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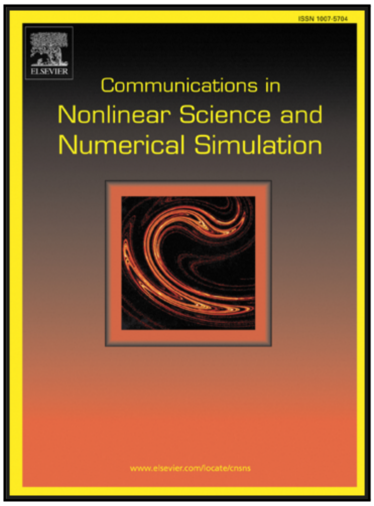

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Highlights

- The distributed delays and discrete delays are considered in the pinning impulsive controllers respectively.
- A more general pinning impulsive algorithm is designed.
- Two criteria for exponential synchronization of coupled reaction-diffusion neural networks are given.
- The relations of impulsive gains, amounts of pinned nodes, impulsive intervals and time delays are presented.

Synchronization of Coupled Reaction-Diffusion Neural Networks: Delay-Dependent Pinning Impulsive Control

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Abstract

This paper studies the synchronization problem of coupled reaction-diffusion neural networks with time-varying delays. A novel pinning impulsive controller is proposed, where distributed delays and discrete delays are taken into account, respectively. By using the Lyapunov-Krasovskii method, the relations among impulsive gains, pinned node numbers, impulsive intervals, impulsive instants and time delays are derived. Exponential synchronization criteria are established for the delayed coupled reaction-diffusion neural networks. Our results show that synchronization of the neural networks can be achieved by controlling a small portion of nodes in the networks via delayed impulses. Numerical examples are provided to demonstrate the effectiveness of the theoretical results.

Keywords: Impulsive control, neural network, pinning control, reaction-diffusion.

1. Introduction

During recent decades, coupled neural networks and other kinds of complex networks have stirred much research interest and can be applied to signal processing, pattern recognition, associative memories, automatic control, combinatorial optimization, etc. (See [1–4].) In particular, the synchronization problem of coupled neural networks is paying appreciable heed from researchers because it has many vital applications in engineering and human cooperation, such as information processing [5], secure communication [6] and biological systems [7].

Various types of control methods have been used to achieve synchronization of coupled neural networks, such as adaptive control [8], pinning control [9], intermittent control [10], sampled-data control [11], and impulsive control [12]. Since a coupled neural network consists of a large number of high-dimensional neural networks, and most of these control methods would be computing-expensive and infeasible to control all the nodes. Therefore, the pinning control method, adding controllers to a small portion of nodes to tame the network dynamics to approach a desired stability or synchronization performance, would be an effective method to reduce the control cost. By the practical consideration, a wealth of interesting pinning impulsive control strategies have been reported for the stability or synchronization of dynamical networks (see [13–18]), and these control strategies have been proved to be advantageous to further reduce the control cost in practical applications [14, 19, 20].

Recently, since the inevitability of time delays in the sampling and transmission of impulsive information, lots of researchers have paid attention to the delayed impulsive control and its potential applications in various kinds of control problems. For example, stability of complex-valued neural networks with delayed impulses [21, 22], master-slave synchronization of time-delay systems using delayed impulses [23], Input-to-state stability of nonlinear systems with distributed-delayed impulses [24]. In [25], the stabilization problem of time-delay neural networks was investigated via pinning delayed impulses. A new pinning impulsive controller with discrete delays was designed to achieve stabilization, and the theoretical analysis about how the delay of impulses to affect the control process was

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also presented. The synchronization problem of complex dynamical networks on time scales was studied in [18]. Based on the theory of time scales and the direct Lyapunov method, the pinning delayed impulsive control scheme was introduced to achieve the synchronization of complex dynamical networks on general time scales. In [26], two pinning delayed impulsive control protocols were proposed for mean square exponential consensus of stochastic multi-agent systems, and sufficient conditions were constructed under the presented strategies. These results focus on the pinning impulsive controller with discrete delays. To the best of our knowledge, very few works about pinning impulsive control with distributed delay have been reported in the literature. The idea of distributed-delayed impulsive control is as follows: instead of relying on the system states at the impulsive instants, or the states at history moments, the decision of the impulsive controller depends on the accumulation (or average) of the system states over a time period. In many practical systems, distributed delays occur more often due to the evolution property of time delays [27]. Hence, it is highly desirable to investigate the pinning impulsive control for dynamical systems with distributed-delayed impulses.

On the other hand, the controlled network models above are described by ordinary differential equations with time-varying state variables, which limit their applications. In practice, the reaction-diffusion effects cannot be avoided in some applications of neural networks due to, for instance, the uneven electromagnetic field in which electrons are moving and the diffusion effects in biological systems [12, 26]. Therefore, it is necessary to consider the state activation that varies in space as well as in time. More recently, the synchronization problem of the reaction-diffusion neural networks with Neumann boundary or Dirichlet boundary conditions have been considered in [12, 28–33]. In [29], two coupled reaction-diffusion neural networks with different dimensions of input and output were addressed, the dissipativity and passivity criteria for the considered systems were established. A general array model of coupled reaction-diffusion neural networks with hybrid coupling was proposed in [31], in which the spatial diffusion coupling and state coupling were considered. Sufficient conditions were presented to guarantee synchronization and H_∞ of the networks. Hence, due to the advantages of the pinning impulsive control and the existence of time-delay, it would be interesting and challenging to study pinning impulsive synchronization of reaction-diffusion neural networks with distributed-delayed impulses and discrete-delayed impulses, which motivates the research of this paper.

This paper investigates the synchronization problem of coupled reaction-diffusion neural networks by delayed pinning impulsive control. A novel pinning impulsive control scheme is proposed, where distributed delays and discrete delays are taken into account, respectively. By employing the distributed-delayed pinning impulsive controller, new criteria on synchronization of the coupled reaction-diffusion neural networks with time-varying delays are given. Under the discrete-delayed impulsive control, sufficient conditions on the system parameters, impulsive gains, pinned node numbers, impulsive instants and time delays are derived, and three corollaries are presented for the special cases.

The main contributions of this paper are given in the following three aspects: (1) A more general pinning impulsive algorithm is designed to synchronize the coupled reaction-diffusion neural networks, in which the amount of pinned nodes and the impulsive gains are impulsive-instant dependent, i.e., at distinct impulsive instants, the amount of pinned nodes and the impulsive gains are different. (2) We introduce a type of Lyapunov-Krasovskii functional candidates, which consist of a function part and a functional part. Based on the Lyapunov-Krasovskii functionals, pinning impulsive synchronization of coupled reaction-diffusion neural network with distributed-delayed impulses and discrete-delayed impulses are achieved, respectively. (3) Appropriate inequalities are applied to deal with the diffusion terms and the integral terms, which provide a tighter estimate on the integrals of these terms and relax the constraints on the sufficient conditions. (4) The relations among impulsive gains, pinned node numbers, impulsive intervals and time delays are derived and further discussed with three corollaries.

The remainder of this paper is as follows. Section 2 gives the synchronization problem of the coupled reaction-diffusion neural networks, and proposes the delayed pinning impulsive scheme. Exponential synchronization criteria are established in Section 3 with further discussions. Numerical examples are given in Section 4 to illustrate the effectiveness of the obtained results. Finally, some conclusions and possible future research topics are drawn in Section 5.

2. Problem Formulation

2.1. Notations

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm. $\mathbb{R}^{n\times n}$ denotes the $n\times n$ real matrices. $P \in \mathbb{R}^{n\times n} \geq$

$0\ (P \in \mathbb{R}^{n\times n} \leq 0)$ means that matrix P is symmetric and semi-positive (semi-negative) definite. I_n denotes the $n \times n$ real identity matrix. The notation T denotes the transpose of a matrix or a vector. $\otimes$ denotes the Kronecker product of two matrices. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix, respectively. $C^m(W)$ represents the set of continuous m-time differentiable real-valued functions on the domain W. $\sharp\mathcal{G}$ denotes the cardinality of set $\mathcal{G}$ (i.e., the number of elements of set $\mathcal{G}$ if $\mathcal{G}$ is finite).

2.2. Preliminaries

Lemma 1. *Let Ω be the cube satisfying $|x_k| < h_k$ $(k = 1, 2, \cdots, q)$ and let $u(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanishes on the boundary $\partial\Omega$ of Ω, i.e., $u(x)|_{\partial\Omega} = 0$. Then*

$$\int_\Omega u^2(x)dx \leq h_k^2 \int_\Omega \left(\frac{\partial u}{\partial x_k}\right)^2 dx, \text{ where } x = \left(x_1, x_2, \cdots, x_q\right)^T.$$

Lemma 2 (*[34]*). *For matrix $M > 0$, any matrix G with an appropriate dimension, $\beta \leq s \leq \alpha$ and $\beta \leq \gamma \leq \alpha$, the following inequalities hold:*

$$F = \begin{bmatrix} M & G \\ * & M \end{bmatrix} \geq 0,$$
$$-(\alpha - \beta) \int_\beta^\alpha x^T(s)Mx(s)ds \leq -\eta^T F\eta,$$

where $\eta = \left[\int_\gamma^\alpha x^T(s)ds,\ \int_\beta^\gamma x^T(s)ds\right]^T$.

Lemma 3 (*Jensen's inequalities [35]*). *For a given matrix $M > 0$, the following inequality holds for all continuously differentiable function x in $[a, b] \to \mathbb{R}^n$:*

$$(b-a)\int_a^b x^T(s)Mx(s)ds \geq \left(\int_a^b x(s)ds\right)^T M\left(\int_a^b x(s)ds\right) + 3\xi^T M\xi,$$
$$\frac{(b-a)^2}{2}\int_a^b\int_k^b x^T(s)Mx(s)dsdk \geq \left(\int_a^b\int_k^b x(s)ds\right)^T M\left(\int_a^b\int_k^b x(s)ds\right),$$

where $\xi = \int_a^b x(s)ds - \frac{2}{b-a}\int_a^b\int_a^s x(u)duds$.

Lemma 4. *For any vectors y, $\tilde{y} \in \mathbb{R}^n$, and matrix $P \in \mathbb{R}^{n\times n}$, the following inequality holds for positive constant $\varepsilon > 0$,*

$$2y^T P\tilde{y} \leq \varepsilon y^T PP^T y + \varepsilon^{-1}\tilde{y}^T\tilde{y}.$$

Remark 1. In this paper, we construct the Lyapunov functional candidates as (12), in order to estimate the derivatives of double integral functional terms, the reciprocally convex combination inequality in Lemma 2 would be used. Under this integral inequality, more relationships among different integral terms would be established and more free matrices are involved, and it is proved to be beneficial to yield less conservative stability criteria for time delayed systems [10, 36].

2.3. Network Model

In this paper, a single reaction-diffusion neural network with time-varying delays is described by

$$\frac{\partial w_m(t,x)}{\partial t} = \sum_{l=1}^q \frac{\partial}{\partial x_l}\left(d_{ml}\frac{\partial w_m(t,x)}{\partial x_l}\right) - a_m w_m(t,x) + \sum_{j=1}^n c_{mj} g_j\left(w_j\left(t-\tau(t),x\right)\right) + J_m, \tag{1}$$

where $m = 1, 2, \cdots, n$, $w_m(t, x) \in \mathbb{R}$ is the state of m-th neuron at time t and in space x. $x = \left(x_1, x_2, \cdots, x_q\right)^T \in \Omega \subset \mathbb{R}^q$ is the space variable with $|x_k| \leq h_k$, $k = 1, 2, \cdots, q$. $g_j(\cdot)$ stands for the activation function of j-th neuron. a_m, c_{mj}

are constants: $a_m > 0$ represents the rate with which the m-th neuron will reset its potential to the resting state when disconnected from the networks and external input J_m; c_{mj} is the connection weight between neurons. J_m is the external bias or input to the m-th neuron. $d_{ml} > 0$ is the transmission diffusion coefficient along the m-th neuron.

Throughout this paper, the following two assumptions on time-varying delays and activation functions hold.

(A1) There exist positive constants τ_1 and τ_2 and η such that

$$0 < \tau_1 \le \tau(t) \le \tau_2, \quad \dot{\tau}(t) \le \eta \le 1,$$

(A2) The function $g_j(\cdot)$, $j = 1, 2, \cdots, n$ satisfies the Lipschitz condition, that is, there exists constant ϑ_j such that

$$|g_j(\xi_1) - g_j(\xi_2)| \le \vartheta_j |\xi_1 - \xi_2|,$$

for all $\xi_1, \xi_2 \in \mathbb{R}$, and $|\cdot|$ is the Euclidean norm.

The initial value and Dirichlet boundary conditions associated with system (1) are given by

$$w_m(t_0 + s, x) = \phi_m(s, x), \text{ for } (s, x) \in [-\tau_2, 0) \times \Omega, \quad (2)$$
$$w_m(t, x) = 0, \text{ for } (t, x) \in [t_0 - \tau_2, +\infty) \times \partial\Omega, \quad (3)$$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C([-\tau_2, 0] \times \Omega, \mathbb{R}^n)$, τ_2 is the upper bound of $\tau(t)$.

Then, the single reaction-diffusion neural networks (1) can be rewritten in the following compact form:

$$\frac{\partial w(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial w(t,x)}{\partial x_l}\right) - Aw(t,x) + Cg(w(t - \tau(t), x)) + J, \quad (4)$$

where $w(t,x) = (w_1(t,x), w_2(t,x), \cdots, w_n(t,x))^T$, $g(w(t,x)) = (g_1(w_1(t,x)), g_2(w_2(t,x)), \cdots, g_n(w_n(t,x)))^T$, $D_l = \mathrm{diag}(d_{1l}, d_{2l}, \cdots, d_{nl})$, $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$, $C = (c_{mj})_{n\times n}$, $J = (J_1, J_2, \cdots, J_n)^T$.

Next, the dynamical network to be considered in this paper, which is composed of N mutually coupled reaction-diffusion neural networks (4), can be described by

$$\frac{\partial z_i(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial z_i(t,x)}{\partial x_l}\right) - Az_i(t,x) + Cg(z_i(t - \tau(t), x)) + J + \alpha \sum_{j=1}^{N} G_{ij} \Gamma z_j(t,x), \quad (5)$$

where $i = 1, 2, \cdots, N$, and N is the number of nodes in the networks. $z_i(t,x) = (z_{i1}(t,x), z_{i2}(t,x), \cdots, z_{in}(t,x))^T \in \mathbb{R}^n$ is the state vector of node i at time t and in space x. $\alpha > 0$ represents the overall coupling strength, $\Gamma \in \mathbb{R}^{n\times n} > 0$ is an inner coupling matrix, $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the coupling strength and topological structure of the network, where G_{ij} is defined as follows: if there exists a connection from neural network i to neural network j, then $G_{ij} > 0$; otherwise, $G_{ij} = 0$ $(i \neq j)$; and the diagonal elements of matrix G are defined by $G_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}$, $i = 1, 2, \cdots, N$. The initial value and boundary value conditions of (5) are given in the following form:

$$z_i(t_0 + s, x) = \varphi_i(s, x), \text{ for } (s, x) \in [-\tau_2, 0) \times \Omega, \ \varphi_i \in \mathbb{R}^n, \quad (6)$$
$$z_i(t, x) = \mathbf{0}, \text{ for } (t, x) \in [t_0 - \tau_2, +\infty) \times \partial\Omega, \ \mathbf{0} \in \mathbb{R}^n, \quad (7)$$

where φ_i is bounded and continuous on Ω. For $z_i(t,x) = (z_{i1}(t,x), z_{i2}(t,x), \cdots, z_{in}(t,x))^T \in C([t_0 - \tau_2, +\infty) \times \Omega, \mathbb{R}^n)$ and a given $t \ge 0$, we define the norm: $\|z_i(t,\cdot)\|_2^2 = \int_\Omega z_i^T(t,x) z_i(t,x) dx$.

2.4. Pinning Impulsive Control Scheme

The objective of this paper is to exponentially synchronize the network (5) with (4) by designing an appropriate delayed pinning impulsive controller $U_i(t,x)$, under which the trajectories of all nodes can be synchronized. Define $e_i(t,x) = z_i(t,x) - w(t,x)$, the impulsive controller is designed as follows:

$$U_i(t,x) = \begin{cases} \sum_{k=1}^{\infty} -q_k \int_{t-d}^{t} e_i(s,x) ds \delta(t - t_k), & i \in \mathcal{D}_k \text{ and } \sharp\mathcal{D}_k = l_k, \\ 0, & i \notin \mathcal{D}_k, \end{cases} \quad (8)$$

where $i = 1, 2, \cdots, N$, q_k are impulsive control gains, d is the distributed delay in the controller. The impulsive instant sequence $\{t_k\}$ satisfies $\{t_k\} \subset \mathbb{R}$, $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$. $\delta(\cdot)$ is the Dirac delta function. l_k denotes the number of nodes to be pinned at each impulsive instant. The index set $\mathcal{D}_k$ is defined as follows: at the impulsive instant t_k, we order the scalar states $e_1(t_k, x), e_2(t_k, x), \cdots, e_N(t_k, x)$ such that $\|e_{p1}(t_k, \cdot)\|_2 \geq \|e_{p2}(t_k, \cdot)\|_2 \geq \cdots \geq \|e_{pl_k}(t_k, \cdot)\|_2 \geq \cdots \geq \|e_{pN}(t_k, \cdot)\|_2$, then we define $\mathcal{D}_k = \{p1, p2, \cdots, pl_k\}$ and $\sharp\mathcal{D}_k = l_k$, where $0 < l_k \leq N$. The pinning impulsive control mechanism is as follows: at each impulsive instant t_k, we only control l_k networks which have larger deviations with trivial state than the rest $N - l_k$ networks.

Furthermore, under the properties of Dirac delta function, the controlled neural networks (5) with the distributed delayed pinning impulsive controller (8) can be rewritten in the following form:

$$\begin{cases} \frac{\partial z_i(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial z_i(t,x)}{\partial x_l}\right) - Az_i(t,x) + Cg\left(z_i(t-\tau(t), x)\right) + J + \alpha \sum_{j=1}^{N} G_{ij}\Gamma z_j(t,x),\ t \neq t_k, \\ \Delta z_i(t_k, x) = -q_k \int_{t_k-d}^{t_k^-} e_i(s,x)ds,\ i \in \mathcal{D}_k,\ \sharp\mathcal{D}_k = l_k,\ k \in \mathbb{N}, \\ z_i(t_0+s, x) = \varphi_i(s,x),\ (s,x) \in [-\tau_2, 0] \times \Omega, \\ z_i(t,x) = \mathbf{0},\ (t,x) \in [t_0-\tau_2, +\infty) \times \partial\Omega,\ \mathbf{0} = [0,0,\cdots,0]^T \in \mathbb{R}^n. \end{cases} \tag{9}$$

Remark 2. Under the assumption (A2), it is shown in Theorem 1 in [37] that system (4) admits a unique mild solution with the Dirichlet boundary condition (3) and the initial condition (2). Meanwhile, since the considered delays in the system are bounded from assumption (A1), the existence of solution to system (5) can be guaranteed by the results of reaction-diffusion equations in [38], and the method of steps. In addition, the case for system (9) with impulses is essentially the same, by an argument using the method of steps over all the impulsive intervals.

By introducing the error vector $e_i(t, x)$, the error system is described as follows:

$$\begin{cases} \frac{\partial e_i(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial e_i(t,x)}{\partial x_l}\right) - Ae_i(t,x) + Cf\left(e_i(t-\tau(t), x)\right) + \alpha \sum_{j=1}^{N} G_{ij}\Gamma e_j(t,x),\ t \neq t_k, \\ \Delta e_i(t_k, x) = -q_k \int_{t_k-d}^{t_k^-} e_i(s,x)ds,\ i \in \mathcal{D}_k,\ \sharp\mathcal{D}_k = l_k,\ k \in \mathbb{N}, \\ e_i(t_0+s, x) = \varphi_i(s,x) - \phi(s,x),\ (s,x) \in [-\gamma, 0] \times \Omega, \\ e_i(t,x) = \mathbf{0},\ (t,x) \in [t_0-\gamma, +\infty) \times \partial\Omega,\ \mathbf{0} = [0,0,\cdots,0]^T \in \mathbb{R}^n, \end{cases} \tag{10}$$

where $\gamma = \max(\tau_2, d)$. We assume that $e_i(t, x)$ is right-continuous at each t_k, $k \in \mathbb{N}$, i.e., $\lim_{t\to t_k^+} e_i(t, x) = e_i(t_k, x)$. $f(e_i(t, x)) = g(z_i(t, x)) - g(w(t, x))$ for $i = 1, 2, \cdots, N$.

Definition 1. *The controlled neural network (5) is said to be globally exponentially synchronized onto the trajectory of (4) if there exist $\mu > 0$ and $M \geq 1$ such that for $i = 1, 2, \cdots, N$, it holds that*

$$\|e_i(t, \cdot)\|_2 \leq Me^{-\mu(t-t_0)} \sup_{s\in[-\hat{\gamma},0]} \|\varphi_i(s, \cdot) - \phi(s, \cdot)\|_2.$$

Remark 3. The key point of applying pinning control approach is the selection of suitable nodes to control. Comparing with the pinning impulsive scheme for synchronization of delayed networks in [13, 39–41], the pinning algorithm in this paper is more general and flexible. In this paper, the pinning strategy (8) is impulsive-instant dependent, i.e., at distinct impulsive instants t_k, the number of pinned nodes l_k and the impulsive gain q_k are maybe different, while the two parameters are assumed to be fixed for all impulsive instants in [13, 39–41]. The pinning impulsive control scheme (8) is inspired by the idea in [30]. The distributed delay is considered in the impulsive controller. Qualitative analysis on how the time delay contributes the dynamics of systems would be presented in Section 3. Furthermore, sufficient conditions on suitable relation among the impulsive gain q_k, the number of pinned nodes l_k, and the length of the impulsive interval would be established.

3. Main Results

In this section, the exponential synchronization criteria for reaction-diffusion neural networks will be derived. By using the Lyapunov-Krasovskii method and the reciprocally convex combination inequality, Subsection 3.1 aims to investigate the pinning impulsive synchronization of reaction-diffusion neural network (10) with distributed-delayed impulses. By estimating the relation between the system states at the impulsive instants and the distributed-delayed system states, two necessary propositions are proposed first, and sufficient conditions are given in Theorem 1 to ensure exponential synchronization of system (10). Moreover, in Subsection 3.2, we will further discuss the pinning impulsive synchronization of reaction-diffusion neural network with discrete-delayed impulses. Exponential synchronization criterion is presented in Theorem 2 for the considered system with discrete-delayed impulses. We also give three corollaries and some detailed discussions regarding the construct of Lyapunov-Krosovskii functional candidates, the affects of time-delay in the controller, and the relationship of the time-delay, pinned node numbers, impulsive gains and the length of impulsive interval.

3.1. Pinning Impulsive Synchronization of Reaction-Diffusion Neural Network with Distributed-Delayed Impulses

By employing the Kronecker product, network (10) can be rewritten in the following compact form:

$$\begin{cases} \frac{\partial e(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left[(I_N\otimes D_l)\frac{\partial e(t,x)}{\partial x_l}\right] - (I_N\otimes A)\,e(t,x) + (I_N\otimes C)\,f\left(e\left(t-\tau(t),x\right)\right) + \alpha\left(G\otimes\Gamma\right)e(t,x),\ t\neq t_k, \\ \Delta e_i\left(t_k,x\right) = -q_k\int_{t_k-d}^{t_k^-} e_i(s,x)ds,\ i\in\mathcal{D}_k,\ \sharp\mathcal{D}_k = l_k,\ k\in\mathbb{N}, \\ e(t_0+s,x) = \varphi(s,x)-\bar{\phi}(s,x),\ (s,x)\in[-\gamma,0]\times\Omega, \\ e(t,x) = \mathbf{1}^N\otimes\mathbf{0},\ (t,x)\in[t_0-\gamma,+\infty)\times\partial\Omega, \mathbf{0} = [0,0,\cdots,0]^T\in\mathbb{R}^n, \end{cases} \tag{11}$$

where $\varphi(s,x) = \left(\varphi_1^T(s,x),\varphi_2^T(s,x),\cdots,\varphi_N^T(s,x)\right)^T$, and $\bar{\phi}(s,x) = \mathbf{1}^N\otimes\phi(s,x)$ with $\mathbf{1}^N = (1,1,\cdots,1)^T\in\mathbb{R}^N$. For convenience, we define the following notations:

$$\rho = \mathrm{diag}\left(\vartheta_1,\vartheta_2,\cdots,\vartheta_n\right),\ \tilde{D} = \sum_{l=1}^{q}\frac{D_l}{h_l^2},\ \lambda_1 = \sqrt{\frac{\lambda_{\max}\left(A^TP_0A\right)}{\lambda_{\min}\left(P_0\right)}} + \sqrt{\frac{\lambda_{\max}\left(\rho C^TP_0C\rho\right)}{\lambda_{\min}\left(P_0\right)}} + \alpha g_1\sqrt{\frac{Nl_k\lambda_{\max}\left(\Gamma^TP_0\Gamma\right)}{\lambda_{\min}\left(P_0\right)}},$$

$$e_i^T = \Big[\underbrace{0_{Nn},\cdots,0_{Nn}}_{i-1},I_{Nn},\underbrace{0_{Nn},\cdots,0_{Nn}}_{9-i}\Big]\in\mathbb{R}^{Nn\times 9Nn},\ \xi(t,x) = -\int_{t-\tau_1}^{t}e(s,x)ds + \frac{2}{\tau_1}\int_{t-\tau_1}^{t}\int_{k}^{t}e(s,x)dsdk,$$

$$\zeta(t,x) = \left[\int_{t-\tau(t)}^{t-\tau_1}e^T(s,x)ds,\ \int_{t-\tau_2}^{t-\tau(t)}e^T(s,x)ds\right]^T,\ \Theta(t,x) = -Ae_i(t,x) + Cf\left(e_i\left(t-\tau(t),x\right)\right) + \alpha\sum_{j=1}^{N}G_{ij}\Gamma e_j(t,x),$$

$$\eta(t,x) = \Big[e^T(t,x),\ e^T\left(t-\tau(t),x\right),\ e^T\left(t-\tau_1,x\right),\ e^T\left(t-\tau_2,x\right),\ \int_{t-\tau_1}^{t}e^T(s,x)ds,\ \int_{t-\tau(t)}^{t-\tau_1}e^T(s,x)ds,$$
$$\int_{t-\tau_2}^{t-\tau(t)}e^T(s,x)ds,\ \int_{t-\tau_1}^{t}\int_{k}^{t}e^T(s,x)dsdk,\ \int_{t-\tau_2}^{t-\tau_1}\int_{k}^{t}e^T(s,x)dsdk\Big]^T,\ g_1 = |\max\left(G_{ij}\right)|.$$

To facilitate the analysis of (11), we construct the Lyapunov-Krasovskii functional candidates along the trajectory of (11) as

$$\begin{aligned} V(t) &= V_1(t)+V_2(t)+V_3(t),\ \text{and}\ \Psi(t) = e^{-ct}V(t), \\ V_1(t) &= \frac{1}{2}\int_{\Omega}e^T(t,x)\left(I_N\otimes P_0\right)e(t,x)dx, \\ V_2(t) &= \varpi\int_{\Omega}\int_{t-\tau_1}^{t}e^T(s,x)\left(I_N\otimes P_1\right)e(s,x)dsdx + \varpi\int_{\Omega}\int_{t-\tau_2}^{t}e^T(s,x)\left(I_N\otimes P_2\right)e(s,x)dsdx \\ &\quad +\varpi\int_{\Omega}\int_{t-\tau(t)}^{t}e^T(s,x)\left(I_N\otimes P_3\right)e(s,x)dsdx, \\ V_3(t) &= \varpi\tau_1\int_{\Omega}\int_{t-\tau_1}^{t}\int_{k}^{t}e^T(s,x)\left(I_N\otimes Q_1\right)e(s,x)dsdkdx \\ &\quad +\varpi\left(\tau_2-\tau_1\right)\int_{\Omega}\int_{t-\tau_2}^{t-\tau_1}\int_{k}^{t}e^T(s,x)\left(I_N\otimes Q_2\right)e(s,x)dsdkdx, \end{aligned} \tag{12}$$

in which $P_0,\ P_1,\ P_2,\ P_3,\ Q_1,\ Q_2 \in \mathbb{R}^{n\times n} > 0$ are positive matrices, ϖ is a positive constant. Denote $w_1 = \frac{1}{2}\lambda_{\min}(P_0)$, $w_2 = \frac{1}{2}\lambda_{\max}(P_0)$, $w_3 = \varpi\left(\tau_1\lambda_{\max}(P_1) + \tau_2\lambda_{\max}(P_2) + \tau_2\lambda_{\max}(P_3) + \frac{1}{2}\tau_1^3\lambda_{\max}(Q_1) + \frac{1}{2}(\tau_2+\tau_1)(\tau_2-\tau_1)^2\lambda_{\max}(Q_2)\right)$. It is obtained that

$$w_1\|e(t,x)\|_2^2 \le V_1(t) \le w_2\|e(t,x)\|_2^2,\ 0 \le V_2(t) + V_3(t) \le w_3 \sup_{-\tau_2\le s\le 0}\|e(t+s,x)\|_2^2. \tag{13}$$

Proposition 1. *Suppose that assumptions (A1) and (A2) hold. If there exist a matrix $Q_3 \in \mathbb{R}^{Nn\times Nn}$, positive constants $\varpi,\ \varepsilon_1,\ c > 0$ such that for all $k \in \mathbb{N}$,*

$$\Upsilon \le 0, \tag{14}$$

$$\mathcal{Q} = \begin{pmatrix} I_N\otimes\varpi Q_2 & Q_3 \\ * & I_N\otimes\varpi Q_2 \end{pmatrix} \ge 0, \tag{15}$$

then along the trajectory of system (11), it is satisfied that $\dot{V}(t) \le cV(t)$ for $t \in [t_k, t_{k+1})$, in which

$$\begin{aligned}\Upsilon = {} & e_1\left[I_N\otimes\left(-P_0\tilde{D} - P_0A + \frac{1}{2}\varepsilon_1^{-1}P_0CC^TP_0^T + \varpi(P_1+P_2+P_3) + \varpi\tau_1^2Q_1 - \frac{c}{2}P_0 + \varpi(\tau_2-\tau_1)^2Q_2\right)\right. \\ & \left.+\alpha(G\otimes P_0\Gamma)\right]e_1^T + e_2\left[I_N\otimes\left(\frac{1}{2}\varepsilon_1\rho^2 - \varpi(1-\eta)P_3\right)\right]e_2^T - e_3(I_N\otimes\varpi P_1)e_3^T - e_4(I_N\otimes\varpi P_2)e_4^T \\ & -e_5\left[I_N\otimes\varpi\left(Q_1 + \frac{c}{\tau_1}P_1\right)\right]e_5^T - \left(-e_5 + \frac{2}{\tau_1}e_8\right)(I_N\otimes 3\varpi Q_1)\left(-e_5^T + \frac{2}{\tau_1}e_8^T\right) - (e_5+e_6)\left(I_N\otimes\frac{\varpi cP_3}{\tau_2}\right)\left(e_5^T+e_6^T\right) \\ & -(e_5+e_6+e_7)\left(I_N\otimes\frac{\varpi cP_2}{\tau_2}\right)\left(e_5^T+e_6^T+e_7^T\right) - (e_6,\ e_7)\begin{pmatrix} I_N\otimes\varpi Q_2 & Q_3 \\ * & I_N\otimes\varpi Q_2\end{pmatrix}\begin{pmatrix} e_6^T \\ e_7^T\end{pmatrix} \\ & -e_8\left(I_N\otimes\frac{2\varpi cQ_1}{\tau_1}\right)e_8^T - e_9\left(I_N\otimes\frac{2\varpi cQ_2}{\tau_2-\tau_1}\right)e_9^T.\end{aligned}$$

Proof. By Green's formula, the Dirichlet boundary condition, Lemma 1, and the definition of $\tilde{D}$, we have

$$\begin{aligned}\int_\Omega\sum_{l=1}^q e^T(t,x)(I_N\otimes D_l)\frac{\partial^2}{\partial x_l^2}(e(t,x))\,dx &= -\int_\Omega\sum_{l=1}^q\left(\frac{\partial e(t,x)}{\partial x_l}\right)^T(I_N\otimes D_l)\frac{\partial e(t,x)}{\partial x_l}dx \\ &\le -\int_\Omega e^T(t,x)\left(I_N\otimes\tilde{D}\right)e(t,x)dx.\end{aligned} \tag{16}$$

For any impulsive interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, differentiate $V(t)$ along the trajectory of system (11). By using Lemma 4 and (16), it yields that

$$\begin{aligned}\dot{V}_1(t) \le {} & \int_\Omega e^T(t,x)\left(-\left(I_N\otimes P_0\tilde{D}\right) - (I_N\otimes A) + \frac{1}{2}\left(I_N\otimes\varepsilon_1^{-1}P_0CC^TP_0^T\right) + \alpha(G\otimes P_0\Gamma)\right)e(t,x)\,dx \\ & +\frac{1}{2}\int_\Omega e^T(t-\tau(t),x)\left(I_N\otimes\varepsilon_1\rho^2\right)e(t-\tau(t),x)\,dx,\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{V}_2(t) = {} & \int_\Omega\left[e^T(t,x)(I_N\otimes\varpi(P_1+P_2+P_3))e(t,x)\,dx - e^T(t-\tau_1,x)(I_N\otimes\varpi P_1)e(t-\tau_1,x)\right. \\ & \left.-e^T(t-\tau_2,x)(I_N\otimes\varpi P_2)e(t-\tau_2,x) - (1-\eta)e^T(t-\tau(t),x)(I_N\otimes\varpi P_3)e(t-\tau(t),x)\right]dx.\end{aligned} \tag{18}$$

By employing the inequalities in Lemma 2 and Lemma 3, and differentiating $V_3(t)$ along the trajectory of system (11), it gives

$$\begin{aligned}\dot{V}_3(t) = {} & \int_\Omega e^T(t,x)\left[I_N\otimes\varpi\left(\tau_1^2Q_1 + (\tau_2-\tau_1)^2Q_2\right)\right]e(t,x)\,dx - \int_\Omega\zeta^T(t,x)\mathcal{Q}\zeta(t,x) \\ & -\int_\Omega\left(\int_{t-\tau_1}^t e(s,x)ds\right)^T(I_N\otimes\varpi Q_1)\left(\int_{t-\tau_1}^t e(s,x)ds\right)dx - 3\int_\Omega\xi^T(t,x)(I_N\otimes\varpi Q_1)\xi(t,x)dx,\end{aligned} \tag{19}$$

where $Q = \begin{pmatrix} I_N \otimes \varpi Q_2 & Q_3 \\ * & I_N \otimes \varpi Q_2 \end{pmatrix} \geq 0$ in (15).

Then, by Lemma 3 and the definition of $V(t)$, it is obtained that

$$-cV_1(t) = -\frac{c}{2}\int_{\Omega} e^T(t,x)(I_N \otimes P_0)e(t,x)dx, \tag{20}$$

$$\begin{aligned} -cV_2(t) \leq & -c\int_{\Omega}\Bigg[\left(\int_{t-\tau_1}^{t} e(s,x)ds\right)^T\left(I_N \otimes \frac{\varpi}{\tau_1}P_1\right)\left(\int_{t-\tau_1}^{t} e(s,x)ds\right) + \left(\int_{t-\tau_2}^{t} e(s,x)ds\right)^T\left(I_N \otimes \frac{\varpi}{\tau_2}P_2\right) \\ & \left(\int_{t-\tau_2}^{t} e(s,x)ds\right) + \left(\int_{t-\tau(t)}^{t} e(s,x)ds\right)^T\left(I_N \otimes \frac{\varpi}{\tau(t)}P_3\right)\left(\int_{t-\tau(t)}^{t} e(s,x)ds\right)\Bigg]dx, \end{aligned} \tag{21}$$

$$\begin{aligned} -cV_3(t) \leq & -c\int_{\Omega}\Bigg[\left(\int_{t-\tau_1}^{t}\int_{k}^{t} e(s,x)dsdk\right)^T\left(I_N \otimes \frac{2\varpi}{\tau_1}Q_1\right)\left(\int_{t-\tau_1}^{t}\int_{k}^{t} e(s,x)dsdk\right) \\ & + \left(\int_{t-\tau_2}^{t-\tau_1}\int_{k}^{t} e(s,x)dsdk\right)^T\left(I_N \otimes \frac{2\varpi}{\tau_2-\tau_1}Q_2\right)\left(\int_{t-\tau_2}^{t-\tau_1}\int_{k}^{t} e(s,x)dsdk\right)\Bigg]dx. \end{aligned} \tag{22}$$

Now calculate the time derivative of $\Psi(t)$ along the trajectory of system (11), combine (17)-(22) with (12), then we have

$$\frac{d}{dt}(\Psi(t)) = e^{-ct}\left(\dot{V}(t) - cV(t)\right) \leq e^{-ct}\int_{\Omega}\eta^T(t,x)\Upsilon\eta(t,x)dx, \tag{23}$$

where $\Upsilon \leq 0$ in (14), and $\eta(t,x)$ is defined before. Consequently, $\dot{V}(t) \leq cV(t)$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$.

Proposition 2. *For the Lyapunov-Krasovskii candidates (12), when $t = t_k$, $k \in \mathbb{N}$, there exist positive constants ϵ_0, ϵ_1, ϵ_2, ϵ_5, ϵ_6, $\epsilon_7 > 0$ and $0 < \rho_{1k} < 1$, $\rho_{2k} \geq 0$ such that along the solution of system (11),*

$$V_1\left(t_k^+\right) \leq \rho_{1k}V_1\left(t_k^-\right) + \rho_{2k}\sup_{s\in[-\gamma-d,0]}\left\{V_1\left(t_k^- + s\right)\right\}, \text{ where for } k \in \mathbb{N}, \tag{24}$$

$\rho_{0k} = (1+\epsilon_0)(1-dq_k)^2 + \Phi_4(1+\epsilon_6)d^2q_k^2/\left(1+\epsilon_6^{-1}\right)$, $\rho_{2k} = \Phi_1 q_k^4 d^4 \varsigma^2 + \frac{1}{3}\Phi_2 q_k^2 d^4 \lambda_1^2 + \Phi_3 q_k^2 d^2$, $\rho_{1k} = 1 - \frac{l_k}{N}(1-\rho_{0k})$,
$\Phi_1 = \left(1+\epsilon_0^{-1}\right)(1+\epsilon_1) + \Phi_4(1+\epsilon_7)$, $\Phi_2 = \Phi_5(1+\epsilon_2) + \Phi_4\left(1+\epsilon_7^{-1}\right)$, $\Phi_3 = \Phi_5\left(1+\epsilon_2^{-1}\right)(1+\epsilon_5)$,
$\Phi_4 = \Phi_5\left(1+\epsilon_2^{-1}\right)\left(1+\epsilon_5^{-1}\right)\left(1+\epsilon_6^{-1}\right)$, $\Phi_5 = \left(1+\epsilon_0^{-1}\right)\left(1+\epsilon_1^{-1}\right)$.
$\varsigma = \left\lfloor \frac{d}{\delta} \right\rfloor$ *denotes the number of impulses that the system (11) subject to on each impulsive interval* $[t_k - d, t_k)$, $\delta = t_{k+1} - t_k$.

Proof. Here for $t = t_k^+$, $k \in \mathbb{N}$, it gives

$$V_1\left(t_k^+\right) = \frac{1}{2}\sum_{i\in\mathcal{D}_k}\int_{\Omega} e_i^T\left(t_k^+, x\right)P_0 e_i(t_k^+, x)dx + \frac{1}{2}\sum_{i\notin\mathcal{D}_k}\int_{\Omega} e_i^T\left(t_k^+, x\right)P_0 e_i(t_k^+, x)dx. \tag{25}$$

For $i \in \mathcal{D}_k$, from the second equation of (11), we have $e_i\left(t_k^+, x\right) = e_i\left(t_k^-, x\right) - q_k\int_{t_k-d}^{t_k^-} e_i(s,x)ds$. Next we should estimate the term $\int_{t_k-d}^{t_k^-} e_i(s,x)ds$. For $t \in [t_k - d, t_k)$, integrating both sides of the first equation in system (10) from t to t_k^-, it yields

$$e_i(t,x) = e_i\left(t_k^-, x\right) - \int_t^{t_k^-}\sum_{l=1}^{q}\frac{\partial}{\partial x_l}\left(D_l\frac{\partial e_i(s,x)}{\partial x_l}\right)ds - \int_t^{t_k^-}\Theta(s,x)ds + \sum_{m=1}^{\varsigma_k} q_k\int_{-d}^{0} e_i(t_{k-m}+s,x)ds, \tag{26}$$

where ς_k denotes the number of impulses activated onto the ith node during the period (t, t_k) and it is satisfied that $\varsigma_k \leq \varsigma$ (Refer to Fig. 2 in [25]). Continue to integrate both sides of (26) from $t_k - d$ to t_k^-, then we have

$$\begin{aligned} \int_{t_k-d}^{t_k^-} e_i(s,x)ds = & \; de_i\left(t_k^-, x\right) - \int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\sum_{l=1}^{q}\frac{\partial}{\partial x_l}\left(D_l\frac{\partial e_i(s,x)}{\partial x_l}\right)dsdt + q_k\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0} e_i(t_{k-m}+s,x)dsdt \\ & - \int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\Theta(s,x)dsdt. \end{aligned} \tag{27}$$

So it follows that

$$e_i(t_k^+,x) = (1-dq_k)\,e_i\left(t_k^-,x\right) - q_k^2\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0} e_i\left(t_{k-m}+s,x\right)dsdt + q_k\int_{t_k-d}^{t_k^-}\int_{t}^{t_k^-}\Theta(s,x)dsdt$$
$$+q_k\int_{t_k-d}^{t_k^-}\int_{t}^{t_k^-}\sum_{l=1}^{q}\frac{\partial}{\partial x_l}\left(D_l\frac{\partial e_i(s,x)}{\partial x_l}\right)dsdt. \tag{28}$$

Let $X_{i1}=(1-dq_k)\,e_i\left(t_k^-,x\right)$, $X_{i2}=-q_k^2\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0}e_i\left(t_{k-m}+s,x\right)dsdt$, $X_{i3}=q_k\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\Theta(s,x)dsdt$, $X_{i4}=q_k\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\sum_{l=1}^{q}\frac{\partial}{\partial x_l}\left(D_l\frac{\partial e_i(s,x)}{\partial x_l}\right)dsdt$. Then, we have

$$\frac{1}{2}\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^T\left(t_k^+,x\right)P_0e_i(t_k^+,x)dx \le \frac{1}{2}\sum_{i\in\mathcal{D}_k}\int_\Omega\Big\{(1+\epsilon_0)\,X_{i1}^TP_0X_{i1}+\left(1+\epsilon_0^{-1}\right)(1+\epsilon_1)\,X_{i2}^TP_0X_{i2}+\left(1+\epsilon_0^{-1}\right)\left(1+\epsilon_1^{-1}\right)$$
$$(1+\epsilon_2)\,X_{i3}^TP_0X_{i3}+\left(1+\epsilon_0^{-1}\right)\left(1+\epsilon_1^{-1}\right)\left(1+\epsilon_2^{-1}\right)X_{i4}^TP_0X_{i4}\Big\}dx. \tag{29}$$

Applying Lemma 3 and Schwarz's inequality, we get

$$\sum_{i\in\mathcal{D}_k}\int_\Omega X_{i2}^TP_0X_{i2}dx = q_k^4\sum_{i\in\mathcal{D}_k}\int_\Omega\left(\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0}e_i\left(t_{k-m}+s,x\right)dsdt\right)^T P_0\left(\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0}e_i\left(t_{k-m}+s,x\right)dsdt\right)dx$$
$$\le q_k^4d^2\sum_{i\in\mathcal{D}_k}\int_\Omega\int_{t_k-d}^{t_k^-}\varsigma_k\sum_{m=1}^{\varsigma_k}\int_{-d}^{0}e_i^T\left(t_{k-m}+s,x\right)P_0e_i\left(t_{k-m}+s,x\right)dsdtdx$$
$$\le q_k^4d^4\varsigma^2\sum_{i\in\mathcal{D}_k}\int_\Omega\sup_{s\in[-2d,0]}\left\{e_i^T\left(t_k^-+s,x\right)P_0e_i\left(t_k^-+s,x\right)\right\}dx. \tag{30}$$

Moreover, by employing the Jensen's inequality twice, we can estimate

$$\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}e_i^T(s,x)dsdt\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}e_i(s,x)dsdt \le \frac{1}{3}d^4\sup_{s\in[-d,0]}\left\{e_i^T\left(t_k^-+s,x\right)e_i\left(t_k^-+s,x\right)\right\}. \tag{31}$$

Similarly, it is obtained that

$$\sum_{i\in\mathcal{D}_k}\int_\Omega X_{i3}^TP_0X_{i3}dx = q_k^2\sum_{i\in\mathcal{D}_k}\int_\Omega\left(\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\Theta(s,x)dsdt\right)^T P_0\left(\int_{t_k-d}^{t_k^-}\int_t^{t_k^-}\Theta(s,x)dsdt\right)dx$$
$$\le \frac{1}{3}q_k^2d^4\sum_{i=1}^{N}\int_\Omega\Bigg[(1+\epsilon_3)\frac{\lambda_{\max}\left(A^TP_0A\right)}{\lambda_{\min}\left(P_0\right)}+\left(1+\epsilon_3^{-1}\right)(1+\epsilon_4)\frac{\lambda_{\max}\left(\rho C^TP_0C\rho\right)}{\lambda_{\min}\left(P_0\right)}$$
$$+\left(1+\epsilon_3^{-1}\right)\left(1+\epsilon_4^{-1}\right)\alpha^2Nl_kg_1^2\frac{\lambda_{\max}\left(\Gamma^TP_0\Gamma\right)}{\lambda_{\min}\left(P_0\right)}\Bigg]\sup_{s\in[-d-\tau_2,0]}\left\{e_i^T\left(t_k^-+s,x\right)P_0e_i\left(t_k^-+s,x\right)\right\}$$
$$= \frac{1}{3}q_k^2d^4\lambda_1^2\sum_{i=1}^{N}\int_\Omega\sup_{s\in[-d-\tau_2,0]}\left\{e_i^T\left(t_k^-+s,x\right)P_0e_i\left(t_k^-+s,x\right)\right\}, \tag{32}$$

with $(\epsilon_3,\epsilon_4)=\left(\sqrt{\frac{\lambda_{\max}(\rho C^TP_0C\rho)}{\lambda_{\max}(A^TP_0A)}}+\alpha g_1\sqrt{\frac{Nl_k\lambda_{\max}(\Gamma^TP_0\Gamma)}{\lambda_{\max}(A^TP_0A)}},\alpha g_1\sqrt{\frac{Nl_k\lambda_{\max}(\Gamma^TP_0\Gamma)}{\lambda_{\max}(\rho C^TP_0C\rho)}}\right)$. From (27), it can be further deduced that

$$\int_{t_k-d}^{t_k^-}\int_{t}^{t_k^-}\sum_{l=1}^{q}\frac{\partial}{\partial x_l}\left(D_l\frac{\partial e_i(s,x)}{\partial x_l}\right)dsdt=-\int_{t_k-d}^{t_k^-}e_i(s,x)ds+de_i\left(t_k^-,x\right)+q_k\int_{t_k-d}^{t_k^-}\sum_{m=1}^{\varsigma_k}\int_{-d}^{0}e_i\left(t_{k-m}+s,x\right)dsdt$$
$$-\int_{t_k-d}^{t_k^-}\int_{t}^{t_k^-}\Theta(s,x)dsdt,$$

$$\sum_{i\in\mathcal{D}_k}\int_\Omega X_{i4}^TP_0X_{i4}dx\le q_k^2\sum_{i\in\mathcal{D}_k}\int_\Omega(1+\epsilon_5)\left(\int_{t_k-d}^{t_k^-}e_i(s,x)ds\right)^TP_0\left(\int_{t_k-d}^{t_k^-}e_i(s,x)ds\right)$$
$$+\left(1+\epsilon_5^{-1}\right)(1+\epsilon_6)d^2e_i^T\left(t_k^-,x\right)P_0e_i\left(t_k^-,x\right)+\left(1+\epsilon_5^{-1}\right)\left(1+\epsilon_6^{-1}\right)(1+\epsilon_7)\frac{1}{q_k^2}X_{i2}^TP_0X_{i2}$$
$$+\left(1+\epsilon_5^{-1}\right)\left(1+\epsilon_6^{-1}\right)\left(1+\epsilon_7^{-1}\right)\frac{1}{q_k^2}X_{i3}^TP_0X_{i3}. \tag{33}$$

So, from (29)-(33), it is obtained that

$$\frac{1}{2}\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^T\left(t_k^+,x\right)P_0e(t_k^+,x)dx\le\frac{1}{2}\rho_{0k}\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^T\left(t_k^-,x\right)P_0e\left(t_k^-,x\right)dx+\rho_{2k}\sup_{s\in[-\gamma,0]}\left\{V_1\left(t_k^-+s\right)\right\}, \tag{34}$$

where ρ_{0k}, ρ_{2k} are defined above. According to the selection of nodes in set $\mathcal{D}_k$, we have

$$\frac{1}{2}\sum_{i\notin\mathcal{D}_k}\int_\Omega e_i^T\left(t_k^-,x\right)P_0e_i(t_k^-,x)dx\le\frac{1}{2}(N-l_k)\frac{1}{l_k}\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^T\left(t_k^-,x\right)P_0e_i\left(t_k^-,x\right)dx. \tag{35}$$

Let $\rho_{1k}=1-\frac{l_k}{N}(1-\rho_{0k})$. Based on (34), (35) and (25), we can conclude that

$$V_1\left(t_k^+\right)\le\rho_{1k}V_1\left(t_k^-\right)+\rho_{2k}\sup_{s\in[-\gamma,0]}\left\{V_1\left(t_k^-+s\right)\right\}.$$

This completes the proof.

Theorem 1. *Suppose that assumptions (A1) and (A2) hold. If there exist positive matrices* $P_0,\ P_1,\ P_2,\ P_3,\ Q_1,\ Q_2\in\mathbb{R}^{n\times n}>0$, *a matrix* $Q_3\in\mathbb{R}^{Nn\times Nn}$, *positive constants* $\varpi,\ \varepsilon_1,\ \epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \epsilon_5,\ \epsilon_6,\ \epsilon_7,\ c,\ \mu>0,\ \rho_{1k}>0,\ \rho_{2k}\ge 0$ *such that for all* $k\in\mathbb{N}$,

$$\ln\left[\rho_{1k}+\rho_{2k}e^{\mu(\gamma+d)}+\frac{w_3}{w_1}e^{\mu\tau_2}\right]\le-(\mu+c)(t_{k+1}-t_k), \tag{36}$$
$$\Upsilon\le 0, \tag{37}$$
$$\begin{pmatrix} I_N\otimes\varpi Q_2 & Q_3\\ * & I_N\otimes\varpi Q_2\end{pmatrix}\ge 0, \tag{38}$$

then the trivial solution of system (9) is globally exponentially synchronized.

Proof. *Step 1*: By employing Proposition 1, it is obtained from (37) and (38) that along the trajectory of system (11),

$$\dot{V}(t)\le cV(t). \tag{39}$$

Step 2: Next we estimate the growth trend of $V(t)$ at impulsive instants, when $t=t_k$, under the pinning impulsive scheme (8), from Proposition 2, we have

$$V_1\left(t_k^+\right)\le\rho_{1k}V_1\left(t_k^-\right)+\rho_{2k}\sup_{s\in[-\gamma-d,0]}\left\{V_1\left(t_k^-+s\right)\right\}. \tag{40}$$

In addition, for the Lyapunov-Krasovskii functional candidates $V_2\left(t_k^+\right),\ V_3\left(t_k^+\right)$, we have

$$V_2\left(t_k^+\right) + V_3\left(t_k^+\right) = V_2\left(t_k^-\right) + V_3\left(t_k^-\right). \tag{41}$$

Step 3: We derive globally exponential synchronization criteria of system (9).

Since $\lim_{k\to\infty} t_k = \infty$, there exists an integer $i \geq 1$ such that $t_i - \gamma - d \geq t_0$, and for $t \in [t_0, t_i)$, we can obtain that

$$V(t) = V(t)e^{\mu(t-t_0)}e^{-\mu(t-t_0)} \leq Me^{-\mu(t-t_0)}, \tag{42}$$

where $M = e^{\mu(t_i-t_0)} \sup_{t\in[t_0,t_i)} \{V(t)\}$. Next, we shall claim that for $t \in [t_k, t_{k+1}),\ k \geq i$,

$$V(t) \leq Me^{-(\mu+c)(t_{k+1}-t_0)}e^{c(t-t_0)}. \tag{43}$$

From the idea in Theorem 1 in [25], here we give a sketch of the proof. First, when $k = i$, we obtain from (42) and (13) that

$$\sup_{s\in[-\gamma-d,0]} \left\{\left\|e(t_i^- + s, \cdot)\right\|_2^2\right\} \leq \sup_{s\in[-\gamma-d,0]} \left\{\frac{1}{w_1} V_1(t_i^- + s)\right\} \leq \frac{1}{w_1} Me^{\mu(\gamma+d)}e^{-(\mu+c)(t_i-t_0)}e^{c(t_i-t_0)}. \tag{44}$$

Next, it gives from (36), (40) and (41) that

$$V(t_i^+) \leq \left(\rho_{1i} + \rho_{2i}e^{\mu(\gamma+d)} + \frac{w_3}{w_1}e^{\mu\tau_2}\right) Me^{-(\mu+c)(t_i-t_0)}e^{c(t_i-t_0)} \leq Me^{-(\mu+c)(t_{i+1}-t_0)}e^{c(t_i-t_0)}. \tag{45}$$

Therefore, we obtain from (39) and (45) that for $t \in [t_i, t_{i+1})$,

$$V(t) \leq V(t_i^+)e^{c(t-t_i)} \leq Me^{-(\mu+c)(t_{i+1}-t_0)}e^{c(t-t_0)}. \tag{46}$$

So (43) holds for $k = i$. Then based on the mathematical induction, we can prove that (43) is true for all $k \geq i$. Therefore, for $t \in [t_k, t_{k+1})$, we have

$$V(t) \leq Me^{-(\mu+c)(t_{k+1}-t_0)}e^{c(t-t_0)} \leq Me^{-(\mu+c)(t-t_0)}e^{c(t-t_0)} = Me^{-\mu(t-t_0)}. \tag{47}$$

Thus, from (42) and (47), we get for $t \geq t_0$, $V(t) \leq Me^{-\mu(t-t_0)}$. It follows that

$$\|e(t,\cdot)\|_2 \leq \bar{M} \sup_{s\in[-\gamma,0]} \|\varphi(s,\cdot) - \bar{\phi}(s,\cdot)\|_2 e^{-\frac{\mu}{2}(t-t_0)}, \tag{48}$$

where $\bar{M} = \frac{\sqrt{M}}{\sqrt{w_1}\sup_{s\in[-\gamma,0]}\left\|\varphi(s,\cdot)-\bar{\phi}(s,\cdot)\right\|_2} > 1$. So from Definition 1, we can conclude that system (9) is globally exponentially synchronized.

Remark 4. The Lyapunov-Krasovskii functional candidates in this section are divided into a function part $V_1(t)$ and the functional parts $V_2(t),\ V_3(t)$. Since the function $V_1(t)$ is a quadratic form, it is straightforward that the function part can be affected instantaneously by the impulses, whereas the impulses can not bring the value of the purely functional parts $V_2(t),\ V_3(t)$ down. So the function $V_1(t)$ plays an important role in describing the dynamic of impulsive behavior. Note that in (39), the constant c is positive, which means that the reaction-diffusion neural networks (11) maybe unstable without the impulsive controller. Therefore, the sufficient conditions in Theorem 1 is applicable to a delayed reaction-diffusion neural network with unstable continuous dynamics.

Remark 5. The most recent results about distributed delay-dependent impulsive control were reported in [24, 42, 43]. However, the authors did not consider the reaction-diffusion effects in the systems. Since the existence of reaction-diffusion effects, the methods in [24, 42, 43] to estimate the states of Lyapunov candidates at impulsive instants are not feasible.

Very recently, there have been some studies on the stability or synchronization analysis issue for various reaction-diffusion neural networks via impulsive control, see e.g. [12, 30] for some recent publications. However, these results have not considered the time-delay existed in the impulsive controller. Actually the existence of time-delay in the impulsive controller brings dramatic difficulties in estimating the relation between the states with and without time-delay. Furthermore, in [44–50], the delayed impulsive stability for network systems have been studied, but among the publications, only discrete-delayed impulses have been handled, and the reaction-diffusion effects have not been considered. To the best of our knowledge, this is the first time to study the pinning impulsive synchronization for the reaction-diffusion neural network with distributed-delayed impulses.

Remark 6. The factor $\frac{w_3}{w_1}$ plays an important role in estimating the functional parts $V_2(t)$ and $V_3(t)$, which eventually leads to condition (36), The length of impulsive interval conditions are depending on the delay size τ_2, d, and the factor $\frac{w_3}{w_1}$, ρ_{1k}, ρ_{2k}. Note that the results in Theorem 1 can be applicable to system with arbitrarily large delay size. Because we can always add the tuning coefficient of Lyapunov functional parts $V_2(t)$ and $V_3(t)$ to make w_3 sufficiently small (ϖ is the tuning coefficient in this paper), such that $\frac{w_3}{w_1}e^{\mu\tau_2} < 1$ and (36) is satisfied. However, by doing so, more burden would be placed on the estimations of ρ_{1k} and ρ_{2k} (because they can become larger), which leads more restrictions on the length of impulsive interval (in [51], similar discussions are given for delayed impulsive system). ρ_{1k} and ρ_{2k} are original from (27) in the estimation of the relation between the state $e_i\left(t_k^-, x\right)$ and the delayed state $\int_{t_k-d}^{t_k^-} e_i(t,x)dt$. It is required that the factor $\frac{w_3}{w_1}$ and the parameters ρ_{1k}, ρ_{2k} should be small enough such that $\rho_{1k} + \rho_{2k}e^{\mu(\gamma+d)} + \frac{w_3}{w_1}e^{\mu\tau_2} < 1$. Therefore, it is the key point to appropriately estimate the derivatives and the growth by impulsive effects to reach balance conditions in terms of the impulsive interval upper bounds. For more details, please see the examples in the next section.

3.2. *Pinning Impulsive Synchronization of Reaction-Diffusion Neural Network with Discrete-Delayed impulses*

Next we will consider the discrete delay in the impulsive controller, then the pinning impulsive controller is given by the following form:

$$U_i(t,x) = \begin{cases} \sum_{k=1}^{\infty} -\bar{q}_k e_i(t-\bar{d},x)\delta(t-t_k), \ i \in \mathcal{D}_k \text{ and } \sharp\mathcal{D}_k = l_k, \\ 0, \ i \notin \mathcal{D}_k, \end{cases} \tag{49}$$

where $i = 1, 2, \cdots, N$, $\bar{q}_k$ is the impulsive gains and $\bar{d}$ is the discrete delay. Under this impulsive controller, the error system is described as follows:

$$\begin{cases} \frac{\partial e_i(t,x)}{\partial t} = \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial e_i(t,x)}{\partial x_l}\right) - Ae_i(t,x) + Cf\left(e_i\left(t-\tau(t),x\right)\right) + \alpha \sum_{j=1}^{N} G_{ij}\Gamma e_j(t,x), \ t \neq t_k, \\ \Delta e_i\left(t_k,x\right) = -\bar{q}_k e_i\left(t_k - \bar{d}, x\right), \ i \in \mathcal{D}_k, \ \sharp\mathcal{D}_k = l_k, \ k \in \mathbb{N}, \\ e_i(t_0+s,x) = \varphi_i(s,x) - \phi(s,x), \ (s,x) \in [-\bar{\gamma}, 0] \times \Omega, \\ e_i(t,x) = \mathbf{0}, \ (t,x) \in [t_0 - \bar{\gamma}, +\infty) \times \partial\Omega, \ \bar{\gamma} = \max\left(\tau_2, \bar{d}\right). \end{cases} \tag{50}$$

Remark 7. Note that the principles of the distributed-delayed impulsive controller (8) and the discrete-delayed impulsive controller (49) are different. For the first case, the distributed-delayed states $\int_{t_k-d}^{t_k^-} e_i(s,x)ds$ can be regarded as the accumulation (or average) of the states over a time interval $[t_k - d, t_k)$. The idea of distributed-delayed impulsive control is as follows: instead of relying on the system states at the impulsive instants t_k^-, or the states at history moments $t_k - d$, the decision of the impulsive controller depends on the accumulation (or average) of the system states over a time period $[t_k - d, t_k)$. Therefore, it is more practical to employ the distributed-delayed states in this case.

As for the second case, since the inevitability of time delays in the sampling and transmission of impulsive information, the delayed states $e_i(t_k - \bar{d}, x)$ are considered. To be more specific, the states $e_i(\tau_k, x)$ are sampled at a set of moments denoted as τ_k, the controller formulates the impulsive signals based on the sampled values. However, there are two types of delays involved in the control process: sampling-to-controller delays and controller-to-actuator delays. Hence, due to the fact that these two kinds of delays co-exist, it is not feasible to instantaneously apply the impulsive control signals at the exact moments τ_k. Rather, there will be a delay term, $\bar{d}$, at each impulse. Therefore the impulsive signals are, in fact, applied at the moments $t_k := \tau_k + \bar{d}$, and the discrete-delayed states $e_i(\tau_k, x) = e_i(t_k - \bar{d}, x)$ would be employed in this case.

Theorem 2. *Suppose that assumptions (A1) and (A2) hold. If there exist positive matrices $P_0,\ P_1,\ P_2,\ P_3,\ Q_1,\ Q_2 \in \mathbb{R}^{n\times n} > 0$, a matrix $Q_3 \in \mathbb{R}^{Nn\times Nn}$, positive constants $\varpi,\ \varepsilon_1,\ \epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \epsilon_5,\ \epsilon_6,\ \epsilon_7,\ c,\ \mu > 0,\ \bar{\rho}_{1k} > 0,\ \bar{\rho}_{2k} \geq 0$ such that for all $k \in \mathbb{N}$,*

$$\ln\left[\bar{\rho}_{1k} + \bar{\rho}_{2k} e^{\mu(\bar{\gamma}+\bar{d})} + \frac{w_3}{w_1} e^{\mu\tau_2}\right] \leq -(\mu + c)(t_{k+1} - t_k), \tag{51}$$

$$\Upsilon \leq 0, \tag{52}$$

$$\begin{pmatrix} I_N \otimes \varpi Q_2 & Q_3 \\ * & I_N \otimes \varpi Q_2 \end{pmatrix} \geq 0, \text{ where for } k \in \mathbb{N}, \tag{53}$$

$\bar{\rho}_{0k} = (1+\epsilon_0)(1-\bar{q}_k)^2 + \Phi_4(1+\epsilon_6)\bar{q}_k^2/\left(1+\epsilon_6^{-1}\right),\ \bar{\rho}_{2k} = \Phi_1\bar{q}_k^4\bar{\varsigma}^2 + \Phi_2\bar{q}_k^2\bar{d}^2\lambda_1^2 + \Phi_3\bar{q}_k^2,\ \bar{\rho}_{1k} = 1 - \frac{l_k}{N}(1-\bar{\rho}_{0k})$, *$\Phi_1,\ \Phi_2,\ \Phi_3,\ \Phi_4,\ \Phi_5$ are the same as in Theorem 1. $\bar{\varsigma} = \left\lfloor \frac{\bar{d}}{\delta} \right\rfloor$ is the number of impulses that the system (50) subject to on each impulsive interval $[t_k - d, t_k)$, $\delta = t_{k+1} - t_k$, then the trivial solution of system (9) is globally exponentially synchronized.*

Proof. Here for $i \in \mathcal{D}_k$ and $t = t_k^+$, we will estimate the relationship between $e_i(t_k^-, x)$ and $e_i\left(t_k - \bar{d}, x\right)$. from the second equation of (50), we have $e_i\left(t_k^+, x\right) = e_i\left(t_k^-, x\right) - \bar{q}_k e_i\left(t_k - \bar{d}, x\right)$. Integrating both sides of first equation in system (50) from $t_k - d$ to t_k^- yields

$$e_i\left(t_k - \bar{d}, x\right) = e_i\left(t_k^-, x\right) - \int_{t_k-\bar{d}}^{t_k^-} \sum_{l=1}^{q} \frac{\partial}{\partial x_l}\left(D_l \frac{\partial e_i(s,x)}{\partial x_l}\right) ds + \sum_{m=1}^{\bar{\varsigma}_k} \bar{q}_k e_i\left(t_{k-m} - \bar{d}, x\right) - \int_{t_k-\bar{d}}^{t_k^-} \Theta(s,x) ds, \tag{54}$$

where $\bar{\varsigma}_k$ denotes the number of impulses activated onto the ith node during the period $(t_k - d, t_k^-)$, and similarly we have $\bar{\varsigma}_k \leq \bar{\varsigma}$. We consider the same Lyapunov-Krasovskii functional candidates as (12). Similarly, by employing Green's formula, the Dirichlet boundary condition (7), Lemma 1 and Lemma 4, we get

$$V_1\left(t_k^+\right) \leq \bar{\rho}_{1k} V_1\left(t_k^-\right) + \bar{\rho}_{2k} \sup_{s\in[-\bar{\gamma}-\bar{d},0]} \left\{V_1\left(t_k^- + s\right)\right\}. \tag{55}$$

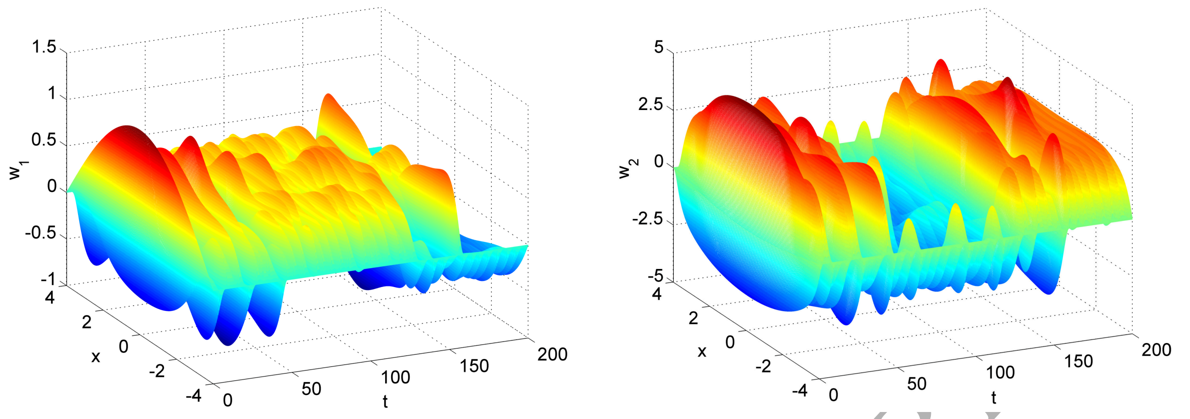
Moreover, from (52), (53) and Proposition 1, we conclude that $\dot{V}(t) \leq cV(t)$ for $t \in [t_k, t_{k+1})$. The rest of the proof is essentially the same as the proof of Theorem 1.

Remark 8. The Dirichlet boundary conditions for reaction-diffusion neural networks are considered in this paper. Actually when considering the Neumannn boundary conditions, which is given by

$$\frac{\partial z_i(t,x)}{\partial v} = \left(\frac{\partial z_i(t,x)}{\partial x_1}, \frac{\partial z_i(t,x)}{\partial x_2} \cdots, \frac{\partial z_i(t,x)}{\partial x_q}\right)^T \text{ for } (t,x) \in [t_0 - \tau_2, +\infty) \times \partial\Omega,$$

the method in Theorem 1 and Theorem 2 are still applicable. Obviously Lemma 1 is not feasible for the Neumannn boundary conditions, but we can employ the Poincaré inequality instead. By using the Poincaré inequality, the parameter h_l in (36) would be replaced by the lowest positive eigenvalue λ_1 of the Neumann boundary problem (5) in [28]. For more details about the explanation and application of Poincaré inequality on reaction-diffusion neural networks, we can refer to [28, 52–54].

Remark 9. In Theorem 1 and Theorem 2, the guidelines of balancing the values among the impulsive gain q_k (or $\bar{q}_k$), the number of pinned nodes l_k, and the length of the impulsive interval are estimated. From inequalities (36) and (51), it is implied that for fixed impulsive gain, reducing the amounts of pinned networks at each impulsive instant would reduce the length of impulsive interval. It is also observed that reducing the absolute value of the impulsive strength will lead to increasing the frequency of the impulsive effects. In order to better understand the effects of the time-delay in the impulses and the ratio $\frac{l_k}{N}$ on the synchronization process, we will give the following three particular cases.



(a) (b)

Fig. 1. Spatiotemporal chaotic behavior of neural network (4) with the given initial conditions $\phi_1(s,x) = 1.2\cos(\pi x/8)$, $\phi_2(s,x) = -0.6\cos(\pi x/8)$ for $s \in [-\tau_2, 0]$ and $x \in \Omega$. (a) State trajectory of w_1. (b) State trajectory of w_2.

Case 1: For $l_k = N$. Then the impulsive controller (49) is reduced to the form: $U_i(t,x) = \sum_{k=1}^{\infty} -\bar{q}_k e_i(t-\bar{d}, x)\delta(t-t_k)$, we have the following stability criterion:

Corollary 1. *Suppose the assumptions (A1) and (A2) hold, and (52)-(53) satisfied, replace the condition (51) by* $\ln\left[\bar{\rho}_{0k} + \bar{\rho}_{2k} e^{\mu(\bar{\gamma}+\bar{d})} + \frac{w_3}{w_1} e^{\mu\tau_2}\right] \leq -(\mu + c)(t_{k+1} - t_k)$, *then the trivial solution of system (9) is globally exponentially synchronized.*

Case 2: For $\bar{d} = 0$. The impulsive controller (49) reduces to be $U_i(t,x) = \begin{cases} \sum_{k=1}^{\infty} -\bar{q}_k e_i(t,x)\delta(t-t_k), & i \in \mathcal{D}_k \\ 0, & i \notin \mathcal{D}_k \end{cases}$. The corresponding stability criterion can be obtained as follows:

Corollary 2. *Suppose the assumptions (A1) and (A2) hold, and (52)-(53) satisfied, replace the condition (51) by* $\ln\left[\bar{\rho}_{1k} + \frac{w_3}{w_1} e^{\mu\tau_2}\right] \leq -(\mu + c)(t_{k+1} - t_k)$ *with* $\bar{\rho}_{0k} = (1 - \bar{q}_k)^2$ *and* $\bar{\rho}_{2k} = 0$, *then the trivial solution of system (9) is globally exponentially synchronized.*

Case 3: For $l_k = N$ and $\bar{d} = 0$. The impulsive controller (49) reduces to be $U_i(t,x) = \sum_{k=1}^{\infty} -\bar{q}_k e_i(t,x)\delta(t-t_k)$. We derive the corresponding stability criterion:

Corollary 3. *Suppose the assumptions (A1) and (A2) hold, and (52)-(53) satisfied, replace the condition (51) by* $\ln\left[\bar{\rho}_{0k} + \frac{w_3}{w_1} e^{\mu\tau_2}\right] \leq -(\mu + c)(t_{k+1} - t_k)$ *with* $\bar{\rho}_{0k} = (1 - \bar{q}_k)^2$ *and* $\bar{\rho}_{2k} = 0$, *then the trivial solution of system (9) is globally exponentially synchronized.*

By comparing Corollary 1 with Corollary 3, the main difference is the term $\bar{\rho}_{2k}$, which relates to the time-delay in the impulsive controller. Therefore, it can be easily observed from (51) that for fixed impulsive control gain, reducing the discrete delay size in the impulses would increase the upper bound of the impulsive interval. Meanwhile, from Corollary 2 with Corollary 3, the difference lies in the term $\bar{\rho}_{1k}$. Since $\bar{\rho}_{1k} < 1$ is required, we have $\bar{\rho}_{0k} < 1$ and $\bar{\rho}_{1k} > \bar{\rho}_{0k}$ if $l_k < N$. Therefore, it can be seen from the results in Corollary 2 with Corollary 3 that increasing the number of nodes to be pinned would lead to reducing the frequency of the impulsive effects, which is in accord with the theoretical analysis given before.

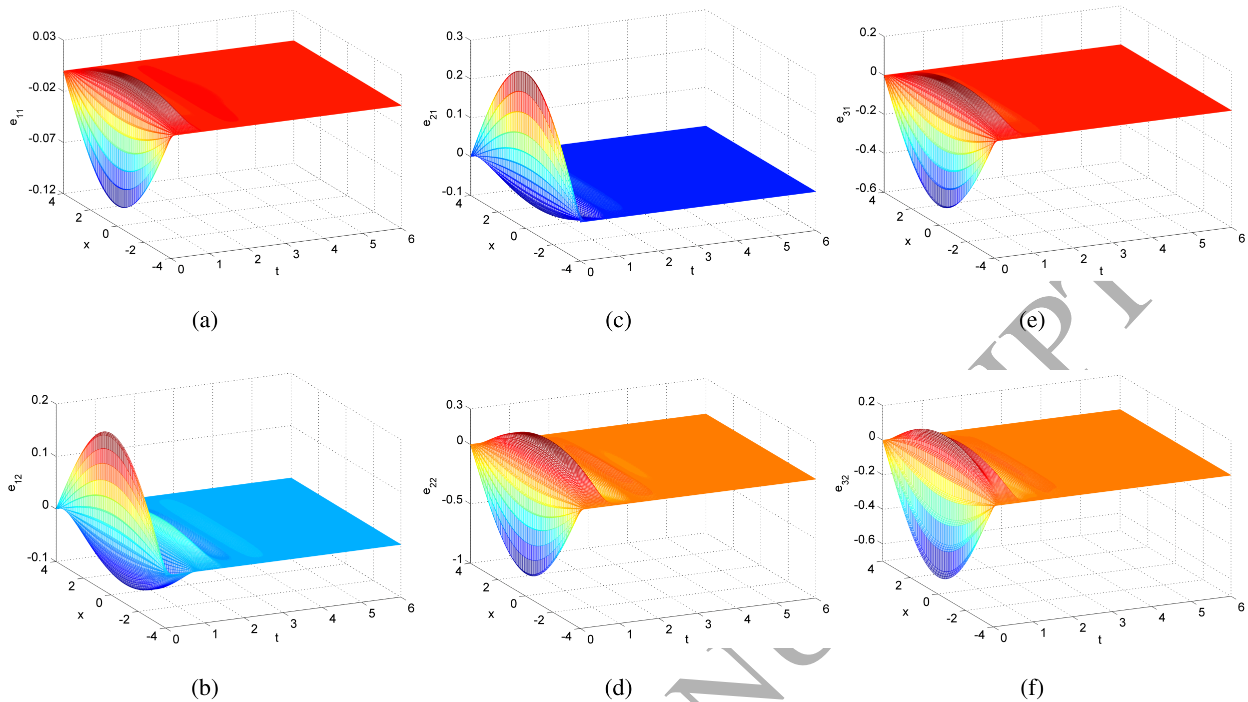


Fig. 2. Synchronization error states of neural network (5) with the initial conditions $\varphi_{11}(s,x) = 1.1\cos(\pi x/8)$, $\varphi_{12}(s,x) = -0.4\cos(\pi x/8)$, $\varphi_{21}(s,x) = 1.5\cos(\pi x/8)$, $\varphi_{22}(s,x) = -1.4\cos(\pi x/8)$, $\varphi_{31}(s,x) = 0.7\cos(\pi x/8)$, $\varphi_{32}(s,x) = 0$ for $s \in [-\gamma, 0]$ and $x \in \Omega$. (a) State trajectory of $e_{11}(t,x)$. (b) State trajectory of $e_{12}(t,x)$. (c) State trajectory of $e_{21}(t,x)$. (d) State trajectory of $e_{22}(t,x)$. (e) State trajectory of $e_{31}(t,x)$. (f) State trajectory of $e_{32}(t,x)$.

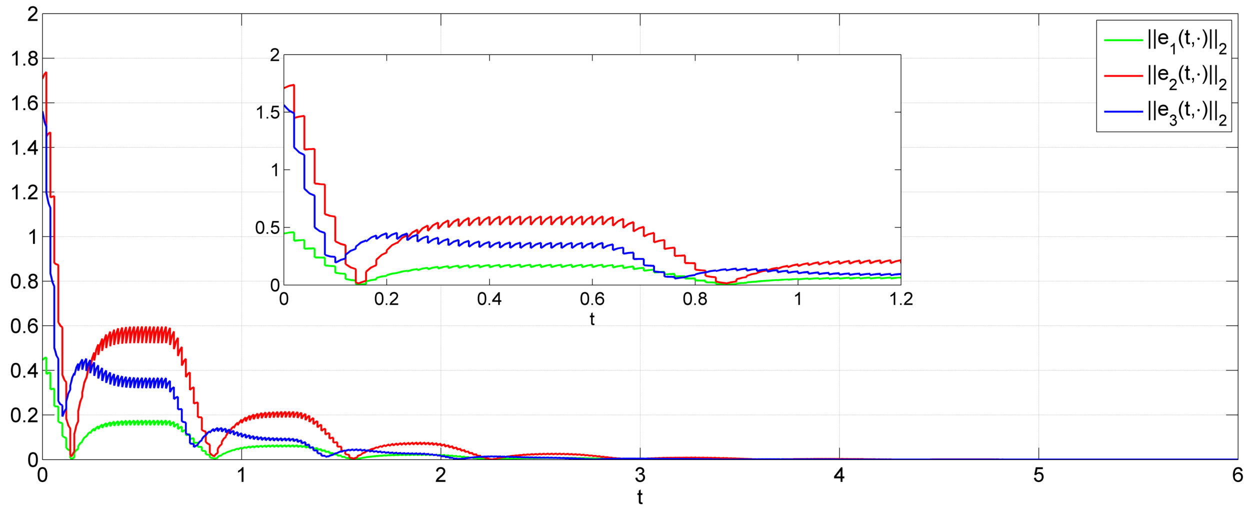
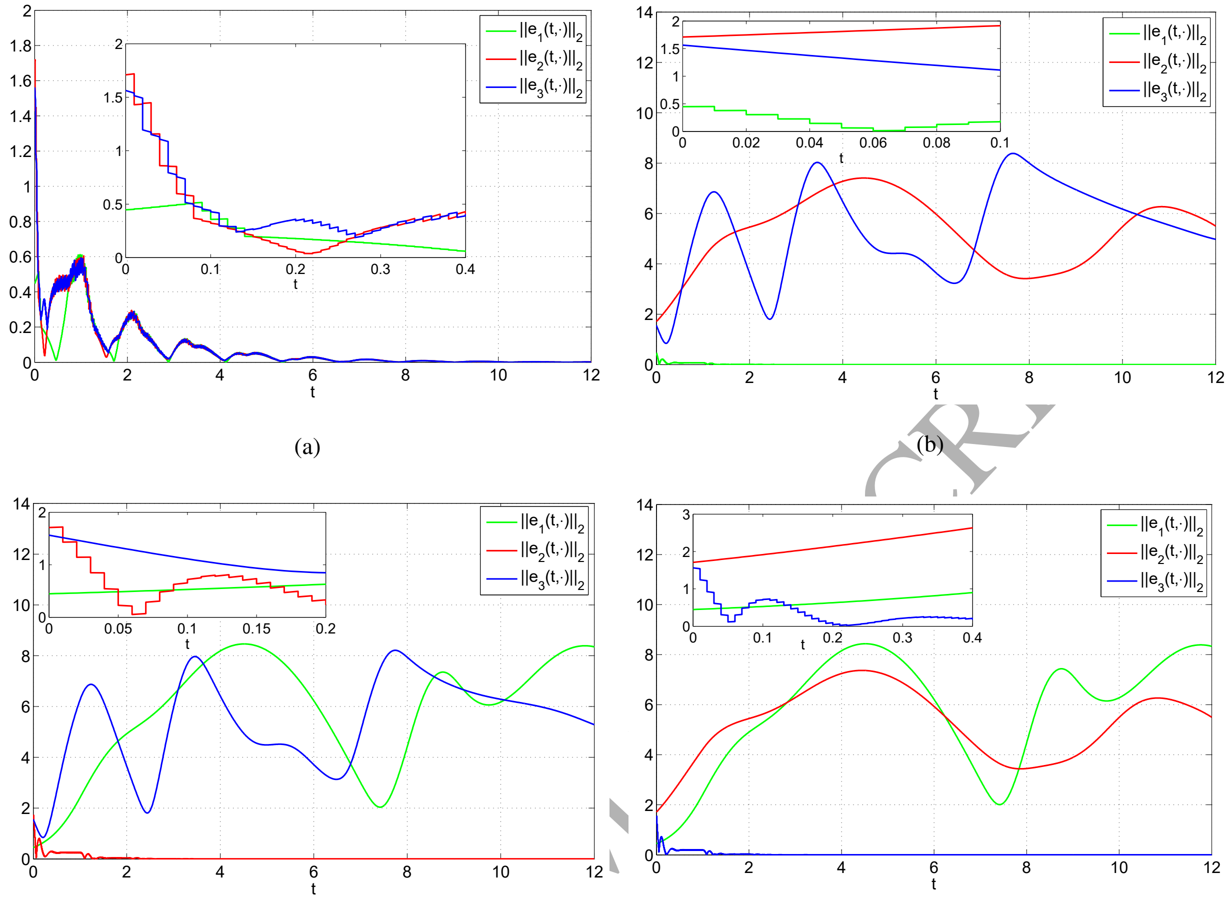


Fig. 3. Synchronization processes of the error states in norm ($\|e_1(t,\cdot)\|_2$, $\|e_2(t,\cdot)\|_2$, $\|e_3(t,\cdot)\|_2$) via the distributed-delayed impulsive controller. The effects of time-delay are shown through the visible serrations.

4. Numerical Examples

Example 1. In this section, two representative examples are presented to demonstrate our main results. In the first example, we consider the synchronization problem of coupled reaction-diffusion neural network with distributed-



(a) (b) (c) (d)

Fig. 4. Synchronization processes of the error states $\|e_1(t,\cdot)\|_2$, $\|e_2(t,\cdot)\|_2$, $\|e_3(t,\cdot)\|_2$ with different pinning algorithms: (a) Pinning impulsive control the neural network (5) with $l_k = 1$. (b) Impulsive control the first node of neural network (5) at each impulsive instants. (c) Impulsive control the second node of neural network (5). (d) Impulsive control the third node of neural network (5).

delayed impulses. Consider the coupled reaction-diffusion neural network (5) consisting of three nodes with hybrid coupling, the single node is described by (4) with the initial condition (2) and Dirichlet boundary condition (3), where $m = 1,2$, $i = 1,2,3$, $t_0 = 0$, $q = 1$, $\Omega = [-4,4]$, $\tau(t) = 0.6 - 0.4e^{-t}$, $g_j(\cdot) = \tanh(\cdot)$, $J_1 = J_2 = 0$. $j = 1,2$, $D_1 = \mathrm{diag}(0.1, 0.1)$, $\alpha = 0.1$, $\Gamma = \mathrm{diag}(0.2, 0.2)$, $A = \mathrm{diag}(1,1)$, and the matrices $C = \left(c_{mj}\right)_{2\times 2}$, $G = \left(G_{ij}\right)_{3\times 3}$ are chosen as

$$C = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix},\ G = \begin{pmatrix} -0.1 & 0.1 & 0 \\ 0.15 & -0.15 & 0 \\ 0 & 0.2 & -0.2 \end{pmatrix}.$$

Obviously, assumption (A1) is satisfied with $\tau_1 = 0.2$, $\tau_2 = 0.6$ and $\eta = 0.4$, $g_j(\cdot)$ ($j = 1,2$) are Lipschitz continuous with $\vartheta_1 = \vartheta_2 = 1$, so $\rho = \mathrm{diag}(1,1)$. Set the initial value of neural network (4) as $\phi_1(s,x) = 1.2\cos(\pi x/8)$, $\phi_2(s,x) = -0.6\cos(\pi x/8)$ for $s \in [-\tau_2, 0]$ and $x \in \Omega$, then the chaotic behavior of neural network (4) is shown in Fig. 1. It can be easily observed that $w_1(t,x)$ and $w_2(t,x)$ are not convergent as time goes by.

Secondly, we consider the coupled reaction-diffusion neural network (5) with the given coupling information. Choosing $\varepsilon_1 = 0.01$, by using MATLAB LMI Toolbox, we can find feasible solutions for (37) and (38) with $c =$

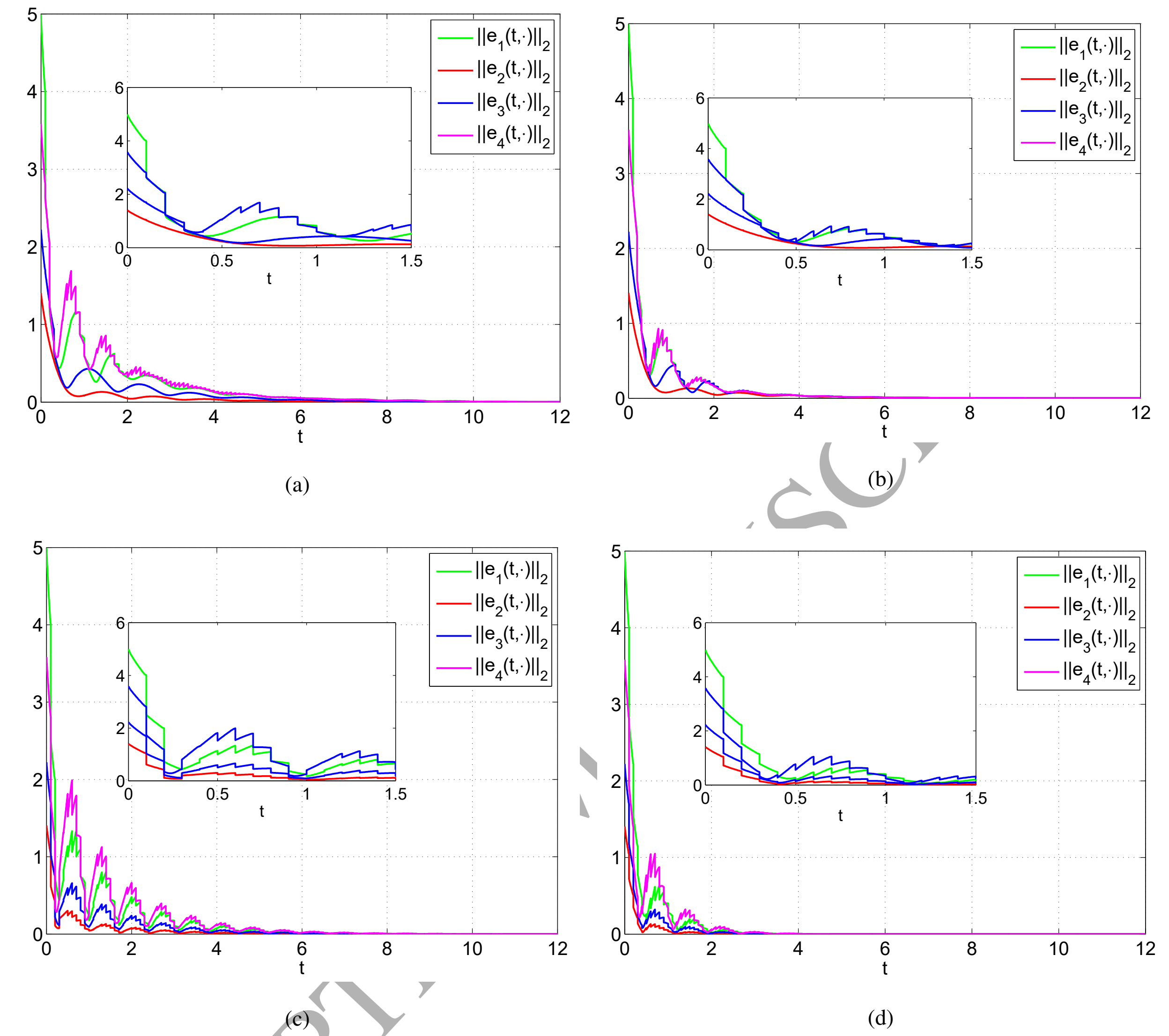


Fig. 5. Synchronization processes of the error states $\|e_1(t,\cdot)\|_2$, $\|e_2(t,\cdot)\|_2$, $\|e_3(t,\cdot)\|_2$ and $\|e_4(t,\cdot)\|_2$. Set the initial value as $\varphi_1^T(s,x) = (0.5\cos(\pi x/2),\ 0.5\cos(\pi x/2))$, $\varphi_2^T(s,x) = (2.5\cos(\pi x/2),\ 0.75\cos(\pi x/2))$, $\varphi_3^T(s,x) = (2\cos(\pi x/2),\ 0.75\cos(\pi x/2))$, $\varphi_4^T(s,x) = (4.5\cos(\pi x/2),\ 1.25\cos(\pi x/2))$, $\phi^T(s,x) = (3\cos(\pi x/2),\ 0.25\cos(\pi x/2))$. Four cases are considered: (a) Pinning impulsive control the neural network (5) with $l_k = 1,\ \bar{d} = 0.15$. (b) $l_k = 1,\ \bar{d} = 0$. (c) $l_k = 4,\ \bar{d} = 0.15$. (d) $l_k = 4,\ \bar{d} = 0$.

35.8882. Then we consider the impulsive controller (8) with $l_k = N = 3$, $d = 0.05$, $t_k - t_{k-1} = 0.02$, and $q_k = 3.5$ for all $k \in \mathbb{N}$, and we can get the following estimations: $\rho_{1k} = 0.8053$, $\rho_{2k} = 0.3948$, $\gamma = 0.6$, $w_1 = 0.1808$ and $w_3 = 0.0127$ (ϖ is sufficiently small). So (36) is satisfied with $\mu = 1.4837$. Therefore, according to the results in Theorem 1, the exponential synchronization of neural network (5) can achieved under the impulsive controller (8). The trajectories of error states are presented in Fig. 2 and Fig. 3. Since the existence of time-delay, the serration phenomenon occurred and can be clearly observed in Fig. 3 when $t < 0.6$. It is indicated through these two figures that under the delayed impulsive controller (8), the synchronization of coupled reaction-diffusion neural network (5) can be achieved.

Next, we consider the pinning impulsive controller with $l_k = 1$, i.e., at each impulsive instant, only one node is under controlled. As mentioned in Remark 3, the selection of pinned nodes is important when applying pinning control approach. We make a comparison with the traditional impulsive control method in [15, 32], in which the pinned nodes are fixed and no conditions about the selection of pinned nodes are given. Set $t_k - t_{k-1} = 0.01$ for $k \in \mathbb{N}$, the other parameters stay the same, the simulation results are shown in Fig. 4. It can be observed in sub-figure 4(a) that different node are controlled at distinct impulsive instants. This is consistent with the idea of our pinning controller, i.e., to control the nodes who have larger state deviation. From the comparison of 4(a) with 4(b), 4(c) and 4(d), we can find that the selected pinning methods in sub-figures 4(b), 4(c) and 4(d) fail to synchronize the network. In sub-figure 4(a), in order to select the appropriate pinned nodes, the dynamical performances of each isolated node, the coupling strategies and their relations are more deeply considered, and it is efficient in synchronizing the network. Therefore, our pinning impulsive method is more efficient to synchronize the reaction-diffusion neural network (5) than the results in [15, 32].

Example 2. To illustrate the effectiveness of Theorem 2, we consider neural network (5) with the pinning impulsive controller (49), where $m = 1, 2$, $i = 1, 2, 3, 4$, $t_0 = 0$, $q = 1$, $\Omega = [-4, 4]$, $\tau(t) = 0.2 + \frac{0.3e^t}{1+e^t}$, $g_j(w_j) = \frac{|w_j+1|-|w_j-1|}{4}$, $J_1 = J_2 = 0$. $j = 1, 2$, $D_1 = \text{diag}(0.2, 0.1)$, $\alpha = 0.2$, $\Gamma = \text{diag}(0.4, 0.6)$, $A = \text{diag}(1, 2)$, and the matrices $C = \left(c_{mj}\right)_{2\times 2}$, $G = \left(G_{ij}\right)_{4\times 4}$ are chosen as

$$C = \begin{pmatrix} -1.5 & -0.5 \\ -1 & -1.5 \end{pmatrix}, \; G = \begin{pmatrix} -0.1 & 0.1 & 0 & 0 \\ 0 & -0.1 & 0.1 & 0 \\ 0 & 0 & -0.1 & 0.1 \\ 0.1 & 0 & 0 & -0.1 \end{pmatrix}.$$

It is obvious that $g_j(\cdot)$ satisfies assumption (A2) with $\rho = \text{diag}(0.5, 0.5)$, assumption (A1) is satisfied with $\tau_1 = 0.2$, $\tau_2 = 0.5$ and $\eta = 0.075$. Based on the results on Theorem 2, Corollary 1, Corollary 2 and Corollary 3, Set $t_k - t_{k-1} = 0.1$, $\bar{q}_k = 0.3$, we consider four different cases, and the numerical results are shown in Fig. 5.

5. Conclusion

This paper has analyzed the exponential synchronization problem of coupled reaction-diffusion neural networks with time-varying delays. The pinning impulsive controllers that take into account both distributed time delays or discrete time delays have been proposed, respectively. By utilizing the Lyapunov-Krasovskii functional method and the pinning impulsive control algorithms, exponential synchronization criteria are derived to design suitable pinning impulsive controllers. Numerical simulations of delayed reaction-diffusion neural networks, pinning impulsive synchronization of delayed reaction-diffusion neural networks with distributed impulses and discrete impulses are given to demonstrate the effectiveness of our theoretical results.

Moreover, possible future research directions may include the following: 1) This paper focuses on the synchronization problem of couple neural networks. It would be interesting to apply our results to the consensus problem of networked multi-agent systems with suitable impulsive protocols (see, e.g., [26, 55]). 2) Since stochastic disturbances always occur in networked systems, our proposed approach can be extended to solve various control problems of networked systems with stochastic disturbance.

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