A Generalization to Signed Graphs of a Theorem of Sergey Norin and Robin Thomas

by

Courtney Horrocks

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2019

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis we characterize the minimal non-planar extensions of a signed graph. We consider the following question: Given a subdivision of a planar signed graph (G, Σ), what are the minimal structures that can be added to the subdivision to make it non-planar? Sergey Norin and Robin Thomas answered this question for unsigned graphs, assuming almost 4-connectivity for G and H. By adapting their proof to signed graphs, we prove a generalization of their result.

Acknowledgments

I would like to acknowledge my supervisor, Bertrand Guenin, for his guidance and advice. I would also like to thank Katie Naismith for her valuable ideas and discussions.

I would like to thank Luke Postle and Bruce Richer for reading my thesis and offering suggestions.

I am forever grateful to my parents, David and Alice Horrocks, for their love and support. Thanks also to Bethany for being the best sister and roommate anyone could ask for.

Table of Contents

List of Figures

Chapter 1

Introduction

In this thesis, graphs are finite and simple unless otherwise specified. Given a planar graph, we are interested in knowing what are the minimal structures that can be added to the graph to make it non-planar.

A graph G is *almost 4-connected* if it is simple, 3-connected, has at least five vertices, and $V(G)$ cannot be partitioned into three sets A, B, C in such a way that $|C| = 3$, $|A| \geq 2$, $|B| \geq 2$, and no edge of G has one end in A and the other end in B. Almost 4-connectivity is a weakening of 4-connectivity that allows for vertices of degree three. A graph S is a *subdivision* of G if S can be obtained by replacing each edge of G by a path of length at least one that has the same ends, where the paths are internally vertex-disjoint from each other. The paths in S that correspond to the edges of G are called the *segments* of S, and the ends of segments are called *branch-vertices*. If S is a subgraph of H and S is a subdivision of G, then we say that S is a *G-subdivision* in H. Suppose that S is a G-subdivision in H. An *S-path* is a path P in H such that P has length at least one and its ends and only its ends belong to $V(S)$.

In [3], Norin and Thomas proved a result for unsigned graphs. The main theorem describes the minimal non-planar extensions of planar graphs if we assume that our graphs are almost 4-connected. The following is their result, stated as (1.1) in [3]:

Theorem 1.1. *Let G be an almost 4-connected planar graph on at least seven vertices, let H be an almost 4-connected non-planar graph, and let there exist a G-subdivision in H. Then there exists a G-subdivision S in H such that one of the following conditions holds:*

(i) *there exists an S-path in H joining two vertices of S not incident with the same face, or*

(ii) *there exist two vertex-disjoint S-paths with ends s_1, t_1 and s_2, t_2 respectively such that the vertices s_1, s_2, t_1, t_2 belong to some face boundary of S in the order listed Moreover, for $i = 1, 2$ the vertices s_i and t_i do not belong to the same segment of S, and if two segments of S include all of s_1, t_1, s_2, t_2, then those segments are vertex-disjoint.*

1.1 Signed Graphs and Rerouting

Our goal in this thesis is to prove a generalization of Theorem 1.1 for signed graphs. A signed graph is a pair (G, Σ) such that G is a graph and Σ is a subset of $E(G)$. The edges of G that are in Σ are *odd* and the other edges are *even.* The parity of a subgraph F of G is defined as the parity of $|E(F) \cap \Sigma|$.

We say that Γ' is a *signature* of a signed graph (H, Γ) if each cycle in H has the same parity in both (H, Γ) and (H, Γ'). Since every cycle uses an even number of edges on a cut, we know that if F is a cut of H, then $\Gamma \,\Delta\, F$ is also a signature of (H, Γ). In fact, Γ' is a signature of (H, Γ) if and only if $\Gamma \,\Delta\, \Gamma'$ is a cut of H.

When we consider a path P in (H, Γ), it will often be helpful to assume that Γ has been replaced by a signature Γ' for which $E(P) \cap \Gamma' = \emptyset$. In that case we say that (H, Γ) has been *resigned* so that every edge of P is even. Since a path is bipartite, this resigning can always be done.

We will now define the concept of subdivisions for signed graphs. Let (G, Σ) and (H, Γ) be signed graphs. Let (S, Λ) be a signed graph such that S is a subdivision of G. Suppose that there exists a signature Σ' of G such that for each edge $e \in E(G)$, the parity of e in (G, Σ') is the same as the parity of the segment Z of (S, Λ) corresponding to e. In those circumstances, we say that (S, Λ) is a *subdivision* of (G, Γ). If S is a subgraph of H and $(S, \Gamma \cap E(S))$ is a subdivision of (G, Σ), then we say that S is a (G, Σ)-*subdivision* in (H, Γ).

It will often be necessary to transform one (G, Σ)-subdivision into another, and for this purpose we will use rerouting. Suppose that there exists a (G, Σ)-subdivision S in (H, Γ). If there exists another (G, Σ) subdivision S' that can be obtained from S by deleting vertices and edges of S and adding vertices and edges from H that are not in S, then we say that S' is obtained from S by *rerouting*. We will introduce two kinds of rerouting that will be especially common.

Let B be a subgraph of H such that $E(B) \subseteq E(H) - E(S)$ and some segment Z of S contains every vertex of $V(B) \cap V(S)$. The *shadow* of B is the minimal subpath of Z that contains all attachments of B. We denote the shadow of B by $\text{shadow}(B)$.

Let Z be a segment of S. By possibly resigning (H, Γ), we may assume that every edge of Z is even. Let P be an even S-path with endpoints on Z. Let S' be obtained from S by replacing $\text{shadow}(P)$ by P. Then S' is a (G, Σ)-subdivision in (H, Γ), and we say S' is obtained from S by rerouting Z along P. See Figure 1.1.

Now let P_1 and P_2 be S-paths with endpoints on Z such that P_1 and P_2 have the same parity. Suppose $\text{shadow}(P_1) \not\subseteq \text{shadow}(P_2)$ and $\text{shadow}(P_2) \not\subseteq \text{shadow}(P_1)$. Let S' be obtained from S by replacing $\text{shadow}(P_1) \,\Delta\, \text{shadow}(P_2)$ by $P_1 \cup P_2$. Then S' is a (G, Σ)-subdivision in (H, Γ), and we say S' is obtained from S by rerouting Z along P_1 and P_2. See Figure 1.2. If S' can be obtained from S by a series of these two kinds of reroutings, then we say that S' is *related to* S.

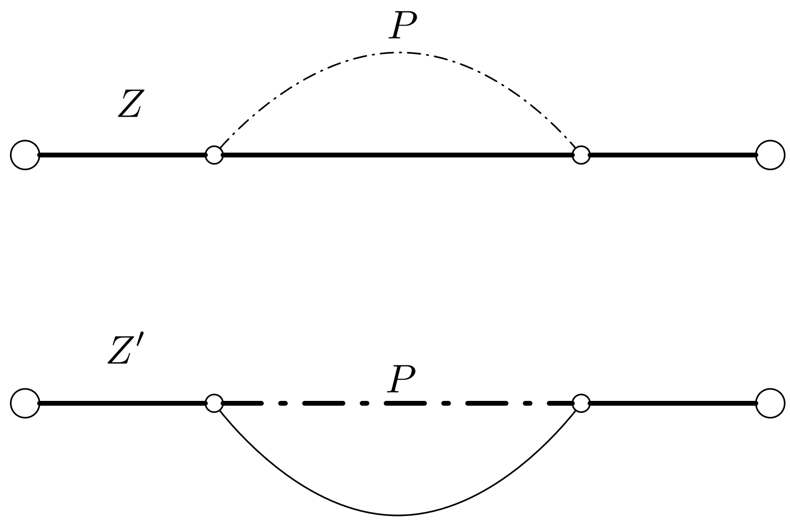

Figure 1.1: Rerouting Z along P. The new segment is Z'.

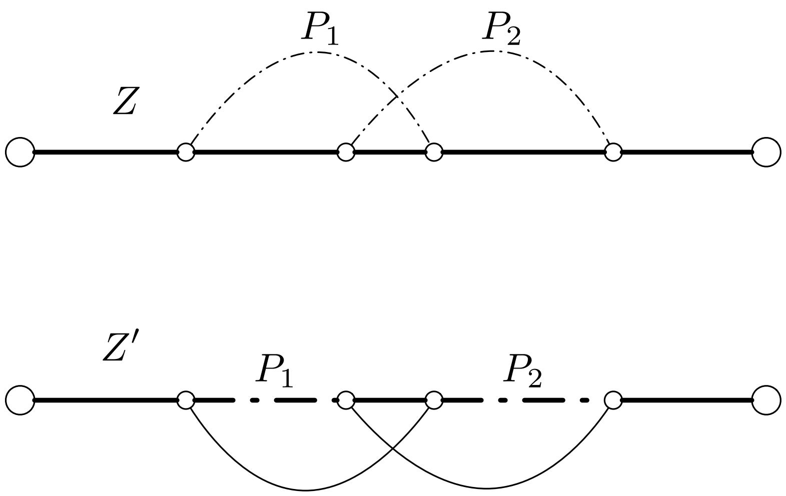

Figure 1.2: Rerouting Z along P_1 and P_2. The new segment is Z'.

1.2 Disk Systems

Instead of working directly with the facial cycles of a planar graph, we will work with a set of cycles that have the specific properties of facial cycles that we need. We will use the concept of disk systems that was used in [3].

A cycle C in a graph G is called *peripheral* if C is an induced subgraph of G and $G - V(C)$ is connected. We will need the following three well-known results about planar graphs [4, 5].

Lemma 1.2.1. *Let G be a subdivision of a 3-connected planar graph, and let C be a cycle in G. Then the following conditions are equivalent:*

(i) the cycle C bounds a face in some planar embedding of G,

(ii) the cycle C bounds a face in every planar embedding of G,

(iii) the cycle C is peripheral.

Lemma 1.2.2. *Let G be a subdivision of a 3-connected planar graph, and let C_1, C_2 be two distinct peripheral cycles in G. Then the intersection of C_1 and C_2 is either null, a one-vertex graph, or a segment.*

Lemma 1.2.3. *Let G be a subdivision of a 3-connected planar graph, let $v \in V(G)$, and let e_1, e_2, e_3 be three distinct edges of G incident with v. If there exist peripheral cycles C_1, C_2, C_3 in G such that $e_i \in E(C_j)$ for all distinct indices i, $j \in \{1, 2, 3\}$, then v has degree three.*

A *weak disk system* in a graph G is a set $\mathcal{C}$ of distinct cycles of G, called *disks*, such that

(X0) every edge of G belongs to exactly two members of $\mathcal{C}$, and

(X1) the intersection of any two distinct members of $\mathcal{C}$ is either null, a one-vertex graph, or a segment.

A weak disk system is a *disk system* if it satisfies (X0), (X1), and

(X2) if e_1, e_2, e_3 are three distinct edges incident with a vertex v of G and there exist disks C_1, C_2, C_3 such that $e_i \in E(C_j)$ for all distinct integers i, $j \in \{1, 2, 3\}$, then v has degree three.

By Lemmas 1.2.1, 1.2.2, and 1.2.3, the peripheral cycles of a subdivision of a 3-connected planar graph form a disk system. If S' is obtained from S by rerouting, then a weak disk system $\mathcal{C}$ in S induces a weak disk system $\mathcal{C}'$ in S'. If $\mathcal{C}$ is a disk system, then so is $\mathcal{C}'$.

1.3 Main Theorem

We will need several definitions before stating our main theorem. For all the definitions below, let (G, Σ) and (H, Γ) be signed graphs, let S be a (G, Σ)-subdivision in (H, Γ), and let $\mathcal{C}$ be a weak disk system in S. If x and y are vertices on a path Z, we use xZy to denote the subpath of Z with ends x and y.

An S-path P is an *S-jump* if no disk in $\mathcal{C}$ includes both ends of P. Suppose that P_1, P_2, P_3 are internally vertex-disjoint S-paths. Let x_i, y_i be the ends of P_i. Suppose C_1 and C_2 are two disks that share a segment Z. If x_3, $y_3 \in V(Z)$, x_1 and x_2 are in the interior of x_3Zy_3, $y_1 \in V(C_1 - Z)$, $y_2 \in V(C_2 - Z)$, and P_3 and x_3Zy_3 have opposite parity, then we say that P_1, P_2, P_3 forms an *interrupted S-jump*. See Figure 1.3.

Let $C \in \mathcal{C}$, and let P_1 and P_2 be two vertex-disjoint S-paths with ends u_1, v_1 and u_2, v_2 respectively, such that u_1, u_2, v_1, v_2 belong to $V(C)$ and occur on C in the order listed. Then we say that the pair P_1, P_2 is an *S-cross* and that the vertices u_1, u_2, v_1, v_2 are its *feet*. We say that the cross P_1, P_2 is *weakly free* if, for $i = 1, 2$, no segment of S includes both ends of P_i. We say that a cross P_1, P_2 is *free* if it is weakly free and no two segments of S that share a vertex include all the feet of the cross.

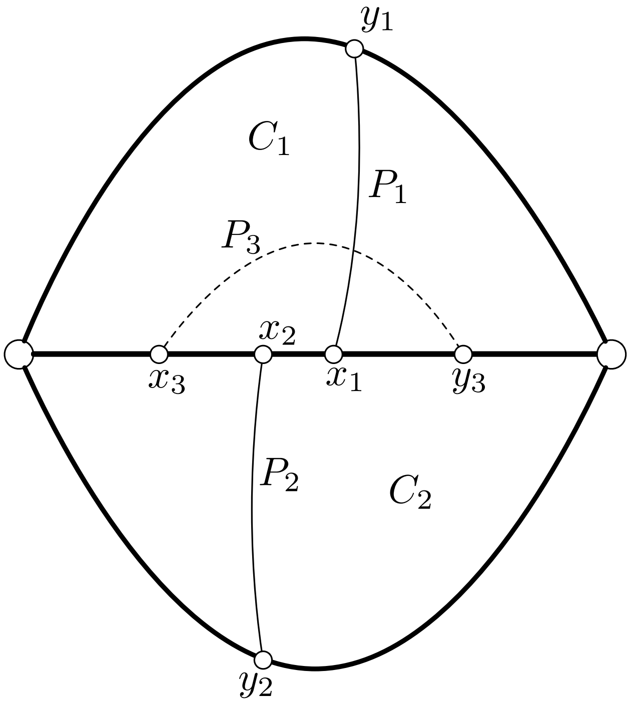

Figure 1.3: An interrupted S-jump. Dotted paths are odd.

Let P_1, P_2 be a weakly free cross in C that is not free. Then there exist two distinct segments Z_1 and Z_2, both incident with a branch-vertex v of S, such that $Z_1 \cup Z_2$ includes all the feet of P_1, P_2. In that case we say that the cross P_1, P_2 is *centred at* v and *based at* Z_1 and Z_2. If v has degree three, let Z_3 be the third segment incident with v. For $i = 1, 2$, let C_i be the other disk containing Z_i. Let v_i be the other end of Z_i. Let x_i and y_i be the ends of P_i. We may assume that x_1, x_2, v occur on Z_1 in the order listed and that y_2, y_1, v occur on Z_2 in the order listed. We may also assume that (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2$ is even. Suppose there exists a path Q_1 with ends s_1 and t_1 such that s_1 is in the interior of vZ_2y_2. Suppose further that $y \in V(C_2 - (Z_2 \cup Z_3))$ if v has degree three and that $y \in V(C_2 - Z_2)$ if v has degree four or greater. Suppose that at least one of P_1 and P_2 is odd. Then we say that P_1, P_2, Q_1 is a *type-1 extended S-cross*. See Figure 1.4.

Now suppose that there exist paths Q_1 and Q_2, where Q_i has endpoints s_i and t_i. Suppose that s_1 is in the interior of vZ_2y_2, $t_1 \in V(y_2Z_2v_2) - \{y_2\}$, $s_2 \in V(y_2Z_2t_1) - \{t_1\}$, and $t_2 \in V(C - (Z_1 \cup Z_2))$. Suppose that at least one of P_1 and P_2 is odd and that Q_1 is odd. Then we say that P_1, P_2, Q_1, Q_2 is a *type-2 extended S-cross*. See Figure 1.5.

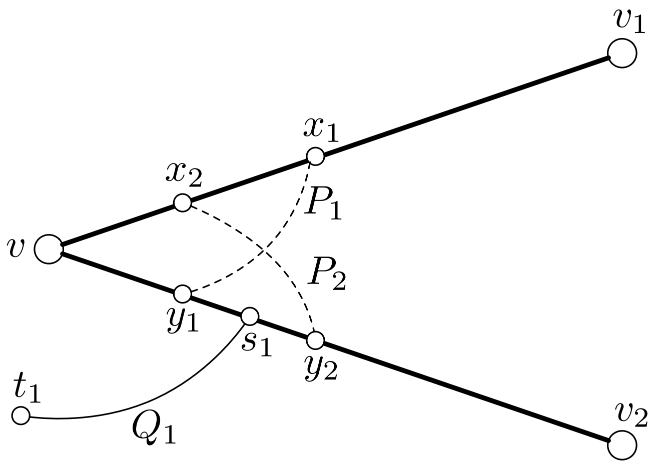

Figure 1.4: A type-1 extended S-cross

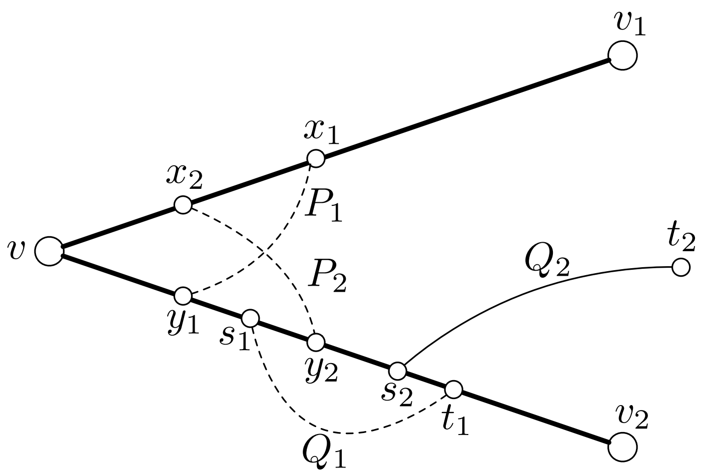

Figure 1.5: A type-2 extended S-cross

Now suppose that there exist paths Q_1, Q_2, Q_3 where Q_i has endpoints s_i and t_i. Suppose that s_1 is in the interior of vZ_2y_2, $t_1 \in V(y_2Z_2v_2) - \{y_2\}$, $s_2 \in V(y_2Z_2t_1) - \{t_1\}$, $t_2 \in V(x_1Z_1v_1) - \{x_1\}$, $s_3 \in V(x_1Z_1t_2) - \{t_2\}$, and $t_3 \in V(C_1 - Z_1)$. Suppose that at least one of P_1 and P_2 is odd and that Q_1 is odd. Then we say that P_1, P_2, Q_1, Q_2, Q_3 is a *type-3 extended S-cross*. See Figure 1.6.

Now suppose that there exist paths Q_1, Q_2, Q_3, Q_4 where Q_i has endpoints s_i and t_i. Suppose that s_1 is in the interior of vZ_2y_2, $t_1 \in V(y_2Z_2v_2) - \{y_2\}$, $s_2 \in V(y_2Z_2t_1) - \{t_1\}$, $t_2 \in V(x_1Z_1v_1) - \{x_1\}$, $s_3 \in V(x_1Z_1t_2) - \{t_2\}$, $t_3 \in V(t_2Z_1v_2) - \{t_2\}$, $s_4 \in V(t_2Z_1s_3) - \{t_3\}$, and $t_4 \in V(C - (Z_1 \cup Z_2))$. Suppose

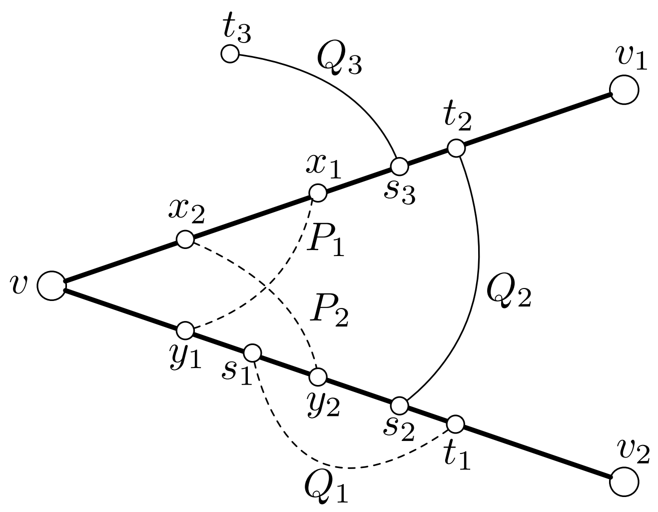

Figure 1.6: A type-3 extended S-cross

that at least one of P_1 and P_2 is odd and that Q_1 and Q_3 are odd. Then we say that P_1, P_2, Q_1, Q_2, Q_3, Q_4 is a *type-4 extended S-cross*. See Figure 1.7.

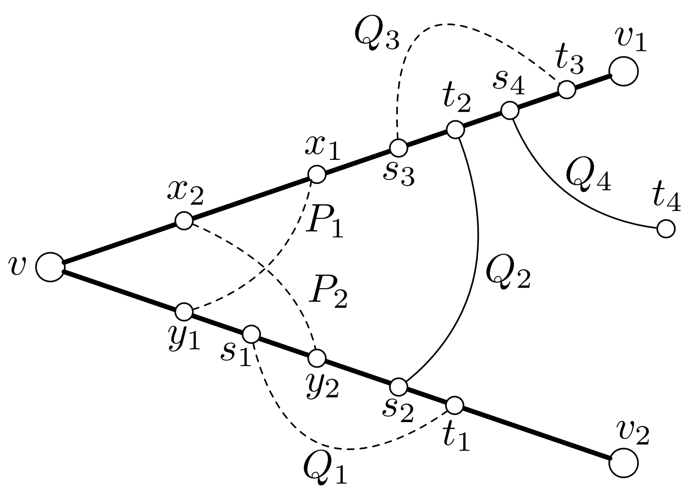

Figure 1.7: A type-4 extended S-cross

Now suppose that there exist paths Q_1, Q_2, Q_3, Q_4, Q_5 where Q_i has end-points s_i and t_i. Suppose that s_1 is in the interior of vZ_2y_2, $t_1 \in V(y_2Z_2v_2) - \{y_2\}$, $s_2 \in V(y_2Z_2t_1) - \{t_1\}$, $t_2 \in V(x_1Z_1v_1) - \{x_1\}$, $s_3 \in V(x_1Z_1t_2) - \{t_2\}$, $t_3 \in V(t_2Z_1v_2) - \{t_2\}$, $s_4 \in V(t_2Z_1s_3) - \{t_3\}$, $t_4 \in V(t_1Z_2v_2) - \{t_1\}$, $s_5 \in V(t_1Z_2t_4) - \{t_4\}$, and $t_5 \in V(C_2 - Z_2)$. Suppose that at least one of P_1 and P_2 is odd and that Q_1 and Q_3 are odd. Then we say that P_1, P_2, Q_1, Q_2, Q_3, Q_4, Q_5 is a *type-5 extended S-cross.* See Figure 1.8.

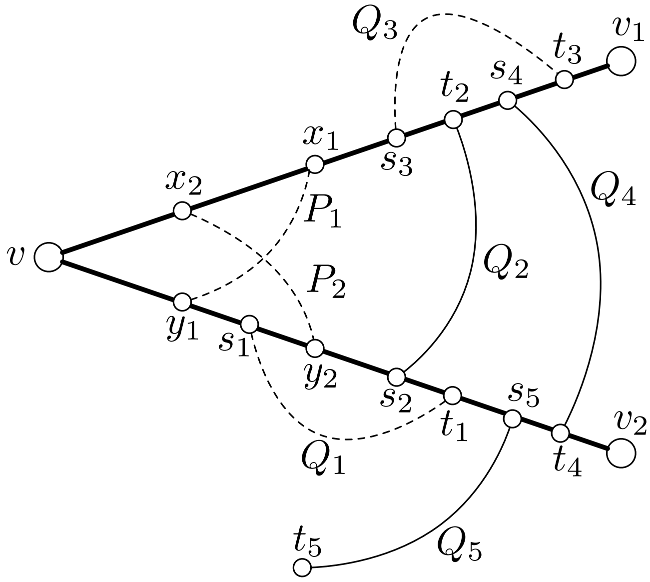

Figure 1.8: A type-5 extended S-cross

Let S be a subgraph of a graph H. An *S-bridge* in H is a connected subgraph B of H such that $E(B) \cap E(S) = \emptyset$ and either $E(B)$ consists of a single edge with both ends in S, or for some component C of $H - V(S)$ the set $E(B)$ consists of all edges of H with at least one end in $V(C)$. The vertices in $V(B) \cap V(S)$ are called the *attachments* of B. Let G be a graph with no vertices of degree two, and let S be a G-subdivision in a graph H. If B is an S-bridge of H, then we say that B is *unstable* if some segment of S includes all the attachments of B; otherwise we say that B is *stable.* If B is an unstable S-bridge of H, then we say B is *S-planar* if B has a planar embedding in a disk with all attachments of B on the boundary of the disk.

Let B be an unstable S-bridge that is not S-planar. Let Z be the segment that contains all the attachments of B, and let C be a disk that contains Z. Let x_1, x_2, x_3, x_4 be distinct attachments of B that occur on Z in that order. Let P_1, P_2 be vertex-disjoint S-paths with endpoints w_1, z_1 and w_2, z_2 respectively such that w_1, $w_2 \in V(x_2Zx_3)$ and z_1, $z_2 \in V(C - Z)$. Suppose that (H, Γ) can be resigned such that every edge of B is even, any one edge of x_2Zx_3 is odd, and every other edge of Z is even. Then we say that B, P_1, P_2 is an *S-umbrella.* See Figure 1.9.

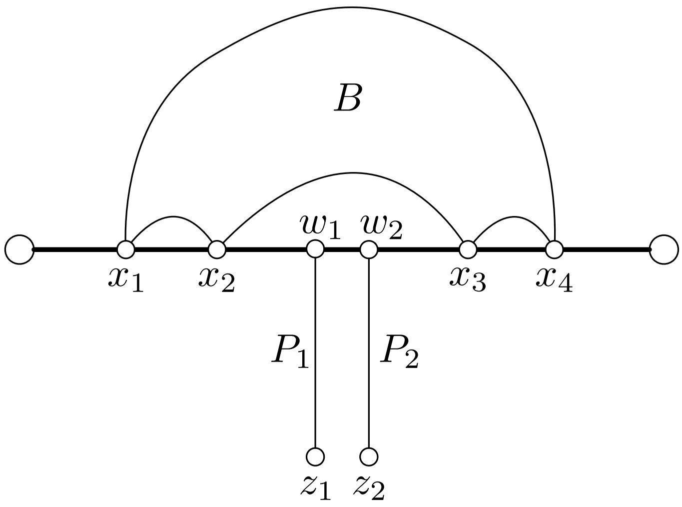

Figure 1.9: An S-umbrella

We are now ready to state the main theorem.

Theorem 1.3.1. *Let (G, Σ) be an almost 4-connected planar signed graph on at least six vertices, let (H, Γ) be an almost 4-connected non-planar signed graph, and let S be a (G, Σ)-subdivision in (H, Γ). Then there exists a (G, Σ)-subdivision S' in (H, Γ) obtained from S by repeated reroutings such that S' and the disk system of peripheral cycles in S' satisfy one of the following conditions:*

(i) there exists an S'-jump,

(ii) there exists a free S'-cross,

(iii) there exists an interrupted S'-jump,

(iv) there exists an S'-umbrella, or

(v) for some $j \in \{1, 2, ..., 5\}$, there exists a type-j extended S'-cross.

1.4 Outcomes

The outcomes in Theorem 1.3.1 that are not present in Theorem 1.1 are there because we have fewer options when rerouting with signed graphs. We will give a brief explanation of why the extra outcomes are not present in Theorem 1.1.

In the case of an interrupted S-jump, P_3 and $x_3 Z y_3$ are required to have opposite parity. If P_3 and $x_3 Z y_3$ were both even, we could reroute Z along P_3 to obtain a new (G, Σ) subdivision S' and an S'-jump.

If $\Sigma = \emptyset$, then there exists a (G, Σ)-subdivision S such that every S-bridge is stable [3]. Thus we can eliminate unstable S-bridges in the unsigned case.

In the case of a weakly free S-cross that is not free, if both of the crossing paths are even, then we can reroute the two segments along the crossing paths in what is described in [3] as an *X-rerouting*. In the case where P_1 and P_2 are both even, we do not need the extended S-cross outcomes. Extended S-crosses become unavoidable outcomes when at least one of P_1 and P_2 is odd.

Chapter 2

Unstable Bridges

2.1 An Unstable Bridge Lemma

In [2], Naismith proved a minimal non-planar extension theorem assuming only 3-connectivity for (G, Σ) and no restrictions on the connectivity of (H, Γ). These results were improved in [1], which has not yet been published. We will be quoting several results from [1].

By eventually assuming almost 4-connectivity for both (G, Σ) and (H, Γ), we will be able to eliminate several outcomes from Naismith's theorem. In order to eliminate most of the outcomes involving unstable bridges, we will need to prove a lemma about the structure of unstable bridges.

First we introduce some terminology used in [1]. Let B_1 and B_2 be unstable S-bridges. If B_1 has an attachment in the interior of shadow(B_2), then we say that B_2 is *over an attachment of* B_1. If B_1 is over an attachment of B_2 and B_2 is over an attachment of B_1, then we say that B_1 and B_2 *cross*.

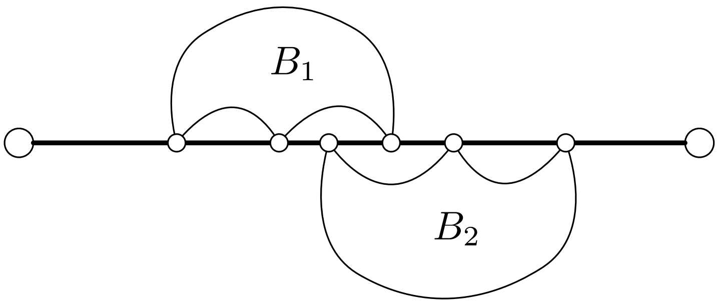

Figure 2.1: B_1 and B_2 are unstable bridges that cross.

Let (G, Σ) and (H, Γ) be signed graphs, and let S be a (G, Σ)-subdivision in (H, Γ). We say that an S-bridge B is *type-0* if B is stable. If B is an unstable S-bridge, B is over an attachment of a type-j S-bridge, and B is not over an attachment of any type-k S-bridge for $k < j$, then B is *type-(j+1)*.

A *separation* of a graph H is a pair (A, B) of subsets of $V(H)$ such that $A \cup B = V(H)$ and there is no edge between $A - B$ and $B - A$. The *order* of a separation is the size of the set $A \cap B$.

A *clump* is a maximal non-empty set C of unstable bridges such that, for every pair B, B' of bridges in C, there exists a sequence B, B_1, ..., B_k, B' of bridges in C such that consecutive bridges in the sequence are crossing. Let $\mathcal{B}$ be a clump of S-bridges on a segment Z. We define the *shadow* of $\mathcal{B}$ to be the minimal subpath of Z that contains the attachments of every bridge in $\mathcal{B}$. Let x, y be the endpoints of shadow($\mathcal{B}$). Let $A = V(\mathcal{B} \cup xZy))$. We say $\mathcal{B}$ is *2-separated* from S in H if $(A, V(H) - (A - \{x, y\}))$ is a separation of H with order at most 2.

Suppose that some S-bridge has an attachment a on Z. Let $\mathcal{B}$ be a set of unstable S-bridges with attachments on Z. We say a is *$\mathcal{B}$-significant* if a is interior to shadow(B) for some S-bridge $B \in \mathcal{B}$.

In [1], Naismith proved a useful result about clump structure.

Lemma 2.1.1. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ). Then there exists a (G, Σ)-subdivision S' related to S such that, for every clump $\mathcal{B}$ of S' bridges on a segment Z, one of the following occurs:*

(a) $\mathcal{B}$ is 2-separated from S'.

(b) A unique vertex w of Z is a $\mathcal{B}$-significant attachment of a stable S'-bridge, and every S'-bridge in $\mathcal{B}$ is over w. Suppose (H, Γ) has been resigned such that every edge of Z is even. There exist distinct vertices y_1, y_2 such that y_1, w, y_2 are distinct and occur on Z in the order stated, and there exist odd S'-paths P_1 and P_2 in S'-bridges of $\mathcal{B}$ such that, for $i = 1$, 2, P_i has endpoints y_i, w. Furthermore, for all $B \in \mathcal{B}$, every S'-path P in B with w in the interior of shadow(P) *is odd, and every S'-path P in B with $w \notin$* shadow(P) *is even.*

(c) Every S'-bridge in $\mathcal{B}$ is over an attachment of a stable S'-bridge. There exist vertices x_1, x_2, x_3, x_4 that occur on Z in the order stated such that

$x_2 \neq x_3$, each of x_1Zx_2, x_3Zx_4 contains an attachment of each bridge in $\mathcal{B}$, and every attachment of $\mathcal{B}$ is in $x_1Zx_2 \cup x_3Zx_4$. Every $\mathcal{B}$-significant attachment of any stable S'-bridge is in x_2Zx_3, and x_2Zx_3 contains at least one such attachment. Furthermore, (H, Γ) can be resigned such that every edge of every S'-bridge in $\mathcal{B}$ is even, any one edge of x_2Zx_3 is odd, and every other edge of Z is even.

(d) A unique vertex w of Z is a $\mathcal{B}$-significant attachment of a stable S'-bridge, and some S'-bridge of $\mathcal{B}$ is not over w. Let C_0 denote the set of all S'-bridges B in $\mathcal{B}$ such that B is over w. Let C_1, ..., C_k denote the maximal non-empty sets of S'-bridges in $\mathcal{B}$ such that, for $i = 1$, ..., k, no member of C_i is over w, and for every pair of S'-bridges B, $B' \in C_i$, there exists a sequence B, B_1, ..., B_j, B' of S'-bridges in C_i such that consecutive S'-bridges in the sequence are crossing. Then $\mathcal{B} = C_0 \cup C_1 \cup ...C_k$.

If G is a graph and X is a subset of $V(G)$, then we use $G[X]$ to denote the graph $G - (V(G) - X)$.

Let G and H be graphs where G has no vertices of degree two, and let S be a G-subdivision in H. A separation (X, Y) of H is called an S-*separation* if the order of (X, Y) is at most three, $X - Y$ includes at most one branch-vertex of S, and the graph $H[X]$ does not have a planar embedding in a disk with $X \cap Y$ drawn on the boundary of the disk. When we introduce almost 4-connectivity for (H, Γ), we will be able to show that (H, Γ) does not contain an S-separation.

Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. We say that (H, Γ) contains a *standard obstruction* if there exists a (G, Σ)-subdivision S in (H, Γ) with a weak disk system $\mathcal{C}$ such that (H, Γ) contains an S-jump, an interrupted S-jump, a weakly free S-cross, or an S-separation.

Now, using Lemma 2.1.1, we will prove a result that restricts the kinds of unstable bridge cases that we will have to consider.

Lemma 2.1.2. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$. Suppose that (H, Γ) does not contain a standard obstruction. Then if S' is the (G, Σ)-subdivision guaranteed by Lemma 2.1.1, (H, Γ) does not contain any crossing unstable S'-bridges.*

Proof. Let S' be the subdivision guaranteed by Lemma 2.1.1. For a contradiction, we assume that there exist two crossing unstable bridges with all their attachments on a segment Z of S'. Let z_1 and z_2 be the endpoints of Z. We consider a clump $\mathcal{B}$ on Z that contains at least two bridges. We know that one of the outcomes of Lemma 2.1.1 holds. If any of (a), (b), or (d) holds, then H contains an S'-separation of order two or three. Thus (c) holds.

Since outcome (c) of Lemma 2.1.1 holds, we may characterize the structure of B as follows. Every S-bridge in $\mathcal{B}$ is over an attachment of a stable S'-bridge. There exist vertices x_1, x_2, x_3, x_4 that occur on Z in the order stated such that $x_2 \neq x_3$, each of x_1Zx_2, x_3Zx_4 contains an attachment of each bridge in $\mathcal{B}$, and every attachment of $\mathcal{B}$ is in $x_1Zx_2 \cup x_3Zx_4$. Every $\mathcal{B}$-significant attachment of any stable S'-bridge is in x_2Zx_3, and x_2Zx_3 contains at least one such attachment. Furthermore, (H, Γ) can be resigned such that every edge of every S'-bridge in $\mathcal{B}$ is even, any one edge of x_2Zx_3 is odd, and every other edge of Z is even. In addition, we may assume that x_1, x_2, x_3, x_4 are chosen with $|x_1Zx_2| + |x_3Zx_4|$ minimum.

Suppose that $z_1 = x_1 = x_2$. Then H contains an S'-separation (X, Y) with $X \cap Y = \{x_1, x_3, x_4\}$. Thus we may assume that $z_1 \neq x_2$ and $z_2 \neq x_3$. If there were only one attachment w of a stable bridge in x_2Zx_3, then (H, Γ) would contain an S'-separation (X, Y) with $X \cap Y = \{x_1, x_4, w\}$. Thus there exist vertex-disjoint paths P_1, P_2 with ends w_1, v_1 and w_2, v_2 respectively, where w_1, $w_2 \in V(x_2Zx_3)$. Since (H, Γ) does not contain an S'-jump, v_1, $v_2 \in (V(C_1) \triangle V(C_2))$ where C_1 and C_2 are the two disks that contain Z. If $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, then (H, Γ) contains an interrupted S'-jump. Thus we may assume that z_1, $z_2 \in V(C_1)$. If w_2, w_1, v_2, v_1 occur on C_1 in that order, then (H, Γ) contains a weakly free S'-cross since $z_1 \neq x_2$ and $z_2 \neq x_3$. Thus we may assume that the vertices w_1, w_2, v_2, v_1 occur on C_1 in that order.

Let B_1, B_2,..., B_n be the bridges in $\mathcal{B}$, listed so that for each $i > 1$, B_i crosses at least one bridge in the set $\{B_1, B_2, ..., B_{i-1}\}$. We may construct this list as follows. Let any bridge in the clump be B_1. For each $i > 1$, there must exist a bridge $B \in \mathcal{B}$ not yet in the list that crosses a bridge in the set $\{B_1, B_2, ..., B_{i-1}\}$; otherwise $\mathcal{B}$ would not be a clump. Let $B_i = B$. Thus we can construct the required listing.

We claim that there exist bridges B_i with attachments s_i, t_i and B_j with attachments s_j, t_j such that s_i, t_i, s_j, t_j are all distinct, s_i and s_j appear on x_1Zx_2 in that order, and t_i and t_j appear on x_3Zx_4 in that order.

Suppose that B_1 has at least two attachments in x_1Zx_2 and at least two attachments in x_3Zx_4. Since B_2 crosses B_1, there must exist s_1, t_1, s_2, t_2 such that the claim holds. Thus, by symmetry, we may assume that B_1 has exactly one attachment $u \in V(x_1Zx_2)$. If B_2 has an attachment in x_1Zx_3 distinct from u, then the claim holds. Thus we may assume that B_2 has exactly one attachment $u \in V(x_1Zx_2)$. Now suppose that for all k from 1 to $i-1$, B_k has u as its only attachment in $V(x_1Zx_2)$. Consider B_i. Let B_j be a bridge in the set $\{B_1, B_2, ..., B_{i-1}\}$ such that B_i crosses B_j. The only attachment of B_j in $V(x_1Zx_2)$ is u. If B_i has an attachment in $V(x_1Zx_3)$ distinct from u, then the claim holds. Thus we may assume that B_2 has exactly one attachment $u \in V(x_1Zx_2)$. Thus $x_1 = x_2 = u$. But then (H, Γ) contains an S'-separation (X, Y) with $X \cap Y = \{u, x_3, x_4\}$. Therefore we may assume that the claim holds. Thus there exist bridges B_i with attachments s_i, t_i and B_j with attachments s_j, t_j such that s_i, t_i, s_j, t_j are all distinct, s_i and s_j appear on x_1Zx_2 in that order, and t_i and t_j appear on x_3Zx_4 in that order.

Let Q_i be a path from s_i to t_i in B_i, and let Q_j be a path from s_j to t_j in B_j. By rerouting Z along Q_1 and Q_2, we obtain a subdivision S'' from S' in which w_2, w_1, z_2, z_1 appear on a disk in that order, and thus P_1, P_2 form a weakly free S''-cross, which is a contradiction. □

2.2 Using the Unstable Bridge Lemma

We will now use Lemma 2.1.2 to narrow down the unstable bridge outcomes to a single non-planar unstable bridge.

Let S be a G-subdivision in H with a weak disk system $\mathcal{C} = \{C_1, C_2, ..., C_k\}$. We say that $\mathcal{C}$ is *locally planar in H* if there exists a partition $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_k$ of the S-bridges such that for each i from 1 to k there is a disk C_i where all the bridges in $\mathcal{P}_i$ have all their attachments in C_i and $C_i \cup \mathcal{P}_i$ has a planar embedding with C_i bounding the infinite face.

The following lemma will deal with the case that the unstable S-bridges are part of the obstruction to planarity.

Lemma 2.2.1. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$. Let $\bar{H}$ be obtained from H by deleting all unstable S-bridges. Suppose $\mathcal{C}$ is locally planar in $\bar{H}$ but is not locally planar in H. Suppose that (H, Γ) does not contain a standard obstruction. Then there exists a (G, Σ)-subdivision S' related to S and an S'-umbrella.*

Proof. Let S' be the (G, Σ)-subdivision such that one of the outcomes of Lemma 2.1.1 holds. Suppose there does not exist an unstable S'-bridge B such that B is not S'-planar. There are no crossing unstable S'-bridges by Lemma 2.1.2.

Let B_1, B_2, ..., B_n be the unstable S'-bridges of H. Suppose that $\mathcal{C}$ is locally planar in $\bar{H} + B_i$ for each i. Thus $\mathcal{C}$ is locally planar in $\bar{H} + B_1$. Now suppose that $\mathcal{C}$ is locally planar in $\bar{H} + B_1 + B_2 + ... + B_j$ for some j. Since B_{j+1} does not cross any bridge in $\{B_1, B_2, ..., B_j\}$, B_{j+1} may be added to $\bar{H} + B_1 + B_2 + ... + B_j$ in a planar way, and thus $\mathcal{C}$ is locally planar in $\bar{H} + B_1 + B_2 + ... + B_{j+1}$. By induction, $\mathcal{C}$ is locally planar in H, which contradicts the hypothesis of the lemma. Therefore there must exist an unstable S'-bridge B such that $\mathcal{C}$ is not locally planar in $\bar{H} + B$.

Let Z be the segment that contains all the attachments of B. We may assume that (H, Γ) has been resigned so that every edge of Z is even. Let x and y be the attachments of B such that $|xZy|$ is maximum. Then there exists an S-path Q in B with endpoints x and y. Let C_1 and C_2 be the disks that contain Z. There must exist an S-path P_1 with endpoints w_1 in the interior of xZy and $z_1 \in V(C_1 - Z)$; otherwise $\bar{H} + B$ has a planar embedding with B drawn inside C_1. There must exist an S-path P_2 with endpoints w_2 in the interior of xZy and $z_2 \in V(C_2 - Z)$; otherwise $\bar{H} + B$ has a planar embedding with B drawn inside C_2. If Q is even, then we may reroute Z along Q to obtain a (G, Σ)-subdivision S'' and an S'' jump. If Q is odd, then (H, Γ) contains an interrupted S'-jump. Therefore we may assume that there exists an unstable S'-bridge B such that B is not S'-planar.

Since there do not exist crossing unstable S'-bridges, there exists a clump $\mathcal{B}$ that contains only B. We apply Lemma 2.1.1 to $\mathcal{B}$. One of the outcomes from that lemma must hold. If any of (a), (b), or (d) holds, then (H, Γ) contains an S'-separation of order two or three. Thus (c) holds.

Since outcome (c) of Lemma 2.1.1 holds, we may characterize the structure of B as follows. B is over an attachment of a stable S'-bridge. There exist vertices x_1, x_2, x_3, x_4 that occur on Z in the order stated such that $x_2 \neq x_3$, each of $x_1 Z x_2$, $x_3 Z x_4$ contains an attachment of B, and every attachment of B is in $x_1 Z x_2 \cup x_3 Z x_4$. Every $\mathcal{B}$-significant attachment of any stable S'-bridge is in $x_2 Z x_3$, and $x_2 Z x_3$ contains at least one such attachment. Furthermore, (H, Γ) can be resigned such that every edge of B is even, any one edge of $x_2 Z x_3$ is odd, and every other edge of Z is even. In addition, we may assume that x_1, x_2, x_3, x_4 are chosen with $|x_1 Z x_2| + |x_3 Z x_4|$ minimum.

If either $x_1 = x_2$ or $x_3 = x_4$, then there exists an S'-separation of order three. Thus B has distinct attachments x_1, x_2, x_3, x_4. If there were only one attachment w of a stable bridge in $x_2 Z x_3$, then (H, Γ) would contain an S-separation (X, Y) with $X \cap Y = \{x_1, x_4, w\}$. Thus there exist vertex-disjoint paths P_1, P_2 with ends w_1, z_1 and w_2, z_2 respectively, where $w_1, w_2 \in V(x_2 Z x_3)$. Since (H, Γ) does not contain an S'-jump, $z_1, z_2 \in (V(C_1) \triangle V(C_2))$ where C_1 and C_2 are the two disks that contain Z. If $z_1 \in V(C_1)$ and $z_2 \in V(C_2)$, then (H, Γ) contains an interrupted S'-jump. Thus we may assume that z_1, $z_2 \in V(C_1)$. Therefore B, P_1, P_2 is an S'-umbrella. □

From now on, we may assume that the non-planarity of H comes from the stable S-bridges.

Chapter 3

An Intermediate Theorem

In this chapter we will introduce and prove a theorem which will serve as our starting point for proving the main theorem in Chapter 4.

3.1 Introducing the Intermediate Theorem

Let G and H be graphs where G has no vertices of deegree two, and let S be a G-subdivision in H with a weak disk system $\mathcal{C}$. Let x_1, x_2, $x_3 \in V(S)$, let $x \in V(H) - V(S)$, and let R_1, R_2, R_3 be three paths in H such that R_i has ends x and x_i, they are pairwise vertex-disjoint except for x, and each is vertex-disjoint from $V(S) - \{x_1, x_2, x_3\}$. Assume further that for each pair x_i, x_j there exists a disk C_{ij} containing both x_i and x_j, but no disk contains all of x_1, x_2, x_3. In those circumstances we say that the triple R_1, R_2, R_3 is an *S-triad*. The vertices x_1, x_2, x_3 are its *feet*.

A subgraph J of S is a *detached K_4-subdivision* if J is isomorphic to a subdivision of K_4, every segment of J is a segment of S, and each of the four cycles of J consisting of precisely three segments is a disk.

The goal of this chapter is to prove the following theorem, which is an analogue of (4.2) in [3]:

Theorem 3.1.1. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$. Then (H, Γ) has a (G, Σ)-subdivision S' obtained from S by repeated reroutings such that S' and the weak disk system $\mathcal{C}'$ in S' induced by $\mathcal{C}$ satisfy one of the following conditions:*

(i) there exists an S'-jump,

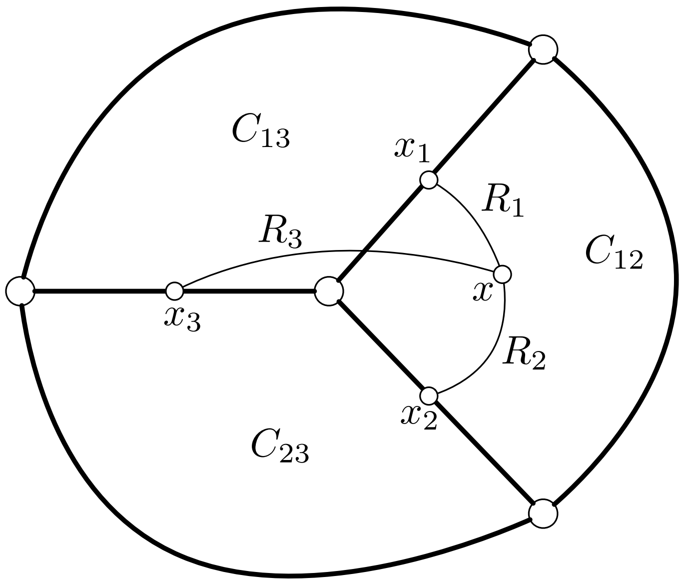

Figure 3.1: An S-triad

(ii) there exists a weakly free S'-cross,

(iii) there exists an S'-separation,

(iv) there exists an S'-triad,

(v) S' has a detached K_4-subdivision J, and H has an S'-bridge B such that the attachments of B are precisely the branch-vertices of J,

(vi) there exists an S'-umbrella,

(vii) there exists an interrupted S'-jump, or

(viii) the weak disk system $\mathcal{C}'$ is locally planar in H.

In Chapter 4 we will improve the above theorem.

We will need the following lemma, stated as (3.2) in [3]:

Lemma 3.1.2. *Let G be a graph, and let C be a cycle in G. Then one of the following conditions holds:*

(i) the graph G has a planar embedding in which C bounds a face,

(ii) there exists a separation (A, B) of G of order at most three such that $V(C) \subseteq A$ and $G[B]$ does not have a drawing in a disk with the vertices in $A \cap B$ drawn on the boundary of the disk,

(iii) there exist two vertex-disjoint paths in G with ends s_1, $t_1 \in V(C)$ and s_2, $t_2 \in V(C)$, respectively, and otherwise vertex-disjoint from C such that the vertices s_1, s_2, t_1, t_2 occur on C in the order listed.

We will also need this lemma, stated as (4.1) in [3]:

Lemma 3.1.3. *Let G be a graph with no vertices of degree two, let S be a G-subdivision in a graph H, let $\mathcal{C}$ be a weak disk system in S, and let B be an S-bridge with at least two attachments such that no disk includes all attachments of B. Then one of the following conditions holds:*

(i) there exists an S-jump,

(ii) there exists an S-triad, or

(iii) S has a detached K_4-subdivision J such that the attachments of B are precisely the branch-vertices of J.

3.2 S-tripods and S-leaps

Lemma 3.2.1. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$ such that none of outcomes (i), (iv), or (v) of Theorem 3.1.1 hold for S. Then for every stable S-bridge B of H there exists a unique disk $C \in \mathcal{C}$ such that $V(C)$ contains all attachments of B.*

Proof. Since B is a stable S-bridge, then by Lemma 3.1.3, there exists a disk C that includes all attachments of B, or else one of (i), (iv), or (v) of Theorem 3.1.1 holds. Since B is stable, no one segment of C contains all the attachments of B. Because two disks do not share more than one segment, C is unique. □

Let (G, Σ) and (H, Γ) be signed graphs. Let S be a (G, Σ)-subdivision in (H, Γ) with weak disk system $\mathcal{C}$, and suppose none of outcomes (i)-(v) of Theorem 3.1.1 holds for S. For every disk C of $\mathcal{C}$, let H_C be the union of C and all stable S-bridges B having all attachments in C. By Lemma 3.2.1, H_C is well-defined. The definition of H_C depends on the choice of S and $\mathcal{C}$, but these will always be clear from context, and so they will not be included in the notation of H_C.

Let Z be a segment of S with ends z and w, and let C be a disk of $\mathcal{C}$ that contains Z. Let P_1, P_2 be two vertex-disjoint S-paths in H with ends x_1, y_1 and x_2, y_2, respectively, such that z, x_1, x_2, y_1, w occur on Z in the order listed, and $y_2 \in V(C-Z)$. Let P_3 be a path vertex-disjoint from $V(S)-\{y_2\}$ with one end $x_3 \in V(P_1)$ and the other $y_3 \in V(P_2)$ and otherwise vertex-disjoint from $P_1 \cup P_2$. We say that the triple P_1, P_2, P_3 is an *S-tripod in C based at Z*, and that x_1, y_1, x_2, y_2 are its *feet*. We say that zZx_1, y_1Zw and $y_3P_2y_2$ are the *legs* of the tripod.

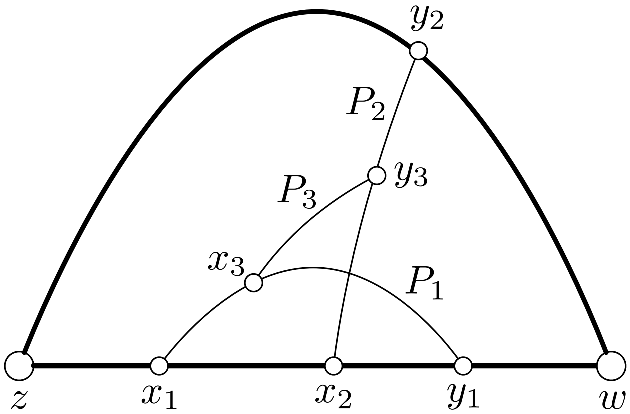

Figure 3.2: An S-tripod

Let Z_1, Z_2 be distinct segments of S with a common end v such that they are both subgraphs of a disk $C \in \mathcal{C}$, and, for $i = 1, 2$, let v_i be the other end of Z_i. Let P_1, P_2, P_3 be paths such that

- the ends of P_i are x_i and y_i,
- v_1, x_1, x_3, v, y_3, y_1, v_2 appear on $Z_1 \cup Z_2$ in the order listed, where v_1 and x_1 may coincide and v_2 and y_1 may coincide, but all other pairs are distinct,
- x_2 is an internal vertex of P_1 and $y_2 = v$,
- the paths P_1, P_2, P_3 share no internal vertices with each other or with S.

In those circumstances we say that P_1, P_2, P_3 is an *S-leap based at Z_1 and Z_2*. We call $x_1Z_1v_1$, $y_1Z_1v_2$ its *legs*, and x_1, x_3, v, y_3, y_1, (in that order) are its *feet*.

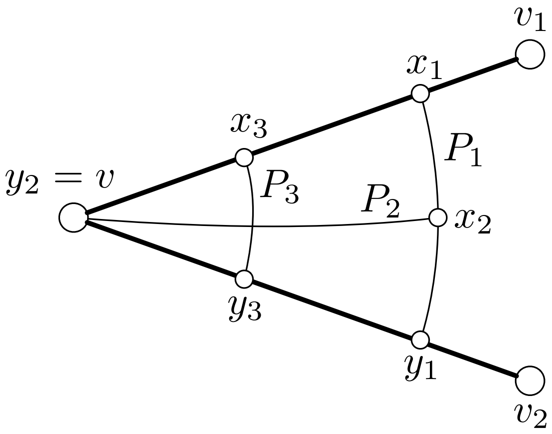

Figure 3.3: An S-leap

Lemma 3.2.2. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$, and suppose none of outcomes (i)-(v) of Theorem 3.1.1 hold for any (G, Σ)-subdivision S' related to S. Let $\bar{H}$ be the graph obtained from H by deleting the unstable S-bridges, and suppose for some disk C of $\mathcal{C}$, $\bar{H}_C$ does not have a planar embedding with C bounding the infinite face. Then (H, Γ) contains an S-tripod or an S-leap.*

Proof. We apply Lemma 3.1.2 to H and C. Because H does not have a planar embedding in which C bounds a face, outcome (i) of Lemma 3.1.2 does not hold. Since outcome (iii) of Theorem 3.1.1 does not hold, outcome (ii) of Lemma 3.1.2 does not hold. Thus outcome (iii) of Lemma 3.1.2 holds, and thus there exists an S-cross P_1, P_2 in C. For $i = 1$, 2, let x_i, y_i be the ends of P_i, and let B_i be the S-bridge that contains P_i. We may assume that there is a segment Z of S such that $x_1, x_2, y_1 \in V(Z)$; otherwise P_1, P_2 is a weakly free S-cross, and outcome (ii) of Theorem 3.1.1 holds. Suppose $y_2 \in V(Z)$. By the definition of $\bar{H}$, B_1 and B_2 are both stable S-bridges. It follows that there exists a path P from $P_1 \cup P_2$ to $C - V(Z)$, where P is internally vertex-disjoint from both $P_1 \cup P_2$ and S. Then $P_1 \cup P_2 \cup P$ contains an S-cross with at least one foot in $V(C - Z)$. So we may assume $y_2 \notin V(Z)$.

If $B_1 = B_2$, then there exists a path P_3 with one end x_3 in the interior of P_1 and the other end y_3 in the interior of P_2, internally vertex-disjoint from P_1, P_2 and S. So P_1, P_2, P_3 is an S-tripod. Now suppose $B_1 \neq B_2$. Since B_1 is a stable S-bridge, there exists a path P_3 in B_1 with one end x_3 in the interior of P_1 and the other end $y_3 \in V(C) - V(Z)$. If $y_3 = y_2$, then P_1, P_2, P_3 is an S-tripod. If $y_3 \neq y_2$, either $P_1 \cup P_2 \cup P_3$ contains a weakly free cross or either x_1, y_2, y_3 or y_1, y_2, y_3 occur on some segment Z' of S in that order. By symmetry we may assume that y_1, y_2, y_3 occur on Z' in that order. Then y_1 is an endpoint of both Z and Z'. Thus $x_1P_1x_3 \cup P_3$, $y_1P_1x_3$, P_2 forms an S-leap. □

In [1], the graph G is allowed to have parallel edges. The disk system axioms are modified to allow for a *trivial disk*, which is a disk bordered by the two segments corresponding to a pair of parallel edges in G. Another outcome called an *S-passage* comes from the introduction of trivial disks. The results from [1] that we will quote originally included S-passages, but if G is simple, then $\mathcal{C}$ contains no trivial disks. Thus S-passages do not occur.

The following two results are from [1], adapted here to the case that G has no parallel edges.

Lemma 3.2.3. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision with a weak disk system $\mathcal{C}$, and suppose none of outcomes (i)-(v) of Theorem 3.1.1 hold for any (G, Σ)-subdivision S' related to S. If there exists an S-leap, then there exists an interrupted S-jump.*

Lemma 3.2.4. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$. Suppose none of outcomes (i)-(v) of Theorem 3.1.1 hold for any (G, Σ)-subdivision S' related to S. If there exists an S-tripod, then for some (G, Σ)-subdivision S' related to S there exists an interrupted S'-jump.*

We now proceed to prove Theorem 3.1.1.

Proof. Suppose for a contradiction that none of the outcomes of Theorem 3.1.1 hold. Thus $\mathcal{C}$ is not locally planar in H. Let $\bar{H}$ be the graph obtained from H by deleting all the unstable S-bridges. If $\mathcal{C}$ is locally planar in $\bar{H}$, then by Lemma 2.2.1 there exists an S-umbrella, and so (vi) holds. Thus we may assume that $\mathcal{C}$ is not locally planar in $\bar{H}$.

By Lemma 3.2.1, for every stable bridge B, there exists a disk C such that C includes all the attachments of B. Since $\mathcal{C}$ is not locally planar in $\bar{H}$, there exists a disk C such that $\bar{H}_C$ does not have a planar embedding with C bounding the infinite face. By Lemma 3.2.2, there exists an S-tripod or an S-leap.

Suppose there exists an S-tripod. By Lemma 3.2.4, there exists a (G, Σ)-subdivision S' related to S and an interrupted S'-jump. Suppose there exists an S-leap. By Lemma 3.2.3, there exists a (G, Σ)-subdivision S' related to S and an interrupted S'-jump. Thus (vii) holds. □

Chapter 4

Main Result

Our goal in this chapter is to improve outcome (ii) of Theorem 3.1.1 and eliminate outcome (iv). If G and H are almost 4-connected, then neither of these outcomes is minimal. Adding either a weakly free cross or a triad to an almost 4-connected graph causes the new graph to contain a separation of order three. Therefore we will be able to further specify the structure of a cross and show that the triad outcome is not necessary.

4.1 Minimality

To improve outcomes (ii) and (iv) of Theorem 3.1.1, we will need to use the minimality of the non-planar extensions. Using rerouting, Norin and Thomas were able to prove their result without making full use of minimality. But in the case of signed graphs, the limited possibilities for rerouting require us to make greater use of a minimality argument.

Let G and H be graphs, and let S be a G-subdivision in H. Suppose that G is planar and that H is non-planar. Suppose that H does not contain an S-separation. Then there exists a subgraph H' of H such that H contains a G-subdivision S', H' is non-planar, and H' does not contain an S'-separation. We may assume that H' is chosen with a minimal number of edges.

In the proofs of several lemmas in this chapter, we will assume that H is a minimal counterexample to the lemma. We use "minimal counterexample" to mean a counterexample having the fewest edges. Then we will show that there exist edges of H that can be deleted so that the resulting graph is also a counterexample to the lemma. If we can do that, then we arrive at a contradiction. The goal is a theorem with a list of outcomes in which each outcome prevents H from being planar, does not cause an S-separation in H, and uses a minimal number of edges.

4.2 Augmenting sequences

We will now introduce the concept of augmenting sequences. An augmenting sequence is a sequence of paths that is needed to preserve a given level of connectivity. For example, adding a weakly free cross to a G-subdivision S introduces an S-separation (X, Y), so we know that H must contain a path with one end in $X - Y$ and the other end in $Y - X$. But if adding that path causes another S-separation, then we must add another path. The process continues until the resulting graph does not contain an S-separation.

Let G be a graph. Let X and Y be disjoint subsets of $V(G)$. Suppose that there are k vertex-disjoint XY-paths P_1, P_2,..., P_k where the endpoints of P_i are $x_i \in X$ and $y_i \in Y$. Let Q_1, Q_2,..., Q_m be internally vertex-disjoint paths in G. Let $f : [1, m-1] \to [1, k]$ be a function. Suppose that the following conditions hold:

- Q_i has ends s_i and t_i,
- $s_1 \in X - \{x_1, x_2, ..., x_k\}$,
- for $i < m$, $t_i \in V(P_{f(i)})$,
- for $i > 1$, $s_i \in V(x_{f(i-1)}P_{f(i-1)}t_{i-1} - \{t_{i-1}\})$,
- for all i, j, Q_i and P_j are internally vertex-disjoint.

In these circumstances, we say that Q_1, Q_2,..., Q_m is a *partial augmenting sequence*. If $t_m \in Y - \{y_1, y_2, ..., y_k\}$, then the partial augmenting sequence is an *augmenting sequence from X to Y*. The vertices s_1 and t_m are the *endpoints* of the sequence.

The function f describes which P_j contains the endpoint t_i of Q_i. In Figure 4.1 below, $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, and $f(4) = 1$.

Let Q_1, Q_2,..., Q_m be an augmenting sequence from X to Y. Suppose that for each i from 1 to $m - 1$, $x_{f(i)}P_{f(i)}t_i$ does not contain s_j for any $j > i + 1$. We call such an augmenting sequence *basic*.

Lemma 4.2.1. *If there exists an augmenting sequence from X to Y, then there exists a basic augmenting sequence with the same endpoints.*

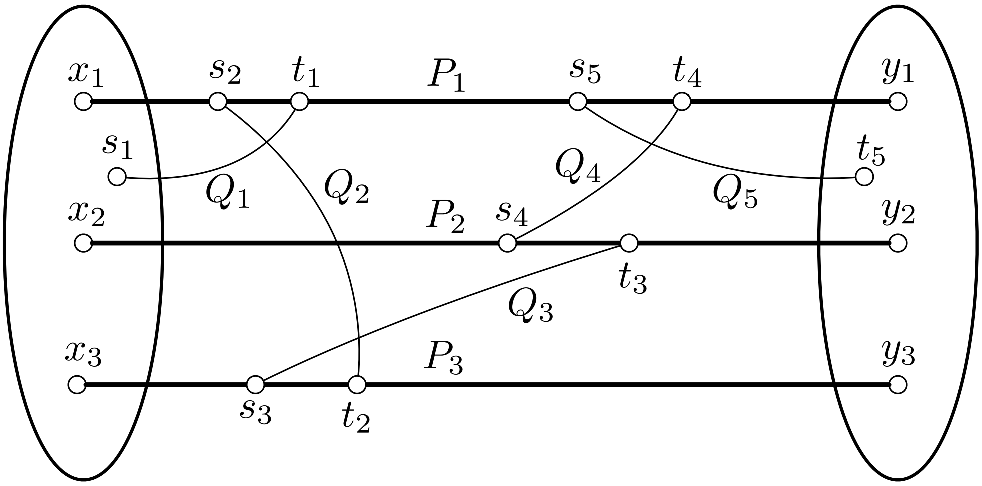

Figure 4.1: An example of a basic augmenting sequence from X to Y

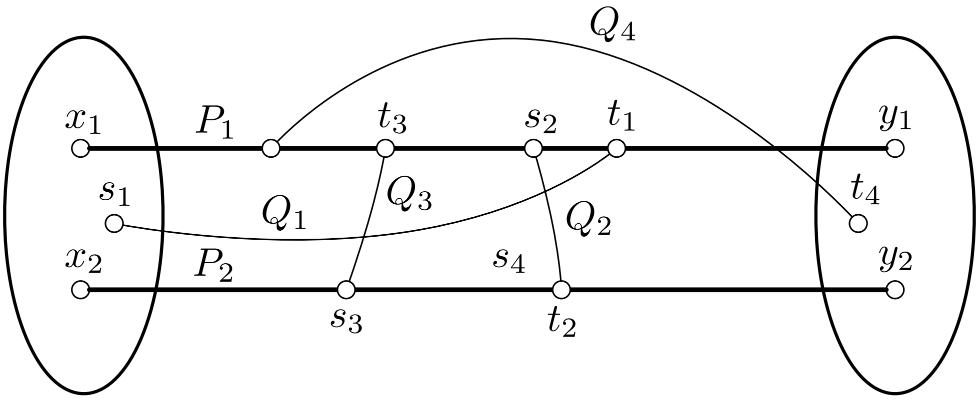

Figure 4.2: An augmenting sequence that is not basic

Proof. We choose the augmenting sequence Q_1, Q_2,..., Q_m from X to Y such that m is minimum. Suppose that the augmenting sequence is not basic. Thus there exist i and j such that $j > i + 1$ and $x_{f(i)}P_{f(i)}t_i$ contains s_j. Then Q_1, Q_2,..., Q_i, Q_{j+1},..., Q_m is a shorter augmenting sequence from X to Y, which contradicts our choice of sequence. Therefore there exists a basic augmenting sequence with endpoints s_1 and t_m. □

Lemma 4.2.2. *Let $G = (V, E)$ be a graph. Let X and Y be disjoint subsets of V. Suppose that there are k vertex-disjoint XY-paths. Then there exist $k + 1$ XY-paths if and only if there exists an augmenting sequence from X to Y.*

Proof. Suppose that there exists an augmenting sequence Q_1, Q_2,..., Q_m from X to Y. By Lemma 4.2.1, we may assume that the augmenting sequence is basic. Let Z be the set of interior vertices of the paths $t_i P_{f(i)} s_{i+1}$ for $i < m$. Let G' be the graph $(P_1 \cup P_2 \cup ... \cup P_k \cup Q_1 \cup Q_2 \cup ... \cup Q_m) - Z$. Let v be a vertex in G'. If v has degree 2 in G, then it also has degree 2 in G'. Note that $v \in \{s_1, x_1, x_2, ..., x_k, t_m, y_1, y_2, ..., y_k\}$ if and only if v has degree 1 in G'. If v has degree 3 in G, then v is the endpoint of Q_i for exactly one i and $v \notin \{s_1, t_m\}$, and thus v has degree 2 in G'. If v has degree 4 in G, then $v = s_i = t_j$ for some $i \neq j$, and thus v has degree 2 in G'. Since the sequence $Q_1, Q_2, ..., Q_m$ is basic, v cannot have degree greater than 4. Therefore all the vertices of G' have degree 2 in G' except for s_1, x_1, x_2,..., x_k, t_m, y_1, y_2,..., y_k which have degree 1. Thus there are $k+1$ vertex-disjoint XY-paths in G.

Now suppose that there is no augmenting sequence from X to Y. Let U be the set of all vertices u such that there is a partial augmenting sequence ending with u or $u \in x_i P_i w$ and there is a partial augmenting sequence ending with w. Let $S = X \cup U$. Let $T = V - S$. Since there is no augmenting sequence, $S \cap Y \subseteq \{y_1, y_2, ..., y_k\}$. For each i from 1 to k, let $z_i \in S$ be the vertex in P_i such that the length of $z_i P_i y_i$ is minimal. We claim that $\{z_1, z_2, ..., z_3\}$ is a k-vertex cut separating $S - \{z_1, z_2, ..., z_3\}$ and T. Suppose that there exists a path R with ends $r_1 \in S - \{z_1, z_2, ..., z_3\}$ and $r_2 \in T$. First suppose that $r_1 \in P_i$ for some i. Since $z_i \in S$, there is a partial augmenting sequence ending with z_i. Now we add R to the sequence, which results in a partial augmenting sequence ending with r_2. Now if there does not exist i such that $r_1 \in P_i$, then $r_1 \in X - \{x_1, x_2, ..., x_k\}$. Thus R is a partial augmenting sequence ending with r_2. Therefore $r_2 \in S$, which is a contradiction. Therefore $\{z_1, z_2, ..., z_3\}$ is a k-vertex cut separating $S - \{z_1, z_2, ..., z_3\}$ and T. Since $X \subseteq S$ and $Y - \{z_1, z_2, ..., z_k\} \subseteq T$, there do not exist $k+1$ vertex-disjoint XY-paths. □

4.3 Weakly Free Crosses

In this section, we improve outcome (ii) of Theorem 3.1.1. We will show that either a weakly free cross is free or else the augmenting sequence that extends the cross can be defined as one of several types.

The proofs proceed differently depending on whether the weakly free cross is centred at a vertex of degree three or degree four and whether the paths that cross are odd or even. We will begin with a preliminary lemma that deals with the case that the cross is centred at a vertex of degree three and one of the crossing paths is even.

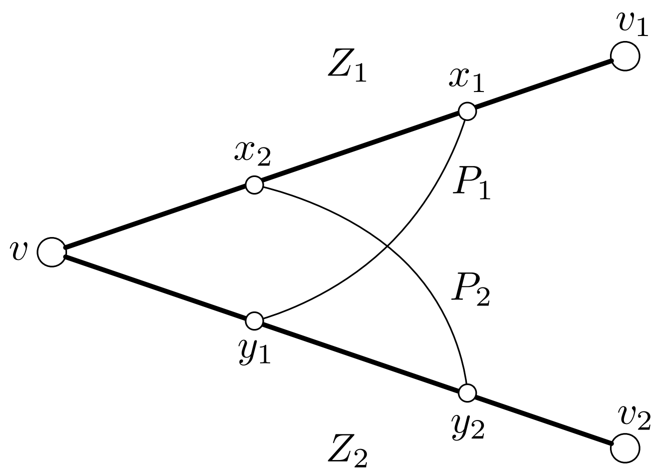

Figure 4.3: A weakly free S-cross

Lemma 4.3.1. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$, let $C \in \mathcal{C}$, let $v \in V(C)$ have degree in S exactly three, and let P_1, P_2 be a weakly free S-cross in C centred at v. Then either P_1 and P_2 are both odd or there exists a (G, Σ)-subdivision S' obtained from S by one rerouting and an S'-triad.*

Proof. If P_1 and P_2 are both odd, then the lemma holds. By symmetry, we may assume that P_2 is even. For $i = 1$, 2 let x_i, y_i be the ends of P_i, and let P_1, P_2 be based at Z_1 and Z_2. Let Z_3 be the third segment of S incident with v, and for $i \in \{1, 2, 3\}$, let v_i be the other end of Z_i. Then we may assume that x_1, x_2, $v \in V(Z_1)$ occur on Z_1 in the order listed; then y_2, y_1, $v \in V(Z_2)$ occur on Z_2 in the order listed. We may assume that (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2$ is even. Let S' be the (G, Σ)-subdivision obtained from S by replacing vZ_2y_2 by P_2. The segments of S' that correspond to Z_1, Z_2, and Z_3 are $Z_1' = x_2Z_1v_1$, $Z_2' = P_2 \cup y_2Z_2v_2$, and $Z_3' = x_2Z_1v \cup Z_3$. Then $x_1 \in V(Z_1')$, $y_2 \in V(Z_2')$, and $v \in V(Z_3')$. Thus P_1, $y_1Z_2y_2$, y_1Z_2v is an S'-triad. □

Now we deal with the case that the cross is centred at a vertex of degree at least four and both of the crossing paths are even.

Lemma 4.3.2. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with a weak disk system $\mathcal{C}$, and assume that H contains a weakly free S-cross centred at a vertex of degree at least four. Then there exists a (G, Σ)-subdivision S' obtained* *from S by repeated reroutings such that S' and the disk system $\mathcal{C}'$ induced in S' by $\mathcal{C}$ satisfy one of the following conditions:*

(i) there exists an S'-jump,

(ii) there exists a free S'-cross,

(iii) there exists an S'-separation, or

(iv) there exists a weakly free S'-cross P_1, P_2 based at the segments Z_1 and Z_2 such that when (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2$ is even, at least one of P_1 and P_2 is odd.

Proof. Let P_1, P_2 be a weakly free S-cross in H centred at a vertex v of degree at least four. Thus there exist two segments Z_1, Z_2 of S, both incident with v, such that Z_1, Z_2 include all the feet of the cross. For $i = 1$, 2, let x_i, y_i be the ends of P_i. We may assume that x_1, x_2, $v \in V(Z_1)$ occur on Z_1 in the order listed; then y_2, y_1, $v \in V(Z_2)$, and they occur on Z_2 in the order listed. For $i = 1, 2$, let v_i be the other end of Z_i and let $L_1 = x_1Z_1v_1$ and $L_2 = y_2Z_2v_2$. We call L_1 and L_2 the *legs* of the cross P_1, P_1. We may assume that (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2$ is even. Then we may also assume that the paths P_1 and P_2 are both even since otherwise (iv) holds.

Consider all triples (S', P'_1, P'_2), where S' is a (G, Σ)-subdivision obtained from S by repeated reroutings and P'_1, P'_2 is a weakly free S'-cross based at Z'_1, Z'_2 where Z'_1, Z'_2 are the segments of S' corresponding to Z_1, Z_2. We may assume that among all such triples the triple (S, P_1, P_2) is chosen with $|V(L_1)| + |V(L_2)|$ minimum.

Let X' be the vertex-set of $P_1 \cup P_2 \cup vZ_1x_1 \cup vZ_2y_2$, and let $Y' = V(S) - (X' - \{v, x_1, y_2\})$. If there is no path in $H - \{v, x_1, y_2\}$ with one end in X' and the other in Y', then there exists a separation (X, Y) of order three with $X' \subseteq X$ and $Y' \subseteq Y$. This separation satisfies (iii), and so we may assume that there exists a path P in $H - \{v, x_1, y_2\}$ with one end $x \in X$ and the other end $y \in Y$. From the symmetry we may assume that x belongs to $V(P_1 \cup vZ_2y_2)$.

If $y \in V(L_1)$, then replacing P_1 by P if $x \notin V(P_1)$ and by $P \cup xP_1y_1$ otherwise produces a cross that has legs yL_1v_1 and L_2. Since $|V(yL_1v_1)| + |V(L_2)| < |V(L_1)| + |V(L_2)|$, this new cross contradicts our choice of the triple (S, P_1, P_2). If $y \in V(L_2)$, then let S' be obtained from S by replacing

$x_2Z_1x_1 \cup y_1Z_2y_2$ with $P_1 \cup P_2$. If $x \in P_1$, then xZ_1x_2, P forms a weakly free S'-cross. If $x \notin P_1$, then xZ_1x_2, $y_1Z_2x_2 \cup P$ forms a weakly free S'-cross. In either case, the cross contradicts our choice of (S, P_1, P_2). Thus $y \notin V(Z_1 \cup Z_2)$.

Let C be the disk that includes both Z_1 and Z_2, and for $i = 1, 2$, let C_i be the other disk that includes Z_i. If $y \in V(C)$, then $P_1 \cup P_2 \cup P$ includes a free S-cross, and so (ii) holds. Thus we may assume that $y \notin V(C)$. If $y \notin V(C_2)$, then $P_1 \cup P$ includes an S-jump with one end y and the other end x or y_1, and so we may assume that $y \in V(C_2)$. Since v has degree at least four, disk system axioms (X1) and (X2) imply that $V(C_1) \cap V(C_2) = \{v\}$. It follows that $y \notin V(C_1)$. Now let S' be obtained from S by replacing $x_2Z_1x_1 \cup y_1Z_2y_2$ with $P_1 \cup P_2$, and let Z'_1, Z'_2 be the segments of S' corresponding to Z_1, Z_2 respectively. Thus $Z'_1 = vZ_1x_1 \cup P_1 \cup y_1Z_2v$ and $Z'_2 = v_2Z_2y_2 \cup P_2 \cup x_2Z_1v$. Now $P \cup xZ_2y_1$ includes an S'-jump with one end y and the other end in the interior of Z'_1, and so (i) holds. □

In the next lemma, we will use Lemmas 4.3.1 and 4.3.2 to prove a strengthening of Theorem 3.1.1. We use augmenting sequences and minimality to show that the paths added to a weakly free cross to prevent an S-separation belong to one of a limited number of types.

Lemma 4.3.3. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two and is not the complete graph on four vertices. Let S be a (G, Σ)-subdivision in (H, Γ) with a disk system $\mathcal{C}$. Then there exists a (G, Σ)-subdivision S' obtained from S by repeated reroutings such that S' and the disk system $\mathcal{C}'$ induced in S' by $\mathcal{C}$ satisfy one of the following conditions:*

(i) there exists an S'-jump,

(ii) there exists a free S'-cross,

(iii) there exists an interrupted S'-jump,

(iv) for some $j \in \{1, 2, ..., 5\}$, there exists a type-j extended S'-cross,

(v) there exists an S'-triad,

(vi) there exists an S'-umbrella,

(vii) there exists an S'-separation, or

(viii) the disk system $\mathcal{C}'$ is locally planar in H.

Proof. Suppose that (H, Γ) is a counterexample to the lemma such that H has a minimal number of edges. (H, Γ) must satisfy one of the outcomes of Theorem 3.1.1. We know that outcome (v) of Theorem 3.1.1 does not hold since $\mathcal{C}$ satisfies disk system axiom (X2) and G is not K_4. Thus outcome (ii) of Theorem 3.1.1 holds, or otherwise the lemma holds. Therefore H contains a weakly free cross P_1, P_2.

For $i = 1, 2$ let x_i, y_i be the ends of P_i, and let P_1, P_2 be centred at v and based at Z_1 and Z_2. We may assume that x_1, x_2, $v \in V(Z_1)$ occur on Z_1 in the order listed; then y_2, y_1, $v \in V(Z_2)$ occur on Z_2 in the order listed. For $i = 1, 2$, let v_i be the other end of Z_i and let $L_1 = x_1Z_1v_1$ and $L_2 = y_2Z_2v_2$. We may assume that (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2$ is even.

Consider all triples (S', P_1', P_2'), where S' is a (G, Σ)-subdivision obtained from S by repeated rerouting and P_1', P_2' is a weakly free S'-cross based at Z_1', Z_2' where Z_1', Z_2' are the branches of S' corresponding to Z_1, Z_2. We may assume that among all such triples the triple (S, P_1, P_2) is chosen with $|V(L_1)| + |V(L_2)|$ minimum.

There are two cases to consider: the case in which v has degree three and the case in which v has degree four or greater. First we assume that v has degree three. Let Z_3 be the third segment incident with v, and let v_3 be the other end of Z_3. One of the outcomes from Lemma 4.3.1 must hold, so we can assume that P_1 and P_2 are both odd since otherwise (v) holds.

Let X be the vertex set of $vZ_1x_1 \cup vZ_2y_2 \cup P_1 \cup P_2$. Let Y be the vertex set of $H - (X \cup Z_1 \cup Z_2 \cup Z_3)$. We know there exist three vertex-disjoint paths from X to Y in H. There must exist a fourth XY-path or else (vii) holds. Thus by Lemma 4.2.2 there must exist an augmenting sequence from X to Y. Let Q_1, Q_2,..., Q_m be an augmenting sequence from X to Y. By Lemma 4.2.1, we may assume that the augmenting sequence is basic. For $i = 1$,2 ,..., m, let s_i and t_i be the endpoints of Q_i. Consider Q_1. By symmetry we may assume that $s_1 \in V(P_1 \cup vZ_2y_2)$.

Suppose that $s_1 \in V(P_1)$. If $t_1 \in V(L_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_1 \in V(Z_3)$, then there exists an S-triad, and so (v) holds. If $t_1 \in V(C - (Z_1 \cup Z_2))$, then there exists a free S'-cross, and so (ii) holds. If $t_1 \notin V(C \cup Z_3)$ then there exists an S-jump, and so (i) holds. If $t_1 \in V(L_2)$, then we consider the parity of $y_1P_1s_1 \cup Q_1$.

If it is even, we reroute Z_2 along $y_1P_1s_1 \cup Q_1$, which results in a cross that contradicts our choice of (S, P_1, P_2). If $y_1P_1s_1 \cup Q_1$ is odd, then we consider Q_2. We know that $s_2 \in y_2Z_2t_1$. If $t_2 \in V(L_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_2 \in V(C_2)$, there exists an interrupted S-jump satisfying (iii). So we assume that $t_2 \in V(t_1Z_2v_2)$.

If Q_2 is even, then we let S' be the (G, Σ)-subdivision formed by rerouting Z_2 along Q_2. Let H' be the graph obtained from H by deleting the interior of $s_2Z_2t_1$. We note that P_1, P_2 form a weakly-free S'-cross in H', so the disk system $\mathcal{C}'$ is not locally planar in H'. Also, $Q_1 \cup t_1Z_2s_2 \cup Q_3 \cup ... \cup Q_m$ contains an augmenting sequence from X to Y, and thus H' does not contain an S'-separation. Therefore H' is also a counterexample to the lemma. But H' has fewer edges than H. This contradicts our choice of H as the counterexample with the fewest edges. Future cases will be similar. We will discover that, after rerouting, a certain portion of H is not necessary for non-planarity or connectivity and thus it may be deleted to produce a smaller counterexample. In those cases, we will simply specify the rerouting and the part of H that may be deleted.

If Q_2 is odd, then we reroute Z_2 along $y_1P_1s_1 \cup Q_1$ and Q_2. Then by deleting the interior of $y_1Z_2y_2$, we obtain a graph that contradicts the minimality of H.

Now suppose that $s_1 \in V(vZ_2y_2)$. If $t_1 \in V(L_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_1 \notin V(Z_1 \cup Z_2 \cup Z_3)$, then there exists either an S-jump or a type-1 extended S-cross, and so either (i) or (iv) holds. Suppose that $t_1 \in V(Z_3)$. If $s_1 \in V(vZ_2y_1)$ and Q_1 is even, we reroute by replacing t_1Z_3v with Q_1 and delete the interior of $t_1Z_3s_2$ to obtain a graph that contradicts the minimality of H. If $s_1 \in V(vZ_2y_1)$ and Q_1 is odd, we reroute by replacing $t_1Z_3v \cup y_1Z_2y_2 \cup x_2Z_1x_1$ with $Q_1 \cup P_1 \cup P_2$ and delete the interior of $t_1Z_3s_2$ to obtain a graph that contradicts the minimality of H. If $s_1 \in V(y_1Z_2y_2)$ and Q_1 is even, we reroute by replacing t_1Z_3v with Q_1 so that (iii) holds. If $s_1 \in V(y_1Z_2y_2)$ and Q_1 is odd, we reroute by replacing $t_1Z_3v \cup s_1Z_2y_2 \cup x_2Z_1x_1$ with $Q_1 \cup P_1 \cup P_2$ and delete the interior of $x_1Z_1x_2$ to obtain a graph that contradicts the minimality of H. Thus we may assume that $t_1 \in V(L_2)$. If Q_1 is even, we reroute Z_2 along Q_1, which results in a cross that contradicts our choice of (S, P_1, P_2). If Q_1 is odd, then we reroute Z_2 along both P_2 and Q_1 and delete the interior of $t_1Z_2s_2$ to obtain a graph that contradicts the minimality of H.

Now we consider the case that v has degree four or greater. One of the outcomes of Lemma 4.3.2 must hold, and thus we may assume that at least one of P_1 and P_2 is odd. Let X be the vertex set of $vZ_1x_1 \cup vZ_2y_2 \cup P_1 \cup P_2$. Let Y be the vertex set of $H - (X \cup Z_1 \cup Z_2)$. We know there exist three vertex-disjoint paths from X to Y in H. There must exist a fourth XY-path or else (vii) holds. Thus by Lemma 4.2.2 there must exist an augmenting sequence from X to Y. Let Q_1, Q_2,..., Q_m be an augmenting sequence from X to Y. By Lemma 4.2.1, we may assume that the augmenting sequence is basic. For $i = 1, 2, ..., m$, let s_1 and t_1 be the endpoints of Q_i. Consider Q_1. By symmetry we may assume that $s_1 \in V(P_1 \cup vZ_2y_2)$.

Suppose that $s_1 \in V(P_1)$. If $t_1 \in V(L_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_1 \in V(C-(Z_1 \cup Z_2))$, then there exists a free S-cross, and so (ii) holds. If $t_1 \notin V(C)$, then (i) holds. If $t_1 \in V(L_2)$, then we consider the parity of $y_1P_1s_1 \cup Q_1$. If $y_1P_1s_1 \cup Q_1$ is even, we reroute Z_2 along $y_1P_1s_1 \cup Q_1$, which results in a cross that contradicts our choice of (S, P_1, P_2). If $y_1P_1s_1 \cup Q_1$ is odd, then we consider Q_2. If $t_2 \in V(L_1)$, then there exists a cross that contradicts our choice of (S, P_1, P_2). If $t_2 \in V(C_2)$, then (iii) holds. So we may assume that $t_2 \in V(t_1Z_2v_2)$. If Q_2 is even, then we reroute Z_2 along Q_2 and delete the interior of $s_2Z_2t_1$ to obtain a graph that contradicts the minimality of H. If Q_2 is odd, then we reroute Z_2 along both $y_1P_1s_1$ and Q_2 and delete the interior of $y_1Z_2y_2$ to obtain a graph that contradicts the minimality of H.

Now suppose that $s_1 \in V(vZ_2y_2)$. If $t_1 \in V(L_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_1 \notin V(Z_1 \cup Z_2)$, then there exists either an S-jump or a type-1 extended S-cross, and so either (i) or (iv) holds. Thus we may assume that $t_1 \in V(L_2)$. If Q_1 is even, we reroute Z_2 along Q_1, which results in an S-cross that contradicts our choice of (S, P_1, P_2). Thus we may assume that Q_1 is odd. We consider Q_2. Suppose that $t_2 \in V(L_2)$. If Q_2 is even, we reroute Z_2 along Q_2 and delete the interior of $s_2Z_2t_1$ to obtain a graph that contradicts the minimality of H. If Q_2 is odd, we reroute Z_2 along both Q_1 and Q_2 and delete the interior of $y_1Z_2y_2$ to obtain a graph that contradicts the minimality of H. If $t_2 \notin V(C)$, then one of (i) or (iii) holds. If $t_2 \in V(C - (Z_1 \cup Z_2)$, then there exists a type-2 extended S-cross, and so (iv) holds. Thus $t_2 \in V(L_1)$. Now we consider Q_3. If $t_3 \in V(C_1 - Z_1)$, then there exists a type-3 extended S-cross, and so (iv) holds. If $t_3 \in V(C - (Z_1 \cup Z_2))$, then (ii) holds. If $t_3 \notin V(C_1 \cup C)$, then (i) holds. If $t_3 \in V(L_2)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). Thus we may assume that $t_3 \in V(L_1)$. If Q_3 is even, we reroute Z_1 along Q_3 and delete the interior of $s_3Z_1t_2$ to obtain a graph that contradicts the minimality of H. If Q_3 is odd, then we consider Q_4. If $t_4 \in V(Z_1)$ and Q_4 is even, we reroute Z_1 along Q_4 and delete the interior of $s_4Z_1t_3$ to obtain a graph that contradicts the minimality of H. If $t_4 \in V(Z_1)$ and Q_4 is odd, we reroute Z_1 along both Q_3 and Q_4 and delete the interior of $s_3Z_1t_2$ to obtain a graph that contradicts the minimality of H. If $t_4 \in V(C_1 - Z_1)$, then (iii) holds. If $t_4 \in V(C - (Z_1 \cup Z_2))$, then there exists a type-4 extended S-cross, and so (iv) holds. If $t_4 \notin V(C_1 \cup C)$, then (i) holds. Thus we may assume that $t_4 \in V(t_1Z_2v_2)$. We consider Q_5. If $t_5V \in V(Z_1)$, then there exists an S-cross that contradicts our choice of (S, P_1, P_2). If $t_5 \in V(C - (Z_1 \cup Z_2))$, then (ii) holds. If $t_5 \in V(Z_2)$ and Q_5 is even, then we reroute Z_2 along Q_5 and delete the interior of $s_5Z_2t_4$ to obtain a graph that contradicts the minimality of H. If $t_5 \in V(Z_2)$ and Q_5 is odd, then we reroute Z_2 along both Q_1 and Q_5 and delete the interior of $y_1Z_2y_2$ to obtain a graph that contradicts the minimality of H. If $t_5 \notin V(C \cup C_2)$, then (i) holds. Thus we may assume that $t_5 \in V(C_2 - Z_2)$, and then there exists a type-5 extended S-cross, and so (iv) holds. □

4.4 Triads

In this section, our goal is to eliminate outcome (v) of Lemma 4.3.3. We say that an S-triad is *local* if there exists a vertex v of S of degree three in S such that each of the three segments of S incident with v includes exactly one foot of the triad. We say that the local S-triad is *centred* at v.

In the main proof of this section, we assume that the S-triad is local. Thus our first task is to show that there are no non-local triads. In order to do this, we will have to assume that G is almost 4-connected.

Lemma 4.4.1. *Let G be an almost 4-connected planar graph, and let S be a G-subdivision in a graph H. Let $\mathcal{C}$ be the disk system in S consisting of peripheral cycles of S. Then every S-triad is local.*

Proof. Suppose that there exists a non-local S-triad, and let F be the set of feet of the triad. Let us fix a drawing of S in the plane. Since each pair of vertices in F belong to a common face, there exists a simple closed curve ϕ intersecting S precisely in the set F. Let I be the interior of the region bounded by ϕ, and let O be the exterior. If I contains no branch vertex of S, then there exists a disk that includes every vertex in F, contrary to the definition of a triad. If I contains exactly one branch vertex of S, then that vertex must be incident with three segments, each of which contains exactly one vertex in F, and thus the triad is local.

Thus we may assume that I contains more that one branch vertex of S. If O contains no branch vertex of S, then there exists a disk that includes every vertex in F, contrary to the definition of a triad. If O contains exactly one branch vertex of S, then that vertex must be incident with three segments whose other ends are the vertices in F, and thus the triad is local. Thus I and O both contain more than one branch vertex of S, contrary to the almost 4-connectivity of G. □

Adding only a local triad to S introduces an S-separation. Thus H must also contain an augmenting sequence with the triad. We show that adding an augmenting sequence to a local triad leads to a non-minimal obstruction to planarity.

Lemma 4.4.2. *Let (G, Σ) and (H, Γ) be signed graphs, where G is almost 4-connected. Let S be a (G, Σ)-subdivision in (H, Γ) with a disk system $\mathcal{C}$. Then there exists a G-subdivision S' obtained from S by repeated reroutings such that S' and the disk system $\mathcal{C}'$ induced in S' by $\mathcal{C}$ satisfy one of the following conditions:*

- *(i) there exists an S'-jump,*
- *(ii) there exists a free S'-cross,*
- *(iii) there exists an interrupted S'-jump,*
- *(iv) for some $j \in \{1, 2, ..., 5\}$, there exists a type-j extended S'-cross,*
- *(v) there exists an S'-umbrella,*
- *(vi) there exists an S'-separation in H, or*
- *(vii) the disk system $\mathcal{C}'$ is locally planar in H.*

Proof. Let (H, Γ) be a minimal counterexample to the lemma such that H has a minimal number of edges. H must satisfy one of the outcomes of Lemma 4.3.3. Thus outcome (v) of Lemma 4.3.3 holds, or otherwise the lemma holds. Thus H contains an S-triad R_1, R_2, R_3.

By Lemma 4.4.1, the triad R_1, R_2, R_3 must be local. Let the triad R_1, R_2, R_3 be centred at v, let its feet be x_1, x_2, x_3, let Z_1, Z_2, Z_3 be the three segments of S incident with v numbered so that $x_i \in V(Z_i)$, and let v_i be the other end of Z_i. We may assume that (H, Γ) has been resigned so that every edge of $Z_1 \cup Z_2 \cup Z_3$ is even. Let L_i be the subpath of Z_i with ends v_i and x_i, and let P_i be the subpath of Z_i with ends v and x_i. We say that the paths L_1, L_2, L_3 are the *legs* of the S-triad. We may assume that R_1, R_2, R_3 are chosen so that there is no S-triad as above such that the sum of the lengths of its legs is strictly smaller than $|E(L_1)| + |E(L_2)| + |E(L_3)|$. Let $X = V(P_1 \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup R_3)$ and $Y = V(S) - (X \cup L_1 \cup L_2 \cup L_3)$.

We know that there exist three vertex-disjoint XY-paths in H. There must exist a fourth XY-path or else (vi) holds. Thus by Lemma 4.2.2 there must exist an augmenting sequence from X to Y. Let Q_1, Q_2,..., Q_m be an augmenting sequence from X to Y. By Lemma 4.2.1, we may assume that the augmenting sequence is basic.

By symmetry, we may assume that $s_1 \in V(P_1 \cup R_1)$. Suppose that $s_1 \in V(R_1)$. If $t_1 \notin V(Z_1 \cup Z_2 \cup Z_3)$, then (i) holds. If $t_1 \in V(L_1 \cup L_2 \cup L_3)$, then there exists an S-triad that contradicts our choice of R_1, R_2, R_3. Thus we may assume that $s_1 \in V(P_1)$. If t_1 is not in either of the disks that include Z_1, then (i) holds. If t_1 is in one of the disks that include Z_1 but $t_1 \notin V(Z_1 \cup Z_2 \cup Z_3)$, then (ii) holds. If $t_1 \in V(Z_3)$, then deleting the interior of R_2 results in a graph that contradicts the minimality of H. If $t_1 \in V(Z_2)$, then deleting the interior of R_3 results in a graph that contradicts the minimality of H. Thus we may assume that $t_1 \in V(Z_1)$.

If Q_1 is even, then rerouting Z_1 along Q_1 and deleting the interior of $s_1Z_1x_1$ results in a graph that contradicts the minimality of H. So we may assume that Q_1 is odd. Now we consider Q_2. If t_2 is not in either of the disks that include Z_1, then (i) holds. If $t_2 \notin V(Z_1)$ but t_2 is in the disk that contains both Z_1 and Z_2, then deleting the interior of R_2 results in a graph that contradicts the minimality of H. If $t_2 \notin V(Z_1)$ but t_2 is in the disk that contains both Z_1 and Z_3, then deleting the interior of R_3 results in a graph that contradicts the minimality of H. Thus we may assume that $t_2 \in V(Z_1)$.

If Q_2 is even, then rerouting Z_1 along Q_2 and deleting the interior of $s_2Z_1t_1$ results in a graph that contradicts the minimality of H. If Q_2 is odd, then rerouting Z_1 along both Q_1 and Q_2 and deleting the interior of $s_1Z_1x_1$ results in a graph that contradicts the minimality of H. □

4.5 Conclusion

Now we are ready to formulate our results in terms of graphs embedded in the plane. From now on, our disk systems will refer to the disk systems consisting of peripheral cycles of 3-connected planar graphs. Terms such as S-jump and S-cross will refer to the disks corresponding to peripheral cycles.

If S is a G-subdivision and S' is another G-subdivision obtained from S by rerouting, then the embedding of S uniquely determines an embedding of S', and the disk system induced in S' by $\mathcal{C}$ consists of the face boundaries of S'.

Lemma 4.5.1. *Let (G, Σ) be an almost 4-connected planar signed graph, let (H, Γ) be a non-planar signed graph, and let S be a (G, Σ)-subdivision in (H, Γ). Then there exists a G-subdivision S' in (H, Γ) obtained from S by repeated reroutings such that S' and the disk system of peripheral cycles in S' satisfy one of the following conditions:*

(i) there exists an S'-jump,

(ii) there exists a free S'-cross,

(iii) there exists an S'-separation,

(iv) there exists an interrupted S'-jump,

(v) there exists an S'-umbrella, or

(vi) for some $j \in \{1, 2, ..., 5\}$, there exists a type-j extended S'-cross.

Proof. By Lemma 4.4.2, one of the outcomes of that lemma holds. If one of outcomes (i)-(vi) of Lemma 4.4.2 holds, then our lemma holds. We know that outcome (vii) of Lemma 4.4.2 does not hold since S does not extend to an embedding of H. Thus the result holds. □

The only outcome still to be eliminated is the S-separation. The almost 4-connectivity of H can be used to show that an S-separation leads to a contradiction. We are now ready to restate and prove Theorem 1.3.1, our main result.

Theorem 1.3.1. *Let (G, Σ) be an almost 4-connected planar signed graph on at least six vertices, let (H, Γ) be an almost 4-connected non-planar signed graph, and let S be a (G, Σ)-subdivision in (H, Γ). Then there exists a (G, Σ)-subdivision S' in (H, Γ) obtained from S by repeated reroutings such that S' satisfies one of the following conditions:*

(i) there exists an S'-jump ,

(ii) there exists a free S'-cross,

(iii) there exists an interrupted S'-jump,

(iv) there exists an S'-umbrella, or

(v) for some $j \in \{1, 2, ..., 5\}$, there exists a type-j extended S'-cross.

Proof. Let (G, Σ), (H, Γ), and S be as stated. By Lemma 4.5.1 there exists a (G, Σ)-subdivision S' in (H, Γ) obtained from S by repeated reroutings such that one of the conclusions of that lemma holds. We may assume that H contains an S' separation (X, Y), for otherwise the theorem holds. Then $|X - Y| \geq 2$ because $H[X]$ does not have a planar embedding in a disk with $X \cap Y$ drawn on the boundary of the disk. The set $X - Y$ includes at most one branch-vertex of S' by the definition of S'-separation. Since S has at least six branch vertices, $|Y - X| \geq 2$. But this contradicts the almost 4-connectivity of H. □

Our main theorem is a generalization of the main theorem in [3]. Suppose that $\Sigma = \Gamma = \emptyset$. Then all the edges of our graphs are even. Some of the outcomes of Theorem 1.3.1 involve odd edges. We have the following corollary:

Corollary 4.5.2. *Let G be an almost 4-connected planar graph on at least six vertices, let H be an almost 4-connected non-planar graph, and let S be a G-subdivision in H. Then there exists a G-subdivision S' in H obtained from S by repeated reroutings such that S' satisfies one of the following conditions:*

(i) there exists an S'-jump in H, or

(ii) there exists a free S'-cross in H on some peripheral cycle of S'.

Proof. We may consider G as (G, Σ) and H as (H, Γ) where $\Sigma = \Gamma = \emptyset$. We apply Theorem 1.3.1 to (G, Σ), (H, Γ), and S. We may assume that one of the outcomes (iii)-(v) of Theorem 1.3.1 holds; otherwise the corollary holds. But these outcomes all require H to have odd edges, which is a contradiction. □

Corollary 4.5.2 is the main theorem in [3].

Chapter 5

Future Work

5.1 One Non-Planar Unstable Bridge

We would like to further characterize outcome (iv) of Theorem 1.3.1. As currently stated, the outcome is not specific enough to be particularly useful in applications. We would like to show what minimal structures an unstable S-bridge must contain if it is not S-planar. In particular, we would like to prove the following conjecture:

Conjecture. *Let (G, Σ) and (H, Γ) be signed graphs where G has no vertices of degree two. Let S be a (G, Σ)-subdivision in (H, Γ) with weak disk system $\mathcal{C}$, and suppose none of outcomes (i)-(v) of Theorem 3.1.1 hold for any (G, Σ)-subdivision S' related to S. Let B, P_1, P_2 be an S-umbrella on a segment Z. Let w_1, z_1 and w_2, z_2 be the ends of P_1 and P_2 respectively, where w_1, $w_2 \in V(Z)$. Then there exist vertex-disjoint paths Q_1 and Q_2 in B with ends s_1, t_1 and s_2, t_2 such that s_1, s_2, t_1, t_2 appear on Z in that order and w_1, $w_2 \in V(s_2 Z t_1)$.*

If this conjecture is true, then we can reroute Z along Q_1 and Q_2 to turn P_1 and P_2 into a weakly free cross. See the proof of Lemma 2.1.2. Thus we could remove outcome (iv) from the statement of Theorem 1.3.1.

5.2 Parallel Edges

If (G, Σ) is a signed graph, it is natural to allow G to have parallel edges, where those parallel edges are in even-odd pairs. We would like to prove a modified version of Theorem 1.3.1 where G is allowed to have some parallel edges. The disk system axioms can be modified to allow for *trivial disks*, which are disks bounded by the two segments corresponding to a pair of parallel edges in G.

If we allow parallel edges, then we have to make some changes to the outcomes. For example, an interrupted S-jump is not necessarily non-planar if it occurs in a trivial disk. See Figure 5.1.

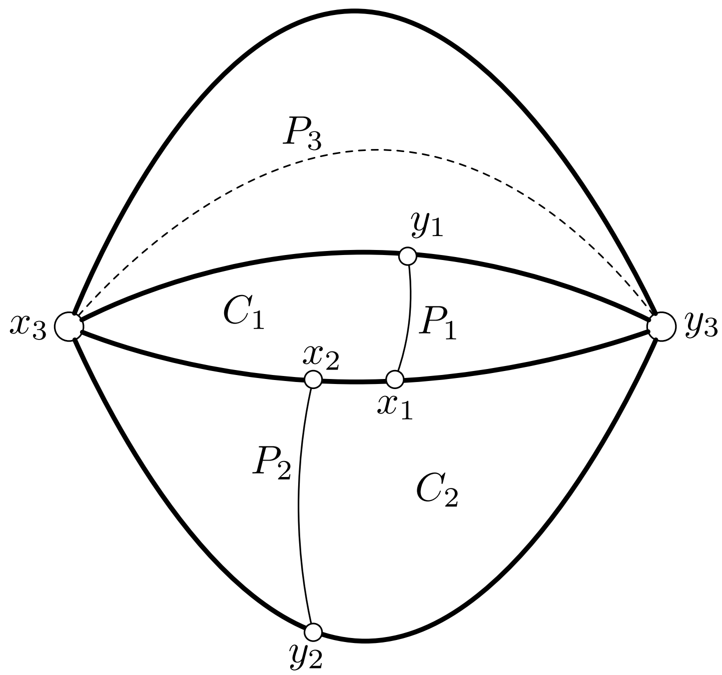

Figure 5.1: A planar drawing of an interrupted jump in a trivial disk

5.3 Minors

We will define the concept of a minor for a signed graph. Let (G, Σ) be a signed graph. Let I, J, be subsets of $E(G)$ such that $I \cap J = \emptyset$. If I does not contain the edges of an odd cycle, then there exists a signature Γ of (G, Σ) such that $\Gamma \cap I = \emptyset$. Then the graph $((G/I) - J, \Gamma - J)$ is a *signed minor* of (G, Σ).

In [3], Norin and Thomas also prove a version of their main result where the conclusion is about minors rather than subdivisions. We say that a graph G is *internally 4-connected* if it is 3-connected and for every separation (A, B) of order three one of $G[A]$, $G[B]$ has at most three edges. If u, $v \in V(G)$ are not adjacent, then we use $G + uv$ to denote the graph obtained from G by adding an edge with ends u and v.

Theorem 5.3.1. *Let G be a triangle-free internally 4-connected planar graph, and let H be an almost 4-connected non-planar graph such that H has a subgraph isomorphic to a subdivision of G. Then there exists a graph G' such that G' is isomorphic to a minor of H, and either*

(i) $G' = G + uv$ for some vertices u, $v \in V(G)$ such that no peripheral cycle of G contains both u and v, or

(ii) $G' = G + u_1v_1 + u_2v_2$ for some distinct vertices u_1, u_2, v_1, $v_2 \in V(G)$ such that u_1, u_2, v_1, v_2 appear on some peripheral cycle of G in the order listed.

We would like to prove an analogue of this result for signed graphs and signed minors.

Bibliography

[1] Katherine Naismith. Escape Paper. Unpublished manuscript. 2019.

[2] Katherine Naismith. Extensions of Signed graphs. Master's thesis. University of Waterloo, 2014.

[3] Sergey Norin and Robin Thomas. Non-planar Extensions of Subdivisions of Planar Graphs. *Journal of Combinatorial Theory*, 121:326–366, 2016.

[4] William Tutte. How to Draw a Graph. *Proc. London Math. Soc.*, 13:742-768, 1963.

[5] Hassler Whitney. Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54:150-168, 1932.