta I \end{bmatrix} : \alpha, \beta, \gamma, x \in \mathbb{C}, M_{ij} \in Q_i\mathbb{M}_nQ_j \right\}\end{aligned}$$

is idempotent compressible.

Having reintroduced our library of examples from Chapter 2, we now present a simple structural requirement for a unital subalgebra of $\mathbb{M}_n$, $n \geq 4$, to admit the projection compression property. This result, together with the structure theory for matrix algebras outlined in §2.3, will provide a strategy for classifying the unital projection compressible algebras that exist in this setting.

Theorem 4.1.2. *Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a unital projection compressible subalgebra of $\mathbb{M}_n$. Suppose there exist mutually orthogonal projections P_1 and P_2 in $\mathbb{M}_n$ such that $P_2\mathcal{A}P_1 = \{0\}$ and rank$(P_i) \geq 2$ for $i = 1, 2$. Then $P_1\mathcal{A}P_1 = \mathbb{C}P_1$ or $P_2\mathcal{A}P_2 = \mathbb{C}P_2$.*

Proof. First assume that $rank(P_1) = rank(P_2) = 2$. By replacing $\mathcal{A}$ with the compression $(P_1 + P_2)\mathcal{A}(P_1 + P_2)$ if necessary, we may also assume that $P_1 + P_2 = I$.

Arguing by contradiction, suppose that $P_1\mathcal{A}P_1 \neq \mathbb{C}P_1$ and $P_2\mathcal{A}P_2 \neq \mathbb{C}P_2$. It follows that $\mathcal{A}$ admits an operator A such that $P_iAP_i \notin \mathbb{C}P_i$ for each $i \in \{1, 2\}$. Indeed, choose operators $A_1, A_2 \in \mathcal{A}$ such that $P_1A_1P_1 \notin \mathbb{C}P_1$ and $P_2A_2P_2 \notin \mathbb{C}P_2$. If $P_2A_1P_2 \notin \mathbb{C}P_2$ or $P_1A_2P_1 \notin \mathbb{C}P_1$, then A_1 or A_2 will satisfy the above requirements. Otherwise, $A \coloneqq A_1 + A_2$ will suffice.

Thus, assume that $A \in \mathcal{A}$ has been chosen such that $P_1AP_1 \notin \mathbb{C}P_1$ and $P_2AP_2 \notin \mathbb{C}P_2$. For each $i \in \{1, 2\}$, choose an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}\right\}$ for $\text{ran}(P_i)$ such that P_iAP_i is not diagonal with respect to $\mathcal{B} = \left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)}\right\}$. By permuting the basis vectors if necessary, we may assume that $\langle Ae_2^{(i)}, e_1^{(i)}\rangle \neq 0$ for each $i \in \{1, 2\}$.

Consider the matrix

$$Q \coloneqq \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

written with respect to $\mathcal{B}$. It is straightforward to check that $\frac{1}{2}Q$ is a projection in $\mathbb{M}_4$ and every $B \in Q\mathcal{A}Q$ satisfies $\langle Be_2^{(1)}, e_1^{(2)}\rangle = 0$. With A as above, however,

$$\langle (QAQ)^2 e_2^{(1)}, e_1^{(2)}\rangle = 8\langle Ae_2^{(1)}, e_1^{(1)}\rangle\langle Ae_2^{(2)}, e_1^{(2)}\rangle \neq 0.$$

Thus, $(QAQ)^2$ does not belong to $Q\mathcal{A}Q$, so $Q\mathcal{A}Q$ is not an algebra. This contradicts the assumption that $\mathcal{A}$ is projection compressible.

Now consider the general case in which each P_i has rank at least 2. One may deduce from the above analysis that for some $i \in \{1, 2\}$, every rank-two subprojection $P \leq P_i$ is such that $P\mathcal{A}P = \mathbb{C}P$. It then follows that $P_i\mathcal{A}P_i = \mathbb{C}P_i$, as required. ■

As we shall see in the coming analysis, this simple observation has significant implications for the classification of projection compressible algebras. Additionally, it highlights a major difference between the classification in this setting and that of $\mathbb{M}_3$. Since $\mathbb{M}_3$ cannot contain projections P_1 and P_2 as described in Theorem 4.1.2, this result may help to explain why there exist certain projection compressible subalgebras of $\mathbb{M}_3$ that do not admit analogues in higher dimensions (see Examples 2.2.7, 2.2.10, and 2.2.13).

The following corollaries to Theorem 4.1.2 provide a more explicit description of the reduced block upper triangular forms that can exist for a unital projection compressible algebra.

Corollary 4.1.3. *Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$. Suppose that there is an orthogonal decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$ with respect to which*

(i) $\mathcal{A}$ is reduced block upper triangular, and

(ii) there is an index $k \in \{1, 2, \ldots, m\}$ such that if Q_1, Q_2, and Q_3 denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively, then

$$(Q_1 + Q_2)\mathcal{A}(Q_1 + Q_2) \neq \mathbb{C}(Q_1 + Q_2) \quad \textit{and} \quad (Q_2 + Q_3)\mathcal{A}(Q_2 + Q_3) \neq \mathbb{C}(Q_2 + Q_3).$$

If $\mathcal{A}$ is projection compressible, then k is unique. When this is the case, $Q_1\mathcal{A}Q_1 = \mathbb{C}Q_1$ and $Q_3\mathcal{A}Q_3 = \mathbb{C}Q_3$.

Proof. Assume that $\mathcal{A}$ is projection compressible. Suppose to the contrary that there were a second index k' together with corresponding projections Q_1', Q_2', and Q_3' such that

$$\begin{aligned}(Q_1' + Q_2')\mathcal{A}(Q_1' + Q_2') &\neq \mathbb{C}(Q_1' + Q_2') \quad \text{and} \\ (Q_2' + Q_3')\mathcal{A}(Q_2' + Q_3') &\neq \mathbb{C}(Q_2' + Q_3').\end{aligned}$$

Assume without loss of generality that $k < k'$. One may verify that $P_1 := Q_1 + Q_2$ and $P_2 := Q_2' + Q_3'$ are projections satisfying the hypotheses of Theorem 4.1.2. Since $P_1\mathcal{A}P_1 \neq \mathbb{C}P_1$ and $P_2\mathcal{A}P_2 \neq \mathbb{C}P_2$, this is a contradiction.

The final claim follows immediately from the uniqueness of k. Indeed, if $Q_1\mathcal{A}Q_1 \neq \mathbb{C}Q_1$, then $k - 1$ would be another such index. If instead $Q_3\mathcal{A}Q_3 \neq \mathbb{C}Q_3$, then one could derive a similar contradiction by considering the index $k + 1$. ■

The following special case of Corollary 4.1.3 describes the situation for algebras whose block-diagonal contains a block of size at least 2.

Corollary 4.1.4. *Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$. Suppose that there is a decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$ with respect to which*

(i) $\mathcal{A}$ is reduced block upper triangular, and

(ii) there is an index $k \in \{1, 2, \ldots, m\}$ such that $\dim \mathcal{V}_k \geq 2$.

If $\mathcal{A}$ is projection compressible, then k is unique. When this is the case, if Q_1, Q_2, and Q_3 denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively, then

$$Q_1 \mathcal{A} Q_1 = \mathbb{C} Q_1, \quad Q_2 \mathcal{A} Q_2 = Q_2 \mathbb{M}_n Q_2, \quad \textit{and} \quad Q_3 \mathcal{A} Q_3 = \mathbb{C} Q_3.$$

Proof. Assume that $\mathcal{A}$ is projection compressible. Since $(Q_1+Q_2)\mathcal{A}(Q_1+Q_2) \neq \mathbb{C}(Q_1+Q_2)$ and $(Q_2+Q_3)\mathcal{A}(Q_2+Q_3) \neq \mathbb{C}(Q_2+Q_3)$, the result is immediate from Corollary 4.1.3. ■

The results presented above provide a strategy for classifying the unital subalgebras of $\mathbb{M}_n$ that exhibit the projection compression property. Indeed, we may use Corollaries 4.1.3 and 4.1.4 to partition the unital subalgebras of $\mathbb{M}_n$ into the following three distinct types determined by their reduced block upper triangular forms:

Type I: $\mathcal{A}$ has a reduced block upper triangular form with respect to an orthogonal decomposition of $\mathbb{C}^n$ such that there does *not* exist an index k as in Corollary 4.1.3;

Type II: $\mathcal{A}$ has a reduced block upper triangular form with respect to an orthogonal decomposition of $\mathbb{C}^n$ such that $BD(\mathcal{A})$ contains a block of size at least 2 (i.e., there is an integer k as in Corollary 4.1.4).

Type III: For each orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular, every block in $BD(\mathcal{A})$ is 1×1, and there is an integer k as in Corollary 4.1.3.

The unital projection compressible algebras of type I, type II, and type III will be classified up to transpose similarity in §4.2, §4.3, and §4.4, respectively.

§4.2 Algebras of Type I

In what follows, the term *type I* will be used to describe a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$, $n \geq 4$, that has a reduced block upper triangular form with respect to an orthogonal decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$ such that there does not exist an integer k as in Corollary 4.1.3. If $\mathcal{A}$ is such an algebra, then it must be the case that $\dim \mathcal{V}_i = 1$ for all i (i.e., $m = n$). For instance, the algebra from Example 4.1.1(i) is of type I if and only if $Q_2 = 0$; or $rank(Q_2) = 1$ and $Q_i = 0$ for some $i \in \{1, 3\}$.

The goal of this section is to determine which type I algebras possess the projection compression property. As we shall see, the type I algebras satisfying this condition are either unitizations of $\mathcal{LR}$-algebras, or unitarily equivalent to the type I algebra from Example 4.1.1(i). In order to demonstrate this systematically, it will be useful to keep a record of the orthogonal decompositions of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ satisfies the definition of type I.

Definition 4.2.1. If $\mathcal{A}$ is an algebra of type I, let $\mathcal{F}_I = \mathcal{F}_I(\mathcal{A})$ denote the set of pairs $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$, where

(i) $\bigoplus_{i=1}^n \mathcal{V}_i$ is an orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular, and

(ii) d is an integer in $\{1, 2, \ldots, n\}$ such that if $Q_{1\Omega}$ denotes the orthogonal projection onto $\bigoplus_{i=1}^d \mathcal{V}_i$, and $Q_{2\Omega}$ denotes its complement $I - Q_{1\Omega}$, then

$$Q_{1\Omega}\mathcal{A}Q_{1\Omega} = \mathbb{C}Q_{1\Omega} \quad \text{and} \quad Q_{2\Omega}\mathcal{A}Q_{2\Omega} = \mathbb{C}Q_{2\Omega}.$$

As an example, let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis for $\mathbb{C}^4$ and consider the algebra $\mathcal{A} = \text{span}\{e_1 \otimes e_4^*, I\}$. It is not difficult to see that $\mathcal{A}$ has a reduced block upper triangular form with respect to the decomposition $\mathbb{C}^4 = \bigoplus_{i=1}^4 \mathcal{V}_i$, where $\mathcal{V}_i = \text{span}\{e_i\}$. Moreover, with respect to this decomposition there does not exist an integer k as in Corollary 4.1.3. Consequently, $\mathcal{A}$ is of type I. One may note that $(d, \bigoplus_{i=1}^4 \mathcal{V}_i)$ belongs to $\mathcal{F}_I(\mathcal{A})$ for each $d \in \{1, 2, 3\}$.

Notation. If $\mathcal{A}$ is a type I algebra and $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ is a pair in $\mathcal{F}_I(\mathcal{A})$, the notation $n_{1\Omega} = d$ and $n_{2\Omega} = n - d$ will be used to refer to the ranks of $Q_{1\Omega}$ and $Q_{2\Omega}$, respectively.

Suppose that $\mathcal{A}$ is a projection compressible algebra of type I and Ω is a pair in $\mathcal{F}_I(\mathcal{A})$. In the language of §2.3, each corner $Q_{i\Omega}\mathcal{A}Q_{i\Omega} = \mathbb{C}Q_{i\Omega}$ is a diagonal algebra comprised of mutually linked 1×1 blocks. Note that the blocks in $Q_{1\Omega}\mathcal{A}Q_{1\Omega}$ may or may not be linked to those in $Q_{2\Omega}\mathcal{A}Q_{2\Omega}$. If these blocks are linked, we will say that the projections $Q_{1\Omega}$ and $Q_{2\Omega}$ are *linked*. Otherwise, we will say that $Q_{1\Omega}$ and $Q_{2\Omega}$ are *unlinked*. Note that the projections $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked for some pair in $\Omega \in \mathcal{F}_I(\mathcal{A})$ if and only if they are linked for every pair in $\mathcal{F}_I(\mathcal{A})$.

It will be important to distinguish between the type I algebras whose projections are linked and those whose projections are unlinked. The projection compressible type I algebras with unlinked projections will be classified in §4.2.1, while those with linked projections will be classified in §4.2.2. Before our analysis splits, however, let us examine one extreme case that will be relevant to the classification in either setting.

Observe that if $\mathcal{A}$ is an algebra of type I and $\mathcal{F}_I(\mathcal{A})$ contains a pair $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ with $d = n$, then $\mathcal{A} = \mathbb{C}I$, and hence $\mathcal{A}$ is idempotent compressible. If instead $d = 1$ or $d = n-1$, then Proposition 4.2.2 indicates that $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra.

Proposition 4.2.2. *Let $\mathcal{A}$ be a type I subalgebra of $\mathbb{M}_n$. If there is a pair $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ in $\mathcal{F}_I(\mathcal{A})$ with $d = 1$ or $d = n-1$, then $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, and hence $\mathcal{A}$ is idempotent compressible.*

Proof. Assume that $\mathcal{F}_I(\mathcal{A})$ contains a pair $\Omega = (n-1, \bigoplus_{i=1}^n \mathcal{V}_i)$. By Theorem 2.3.9, there exists an invertible upper triangular matrix S such that $\mathcal{A}_0 \coloneqq S^{-1}\mathcal{A}S$ is unhinged with respect to the decomposition $\mathbb{C}^n = \bigoplus_{i=1}^n \mathcal{V}_i$. Thus, since the class of $\mathcal{LR}$-algebras is invariant under similarity, it suffices to prove that $\mathcal{A}_0$ is the unitization of an $\mathcal{LR}$-algebra.

Note that by Theorem 2.1.12, there is a subprojection $Q_1' \leq Q_{1\Omega}$ such that

$$Q_{1\Omega}\mathcal{A}_0 Q_{2\Omega} = Q_{1\Omega} Rad(\mathcal{A}_0) Q_{2\Omega} = Q_1' \mathbb{M}_n Q_{2\Omega}.$$

Thus, either $\mathcal{V}_n$ is linked to the other $\mathcal{V}_i$'s, in which case $\mathcal{A}_0 = Q_1'\mathbb{M}_n Q_{2\Omega} + \mathbb{C}I$; or $\mathcal{V}_n$ is not linked to the other $\mathcal{V}_i$'s, in which case $\mathcal{A}_0 = (Q_1' + Q_{2\Omega})\mathbb{M}_n Q_{2\Omega} + \mathbb{C}I$. In either scenario, $\mathcal{A}_0$ is the unitization of an $\mathcal{LR}$-algebra.

Suppose instead that $\mathcal{F}_I(\mathcal{A})$ contains a pair whose first entry is 1. It follows that $\mathcal{F}_I(\mathcal{A}^{aT})$ contains a pair whose first entry is $n-1$. The above analysis then shows that $\mathcal{A}^{aT}$ is the unitization of an $\mathcal{LR}$-algebra, and thus so too is $\mathcal{A}$. ■

§4.2.1 Type I Algebras with Unlinked Projections

In this section we consider the type I algebras $\mathcal{A}$ for which the pairs $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ in $\mathcal{F}_I(\mathcal{A})$ are such that $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked. In light of Proposition 4.2.2 and its preceding remarks, we may assume that $1 < d < n-1$ for all pairs Ω. Thus, if Ω is any such pair, then $\min(d, n-d) \geq 2$. That is, the corresponding projections $Q_{1\Omega}$ and $Q_{2\Omega}$ have ranks $n_{1\Omega} \geq 2$ and $n_{2\Omega} \geq 2$, respectively.

It will be shown in Theorem 4.2.11 that every projection compressible type I algebra satisfying the above assumptions is unitarily equivalent to the type I algebra from Example 4.1.1(i). The majority of the work leading to this classification, however, occurs in Lemma 4.2.8. The proof of Lemma 4.2.8 itself relies on several intermediate results concerning the structure of the radical of a projection compressible type I algebra.

It should be noted that while Lemmas 4.2.3, 4.2.4, and 4.2.5 are presented here in the context of type I algebras with unlinked projections, these results are also applicable to type I algebras whose projections are linked.

Lemma 4.2.3. *Let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_n$, and suppose that $\Omega = (d, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ is a pair in $\mathcal{F}_I(\mathcal{A})$ with $1 < d < n-1$. Suppose further that there are orthonormal bases $\left\{e_i^{(1)}\right\}_{i=1}^{n_{1\Omega}}$ for $\operatorname{ran}(Q_{1\Omega})$ and $\left\{e_i^{(2)}\right\}_{i=1}^{n_{2\Omega}}$ for $\operatorname{ran}(Q_{2\Omega})$, as well as indices i_0 and j_0 such that*

$$\langle Re_{j_0}^{(2)}, e_{i_0}^{(1)} \rangle = 0 \text{ for all } R \in Rad(\mathcal{A}).$$

Then $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, and either $\langle Re_{j_0}^{(2)}, e_k^{(1)} \rangle = 0$ for all $k \in \{1, 2, \ldots, n_{1\Omega}\}$, or $\langle Re_k^{(2)}, e_{i_0}^{(1)} \rangle = 0$ for all $k \in \{1, 2, \ldots, n_{2\Omega}\}$.

Proof. Suppose to the contrary that $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked. By considering a suitable principal compression of $\mathcal{A}$ to a subalgebra of $\mathbb{M}_4$, we may assume without loss of generality that $d = n_{1\Omega} = n_{2\Omega} = 2$. Furthermore, we may reorder the bases if necessary to assume that $\langle Re_1^{(2)}, e_2^{(1)} \rangle = 0$ for all $R \in Rad(\mathcal{A})$.

Since $\mathcal{A}$ is similar to $BD(\mathcal{A}) \dotplus Rad(\mathcal{A})$ via an upper triangular similarity, there is a fixed matrix $M \in Q_{1\Omega}\mathcal{A}Q_{2\Omega}$ such that with respect to the basis $\left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)}\right\}$, every A in $\mathcal{A}$ has the form

$$A = \left[\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ & \alpha & 0 & 0 \\ \hline & & \beta & 0 \\ & & & \beta \end{array}\right] + (\beta - \alpha)M + R$$

for some $\alpha, \beta \in \mathbb{C}$ and $R \in Rad(\mathcal{A})$.

For each $i, j \in \{1, 2\}$ define $m_{ij} = \langle Me_j^{(2)}, e_i^{(1)} \rangle$. Furthermore, for each $k \in \mathbb{R}$ let P_k denote the matrix

$$P_k := \begin{bmatrix} k^2+1 & 0 & 0 & 0 \\ 0 & k^2 & 0 & -k \\ 0 & 0 & k^2+1 & 0 \\ 0 & -k & 0 & 1 \end{bmatrix},$$

so that $\frac{1}{k^2+1}P_k$ is a projection in $\mathbb{M}_4$. By direct computation, one may verify that every element $B = (b_{ij})$ in $P_k\mathcal{A}P_k$ satisfies the equation

$$(k^2+1)b_{23} - m_{21}k^2(b_{33} - b_{11}) = 0.$$

If, however, A is as above with $\alpha = 0$, $\beta = 1$, and $R = 0$, then for $(P_kAP_k)^2 = (c_{ij})$, we have

$$(k^2+1)c_{23} - m_{21}k^2(c_{33} - c_{11}) = m_{21}k^2(k^2+1)^3(1-km_{22}).$$

The fact that $\mathcal{A}$ is projection compressible implies that $(P_kAP_k)^2$ belongs to $P_k\mathcal{A}P_k$, and hence the right-hand side of the above expression must be 0 for all k. We therefore deduce that $m_{21} = \langle Me_1^{(2)}, e_2^{(1)}\rangle = 0$.

It now follows that $\langle Ae_1^{(2)}, e_2^{(1)}\rangle = 0$ for all $A \in \mathcal{A}$. So with respect to the basis $\left\{e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}\right\}$ for $\mathbb{C}^4$, every $A \in \mathcal{A}$ may be expressed as

$$A = \left[\begin{array}{cc|cc} \alpha & (\beta-\alpha)m_{11} + r_{11} & 0 & (\beta-\alpha)m_{12} + r_{12} \\ & \beta & 0 & 0 \\ \hline & & \alpha & (\beta-\alpha)m_{22} + r_{22} \\ & & & \beta \end{array}\right]$$

for some α, β, and r_{ij} in $\mathbb{C}$. Since α and β may be chosen arbitrarily, this contradicts Theorem 4.1.2. Thus, $Q_{1\Omega}$ and $Q_{2\Omega}$ must be linked.

For the final claim, first note that $BD(\mathcal{A}) = \mathbb{C}I$ as $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked. By the remarks following Theorem 2.3.9, we have that $\mathcal{A} = \mathbb{C}I \dotplus Rad(\mathcal{A})$, and hence

$$\langle Ae_{j_0}^{(2)}, e_{i_0}^{(1)}\rangle = 0 \text{ for all } A \in \mathcal{A}.$$

Suppose for the sake of contradiction that there exist indices $k_1 \in \{1, 2, \ldots, n_{1\Omega}\} \setminus \{i_0\}$ and $k_2 \in \{1, 2, \ldots, n_{2\Omega}\} \setminus \{j_0\}$ such that for some operators $A_1, A_2 \in \mathcal{A}$, $\langle A_1e_{j_0}^{(2)}, e_{k_1}^{(1)}\rangle \neq 0$ and $\langle A_2e_{k_2}^{(2)}, e_{i_0}^{(1)}\rangle \neq 0$. Let P_1 and P_2 denote the orthogonal projections onto $\text{span}\{e_{k_1}^{(1)}, e_{j_0}^{(2)}\}$ and $\text{span}\{e_{i_0}^{(1)}, e_{k_2}^{(2)}\}$, respectively. It is easy to see that $P_1\mathcal{A}P_1 \neq \mathbb{C}P_1$, $P_2\mathcal{A}P_2 \neq \mathbb{C}P_2$, and $P_2\mathcal{A}P_1 = \{0\}$. Thus, Theorem 4.1.2 indicates that $\mathcal{A}$ is not projection compressible—a contradiction. ■

Lemma 4.2.4. *Let $n \geq 4$ be an even integer, and let $\mathcal{A}$ be a projection compressible subalgebra of $\mathbb{M}_n$. Let Q_1 be a projection in $\mathbb{M}_n$ of rank $n/2$ and define $Q_2 := I - Q_1$. If $E \in \mathbb{M}_n$ is a partial isometry satisfying $E^*E = Q_1$ and $EE^* = Q_2$, then the linear space*

$$\mathcal{A}_0 := \{Q_1AQ_1 + E^*AQ_1 + Q_1AE + E^*AE : A \in \mathcal{A}\}$$

is an algebra.

Proof. The assumptions on E imply that the operator $P := \frac{1}{2}(I+E+E^*)$ is a projection in $\mathbb{M}_n$, and hence $P\mathcal{A}P$ is an algebra. One may verify that with respect to the decomposition $\mathbb{C}^n = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2)$, we have

$$P\mathcal{A}P = \left\{ \begin{bmatrix} X & X \\ X & X \end{bmatrix} : X \in \mathcal{A}_0 \right\}.$$

It follows that for any X and Y in $\mathcal{A}_0$,

$$\begin{bmatrix} X & X \\ X & X \end{bmatrix} \begin{bmatrix} Y & Y \\ Y & Y \end{bmatrix} = 2 \begin{bmatrix} XY & XY \\ XY & XY \end{bmatrix} \in P\mathcal{A}P,$$

and hence XY belongs to $\mathcal{A}_0$ as well. Thus, $\mathcal{A}_0$ is an algebra. ∎

Lemma 4.2.5. *Let $\mathcal{A}$ be a type I subalgebra of $\mathbb{M}_4$. If $Rad(\mathcal{A})$ is 3-dimensional and $\mathcal{F}_I(\mathcal{A})$ contains a pair $\Omega = (d, \bigoplus_{i=1}^4 \mathcal{V}_i)$ with $d = 2$, then $\mathcal{A}$ is not projection compressible.*

Proof. Suppose that $\dim Rad(\mathcal{A}) = 3$ and Ω is a pair in $\mathcal{F}_I(\mathcal{A})$ as described above. Write $\mathcal{A} = \mathcal{S} \dotplus Rad(\mathcal{A})$, where $\mathcal{S}$ is similar to $BD(\mathcal{A})$ via a block upper triangular similarity. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, then $\mathcal{A} = \{\alpha I : \alpha \in \mathbb{C}\} \dotplus Rad(\mathcal{A})$. If instead $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, there is a matrix $M \in Q_{1\Omega}\mathbb{M}_4 Q_{2\Omega}$ such that

$$\mathcal{A} = \{\alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha)M : \alpha, \beta \in \mathbb{C}\} \dotplus Rad(\mathcal{A}).$$

Note that the only distinctions between the linked and unlinked settings are the presence of the matrix M and the freedom to choose α and β independently. In the arguments that follow, we treat the entries of M as arbitrary constants (possibly zero), and make no attempt to choose independent values for α and β. Thus, these arguments are applicable to both cases.

For each $i \in \{1, 2\}$, let $\left\{e_1^{(i)}, e_2^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$. Since $Rad(\mathcal{A})$ is a 3-dimensional subspace of $Q_{1\Omega}\mathbb{M}_4 Q_{2\Omega}$, there is a non-zero matrix $\Gamma \in Q_{1\Omega}\mathbb{M}_4 Q_{2\Omega}$ such that $\operatorname{Tr}(\Gamma^* R) = 0$ for all R in $Rad(\mathcal{A})$. By reordering the bases for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$ if necessary, we may assume that $\langle \Gamma e_1^{(2)}, e_1^{(1)} \rangle$ is non-zero. From this it follows that there exist $\gamma_{12}, \gamma_{21}, \gamma_{22} \in \mathbb{C}$ such that

$$Rad(\mathcal{A}) = \left\{ \left[\begin{array}{cc|cc} 0 & 0 & \gamma_{12}r_{12} + \gamma_{21}r_{21} + \gamma_{22}r_{22} & r_{12} \\ 0 & 0 & r_{21} & r_{22} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] : r_{12}, r_{21}, r_{22} \in \mathbb{C} \right\}$$

with respect to the basis $\left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)}\right\}$ for $\mathbb{C}^4$.

To see that $\mathcal{A}$ is not projection compressible, consider the matrix

$$P := \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

and note that $\frac{1}{2}P$ is a projection in $\mathbb{M}_4$. One may verify that every operator $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation

$$\begin{aligned} b_{13} - 4\gamma_{22}b_{24} - 2\gamma_{21}b_{23} - 2\gamma_{12}b_{14} - (\gamma_{12}m_{12} + \gamma_{21}m_{21} - \gamma_{22}(1 - m_{22}) - m_{11})b_{11} \\ + (\gamma_{12}m_{12} + \gamma_{21}m_{21} + \gamma_{22}(1 + m_{22}) - m_{11})b_{33} = 0, \end{aligned}$$

where for each $i, j \in \{1, 2\}$, we define $m_{ij} = \langle Me_j^{(2)}, e_i^{(1)} \rangle$. If, however, A is the element of $\mathcal{A}$ obtained by setting $\alpha = \beta = r_{12} = r_{21} = 1$ and $r_{22} = 0$, then $B := (PAP)^2$ produces a value of 8 on the left-hand side of the above equation. Consequently, $(PAP)^2$ does not belong to $P\mathcal{A}P$, so $P\mathcal{A}P$ is not an algebra. ■

The following classical theorem from linear algebra will be applied in the proof of Lemma 4.2.8 and used extensively throughout §4.3. For reference, see [12, Theorem 2.6.3].

Theorem 4.2.6 (Singular Value Decomposition)**.** *Let n and p be positive integers, and let A be a complex $n \times p$ matrix.*

(i) If $n \le p$, then there are unitaries $U \in \mathbb{M}_n$ and $V \in \mathbb{M}_p$, and a positive semi-definite diagonal matrix $D \in \mathbb{M}_n$ such that

$$U^*AV = \begin{bmatrix} D & 0 \end{bmatrix}.$$

(ii) If $n \ge p$, then there are unitaries $U \in \mathbb{M}_n$ and $V \in \mathbb{M}_p$, and a positive semi-definite diagonal matrix $D \in \mathbb{M}_p$ such that

$$U^*AV = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

The principal application of Theorem 4.2.6 will be in simplifying the structure of the semi-simple part of an algebra $\mathcal{A}$ in reduced block upper triangular form. Indeed, suppose that $\mathcal{A} = \mathcal{S} \dotplus Rad(\mathcal{A})$ is a type I subalgebra of $\mathbb{M}_n$ where $\mathcal{S}$ is semi-simple. Let $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ be a pair in $\mathcal{F}_I(\mathcal{A})$, and assume that the projections $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked. For each $i \in \{1, 2\}$, let $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$. As a consequence of Theorem 2.3.9, there is a matrix $M \in Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ such that

$$\mathcal{S} = \{\alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha)M : \alpha, \beta \in \mathbb{C}\}.$$

It then follows from Theorem 4.2.6 that there is a unitary $U \in \mathbb{M}_n$ such that $Q_{1\Omega}UQ_{2\Omega} = 0$, $Q_{2\Omega}UQ_{1\Omega} = 0$, and $\langle U^*MUe_j^{(2)}, e_i^{(1)}\rangle = 0$ whenever $i \neq j$.

Finally, the proof of Lemma 4.2.8 will require the following result of Azoff concerning the minimum dimension of a transitive space of linear operators.

Theorem 4.2.7. [2, Proposition 4.7] *If $\mathcal{L}$ is a transitive space of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, then the dimension of $\mathcal{L}$ is at least $m + n - 1$.*

We are now prepared to state and prove Lemma 4.2.8. This result indicates that under certain restrictive assumptions, a projection compressible type I algebra with unlinked projections is unitarily equivalent to the type I algebra from Example 4.1.1(i). Loosening these assumptions will require a refinement of Theorem 4.2.7 to specific classes of transitive spaces of operators.

Lemma 4.2.8. *Let $n \geq 4$ be an even integer, and let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_n$. Suppose that $\mathcal{F}_I(\mathcal{A})$ contains a pair $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ with $d = n/2$. If the projections $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, then $\mathcal{A}$ is unitarily equivalent to*

$$\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{2\Omega} + Q_{1\Omega}\mathbb{M}_n Q_{2\Omega},$$

the type I algebra from Example 4.1.1(i). Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. For each $i \in \{1, 2\}$, let $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_d^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$. As a consequence of Theorem 2.3.9, there is a matrix M in $Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ such that

$$\mathcal{A} = \{\alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha)M : \alpha, \beta \in \mathbb{C}\} \dotplus Rad(\mathcal{A}).$$

In fact, one may assume by Theorem 4.2.6 and its subsequent remarks that there are constants $m_{ij} \geq 0$ such that $\langle Me_j^{(2)}, e_i^{(1)}\rangle = \delta_{ij} m_{ij}$ for all i and j.

Let $E \in \mathbb{M}_n$ denote the partial isometry satisfying $Ee_i^{(1)} = e_i^{(2)}$ and $Ee_i^{(2)} = 0$ for all $i \in \{1, 2, \dots, d\}$. Since $\mathcal{A}$ is projection compressible, Lemma 4.2.4 implies that

$$\mathcal{A}_0 \coloneqq \{(\alpha + \beta)Q_{1\Omega} + (\beta - \alpha)ME + RE : \alpha, \beta \in \mathbb{C}, R \in Rad(\mathcal{A})\}$$

is a subalgebra of $Q_{1\Omega}\mathbb{M}_nQ_{1\Omega}$. If this subalgebra were proper, then by Burnside's theorem, we may change the orthonormal basis for $\operatorname{ran}(Q_{1\Omega})$ if necessary to assume that $\langle Ae_1^{(1)}, e_d^{(1)}\rangle = 0$ for all $A \in \mathcal{A}_0$. In this case, one may change the orthonormal basis for $\operatorname{ran}(Q_{2\Omega})$ accordingly and assume that $\langle Re_1^{(2)}, e_d^{(1)}\rangle = 0$ for all $R \in Rad(\mathcal{A})$. Since $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, an application of Lemma 4.2.3 demonstrates that $\mathcal{A}$ lack the projection compression property—a contradiction.

We may therefore assume that $\mathcal{A}_0$ is equal to $Q_{1\Omega}\mathbb{M}_nQ_{1\Omega}$. This means that $Rad(\mathcal{A})E$ can be enlarged to a d^2-dimensional space by adding

$$\{\alpha(Q_{1\Omega} - ME) + \beta(Q_{1\Omega} + ME) : \alpha, \beta \in \mathbb{C}\},$$

the linear span of two diagonal matrices in $Q_{1\Omega}\mathbb{M}_nQ_{1\Omega}$. It follows that

$$\dim Rad(\mathcal{A})E = \dim Rad(\mathcal{A}) \geq d^2 - 2,$$

and any entries in $Rad(\mathcal{A})E$ that depend linearly on other entries must be located on the diagonal. Our goal is to show that $\dim Rad(\mathcal{A}) = d^2$, and hence $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_nQ_{2\Omega}$.

Let us begin by addressing the case in which $n = 4$, and hence $d = 2$. If $\dim Rad(\mathcal{A})$ is strictly less than $d^2 = 4$, then $Rad(\mathcal{A})$ is 2- or 3-dimensional by the analysis above. If $\dim Rad(\mathcal{A}) = 2$, then by Theorem 4.2.7, $Rad(\mathcal{A})$ is not transitive as a space of linear maps from $\operatorname{ran}(Q_{2\Omega})$ to $\operatorname{ran}(Q_{1\Omega})$. In this case there exist unit vectors $v \in \operatorname{ran}(Q_{1\Omega})$ and $w \in \operatorname{ran}(Q_{2\Omega})$ such that $Rw \in \mathbb{C}v$ for every $R \in Rad(\mathcal{A})$. Choose unit vectors $v' \in \operatorname{ran}(Q_{1\Omega}) \cap (\mathbb{C}v)^\perp$ and $w' \in \operatorname{ran}(Q_{2\Omega}) \cap (\mathbb{C}w)^\perp$, and replace the orthonormal bases for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$ with $\{v, v'\}$ and $\{w, w'\}$, respectively. Since

$$\langle Rw, v'\rangle = \langle \lambda v, v'\rangle = 0 \quad \text{for all } R \in Rad(\mathcal{A}),$$

$\mathcal{A}$ lacks the projection compression property by Lemma 4.2.3—a contradiction. Using Lemma 4.2.5, one may also obtain a contradiction in the case that $\dim Rad(\mathcal{A}) = 3$.

Assume now that $n > 4$. By the above analysis, there are at most two entries from $Rad(\mathcal{A})E$ which cannot be chosen arbitrarily, and these entries necessarily occur on the diagonal. By reordering the bases for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$, we may relocate the linearly dependent entries to the $(1, n-1)$ and $(2, n)$ positions of $Rad(\mathcal{A})$, respectively. That is, we may assume that with respect to the decomposition

$$\mathbb{C}^n = \vee \left\{ e_1^{(1)}, e_2^{(1)}, \ldots, e_{d-1}^{(1)} \right\} \oplus \vee \left\{ e_d^{(1)}, e_1^{(2)} \right\} \oplus \vee \left\{ e_2^{(2)}, e_3^{(2)}, \ldots, e_d^{(2)} \right\},$$

each $A \in \mathcal{A}$ can be represented by a matrix of the form

$$A = \left[\begin{array}{ccccc|cc|cccccc} \alpha & & & & & 0 & t_{11} & t_{12} & \cdots & t_{1,d-2} & \gamma_1 & t_{1d} \\ & \alpha & & & & 0 & t_{21} & t_{22} & \cdots & t_{2,d-2} & t_{2,d-1} & \gamma_2 \\ & & \alpha & & & 0 & t_{31} & t_{32} & \cdots & t_{3,d-2} & t_{3,d-1} & t_{3d} \\ & & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & \alpha & 0 & t_{d-1,1} & t_{d-1,2} & \cdots & t_{d-1,d-2} & t_{d-1,d-1} & t_{d-1,d} \\ \hline & & & & & \alpha & t_{d1} & t_{d2} & \cdots & t_{d,d-2} & t_{d,d-1} & t_{dd} \\ & & & & & & \beta & 0 & \cdots & 0 & 0 & 0 \\ \hline & & & & & & & \beta & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & \beta & & \\ & & & & & & & & & & \beta & \\ & & & & & & & & & & & \beta \end{array} \right],$$

where α, β, and t_{ij} can be chosen arbitrarily, and γ_1 and γ_2 may depend linearly on these entries. We will demonstrate that, in fact, γ_1 and γ_2 can be chosen arbitrarily and independently of the remaining terms.

Consider the matrix

$$P = \left[\begin{array}{ccccc|cc|ccccc} 2 & & & & & & & & & & & \\ & 2 & & & & & & & & & & \\ & & 2 & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & & 2 & & & & & & & \\ \hline & & & & & 1 & 1 & & & & & \\ & & & & & 1 & 1 & & & & & \\ \hline & & & & & & & 2 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 2 & & \\ & & & & & & & & & & 2 & \\ & & & & & & & & & & & 2 \end{array} \right]$$

written with respect to the decomposition above. Observe that $\frac{1}{2}P$ is a projection in $\mathbb{M}_n$. Direct computations show that with A as above, PAP is given by

$$\left[\begin{array}{ccccc|cc|ccccc}
4\alpha & & & & & 2t_{11} & 2t_{11} & 4t_{12} & \cdots & 4t_{1,d-2} & 4\gamma_1 & 4t_{1d} \\
 & 4\alpha & & & & 2t_{21} & 2t_{21} & 4t_{22} & \cdots & 4t_{2,d-2} & 4t_{2,d-1} & 4\gamma_2 \\
 & & 4\alpha & & & 2t_{31} & 2t_{31} & 4t_{32} & \cdots & 4t_{3,d-2} & 4t_{3,d-1} & 4t_{3d} \\
 & & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & & & & 4\alpha & 2t_{d-1,1} & 2t_{d-1,1} & 4t_{d-1,2} & \cdots & 4t_{d-1,d-2} & 4t_{d-1,d-1} & 4t_{d-1,d} \\
\hline
 & & & & & \alpha+\beta+t_{d1} & \alpha+\beta+t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
 & & & & & \alpha+\beta+t_{d1} & \alpha+\beta+t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
\hline
 & & & & & & & 4\beta & & & & \\
 & & & & & & & & \ddots & & & \\
 & & & & & & & & & 4\beta & & \\
 & & & & & & & & & & 4\beta & \\
 & & & & & & & & & & & 4\beta
\end{array}\right].$$

Hence, it suffices to prove that $e_1^{(1)} \otimes e_{d-1}^{(2)*}$ and $e_2^{(1)} \otimes e_d^{(2)*}$ belong to $P\mathcal{A}P$.

To see that this is the case, let A be as above with $t_{11} = t_{d,d-1} = 1$ and $\alpha = \beta = t_{ij} = 0$ for all other indices i and j. It is straightforward to verify that

$$(PAP)^2 = 8e_1^{(1)} \otimes e_{d-1}^{(2)^*}.$$

Consequently, $e_1^{(1)} \otimes e_{d-1}^{(2)*}$ belongs to $P\mathcal{A}P$, so γ_1 can indeed be chosen arbitrarily. By reordering the basis to interchange the positions of γ_1 and γ_2, one may repeat this process to show that γ_2 may be chosen arbitrarily as well. ■

Observe that the success of Lemma 4.2.8 relied heavily on the existence of the pair $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ with $d = n/2$. Indeed, without such a pair, one would be unable to directly apply Lemma 4.2.4 or Burnside's Theorem to infer that $\dim Rad(\mathcal{A}) \geq d^2 - 2$.

Our final goal of this section is to generalize Lemma 4.2.8 to type I algebras $\mathcal{A}$ that may not admit a pair Ω as describe above. We will accomplish this goal by applying Lemma 4.2.8 to study the structure of the radical of certain principal compressions of $\mathcal{A}$. It will then follow from [8, Theorem 1.2], an extension of Theorem 4.2.7, that $\mathcal{A}$ is unitarily equivalent to the type I algebra from Example 4.1.1(i). In order to introduce this extension, we first present the following definition.

Definition 4.2.9. Let $\mathcal{L}$ be a vector space of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, and let k be a positive integer. We say that $\mathcal{L}$ is *k-transitive* if for every choice of k

linearly independent vectors $x_1, x_2, \ldots, x_k$ in $\mathbb{C}^n$, and every choice of k arbitrary vectors $y_1, y_2, \ldots, y_k$ in $\mathbb{C}^m$, there is an element $A \in \mathcal{L}$ such that $Ax_i = y_i$ for all $i \in \{1, 2, \ldots, k\}$.

Theorem 4.2.10. [8, Theorem 1.2] *If $\mathcal{L}$ is a k-transitive space of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, then the dimension of $\mathcal{L}$ is at least $k(m+n-k)$.*

Note that a space of linear operators is transitive if and only if it is 1-transitive, and that the bounds from Theorems 4.2.7 and 4.2.10 coincide when $k = 1$.

We are now prepared to prove the classification in the general case of type I algebras with unlinked projections.

Theorem 4.2.11. *Let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_n$, and let $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ be a pair in $\mathcal{F}_I(\mathcal{A})$ with $1 < d < n-1$. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, then $\mathcal{A}$ is unitarily equivalent to*

$$\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{2\Omega} + Q_{1\Omega}\mathbb{M}_n Q_{2\Omega},$$

the type I algebra from Example 4.1.1(i). Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $d \leq n-d$. That is, $n_{1\Omega} \leq n_{2\Omega}$. We will demonstrate that $Rad(\mathcal{A})$ has dimension $d(n-d)$, and hence must be equal to $Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$. Of course, it is clear that $\dim Rad(\mathcal{A}) \leq d(n-d)$.

Note that $Rad(\mathcal{A})$ is d-transitive as a space of linear maps from $\operatorname{ran}(Q_{2\Omega})$ to $\operatorname{ran}(Q_{1\Omega})$. Indeed, let S be a linearly independent d-element subset of $\operatorname{ran}(Q_{2\Omega})$, and let Q_S denote the orthogonal projection onto the span of S. Since $Q_{1\Omega}$ and Q_S are both of rank d, Lemma 4.2.8 implies that the radical of $\mathcal{A}_0 := (Q_{1\Omega}+Q_S)\mathcal{A}(Q_{1\Omega}+Q_S)$ is equal to $Q_{1\Omega}\mathbb{M}_n Q_S$. As a result, the vectors in S can be mapped anywhere in $\operatorname{ran}(Q_{1\Omega})$ by elements of $Rad(\mathcal{A})$. We conclude that $Rad(\mathcal{A})$ is d-transitive.

The proof ends with an application of Theorem 4.2.10. Since $Rad(\mathcal{A})$ is a d-transitive subspace of $Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$, we have that $\dim Rad(\mathcal{A}) \geq d(d + (n-d) - d) = d(n-d)$. ∎

§4.2.2 Type I Algebras with Linked Projections

We now wish to describe the projection compressible type I algebras $\mathcal{A}$ for which the pairs $\Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i)$ in $\mathcal{F}_I(\mathcal{A})$ are such that $Q_{1\Omega}$ is linked to $Q_{2\Omega}$. An inductive argument in Theorem 4.2.13 will demonstrate that every such algebra is the unitization of an $\mathcal{LR}$-algebra. The base case of this argument will require the following lemma.

Lemma 4.2.12. *Let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_4$, and suppose that $\mathcal{F}_I(\mathcal{A})$ contains a pair $\Omega = (d, \bigoplus_{i=1}^{4} \mathcal{V}_i)$ with $d = 2$. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, then there are projections $Q_1' \leq Q_{1\Omega}$ and $Q_2' \leq Q_{2\Omega}$ such that $Rad(\mathcal{A}) = Q_1'\mathbb{M}_4 Q_2'$. In this case $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, so $\mathcal{A}$ is idempotent compressible.*

Proof. Let Ω be a pair in $\mathcal{F}_I(\mathcal{A})$ as above, and assume that $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked. By the observations following Theorem 2.3.9, $\mathcal{A} = \mathbb{C}I \dotplus Rad(\mathcal{A})$.

For each $i \in \{1, 2\}$, let $\left\{e_1^{(i)}, e_2^{(i)}\right\}$ be a fixed orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$. Furthermore, let $E \in \mathbb{M}_n$ denote the partial isometry satisfying $Ee_i^{(1)} = e_i^{(2)}$ and $Ee_i^{(2)} = 0$ for each $i \in \{1, 2\}$. By Lemma 4.2.4,

$$\mathcal{A}_0 \coloneqq \mathbb{C}Q_{1\Omega} + Rad(\mathcal{A})E$$

is a subalgebra of $Q_{1\Omega}\mathbb{M}_4 Q_{1\Omega}$. If this subalgebra $\mathcal{A}_0$ is proper, then by Burnside's Theorem, we may change the orthonormal basis for $\operatorname{ran}(Q_{1\Omega})$ if required and assume that $\langle Ae_1^{(1)}, e_2^{(1)}\rangle = 0$ for all $A \in \mathcal{A}_0$. In this case we may adjust the orthonormal basis for $\operatorname{ran}(Q_{2\Omega})$ accordingly and assume that $\langle Re_1^{(2)}, e_2^{(1)}\rangle = 0$ for all $R \in Rad(\mathcal{A})$. Thus, by Lemma 4.2.3, either $\langle Re_1^{(2)}, e_1^{(1)}\rangle = 0$ for all $R \in Rad(\mathcal{A})$, or $\langle Re_2^{(2)}, e_2^{(1)}\rangle = 0$ for all $R \in Rad(\mathcal{A})$. The fact that $Rad(\mathcal{A})$ has the required form now follows from Theorem 2.1.12.

Suppose instead that $\mathbb{C}Q_{1\Omega} + Rad(\mathcal{A})E$ is equal to $Q_{1\Omega}\mathbb{M}_4 Q_{1\Omega}$. It follows that $Rad(\mathcal{A})$ is at least 3-dimensional. If $\dim Rad(\mathcal{A}) = 3$, then $\mathcal{A}$ is of the form described in Lemma 4.2.5, and hence $\mathcal{A}$ is not projection compressible. We therefore have that $\dim Rad(\mathcal{A}) = 4$, so $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_4 Q_{2\Omega}$. ∎

Theorem 4.2.13. *Let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_n$, and let $\Omega = (d, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ be a pair in $\mathcal{F}_I(\mathcal{A})$. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, then there are projections $Q_1' \leq Q_{1\Omega}$ and $Q_2' \leq Q_{2\Omega}$ such that $Rad(\mathcal{A}) = Q_1'\mathbb{M}_n Q_2'$. Thus, $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, so $\mathcal{A}$ is idempotent compressible.*

Proof. We will proceed by induction on n. By definition of a type I algebra, our base case occurs when $n = 4$. That said, let $\mathcal{A}$ be a projection compressible type I subalgebra of $\mathbb{M}_4$, and suppose that $\Omega = (d, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ is a pair in $\mathcal{F}_I(\mathcal{A})$ with $Q_{1\Omega}$ linked to $Q_{2\Omega}$. If $d = 1$ or $d = 3$, then Proposition 4.2.2 guarantees that $Rad(\mathcal{A})$ admits the required form. If instead $d = 2$, then $\mathcal{A}$ and Ω are as in Lemma 4.2.12. Once again $Rad(\mathcal{A})$ is of the correct form.

Now fix an integer $N \geq 5$. Assume that for every positive integer $n < N$, if $\mathcal{A}$ is a projection compressible type I subalgebra of $\mathbb{M}_n$ and Ω is a pair in $\mathcal{F}_I(\mathcal{A})$ with $Q_{1\Omega}$ linked to $Q_{2\Omega}$, then $Rad(\mathcal{A}) = Q_1'\mathbb{M}_nQ_2'$ for some subprojections $Q_1' \leq Q_{1\Omega}$ and $Q_2' \leq Q_{2\Omega}$. We claim that this is also the case for every such subalgebra $\mathcal{A}$ of $\mathbb{M}_N$ and pair $\Omega \in \mathcal{F}_I(\mathcal{A})$. Indeed, fix a subalgebra $\mathcal{A}$ of $\mathbb{M}_N$ and pair $\Omega = (d, \bigoplus_{i=1}^N \mathcal{V}_i)$ in $\mathcal{F}_I(\mathcal{A})$ as in the statement of the theorem. If $d = 1$ or $d = N-1$, then Proposition 4.2.2 ensures that $Rad(\mathcal{A})$ is of the desired form. Thus, we will assume that $1 < d < N-1$. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we will also assume that $d \leq N - d$.

First consider the possibility that N is even and $d = N - d = N/2$. Fix orthonormal bases $\left\{e_1^{(1)}, e_2^{(1)}, \ldots, e_d^{(1)}\right\}$ and $\left\{e_1^{(2)}, e_2^{(2)}, \ldots, e_d^{(2)}\right\}$ for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$, respectively. Let $E \in \mathbb{M}_n$ denote the partial isometry satisfying $Ee_i^{(1)} = e_i^{(2)}$ and $Ee_i^{(2)} = 0$ for each $i \in \{1, 2, \ldots, d\}$. Arguing as in the proof of Lemma 4.2.8, either $\mathbb{C}Q_{1\Omega} + Rad(\mathcal{A})E$ is equal to $Q_{1\Omega}\mathbb{M}_NQ_{1\Omega}$, or Burnside's Theorem may be used to assume that

$$\langle Re_1^{(2)}, e_d^{(1)} \rangle = 0 \text{ for all } R \in Rad(\mathcal{A}).$$

If the latter holds, then by Lemma 4.2.3, $Rad(\mathcal{A})$ contains a permanent row or column of zeros. Consider the algebra $\mathcal{A}_0$ obtained by deleting this row and its corresponding column from $\mathcal{A}$. We have that $\mathcal{A}_0$ is a projection compressible type I subalgebra of $\mathbb{M}_{N-1}$, so $Rad(\mathcal{A}_0)$ admits the the required form by the inductive hypothesis. Upon reintroducing the removed row and column, one can see that $Rad(\mathcal{A})$ is also of the required form. We may therefore assume that $\mathbb{C}Q_{1\Omega} + Rad(\mathcal{A})E = Q_{1\Omega}\mathbb{M}_NQ_{1\Omega}$.

Since $Rad(\mathcal{A})E$ can be enlarged to a d^2-dimensional space by adding $\mathbb{C}Q_{1\Omega}$, it must be that $\dim Rad(\mathcal{A}) \geq d^2 - 1$. We claim that in fact, $\dim Rad(\mathcal{A}) = d^2$, and hence $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_nQ_{2\Omega}$. To see this is the case, reorder the bases for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$ if necessary to assume that with respect to the decomposition

$$\mathbb{C}^N = \vee\left\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{d-1}^{(1)}\right\} \oplus \vee\left\{e_d^{(1)}, e_1^{(2)}\right\} \oplus \vee\left\{e_2^{(2)}, e_3^{(2)}, \ldots, e_d^{(2)}\right\},$$

each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$
A = \left[\begin{array}{ccccc|cc|ccccc}
\alpha & & & & & 0 & t_{11} & t_{12} & \cdots & t_{1,d-2} & \gamma & t_{1d} \\
& \alpha & & & & 0 & t_{21} & t_{22} & \cdots & t_{2,d-2} & t_{2,d-1} & t_{2d} \\
& & \alpha & & & 0 & t_{31} & t_{32} & \cdots & t_{3,d-2} & t_{3,d-1} & t_{3d} \\
& & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
& & & & \alpha & 0 & t_{d-1,1} & t_{d-1,2} & \cdots & t_{d-1,d-2} & t_{d-1,d-1} & t_{d-1,d} \\
\hline
& & & & & \alpha & t_{d1} & t_{d2} & \cdots & t_{d,d-2} & t_{d,d-1} & t_{dd} \\
& & & & & & \alpha & 0 & \cdots & 0 & 0 & 0 \\
\hline
& & & & & & & \alpha & & & & \\
& & & & & & & & \ddots & & & \\
& & & & & & & & & \alpha & & \\
& & & & & & & & & & \alpha & \\
& & & & & & & & & & & \alpha
\end{array}\right].
$$

Here, α and t_{ij} are arbitrary values in $\mathbb{C}$, and γ may depend linearly on these entries.

It will be shown that γ is in fact, independent of the other terms. Indeed, let P denote the matrix from the proof of Lemma 4.2.8, so that $\frac{1}{2}P$ is a projection in $\mathbb{M}_N$. Proceed now as in the proof of that lemma by noting that with A as above, PAP is given by

$$
\left[\begin{array}{ccccc|cc|ccccc}
4\alpha & & & & & 2t_{11} & 2t_{11} & 4t_{12} & \cdots & 4t_{1,d-2} & 4\gamma & 4t_{1d} \\
& 4\alpha & & & & 2t_{21} & 2t_{21} & 4t_{22} & \cdots & 4t_{2,d-2} & 4t_{2,d-1} & 4t_{2d} \\
& & 4\alpha & & & 2t_{31} & 2t_{31} & 4t_{32} & \cdots & 4t_{3,d-2} & 4t_{3,d-1} & 4t_{3d} \\
& & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
& & & & 4\alpha & 2t_{d-1,1} & 2t_{d-1,1} & 4t_{d-1,2} & \cdots & 4t_{d-1,d-2} & 4t_{d-1,d-1} & 4t_{d-1,d} \\
\hline
& & & & & 2\alpha + t_{d1} & 2\alpha + t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
& & & & & 2\alpha + t_{d1} & 2\alpha + t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
\hline
& & & & & & & 4\alpha & & & & \\
& & & & & & & & \ddots & & & \\
& & & & & & & & & 4\alpha & & \\
& & & & & & & & & & 4\alpha & \\
& & & & & & & & & & & 4\alpha
\end{array}\right].
$$

It now suffices to prove that $e_1^{(1)} \otimes e_{d-1}^{(2)*}$ belongs to $P\mathcal{A}P$. But if A denotes the particular element of $\mathcal{A}$ obtained by taking $t_{11} = t_{d,d-1} = 1$ and $\alpha = t_{ij} = 0$ for all other indices i and j, then

$$(PAP)^2 = 8e_1^{(1)} \otimes e_{d-1}^{(2)*}.$$

Since $\mathcal{A}$ is projection compressible, this element belongs to $P\mathcal{A}P$. We conclude that $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_N Q_{2\Omega}$, and hence the proof of the $d = N - d$ case is complete.

Let us now turn to the case in which $d < N - d$. As above, let $\left\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{n_{1\Omega}}^{(1)}\right\}$ and $\left\{e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_{2\Omega}}^{(2)}\right\}$ be fixed orthonormal bases for $\operatorname{ran}(Q_{1\Omega})$ and $\operatorname{ran}(Q_{2\Omega})$, respectively. For each linearly independent d-element subset S of $\operatorname{ran}(Q_{2\Omega})$, let Q_S denote the orthogonal projection onto the span of S, and define $P_S \coloneqq Q_{1\Omega} + Q_S$. Let $\mathcal{A}_S$ denote the compression $P_S\mathcal{A}P_S$, which we regard as a subalgebra of $\mathbb{C}I \dotplus Q_{1\Omega}\mathbb{M}_{2d}Q_S$.

If each compression $\mathcal{A}_S$ is equal to $\mathbb{C}I \dotplus Q_{1\Omega}\mathbb{M}_{2d}Q_S$, then $Rad(\mathcal{A})$ is a d-transitive space of linear maps from $\operatorname{ran}(Q_{2\Omega})$ into $\operatorname{ran}(Q_{1\Omega})$. In this case we may apply Theorem 4.2.10 to conclude that $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_N Q_{2\Omega}$, as desired. Instead, suppose that one of the sets S is such that the radical of $\mathcal{A}_S$ is properly contained in $Q_{1\Omega}\mathbb{M}_{2d}Q_S$. For such an S, the inductive hypothesis gives rise to subprojections $Q_1' \leq Q_{1\Omega}$ and $Q_S' \leq Q_S$ such that

$$Rad(\mathcal{A}_S) = Q_1'\mathbb{M}_{2d}Q_S'.$$

At least one of these subprojections must be proper.

If $Q_S' \neq Q_S$ or $Q_1' = 0$, then there is an orthonormal basis for $\mathbb{C}^{2d}$ with respect to which $Rad(\mathcal{A}_S)$ has a permanent column of zeros. One may then extend this basis to an orthonormal basis for $\mathbb{C}^N$ with respect to which $Rad(\mathcal{A})$ also admits a permanent column of zeros. By deleting this column and its corresponding row from $\mathcal{A}$, we obtain a projection compressible type I subalgebra of $\mathbb{M}_{N-1}$. The inductive hypothesis then implies that the radical of this compression is of the desired form. Upon reintroducing the column and row deleted from $\mathcal{A}$, it is easy to see that $Rad(\mathcal{A})$ is of the desired form as well.

On the other hand, if $Q_S = Q_S'$ and Q_1' is a proper non-zero subprojection of $Q_{1\Omega}$, then it must be the case that $Rad(\mathcal{A}_S)$ has a permanent row of zeros, but not a permanent column of zeros. Thus, $Rad(\mathcal{A})$ has a permanent row of zeros by Lemma 4.2.3. By removing this row and its corresponding column from $\mathcal{A}$, we obtain a projection compressible type I subalgebra of $\mathbb{M}_{N-1}$. The radical of this algebra is of the correct form by the inductive hypothesis, and hence so too is $Rad(\mathcal{A})$. ■

§4.3 Algebras of Type II

The term *type II* will be used to describe a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$, $n \geq 4$, that has a reduced block upper triangular form with respect to an orthogonal decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$

of $\mathbb{C}^n$ such that $\dim \mathcal{V}_k \geq 2$ for some k. For example, the algebra from Example 4.1.1(i) is of type II if and only if $rank(Q_2) \geq 2$. Importantly, it follows from this definition that every type II algebra satisfies the assumptions of Corollary 4.1.4.

The purpose of this section is to classify the type II algebras that afford the projection compression property. It will be shown that every projection compressible algebra of type II is either the unitization of an $\mathcal{LR}$-algebra, or is unitarily equivalent to the type II algebra from Example 4.1.1(i).

As in the case of type I algebras, it will be helpful to keep a record of all orthogonal decompositions of $\mathbb{C}^n$ that satisfy the conditions of Corollary 4.1.4 for a given type II algebra $\mathcal{A}$. Thus, we make the following definition.

Definition 4.3.1. If $\mathcal{A}$ is an algebra of type II, let $\mathcal{F}_{II} = \mathcal{F}_{II}(\mathcal{A})$ denote the set of triples $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ that satisfy the following conditions:

(i) $\bigoplus_{i=1}^m \mathcal{V}_i$ is an orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular;

(ii) d and k are integers such that $d \geq 2$, $k \in \{1, 2, \ldots, m\}$, and $\dim \mathcal{V}_k = d$.

Notation. If $\mathcal{A}$ is an algebra of type II and Ω is a triple in $\mathcal{F}_{II}(\mathcal{A})$, let $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively. Furthermore, for each $i \in \{1, 2, 3\}$, let $n_{i\Omega}$ denote the rank of $Q_{i\Omega}$.

Observe that if $\mathcal{A}$ is a projection compressible type II subalgebra of $\mathbb{M}_n$ and $\mathcal{F}_{II}(\mathcal{A})$ contains a triple $\Omega = (d, \bigoplus_{i=1}^m \mathcal{V}_i)$, then Corollary 4.1.4 implies that $Q_{2\Omega}\mathcal{A}Q_{2\Omega} = Q_{2\Omega}\mathbb{M}_n Q_{2\Omega}$ and $Q_{i\Omega}\mathcal{A}Q_{i\Omega} = \mathbb{C}Q_{i\Omega}$ for each $i \in \{1, 3\}$. In this case, $n_{1\Omega} = k - 1$, $n_{2\Omega} = d$, and $n_{3\Omega} = n - d - k + 1$.

We will begin by considering the extreme case in which a type II algebra $\mathcal{A}$ admits a triple $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ with $k = 1$ or $k = m$. The projection compressible algebras of this form can be easily identified using Theorem 2.1.12.

Proposition 4.3.2. *Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_n$. If there is a triple $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ in $\mathcal{F}_{II}(\mathcal{A})$ with $k = 1$ or $k = m$, then $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra. Consequently, $\mathcal{A}$ is idempotent compressible.*

Proof. Let $\Omega \in \mathcal{F}_{II}(\mathcal{A})$ be as in the statement above. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $k = m$. Furthermore, since any algebra similar to an $\mathcal{LR}$-algebra is again an $\mathcal{LR}$-algebra, we may assume that $\mathcal{A}$ is unhinged with respect to $\bigoplus_{i=1}^m \mathcal{V}_i$.

Since $Rad(\mathcal{A})$ is a right $\mathbb{M}_d$-module, Theorem 2.1.12 indicates that $Rad(\mathcal{A}) = Q_1'\mathbb{M}_n Q_2$ for some projection $Q_1' \leq Q_{1\Omega}$. It follows that,

$$\mathcal{A} = BD(\mathcal{A}) \dot{+} Rad(\mathcal{A}) = (Q_1' + Q_{2\Omega})\mathbb{M}_n Q_{2\Omega} + \mathbb{C}I,$$

and hence $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra. ■

In light of Proposition 4.3.2, it suffices to consider the type II algebras for which the triples $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ in $\mathcal{F}_{II}$ are such that $1 < k < m$. For such an algebra $\mathcal{A}$ and triple Ω, the projections $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ are all non-zero. In the language of Theorem 2.3.3 and the remarks that follow, the corners $Q_{1\Omega}\mathcal{A}Q_{1\Omega}$ and $Q_{3\Omega}\mathcal{A}Q_{3\Omega}$ are diagonal algebras, each comprised of mutually linked 1×1 blocks. Note that the blocks in $Q_{1\Omega}\mathcal{A}Q_{1\Omega}$ may be linked to those in $Q_{3\Omega}\mathcal{A}Q_{3\Omega}$. If this is the case, we will say that $Q_{1\Omega}$ and $Q_{3\Omega}$ are *linked*. Otherwise, we will say that $Q_{1\Omega}$ and $Q_{3\Omega}$ are *unlinked*. In either case, dimension considerations imply that neither $Q_{1\Omega}$ nor $Q_{3\Omega}$ is linked to $Q_{2\Omega}$. As in our analysis of type I algebras, it will be important to distinguish between these settings.

§4.3.1 Type II Algebras with Unlinked Projections

Let us first consider the type II algebras $\mathcal{A}$ for which the triples $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ in $\mathcal{F}_{II}(\mathcal{A})$ are such that $Q_{1\Omega}$ and $Q_{3\Omega}$ are unlinked. We aim to show that the only such algebras with the projection compression property are those that are unitarily equivalent to the type II algebra in Example 4.1.1(i). To accomplish this goal, we will first show in Lemma 4.3.3 that the result holds in the $\mathbb{M}_4$ setting. An extension to larger type II algebras will be made in Theorem 4.3.4 by applying Lemma 4.3.3 to their 4×4 compressions.

As stated above, if $\mathcal{A}$ is a type II algebra and Ω is a triple in $\mathcal{F}_{II}(\mathcal{A})$, then $Q_{2\Omega}$ is necessarily unlinked from $Q_{1\Omega}$ and $Q_{3\Omega}$. Thus, type II algebras satisfy the assumptions of Lemma 2.3.10. We will apply this fact in the proofs of Lemma 4.3.3 and Theorem 4.3.4.

Lemma 4.3.3. *Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_4$. Assume that $\mathcal{F}_{II}(\mathcal{A})$ contains a pair $\Omega = (d, k, \bigoplus_{i=1}^3 \mathcal{V}_i)$ such that $d = k = 2$. If $Q_{1\Omega}$ and $Q_{3\Omega}$ are unlinked, then $\mathcal{A}$ is unitarily equivalent to*

$$\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{3\Omega} + (Q_{1\Omega} + Q_{2\Omega})\mathbb{M}_4(Q_{2\Omega} + Q_{3\Omega}),$$

the type II algebra from Example 4.1.1(i). Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. Suppose to the contrary that $\mathcal{A}$ is not unitarily equivalent to the algebra described above. Lemma 2.3.10 (ii) then implies that

$$\begin{aligned} &Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} \neq Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega} \quad \text{or} \\ &Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} \neq Q_{2\Omega} \mathbb{M}_4 Q_{3\Omega}. \end{aligned}$$

By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} \neq Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega}$. Consequently, $Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} = \{0\}$ by Theorem 2.1.12.

An application of Theorem 2.3.9 provides a precise description of $Q_{1\Omega} \mathcal{A} Q_{2\Omega}$. Since $\mathcal{A}$ is similar to $BD(\mathcal{A}) \dot{+} Rad(\mathcal{A})$ via a block upper triangular similarity, there is a fixed element $T \in Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega}$ such that

$$Q_{1\Omega} A Q_{2\Omega} = (Q_{1\Omega} A Q_{1\Omega}) T - T (Q_{2\Omega} A Q_{2\Omega}) \ \text{ for every } A \in \mathcal{A}.$$

For each $i \in \{1, 2, 3\}$, fix an orthonormal basis $\left\{ e_1^{(i)}, e_2^{(i)}, \dots, e_{n_{i\Omega}}^{(i)} \right\}$ for $\operatorname{ran}(Q_{i\Omega})$. To simplify matters, we may use Theorem 4.2.6 and the remarks that follow to assume that $\langle T e_2^{(2)}, e_1^{(1)} \rangle = 0$. That is, with respect to the basis $\left\{ e_1^{(1)}, e_1^{(2)}, e_2^{(2)}, e_1^{(3)} \right\}$ for $\mathbb{C}^4$, each $A \in \mathcal{A}$ may be expressed as

$$A = \left[\begin{array}{c|cc|c} a_{11} & a_{11} t - t a_{22} & -t a_{23} & a_{14} \\ \hline & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ \hline & & & a_{44} \end{array} \right],$$

where $a_{ij} \in \mathbb{C}$ and $t := \langle T e_1^{(2)}, e_1^{(1)} \rangle$. Here, the entries on the block-diagonal may be selected arbitrarily.

To reach a contradiction, consider the matrices

$$P_0 := \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \ P_1 := \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \ \text{ and } \ P_2 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Observe that for each i, $\frac{1}{2} P_i$ is a projection in $\mathbb{M}_4$. Through direct computation, one may verify that

$$\langle B e_2^{(2)}, e_1^{(1)} \rangle + 2t \langle B e_2^{(2)}, e_1^{(2)} \rangle = 0 \text{ for all } B \in P_0 \mathcal{A} P_0.$$

Yet with A as above and $B_0 := (P_0 A P_0)^2$, we have

$$\langle B_0 e_2^{(2)}, e_1^{(1)} \rangle + 2t \langle B_0 e_2^{(2)}, e_1^{(2)} \rangle = 8 a_{23} \left(a_{14} - t (a_{11} - a_{44} - a_{24}) \right).$$

It follows that $a_{23} = 0$ for all $A \in \mathcal{A}$ or $a_{14} = t(a_{11} - a_{44} - a_{24})$ for all $A \in \mathcal{A}$. Indeed, it is clear that every element of A must satisfying at least one of these equations. But if $\mathcal{A}$ contained an operator A_1 satisfying only the first equation and an operator A_2 satisfying only the second, then neither equation would hold for $A_1 + A_2$. Since a_{23} may be selected arbitrarily, it must be that $a_{14} = t(a_{11} - a_{44} - a_{24})$ for every $A \in \mathcal{A}$.

One may now derive similar relations using P_1 and P_2. Indeed, it is straightforward to check that for $j \in \{1, 2\}$, the equation

$$t\langle P_j A P_j e_2^{(2)}, e_1^{(2)}\rangle + 2\langle P_j A P_j e_2^{(2)}, e_1^{(1)}\rangle = 0$$

holds for every $A \in \mathcal{A}$. Yet if A_0 denotes any element of $\mathcal{A}$ of the above form satisfying $a_{11} = a_{23} = 1$ and $a_{44} = 0$, then for $B_j := (P_j A_0 P_j)^2$,

$$\left(t\langle B_1 e_2^{(2)}, e_1^{(2)}\rangle + 2\langle B_1 e_2^{(2)}, e_1^{(1)}\rangle\right) - \left(t\langle B_2 e_2^{(2)}, e_1^{(2)}\rangle + 2\langle B_2 e_2^{(2)}, e_1^{(1)}\rangle\right) = 16t^2.$$

Since B_1 and B_2 belong to $P_1\mathcal{A}P_1$ and $P_2\mathcal{A}P_2$, respectively, we conclude that $t = 0$. That is, $Q_{1\Omega}\mathcal{A}Q_{2\Omega} = \{0\}$. It follows that with respect to the basis $\left\{e_1^{(2)}, e_2^{(2)}, e_1^{(1)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$, each $A \in \mathcal{A}$ may be written as

$$A = \left[\begin{array}{cc|c|c} a_{22} & a_{23} & 0 & a_{24} \\ a_{32} & a_{33} & 0 & a_{34} \\ \hline & & a_{11} & 0 \\ \hline & & & a_{44} \end{array}\right]$$

for some $a_{ij} \in \mathbb{C}$. Theorem 4.1.2 now demonstrates that $\mathcal{A}$ is not projection compressible, as the entries in $BD(\mathcal{A})$ may be chosen arbitrarily. This is a contradiction. ∎

Theorem 4.3.4. *Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_n$, and assume that $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ is a triple in $\mathcal{F}_{II}(\mathcal{A})$ with $1 < k < m$. If $Q_{1\Omega}$ and $Q_{3\Omega}$ are unlinked, then $\mathcal{A}$ is unitarily equivalent to*

$$\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{3\Omega} + (Q_{1\Omega} + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_{3\Omega}),$$

the type II algebra from Example 4.1.1(i). Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. Suppose to the contrary that $\mathcal{A}$ is not unitarily equivalent to the algebra described above. As in the proof of the previous result, we may appeal to Lemma 2.3.10 (ii) and assume without loss of generality that $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} \neq Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$. Thus, Theorem 2.1.12 gives rise to a proper subprojection Q_1' of $Q_{1\Omega}$ satisfying

$$Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_1'\mathbb{M}_n Q_{2\Omega}.$$

Define $Q_1'' \coloneqq Q_{1\Omega} - Q_1'$, and let $\left\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{n_{1\Omega}}^{(1)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{1\Omega})$ such that

$$\operatorname{ran}(Q_1'') = \vee\left\{e_1^{(1)}, e_2^{(1)}, \ldots, e_\ell^{(1)}\right\}$$

for some integer $1 \leq \ell \leq n_{1\Omega}$. Since $\mathcal{A}$ is similar to $BD(\mathcal{A}) \dotplus Rad(\mathcal{A})$ via a matrix that is block upper triangular with respect to $\mathbb{C}^n = \operatorname{ran}(Q_{1\Omega}) \oplus \operatorname{ran}(Q_{2\Omega}) \oplus \operatorname{ran}(Q_{3\Omega})$, there is an operator $T \in Q_1''\mathbb{M}_n Q_{2\Omega}$ such that

$$Q_1''AQ_{2\Omega} = (Q_1''AQ_1'')T - T(Q_{2\Omega}AQ_{2\Omega}) \quad \text{for all } A \in \mathcal{A}.$$

By Theorem 4.2.6, one may choose a suitable orthonormal basis $\left\{e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_{2\Omega}}^{(2)}\right\}$ for $\operatorname{ran}(Q_{2\Omega})$ and adjust the basis for $\operatorname{ran}(Q_1'')$ if necessary to impose additional structure on T. Specifically, one may assume that $\langle Te_j^{(2)}, e_i^{(1)}\rangle = 0$ whenever $i \neq j$.

Let $e_1^{(3)}$ be any non-zero vector in $\operatorname{ran}(Q_{3\Omega})$, and define $\mathcal{B} = \left\{e_1^{(1)}, e_1^{(2)}, e_2^{(2)}, e_1^{(3)}\right\}$. Let P denote the orthogonal projection onto the span of $\mathcal{B}$, and consider the compression $\mathcal{A}_0 \coloneqq P\mathcal{A}P$. It is easy to see that $\mathcal{A}_0$ is a projection compressible type II subalgebra of $\mathbb{M}_4$. Moreover, if

$$\mathcal{W}_1 \coloneqq \mathbb{C}e_1^{(1)}, \quad \mathcal{W}_2 \coloneqq \vee\left\{e_1^{(2)}, e_2^{(2)}\right\}, \quad \text{and} \quad \mathcal{W}_3 \coloneqq \mathbb{C}e_1^{(3)},$$

then the triple $\Omega' = (2, 2, \bigoplus_{i=1}^3 \mathcal{W}_i)$ belongs to $\mathcal{F}_{II}(\mathcal{A}_0)$. Since $Q_{1\Omega'}$ and $Q_{3\Omega'}$ are unlinked, $\mathcal{A}_0$ is among the class of algebras addressed in Lemma 4.3.3. With respect to the basis $\mathcal{B}$ for $\operatorname{ran}(P)$, however, every element of $\mathcal{A}_0$ may be expressed as a matrix of the form

$$A = \left[\begin{array}{c|cc|c} a_{11} & a_{11}t - ta_{22} & -ta_{23} & a_{14} \\ \hline & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ \hline & & & a_{44} \end{array}\right],$$

where $t \coloneqq \langle Te_1^{(2)}, e_1^{(1)}\rangle$. Since $\mathcal{A}_0$ is not of the form prescribed by Lemma 4.3.3, it follows that $\mathcal{A}_0$ is not projection compressible—a contradiction. ∎

§4.3.2 Type II Algebras with Linked Projections

Consider now the type II algebras $\mathcal{A}$ for which the triples $\Omega = (d, k, \bigoplus_{i=1}^{m} \mathcal{V}_i)$ in $\mathcal{F}_{II}(\mathcal{A})$ are such that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked. It will be shown in Theorem 4.3.6 that all projection compressible algebras of this form are unitizations of $\mathcal{LR}$-algebras. The proof of this result requires a careful analysis of the upper triangular blocks in the semi-simple part of the algebra. The following lemma is the crux of this analysis.

Lemma 4.3.5. *Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_4$. Assume that $\mathcal{F}_{II}(\mathcal{A})$ contains a triple $\Omega = (d, k, \bigoplus_{i=1}^{3} \mathcal{V}_i)$ with $d = k = 2$, and such that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked.*

(i) If there are a constant $t \in \mathbb{C}$ and for each $i \in \{1, 2, 3\}$, an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$ such that

$$\begin{aligned} \langle Ae_1^{(2)}, e_1^{(1)}\rangle &= \quad t\left(\langle Ae_1^{(1)}, e_1^{(1)}\rangle - \langle Ae_1^{(2)}, e_1^{(2)}\rangle\right) \quad \textit{and} \\ \langle Ae_2^{(2)}, e_1^{(1)}\rangle &= -t\langle Ae_2^{(2)}, e_1^{(2)}\rangle \end{aligned}$$

for all $A \in \mathcal{A}$, then $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$ for all $A \in \mathcal{A}$.

(ii) If there are a constant $t \in \mathbb{C}$ and for each $i \in \{1, 2, 3\}$, an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$ such that

$$\begin{aligned} \langle Ae_1^{(3)}, e_1^{(2)}\rangle &= t\left(\langle Ae_1^{(2)}, e_1^{(2)}\rangle - \langle Ae_1^{(3)}, e_1^{(3)}\rangle\right) \quad \textit{and} \\ \langle Ae_1^{(3)}, e_2^{(2)}\rangle &= t\langle Ae_1^{(2)}, e_2^{(2)}\rangle \end{aligned}$$

for all $A \in \mathcal{A}$, then $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = t\langle Ae_1^{(2)}, e_1^{(1)}\rangle$ for all $A \in \mathcal{A}$.

Proof. We will begin with the proof of (i). Suppose that there are a constant $t \in \mathbb{C}$ and for each $i \in \{1, 2, 3\}$, an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$ as described above. Then with respect to the basis $\left\{e_1^{(1)}, e_1^{(2)}, e_2^{(2)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$, each $A \in \mathcal{A}$ can be written as

$$A = \left[\begin{array}{c|cc|c} a_{11} & t(a_{11} - a_{22}) & -ta_{23} & a_{14} \\ \hline & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ \hline & & & a_{11} \end{array}\right]$$

for some $a_{ij} \in \mathbb{C}$. Since $\mathcal{A}$ is in reduced block upper triangular form, the entries on the block-diagonal may be chosen arbitrarily.

Consider the matrix

$$P \coloneqq \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

and note that $\frac{1}{2}P$ is a projection in $\mathbb{M}_4$. One may verify that

$$t\langle Be_2^{(2)}, e_2^{(1)}\rangle + 2\langle Be_2^{(2)}, e_1^{(1)}\rangle = 0 \text{ for all } B \in P\mathcal{A}P.$$

But with A as above and $B \coloneqq (PAP)^2$, we see that

$$t\langle Be_2^{(2)}, e_1^{(2)}\rangle + 2\langle Be_2^{(2)}, e_1^{(1)}\rangle = -8ta_{23}(a_{14} + ta_{24}).$$

The projection compressibility of $\mathcal{A}$ implies that B belongs to $P\mathcal{A}P$. Consequently, $ta_{23}(a_{14} + ta_{24}) = 0$ for all $A \in \mathcal{A}$.

If $t \neq 0$, then either $a_{23} = 0$ for all $A \in \mathcal{A}$ or $a_{14} = -ta_{24}$ for all $A \in \mathcal{A}$. Indeed, it is clear that every operator in $\mathcal{A}$ must satisfy at least one of these equation. If, however, $\mathcal{A}$ contained an operator A_1 satisfying the first equation but not the second, as well as an operator A_2 satisfying the second but not the first, then neither equation would hold for $A_1 + A_2$. Finally, since a_{23} can be selected arbitrarily, we conclude that either $t = 0$ or $a_{14} = -ta_{24}$ for all A.

If the latter holds, then every $A \in \mathcal{A}$ satisfies the equation $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$, as required. If instead $t = 0$, then with respect to the basis $\left\{e_1^{(2)}, e_2^{(2)}, e_1^{(1)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$, each $A \in \mathcal{A}$ may be expressed as a matrix of the form

$$A = \left[\begin{array}{cc|c|c} a_{22} & a_{23} & 0 & a_{24} \\ a_{32} & a_{33} & 0 & a_{34} \\ \hline & & a_{11} & a_{14} \\ \hline & & & a_{11} \end{array}\right]$$

for some $a_{ij} \in \mathbb{C}$. It follows from Theorem 4.1.2 that $a_{14} = \langle Ae_1^{(3)}, e_1^{(1)}\rangle = 0$ for all A, and hence the equation $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$ holds in this case as well.

In the context of (ii), note that every $A \in \mathcal{A}$ may be expressed as a matrix of the form

$$\left[\begin{array}{c|cc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline & a_{22} & a_{23} & t(a_{22}-a_{11}) \\ & a_{32} & a_{33} & ta_{32} \\ \hline & & & a_{11} \end{array}\right]$$

with respect to the basis $\left\{e_1^{(1)}, e_1^{(2)}, e_2^{(2)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$. Since this matrix is transpose equivalent to

$$\left[\begin{array}{c|cc|c} a_{11} & t(a_{22}-a_{11}) & ta_{32} & a_{14} \\ \hline & a_{22} & a_{32} & a_{12} \\ & a_{23} & a_{33} & a_{13} \\ \hline & & & a_{11} \end{array}\right],$$

we conclude from (i) that $a_{14} = ta_{12}$. That is, $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = t\langle Ae_1^{(2)}, e_1^{(1)}\rangle$ for all $A \in \mathcal{A}$. ■

Theorem 4.3.6. *Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_n$, and let $\Omega = (d, k, \bigoplus_{i=1}^m \mathcal{V}_i)$ be a triple in $\mathcal{F}_{II}(\mathcal{A})$. If $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked, then $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra. Consequently, $\mathcal{A}$ is idempotent compressible.*

Proof. Let Ω be as above, and assume that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked. Note that if $k = 1$ or $k = m$, then $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra by Proposition 4.3.2. Thus, we will assume that $1 < k < m$. In this case, Theorem 2.1.12 gives rise to subprojections $Q_1' \leq Q_{1\Omega}$ and $Q_3' \leq Q_{3\Omega}$ such that

$$\begin{aligned} Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} &= Q_1' \mathbb{M}_n Q_{2\Omega} \quad \text{and} \\ Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} &= Q_{2\Omega} \mathbb{M}_n Q_3'. \end{aligned}$$

Our goal is to show that $\mathcal{A}$ is similar to

$$\mathcal{A}_0 := (Q_1' + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_3') + \mathbb{C}I.$$

Since $\mathcal{A}_0$ is the unitization of an $\mathcal{LR}$-algebra, this will demonstrate that so too is $\mathcal{A}$. We will accomplish this task by first determining the structure of $Q_{1\Omega}\mathcal{A}Q_{3\Omega}$.

Define $Q_1'' := Q_{1\Omega} - Q_1'$ and $Q_3'' := Q_{3\Omega} - Q_3'$. For each $i \in \{1, 2, 3\}$, let $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$ such that if $Q_i'' \neq 0$, then

$$\operatorname{ran}(Q_i'') = \vee \left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{\ell_i}^{(i)}\right\}$$

for some $\ell_i \in \{1, 2, \ldots, n_{i\Omega}\}$. Since $\mathcal{A}$ is similar to $BD(\mathcal{A}) \dotplus Rad(\mathcal{A})$ via a matrix that is block upper triangular with respect to $\mathbb{C}^n = \operatorname{ran}(Q_{1\Omega}) \oplus \operatorname{ran}(Q_{2\Omega}) \oplus \operatorname{ran}(Q_{3\Omega})$, there are operators $T_1 \in Q_1'' \mathbb{M}_n Q_{2\Omega}$ and $T_2 \in Q_{2\Omega} \mathbb{M}_n Q_3''$ such that every $A \in \mathcal{A}$ satisfies

$$\begin{aligned} Q_1'' A Q_{2\Omega} &= (Q_1'' A Q_1'') T_1 - T_1 (Q_{2\Omega} A Q_{2\Omega}) \quad \text{and} \\ Q_{2\Omega} A Q_3'' &= (Q_{2\Omega} A Q_{2\Omega}) T_2 - T_2 (Q_3'' A Q_3''). \end{aligned}$$

We will begin by using Lemma 4.3.5 to identify the structure of $Q_1'' \mathcal{A} Q_{3\Omega}$. Of course, there is little to be said when $Q_1'' = 0$, so assume for now that $Q_1'' \neq 0$. By Theorem 4.2.6 and its subsequent remarks, one may change the orthonormal bases for $\operatorname{ran}(Q_1'')$ and $\operatorname{ran}(Q_{2\Omega})$ if required and assume that

$$t_{ij}^{(1)} := \langle T_1 e_j^{(2)}, e_i^{(1)} \rangle = 0 \text{ for all } i \neq j.$$

Let i and i' be arbitrary indices from $\{1, 2, \ldots, \ell_1\}$ and $\{1, 2, \ldots, n_{3\Omega}\}$, respectively. Define

$$j = \begin{cases} i & \text{if } i \leq n_{2\Omega}, \\ 1 & \text{otherwise,} \end{cases}$$

and fix an index $j' \in \{1, 2, \ldots, n_{2\Omega}\} \setminus \{j\}$. Let P denote the orthogonal projection onto the span of $\mathcal{B} := \left\{ e_i^{(1)}, e_j^{(2)}, e_{j'}^{(2)}, e_{i'}^{(3)} \right\}$, and consider the algebra $P\mathcal{A}P$. If $i > n_{2\Omega}$, then for each $A \in \mathcal{A}$, PAP may be expressed as a matrix of the form

$$PAP = \left[\begin{array}{c|cc|c} a_{11} & 0 & 0 & a_{14} \\ \hline & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ \hline & & & a_{11} \end{array} \right]$$

with respect to $\mathcal{B}$. In this case, $P\mathcal{A}P$ is an algebra of the form described in Lemma 4.3.5 (i) with $t = 0$. Thus, this result implies that

$$a_{14} = \langle A e_{i'}^{(3)}, e_i^{(1)} \rangle = 0 \text{ for all } A \in \mathcal{A}.$$

Suppose instead that $i \leq n_{2\Omega}$. We then have that for each $A \in \mathcal{A}$, PAP can be written as a matrix of the form

$$PAP = \left[\begin{array}{c|cc|c} a_{11} & a_{11} t_{ii}^{(1)} - t_{ii}^{(1)} a_{22} & -t_{ii}^{(1)} a_{23} & a_{14} \\ \hline & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ \hline & & & a_{11} \end{array} \right]$$

with respect to $\mathcal{B}$. It follows that $P\mathcal{A}P$ is of the form described in Lemma 4.3.5 (i) with $t = t_{ii}^{(1)}$, and hence

$$a_{14} = \langle Ae_{i'}^{(3)}, e_i^{(1)} \rangle = -t_{ii}^{(1)} \langle Ae_{i'}^{(3)}, e_i^{(2)} \rangle \text{ for all } A \in \mathcal{A}.$$

Since our choice of indices was arbitrary, these conclusions hold for all $i \in \{1, 2, \ldots, \ell_1\}$ and all $i' \in \{1, 2, \ldots, n_{3\Omega}\}$. Consequently,

$$Q_1'' A Q_{3\Omega} = -T_1 Q_{2\Omega} A Q_{3\Omega} \text{ for all } A \in \mathcal{A}.$$

We now wish to obtain information on the structure of $Q_{1\Omega}\mathcal{A}Q_3''$. As in the analysis above, it will be convenient to simplify the description of T_2 by choosing suitable bases for $\operatorname{ran}(Q_{2\Omega})$ and $\operatorname{ran}(Q_{3\Omega})$. Specifically, Theorem 4.2.6 gives rise to operators $V \in Q_{2\Omega}\mathbb{M}_n Q_{2\Omega}$, $W \in Q_3''\mathbb{M}_n Q_3''$, and a unitary $U \in \mathbb{M}_n$ such that

$$\begin{aligned}(Q_{1\Omega} + Q_3')U(Q_{1\Omega} + Q_3') &= Q_{1\Omega} + Q_3',\\ (Q_{2\Omega} + Q_3'')U(Q_{2\Omega} + Q_3'') &= V + W,\end{aligned}$$

and

$$\langle U^* T_2 U e_j^{(3)}, e_i^{(2)} \rangle = \langle V^* T_2 W e_j^{(3)}, e_i^{(2)} \rangle = 0 \text{ for all } i \neq j.$$

By considering the algebra $U^*\mathcal{A}U$ and arguing as above, one may deduce that

$$(Q_{1\Omega} A Q_3'') = (Q_{1\Omega} A Q_{2\Omega}) T_2 \text{ for all } A \in \mathcal{A}.$$

Our findings thus far indicate that with respect to the decomposition

$$\mathbb{C}^n = \operatorname{ran}(Q_1'') \oplus \operatorname{ran}(Q_1') \oplus \operatorname{ran}(Q_{2\Omega}) \oplus \operatorname{ran}(Q_3'') \oplus \operatorname{ran}(Q_3'),$$

each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \left[\begin{array}{c|c|c|c|c} a_{11}I & 0 & a_{11}T_1 - T_1 M & -T_1(MT_2 - a_{11}T_2) & -T_1 J_2 \\ \hline & a_{11}I & J_1 & J_1 T_2 & A_{25} \\ \hline & & M & MT_2 - a_{11}T_2 & J_2 \\ \hline & & & a_{11}I & 0 \\ \hline & & & & a_{11}I \end{array}\right]$$

for some $a_{11} \in \mathbb{C}$ and operators $M \in Q_{2\Omega}\mathbb{M}_n Q_{2\Omega}$, $J_1 \in Q_1' Rad(\mathcal{A}) Q_{2\Omega}$, $J_2 \in Q_{2\Omega} Rad(\mathcal{A}) Q_3'$, and $A_{25} \in Q_1' \mathbb{M}_n Q_3'$. With this description in hand we are prepared to show that $\mathcal{A}$ is similar to $\mathcal{A}_0$, and hence is the unitization of an $\mathcal{LR}$-algebra.

Consider the operator $S \coloneqq I - T_1 - T_2$. This S is invertible with $S^{-1} = I + T_1 + T_2 + T_1 T_2$. Moreover, for each $A \in \mathcal{A}$ as above, we have that

$$S^{-1}AS = \left[\begin{array}{c|c|c|c|c} a_{11}I & 0 & 0 & 0 & 0 \\ \hline & a_{11}I & J_1 & 0 & A_{25} \\ \hline & & M & 0 & J_2 \\ \hline & & & a_{11}I & 0 \\ \hline & & & & a_{11}I \end{array}\right].$$

From here it is easy to see that $S^{-1}\mathcal{A}S$ is a type II algebra that has a reduced block upper triangular form with respect to the above decomposition. Moreover,

$$\begin{aligned} Q_{1\Omega} Rad(S^{-1}\mathcal{A}S) Q_{2\Omega} &= Q_1' Rad(\mathcal{A}) Q_{2\Omega} = Q_1' \mathbb{M}_n Q_{2\Omega} \quad \text{and} \\ Q_{2\Omega} Rad(S^{-1}\mathcal{A}S) Q_{3\Omega} &= Q_{2\Omega} Rad(\mathcal{A}) Q_3' = Q_{2\Omega} \mathbb{M}_n Q_3'. \end{aligned}$$

Thus, Lemma 2.3.10 (ii) implies that

$$S^{-1}\mathcal{A}S = (Q_1' + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_3') + \mathbb{C}I = \mathcal{A}_0,$$

as claimed. ∎

§4.4 Algebras of Type III

We now begin the final stage of our classification of unital projection compressible subalgebras of $\mathbb{M}_n$ when $n \geq 4$. The term *type III* will be used to describe a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$, $n \geq 4$, such that for every orthogonal decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular, $\dim \mathcal{V}_i = 1$ for all i (i.e., $m = n$), and there is an integer k as in Corollary 4.1.3. It is obvious that such a k must lie strictly between 1 and n.

As in the preceding sections, it will be important to maintain a record of the integers k and decompositions of $\mathbb{C}^n$ that satisfy the assumptions of Corollary 4.1.3 for a given type III algebra $\mathcal{A}$.

Definition 4.4.1. If $\mathcal{A}$ is an algebra of type III, let $\mathcal{F}_{III} = \mathcal{F}_{III}(\mathcal{A})$ denote the set of pairs $\Omega = (k, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ that satisfy the following conditions:

(i) $\bigoplus_{i=1}^{n} \mathcal{V}_i$ is an orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular;

(ii) k is an integer in $\{2, \ldots, n-1\}$ such that if $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively, then for each $i \in \{1, 3\}$,

$$(Q_{i\Omega} + Q_{2\Omega})\mathcal{A}(Q_{i\Omega} + Q_{2\Omega}) \neq \mathbb{C}(Q_{i\Omega} + Q_{2\Omega}).$$

Notation. If $\mathcal{A}$ is an algebra of type III and $\Omega = (k, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ is a pair in $\mathcal{F}_{III}(\mathcal{A})$, let $n_{1\Omega} = k-1$, $n_{2\Omega} = 1$, and $n_{3\Omega} = n-k$ denote the ranks of $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$, respectively. Note that since $n_{2\Omega} = 1$ and $n \geq 4$, we necessarily have $\max\{n_{1\Omega}, n_{3\Omega}\} \geq 2$.

If $\mathcal{A}$ is a projection compressible algebra of type III with pair $\Omega \in \mathcal{F}_{III}(\mathcal{A})$, then $Q_{i\Omega}\mathcal{A}Q_{i\Omega} = \mathbb{C}Q_{i\Omega}$ for each $i \in \{1, 2, 3\}$. Thus, each corner $Q_{i\Omega}\mathcal{A}Q_{i\Omega}$ is a diagonal algebra comprised of mutually linked 1×1 blocks. Of course, the blocks in $Q_{i\Omega}\mathcal{A}Q_{i\Omega}$ may or may not be linked to those in $Q_{j\Omega}\mathcal{A}Q_{j\Omega}$. If there is linkage between these blocks, we will say that the projections $Q_{i\Omega}$ and $Q_{j\Omega}$ are *linked*; otherwise, we will say that they are *unlinked*.

Unlike in §4.3, it is now entirely possible that $Q_{2\Omega}$ is linked to $Q_{1\Omega}$ or $Q_{3\Omega}$. As the following result demonstrates, however, there do not exist projection compressible algebras of type III for which all projections $Q_{i\Omega}$ are mutually linked.

Proposition 4.4.2. *Let $\mathcal{A}$ be a projection compressible algebra of type III, and let Ω be a pair in $\mathcal{F}_{III}(\mathcal{A})$.*

(i) If $Q_{2\Omega}$ is linked to $Q_{1\Omega}$, then $n_{1\Omega} = 1$ and $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$

(ii) If $Q_{2\Omega}$ is linked to $Q_{3\Omega}$, then $n_{3\Omega} = 1$ and $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$.

Consequently, $Q_{2\Omega}$ cannot be linked to both $Q_{1\Omega}$ and $Q_{3\Omega}$.

Proof. Clearly (ii) follows from (i) by replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$. Thus, it suffices to prove (i).

Suppose to the contrary that $n_{1\Omega} \geq 2$. For each $i \in \{1, 2, 3\}$, let $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ be an orthonormal basis for $\mathrm{ran}(Q_{i\Omega})$. For each index j in $\{1, 2, \ldots, n_{3\Omega}\}$, let P_j denote the orthogonal projection onto the span of $\mathcal{B}_j := \left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_j^{(3)}\right\}$. Furthermore, define P_j' to be the operator

$$P_j' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

acting on $\operatorname{ran}(P_j)$ and written with respect to the basis $\mathcal{B}_j$. It is clear that $\frac{1}{2}P_j'$ is a subprojection of P_j.

One may verify that every $B \in P_j'\mathcal{A}P_j'$ satisfies the equation $\langle Be_1^{(2)}, e_1^{(2)}\rangle = \langle Be_2^{(1)}, e_2^{(1)}\rangle$. But if A belongs to $\mathcal{A}$ and $C := (P_j'AP_j')^2$, then

$$\langle Ce_1^{(2)}, e_1^{(2)}\rangle - \langle Ce_2^{(1)}, e_2^{(1)}\rangle = 8\langle Ae_1^{(2)}, e_1^{(1)}\rangle\langle Ae_j^{(3)}, e_1^{(2)}\rangle.$$

Since C is an element of $P_j'\mathcal{A}P_j'$, the right-hand side of this equation must be zero. To obtain a contradiction, it therefore suffices to exhibit an element A in $\mathcal{A}$ such that for some $j \in \{1, 2, \ldots, n_{3\Omega}\}$, both $\langle Ae_1^{(2)}, e_1^{(1)}\rangle$ and $\langle Ae_j^{(3)}, e_1^{(2)}\rangle$ are non-zero.

First suppose that the projections $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ are mutually linked. By definition of Ω as a pair in $\mathcal{F}_{III}(\mathcal{A})$, there exist $i \in \{1, 2, \ldots, n_{1\Omega}\}$ and $j \in \{1, 2, \ldots, n_{3\Omega}\}$, as well as $A_1, A_2 \in \mathcal{A}$, such that $\langle A_1e_1^{(2)}, e_i^{(1)}\rangle \neq 0$ and $\langle A_2e_j^{(3)}, e_1^{(2)}\rangle \neq 0$. By reordering the basis for $\operatorname{ran}(Q_{1\Omega})$ if necessary, we may assume that $i = 1$. If $\langle A_2e_1^{(2)}, e_1^{(1)}\rangle \neq 0$ or $\langle A_1e_j^{(3)}, e_1^{(2)}\rangle \neq 0$, then we obtain the required contradiction. Otherwise, $A := A_1 + A_2$ is such that $\langle Ae_1^{(2)}, e_1^{(1)}\rangle \neq 0$ and $\langle Ae_j^{(3)}, e_1^{(2)}\rangle \neq 0$, as desired.

Now suppose that $Q_{3\Omega}$ is unlinked from $Q_{1\Omega}$ and $Q_{2\Omega}$. By reordering the basis for $\operatorname{ran}(Q_{1\Omega})$ if necessary, we may obtain an element $A_1 \in \mathcal{A}$ such that $\langle A_1e_1^{(2)}, e_1^{(1)}\rangle \neq 0$. If there is an element $A_2 \in \mathcal{A}$ such that $\langle A_2e_j^{(3)}, e_1^{(2)}\rangle \neq 0$ for some $j \in \{1, 2, \ldots, n_{3\Omega}\}$, then arguments similar to those in the linked case above provide the required contradiction. Of course, it is now entirely possible that no such A_2 exists, as $Q_{2\Omega}$ and $Q_{3\Omega}$ are unlinked. That is, it may be that $Q_{2\Omega}\mathcal{A}Q_{3\Omega} = \{0\}$. Assume that this is the case.

Let $\mathcal{B} = \left\{e_1^{(1)}, e_2^{(1)}, e_1^{(3)}, e_1^{(2)}\right\}$, and define P to be the orthogonal projection onto the span of $\mathcal{B}$. Note that with respect to the basis $\mathcal{B}$ for $\operatorname{ran}(P)$, each $A \in P\mathcal{A}P$ may be written as

$$A = \left[\begin{array}{cc|c|c} \alpha & 0 & a_{13} & a_{14} \\ & \alpha & a_{23} & a_{24} \\ \hline & & \beta & 0 \\ \hline & & & \alpha \end{array}\right]$$

for some α, β, and $a_{ij} \in \mathbb{C}$. Consider the operator

$$P' = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

acting on $\operatorname{ran}(P)$ and written with respect to $\mathcal{B}$. It is easy to see that $\frac{1}{3}P'$ is a subprojection of P. Moreover, one may verify that every element $B = (b_{ij})$ in $P'\mathcal{A}P'$ satisfies the equation $b_{33} + 2b_{31} - b_{43} - 2b_{41} - b_{22} = 0$. But if A is as above and we define $(P'AP')^2 = (c_{ij})$, then

$$c_{33} + 2c_{31} - c_{43} - 2c_{41} - c_{22} = 27a_{14}(\beta - \alpha).$$

Since α and β may be chosen arbitrarily, it must be that $a_{14} = \langle Ae_1^{(2)}, e_1^{(1)} \rangle = 0$ for all A. This is a contradiction, as $\langle A_1 e_1^{(2)}, e_1^{(1)} \rangle \neq 0$. We therefore conclude that $n_{1\Omega} = 1$.

Since $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, yet $(Q_{1\Omega} + Q_{2\Omega})\mathcal{A}(Q_{1\Omega} + Q_{2\Omega}) \neq \mathbb{C}(Q_{1\Omega} + Q_{2\Omega})$ by definition of Ω as a pair in $\mathcal{F}_{III}(\mathcal{A})$, it follows that $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} \neq \{0\}$. Consequently, $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ as $n_{1\Omega} = n_{2\Omega} = 1$.

The final claim now follows from the fact that $\max\{n_{1\Omega}, n_{3\Omega}\} \geq 2$. ■

The above result indicates that if $\mathcal{A}$ is a projection compressible algebra of type III and Ω is a pair in $\mathcal{F}_{III}(\mathcal{A})$, then there is a projection $Q_{i\Omega}$ that is unlinked from $Q_{2\Omega}$. In the case that this $Q_{i\Omega}$ is also unlinked from the remaining projection $Q_{j\Omega}$, one can say more about the structure of $\mathcal{A}$.

Proposition 4.4.3. *Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$, and let Ω be a pair in $\mathcal{F}_{III}(\mathcal{A})$.*

(i) If $Q_{3\Omega}$ is unlinked from $Q_{1\Omega}$ and $Q_{2\Omega}$, then either $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$; or $n_{3\Omega} = 1$ and $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = \{0\}$.

(ii) If $Q_{1\Omega}$ is unlinked from $Q_{2\Omega}$ and $Q_{3\Omega}$, then either $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$; or $n_{1\Omega} = 1$ and $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = \{0\}$.

Proof. As in the previous proof it is easy that (ii) follows from (i) by replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$. Thus, it suffices to prove (i).

Assume that $Q_{3\Omega}$ is unlinked from both $Q_{1\Omega}$ and $Q_{2\Omega}$. Suppose for the sake of contradiction that $n_{3\Omega} \geq 2$ and $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} \neq Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$. For each $i \in \{1, 2, 3\}$, let

$\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$, and assume that the basis for $\operatorname{ran}(Q_{3\Omega})$ is chosen so that $\langle Re_1^{(3)}, e_1^{(2)}\rangle = 0$ for all $R \in Rad(\mathcal{A})$.

Define $\mathcal{B} = \left\{e_1^{(1)}, e_1^{(2)}, e_1^{(3)}, e_2^{(3)}\right\}$, let P denote the orthogonal projection onto the span of $\mathcal{B}$, and consider the compression $\mathcal{A}_0 \coloneqq P\mathcal{A}P$. As a consequence of Theorem 2.3.9, there is a constant $t \in \mathbb{C}$ such that with respect to the basis $\mathcal{B}$ for $\operatorname{ran}(P)$, each A in $\mathcal{A}_0$ admits a matrix of the form

$$A = \left[\begin{array}{c|c|cc} \alpha & a_{12} & a_{13} & a_{14} \\ \hline & \beta & t(\beta-\gamma) & a_{24} \\ \hline & & \gamma & 0 \\ & & & \gamma \end{array}\right]$$

for some α, β, γ, and a_{ij} in $\mathbb{C}$. Note that in the case that $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, α and β must coincide for each $A \in \mathcal{A}_0$. In the case that they are unlinked, these values may be chosen independently. With this in mind, the following arguments are applicable to either setting.

Consider the matrices

$$P_1 \coloneqq \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 \coloneqq \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

acting on $\operatorname{ran}(P)$ and written with respect to the basis $\mathcal{B}$. It is easy to see that $\frac{1}{2}P_1$ and $\frac{1}{2}P_2$ are subprojections of P. In addition, one may verify that every $B \in P_1\mathcal{A}_0P_1$ satisfies the equation

$$\langle Be_1^{(3)}, e_1^{(2)}\rangle - t\langle Be_1^{(2)}, e_1^{(2)}\rangle + t\langle Be_1^{(3)}, e_1^{(3)}\rangle = 0.$$

Thus, if A belongs to $\mathcal{A}_0$ and $C \coloneqq (P_1AP_1)^2$, then

$$\langle Ce_1^{(3)}, e_1^{(2)}\rangle - t\langle Ce_1^{(2)}, e_1^{(2)}\rangle + t\langle Ce_1^{(3)}, e_1^{(3)}\rangle = 8\langle Ae_2^{(3)}, e_1^{(2)}\rangle \left(\langle Ae_1^{(3)}, e_1^{(1)}\rangle - t\langle Ae_1^{(2)}, e_1^{(1)}\rangle\right)$$

must be zero. It follows that $\langle Ae_2^{(3)}, e_1^{(2)}\rangle = 0$ for all $A \in \mathcal{A}_0$, or $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = t\langle Ae_1^{(2)}, e_1^{(1)}\rangle$ for all $A \in \mathcal{A}_0$. Indeed, it is clear that every member of $\mathcal{A}_0$ must satisfy at least one of these equations. If, however, there were elements A_1 and A_2 in $\mathcal{A}_0$ such that $\langle A_1e_2^{(3)}, e_1^{(2)}\rangle \neq 0$ and $\langle A_2e_1^{(3)}, e_1^{(1)}\rangle \neq t\langle A_2e_1^{(2)}, e_1^{(1)}\rangle$, then neither equation would be satisfied by their sum.

If it were the case that $\langle Ae_2^{(3)}, e_1^{(2)}\rangle = 0$ for every $A \in \mathcal{A}_0$, then by viewing $\mathcal{A}_0$ as an algebra of matrices with respect to the reordered basis $\left\{e_1^{(1)}, e_2^{(3)}, e_1^{(2)}, e_1^{(3)}\right\}$ for $\operatorname{ran}(P)$,

$\mathcal{A}_0$ would be seen to lack the projection compression property by Theorem 4.1.2. This is clearly a contradiction, so it must be that

$$\langle Ae_1^{(3)}, e_1^{(1)} \rangle = t\langle Ae_1^{(2)}, e_1^{(1)} \rangle \text{ for all } A.$$

From here one may verify that every $B \in P_2\mathcal{A}_0P_2$ satisfies the equation

$$2\langle Be_1^{(3)}, e_1^{(2)} \rangle - t\langle Be_1^{(2)}, e_1^{(2)} \rangle + t\langle Be_2^{(3)}, e_2^{(3)} \rangle = 0.$$

In particular, if $A \in \mathcal{A}_0$ is as above, then this equation must also hold for $D := (P_2AP_2)^2$. Since

$$2\langle De_1^{(3)}, e_1^{(2)} \rangle - t\langle De_1^{(2)}, e_1^{(2)} \rangle + t\langle De_2^{(3)}, e_2^{(3)} \rangle = 8t(\beta - \gamma)(\alpha - \gamma)$$

and γ may be selected independently from α and β, we deduce that $t = 0$. It is now evident that every $A \in \mathcal{A}_0$ can be expressed as a matrix of the form

$$A = \left[\begin{array}{cc|cc} \alpha & 0 & a_{12} & a_{14} \\ & \gamma & 0 & 0 \\ \hline & & \beta & a_{24} \\ & & & \gamma \end{array}\right]$$

with respect to the basis $\left\{e_1^{(1)}, e_1^{(3)}, e_1^{(2)}, e_2^{(3)}\right\}$ for $\mathrm{ran}(P)$. Thus, Theorem 4.1.2 provides the required contradiction.

It must therefore be the case that $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_nQ_{3\Omega}$ or $n_{3\Omega} = 1$. Of course, in the event that $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} \neq Q_{2\Omega}\mathbb{M}_nQ_{3\Omega}$ and hence $n_{3\Omega} = 1$, it follows immediately that $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = \{0\}$. ■

The preceding propositions will be key ingredients in our treatment of projection compressible algebras of type III. Our analysis will proceed in the same spirit as those for algebras of types I or II. We will begin in §4.4.1 by classifying the projection compressible type III algebras for which the projections $Q_{i\Omega}$ are mutually unlinked. In §4.4.2, we will classify the projection compressible type III algebras for which exactly two distinct projections $Q_{i\Omega}$ and $Q_{j\Omega}$ are linked.

§4.4.1 Type III Algebras with Unlinked Projections

In this section we present a classification of the projection compressible type III algebras for which the pairs Ω in $\mathcal{F}_{III}$ are such that no two distinct projections $Q_{i\Omega}$ and $Q_{j\Omega}$ are linked. Such algebras include the algebra from Example 4.1.1(i) when $Q_1 \neq 0, Q_3 \neq 0$ and $\dim Q_2 = 1$; and the algebra from Example 4.1.1(ii). As the following theorem demonstrates, every projection compressible type III algebra with mutually unlinked projections is either transpose equivalent to the former, or transpose similar to the latter.

Theorem 4.4.4. *Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$. If there is a pair Ω in $\mathcal{F}_{III}(\mathcal{A})$ such that no two distinct projections $Q_{i\Omega}$ and $Q_{j\Omega}$ are linked, then $\mathcal{A}$ is transpose equivalent to the type III algebra from Example 4.1.1(i), or transpose similar to the algebra from Example 4.1.1(ii). Consequently, $\mathcal{A}$ is idempotent compressible.*

Proof. Let $\Omega = (k, \bigoplus_{i=1}^n \mathcal{V}_i)$ be a pair in $\mathcal{F}_{III}(\mathcal{A})$ as in the statement of the theorem. For each i in $\{1, 2, 3\}$, fix an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$.

Note that if $Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ and $Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$, then by Lemma 2.3.10 (ii),

$$Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega} \dotplus Q_{1\Omega}\mathbb{M}_n Q_{3\Omega} \dotplus Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}.$$

In this case, $\mathcal{A}$ is the type III algebra from Example 4.1.1(i), so $\mathcal{A}$ is idempotent compressible. It therefore suffices to consider the case in which $Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} \neq Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ or $Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} \neq Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$.

By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume without loss of generality that $Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} \neq Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$. It then follows from Proposition 4.4.3 (i) that $n_{3\Omega} = 1$ and $Q_{2\Omega} Rad(\mathcal{A}) Q_{3\Omega} = \{0\}$. Consequently, $n_{1\Omega} \geq 2$ and hence $Q_{1\Omega} Rad(\mathcal{A}) Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ by Proposition 4.4.3 (ii).

The above observations imply that for every $X \in Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$, there exists an element $Y_X \in Q_{1\Omega}\mathbb{M}_n Q_{3\Omega}$ such that $X + Y_X \in Rad(\mathcal{A})$. Additionally, as a consequence of Theorem 2.3.9, there is a constant $t \in \mathbb{C}$ such that

$$\langle Ae_1^{(3)}, e_1^{(2)} \rangle = t\left(\langle Ae_1^{(2)}, e_1^{(2)} \rangle - \langle Ae_1^{(3)}, e_1^{(3)} \rangle\right) \text{ for all } A \in \mathcal{A}.$$

It therefore suffices to prove that $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_n(Q_{2\Omega} + Q_{3\Omega})$. Indeed, when this is the case, consider the operator $S := I - te_1^{(2)} \otimes e_1^{(3)*} \in \mathbb{M}_n$. One may verify that S is invertible with $S^{-1} = I + te_1^{(2)} \otimes e_1^{(3)*}$, and $S^{-1}\mathcal{A}S$ is the anti-transpose of the type III algebra from Example 4.1.1(ii).

To this end, note that since $Q_{1\Omega}, Q_{2\Omega}$, and $Q_{3\Omega}$ are mutually unlinked, there is an element $A_1 \in \mathcal{A}$ such that $Q_{2\Omega} A_1 Q_{2\Omega} = Q_{2\Omega}$ and $Q_{1\Omega} A_1 Q_{1\Omega} = Q_{3\Omega} A_1 Q_{3\Omega} = 0$. With respect to the decomposition $\mathbb{C}^n = \mathrm{ran}(Q_{1\Omega}) \oplus \mathrm{ran}(Q_{2\Omega}) \oplus \mathrm{ran}(Q_{3\Omega})$, we may write

$$A_1 = \left[\begin{array}{c|c|c} 0 & A_{12} & A_{13} \\ \hline & 1 & t \\ \hline & & 0 \end{array}\right]$$

for some $A_{12} \in Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ and $A_{13} \in Q_{1\Omega}\mathbb{M}_n Q_{3\Omega}$. Thus, for any $X \in Q_{1\Omega}\mathcal{A}Q_{2\Omega}$, there exists $Y_X \in Q_{1\Omega}\mathcal{A}Q_{3\Omega}$ such that $Rad(\mathcal{A})$ contains

$$(X + Y_X)A_1 = \left[\begin{array}{c|c|c} 0 & X & Y_X \\ \hline & 0 & 0 \\ \hline & & 0 \end{array}\right] \left[\begin{array}{c|c|c} 0 & A_{12} & A_{13} \\ \hline & 1 & t \\ \hline & & 0 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & X & tX \\ \hline & 0 & 0 \\ \hline & & 0 \end{array}\right].$$

We conclude that $Rad(\mathcal{A}) = \mathcal{R}^{(1)} \dotplus \mathcal{R}^{(2)}$ where

$$\mathcal{R}^{(1)} := \left\{ \left[\begin{array}{c|c|c} 0 & X & tX \\ \hline & 0 & 0 \\ \hline & & 0 \end{array}\right] : X \in \mathbb{M}_{(k-1)\times 1} \right\}.$$

and $\mathcal{R}^{(2)} := Rad(\mathcal{A}) \cap Q_{1\Omega}\mathbb{M}_n Q_{3\Omega}$.

We claim that $\mathcal{R}^{(2)}$ must be equal to $Q_{1\Omega}\mathbb{M}_n Q_{3\Omega}$. Suppose to the contrary that this is not the case. By changing the orthonormal basis for $\mathrm{ran}(Q_{1\Omega})$ if necessary, we may assume that

$$\langle Y e_1^{(3)}, e_1^{(1)} \rangle = 0 \text{ for all } Y \in \mathcal{R}^{(2)}.$$

Consider the set $\mathcal{B} = \left\{ e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)} \right\}$ and let P denote the orthogonal projection onto the span of $\mathcal{B}$. Define $\mathcal{A}_0$ to be the compression $P\mathcal{A}P$, and accordingly, define

$$\mathcal{R}_0^{(1)} := P\mathcal{R}^{(1)}P \quad \text{and} \quad \mathcal{R}_0^{(2)} := P\mathcal{R}^{(2)}P.$$

Since $\mathcal{A}_0 = \mathcal{S} \dotplus Rad(\mathcal{A}_0)$ where $\mathcal{S}$ is similar to $BD(\mathcal{A}_0)$ via a block upper triangular similarity, there are constants $u_1, u_2, v_1, v_2 \in \mathbb{C}$ such that each $A \in \mathcal{A}_0$ can be written as

$$A = \left[\begin{array}{cc|c|c} \alpha & 0 & v_1(\alpha-\beta) & u_1(\alpha-\gamma) - tv_1(\beta-\gamma) \\ & \alpha & v_2(\alpha-\beta) & u_2(\alpha-\gamma) - tv_2(\beta-\gamma) \\ \hline & & \beta & t(\beta-\gamma) \\ \hline & & & \gamma \end{array}\right] + \left[\begin{array}{cc|c|c} 0 & 0 & x_1 & tx_1 \\ & 0 & x_2 & tx_2 \\ \hline & & 0 & 0 \\ \hline & & & 0 \end{array}\right] + \left[\begin{array}{cc|c|c} 0 & 0 & 0 & 0 \\ & 0 & 0 & y \\ \hline & & 0 & 0 \\ \hline & & & 0 \end{array}\right].$$

where the above summands are expressed with respect to the basis $\mathcal{B}$ for $\mathrm{ran}(P)$, and belong to $\mathcal{S}$, $\mathcal{R}_0^{(1)}$, and $\mathcal{R}_0^{(2)}$, respectively. We will obtain a contradiction by showing that a certain compression of $\mathcal{A}_0$ violates Theorem 4.1.2. To accomplish this goal, it will first be necessary to prove that $t = u_1 = 0$.

With this in mind, consider the matrices

$$P_1 \coloneqq \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad P_2 \coloneqq \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad P_3 \coloneqq \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

acting on $\operatorname{ran}(P)$ and written with respect to the basis $\mathcal{B}$. It is clear that for each i, $\frac{1}{2}P_i$ is a subprojection of P. One may verify that if $B_1 = (b_{ij}^{(1)})$ and $B_2 = (b_{ij}^{(2)})$ belong to $P_1\mathcal{A}_0P_1$ and $P_2\mathcal{A}_0P_2$, respectively, then their entries satisfy the equations

$$4tb_{14}^{(1)} + 2(tv_1 - u_1 + 1)b_{34}^{(1)} - 2t^2b_{13}^{(1)} + t(tv_1 - u_1 - 1)b_{22}^{(1)} - t(tv_1 - u_1 + 1)b_{33}^{(1)} = 0, \quad \text{and}$$

$$4tb_{14}^{(2)} + 2(tv_1 - u_1 - 1)b_{34}^{(2)} - 2t^2b_{13}^{(2)} + t(tv_1 - u_1 + 1)b_{22}^{(2)} - t(tv_1 - u_1 - 1)b_{33}^{(2)} = 0.$$

Let A_0 denote the element of $\mathcal{A}_0$ obtained by setting $\alpha = \beta = x_2 = y = 0$ and $\gamma = x_1 = 1$. That is,

$$A_0 = \left[\begin{array}{cc|c|c} 0 & 0 & 1 & tv_1 - u_1 + t \\ & 0 & 0 & tv_2 - u_2 \\ \hline & & 0 & -t \\ \hline & & & 1 \end{array}\right].$$

Since $\mathcal{A}$ is projection compressible, $C_1 \coloneqq (P_1A_0P_1)^2$ must satisfy the first equation above, while $C_2 \coloneqq (P_2A_0P_2)^2$ must satisfy the second. But with $C_1 = (c_{ij}^{(1)})$ and $C_2 = (c_{ij}^{(2)})$, we have

$$\begin{aligned} 4tc_{14}^{(1)} + 2(tv_1 - u_1 + 1)c_{34}^{(1)} - 2t^2c_{13}^{(1)} & \\ + t(tv_1 - u_1 - 1)c_{22}^{(1)} - t(tv_1 - u_1 + 1)c_{33}^{(1)} &= 8t^2(tv_1 - u_1 - 1), \quad \text{and} \\ 4tc_{14}^{(2)} + 2(tv_1 - u_1 - 1)c_{34}^{(2)} - 2t^2c_{13}^{(2)} & \\ + t(tv_1 - u_1 + 1)c_{22}^{(2)} - t(tv_1 - u_1 - 1)c_{33}^{(2)} &= -8t^2(tv_1 - u_1 + 1). \end{aligned}$$

Adding these equations, it becomes evident that $t = 0$. Consequently, $Q_{1\Omega}\mathcal{R}_0^{(1)}Q_{3\Omega} = \{0\}$.

We now prove that $u_1 = 0$. Let A_0' denote the element of $\mathcal{A}_0$ obtained by setting $\alpha = \beta = x_1 = 1$ and $\gamma = x_2 = y = 0$. That is,

$$A_0' = \left[\begin{array}{cc|c|c} 1 & 0 & 1 & u_1 \\ & 1 & 0 & u_2 \\ \hline & & 1 & 0 \\ \hline & & & 0 \end{array}\right].$$

Since any element $B_3 = (b_{ij}^{(3)})$ in $P_3\mathcal{A}_0P_3$ satisfies the equation $2b_{14}^{(3)} - u_1(b_{22}^{(3)} - b_{44}^{(3)}) = 0$, it must be the case that the element $C_3 := (P_3A_0'P_3)^2$ satisfies this equation as well. But if $C_3 = (c_{ij}^{(3)})$, then $2c_{14}^{(3)} - u_1(c_{22}^{(3)} - c_{44}^{(3)}) = 8u_1$. Therefore, $u_1 = 0$.

We deduce that every element in $\mathcal{A}_0$ admits a matrix representation of the form

$$\left[\begin{array}{cc|cc} \alpha & u_2(\alpha-\gamma)+y & 0 & v_2(\alpha-\beta)+x_2 \\ & \gamma & 0 & 0 \\ \hline & & \alpha & v_1(\alpha-\beta)+x_1 \\ & & & \beta \end{array}\right]$$

with respect to the reordered basis $\left\{e_2^{(1)}, e_1^{(3)}, e_1^{(1)}, e_1^{(2)}\right\}$ for $\operatorname{ran}(P)$. Since the values of α, β, and γ can be selected arbitrarily, an application of Theorem 4.1.2 shows that $\mathcal{A}_0$ is not projection compressible—a contradiction.

The arguments above demonstrate that $\mathcal{R}^{(2)} = Q_{1\Omega}\mathbb{M}_nQ_{3\Omega}$. We therefore conclude that $Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_n(Q_{2\Omega} + Q_{3\Omega})$, and thus the proof is complete. ■

§4.4.2 Type III Algebras with Linked Projections

Let us now consider the projection compressible type III algebras that admit pairs $\Omega \in \mathcal{F}_{III}$ with distinct mutually linked projections. By Proposition 4.4.2, it cannot be the case that all three projections $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ are mutually linked.

We begin with the case in which there is a pair $\Omega \in \mathcal{F}_{III}$ with $Q_{2\Omega}$ linked to $Q_{1\Omega}$ or $Q_{3\Omega}$. One example of such an algebra is given by the type III algebra from Example 4.1.1(iii). The following theorem demonstrates that this algebra is in fact, the only example up to transpose equivalence.

Theorem 4.4.5. *Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$. If there is a pair Ω in $\mathcal{F}_{III}(\mathcal{A})$ such that $Q_{2\Omega}$ is linked to $Q_{1\Omega}$ or $Q_{3\Omega}$, then $\mathcal{A}$ is transpose equivalent to the algebra from Example 4.1.1(iii). Consequently, $\mathcal{A}$ is idempotent compressible.*

Proof. Let Ω be as in the statement of the theorem. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume without loss of generality that $Q_{1\Omega}$ is the projection that is linked to $Q_{2\Omega}$. In this case, Proposition 4.4.2 (i) implies that $n_{1\Omega} = 1$ and $Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$. It follows that $n_{3\Omega} \geq 2$, and hence $Q_{3\Omega}$ is unlinked from $Q_{1\Omega}$ and $Q_{2\Omega}$ by Proposition 4.4.2 (ii). Finally, Proposition 4.4.3 implies that $Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$.

Fix operators $T_1 \in Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$ and $T_2 \in Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$, the above observations imply that there exist $R_1, R_2 \in Rad(\mathcal{A})$ such that $Q_{1\Omega}R_1Q_{2\Omega} = T_1$ and $Q_{2\Omega}R_2Q_{3\Omega} = T_2$. With respect to the decomposition $\mathbb{C}^n = \operatorname{ran}(Q_{1\Omega}) \oplus \operatorname{ran}(Q_{2\Omega}) \oplus \operatorname{ran}(Q_{3\Omega})$, we may write

$$R_1 = \begin{bmatrix} 0 & T_1 & R_{13}^{(1)} \\ 0 & 0 & R_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} 0 & R_{12}^{(2)} & R_{13}^{(2)} \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix}$$

for some operators $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$. From here it is easy to see that $R_1R_2 = T_1T_2 \in Rad(\mathcal{A})$. Since T_1 and T_2 were arbitrary, we conclude that $Rad(\mathcal{A})$ contains $Q_{1\Omega}\mathbb{M}_n Q_{3\Omega}$.

It will now be shown that each block $Q_{i\Omega}Rad(\mathcal{A})Q_{j\Omega}$ exists independently in $Rad(\mathcal{A})$. First, write $\mathcal{A} = \mathcal{S} \dotplus Rad(\mathcal{A})$ where $\mathcal{S}$ is semi-simple. Since $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, $\mathcal{S}$ is similar to $\mathbb{C}(Q_{1\Omega} + Q_{2\Omega}) + \mathbb{C}Q_{3\Omega}$ via an upper triangular similarity. From this it follows that $Q_{1\Omega}\mathcal{S}Q_{2\Omega} = \{0\}$, and hence $\mathcal{S}$ contains an element A of the form

$$A = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & I \end{bmatrix}.$$

Using the fact that $Q_{1\Omega}\mathbb{M}_n Q_{3\Omega} \subseteq Rad(\mathcal{A})$, we deduce that $T_2 = R_2A - Q_{1\Omega}R_2AQ_{3\Omega}$ belongs to $Rad(\mathcal{A})$. Since T_2 was arbitrary, $Rad(\mathcal{A})$ contains $Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}$. Consequently, $T_1 = R_1 - Q_{1\Omega}R_1Q_{3\Omega} - Q_{2\Omega}R_1Q_{3\Omega}$ belongs to $Rad(\mathcal{A})$. This proves that $Rad(\mathcal{A})$ contains $Q_{1\Omega}\mathbb{M}_n Q_{2\Omega}$, and therefore

$$Rad(\mathcal{A}) = Q_{1\Omega}\mathbb{M}_n Q_{2\Omega} \dotplus Q_{1\Omega}\mathbb{M}_n Q_{3\Omega} \dotplus Q_{2\Omega}\mathbb{M}_n Q_{3\Omega}.$$

We conclude that $\mathcal{A} = \mathbb{C}(Q_{1\Omega} + Q_{2\Omega}) + \mathbb{C}Q_{3\Omega} \dotplus Rad(\mathcal{A})$ is the algebra from Example 4.1.1(iii), as claimed. ■

With the proof of Theorem 4.4.5 complete, we are left only to classify the projection compressible type III algebras such that $\mathcal{F}_{III}$ contains a pair Ω in which $Q_{1\Omega}$ and $Q_{3\Omega}$ linked, yet neither of these projections is linked to $Q_{2\Omega}$. It will be shown in Theorem 4.4.7

that such an algebra is necessarily the unitization of an $\mathcal{LR}$-algebra. Unsurprisingly, the proof of this result shares many similarities with that of Theorem 4.3.6, the analogous result for algebras of type II. One must modify the arguments in the type III case, however, to reflect the absence of a block in $BD(\mathcal{A})$ of size 2 or greater.

The first step in this direction is the following adaptation of Lemma 4.3.5 to the type III setting.

Lemma 4.4.6. *Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_4$, and suppose that $\mathcal{F}_{III}(\mathcal{A})$ contains a pair $\Omega = (k, \bigoplus_{i=1}^4 \mathcal{V}_i)$ with $k = 3$. Assume that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked.*

(i) If there exist a constant $t \in \mathbb{C}$ and for each $i \in \{1,2,3\}$, an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$ such that

$$\langle Ae_1^{(2)}, e_1^{(1)}\rangle = t\left(\langle Ae_1^{(1)}, e_1^{(1)}\rangle - \langle Ae_1^{(2)}, e_1^{(2)}\rangle\right) \text{ for all } A \in \mathcal{A},$$

then $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$ for every $A \in \mathcal{A}$.

(ii) If there exist a constant $t \in \mathbb{C}$ and for each $i \in \{1,2,3\}$, an orthonormal basis $\left\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{i\Omega}}^{(i)}\right\}$ for $\operatorname{ran}(Q_{i\Omega})$ such that

$$\langle Ae_1^{(3)}, e_1^{(2)}\rangle = t\left(\langle Ae_1^{(2)}, e_1^{(2)}\rangle - \langle Ae_1^{(3)}, e_1^{(3)}\rangle\right) \text{ for all } A \in \mathcal{A},$$

then $\langle Ae_1^{(3)}, e_i^{(1)}\rangle = t\langle Ae_1^{(2)}, e_i^{(1)}\rangle$ for every $A \in \mathcal{A}$ and each $i \in \{1,2\}$.

Proof. First note that since $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked, Proposition 4.4.2 implies that neither of these projections is linked to $Q_{2\Omega}$.

We begin by considering the situation of (i). With respect to the basis $\mathcal{B} = \left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$, each A in $\mathcal{A}$ can be expressed as a matrix of the form

$$A = \left[\begin{array}{cc|c|c} \alpha & 0 & t(\alpha-\beta) & a_{14} \\ & \alpha & a_{23} & a_{24} \\ \hline & & \beta & a_{34} \\ \hline & & & \alpha \end{array}\right]$$

for some α, β, and a_{ij} in $\mathbb{C}$. Consider the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

It is straightforward to check that $\frac{1}{2}P$ is a projection in $\mathbb{M}_4$ and every element $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation $2b_{13} - t(b_{22} - b_{33}) = 0$. But if $A \in \mathcal{A}$ is as above, and $C = (c_{ij})$ denotes the operator $(P\!AP)^2$, then

$$2c_{13} - t(c_{22} - c_{33}) = 8t(ta_{34} + a_{14})(\alpha - \beta).$$

Since $\mathcal{A}$ is projection compressible, C belongs to $P\mathcal{A}P$, and hence the right-hand side of this equation must be 0 for all $\mathcal{A}$. Since α and β may be chosen arbitrarily, it follows that either $t = 0$ or $a_{14} = -ta_{34}$ for all A in $\mathcal{A}$.

If $t = 0$, then each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \left[\begin{array}{cc|cc} \alpha & a_{23} & 0 & a_{24} \\ & \beta & 0 & a_{34} \\ \hline & & \alpha & a_{14} \\ & & & \alpha \end{array}\right]$$

with respect to the reordered basis $\left\{e_2^{(1)}, e_1^{(2)}, e_1^{(1)}, e_1^{(3)}\right\}$ for $\mathbb{C}^4$. In this case, Theorem 4.1.2 demonstrates that $a_{14} = \langle Ae_1^{(3)}, e_1^{(1)}\rangle = 0$ for all A. Thus, the equation $a_{14} = -ta_{34}$ holds in either case. That is, $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$ for all $A \in \mathcal{A}$.

We now turn our attention to the proof of (ii). In this setting, every A in $\mathcal{A}$ admits a matrix representation of the form

$$A = \left[\begin{array}{cc|c|c} \alpha & 0 & a_{13} & a_{14} \\ & \alpha & a_{23} & a_{24} \\ \hline & & \beta & t(\beta - \alpha) \\ \hline & & & \alpha \end{array}\right]$$

with respect to the basis $\mathcal{B} = \left\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\right\}$. With P as in (i), every element $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation $2b_{34} - t(b_{33} - b_{22}) = 0$. It can be verified, however, that if A is as above and $C := (P\!AP)^2 = (c_{ij})$, then

$$2c_{34} - t(c_{33} - c_{22}) = 8t(ta_{13} - a_{14})(\alpha - \beta).$$

Once again, it follows that either $t = 0$ or $a_{14} = ta_{13}$ for all $A \in \mathcal{A}$.

Suppose first that $t = 0$. Let P' denote the matrix

$$P' = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

written with respect to the basis $\mathcal{B}$, so $\frac{1}{3}P'$ is a projection in $\mathbb{M}_4$. Direct computations show that if $B = (b_{ij})$ belongs to $P'\mathcal{A}P'$, then $b_{33} + 2b_{31} - b_{43} - 2b_{41} - b_{22} = 0$. But with A as above and $C \coloneqq (P'AP')^2 = (c_{ij})$, we have

$$c_{33} + 2c_{31} - c_{43} - 2c_{41} - c_{22} = 27a_{14}(\beta - \alpha).$$

Since α and β may be selected arbitrarily, it follows that $a_{14} = \langle Ae_1^{(3)}, e_1^{(1)} \rangle = 0$ for all A in $\mathcal{A}$. Thus, the equation $a_{14} = ta_{13}$ holds in either case. That is,

$$\langle Ae_1^{(3)}, e_1^{(1)} \rangle = t\langle Ae_1^{(2)}, e_1^{(1)} \rangle \text{ for all } A \in \mathcal{A}.$$

Finally, by switching the order of the first two vectors in $\mathcal{B}$ and repeating the above analysis with respect to this reordered basis, one may deduce that

$$\langle Ae_1^{(3)}, e_2^{(1)} \rangle = t\langle Ae_1^{(2)}, e_2^{(1)} \rangle \text{ for all } A \in \mathcal{A}.$$

Thus, the proof is complete. ∎

Theorem 4.4.7. *Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$. If there is a pair Ω in $\mathcal{F}_{III}(\mathcal{A})$ such that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked, then $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra. Consequently, $\mathcal{A}$ is idempotent compressible.*

Proof. Let Ω be a pair in $\mathcal{F}_{III}(\mathcal{A})$ such that $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we will assume that $n_{1\Omega} = \max\{n_{1\Omega}, n_{3\Omega}\} \geq 2$. Note that by Proposition 4.4.2, neither of these projections is linked to $Q_{2\Omega}$.

By Theorem 2.1.12, there are subprojections $Q_1' \leq Q_{1\Omega}$ and $Q_3' \leq Q_{3\Omega}$ such that

$$Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q_1'\mathbb{M}_nQ_{2\Omega} \quad \text{and}$$
$$Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}\mathbb{M}_nQ_3'.$$

As in the proof of Theorem 4.3.6, we will show that $\mathcal{A}$ is similar to

$$\mathcal{A}_0 := (Q_1' + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_3') + \mathbb{C}I,$$

and hence that $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra. To show that this is the case, we must first determine the structure of $Q_{1\Omega}\mathcal{A}Q_{3\Omega}$.

Define projections $Q_1'' := Q_{1\Omega} - Q_1'$ and $Q_3'' := Q_{3\Omega} - Q_3'$. For each $i \in \{1, 3\}$, let $\left\{e_1^{(i)}, e_2^{(i)}, \dots, e_{n_{i\Omega}}^{(i)}\right\}$ be an orthonormal basis for $\operatorname{ran}(Q_{i\Omega})$ such that if $Q_i'' \neq 0$, then

$$\operatorname{ran}(Q_i'') = \vee \left\{e_1^{(i)}, e_2^{(i)}, \dots, e_{\ell_i}^{(i)}\right\}$$

for some index $\ell_i \in \{1, 2, \dots, n_{i\Omega}\}$. Furthermore, let $e_1^{(2)}$ be a unit vector in $\operatorname{ran}(Q_{2\Omega})$. Since $\mathcal{A}$ is similar to $BD(\mathcal{A}) \dot{+} Rad(\mathcal{A})$ via an upper triangular similarity, there are matrices $T_1 \in Q_1''\mathbb{M}_n Q_{2\Omega}$ and $T_2 \in Q_{2\Omega}\mathbb{M}_n Q_3''$ such that for each $A \in \mathcal{A}$,

$$\begin{aligned} Q_1'' A Q_{2\Omega} &= (Q_1'' A Q_1'')T_1 - T_1(Q_{2\Omega} A Q_{2\Omega}) \quad \text{and} \\ Q_{2\Omega} A Q_3'' &= (Q_{2\Omega} A Q_{2\Omega})T_2 - T_2(Q_3'' A Q_3''). \end{aligned}$$

We may obtain information on the structure of $Q_1''\mathcal{A}Q_{3\Omega}$ by appealing to Lemma 4.4.6. Of course, there is little to be said when $Q_1'' = 0$. If instead $Q_1'' \neq 0$, fix arbitrary indices $i \in \{1, 2, \dots, \ell_1\}$, $i' \in \{1, 2, \dots, n_{1\Omega}\} \setminus \{i\}$, and $j \in \{1, 2, \dots, n_{3\Omega}\}$. Define $\mathcal{B} = \left\{e_i^{(1)}, e_{i'}^{(1)}, e_1^{(2)}, e_j^{(3)}\right\}$ and let P denote the orthogonal projection onto the span of $\mathcal{B}$. With respect to the basis $\mathcal{B}$ for $\operatorname{ran}(P)$, every member of $P\mathcal{A}P$ can be written as a matrix of the form

$$\left[\begin{array}{cc|c|c} \alpha & 0 & t_i^{(1)}(\alpha - \beta) & a_{14} \\ & \alpha & a_{23} & a_{24} \\ \hline & & \beta & a_{34} \\ \hline & & & \alpha \end{array}\right],$$

where $t_i^{(1)} := \langle T_1 e_1^{(2)}, e_i^{(1)} \rangle$. Thus, an application Lemma 4.4.6 (i) demonstrates that

$$\langle A e_j^{(3)}, e_i^{(1)} \rangle = -t_i^{(1)} \langle A e_j^{(3)}, e_1^{(2)} \rangle \text{ for all } A \in \mathcal{A}.$$

Since the indices i, i', and j were selected arbitrarily, it follows that

$$Q_1'' A Q_{3\Omega} = -T_1 Q_{2\Omega} A Q_{3\Omega} \text{ for all } A \in \mathcal{A}.$$

A similar argument can be used to determine the structure of $Q_{1\Omega}\mathcal{A}Q_3''$. Indeed, there is nothing to be said when $Q_3''=0$. If instead $Q_3''\neq 0$, choose distinct indices i and i' in $\{1,2,\ldots,n_{1\Omega}\}$, and let $j\in\{1,2,\ldots,\ell_3\}$ be arbitrary. Define $\mathcal{C}=\left\{e_i^{(1)},e_{i'}^{(1)},e_1^{(2)},e_j^{(3)}\right\}$, and let P' denote the orthogonal projection onto the span of $\mathcal{C}$. The compression $P'\mathcal{A}P'$ is an algebra of the form described in Lemma 4.4.6 (ii), and hence this result indicates that each $A\in\mathcal{A}$ satisfies the equation

$$\langle Ae_j^{(3)},e_i^{(1)}\rangle = t_j^{(2)}\langle Ae_1^{(2)},e_i^{(1)}\rangle,$$

where $t_j^{(2)}\coloneqq\langle T_2e_j^{(3)},e_1^{(2)}\rangle$. Again, the fact that i, i', and j were chosen arbitrarily implies that $Q_{1\Omega}AQ_3''=Q_{1\Omega}AQ_{2\Omega}T_2$ for all $A\in\mathcal{A}$.

Our findings thus far indicate that with respect to the decomposition

$$\mathbb{C}^n=\operatorname{ran}(Q_1'')\oplus\operatorname{ran}(Q_1')\oplus\operatorname{ran}(Q_{2\Omega})\oplus\operatorname{ran}(Q_3'')\oplus\operatorname{ran}(Q_3'),$$

each A in $\mathcal{A}$ can be expressed as a matrix of the form

$$A=\left[\begin{array}{c|c|c|c|c} \alpha I & 0 & (\alpha-\beta)T_1 & A_{14} & A_{15}\\ \hline & \alpha I & J_1 & A_{24} & A_{25}\\ \hline & & \beta & (\beta-\alpha)T_2 & J_2\\ \hline & & & \alpha I & 0\\ \hline & & & & \alpha I \end{array}\right],$$

for some $\alpha,\beta\in\mathbb{C}$, $J_1\in Q_1'Rad(\mathcal{A})Q_{2\Omega}$, $J_2\in Q_{2\Omega}Rad(\mathcal{A})Q_3'$, and operators A_{ij} satisfying the equations

$$\left[\begin{array}{c|c} A_{14} & A_{15}\end{array}\right]=-T_1\left[\begin{array}{c|c}(\beta-\alpha)T_2 & J_2\end{array}\right]\quad\text{and}\quad\left[\begin{array}{c}A_{14}\\ \hline A_{24}\end{array}\right]=\left[\begin{array}{c}(\alpha-\beta)T_1\\ \hline J_1\end{array}\right]T_2.$$

To see that $\mathcal{A}$ is similar to $\mathcal{A}_0=(Q_1'+Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega}+Q_3')+\mathbb{C}I$, and hence is the unitization of an $\mathcal{LR}$-algebra, consider the operator $S\coloneqq I-T_1-T_2$. This map is invertible with $S^{-1}=I+T_1+T_2+T_1T_2$. In addition, we have that for A as above,

$$S^{-1}AS=\left[\begin{array}{c|c|c|c|c} \alpha I & 0 & 0 & 0 & 0\\ \hline & \alpha I & J_1 & 0 & A_{25}\\ \hline & & \beta & 0 & J_2\\ \hline & & & \alpha I & 0\\ \hline & & & & \alpha I \end{array}\right].$$

It is now apparent that $S^{-1}\mathcal{A}S$ is a type III algebra that admits a reduced block upper triangular form with respect to the above decomposition. Since

$$Q_{1\Omega}Rad(S^{-1}\mathcal{A}S)Q_{2\Omega} = Q_1'Rad(\mathcal{A})Q_{2\Omega} = Q_1'\mathbb{M}_nQ_{2\Omega} \quad \text{and}$$
$$Q_{2\Omega}Rad(S^{-1}\mathcal{A}S)Q_{3\Omega} = Q_{2\Omega}Rad(\mathcal{A})Q_3' = Q_{2\Omega}\mathbb{M}_nQ_3',$$

it follows from Lemma 2.3.10 (ii) that $S^{-1}\mathcal{A}S = (Q_1' + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_3') + \mathbb{C}I = \mathcal{A}_0$. ■

§4.5 Main Results and Applications

The analysis of unital projection compressible subalgebras of $\mathbb{M}_n$, $n \geq 4$, carried out in §4.1-4.4 provides a complete description of these algebras up to transpose similarity. In addition, it was observed that every unital projection compressible algebra in this setting also admits the idempotent compression property. We therefore obtain the following theorem, the main result of this chapter.

Theorem 4.5.1. *Let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$ for some integer $n \geq 4$. The following are equivalent.*

(i) $\mathcal{A}$ is projection compressible;

(ii) $\mathcal{A}$ is idempotent compressible;

(iii) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose similar to one of the algebras from Example 4.1.1.

Combining Theorems 3.2.7 and 4.5.1, we conclude that the two notions of compressibility coincide for all unital algebras.

Theorem 4.5.2. *A unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$, $n \geq 2$, is projection compressible if and only if it is idempotent compressible.*

In light of Theorem 4.5.2, we make the following definition.

Definition 4.5.3. A unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is *compressible* if $\mathcal{A}$ is projection compressible (equivalently, if $\mathcal{A}$ is idempotent compressible).

It is worth noting that nearly all of the classification results from §4.1-4.4 describe the various unital compressible subalgebras of $\mathbb{M}_n$ up to transpose *equivalence*, not just transpose similarity. Indeed, the only instance in which a description up to transpose equivalence was not achieved was in Theorem 4.4.4. There it was shown that a projection compressible type III algebra is either transpose equivalent to the type III algebra from Example 4.1.1(i), or transpose similar to the algebra from Example 4.1.1(ii).

The following proposition describes the similarity orbit of the algebra from Example 4.1.1(ii) up to unitary equivalence, thereby providing a characterization of the (unital) compressible subalgebras of $\mathbb{M}_n$, $n \geq 4$, up to transpose equivalence.

Proposition 4.5.4. *Let $n \geq 3$ be an integer, let Q_1 and Q_2 be mutually orthogonal rank-one projections in $\mathbb{M}_n$, and define $Q_3 := I - Q_1 - Q_2$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that $e_1 \in \operatorname{ran}(Q_1)$, $e_2 \in \operatorname{ran}(Q_2)$, and $e_i \in \operatorname{ran}(Q_3)$ for all $i \geq 3$. If*

$$\begin{aligned}\mathcal{A} &:= \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_n Q_3 \\ &= \left\{ \begin{bmatrix} \alpha & 0 & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\}\end{aligned}$$

denotes the compressible algebra from Example 4.1.1(ii), and $\mathcal{B}$ is an algebra that is similar to $\mathcal{A}$, then there is some $t \in \mathbb{C}$ such that $\mathcal{B}$ is unitarily equivalent to

$$\begin{aligned}\mathcal{A}_t &:= \{A + t\left(\langle Ae_1, e_1\rangle - \langle Ae_2, e_2\rangle\right) e_1 \otimes e_2^* : A \in \mathcal{A}\} \\ &= \left\{ \begin{bmatrix} \alpha & t(\alpha - \beta) & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in Q_i \mathbb{M}_n Q_j \right\}.\end{aligned}$$

Proof. Suppose that $\mathcal{B} = S^{-1}\mathcal{A}S$ for some invertible $S \in \mathbb{M}_n$. For all indices $i \in \{1, 2\}$ and $j \in \{3, 4, \ldots, n\}$, define $E_{ij} := e_i \otimes e_j^*$ and $E'_{ij} := S^{-1}E_{ij}S$. Furthermore, define $Q'_i := S^{-1}Q_i S$ for $i \in \{1, 2, 3\}$. Observe that

$$\mathcal{B} = S^{-1}\mathcal{A}S = \operatorname{span}\left\{Q'_1, Q'_2, Q'_3, E'_{ij} : i \in \{1, 2\}, j \in \{3, 4, \ldots, n\}\right\}.$$

Let $\{f_1, f_2, \ldots, f_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that f_1 and f_2 belong to $\ker(Q'_3)$. Let P_1, P_2, and P_3 denote the orthogonal projections onto $\mathbb{C}f_1$, $\mathbb{C}f_2$, and $\operatorname{span}\{f_i : i \geq 3\} = \ker(Q'_3)^\perp$, respectively. Since $P_3 Q'_3 P_3 = P_3$ and $Q'_3 Q'_1 = Q'_3 Q'_2 = 0$, we have that $Q'_1 = (P_1 + P_2)Q'_1$ and $Q'_2 = (P_1 + P_2)Q'_2$.

Note that since $Q_1'Q_2' = Q_2'Q_1' = 0$, we may adjust the first two basis vectors if necessary to assume that Q_1' and Q_2' are upper triangular with respect to $\{f_1, f_2, \ldots, f_n\}$, and $\langle Q_i' f_j, f_j\rangle = \delta_{ij}$ for $i, j \in \{1, 2\}$. Thus, there are matrices X_{ij}, Y_{ij}, and Z_{ij}, and a constant $t \in \mathbb{C}$ such that with respect to the decomposition $\mathbb{C}^n = \operatorname{ran}(P_1) \oplus \operatorname{ran}(P_2) \oplus \operatorname{ran}(P_3)$,

$$Q_1' = \left[\begin{array}{c|c|c} 1 & t & X_{13} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad Q_2' = \left[\begin{array}{c|r|c} 0 & -t & Y_{13} \\ \hline 0 & 1 & Y_{23} \\ \hline 0 & 0 & 0 \end{array}\right], \quad \text{and} \quad Q_3' = \left[\begin{array}{c|c|c} 0 & 0 & Z_{13} \\ \hline 0 & 0 & Z_{23} \\ \hline 0 & 0 & I \end{array}\right].$$

Finally, since $E_{ij}' = (Q_1' + Q_2')E_{ij}'Q_3'$, we have that $E_{ij}' = (P_1 + P_2)E_{ij}'P_3$ for all indices i and j. Dimension considerations then imply that

$$\operatorname{span}\left\{E_{ij}' : i \in \{1, 2\}, j \in \{3, 4, \ldots, n\}\right\} = (P_1 + P_2)\mathbb{M}_n P_3,$$

and therefore

$$\mathcal{B} = \{B + t\left(\langle Bf_1, f_1\rangle - \langle Bf_2, f_2\rangle\right) f_1 \otimes f_2^* : B \in \mathbb{C}P_1 + \mathbb{C}P_2 + \mathbb{C}P_3 + (P_1 + P_2)\mathbb{M}_n P_3\}.$$

We conclude that $\mathcal{A}_t = U^*\mathcal{B}U$ where $U \in \mathbb{M}_n$ is the unitary satisfying $Ue_i = f_i$. ∎

Corollary 4.5.5. *Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$. The following are equivalent.*

(i) $\mathcal{A}$ is compressible;

(ii) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose equivalent to the algebra from Example 4.1.1(i), the algebra from Example 4.1.1(iii), or the algebra $\mathcal{A}_t$ from Proposition 4.5.4.

Remark 4.5.6. The above result, together with Theorem 4.5.1, implies that if $\mathcal{A}$ is transpose similar to an algebra from Theorem 4.5.1 (iii), then $\mathcal{A}$ is transpose equivalent to an algebra from Corollary 4.5.5 (ii). Indeed, Proposition 4.5.4 makes this fact explicit for the algebra in Example 4.1.1(ii), while in §2.1 it was shown that the class of $\mathcal{LR}$-algebras is invariant under transpose similarity. Arguments akin to those in the proof of Proposition 4.5.4 can be used to show that any algebra transpose similar to the algebra from Example 4.1.1(i) (resp. Example 4.1.1(iii)) is in fact, transpose equivalent to it.

§4.5.1 Applications

Here we investigate some of the applications of the classification of unital compressible algebras. It follows from Theorem 4.5.2 that the class of all such algebras is invariant under similarity and transposition. Using this fact, it is relatively straightforward to determine which unital semi-simple algebras admit the compression property.

Corollary 4.5.7. *Let $n \geq 2$ be an integer, and let $\mathcal{A}$ be a unital, semi-simple subalgebra of $\mathbb{M}_n$. The following are equivalent:*

(i) $\mathcal{A}$ is compressible;

(ii) $\mathcal{A} = \mathbb{C}I$ or $\mathcal{A}$ is similar to $\mathbb{M}_k \oplus \mathbb{C}I_{n-k}$ for some positive integer k.

Proof. Since $\mathbb{C}I$ and $\mathbb{M}_k \oplus \mathbb{C}I_{n-k}$ are unitizations of $\mathcal{LR}$-algebras, it is obvious that (ii) implies (i). Assume now that (i) holds, so $\mathcal{A}$ is a unital, semi-simple subalgebra of $\mathbb{M}_n$ that admits the compression property. Assume as well that $\mathcal{A}$ is in reduced block upper triangular form with respect to some orthogonal decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$. By Theorem 2.3.9, $\mathcal{A}$ is similar to $\mathcal{B} := BD(\mathcal{A})$. It therefore suffices to prove that $\mathcal{B}$ is similar to an algebra of the form prescribed in (ii).

If $n = 2$, then $\mathcal{B}$ is equal to $\mathbb{C}I$, $\mathbb{C} \oplus \mathbb{C}$, or $\mathbb{M}_2$, and hence $\mathcal{B}$ is of the desired form. If instead $n = 3$, then either $\mathcal{B}$ is equal to $\mathbb{C}I$ or $\mathbb{M}_3$, or $\mathcal{B}$ is unitarily equivalent to $\mathbb{C} \oplus \mathbb{C}I_2$ or $\mathbb{M}_2 \oplus \mathbb{C}$. Indeed, the only other block diagonal subalgebra of $\mathbb{M}_3$ is the algebra of all 3×3 diagonal matrices. This algebra was shown to lack the compression property in Theorem 3.2.6, and hence cannot be similar to $\mathcal{B}$. Again we see that (ii) holds.

Suppose now that $n \geq 4$. By Theorem 4.1.2, there is at most one space $\mathcal{V}_i$ of dimension 2 or greater. If such a space exists, we may reindex the sum $\bigoplus_{i=1}^m \mathcal{V}_i$ if necessary and assume that $\dim(\mathcal{V}_1) = k \geq 2$. Theorem 4.1.2 then implies that $\mathcal{V}_i$ is linked to $\mathcal{V}_j$ for all $i, j \geq 2$, so $\mathcal{B} = \mathbb{M}_k \oplus \mathbb{C}I_{n-k}$. If instead $\dim \mathcal{V}_i = 1$ for all i, then Theorem 4.1.2 indicates that with at most one exception, all spaces $\mathcal{V}_i$ are mutually linked. Thus, $\mathcal{B}$ is equal to $\mathbb{C}I$ or is unitarily equivalent to $\mathbb{C} \oplus \mathbb{C}I_{n-1}$. ■

Theorem 4.5.1 can also be used to quickly identify the operators $T \in \mathbb{M}_n$ such that $Alg(T, I)$—the unital algebra generated by T—is compressible.

Corollary 4.5.8. *Let $n \geq 2$ be an integer, and let $T \in \mathbb{M}_n$. The following are equivalent:*

(i) $Alg(T, I)$ is compressible;

(ii) $Alg(T, I)$ *is the unitization of an* $\mathcal{LR}$*-algebra;*

(iii) $T \in span\{I, R\}$ *for some* $R \in \mathbb{M}_n$ *of rank* 1.

Proof. It is clear that (ii) implies (i). To see that (i) implies (iii), assume that $Alg(T, I)$ is compressible. It follows that $Alg(S^{-1}TS, I) = S^{-1}Alg(T, I)S$ is compressible for all invertible $S \in \mathbb{M}_n$; hence we may assume that T is in Jordan canonical form with respect to the standard basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$.

If T has a Jordan block of size at least 3, then $Alg(T, I)$ admits a principal compression of the form

$$\left\{ \begin{bmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix} : x, y, z \in \mathbb{C} \right\}.$$

Since this algebra was shown to lack the compression property in Theorem 3.2.4, it must be the case that each Jordan block of T has size at most 2. Note as well that if two or more Jordan blocks of size 2 were present, then $Alg(T, I)$ would lack the compression property by Theorem 4.1.2. Consequently, T has at most one Jordan block of size 2, and the remaining blocks have size 1.

If a Jordan block of size 2 occurs, then T cannot have two or more distinct eigenvalues. Indeed, if T had at least two distinct eigenvalues, then $Alg(T, I)$ would admit a principal compression that is unitarily equivalent to

$$\left\{ \begin{bmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{bmatrix} : x, y, z \in \mathbb{C} \right\}.$$

By Theorem 3.2.2, this algebra is not compressible—a contradiction. Thus, T must be unitarily equivalent to $e_1 \otimes e_2^* + \alpha I$ for some $\alpha \in \mathbb{C}$. We conclude that $T = \alpha I + R$ for some R in $\mathbb{M}_n$ of rank 1.

Suppose now that every Jordan block of T is 1×1, so T is diagonal. If T had at least three distinct eigenvalues, then the algebra $\mathcal{D}$ of all 3×3 diagonal matrices could be obtained as a principal compression of $Alg(T, I)$. Since no algebra similar to $\mathcal{D}$ is projection compressible by Theorem 3.2.6, this is not possible. Therefore, T has at most two distinct eigenvalues. By Theorem 4.1.2, one of the eigenvalues must have multiplicity 1. We deduce that either T has exactly one eigenvalue, and hence is a multiple of the identity; or T has exactly two eigenvalues, and hence is a rank-one perturbation of a multiple of the identity. Thus, (iii) holds in this case as well.

Finally, we will show that (iii) implies (ii). Suppose that $T \in \text{span}\{I, R\}$ for some rank-one operator $R \in \mathbb{M}_n$. That is, $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$. If $\beta = 0$, then $Alg(T, I) = \mathbb{C}I$. Otherwise, βR has rank 1, and hence $Alg(T, I) = Alg(\beta R) + \mathbb{C}I$ is the unitization of an $\mathcal{LR}$-algebra by Proposition 2.1.15. ■

It is interesting to note that in the 3-dimensional case, the matrices of the form $\alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$ and $R \in \mathbb{M}_3$ of rank one are exactly those with two or more Jordan blocks corresponding to a common eigenvalue. Such matrices are said to be *derogatory* [12, Definition 1.4.4]. One may therefore view Corollary 4.5.8 as a higher-dimensional analogue of Corollary 3.1.3.

§4.6 Future Directions and Open Questions

Throughout this exposition we have devoted our attention almost exclusively to unital subalgebras of $\mathbb{M}_n$. Of course, it is reasonable to ask which non-unital algebras admit the projection or idempotent compression properties. In particular, it would be interesting to know whether or not the equivalence of these notions established in this manuscript in the unital case extends to non-unital algebras as well.

Question 1. *For each integer $n \geq 3$, which non-unital subalgebras of $\mathbb{M}_n$ admit the projection compression property? Is it true that the notions of projection compressibility and idempotent compressibility coincide for non-unital subalgebras of $\mathbb{M}_n$?*

By Proposition 2.1.9, if a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ admits the projection (resp. idempotent) compression property, then so too does its unitization. Theorems 3.2.7 and 4.5.1 therefore offer considerable insight into which non-unital projection (resp. idempotent) compressible algebras can exist in $\mathbb{M}_n$. For instance, Theorem 4.5.1 indicates that if $\mathcal{A}$ is a projection compressible subalgebra of $\mathbb{M}_n$, $n \geq 4$, then $\widetilde{A}$ is the unitization of an $\mathcal{LR}$-algebra or transpose similar to one of the algebras from Example 4.1.1.

The approach to studying the compression properties of a non-unital algebra by examining these properties for its unitization is used in the proof of the following proposition. This result is a non-unital analogue of Corollary 4.5.8.

Proposition 4.6.1. *Let $n \geq 3$ be an integer, and let $T \in \mathbb{M}_n$. The following are equivalent:*

(i) $Alg(T)$ is projection compressible;

(ii) $Alg(T)$ is idempotent compressible;

(iii) $Alg(T)$ is an $\mathcal{LR}$-algebra, or the unitization thereof;

(iv) $T \in span\{I, R\}$ for some $R \in \mathbb{M}_n$ of rank 1, and 0 does not occur as an eigenvalue of T with algebraic multiplicity 1.

Proof. It is clear that (iii) implies (ii), and (ii) implies (i).

To see that (i) implies (iv), note that if $Alg(T)$ is projection compressible, then so too is $Alg(T, I)$. By Corollary 4.5.8, there is a rank-one operator $R \in \mathbb{M}_n$ such that $T \in \text{span}\{I, R\}$. For the final claim, write $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$, and suppose to the contrary that $\lambda = 0$ is an eigenvalue of T with algebraic multiplicity 1. Since $rank(R) = 1$, there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$ with respect to which

$$\beta R = \gamma_1 e_1 \otimes e_1^* + \gamma_2 e_1 \otimes e_2^*$$

for some constants $\gamma_1, \gamma_2 \in \mathbb{C}$. Thus, when expressed as a matrix with respect to this basis, T is upper triangular with diagonal entries $\alpha + \gamma_1$ with multiplicity 1, and α with multiplicity $n - 1$. It must therefore be the case that $\alpha + \gamma_1 = 0$ and $\alpha \neq 0$.

Let P denote the orthogonal projection onto span$\{e_1, e_2, e_3\}$ and define $T' := PTP$. With respect to the ordered basis $\mathcal{B} = \{e_1, e_2, e_3\}$ for ran(P),

$$T' = \begin{bmatrix} 0 & \gamma_2 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$

Thus, since $Alg(T) = \mathbb{C}T$ is projection compressible, $P' Alg(T) P' = \mathbb{C}P'T'P'$ is an algebra for all projections $P' \leq P$. Consider the matrix

$$P' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

written with respect to the basis $\mathcal{B}$. It is easy to see that $\frac{1}{2}P'$ is a subprojection of P. Moreover, it is straightforward to show that $\langle Be_2, e_2 \rangle = 4\langle Be_1, e_1 \rangle$ for all $B \in P' Alg(T) P'$. One may verify, however, that

$$\langle (P'TP')^2 e_2, e_2 \rangle - 4\langle (P'TP')^2 e_1, e_1 \rangle = 8\alpha^2 \neq 0,$$

and thus $(P'TP')^2 \notin P' Alg(T) P'$. This is clearly a contradiction.

It remains to show that (iv) implies (iii). To this end, let T and R be as in (iv), and write $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ with respect to which

$$\beta R = \gamma_1 e_1 \otimes e_1^* + \gamma_2 e_1 \otimes e_2^*$$

for some $\gamma_1, \gamma_2 \in \mathbb{C}$.

First suppose that $\alpha = 0$, so $Alg(T) = Alg(\beta R)$. If $\beta = 0$ then this algebra is trivial. Otherwise, $Alg(T)$ is an $\mathcal{LR}$-algebra by Proposition 2.1.15. If instead $\alpha \neq 0$, then our assumptions on T imply that $\alpha + \gamma_1 \neq 0$. Consequently,

$$I = \left(\frac{1}{\alpha} + \frac{1}{\alpha + \gamma_1}\right) T - \frac{1}{\alpha(\alpha + \gamma_1)} T^2 \in Alg(T).$$

It follows that $Alg(T) = Alg(T, I)$, so $Alg(T)$ is the unitization of an $\mathcal{LR}$-algebra by Corollary 4.5.8. ∎

The following result may be a useful tool in the classification of non-unital projection compressible subalgebras of $\mathbb{M}_n$. It outlines a simple necessary condition that may prevent several non-unital algebras from admitting the projection compression property.

Proposition 4.6.2. *Let $\mathcal{B} = \{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{C}^3$. Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$ consisting of matrices that are upper triangular with respect to $\{e_1, e_2, e_3\}$, and such that $\langle Ae_2, e_2\rangle = 0$ for all $A \in \mathcal{A}$. If there exists an element $A \in \mathcal{A}$ such that $\langle Ae_1, e_1\rangle \neq \langle Ae_2, e_1\rangle$ and $\langle Ae_3, e_3\rangle \neq \langle Ae_3, e_2\rangle$, then $\mathcal{A}$ is not projection compressible.*

Proof. Let A be as in the statement above. Consider the matrix

$$P = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

written with respect to $\mathcal{B}$, and note that $\frac{1}{3}P$ is a projection in $\mathbb{M}_3$. It is routine to verify that any $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation $b_{21} - b_{31} - b_{22} + b_{32} = 0$. But if $C := (PAP)^2 = (c_{ij})$, then

$$c_{21} - c_{31} - c_{22} + c_{32} = 27\left(\langle Ae_1, e_1\rangle - \langle Ae_2, e_1\rangle\right)\left(\langle Ae_3, e_3\rangle - \langle Ae_3, e_2\rangle\right) \neq 0$$

Consequently, C does not belong to $P\mathcal{A}P$, so $\mathcal{A}$ is not projection compressible. ∎

As an immediate consequence of the above result, the subalgebra of $\mathbb{M}_3$ consisting of all matrices that are strictly upper triangular with respect to the standard basis is not projection compressible.

The notions of projection compressibility and idempotent compressibility can be naturally extended to algebras of bounded linear operators acting on a Hilbert space $\mathcal{H}$ of arbitrary dimension. It would therefore be interesting to determine whether or not analogues of our results can be established in this setting.

Question 2. *Let $\mathcal{H}$ be complex infinite-dimensional Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. Which subalgebras of $\mathcal{B}(\mathcal{H})$ admit the projection compression property? Is it true that every projection compressible algebra is in fact, idempotent compressible?*

Certain elementary results on compressible subalgebras of $\mathbb{M}_n$ are easily seen to generalize to subalgebras of $\mathcal{B}(\mathcal{H})$. For instance, it is clear that both the projection and idempotent compression properties pass to unitizations and are enjoyed by (not necessarily closed) one- or two-sided ideals of $\mathcal{B}(\mathcal{H})$. We note, however, that much of our analysis from Chapters 3 and 4 will not immediately carry over to the infinite-dimensional setting, as there does not exist a direct analogue of Burnside's Theorem for general operator algebras.

One approach to understanding the structure of a projection (resp. idempotent) compressible operator algebra $\mathcal{A}$ would be to apply Theorems 3.2.7 and 4.5.1 to the unital compressions $P\widetilde{\mathcal{A}}P$, where P is a projection (resp. idempotent) of finite rank. This technique may have its limits, however, as there could exist operator algebras $\mathcal{A}$ that lack the projection compression property, yet such that $P\mathcal{A}P$ is an algebra for all finite-rank projections P.

With this in mind, the most viable avenue for understanding the compression properties in this setting may be to first obtain an explanation as to why these notions coincide for unital subalgebras of $\mathbb{M}_n$. Despite a considerable amount of effort, we have been unable to provide a direct proof of this fact that does not involve characterizing each class of algebras. Such a proof might help to explain why these algebras have the particular structures seen throughout Chapters 3 and 4.

Chapter 5

The Distance from a Rank $n-1$ Projection to the Nilpotents

In this chapter we address the problem of computing $\nu_{n-1,n}$, the distance from a projection of rank $n-1$ to the set of nilpotent operators acting on $\mathbb{C}^n$. We begin by establishing the notation that will be used throughout.

Fix a positive integer $n \geq 3$. As in previous chapters, let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ denote the algebra of all complex $n \times n$ matrices. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$, and define $\mathcal{T}_n$ to be the subalgebra of $\mathbb{M}_n$ consisting of all matrices that are strictly upper triangular with respect to $\{e_i\}_{i=1}^n$. Finally, let $E_0 = 0$, and define

$$E_k \coloneqq \sum_{i=1}^{k} e_i \otimes e_i^* \quad \text{for } k \in \{1, 2, \ldots, n\}.$$

In Chapter 1 it was observed that the distance $\nu_{n-1,n}$ is always achieved by some projection-nilpotent pair. Thus, since one may assume without loss of generality that a given nilpotent is upper triangular with respect to $\{e_i\}_{i=1}^n$, we have that

$$\nu_{n-1,n} = \min \left\{ \|Q - N\| : Q \in \mathcal{P}(\mathbb{C}^n),\, rank(Q) = n-1, N \in \mathcal{T}_n \right\}.$$

With this in mind, fix a projection $Q = (q_{ij})$ of rank $n-1$ that is of distance $\nu_{n-1,n}$ to $\mathcal{T}_n$. In addition, let $P = (p_{ij})$ denote the rank-one projection $I - Q$.

As was the case in MacDonald's computation of $\nu_{1,n}$, the Arveson Distance Formula will play a key role in our determination of $\nu_{n-1,n}$. This result was introduced in Chapter 1, though we restate it below for convenience.

Theorem 5.0.3 (Arveson Distance Formula). *The distance from an operator $A \in \mathbb{M}_n$ to $\mathcal{T}_n$ is*

$$\operatorname{dist}(A, \mathcal{T}_n) = \max_{1 \le i \le n} \|E_{i-1}^{\perp} A E_i\|.$$

The key application of the Arveson Distance Formula occurs in §5.1. There we derive closed-form expressions for the norms $\|E_{i-1}^{\perp} Q E_i\|$ in terms of the entries of Q. Next, we use the minimality of $\operatorname{dist}(Q, \mathcal{T}_n)$, together with the expressions for $\|E_{i-1}^{\perp} Q E_i\|$, to prove that all such norms necessarily share a common value. In particular, this will imply that $\|E_{i-1}^{\perp} Q E_i\| = \nu_{n-1,n}$ for all i.

In §5.2, we use the equations $\|E_{i-1}^{\perp} Q E_i\| = \nu_{n-1,n}$, together with some basic algebraic properties of Q, to derive a finite list of candidates for $\nu_{n-1,n}$. A closer examination of these candidates reveals that exactly one of them satisfies a certain necessary norm inequality from [17], and hence this value must be $\nu_{n-1,n}$.

In §5.3, we present a construction of all closest projection-nilpotent pairs. Additionally, we prove that any two such pairs are unitarily equivalent.

Finally, in §5.4, we discuss possible extensions of these results to projections of intermediate ranks. In particular, we conjecture a general formula for $\nu_{r,n}$ which appears to closely resemble estimates for $\nu_{r,n}$ for small values of n.

§5.1 Equality in the Arveson Distance Formula

The goal of this section is to derive a sequence of equations relating the entries of Q to the distance $\nu_{n-1,n}$. Our strategy will be to use the algebraic relations satisfied by the entries of Q to derive closed-form expressions for the norms $\|E_{i-1}^{\perp} Q E_i\|$. Next, we will relate these expression to $\nu_{n-1,n}$ through the Arveson Distance Formula.

We begin with a few important observations regarding the structure of the projections P and Q. Note that since P has rank one, any 2×2 principal compression of P must be singular. It follows that the determinant of

$$\begin{bmatrix} p_{ii} & p_{ij} \\ \overline{p_{ij}} & p_{jj} \end{bmatrix}$$

is zero for any choice of distinct indices i and j, and thus there are complex numbers z_{ij} of modulus 1 such that

$$p_{ij} = z_{ij} \sqrt{p_{ii} p_{jj}}.$$

As a result, the entries of Q satisfy

$$q_{ij} = -z_{ij}\sqrt{(1-q_{ii})(1-q_{jj})} \quad \text{for all } i \neq j.$$

It would be cumbersome to keep track of the complex numbers z_{ij} throughout the coming analysis. Fortunately, however, the following result indicates that one may assume without loss of generality that each z_{ij} is equal to 1.

Lemma 5.1.1. *If $R \in \mathbb{M}_n$ is a rank-one projection, then there is a diagonal unitary $U \in \mathbb{M}_n$ such that the entries of $S := U^*RU$ are non-negative real numbers. Furthermore, $\|E_{i-1}^{\perp}SE_i\| = \|E_{i-1}^{\perp}RE_i\|$ for all i.*

Proof. Suppose that $R = (r_{ij})$ with respect to the standard basis, and choose an index k so that $r_{kk} \neq 0$. Clearly such a k exists as $\mathrm{Tr}(R) > 0$. Let $U = \mathrm{diag}(u_1, u_2, \ldots, u_n)$ denote the diagonal unitary in $\mathbb{M}_n$ obtained by setting $u_k = 1$ and

$$u_j = \begin{cases} |r_{kj}|/r_{kj} & \text{if } r_{kj} \neq 0 \\ 1 & \text{if } r_{kj} = 0 \end{cases} \quad \text{for } j \geq 2.$$

If $S := U^*RU = (s_{ij})$, then $s_{ij} = \overline{u_i}u_j r_{ij}$. In particular, $s_{jj} = r_{jj}$ and $s_{kj} = |r_{kj}|$ for all j. Note that since S has rank 1 and $s_{kk} = r_{kk} \neq 0$, every row of S is a multiple of the k^{th} row. But S has a non-negative diagonal and non-negative k^{th} row, so every row of S must be a *non-negative* multiple of the k^{th} row. Finally, it is evident that $\|E_{i-1}^{\perp}SE_i\| = \|E_{i-1}^{\perp}RE_i\|$ for all i, as each projection E_i commutes with U. ■

By Lemma 5.1.1, one may assume that the rank $n-1$ projection Q of minimal distance to $\mathcal{T}_n$ is such that every entry of $P = I - Q$ is a non-negative real number. It then follows that the entries p_{ij} and q_{ij} satisfy the relations

$$p_{ij} = \sqrt{p_{ii}p_{jj}} \quad \text{and} \quad q_{ij} = -\sqrt{(1-q_{ii})(1-q_{jj})} \quad \text{for all } i \neq j. \tag{5.1}$$

These equations quickly lead to the useful identities

$$p_{ij}p_{ik} = p_{ii}p_{jk} \quad \text{and} \quad q_{ij}q_{ik} = -q_{jk}(1-q_{ii}) \quad \text{for all } i,j,k \text{ distinct.} \tag{5.2}$$

In the case of rank-one projections, MacDonald derived the distance formula of Theorem 1.0.2 by analysing a certain sequence $\{a_i\}_{i=0}^n$ associated to such a projection. For $P = (p_{ij})$, this sequence is defined by setting $a_0 = 0$ and

$$a_k = \sum_{i=1}^{k} p_{ii} = k - \sum_{i=1}^{k} q_{ii}, \quad k \in \{1, 2, \ldots, n\}. \tag{5.3}$$

When the entries of P are non-negative, P and Q are entirely determined by this sequence. Indeed,

$$P = e \otimes e^* \quad \text{and} \quad Q = I - e \otimes e^*$$

where $e = \begin{bmatrix} \sqrt{a_1 - a_0} & \sqrt{a_2 - a_1} & \cdots & \sqrt{a_n - a_{n-1}} \end{bmatrix}^T$. In particular, the diagonal entries of P and Q are given by

$$p_{kk} = a_k - a_{k-1} \quad \text{and} \quad q_{kk} = 1 - (a_k - a_{k-1}), \quad k \in \{1, 2, \ldots, n\}. \tag{5.4}$$

Note that $\{a_i\}_{i=0}^n$ increases monotonically from $a_0 = 0$ to $a_n = \mathrm{Tr}(P) = 1$. Moreover, by considering the projection $P = e \otimes e^*$ with e as above, it follows that any sequence increasing monotonically from 0 to 1 can be obtained in this way. We record this fact below for future reference.

Lemma 5.1.2. *If $\{a_i\}_{i=0}^n$ is a sequence that increases monotonically from $a_0 = 0$ to $a_n = 1$, then there is a rank-one projection $R = (r_{ij})$ in $\mathbb{M}_n$ such that $r_{ij} \geq 0$ for all i and j, and $a_k = \sum_{i=1}^k r_{ii}$ for each $k \in \{1, 2, \ldots, n\}$.*

In [16], MacDonald computed the values of $\|E_{i-1}^\perp P E_i\|$ in terms of the sequence $\{a_i\}_{i=0}^n$ and subsequently proved that all such norms must be equal when P is of minimal distance to $\mathcal{T}_n$. Our goal is to translate MacDonald's arguments to the case in which Q is of minimal distance to $\mathcal{T}_n$. Namely, we wish to obtain a formula for $\|E_{i-1}^\perp Q E_i\|$ in terms of $\{a_i\}_{i=0}^n$ and demonstrate that when Q is of minimal distance to $\mathcal{T}_n$, these norms share a common value.

The first step in this direction occurs in Lemma 5.1.5 wherein we analyse the singular values of $E_{k-1}^\perp Q E_k$. The proof of this result requires the following classical theorem of Cauchy (see [12, Theorem 4.3.17]).

Theorem 5.1.3 (Cauchy's Interlacing Theorem). *Let B be a self-adjoint matrix in $\mathbb{M}_n$. Fix an integer $k \in \{1, 2, \ldots, n\}$, and let $\widehat{B} \in \mathbb{M}_{n-1}$ be the self-adjoint matrix obtained by deleting the k^{th} row and k^{th} column from B. If $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of B, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ are the eigenvalues of $\widehat{B}$, then*

$$\lambda_j \leq \mu_j \leq \lambda_{j+1}$$

for all $j \in \{1, 2, \ldots, n-1\}$.

The following corollary to Theorem 5.1.3 highlights an important fact concerning self-adjoint operators with repeated eigenvalues. This fact will be a key tool in the proof of Lemma 5.1.5.

Corollary 5.1.4. *Let B be a self-adjoint matrix in $\mathbb{M}_n$. Fix an integer $k \in \{1, 2, \ldots, n\}$, and let $\widehat{B} \in \mathbb{M}_{n-1}$ be the self-adjoint matrix obtained by deleting the k^{th} row and k^{th} column from B.*

(i) If λ is an eigenvalue of B with multiplicity $m \geq 2$, then λ is an eigenvalue of $\widehat{B}$ with multiplicity at least $m-1$.

(ii) If λ is an eigenvalue of $\widehat{B}$ with multiplicity $m \geq 2$, then λ is an eigenvalue of B with multiplicity at least $m-1$.

We are now prepared to give a description of the singular values of $E_{k-1}^{\perp} Q E_k$, and therefore a description of $\|E_{k-1}^{\perp} Q E_k\|$, $k \in \{1, 2, \ldots, n\}$.

Lemma 5.1.5. *Let $Q = (q_{ij})$ be a projection in $\mathbb{M}_n$ of rank $n-1$, and let $\{a_i\}_{i=0}^n$ denote the non-decreasing sequence from equation (5.3). For $k \in \{1, 2, \ldots, n\}$, define $Q_k \coloneqq E_{k-1}^{\perp} Q E_k$, and let B_k denote the restriction of $Q_k^* Q_k$ to the range of E_k.*

(i) If $q_{ij} \leq 0$ for all $i \neq j$, then the entries of $B_k = (b_{ij})$ are given by

$$b_{ij} = \begin{cases} q_{kk} - a_{k-1}(1 - q_{kk}) & \text{if } i = j = k, \\ (1 - a_{k-1})(1 - q_{ii}) & \text{if } i = j \neq k, \\ -(1 - a_{k-1}) q_{ij} & \text{if } i, j, k \text{ are distinct}, \\ a_{k-1} q_{ij} & \text{otherwise.} \end{cases}$$

(ii) If $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$ are the eigenvalues of B_k, then

$$\lambda_i = \begin{cases} \frac{\operatorname{Tr}(B_k) + \sqrt{2\operatorname{Tr}(B_k^2) - \operatorname{Tr}(B_k)^2}}{2} & \text{if } i = k, \\ \frac{\operatorname{Tr}(B_k) - \sqrt{2\operatorname{Tr}(B_k^2) - \operatorname{Tr}(B_k)^2}}{2} & \text{if } i = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\|Q_k\|^2 = \frac{\operatorname{Tr}(B_k) + \sqrt{2\operatorname{Tr}(B_k^2) - \operatorname{Tr}(B_k)^2}}{2}.$$

Proof. First, suppose that $q_{ij} \leq 0$ for all $i \neq j$. Since Q is idempotent, its entries q_{ij} satisfy the equation

$$q_{ij} = \sum_{\ell=1}^{n} q_{i\ell} q_{\ell j}.$$

This equation, together with the identities from (5.2), allows one to compute the entries of B_k directly. Indeed,

$$\begin{aligned} b_{kk} &= q_{kk}^2 + q_{k+1,k}^2 + \cdots + q_{nk}^2 \\ &= q_{kk} - q_{1k}^2 - q_{2k}^2 - \cdots - q_{k-1,k}^2 \\ &= q_{kk} - \sum_{\ell=1}^{k-1} (1 - q_{\ell\ell})(1 - q_{kk}) \;=\; q_{kk} - a_{k-1}(1 - q_{kk}), \end{aligned}$$

and if $i \neq k$, then

$$\begin{aligned} b_{ii} &= q_{ki}^2 + q_{k+1,i}^2 + \cdots + q_{ni}^2 \\ &= q_{ii} - q_{1i}^2 - q_{2i}^2 - \cdots - q_{k-1,i}^2 \\ &= q_{ii} - q_{ii}^2 - \sum_{\ell=1,\ell\neq i}^{k-1} (1 - q_{\ell\ell})(1 - q_{ii}) \\ &= (1 - q_{ii})\left((k-2) - \sum_{\ell=1}^{k-1} q_{\ell\ell}\right) \;=\; (1 - a_{k-1})(1 - q_{ii}). \end{aligned}$$

If i, j, and k are all distinct, then

$$\begin{aligned} b_{ij} &= q_{ki} q_{kj} + q_{k+1,i} q_{k+1,j} + \cdots + q_{ni} q_{nj} \\ &= q_{ij} - q_{1i} q_{1j} - q_{2i} q_{2j} - \cdots - q_{k-1,i} q_{k-1,j} \\ &= q_{ij} - q_{ii} q_{ij} - q_{ji} q_{jj} + \sum_{\ell=1,\ell\neq i,j}^{k-1} q_{ij}(1 - q_{\ell\ell}) \\ &= q_{ij}\left((k-2) - \sum_{\ell=1}^{k-1} q_{\ell\ell}\right) \;=\; -(1 - a_{k-1}) q_{ij}. \end{aligned}$$

Lastly, either $i < j = k$ or $j < i = k$. Since $B_k = B_k^*$, it suffices to establish the formula for b_{ij} in the case that $i < j = k$. We have

$$\begin{aligned} b_{ik} &= q_{ki}q_{kk} + q_{k+1,i}q_{k+1,k} + \cdots + q_{ni}q_{nk} \\ &= q_{ik} - q_{1i}q_{1k} - q_{2i}q_{2k} - \cdots - q_{k-1,i}q_{k-1,k} \\ &= q_{ik} - q_{ii}q_{ik} + \sum_{\ell=1,\ell\neq i}^{k-1} q_{ik}(1 - q_{\ell\ell}) \\ &= q_{ik}\left((k-1) - \sum_{\ell=1}^{k-1} q_{\ell\ell}\right) = a_{k-1}q_{ik}. \end{aligned}$$

We now turn our attention to the proof of (ii). By Lemma 5.1.1, one may conjugate Q by a diagonal unitary if necessary to assume that $q_{ij} \leq 0$ for all $i \neq j$. Since the eigenvalues of B_k are invariant under such a transformation, this assumption imposes no loss of generality.

From the description of the entries b_{ij} in (i), it is apparent that if $\widehat{B_k} \in \mathbb{M}_{k-1}$ denotes the matrix obtained by deleting the final row and column of B_k, then

$$\widehat{B_k} = (1 - a_{k-1})(I - \widehat{Q}),$$

where $\widehat{Q} \in \mathbb{M}_{k-1}$ denotes the $(k-1)^{th}$ leading principal submatrix of Q. Since Q is a projection of rank $n-1$, Corollary 5.1.4 (i) ensures that $\lambda = 1$ is an eigenvalue of $\widehat{Q}$ of multiplicity at least $k-2$. Thus, $\lambda = 0$ is an eigenvalue of $\widehat{B_k}$ of multiplicity at least $k-2$. It follows that the remaining eigenvalue of $\widehat{B_k}$ is given by

$$\mathrm{Tr}(\widehat{B_k}) = (1 - a_{k-1})\sum_{i=1}^{k-1}(1 - q_{ii}) = a_{k-1}(1 - a_{k-1}).$$

This information can now be used to analyse the eigenvalues of B_k. By Corollary 5.1.4 (ii), $\lambda = 0$ is an eigenvalue of B_k with multiplicity no less than $k-3$. Furthermore, Theorem 5.1.3 indicates that the remaining eigenvalues λ_1, λ_{k-1}, and λ_k are such that

$$\lambda_1 \leq 0 \leq \lambda_{k-1} \leq a_{k-1}(1 - a_{k-1}) \leq \lambda_k.$$

Since $B_k \geq 0$, we have that $\lambda_1 = 0$. The final two eigenvalues can be recovered by examining the traces of B_k and B_k^2. In particular, one may solve the system of equations

$$\left\{ \begin{array}{rcl} \lambda_{k-1} + \lambda_k &=& \mathrm{Tr}(B_k) \\ \lambda_{k-1}^2 + \lambda_k^2 &=& \mathrm{Tr}(B_k^2) \end{array} \right. ,$$

to obtain the values in (ii). This completes the proof. ■

Theorem 5.1.6. *Let $Q = (q_{ij})$ be a projection in $\mathbb{M}_n$ of rank $n-1$, and let $\{a_i\}_{i=0}^n$ denote the non-decreasing sequence from equation (5.3). If $f : [0,1] \times [0,1] \to \mathbb{R}$ denotes the function*

$$f(x,y) = \frac{\sqrt{x^2y^2 - 4x^2y + 2xy^2 + 4x^2 - 2xy + y^2 - 2y + 1} - xy - y + 2x + 1}{2},$$

then for each $k \in \{1, 2, \dots, n\}$, $\|E_{k-1}^{\perp} Q E_k\|^2 = f(a_{k-1}, a_k)$.

Proof. By Lemma 5.1.1 we may assume without loss of generality that $q_{ij} \leq 0$ for all $i \neq j$. Fix an integer $k \in \{1, 2, \dots, n\}$, define $Q_k := E_{k-1}^{\perp} Q E_k$, and let $B_k = (b_{ij})$ denote the restriction of $Q_k^* Q_k$ to the range of E_k. By Lemma 5.1.5 (ii), we have that

$$\|Q_k\|^2 = \frac{\mathrm{Tr}(B_k) + \sqrt{2\mathrm{Tr}(B_k^2) - \mathrm{Tr}(B_k)^2}}{2}.$$

If $\widehat{B_k} \in \mathbb{M}_{k-1}$ denotes the matrix obtained by deleting the final row and column of B_k as in the proof of Lemma 5.1.5, then

$$\begin{aligned} \mathrm{Tr}(B_k) &= \mathrm{Tr}(\widehat{B_k}) + b_{kk} \\ &= a_{k-1}(1 - a_{k-1}) + q_{kk} - a_{k-1}(1 - q_{kk}) \\ &= q_{kk} + a_{k-1}(q_{kk} - a_{k-1}) \\ &= q_{kk} + a_{k-1}(1 - a_k). \end{aligned}$$

Moreover, if $B_k^2 = (c_{ij})$, then

$$\begin{aligned} c_{kk} &= b_{kk}^2 + \sum_{\ell=1}^{k-1} b_{k\ell}^2 \\ &= (q_{kk} - a_{k-1}(1 - q_{kk}))^2 + \sum_{\ell=1}^{k-1} a_{k-1}^2 q_{k\ell}^2 \\ &= (q_{kk} - a_{k-1}(1 - q_{kk}))^2 + \sum_{\ell=1}^{k-1} a_{k-1}^2 (1 - q_{kk})(1 - q_{\ell\ell}) \\ &= (q_{kk} - a_{k-1}(1 - q_{kk}))^2 + a_{k-1}^3 (1 - q_{kk}), \end{aligned}$$

and for $i \leq k-1$,

$$\begin{aligned}
c_{ii} &= b_{ii}^2 + b_{ik}^2 + \sum_{\ell=1,\ell\neq i}^{k-1} b_{i\ell}^2 \\
&= (1-a_{k-1})^2(1-q_{ii})^2 + a_{k-1}^2 q_{ik}^2 + \sum_{\ell=1,\ell\neq i}^{k-1} (1-a_{k-1})^2 q_{i\ell}^2 \\
&= (1-a_{k-1})^2(1-q_{ii})^2 + a_{k-1}^2(1-q_{ii})(1-q_{kk}) + \sum_{\ell=1,\ell\neq i}^{k-1} (1-a_{k-1})^2(1-q_{ii})(1-q_{\ell\ell}) \\
&= a_{k-1}(1-q_{ii})\left((1-a_{k-1})^2 + a_{k-1}(1-q_{kk})\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{Tr}(B_k^2) &= c_{kk} + \sum_{\ell=1}^{k-1} a_{k-1}(1-q_{\ell\ell})\left((1-a_{k-1})^2 + a_{k-1}(1-q_{kk})\right) \\
&= (q_{kk} - a_{k-1}(1-q_{kk}))^2 + a_{k-1}^3(1-q_{kk}) + a_{k-1}^2\left((1-a_{k-1})^2 + a_{k-1}(1-q_{kk})\right).
\end{aligned}$$

These descriptions of $\operatorname{Tr}(B_k)$ and $\operatorname{Tr}(B_k^2)$ allow one to express $\|Q_k\|^2$ as a function of a_{k-1}, a_k, and q_{kk}. The desired formula for $\|Q_k\|^2$ may now be obtained by writing $q_{kk} = 1-(a_k - a_{k-1})$ as in equation (5.4). ■

Our first goal of this section is now complete: we have derived a closed-form expression for each norm $\|E_{k-1}^{\perp} Q E_k\|$. In order to show that every such norm is equal to $\nu_{n-1,n}$, we must first investigate the properties of the function f from Theorem 5.1.6.

Lemma 5.1.7. *If $f:[0,1]\times[0,1]\to\mathbb{R}$ denotes the function*

$$f(x,y) = \frac{\sqrt{x^2y^2 - 4x^2y + 2xy^2 + 4x^2 - 2xy + y^2 - 2y + 1} - xy - y + 2x + 1}{2},$$

then f is increasing in x and decreasing in y. Moreover, $0 \leq f(x,y) \leq 1$ whenever $0 \leq x \leq y \leq 1$.

Proof. Define $g:[0,1]\times[0,1]\to\mathbb{R}$ by

$$g(x,y) = x^2y^2 - 4x^2y + 2xy^2 + 4x^2 - 2xy + y^2 - 2y + 1,$$

so $f(x,y) = \frac{1}{2}(\sqrt{g(x,y)} - xy - y + 2x + 1)$. We begin by proving that $g(x,y)$ is non-negative on its domain and zero only at $(0,1)$. We will therefore verify that f is well-defined on $[0,1] \times [0,1]$, and that the partial derivatives of f exist at all points $(x,y) \neq (0,1)$.

Observe that for each fixed $y \in [0,1]$, the map

$$x \mapsto g(x,y) = (2-y)^2x^2 - 2y(1-y)x + (1-y)^2$$

defines a convex quadratic on $[0,1]$ with vertex at $x_0 = y(1-y)/(2-y)^2$. If $y \in [0,1)$, then

$$g(x_0,y) = g\left(\frac{y(1-y)}{(2-y)^2}, y\right) = \frac{4(1-y)^3}{(2-y)^2} > 0.$$

Consequently, $g(x,y) > 0$ for all $(x,y) \in [0,1] \times [0,1)$. Note as well that at $y = 1$ we have $g(x,1) = x^2$. It follows that $g(0,1) = 0$ and $g(x,y) > 0$ for all other values of $(x,y) \in [0,1] \times [0,1]$. Thus, f is well-defined, and the partial derivatives

$$f_x(x,y) = \frac{g_x(x,y)}{4\sqrt{g(x,y)}} + \frac{2-y}{2}, \qquad f_{xx}(x,y) = \frac{2(1-y)^3}{(g(x,y))^{3/2}},$$

$$f_y(x,y) = \frac{g_y(x,y)}{4\sqrt{g(x,y)}} - \frac{x+1}{2}, \qquad f_{yy}(x,y) = \frac{2x^3}{(g(x,y))^{3/2}}$$

exist for all $(x,y) \neq (0,1)$.

Our next task is to prove that $f(x,y)$ is increasing in x. First observe that $f(x,1) = x$ is clearly increasing. Furthermore, for every fixed $y \in [0,1)$, $f_{xx}(x,y)$ is well-defined and strictly positive for all x. Hence,

$$f_x(x,y) = \frac{xy^2 - 4xy + y^2 + 4x - y}{2\sqrt{g(x,y)}} + \frac{2-y}{2}$$

is an increasing function of x. We conclude that $f_x(x,y) \geq f_x(0,y) = 1 - y > 0$ for every $x \in [0,1]$. Thus, f is an increasing function of x on $[0,1]$.

We now use a similar argument to show that f is a decreasing function of y. For $x = 0$, we have that $f(0,y) = 1 - y$ is clearly decreasing. Now given a fixed $x \in (0,1]$, it is clear from above that $f_{yy}(x,y)$ is well-defined and strictly positive for all y. It follows that the partial derivative

$$f_y(x,y) = \frac{x^2y - 2x^2 + 2xy - x + y - 1}{2\sqrt{g(x,y)}} - \frac{x+1}{2}$$

is an increasing function of y on $[0, 1]$. Hence $f_y(x, y) \leq f_y(x, 1) = -x < 0$ for every $y \in [0, 1]$. This proves that f is a decreasing function of y on $[0, 1]$, as desired.

For the final claim suppose that $0 \leq x \leq y \leq 1$, and consider the sequence $\{a_k\}_{k=0}^3$ defined by $a_0 = 0$, $a_1 = x$, $a_2 = y$, and $a_3 = 1$. By Lemma 5.1.2, there is a rank-two projection $Q = (q_{ij})$ in $\mathbb{M}_3$ that is defined by $\{a_k\}_{k=0}^3$ in the sense of equation (5.3). Turning to Theorem 5.1.6, we have that

$$f(x, y) = f(a_1, a_2) = \|E_1^\perp Q E_2\|^2,$$

and hence $0 \leq f(x, y) \leq 1$. ∎

Theorem 5.1.8. *If $Q \in \mathbb{M}_n$ is a projection of rank $n - 1$ that is of minimal distance to $\mathcal{T}_n$, then $\|E_{i-1}^\perp Q E_i\| = \|E_{j-1}^\perp Q E_j\|$ for all i and j.*

Proof. By Lemma 5.1.1, we may assume without loss of generality that $Q = (q_{ij})$ is such that $q_{ij} \leq 0$ whenever $i \neq j$. Let $\{a_i\}_{i=0}^n$ denote the non-decreasing sequence from equation (5.3), and for each $i \in \{1, 2, \ldots, n\}$, define $Q_i \coloneqq E_{i-1}^\perp Q E_i$. Suppose to the contrary that not all values of $\|Q_i\|$ are equal. Define

$$\mu \coloneqq \max_{1 \leq i \leq n} \|Q_i\|,$$

and let j denote the largest index in $\{1, 2, \ldots, n\}$ such that $\|Q_j\| = \mu$.

First consider the case in which $j = n$. Let k denote the largest index in $\{1, 2, \ldots, n-1\}$ such that $\|Q_k\| < \mu$. With f as in Theorem 5.1.6, we have that

$$f(a_{k-1}, a_k) = \|Q_k\|^2 < \|Q_{k+1}\|^2 = f(a_k, a_{k+1}).$$

Thus, if $g : [a_{k-1}, a_k] \to \mathbb{R}$ is given by

$$g(x) = f(a_{k-1}, x) - f(x, a_{k+1}),$$

then $g(a_k) = f(a_{k-1}, a_k) - f(a_k, a_{k+1}) < 0$, while $g(a_{k-1}) = 1 - f(a_{k-1}, a_{k+1}) \geq 0$ by Lemma 5.1.7. Since g is continuous on its domain, the Intermediate Value Theorem gives rise to some $a_k' \in [a_{k-1}, a_k]$ such that $g(a_k') = 0$. By replacing a_k with a_k' in the sequence $\{a_i\}_{i=0}^n$, one may equate $\|Q_k\|$ and $\|Q_{k+1}\|$ while leaving the remaining norms $\|Q_i\|$ unchanged. Most importantly, since $a_k' \leq a_k$, Lemma 5.1.7 implies that the new common value of $\|Q_k\|$ and $\|Q_{k+1}\|$ is strictly less than μ.

This argument may now be repeated to successively reduce the norms $\|Q_i\|$ for $i > k$ to values strictly less than μ. At the end of this process, either the new largest index j at which the maximum norm occurs is strictly less than n, or the maximum μ decreases. Of course, the latter cannot happen as Q was assumed to be of minimal distance to $\mathcal{T}_n$.

Thus, we may assume that the largest index j at which μ occurs is strictly less than n. In this case we have that

$$f(a_j, a_{j+1}) = \|Q_{j+1}\|^2 < \|Q_j\|^2 = f(a_{j-1}, a_j).$$

As above, we may invoke the Intermediate Value Theorem to obtain a root a_j' of the continuous function

$$h(x) := f(a_{j-1}, x) - f(x, a_{j+1})$$

on the interval $[a_j, a_{j+1}]$. By replacing a_j with a_j' in the sequence $\{a_i\}_{i=0}^n$, one may equate $\|Q_j\|$ and $\|Q_{j+1}\|$ while preserving all other norms $\|Q_i\|$. Since $a_j' \geq a_j$, Lemma 5.1.7 demonstrates that the new common value of $\|Q_j\|$ and $\|Q_{j+1}\|$ is strictly less than μ. Thus, this process either decreases the largest index j at which the maximum norm occurs, or reduces the value of μ. Since this argument may be repeated for smaller and smaller values of j, eventually μ must decrease—a contradiction. ■

§5.2 Computing the Distance

We will now utilize the results of §5.1 to determine the precise value of $\nu_{n-1,n}$. The first step in this direction is the following proposition, which applies Theorem 5.1.8 to obtain a recursive description of the sequence $\{a_i\}_{i=0}^n$.

Proposition 5.2.1. *Let $Q \in \mathbb{M}_n$ be a projection of rank $n-1$ that is of minimal distance to $\mathcal{T}_n$. If $\{a_i\}_{i=0}^n$ denotes the non-decreasing sequence from equation (5.3), then*

$$a_k = \frac{-\nu_{n-1,n}^4 + 2\nu_{n-1,n}^2 a_{k-1} + \nu_{n-1,n}^2 - a_{k-1}}{\nu_{n-1,n}^2 a_{k-1} + \nu_{n-1,n}^2 - a_{k-1}}$$

for each $k \in \{1, 2, \ldots, n\}$.

Proof. Since the distance from Q to $\mathcal{T}_n$ is minimal, Theorems 5.0.3 and 5.1.8 imply that $\|E_{k-1}^\perp Q E_k\| = \nu_{n-1,n}$ for all $k \in \{1, 2, \ldots, n\}$. Thus, with f as in Theorem 5.1.6, we have that

$$f(a_{k-1}, a_k) = \|E_{k-1}^\perp Q E_k\|^2 = \nu_{n-1,n}^2.$$

The desired formula can now be obtained by solving this equation for a_k. ∎

The recursive formula for a_k described in Proposition 5.2.1 will be the key to computing $\nu_{n-1,n}$. Our goal will be to use this formula and some basic properties of the sequence $\{a_i\}_{i=0}^n$ to determine a list of candidates for $\nu_{n-1,n}^2$. A careful analysis of these candidates will reveal that exactly one of them satisfies a certain necessary norm inequality from [17]. This value must therefore be $\nu_{n-1,n}^2$.

To simplify notation, let $t = \nu_{n-1,n}^2$ and define the function $h_t : [0,1] \to \mathbb{R}$ by

$$h_t(x) \coloneqq \frac{-t^2 + 2tx + t - x}{tx + t - x}. \tag{5.5}$$

Proposition 5.2.1 states that for each $k \in \{1, 2, \ldots, n\}$,

$$a_k = \frac{-t^2 + 2ta_{k-1} + t - a_{k-1}}{ta_{k-1} + t - a_{k-1}} = h_t(a_{k-1}).$$

Since $h_t(0) = (t - t^2)/t = 1 - t = a_1$, this formula may be expressed as $a_k = h_t^{(k)}(0)$ for all $k \in \{1, 2, \ldots, n\}$. Upon taking into account the condition $a_n = 1$, we are interested in identifying the values of $t \in \left[\frac{1}{4}, 1\right]$ that satisfy the equation $h_t^{(n)}(0) = 1$.

Notice that each expression $h_t^{(k)}(0)$ is a rational function of t. For each $k \geq 1$, let $p_{k-1}(t)$ and $q_{k-1}(t)$ denote polynomials in t such that

$$h_t^{(k)}(0) = \frac{p_{k-1}(t)}{q_{k-1}(t)}.$$

It then follows that

$$\frac{p_k(t)}{q_k(t)} = h_t\left(h_t^{(k)}(0)\right) = h_t\left(\frac{p_{k-1}(t)}{q_{k-1}(t)}\right) = \frac{-t^2 q_{k-1}(t) + 2tp_{k-1}(t) + tq_{k-1}(t) - p_{k-1}(t)}{tp_{k-1}(t) + tq_{k-1}(t) - p_{k-1}(t)},$$

and hence we obtain the relations

$$p_k(t) = t(1-t)q_{k-1}(t) + (2t-1)p_{k-1}(t), \tag{5.6}$$

$$q_k(t) = tq_{k-1}(t) - (1-t)p_{k-1}(t). \tag{5.7}$$

We may replace $p_{k-1}(t)$ in (5.7) using equation (5.6), thereby leading to a recurrence expressed only in the $q_k(t)$'s. Specifically, we have that

$$\begin{aligned} q_k(t) &= tq_{k-1}(t) - (1-t)p_{k-1}(t) \\ &= tq_{k-1}(t) - (1-t)\left[t(1-t)q_{k-2}(t) + (2t-1)p_{k-2}(t)\right] \\ &= tq_{k-1}(t) - t(1-t)^2 q_{k-2}(t) - (2t-1)\left[tq_{k-2}(t) - q_{k-1}(t)\right] \\ &= (3t-1)q_{k-1}(t) - t^3 q_{k-2}(t) \end{aligned}$$

for all $k \geq 2$. We may extend this recurrence relation to include $k = 1$ by choosing a suitable expression for $q_{-1}(t)$. Indeed, note that

$$\frac{p_0(t)}{q_0(t)} = h_t(0) = 1 - t \quad \text{and} \quad \frac{p_1(t)}{q_1(t)} = h_t(h_t(0)) = \frac{-3t^2 + 4t - 1}{-t^2 + 3t - 1},$$

so $q_0(t) = 1$, and $q_1(t) = -t^2 + 3t - 1$. Thus, we may write $q_1(t) = (3t-1)q_0(t) - t^3 q_{-1}(t)$ by defining $q_{-1}(t) := t^{-1}$.

The requirement that $h_t^{(n)}(0) = 1$ is equivalent to asking that $p_{n-1}(t) = q_{n-1}(t)$. Using the relations above, this equation can be restated as $tq_{n-2}(t) = p_{n-2}(t)$, or equivalently $q_{n-1}(t) = t^2 q_{n-2}(t)$ by (5.7). Thus, we wish to determine the values of $t \in \left[\frac{1}{4}, 1\right]$ that satisfy

$$q_{n-1}(t) = t^2 q_{n-2}(t),$$

where

$$q_{-1}(t) = t^{-1}, \quad q_0(t) = 1, \quad \text{and} \quad q_k(t) = (3t-1)q_{k-1}(t) - t^3 q_{k-2}(t) \ \text{ for } k \geq 1.$$

A solution to this problem will require closed-form expressions for the polynomials $q_{n-1}(t)$ and $q_{n-2}(t)$. In order to obtain such expressions, we will first rewrite the recurrence relation defining these polynomials in terms of matrix multiplication:

$$\begin{bmatrix} q_k(t) \\ q_{k-1}(t) \end{bmatrix} = \begin{bmatrix} 3t-1 & -t^3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{k-1}(t) \\ q_{k-2}(t) \end{bmatrix} = \begin{bmatrix} 3t-1 & -t^3 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} q_0(t) \\ q_{-1}(t) \end{bmatrix}.$$

One may therefore obtain a description of $q_{n-1}(t)$ and $q_{n-2}(t)$ by diagonalizing the matrix

$$A := \begin{bmatrix} 3t-1 & -t^3 \\ 1 & 0 \end{bmatrix}.$$

Routine computations show that the eigenvalues of A are given by

$$\lambda_1 = \frac{3t-1+(1-t)\sqrt{1-4t}}{2} = \frac{3t-1+(1-t)iy}{2}$$

and

$$\lambda_2 = \frac{3t-1-(1-t)\sqrt{1-4t}}{2} = \frac{3t-1-(1-t)iy}{2},$$

where $y \coloneqq \sqrt{4t-1}$. Furthermore, the columns of the matrix $P \coloneqq \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ form a basis of eigenvectors corresponding to λ_1 and λ_2, respectively. By computing

$$P^{-1} = \frac{1}{(1-t)iy} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

and setting $D \coloneqq \operatorname{diag}(\lambda_1, \lambda_2)$, we have that $A = PDP^{-1}$. Consequently,

$$\begin{bmatrix} q_{n-1}(t) \\ q_{n-2}(t) \end{bmatrix} = PD^{n-1}P^{-1} \begin{bmatrix} q_0(t) \\ q_{-1}(t) \end{bmatrix} = \frac{1}{t(1-t)iy} \begin{bmatrix} t\left(\lambda_1^n - \lambda_2^n\right) - \lambda_2\lambda_1^n + \lambda_1\lambda_2^n \\ t\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right) - \lambda_2\lambda_1^{n-1} + \lambda_1\lambda_2^{n-1} \end{bmatrix}.$$

The expressions for $q_{n-1}(t)$ and $q_{n-2}(t)$ derived above can now be used to identify the desired values of t. Indeed, when $q_{n-1}(t) = t^2 q_{n-2}(t)$, we have that

$$\begin{aligned} t\left(\lambda_1^n - \lambda_2^n\right) - \lambda_2\lambda_1^n + \lambda_1\lambda_2^n &= t^2\left(t\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right) - \lambda_2\lambda_1^{n-1} + \lambda_1\lambda_2^{n-1}\right) \\ \Rightarrow \quad \lambda_1^n(t-\lambda_2) - \lambda_2^n(t-\lambda_1) &= t^2\left(\lambda_1^{n-1}(t-\lambda_2) - \lambda_2^{n-1}(t-\lambda_1)\right) \\ \Rightarrow \quad \lambda_2^{n-1}(t^2-\lambda_2)(t-\lambda_1) &= \lambda_1^{n-1}(t^2-\lambda_1)(t-\lambda_2), \end{aligned}$$

and therefore

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{n-1} \left(\frac{t^2-\lambda_2}{t^2-\lambda_1}\right) \left(\frac{t-\lambda_1}{t-\lambda_2}\right) = 1. \tag{5.8}$$

This equation may be simplified using the following identities that relate the values of t, λ_1, and λ_2.

Lemma 5.2.2. *If* $y = \sqrt{4t-1}$, $\lambda_1 = (3t-1+(1-t)iy)/2$, *and* $\lambda_2 = (3t-1-(1-t)iy)/2$, *then* *(i)* $t - \lambda_1 = (1-t)\left(\dfrac{1-iy}{2}\right)$ *and* $t - \lambda_2 = (1-t)\left(\dfrac{1+iy}{2}\right)$.

(ii) $t^2 - \lambda_1 = (1-t)\left(\dfrac{1-2t-iy}{2}\right)$ *and* $t^2 - \lambda_2 = (1-t)\left(\dfrac{1-2t+iy}{2}\right)$.

(iii) $\dfrac{1+iy}{1-iy} = \dfrac{1-2t+iy}{2t}$ *and* $\dfrac{1-iy}{1+iy} = \dfrac{1-2t-iy}{2t}$.

(iv) $\dfrac{\lambda_2}{\lambda_1} = \left(\dfrac{1+iy}{1-iy}\right)^3$.

Proof. Verification of statements (i)-(iii) is straightforward, and thus their proofs are left to the reader. For (iv), an application of the Binomial Theorem demonstrates that

$$\begin{aligned}(1+iy)^3 &= 1 + 3iy - 3y^2 - iy^3 \\ &= (1-3y^2) + iy(3-y^2) \\ &= 4(1-3t) + 4(1-t)iy \quad = \quad -8\lambda_2.\end{aligned}$$

From this we deduce that $(1-iy)^3 = \overline{(1+iy)^3} = -8\overline{\lambda_2} = -8\lambda_1$, and thus the result holds. ∎

One may apply the identities above to simplify equation (5.8) as follows:

$$\begin{aligned}1 &= \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}\left(\frac{t^2-\lambda_2}{t^2-\lambda_1}\right)\left(\frac{t-\lambda_1}{t-\lambda_2}\right) \\ &= \left(\frac{1+iy}{1-iy}\right)^{3(n-1)}\left(\frac{1-2t+iy}{1-2t-iy}\right)\left(\frac{1-iy}{1+iy}\right) \\ &= \left(\frac{1+iy}{1-iy}\right)^{3n-3}\left(\frac{1+iy}{1-iy}\right)^{2}\left(\frac{1-iy}{1+iy}\right) \\ &= \left(\frac{1+iy}{1-iy}\right)^{3n-2}.\end{aligned}$$

We therefore conclude that

$$\frac{1+iy}{1-iy} = \rho_m^k,$$

where $m := 3n-2$, $\rho_m := e^{2\pi i/m}$, and k is an integer.

We are now in a position to determine the possible values of t. By solving for y in the equation above, we obtain

$$\begin{aligned} y &= \frac{1}{i}\frac{\rho_m^k - 1}{\rho_m^k + 1} = \frac{1}{i}\frac{\rho_m^{k/2}\left(\rho_m^{k/2} - \rho_m^{-k/2}\right)}{\rho_m^{k/2}\left(\rho_m^{k/2} + \rho_m^{-k/2}\right)} \\ &= \frac{\rho_m^{k/2} - \rho_m^{-k/2}}{2i}\frac{2}{\rho_m^{k/2} + \rho_m^{-k/2}} \\ &= \frac{\sin(k\pi/m)}{\cos(k\pi/m)} = \tan\left(\frac{k\pi}{m}\right). \end{aligned}$$

Since $y = \sqrt{4t-1}$, we have

$$t = \frac{1}{4}\left(\tan^2\left(\frac{k\pi}{m}\right) + 1\right) = \frac{1}{4}\sec^2\left(\frac{k\pi}{3n-2}\right) \text{ for some } k \in \mathbb{Z}.$$

That is, the distance $\nu_{n-1,n}$ from Q to $\mathcal{T}_n$ must belong to the set

$$\left\{\frac{1}{2}\sec\left(\frac{k\pi}{3n-2}\right) : k \in \mathbb{Z}\right\}.$$

It remains to determine which element of this set represents $\nu_{n-1,n}$. We will accomplish this task by appealing to the following result of MacDonald concerning a lower bound on the distance from a projection to a nilpotent.

Proposition 5.2.3. [17, Lemma 3.3] *If $P \in \mathbb{M}_n$ is a projection of rank r and $N \in \mathbb{M}_n$ is nilpotent, then*

$$\|P - N\| \geq \sqrt{\frac{r}{2n}\left(1 + \frac{r}{n}\right)}.$$

In the analysis that follows, we will demonstrate that the only value in

$$\left\{\frac{1}{2}\sec\left(\frac{k\pi}{3n-2}\right) : k \in \mathbb{Z}\right\}$$

that respects the lower bound of Proposition 5.2.3 for projections of rank $r = n-1$ occurs when $k = n-1$. We begin with the following lemma, which proves that MacDonald's lower bound is indeed satisfied for this choice of k.

Lemma 5.2.4. *For every integer $n \geq 3$,*

$$\frac{n-1}{2n}\left(1+\frac{n-1}{n}\right) \leq \frac{1}{4}\sec^2\left(\frac{(n-1)\pi}{3n-2}\right) \leq 1$$

Proof. Define $\alpha_n \coloneqq (3n-2)/(n-1)$. By considering reciprocals, this problem is equivalent to that of establishing the inequalities

$$\frac{1}{4} \leq \cos^2\left(\frac{\pi}{\alpha_n}\right) \leq \frac{n^2}{2(n-1)(2n-1)}$$

for all $n \geq 3$. In the computations that follow, it will be helpful to view n as a continuous variable on $[3, \infty)$.

To establish the inequality $1/4 \leq \cos^2(\pi/\alpha_n)$, simply note that π/α_n is an increasing function of n tending to $\pi/3$, $\cos(x)$ is decreasing on $[0, \pi/3]$, and $\cos(\pi/3) = 1/2$. The second inequality will require a bit more work. Since $(2n - \frac{3}{2})^2 \geq 2(n-1)(2n-1)$ for all n, it suffices to prove that

$$\cos^2\left(\frac{\pi}{\alpha_n}\right) \leq \frac{n^2}{\left(2n-\frac{3}{2}\right)^2}.$$

This inequality can be reduced further by taking square roots. Indeed, the above holds if and only if

$$f(n) \coloneqq \frac{2n}{4n-3} - \cos\left(\frac{\pi}{\alpha_n}\right) \geq 0 \quad \text{for } n \in [3, \infty).$$

We will prove that $f'(n) < 0$ for all $n \in [3, \infty)$, so that f is monotonically decreasing on this interval. Since

$$\lim_{n\to\infty} f(n) = 0 \quad \text{and} \quad f(3) = \frac{2}{3} - \cos\left(\frac{2\pi}{7}\right) \approx 0.043 > 0,$$

this will demonstrate that $f(n) \geq 0$ for all $n \geq 3$. To this end, we compute

$$f'(n) = \frac{16\pi\sin\left(\frac{\pi}{\alpha_n}\right)n^2 - 24\pi\sin\left(\frac{\pi}{\alpha_n}\right)n + 9\pi\sin\left(\frac{\pi}{\alpha_n}\right) - 54n^2 + 72n - 24}{(4n-3)^2(3n-2)^2}.$$

Of course $(4n-3)^2(3n-2)^2 \geq 0$, so the sign of $f'(n)$ depends only on the sign of

$$g(n) \coloneqq 16\pi\sin\left(\frac{\pi}{\alpha_n}\right)n^2 - 24\pi\sin\left(\frac{\pi}{\alpha_n}\right)n + 9\pi\sin\left(\frac{\pi}{\alpha_n}\right) - 54n^2 + 72n - 24.$$

But since $\pi/\alpha_n \in [\pi/4, \pi/3]$ for $n \geq 3$, we have that $\sin(\pi/\alpha_n) \in [\sqrt{2}/2, \sqrt{3}/2]$ for all such n, and hence

$$\begin{aligned} g(n) &\leq 16\pi\left(\frac{\sqrt{3}}{2}\right)n^2 - 24\pi\left(\frac{\sqrt{2}}{2}\right)n + 9\pi\left(\frac{\sqrt{3}}{2}\right) - 54n^2 + 72n - 24 \\ &= \left(8\sqrt{3}\pi - 54\right)n^2 - \left(12\sqrt{2} - 72\right)n + \left(\frac{9\sqrt{3}}{2} - 24\right). \end{aligned}$$

This upper bound for g is a concave quadratic whose larger root occurs at $n \approx 1.8105$. It follows that g is negative on $[3, \infty)$, and therefore so too is f'. ■

Lemma 5.2.5. *For any integer $n \geq 3$, the set*

$$\left\{\frac{1}{4}\sec^2\left(\frac{k\pi}{3n-2}\right) : k \in \mathbb{Z}\right\}$$

contains exactly one value in $\left[\frac{n-1}{2n}\left(1 + \frac{n-1}{n}\right), 1\right]$*, and it occurs when $k = n - 1$.*

Proof. Fix an integer $n \geq 3$. We wish to prove that

$$\mathcal{A} \coloneqq \left\{\cos^2\left(\frac{k\pi}{3n-2}\right) : k \in \mathbb{Z}\right\}$$

contains exactly one value in the interval

$$\mathcal{I} \coloneqq \left[\frac{1}{4}, \frac{n^2}{2(n-1)(2n-1)}\right].$$

Since Lemma 5.2.4 demonstrates that this is the case when $k = n - 1$, it suffices to show that no other values in $\mathcal{A}$ are within distance

$$\beta(n) \coloneqq \frac{n^2}{2(n-1)(2n-1)} - \frac{1}{4}$$

of $\cos^2((n-1)\pi/(3n-2))$.

Note, however, that not all values of $k \in \mathbb{Z}$ need to be considered. In particular, since the function $k \mapsto \cos^2(k\pi/(3n-2))$ is periodic, it suffices to check only its values at the integers $k \in \{0, 1, \ldots, 3n-2\}$. Additionally, since

$$\cos^2\left(\frac{((3n-2)-k)\pi}{3n-2}\right) = \cos^2\left(\frac{k\pi}{3n-2}\right) \text{ for all } k,$$

we may restrict our attention to $k \in \{0, 1, 2, \ldots, \lfloor(3n-2/2)\rfloor\}$.

Although we are solely concerned with the integer values of k described above, it will be useful to view k as a continuous real variable. With this in mind, define the function $f_n : [0, (3n-2)/2] \to \mathbb{R}$ by

$$f_n(k) := \sin\left(\frac{(n-k-1)\pi}{3n-2}\right)\sin\left(\frac{(n+k-1)\pi}{3n-2}\right).$$

It follows from the identity $\cos^2(x) - \cos^2(y) = -\sin(x-y)\sin(x+y)$ that

$$\left|\cos^2\left(\frac{k\pi}{3n-2}\right) - \cos^2\left(\frac{(n-1)\pi}{3n-2}\right)\right| < \beta(n) \iff |f_n(k)| < \beta(n).$$

Notice, however, that

$$f_n'(k) = \left(\frac{-\pi}{3n-2}\right)\sin\left(\frac{2k\pi}{3n-2}\right),$$

so $f_n'(k) < 0$ on $[0, (3n-2)/2]$, and hence f_n is decreasing on its domain. Since $f_n(n-1) = 0$, it therefore suffices to prove that

$$f_n(n-2) > \beta(n) \quad \text{and} \quad -f_n(n) > \beta(n).$$

We will demonstrate that these inequalities hold via application of Taylor's Theorem.

Consider the approximation of $\sin(x)$ by $x - x^3/6$, its third degree MacLauren polynomial. On $[0, \pi/6]$, the error in this approximation is at most

$$E(x) = \frac{\sin(\pi/6)}{4!}|x|^4 = \frac{x^4}{48}.$$

Thus, since $1/n \leq \pi/(3n-2) \leq \pi/6$, we have

$$\sin\left(\frac{\pi}{3n-2}\right) \geq \sin\left(\frac{1}{n}\right) \geq \left(\frac{1}{n} - \frac{1}{6n^3} - E\left(\frac{1}{n}\right)\right) \geq \left(\frac{1}{n} - \frac{1}{6n} - \frac{1}{48n}\right) = \frac{13}{16n}.$$

It is routine to verify that $\sin((2x-1)\pi/(3x-2))$ is an increasing function of x on $[3,\infty)$. Consequently, this function is bounded below by $\sin(5\pi/7)$, its value at $x=3$. We deduce that

$$-f_n(n) = \sin\left(\frac{\pi}{3n-2}\right)\sin\left(\frac{(2n-1)\pi}{3n-2}\right) \geq \frac{13}{16n}\sin\left(\frac{5\pi}{7}\right) \geq \frac{13}{16n}\cdot\frac{3}{4} = \frac{39}{64n}.$$

Lastly, one may show directly that

$$\frac{39}{64n} > \beta(n) \text{ whenever } n > \frac{101+\sqrt{5521}}{60} \approx 2.9217,$$

and hence $-f_n(n) > \beta(n)$ for our fixed integer $n \geq 3$.

A similar analysis may now be used to prove that $f_n(n-2) > \beta(n)$. Indeed, it is straightforward to verify that $\sin((2n-3)\pi/(3n-2))$ is bounded below by $\sin(2\pi/3)$, and therefore

$$\begin{aligned} f_n(n-2) &= \sin\left(\frac{\pi}{3n-2}\right)\sin\left(\frac{(2n-3)\pi}{3n-2}\right) \\ &\geq \frac{13}{16n}\sin\left(\frac{2\pi}{3}\right) = \frac{13}{16n}\cdot\frac{\sqrt{3}}{2} \geq \frac{13}{16n}\cdot\frac{3}{4} = \frac{39}{64n}. \end{aligned}$$

It now follows from the arguments of the previous case that $f_n(n-2) > \beta(n)$. ■

With the above analysis complete, we may now present the main result of this chapter: the distance from a projection in $\mathbb{M}_n$ of rank $n-1$ to the set $\mathcal{N}(\mathbb{C}^n)$ is

$$\nu_{n-1,n} = \frac{1}{2}\sec\left(\frac{(n-1)\pi}{3n-2}\right).$$

Interestingly, this expression can be rewritten to bear an even stronger resemblance to MacDonald's formula in the rank-one case.

Theorem 5.2.6. *For every integer $n \geq 2$, the distance from the set of projections in $\mathbb{M}_n$ of rank $n-1$ to $\mathcal{N}(\mathbb{C}^n)$ is*

$$\nu_{n-1,n} = \frac{1}{2}\sec\left(\frac{\pi}{\frac{n}{n-1}+2}\right).$$

§5.3 Closest Projection-Nilpotent Pairs

Given a projection Q in $\mathbb{M}_n$ of rank $n-1$ that is of distance $\nu_{n-1,n}$ to $\mathcal{T}_n$, the following theorem provides a means for determining an element $T \in \mathcal{T}_n$ that is closest to Q. As we will see in Theorem 5.3.2, this element of $\mathcal{T}_n$ is unique to Q.

Theorem 5.3.1. [3, 17] *Fix $\gamma \in [0, \infty)$. An operator $A \in \mathbb{M}_n$ satisfies $\|E_{i-1}^{\perp} A E_i\| = \gamma$ for all $i \in \{1, 2, \ldots, n\}$ if and only if there exist $T \in \mathcal{T}_n$ and a unitary $U \in \mathbb{M}_n$ such that $A - T = \gamma U$.*

Furthermore, if $\|E_{i-1}^{\perp} A E_i\| = \gamma$ and $\|E_i^{\perp} A E_i\| < \gamma$ for all $i \in \{1, 2, \ldots, n-1\}$, then the operators T and U are unique.

With this result in hand, we are now able to describe all closest pairs (Q, N) where Q is a projection of rank $n-1$ and $N \in \mathcal{N}(\mathbb{C}^n)$.

Theorem 5.3.2. *Fix a positive integer $n \geq 2$. Let $\{a_i\}_{i=0}^n$ be the sequence given by $a_0 = 0$ and*

$$a_k = \frac{-\nu_{n-1,n}^4 + 2\nu_{n-1,n}^2 a_{k-1} + \nu_{n-1,n}^2 - a_{k-1}}{\nu_{n-1,n}^2 a_{k-1} + \nu_{n-1,n}^2 - a_{k-1}} \quad \text{for } k \geq 1.$$

Let $\{z_i\}_{i=1}^n$ be a sequence of complex numbers of modulus 1, define

$$e = \begin{bmatrix} z_1\sqrt{a_1 - a_0} & z_2\sqrt{a_2 - a_1} & \cdots & z_n\sqrt{a_n - a_{n-1}} \end{bmatrix}^T,$$

and let $Q = I - e \otimes e^*$.

(i) Q is a projection of rank $n-1$ such that $\operatorname{dist}(Q, \mathcal{T}_n) = \nu_{n-1,n}$. Moreover, every projection of rank $n-1$ that is of minimal distance to $\mathcal{T}_n$ is of this form.

(ii) There is a unique operator $T \in \mathcal{T}_n$ of minimal distance to Q, and this T is such that $Q - T = \nu_{n-1,n} U$ for some unitary $U \in \mathbb{M}_n$. Thus, if $q_k = Qe_k$ and $t_k = Te_k$ denote the columns of Q and T, respectively, then one can iteratively determine columns t_k by solving the system of linear equations

$$\begin{cases} \langle q_1 - t_1, q_k - t_k \rangle &= 0 \\ \langle q_2 - t_2, q_k - t_k \rangle &= 0 \\ \vdots & \vdots \\ \langle q_{k-1} - t_{k-1}, q_k - t_k \rangle &= 0 \end{cases}$$

for $k \in \{2, 3, \ldots, n\}$.

Proof. Statement (i) follows immediately from the results of §5.1 and §5.2. For statement (ii), the existence of T and U is guaranteed by Theorems 5.1.8 and 5.3.1. All that remains to show is the uniqueness of these operators.

To accomplish this task, note that it suffices to prove uniqueness in the case that $z_i = 1$ for all i (i.e., when $q_{ij} \leq 0$ for all $i \neq j$). For $k \in \{1, 2, \ldots, n\}$, let Q_k denote the restriction of $E_{k-1}^{\perp} Q E_k$ to the range of E_k, and define $B_k \coloneqq Q_k^* Q_k$. Let $Q_k' = E_k^{\perp} Q_k$, so that

$$Q_k = \left[\begin{array}{c} v_k^* \\ \hline Q_k' \end{array} \right],$$

where $v_k \coloneqq \begin{bmatrix} q_{k1} & q_{k2} & \ldots & q_{kk} \end{bmatrix}^T$.

We will demonstrate that $\|Q_k'\| < \|Q_k\|$ for all $k \in \{1, 2, \ldots, n-1\}$, and therefore obtain the uniqueness of T and U via Theorem 5.3.1. Observe that this inequality holds when $k = 1$, as

$$\|Q_1\|^2 - \|Q_1'\|^2 = q_{11}^2 = \nu_{n-1,n} > 0.$$

Suppose now that $k \in \{2, 3, \ldots, n-1\}$ is fixed, and define $B_k' \coloneqq {Q_k'}^* Q_k' = B_k - v_k v_k^*$. One may determine the entries of $B_k' = (b_{ij}')$ using the formulas for the entries of $B_k = (b_{ij})$ from Lemma 5.1.5 (i). Indeed,

$$\begin{aligned} b_{kk}' &= b_{kk} - q_{kk}^2 \\ &= q_{kk} - a_{k-1}(1 - q_{kk}) - q_{kk}^2 \\ &= (q_{kk} - a_{k-1})(1 - q_{kk}) \;=\; (1 - a_k)(1 - q_{kk}), \end{aligned}$$

and for if $i < k$,

$$\begin{aligned} b_{ii}' &= b_{ii} - q_{ki}^2 \\ &= (1 - a_{k-1})(1 - q_{ii}) - (1 - q_{kk})(1 - q_{ii}) \\ &= (q_{kk} - a_{k-1})(1 - q_{ii}) \;=\; (1 - a_k)(1 - q_{ii}). \end{aligned}$$

If i, j, and k are all distinct, then

$$\begin{aligned} b_{ij}' &= b_{ij} - q_{ki} q_{kj} \\ &= -(1 - a_{k-1}) q_{ij} + q_{ij}(1 - q_{kk}) \\ &= -(q_{kk} - a_{k-1}) q_{ij} \;=\; -(1 - a_k) q_{ij}. \end{aligned}$$

Finally, either $i < j = k$ or $j < i = k$. In the case of former, we have

$$\begin{aligned} b'_{ik} &= b_{ik} - q_{ki}q_{kk} \\ &= a_{k-1}q_{ik} - q_{ik}q_{kk} \\ &= -(q_{kk} - a_{k-1})q_{ik} \;\;=\;\; -(1-a_k)q_{ik}. \end{aligned}$$

The fact that B'_k is self-adjoint implies that $b'_{kj} = -(1-a_k)q_{kj}$ for all $j < k$ as well.

The above expressions for the entries b'_{ij} reveal that $B'_k = (1-a_k)(I-\widehat{Q})$, where $\widehat{Q} \in \mathbb{M}_k$ denotes the k^{th} leading principal submatrix of Q. Since Q has rank $n-1$, Corollary 5.1.4 (i) implies that $\lambda = 1$ occurs as an eigenvalue of $\widehat{Q}$ with multiplicity at least $k-1$, and hence 0 occurs as an eigenvalue of B'_k with multiplicity at least $k-1$. It follows that

$$\|B'_k\| = \mathrm{Tr}(B'_k) = \sum_{\ell=1}^{k}(1-a_k)(1-q_{\ell\ell}) = a_k(1-a_k).$$

Now let $f : [0,1] \times [0,1] \to \mathbb{R}$ denote the function from Theorem 5.1.6, so that $\|Q_k\|^2 = f(a_{k-1}, a_k)$. Suppose for the sake of contradiction that $\|B_k\| = \|B'_k\|$, and hence $f(a_{k-1}, a_k) = a_k(1-a_k)$. One may verify that for this equation to hold, we necessarily have

$$(1-a_k)^3(a_k - a_{k-1}) = 0.$$

Thus, either $a_k = 1$ or $a_k = a_{k-1}$.

If the former is true, then $a_j = 1$ for all $j \geq k$. In particular, $a_{n-1} = a_n$. It then follows that $q_{nn} = 1 - (a_n - a_{n-1}) = 1$ by equation (5.4), and hence $\|Q_n\| \geq 1$. This contradicts the minimality of $\mathrm{dist}(Q, \mathcal{T}_n)$. If instead $a_k = a_{k-1}$, then $q_{kk} = 1$, and hence $\|Q_k\| \geq 1$. Again we reach a contradiction. We therefore conclude that $\|B'_k\| < \|B_k\|$, and thus $\|Q'_k\| < \|Q_k\|$. ■

To save the reader from lengthy computations, we have included a few examples of pairs (Q, T) where $Q \in \mathbb{M}_n$ is a projection of rank $n-1$, T belongs to $\mathcal{T}_n$, and $\|Q - T\| = \nu_{n-1,n}$. Theorem 5.3.2 implies that if (Q', T') is any other projection-nilpotent pair such that $rank(Q') = n-1$ and $\|Q' - T'\| = \nu_{n-1,n}$, then there is a unitary $V \in \mathbb{M}_n$ such that $Q' = V^*QV$ and $T' = V^*TV$.

$\underline{n = 3}$

$$Q = \begin{bmatrix} 0.64310 & -0.31960 & -0.35689 \\ -0.31960 & 0.71379 & -0.31960 \\ -0.35689 & -0.31960 & 0.64310 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & -0.49697 & -0.80194 \\ 0 & 0 & -0.49697 \\ 0 & 0 & 0 \end{bmatrix};$$

$\underline{n = 4}$

$$Q = \begin{bmatrix} 0.72361 & -0.24860 & -0.24860 & -0.27639 \\ -0.24860 & 0.77639 & -0.22361 & -0.24860 \\ -0.24860 & -0.22361 & 0.77639 & -0.24860 \\ -0.27639 & -0.24860 & -0.24860 & 0.72361 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & -0.34356 & -0.46094 & -0.65836 \\ 0 & 0 & -0.34164 & -0.46094 \\ 0 & 0 & 0 & -0.34356 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$\underline{n = 5}$

$$Q = \begin{bmatrix} 0.77471 & -0.20512 & -0.19907 & -0.20512 & -0.22528 \\ -0.20512 & 0.81324 & -0.18126 & -0.18676 & -0.20512 \\ -0.19907 & -0.18126 & 0.82409 & -0.18126 & -0.19907 \\ -0.20512 & -0.18676 & -0.18126 & 0.81324 & -0.20512 \\ -0.22528 & -0.20512 & -0.19907 & -0.20512 & 0.77472 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & -0.26477 & -0.32678 & -0.41846 & -0.55566 \\ 0 & 0 & -0.26373 & -0.32453 & -0.41846 \\ 0 & 0 & 0 & -0.26373 & -0.32678 \\ 0 & 0 & 0 & 0 & -0.26477 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is interesting to note that each projection above is symmetric about its anti-diagonal. This symmetry is in fact, always present in the optimal projection $Q = (q_{ij})$ from Theorem 5.3.2 obtained by taking $z_i = 1$ for all i. To see this, first observe that the function h_t from equation (5.5) satisfies the identity

$$h_t(x) + h_t^{-1}(1-x) = 1, \quad x \in [0,1].$$

From here we have that $a_1 + a_{n-1} = h_t(0) + h_t^{-1}(1) = 1$, and by induction,

$$a_k + a_{n-k} = h_t(a_{k-1}) + h_t^{-1}(a_{n-k+1}) = h_t(a_{k-1}) + h_t^{-1}(1 - a_{k-1}) = 1$$

for all $k \in \{1, 2, \dots, n\}$. Consequently,

$$\begin{aligned} q_{kk} &= 1 - (a_k - a_{k-1}) \\ &= a_{n-k} + a_{k-1} \\ &= a_{n-k} + (1 - a_{n-k+1}) \\ &= 1 - (a_{n-k+1} - a_{n-k}) = q_{n-k+1,n-k+1} \end{aligned}$$

for all k. We now turn to the identity $q_{ij} = -\sqrt{(1-q_{ii})(1-q_{jj})}$ to conclude that that $q_{ij} = q_{n-j+1,n-i+1}$ for all i and j, which is exactly the statement that Q is symmetric about its anti-diagonal. An analogous argument using the formulas from [16] demonstrates a similar phenomenon for optimal projections of rank 1.

§5.4 Future Directions and Open Questions

The distance $\nu_{r,n}$ from the set of projections in $\mathbb{M}_n$ of rank r to the set of nilpotent operators on $\mathbb{C}^n$, as well as the corresponding closest projection-nilpotent pairs, are now well understood when $r = 1$ or $r = n-1$. Of course, it is natural to wonder about the value of $\nu_{r,n}$ for r strictly between 1 and $n-1$.

The difficulty in extending the above arguments to projections P of intermediate ranks lies in deriving closed-form expressions for $\|E_{i-1}^{\perp} P E_i\|$. Computing these norms for projections of rank $r = 1$ or $r = n-1$ was made possible by the rigid structure afforded by such projections. In particular, we made frequent use of equations (5.1) and (5.2) throughout the proofs of Lemma 5.1.5 and Theorem 5.1.6. These equations—which describe the algebraic relations satisfied by the entries of a projection of rank 1 or $n-1$—become considerably more complex for projections of intermediate ranks.

For small values of r and n, the mathematical programming software *Maple* was used to construct examples of rank r projections $P_{r,n}$ in $\mathbb{M}_n$ which we believe are of minimal distance to $\mathcal{T}_n$. To ease the computations, the program was tasked with minimizing the maximum norm $\|E_{i-1}^{\perp} P E_i\|$ over all projections P of rank r with real entries and symmetry about the anti-diagonal. While it may not always be possible for such conditions to be met by an optimal projection of rank r, the computations that follow may still shed light on a potential formula for $\nu_{r,n}$.

The smallest value of n for which $\mathcal{P}(\mathbb{C}^n)$ contains projections of intermediate ranks is $n = 4$. In this case, the intermediate-rank projections are those of rank 2. We found that

$$P_{2,4} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

is an optimal projection of rank 2 satisfying the conditions above. It is easy to see that

$$\|E_{i-1}^{\perp} P_{2,4} E_i\| = 1/\sqrt{2} = \nu_{1,2} \quad \text{for all } \; i,$$

and hence $P_{2,4}$ is a direct sum of optimal rank-one projections in $\mathbb{M}_2$.

In $\mathbb{M}_5$, the intermediate-rank projections are those of rank $r = 2$ or $r = 3$. For such r, we obtained

$$P_{2,5} = \begin{bmatrix} 0.42602 & -0.07632 & 0.22568 & 0.42334 & -0.09248 \\ -0.07632 & 0.42127 & 0.23481 & -0.06022 & 0.42334 \\ 0.22568 & 0.23481 & 0.30541 & 0.23481 & 0.22568 \\ 0.42334 & -0.06022 & 0.23481 & 0.42127 & -0.07632 \\ -0.09248 & 0.42334 & 0.22568 & -0.07632 & 0.42602 \end{bmatrix} \quad \text{and}$$

$$P_{3,5} = \begin{bmatrix} 0.58296 & -0.29271 & -0.10684 & 0.12213 & 0.36209 \\ -0.29271 & 0.62479 & -0.33169 & -0.15433 & 0.12213 \\ -0.10684 & -0.33169 & 0.58448 & -0.33169 & -0.10684 \\ 0.12213 & -0.15433 & -0.33169 & 0.62479 & -0.29271 \\ 0.36209 & 0.12213 & -0.10684 & -0.29271 & 0.58296 \end{bmatrix}.$$

Again, the norms $\|E_{i-1}^{\perp} P_{r,n} E_i\|$ share a common value, with

$$\|E_{i-1}^{\perp} P_{2,5} E_i\| = 0.65270 \approx \frac{1}{2} \sec\left(\frac{\pi}{\frac{5}{2} + 2}\right) \quad \text{for all } i, \quad \text{and}$$

$$\|E_{i-1}^{\perp} P_{3,5} E_i\| = 0.76352 \approx \frac{1}{2} \sec\left(\frac{\pi}{\frac{5}{3} + 2}\right) \quad \text{for all } i.$$

In light of these findings, as well as the distance formulas that exist for projections of rank 1 or $n-1$, we propose the following generalized distance formula for projections of arbitrary rank.

Conjecture 5.4.1. *For every $n \in \mathbb{N}$ and each $r \in \{1, 2, \dots, n\}$, the distance from the set of projections in $\mathbb{M}_n$ of rank r to $\mathcal{N}(\mathbb{C}^n)$ is*

$$\nu_{r,n} = \frac{1}{2} \sec \left(\frac{\pi}{\frac{n}{r} + 2} \right).$$

Using a random walk process implemented by the computer algebra system *PARI/GP*, we estimated the values of $\nu_{r,n}$ for all $r \leq n \leq 10$ without the additional assumptions described above. We observed only minute differences between these estimates and the expression from Conjecture 5.4.1. In many cases, these quantities differed by no more than 1×10^{-3}.

The proposed formula from Conjecture 5.4.1 merits several interesting consequences. Firstly, this formula suggests that $\nu_{r,n} = \nu_{kr,kn}$ for every positive integer k, meaning that a closest projection of rank kr to $\mathcal{T}_{kn}$ could be obtained as a direct sum of k closest projections of rank r to $\mathcal{T}_n$. Notice as well that if the equation $\nu_{r,n} = \nu_{kr,kn}$ were true, it would follow that

$$\nu_{1,n} = \nu_{r,rn} \leq \nu_{r,n}$$

for each n and r. Thus, a proof of Conjecture 5.4.1—or of the formula $\nu_{r,n} = \nu_{kr,kn}$—would validate Conjecture 1.0.3.

Conjecture 5.4.2. *If n and r are positive integers with $r \leq n$, then*

$$\nu_{r,n} = \nu_{kr,kn} \text{ for all } k \in \mathbb{N}.$$

Despite considerable effort, little headway has been made in proving Conjecture 5.4.1. As discussed above, obtaining expressions for the norms $\|E_{i-1}^{\perp} P E_i\|$ appears to be a formidable task when P is a projection of rank $r \neq 1, n-1$. Thus, rather than focussing on deriving explicit formulas for these norms, we have sought to determine whether or not they necessarily share a common value when P is of minimal distance to $\mathcal{T}_n$. This fact alone may be useful in verifying Conjecture 5.4.1 for small values of n.

Question 1. *If $P \in \mathbb{M}_n$ is a projection of rank r that is of distance $\nu_{r,n}$ to $\mathcal{T}_n$, is it necessarily true that $\|E_{i-1}^{\perp} P E_i\| = \|E_{j-1}^{\perp} P E_j\|$ for all i, j?*

As discussed in §5.3, Question 1 has an affirmative answer when $r = 1$ [16, Lemma 3], as well as when $r = n-1$ (Theorem 5.1.8). Moreover, the Maple estimates described above suggest a similar phenomenon for intermediate-rank projections in $\mathbb{M}_4$ and $\mathbb{M}_5$.

Adapting the arguments from [16, Lemma 3] and Theorem 5.1.8 may be a viable approach to answering Question 1. In the proofs of these results, one first assumes for the sake of contradiction that there is a projection P of minimal distance to $\mathcal{T}_n$ such that not all norms $\|E_{j-1}^{\perp}PE_j\|$ are equal. Next, by adjusting individual terms of the sequence $\{a_i\}_{i=0}^{n}$, one shifts the weight between the norms of adjacent corners $E_{j-1}^{\perp}PE_j$ and $E_j^{\perp}PE_{j+1}$ until the largest such norm has decreased. These adjustments are implemented via unitary conjugation by operators that act as the identity on $\operatorname{span}\{e_j, e_{j+1}\}^{\perp}$. Eventually this process yields an equivalent projection P' that is closer to $\mathcal{T}_n$ than P, thereby giving the required contradiction.

Motivated by the success of the above arguments, one may hope that a similar result could be obtained for projections Q of arbitrary rank using unitary conjugations of this form. We believe that this will indeed be the case. It is straightforward to prove that if U is a unitary such that $U\big|_{\operatorname{span}\{e_j,e_{j+1}\}^{\perp}} = I$, then

$$\|E_{i-1}^{\perp}U^*QUE_i\| = \|E_{i-1}^{\perp}QE_i\| \quad \text{for all} \quad i \neq j, j+1.$$

Without a closed-form expression for $\|E_{i-1}^{\perp}QE_i\|$, however, it becomes difficult to determine whether or not there is a transformation of this form that decreases the maximum of $\|E_{j-1}^{\perp}QE_j\|$ and $\|E_j^{\perp}QE_{j+1}\|$. Thus, this approach may have its limitations until the monotonicity properties the norms $\|E_{i-1}^{\perp}QE_i\|$ under such a transformation are better understood.

Another interesting open problem concerns the converse to Question 1.

Question 2. *If $P \in \mathbb{M}_n$ is a projection of rank r and $\|E_{i-1}^{\perp}PE_i\| = \|E_{j-1}^{\perp}PE_j\|$ for all i and j, is P of distance $\nu_{r,n}$ to $\mathcal{T}_n$?*

An affirmative answer to Question 2 would imply that if P_1 and P_2 are unitarily equivalent projections with the property that for each i, $\|E_{i-1}^{\perp}P_kE_i\| = \gamma_k$ for some constants $\gamma_1, \gamma_2 \geq 0$, then $\gamma_1 = \gamma_2$. Furthermore, it is easy to see that affirmation of both Questions 1 and 2 would validate Conjecture 5.4.2. Indeed, a positive answer to Question 1 would imply that every projection P of rank r closest to $\mathcal{T}_n$ is such that $\|E_{i-1}^{\perp}PE_i\| = \nu_{r,n}$ for all i. As a result, the k-fold direct sum $Q = P \oplus P \oplus \cdots \oplus P$ would define a projection in $\mathbb{M}_{kn}$ of rank kr with the property that $\|E_{i-1}^{\perp}QE_i\| = \nu_{r,n}$ for all i. From here, a positive answer to Question 2 would imply that $\nu_{r,n} = \nu_{kr,kn}$.

Although Question 2 remains open, a natural analogue of this problem is known to fail for unitary orbits of self-adjoint matrices. Specifically, there are examples of unitarily equivalent self-adjoint matrices A and B, as well as positive constants $\gamma_1 < \gamma_2$, such that $\|E_{i-1}^{\perp} A E_i\| = \gamma_1$ and $\|E_{i-1}^{\perp} B E_i\| = \gamma_2$ for all i. Thus, while the norms $\|E_{i-1}^{\perp} B E_i\|$ share a common value, there exist matrices in the unitary orbit of B that are strictly closer to $\mathcal{T}_n$.

For one such example, consider the operators

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1/2 & 0 & 3/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 3/2 & 0 & 1/2 & 0 \end{bmatrix}.$$

It is straightforward to check that $B = U^* A U$, where

$$U \coloneqq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Moreover, we have that $\|E_{i-1}^{\perp} A E_i\| = 1$, while $\|E_{i-1}^{\perp} B E_i\| = \frac{\sqrt{10}}{2}$ for each $i \in \{1, 2, 3, 4\}$.

Note that the eigenvalues of A and B are $\pm\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Thus, while this example shows that the common norm condition is not sufficient for a self-adjoint operator to be closest to $\mathcal{T}_n$ among the operators in its unitary orbit, it does not indicate that such a phenomenon can occur for positive operators. It would therefore be interesting to determine whether or not the analogue of Question 5.4 fails for positive operators as well. Of course, since the norms $\|E_{i-1}^{\perp} A E_i\|$ behave poorly under translation of A by λI, one cannot hope to obtain a counterexample by simply adding a multiple of the identity to the operators above.

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