# Born Geometry

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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## Abstract

In this thesis, we summarize the work which we authored or co-authored during our PhD studies and also present additional details and ideas that are not found elsewhere. The main topic is the study of T-duality in theoretical physics through the lens of para-Hermitian geometry and Born geometry as well as the description of mathematical aspects of said geometries. In the summary portion, we introduce the D-bracket and a related notion of torsion on para-Hermitian manifolds, consequently using these geometric elements to define a unique connection with canonical properties analogous to the Levi-Civita connection in Riemannian geometry. We then discuss para-Hermitian geometry and Born geometry in the framework of generalized geometry, showing that both arise naturally in this context. We also show that the D-bracket can be recovered from the small and large Courant algebroids of the para-Hermitian manifold using the formalism of generalized geometry. Lastly, we discuss applications to theoretical physics beyond the immediate context of T-duality, showing that our generalized-geometric formulations of para-Hermitian geometry and Born geometry correspond to extended symmetries of two-dimensional non-linear sigma models. We also introduce the notion of para-Calabi-Yau manifolds and use this new geometry to study the semi-flat mirror symmetry. We show, in particular, that both the mirror manifolds carry Born structures and that the mirror map relates the symplectic moduli space of the Born geometry on one side to the complex and para-complex moduli on the other side. Additionally, we discuss the para-Hermitian geometry underlying the topological T-duality of Bouwknegt, Evslin and Mathai and present various new discussions and reformulations of known results.

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## Dedication

To my parents Jiří and Viera, my brother Marek, and my girlfriend Kristýna, none of whom understand much of my work but still support it unconditionally.

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# Chapter 1

# Introduction

The aim of this thesis is to unify the differential geometry that underlies the concept of T-duality in theoretical physics, particularly string theory, under the roof of *para-Hermitian geometry*, and its special case, *Born geometry*. For this, we start with a brief review of the appearance of T-duality in physics. The following paragraphs should not, however, be understood as a full account of the notion of T-duality, but rather as an introduction to the particular cases we will consider in this work.

In short, T-duality is an equivalence of two physical theories that arises in string theory and more generally quantum field theory. A particular class of quantum field theories that give rise to T-duality are $2D$ $\sigma$-models, which are theories of maps from a Riemann surface, called the *worldsheet*, into a target manifold (usually also Riemannian), called the *target space*. What is meant by *equivalence of physical theories* is vastly dependent on the given context, but for the purpose of this work we shall say that two theories are equivalent if they yield the same equations of motion.

Let now **Geo1** and **Geo2** denote the placeholders referring generically to the geometric data associated to the two T-dual theories. We will be mostly concerned with studying the *geometric realization* of T-duality, i.e. the purely geometric relationship between **Geo1** and **Geo2**. The philosophy we employ is that any time T-duality exists, the two geometries **Geo1** and **Geo2** can be recovered from a shared geometric origin **Geo**, which should then be understood as the fundamental geometric picture and **Geo1** and **Geo2** as its physical realizations, or limits:

$$\begin{array}{ccc} & \textbf{Geo} & \\ \swarrow & & \searrow \\ \textbf{Geo1} & \overset{\text{T-duality}}{\longleftrightarrow} & \textbf{Geo2} \end{array} \quad . \tag{1.0.1}$$

The very appearance of T-duality is then seen as a consequence of the existence of the overarching geometry **Geo**. This geometry is the main object of interest in this thesis and we propose its description in terms of Born geometry.

The above philosophy can also be realized at the level of physical theories, following the natural belief that there exists a unifying theory "upstairs" corresponding to **Geo**. There, T-duality should act simply as a symmetry, exchanging the arrows to **Geo1** and **Geo2**. This research program is generally called **Double Field Theory** after the pioneering work of Hull and Zwiebach [1], which studies T-duality covariant string field theory on tori. The name is derived from the fact that the fields of the theory on **Geo** carry degrees of freedom corresponding to both **Geo1** and **Geo2** and therefore are *doubled*. A similar approach, where Born geometry first appeared, is the **Metastring Theory** [2, 3] of Freidel, Leigh and Minic, where **Geo** is also given an additional physical interpretation of *phase space* of what the authors call the fundamental string.

The case we will mostly study is when **Geo1** and **Geo2** are two different target space geometries for a shared worldsheet. Then, **Geo1** refers to a particular manifold $M$ with additional data, for example a Riemannian metric $g$ and a closed three-form $H$. Similarly, **Geo2** is then given by a triple $(\tilde{M}, \tilde{g}, \tilde{H})$ of the same type of data on a different manifold $\tilde{M}$. The simplest naive model for **Geo** is given by the manifold $\mathbb{M} = M \times \tilde{M}$ with the diagonal metric $G = g \times \tilde{g}$ and the three-form $p^* H - \tilde{p}^* \tilde{H}$, where $p : \mathbb{M} \to M$ and $\tilde{p} : \mathbb{M} \to \tilde{M}$ are the obvious projections. Because $M$ and $\tilde{M}$ are understood as the physical spaces of the T-dual theories, we call $\mathbb{M}$ the **extended spacetime** (or extended geometry/space). The geometry of $\mathbb{M}$ can be more intricate and in particular will not be globally a product $M \times \tilde{M}$ but, locally, it will (at least for the cases we study) always have this form. The manifolds $M$ and $\tilde{M}$ will also be recovered from $\mathbb{M}$ in a more subtle way, for example by taking certain quotients.

Typically, there will be a free action by dual tori[1] $T^d$ and $\tilde{T}^d$ on $M$ and $\tilde{M}$, respectively. If the action is also transitive, then $M = T^d$ and $\tilde{M} = \tilde{T}^d$, but in general $M$ and $\tilde{M}$ are only

[1] By dual torus we mean a torus defined by the dual lattice.

torus fibrations:

$$\begin{array}{ccc} T^d & \hookrightarrow & M \\ & & \downarrow{\scriptstyle p} \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{T}^d & \hookrightarrow & \tilde{M} \\ & & \downarrow{\scriptstyle \tilde{p}} \\ & & \tilde{B} \end{array} .$$

When $B = \tilde{B}$, we can take $\mathbb{M} = M \times_B \tilde{M}$ to be the fiber product and understand T-duality as the exchange of the torus fibres. This geometric scenario is typical of **topological T-duality** [4, 5] and a similar picture also arises in **SYZ mirror symmetry** [6]. In general, we call this **abelian T-duality** because the group actions on $M$ and $\tilde{M}$ are abelian. This picture can be generalized to the case of a free action of non-abelian groups, carrying the name **non-abelian** [7] or **Poisson-Lie T-duality** [8, 9]. In this case, the duality between the tori is replaced by the requirement that the Lie algebras of the groups $G$ and $\tilde{G}$ acting on $M$ and $\tilde{M}$, respectively, are dual as vector spaces.

We will now illustrate how para-Hermitian and Born geometries arise in the simplest possible T-duality scenario, where both $M$ and $\tilde{M}$ are circles $S^1$ and $\mathbb{M} = S^1 \times S^1 = T^2$ is the two-dimensional torus.

## 1.1 Example: Born geometry toy model

Consider a string theory on a circle $S^1_R$ with radius $R$, i.e. a theory of maps from a Riemann surface into the circle $S^1_R$. One of the important physical quantities one might wish to calculate is the center of mass energy of the string sitting in this circle. Because the string is allowed to wrap around the circle multiple times, the energy depends on an integer $m$ that counts this, called the *winding number*. Of course, the energy also depends on a momentum $p$, which must take only discrete values because the space is circular and is proportional to $n/R$, where $n$ is an integer called the momentum number and $R$ is the radius of the circle. Calculating the total energy, one gets with appropriate choice of physical constants,

$$E^2 \sim (mR)^2 + \left(\frac{n}{R}\right)^2 + \cdots,$$

where $(\cdots)$ represents terms independent of $m, n$ and $R$. Here we can observe the simplest incarnation of T-duality: the energy of the string is invariant under the exchange $(m, n, R) \leftrightarrow (n, m, \frac{1}{R})$. This tells us that a string propagating on a circle with a radius $R$, momentum number $n$ and winding number $m$ will have the exact same energy as a string propagating on a circle with a radius $R' = 1/R$, momentum number $m$ and winding number $n$.

We will now aim to formulate this feature geometrically, i.e. look for the unifying extended geometry from which both T-dual pictures can be recovered as in (1.0.1). To do so, we interpret the winding number $m$ attached to the circle $S^1_R$ as a momentum along some *dual* circle with radius $R' = 1/R$. In this way, we are replacing a string wrapped $m$ times around the circle $S^1_R$ and propagating with a momentum number $n$ by a string propagating on the *extended space* $\mathbb{S} = S^1_R \times S^1_{R'}$ with momentum numbers $(n,m)$ in each of the circular directions. Note that we can recover the two T-dual pictures from $\mathbb{S}$ equally well by interpreting one of the circles as the space-time direction with momenta tangent to it, while identifying the momenta along the other circle as the winding modes.

The extended space $\mathbb{S} = S^1_R \times S^1_{R'}$ of the toy model is in fact the first instance of Born geometry and we will now use it to illustrate the defining properties of this geometry. First, we recall the fact that while the momenta along $S^1_R$ are represented by vectors, the winding modes around $S^1_R$ are given by covectors. Because T-duality exchanges winding modes around $S^1_R$ and momenta along $S^1_{R'}$ (and vice versa), it can be written as a map

$$\begin{aligned} \mathrm{T}:\ & TS^1_R \oplus T^*S^1_R \to T^*S^1_{R'} \oplus TS^1_{R'} \\ & (\partial_\theta, d\theta) \mapsto (d\tilde{\theta}, \partial_{\tilde{\theta}}), \end{aligned}$$

where $(\theta, \tilde{\theta})$ are coordinates on $S^1_R \times S^1_{R'}$. From our previous discussion it is clear that T-duality also maps in the opposite direction and we denote this map by T as well. This defines a metric $\eta$ on $\mathbb{S}$ by

$$\eta(X,Y) = \langle \mathrm{T}X, Y\rangle,$$

where $\langle\ ,\ \rangle$ is the duality pairing between $T\mathbb{S}$ and $T^*\mathbb{S}$ and $X,Y$ are vector fields on $\mathbb{S}$. T acts on $X$ as $TX = T\mathbf{x} + T\tilde{\mathbf{x}}$, where $X = \mathbf{x} + \tilde{\mathbf{x}}$ is the splitting into components tangent to $S^1_R$ and $S^1_{R'}$, respectively. The metric $\eta$ is of signature $(1,1)$ and satisfies

$$\eta(\partial_\theta, \partial_\theta) = \eta(\partial_{\tilde{\theta}}, \partial_{\tilde{\theta}}) = 0 \quad \text{and} \quad \eta(\partial_\theta, \partial_{\tilde{\theta}}) = \eta(\partial_{\tilde{\theta}}, \partial_\theta) = 1.$$

We also observe that there is a natural endomorphism $K \in \Gamma(\mathrm{End}(T\mathbb{S}))$, defined by

$$K\partial_\theta = \partial_\theta \quad \text{and} \quad K\partial_{\tilde{\theta}} = -\partial_{\tilde{\theta}}, \tag{1.1.1}$$

and satisfying

$$K^2 = \mathbb{1}, \quad \eta(KX, KY) = -\eta(X, Y). \tag{1.1.2}$$

The pair $(\eta, K)$ defines a *para-Hermitian structure* on $\mathbb{S}$, which can be seen as the basic building block of the Born geometry on $\mathbb{S}$. In order to get the full Born geometry, we define a metric $g$ on $S^1_R$ by

$$g(\partial_\theta, \partial_\theta) = 1,$$

which in turn defines a metric $\tilde{g}$ on $S^1_{R'}$:

$$\tilde{g}(\partial_{\tilde{\theta}}, \partial_{\tilde{\theta}}) = g^{-1}(\mathrm{T}\partial_{\tilde{\theta}}, \mathrm{T}\partial_{\tilde{\theta}}).$$

The pair $(g, \tilde{g})$ then defines a diagonal Riemannian metric $\mathcal{H}$ on $\mathbb{S}$ by $\mathcal{H} = g \oplus \tilde{g}$. In summary, the Born geometry on $\mathbb{S}$ is given by the triple $(\eta, K, \mathcal{H})$, which in the coordinate frame $(\partial_\theta, \partial_{\tilde{\theta}})$ is given by the matrices

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These structures satisfy a wealth of compatibility conditions characteristic for Born geometry. In particular, there is a symplectic structure $\omega = \eta^{-1}K$, a complex structure $I = \eta^{-1}\omega$ and a chiral involution $J = \eta^{-1}\mathcal{H}$, which satisfy the *para-quaternionic algebra*

$$-I^2 = J^2 = K^2 = \mathbb{1}, \quad \{I, J\} = \{J, K\} = \{I, K\} = 0, \quad IJK = -\mathbb{1}.$$

Even though this toy example is very simple and non-physical due to only one compact spatial direction, the main intuitive idea remains the same in more complicated geometric settings where there are both compact and non-compact spatial directions and T-duality acts only on certain cycles of the manifold. The corresponding geometry is then a fibration where T-duality only acts on the fibres and the radius of the circle is generalized to a measure given by a fiber metric $g$. One can also consider a non-zero NS-NS flux given by a closed three-form $H$ which is locally specified by a two-form $b$ called the $b$-field and satisfying $H = \mathrm{d}b$. If one then performs T-duality as an exchange of two fiber directions (with coordinates labelled by $x$ and $\tilde{x}$), the components of $g$ and $b$ are transformed according to the Buscher rules (due to T. H. Buscher [10, 11]) and new quantities $\tilde{g}$ and $\tilde{b}$ are obtained:

$$\begin{aligned} \tilde{g}_{\tilde{x}\tilde{x}} &= \frac{1}{g_{xx}} & \tilde{b}_{\tilde{x}\mu} &= \frac{g_{x\mu}}{g_{xx}} \\ \tilde{g}_{\tilde{x}\mu} &= \frac{b_{x\mu}}{g_{xx}} & \tilde{b}_{\mu\nu} &= b_{\mu\nu} + \frac{g_{\tilde{x}\mu} b_{x\nu} - b_{x\mu} g_{x\nu}}{g_{xx}} \\ \tilde{g}_{\mu\nu} &= g_{\mu\nu} + \frac{b_{\tilde{x}\mu} b_{x\nu} - g_{x\mu} g_{x\nu}}{g_{xx}} \end{aligned} \tag{1.1.3}$$

where the directions labelled by Greek letters are not acted upon by T-duality. We observe that when $b = 0$ and the metric $g$ is diagonal, the Buscher rules tell us that the metric component in the T-dualized direction is replaced by the inverse metric, which is in accordance to our initial toy model example where the change in geometry was represented by the replacement $R \to 1/R$.

The purpose of this work is to generalize the above example to more complicated settings and explore the properties of Born geometry in detail, relating them to various applications in physics, both in string theory and beyond.

The thesis is organised as follows. In Chapter 2, we introduce para-Hermitian geometry, which is the most important building block of Born geometry because it describes the fundamental T-duality splitting. As such, it should be thought of as the background or kinematical component of Born geometry. It is defined by a triple $(\mathbb{M}, \eta, K)$, where $\mathbb{M}$ is an even-dimensional manifold, $K$ is a tangent bundle endomorphism satisfying $K^2 = +\mathbb{1}$ and $\eta$ is a split-signature metric. $K$ defines a para-complex structure on $\mathbb{M}$, whose two eigenbundles define as splitting of the tangent bundle, and $\eta$ captures the duality between the two eigenbundles of $K$. The two dual eigenbundles are then physically understood as directions tangent to mutually T-dual *physical* and *winding* (local) directions. A special case of this geometry, called para-Kähler geometry, arises when the non-degenerate two-form $\omega = \eta K$ is closed. We provide several examples of para-Hermitian manifolds, showing how the action of T-duality is realized on them. We also relate our description of T-duality in terms of para-Hermitian geometry to that of Bouwknegt, Evslin and Mathai, called *topological T-duality*. We then introduce a differentiable structure on $\mathbb{M}$ that is adapted to the para-Hermitian T-duality splitting and is given by the D-bracket, a bracket operation on the vector fields of the manifold. We conclude the chapter with the definition of para-Calabi-Yau manifolds, which are a subclass of para-Kähler manifolds for which the holonomy group of the Levi-Civita connection of $\eta$ is contained in $SL(\dim(\mathbb{M})/2)$.

In Chapter 3, we introduce Born geometry, which is equivalent to adding an extra metric structure on top of the existing background para-Hermitian data on $\mathbb{M}$, and therefore defines the dynamical structure on the manifold. Another part of this dynamical structure is given by a unique canonical connection, called the Born connection, which parallelizes all the defining structures of Born geometry and is also compatible with the D-bracket of the underlying para-Hermitian manifold. This connection is therefore the Born geometry analogue of the Levi-Civita connection in Riemannian geometry and, as we show, it also reduces to the Levi-Civita connection of the spacetime manifold inside the extended space $\mathbb{M}$. In the last section of this chapter, we present a new example of Born geometry arising in the context of semi-flat mirror symmetry, which in this case provides a mirror map between Born structures on the total spaces of the tangent and the cotangent bundles of an affine manifold. We also show that the mirror map relates the symplectic moduli space of the tangent bundle to *both* the complex and para-complex moduli spaces of the cotangent bundle and vice versa.

Chapters 4 and 5 are dedicated to the relationship between para-Hermitian and Born geometries and the framework of generalized geometry, which studies geometric structures on the bundle $(T \oplus T^*)M$. In Chapter 4, we first review two building blocks of generalized geometry, Dirac geometry and generalized structures. Then, we present basic facts about generalized para-Kähler and generalized chiral structures, which are examples of commuting pairs of generalized structures, and are the generalized-geometric versions of para-Hermitian and Born geometry, respectively. Lastly, we study how generalized Bismut connections can be used to study the integrability of the commuting pairs of generalized structures.

Finally, in the Chapter 6, we discuss applications of the geometric structures discussed in this thesis beyond T-duality, concretely to nonlinear two-dimensional supersymmetric $\sigma$-models. There, both generalized para-Kähler and generalized chiral geometries correspond to additional symmetries of the said $\sigma$-models.

## 1.2 Literature overview

Most of the material and ideas presented in this thesis previously appeared in other works and we will review these here. Most importantly, the presentation of extended geometry through the formalism of para-Hermitian and Born manifolds was developed in the works of the author in collaboration with Freidel and Rudolph. Section 2.4 about the D-bracket on para-Hermitian manifolds is based on [12, 13, 14] and most of the Chapter 3, particularly the discussion on Born geometry is taken from [14]. A review of this portion of the thesis from conference proceedings can be also found in [15]. These works build on the original ideas of Vaisman [16, 17] and have been developed further in the works of Chatzistavrakidis, Jonke, Marotta, Pezzella, Szabo, Vitale, and others [18, 19, 20, 21]. From the physics point of view, the idea of doubling the spacetime directions to accommodate T-duality originates in the work of Tsytlin and Siegel [22, 23, 24, 25] and then in the seminal works on Double Field Theory of Hohm, Hull and Zwiebach [1, 26, 27]. A comprehensive overview with additional references can be found in [28, 29]. Born geometry was in this context first studied by Freidel, Leigh and Minic in [2, 3], where the name was also coined. See also Boulter's masters thesis [30], where examples of Born structures on compact complex surfaces are given along with important mathematical results.

The interpretation of para-Hermitian and Born geometry in terms of generalized geometry in Chapter 4 follows almost exclusively the author's work with Hu and Moraru [31]. A lot of the present ideas are analogous to the generalized complex and generalized Kähler geometry of Gualtieri [32, 33]. A good source of information about generalized geometry are also lecture notes found on Gualtieri's homepage [34].

Finally, some of the discussion in Chapter 5 is taken from [13] and from [31]. The ideas presented in Chapter 6 were mostly presented in this form in [31] and some will also appear in joint work with Williams [35]. The relationship between para-Hermitian geometry and para-supersymmetry was first observed be Abou-Zeid and Hull in [36]. The relationship between chiral geometry and the splitting of the $(1,1)$ superconformal algebra was studied by Stojevic in [37].

# Chapter 2

# Para-Hermitian Geometry

We now introduce the main mathematical building block of Born geometry, which is para-Hermitian geometry. As we will see, para-Hermitian geometry naturally arises in the description of a phenomenon called T-duality, appearing in physics and in particular string theory, that relates two a priori different physical theories, which are then called *T-dual*. The geometric picture we will invoke here is that of Double Field Theory (DFT) and Metastring Theory, which consider an *extended space* $\mathbb{M}$ that is locally decomposed into two sets of canonical directions. These directions allow for two descriptions that correspond to the two T-dual theories: In one of them, half of the directions are assigned the physical interpretation of the *space of positions* and the other half is understood as the *space of winding modes*, while in the other, the roles are exchanged. Geometrically, the action of T-duality is therefore realized as the exchange of the canonical directions.

Para-Hermitian geometry $(\mathbb{M}, K, \eta)$ is given by an even-dimensional manifold $\mathbb{M}$, a tangent bundle endomorphism $K$ satisfying $K^2 = \mathbb{1}$ and a split-signature metric $\eta$. This triple describes the extended space as follows. The eigenbundles of the endomorphism $K$ corresponding to the $\pm 1$-eigenvalues distinguish the mutually T-dual directions within the extended space, on the level of the tangent bundle. On the level of the manifold $\mathbb{M}$ itself, the interpretation is more subtle. The simplest scenario arises when both the eigenbundles are integrable, in which case there exist two complimentary foliations of $\mathbb{M}$, which we denote $M$ and $\tilde{M}$. Locally, this translates to the existence canonical local coordinates, parametrizing the physical and winding directions, on every patch of $\mathbb{M}$. However, in typical physical settings, at least one of the eigenbundles is not integrable, and the intuitive global description in terms of two foliations cannot be recovered. If at least one of the foliations is still present, the physical space can be identified with the leaf space of this foliation. In this thesis, this quotient approach will not be discussed in a great depth, but an interested reader may consult [21], where a significant progress in this direction has been made.

Therefore, $K$ fixes what we call the **T-duality frame** of the tangent bundle and consequently also determines the local splitting of $\mathbb{M}$. The metric $\eta$ then captures the fact that the physical and winding tangent directions are linearly dual to one another, giving the $2d$-dimensional extended space an $O(d,d)$ structure. Hence, the defining data $(\eta, K)$ is a geometric repackaging of the information we have about the T-duality set-up and should be understood as a *background* or *kinematic* structure of the extended geometry. The only extra piece of data one needs to define Born geometry is then a choice of a metric structure on $M$, which in turn defines a metric on $\tilde{M}$ (and vice versa). This metric, along with a choice of a compatible connection, is from the physical point of view understood as the *dynamical* component of the geometry.

In what follows, we will lay out basic definitions and properties of para-Hermitian geometry illustrated on various examples, mostly taken from the literature on T-duality. In the remainder of the section, we will discuss a construction of a new differentiable structure on para-Hermitian manifolds: the D-bracket. This bracket operation on vector fields of the extended spacetime is necessary for the applications to T-duality because of the fact that vectors tangent to the physical and winding directions $M$ and $\tilde{M}$, respectively, behave as duals of each other and we would like the bracket operation to respect this property. We will see that the D-bracket is uniquely fixed by the underlying para-Hermitian structure and in later sections we will discuss deformations of this construction in the presence of a $B$- and $\beta$-field, which give rise to a twisting of the bracket by geometric and non-geometric fluxes.

## 2.1 Para-Complex Geometry

As the name suggests, para-Hermitian geometry is closely related to Hermitian geometry, with the crucial difference being that the underlying structure is not complex but para-complex. Instead of the Hermitian pair $(I, g)$, where $I$ is a complex structure and $g$ a compatible metric (i.e. $I$ is an isometry of $g$), we consider a pair $(K, \eta)$, where $K$ is a para-complex structure that is an anti-isometry of the metric $\eta$ as in (1.1.2). Therefore, we first discuss para-complex geometry and then continue by adding the para-Hermitian metric $\eta$ to the picture.

**Definition 2.1.1.** *Let $E \to \mathbb{M}$ be a vector bundle. A* ***para-complex structure*** *on $E$ is a product structure, i.e. a fiber-wise linear endomorphism $K \in \Gamma(\mathrm{End}(E))$ that satisfies $K^2 = \mathbb{1}_E$, whose $+1$ and $-1$ eigenbundles, denoted $L$ and $\tilde{L}$, respectively, have the same rank. An almost para-complex manifold is a manifold equipped with a para-complex structure on its tangent bundle.*

*Remark.* In the context of T-duality, we refer to the splitting of the tangent bundle of a para-Hermitian manifold $T\mathbb{M} = L \oplus \tilde{L}$ as the **T-duality frame**.

An (almost) para-complex manifold is therefore a special case of an (almost) product manifold such that the two real eigenbundles have the same rank. A direct consequence of this is that any almost para-complex manifold is even-dimensional. The use of the word *almost* as usual refers to integrability of the endomorphism and is used for para-complex structures on the tangent bundle, where integrability can be defined in terms of the Lie bracket. That is, we omit the word almost, or call the structure $K$ *integrable* for emphasis, whenever its eigenbundles are involutive under the Lie bracket and therefore each define a foliation of the underlying manifold. In that case, the base manifold $\mathbb{M}$ is called para-complex and denoted $(\mathbb{M}, K)$. The Frobenius integrability condition can be expressed in terms of the **Nijenhuis tensor** $N_K$:

$$\begin{aligned} N_K(X,Y) &\coloneqq [X,Y] + [KX,KY] - K([KX,Y] + [X,KY]) \\ &= (\nabla_{KX}K)Y + (\nabla_X K)KY - (\nabla_{KY}K)X - (\nabla_Y K)KX \\ &= 4(P[\tilde{P}X,\tilde{P}Y] + \tilde{P}[PX,PY]), \end{aligned} \tag{2.1.1}$$

which vanishes if and only if $K$ is integrable. Here, $X, Y \in \Gamma(T\mathbb{M})$, $\nabla$ is any torsionless connection and

$$P \coloneqq \frac{1}{2}(\mathbb{1} + K), \quad \text{and} \quad \tilde{P} \coloneqq \frac{1}{2}(\mathbb{1} - K), \tag{2.1.2}$$

are projections onto the $\pm1$-eigenbundles. From (2.1.1), it is apparent that $K$ is integrable if and only if *both* its eigenbundles are simultaneously Frobenius integrable (that is, involutive distributions in $T\mathbb{M}$); the integrability of one of the eigenbundles is, however, not tied to the integrability of the other. This is one of the main differences between complex geometry and para-complex geometry: while in the complex case the eigenbundles are complex bundles related by complex conjugation, here the eigenbundles are real and therefore one can be integrable while the other is not. We call this phenomenon **half-integrability**. More on this can be found for example in [12, 13] or in [38], where examples of half-integrable para-complex structures motivated by physics are given. We should also emphasize here that in most of the physical applications the para-Hermitian structure is **not fully integrable**.

*Notation.* In the following we will use the following notation for the splitting of a vector field $X \in \Gamma(T\mathbb{M})$ into its components in $L$ and $\tilde{L}$:

$$X = \mathbf{x} + \tilde{\mathbf{x}}, \quad \mathbf{x} = P(X) \in \Gamma(L), \;\; \tilde{\mathbf{x}} = \tilde{P}(X) \in \Gamma(\tilde{L}). \tag{2.1.3}$$

We conclude this introductory portion with the simplest example of a para-complex manifold, which is the product of two $d$-dimensional manifolds:

**Example 2.1.2** (Manifold product)**.** Let $M$ and $\tilde{M}$ be two $d$-dimensional manifolds. Then $\mathbb{M} = M \times \tilde{M}$ is a para-complex manifold. The eigenbundles $L$ and $\tilde{L}$ of the (integrable) endomorphism $K$ are defined, over every point $(p, \tilde{p}) \in \mathbb{M} = M \times \tilde{M}$, by setting

$$L_{(p,\tilde{p})} = T_p M \quad \text{and} \quad \tilde{L}_{(p,\tilde{p})} = T_{\tilde{p}} \tilde{M}.$$

In particular, any smooth manifold $M$ gives rise to a para-complex manifold $\mathbb{M} = M \times M$. ◁

### 2.1.1 Adapted coordinates and the Dolbeault complex

Let now $(\mathbb{M}, K)$ be an almost para-complex manifold. If $K$ is integrable, a local neighborhood $\mathbb{U} \subset \mathbb{M}$ locally splits as $\mathbb{U} = U \times \tilde{U}$ with corresponding set of $2n$ coordinates $(x^i, \tilde{x}_i)$ called **adapted coordinates**, with respect to which $K$ satisfies (see for example [39, 40])

$$dx^i \circ K = dx^i, \quad \text{and} \quad d\tilde{x}_i \circ K = -d\tilde{x}_i. \tag{2.1.4}$$

Therefore, a para-complex structure – similarly to a complex structure – can be equivalently specified either by an integrable endomorphism $K$, or by a choice of adapted coordinates $(x^i, \tilde{x}_i)$ on every neighborhood. These coordinates must then transform on patch overlaps as

$$(x^i, \tilde{x}_i) \mapsto (y^j(x^i), \tilde{y}_j(\tilde{x}_i)). \tag{2.1.5}$$

Even when $K$ is not integrable, the splitting of the tangent bundle $T\mathbb{M} = L \oplus \tilde{L}$ gives rise to a decomposition of tensors analogous to the $(p,q)$-decomposition in almost complex geometry. Denote $\Lambda^{(k,0)}(T^*\mathbb{M}) \coloneqq \Lambda^k(L^*)$ and $\Lambda^{(0,k)}(T^*\mathbb{M}) \coloneqq \Lambda^k(\tilde{L}^*)$. The splitting is then

$$\Lambda^k(T^*\mathbb{M}) = \bigoplus_{k=p+q} \Lambda^{(p,q)}(T^*\mathbb{M}), \tag{2.1.6}$$

with corresponding sections denoted as $\Omega^{(p,q)}(\mathbb{M})$. The bigrading (2.1.6) then yields the natural projections

$$\Pi^{(p,q)} : \Lambda^k(T^*\mathbb{M}) \to \Lambda^{(p,q)}(T^*\mathbb{M}).$$

When $K$ is integrable, the de-Rham differential splits as $\mathrm{d} = \partial + \tilde{\partial}$, where

$$\begin{aligned} \partial &\coloneqq \Pi^{(p+1,q)} \circ \mathrm{d} \\ \tilde{\partial} &\coloneqq \Pi^{(p,q+1)} \circ \mathrm{d}, \end{aligned}$$

are the **para-complex Dolbeault operators**, acting on forms as

$$\begin{aligned} \partial &: \Omega^{(p,q)}(\mathbb{M}) \to \Omega^{(p+1,q)}(\mathbb{M}) \\ \tilde{\partial} &: \Omega^{(p,q)}(\mathbb{M}) \to \Omega^{(p,q+1)}(\mathbb{M}), \end{aligned} \tag{2.1.7}$$

such that when $K$ is integrable, we have

$$\partial^2 = 0, \quad \tilde{\partial}^2 = 0, \quad \text{and} \quad \partial\tilde{\partial} + \tilde{\partial}\partial = 0.$$

We also introduce the *twisted differential*:

**Definition 2.1.3.** *Let* $(\mathbb{M}, K)$ *be a paracomplex manifold. The* **twisted differential** $\mathrm{d}^P$ *is defined for an arbitrary* $k$*-form* $\alpha$ *by*

$$\mathrm{d}^P\alpha \coloneqq (\Lambda^{k+1}K) \circ \mathrm{d} \circ (\Lambda^k K)\alpha,$$

*where* $\Lambda^k K$ *denotes the* $k$*-the exterior power of the endomorphism* $K$*:*

$$(\Lambda^k K)\alpha(\overbrace{\cdot, \,\cdots, \cdot}^{k \text{ entries}}) = \alpha(K\cdot, \,\cdots, K\cdot).$$

The twisted differential can be simply expressed in terms of the Dolbeault operators in the following way:

**Lemma 2.1.4.** *Let* $(\mathbb{M}, K)$ *be a paracomplex manifold with* $\partial$ *and* $\tilde{\partial}$ *the para-complex Dolbeault operators and* $\mathrm{d}^P$ *the twisted differential. Then the following identity holds*

$$\mathrm{d}^P = \partial - \tilde{\partial}. \tag{2.1.8}$$

*Proof.* Let $\alpha \in \Omega^{(p,q)}(\mathbb{M})$ with $p + q = k$. Then we have

$$\mathrm{d}^P \alpha = (-1)^q (\Lambda^{p+q+1} K) \mathrm{d}\alpha = (-1)^{2q} \partial\alpha + (-1)^{2q+1} \tilde{\partial}\alpha = (\partial - \tilde{\partial})\alpha.$$

□

### 2.1.2 Foliations of a para-complex manifold

When $(\mathbb{M}, K)$ is a para-complex manifold of dimension $2d$, the distributions $L$ and $\tilde{L}$ define $d$-dimensional foliations of $\mathbb{M}$, which we call $M$ and $\tilde{M}$ and are defined by the property that they integrate $L$ and $\tilde{L}$, respectively:

$$T_p M = L_p \quad \text{and} \quad T_p \tilde{M} = \tilde{L}_p,$$

for all points $p \in \mathbb{M}$. We call such foliations the **fundamental foliations** of the para-complex manifold $\mathbb{M}$.

For the definition of foliation we are using, we refer the reader to [41, Def. 1.1].

*Notation.* Throughout this thesis, we may equivalently refer to the pair $(\mathbb{M}, K)$ by the ordered triple $(\mathbb{M}, M, \tilde{M})$, which explicitly specifies the fundamental foliations.

Each of the two foliations can be understood as a decomposition of $\mathbb{M}$ into *leaves*, which are immersed submanifolds of $\mathbb{M}$. Since all the leaves have the same dimension (because the eigenbundles have constant rank), the foliation is called **regular**. We therefore get a set of $d$-dimensional submanifolds $M_i \subset \mathbb{M}$ such that, for every point $p \in \mathbb{M}$, there is a unique $M_i$ passing through this point. Typically, there are infinitely many such submanifolds. We therefore have

$$M = \bigcup_i M_i, \quad \text{and} \quad \tilde{M} = \bigcup_j \tilde{M}_j,$$

where the indices $i$ and $j$ are running over the (potentially uncountable) sets labelling the individual leaves.

Both $M$ and $\tilde{M}$ are therefore disjoint unions of $d$-dimensional manifolds and can also be given a topology with respect to which they are themselves d-dimensional manifolds (see [41] or [42]), called the *leaf topology*. For $M$, this is done by considering a subset $U \subseteq M$ to be open if and only if $U$ is open in $M_i$ for some $i$. In this topology, each $M_i$ are in particular open subsets in $M$. The topology of $\tilde{M}$ is defined analogously. Because every point $p \in \mathbb{M}$ lies on exactly one leaf of $M$ and also on exactly one leaf of $\tilde{M}$, we see that as sets of points, all three manifolds $\mathbb{M}$, $M$ and $\tilde{M}$ are the same, but the topology is different for each $\mathbb{M}$, $M$ and $\tilde{M}$. As a result, the topology of the foliations never has a countable basis and typically has uncountably many connected components.

For this reason, we will understand foliations in terms of their individual leaves, which are immersed submanifolds of $\mathbb{M}$. This will become useful for example in Section 5.1, where we will discuss morphisms between bundles over $\mathbb{M}$ and $M$ or $\tilde{M}$. Interested reader may also consult [21], where foliations and quotients in the context of para-Hermitian geometry are discussed in more detail.

**Example 2.1.5** ($\mathbb{R}^2$)**.** Take the manifold $\mathbb{R}^2$ with the standard coordinates $(x,y)$. Then

$$K\partial_x = \partial_x \quad \text{and} \quad K\partial_y = -\partial_y,$$

defines a para-complex structure and the two corresponding foliations are simply the horizontal lines $y = const.$ for $L$ and the vertical lines $x = const.$ for $\tilde{L}$. Both the para-complex structure and the foliations also descend to the torus $T^2$, which we get by identifying $(x,y) \sim (x+1,y) \sim (x,y+1)$, where the leaves become circles. ◁

### 2.1.3 Recovering the physical space

From the physics point of view, we wish to understand $\mathbb{M}$ as the extended manifold, which in particular describes simultaneously the two T-dual manifolds of half the dimension. For this reason, we must geometrically describe how such manifolds are recovered from $\mathbb{M}$. The para-complex splitting $T\mathbb{M} = L \oplus \tilde{L}$ gives rise to two different ways of how to achieve this – either by the integration map, which assigns to $\mathbb{M}$ the pair of $d$-dimensional foliations $M$ and $\tilde{M}$, or by the quotient map. In the case of the torus $T^2$ described in Example 2.1.5, both $M$ and $\tilde{M}$ are sets of circles parametrized by a circle, while the quotient map recovers in both cases just a single circle. Physically, it is natural for the physical space to have only one connected component, and for the winding space to have the structure of a fiber bundle, i.e. a space of windings attached to every point of the spacetime. Therefore, from an intuitive point of view, the quotient map $\mathbb{M} \to \mathbb{M}/\tilde{M}$ recovers a geometry more suitable for a spacetime description, while the integration map $\mathbb{M} \to \tilde{M}$ is more suitable for the dual, winding description. Note that, T-duality – which exchanges what is seen as the physical space and what is understood as the winding space – switches the two pictures. Note also that for both the space-time $M_{phys.} = \mathbb{M}/\tilde{M}$ and the winding space $\tilde{M}$, only $\tilde{L}$ must be integrable, while for the T-dual picture we need the integrability of $L$. Such description of the physical and winding spaces is used in many physical examples where only one of the eigenbundles is integrable, arising especially in the presence of fluxes. For more in-depth discussion on the space-time interpretations of the extended space see for example [2, 3, 21].

A large class of (almost-)para-complex manifolds for which only one of the eigenbundle is generically integrable and which are not globally given by a product of two manifolds, are fiber bundles:

**Example 2.1.6** (Fiber bundle)**.** Let $\mathbb{M} \xrightarrow{\pi} M$ be a fiber bundle with $d$-dimensional fibres over a $d$-dimensional manifold $M$ and consider the following exact sequence of vector bundles over $\mathbb{M}$:

$$0 \longrightarrow V = \mathrm{Ker}(\pi_*) \longrightarrow T\mathbb{M} \longrightarrow \pi^* TM \longrightarrow 0,$$

where $V$ is called the *vertical* distribution of $\pi$ and maps into $T\mathbb{M}$ by inclusion. A splitting of the above exact sequence amounts to a choice of an Ehresmann connection, i.e. a choice of a *horizontal* subbundle $H \subset T\mathbb{M}$, such that $T\mathbb{M} = H \oplus V$. One then obtains an almost para-complex structure $K$ on $\mathbb{M}$ defined by

$$K\,|_H = \mathbb{1}, \quad \text{and} \quad K\,|_V = -\mathbb{1}.$$

While the distribution $V$ is always integrable and the integral foliation is given by the fibres of $\mathbb{M} \to M$, the distribution $H$ is in general not integrable and its obstruction to integrability can be taken as the definition of the *curvature* of the chosen connection. In other words, the para-complex structure $K$ is always half-integrable and full integrability is equivalent to a choice of a flat Ehresmann connection. ◁

### 2.1.4 Para-holomorphic functions and bundles

We will now discuss the para-holomorphic structure of para-complex manifolds, and give important examples of para-holomorphic vector bundles. As usual, a map of para-complex manifolds is called para-holomorphic if its pushforward commutes with the respective para-complex structures:

**Definition 2.1.7.** *Let* $(\mathbb{M}, K_{\mathbb{M}})$ *and* $(\mathbb{N}, K_{\mathbb{N}})$ *be para-complex manifolds. A map* $F : \mathbb{M} \to \mathbb{N}$ *is called* **para-holomorphic** *if*

$$K_{\mathbb{N}} \circ F_* = F_* \circ K_{\mathbb{M}}. \tag{2.1.9}$$

In the following we will omit the prefix "para-" in para-holomorphic whenever no confusion with complex holomorphicity is possible. Locally, the map $F : \mathbb{M} \to \mathbb{N}$ of para-complex manifolds can be understood via local coordinates as

$$\begin{aligned} F &: \mathbb{U} \to \mathbb{V} \\ F &= (f^i, \tilde{f}_j) = (y^i(x^k, \tilde{x}_l), \tilde{y}_j(x^k, \tilde{x}_l))^{i,j=1,\cdots,n}_{k,l=1,\cdots,m}, \end{aligned}$$

where $(x^k, \tilde{x}_l)$ and $(y^i, \tilde{y}_j)$ are adapted local coordinates on $\mathbb{U} \subset \mathbb{M}$ and $\mathbb{V} \subset \mathbb{N}$, respectively. It is easy to check from (2.1.9) that $F$ is a holomorphic map if and only if the components satisfy the para-complex Cauchy-Riemann equations

$$\frac{\partial}{\partial \tilde{x}_i} f^j = \frac{\partial}{\partial x^i} \tilde{f}_j = 0. \tag{2.1.10}$$

The conditions (2.1.10) tell us that the holomorphic function $F$ is given by a pair $(f, \tilde{f})$ of functions between the fundamental foliations of the para-complex manifolds $\mathbb{M}$ and $\mathbb{N}$. This also means that the transition functions on a para-complex manifold (2.1.5) are holomorphic since the foliations $M$ and $\tilde{M}$ must be preserved.

A holomorphic vector bundle $E \xrightarrow{\pi} \mathbb{M}$ over the para-complex manifold $(\mathbb{M}, K)$ is then defined analogously to complex geometry as a para-complex vector bundle whose total space is a para-complex manifold with $\pi$ a holomorphic map. The form of the transition functions (2.1.5) then gives us an intuition of what holomorphic bundles and their holomorphic sections look like. For example, the tangent bundle $T\mathbb{M}$ is itself a holomorphic bundle (see for example

[40, 31]); this is simply because the tangent bundle is glued together on patch overlaps by the pushforwards $T_{x\to y}$ of the transition functions (2.1.5):

$$T_{x\to y} = \begin{pmatrix} \frac{\partial y^j(x)}{\partial x^i} & 0 \\ 0 & \frac{\partial \tilde{y}_j(\tilde{x})}{\partial \tilde{x}^i} \end{pmatrix},$$

which defines a holomorphic structure on $T\mathbb{M}$ with local canonical coordinates[1] $(dx^i, d\tilde{x}_j)$ on the fibres. The holomorphic sections of $T\mathbb{M}$ – the holomorphic vector fields – are locally of the form

$$X = X^i(x)\partial_i + \tilde{X}_i(\tilde{x})\tilde{\partial}^i,$$

i.e. the components in $L$ and $\tilde{L}$ individually define vector fields on the foliations $M$ and $\tilde{M}$, respectively. By a similar argument, the cotangent bundle is also a holomorphic bundle. The following example illustrates the intuition behind the local structure of holomorphic bundles and their sections

**Example 2.1.8.** Let $E \to M$ and $\tilde{E} \to \tilde{M}$ be arbitrary vector bundles over $n$-dimensional manifolds $M$ and $\tilde{M}$ and let $s : M \to E$ and $\tilde{s} : \tilde{M} \to \tilde{E}$ be smooth sections. Then $\mathbb{E} = p^* E \oplus \tilde{p}^* \tilde{E} \to M \times \tilde{M}$, where $p : M \times \tilde{M} \to M$ and $\tilde{p} : M \times \tilde{M} \to \tilde{M}$ are the natural projections, is a holomorphic vector bundle over the para-complex manifold $(\mathbb{M} = M \times \tilde{M}, K)$ of Example 2.1.2 and $\mathbf{s} = p^* s + \tilde{p}^* \tilde{s}$ is a holomorphic section of $\mathbb{E}$. ◁

## 2.2 Para-Hermitian Geometry

We now introduce a metric $\eta$ compatible with the para-complex structure $K$, which by contraction $\eta \circ K$ induces a non-degenerate two-form $\omega$. When $\omega$ is closed, we get para-Kähler geometry.

**Definition 2.2.1.** *Let $(\mathbb{M}, K)$ be a para-complex manifold and let $\eta$ be a pseudo-Riemannian metric that satisfies $\eta(K\cdot, K\cdot) = -\eta$. Then we call $(\mathbb{M}, K, \eta)$ a* ***para-Hermitian manifold***[2]*. The* **fundamental form** *$\omega$ of a para-Hermitian manifold is a tensor defined by the contraction $\omega \coloneqq \eta K$.*

[1] Here, $dx^i$ and $d\tilde{x}_j$ are understood as fiber-wise linear functions $T\mathbb{M}\,|_p \to C^\infty(\mathbb{M})$ for every $p \in \mathbb{M}$.

[2] If $K$ is not integrable, i.e. $(\mathbb{M}, K)$ is almost para-complex, we would call $(\mathbb{M}, K, \eta)$ an almost para-Hermitian manifold.

The above definition implies that $\omega$ is skew, i.e. for any $X, Y \in \mathfrak{X}(\mathbb{M})$, we have

$$\omega(X,Y) = \eta(KX,Y) = -\eta(X,KY) = -\omega(Y,X).$$

$\omega$ is also nondegenrate (because $\eta$ is nondegenerate), which makes it an almost symplectic form, sometimes called the **fundamental form**. From $K^2 = \mathbb{1}$ we also have $K = \eta^{-1}\omega = \omega^{-1}\eta$. Another observation is that since the eigenbundles of $K$ have the same rank, $\eta$ has neutral (or split) signature. Furthermore, the eigenbundles of $K$ are isotropic with respect to both $\eta$ and $\omega$. This means that the almost symplectic form $\omega$ is of type $(1,1)$: $\omega \in \Omega^{(1,1)}(\mathbb{M})$.

*Remark.* As shown above, the data $(\mathbb{M}, K, \eta)$, $(\mathbb{M}, \eta, \omega)$ and $(\mathbb{M}, K, \omega)$ are equivalent on a para-Hermitian manifold and so we use the different triples interchangeably to refer to a para-Hermitian manifold. Additionally, we may again replace $K$ by the pair of fundamental foliations $M$ and $\tilde{M}$ as described in Section 2.1.

**Example 2.2.2** (Local structure)**.** Almost para-Hermitian structures all look the same locally. Indeed, let $(\mathbb{M}, K, \eta)$ be a $2d$-dimensional almost para-Hermitian manifold. Bejan then shows [43] that there exist local frames of $T\mathbb{M}$ with respect to which

$$K = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

or

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{2.2.1}$$

where the matrix blocks have dimension $d$. The second choice of frame that diagonalizes $K$ is called **adapted** because it corresponds to the splitting of the tangent bundle $T\mathbb{M} = L \oplus \tilde{L}$ into the eigenbundles of $K$; we denote such frame by $(e_i, \tilde{e}^i)$ and the dual frame by $(e^i, \tilde{e}_i)$:

$$\begin{gathered} \langle e^i, e_j \rangle = e^i(e_j) = \delta^i_j, \quad \langle \tilde{e}_i, \tilde{e}^j \rangle = \tilde{e}_i(\tilde{e}^j) = \delta^j_i. \\ \langle e^i, \tilde{e}_j \rangle = \langle \tilde{e}^i, e_j \rangle = 0. \end{gathered} \tag{2.2.2}$$

◁

The structure group of a para-Hermitian manifold is the para-unitary group, isomorphic to $Gl(d)$:

$$pU(2d) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \middle| \; A \in Gl(d) \right\}, \tag{2.2.3}$$

and since it preserves both $\eta$ and $\omega$, we have $pU(2d) = GL(d) = O(d,d) \cap Sp(2d)$.

A para-Hermitian vector bundle is a para-complex vector bundle $(E, K)$ endowed with a split fibre metric $\eta$, with respect to which the para-complex structure $K$ is anti-orthogonal. The following is an important example of such vector bundle, which will frequently appear in this work.

**Example 2.2.3.** (Extended tangent bundle) Consider the *extended tangent bundle* $(T \oplus T^*)M$ over an arbitrary manifold $M$. We get a constant, linear para-Hermitian structure on every fibre, given by the following pair $(K, \eta)$:

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} T \\ T^* \end{pmatrix}, \quad \text{and} \quad \eta(X + \alpha, Y + \beta) \coloneqq \langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X),$$

where $X + \alpha$ and $Y + \beta$ are sections of $(T \oplus T^*)M$ and $\langle \ , \ \rangle$ is the duality pairing. $\triangleleft$

### 2.2.1 Para-Kähler Manifolds

We will now summarize some important properties of a special class of para-Hermitian manifolds, para-Kähler manifolds.

**Definition 2.2.4.** *Let* $(\mathbb{M}, \eta, \omega)$ *be a para-Hermitian manifold with* $\mathrm{d}\omega = 0$. *We call* $(\mathbb{M}, \eta, \omega)$ *a* ***para-Kähler manifold****.*

*Remark.* As a consequence of the compatibility $\omega(K\cdot, K\cdot) = -\omega(\cdot,\cdot)$, the eigenbundles $L$ and $\tilde{L}$ are Lagrangian with respect to $\omega$. Therefore, a para-Kähler manifold $(\mathbb{M}, \eta, \omega)$ can be seen as a symplectic manifold with a preferred choice of Lagrangian distributions $L$ and $\tilde{L}$, the unique Lagrangians of $\omega$ isotropic with respect to $\eta$. Such symplectic manifolds are called bi-Lagrangian. For more details, see [44, 45].

**Lemma 2.2.5.** *Let* $(\mathbb{M}, K, \eta)$ *be an almost para-Hermitian manifold. Then* $(\mathbb{M}, K, \eta)$ *is para-Kähler if and only if* $\mathring{\nabla} K = 0$ *(or equivalently* $\mathring{\nabla}\omega = 0$*), where* $\mathring{\nabla}$ *is the Levi-Civita connection of* $\eta$*.*

*Proof.* The idea of the proof is entirely in parallel to the analogous statement in complex geometry, see for example [46, Theorem 5.5]. □

The structure of para-Kähler manifolds is, similarly to Kähler manifolds, locally given only in terms of a real, smooth function:

**Proposition 2.2.6.** *Let $(\mathbb{M}, K, \eta)$ be a para-Kähler manifold. Then for every point $p \in \mathbb{M}$, there exists a smooth real function $f$ in a neighborhood $U$ around $p$, such that $\omega = \partial\tilde{\partial} f$.*

*Proof.* By the Poincaré lemma, there exists a neighborhood $V$ around $p$ and a one-form $\alpha$ such that $\omega = d\alpha$. Furthermore, because $\omega$ is of type $(1,1)$, the components $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$ satisfy

$$\omega = \tilde{\partial}\alpha^{(1,0)} + \partial\alpha^{(0,1)}, \quad \text{and} \quad \partial\alpha^{(1,0)} = \tilde{\partial}\alpha^{(0,1)} = 0.$$

This further implies (by a local exactness of the para-complex Dolbeault operators, see [39]) that $\alpha^{(1,0)} = \partial u$ and $\alpha^{(0,1)} = \tilde{\partial} v$ for some $u, v \in C^\infty(U)$, where $U$ is a neighborhood around $p$, possibly smaller than $V$. Finally, defining $f \coloneqq v - u$ we get

$$\omega = \tilde{\partial}\alpha^{(1,0)} + \partial\alpha^{(0,1)} = \partial\tilde{\partial}(-u + v) = \partial\tilde{\partial} f.$$

□

### 2.2.2 The Canonical Connection

On any almost para-Hermitian manifold, one can define the *canonical connection* which will play an important role in our constructions later on:

**Definition 2.2.7.** *Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold and $\mathring{\nabla}$ the Levi-Civita connection of $\eta$. We define the **canonical connection** $\nabla^c$ by*

$$\eta(\nabla^c_X Y, Z) = \eta(\mathring{\nabla}_X Y, Z) - \frac{1}{2}\mathring{\nabla}_X \omega(Y, KZ). \tag{2.2.4}$$

*Remark.* The canonical connection appears in [47], where the authors introduce a class of para-Hermitian connections $\nabla^t$ parametrized by $t \in \mathbb{R}$. This class also includes the Chern connection and the Bismut connection of a para-Hermitian manifold. The canonical connection is given by $\nabla^{t=0}$ and all connections in this class degenerate to the canonical connection on a class of para-Hermitian manifolds called *nearly para-Kähler*.

The canonical connection is a para-Hermitian connection (i.e. $\nabla^c\eta = \nabla^c\omega = 0$). This implies that $\nabla^c$ preserves the eigenbundles of $K$:

$$\nabla^c \mathbf{y} \in \Gamma(L) \quad \text{and} \quad \nabla^c \tilde{\mathbf{y}} \in \Gamma(\tilde{L}),$$

where $\mathbf{y} \in \Gamma(L)$ and $\tilde{\mathbf{y}} \in \Gamma(\tilde{L})$. This property becomes obvious when $\nabla^c$ is rewritten in the following form:

$$\nabla^c Y = \nabla^c(\mathbf{y} + \tilde{\mathbf{y}}) = P\mathring{\nabla}\mathbf{y} + \tilde{P}\mathring{\nabla}\tilde{\mathbf{y}}. \tag{2.2.5}$$

From (2.2.4) we also note that when $(\mathbb{M}, \eta, K)$ is para-Kähler, we get $\nabla^c = \mathring{\nabla}$.

We note here that para-Kähler manifolds whose metric $\eta$ is flat will locally have adapted coordinates such that the forms of $\eta$ and $K$ reduce to (2.2.1):

**Proposition 2.2.8.** *Let $(\mathbb{M}, \eta, K)$ be a para-Kähler manifold. Then $\eta$ is a flat metric if and only if there locally exists an adapted coordinate system $(x^i, \tilde{x}_i)$ in which $(K, \eta, \omega)$ take the form*

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

*Proof.* This is a direct consequence of [48, Thm. 1.2.1], see also [44, Thm. 14], because bi-Lagrangian is equivalent to para-Kähler and $\mathring{\nabla}$ coincides with the bi-Lagrangian connection of the bi-Lagrangian manifold (see [44]). □

### 2.2.3 Examples of para-Hermitian manifolds

We now present basic examples of (almost) para-Hermitian and para-Kähler manifolds.

**Example 2.2.9** (Doubled Tori)**.** The simplest example comes from a straightforward generalization of the toy model given in the introduction. Consider a $d$-dimensional torus $T^d = \mathbb{R}^d/\Lambda$ and its dual, $(T^d)^* = (\mathbb{R}^d)^*/(\Lambda)^*$, $(\Lambda)^*$ being the lattice dual to $\Lambda$. Then $\mathbb{M} = T^d \times (T^d)^*$ is para-Kähler with the para-complex structure given explicitly by the global product structure (see Example 2.1.2) and the $O(d,d)$ pairing is induced by the duality between the two lattices. ◁

**Example 2.2.10** (Tangent bundles)**.** We will now study a special case of the Example 2.1.6 when the fiber bundle is the tangent bundle of a manifold, $\mathbb{M} = TM \xrightarrow{\pi} M$, and augment this by a compatible para-Hermitian metric. Consider a choice of a linear connection $\nabla$ on $TM$ that defines the splitting $T\mathbb{M} = TTM = H \oplus V$ to the horizontal and vertical subbundles of $T\mathbb{M}$, giving rise to a half-integrable para-complex structure $K$. If $\{x^i\}_{i=1\cdots n}$ are local coordinates on $M$, $\{v^i\}_{i=1\cdots n}$ the fiber coordinates and $\Gamma^k_{ij} = (\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})^k$ are the connection coefficients, then $H$ is spanned by the vector fields:

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij} v^j \frac{\partial}{\partial v^k},$$

and $V$ is simply spanned by the vertical vector fields $V^i = \frac{\partial}{\partial v^i}$. The dual frame is then given by the one-forms $(dx^i, \tau_i)$, where

$$\tau^i = dv^i + \Gamma^i_{jk} v^k dx^j.$$

We also get two maps from the tangent bundle of the base $TM$ into $V$ and $H$ called the **horizontal** and **vertical lift**:

$$\begin{aligned} \textbf{Vertical lift:} \quad & v: \ TM \to V : X \mapsto X^v, \quad X^v[\alpha] \coloneqq \alpha(X) \circ \pi, \\ \textbf{Horizontal lift:} \quad & h: \ TM \to H : X \mapsto X^h, \quad X^h[\alpha] \coloneqq \nabla_X \alpha, \end{aligned}$$

where $\alpha$ and $\nabla_X \alpha$ are one-forms on $M$ regarded as functions on $TM$. Now, choose a Riemannian metric $g$ on $M$. This allows us to define an $O(d,d)$ metric $\eta$ on $\mathbb{M} = TM$ by

$$\eta(X^h, Y^v) = g(X,Y), \quad \text{and} \quad \eta(X^h, Y^h) = \eta(X^v, Y^v) = 0.$$

Explicitly, this is

$$K = dx^i \otimes H_i + \tau^i \otimes \frac{\partial}{\partial v_i}, \quad \text{and} \quad \eta = g_{ij}(dx^i \otimes \tau^j + \tau^i \otimes dx^j).$$

Clearly, $(\eta, K)$ are compatible and define a half integrable para-Hermitian structure on $\mathbb{M}$. In [12], it is shown that whenever $\nabla$ is torsionless, $(\eta, K)$ is almost para-Kähler, i.e. $\omega = \eta K$ is closed. Therefore, if one takes for example $\Gamma$ to be the Levi-Civita connection, the above construction yields a half-integrable para-Kähler structure which is fully integrable whenever $g$ is a flat metric. ◁

**Example 2.2.11** (Cotangent bundles)**.** We will now define a para-Hermitian structure compatible with the para-complex structure on a cotangent bundle $\mathbb{M} = T^*M$ induced by a choice of a connection as in Example 2.1.6. Here we will consider a much more general construction of the para-complex structure $K$ than in the Example 2.2.10, because we will consider a generic Ehresmann connection, i.e. any splitting of $T(T^*M) = H \oplus V$ without any assumption on the local connection coefficients. Such para-Hermitian structures were described in [18, 21].

We start by choosing a local Darboux chart $(q^i, p_i)$ on $T^*M$. Then $V$ is locally spanned by the vectors $v^i = \frac{\partial}{\partial p_i}$ and $H$ is spanned by $h_i = \frac{\partial}{\partial q^i} + C_{ij} v^j$ for some local coefficients $C_{ij}$. In the special case when the connection is chosen to be a linear connection $\nabla$ with connection coefficients $\Gamma^i_{jk} = (\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k})^i$, we have $C_{ij} = \Gamma^k_{ij} p_k$.

In general, the coefficients $C_{ij}$ can be arbitrary and when they are not linear in the coordinates $p$, the connection is called non-linear. The canonical symplectic form $\omega_0 = dp_i \wedge dq^i$ is compatible with the para-complex structure $K$ defined by the splitting $T\mathbb{M} = H \oplus V$ if and only if $C_{ij} = C_{ji}$ [18, 21]. Therefore, when this condition is satisfied, the Ehresmann connection, which is uniquely determined by the collection of local coefficients $\{C_{ij}\}$, defines a para-Kähler (since $\mathrm{d}\omega_0 = 0$) structure $(\omega_0, K)$ if and only if $C_{ij}$ are symmetric functions. $\triangleleft$

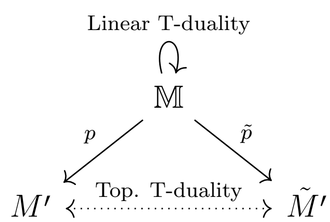
In Section 3.3, we will present a different way of constructing para-Kähler structures on the tangent and cotangent bundle of an affine manifold.

**Example 2.2.12** (Drinfel'd doubles)**.** There is a natural para-Hermitian structure on every (classical) Drinfel'd double, defined as a $2d$-dimensional Lie group $\mathbf{D}$ whose Lie algebra $\mathfrak{d}$ splits as $\mathfrak{d} = \mathfrak{l} \bowtie \tilde{\mathfrak{l}}$ into two dual subalgebras $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ [49]. The duality between $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ gives rise to a signature $(d, d)$ invariant pairing on $\mathfrak{d}$, with respect to which $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ are isotropic, and the triple $(\mathfrak{d}, \mathfrak{l}, \tilde{\mathfrak{l}})$ is called a *Manin triple* on $\mathfrak{d}$. Because there is a split-signature pairing on $\mathfrak{d}$ together with a pair of maximally isotropic subspaces, $\mathfrak{d}$ – the tangent fiber at the identity of $\mathbf{D}$ – has the structure of a para-Hermitian vector space. Therefore, a Manin triple is equivalent to a para-Hermitian structure on $\mathfrak{d}$. Now, the fact that $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ are Lie subalgebras means that there exist Lie subgroups $\mathbf{G}$ and $\tilde{\mathbf{G}}$ such that $\mathbf{D} = \mathbf{G} \bowtie \tilde{\mathbf{G}}$, which shows that there is in fact a global para-Hermitian structure on the whole $\mathbf{D}$ with the fundamental foliations given by $\mathbf{G}$ and $\tilde{\mathbf{G}}$. Explicit details of this construction can be found in [18]. There, it is shown for example that such a para-Hermitian structure is para-Kähler if and only if both $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are abelian. $\triangleleft$

## 2.3 T-duality and physical interpretation

The main reason why para-Hermitian geometry is important in the context of Born geometry is that it describes a geometric setting that facilitates T-duality. In this work we will distinguish between the **linear** T-duality and the T-duality of the **underlying manifolds**. The latter is the correspondence between two T-dual manifolds $M'$ and $\tilde{M}'$, while the former is the corresponding linear map on bundles – typically the tangent and cotangent bundles and their direct sum – over the extended, para-Hermitian manifold $\mathbb{M}$, from which $M'$ and $\tilde{M}'$ can be recovered as discussed in Section 2.1.3. $M'$ and $\tilde{M}'$ are therefore not necessarily the fundamental foliations $M$ and $\tilde{M}$ of $\mathbb{M}$, but can be for example the quotients $M' = \mathbb{M}/\tilde{M}$ and $\tilde{M}' = \mathbb{M}/M$.

The manifold $\mathbb{M}$ should therefore be understood as a *correspondence space* between $M'$ and $\tilde{M}'$, in the sense that it maps into each of the two manifolds, even though there might be no explicit map between them:




(2.3.1)

### 2.3.1 Linear T-duality

Let us assume for clarity of the notation that the conditions of Proposition 2.2.8 are satisfied (i.e. $\mathbb{M}$ is para-Kähler and $\eta$ is flat) so that the adapted frame and its dual are given by $(\partial_i, \tilde{\partial}^i)$ and $(dx^i, d\tilde{x}_i)$. The linear T-duality is a fiberwise linear map that acts as an exchange of tangent vectors of $M$ with cotangent vectors of $\tilde{M}$ [50], which can be locally expressed in the adapted frames as

$$\begin{aligned} \mathrm{T}:\ & TM \oplus T^*M \to T^*\tilde{M} \oplus T\tilde{M} \\ & (\partial_i, dx^j) \mapsto (d\tilde{x}_i, \tilde{\partial}^j). \end{aligned}$$

This is equivalent to the existence of the metric $\eta$, which is defined using T and the natural duality pairing $\langle\ ,\ \rangle$ between $T\mathbb{M}$ and $T^*\mathbb{M}$:

$$\eta(\mathbf{x}, \tilde{\mathbf{y}}) \coloneqq \langle \mathrm{T}\mathbf{x}, \tilde{\mathbf{y}} \rangle = \langle \mathbf{x}, \mathrm{T}\tilde{\mathbf{y}} \rangle, \quad \eta(\mathbf{x}, \mathbf{y}) \coloneqq \langle \mathrm{T}\mathbf{x}, \mathbf{y} \rangle = 0 = \eta(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \coloneqq \langle \mathrm{T}\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \rangle, \tag{2.3.2}$$

where we used the notation (2.1.3) and denote the inverse of T by the same symbol.

Via $\eta$, the directions tangent to $M$ and $\tilde{M}$, respectively, are then naturally dual to each other, i.e. in the adapted frame we have

$$\eta(\partial_i, \tilde{\partial}^j) = \delta_i^j \quad \text{and} \quad \eta(\partial_i, \partial_j) = \eta(\tilde{\partial}^i, \tilde{\partial}^j) = 0.$$

This is also the reason for our choice of the index notation, denoting the vectors tangent to $M$ and $\tilde{M}$ by $\partial_i$ and $\tilde{\partial}^i$, respectively. We see that the duality between $\partial_i$ and $\tilde{\partial}^i$ is a consequence of the T-duality between $M$ and $\tilde{M}$ and the choice of $\eta$ on the para-complex manifold $\mathbb{M}$ is equivalent to the choice of the T-duality map T. The linear T-duality map T can then be repackaged as an endomorphism of $(T \oplus T^*)\mathbb{M}$

$$\mathrm{T} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \in \Gamma(\mathrm{End}((T \oplus T^*)\mathbb{M})), \tag{2.3.3}$$

which, as we will see in Section 4.2.4, defines a *generalized metric* on the para-Hermitian manifold.

**Linear T-duality as an $O(d,d)$ transformation** We will now describe the action of the T-duality map (2.3.3) on the para-Hermitian structure $(\eta, K)$. First, we note that $K$ naturally acts on $(T \oplus T^*)\mathbb{M}$ via its diagonal action, denoted as K:

$$\mathrm{K} = \begin{pmatrix} K & 0 \\ 0 & K^* \end{pmatrix}.$$

The action of the linear T-duality map T is then straightforward to compute, $\mathrm{T}(\mathrm{K}) = \mathrm{T}\mathrm{K}\mathrm{T}^{-1} = \mathrm{T}\mathrm{K}\mathrm{T} = -\mathrm{K}$. Therefore, the action of T-duality on $K$ itself is simply given by

$$K \overset{\mathrm{T}}{\longmapsto} -K.$$

In order to find out how T-duality acts on $\eta$, we simply recall the definition of $\eta$ via T

(2.3.2) and the fact that $\mathrm{T}^2 = \mathbb{1}$:

$$\eta(\mathrm{T}\cdot, \mathrm{T}\cdot) = \langle \mathrm{T}^2\cdot, \mathrm{T}\cdot\rangle = \langle \cdot, \mathrm{T}\cdot\rangle = \eta(\cdot,\cdot).$$

Therefore, T-duality is an isometry of $\eta$, making it an element of the $O(d,d)$ group. In DFT, this is typically taken as one of the defining properties of T-duality transformations and in [21], **all** $O(d,d)$ transformations on a given para-Hermitian manifold are called **generalized T-duality transformations**. In Section 5.3, we will discuss another important $O(d,d)$ transformation, the B-transformation.

### 2.3.2 T-duality of underlying manifolds

T-duality of the underlying manifolds $M'$ and $\tilde{M}'$ is given by the correspondence via $\mathbb{M}$ (2.3.1) and in particular the maps $\mathbb{M} \overset{p}{\to} M'$ and $\mathbb{M} \overset{\tilde{p}}{\to} \tilde{M}'$. In our case, $\mathbb{M}$ is an (almost-) para-Hermitian manifold and $p$ and $\tilde{p}$ are either the quotient maps,

$$p : \mathbb{M} \to \mathbb{M}/\tilde{M} = M', \quad \text{and} \quad \tilde{p} : \mathbb{M} \to \mathbb{M}/M \simeq \tilde{M}', \tag{2.3.4}$$

or the integration maps

$$p : \mathbb{M} \to M, \quad \text{and} \quad \tilde{p} : \mathbb{M} \to \tilde{M}. \tag{2.3.5}$$

When all four of the above maps exist, T-duality acts on the para-Hermitian manifold $\mathbb{M}$ as the exchange of the ordered triples

$$(\mathbb{M}, M, \tilde{M}) \leftrightarrow (\mathbb{M}, \tilde{M}, M), \tag{2.3.6}$$

exchanging $p$ and $\tilde{p}$ in both (2.3.4) and (2.3.5), and consequently also $M'$ and $\tilde{M}'$.

When $\mathbb{M}$ is globally a product of the two T-dual manifolds $\mathbb{M} = M' \times \tilde{M}'$, the relationship between the two pictures (2.3.4) and (2.3.5) is clear. In such case, the quotients yield $M'$ and $\tilde{M}'$ and the fundamental foliations are of the form $M = \bigcup_{\tilde{p}\in\tilde{M}'} M' \times \{\tilde{p}\}$ and $\tilde{M} = \bigcup_{p\in M'} \tilde{M}' \times \{p\}$. Therefore, $M$ is the union of copies of $M'$ labelled by points in $\tilde{M}$, and similarly for $\tilde{M}$ and $\tilde{M}'$. Because $\mathbb{M}$ is always a product locally, we always have such description at least in a local sense.

Note that the linear T-duality is weaker than the T-duality of the underlying manifolds in the sense that the linear T-duality identifies the T-dual directions (in our case, the eigenbundles of the para-Hermitian structure $K$) only on the level of the bundles on $\mathbb{M}$, but there might not always be good notions of what the T-dual manifolds $M$ and $\tilde{M}$ are, for example when the para-Hermitian structure is not integrable. Such scenarios arise in string theory, where they correspond to so-called *non-geometric backgrounds*. Those are situations when the linear T-duality identifies directions T-dual to $M'$ within $\mathbb{M}$ for which there is no corresponding T-dual manifold $\tilde{M}'$ [18, 21, 38]. However, those situations are still physically perfectly valid and even though $\tilde{M}'$ does not exist as a smooth manifold (in [51, 52], this object is called a *T-fold*), $\mathbb{M}$ is still a good geometric description of the full doubled space. One can even consider a scenario where none of the para-Hermitian eigenbundles are integrable, in which case there exists no description in terms of a half-dimensional "physical space" geometry and $\mathbb{M}$ is then called an *essential doubling* [21].

### 2.3.3 The Full and Partial T-duality

So far, we have been describing T-duality on a para-Hermitian manifold $(\mathbb{M}, M, \tilde{M}, \eta)$ as the exchange between the two foliation manifolds $M$ and $\tilde{M}$. In that case, we call the T-duality **full**, because the whole manifold $\mathbb{M}$ locally splits only to the two sets of T-dual directions. Typically, the full T-duality arises from a free and transitive action of the Drinfel'd double $\mathbf{D} = \mathbf{G} \bowtie \tilde{\mathbf{G}}$ (see Example 2.2.12), giving rise to a Poisson-Lie T-duality:

**Example 2.3.1** (Poisson-Lie T-duality). The para-Hermitian structures of Example 2.2.12 give rise to a notion of T-duality called the Poisson-Lie T-duality[3] [8, 9]. There, the quotient maps are $\mathbf{D} \xrightarrow{p} \mathbf{D}/\tilde{\mathbf{G}} = \mathbf{G}$ and $\mathbf{D} \xrightarrow{\tilde{p}} \mathbf{D}/\mathbf{G} = \tilde{\mathbf{G}}$. The two can be also related by linear T-duality, which acts on the level of the Lie algebra $\mathfrak{d}$ as the exchange between the Manin triples $(\mathfrak{d}, \mathfrak{l}, \tilde{\mathfrak{l}})$ and $(\mathfrak{d}, \tilde{\mathfrak{l}}, \mathfrak{l})$, i.e. the mapping $K \mapsto -K$. $\triangleleft$

In many situations, it is nevertheless desirable to consider **partial T-duality**, which in its linear form acts as an exchange of only subspaces of the tangent and cotangent bundles of the base manifold. The geometric setting needed for the partial T-duality is a pair $M \to B$ and $\tilde{M} \to B$ of fiber bundles over some common base $B$ and typical fibres $M'$ and $\tilde{M}'$, respectively, such that each fiber of the fiber product $\mathbb{M} = M \times_B \tilde{M}$ is a para-Hermitian manifold. This means that over every $b \in B$, the fiber $M' \times \tilde{M}'$ carries a non-degenerate split-signature symmetric pairing $\eta$ compatible with the para-complex structure defined by

[3] This name stems from the fact that in this picture both $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are Poisson-Lie groups.

the product, so that $(M' \times \tilde{M}', \eta)$ is para-Hermitian:

$$\begin{array}{ccc} M' \times \tilde{M}' & \hookrightarrow & \mathbb{M} \\ & & \downarrow \\ & & B \end{array} \quad .$$

The partial T-duality then acts as the exchange of the fibres of $M$ and $\tilde{M}$, i.e. as a full T-duality on every fiber $M' \times \tilde{M}'$.

The full manifold $\mathbb{M}$ could also be a para-Hermitian manifold, but only the fibres are necessarily para-Hermitian because that is where the T-duality that ensures the existence of the para-Hermitian metric via (2.3.2) acts. After all, in the most typical string theory scenario, the string spacetime $M$ is given by a *compactification* of the form $M = B \times M'$, where $B$ is a 4-dimensional Minkowski spacetime, $M'$ is some compact manifold and the T-duality acts only on $M'$. Denoting the T-dual to $M'$ by $\tilde{M}'$, the full extended manifold for this T-duality scenario is therefore $\mathbb{M} = B \times M \times \tilde{M}'$ and so the para-Hermitian manifold $M' \times \tilde{M}'$ is the compact subsector of the full string theory geometry $\mathbb{M}$.

An example of partial T-duality on a manifold which is para-Hermitian also away from the T-dualized directions is given for example by the doubled torus:

**Example 2.3.2** (Doubled torus)**.** Consider the para-Hermitian doubled torus $\mathbb{M} = T^d \times (T^d)^*$ from Example 2.2.9. There, T-duality is simply a choice of a $d$-dimensional torus $\hat{T} \subset \mathbb{M}$, called a *polarization*. The two extremal choices $\hat{T} = T^d$ and $\hat{T} = (T^d)^*$ correspond to the two fully T-dual pictures and the way we recover $\hat{T}$ is as a quotient by the complementary torus action, i.e. $T^d = \mathbb{M}/(T^d)^*$ and $(T^d)^* = \mathbb{M}/T^d$. One can also consider a partial T-duality for example in one direction, viewing $\mathbb{M}$ as the fibration

$$\begin{array}{ccc} T^2 & \hookrightarrow & \mathbb{M} \\ & & \downarrow \\ & & T^{d-1} \times (T^{d-1})^* \end{array} \quad ,$$

in which case the linear T-duality is no longer given by (2.3.3), but by the **partial T-duality**

**matrix** along the torus $T^2$

$$\mathrm{T}_{T^2} = \left(\begin{array}{cc|cc} \mathbb{1}_{(d-1)\times(d-1)} & & & \\ & 0 & & 1 \\ \hline & & \mathbb{1}_{(d-1)\times(d-1)} & \\ & 1 & & 0 \end{array}\right), \tag{2.3.7}$$

which now defines a para-Hermitian pairing via (2.3.2) only on the $T^2$-fibres where $\mathrm{T}_{T^2}$ acts non-trivially. ◁

We conclude by describing how T-duality can be understood on the tangent and cotangent bundles.

**Example 2.3.3** (Tangent and cotangent bundles)**.** In Examples 2.2.10 and 2.2.11, we described half-integrable para-Hermitian and para-Kähler structures on the tangent and cotangent bundles of an arbitrary manifold, $\mathbb{M} = TM$ and $\mathbb{M} = T^*M$. There, because the structures are generally only half-integrable, T-duality can be understood as a correspondence between the foliation of $\mathbb{M}$ by the fibres $\tilde{M}$ and the corresponding leaf space given by the original manifold $M \simeq \mathbb{M}/\tilde{M}$. Therefore, in this case $p$ and $\tilde{p}$ in (2.3.1) are given by the quotient map $p : \mathbb{M} \overset{/\tilde{M}}{\to} M$ and the integration map $\tilde{p} : \mathbb{M} \overset{|_{\tilde{M}}}{\to} \tilde{M}$.

Another option, more typically employed in the literature on T-duality, is to consider the bundle $\mathbb{M} = (T \oplus T^*)M$ with the obvious para-Hermitian structure on the fibres (Example 2.2.3). The T-duality is then fibre-wise over every point in $M$ and exchanges the $TM$ and $T^*M$ dual fibres. We also employ this approach in Section 3.3, where the T-duality is given by a mirror map between the manifolds $TM$ and $T^*M$. ◁

### 2.3.4 Topological T-duality

We will now turn to perhaps the most studied case of T-duality, which is in the literature referred to as the *topological* T-duality [4, 5] and show how para-Hermitian geometry fits into this picture.

Let $M \overset{\pi}{\to} B$ and $\tilde{M} \overset{\tilde{\pi}}{\to} B$ be two principal torus bundles over the same base $B$ and denote by $H$ and $\tilde{H}$ closed 3-forms on $M$ and $\tilde{M}$, respectively. Consider the following diagram:

$$\begin{array}{ccccc} & & (\mathbb{M} = M \times_B \tilde{M}, p^* H - \tilde{p}^* \tilde{H}) & & \\ & \overset{p}{\swarrow} & & \overset{\tilde{p}}{\searrow} & \\ (M, H) & & & & (\tilde{M}, \tilde{H}) \\ & \underset{\pi}{\searrow} & & \underset{\tilde{\pi}}{\swarrow} & \\ & & B & & \end{array} \quad . \tag{2.3.8}$$

We say that $(M, H)$ and $(\tilde{M}, \tilde{H})$ are T-dual if there exists a two-form $\omega$ on $\mathbb{M}$ such that

- $\mathrm{d}\omega = p^* H - \tilde{p}^* \tilde{H}$,
- $\omega$ is invariant under both torus actions on $M$ and $\tilde{M}$,
- $(\omega_0)_i{}^j = \omega(\partial_{\theta^i}, \partial_{\tilde{\theta}_j})$ is a non-degenerate matrix, $\theta^i$ and $\tilde{\theta}_j$ being the coordinates on the dual torus fibres.

From the above definition we immediately see that we obtain a para-Hermitian structure $(\eta, K)$ on every fiber of $\mathbb{M} = M \times_B \tilde{M}$ from the above data: the para-complex structure simply acts as the identity in the directions of the $M$-fibres and negative identity in the directions of the $\tilde{M}$-fibres, and the split pairing is defined by $\eta = \omega_0 K$, where $\omega_0$ is the fundamental form defined by the coefficients $(\omega_0)_i{}^j = \omega(\partial_{\theta^i}, \partial_{\tilde{\theta}_j})$.

Generally, topological T-duality only defines a para-Hermitian fibration over the base $B$ and neither the correspondence space $M \times_B \tilde{M}$ nor the full product space $M \times \tilde{M}$ are endowed with a natural para-Hermitian structure. As such, topological T-duality is therefore an example of partial T-duality and one cannot a priori expect that the base directions of the fibration will be para-Hermitian or even even-dimensional.

**Example 2.3.4** (Topological T-duality on $S^3$)**.** Consider the topological T-duality setting (2.3.8), where $M$ is the three-sphere $S^3$ seen as the circle Hopf fibration over $S^2$

$$\begin{array}{ccc} S & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

with $H = 0$. It can be checked that according to (2.3.8), the T-dual $\tilde{M}$ of this circle bundle is the trivial circle bundle $S^2 \times S^1 \xrightarrow{\tilde{\pi}} S^2$ with $\tilde{H} = \sigma \wedge d\tilde{\theta}$, where $\sigma$ is the volume form on $S^2$.

If we parametrize $S^2$ by the angle coordinates $(\phi, \psi) \in [0, 2\pi] \times [0, \pi]$, we get $\sigma = d\phi \wedge d\psi$ and in this case $\omega = (d\theta - \phi d\psi) \wedge d\tilde{\theta}$ and $\omega_0 = d\theta \wedge d\tilde{\theta}$, $(\theta, \tilde{\theta})$ being the T-dualized circle coordinates. ◁

*Remark.* As we noted above, the correspondence space $\mathbb{M} = M \times_B \tilde{M}$ in Example 2.3.4 is a $T^2$-fibration over $S^2$ and by construction, it is not a para-Hermitian manifold, but rather a para-Hermitian fibration, i.e. the fiber over any point of the base $S^2$ is a para-Hermitian manifold.

As we have seen above, the relationship between topological T-duality and the full T-duality on para-Hermitian manifolds, which exchanges the whole of $M$ and $\tilde{M}$, is that the former degenerates to the latter when the base is a point:

$$\begin{array}{ccccc} & & \mathbb{M} = M \times \tilde{M} & & \\ & \swarrow^{p} & & \searrow^{\tilde{p}} & \\ M & & & & \tilde{M} \\ & \searrow_{\pi} & & \swarrow_{\tilde{\pi}} & \\ & & \{*\} & & \end{array} . \tag{2.3.9}$$

In this case, it is easy to see from the defining properties of topological T-duality (2.3.8) that both $H$ and $\tilde{H}$ must vanish as the property $p^* H - \tilde{p}^* \tilde{H} = \mathrm{d}\omega$ forces $H$ and $\tilde{H}$ to satisfy

$$H(\mathbf{x}, \mathbf{y}) = 0, \quad \text{and} \quad \tilde{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = 0,$$

where we used the notation (2.1.3). For $H$, this follows from $\omega$ being invariant and therefore $\mathrm{d}\omega(\mathbf{x}, \mathbf{y}) = 0$ and additionally $\tilde{p}^* \tilde{H}(\mathbf{x}, \cdot, \cdot) = 0$. For $\tilde{H}$, the argument is analogous.

However, there is still a way of including the $H$-flux (and other fluxes as well) in the T-duality setting on para-Hermitian manifolds (2.3.9) and we will discuss this in the Section 5.3.

## 2.4 Differentiable Structure and The D-bracket

We will now describe a new differentiable structure on a para-Hermitian manifold that arises from T-duality considerations. In the following, $(\mathbb{M}, \eta, K)$ is a para-Hermitian manifold with $(M, \tilde{M})$ the fundamental foliations and we will again assume that the conditions of

Proposition 2.2.8 are satisfied so that there locally exists a holonomic frame in which $\eta$ and $K$ are constant. This is merely for notational convenience and for making the connection with formulas appearing in physics (where this is usually the case) more explicit.

In the usual geometry, the differentiable structure of the manifold is encoded in the Lie bracket $[\ ,\ ]$ of the vector fields, which then extends to the Cartan calculus on the manifold, i.e Lie derivative $\mathcal{L}$, de-Rham differential d etc. This gives rise to a natural notion of integrability for different geometric structures, such as closedness of a symplectic form under d, integrability of a (complex) endomorphism defined by the Lie bracket or the Poisson condition $[\beta, \beta] = 0$ for a bi-vector in terms of the Schouten bracket. The differentiable structure also enters for example in the definition of the torsion tensor of a connection $\nabla$, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, which measures how the skew-symmetrization of the corresponding covariant derivative differs from the Lie bracket.

When describing the physics setting of T-duality in terms of a para-Hermitian manifold, the usual differentiable structure is, however, not an appropriate choice. This is because the Lie bracket does not respect the duality between physical and winding directions. We have seen that on para-Hermitian manifolds, T-duality relates the vector field $\tilde{\partial}^i$ to $dx^i$, and therefore $\tilde{\partial}^i$ should transform under the diffeomorphisms of $M$ as a one-form and this should be reflected in the action of a new bracket operation on vector fields on $\mathbb{M}$ that we will denote by $[\![\ ,\ ]\!]$.

We will now derive the form of the new bracket from physics heuristics. First, since the para-Hermitian splitting $T\mathbb{M} = L \oplus \tilde{L}$ defines the T-duality frame and both $L$ and $\tilde{L}$ are tangent to the physical directions in the two T-dual pictures, the bracket $[\![\ ,\ ]\!]$ should restrict to the usual Lie bracket on both $L$ and $\tilde{L}$:

$$\begin{aligned} [\![\mathbf{x}, \mathbf{y}]\!] &= [\mathbf{x}, \mathbf{y}] = (\mathbf{x}^i \partial_i(\mathbf{y}^j) - \mathbf{y}^i \partial_i(\mathbf{x}^j))\partial_j, \\ [\![\tilde{\mathbf{x}}, \tilde{\mathbf{y}}]\!] &= [\tilde{\mathbf{x}}, \tilde{\mathbf{y}}] = (\tilde{\mathbf{x}}_i \tilde{\partial}^i(\tilde{\mathbf{y}}_j) - \tilde{\mathbf{y}}_i \tilde{\partial}^i(\tilde{\mathbf{x}}_j))\tilde{\partial}^j, \end{aligned} \tag{2.4.1}$$

where we used the notation for vector fields on $\mathbb{M}$ and their splitting into components in $L$ and $\tilde{L}$ (2.1.3):

$$X = \mathbf{x} + \tilde{\mathbf{x}} = \mathbf{x}^i \partial_i + \tilde{\mathbf{x}}_j \tilde{\partial}^j, \quad X \in \Gamma(T\mathbb{M}),\ \mathbf{x} \in \Gamma(L),\ \tilde{\mathbf{x}} \in \Gamma(\tilde{L}).$$

Consider now the T-dual picture in which $L$ generates the translations along the physical space and $\tilde{L}$ represents the (linearly dual) winding directions. Because the sections of $\tilde{L}$ in this case represent one-forms, they should transform under the infinitesimal diffeomorphisms along $L$ accordingly:

$$[\![\mathbf{x},\tilde{\mathbf{y}}]\!] = (\mathbf{x}^i\partial_i(\tilde{\mathbf{y}}_j) + \tilde{\mathbf{y}}_i\partial_j(\mathbf{x}^i))\tilde{\partial}^j.$$

The analogous, T-dual statement, reads

$$[\![\tilde{\mathbf{x}},\mathbf{y}]\!] = (\tilde{\mathbf{x}}_i\tilde{\partial}^i(\mathbf{y}^j) + \mathbf{y}^i\tilde{\partial}^j(\tilde{\mathbf{x}}_i))\partial_j.$$

Putting these formulas together while using the capital index notation,

$$X = \mathbf{x}^i\partial_i + \tilde{\mathbf{x}}_j\tilde{\partial}^j = X^I\partial_I,$$

we get

$$[\![X,Y]\!] = \left(X^I\partial_I Y^J - Y^I\partial_I X^J + \eta_{IL}\eta^{KJ}Y^I\partial_K X^L\right)\partial_J, \tag{2.4.2}$$

which is a local coordinate expression for a bracket operation well-known in the literature of DFT under the name **D-bracket**. Note that we derived this expression for the simple case of a flat para-Kähler manifold. Our aim will now be to formalize the definition of the D-bracket for *any almost para-Hermitian manifold* and find its coordinate-free description.

We start with the formal definition of the D-bracket. The choice for the axioms might not be very intuitive at a first glance, but we will subsequently show that this definition corresponds well to the above discussion.

**Definition 2.4.1.** *Let $(\mathbb{M},\eta,K)$ be an almost para-Hermitian manifold. We define the* **D-bracket** *to be a bracket operation on the space of vector fields*

$$[\![\ ,\ ]\!] : \mathfrak{X}(\mathbb{M})\times\mathfrak{X}(\mathbb{M})\to\mathfrak{X}(\mathbb{M}),$$

*satisfying the following properties:*

1. *Leibniz property*

$$[\![X,fY]\!] = f[\![X,Y]\!] + X[f]Y,$$

2. *Compatibility with $\eta$*

$$X[\eta(Y,Z)] = \eta([\![X,Y]\!],Z) + \eta(Y,[\![X,Z]\!])$$
$$\eta(Y,[\![X,X]\!]) = \eta([\![Y,X]\!],X),$$

3. *Compatibility with $K$: vanishing generalized Nijenhuis tensor*

$$\begin{aligned}\mathcal{N}_K &= [\![X,Y]\!] + [\![KX,KY]\!] - K\left([\![KX,Y]\!] + [\![X,KY]\!]\right)\\ &= 4\left(P[\![\tilde{P}X,\tilde{P}Y]\!] + \tilde{P}[\![PX,PY]\!]\right) = 0.\end{aligned}$$

*4. Relationship with the Lie bracket*

$$\begin{aligned}[\![PX,PY]\!] &= P([PX,PY]),\\ [\![\tilde{P}X,\tilde{P}Y]\!] &= \tilde{P}([\tilde{P}X,\tilde{P}Y]),\end{aligned}$$

*for any* $X,Y,Z \in \mathfrak{X}(\mathbb{M})$ *and* $f \in C^\infty(\mathbb{M})$

Let us now verify that the above is a good definition of the D-bracket, i.e. that (2.4.2) satisfies conditions 1. – 4. of Definition 2.4.1.

**Lemma 2.4.2.** *Let* $(\mathbb{M},\eta,K)$ *be a para-Kähler manifold with* $\eta$ *a flat metric. Then* (2.4.2) *is a local expression for a D-bracket on* $(\mathbb{M},\eta,K)$ *in adapted coordinates, i.e. satisfies properties* 1. – 4. *of Definition 2.4.1.*

*Proof.* The property 1. is straightforward to verify and properties 3. and 4. are verified by (2.4.1). For property 2., we rewrite (2.4.2)

$$\eta([\![X,Y]\!],Z) = Z^K\eta_{JK}\left(X^I\partial_I Y^J - Y^I\partial_I X^J\right) + Z^K\eta_{IJ}\left(Y^I\partial_K X^J\right),$$

so that $\eta([\![Y,X]\!],X) = Y^K\eta_{IJ}(X^I\partial_K X^J) = \eta([\![X,X]\!],Y)$, verifying the second line of property 2. To verify the first line, we use the fact that the components of $\eta$ are constant:

$$\begin{aligned}X[\eta(Y,Z)] &= \eta_{IJ}X^K(Y^I\partial_K Z^J + Z^I\partial_K Y^J)\\ \eta([\![X,Y]\!],Z) + \eta(Y,[\![X,Z]\!]) &= Z^K\eta_{JK}\left(X^I\partial_I Y^J - Y^I\partial_I X^J\right) + Z^K\eta_{IJ}\left(Y^I\partial_K X^J\right)\\ &\quad + Y^K\eta_{JK}\left(X^I\partial_I Z^J - Z^I\partial_I X^J\right) + Y^K\eta_{IJ}\left(Z^I\partial_K X^J\right),\end{aligned}$$

where in the second expression only the first terms on each line survive after cancellations, proving the equality and completing the proof. □

We have therefore shown that the D-bracket exists at least on flat para-Kähler manifolds. It is not, however, immediately clear that such a bracket will exist on any almost para-Hermitian manifold and if it does exist, we would like to know if it is unique. The following statement answers both of these questions:

**Theorem 2.4.3.** *Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold and $\nabla^c$ be its canonical connection. Then the following expression defines a D-bracket:*

$$\eta([\![X, Y]\!], Z) \coloneqq \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y). \tag{2.4.3}$$

*Moreover, the D-bracket is unique.*

*Proof.* First, in order to check all the properties of the Definition 2.4.1, we simply use the definition of $\nabla^c$, particularly its form (2.2.5).

In order to prove uniqueness, we consider two D-brackets $[\![\ ,\ ]\!]$ and $[\![\ ,\ ]\!]'$ associated to the same almost para-Hermitian manifold $(\mathbb{M}, \eta, K)$, denoting their difference by

$$[\![X, Y]\!]' = [\![X, Y]\!] + \Delta(X, Y) \quad \text{and} \quad \Delta(X, Y, Z) = \eta(\Delta(X, Y), Z)$$

From the $\eta$-compatibility properties of the D-bracket (property 2. of Definition 2.4.1), it follows that $\Delta(X, Y, Z)$ is fully skew, while Leibniz property (property 1.) tells us that $\Delta$ is tensorial, meaning $\Delta$ is a three-form on $\mathbb{M}$. Furthermore, the relationships with the Lie bracket (property 4.) implies that $\Delta(PX, PY) = 0 = \Delta(\tilde{P}X, \tilde{P}Y)$, which in turn means that $\Delta(PX, PY, Z) = 0$ and $\Delta(\tilde{P}X, \tilde{P}Y, Z) = 0$ for all $Z \in \mathfrak{X}(\mathbb{M})$. Since $\Delta$ is a three-form, this concludes that $\Delta = 0$ identically. □

We note a useful form of the D-bracket (2.4.3) in terms of the Levi-Civita connection instead of the canonical connection, which follows from (2.2.4):

$$\begin{aligned} \eta([\![X, Y]\!], Z) &= \eta(\mathring{\nabla}_X Y - \mathring{\nabla}_Y X, Z) + \eta(\mathring{\nabla}_Z X, Y) \\ &- \frac{1}{2}[\mathrm{d}\omega^{(3,0)}(X, Y, Z) + \mathrm{d}\omega^{(2,1)}(X, Y, Z) - \mathrm{d}\omega^{(1,2)}(X, Y, Z) - \mathrm{d}\omega^{(0,3)}(X, Y, Z)]. \end{aligned} \tag{2.4.4}$$

We conclude the discussion with several remarks. The properties 1. and 2. in Definition 2.4.1 define a *metric algebroid* [53] $(\eta, [\![\ ,\ ]\!], a = \mathbb{1})$. It is also easy to see, for example from (2.4.3), that $[\![\ ,\ ]\!]$ is not skew and satisfies

$$[\![fX, Y]\!] = f[\![X, Y]\!] - Y[f]X + \eta(X, Y)\mathcal{D}f, \quad \mathcal{D}f = \eta^{-1}\mathrm{d}f. \tag{2.4.5}$$

Moreover, the D-bracket does not satisfy the Jacobi identity. However, it can be split in two parts, each of which does satisfy the Jacobi identity independently. We will discuss this in Section 5.1.

### 2.4.1 The D-Torsion

Note that even though the D-bracket itself is unique, the expression (2.4.3) is not, i.e. there are many different connections apart from the canonical conenction $\nabla^c$ that can be used to define the D-bracket via (2.4.3). This is exactly analogous to the Lie bracket, which is defined on vector fields $X$ and $Y$ by

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

for any $\nabla$ with vanishing torsion. This leads to the analogous notion for the D-bracket, the D-torsion:[4]

**Definition 2.4.4.** *Let* $(\mathbb{M}, \eta, K)$ *be an almost para-Hermitian manifold,* $[\![\ ,\ ]\!]$ *its D-bracket and* $\nabla$ *a connection on* $T\mathbb{M}$*. The* **D-torsion** *of* $\nabla$ *is defined as*

$$\mathcal{T}_\nabla(X, Y, Z) := \eta(\nabla_X Y - \nabla_Y X, Z) + \eta(\nabla_Z X, Y) - \eta([\![X, Y]\!], Z). \tag{2.4.6}$$

The meaning of the D-torsion is that it measures how much the bracket $[\![\ ,\ ]\!]^\nabla$ associated to the connection $\nabla$, given by

$$\eta([\![X, Y]\!]^\nabla, Z) := \eta(\nabla_X Y - \nabla_Y X, Z) + \eta(\nabla_Z X, Y), \tag{2.4.7}$$

deviates from the D-bracket. The D-torsion therefore vanishes precisely when $[\![\ ,\ ]\!]^\nabla$ is the D-bracket. In [13], the connections with a vanishing D-torsion are called *adapted*.

The D-torsion has the following properties:

**Lemma 2.4.5.** *The D-torsion* $\mathcal{T}_\nabla(X, Y, Z)$ *of a connection* $\nabla$ *is a three-form – i.e. it is tensorial and fully skew – if and only if* $\nabla$ *is compatible with* $\eta$*. When this is the case the bracket* $[\![\ ,\ ]\!]^\nabla$ *satisfies the Property* 2 *of Definition 2.4.1. Moreover, if* $\nabla$ *is para-Hermitian – i.e. it preserves* $(\eta, K)$ *– the* $(3,0)$ *and* $(0,3)$ *components of* $\mathcal{T}_\nabla(X, Y, Z)$ *vanish and* $[\![\ ,\ ]\!]^\nabla$ *satisfies properties* $1. - 3.$ *of Definition* (2.4.1)*.*

*Proof.* We introduce the contorsion tensor $\Omega$ associated with $\nabla$ which measures its deviation

[4] In [14], this quantity is called the generalized torsion but here we choose the name D-torsion and reserve the name generalized torsion for its more common usage in the context of generalized geometry. In later sections we will see that the two notions are closely related.

from the canonical connection,

$$\eta(\nabla_X Y, Z) = \eta(\nabla^c_X Y, Z) + \Omega(X, Y, Z).$$

The D-torsion in terms of the contorsion reads

$$\mathcal{T}_\nabla(X, Y, Z) = \Omega(X, Y, Z) - \Omega(Y, X, Z) + \Omega(Z, X, Y).$$

It is easy to check that the contorsion tensor is skew-symmetric in its last last two entries if and only if $\nabla$ preserves $\eta$,

$$\Omega(X, Y, Z) = -\Omega(X, Z, Y) \iff \nabla\eta = 0.$$

Using the skew-symmetry property of $\Omega$, we get

$$\mathcal{T}_\nabla(X, Y, Z) = \Omega(X, Y, Z) + \Omega(Y, Z, X) + \Omega(Z, X, Y) = \sum_{(X,Y,Z)} \Omega(X, Y, Z).$$

The D-torsion $\mathcal{T}_\nabla(X, Y, Z)$ is therefore invariant under cyclic permutations. The fact that is is totally skew follows from skewness of $\Omega$ in the last two entries.

For the converse statement, we use that

$$\mathcal{T}_\nabla(X, Y, Z) + \mathcal{T}_\nabla(Y, X, Z) = \Omega(X, Y, Z) + \Omega(X, Z, Y),$$

which vanishes if $\mathcal{T}_\nabla$ is fully skew. This yields $\Omega(X, Y, Z) + \Omega(X, Z, Y) = 0$, which implies (after a brief calculation) that $\nabla$ must be compatible with $\eta$.

If $\nabla$ is a para-Hermitian connection, i.e. $\nabla P = P\nabla$, then

$$\eta(\nabla_{PX} PY, PZ) = \eta(P\nabla_{PX} PY, PZ) = 0.$$

From this it is clear that the contorsion satisfies $\Omega(PX, PY, PZ) = 0$ and that the $(3,0)$ component of $\mathcal{T}_\nabla$ also vanishes. The same argument applies for $\tilde{P}$. Therefore, the D-torsion of a para-Hermitian connection is a $(2,1) + (1,2)$-form. This means that $[\![\ ,\ ]\!]^\nabla$ satisfies the properties $1.-3.$ of the Definition 2.4.1, while 4., which fixes the $(2,1) + (1,2)$ tensorial part of $[\![\ ,\ ]\!]$, is in general violated. □

**Example 2.4.6** (D-torsion of the Levi-Civita connection)**.** A counter-intuitive fact following from the definition of the D-torsion is that the Levi-Civita connection of $\eta$ now has torsion (with respect to $[\![\ ,\ ]\!]$) unless $\omega$ is closed. This is easy to see from (2.2.4). Using the formula

$$\mathrm{d}\omega(X,Y,Z) = \sum_{(X,Y,Z)} \mathring{\nabla}_X \omega(Y,Z),$$

we find that the D-torsion of the Levi-Civita connection of $\eta$ is given by

$$\mathcal{T}_{\mathring{\nabla}}(Y,X,Z) = \frac{1}{2}[\mathrm{d}\omega^{(3,0)} + \mathrm{d}\omega^{(2,1)} - \mathrm{d}\omega^{(1,2)} - \mathrm{d}\omega^{(0,3)}](X,Y,Z),$$

where the superscript $(p,q)$ denotes the para-Hermitian bigrading of forms. ◁

## 2.5 Para-Calabi-Yau manifolds

In this section, we introduce the notion of para-Calabi-Yau manifolds. As the name suggests, these manifolds should play the role of the para-complex version of Calabi-Yau manifolds in complex geometry. We introduce this notion for its importance in section 3.3, where this geometry underlies a key example of Born geometry that appears in the context of mirror symmetry.

We start with a recollection of Calabi-Yau manifolds. For more details, the reader may consult for example the book [54]. There are many different ways to define what a Calabi-Yau manifold is, but for the purpose of our discussion we choose the following:

**Definition 2.5.1.** *A* **Calabi-Yau manifold** *is a Kähler manifold $(M,g,I)$ of complex dimension $d$ such that the holonomy group of the underlying Riemannian metric $g$ is $Hol(g) \subseteq SU(d)$.*

Typically, a part of the definition of a Calabi-Yau manifold is the requirement of compactness of $M$ and sometimes, one requires that $\mathrm{Hol}(g)$ is exactly equal to $SU(d)$.

Recall that on any Kähler manifold we have $\mathrm{Hol}(g) \subseteq U(d)$. A consequence of the property $\mathrm{Hol}(g) \subseteq SU(d) \subset U(d)$ is that there exists a covariantly constant, non-vanishing section $\Omega \in \Omega^{(d,0)}(M)$ called the *holomorphic volume form*. On $\mathbb{C}^d$, the structures $(g,\omega,\Omega)$

in the canonical holomorphic coordinates take the form:

$$g = \sum_{i=1}^{d} dz^i \otimes d\bar{z}^i, \quad \omega = \frac{i}{2}\sum_{i=1}^{d} dz^i \wedge d\bar{z}^i, \quad \text{and} \quad \Omega = dz^1 \wedge \cdots \wedge dz^d.$$

We can describe $\Omega$ in a coordinate-independent way as follows:

**Lemma 2.5.2.** *Let $(M, g, I)$ be a Calabi-Yau manifold of complex dimension $d$ and $\omega = gI$ the fundamental form. There exists a nowhere vanishing section $\Omega \in \Omega^{(d,0)}(M)$ that is holomorphic, that is, $\bar{\partial}\Omega = 0$, and satisfies*

$$\frac{\omega^d}{d!} = (-1)^{\frac{d(d-1)}{2}} \left(\frac{i}{2}\right)^d \Omega \wedge \bar{\Omega}.$$

We now turn to the analogue of the above notions in the para-complex setting. Let $(\mathbb{M}, K, \eta, \omega)$ be a $2d$-dimensional para-Kähler manifold with $\mathring{\nabla}$ the Levi-Civita connection of $\eta$. From $\mathring{\nabla}\omega = \mathring{\nabla}\eta = 0$, we have that the holonomy group of a para-Kähler manifold is $GL(d) = O(d,d) \cap Sp(2d)$. For a para-Calabi-Yau manifold, instead of the reduction from $U(d)$ to $SU(d)$, we force the holonomy group to be in $SL(d)$:

**Definition 2.5.3.** *A* **para-Calabi-Yau manifold** *is a para-Kähler manifold $(\mathbb{M}, K, \eta)$ of dimension $2d$ such that the holonomy group of $\eta$ is $Hol(\eta) \subseteq SL(d)$.*

*Remark.* Note the interesting property that both $SU(d)$ and $SL(d)$ are real forms of $SL_{\mathbb{C}}(d)$.

Let us now investigate what structure on para-Calabi-Yau manifolds is analogous to the holomorphic volume form. Consider the canonical para-Kähler structure on $\mathbb{R}^{2d}$, given by

$$\eta = dx^i \otimes d\tilde{x}_i + d\tilde{x}_i \otimes dx^i \quad \text{and} \quad \omega = dx^i \wedge d\tilde{x}_i,$$

which is preserved by $GL(d)$. Introducing the following constant tensor

$$\hat{\Omega} = dx^1 \wedge \cdots \wedge dx^d + d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_d = \Omega + \tilde{\Omega},$$

we see that the triple $(\eta, \omega, \hat{\Omega})$ is preserved exactly by $SL(d)$ and $\hat{\Omega}$ satisfies

$$\frac{\omega^d}{d!} = (-1)^{\frac{d(d-1)}{2}} \Omega \wedge \tilde{\Omega}.$$

Therefore, we have the following analogue of Lemma 2.5.2:

**Lemma 2.5.4.** *Let* $(\mathbb{M}, \eta, K)$ *be a para-Calabi-Yau manifold of dimension* $2d$ *and* $\omega = \eta K$ *the fundamental form. There exists a nowhere vanishing section* $\hat{\Omega} = (\Omega, \tilde{\Omega}) \in \Omega^{(d,0)}(\mathbb{M}) \oplus \Omega^{(0,d)}(\mathbb{M})$ *that is para-holomorphic, that is,* $\tilde{\partial}\Omega = \partial\tilde{\Omega} = 0$*, and satisfies*

$$\frac{\omega^d}{d!} = (-1)^{\frac{d(d-1)}{2}} \Omega \wedge \tilde{\Omega}.$$

*Proof.* The proof is exactly analogous to the complex case. See for example [55, Lemma 4.4]. □

Recall that on a para-complex manifold $(\mathbb{M}, M, \tilde{M})$ of dimension $2d$, the appropriate analogue of the holomorphic forms are the para-holomorphic forms, given by pairs $(\alpha, \tilde{\alpha}) \in \Omega^{(k,0)+(0,k)}(\mathbb{M})$ (here the bigrading is with respect to the para-complex structure), satisfying $\tilde{\partial}\alpha = \partial\tilde{\alpha} = 0$. Therefore, the para-holomorphic volume form is exactly the correct analogue of the holomorphic volume form in the case of the Calabi-Yau manifolds.

Furthermore, the fact that $\hat{\Omega}$ is para-holomorphic (particularly $\tilde{\partial}\Omega = 0$) implies that the section $\Omega$ locally takes the form

$$\Omega = \Omega(x)\ dx^1 \wedge \cdots \wedge dx^d,$$

defining a volume form on $M$. Similarly, $\tilde{\Omega}$ defines a volume form on $\tilde{M}$ and we see that the para-Calabi-Yau structure on $\mathbb{M}$ augments the fundamental foliations $M$ and $\tilde{M}$ of the para-complex manifold with volume forms and both the foliation manifolds are consequently orientable.

*Remark.* One of the unresolved problems in the para-Hermitian framework for DFT [12, 14, 13], is the interpretation of the dilaton field $\phi$ in terms of the para-Hermitian geometry. The dilaton is crucial for string theory in that it defines an integration measure $\mu = e^{2\phi}$ on the space-time manifold $M$ and consequently also the notion of divergence. On a para-Calabi-Yau manifold, there is a natural integration measure on both fundamental foliations $M$ and $\tilde{M}$, given by the volume forms $\Omega$ and $\tilde{\Omega}$. Para-Calabi-Yau manifolds, or their generalization (for example to the para-Hermitian case $\mathrm{d}\omega \neq 0$) could therefore serve as the geometric model that adresses the dilaton issue.

# Chapter 3

# Born Geometry

We have seen that para-Hermitian geometry consists of a $2d$-dimensional para-complex manifold $(\mathbb{M}, K)$ with a compatible metric $\eta$ of signature $(d, d)$ and that this data represents what we call the T-duality frame on the tangent bundle and therefore serves as the background kinematic structure on the manifold. In this chapter, we will take the next step by adding the dynamical data to the picture, which is encoded in another compatible metric $\mathcal{H}$ of signature $(2d, 0)$, giving rise to what has been named Born geometry [2]. The additional Riemannian structure $\mathcal{H}$ defines two more endomorphisms of the tangent bundle which – along with the para-complex structure $K$ already in place – form a para-hypercomplex structure.

**Definition 3.0.1.** *Let* $(\mathbb{M}, \eta, \omega)$ *be a para-Hermitian manifold and let* $\mathcal{H}$ *be a Riemannian metric satisfying*

$$\eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta, \quad \omega^{-1}\mathcal{H} = -\mathcal{H}^{-1}\omega. \tag{3.0.1}$$

*Then we call the triple* $(\eta, \omega, \mathcal{H})$ *a* **Born structure** *on* $\mathbb{M}$ *where* $\mathbb{M}$ *is called a* **Born manifold** *and* $(\mathbb{M}, \eta, \omega, \mathcal{H})$ *a* **Born geometry**.

*Remark.* The condition on signature of $\mathcal{H}$ can be relaxed without changing any general properties of Born geometry discussed in this work. Indeed, when $\mathbb{M}$ is considered to be an extended spacetime (as opposed to extended space), $\mathcal{H}$ is usually taken to have the signature $(2d - 2, 2)$. We also note here that in [56], Born geometry was incorrectly called para-hyperKähler.

We now review the three fundamental structures of Born geometry. First, as we have seen, it contains an almost *para-Hermitian* structure $(\omega, K)$ with compatibility

$$K^2 = \mathbb{1}, \qquad \omega(KX, KY) = -\omega(X, Y).$$

Next, the compatibility between $\eta$ and $\mathcal{H}$ implies that $J = \eta^{-1}\mathcal{H} \in \Gamma(\mathrm{End}(T\mathbb{M}))$ defines what we refer to as a *chiral* structure $(\eta, J)$ on $\mathbb{M}$[1]:

**Definition 3.0.2.** *Let* $(\mathbb{M}, J)$ *be an almost para-complex manifold and let* $\eta$ *be a pseudo-Riemannian metric that satisfies* $\eta(JX, JY) = \eta(X, Y)$*. Then we call* $(J, \eta)$ *a* ***chiral structure***[2] *on* $\mathbb{M}$*.*

Finally, the compatibility between $\mathcal{H}$ and $\omega$ defines an almost *Hermitian* structure $(\mathcal{H}, I)$ on $\mathbb{M}$

$$I = \mathcal{H}^{-1}\omega, \quad I^2 = -\mathbb{1}, \quad \mathcal{H}(IX, IY) = \mathcal{H}(X, Y).$$

One easily verifies that the three endomorphisms $I, J, K$, satisfy $KJI = \mathbb{1}$, which means that the triple $(I, J, K)$ obeys the para-quaternionic algebra

$$-I^2 = J^2 = K^2 = \mathbb{1}, \qquad \{I, J\} = \{J, K\} = \{K, I\} = 0, \qquad KJI = \mathbb{1}, \tag{3.0.2}$$

where $\{\ ,\ \}$ is the anti-commutator, forming a **para-hypercomplex** structure. The key relations between the structures of Born geometry are summarized in Table 3.1.

**Integrability** The integrability of Born geometry is very subtle, since we can consider integrability of three separate endomorphisms and two of these endomorphisms are para-complex, admitting half-integrability. The relationship between the integrability of the para-hypercomplex structure at hand has been explored in [57], where the authors found that whenever two out of three structures are integrable, the third is integrable as well. Unless explicitly stated, we will therefore always assume the endomorphisms $I, J, K$ to be almost (para-)complex. In Section 4.3.3, we will also introduce a different notion of integrability for Born geometry motivated by generalized geometry.

[1] In our context the endomorphism $J$ underlying the chiral structure is always para-complex but one can relax the condition on the ranks of the eigenbundles, in which case $J$ would only be a product structure.

[2] In mathematics it is customary to call this structure a *pseudo-Riemannian almost product structure*.

| | | |
|---|---|---|
| $I = \mathcal{H}^{-1}\omega = -\omega^{-1}\mathcal{H}$ | $J = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta$ | $K = \eta^{-1}\omega = \omega^{-1}\eta$ |
| $-I^2 = J^2 = K^2 = \mathbb{1}$ | $\{I, J\} = \{J, K\} = \{K, I\} = 0$ | $IJK = -\mathbb{1}$ |
| $\mathcal{H}(IX, IY) = \mathcal{H}(X, Y)$ | $\eta(IX, IY) = -\eta(X, Y)$ | $\omega(IX, IY) = \omega(X, Y)$ |
| $\mathcal{H}(JX, JY) = \mathcal{H}(X, Y)$ | $\eta(JX, JY) = \eta(X, Y)$ | $\omega(JX, JY) = -\omega(X, Y)$ |
| $\mathcal{H}(KX, KY) = \mathcal{H}(X, Y)$ | $\eta(KX, KY) = -\eta(X, Y)$ | $\omega(KX, KY) = -\omega(X, Y)$ |

Table 3.1: Summary of structures in Born geometry.

**Example 3.0.3** (Local structure)**.** Similarly to para-Hermitian structures, Born structures also have a canonical local form. Let $(\mathbb{M}, K, \eta)$ be a $2n$-dimensional almost para-Hermitian manifold and choose the adapted frame $(e_i, \tilde{e}^i)$, in which we have (using the $d$-dimensional block notation)

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.0.3}$$

Now, from the properties (3.0.1) and the fact that $\mathcal{H}$ is symmetric, we find that $\mathcal{H}$ must take the form

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

for some symmetric, invertible $d$-dimensional matrix $g$. The para-unitary group $pU(2d) = GL(d)$ that preserves $(\eta, \omega, K)$, acts on $\mathcal{H}$ by

$$\mathcal{H} \mapsto \begin{pmatrix} A^t g A & 0 \\ 0 & (A^t g A)^{-1} \end{pmatrix},$$

which means that we can choose $A$ that diagonalizes $g$, $A^t g A = \mathbb{1}$ so that we get a new adapted frame $(e'_i, \tilde{e}'^i) = (Ae_i, (A^t)^{-1}\tilde{e}^i)$, in which $\mathcal{H}$ takes the form

$$\mathcal{H} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

◁

The above example shows three important facts about Born geometry, which are summarized in the following statement:

**Proposition 3.0.4** ([14]). *There exists a Born structure on any almost para-Hermitian manifold* $(\mathbb{M}, K, \eta)$*, and it is equivalent to a choice of a metric* $g$ *on* $L$*, the* $+1$ *eigenbundle of* $K$*, i.e. a non-degenerate symmetric tensor* $g \in \Gamma(L^* \otimes L^*)$*. Moreover, there exists a local frame in which the born structure* $(\eta, \omega, \mathcal{H})$ *takes the canonical form*

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{3.0.4}$$

*Lastly, the structure group of Born geometry is*

$$O(d,d) \cap Sp(2d) \cap O(2d) = O(d).$$

The way in which the metric $g$ on $L$ defines the compatible Riemannian metric $\mathcal{H}$ on $\mathbb{M}$ is the following. The fact that $L$ and $\tilde{L}$ are maximally isotropic with respect to $\eta$ implies that $\eta$ defines by contraction the bundle isomorphisms

$$\eta : \begin{cases} L \to \tilde{L}^* \\ \tilde{L} \to L^* \end{cases}.$$

Understanding $g$ also as an isomorphism $g: \ L \to L^*$, we can construct the isomorphism $\tilde{g}: \ \tilde{L} \to \tilde{L}^*$ given by the composition of maps $\tilde{g} = \eta \circ g^{-1} \circ \eta$, which in turn defines a metric on $\tilde{L}$ that we also denote by $\tilde{g}$. Then $\mathcal{H}$ takes the diagonal form

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & \tilde{g} \end{pmatrix}.$$

The above statement makes a perfect sense in the context of the extended spacetime $(\mathbb{M}, M, \tilde{M})$: the choice of a Born structure is simply a choice of a metric on tangent directions along the physical space $M$, reducing the structure group to the group of orthogonal transformations of $M$, $O(d)$. Furthermore, this tells us that all examples of para-Hermitian manifolds listed in Section 2.2.3 give rise in a simple way to Born geometries simply by choosing a metric on one of the eigenbundles. We demonstrate this on the following example from [14, 18] that naturally arises in Lagrangian mechanics:

**Example 3.0.5** (Born structures on the tangent bundle)**.** Let $\mathbb{M} = TM$ be the tangent bundle of a Riemannian manifold $(M, g)$ and choose a connection $\Gamma$, which defines a (half-)integrable para-Hermitian structure $(K, \eta)$ on $\mathbb{M}$, as described in Example 2.2.10. Pulling back $g$ by the fiber projection $TM \xrightarrow{\pi} M$ defines a metric on the horizontal subbundle $H$ and also induces a metric $\tilde{g}$ on the vertical bundle $V$ as described above: $\tilde{g} = \eta g^{-1}\eta$. In turn, this defines the diagonal Riemannian metric on $\mathbb{M} = TM$ called the Sasaki metric

$$\mathcal{H} = g_{ij}(dx^i \otimes dx^j + \tau^i \otimes \tau^j),$$

where $\tau^i$ are the one-forms in $V^*$ described in the Example 2.2.10. $\triangleleft$

## 3.1 Perspectives on Born geometry

Depending on the perspective one wants to emphasize, there are multiple ways to present Born geometry, given the various different structures it defines. So far, the emphasis was put on the para-Hermitian triple $(\eta, \omega, K)$ which defines the underlying T-duality frame and differentiable structure compatible with this frame. However, there are many different ways one can present Born geometry and we will discuss them in the following sections.

### 3.1.1 Chiral perspective

In DFT, the extended geometry is usually presented in terms of the chiral structure $(\eta, \mathcal{H}, J)$ and the para-Hermitian structure only arises implicitly as a choice of the T-duality frame, sometimes called the *polarization*. This point of view can be summarized in the following way:

**Lemma 3.1.1.** *The Born structure $(\eta, \omega, \mathcal{H})$ is equivalent to the chiral triple $(\eta, \mathcal{H}, J)$ satisfying*

$$\begin{aligned} J &= \eta^{-1}\mathcal{H}, & J^2 &= \mathbb{1}, \\ \eta(JX, JY) &= \eta(X, Y), & \mathcal{H}(JX, JY) &= \mathcal{H}(X, Y), \end{aligned} \tag{3.1.1}$$

*along with a compatible almost symplectic form $\omega$ fixing the polarization*

$$\omega^{-1}\eta = \eta^{-1}\omega, \qquad \omega^{-1}\mathcal{H} = -\mathcal{H}^{-1}\omega. \tag{3.1.2}$$

*Proof.* We have already shown that starting with a para-Hermitian manifold and a compatible Riemannian metric $\mathcal{H}$ yields the compatible chiral structure. The converse statement follows from observing that $\omega^{-1}\eta = \eta^{-1}\omega$ implies $K = \eta^{-1}\omega$ is a para-Hermitian structure compatible with $\eta$. □

The chiral structure $J$ plays an important role in subsequent constructions. For this reason, we introduce the chiral subspaces $C_\pm$ which are the $\pm1$-eigenbundles of $J$. Together these subspaces span the tangent space of our Born manifold

$$T\mathbb{M} = C_+ \oplus C_-. \tag{3.1.3}$$

We now also have another set of projection operators onto these subspaces, which we denote by $P_\pm$. Additionally, we introduce the following notation for the splitting of a vector field on $\mathbb{M}$ into the eigenbundles of $J$:

$$X_\pm = P_\pm(X), \quad P_\pm = \frac{1}{2}(\mathbb{1} \pm J). \tag{3.1.4}$$

The compatibility conditions of $\eta$ and $J$ further imply that the eigenbundles $C_\pm$ are orthogonal with respect to $\eta$

$$\eta(X_\pm, Y_\mp) = 0, \tag{3.1.5}$$

for any $X_\pm, Y_\pm \in \Gamma(C_\pm)$. The following property will be used later repeatedly:

**Lemma 3.1.2.** *The para-Hermitian structure $K$ on a Born manifold maps the chiral subspaces into each other isomorphically,*

$$K : C_\pm \to C_\mp \tag{3.1.6}$$

*such that $K(X_\pm) = (KX)_\mp$.*

*Proof.* Because $\{J, K\} = 0$, $K(X_\pm) = (KX)_\mp$ follows immediately from (3.1.4). Since $K$ is clearly invertible, the map is an isomorphism. □

Even though chiral structures are similar to para-Hermitian structures in the sense that they are defined by a real endomorphism $J$ that is compatible with a metric $\eta$, the fact that $J$ is orthogonal with respect to $\eta$ forces many qualitative differences. The eigenbundles of $J$ are not isotropic with respect to either of the metrics $\eta$ or $\mathcal{H} = \eta J$. Moreover, the Nijenhuis tensor of $J$,

$$N_J(X, Y, Z) \coloneqq \eta(N_J(X, Y), Z),$$

is of type $(2,1) + (1,2)$ in the bigrading of $J$, as opposed to the Nijenhuis tensor

$$N_K(X, Y, Z) = \eta(N_K(X, Y), Z)$$

of an almost para-Hermitian structure (2.1.1), which is of type $(3,0)+(0,3)$ (in the bigrading defined by $K$). Because the contraction $\eta J = \mathcal{H}$ is now a metric, the condition analogous to the "para-Kähler" condition $\mathrm{d}\omega = 0$ is different. Instead, one can classify the chiral structures with respect to the **fundamental tensor** $F$:

$$F(X, Y, Z) \coloneqq \mathring{\nabla}_X \mathcal{H}(Y, Z) = \eta((\mathring{\nabla}_X J)Y, Z). \tag{3.1.7}$$

The full classification in terms of 36 classes was done in [58] but here we recall only two subclasses, $\mathcal{W}_3$ and $\mathcal{W}_0$.

**Definition 3.1.3.** *Let $(\eta, J)$ be an almost chiral structure. We say $(\eta, J)$ is of class $\mathcal{W}_3$ if*

$$\sum_{Cycl.\ X,Y,Z} F(X, Y, Z) = 0. \tag{3.1.8}$$

*If $F = 0$ identically, we say $(\eta, J)$ is of class $\mathcal{W}_0$.*

Note that the condition (3.1.8) is the chiral geometry analogue of the condition $\mathrm{d}\omega = 0$ in para-Hermitian geometry via the formula

$$\mathrm{d}\omega(X, Y, Z) = \sum_{Cycl.\ X,Y,Z} \mathring{\nabla}_X \omega(Y, Z),$$

which is the para-Hermitian version of (3.1.8) once we plug in the definition of $F$ (3.1.7). Therefore, the class $\mathcal{W}_3$ should be thought of as the closest analogue of para-Kähler structures one can define for chiral structures.

### 3.1.2 Para-hypercomplex perspective

The para-hypercomplex perspective, which considers the triple $\{I, J, K\}$ satisfying the para-quaternionic algebra (3.0.2) as the starting point of Born geometry, is another natural point of view. First, Boulter proves in his work [30] that any para-hypercomplex triple in fact admits a Born structure, meaning there always exists a metric $\eta$ that is appropriately compatible according to Table 3.1.

We now show that Born geometry is in fact one of only two options of adding a metric $\eta$[3] compatible with the para-hypercomplex triple $\{I, J, K\}$. Because there are two options for the choices of orthogonality for each, we could in theory get $2^3 = 8$ options. However, because $K$ and $J$ are both structures of the same type and all three structures anticommute, this already restricts the options to 3 cases

1. $I$ is orthogonal $\Rightarrow J, K$ have the same orthogonality:
   - Both $J, K$ are orthogonal,
   - Both $J, K$ are anti-orthogonal.
2. $I$ is anti-orthogonal $\Rightarrow J, K$ have different orthogonality.

As we will now explain, two of the above options in fact give the same geometry and therefore there are only two inequivalent options of how to choose the orthogonality of a para-hypercomplex triple with respect to a metric.

First, we start with the case when $I$ is orthogonal and both $J$ and $K$ are anti-orthogonal. In this situation all three contractions

$$\omega_I = \eta I, \quad \omega_J = \eta J, \quad \omega_K = \eta K,$$

define a non-degenerate two-form. This geometry is called **para-hyper-Hermitian** and has been studied for example in [59, 60] and other works, including in physics [61, 62].

Now, let all $I, J, K$ be orthogonal with respect to $\eta$. From the first column of the metric compatibility relationships in Table 3.1, we see that upon identifying our $\eta$ with $\mathcal{H}$, $(I, J, K, \mathcal{H} = \eta)$ is Born geometry.

[3] No conditions on the signature of $\eta$ are a priori assumed.

Lastly, when $I$ is anti-orthogonal and $J$ and $K$ have opposite orthogonality with respect to $\eta$, we choose without a loss of generality that $K$ is anti-orthogonal and $J$ is orthogonal, which again recovers Born geometry, this time described in the second column of Table 3.1. We conclude that apart from para-hyper-Hermitian geometry, Born geometry describes the only other way one can choose a metric compatible with a para-hypercomplex structure $\{I, J, K\}$. In particular, since para-hyper-Hermitian structures only exist on manifolds of dimension $4n$ (see for example [30, Remark 3.3]), Born geometry is *the only* such choice on manifolds of dimension $2(2n-1)$.

### 3.1.3 Bi-Chiral perspective

In the above discussion we have neglected one remaining structure that one gets as a contraction of all $\eta$, $\omega$ and $\mathcal{H}$, $\hat{\eta} \coloneqq \eta\mathcal{H}^{-1}\omega$. Using the properties of Table 3.1, one easily verifies that the orthogonality of the para-complex structures $J, K$ with respect to $\hat{\eta}$ is swapped compared to $\eta$:

$$\hat{\eta}(JX, JY) = -\hat{\eta}(X, Y), \quad \text{and} \quad \hat{\eta}(KX, KY) = \hat{\eta}(X, Y).$$

This means that the data of Born geometry can be equivalently specified by a bi-chiral structure – i.e. a pair of chiral structures $(\eta, J)$, $(\hat{\eta}, K)$, which share the same metric $\mathcal{H} = \eta J = \hat{\eta} K$ – that mutually anti-commute: $\{J, K\} = 0$. We will come back to this point of view in Section 4.3.2, where Born geometry is defined in terms of a commuting pair of generalized structures, i.e. endomorphisms of the bundle $(T \oplus T^*)\mathbb{M}$.

## 3.2 The Born connection

In the previous sections, we introduced Born geometry and argued that it represents a choice of *dynamical metric* $\mathcal{H}$ on a para-Hermitian manifold $(\mathbb{M}, \eta, K)$. More precisely, it corresponds to a choice of a "physical space" metric $g$ on $L$, which consequently also defines the T-dual metric $\tilde{g}$ on $\tilde{L}$, so that both $L$ and $\tilde{L}$ are Riemannian vector bundles.

From the physics point of view, the metric represents gravity. As in general relativity, we need to accompany the metric structure by an appropriate compatible connection on $\mathbb{M}$. A naive choice would be the Levi-Civita connection of $\mathcal{H}$, $\nabla^{\mathcal{H}}$. If we follow our reasoning from

Section 2.4, we realize that this is not the right choice, simply because $\nabla^{\mathcal{H}}$ is not compatible with the T-duality frame and the underlying differentiable structure given by the D-bracket.

Instead, we are looking for a connection $\nabla^B$ that satisfies the following properties:

- Compatibility with the T-duality splitting: $\nabla^B\omega = \nabla^B\eta = \nabla^B K = 0$,
- Compatibility with the dynamical data: $\nabla^B\mathcal{H} = 0$, and
- Compatibility with the D-bracket: vanishing of the D-torsion $\mathcal{T}_{\nabla^B} = 0$.

As we show in [14], such a connection not only exists, but is also unique:

**Theorem 3.2.1** ([14])**.** *Let $(\mathbb{M}, \eta, \omega, \mathcal{H})$ be a Born geometry. Then there exists a unique connection $\nabla^{\mathrm{B}}$ on $T\mathbb{M}$ called the Born connection which parallelizes the Born structure $(\eta, \omega, \mathcal{H})$ on $\mathbb{M}$ and has a vanishing D-torsion, $\mathcal{T}_{\nabla^{\mathrm{B}}} = 0$. Moreover, $\nabla^{\mathrm{B}}$ can be expressed in terms of the para-Hermitian structure $K = \eta^{-1}\omega$ and the chiral projections $X_\pm = P_\pm(X)$ as*

$$\nabla^{\mathrm{B}}_X Y = [\![X_-, Y_+]\!]_+ + [\![X_+, Y_-]\!]_- + (K[\![X_+, KY_+]\!])_+ + (K[\![X_-, KY_-]\!])_-. \tag{3.2.1}$$

*Proof.* Here we sketch the proof of the theorem, a detailed version of the proof can be found in [14]. First, we show that the expression (3.2.1) indeed defines a connection and has the listed properties, i.e. it is compatible with all the structures of Born geometry and its D-torsion vanishes. For the tensoriality, we use the property (2.4.5):

$$\begin{aligned}\nabla^{\mathrm{B}}_{fX} Y = f\nabla^{\mathrm{B}}_X Y &- Y_+[f](X_-)_+ - Y_-[f](X_+)_- \\ &- K(Y_+)[f](KX_+)_+ - K(Y_-)[f](KX_-)_- \\ &+ \eta(X_-, Y_+)(\mathcal{D}f)_+ + \eta(X_+, Y_-)(\mathcal{D}f)_- \\ &+ \eta(X_+, KY_+)(K\mathcal{D}f)_+ + \eta(X_-, KY_-)(K\mathcal{D}f)_- \\ = f\nabla^{\mathrm{B}}_X Y&,\end{aligned} \tag{3.2.2}$$

where $\mathcal{D} = \eta^{-1}\mathrm{d}$ and we made use of Lemma 3.1.2 along with the fact that $C_\pm$ are orthogonal with respect to $\eta$ (3.1.5) and complementary to each other. Similarly, the derivation property in the second argument follows by

$$\begin{aligned}\nabla^{\mathrm{B}}_X fY = f\nabla^{\mathrm{B}}_X Y &+ X_-[f](Y_+)_+ + X_+[f](Y_-)_- \\ &+ X_+[f](K^2Y_+)_+ + X_-[f](K^2Y_-)_- \\ = f\nabla^{\mathrm{B}}_X Y &+ X[f]Y,\end{aligned} \tag{3.2.3}$$

which shows that $\nabla^{\mathrm{B}}$ indeed is a connection.

The compatibility of $\nabla^{\mathrm{B}}$ with $\eta$ follows from the property 2. in Definition 2.4.1 of the D-bracket and the orthogonality properties of $J$ and $K$ with respect to $\eta$. The fact that $\nabla^{\mathrm{B}}$ parallelizes $J$ follows directly from the expression (3.2.1) which manifestly preserves the eigenbundles of $J$. Lastly, $\nabla^{\mathrm{B}} K = 0$ follows from Lemma 3.1.2, which can be used to show that $\nabla^{\mathrm{B}}_X(KY) = K(\nabla^{\mathrm{B}}_X Y)$ by directly plugging into (3.2.1).

To show that $\mathcal{T}_{\nabla^{\mathrm{B}}} = 0$, we use the definition of the D-torsion 2.4.4:

$$\mathcal{T}_{\nabla^{\mathrm{B}}}(X,Y,Z) = \eta(\nabla^{\mathrm{B}}_X Y - \nabla^{\mathrm{B}}_Y X, Z) + \eta(\nabla^{\mathrm{B}}_Z X, Y) - \eta([\![X,Y]\!], Z),$$

and again use the fact that $\nabla^{\mathrm{B}}$ itself can be written using $[\![\ ,\ ]\!]$. Carrying out the calculation eventually yields

$$\mathcal{T}_{\nabla^{\mathrm{B}}}(X,Y,Z) = -\sum_{\pm} \eta(\mathcal{N}_K(X_\pm, Y_\pm), Z_\pm),$$

which vanishes as a result of the axiom 3. in Definition 2.4.1.

For the uniqueness, we assume $\tilde{\nabla}$ is another connection satisfying the properties listed in Theorem 3.2.1. We then decompose $\tilde{\nabla}$ into the four chiral components:

$$\eta(\tilde{\nabla}_X Y, Z) = \sum_{\pm} \left[\eta(\tilde{\nabla}_{X_\pm} Y_\mp, Z) + \eta(\tilde{\nabla}_{X_\pm} Y_\pm, Z)\right], \tag{3.2.4}$$

and show that the properties of $\tilde{\nabla}$ imply that $\tilde{\nabla} = \nabla^{\mathrm{B}}$ for each of the four components. For example,

$$\begin{aligned}\eta(\tilde{\nabla}_{X_+} Y_-, Z) &\overset{\tilde{\nabla} J=0}{=} \eta(\tilde{\nabla}_{X_+} Y_-, Z_-) \overset{\tilde{\mathcal{T}}=0}{=} \eta([\![X_+, Y_-]\!], Z_-) + \eta(\tilde{\nabla}_{Y_-} X_+, Z_-) - \eta(\tilde{\nabla}_{Z_-} X_+, Y_-) \\ &\overset{\tilde{\nabla} J=0}{=} \eta([\![X_+, Y_-]\!], Z_-) = \eta([\![X_+, Y_-]\!]_-, Z).\end{aligned} \tag{3.2.5}$$

The rest follows analogously, which completes the proof of the uniqueness as well as of the whole Theorem 3.2.1. □

### 3.2.1 The Levi-Civita connection on $L$

In Proposition 3.0.4, we presented the point of view that Born geometry augments para-Hermitian geometry with a metric $g$ on the eigenbundle $L$, which is then (when certain integrability conditions are satisfied) identified with a space-time metric on the manifold $M = \mathbb{M}/\tilde{M}$. This interpretation of Born geometry is completed by the Born connection $\nabla^B$, which on $L$ restricts to the Levi-Civita connection of $g$. This means that $(\mathcal{H}, \nabla^B)$ is an appropriate extension of the gravitational dynamics on $M$, given by the metric $g$ and its Levi-Civita connection, to the extended space $\mathbb{M}$. Moreover, the next result shows that the existence and uniqueness of the Born connection can be understood as the consequence of the existence and uniqueness of the Levi-Civita connection of $g$.

**Theorem 3.2.2.** *Let $(\mathbb{M}, \eta, K, \mathcal{H})$ be a Born manifold with $\mathcal{H}$ defined by a metric $g$ on the $+1$-eigenbundle $L$ of $K$, which is integrable:*

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & \tilde{g} \end{pmatrix},$$

*where $\tilde{g} = \eta g^{-1} \eta$. Then the partial connection*

$$\begin{aligned} \nabla^B_{P\bullet} :\ \Gamma(L) \times \mathfrak{X}(\mathbb{M}) &\to \mathfrak{X}(\mathbb{M}), \\ (\mathbf{x}, Y) &\mapsto \nabla^B_{\mathbf{x}} Y, \end{aligned}$$

*with $\nabla^B$ the Born connection, takes in the splitting $T\mathbb{M} = L \oplus \tilde{L}$ the form*

$$\nabla^B_{\mathbf{x}} = \begin{pmatrix} \nabla^g_{\mathbf{x}} & 0 \\ 0 & \nabla^{g*}_{\mathbf{x}} \end{pmatrix},$$

*where $\nabla^g$ is the Levi-Civita connection of $g$ and $\nabla^{g*}$ is the dual connection on $\tilde{L}$ defined by $\eta(\nabla^{g*}_{\mathbf{x}} \tilde{\mathbf{y}}, \mathbf{z}) = \mathbf{x}\eta(\tilde{\mathbf{y}}, \mathbf{z}) - \eta(\tilde{\mathbf{y}}, \nabla^g_{\mathbf{x}} \mathbf{z})$.*

*Proof.* The proof amounts to checking that the restriction of $\nabla^B$ to $L$ preserves the metric $g$ and has a vanishing torsion along $L$, therefore showing it must be the unique Levi-Civita connection of $g$ with these properties. The former is true due to $\nabla^B \mathcal{H} = 0$ and the fact that $\mathcal{H}\,|_L = g$:

$$0 = (\nabla^B_{\mathbf{x}} \mathcal{H})(\mathbf{y}) = \nabla^B_{\mathbf{x}}(\mathcal{H}(\mathbf{y})) - \mathcal{H}(\nabla^B_{\mathbf{x}} \mathbf{y}) = \nabla^B_{\mathbf{x}}(g(\mathbf{y})) - g(\nabla^B_{\mathbf{x}} \mathbf{y}) = (\nabla^B_{\mathbf{x}} g)(\mathbf{y}),$$

where we also implicitly used that $\nabla^B$ preserves $L$. The torsionlessness along $L$ follows from the fact that the D-torsion of $\nabla^B$ vanishes identically and $[\![\ ,\ ]\!]$ restricts to a Lie bracket on $L$

$$0 = \mathcal{T}_{\nabla^{\mathrm{B}}}(\mathbf{x},\mathbf{y},\tilde{\mathbf{z}}) = \eta(\nabla^{\mathrm{B}}_{\mathbf{x}}\mathbf{y} - \nabla^{\mathrm{B}}_{\mathbf{y}}\mathbf{x},\tilde{\mathbf{z}}) + \overbrace{\eta(\nabla^{\mathrm{B}}_{\tilde{\mathbf{z}}}\mathbf{x},\mathbf{y})}^{=0} - \eta([\mathbf{x},\mathbf{y}],\tilde{\mathbf{z}}) = \eta(T^{\nabla^{\mathrm{B}}}(\mathbf{x},\mathbf{y}),\tilde{\mathbf{z}}).$$

Lastly, we find the component of $\nabla^{\mathrm{B}}$ along $\tilde{L}$ by $\eta$-compatibility

$$\eta(\nabla^{\mathrm{B}}_{\mathbf{x}}\tilde{\mathbf{y}},\mathbf{z}) = \mathbf{x}\eta(\tilde{\mathbf{y}},\mathbf{z}) - \eta(\tilde{\mathbf{y}},\nabla^{\mathrm{B}}_{\mathbf{x}}\mathbf{z}) = \mathbf{x}\eta(\tilde{\mathbf{y}},\mathbf{z}) - \eta(\tilde{\mathbf{y}},\nabla^{g}_{\mathbf{x}}\mathbf{z}).$$

□

Similarly, we find an analogous form of the partial connection along $\tilde{L}$ defined by $\nabla^{\mathrm{B}}_{\tilde{P}\bullet}$:

$$\nabla^{\mathrm{B}}_{\tilde{\mathbf{x}}} = \begin{pmatrix} \nabla^{\tilde{g}}_{\tilde{\mathbf{x}}} & 0 \\ 0 & \nabla^{\tilde{g}*}_{\tilde{\mathbf{x}}} \end{pmatrix} : \begin{cases} \Gamma(\tilde{L}) \times \mathfrak{X}(\mathbb{M}) \to \mathfrak{X}(\mathbb{M}) \\ (\tilde{\mathbf{x}}, Y) \mapsto \nabla^{B}_{\tilde{\mathbf{x}}} Y \end{cases},$$

where $\nabla^{\tilde{g}}$ is the Levi-Civita connection of $\tilde{g}$ on $\tilde{L}$ and $\nabla^{\tilde{g}*}$ is again the dual connection defined by $\eta(\nabla^{\tilde{g}*}_{\tilde{\mathbf{x}}}\mathbf{y},\tilde{\mathbf{z}}) = \tilde{\mathbf{x}}\eta(\mathbf{y},\tilde{\mathbf{z}}) - \eta(\mathbf{y},\nabla^{\tilde{g}}_{\tilde{\mathbf{x}}}\tilde{\mathbf{z}})$.

## 3.3 Example: Born geometry and mirror symmetry

We will now turn to an example of Born geometry arising in one of the most elementary geometric setting of so-called SYZ mirror symmetry [6], the **semi-flat mirror symmetry**. Even though this example of mirror symmetry is relatively simple and easy to understand in elementary geometric terms, it already shows a non-trivial relationship between moduli of geometric structures on the mirror manifolds. The goal of this section is to show that:

1. Both sides of the semi-flat mirror symmetry admit Born geometries.
2. The mirror map relates the moduli spaces of the symplectic structures on one side to the moduli of para-complex structures on the other side, on top of the well-known relationship of symplectic and complex moduli.

We follow the discussions in [63, 64, 65], which are the most relevant references for the material discussed in this section and we now briefly review the most important facts.

Broadly speaking, mirror symmetry is a correspondence between two sets of mathematical data that was originally discovered in physics as an equivalence of supersymmetric nonlinear

$\sigma$-models, but has been vastly generalized beyond that context since then. In this section, we will be concerned with the SYZ mirror symmetry, where the correspondence is between pairs of Calabi-Yau manifolds $M$ and $\tilde{M}$, specifically ones that admit a particular type of fibration. The mirror symmetry in this case acts geometrically as an exchange of these fibrations, which are linear duals of each other. This corresponds well with the picture of T-duality, which is usually locally described precisely by such an exchange, giving rise to the motto of SYZ mirror symmetry: “mirror symmetry is T-duality” [6]. One of the hallmarks of mirror symmetry is the feature that it relates the moduli space of symplectic structures on $M$ with the moduli space of complex structures on $\tilde{M}$. In the physics language, this exchange is realized as a correspondence between an $A$-model on $M$ and a $B$-model on $\tilde{M}$.

In the following paragraphs, we will study the semi-flat case of SYZ mirror symmetry, where the pair of Kähler Calabi-Yau manifolds is given by $M = TB$ and $\tilde{M} = T^*B$, the tangent and cotangent bundles of an affine manifold $B$. Our result is that this setting admits an equally natural description in terms of para-Kähler and para-Calabi-Yau geometry, where the mirror map exchanges the symplectic moduli on $M$ with the para-complex moduli on $\tilde{M}$. The underlying symplectic fundamental forms of both the Kähler and para-Kähler geometries coincide, while the complex and para-complex structures anti-commute and therefore define Born geometry. Consequently, the mirror map in the semi-flat case relates Born geometries on $M$ and $\tilde{M}$, mapping between the symplectic moduli on one side and the para-complex and complex moduli on the other side.

*Remark.* Note that even though each of the manifolds $M = TB$ and $\tilde{M} = T^*B$ are individually para-Hermitian (particularly, they are para-Calabi-Yau), the para-Hermitian structures do not describe the T-duality facilitated by the mirror map. Instead, the T-duality described in this section is partial and happens on the fibres of the total space of the manifold $\mathbb{M} = TB \times_B T^*B = (T \oplus T^*)B$, which are para-Hermitian as well.

**The para-complex point of view** We start with the discussion of the para-Calabi-Yau geometry on the side of $M$ and then contrast it with the usual Calabi-Yau geometry of $M$. Consider an affine manifold $B$ and take a neighbourhood $U \subset B$ with local coordinates $u^i$. Let $U \times \mathbb{R}^n$ with the coordinates $(u^i, v^i)$, $v^i$ denoting the coordinates on $\mathbb{R}^n$, be the local model for $M = TB$. We define the para-complex structure on $M$ in terms of its adapted coordinates, which we choose to be $(x^i = u^i + v^i, \tilde{x}^j = u^k - v^k)$. The fact that $M$ is affine means that $M$ admits an atlas with affine transition functions in $GL(n) \ltimes \mathbb{R}^n$, i.e. of the form

$$u^i \mapsto u'^i(u^i) = A^i{}_j u^j + B^i, \quad A \in GL(n),\ B \in \mathbb{R}^n.$$

We therefore choose $(U, u^i)$ from this atlas. This implies that the natural fiber coordinates of $TM$, $v^i = du^i$[4], transform as $v^i \mapsto A^i{}_j v^j$, so that the adapted coordinates $(x^i = u^i + v^i, \tilde{x}^j = u^k - v^k)$ transform by

$$(x^i, \tilde{x}^j) \mapsto (A^i{}_k x^k + B^i, A^j{}_k \tilde{x}^k + B^j),$$

which shows that the total space of $TB$ is an affine manifold with $(x^i, \tilde{x}^j)$ affine coordinates. In particular $(x^i, \tilde{x}^j)$ are adapted coordinates of an integrable para-complex structure on $TB$, because the transition functions are para-holomorphic.

We now endow $M$ with a para-Calabi-Yau structure defined by the fundamental form $\omega$ and the para-holomorphic volume form $\hat{\Omega}$:

$$\begin{aligned}\omega &= \omega_{ij}\ dx^i \wedge d\tilde{x}^j = -2\omega_{ij}\ du^i \wedge dv^j,\\ \hat{\Omega} &= \frac{1}{2}(dx^1 \wedge \dots \wedge dx^n + d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n) = \Omega + \tilde{\Omega}.\end{aligned}$$

We also assume that the local para-Kähler potential $\phi$ on $U \subset M$, which defines $\omega$ via

$$\omega_{ij} = \partial_i \tilde{\partial}^j \phi = \left(\frac{\partial^2}{\partial u^i \partial u^j} - \frac{\partial^2}{\partial v^i \partial v^j}\right)\phi,$$

is of the form $\phi(u,v) = \phi(u)$. This means it is invariant under the translations in the fiber directions of $TB$ and the geometry is therefore *semi-flat*, hence the name semi-flat mirror symmetry. The fact that the pair $(\omega, \hat{\Omega})$ defines a para-Calabi-Yau structure means that it satisfies the Monge-Ampere equation:

$$\omega^n = C\Omega \wedge \tilde{\Omega} \iff \det\left(\frac{\partial^2 \phi}{\partial u^i \partial u^j}\right) = C.$$

The above has a unique solution for $\phi$ [66] assuming $\phi\,|_{\partial U} = 0$ and $\phi$ is convex $(\partial_i \partial_j \phi > 0)$. For the Monge-Ampere equation to be invariant under the affine transformations, we must have $\det(A) = 1$, which is a requirement that defines a *special affine manifold*. Therefore, we will from now on require that $B$ is special affine. For completeness, we note that the

[4]Here $du$ is understood as a fiber-wise linear function $TM\,|_p \to \mathbb{R}$, for $p \in U$.

para-Kähler metric $\eta$ takes the form

$$\eta = \omega_{ij}(dx^i \otimes d\tilde{x}^j + d\tilde{x}^i \otimes dx^j) = 2\omega_{ij}(du^i \otimes du^j - dv^i \otimes dv^j).$$

**The complex point of view** In the usual discussion of the semi-flat mirror symmetry, one considers a complex structure defined by the holomorphic coordinates $z^i = u^i + iv^i$ instead of the para-complex structure above. The fundamental form is then taken to be $\omega_I = i\partial\bar{\partial}\phi$, which in the semi-flat case $\phi(u,v) = \phi(u)$ coincides with $\omega$. The Riemannian Kähler metric is then

$$g = \omega_{ij}(du^i \otimes du^j + dv^i \otimes dv^j).$$

It is easy to show that $I$ and $K$ anticommute, and because $\omega_I = \omega$, we see that this geometric setting defines Born geometry.

**Legendre transform and the mirror map** So far we described the geometry on the side of $TB$. Now, we will describe the corresponding mirror geometry on $T^*B$. We start by considering new coordinates $\hat{u}_i$ on $U \subset B$ given by

$$\frac{\partial \hat{u}_i(u)}{\partial u^j} = \omega_{ij} = \frac{\partial^2 \phi}{\partial u^i \partial u^j},$$

with the inverse transformation $\frac{\partial u^i(\hat{u})}{\partial \hat{u}_j} = \omega^{ij} = (\omega_{ij})^{-1}$. Integrating this, we can write the relationship between the coordinates

$$u^i(\hat{u}) = \frac{\partial \psi(\hat{u})}{\partial \hat{u}_i},$$

for some local function $\psi(\hat{u})$ so that $\omega^{ij} = \frac{\partial \psi(\hat{u})}{\partial \hat{u}_i \partial \hat{u}_j}$ and $\phi(u)$ and $\psi(\hat{u})$ are Legendre transforms of one another:

$$\psi(\hat{u}) = u^i \hat{u}_i - \phi(u).$$

This ensures that $\psi$ also satisfies the Monge-Ampere equation, but with the inverse constant

$$\det\left(\frac{\partial^2 \psi}{\partial \hat{u}_i \partial \hat{u}_j}\right) = C^{-1}. \tag{3.3.1}$$

We continue by discussing the cotangent bundle $\tilde{M} = T^*B$. The metric $\eta$ on $M = TB$ defines a negative-definite metric $-2\omega_{ij} dv^i \otimes dv^j$ on each fiber, which can be inverted to give a negative-definite metric on fibres of $T^*B$ so that there is a metric on $\tilde{M}$:

$$\hat{\eta} = 2(\omega_{ij} du^i \otimes du^j - \omega^{ij} d\hat{v}_i \otimes d\hat{v}_j),$$

where we denoted the fiber coordinates dual to $v^i = du^i$ by $\hat{v}_i = \frac{\partial}{\partial u^i}$. Using the canonical symplectic form on $T^*B$,

$$\hat{\omega} = du^i \wedge d\hat{v}_i$$

we define a para-Kähler structure $\hat{K}$ via

$$\hat{K} := \hat{\eta}^{-1} \hat{\omega}.$$

It can be checked that $\hat{K}$ is again integrable and the corresponding adapted coordinates are $(\hat{x}_i = \hat{u}_i + \hat{v}_i, \hat{\tilde{x}}_j = \hat{u}_j - \hat{v}_j)$, in terms of which we get

$$\begin{aligned} \hat{\omega} &= \hat{\omega}^{ij} d\hat{x}_i \wedge d\hat{\tilde{x}}_j \\ \hat{\eta} &= \hat{\omega}^{ij} (d\hat{x}_i \otimes d\hat{\tilde{x}}_j + d\hat{\tilde{x}}_i \otimes d\hat{x}_j), \end{aligned}$$

where $\hat{\omega}^{ij} = \frac{\partial^2 \psi}{\partial \hat{u}_i \partial \hat{u}_j}$. Crucially, $\tilde{M}$ is also a para-Calabi-Yau manifold because $\psi$ satisfies the Monge-Ampere equation (3.3.1). Moreover, we get a relationship between the moduli of symplectic structures on $M$ and para-complex structures on $\tilde{M}$, since varying the symplectic structure on $M$ corresponds to a change of $\phi$, which also changes $\psi$ as well as the coordinates $\hat{u}_i$. This in turn changes the para-complex structure $\hat{K}$ defined by its adapted coordinates, and in particular depending on $\hat{u}_i$.

We find that in this case the mirror symmetry not only relates the symplectic and complex moduli of the mirror manifolds $M$ and $\tilde{M}$ (details about this statement can be found for example in [63]), but also relates the symplectic and para-complex moduli in an analogous way. Therefore, the semi-flat mirror symmetry defines a map between the Born geometries on $M$ and $\tilde{M}$, where the variation of the symplectic structure on one side corresponds to a variation of the complex and para-complex structures on the other side.

# Chapter 4

# Born Geometry and Generalized Geometry

In this section we introduce the framework of generalized geometry, which we shall use to describe many geometric objects discussed so far, from a different point of view. In Section 4.3, we define *generalized* para-Kähler geometry and show how it relates to para-Kähler and para-Hermitian geometry. We also define generalized chiral geometry and discuss its subclass that is equivalent to Born geometry. A different point of view on the relationship between para-Hermitian manifolds and generalized geometry is presented in Section 5. There, we introduce *small* Courant algebroids, which are objects attached to the fundamental foliations $M$ and $\tilde{M}$ of a para-Hermitian manifold $(\mathbb{M}, M, \tilde{M}, \eta)$, and use them to explicitly construct the D-bracket.

The central role in generalized geometry is played by the *extended tangent bundle* $TN \oplus T^*N = (T \oplus T^*)N$ over, for the time being, an arbitrary manifold $N$ and geometric structures defined on this bundle. Later on, $N$ will be taken to be either $M$ (resp. $\tilde{M}$) or $\mathbb{M}$, where $M$ and $\tilde{M}$ are the fundamental foliations of a para-Hermitian manifold $(\mathbb{M}, M, \tilde{M}, \eta)$. Because the present considerations hold for any $N$, we shall abbreviate $(T \oplus T^*)N$ by $T \oplus T^*$.

The appeal of studying the extended tangent bundle is that many different geometric objects can be elegantly understood as objects on $T \oplus T^*$ in a unified way. For example, all of complex, symplectic and Poisson structures can be recast as subbundles of $T \oplus T^*$ called Dirac structures and their different integrability conditions all boil down to an involutivity condition of the corresponding Dirac structures under an appropriate bracket. Another reason to study the extended tangent bundle in our case is that it is the simplest para-

Hermitian vector bundle (see Example 2.2.3) and the extended tangent bundle $T \oplus T^*$ on a para-Hermitian manifold $(\mathbb{M}, M, \tilde{M}, \eta)$ is closely related to the extended space $\mathbb{M}$ with its split tangent bundle $T\mathbb{M} = L \oplus \tilde{L}$.

We first review basic facts from *Dirac geometry*, which studies the bundle $T \oplus T^*$ itself and its natural Courant algebroid structure. Then we will focus on the *generalized structures*, which are endomorphisms on $T \oplus T^*$ compatible with the underlying Dirac geometry. A special non-degenerate type of generalized structures give rise to *generalized metrics*, which we will also discuss along with interesting connections and bracket operations they induce on $T \oplus T^*$.

## 4.1 Dirac Geometry Review

The natural **Courant algebroid** structure [67] on $T \oplus T^*$ is given by the following data. The symmetric pairing,

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X),$$

the **Dorfman bracket**

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y \mathrm{d}\alpha, \tag{4.1.1}$$

and the anchor $\pi : X + \alpha \mapsto X$. The three structures are compatible in the following way

$$\pi(X + \alpha)\langle Y + \beta, Z + \gamma \rangle = \langle [X + \alpha, Y + \beta], Z + \gamma \rangle + \langle Y + \beta, [X + \alpha, Z + \gamma] \rangle. \tag{4.1.2}$$

In the above, $X + \alpha$ denotes a section of $T \oplus T^*$. The Dorfman bracket can be thought of as an extension of the Lie bracket from $T$ to $T \oplus T^*$ and therefore we opt to use the same notation for both brackets; the expression $[X, Y]$ is always the Lie bracket of vector fields whether we think of $[\ ,\ ]$ as the Lie bracket or the Dorfman bracket and no confusion is therefore possible.

*Remark.* The Courant algebroid structure can be equivalently given by the **Courant bracket**, which is just a skew-symmetrization of [ , ]:

$$[X+\alpha, Y+\beta]_{\text{Cour.}} = \frac{1}{2}([X+\alpha, Y+\beta] - [Y+\beta, X+\alpha])$$
$$= [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\mathrm{d}(\imath_X\beta - \imath_Y\alpha).$$

The inverse relationship is given by

$$[X+\alpha, Y+\beta] = [X+\alpha, Y+\beta]_{\text{Cour.}} + \mathrm{d}\langle X+\alpha, Y+\beta\rangle.$$

While $[\ ,\ ]_{\text{Cour.}}$ is conveniently skew-symmetric, it does not satisfy the Jacobi identity, which [ , ] does. Instead, the Jacobi identity of $[\ ,\ ]_{\text{Cour.}}$ is violated by an exact non-vanishing 3-product, which is why Courant algebroids are Lie 2-algebroids (or, Lie algebroids up to homotopy).

The Courant algebroid on $T \oplus T^*$ is exact, meaning that the associated sequence of vector bundles

$$0 \longrightarrow T^* \xrightarrow{\pi^t} T \oplus T^* \xrightarrow{\pi} T \longrightarrow 0, \tag{4.1.3}$$

is exact. Here, $\pi^t$ is the transpose of $\pi$ with respect to the pairing $\langle\ ,\ \rangle$,

$$\langle \pi^t(\alpha), Y+\beta\rangle = \langle \alpha, \pi(Y+\beta)\rangle = \langle \alpha, Y\rangle$$

i.e. $\pi^t : \alpha \mapsto \alpha + 0$. In fact, all exact Courant algebroid structures on $T \oplus T^*$ are parametrized by a closed three-form $H \in \Omega^3_{cl}$ [68], sometimes called the $H$-flux[1] or Ševera form, which enters the definition of the bracket (4.1.1), changing it to a **twisted** Dorfman bracket

$$[X+\alpha, Y+\beta]_H = [X,Y] + \mathcal{L}_X\beta - \imath_Y\mathrm{d}\alpha + \imath_Y\imath_X H. \tag{4.1.4}$$

*Remark.* In the following text we tend to omit the word *twisted* and it should be assumed we mean "twisted Dorfman bracket" whenever we say only "Dorfman bracket" unless specified otherwise.

[1] Flux is a term used mainly in physics, in this context simply meaning the "tensorial contribution to the bracket".

**The b-field transformation.** Any isotropic splitting $s : T \to T \oplus T^*$ of (4.1.3) is given by a two-form $b$, such that $X \overset{s}{\mapsto} X + b(X)$. This is equivalent to an action of a $b$-field transformation on $T \oplus T^*$[2]

**Definition 4.1.1.** *Let $b$ be an arbitrary two-form. A* **b-field transformation** *is an endomorphism of $T \oplus T^*$ given by*

$$\begin{aligned} e^b &= \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}(T \oplus T^*)) \\ u = X + \alpha &\mapsto e^b(u) = X + b(X) + \alpha \end{aligned} \tag{4.1.5}$$

The map $e^b$ satisfies $\langle e^b \cdot, e^b \cdot \rangle = \langle \cdot, \cdot \rangle$ and acts on the (twisted) Dorfman bracket as

$$[e^b(X + \alpha), e^b(Y + \beta)]_H = e^b([X + \alpha, Y + \beta]_{H + \mathrm{d}b}), \tag{4.1.6}$$

which implies that when $H$ is trivial in cohomology, then a choice of a $b$-field transformation such that $\mathrm{d}b = -H$ brings the twisted bracket $[\ ,\ ]_H$ into the standard form (4.1.1). When $H$ is cohomologically non-trivial this can be done at least locally. This also means that any choice of splitting with a non-trivial $b$-field can be absorbed into the Dorfman bracket in terms of the flux $\mathrm{d}b$.

We remark here that all the results in this thesis remain valid for any exact Courant algebroid $E$ (i.e. $E$ fits in the sequence (4.1.3)), which can be always identified with $T \oplus T^*$ by the choice of splitting equivalent to a choice of a representative $H \in \Omega^3_{cl}$. This also amounts to setting $b = 0$ in all formulas since the $b$-field appears as a difference of two splittings.

**Dirac Structures** An important class of objects in Dirac geometry are (almost) Dirac structures, which are subbundles $\mathbb{L} \subset T \oplus T^*$ with special properties.

**Definition 4.1.2.** *An* **almost Dirac structure** *$\mathbb{L}$ is a maximally isotropic subbundle of $T \oplus T^*$, i.e. $\langle u, v \rangle = 0$ for any $u, v \in \Gamma(\mathbb{L})$ and rank$(\mathbb{L})$ = rank$(T)$. When $\mathbb{L}$ is involutive under the Dorfman bracket, i.e. it satisfies $[\mathbb{L}, \mathbb{L}] \subset \mathbb{L}$, we call $\mathbb{L}$ simply a* **Dirac structure**.

An important fact we will repeatedly use is that the Dorfman bracket becomes fully skew when restricted to sections of a Dirac structure $\mathbb{L}$ and in particular becomes a Lie

[2]Here we are using the term $b$-field transformation more liberally as it is customary to use the term only in the cases when $\mathrm{d}b = 0$ so that $e^b$ is a symmetry of $[\ ,\ ]$.

algebroid bracket. $\mathbb{L}$ then inherits a Lie algebroid structure given by $([\ ,\ ]|_{\mathbb{L}}, \pi_T)$, $\pi_T$ being the projection to the tangent bundle $T$. More details about Dirac structures can be found in [69, 70, 71].

We conclude this section with a useful formula for $[\ ,\ ]$ [13, Prop. 2.7]

$$\begin{aligned}\langle [X+\alpha, Y+\beta], Z+\gamma\rangle &= \langle \nabla_X(Y+\beta) - \nabla_Y(X+\alpha), Z+\gamma\rangle \\ &\quad + \langle \nabla_Z(X+\alpha), Y+\beta\rangle,\end{aligned} \tag{4.1.7}$$

where $\nabla$ is any torsionless connection.

## 4.2 Generalized Structures

We continue by introducing generalized structures, i.e. endomorphisms of the extended tangent bundle $T\oplus T^*$ that square to $\pm\mathbb{1}$ and are (anti-)orthogonal with respect to the natural pairing $\langle\ ,\ \rangle$ on $T\oplus T^*$. This involves four different choices, but in this thesis we will only discuss the two *real* structures that square to $+\mathbb{1}$: **generalized para-complex** and **generalized product structures**. For more details on their complex counterparts, **generalized complex (GC)** and **anti-complex** structures, see for example [32] and [31], respectively.

### 4.2.1 Generalized para-complex structures

In [72, 73], the notion of generalized para-complex (GpC) geometry along with basic integrability conditions and examples was introduced. Here we review the properties of GpC structures relevant to our discussion. A more complete overview can be also found in [31].

**Definition 4.2.1.** *A* ***generalized para-complex*** *(GpC) structure $\mathcal{K}$ is an endomorphism of $T\oplus T^*$, such that $\mathcal{K}^2 = \mathbb{1}$ and $\langle\mathcal{K}\cdot,\mathcal{K}\cdot\rangle = -\langle\cdot,\cdot\rangle$, whose generalized Nijenhuis tensor*

$$\mathcal{N}_{\mathcal{K}}(u,v) = [\mathcal{K}u,\mathcal{K}v] + \mathcal{K}^2[u,v] - \mathcal{K}([\mathcal{K}u,v] + [u,\mathcal{K}v]), \tag{4.2.1}$$

*vanishes for any $u, v \in T\oplus T^*$.*

Similarly to usual endomorphisms of the tangent bundle, we use the name almost whenever we want to emphasize that integrability of $\mathcal{K}$ is not concerned. Moreover, also in direct analogy to tangent bundle geometry, the generalized Nijenhuis tensor (4.2.1) can be rewritten as

$$\mathcal{N}_{\mathcal{K}}(u,v) = 4(\mathcal{P}[\tilde{\mathcal{P}}u,\tilde{\mathcal{P}}v] + \tilde{\mathcal{P}}[\mathcal{P}u,\mathcal{P}v]),\ \mathcal{P} = \frac{1}{2}(\mathbb{1}+\mathcal{K}),\ \tilde{\mathcal{P}} = \frac{1}{2}(\mathbb{1}-\mathcal{K}),$$

which tells us that the integrability of $\mathcal{K}$ is equivalent to the involutivity of the eigenbundles of $\mathcal{K}$ under $[\ ,\ ]$. From the definition of $\mathcal{K}$ we can infer that its eigenbundles are almost Dirac structures and so $\mathcal{K}$ is integrable exactly when both its eigenbundles are Dirac structures. Moreover, a splitting of $T\oplus T^*$ into a pair of transversal Dirac structures $T\oplus T^* = \mathbb{L}\oplus\widetilde{\mathbb{L}}$ defines a generalized para-complex structure by setting $\mathcal{K}\,|_{\mathbb{L}} = \mathbb{1}$ and $\mathcal{K}\,|_{\widetilde{\mathbb{L}}} = -\mathbb{1}$:

**Theorem** ([72])**.** *There is a one-to-one correspondence between generalized para-complex structures on $M$ and pairs of transversal Dirac subbundles of $T\oplus T^*$.*

*Remark.* In the present context, it is useful to view the GpC structures as the generalized geometry analogue of para-Hermitian structures: the endomorphism $\mathcal{K}$ defines a para-complex structure on the bundle $T\oplus T^*$, and the natural metric $\langle\ ,\ \rangle$ is the para-Hermitian $O(d,d)$ structure ($d$ being the dimension of the underlying manifold). We will make this analogy more precise in Section 5.1.

The most general form of an almost GpC structure is given by

$$\mathcal{K} = \begin{pmatrix} A & \Pi \\ \Omega & -A^* \end{pmatrix}, \text{ such that } \begin{cases} A^2 + \Pi\Omega & = \mathbb{1} \\ A\Pi - \Pi A^* & = 0 \\ \Omega A - A^*\Omega & = 0 \end{cases}. \tag{4.2.2}$$

where $A\in\Gamma(\mathrm{End}(T))$ and $\Omega\in\Omega^2(M)$, $\Pi\in\Gamma(\Lambda^2 T)$ are skew tensors. We now present main examples. More can be found in [72].

**Example 4.2.2** (The trivial structure and its deformations)**.** Any manifold supports the following GpC structure

$$\mathcal{K}_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

that has eigenbundles $T$ and $T^*$ and is always integrable. The following two GpC structures can be seen as deformations of $\mathcal{K}_0$ by either a two-form $b$ or a bi-vector $\beta$:

$$\mathcal{K}_b = \begin{pmatrix} \mathbb{1} & 0 \\ 2b & -\mathbb{1} \end{pmatrix}, \quad \mathcal{K}_\beta = \begin{pmatrix} \mathbb{1} & 2\beta \\ 0 & -\mathbb{1} \end{pmatrix}.$$

$\mathcal{K}_b$ is integrable if and only if $\mathrm{d}b = 0$, i.e. $b$ is presymplectic and its eigenbundles are $\mathbb{L}_b = \mathrm{graph}(b) = \{X + b(X) \mid X \in \mathfrak{X}\}$ and $\widetilde{\mathbb{L}} = T^*$. Similarly, $\mathcal{K}_\beta$ is integrable if and only if $\beta$ is Poisson and its eigenbundles are $\mathbb{L} = T$ and $\widetilde{\mathbb{L}} = \mathrm{graph}(-\beta) = \{\alpha - \beta(\alpha)\} \mid \alpha \in \Omega\}$. ◁

**Example 4.2.3** (Para-complex structures)**.** A para-complex structure $K \in \Gamma(\mathrm{End}(T))$, defines the diagonal generalized para-complex structure:

$$\mathcal{K}_K = \begin{pmatrix} K & 0 \\ 0 & -K^* \end{pmatrix}.$$

The corresponding Dirac structures are given by $\mathbb{L} = L \oplus \tilde{L}^*$ and $\widetilde{\mathbb{L}} = \tilde{L} \oplus L^*$. The integrability of $\mathcal{K}_K$ is equivalent to Frobenius integrability of $K$ , i.e. vanishing of the Nijenhuis tensor of $K$. ◁

**Example 4.2.4** (Symplectic structures)**.** A symplectic form $\omega$ defines the anti-diagonal GpC structure

$$\mathcal{K}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The $\pm 1$ eigenbundles are given by $\mathrm{graph}(\pm\omega) = \{X \pm \omega(X) \mid X \in \mathfrak{X}\}$, and the integrability of $\mathcal{K}_\omega$ is equivalent to $\mathrm{d}\omega = 0$. ◁

### 4.2.2 Generalized product structures

**Definition 4.2.5.** *A* **generalized product structure** *(GP) is an endomorphism* $\mathcal{J} \in \Gamma(\mathrm{End}(T \oplus T^*))$ *such that* $\mathcal{J}^2 = \mathbb{1}$ *and* $\langle \mathcal{J}, \mathcal{J} \rangle = \langle \cdot, \cdot \rangle$.

For the GP structures, integrability is not well defined via the Dorfman bracket, because their eigenbundles are not isotropic with respect to the pairing $\langle\ ,\ \rangle$ and as a result the involutivity under the Dorfman bracket is not well-defined. This can be seen for example from the fact that the expression (4.2.1) is not tensorial for the GP structures. We will tackle this issue in Section 4.3.3, where we define a notion of integrability that is applicable to GP structures as well.

*Remark.* If we understand GpC structures as para-Hermitian structures on $T \oplus T^*$, the GP structures can in the same way be seen as the generalized geometry analogue of chiral structures: they are defined by a real endomorphism $\mathcal{J}$ that is an isometry of the $O(d,d)$ structure $\langle\ ,\ \rangle$ (compare Definitions 4.2.5 and 3.0.2).

A general form of GP structures is the following

$$\mathcal{J} = \begin{pmatrix} A & \tau \\ \sigma & A^* \end{pmatrix} \text{ with } \begin{cases} A^2 + \tau\sigma &= \mathbb{1}, \\ A\tau + \tau A^* &= 0, \\ \sigma A + A^*\sigma &= 0, \end{cases} \tag{4.2.3}$$

where $A \in \Gamma(\mathrm{End}(T))$ and $\tau \in \Gamma(T \otimes T)$, $\sigma \in \Gamma(T^* \otimes T^*)$ are symmetric tensors. The main examples are the following:

**Example 4.2.6** (Para-complex structures)**.** Any almost para-complex structure $J \in \Gamma(\mathrm{End}(T))$ induces a GP structure

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},$$

whose $+1$- and $-1$-eigenbundles are $\mathbb{C}_+ = C_+ \oplus C_+^*$ and $\mathbb{C}_- = C_- \oplus C_-^*$, respectively, where $C_\pm$ are eigenbundles of $J$. ◁

**Example 4.2.7** (Pseudo-Riemannian structures)**.** Any pseudo-Riemannian metric $\eta$ defines a GP structure

$$\mathcal{J}_\eta := \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix}$$

whose $\pm1$-eigenbundles are $\mathrm{graph}(\pm\eta) \subset T \oplus T^*$, implying they are isomorphic to the tangent bundle $T$. Such generalized product structures are called *generalized metrics* and are discussed in detail in Section 4.2.4. ◁

### 4.2.3 Action of b-field transformation

The $b$-field transformation (4.1.5) induces an action on endomorphisms of $T \oplus T^*$ by conjugation:

$$\begin{aligned} e^b : \mathrm{End}(T \oplus T^*) &\to \mathrm{End}(T \oplus T^*) \\ \mathcal{A} &\mapsto e^b(\mathcal{A}) = e^b \circ \mathcal{A} \circ e^{-b}. \end{aligned}$$

The properties of $e^b$ then ensure that it preserves the type of a generalized structure:

**Proposition 4.2.8** ([31])**.** *The b-field transformation preserves the type of a generalized almost structure $\mathcal{A}$ for any two-form b. This means that if $\mathcal{A}^2 = \pm\mathbb{1}$, then $[e^b(\mathcal{A})]^2 = \pm\mathbb{1}$ and if $\langle \mathcal{A}\cdot, \mathcal{A}\cdot\rangle = \pm\langle\cdot,\cdot\rangle$, then also $\langle e^b(\mathcal{A}), e^b(\mathcal{A})\rangle = \pm\langle\cdot,\cdot\rangle$. Additionally, if $\mathrm{d}b = 0$, $e^b$ also preserves the integrability of an isotropic structure $\mathcal{A}$.*

*Proof.* The fact that $e^b$ preserves type is straightforward to check:

$$\begin{aligned} e^b(\mathcal{A})e^b(\mathcal{A}) &= e^b\mathcal{A}e^{-b}e^b\mathcal{A}e^{-b} = e^b(\mathcal{A}^2) \\ \langle e^b(\mathcal{A})\cdot, e^b(\mathcal{A})\cdot\rangle &= \langle e^b\mathcal{A}e^{-b}\cdot, e^b\mathcal{A}e^{-b}\cdot\rangle = \langle \mathcal{A}e^{-b}\cdot, \mathcal{A}e^{-b}\cdot\rangle = \pm\langle e^{-b}\cdot, e^{-b}\cdot\rangle = \pm\langle\cdot,\cdot\rangle \\ &= \langle \mathcal{A}\cdot, \mathcal{A}\cdot\rangle. \end{aligned}$$

We now prove the statement about the integrability for $\mathcal{A}$ a GpC structure, for GC structures the proof is analogous except the appearing bundles are complexified. Let now $\mathcal{A}$ be integrable and $u, v \in \Gamma(T \oplus T^*)$ be $+1$ eigenvectors of $\mathcal{A}$. Then $e^b(u)$ and $e^b(v)$ are $+1$ eigenvectors of $e^b(\mathcal{A})$. Using (4.1.6) and $\mathrm{d}b = 0$:

$$[e^b(u), e^b(v)]_H = e^b[u, v]_H,$$

so that the $+1$ eigenbundle of $e^b(\mathcal{A})$ is involutive. Similar argument shows involutivity of the $-1$ eigenbundle of $e^b(\mathcal{A})$. □

### 4.2.4 Generalized metrics and related structures

In this section, we discuss generalized metrics, which are a special, *non-degenerate* case of generalized product structures. Such structures are generically given by $b$-field transformations of the structure $\mathcal{J}_\eta$ in Example 4.2.7. We also recall some properties of the generalized

Bismut connection, which is a generalized connection on $T \oplus T^*$ that one can naturally associate to any generalized metric.

**Definition 4.2.9.** *A* **generalized (indefinite) metric** $\mathcal{G}$ *is a non-degenerate generalized product structure, which means that* $\mathcal{G}$ *defines a metric (that is, a non-degenerate symmetric tensor) on* $T \oplus T^*$ *by*

$$G(u, v) \coloneqq \langle \mathcal{G}u, v \rangle$$

*for all* $u, v \in \Gamma(T \oplus T^*)$.

*Remark.* The name generalized metric is typically used in the literature when $G$ is positive-definite, but here we use the term for indefinite metrics as well, emphasizing this fact by the name "indefinite generalized metric" whenever necessary. We also note that the discussion below was first presented in the positive definite case in [32].

Let us now describe what non-degeneracy implies for the general form (4.2.3) of generalized product structures. It is easy to show that for a GP structure $\mathcal{J}$ to be non-degenerate, its upper right corner has to be an invertible map $T^* \to T$. Whenever this is the case, the system of equations in (4.2.3) can be solved explicitly in terms of a pseudo-Riemannian metric $\eta \coloneqq \tau^{-1}$ and a two-form $b \coloneqq -\eta A$. The structure $\mathcal{J}$ is then simply the b-transform of the GP structure $\mathcal{J}_\eta$ from Example 4.2.7:

$$\mathcal{J} = \mathcal{G}(\eta, b) \coloneqq e^b(\mathcal{J}_\eta) = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}. \tag{4.2.4}$$

The eigenbundles of $\mathcal{G}(\eta, b)$ are $\mathbb{C}_\pm = \mathrm{graph}(b \pm \eta)$ and are therefore isomorphic to $T$. We denote these isomorphisms by $\pi_\pm : \mathbb{C}_\pm \xrightarrow{\simeq} T$ so that

$$\begin{aligned} \pi_\pm(X + \alpha) &= X, \quad X + \alpha \in \Gamma(\mathbb{C}_\pm), \\ \pi_\pm^{-1}(X) &= X + (b \pm \eta)X, \quad X \in \Gamma(T). \end{aligned} \tag{4.2.5}$$

We also recall the following useful formula that recovers the metric $\eta$ from $\mathcal{G} = \mathcal{G}(\eta, b)$:

$$\eta(X, Y) = \frac{1}{2}\langle \mathcal{G}\pi_\pm^{-1}X, \pi_\pm^{-1}Y \rangle = \pm\frac{1}{2}\langle \pi_\pm^{-1}X, \pi_\pm^{-1}Y \rangle \tag{4.2.6}$$

for all $X, Y \in \Gamma(T)$.

**Generalized Bismut Connection.** To any generalized metric, one can associate a generalized connection called the *generalized Bismut connection*, which plays a central role in Section 4.3.3 where it is used to define integrability of certain non-isotropic generalized structures.

We start with the definition of a generalized connection (in the context of the exact Courant algebroid $T \oplus T^*$) and its generalized torsion from [33]:

**Definition 4.2.10.** *Consider the $H$-twisted Courant algebroid structure on $T \oplus T^* \xrightarrow{\pi} T$ and let $u, v, w \in \Gamma(T \oplus T^*)$ be arbitrary sections. A* **generalized connection** *$D$ on this Courant algebroid is a map*

$$\begin{aligned} D: \ \Gamma(T \oplus T^*) \times \Gamma(T \oplus T^*) &\longrightarrow \Gamma(T \oplus T^*) \\ (u, v) &\longmapsto D_u v, \end{aligned}$$

*satisfying $D_{fu} v = f D_u v$ and $D_u f v = f D_u v + \pi(u)[f] v$ for any $f \in C^\infty$. Moreover, $D$ is compatible with the pairing $\langle \ , \ \rangle$[3]:*

$$\langle D_u v, w \rangle + \langle v, D_u w \rangle = \pi(u)[\langle v, w \rangle].$$

*The* **generalized torsion** *of $D$ is defined with respect to the $H$-twisted Dorfman bracket as*

$$T^D(u, v, w) = \langle D_u v - D_v u - [u, v]_H, w \rangle + \langle D_w u, v \rangle. \tag{4.2.7}$$

Therefore, Definition 4.2.10 generalizes the notion of an ordinary connection on the bundle $T \oplus T^*$ to a setting where the linear slot of the connection $D_\bullet$ can be contracted with a section of $T \oplus T^*$ instead of just a vector field. Using the splitting $u = X + \alpha$, we can separate $D$ into two parts:

$$D_{X+\alpha} = \nabla_X + \chi(\alpha), \tag{4.2.8}$$

where $\nabla$ is a regular connection on the bundle $T \oplus T^*$ and $\chi$ is a $\mathfrak{so}(T \oplus T^*)$-valued vector field. Therefore, we are taking a derivative along the vector field part $X$ just like in the case of an ordinary connection, but we are additionally allowing for an endomorphism of $T \oplus T^*$ that depends on the one-form part $\alpha$ of the section $u$.

[3]Sometimes this property is taken to be an extra *unitarity* requirement on the generalized connection

Just like the D-torsion, the generalized torsion is then a straightforward analogue of the usual torsion on the bundle $T \oplus T^*$, where the role of the usual Lie bracket is replaced by the Dorfman bracket, i.e. the generalized torsion $T^D$ of a generalized connection $D$ vanishes if and only if $D$ defines the $H$-twisted Dorfman bracket by the formula

$$\langle [u,v]_H, w\rangle = \langle D_u v - D_v u, w\rangle + \langle D_w u, v\rangle. \tag{4.2.9}$$

We will discuss the relationship between the D-torsion and the generalized torsion in the Section 5.1. For now, we notice that when the generalized connection is given by the diagonal action of the regular connection, its generalized torsion is simply a cyclic sum of its torsion (modulo $H$-flux):

**Lemma 4.2.11.** *Let $D = \nabla$ be a generalized connection on $T \oplus T^*$ given by the diagonal action of a regular connection on $T \oplus T^*$, i.e. $\chi = 0$ in (4.2.8). Then the generalized torsion of $D$ is given by the (ordinary) torsion $T$ of $\nabla$ and the $H$-flux of $T \oplus T^*$:*

$$T^D(u,v,w) = -H(X,Y,Z) + \langle T(X,Y),\gamma\rangle + \langle T(Z,X),\beta\rangle + \langle T(Y,Z),\alpha\rangle,$$

*where $u = X + \alpha$, $v = Y + \beta$ and $w = Z + \gamma$.*

*Proof.* The formula is easily proved using (4.1.7). □

We continue our discussion by recalling the definition of a generalized Bismut connection from [74], extending it to indefinite metrics:

**Definition 4.2.12.** *Let $\mathcal{G} = \mathcal{G}(\eta, b) \in \Gamma(\mathrm{End}(T \oplus T^*))$ be a generalized metric and denote its eigenbundles by $\mathbb{C}_\pm$. We split sections $u \in \Gamma(T \oplus T^*)$ accordingly and denote the projections by subscripts, $u = u_+ + u_-$ with $u_\pm \in \mathbb{C}_\pm$. The following expression then defines a generalized connection parallelizing $\mathcal{G}$ called the* ***generalized Bismut connection****:*

$$D^H_u v = [u_-, v_+]_{H+} + [u_+, v_-]_{H-} + [\mathcal{C}u_-, v_-]_{H-} + [\mathcal{C}u_+, v_+]_{H+} \tag{4.2.10}$$

*for all $u, v \in \Gamma(T \oplus T^*)$, where $[\ ,\ ]_H$ is the twisted Dorfman bracket and $\mathcal{C}$ is the almost generalized para-complex structure*

$$\mathcal{C} = \begin{pmatrix} \mathbb{1} & 0 \\ 2b & -\mathbb{1} \end{pmatrix} = e^b \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} e^{-b} \in \Gamma(\mathrm{End}(T \oplus T^*)),$$

*which maps* $\mathbb{C}_\pm \overset{\mathcal{C}}{\mapsto} \mathbb{C}_\mp$.

The generalized Bismut connection has a generalized torsion [31] $T^D = 2\pi_+^* H_b + 2\pi_-^* H_b$. Additionally, it is related to two "usual" connections $\nabla^\pm$ via the isomorphisms $\pi_\pm$:

$$\begin{aligned} D_u v &= \pi_+^{-1} \nabla^+_{\pi(u)} \pi_+ v_+ + \pi_-^{-1} \nabla^-_{\pi(u)} \pi_- v_-, \\ \nabla^\pm &= \mathring{\nabla} \pm \frac{1}{2} \eta^{-1} H_b, \end{aligned} \tag{4.2.11}$$

where $\mathring{\nabla}$ is the Levi-Civita connection of $\eta$ in $\mathcal{G}(\eta, b)$ and $H_b$ is the $H$-flux of the Courant algebroid with the $b$-field absorbed, $H_b = H + \mathrm{d}b$.

The connections $\nabla^\pm$ appear in physics as the natural connections in the context of supersymmetry, particularly $2D$ $(2,2)$ supersymmetric $\sigma$-models. The reason for this is that they parallelize the metric $\eta$ and have fully skew torsion equal to $T^{\nabla^\pm} = \pm H_b$.

## 4.3 Commuting Pairs of Generalized Structures

We will now study pairs of generalized para-complex and chiral structures $(\mathcal{J}_+, \mathcal{J}_-)$ that commute and their product is non-degenerate in the sense that it defines a generalized metric[4]. Let $(\mathcal{J}_\pm)$ be such pair and denote $\mathcal{G} = \mathcal{J}_+ \mathcal{J}_-$, then any pair out of the three endomorphisms $(\mathcal{G}, \mathcal{J}_\pm)$ commute and we will call such pairs **commuting pairs**. Our discussion will follow [31], where more details can be found. Several of the constructions below are also analogous to ones well known in the context of generalized Kähler (GK) geometry. To consult the classical literature on GK geometry, see [32, 33].

### 4.3.1 Generalized para-Kähler geometry

We start with the definition:

**Definition 4.3.1.** *An* ***almost generalized para-Kähler structure*** *(GpK) is a commuting pair* $(\mathcal{G}, \mathcal{K}_+)$ *of a split signature generalized metric* $\mathcal{G} = \mathcal{G}(\eta, b)$ *and a GpC structure* $\mathcal{K}_+$*. If additionally both* $\mathcal{K}_+$ *and* $\mathcal{K}_- \coloneqq \mathcal{G}\mathcal{K}_+$ *are integrable with respect to the (twisted) Dorfman bracket, we call* $(\mathcal{G}, \mathcal{K}_+)$ *a (twisted) GpK structure.*

[4] Notice that any product of two generalized structures that commute will define an endomorphism of $T \oplus T^*$ of the appropriate type, i.e. it will square to $\mathbb{1}$ and be orthogonal with respect to $\langle\ ,\ \rangle$.

Since any two structures in the triple $(\mathcal{G},\mathcal{K}_+,\mathcal{K}_-)$ determine the third, we may refer to the GpK structure $(\mathcal{G},\mathcal{K}_+)$ by the pair $(\mathcal{K}_+,\mathcal{K}_-)$, in particular when integrability – which is tied with $\mathcal{K}_\pm$ – is discussed. As the name suggests, GpK structures generalize para-Kähler geometry:

**Example 4.3.2** (Para-Kähler manifolds)**.** Let $(\eta, K)$ be an almost para-Hermitian structure, with $\omega = \eta K$ the fundamental form. Then

$$\mathcal{K}_+ = \begin{pmatrix} K & 0 \\ 0 & -K^* \end{pmatrix},\ \mathcal{K}_- = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix},\ \mathcal{G} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix},$$

gives an almost generalized para-Kähler structure that is integrable if and only if $(\eta, K)$ is para-Kähler. ◁

Let $\mathbb{C}_\pm$ be the eigenbundles of $\mathcal{G}(\eta, b)$. From the fact that $\mathcal{G} = \mathcal{K}_+\mathcal{K}_-$, we see that $\mathcal{K}_+ |_{\mathbb{C}_\pm} = \pm\mathcal{K}_- |_{\mathbb{C}_\pm}$, allowing us to define two para-complex structures $K_\pm$ as follows:

$$K_+ = \pi_+\mathcal{K}_\pm\pi_+^{-1} \quad K_- = \pm\pi_-\mathcal{K}_\pm\pi_-^{-1}. \tag{4.3.1}$$

Using (4.2.6), it can be easily checked that $\eta(K_\pm X, K_\pm Y) = -\eta(X,Y)$ and $\eta K_\pm \coloneqq \omega_\pm$ defines two almost symplectic forms, therefore $(\eta, K_\pm)$ are two almost para-Hermitian structures. We therefore see that any (almost) generalized para-Kähler structure defines an (almost) bi-para-Hermitian structure $(\eta, K_\pm)$ with extra data given by the two-form $b$. The converse is also true; given $(K_\pm, \eta, b)$ we reconstruct the isomorphisms $\pi_\pm$ (4.2.5) and use them to define a pair of commuting structures $\mathcal{K}_\pm$ using $K_\pm$:

$$\mathcal{K}_\pm = \pi_+^{-1}K_+\pi_+P_{\mathbb{C}_+} \pm \pi_-^{-1}K_-\pi_-P_{\mathbb{C}_-}, \tag{4.3.2}$$

where $P_{\mathbb{C}_\pm}$ are the projections onto $\mathbb{C}_\pm$ given by $P_{\mathbb{C}_\pm} = \frac{1}{2}(\mathbb{1} \pm \mathcal{G})$. In matrix form, this yields an expression similar to one well-known from GK geometry

$$\mathcal{K}_\pm = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix}\begin{pmatrix} K_+ \pm K_- & \omega_+^{-1} \mp \omega_-^{-1} \\ \omega_+ \mp \omega_- & -(K_+^* \pm K_-^*) \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}. \tag{4.3.3}$$

Clearly, this recovers Example 4.3.2 in the limit $b = 0$ and $K_+ = \pm K_-$. We saw that in such limit the integrability of the simple GpK structure is equivalent $(\eta, K)$ being para-Kähler. In the general bi-para-Hermitian case, the statement is more complicated, as we showed in

[31]:

**Theorem 4.3.3** ([31])**.** *A generalized almost para-Kähler structure* $(\mathcal{K}_+, \mathcal{K}_-)$*, given alternatively by the induced bi-para-Hermitian data* $(K_+, K_-, \eta, b)$*, is integrable if and only if the following conditions are simultaneously satisfied:*

1. $K_\pm$ *are integrable para-Hermitian structures, that is, their Nijenhuis tensors vanish,*
2. $\mathrm{d}^P_+\omega_+ = -\mathrm{d}^P_-\omega_- = -(H + \mathrm{d}b)$*,*

*where* $\mathrm{d}^P_\pm = (\partial_\pm - \tilde{\partial}_\pm)$ *are the* $\mathrm{d}^P$ *operators* (2.1.8) *of* $K_\pm$*.*

The proof of this statement can be found in [31]. We conclude this section with an explicit non-trivial example of a GpK structure taken also from [31].

**Example 4.3.4** ([31])**.** The para-quaternions are defined as

$$\mathbb{H}' = \{q = x_1 + x_2 i + x_3 j + x_4 k : -i^2 = j^2 = k^2 = 1, k = ij, ij = -ji\}.$$

$\mathbb{H}'$ is therefore a 4-dimensional vector space with six natural endomorphisms given by the left/right multiplications by $i, j, k$. We denote $K_-$ $(K_+)$ the para-complex structures defined by the left (right) multiplication by $k$.

Consider now the following quotient:

$$\mathbb{M} = \left(\mathbb{H}' \backslash \{x_1^2 + x_2^2 = x_3^2 + x_4^2\}\right) \Big/ \sim,$$

where $q \sim 2q$ for all $q \in \mathbb{H}' \backslash \{x_1^2 + x_2^2 = x_3^2 + x_4^2\}$. The structures $K_\pm$ described above descend to the quotient $\mathbb{M}$ and we also get a signature $(2,2)$ metric

$$\eta = \frac{1}{|q|^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3 - dx_4 \otimes dx_4),$$

where

$$|q|^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2,$$

is the pseudo-norm of $q \in \mathbb{M}$. One can check that $\eta(K_\pm \cdot, K_\pm \cdot) = -\eta(\cdot, \cdot)$ and also

$$\mathrm{d}^P_{K_\pm}\omega_{K_\pm} = \pm H,$$

where

$$H = \frac{2}{|q|^4}(x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3).$$

Clearly, $dH = 0$, which means that $(K_\pm, \eta)$ is a GpK structure on $\mathbb{M}$ with a non-zero flux $H$. In [31], it is also shown that there is a similar GpK structure on $\mathbb{M}$ associated to the multiplication by $j$ on $\mathbb{H}'$. ◁

### 4.3.2 Generalized Chiral Structures

In this section, we explore a generalization of chiral geometry discussed in Chapter 3 and consequently show that a special case of this geometry is equivalent to Born geometry.

**Definition 4.3.5.** *An* ***almost generalized chiral structure*** *(GCh) is a commuting pair $(\mathcal{G}, \mathcal{J}_+)$ consisting of a generalized metric $\mathcal{G} = \mathcal{G}(g, b)$ and a GP structure $\mathcal{J}_+$.*

Note that given an almost generalized chiral structure $(\mathcal{G}, \mathcal{J}_+)$, the product $\mathcal{J}_- \coloneqq \mathcal{G}\mathcal{J}_+$ is another GP structure. The generalized almost structures defining a generalized chiral structure therefore all have non-isotropic eigenbundles, and so there is no notion of integrability for such structures in terms of the Dorfman bracket as in GK/GpK geometry. We nonetheless tackle the issue of integrability for these structures in Section 4.3.3.

The canonical example of an almost generalized chiral structure is given by usual chiral geometry (see Chapter 3):

**Example 4.3.6** (Chiral geometry)**.** Let $(\eta, J)$ be an almost chiral structure with $\mathcal{H} \coloneqq \eta J$ as in Chapter 3. Define

$$\mathcal{G}(\eta) = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix}, \quad \mathcal{G}(\mathcal{H}) = \begin{pmatrix} 0 & \mathcal{H}^{-1} \\ \mathcal{H} & 0 \end{pmatrix}, \quad \mathcal{J}_+ = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}.$$

Then, both $(\mathcal{G}(\mathcal{H}), \mathcal{J}_+)$ and $(\mathcal{G}(\eta), \mathcal{J}_+)$ define GCh structures such that $\mathcal{J}_- = \mathcal{G}(\mathcal{H})\mathcal{J}_+ = \mathcal{G}(\eta)$. ◁

Let now $(\mathcal{G} = \mathcal{G}(\mathcal{H}, b), \mathcal{J}_+)$ be a generalized chiral structure and denote the eigenbundles of $\mathcal{G}$ by $\mathbb{C}_\pm$. Similarly to GpK geometry, the GCh structure corresponds to a pair of endomorphisms $J_\pm$ via the maps $\pi_\pm$ (4.2.5):

$$J_+ = \pi_+ \mathcal{J}_\pm \pi_+^{-1}, \quad J_- = \pm \pi_- \mathcal{J}_\pm \pi_-^{-1}, \tag{4.3.4}$$

such that $(J_\pm, \mathcal{H})$ is a pair of almost chiral structures on the tangent bundle. Conversely, the formula that recovers the generalized chiral data from $(J_\pm, \mathcal{H}, b)$ is given by

$$\mathcal{J}_\pm = \pi_+^{-1} J_+ \pi_+ P_{\mathbb{C}_+} \pm \pi_-^{-1} J_- \pi_- P_{\mathbb{C}_-}, \tag{4.3.5}$$

where $P_{\mathbb{C}_\pm} = \frac{1}{2}(\mathbb{1} \pm \mathcal{G})$ are the projections onto $\mathbb{C}_\pm$. The usual expressions in the matrix form are

$$\mathcal{J}_\pm = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & \eta_+^{-1} \mp \eta_-^{-1} \\ \eta_+ \mp \eta_- & J_+^* \pm J_-^* \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}, \tag{4.3.6}$$

where $\eta_\pm := \mathcal{H} J_\pm$ denote the two metrics associated to $(J_\pm, \mathcal{H})$.

**Born Geometry as a Generalized Chiral Structure** We now explain how Born geometry fits in the picture of commuting pairs as a generalized chiral structure with anti-commuting tangent bundle data. We stated the following result in [31].

**Proposition 4.3.7** ([31]). *Let $(\mathcal{G}(\mathcal{H}, b), \mathcal{J})$ be a generalized chiral structure and let $(J_\pm, \mathcal{H})$ be the corresponding tangent bundle data. Then $\{J_+, J_-\} = 0$ is equivalent to $(I = J_+ J_-, J = J_+, K = J_-, \mathcal{H})$ defining an (almost) Born structure.*

*Proof.* From Section 3.1.3 we know that the data $(\eta, I, J, K)$ of an (almost) Born structure induces a pair of chiral structures $(J, \mathcal{H})$ and $(K, \mathcal{H})$ with $\{J, K\} = 0$, where $\mathcal{H} = \eta J$. This pair is enough to construct the generalized chiral structure $(\mathcal{G}(\mathcal{H}, b), \mathcal{J})$ with arbitrary $b$. The converse follows once we check the appropriate compatibility properties showing that the tangent bundle structures $J_\pm$ and $\mathcal{H}$ with $\{J_+, J_-\} = 0$ define an almost Born structure with $I = J_+ J_-$, $J = J_+$, $K = J_-$ and $\eta = \mathcal{H} J$, $\omega = \eta K$. □

### 4.3.3 Integrability via generalized Bismut connections

We will now show that the integrability of a generalized para-Kähler structure $(\mathcal{G}, \mathcal{K}_\pm)$ can be formulated in terms of the generalized Bismut connection of $\mathcal{G}$ and that in this way a notion of integrability can be defined for generalized chiral structures as well. We present the statements without proofs and technical lemmas, all of which can be found in our work with Hu and Moraru [31].

In the following, we use and extend the idea of [74], where it is proved that an almost $GK$ structure $(\mathcal{G}, \mathcal{I}_\pm)$ is integrable if and only if the generalized Bismut connection $D$ of $\mathcal{G}$

parallelizes the GC structures $\mathcal{I}_\pm$, and its generalized torsion is of the type $(2,1)+(1,2)$ with respect to both the GC structures $\mathcal{I}_\pm$. The idea of this section is to show that an analogous statement holds for the generalized para-Kähler structure $(\mathcal{G},\mathcal{K}_\pm)$ (Definition 4.3.1) and works well as a definition of integrability for the generalized chiral structures $(\mathcal{G},\mathcal{J}_\pm)$ (Definition 4.3.5). As an intermediate step, we define a notion of *weak* integrability of these structures by requiring only $D\mathcal{K}_\pm = 0$ or $D\mathcal{J}_\pm = 0$; further restrictions on the type of the generalized torsion of $D$ then defines *full* integrability. In the case of GpK geometry we require that the type is $(2,1)+(1,2)$ with respect to both $\mathcal{K}_\pm$ (analogously to GK geometry), while in the case of generalized chiral structures we require that the type is $(3,0)+(0,3)$ with respect to $\mathcal{J}_\pm$. In this way, we can talk about integrability of generalized structures even if their eigenbundles are not isotropic, which is in particular the case for generalized chiral structures.

An additional advantage of this approach is that it provides a natural way to weaken the integrability. As we will see, weak integrability relaxes the Frobenius integrability of the corresponding tangent bundle bi-para-Hermitian (4.3.1) or bi-chiral (4.3.4) structures, which can sometimes be desirable from the point of view of physics. For example, the para-Hermitian geometry of DFT may not always be fully integrable and various DFT fluxes enter as an obstruction to integrability. Moreover, in applications to non-linear supersymmetric $\sigma$-models, where the generalized geometry typically enters in the form of the tangent bundle endomorphisms $K_\pm$ (4.3.1) and $J_\pm$ (4.3.4), it has been observed that sometimes only the requirement that these endomorphisms are parallelized by the connections $\nabla^\pm$ (4.2.11), might be sufficient [75, 37]. As we will show in Proposition 4.3.9, this is exactly equivalent to our condition of weak integrability.

The definition of weak integrability of generalized para-Kähler/chiral structures is given by the following

**Definition 4.3.8.** *Let $\mathcal{G}$ be a generalized (pseudo-)metric, $D$ its generalized Bismut connection and let $\mathcal{A}$ be a generalized almost para-complex or product structure that commutes with $\mathcal{G}$. We say $\mathcal{A}$ is* **weakly integrable** *if $D\mathcal{A} = 0$.*

Note that it follows that in the above definition when $\mathcal{A}$ is weakly integrable then also $\mathcal{A}' = \mathcal{G}\mathcal{A}$ is weakly integrable. We will now analyse what the condition $D\mathcal{A} = 0$ means in terms of the tangent bundle data corresponding to $(\mathcal{G},\mathcal{A})$. Recall that for any commuting pair $(\mathcal{G},\mathcal{A})$, where $\mathcal{G}$ is an (indefinite) generalized metric, we get a pair of tangent bundle endomorphisms via $A_\pm = \pm\pi_\pm\mathcal{A}\pi_\pm^{-1}$.[5] This can be inverted into a formula for $\mathcal{A}$ in terms of $A_\pm$:

$$\mathcal{A} = \pi_+^{-1}A_+\pi_+P_+ + \pi_-^{-1}A_-\pi_-P_-, \tag{4.3.7}$$

where $P_\pm = \frac{1}{2}(\mathbb{1} \pm \mathcal{G})$ projects from $T \oplus T^*$ to $\mathbb{C}_\pm$. Using (4.3.7) and (4.2.11) we can now rephrase the weak integrability condition $D\mathcal{A} = 0$ in terms of $\nabla^\pm$ and $A_\pm$:

**Proposition 4.3.9** ([31]). *Let $(\mathcal{G}, \mathcal{A})$ be a commuting pair with $\mathcal{G}$ a (indefinite) generalized metric and $D$ the generalized Bismut connection of $\mathcal{G}$ given by (4.2.11). Then $D\mathcal{A} = 0$ if and only if $\nabla^\pm A_\pm = 0$, $A_\pm$ being the tangent bundle endomorphisms corresponding to $\mathcal{A}$.*

As we mentioned above, the weak integrability condition is for the $GpK$ structure simply a weakening of the usual integrability conditions (Theorem 4.3.3), which is the content of the following statement proved in [31]:

**Proposition 4.3.10** ([31]). *Let $(\mathcal{G}, \mathcal{K}_\pm)$ be an almost GpK structure. Then $\mathcal{K}_\pm$ are weakly integrable if and only if the fundamental forms $\omega_\pm$ of the induced bi-para-Hermitian data $(K_\pm, \eta)$ and the corresponding Nijenhuis tensors $N_{K_\pm}$ are related to the flux $H_b = H + \mathrm{d}b$ by*

$$\begin{aligned} \mathrm{d}\omega_\pm^{(3,0)_\pm} &= \mp 3H_b^{(3,0)_\pm} = \pm\frac{3}{4}N_{K_\pm}^{(3,0)_\pm}, \qquad \mathrm{d}\omega_\pm^{(2,1)_\pm} = \mp H_b^{(2,1)_\pm}, \\ \mathrm{d}\omega_\pm^{(0,3)_\pm} &= \pm 3H_b^{(0,3)_\pm} = \mp\frac{3}{4}N_{K_\pm}^{(0,3)_\pm}, \qquad \mathrm{d}\omega_\pm^{(1,2)_\pm} = \pm H_b^{(1,2)_\pm}, \end{aligned} \tag{4.3.8}$$

*where $(k,l)_\pm$ denotes the bigrading associated to $K_\pm$.*

To see that the equations (4.3.8) describe a weakening of the usual integrability conditions, we simply notice that whenever the $(3,0)_\pm + (0,3)_\pm$ components of $\mathrm{d}\omega_\pm$ or equivalently of $H_b$ vanish, the almost para-complex structures $K_\pm$ are integrable and the equations for the $(2,1) + (1,2)$ components can then be simply rewritten as

$$\mathrm{d}_\pm^P\omega_\pm = \pm H_b,$$

recovering the integrability conditions of Theorem 4.3.3. Intuitively, this also shows why one needs to impose a restriction on the type of the generalized torsion of $D$ in order to get the full integrability of $(\mathcal{G}, \mathcal{K}_\pm)$:

[5] In terms of the concrete notation used previously, $\mathcal{A} = \mathcal{K}$ or $\mathcal{A} = \mathcal{J}$ and correspondingly, $A_\pm = K_\pm/J_\pm$.

**Theorem 4.3.11** ([31]). *An almost generalized para-Kähler structure* $(\mathcal{G}, \mathcal{K}_+)$ *is integrable and in particular both* $\mathcal{K}_\pm$ *are generalized para-complex structures if and only if* $D\mathcal{K}_\pm = 0$ *and* $T^D$ *is of type* $(2,1) + (1,2)$ *with respect to both* $\mathcal{K}_\pm$*.*

We now turn to generalized chiral structures. In this case, we know that the results cannot be fully analogous because the corresponding tangent bundle geometry is very different; for example, the fundamental tensor of the tangent bundle chiral structure, $F(X,Y,Z) = \eta((\mathring{\nabla}_X J)Y, Z)$, is not fully skew and is of type $(2,1) + (1,2)$ (with respect to $J$) and so is the Nijenhuis tensor $N_J(X,Y,Z) = \eta(N_J(X,Y),Z)$. However, they can still be related to the flux $H_b$:

**Proposition 4.3.12** ([31]). *An almost generalized chiral structure* $(\mathcal{G}, \mathcal{J})$ *is weakly integrable if and only if the fundamental tensors* $F_\pm$ *of the corresponding tangent bundle structures* $(\eta, J_\pm)$ *are related to the* $H_b$*-flux by*

$$F_\pm(X,Y,Z) = \mp\frac{1}{2}\left(H_b(X, J_\pm Y, Z) - H_b(X, Y, J_\pm Z)\right), \tag{4.3.9}$$

*Equivalently, both* $(\eta, J_\pm)$ *are of type* $\mathcal{W}_3$ *almost product pseudo-Riemannian structures whose Nijenhuis tensors* $N_{J_\pm}$ *are related to* $H$ *by*

$$N_{J_\pm}(X,Y,Z) = \pm 2\left(H_b^{(2,1)_\pm+(1,2)_\pm}(J_\pm X, Y, J_\pm Z) + H_b^{(2,1)_\pm+(1,2)_\pm}(X, J_\pm Y, J_\pm Z)\right)$$

The properties of the fundamental tensor $F$ (see for example [76]) imply that $H_b$ determines all non-zero components of $F$. Furthermore, in contrast to the G(p)K geometry where the weak integrability relates all components of $H_b$ to components of the fundamental forms $\omega_\pm$ and integrability of the tangent bundle structures is controlled by the $(3,0)+(0,3)$ parts, in the generalized chiral case the weak integrability condition $D\mathcal{J}_\pm = 0$ only fixes the $(2,1)+(1,2)$ components of $H_b$, which are also the components tied to integrability of the tangent bundle structures.

We therefore introduce the following definition of the (full) integrability for generalized chiral structures:

**Definition 4.3.13.** *Let* $(\mathcal{G}, \mathcal{J}_\pm)$ *be an almost generalized chiral structure and* $D$ *the generalized Bismut connection of* $\mathcal{G}$*. We say* $(\mathcal{G}, \mathcal{J}_\pm)$ *is* ***integrable*** *when it is weakly integrable and the generalized torsion of* $D$ *is of type* $(3,0)+(0,3)$ *with respect to both* $\mathcal{J}_\pm$*.*

We then have the following statement:

**Theorem 4.3.14** ([31]). *An almost generalized chiral structure* $(\mathcal{G}, \mathcal{J}_+)$ *is integrable if and only if the corresponding tangent bundle data* $(\eta, J_\pm)$ *are integrable type* $\mathcal{W}_0$ *chiral structures.*

The notion of integrability for generalized chiral structures therefore forces the corresponding tangent bundle structures to be both integrable and of type $\mathcal{W}_0$, which according to our discussion in Section 3.1.1 is the chiral geometry analogue of integrable para-Kähler structures.

# Chapter 5

# The Courant algebroids of a para-Hermitian manifold

So far, we have discussed generic properties of the bundle $(T \oplus T^*)N$ over an arbitrary differentiable manifold $N$ and showed that certain structures on this bundle render the base manifold para-Hermitian or Born. In such case, we would call $(T \oplus T^*)\mathbb{M}$ the standard (or large) Courant algebroid of $\mathbb{M}$. If the para-Hermitian structure on $\mathbb{M}$ is integrable, the foliation manifolds give rise to an additional pair of Courant algebroids called small:

**Definition 5.0.1.** *Let* $(\mathbb{M}, M, \tilde{M}, \eta)$ *be a para-Hermitian manifold. We call* $(T \oplus T^*)\mathbb{M}$ *the* **standard** *or* **large Courant algebroid** *of* $\mathbb{M}$*, while* $(T \oplus T^*)M$ *and* $(T \oplus T^*)\tilde{M}$ *are called the* **small Courant algebroids** *of* $\mathbb{M}$*.*

The large Courant algebroid (CA) is the standard CA over the whole manifold $\mathbb{M}$ and the small CAs are the standard CAs over the half-dimensional foliations $M$ and $\tilde{M}$. Therefore, while the large CA always exists, the small CAs require integrability of $L$ or $\tilde{L}$, which by itself are sufficient conditions for the existence of the small CAs on $M$ or $\tilde{M}$, respectively. In the forthcoming discussion we will explore the relationship between the large and small CAs as well as each of the pictures separately.

We also show that there are natural maps between the small CAs and the tangent bundle of $\mathbb{M}$. These maps are closely related to T-duality on the para-Hermitian manifold and allow us to realize the D-bracket as a sum of the Dorfman brackets on the respective small CAs. Moreover, structures on the tangent bundle $T\mathbb{M}$ such as the Born structures and the Born connection can be viewed as well-known generalized structures on the small CAs.

## 5.1 The D-bracket from the small Courant algebroids

Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold. We can define the following vector bundle isomorphisms

$$\begin{array}{rl|rl} \rho : T\mathbb{M} = L \oplus \tilde{L} \to L \oplus L^* & & \tilde{\rho} : T\mathbb{M} = L \oplus \tilde{L} \to \tilde{L} \oplus \tilde{L}^* & \\ X = \mathbf{x} + \tilde{\mathbf{x}} \mapsto \mathbf{x} + \eta(\tilde{\mathbf{x}}) & & X = \mathbf{x} + \tilde{\mathbf{x}} \mapsto \tilde{\mathbf{x}} + \eta(\mathbf{x}) \ , & \end{array} \tag{5.1.1}$$

where $\eta(X)$ is a one-form defined by $\eta(X, \cdot)$. The fact that $\eta(\tilde{\mathbf{x}})$ is an element of $L^*$ follows from the fact that $\eta(\tilde{\mathbf{x}})$ only contracts with a vector in $L$, which is a result of $L$ being isotropic with respect to $\eta$ (and similarly for $\eta(\mathbf{x})$ and $\tilde{L}$).

Assume now that the para-Hermitian structure on $\mathbb{M}$ is integrable and let $M$ and $\tilde{M}$ be the corresponding fundamental foliations of $\mathbb{M}$. We have by definition $L_p = T_p M$ and $L_p^* = T_p^* M$ for every point $p \in \mathbb{M}$, which means $T\mathbb{M}|_M \overset{\rho}{\simeq} (T \oplus T^*)M$. An analogous statement holds for $(T \oplus T^*)\tilde{M}$ as well. In fact, each statement holds separately for half-integrable para-Hermitian structures on $\mathbb{M}$. To summarize, we have

**Proposition 5.1.1.** *Let $(\mathbb{M}, M, \tilde{M}, \eta)$ be a para-Hermitian manifold and let $\rho$ and $\tilde{\rho}$ be the maps defined by (5.1.1). Then the restrictions of $T\mathbb{M}$ to $M$ and $\tilde{M}$ are isomorphic to the bundles $(T \oplus T^*)M$ and $(T \oplus T^*)\tilde{M}$, respectively.*

There is an important caveat to the above statement; in order to make the bundles $T\mathbb{M}\,|_M$ and $(T \oplus T^*)M$ (and similarly for $\tilde{M}$) isomorphic and $\rho$ a proper bundle map, one must to construct the isomorphism separately over each leaf $M_i$ of the foliation $M$. Then, $\rho$ is a shorthand for the collection of isomorphisms

$$\rho_i : \ T\mathbb{M}\,|_{M_i} \to (T \oplus T^*)M_i.$$

For more details, see the discussions in [77, Example 4.17] and [21, Remark 2.35].

The goal of our discussion is to show that the maps $\rho$ and $\tilde{\rho}$ are not only vector bundle isomorphisms, but they are also isomorphisms of Courant algebroids. First, we introduce some notation:

**Definition 5.1.2.** *Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold with an associated D-bracket $[\![\ ,\ ]\!]$ given by the formula (2.4.3). We define the $L$- and $\tilde{L}$-* **projected** *D-brackets*

*$[\![\ ,\ ]\!]_L$ and $[\![\ ,\ ]\!]_{\tilde{L}}$, respectively, by the formulas*

$$\begin{aligned} \eta([\![X,Y]\!]_L, Z) &\coloneqq \eta(\nabla^c_{P(X)}Y - \nabla^c_{P(Y)}X, Z) + \eta(\nabla^c_{P(Z)}X, Y), \\ \eta([\![X,Y]\!]_{\tilde{L}}, Z) &\coloneqq \eta(\nabla^c_{\tilde{P}(X)}Y - \nabla^c_{\tilde{P}(Y)}X, Z) + \eta(\nabla^c_{\tilde{P}(Z)}X, Y), \end{aligned} \tag{5.1.2}$$

*where $\nabla^c$ is the canonical connection of the almost para-Hermitian manifold.*

It is clear that the projected brackets sum up to the D-bracket:

$$[\![\ ,\ ]\!] = [\![\ ,\ ]\!]_L + [\![\ ,\ ]\!]_{\tilde{L}}.$$

Another important fact is that the projected brackets, unlike the D-bracket, satisfy the Jacobi identity. Instead of proving this, we state a stronger result, which we proved in [13]:

**Theorem 5.1.3** ([13])**.** *Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold. Whenever $L$ is integrable, the projected bracket $[\![\ ,\ ]\!]_L$ defines a Courant algebroid structure on $T\mathbb{M}$, along with the projection $P = \frac{1}{2}(\mathbb{1} + K)$ and pairing $\eta$. Moreover, this Courant algebroid restricts to the integral foliation $M$ of $L$ and this restriction is isomorphic to the small Courant algebroid $(T \oplus T^*)M$ via $\rho$:*

$$\rho : (T\mathbb{M}|_M, [\![\ ,\ ]\!]_L, P, \eta) \to ((T \oplus T^*)M, [\ ,\ ]_M, \pi_{TM}, \langle\ ,\ \rangle),$$

*where $((T \oplus T^*)M, [\ ,\ ]_M, \pi_{TM}, \langle\ ,\ \rangle)$ denotes the standard Courant algebroid on $M$, i.e. $[\ ,\ ]_M$ is the Dorfman bracket on $M$, $\pi_{TM}$ the projection $(T \oplus T^*)M \to M$ and $\langle\ ,\ \rangle$ the duality pairing. An analogous statements holds for $\tilde{L}$, $\tilde{M}$, $[\![\ ,\ ]\!]_{\tilde{L}}$ and $\tilde{P}$.*

We therefore see that the small Courant algebroids induce a pair of Courant algebroid structures on the tangent bundle $T\mathbb{M}$ via the maps $\rho$ and $\tilde{\rho}$ with the corresponding Courant algebroid brackets given by the projected D-brackets.

Instead of reproducing the complete proof of the Theorem 5.1.3 from [13], we shall explain why this statement holds true on a more intuitive level. In doing so, we will also motivate why the expression for the D-bracket takes the form (2.4.2), which exhibits a clear similarity with the formula (4.1.7) for the standard Dorfman bracket in terms of a torsionless connection $\nabla$.

Because $T\mathbb{M}|_M$ is isomorphic to $(T \oplus T^*)M$ via $\rho$, we can simply define a Courant algebroid bracket on $T\mathbb{M}|_M$ by $\rho^{-1}[\rho X, \rho Y]_M$, $[\ ,\ ]_M$ being the Dorfman bracket on $M$. This can be extended to a bracket $[\![\ ,\ ]\!]'_L$ on the whole bundle $T\mathbb{M}$, since any vector field $X \in \mathfrak{X}(\mathbb{M})$ in particular defines a section $X_M \in \Gamma(T\mathbb{M}|_M)$ by leaf-wise restriction, $X|_M = X_M$. Therefore, because any point $p \in \mathbb{M}$ lies in a unique leaf $M_i$ of $M$, we can define the bracket $[\![\ ,\ ]\!]'_L$ point-wise by

$$[\![X, Y]\!]'_{L,p} = \rho^{-1}[\rho X_M, \rho Y_M]_{M,p}. \tag{5.1.3}$$

An analogous argument for $\tilde{M}$ yields the bracket $[\![\ ,\ ]\!]'_{\tilde{L}}$. The goal now is to show that these brackets coincide with (5.1.2), $[\![\ ,\ ]\!]'_{L/\tilde{L}} = [\![\ ,\ ]\!]_{L/\tilde{L}}$. To do this, we write the bracket $[\ ,\ ]_M$ using the formula (4.1.7):

$$\begin{aligned}\langle [\mathbf{x} + \alpha, \mathbf{y} + \beta]_M, \mathbf{z} + \gamma \rangle = \langle \nabla^M_{\mathbf{x}}(\mathbf{y} + \beta) - \nabla^M_{\mathbf{y}}(\mathbf{x} + \alpha), \mathbf{z} + \gamma \rangle \\ + \langle \nabla^M_{\mathbf{z}}(\mathbf{x} + \alpha), \mathbf{y} + \beta \rangle,\end{aligned} \tag{5.1.4}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vector fields in $TM$, $\alpha, \beta, \gamma$ are one-forms in $T^*M$ and $\nabla^M$ is a torsionless connection on $M$. In order to express this as the bracket (5.1.3) on $TM$, we choose the sections of $(T \oplus T^*)M$ such that there are vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{M})$ satisfying

$$\rho X|_M = \mathbf{x} + \alpha, \quad \rho Y|_M = \mathbf{y} + \beta, \quad \text{and} \quad \rho Z|_M = \mathbf{z} + \gamma.$$

Using (5.1.4), we express (5.1.3) in the following form

$$\eta([\![X, Y]\!]'_L, Z) = \eta\big(\rho^{-1}(\nabla^M_{\mathbf{x}}(\mathbf{y} + \beta) - \nabla^M_{\mathbf{y}}(\mathbf{x} + \alpha)), Z\big) + \eta\big(\rho^{-1}\nabla^M_{\mathbf{z}}(\mathbf{x} + \alpha), Y\big). \tag{5.1.5}$$

In order to see that (5.1.5) equals $[\![\ ,\ ]\!]_L$, which is given by

$$\eta([\![X, Y]\!]_L, Z) = \eta(\nabla^c_{\mathbf{x}} Y - \nabla^c_{\mathbf{y}} X, Z) + \eta(\nabla^c_{\mathbf{z}} X, Y), \tag{5.1.6}$$

we notice that both the connections $\nabla^M$ on $(T \oplus T^*)M$ and $\nabla^c$ on $T\mathbb{M} = L \oplus \tilde{L}$ act by diagonal actions

$$\nabla^M = \begin{pmatrix} \nabla^M & \\ & (\nabla^M)^* \end{pmatrix} \begin{pmatrix} T \\ T^* \end{pmatrix}, \quad \text{and} \quad \nabla = \begin{pmatrix} \nabla^c & \\ & \nabla^c \end{pmatrix} \begin{pmatrix} L \\ \tilde{L} \end{pmatrix}. \tag{5.1.7}$$

For $\nabla^M$ this is obvious simply because a linear connection preserves the tensor type and for $\nabla^c$ this follows from the fact that it preserves the eigenbundles of $K$. Moreover, $\nabla^M$ is torsionless and $\nabla^c$ has a vanishing torsion components along $L$. We can therefore take $\nabla^M$

to be given by

$$\nabla^c_{\mathbf{x}}\rho^{-1}(\mathbf{y}+\beta) = \rho^{-1}\nabla^M_{\mathbf{x}}(\mathbf{y}+\beta),$$

which achieves the desired result and renders $[\![\ ,\ ]\!]'_L = [\![\ ,\ ]\!]_L$.

We have therefore shown that the projected brackets are mapped via $\rho/\tilde{\rho}$ to the standard Dorfman brackets on $M$ and $\tilde{M}$ and the reason they are exactly matched is the choice of the connection $\nabla^c$ to define the projected brackets in (5.1.2). Concretely, the crucial property of $\nabla^c$ is that the partial connections $\nabla^c_{P(\bullet)}$ and $\nabla^c_{\tilde{P}(\bullet)}$ define generalized connections on $(T\oplus T^*)M$ and $(T\oplus T^*)\tilde{M}$ (again, via $\rho/\tilde{\rho}$) with vanishing generalized torsion. This property is, however, simply a consequence of the fact that $\nabla^c$ has vanishing D-torsion (by Definition 2.4.1):

**Proposition 5.1.4.** *Let $(\mathbb{M}, M, \tilde{M}, \eta)$ be a para-Hermitian manifold and $\nabla$ a para-Hermitian connection with a vanishing D-torsion. Then the partial connections $\nabla_{P(\bullet)}$ and $\nabla_{\tilde{P}(\bullet)}$ define generalized connections $D$ and $\tilde{D}$ on $(T\oplus T^*)M$ and $(T\oplus T^*)\tilde{M}$ by*

$$\rho(\nabla_{P(X)}Y) = D_{P(X)}\rho(Y), \quad \text{and} \quad \tilde{\rho}(\nabla_{\tilde{P}(X)}Y) = D_{P(X)}\tilde{\rho}(Y),$$

*whose generalized torsions $T^D$ and $T^{\tilde{D}}$ vanish.*

*Proof.* By $\nabla_X = \nabla_{P(X)} + \nabla_{\tilde{P}(X)}$ and $[\![\ ,\ ]\!] = [\![\ ,\ ]\!]_L + [\![\ ,\ ]\!]_{\tilde{L}}$, we find from the requirement that the D-torsion vanishes that $0 = T^D + T^{\tilde{D}}$. Since $\nabla$ is para-Hermitian, it acts in the splitting $T\mathbb{M} = L\oplus\tilde{L}$ diagonally and Lemma 4.2.11 tells us that $T^D$ is in the para-Hermitian bigrading of type $(2,1)$, while $T^{\tilde{D}}$ is of type $(1,2)$, therefore both have to vanish separately. □

**The strong section condition of DFT** In DFT, it is observed that even though the D-bracket is not a Courant algebroid bracket, this can be fixed by imposing the **strong section condition**. This amounts to choosing local coordinates $(x^i, \tilde{x}_i)$ on a patch $\mathbb{U}$ of the extended manifold $\mathbb{M}$ and restricting the dependence of the vector fields (and other sections) to only half of the coordinates $(x^i, \tilde{x}_i)$. This subset of coordinates then defines a local polarisation. Then, the local expression for the D-bracket in the coordinates $(x^i, \tilde{x}_i)$ restricts to the Dorfman bracket for sections locally constant along certain coordinates. The two most commonly discussed polarisations are defined by setting $\{\tilde{\partial}^i = 0\}_{i=1,\cdots,d}$ or $\{\partial_i = 0\}_{i=1,\cdots,d}$ and should be viewed as the two T-dual local polarisations. For this to be defined globally, the choice of local coordinates must be made consistent on every patch $\mathbb{U}' \subset \mathbb{M}$ and when $(x^i, \tilde{x}_i)$ and $(x'^i, \tilde{x}'_i)$ are two sets of coordinates on $\mathbb{U}$ and $\mathbb{U}'$, they must be related on $\mathbb{U} \cap \mathbb{U}'$ by

$$(x'^i, \tilde{x}'_i) = (x'^i(x), \tilde{x}'_i(\tilde{x})).$$

This is nothing but the requirement that the manifold $\mathbb{M}$ is para-complex and the local coordinates are adapted, parametrizing the fundamental foliations $M$ and $\tilde{M}$. In light of this realization, the section condition is therefore interpreted as the restriction of $T\mathbb{M}$ (and other bundles, whose sections we wish to study) to $M$ or $\tilde{M}$. The sheaves of sections of the resulting bundles $T\mathbb{M}|_M$ ($T\mathbb{M}|_{\tilde{M}}$) are then $C^\infty(M)$ ($C^\infty(\tilde{M})$)-modules, i.e. when expressed locally, they are independent of the $\tilde{x}_i$ ($x^i$) coordinates. Such sections are then *foliated* (see [41]), i.e. behave well under the foliation quotients $\mathbb{M} \xrightarrow{/\tilde{M}} M$ or $\mathbb{M} \xrightarrow{/M} \tilde{M}$, as discussed in [21].

In the present framework, there is, nevertheless, no need for the restrictions or quotients as we can acquire the Courant algebroid brackets on the level of the full doubled space $\mathbb{M}$. According to Theorem 5.1.3, the appropriate procedure for recovering a bracket operation that satisfies the Jacobi identity from the D-bracket is by introducing the projected D-brackets (5.1.2). In [13], we show the following

**Proposition 5.1.5** ([13]). *Let $(\mathbb{M}, M, \tilde{M}, \eta)$ be a flat para-Hermitian manifold and let $X$ and $Y$ be vector fields parallel along $\tilde{M}$. Then*

$$[\![X, Y]\!] = [\![X, Y]\!]_L,$$

*or equivalently,* $[\![X, Y]\!]_{\tilde{L}} = 0$.

Because any vector fields $X$ and $Y$ on $\mathbb{M}$ restricted to $M$ are in particular parallel along $\tilde{M}$, the above statement shows that the procedure of projecting the bracket $[\![\ ,\ ]\!] \to [\![\ ,\ ]\!]_L$ can be, on a flat manifold, understood as a generalization of the restriction of $T\mathbb{M}$ to $T\mathbb{M}|_M$. Of course, a similar statement again holds for $\tilde{L}$ and $\tilde{M}$.

### 5.1.1 Generalized structures

The maps (5.1.1) allow us to think about the generalized endomorphisms of the small Courant algebroids $(T \oplus T^*)M$ ($(T \oplus T^*)\tilde{M}$) as endomorphisms of $T\mathbb{M}$. We will now describe this correspondence for $(T \oplus T^*)M$, but for $\tilde{M}$ the discussion is identical (upon replacing $\rho$ with $\tilde{\rho}$).

Let $\mathcal{J}$ be a generalized structure on $(T \oplus T^*)M$ that satisfies $\mathcal{J}^2 = \epsilon \mathbb{1}$ and $\langle \mathcal{J}\cdot, \mathcal{J}\cdot \rangle = \delta \langle \ , \ \rangle$ with $\epsilon$ and $\delta$ being either 1 or $-1$. Then

$$J = \rho \mathcal{J} \rho^{-1} \tag{5.1.8}$$

defines an endomorphism of $T\mathbb{M}$ (upon pulling it back by the restriction map $\mathbb{M} \to M$), which satisfies

$$J^2 = \epsilon \mathbb{1} \quad \text{and} \quad \eta(J\cdot, J\cdot) = \delta \eta(\cdot, \cdot).$$

Therefore, we in particular get the following correspondences

- $\mathcal{J}$ is (almost) generalized complex structure on $(T \oplus T^*)M$ if and only if $J$ is (almost) Hermitian structure on $(T\mathbb{M}, \eta)$,
- $\mathcal{J}$ is (almost) generalized para-complex structure on $(T \oplus T^*)M$ if and only if $J$ is (almost) para-Hermitian structure on $(T\mathbb{M}, \eta)$ and
- $\mathcal{J}$ is (almost) generalized product structure on $(T \oplus T^*)M$ if and only if $J$ is (almost) chiral structure on $(T\mathbb{M}, \eta)$,

the fourth option being $\epsilon = -1$, $\delta = -1$ and giving the correspondence between generalized anti-complex and anti-Hermitian structures which we have not discussed, but interested reader can consult for example [31] for basic definitions and properties.

The above defines a correspondence of the linear structures; for the integrability, the discussion is more subtle. One immediate result is the following

**Proposition 5.1.6.** *Let $(\mathbb{M}, M, \tilde{M}, \eta)$ be a para-Hermitian manifold and let $(\mathcal{J}, J)$ be a pair of endomorphisms related by equation (5.1.8), i.e. $\mathcal{J} \in \Gamma(\mathrm{End}((T \oplus T^*)M))$ and $J \in \Gamma(\mathrm{End}(T\mathbb{M}))$. Then $\mathcal{J}$ is integrable in the generalized sense if and only if the eigenbundles of $J$ are involutive under the projected bracket $[\![ \ , \ ]\!]_L$.*

*Proof.* This is a consequence of Theorem 5.1.3, which states that $[\![ \ , \ ]\!]_L$ is related to the Dorfman bracket on $(T \oplus T^*)M$ exactly via the map $\rho$, which means that $\rho$ also relates the two Nijenhuis tensors governing the involutivity under $[ \ , \ ]_M$ on $(T \oplus T^*)M$ and $[\![ \ , \ ]\!]_L$ on $T\mathbb{M}$. □

*Remark.* From the above result it is not clear what is the relationship between the usual integrability of $J$ and generalized integrability of $\mathcal{J}$, and what can be inferred about the endomorphism $J$ when it is involutive under $[\![\ ,\ ]\!]_L$. In particular, we do not know of any interpretation of the involutivity under the projected brackets (or even the full D-bracket) in terms of the geometry of $\mathbb{M}$, such as an existence of special local coordinates. This is desirable from the point of view of the philosophy that the D-bracket serves as a replacement for the standard Lie bracket, for which such an interpretation exists in terms of Frobenius integrability and consequently the Newlander–Nirenberg theorem in case of the (para-)complex geometry.

**Relationship with Born Geometry** The Born structure $(\eta, K, \mathcal{H})$ on an integrable para-Hermitian manifold $(\mathbb{M}, \eta, K)$ can also be understood in terms of endomorphisms of $(T \oplus T^*)M$. First, the metric $\eta$ is identified as before with the pairing $\langle\ ,\ \rangle$ on $(T \oplus T^*)M$ and the para-complex structure $K$ defines the simplest generalized para-complex structure $\mathcal{K}$ on $(T \oplus T^*)M$:

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \overset{\rho}{\longmapsto} \mathcal{K} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{5.1.9}$$

The chiral structure $J$ on $T\mathbb{M}$ then similarly defines a generalized product structure on $(T \oplus T^*)M$, which – as a consequence of the Born compatibility condition $\{J, K\} = 0$ – takes the form

$$J = \begin{pmatrix} 0 & g^{-1}\eta \\ \eta g & 0 \end{pmatrix} \overset{\rho}{\longmapsto} \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

i.e. $\mathcal{G}$ is a generalized metric with a vanishing $b$-field. This is the reason why the metric $\mathcal{H} = \eta J$ is sometimes with a slight abuse of language referred to as a generalized metric.

Having discussed the generalized structures on $(T \oplus T^*)M$ induced by the Born geometry on $T\mathbb{M}$, we now turn our attention to the Born connection, which we discussed in Section 3.2, and which has an interpretation in terms of a well-known structure on $(T \oplus T^*)M$ as well:

**Proposition 5.1.7.** *Let $(\mathbb{M}, \eta, K, \mathcal{H})$ be a Born manifold with $\mathcal{H} = \eta J$ defined by a metric $g$ on the $+1$-eigenbundle $L$ of $K$ as above and let also $\nabla^B$ be the Born connection. Then the* *generalized connection $D$ on $(T \oplus T^*)M$ defined by*

$$D_{\rho(X)}\rho(Y) = \rho(\nabla^B_{\mathbf{x}} Y), \quad \mathbf{x} = PX, \tag{5.1.10}$$

*is the generalized Bismut connection associated to the generalized metric $\mathcal{G}(g) = \rho J \rho^{-1}$.*

*Proof.* The statement is a consequence of Theorem 3.2.2, which tells us the explicit form of $\nabla^B_{\mathbf{x}}$ in terms of the Levi-Civita connection $\nabla^g$ of $g$ and the fact that the generalized Bismut connection with $b = H = 0$ takes the same form [74]:

$$D_u = \begin{pmatrix} \nabla^g_{\mathbf{x}} & 0 \\ 0 & \nabla^{g*}_{\mathbf{x}} \end{pmatrix}, \quad \mathbf{x} = \pi(u),$$

where $\nabla^{g*}$ denotes the usual linear dual connection on $T^*M$ and $\pi : (T \oplus T^*)M \to TM$ is the canonical projection. Nevertheless, $D_{\mathbf{x}}$ and $\nabla^{\mathrm{B}}_{\mathbf{x}}$ are easily seen to be mapped onto each other by $\rho$ as claimed in the statement. □

There is another way in which we could *project* the Born connection $\nabla^{\mathrm{B}}$ in order to obtain the generalized Bismut connection on the bundle $(T \oplus T^*)M$. Recall that $\nabla^{\mathrm{B}}$ can be expressed purely in terms of the D-bracket (3.2.1), which can be split to the projected brackets (5.1.2) and we found in Theorem 5.1.3 that the projected brackets are mapped onto the standard Dorfman brackets on the small CAs via $\rho[\![\ ,\ ]\!]_L = [\rho^{-1}\ ,\rho^{-1}\ ]_M$. The definition of the generalized Bismut connection (4.2.10) (with $H = 0$)

$$D_u v = [u_-, v_+]_+ + [u_+, v_-]_- + [\mathcal{C}u_-, v_-]_- + [\mathcal{C}u_+, v_+]_+ \ ,$$

is formally very similar to our definition of the Born connection (3.2.1):

$$\nabla^{\mathrm{B}}_X Y = [\![X_-, Y_+]\!]_+ + [\![X_+, Y_-]\!]_- + (K[\![X_+, KY_+]\!])_+ + (K[\![X_-, KY_-]\!])_-.$$

Moreover, we know that $\rho$ maps $\mathcal{G}$ onto $J$, therefore the eigenbundles of $\mathcal{G}$ and $J$, which are on both sides of the mapping denoted by the $\pm$ subscripts, are related via $\rho$. Additionally, when $b = 0$, the map $\mathcal{C}$ exactly coincides with the image of $K$ under $\rho$. Finally, using the property of the map $\mathcal{C}$ noted in [74]

$$\mathcal{C}[u, v]_H = [\mathcal{C}u, \mathcal{C}v]_{-H},$$

which in our case of vanishing $H$-flux simply means $[\mathcal{C}u, v] = \mathcal{C}[u, \mathcal{C}v]$, we find that the generalized Bismut connection $D_u v$ is mapped via $\rho$ to an expression that coincides with $\nabla^B$, except the D-bracket is replaced by the projected bracket $[\![\ ,\ ]\!]_L$.

**Proposition 5.1.8.** *Assume the setting of Proposition 5.1.7 and define the L-projected Born connection* $\nabla^{B,L}$

$$\nabla^{B,L}_X Y = [\![X_-, Y_+]\!]_{L+} + [\![X_+, Y_-]\!]_{L-} + (K[\![X_+, KY_+]\!]_L)_+ + (K[\![X_-, KY_-]\!]_L)_-.$$

*Then we have the relationship between the projected Born connection and the Bismut connection of* $\mathcal{G}(g)$

$$\rho(\nabla^{B,L}_X Y) = D_{\rho(X)}\rho(Y).$$

*Consequently, taking into account (5.1.10), we find that the projected Born connection* $\nabla^{B,L}$ *coincides with the partial connection* $\nabla^B_{P\bullet}$*:*

$$\nabla^B_{PX} = \nabla^{B,L}_X.$$

*Remark.* The correspondence between generalized structures and tangent bundle structures defined by $\rho$ is qualitatively different from the construction using the isomorphisms $\pi_\pm$ described for generalized para-Kähler and chiral geometries. There, the pair of generalized endomorphisms gives rise to a pair of tangent bundle endomorphisms over the *same base manifold*, while here we have a correspondence between $T\mathbb{M}$ and $(T \oplus T^*)M$.

## 5.2 T-duality of the small Courant algebroids

We observe that the large CA of a para-Hermitian manifold $(\mathbb{M}, \eta, K)$ splits as the direct sum of the bundles that correspond to the small CAs. To see this, we split $(T \oplus T^*)\mathbb{M}$ into the eigenbundles of $K$ (and $K^*$):

$$(T \oplus T^*)\mathbb{M} = (L \oplus L^*) \oplus (\tilde{L} \oplus \tilde{L}^*),$$

and the correspondence with the small CAs is

$$(T \oplus T^*)M \simeq (L \oplus L^*)\,|_M, \quad \text{and} \quad (T \oplus T^*)\tilde{M} \simeq (\tilde{L} \oplus \tilde{L}^*)\,|_{\tilde{M}}\,.$$

Now, recall that linear T-duality is facilitated by the contraction with the para-Hermitian metric $\eta$, which maps $L \to \tilde{L}^*$ and $\tilde{L} \to L^*$. On the small CAs, this yields the full T-duality map (2.3.3), which is given by the generalized metric $\mathcal{G}(\eta)$:

$$\mathcal{G}(\eta) = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} : \left\{ \begin{array}{c|c} L \oplus L^* \to \tilde{L} \oplus \tilde{L}^* & \mathbf{x} + \eta(\tilde{\mathbf{x}}) \mapsto \tilde{\mathbf{x}} + \eta(\mathbf{x}) \\ \tilde{L} \oplus \tilde{L}^* \to L \oplus L^* & \tilde{\mathbf{x}} + \eta(\mathbf{x}) \mapsto \mathbf{x} + \eta(\tilde{\mathbf{x}}) \end{array} \right. . \tag{5.2.1}$$

Note that we can factorize this map as $\mathcal{G}(\eta) = \tilde{\rho} \circ \rho^{-1}$, i.e. the following diagram commutes

$$\begin{array}{ccc} (T \oplus T^*)M & \xrightarrow{\mathcal{G}(\eta)} & (T \oplus T^*)\tilde{M} \\ \big\downarrow{\scriptstyle \rho^{-1}} & & \big\downarrow{\scriptstyle \tilde{\rho}^{-1}} \\ T\mathbb{M} & \xrightarrow{\quad \mathbb{1} \quad} & T\mathbb{M} \end{array} .$$

Therefore, the T-duality map is simply the identity on $T\mathbb{M}$, which is another manifestation of the T-duality covariance of our setup: when the sections of the small CAs are understood as vector fields on the extended space $\mathbb{M}$, T-duality acts trivially. Additionally, when we realize the map (5.2.1) on the whole $(T \oplus T^*)\mathbb{M}$, we find that the eigenbundles $C_\pm$ of $\mathcal{G}(\eta)$ (which are also isomorphic to $T\mathbb{M}$) are invariant under the T-duality map as well.

**Example 5.2.1** (T-duality of generalized metrics)**.** We can demonstrate the idea that endomorphisms of the small CAs are acted upon trivially by T-duality once they are realized as objects on the extended tangent bundle. Consider a generalized metric on one of the small CAs, $\mathcal{G}(g, b) \in \Gamma(\mathrm{End}((T \oplus T^*)M))$:

$$\mathcal{G}(g, b) = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} .$$

We can understand $\mathcal{G}(g, b)$ also as an (almost) chiral endomorphism $J(\hat{g}, \hat{b}) \in \Gamma(\mathrm{End}(T\mathbb{M}))$ via the map $\rho$:

$$J(\hat{g}, \hat{b}) = \rho^{-1} \circ \mathcal{G}(g, b) \circ \rho = \begin{pmatrix} -\hat{g}^{-1}\hat{b} & \hat{g}^{-1} \\ \hat{g} - \hat{b}\hat{g}^{-1}\hat{b} & \hat{b}\hat{g}^{-1} \end{pmatrix}, \quad \begin{array}{l} \hat{g} = \eta^{-1} g \\ \hat{b} = \eta^{-1} b \end{array} .$$

This endomorphism can in turn be identified with an endomorphism $\tilde{\mathcal{G}}(\tilde{g}, \beta) \in \Gamma(\mathrm{End}((T \oplus$

$T^*)\tilde{M})$) on the T-dual small CA:

$$\tilde{\mathcal{G}}(\tilde{g},\beta) = \tilde{\rho}\circ J(\hat{g},\hat{b})\circ\tilde{\rho}^{-1} = \begin{pmatrix} \beta\tilde{g} & \tilde{g}^{-1}-\beta\tilde{g}\beta \\ \tilde{g} & -\tilde{g}\beta \end{pmatrix}, \qquad \begin{matrix} \tilde{g} = \eta g^{-1}\eta \\ \beta = \eta^{-1} b\eta^{-1} \end{matrix} .$$

We see that while $\mathcal{G}$ is given in terms of a metric and a two-form on $L$, $\tilde{\mathcal{G}}$ is defined by a metric and a bi-vector on $\tilde{L}$. This corresponds to the well-known fact in string theory that a 2-form $b$-field is via T-duality mapped onto a $\beta$-field given by a bivector. To summarize, we described the correspondence

$$\begin{array}{ccc} & J(\hat{g},\hat{b}) & \\ {}^{\rho}\swarrow & & \searrow^{\tilde{\rho}} \\ \mathcal{G}(g,b)\ \begin{cases} g = \eta\hat{g} \\ b = \eta\hat{b} \end{cases} & \xleftrightarrow{\text{T-duality}} & \tilde{\mathcal{G}}(\tilde{g},\beta)\ \begin{cases} \tilde{g} = \eta\hat{g}^{-1} \\ \beta = \hat{b}\eta^{-1} \end{cases} \end{array} ,$$

and while the T-duality between the small CAs acts non-trivially, the action is trivial on the level of the tangent bundle $T\mathbb{M}$. ◁

The T-duality map above of course induces a map of the generalized endomorphisms of the small CAs:

$$\begin{aligned} \mathcal{G}(\eta):\ \mathrm{End}((T\oplus T^*)M) &\to \mathrm{End}((T\oplus T^*)\tilde{M}) \\ \mathcal{J} &\mapsto \mathcal{G}(\eta)\mathcal{J}\mathcal{G}(\eta), \end{aligned} \tag{5.2.2}$$

and clearly preserves the type of the generalized structure.

**Example 5.2.2.** Recall from Section 2.3.4 that the full T-duality on a para-Hermitian manifold $\mathbb{M} = T^d\times(T^d)^*$ can be understood as a topological T-duality of two torus fibrations $T^d\to\{*\}$ and $(T^d)^*\to\{*\}$ over a point. In particular, this is also true of the T-duality map of Courant algebroids described in [5], which is now only linear since the base is zero-dimensional and there is no $H$-flux allowed according to the geometric setting of topological T-duality. One can also understand the different interpretations of T-duality presented in [5] in the para-Hermitian framework. For example, T-duality can be interpreted in the language of generalized submanifolds: the authors in [5] show that if $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are generalized complex structures on $M$ and $\tilde{M}$, then $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are T-dual if and only if the correspondence space $M\times_B\tilde{M}$ is a generalized complex submanifold of $M\times\tilde{M}$ endowed with the generalized complex structure $(\mathcal{J}, \mathcal{C}\tilde{\mathcal{J}}\mathcal{C}^{-1})$ on the product, where $M$ and $\tilde{M}$ are as in (2.3.8) and $\mathcal{C}$ is the trivial generalized para-complex structure on $(T \oplus T^*)\tilde{M}$:

$$\mathcal{C} = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}.$$

In our setting we have $B = \{*\}$, so that $M \times_B \tilde{M} = M \times \tilde{M} = \mathbb{M}$. The above statement then says that the generalized complex structures $\mathcal{J}$ and $\tilde{\mathcal{J}}$ on $M$ and $\tilde{M}$ are T-dual if and only if the following subbundle of $(T \oplus T^*)\mathbb{M}$ ($\omega = \eta K$ as usual)

$$\mathrm{graph}(-\omega) = \{X - \omega(X) \mid X \in \mathfrak{X}(\mathbb{M})\} \subset (T \oplus T^*)\mathbb{M},$$

is invariant under the generalized complex structure $(\mathcal{J}, \mathcal{C}\tilde{\mathcal{J}}\mathcal{C}^{-1})$ on $\mathbb{M} = M \times \tilde{M}$, which happens precisely when $\tilde{\mathcal{J}} = \mathcal{G}(\eta)\mathcal{J}\mathcal{G}(\eta)$ as in (5.2.2). $\lhd$

## 5.3 The B-transformation and (non-)geometric Fluxes

We now introduce the B-transformation of a para-Hermitian structure $(\eta, K)$, which is a (finite) deformation $K_B$ of the endomorphism $K$ that preserves one of its eigenbundles and rotates the other in such a way that the transformed endomorphism $K_B$ is still orthogonal with respect to $\eta$. When the invariant bundle is $\tilde{L}$, such deformation then via $\rho$ corresponds to a $b$-field transformation (4.1.5) of the small Courant algebroid $(T \oplus T^*)M$ (similarly for $(T \oplus T^*)\tilde{M}$ when $L$ is invariant). The B-transformation should therefore be understood as a $T\mathbb{M}$-analogue of the $b$-field transformation (4.1.5). Consequently, the D-bracket $[\![\ ,\ ]\!]^B$ associated to $K_B$ can be seen as a deformation of the D-bracket of $K$.

In the special case when the underlying structure $(\eta, K)$ is para-Kähler, the D-bracket corresponding to $K_B$ is the *twisted D-bracket* and the difference between $[\![\ ,\ ]\!]$ and $[\![\ ,\ ]\!]^B$ – which is necessarily tensorial – is given by *fluxes* described in the DFT literature. Moreover, because the B-transformation of $(\eta, K)$ generally spoils both the integrability and the closedness of the fundamental form, this give the DFT fluxes a clear geometric interpretation.

This shows that the language of fluxes used in DFT (and in general in string theory) can be included in the framework of para-Hermitian manifolds and that the twisted bracket arises as a consequence of a deformation of the underlying geometry. For works discussing the fluxes and twisted brackets from a different point of view, see [78, 79, 80, 29, 28] and references therein. A mathematical overview of related concepts is given in [81].

### 5.3.1 B-transformation of para-Hermitian manifolds

We first define the notion of a B-transformation for any almost para-Hermitian manifold:

**Definition 5.3.1.** *Let $(\mathbb{M}, \eta, K)$ be an almost para-Hermitian manifold. A* ***B-transformation*** *is an endomorphism of $T\mathbb{M} = L \oplus \tilde{L}$, given by*

$$e^B \coloneqq \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}(T\mathbb{M})) \tag{5.3.1}$$

*where $B : L \to \tilde{L}$ is a skew map, meaning $\eta(BX, Y) = -\eta(X, BY)$ holds. The induced map on the endomorphisms of $T\mathbb{M}$ by conjugation is also called a $B$-transformation and in particular the $B$-transformation of $K$ is given by*

$$K \xmapsto{e^B} K_B = e^B K e^{-B}.$$

*We also say that $(\eta, K_B)$ is the $B$**-transformation** of $K$.*

It is easy to see that $B$ can be expressed in terms of either a two-form $b$ or a bivector $\beta$,

$$\eta(BX, Y) = b(X, Y) = \beta(\eta(X), \eta(Y)), \tag{5.3.2}$$

where $b$ is of type $(2, 0)$ and $\beta$ is of type $(0, 2)$, so we can write $b(X, Y) = b(\mathbf{x}, \mathbf{y})$.

Similarly, we can define a map $\tilde{B} : \tilde{L} \to L$ given by a type $(0, 2)$ two-form $\tilde{b}$ or a $(2, 0)$ bivector $\tilde{\beta}$. Such $B$-transformation then takes the form

$$e^{\tilde{B}} \coloneqq \begin{pmatrix} \mathbb{1} & \tilde{B} \\ 0 & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}(T\mathbb{M})). \tag{5.3.3}$$

The case when both the transformations (5.3.1) and (5.3.3) are performed simultaneously was worked out in [18]. The new para-Hermitian structure then reads

$$K_{B,\tilde{B}} = \begin{pmatrix} \mathbb{1} - 2\tilde{B}B & 2(\tilde{B} - \tilde{B}B\tilde{B}) \\ 2B & -\mathbb{1} + 2B\tilde{B} \end{pmatrix}. \tag{5.3.4}$$

An important property of the B-transformation (and the compositions of thereof) is that it is an $O(d,d)$ transformation, i.e. it preserves the $O(d,d)$ metric $\eta$:

$$\eta(e^B\cdot, e^B\cdot) = \eta(\cdot,\cdot) \ .$$

Because the simultaneous transformation by both $B$ and $\tilde{B}$ makes the notation rather cluttered but conceptually adds very little, we will continue discussing the case when only one of the pair $(B,\tilde{B})$ is non-zero. Without loss of generality, we choose $\tilde{B} = 0$ so that only the B-transformation (5.3.1) is present.

In the splitting $T\mathbb{M} = L \oplus \tilde{L}$ associated to the original $K$, $K_B$ is given by

$$K_B = \begin{pmatrix} \mathbb{1} & 0 \\ 2B & -\mathbb{1} \end{pmatrix}. \tag{5.3.5}$$

It is easy to check that $K_B^2 = \mathbb{1}$ as well as $\eta(K_B\cdot, K_B\cdot) = -\eta$ and therefore $K_B$ is another almost para-Hermitian structure on $\mathbb{M}$. The action of $e^B$ on $K$ can also be seen as a shift of the fundamental form:

$$\omega \xmapsto{e^B} \omega_B = \eta K_B = \omega + 2b. \tag{5.3.6}$$

The projections $P_B/\tilde{P}_B \coloneqq \frac{1}{2}(\mathbb{1} \pm K_B)$ associated to $K_B$ act as:

$$P_B(X) = \mathbf{x} + B(\mathbf{x}), \quad \tilde{P}_B(X) = \tilde{\mathbf{x}} - B(\mathbf{x}).$$

Because $B$ maps $L \to \tilde{L}$, we see that $\mathrm{Im}(\tilde{P}_B) = \tilde{L}$, but $\mathrm{Im}(P_B) \neq L$, i.e $K$ and $K_B$ share the $-1$ eigenbundle, but the $+1$ eigenbundles are different. This means that even if $K$ is integrable, $K_B$ needs not be. Similarly, even if $K$ is para-Kähler, $\omega_B$ (5.3.6) is in general not closed and so we can view $K_B$ as a deformation of $K$. As we will see, the D-bracket gives rise to a Maurer-Cartan type equation relevant to this deformation problem.

It is also interesting to view the fact that the B-transformation spoils the integrability and closedness of $\omega$ from the opposite point of view: given a (half-)integrable para-Hermitian manifold $(\mathbb{M},\eta,K)$, we can ask what are the conditions on the data $(\mathbb{M},\eta,K)$ such that is it possible to find a B-transformation of $K$ into an integrable and/or para-Kähler structure $(\eta, K_B)$? Clearly, such situations do exist but it is a priori hard to determine what properties $K$ needs to have for such deformations to exist. Notice this is a very different deformation problem than the usual one, where one is looking for deformations of some integrable structure such that the integrability is preserved under the deformation.

We now present an example which was studied in great detail in [19] (see also [18, 21]). This example is customarily discussed as a $\tilde{B}$-transformation of a para-Hermitian structure, although upon replacing $K$ with $-K$ (the fully T-dual picture), one can see this example also as a $B$-transformation, which we discussed so far.

**Example 5.3.2** ($SL(2,\mathbb{C})$)**.** Consider the real Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ generated by the basis elements $e_i = \frac{i}{2}\{\sigma_1, -\sigma_2, \sigma_3\}$ and $\tilde{e}^i = -i\delta^{ij}e_j$, $i = 1, \cdots, 3$, $\sigma_i$ being the Pauli matrices:

$$[e_i, e_j] = \epsilon_{ij}{}^k e_k, \quad [\tilde{e}^i, \tilde{e}^j] = -\epsilon^{ijk} e_k, \quad \text{and} \quad [\tilde{e}^i, e_j] = \epsilon^i{}_{jk}\tilde{e}^k.$$

We see that $\{e^i\}_{i=1\cdots 3}$ generates the subalgebra $\mathfrak{su}(2)$, while $\{\tilde{e}^i\}_{i=1\cdots 3}$ do not close to form a subalgebra. We denote the respective (real) spans by $L$ and $\tilde{L}$, so that $\mathfrak{sl}(2,\mathbb{C}) = L \oplus \tilde{L}$. There are two natural invariant scalar products on $\mathfrak{sl}(2,\mathbb{C})$ defined by:

$$\eta(a,b) = 2\mathrm{Im}(\mathrm{Tr}(ab)), \quad \text{and} \quad (a,b) = -2\mathrm{Re}(\mathrm{Tr}(ab)),$$

for $a, b \in \mathfrak{sl}(2,\mathbb{C})$. It can be checked that

$$\begin{aligned} \eta(e_i, e_j) = \eta(\tilde{e}^i, \tilde{e}^j) = 0, &\quad \text{and} \quad \eta(e_i, \tilde{e}^j) = \delta_i^j, \\ -(\tilde{e}^i, \tilde{e}^j) = (e_i, e_j) = \delta_{ij}, &\quad \text{and} \quad (e_i, \tilde{e}^j) = 0, \end{aligned}$$

so that the above splitting of $\mathfrak{sl}(2,\mathbb{C})$ defines a half-integrable para-Hermitian structure $(K, \eta)$, where

$$Ke_i = e_i, \quad \text{and} \quad K\tilde{e}^i = -\tilde{e}^i,$$

with the integral foliation of $L$ being $SU(2) \subset SL(2,\mathbb{C})$. Moreover, $(\ ,\ )$ defines a Born structure $(K, \eta, \mathcal{H})$ by setting

$$\mathcal{H}(e_i, e_j) = (e_i, e_j), \quad \text{and} \quad \mathcal{H}(\tilde{e}^i, \tilde{e}^j) = -(\tilde{e}^i, \tilde{e}^j).$$

In the frame $(e_i, \tilde{e}^j)$, $(K, \eta, \mathcal{H})$ is the canonical Born structure

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{H} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Now, we will $\tilde{B}$-transform the above $K$ by a bi-vector on $L$ (or, equivalently, a 2-form on $\tilde{L}$) $\beta^{ij} = \epsilon^{3ij}$ which changes the $\tilde{e}^i$ generators, keeping $e_i$ the same:

$$\tilde{b}^i = \tilde{e}^i - \epsilon^{3ij} e_j.$$

The generators satisfy

$$[\tilde{b}^3, \tilde{b}^1] = \tilde{b}^1, \quad [\tilde{b}^3, \tilde{b}^2] = \tilde{b}^2, \quad \text{and} \quad [\tilde{b}^1, \tilde{b}^2] = 0,$$

and therefore form a subalgebra, called the Borel subalgebra $\mathfrak{sb}(2,\mathbb{C})$. Consequently, the splitting[1] $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{sb}(2,\mathbb{C}) = L \oplus \tilde{L}_{\tilde{B}}$ is integrable and defines the Manin triple $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ with the corresponding Drinfel'd double

$$SL(2,\mathbb{C}) = SU(2) \bowtie SB(2,\mathbb{C}).$$

This is an interesting scenario in which the seemingly more natural splitting given by $(e_i, \tilde{e}^j)$, in which the natural pairings $\eta$ and $( \ , \ )$ take a canonical form, is not fully integrable and one obtains an integrable splitting by $\tilde{B}$-transforming it. ◁

### 5.3.2 B-transformation and generalized T-duality transformations

In Section 2.3 we introduced T-duality on para-Hermitian manifolds as a change of the para-Hermitian structure $K \mapsto K'$, which in the case of the full T-duality maps $K \mapsto -K$. We also observed that the T-duality map T is an $O(d,d)$ transformation, i.e. it preserves the signature $(d,d)$ metric $\eta$:

$$\eta(\mathrm{T}\cdot, \mathrm{T}\cdot) = \eta(\cdot,\cdot) \ ,$$

which is a property T-duality shares with the B-transformation $K \mapsto K_B$. For this reason, both the T-duality and the B-transformation (as well as their combinations) are in [21] collec-

[1] Here, the splitting is as a vector space, not as a Lie algebra.

tively called *generalized T-duality transformations*. From this point of view, a (generalized) T-duality transformation on a para-Hermitian manifold $(\mathbb{M}, \eta, K)$ is therefore any change of the para-complex structure $K$ that simultaneously preserves the underlying $O(d,d)$ structure $\eta$.

### 5.3.3 Relationship to $b$-field and $\beta$-field transformations

In generalized geometry, the $b$-field transformation (4.1.5) is well studied for its desirable properties; it preserves the type of a generalized endomorphism – i.e. whenever $\mathcal{J}$ is an (almost) generalized complex/para-complex/chiral structure then also $e^B(\mathcal{J}) = e^B \mathcal{J} e^{-B}$ is (almost) generalized complex/para-complex/chiral – and when $\mathrm{d}b = 0$, it is also a symmetry of the Dorfman bracket (4.1.6) and therefore preserves integrability. There is in some sense a dual notion called the $\beta$-transformation, which is given by a bi-vector instead of a two-form:

$$e^\beta = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}(T \oplus T^*)).$$

Contrary to $e^b$, $e^\beta$ is not a symmetry of the Dorfman bracket but instead satisfies the following property:

$$[e^\beta(X+\alpha), e^\beta(Y+\gamma)] = e^\beta([X+\alpha, Y+\gamma] + [\alpha,\gamma]^\beta) + [\beta,\beta](\alpha,\gamma),$$

where $[\ ,\ ]^\beta$ is the Poisson Lie algebroid bracket [82]:

$$[\alpha,\gamma]^\beta = \mathcal{L}_{\beta(\alpha)}\gamma - \mathcal{L}_{\beta(\gamma)}\alpha - \mathrm{d}\beta(\alpha,\gamma),$$

and $[\beta,\beta]$ is the Schouten commutator of $\beta$ with itself, which is simply the Lie bracket of vector fields extended to bi-vectors via the derivation property. The commutator $[\beta,\beta]$ has the important property that it vanishes if and only if $\beta$ is Poisson bivector. The bracket that is given by the sum of the Dorfman bracket and the bracket $[\ ,\ ]^\beta$ is also a Courant algebroid bracket on $T \oplus T^*$ and the corresponding Courant algebroid is called the Poisson Courant algebroid. In summary, whenever $\beta$ is Poisson, the $\beta$-field transformation is a morphism between the standard Courant algebroid and the Poisson Courant algebroid.

The B-transformation is related to both the $b$-field and $\beta$-field transformations of the small Courant algebroids in the following way. Consider a para-Hermitian manifold whose eigenbundle $L$ is integrable and its integral foliation is $M$. Then, $e^B$ given by (5.3.1) is related to the $b$-field transformation of $(T \oplus T^*)M$ by:

$$e^b = \rho \, e^B \rho^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}((T \oplus T^*)M)),$$

where $b$ is the $(2,0)$-form given by $b = \eta B$. Similarly, the *dual* B-transformation (5.3.3) defines a $\beta$-field transformation of $(T \oplus T^*)M$:

$$e^\beta = \rho \, e^{\tilde{B}} \rho^{-1} = \begin{pmatrix} \mathbb{1} & \tilde{\beta} \\ 0 & \mathbb{1} \end{pmatrix} \in \Gamma(\mathrm{End}((T \oplus T^*)M)),$$

where now $\tilde{\beta}$ is the $(2,0)$-bivector $\tilde{\beta} = \tilde{B}\eta$. Of course, one can also think of $e^B$ and $e^{\tilde{B}}$ as endomorphisms of $(T \oplus T^*)\tilde{M}$ whenever $\tilde{L}$ is integrable, except $e^B$ then corresponds to a $\beta$-field transformation by a $(0,2)$-bivector $\beta = B\eta$ and $e^{\tilde{B}}$ corresponds to a $B$-field transformation by a $(0,2)$-form $\tilde{b} = \eta\tilde{B}$.

Note that this is consistent with the picture of T-duality given by (5.2.1) and Example 5.2.1, that factorizes into the map of the two small Courant algebroids. Via this map, the $b$-field and $\beta$-field transformed small CAs are mapped onto each other and so the pairs $(b, \beta = \eta^{-1} b \eta^{-1})$ and $(\tilde{b}, \tilde{\beta} = \eta^{-1} \tilde{b} \eta^{-1})$ are seen as T-dual. In other words, the same map $e^B$ (5.3.1) on $T\mathbb{M}$ is realized in the two T-dual pictures as a $b$-field transformation on one hand, and as a $\beta$-field transformation on the other. This corresponds to the fact that T-duality maps between the $b$-field and $\beta$-field deformations [83, 50].

The splitting of $T\mathbb{M}$ given by $K_B$ gives rise to an $H$-twisted Courant algebroid structure on $M$ with $H = \partial b$[2]. Conversely, consider the $H$-twisted Courant algebroid on $M$ with $H$ a closed three-form on $M$. In general, $H$ is not exact, but is only locally given by a two-form potential. Using this local data, we can define a patch-wise B-transformation of the underlying para-Hermitian structure that corresponds to the $H$-twisted Courant algebroid (see Example 5.3.3).

*Remark.* In [84], the para-Hermitian geometry, including the B-transformation, was described in terms of a stack formalism of a *higher Kaluza-Klein geometry*. The author further argues why no global object in para-Hermitian geometry can recover a cohomogically non-trivial $H$-flux on the physical space manifold $M$ given by an integral foliation. This is in accor-

[2] Here $\partial$ is the Dolbeault differential along $L$, which coincides with the de-Rham differential on $M$ when $L$ is integrable.

dance with our above observation that in order to achieve this, we would have to consider a collection of local B-transformations. On the other hand, in [21] it is shown that a non-trivial $H$-flux can be achieved if one instead considers the physical space to be given by the foliation $M_{phys.} = \mathbb{M}/\tilde{M}$. In such case, the para-Hermitian fundamental form $\omega$, which is *globally defined* on $\mathbb{M}$, maps to a local 2-form on the quotient $M_{phys.}$ and therefore gives rise to cohomologically non-trivial fluxes as well.

### 5.3.4 B-transformation of Born structures

Let $(\mathbb{M}, \eta, K, \mathcal{H})$ be a Born manifold and we will again use the notation $M$ and $\tilde{M}$ for the fundamental foliations. The B-transformation acts naturally by conjugation not only on $K$, but also on all other tensors of the Born structure, leaving only $\eta$ invariant:

$$(\eta, K, \mathcal{H}) \xmapsto{e^B} (\eta, K_B, \mathcal{H}_B),$$

and because $e^B$ clearly preserves the compatibility conditions, $(\eta, K_B, \mathcal{H}_B)$ defines another Born structure. In the splitting defined by $K_B$, all the B-transformed structures again take the canonical form, for example $\mathcal{H}_B$ is diagonal, but defined by a different metric

$$\mathcal{H}_B = \begin{pmatrix} g_B & 0 \\ 0 & \tilde{g}_B \end{pmatrix}_{K_B}, \quad \begin{cases} g_B = g(1-B) \\ \tilde{g}_B = \eta(1+B)g^{-1}\eta \end{cases},$$

where $g$ is the metric on $L$ that defines the original $\mathcal{H}$. Note that the metric $g_B = g(1-B)$ on $L$ satisfies

$$g_B(\mathbf{x} + B\mathbf{x}, \mathbf{y} + B\mathbf{y}) = g(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Gamma(L).$$

In the splitting of $K$, where $K_B$ takes the form (5.3.5), the $B$-transformation of the chiral structure $J = \eta\mathcal{H}$ yields the familiar matrix expression for a generalized metric

$$J \xmapsto{e^B} J_B = e^B J e^{-B} = \begin{pmatrix} -\hat{g}^{-1}B & \hat{g}^{-1} \\ \hat{g} - B\hat{g}^{-1}B & B\hat{g}^{-1} \end{pmatrix}_K, \quad \hat{g} = \eta^{-1}g,$$

which comes as no surprise, since B-transformation and the b-field transformation on the small CA $(T \oplus T^*)M$ are related by $\rho$, and $J$ gives rise to the generalized metric $\mathcal{G}(g, b)$ via $\rho$ (5.1.8) as well.

**Example 5.3.3** (Doubled torus with H-flux)**.** In Example 2.2.9, we described the standard para-Hermitian structure $(\eta, K)$ on the doubled torus $T^d \times (T^d)^*$. We now describe a Born structure $(\eta, K, \mathcal{H})$ on the doubled torus $\mathbb{M} = T^3 \times (T^3)^*$ and show that in case when there is additionally a constant 3-form $H$-flux on the "space-time manifold" $M = T^3$, it can be absorbed in the Born structure in terms of a local B-transformation of $(\eta, K, \mathcal{H})$. We then perform a partial T-duality along a pair of cycles, and observe that locally, such T-duality transformation recovers the famous Buscher rules. Our discussion here is local and we will discuss global change of topology corresponding to this particular example later in Example 5.3.7.

As we know, choosing a Born structure on $\mathbb{M} = T^3 \times (T^3)^*$ is equivalent to a choice of a metric $g$ on $M = T^3$ and so in the adapted coordinates $(x, y, z, \tilde{x}, \tilde{y}, \tilde{z})$, we choose the simplest metric $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$, which yields the canonical Born structure (3.0.4). Now, consider a constant $H$-flux on $T^3$, $H = k\ dx \wedge dy \wedge dz$. Locally, we can write $H = \partial b$ for $b = kx\ dy \wedge dz$ and use this $b$ to transform the Born structure by

$$B = \eta^{-1} b = kx\ (\tilde{\partial}^z \otimes dy - \tilde{\partial}^y \otimes dz) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -kx \\ 0 & kx & 0 \end{pmatrix}.$$

This yields the new metric $\mathcal{H}_B$, expressed in the original frame corresponding to $K$ as

$$\mathcal{H}_B = e^B \mathcal{H} e^{-B} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+(kx)^2 & 0 & 0 & 0 & -kx \\ 0 & 0 & 1 & 0 & kx & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & kx & 0 & 1 & 0 \\ 0 & -kx & 0 & 0 & 0 & 1 \end{array}\right)_K .$$

Now, we will T-dualize along the $(z, \tilde{z})$-cycles, meaning we will define a new para-Hermitian structure $K_{z\tilde{z}}$, which has the eigendirections tangent to $(z, \tilde{z})$-directions swapped:

$$K_{z\tilde{z}} = \left(\begin{array}{cc|cc} \mathbb{1}_{2\times 2} & & & \\ & -1 & & \\ \hline & & -\mathbb{1}_{2\times 2} & \\ & & & +1 \end{array}\right)_K .$$

Here, $K_{z\tilde{z}}$ can be seen as a particular $O(d,d)$ transformation of the original structure $K$. In the frame corresponding to $K_{z\tilde{z}}$ (whose +1-eigenbundle defines the T-dualized spacetime), we get

$$\mathcal{H}_B = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+(kx)^2 & -kx & 0 & 0 & 0 \\ 0 & -kx & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & kx \\ 0 & 0 & 0 & 0 & kx & 1 \end{array}\right)_{K_{z\tilde{z}}} .$$

From vanishing off-diagonal terms, we infer that there is no $b$-field on the T-dualized spacetime $M'$ locally given by coordinates $(x, y, \tilde{z})$. Moreover, we read off the metric

$$g_{M'} = dx^2 + dy^2 + (d\tilde{z} - kxdy)^2,$$

This result is in perfect accordance with the Buscher rules (1.1.3). The metric $g_{M'}$ is a metric on a twisted torus or nilmanifold, as we will see in Example 5.3.7. ◁

### 5.3.5 (Non-)Geometric Fluxes

We once again turn to the scenario most commonly encountered in physics, which is when the underlying para-Hermitian structure $(\eta, K)$ is para-Kähler. In such case, the D-bracket associated to the B-transformed para-Hermitian structure $K_B$ differs from the original one by tensorial quantities called *fluxes* that are known in the physics literature on T-duality covariant formulations of String Theory. These fluxes are related to the notion of integrability defined by the D-bracket that we call D-integrability:[3]

**Definition 5.3.4.** *Let* $(\mathbb{M}, \eta, K)$ *be an almost para-Hermitian manifold and* $[\![\ ,\ ]\!]^D$ *the associated D-bracket. We say an* $\eta$*-isotropic distribution* $\mathcal{D} \subset T\mathbb{M}$ *is* ***D-integrable*** *(with respect to* $K$*) if it is involutive under the D-bracket of* $K$*, i.e.*

$$[\![\mathcal{D}, \mathcal{D}]\!]^D \subset \mathcal{D}.$$

[3] In [13], this property is called weak integrability but here we use the name D-integrability, which clearly reflects the association to the D-bracket.

From the Property 3. of the Definition 2.4.1 of the D-bracket, it follows that the eigenbundles of $K$ are always D-integrable with respect to the D-bracket of $K$. Whenever it happens that a different (almost-) para-Hermitian structure $K'$ is also D-integrable with respect to this D-bracket, we say that $K$ and $K'$ are **compatible**. We then have the following result that we derived in [13]:

**Proposition 5.3.5** ([13]). *Let $(K_B, \eta)$ be a B-transformation of a para-Hermitian structure $(K, \eta)$ defined by (5.3.1) and denote $b = \eta B$, $\beta = B\eta^{-1}$. Then $K_B$ is compatible with $K$ iff*

$$\partial b + (\Lambda^3 \eta)[\beta, \beta] = 0, \tag{5.3.7}$$

*where $\partial$ is the Dolbeault differential along $L$, $[\ ,\ ]$ is the Schouten bracket on polyvector fields along $\tilde{L}$ and $\Lambda^3\eta$ denotes the third wedge power of the isomorphism defined by the contraction with the metric $\eta$.*

The equation (5.3.7) takes the form of the Maurer-Cartan equation and can be interpreted in the following way. The eigenbundle $L_B = e^B(L)$ of $K_B$ is via the maps $\rho$ and $\tilde{\rho}$ (5.1.1) mapped to an almost Dirac structure on both the small Courant algebroids of $K$. On $(T \oplus T^*)M$, $L_B$ defines a graph of $b$:

$$\mathrm{Graph}(b) = \{\mathbf{x} + b(\mathbf{x}) \mid \mathbf{x} \in \mathfrak{X}(M)\} \subset (T \oplus T^*)M,$$

which is integrable as a Dirac structure on this small CA (i.e. involutive under the Dorfman bracket on $(T \oplus T^*)M$) if and only if $\partial b = 0$, i.e. $b$ is closed on $M$. On the other hand, on $T \oplus T^*(\tilde{M})$, $L_B$ defines a graph of $\beta$:

$$\mathrm{Graph}(\beta) = \{\tilde{\alpha} + \beta(\tilde{\alpha}) \mid \tilde{\alpha} \in \Omega^1(\tilde{M})\} \subset (T \oplus T^*)\tilde{M},$$

which is integrable when $[\beta, \beta] = 0$, i.e. $\beta$ is Poisson on $\tilde{M}$. Therefore, we see that in order for (5.3.7) to vanish, it is sufficient for $L_B$ to define an integrable Dirac structure on both small CAs, in which case both terms in the equation (5.3.7) vanish identically. This is, however, not a necessary condition as the two terms can be non-zero and still be cancelled as they both take values in $L$.

We now state one of our main results in [13], which we leave without proof since it requires several additional technical lemmas that are not important for the current discussion:

**Proposition 5.3.6** ([13]). *Let $K_B$ be a B-transformation of a para-Kähler structure $(\mathbb{M}, \eta, K)$. Then the D-bracket associated to $K_B$ is given by*

$$\eta([\![X,Y]\!]^{D,B}, Z) = \eta([\![X,Y]\!]^D, Z) - (\mathrm{d}b)(X,Y,Z). \tag{5.3.8}$$

*where $[\![\ ,\ ]\!]^D$ denotes the D-bracket of $K$.*

From the expression (5.3.8) it is not immediately clear how the extra tensorial part $\mathrm{d}b$, called the *twist* of the bracket, corresponds to various string theory fluxes. To see this, we must examine the different components of $\mathrm{d}b$ in the bigrading corresponding to $K_B$. While the frame of $T\mathbb{M}$ diagonalizing $K_B$ is $\{\partial_i^B = \partial_i + b_{ij}\tilde{\partial}^j, \tilde{\partial}_j\}$, the dual frame of $T^*\mathbb{M}$ is $\{dx^i, d\tilde{x}_i^B = d\tilde{x}_i + b_{ij}dx^j\}$:

$$\begin{aligned}\mathrm{d}b &= \partial_i b_{jk} dx^i \wedge dx^j \wedge dx^k + \tilde{\partial}^i b_{jk} d\tilde{x}_i \wedge dx^j \wedge dx^k \\ &= \partial_i b_{jk} dx^i \wedge dx^j \wedge dx^k + \tilde{\partial}^i b_{jk} d\tilde{x}_i^B \wedge dx^j \wedge dx^k + b_{il}\tilde{\partial}^l b_{jk} dx^i \wedge dx^j \wedge dx^k.\end{aligned}$$

- The $(3,0)_B$ component of $\mathrm{d}b$ is given by

  $$\mathrm{d}_+ b^{(3,0)_B} = \mathrm{d}_+ b + (\Lambda^3\eta)[\beta,\beta]$$

  where $\mathrm{d}b$ is also the $(3,0)$ component of $\mathrm{d}b$ with respect to $K$. The two terms combine to what is in DFT called a **covariantized H-flux** (or DFT H-flux), while the individual terms correspond to the well known **H-flux** and what can be seen as a (dual) **R-flux**:

  $$\begin{aligned}\mathrm{d}_+ b = H &= \partial_i b_{jk} dx^i \wedge dx^j \wedge dx^k, \\ (\Lambda^3\eta)[\beta,\beta] = \tilde{R} &= b_{il}\tilde{\partial}^l b_{jk} dx^i \wedge dx^j \wedge dx^k,\end{aligned}$$

  The H-flux is a 3-form on $M$, while $[\beta,\beta]$ is a three-vector on $\tilde{M}$. In physics the R-flux is usually a three-vector on the space-time manifold (in our case $M$), which is why we call $(\Lambda^3\eta)[\beta,\beta]$ the dual R-flux. The usual R-flux on $M$ would then be a result of the $\tilde{B}$-transformation (5.3.3) corresponding to a bivector on $L$.

- The $(2,1)_B$ component of $\mathrm{d}b$ reads

  $$\mathrm{d}b^{(2,1)_B} = \tilde{\partial}^i b_{jk} d\tilde{x}_i^B \wedge dx^j \wedge dx^k,$$

In this expression we recognize the (dual) **Q-flux**. We again see that this expression has the opposite index structure to the usual Q-flux due to the fact that $\beta$ is a bivector on $\tilde{M}$ as opposed to $M$, hence the name dual Q-flux.

The remaining components of $\mathrm{d}b$ vanish. We notice that the $H$ and $\tilde{R}$ fluxes are related, giving the $(3,0)_B$ part of $\mathrm{d}b$ (and therefore $\mathrm{d}\omega_B$, by (5.3.6) and $\mathrm{d}\omega = 0$), and therefore the obstruction to compatibility of $K_B$ with $K$. This obstruction,

$$\mathcal{H}_{ijk} = \partial_{[i}b_{jk]} + b_{[il}\tilde{\partial}^l b_{jk]},$$

is in the physics literature sometimes called the *covariantized H-flux* or an *H-flux without section condition* [78].

We have seen that all the fluxes correspond to the same data – the map $B$, which can be seen either as a two-form or as a bivector – and the resulting fluxes are just different differential operations on $b$. This relationship between fluxes reflects what is observed in physics, where the $H$, $Q$ and $R$ fluxes are related by *T-duality*. In our setting, this amounts to the exchange of the individual $x_i$ and $\tilde{x}^i$ coordinates. For example, if one starts with the $H_{123}$ component of $H$, i.e. the component of $H$ along $x^1$, $x^2$ and $x^3$, after performing T-duality along each of these coordinates, one ends up with an R-flux along the T-dual coordinates $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$, $R_{123}$. On the level of the corresponding bundles, this relationship is realized by the isomorphism of $\eta$ (and relabelling of coordinates).

**Example 5.3.7** (Doubled Torus with H-flux)**.** Consider the doubled torus setting $\mathbb{M} = T^6 = T^3 \times \tilde{T}^3$ from Example 5.3.3, i.e. $\mathbb{M}$ is endowed with the standard para-Hermitian structure described in the Example 2.2.9, $(x, y, z, \tilde{x}, \tilde{y}, \tilde{z})$ are the adapted coordinates and we also consider a 3-form flux given by an integer multiple of the volume form on $T^3$, $H = k\ dx \wedge dy \wedge dz$. We will now describe how the topology of $M$ changes from a torus to a nilmanifold $M'$ by performing T-duality along one pair of cycles. We first describe the standard construction of topological T-duality and then reinterpret the topology change in the language of para-Hermitian manifolds.

Performing topological T-duality along the $(z, \tilde{z})$ coordinates according to (2.3.8), we obtain a non-trivial fibration of the $\tilde{z}$-circles over the $(x, y)$ coordinates. This bundle is constructed as follows: choose a connection $\theta = dz$ on the original circle fibration locally given by $(x, y, z) \to (x, y)$. This means $k\ dx \wedge dy = \mathrm{d}\tilde{\theta}$ must be the curvature of the dual bundle $(x, y, \tilde{z}) \to (x, y)$ with $\tilde{\theta}$ its connection. We therefore locally choose $\tilde{\theta} = d\tilde{z} + kx dy$ and we have $\mathrm{d}\omega = H$ $(\tilde{H} = 0)$ with $\omega = -\theta \wedge \tilde{\theta} = (d\tilde{z} + kx\ dy) \wedge dz$. The dual bundle is therefore specified by the connection, in particular the cocycle condition for the identification $x \sim x+1$ is given by $\tilde{\theta}(x+1) - \tilde{\theta}(x) = -\mathrm{d}f$ for $f$ the gluing function along the $\tilde{z}$ fiber. This yields $f = -ky$ and we get the identifications for the T-dual circle bundle

$$(x, y, \tilde{z}) \sim (x+1, y, \tilde{z} - ky) \sim (x, y+1, \tilde{z}) \sim (x, y, \tilde{z}+1), \tag{5.3.9}$$

which define a *nilmanifold*. Therefore, performing T-duality along the pair $(z, \tilde{z})$ maps the trivial circle bundle given by $(x, y, z) \to (x, y)$ with $H$ to a nilmanifold given by $(x, y, \tilde{z}) \to (x, y)$ with identifications (5.3.9).

We now describe the example with non-trivial H-flux in terms of para-Hermitian geometry on the doubled torus. The existence of the $H$-flux is realized through the $B$-transformation corresponding to its local 2-form potential $b = kx\ dy \wedge dz$ reflected in the shift of the para-Hermitian structure $K_0 \mapsto K_B = K + 2B$, where $B = \eta b$. The frames of the eigenbundles get transformed as

$$\begin{aligned}
(\partial_x, \partial_y, \partial_z) &\longmapsto (\partial_x, \partial_y + kx\tilde{\partial}^z, \partial_z - kx\tilde{\partial}^y) = (e^B_x, e^B_y, e^B_z) \\
(\tilde{\partial}^x, \tilde{\partial}^y, \tilde{\partial}^z) &\longmapsto (\tilde{\partial}^x, \tilde{\partial}^y, \tilde{\partial}^z),
\end{aligned}$$

where we used the notation $\partial_x = \frac{\partial}{\partial x}$ and $\tilde{\partial}^x = \frac{\partial}{\partial \tilde{x}}$. The dual transformed frames are

$$(dx, dy, dz), \quad \text{and} \quad (d\tilde{x}, d\tilde{y} + kxdz, d\tilde{z} - kxdy).$$

Note that this splitting is not integrable, as

$$\begin{aligned}
[e^B_x, e^B_y] &= [\partial_x, \partial_y + kx\tilde{\partial}^z] = k\ \tilde{\partial}^z, \\
[e^B_x, e^B_z] &= [\partial_x, \partial_z - kx\tilde{\partial}^y] = -k\ \tilde{\partial}^z,
\end{aligned}$$

and the corresponding Nijenhuis tensor is $N = k\ dx \wedge (dy \otimes \tilde{\partial}^z - dz \otimes \tilde{\partial}^y)$. Now, we perform the T-duality, which is realized by the exchange of the transformed frame vectors in the $(z, \tilde{z})$ directions:

$$\begin{aligned}
e^B_z = \partial_z - kx\tilde{\partial}^y &\longleftrightarrow \tilde{\partial}^z \\
dz &\longleftrightarrow d\tilde{z} - kxdy.
\end{aligned}$$

We can now confirm that after $T$-duality, the +1-eigenbundle spanned by $(e_x^B, e_y^B, \tilde{\partial}^z)$ is now integrable. The dual frame is $(dx, dy, d\tilde{z} - kxdy)$ and since we still have $x \sim x+1$ and $y \sim y+1$, the frame is globally defined exactly when the coordinates $(x, y, \tilde{z})$ satisfy (5.3.9). Therefore, the T-dual is the same nilmanifold as before. To summarize, one can repackage the data of the doubled torus with H-flux and para-Hermitian structure $(\eta, K_0)$ as the non-integrable para-Hermitian structure $(\eta, K_B)$, which after T-duality along the $(z, \tilde{z})$ directions defines a nilmanifold as the integral manifold of its +1-eigenbundle.

This example can be extended to a T-duality between two nilmanifolds by also considering an $H$-flux $\tilde{H} = j\ dx \wedge dy \wedge dz$, where the T-duality acts as the exchange of $H$-fluxes, $k \leftrightarrow j$, together with the exchange of coordinates $z$ and $\tilde{z}$. ◁

**The Full set of fluxes** In the above we only discussed the fluxes when the $B$-transformation (5.3.1) is present. We will now recall the results from [18], where the most general set of fluxes is obtained by simultaneously applying the transformations (5.3.1) and (5.3.3). In such case, one gets as the different obstructions to D-integrability the following set of fluxes with $B = \eta b$ and $\tilde{B} = \eta\beta$:

$$
\begin{aligned}
\mathcal{H}_{ijk} &= \partial_{[i} b_{jk]} + b_{[il} \tilde{\partial}^l b_{jk]}, \\
\mathcal{F}_{ij}{}^{k} &= \tilde{\partial}^k b_{ij} + \beta^{km} \mathcal{H}_{mij}, \\
\mathcal{Q}_k{}^{ij} &= \partial_k \beta^{ij} + \beta^{im} \beta^{jl} \mathcal{H}_{mlk} + b_{km} \tilde{\partial}^m \beta^{ij} + 2\beta^{p[i} \tilde{\partial}^{j]} b_{pk}, \\
\mathcal{R}^{ijk} &= 3\tilde{\partial}^{[i} \beta^{jk]} + 3\beta^{[il} \beta^{jm} \tilde{\partial}^{k]} b_{ml} + 3\beta^{[im} \partial_m \beta^{jk]} + 3b_{lm} \beta^{[il} \tilde{\partial}^m \beta^{jk]} + \beta^{il} \beta^{jm} \beta^{kn} \mathcal{H}_{lmn}.
\end{aligned}
$$

This is the full set of fluxes one obtains in DFT, see for example [29, eq. 5.88].

## 5.4 D-bracket from The Large Courant Algebroid

In Section 5.1, we realized the D-bracket as a sum of the two small Courant algebroid brackets. We will now show that in the context of generalized para-Kähler geometry, one can also construct the D-bracket by restricting the Dorfman bracket of the large CA to the eigenbundles $\mathbb{C}_\pm$ of the generalized metric corresponding to $\eta$. This construction is originally due to [20] where this relationship was observed for para-Kähler manifolds. Later in [31], the idea was fully formalized for any generalized para-Kähler manifolds as well.

First, we observe that on any pseudo-Riemannian manifold $(\mathbb{M}, \eta)$, we get a tangent bundle bracket operation similar to (2.4.4) from a restriction of the (twisted) Dorfman bracket to the eigenbundles of a generalized metric $\mathcal{G}(\eta, b)$ with an arbitrary $b$-field:

**Proposition 5.4.1** ([31]). *Let $\mathcal{G}(\eta, b)$ be a generalized metric on a pseudo-Riemannian manifold $(\mathbb{M}, \eta)$ with eigenbundles $\mathbb{C}_\pm$, $H$ a closed 3-form and denote $H_b = H + \mathrm{d}b$. The restrictions of the $H_b$-twisted Dorfman bracket to $\Lambda^3\mathbb{C}_\pm$ yield a bracket operation on the tangent bundle called the* **almost D-bracket** *with a flux $\pm H_b$:*

$$\pm\frac{1}{2}\langle[\pi_\pm^{-1}X, \pi_\pm^{-1}Y], \pi_\pm^{-1}Z\rangle = \eta([\![X, Y]\!]^{\eta, \pm H_b}, Z) = \eta([\![X, Y]\!]^{\mathring{\nabla}}, Z) \pm \frac{1}{2}H_b(X, Y, Z), \qquad (5.4.1)$$

*where $\mathring{\nabla}$ denotes the Levi-Civita connection of $\eta$ and $[\![\ ,\ ]\!]^{\mathring{\nabla}}$ is defined by*

$$\eta([\![X, Y]\!]^{\mathring{\nabla}}, Z) = \eta(\mathring{\nabla}_X Y - \mathring{\nabla}_Y X, Z) + \eta(\mathring{\nabla}_Z X, Y).$$

*Proof.* Using (4.1.7), we get

$$\begin{aligned}\langle[\pi_\pm^{-1}X, \pi_\pm^{-1}Y], \pi_\pm^{-1}Z\rangle &= \langle\mathring{\nabla}_X\pi_\pm^{-1}(Y) - \mathring{\nabla}_Y\pi_\pm^{-1}(X), \pi_\pm^{-1}(Z)\rangle \\ &\quad + \langle\mathring{\nabla}_Z\pi_\pm^{-1}(X), \pi_\pm^{-1}(Y)\rangle + H(X, Y, Z),\end{aligned}$$

which after a straightforward calculation leads to the result. □

The almost D-bracket of a pseudo-Riemannian manifold in fact defines a metric algebroid [53] with anchor the identity, i.e. it satisfies the properties 1. and 2. in Definition 2.4.1. Moreover, here we see that it arises via the isomorphisms (4.2.5) between the tangent bundle $T\mathbb{M}$ and $\mathbb{C}_\pm \subset (T \oplus T^*)\mathbb{M}$. Recalling (4.2.6), we see that $\pi_\pm$ are isomorphisms of metric algebroids

**Proposition 5.4.2.** *Consider the setting of Proposition 5.4.1 and denote $G(\cdot, \cdot) = \frac{1}{2}\langle\mathcal{G}(\eta, b)\cdot, \cdot\rangle$. Then the maps $\pi_\pm$ given by (4.2.5) define an isomorphism of the following metric algebroids over $\mathbb{M}$:*

$$(T\mathbb{M}, \eta, \mathbb{1}, [\![\ ,\ ]\!]^{\eta, \pm H_b}) \xleftrightarrow{\pi_\pm} (C_\pm, G, \pi_\pm, [\ ,\ ]).$$

The above statement is simply the rephrasing of the fact that the following relationships are satisfied:

$$\eta(X,Y) = G(\pi_\pm^{-1}X, \pi_\pm^{-1}Y), \quad [\![X,Y]\!]^{\eta,\pm H_b} = \pi_\pm[\pi_\pm^{-1}X, \pi_\pm^{-1}Y],$$

and that the anchors of the two metric algebroids are compatible, which is satisfied trivially: $\pi_\pm \pi_\pm^{-1} = \mathbb{1}_{T\mathbb{M}}$.

The final step is now to show that whenever the generalized metric $\mathcal{G}(\eta, b)$ in question is a part of a generalized para-Kähler structure $(\mathcal{G}(\eta,b), \mathcal{K}_\pm)$, the brackets $[\![\ ,\ ]\!]^{\eta,\pm H_b}$ become the D-brackets associated to the corresponding para-Hermitian structures $(\eta, K_\pm)$. We stated this result in [31].

**Theorem 5.4.3** ([31])**.** *Let $(\mathcal{G}, \mathcal{K})$ be a GpK structure on $\mathbb{M}$ with a flux $H$ and $(\eta, K_\pm)$ the corresponding bi-para-Hermitian data. Then the D-brackets $[\![\ ,\ ]\!]_\pm$ associated to the para-Hermitian structures coincide with the brackets $[\![\ ,\ ]\!]^{\eta,\pm H_b}$ associated to the generalized metric $\mathcal{G}$. In other words,*

$$\eta([\![X,Y]\!]_\pm, Z) = \pm\frac{1}{2}\langle[\pi_\pm^{-1}X, \pi_\pm^{-1}Y], \pi_\pm^{-1}Z\rangle. \tag{5.4.2}$$

*Proof.* From Proposition 5.4.1 it follows that

$$\pm\frac{1}{2}\langle[\pi_\pm^{-1}X, \pi_\pm^{-1}Y], \pi_\pm^{-1}Z\rangle = g(\mathring{\nabla}_X Y - \mathring{\nabla}_Y X, Z) + g(\mathring{\nabla}_Z X, Y) \\ \pm \frac{1}{2}H_b(X,Y,Z).$$

It remains to relate this to the expressions for the $D$-bracket (2.4.4) associated to $K_\pm$. Theorem 4.3.3 tells us that $K_\pm$ are necessarily integrable and therefore the $(3,0)$ and $(0,3)$ components of $\mathrm{d}\omega_\pm$ in (2.4.2) vanish. Because $\mathrm{d}^P\omega = \mathrm{d}\omega^{(2,1)} - \mathrm{d}\omega^{(1,2)}$, the $(2,1)$ and $(1,2)$ components then get matched (recalling again Theorem 4.3.3) by equation $\mathrm{d}_\pm^P\omega_\pm = \mp(H + \mathrm{d}b)$. This completes the proof. □

The above result can be understood from the point of view of T-duality in the following way. The eigenbundles $\mathbb{C}_\pm$ on a GpK manifold, which play a central role in the construction, were found in Section 5.2 to be the subbundles of $(T \oplus T^*)\mathbb{M}$ that are left invariant under the linear T-duality operation. Therefore, the D-bracket can be interpreted as the Dorfman bracket on $(T \oplus T^*)\mathbb{M}$ restricted to the T-duality invariant bundles $\mathbb{C}_\pm$. The reason for this is that under T-duality, the *projected* brackets (5.1.2) are mapped onto each other, but the whole D-bracket (which is their sum) is left invariant.

# Chapter 6

# Applications to $2D$ $\sigma$-models

In this section we explain how the generalized para-Kähler and chiral geometries introduced Section 4.3 arise in physics in the context of 2D supersymmetric non-linear $\sigma$-models. For basics of supersymmetry (SUSY) and other details the reader can consult for example [85]. For details about $2D$ $\sigma$-models, in particular their $(2,2)$ supersymmetric version, see [86, 36] as well as the thesis [87] containing many useful details and calculations.

First, we introduce the $2D$ $\sigma$-models that we will study. They are given by the action functional

$$S_{(1,1)}(\Phi) = \int_{\Sigma^{2|2}} [g(\Phi) + b(\Phi)]_{ij} D^1_+ \phi^i D^1_- \phi^j, \tag{6.0.1}$$

where $\Phi = \{\Phi_i\}_{i=1\cdots n}$ are *superfields*, i.e. maps $\Phi : \Sigma^{2|2} \to (M,g)$. Here $\Sigma^{2|2}$ is a super-Riemann surface with two real and two formal odd coordinates $(x_\pm, \theta^\pm_1)$, $(M,g)$ is (for now) arbitrary pseudo-Riemannian manifold and $b$ denotes a *local* two-form. The symbols $D^1_\pm$ denote *superderivatives*, i.e. derivatives on the space of fields $\Phi$ defined by

$$D^1_\pm = \frac{\partial}{\partial \theta^\pm_1} - \theta^\pm_1 \partial_\pm. \tag{6.0.2}$$

The superfields $\Phi$ can be written in terms of their polynomial expansion in the odd coordinates $\theta^\pm$:

$$\Phi^i(x_\pm, \theta^\pm) = \phi^i(x_\pm) + \theta^+_1 \psi^i_+(x_\pm) + \theta^-_1 \psi_-(x_\pm) + \theta^+_1 \theta^-_1 F^i(x_\pm), \tag{6.0.3}$$

and the expressions $g(\Phi)$, $b(\Phi)$ then represent the formal Taylor series expanded around

$\theta_1^\pm = 0$, for example the first two terms in this expansion for $g$ read

$$g(\Phi)_{ij} = g(\phi)_{ij} + \partial_k g(\phi)_{ij}(\theta_1^+\psi_+^k + \theta_1^-\psi_-^k) + \cdots .$$

The action (6.0.1) has the important property that it is invariant under the action of the following pair of derivative operators called supercharges

$$Q_\pm^1 = \frac{\partial}{\partial\theta_1^\pm} + \theta^\pm\partial_\pm, \tag{6.0.4}$$

which obey the anti-commutation relations

$$\{Q_\pm^1, Q_\pm^1\} = 2\partial_\pm, \tag{6.0.5}$$

$\partial_\pm = \frac{\partial}{\partial x_\pm}$ being the derivatives on the even part of $\Sigma^{2|2}$. It can be shown that $Q_\pm^1$ define an extension of the Poincaré algebra to a superalgebra, where the only non-trivial odd brackets are given by (6.0.5). Such a superalgebra is called a $(1,1)$ supersymmetry algebra and the action (6.0.1) is then said to carry a $(1,1)$ supersymmetry (SUSY).

An interesting observation that gives a first clue about how the generalized geometry enters the description of the $2D$ SUSY $\sigma$-models is that any *generalized metric* $\mathcal{G}$ (4.2.4) defines a $(1,1)$ action (6.0.1), because it corresponds to the data of a pseudo-Riemannian metric $g$ and a two-form $b$. In the following discussion, we will argue that whenever the target manifold $M$ with a generalized metric $\mathcal{G}(g,b)$ carries an additional generalized para-complex or chiral structure that commutes with $\mathcal{G}$, the $\sigma$-model defined by $\mathcal{G}(g,b)$ exhibits extra superspace symmetries.

*Remark.* There is an analogous well-known statement in the case when $(M,\mathcal{G})$ is generalized Kähler, i.e. on top of the data of the generalized metric $\mathcal{G}(g,b)$, there is a generalized complex structure $\mathcal{I}$ that commutes with $\mathcal{G}$ and such that $\mathcal{I}' = \mathcal{I}\mathcal{G}$ is also generalized complex. In such case, one obtains a $(2,2)$ SUSY $\sigma$-model.

## 6.1 (2,2) para-SUSY and GpK Geometry

In this subsection we explain that whenever there is an integrable generalized para-Kähler structure $(\mathcal{G},\mathcal{K}_\pm)$ on the target $M$ of the $\sigma$-model (6.0.1) defined by the generalized metric $\mathcal{G} = \mathcal{G}(g = \eta, b)$, the $\sigma$-model acquires a $(2,2)$ *para-supersymmetry*. This result has been obtained in [36] in terms of the bi-para-Hermitian geometry $(\eta, K_\pm, b)$ corresponding to the GpK structure $(\mathcal{G}, \mathcal{K}_\pm)$. This work can also be used as a reference for details about para-supersymmetry and more extensive study of said $\sigma$-models.

*Remark.* The original name for *para-supersymmetry* used in [36] is *twisted* supersymmetry while some other works also used the name *pseudo* supersymmetry. Here we use the name para-supersymmetry in order to avoid the confusion with topologically twisted $\sigma$-models, which are frequently discussed in the context of $(2,2)$ $\sigma$-models and to more intuitively reflect the relationship to para-complex geometry.

Recall that the $\sigma$-model (6.0.1) always carries $(1,1)$ SUSY, which means that it is invariant under the action given by the infinitesimal generators (6.0.4), which satisfy the $(1,1)$ SUSY algebra (6.0.5). We now wish to extend this algebra to $(2,2)$ para-SUSY algebra and show that such extension only exists if the target is generalized para-Kähler. The $(2,2)$ para-SUSY algebra is an extension of the $(1,1)$ algebra by additional supercharges $Q^2_\pm$ that satisfy the relations opposite to (6.0.5). In other words, instead of two supercharges, there are four and the only non-zero anti-brackets are:

$$\{Q^1_\pm, Q^1_\pm\} = 2\partial_\pm \quad \text{and} \quad \{Q^2_\pm, Q^2_\pm\} = -2\partial_\pm. \tag{6.1.1}$$

The additional supercharges $Q^2_\pm$ have to (for dimensional reasons, see [36]) necessarily act on the fields $\Phi$ by

$$Q^2_\pm \Phi^i = (K_\pm(\Phi))^i_j D^1_\pm \Phi^j, \tag{6.1.2}$$

for some (for now unspecified) target space tensors $K_\pm$. The requirement that the action (6.0.1) is invariant under $Q^2_\pm$ forces the compatibility between $(g+b)$ and $K_\pm$:

$$g(K_\pm \cdot, \cdot) + g(\cdot, K_\pm \cdot) = 0,$$
$$b(K_\pm \cdot, \cdot) + b(\cdot, K_\pm \cdot) = 0,$$

along with the condition

$$\nabla^\pm K_\pm = 0,$$

where $\nabla^{\pm}$ are the connections defined in (4.2.11),

$$\nabla^{\pm} = \mathring{\nabla} \pm \frac{1}{2} H.$$

Here $\mathring{\nabla}$ is the Levi-Civita connection of $g$ and $H$ is a closed global three-form, such that $b$ is locally its potential, $\mathrm{d}b = H$[1].

Next, one must also ensure that the transformations (6.1.2) indeed extend (6.0.5) to a $(2,2)$ para-supersymmetry. This is equivalent to the conditions

$$K_{\pm}^{2} = \mathbb{1} \quad \text{and} \quad N_{K_{\pm}} = 0,$$

rendering $(g, K_{\pm}, b)$ a bi-para-Hermitian geometry, or equivalently, $M$ to be a GpK manifold.

When we require that the theory is *parity-symmetric*, we find that the $b$-field term in (6.0.1) has to vanish and additionally $K_{+} = K_{-} = K$, which gives the para-Kähler limit of the geometry. Additionally, one might require additional supersymmetry, which requires additional para-complex structure that anti-commutes with $K$, which is therefore described by the para-hyper-Kähler limit of GpK geometry. Various other heterotic supersymmetries can be realized as well, all as special cases of the GpK geometry.

*Remark.* The integrability of $K_{\pm}$ can sometimes be relaxed [75], giving the GpK (or GK in the case of usual SUSY) geometries which are only integrable in the weaker sense introduced in Section 4.3.3.

**Example 6.1.1** (The para-Kähler model)**.** The most famous model that carries the *usual* $(2,2)$ SUSY is the Kähler model, which is a $\sigma$-model with the target a Kähler manifold and the action is simply given by the local Kähler potential [88]. Here we present an analogous model for the para-Kähler geometry [36].

To do this, we must first introduce a full $(2,2)$ formalism, which means representing the $(2,2)$ para-SUSY on a super-Riemann surface $\Sigma^{2|4}$ with 4 odd coordinates instead of 2. This consequently introduces a much larger space of fields

$$\Phi : \ \Sigma^{2|4} \to (\mathbb{M}, \eta, K),$$

where we also already introduced the para-Kähler target $(\mathbb{M}, \eta, K)$. We denote the 4 odd coordinates by $(\theta^{\pm}, \tilde{\theta}^{\pm})$ and introduce a new basis $(Q_{\pm}, \tilde{Q}_{\pm})$ of the $(2,2)$ para-SUSY algebra,

[1]The expression (6.0.1) is local, which is the reason why the local two-form $b$ appears

which is given by

$$Q_\pm = \frac{1}{\sqrt{2}}(Q^2_\pm - Q^1_\pm), \quad \text{and} \quad \widetilde{Q}_\pm = \frac{1}{\sqrt{2}}(Q^2_\pm + Q^1_\pm),$$

and satisfying the relations

$$\{Q_\pm, \tilde{Q}_\pm\} = -2\partial_\pm.$$

The representation of these charges on $\Sigma^{2|4}$ is:

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} - \tilde{\theta}^\pm\partial_\pm \quad \text{and} \quad \tilde{Q}_\pm = \frac{\partial}{\partial\tilde{\theta}^\pm} - \theta^\pm\partial_\pm, \tag{6.1.3}$$

and we also define the differential operators

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \tilde{\theta}^\pm\partial_\pm \quad \text{and} \quad \tilde{D}_\pm = \frac{\partial}{\partial\tilde{\theta}^\pm} + \theta^\pm\partial_\pm,$$

which can be used to define *para-chiral* fields $(\chi^i, \tilde{\chi}^j)_{i,j=1,\cdots,d}$, which are fields constrained by the differential conditions

$$D_\pm\tilde{\chi}^i = \tilde{D}_\pm\chi^j = 0,$$

for all $i$ and $j$. It is easy to see that the fields have to have the following expansion in the odd coordinates analogous (6.0.3):

$$\begin{aligned}
\chi^i &= \phi^i(y^\pm) + \psi^i_+(y^\pm)\theta^+ + \psi^i_-(y^\pm)\theta^- + F^i(y^\pm)\theta^+\theta^- \\
\tilde{\chi}^i &= \tilde{\phi}^i(\tilde{y}^\pm) + \tilde{\psi}^i_+(\tilde{y}^\pm)\tilde{\theta}^+ + \tilde{\psi}^i_-(\tilde{y}^\pm)\tilde{\theta}^- + \tilde{F}^i(\tilde{y}^\pm)\tilde{\theta}^+\tilde{\theta}^-,
\end{aligned}$$

where $(\phi, \psi_\pm, F)$ and $(\tilde{\phi}, \tilde{\psi}_\pm, \tilde{F})$ are some functions of $y^\pm = x^\pm + \theta^\pm\tilde{\theta}^\pm$ and $\tilde{y}^\pm = x^\pm - \theta^\pm\tilde{\theta}^\pm$. Moreover, the bosonic parts $(\phi^i, \tilde{\phi}^j)$ of the para-chiral fields parametrize the directions of the adapted coordinates $(x^i, \tilde{x}^j)$ of the target para-Kähler manifold and therefore parametrize the fundamental foliations.

Given all this data, the **para-Kähler** model on the para-Kähler manifold $(\mathbb{M}, \eta, K)$ is then given by the action functional:

$$S(\chi, \tilde{\chi}) = \int_{\Sigma^{4|2}} f(\chi, \tilde{\chi}),$$

where $f$ is the local para-Kähler potential for the para-Kähler geometry. It is easy to check that this action is invariant under the $(2,2)$ para-SUSY generators (6.1.3) and is also well-defined on the whole $\mathbb{M}$ due to the fact that $f$ is the para-Kähler potential [36].

We end this example with the natural conjecture that any $(2,2)$ para-SUSY $\sigma$-model on an arbitrary GpK manifold is given by some local scalar *generalized para-Kähler potential* similarly to the case of ordinary $(2,2)$ SUSY described locally by the generalized para-Kähler potential [89]. ◁

### 6.1.1 Topologically twisted theories and mirror symmetry

There is a construction that extracts a topological field theory from any $(2,2)$ $\sigma$-model called **topological twisting**. For ordinary $(2,2)$ SUSY, this has been described by Witten in [90] for the Kähler $\sigma$-model, where there are two distinct twists and one obtains the famous $A$- and $B$-models. Later on, Kapustin and Li [91] generalized this result for arbitrary generalized Kähler target. For a $(2,2)$ para-SUSY, we describe the topological twists in a joint work with Williams [35].

Another fact that is very well known and studied in the context of the usual $(2,2)$ SUSY but also works equally well for $(2,2)$ para-SUSY, is that there exists a $\mathbb{Z}_2$ outer endomorphism of the $(2,2)$ algebra, acting as a certain exchange of the charges. This operation then relates two a priori different $(2,2)$ $\sigma$-models, rendering them in a certain sense equivalent and this equivalence is the physical statement of mirror symmetry. In particular, when applied to the topological twists of the $(2,2)$ theory, it exchanges the two twists. In the case of the Kähler model this means an exchange of the $A$- and $B$-models, which on the side of the underlying Kähler geometry means an exchange of the symplectic and complex geometries. In [35], we derive the analogous statements for (generalized) para-Kähler geometry and $(2,2)$ para-SUSY $\sigma$-models.

In our case of the $(2,2)$ para-SUSY, the $\mathbb{Z}_2$ action on the algebra (6.1.3) is given by the exchange

$$Q_- \longleftrightarrow \widetilde{Q}_-.$$

We conjecture here that this gives rise to a notion of mirror symmetry for para-complex geometry, which in the case of para-Kähler manifolds relates the para-complex and symplectic moduli, as illustrated in Section 3.3.

## 6.2 (1,1) Superconformal Algebra and Generalized Chiral Geometry

In [37], it has been shown that the chiral geometry also plays an important role in introducing additional symmetries to $(1,1)$ $\sigma$-models (6.0.1). While pairs of Hermitian and para-Hermitian structures naturally arise when considering an extended supersymmetry, pairs of chiral structures have different physical interpretation in terms of $\sigma$-models – they correspond to introduction of additional copies of the $(1,1)$ superconformal algebra. Here we briefly review the results of [37].

Consider a $\sigma$-model on a target $(M, g)$ given by the action (6.0.1). For every such $\sigma$-model, there are so-called superconformal symmetries stemming from the fact that $M$ carries the metric $g$. The symmetries close to form an algebra, called a superconformal algebra. Now, it is shown in [37] that when $M$ admits two (almost-)product structures $J_\pm$ orthogonal with respect to $g$, that are also covariantly constant with respect to $\nabla^\pm$ (4.2.11),

$$g(J_\pm \cdot, J_\pm \cdot) = g, \quad \nabla^\pm J_\pm = 0, \tag{6.2.1}$$

one can introduce additional symmetries $\delta_{P_\pm}$ and $\delta_{Q_\pm}$ associated to[2] the $+1$ and $-1$ projectors $P_\pm$ and $Q_\pm$, respectively

$$P_\pm = \frac{1}{2}(\mathbb{1} + J_\pm), \quad Q_\pm = \frac{1}{2}(\mathbb{1} - J_\pm).$$

The symmetries $\delta_{P_\pm}$ and $\delta_{Q_\pm}$ then form copies of the $(1,1)$ superconformal algebra. The conditions (6.2.1) are the only conditions on the tensors $J_\pm$, in particular there are no further requirements on integrability of $J_\pm$. By results of Section 4.3.2 and Proposition 4.3.9, this means that $(J_\pm, g, b)$ defines a generalized chiral structure that is weakly integrable.

Because the additional symmetries $\delta_{P_\pm}$ and $\delta_{Q_\pm}$ form a superconformal algebra even when $J_\pm$ are not integrable, they lack a spacetime description in terms of a corresponding Riemannian manifold, contrary to the original algebra associated to $(M, g)$. The author of [37] then relates this fact to the existence of non-geometric string backgrounds.

[2] We will not explain here how the symmetries are associated to the projectors $P_\pm$ and $Q_\pm$; we merely remark that the projectors are the only additional geometrical data entering the definitions of $\delta_{P_\pm}$ and $\delta_{Q_\pm}$.

# Conclusion

In this thesis we described various aspects and applications of Born geometry with a main focus on its relationship to T-duality in physics, in particular to Double Field Theory (DFT) and the notion of extended space originating therein. Let us now briefly summarize our results and highlight some of the most interesting and important questions for future research.

In Chapter 2 of the thesis we discussed the basics of para-Hermitian geometry and argued that this geometry naturally describes the extended space of DFT and as such can be understood as the main building block of Born geometry when seen through the scope of T-duality. This is because the para-Hermitian geometry is fully fixed by the T-duality setting and therefore plays the role of the background or kinematical structure of the extended space. One of the important components of this kinematical structure is the D-bracket, which is a new bracket operation on vector fields on the extended space, appearing in DFT as the replacement of the Lie bracket, and which – as we showed – is naturally defined in terms of the para-Hermitian geometry. We further discussed the mechanisms through which para-Hermitian geometry facilitates T-duality and provided several examples, including the important case of Topological T-duality. In the final section of the first part, we defined the notion of para-Calabi-Yau manifolds, which are a special type of para-Hermitian manifolds that carry a compatible para-holomorphic volume form and can be understood as the para-complex analogue of Calabi-Yau manifolds.

In Chapter 3 we first elaborated on different aspects, definitions and points of view on Born geometry, establishing the fact that the data of Born geometry is equivalent to a choice of a $d$-dimensional metric structure on the $2d$-dimensional para-Hermitian extended space, understood as the physical space metric. This aligns with our point of view of para-Hermitian geometry as the background structure of Born geometry and additionally shows that in order to define a full Born geometry, one only needs to choose the dynamical data in the form of a metric. Furthermore, we showed that there exists a unique connection analogous to the

Levi-Civita connection in Riemannian geometry – the Born connection – that satisfies two main properties. First, it parallelizes all the structures of the Born geometry, and second, its D-torsion, which is an altered notion of the usual torsion corresponding to the replacement of the Lie bracket by the D-bracket in DFT, vanishes. In the final section we then discussed the perhaps most notable new contribution of this thesis, which is the example of Born geometry in the context of semi-flat mirror symmetry. This example shows that the canonical and well understood example of mirror symmetry between the tangent and cotangent bundles of an affine manifold (which are both Calabi-Yau manifolds) supports para-Calabi-Yau geometries on both sides of the mirror map. Moreover, the Calabi-Yau and para-Calabi-Yau geometries are compatible, so that they define a Born geometry, again, on each of the mirror manifolds. This uncovers a new aspect of the mirror map – famously understood as the exchange of symplectic and complex geometries – showing that in this case, it exchanges the symplectic and para-complex geometries as well.

In Chapters 4 and 5 we explored the different ways the para-Hermitian and Born geometries are related to the mathematical framework of generalized geometry. After reviewing mostly well known aspects of generalized geometry in the beginning of Chapter 4, we showed that a generalized para-Kähler structure, which is a commuting pair of generalized para-complex structures satisfying a certain non-degeneracy condition, is in one-to-one correspondence to a pair of compatible para-Hermitian structures sharing the same metric. Similarly, a commuting pair of compatible generalized product structures, called a generalized chiral structure, is in one-to-one correspondence to a pair of (usual) chiral structures. If the two tangent bundle chiral structures additionally anti-commute, one recovers Born geometry.

In Chapter 5, we explored a relationship between the para-Hermitian geometry and generalized geometry of a different flavor. Using the fact that an integrable para-Hermitian manifold is equipped with a pair of transversal half-dimensional foliations, we showed that there is a natural construction of the D-bracket in para-Hermitian geometry via the Courant algebroids of these two foliations, called small Courant algebroids. We then also demonstrated that using this point of view, one can naturally incorporate simple fluxes of DFT into the D-bracket as well as clarify their relationship with the $b$- and $\beta$- transformations of generalized geometry. In the last part of the chapter, we also showed that the D-bracket can also be easily recovered from the data of generalized para-Kähler geometry, introduced in Chapter 4.

Finally, in Chapter 6, we discussed an a priori unrelated way the para-Hermitian and

Born geometries appear in physics, when considering $2D$ $(1,1)$ supersymmetric (SUSY) $\sigma$-models with extra superspace symmetries. First, we showed that para-Hermitian geometry describes target spaces of $2D$ $\sigma$-models that exhibit a $(2,2)$ extended para-SUSY in much the same way Hermitian geometry describes targets of the usual $(2,2)$ SUSY $\sigma$-models. In fact, the most general geometry of the $(2,2)$ para-SUSY models is given by a pair of para-Hermitian geometries discussed in Chapter 4, or equivalently by a generalized para-Kähler geometry. We then briefly discussed the topological twists of such $\sigma$-models and based on the analogies with the usual $(2,2)$ SUSY and our results of Section 3.3, we conjectured that the para-SUSY $\sigma$-models should exhibit a new type of mirror symmetry exchanging the symplectic and para-complex geometric data. Lastly, we described that Born geometry appears as the target space of the $2D$ $(1,1)$ $\sigma$-models when one considers a splitting of the $(1,1)$ superconformal algebra into two different copies.

Let us now discuss some of the most important future research directions continuing the ideas presented in this thesis. First, we would like to fully incorporate the language of DFT fluxes into the formalism of para-Hermitian geometry, going beyond our discussion in Section 5.3. In our presentation, we mostly focused on the process of recovering the physical space via the canonical foliations of the para-Hermitian manifold, but this approach cannot recover cohomologically non-trivial fluxes on the physical space in terms of global objects on the extended space, as discussed in [84]. One must instead invoke the construction of physical space through quotients, where global objects on the extended space in some cases indeed give rise to cohomologically non-trivial fluxes on the quotient [21]. Additionally, as we pointed out in Section 2.1.3, the description of a physical space in terms of a foliation is problematic from the interpretational point of view. This is because such a physical space would typically have uncountably many connected components and one must consequently either make a non-canonical choice of a particular leaf of the foliation, or identify the physical space with the leaf space of the transverse para-Hermitian foliation.

Both of the above shortcomings of our approach therefore suggest that one should instead consider the quotient by the transverse foliation as the model for the physical space. As we noted, a lot of progress in this direction has been made in [21], but various ideas introduced in this thesis still remain to be discussed in the context of the quotient paradigm. This includes for example studying the small Courant algebroids of a para-Hermitian manifold as well as the (almost) generalized structures they can carry, and their reductions under the action of the quotient. Moreover, we hope to reconcile the relationship between the small Courant algebroids of the physical space and the large Courant algebroid of the whole extended space through the reductions of the latter, in particular by invoking a reduction construction known in generalized Kähler geometry and applying it to the case of generalized para-Kähler geometry.

A related question that has not yet been answered in a satisfying way is the geometric interpretation of the D-bracket and the notion of integrability it gives rise to, in order to complete the picture of the D-bracket replacing the ordinary Lie bracket on the extended space. In particular, it is desirable to understand the D-bracket integrability in terms of theorems analogous to the Frobenius theorem and consequently the Newlander–Nirenberg theorems for (para-)complex geometry. From our brief discussion in the Section 5.1.1, it is clear that one can expect the D-bracket integrability to be closely related to the integrability in terms of the Dorfman bracket on the foliation quotients, but one would also hope for an interpretation intrinsic to the extended space as a whole. After all, the D-bracket can be defined on any almost para-Hermitian manifold with no reference to the foliations.

The idea to replace the Lie bracket by the D-bracket as the physically more natural choice on the extended space should also be taken further and the whole machinery of Riemannian geometry should be mimicked using the D-bracket as well. While the first step of replacing the torsion tensor by the D-torsion is already well understood, a fitting replacement of tools indispensable for the formulation of a gravity-like theory on the extended space, such as the curvature tensors and the corresponding scalar, has not been found yet. This is crucial for the applications in DFT, where one would like to write a full action functional intrinsic for the extended space, which in the language of DFT means partially or entirely relaxing the section condition. Many results are known in scenarios where the section condition is present and the Riemann tensor is defined on the extended space using the knowledge of the reduced Riemann tensor of the physical space, but a top-down approach to this problem without an a priori reference to the half-dimensional physical space has so far been elusive. A particular task in this pursuit is to define a proper definition of the dilaton field, which is sufficiently natural in the para-Hermitian framework and also satisfies the top-down criterion outlined above. As discussed at the end of Section 2.5, because the dilaton field is closely related to the existence of a volume form on the physical space, we believe that the notion of para-Calabi-Yau geometry (or a relaxation thereof) could prove to be important for understanding of this problem.

Lastly, there are many unanswered questions related to the appearance of the para-Hermitian and Born geometries in the framework of $2D$ $\sigma$-models. First, we would like to understand if the appearance of the same geometric objects in both a priori different physical models is a consequence of an underlying relationship between the two, or a mere coincidence. If the formal similarity is indeed not coincidental, then it means that the para-SUSY of the $\sigma$-model is closely related to the T-duality in DFT. This is because in both cases, these features are equivalent to the presence of the para-Hermitian geometry. Understanding this relationship could then provide new insights in both DFT and the para-SUSY $\sigma$-models. On a more hypothetical level, one could study how the topological twists of the para-SUSY $\sigma$-models fit into the DFT framework, as well as the relationship between the conjectured mirror symmetry for the $\sigma$-models on one side and the T-duality in DFT on the other.

The topic of para-SUSY $\sigma$-models opens up numerous interesting questions on its own, regardless of its link to DFT. As we already pointed out, the formulation of mirror symmetry for such $\sigma$-models and more generally for para-complex geometry has not yet been established. Given the extremely wide applicability of mirror symmetry in complex geometry to countless areas of mathematics and physics, we hope that a closer inspection of this topic will give rise to many new results and exciting research directions as well.

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