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# The number of valid factorizations of Fibonacci prefixes

Pierre Bonardo<sup>a</sup>, Anna E. Frid<sup>a</sup>, Jeffrey Shallit<sup>b</sup>

<sup>a</sup>*Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France*

<sup>b</sup>*School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada*

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## Abstract

We establish several recurrence relations and an explicit formula for  $V(n)$ , the number of factorizations of the length- $n$  prefix of the Fibonacci word into a (not necessarily strictly) decreasing sequence of standard Fibonacci words. In particular, we show that the sequence  $V(n)$  is the shuffle of the ceilings of two linear functions of  $n$ .

*Keywords:* numeration systems, Fibonacci numeration system, Fibonacci word

*2010 MSC:* 68R15, 11B39

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## 1. Introduction

In the classical Fibonacci, or Zeckendorf, numeration system [6, 11], a positive integer is represented as a sum of Fibonacci numbers:

$$n = F_{m_k} + F_{m_{k-1}} + \cdots + F_{m_0},$$

- 2 where  $m_k > m_{k-1} > \cdots > m_0 \geq 2$  and, as usual,  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_{m+2} =$   
 3  $F_{m+1} + F_m$  for all  $m \geq 0$ . For example,  $16 = 13 + 3 = F_7 + F_4 = [100100]_F$ ,  
 4 where a digit in brackets is 1 if the respective Fibonacci number appears in the  
 5 sum, and 0 otherwise. Here a representation ends by the digits corresponding  
 6 to  $F_4 = 3$ ,  $F_3 = 2$  and  $F_2 = 1$ .

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*Email addresses:* [anna.e.frid@gmail.com](mailto:anna.e.frid@gmail.com) (Anna E. Frid), [shallit@uwaterloo.ca](mailto:shallit@uwaterloo.ca) (Jeffrey Shallit)

*URL:* <http://iml.univ-mrs.fr/~frid/> (Anna E. Frid),  
<https://cs.uwaterloo.ca/~shallit/> (Jeffrey Shallit)

7 Under the condition that  $m_i$  and  $m_{i+1}$  are never consecutive, that is,  $m_{i+1} -$   
 8  $m_i \geq 2$ , or, equivalently, that the Fibonacci numbers  $F_i$  are chosen greedily,  
 9 such a *canonical* representation is unique, and the language  $L_V$  of all canonical  
 10 representations is given by the regular expression  $\epsilon + 1(0 + 01)^*$ , where the  
 11 empty word  $\epsilon$  is the representation of 0. At the same time, if consecutive  
 12 Fibonacci numbers are allowed, but at most once each, the number of such *legal*  
 13 representations of  $n$  is the well-known integer sequence [A000119](#) from the Online  
 14 Encyclopedia of Integer Sequences (OEIS) [8]. Its values oscillate between 1 (on  
 15 numbers of the form  $F_i - 1$ ) and  $\sqrt{n + 1}$  (on numbers of the form  $n = F_i^2 - 1$ )  
 16 [10].

For example, since

$$\begin{aligned} 16 &= 13 + 3 = 8 + 5 + 3 = 8 + 5 + 2 + 1 = 13 + 2 + 1 \\ &= [100100]_F = [11100]_F = [11011]_F = [100011]_F, \end{aligned}$$

the number of legal representations of 16 is 4. Each legal representation of  $n$   
 can be obtained from a canonical one by a series of replacements

$$\dots 100 \dots \longleftrightarrow \dots 011 \dots,$$

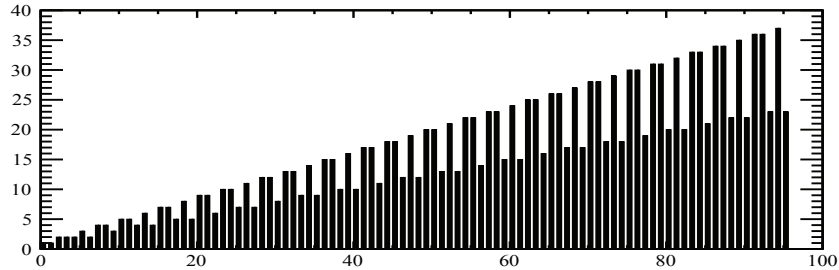
17 corresponding to the replacement of a Fibonacci number  $F_{m+2}$  by  $F_{m+1} + F_m$ .

In this paper, we allow even more freedom in Fibonacci representations of  
 $n$ , allowing the transformations

$$\dots k0l \dots \longleftrightarrow \dots (k-1) 1(l+1) \dots \quad (1)$$

18 for all  $k > 0$ ,  $l \geq 0$ . Note that the introduced transformation corresponds to  
 19 passing from a sum of the form  $kF_{m+1} + lF_{m-1}$  to the sum  $(k-1)F_{m+1} + F_m +$   
 20  $(l+1)F_{m-1}$ , and, in particular, does not change the represented number.

The representations that can be obtained from the canonical one by a series  
 of transformations as in (1) are called *valid*, and were introduced in [4] in a more  
 general setting because of their link to the Fibonacci word and factorizations of  
 its prefixes, as explained below. Clearly, each legal representation is valid, but

Figure 1: First 100 values of  $V(n)$ 

the opposite is not true. For example, starting from the legal representation  $16 = [11011]_F$ , we can find two more valid representations

$$16 = [10121]_F = [1221]_F,$$

and starting from the legal representation  $16 = [11100]_F$ , we find a new representation

$$16 = [20000]_F,$$

21 so that the total number of valid representations of 16 is 7.

22 Let  $V(n)$  denote the number of valid representations of  $n$ . The goal of this  
 23 paper is to prove a precise formula for  $V(n)$ , given below in Theorem 1. Our  
 24 formula demonstrates that the values of  $V(n)$  are determined by the shuffle of  
 25 two straight lines of irrational slope; see Fig. 1.

## 26 2. Notation and Sturmian representations

27 We use notation common in combinatorics on words; the reader is referred,  
 28 for example, to [3] for an introduction. Given a finite word  $u$ , we denote its  
 29 length by  $|u|$ . The power  $u^k$  just means the concatenation  $u^k = \underbrace{u \cdots u}_k$ . The  $i$ 'th  
 30 symbol of a finite or infinite word  $u$  is denoted by  $u[i]$ , so that  $u = u[1]u[2] \cdots$ . A  
 31 factor  $w[i+1]w[i+2] \cdots w[j]$  of a finite or infinite word  $w$ , or, more precisely, its  
 32 occurrence starting from position  $i+1$  of  $w$ , is denoted by  $w(i..j)$ . In particular,  
 33 for  $j \geq 0$ , the word  $w(0..j)$  is the prefix of  $w$  of length  $j$ .

The *standard Fibonacci sequence*  $(f_n)$  of words over the binary alphabet  $\{a, b\}$  is defined as follows:

$$f_{-1} = b, \quad f_0 = a, \quad f_{n+1} = f_n f_{n-1} \text{ for all } n \geq 0. \quad (2)$$

34 The word  $f_n$  is called also the *standard word of order  $n$* . In particular,  $f_1 = ab$ ,  
 35  $f_2 = aba$ ,  $f_3 = abaab$ ,  $f_4 = abaababa$ , and so on. From the definition, we easily  
 36 see that the length of  $f_n$  is the Fibonacci number  $F_{n+2}$ .

The infinite word

$$\mathbf{f} = \lim_{n \rightarrow \infty} f_n = abaababaabaababaababa \dots$$

37 is called the Fibonacci infinite word. Here we index it starting with  $\mathbf{f}[1] = a$ .

In the *Fibonacci*, or *Zeckendorf numeration system*, a non-negative integer  $N < F_{n+3}$  is represented as a sum of Fibonacci numbers

$$N = \sum_{0 \leq i \leq n} k_i F_{i+2}, \quad (3)$$

where  $k_i \in \{0, 1\}$  for  $i \geq 0$ . In the canonical version of the definition, the following condition holds:

$$\text{for } i \geq 1, \text{ if } k_i = 1, \text{ then } k_{i-1} = 0. \quad (4)$$

38 Under this nonadjacency condition, the representation of  $N$  is unique up to  
 39 leading zeros. However, by removing the nonadjacency condition, we can get  
 40 multiple representations: for example,  $14 = F_7 + F_2 = F_6 + F_5 + F_2 = F_6 +$   
 41  $F_4 + F_3 + F_2$ . We call such representations *legal* and denote a representation  
 42  $N = \sum_{0 \leq i \leq n} k_i F_{i+2}$  by  $N = [k_n \dots k_0]_F$ . If the condition (4) holds, we call the  
 43 representation *canonical*.

44 Let  $L(n)$  denote the number of legal representations of  $n$ . The sequence  
 45  $(L(n))$  is well-studied (see, e.g., [2]) and listed in the OEIS as sequence [A000119](#).  
 46 In particular,  $1 \leq L(n) \leq \sqrt{n+1}$ , and both bounds are precise [10].

47 The following lemma is a particular case of [4, Prop. 2].

48 **Lemma 1.** *For all  $k_0, \dots, k_n$  such that  $k_i \in \{0, 1\}$ , the word  $f_n^{k_n} f_{n-1}^{k_{n-1}} \dots f_0^{k_0}$   
 49 is a prefix of the Fibonacci word  $\mathbf{f}$ .*

50 So  $L(n)$  is also the number of ways to factor the prefix  $\mathbf{f}(0..n]$  of the Fibonacci  
 51 word as a sequence of standard words in strictly decreasing order.

52 To expand this definition, in this note we consider all factorizations of Fi-  
 53 bonacci prefixes  $\mathbf{f}(0..n]$  as a concatenation of standard words in (non-strictly)  
 54 decreasing order. We write  $N = [k_n \cdots k_0]_F$  and call this representation of  $N$   
 55 *valid* if  $k_i \geq 0$  for all  $i$  and  $\mathbf{f}(0..N] = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$ . Note that according  
 56 to the previous lemma, every legal representation is valid, but not the other  
 57 way around. For example,  $\mathbf{f}(0..14] = (abaab)(aba)(aba)(aba)$ , making the rep-  
 58 resentation  $14 = [1300]_F$  valid. Theorem 1 of [4] says, in particular, that valid  
 59 representations are exactly those that can be obtained from the canonical one  
 60 by a series of transformations (1).

61 Note that a digit of a valid representation cannot exceed 3 since the Fibonacci  
 62 word does not contain a factor of the form  $u^4$  for any non-empty word  $u$  [5].

63 The number of valid representations of  $N$  is denoted by  $V(N)$ , and this note  
 64 is devoted to the study of the sequence  $(V(n))$ , recently listed in the OEIS as  
 65 sequence [A300066](#). Clearly,  $V(n) \geq L(n)$ , and moreover, we prove an explicit  
 66 formula for  $V(n)$  that implies its linear growth.

### 67 3. Result

As is well-known, the Fibonacci infinite word

$$\mathbf{f} = abaababa \cdots$$

is the fixed point of the Fibonacci morphism  $\mu : a \rightarrow ab, b \rightarrow a$ ; moreover, for  
 each  $n \geq 1$ , we have  $f_n = \mu(f_{n-1})$ . Consequently, if  $N = [k_n \cdots k_0]_F$ , then  
 Lemma 1 implies that

$$\mu(\mathbf{f}(0..N]) = \mu(\mathbf{f}(0..[k_n \cdots k_0]_F]) = \mu(f_n^{k_n} \cdots f_0^{k_0}) = f_{n+1}^{k_n} \cdots f_1^{k_0} = \mathbf{f}(0..[k_n \cdots k_0]_F]).$$

Let  $\varphi$  denote the golden ratio:  $\varphi = \frac{1+\sqrt{5}}{2}$ . It is important that the Fibonacci  
 word is a Sturmian word of slope  $1/(\varphi + 1) = 1/\varphi^2$  and zero intercept (see

Example 2.1.24 of [3]), that is, for all  $n$ , we have

$$\mathbf{f}[n] = \begin{cases} a, & \text{if } \{n/\varphi^2\} < 1 - 1/\varphi^2; \\ b, & \text{otherwise.} \end{cases} \quad (5)$$

68 Here  $\{x\} = x - \lfloor x \rfloor$  denotes the fractional part of  $x$ .

69 **Proposition 1.** *If  $\mathbf{f}[n] = a$ , all valid representations of  $n$  end with an even*  
70 *number of 0s. If  $\mathbf{f}[n] = b$ , all of them end with an odd number of 0s.*

71 *Proof.* It suffices to consult the definition of a valid representation and notice  
72 that  $f_i$  ends with  $a$  if and only if  $i$  is even.  $\square$

73 We now state our main result.

74 **Theorem 1.** *If  $\mathbf{f}[n] = a$ , then  $V(n) = \lceil n/\varphi^2 \rceil$ , or, equivalently,  $V(n)$  is equal*  
75 *to the number of occurrences of  $b$  in  $\mathbf{f}(0..n)$ , plus one. If  $\mathbf{f}[n] = b$ , then  $V(n) =$*   
76  *$\lceil n/\varphi^3 \rceil$ , or, equivalently,  $V(n)$  is equal to the number of occurrences of  $aa$  in*  
77  *$\mathbf{f}(0..n)$ , plus one.*

78 To prove the theorem, we will need several more propositions.

79 **Proposition 2.**

80 (a)  $V([r0]_F) \geq V([r]_F)$  for all  $r \in \{0, 1\}^*$ .

81 (b) For all  $k \geq 0$  and all  $r' \in \{0, 1\}^*$ , we have  $V([r'10^{2k+1}]_F) = V([r'10^{2k}]_F)$ .

82 *Proof.* (a): Consider a factorization  $\mathbf{f}(0..[r]_F) = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$ . Applying  
83 the Fibonacci morphism  $\mu$  to both sides, we get the factorization  $\mathbf{f}(0..[r0]_F) =$   
84  $f_{n+1}^{k_n} f_n^{k_{n-1}} \cdots f_1^{k_0}$ . So the number of factorizations of  $\mathbf{f}(0..[r0]_F)$  (which is equal  
85 to  $V([r0]_F)$ ) is at least as large as the number of factorizations of  $\mathbf{f}(0..[r]_F)$   
86 (which is equal to  $V([r]_F)$ ).

87 (b) If, in addition  $r = r'10^{2k}$  for some  $k \geq 0$ , we see that  $\mathbf{f}(0..[r]_F)$  ends with  
88  $f_{2k}$  and  $\mathbf{f}(0..[r0]_F)$  ends with  $f_{2k+1}$ , which in turn ends with  $b$ . From Proposition  
89 1, no factorization of  $\mathbf{f}(0..[r0]_F)$  ends with  $f_0$ ; that is, such a factorization must  
90 be of the form  $\mathbf{f}(0..[r0]_F) = f_{n+1}^{k_n} f_n^{k_{n-1}} \cdots f_1^{k_0}$ . Taking the  $\mu$ -preimage, we get

91 the factorization  $\mathbf{f}(0..[r]_F) = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$ , thus establishing a bijection and  
 92 the equality  $V([r'10^{2k+1}]_F) = V([r'10^{2k}]_F)$ .  $\square$

**Proposition 3.** *We have*

$$V([z10^{2k}]_F) = V([z10^{2k-2}]_F) + V([z(01)^k]_F).$$

93 for all  $z \in \{0, 1\}^*$  and all  $k \geq 1$ .

94 *Proof.* Proposition 1 tells us that  $\mathbf{f}([z10^{2k}]_F) = a$ , and moreover, since  $k > 0$ ,  
 95 the prefix of length  $[z10^{2k}]_F$  of  $\mathbf{f}$  ends with  $aba$ , which is a suffix of  $f_{2k}$ . Consider  
 96 a valid factorization  $\mathbf{f}(0..[z10^{2k}]_F) = f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_0^{k_0}$ . If  $k_0 = 0$ , then  $k_1 = 0$   
 97 since  $f_1$  ends with  $b$ , so the factorization is of the form  $f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_2^{k_2}$ . Taking  
 98 the  $\mu^2$ -preimage, we get a factorization  $f_{n-2}^{k_{n-2}} f_{n-3}^{k_{n-3}} \cdots f_0^{k_2}$  of  $\mathbf{f}(0..[z10^{2k-2}]_F)$ .  
 99 Moreover,  $\mu^2$  is a bijection between all the factorizations of  $\mathbf{f}(0..[z10^{2k-2}]_F)$  and  
 100 the factorizations of  $\mathbf{f}(0..[z10^{2k}]_F)$  with  $k_0 = k_1 = 0$ .

101 On the other hand, if  $k_0 \neq 0$ , then  $k_0 = 1$  since the word that we factor  
 102 ends with  $aba$ . Removing this last occurrence of  $f_0 = a$ , we get the prefix of  $\mathbf{f}$   
 103 of length  $[z10^{2k}]_F - 1 = [z(01)^k 0]_F$ . From Proposition 2, the number of valid  
 104 factorizations of  $\mathbf{f}(0..[z(01)^k 0]_F)$  is equal to that of  $\mathbf{f}(0..[z(01)^k]_F)$ . Combining  
 105 the two possibilities, we get the statement of the proposition.  $\square$

**Proposition 4.** *For all  $z \in \{0, 1\}^*$  and for all  $k \geq 1$ , we have*

$$V([z10^k 1]_F) = \begin{cases} V([z10^{k+1}]_F), & \text{if } k \text{ is odd;} \\ V([z10^k]_F) + V([z(01)^{k/2}]_F), & \text{if } k \text{ is even.} \end{cases}$$

106 *Proof.* If  $k$  is odd, then  $[z10^k 1]_F = [z10^{k+1}]_F + 1$ , and the prefix  $\mathbf{f}(0..[z10^{k+1}]_F)$   
 107 was considered in the previous proposition. It ends with  $aba$ , and the symbol  
 108 added to get  $\mathbf{f}(0..[z10^k 1]_F)$  is also  $a$ . So  $\mathbf{f}(0..[z10^k 1]_F)$  ends with  $abaa$ , and all  
 109 valid factorizations end with  $f_0$ . This means that the number of valid factoriza-  
 110 tions of  $\mathbf{f}(0..[z10^k 1]_F)$  is equal to that of  $\mathbf{f}[0..[z10^{k+1}]_F]$ ; that is,  $V([z10^k 1]_F) =$   
 111  $V([z10^{k+1}]_F)$ .

If  $k$  is even,  $k > 0$ , then  $\mathbf{f}(0..[z10^k 1]_F)$  ends with  $f_3 f_0 = abaaba$ . In partic-  
 ular, the last factor of any valid factorization of  $\mathbf{f}(0..[z10^k 1]_F)$  is either  $f_0 = a$ ,



or  $f_2 = aba$ . Indeed,  $f_4 = abaababa$  and thus for all  $l > 2$  the  $f_{2l}$  do not have a common suffix with  $\mathbf{f}(0..[z10^k 1]_F)$ . So, letting  $V_2(n)$  denote the number of factorizations of  $\mathbf{f}(0..n]$  of the form  $f_n^{k_n} f_{n-1}^{k_{n-1}} \cdots f_2^{k_2}$ , we get

$$\begin{aligned} V([z10^k 1]_F) &= V([z10^k 1]_F - 1) + V_2([z10^k 1]_F - 3) \\ &= V([z10^{k+1}]_F) + V_2([z(01)^{k/2} 00]_F) \\ &= V([z10^k]_F) + V([z(01)^{k/2}]_F). \end{aligned}$$

112 Here the last equality follows from Proposition 2 (for the first addend) and by  
113 taking  $\mu^{-2}$  of each factorization (for the second one).  $\square$

114 Propositions 2 to 4 give a full list of recurrence relations sufficient to compute  
115  $V(n)$  for every  $n > 1$ , starting from  $V(1) = 1$ . Before using them to prove the  
116 main theorem, we consider two particular cases.

**Corollary 1.** *For all  $k \geq 1$  we have*

$$V(F_{2k+1} - 1) = V(F_{2k+1} - 2) = F_{2k-1}$$

and

$$V(F_{2k+2} - 2) = F_{2k}$$

*Proof.* For  $k = 1$ , the equalities can be easily checked:  $V(F_3 - 1) = V(1) = V(F_3 - 2) = V(0) = 1 = F_1$ , and  $V(F_4 - 2) = V(1) = 1 = F_2$ . We also observe that  $F_{2k+1} - 1 = [(10)^{k-1} 1]_F$ ,  $F_{2k+1} - 2 = [(10)^{k-1} 0]_F$ , and  $F_{2k+2} - 2 = [(10)^{k-1} 01]_F$ . Now we assume that the equalities hold for  $k$ , and use Propositions 3 and 4 to prove they hold for  $k + 1$ :

$$\begin{aligned} V(F_{2k+3} - 2) &= V([(10)^k 0]_F) = V([(10)^{k-1} 1]_F) + V([(10)^{k-1} 01]_F) \\ &= V(F_{2k+1} - 1) + V(F_{2k+2} - 2) = F_{2k-1} + F_{2k} = F_{2k+1}, \\ V(F_{2k+3} - 1) &= V([(10)^k 1]_F) = V([(10)^k 0]_F) = V(F_{2k+3} - 2) = F_{2k+1}, \\ V(F_{2k+4} - 2) &= V([(10)^k 01]_F) = V([(10)^k 0]_F) + V([(10)^{k-1} 01]_F) \\ &= V(F_{2k+3} - 2) + V(F_{2k+2} - 2) = F_{2k+1} + F_{2k} = F_{2k+2}. \end{aligned}$$

117

$\square$

**Corollary 2.** For all  $k \geq 1$ , we have

$$V(F_{2k}) = V(F_{2k+1}) = F_{2k-2} + 1.$$

*Proof.* For  $k = 1$ , the equalities can be easily checked:  $V(F_2) = V(1) = V(F_3) = V(2) = 1 = F_0 + 1$ . Suppose the equalities hold for  $k$ ; let us prove them for  $k + 1$ . With Proposition 3, we have

$$V(F_{2k+2}) = V([10^{2k}]_F) = V([10^{2k-2}]_F) + V([(10)^{k-1}1]_F) = F_{2k-2} + 1 + F_{2k-1} = F_{2k} + 1,$$

and with Proposition 2, we have

$$V(F_{2k+3}) = V([10^{2k+1}]_F) = V([10^{2k}]_F) = V(F_{2k+2}) = F_{2k} + 1.$$

118

□

**Proposition 5.** Let  $n = [z]_F$  and  $n' = [z0]_F$  be such that  $\mathbf{f}[n] = a$ . Then  
119  $[n/\varphi^2] = [n'/\varphi^3]$ .  
120

*Proof.* Let us write the canonical Fibonacci representation of  $n$  as  $\sum_{1 \leq i \leq l} F_{m_i}$ ,  
121 where  $2 \leq m_1 < m_2 < \dots < m_l$ . Since  $\mathbf{f}[n] = a$ , from Proposition 1 we get that  
122  $m_1$  is even.  
123

Now  $F_k = \frac{1}{\sqrt{5}}(\varphi^k - \psi^k)$ , where  $\psi = \frac{1-\sqrt{5}}{2}$ ,  $-1 < \psi < 0$ . So

$$n = \sum_{1 \leq i \leq l} F_{m_i} = \frac{1}{\sqrt{5}} \left( \sum_{1 \leq i \leq l} \varphi^{m_i} - \sum_{1 \leq i \leq l} \psi^{m_i} \right)$$

and

$$n' = \sum_{1 \leq i \leq l} F_{m_i+1} = \frac{1}{\sqrt{5}} \left( \sum_{1 \leq i \leq l} \varphi^{m_i+1} - \sum_{1 \leq i \leq l} \psi^{m_i+1} \right),$$

implying that

$$\frac{n'}{\varphi} = \frac{1}{\sqrt{5}} \left( \sum_{1 \leq i \leq l} \varphi^{m_i} - \frac{1}{\varphi} \sum_{1 \leq i \leq l} \psi^{m_i+1} \right).$$

The difference between the two values is

$$\frac{n'}{\varphi} - n = \frac{1}{\sqrt{5}} \left( 1 - \frac{\psi}{\varphi} \right) S,$$

where

$$S = \sum_{1 \leq i \leq l} \psi^{m_i} = \psi^{m_1} \sum_{1 \leq i \leq l} \psi^{m_i - m_1}.$$

Let us estimate  $S$ . Since  $m_1 \geq 2$ ,  $m_1$  is even and  $0 < \psi^{m_1} < \psi^2$ , an upper bound for  $S$  is

$$S < \psi^{m_1} \sum_{k=0}^{\infty} \psi^{2k} = \frac{\psi^{m_1}}{1 - \psi^2} \leq \frac{\psi^2}{1 - \psi^2},$$

whereas a lower bound is

$$S > \psi^{m_1} \left( 1 + \sum_{k=1}^{\infty} \psi^{2k+1} \right) > \psi^{m_1} \left( 1 + \sum_{k=0}^{\infty} \psi^{2k+1} \right) = \psi^{m_1} \left( 1 + \frac{\psi}{1 - \psi^2} \right) = 0.$$

So

$$0 < \frac{n'}{\varphi} - n < \frac{\psi^2}{\sqrt{5}} \left( 1 - \frac{\psi}{\varphi} \right) \frac{1}{1 - \psi^2} = \frac{1}{\varphi^2}.$$

Dividing by  $\varphi^2$ , we get

$$0 < \frac{n'}{\varphi^3} - \frac{n}{\varphi^2} < \frac{1}{\varphi^4} < \frac{1}{\varphi^2}.$$

124 Together with (5), meaning that  $\{n/\varphi^2\} < 1 - 1/\varphi^2$ , the last inequality implies  
125 the statement of the Proposition.  $\square$

126 *Proof of Theorem 1.* Let us start with the case of  $\mathbf{f}[n] = a$  and proceed by  
127 induction starting with  $V(1) = 1$ . For  $n > 1$ , there are three subcases:

128 (a)  $n = [z10^{2k}]_F$ ,  $k > 0$ ;

129 (b)  $n = [z10^k 1]_F$ ,  $k$  odd;

130 (c)  $n = [z10^k 1]_F$ ,  $k$  even.

131 From now on we suppose that the statement of the theorem holds for all  $n', n'' <$   
132  $n$ .

133 (a) Since  $n = [z10^{2k}]_F$  and  $k > 0$ , Proposition 3 gives  $V(n) = V([z10^{2k}]_F) =$   
134  $V([z10^{2k-2}]_F) + V([z(01)^k]_F)$ . Write  $[z10^{2k-2}]_F = n'$  and  $[z(01)^k]_F = n''$ . Note  
135 that Proposition 1 gives  $\mathbf{f}[n'] = \mathbf{f}[n''] = a$ . At the same time,  $n'' + 1 = [z10^{2k-1}]_F$   
136 and thus  $\mathbf{f}[n'' + 1] = b$ . Now (5) implies that  $\{n'/\varphi^2\} \in (0, 1 - 1/\varphi^2)$  and  
137  $\{(n'' + 1)/\varphi^2\} \in (1 - 1/\varphi^2, 1)$ . Also, the Fibonacci representation of  $n'$  is

138 obtained from that of  $n'' + 1$  by a one-symbol shift to the left. So, summing up  
 139  $n'$  and  $n'' + 1$ , due to the Fibonacci recurrence relation, we get the number with  
 140 the same representation but shifted to the left yet another position, meaning  
 141 that  $n' + n'' + 1 = n$ .

Let us consider the sum  $t = \{n'/\varphi^2\} + \{(n'' + 1)/\varphi^2\}$ . From the inclusions  
 above, we see that  $t$  belongs to the interval  $(1 - 1/\varphi^2, 2 - 1/\varphi^2)$ . But we also  
 know that  $\{n/\varphi^2\} = \{(n' + n'' + 1)/\varphi^2\} \in (0, 1 - 1/\varphi^2)$ , since  $\mathbf{f}[n] = a$ . So

$$\{n/\varphi^2\} = \{n'/\varphi^2\} + \{(n'' + 1)/\varphi^2\} - 1,$$

which is equivalent to  $\lfloor n/\varphi^2 \rfloor = \lfloor n'/\varphi^2 \rfloor + \lfloor (n'' + 1)/\varphi^2 \rfloor + 1$  and to  $\lfloor n/\varphi^2 \rfloor =$   
 $\lfloor n'/\varphi^2 \rfloor + \lfloor n''/\varphi^2 \rfloor + 1$  (since  $\lfloor n''/\varphi^2 \rfloor = \lfloor (n'' + 1)/\varphi^2 \rfloor$ ). Since all the numbers  
 under consideration are irrational, and thus every ceiling is just the floor plus  
 1, we get

$$\lceil n/\varphi^2 \rceil = \lceil n'/\varphi^2 \rceil + \lceil n''/\varphi^2 \rceil.$$

142 To establish the statement of the theorem for this subcase, it is sufficient to  
 143 use Proposition 3 and the induction hypothesis:  $V(n') = \lceil n'/\varphi^2 \rceil$  and  $V(n'') =$   
 144  $\lceil n''/\varphi^2 \rceil$ .

145 (b): Here  $n = [z10^{2k-1}1]_F$  and  $k > 0$ . It suffices to refer to the previous  
 146 subcase and to Proposition 4:  $V(n) = V(n - 1) = V([z10^{2k}]_F) = \lceil (n - 1)/\varphi^2 \rceil$ .  
 147 It remains to notice that  $\lceil (n - 1)/\varphi^2 \rceil = \lceil n/\varphi^2 \rceil$ , since  $\mathbf{f}[n - 1] = a$ .

148 (c): Here  $n = [z10^{2k}1]_F$  and  $k > 0$ . We use Proposition 4:  $V([z10^{2k}1]_F) =$   
 149  $V([z10^{2k}]_F) + V([z(01)^k]_F)$ . As above, write  $n' = [z10^{2k}]_F$  and  $n'' = [z(01)^k]_F$ ;  
 150 then  $n = n' + n'' + 2$ , whereas  $V(n) = V(n') + V(n'')$ . By the induction  
 151 hypothesis,  $V(n') = \lceil n'/\varphi^2 \rceil$  and  $V(n'') = \lceil n''/\varphi^2 \rceil$ .

We have  $\mathbf{f}[n] = a$  and  $\mathbf{f}[n - 1] = b$ , implying from (5) that  $\{(n - 1)/\varphi^2\} \in$   
 $(1 - 1/\varphi^2, 1)$  and thus  $\{n/\varphi^2\} \in (0, 1/\varphi^2)$ . At the same time,  $\mathbf{f}[n'] = \mathbf{f}[n''] = a$   
 implies  $\{n'/\varphi^2\}, \{n''/\varphi^2\} \in (0, 1 - 1/\varphi^2)$  and thus

$$\{n'/\varphi^2\} + \{n''/\varphi^2\} + \{2/\varphi^2\} \in (2/\varphi^2, 2).$$

Comparing it to  $\{n/\varphi^2\} = \{(n' + n'' + 2)/\varphi^2\} \in (0, 1/\varphi^2)$ , we see that

$$\{n/\varphi^2\} = \{n'/\varphi^2\} + \{n''/\varphi^2\} + \{2/\varphi^2\} - 1.$$

But since  $n = n' + n'' + 2$  and  $x = [x] + \{x\}$  for every  $x$ , this also means that

$$[n/\varphi^2] = [n'/\varphi^2] + [n''/\varphi^2] + 1.$$

Finally, since  $k/\varphi^2$  is not an integer for any integer  $k > 0$ , we have  $\lceil k/\varphi^2 \rceil = \lfloor k/\varphi^2 \rfloor + 1$ , so that

$$\lceil n/\varphi^2 \rceil = \lceil n'/\varphi^2 \rceil + \lceil n''/\varphi^2 \rceil.$$

It remains to use the induction hypothesis to establish

$$V(n) = V(n') + V(n'') = \lceil n'/\varphi^2 \rceil + \lceil n''/\varphi^2 \rceil = \lceil n/\varphi^2 \rceil,$$

152 which was to be proved.

153 To complete the part of the proof concerning  $\mathbf{f}[n] = a$ , it remains to notice  
154 that  $\lceil n/\varphi^2 \rceil$  is equal to the number of  $bs$  in  $\mathbf{f}(0..n)$  plus one, due to (5).

Now for  $\mathbf{f}[n] = b$ , it is sufficient to combine Propositions 1, 2 and 5: if  
 $\mathbf{f}[n] = b$ , then  $n = [r0]_F$ , where  $m = [r]_F$  and  $\mathbf{f}[m] = a$ . Then

$$V(n) = V(m) = \lceil m/\varphi^2 \rceil = \lceil n/\varphi^3 \rceil.$$

155 Here  $\mathbf{f}(0..n) = \mu(\mathbf{f}(0..m))$ , and so the occurrences of  $aa$  in  $\mathbf{f}(0..n)$  correspond  
156 exactly to occurrences of  $b$  in  $\mu(\mathbf{f}(0..m))$ . The theorem is proved.  $\square$

157 The theorem ensures that the sequence  $(V(n))$  grows as depicted in Fig. 1.  
158 The two visible straight lines correspond to the symbols of the Fibonacci word  
159 equal to  $a$  (the upper line) or  $b$  (the lower line).

#### 160 4. Fibonacci-regular representation

A sequence  $(s(n))_{n \geq 0}$  is said to be *Fibonacci-regular* if there exist an integer  
 $k$ , a row vector  $v$  of dimension  $k$ , a column vector  $w$  of dimension  $k$ , and a  $k \times k$   
matrix-valued morphism  $\rho$  on  $\{0, 1\}^*$  such that

$$s([z]_F) = v\rho(z)w$$

161 for all canonical Fibonacci representations  $z \in L_V$ . The triple  $(v, \rho, w)$  is called  
 162 a *linear representation*; see, for example, [7].

Berstel [2] gave the following linear representation for the function  $L(n)$  we mentioned previously in Section 2:

$$v = [1\ 0\ 0\ 0], \quad \rho(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \rho(1) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

163 Hence  $L(n)$  is Fibonacci-regular.

164 We can find a similar representation for the function  $V(n)$ . For technical  
 165 reasons it is easier to deal with the reversed Fibonacci representation; one can  
 166 then obtain the ordinary linear representation by interchanging the roles of the  
 167 vectors and taking the transposes of the matrices.

**Theorem 2.**  $V(n)$  has the reversed linear representation  $(t, \gamma, u)$ , where

$$t = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0], \quad u = [1\ 1\ 1\ 1\ 1\ 2\ 1\ 4]^T$$

$$\gamma(0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -3 & 3 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 & 3 & 0 & 1 & 0 \end{bmatrix}, \quad \gamma(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Proof.* Define  $g(x) = V([x]_F)$  if  $x$  is a valid canonical representation (that is, containing no leading zeroes, and no two consecutive 1's), and 0 otherwise. It

suffices to show, for all  $x \in \{0, 1\}^*$  and  $i \in \{0, 1\}$ , that

$$\begin{bmatrix} g(xi) \\ g(xi0) \\ g(xi1) \\ g(xi00) \\ g(xi000) \\ g(xi100) \\ g(xi0000) \\ g(xi10000) \end{bmatrix} = \gamma(i) \begin{bmatrix} g(x) \\ g(x0) \\ g(x1) \\ g(x00) \\ g(x000) \\ g(x100) \\ g(x0000) \\ g(x10000) \end{bmatrix}. \quad (6)$$

168 Once we prove this, it is then easy to see (using induction on  $|z|$ ) that, if  $z$  is the  
169 Fibonacci representation of  $n$ , then  $t\gamma(z^R)u = V(n)$ , where  $z^R$  is the reversal  
170 of  $z$ .

Thus it suffices to verify Eq. (6). This is equivalent to proving the following identities for  $x$ .

$$g(x01) = -g(x) + g(x0) + g(x00) \quad (7)$$

$$g(x10) = g(x1) \quad (8)$$

$$g(x0100) = -g(x) + 2g(x00) + g(x000) \quad (9)$$

$$g(x1000) = g(x100) \quad (10)$$

$$g(x010000) = -g(x) - g(x0) + 2g(x00) + 3g(x000) + g(x0000) \quad (11)$$

$$g(x00000) = g(x) - g(x0) - 3g(x00) + 3g(x000) + g(x0000). \quad (12)$$

171 Identities (8) and (10) are particular cases of Proposition 2 (b).

To prove (7), consider separately two cases: if  $x$  ends with an even number of zeros, then  $g(x) = g(x0)$  due to Proposition 2 (b) and  $g(x00) = g(x01)$  due to Proposition 4, so the identity holds. If  $x$  ends with an odd number of zeros,  $x = z10^{2k+1}$ ,  $k \geq 0$ , then due to Proposition 4,

$$g(x01) = g(z10^{2k+2}1) = g(z10^{2k+2}) + g(z(01)^{k+1}) = g(x0) + g(z(01)^{k+1}).$$

On the other hand, due to Propositions 2 and 3,

$$g(x00) = g(x0) = g(z10^{2k+2}) = g(z10^{2k}) + g(z(01)^{k+1}) = g(x) + g(z(01)^{k+1}).$$

172 Comparing these equalities, we get (7).

To prove (9), it is sufficient to use Proposition 3 to get

$$g(x0100) = g(x01) + g(x001),$$

173 and then to use (7) twice, for  $g(x01)$  and for  $g(x001)$ .

To prove (11), it is sufficient to use Propositions 3 and 2 to get

$$g(x010000) = g(x0100) + g(x00101) = g(x0100) + g(x00100).$$

174 Now (11) is obtained immediately by summing up (9) applied to  $x$  and to  $x0$ .

Finally, to prove (12), we again have to consider two cases. If  $x = z10^{2k}$ ,  $k \geq 0$ , then due to Proposition 2,  $g(x0^5) = g(x0000)$ ,  $g(x000) = g(x00)$ ,  $g(x0) = g(x)$ , and the equality holds. If now  $x = z10^{2k+1}$ ,  $k \geq 0$ , then (12) immediately simplifies with Proposition 2 as

$$g(z10^{2k+6}) - g(z10^{2k+4}) = 3[g(z10^{2k+4}) - g(z10^{2k+2})] - [g(z10^{2k+2}) - g(z10^{2k})].$$

Applying Proposition 3, we reduce it to

$$g(z(01)^{k+3}) = 3g(z(01)^{k+2}) - g(z(01)^{k+1}),$$

or, writing  $y = z(01)^{k+1}$  and applying Proposition 4 again,

$$g(y0100) = 3g(y00) - g(y).$$

175 But this is exactly (9) since  $g(y00) = g(y000)$ .

176

□

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