Late-lumping backstepping control of partial differential equations

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Abstract

We consider in this paper three different partial differential equations (PDEs) that can be exponentially stabilized using backstepping controllers. For implementation, a finite-dimensional controller is generally needed. The backstepping controllers are approximated and it is proven that the finite-dimensional approximated controller stabilizes the original system if the order is high enough. This approach is known as late-lumping. The other approach to controller design for PDE's first approximates the PDE and then a controller is designed; this is known as early-lumping. Simulation results comparing the performance of late-lumping and early-lumping controllers are provided.

Key words: partial differential equations; stabilization; backstepping, late-lumping.

1 Introduction

Controller design for partial differential equations (PDEs) typically needs to be done using a finite-dimensional, or lumped, approximation of the PDE. This approach is known as early-lumping. It introduces questions of stability and performance of the designed system [46]. However, for some PDEs, backstepping controllers can be directly designed using the PDE. Introduced in [53,54] for a general 1-D linear reaction-diffusion-advection PDE, it has been extended to a large number of boundary control problems: flow control [2,3,64], parabolic PDEs [7,21,58,59], or hyperbolic PDEs [6,18,22–24,27]. A complete history of the backstepping method and of its extensions has recently been given in [60]. The resulting controllers are explicit, in the sense that they are expressed as a linear functional of the distributed state at each instant. The (distributed) gains can be computed offline. Considering application of such controllers to industrial problems, in most cases, only an approximation of the state is available for controller design and the controller needs to be approximated.

This direct controller design approach is sometimes referred to as late lumping since the last step in the design is to approximate the controller by a finite-dimensional, or lumped parameter, system. The other approach is early-lumping; in this approach the controller design is based on a finite-dimensional approximation of the PDE. Numerous results ensuring the convergence of early-lumping controllers can be found in the literature; see for example[9,10,37,38,43,44] and the tutorial paper [46]. However, the question of the convergence of late-lumping backstepping controllers has not been well-investigated, contrary to the approximation of the kernels themselves, e.g. in [30] using a trapezoidal rule or in [4] using a sum-of-squares approach. In [63], a method for computing the bounded part of the control operator is proposed. It relies on a finite-dimensional approximation of the state and enables efficient computing of the feedback law. However, the unbounded part of the operator is not approximated and no guarantees of convergence are provided.

In this paper late lumping control is considered for three different systems that can be stabilized using backstepping control laws. The main contribution of this paper is to give sufficient conditions guaranteeing the convergence of backstepping-based late-lumping state feedback controllers for various systems: an unstable heat equation [54], a wave equation [55] and a system of linear hyperbolic PDEs [18]. For each example, we consider an

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approximation of the state (that satisfies some specific assumptions) to design the control law. The resulting feedback system is mapped to a simpler target system using backstepping-like transformations. An explicit Lyapunov function is used to prove exponential stability. The design is based on the boundary control formulation; the system is not converted to state space form. The performance of these late-lumping controllers are compared in terms of performance and control effort to early-lumping controllers in simulations. A high order approximation of the PDE is used as the system.

The paper is organized as follows. Section 2 provides the general framework and recalls existing results for early-lumping and late-lumping control. Some crucial assumptions concerning the state space and the approximating space are also given. We then prove for various examples (for which backstepping control laws have already been derived), that the approximated control laws still guarantee exponential stabilization. An unstable heat equation is considered in Section 3, the wave equation in Section 4 and a general class of hyperbolic PDEs in Section 5. For each example, the late-lumping controller is compared in terms of performance and control effort with early-lumping controllers. Finally, in Section 6, we give some perspectives about the design of an late-lumping output-feedback control law.

2 Presentation of the method

All the systems considered in this paper are boundary control systems [51]

$$\begin{aligned} &\frac{dz}{dt} = \mathfrak{A}z(t), \quad z(0) = z_0, \quad t \in [0, T] \\ &\mathfrak{B}z(t) = u(t), \end{aligned} \tag{1}$$

where $\mathfrak{A} \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $\mathfrak{B} \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ with $\mathcal{Z}$ and $\mathcal{U}$ separable Hilbert spaces. The space $\mathcal{Z}$ is a dense subspace of $\mathcal{H}$ with continuous, injective embedding $i_{\mathcal{Z}}$. We assume that the boundary control system (1) satisfies the following assumptions [51]:

- The operator $\mathfrak{B}$ is onto, its kernel is dense in $\mathcal{H}$ and there exists $\mu \in \mathbb{R}$ such that $\ker(\mu\mathfrak{I} - \mathfrak{A}) \cap \ker \mathfrak{B} = 0$, and $\mu\mathfrak{I} - \mathfrak{A}$ is onto $\mathcal{H}$ (where $\mathfrak{I}$ is the identity operator).
- For any $z_0 \in \mathcal{Z}$ with $\mathfrak{B}z_0 = 0$, there exists a unique solution to (1) in $C^1([0, T; \mathcal{H}]) \cap C([0, T; \mathcal{Z}])$ depending continuously on z_0 (where T is a positive time).

The initial condition z_0 is assumed to belong to $\mathcal{Z}$.

These systems can be rewritten in an abstract state space form, generally using unbounded control operators; that is, a control operator bounded to some Hilbert space larger than the state space and an observation operator bounded from a Hilbert space smaller than the state space [51]. There is an extensive literature dealing with systems having unbounded control operators; see for instance [19,26,36,51,62]).

It is not necessary to convert the boundary control formulation (1) to state space form [17]. The backstepping approach uses the boundary control formulation given by (1) and this formulation is used in approximation of the backstepping controller.

In this paper, the space $\mathcal{Z}$ must satisfy the following assumption.

Assumption 1 *The space $\mathcal{Z}$ satisfies $\mathcal{Z} \subset (\mathcal{H}^1([0, 1]))^p$ where p is a positive integer.*

The value of p depends on the particular PDE. Since the space $\mathcal{H}^1([0, 1])$ is embedded in the Holder space $\mathcal{C}^{0,\frac{1}{2}}([0, 1])$, using Morrey's inequality (see e.g [14, Theorem 9.12]), a direct consequence of Assumption 1 is the existence of a constant $\alpha > 0$ such that for all $z \in \mathcal{Z}$, for all $1 \le i \le p$,

$$\sup_{x\in[0,1]} |z_i(x)| \le \alpha(||z_i||_{\mathcal{H}^1([0,1])}) \tag{2}$$

Definition 1 *The system* (1) *is **exponentially stabilizable** if there exists $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that if $u(t) = Kz(t)$ the semigroup $\mathcal{S}$ associated to* (1) *is exponentially stable semigroup, i.e there exist $M \ge 1$ and $\omega > 0$ such that*

$$||\mathcal{S}(t)|| \le Me^{-\omega t} \tag{3}$$

The early-lumping approach (also known as indirect controller design consists in approximating the original PDE (1) using standard methods (such as finite elements for instance). This yields a system of ordinary differential equations. Controller design is based on this finite-dimensional approximation. Consider finite-dimensional subspace $\mathcal{Z}_n$ of the state-space $\mathcal{Z}$ and P_n the orthogonal projection $P_n : \mathcal{Z} \to \mathcal{Z}_n$ such that

$$\forall\ z\ \in \mathcal{Z},\ \lim_{n\to\infty} ||P_n z - z|| = 0. \tag{4}$$

Although the orthogonality of the operators P_n is not necessary, it is convenient as it makes the computation easier; that is, the reconstruction of the full state from its projections.The subspaces $\mathcal{Z}_n$ are equipped with the norm inherited from $\mathcal{Z}$. Considering this approximation scheme and defining the operator $\mathfrak{A}_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ by some method while $\mathfrak{B}_n = \mathfrak{B}P_n$, this leads to the following finite-dimensional approximation:

$$\frac{d\tilde{z}}{dt} = \mathfrak{A}_n \tilde{z}(t), \quad \tilde{z}(0) = P_n z_0, \quad t \in [0,T].$$
$$\mathfrak{B}_n z(t) = u(t) \tag{5}$$

Define the operator on $\mathcal{H}$

$$Az = \mathfrak{A}z, \quad D(A) = \{z \in \mathcal{Z}; z \in \ker \mathfrak{B}\},$$

and let $S(t)$ be the C_0-semigroup generated on $\mathcal{H}$ by A. Denote similarly by $\mathcal{S}_n(t)$ the semigroups generated by $A_n : D(A_n) \mapsto \mathcal{Z}_n$ with $D(A_n) = \mathcal{Z}_n \cap \ker \mathfrak{B}_n$ and $A_n z = \mathfrak{A}_n z$ for $z \in \mathcal{Z}_n$. We make the following classical assumption that ensure the uniform convergence on bounded intervals of the open-loop approximating state $\tilde{z}(t)$ to the exact state: for each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} ||\mathcal{S}_n(t)P_n z - \mathcal{S}(t)z|| = 0. \tag{6}$$

Equation (6), which is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold (see [28,48]), along with equation (4) imply open loop convergence of the approximatng systems.

However (6) is not sufficient to guarantee that a control sequence u_n that stabilizes the approximations (5) wlll stabilize the original system and provide good performance (see [15,45,46]). For bounded control operators, a large number of tools and techniques are available for controller design using this approach; see for example [10,37,38,43,45] and the tutorial paper [46]). However, boundary control typically leads to an unbounded control operator when put in state space form and only a few results can be found in the literature[9,35,37,38]. We do not provide in this paper conditions guaranteeing the convergence of early-lumping controllers for an unbounded control operator. However, to compare the results we obtain for late-lumping controller we derive for each example, without proving convergence or stabilization, two early-lumping controllers: a backstepping-like controller (that is, an early-lumping control law that places the first poles of the closed-loop system at the same location as the late-lumping controller) and a LQR controller.

Late-lumping control

For numerous systems, it is possible to directly derive from the PDE a stabilizing infinite-dimensional state feedback, that is, to find an operator $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that the semigroup associated to (1) along with the control law $u(t) = Kz(t)$ is stable. Examples include the backstepping controllers derived in [6,18,32,54], the flatness-based controllers derived in [42,50], the optimization controllers in [39], the controller in [40] based on a frequency-domain approach.

In design of a backstepping controller, an integral transformation is used to map the original system to a target system with desirable properties, including stability. The control law ensuring the stabilization of the original system is then derived using this transformation. For application of these controllers to industrial problems for which sensors cannot be placed all along the system, it is necessary to derive an observer. However in this paper, we only focus on the control aspects, neglecting the design of the observer. However, to reflect the fact that we do not have fully-distributed measurements, we assume that only an approximation of the state is available to synthesize the control law. More precisely, considering a stabilizing control law $u(t) = Kz$, the late-lumping assumption implies that the real control law that will be used is

$$u(t) = K(P_n z) = Kz^n, \tag{7}$$

where $z^n = P_n z$ and P_n is as in (4). Our main contribution is to prove the uniform convergence of the late-lumping controller for different examples. Our proofs rely on the following assumption on the approximation sequence.

Assumption 2 *Let p be the integer in Assumption 1. There exists a sequence C_n such that $\forall z \in \mathcal{Z} \subset (\mathcal{H}^1([0,1]))^p$,*

(1) $\lim_{n\to\infty} C_n = 0,$
(2) $\forall n \in \mathbb{N},\ ||KP_n z - Kz|| \leq C_n ||z||_{(\mathcal{H}^1([0,1]))^p}.$

This assumption means that the approximation scheme has to be chosen accurate enough to ensure the uniform convergence of the approximated control operator to the real one. The fact that C_n does not depend on z is crucial to ensure this uniform convergence. It will be shown throughout the next sections, that common approximation methods, such as finite elements, satisfy this assumption.

3 Unstable heat equation

We consider in this section the example of heat conduction in a rod of small cross-section. The rod is assumed thin enough so that the temperature can be assumed uniform across the section. We assume that the effects of heat loss and heat generation inside the rod are significant and have to be modeled (these terms can come from radiation, electrical resistivity). Moreover, we assume that the heat generation dominates the heat loss which makes the system unstable. Stabilization is achieved by applying a Neumann boundary control on one end and insulating the other. This yields (see [13,16,25]) the following parabolic PDE, an unstable heat equation:

$$z_t(t,x) = z_{xx}(t,x) + \lambda z(t,x), \quad z(0,x) = z_0 \tag{8}$$

evolving in $\{(t,x)| \quad t>0,\ x\in[0,1]\}$, with Neumann boundary conditions

$$z_x(t,0)=0, \quad z_x(t,1)=u(t). \tag{9}$$

The parameter λ is assumed strictly positive so that the open-loop system (8)-(9) is unstable. The initial condition denoted z_0 is assumed to belong to $\mathcal{H}^1([0,1])$. For this system, various control laws ensuring exponential stabilization have already been designed (see [8,12,54]). In particular, in [54] a feedback control law is derived using the backstepping approach.

Late-lumping controller. We recall the main results of [54] in which is derived a control law that stabilizes the original infinite-dimensional system (8)-(9) using the backstepping method [33]. We assume then that only an approximation of the state is available for control design (late-lumping) and prove that the resulting control law stabilizes the original system. Let us consider the Volterra transformation

$$w(t,x)=z(t,x)-\int_0^x L(x,\xi)z(t,\xi)d\xi, \tag{10}$$

where the kernel $L(x,y)$ is defined on $\mathcal{T}=\{(x,y)\in[0,1]^2| \quad y\le x\}$ by

$$L(x,y)=\begin{cases} -(\lambda+c)x\frac{I_1(\sqrt{(\lambda+c)(x^2-y^2)})}{\sqrt{(\lambda+c)(x^2-y^2)}}, & \text{if } x\ne y\\ -\frac{(\lambda+c)}{2}x \quad \text{if } x=y, \end{cases} \tag{11}$$

and where c is an arbitrary strictly positive constant. The function I_1 is the first modified Bessel function. The function L is two times differentiable on $\mathcal{T}$. The kernel $L(x,y)$ satisfies the following hyperbolic PDE (given in [54])

$$L_{xx}(x,y)-L_{yy}(x,y)=(\lambda+c)L(x,y), \tag{12}$$

along with the boundary conditions

$$L_y(x,0)=0, \quad L(x,x)=-\frac{1}{2}(\lambda+c)x. \tag{13}$$

In the following, we denote by R (bounded on $\mathcal{T}$) the derivative of L with respect to x, $R:=L_x$.

Lemma 2 *[54, Theorems 5,8] There exist two constants C_1 and C_2 such that*

$$C_1||w||_{\mathcal{H}^1([0,1])}\le||z||_{\mathcal{H}^1([0,1])}\le C_2||w||_{\mathcal{H}^1([0,1])} \tag{14}$$

Defining K_{BS} by

$$K_{BS}z=-\frac{(\lambda+c)}{2}z(1)+\int_0^1 R(1,\xi)z(\xi)d\xi, \tag{15}$$

we define the control law $u(t)$

$$u_{BS}(t)=K_{BS}z(t). \tag{16}$$

The transformation (10) along with the control law (16) maps the original system (8)-(9) to the stable target system

$$w_t(t,x)=w_{xx}(t,x)-cw(t,x), \tag{17}$$
$$w_x(t,0)=0, \quad w_x(t,1)=0. \tag{18}$$

Thus, for any initial condition $z_0\in\mathcal{H}^1([0,1])$, the system (8)-(9) with the control law (16) has a unique classical solution $z(t,x)\in C^{2,1}(([0,1])\times(0,\infty))$ and is exponentially stable at the origin, $u(t,x)\equiv 0$ in the $\mathcal{L}^2([0,1])$ and $\mathcal{H}^1([0,1])$ norm. The control $u(t)=K_{BS}z(t)$ exponentially stabilizes the system (8)-(9).

Let us now consider an approximation scheme satisfying Assumption 2 and assume that only the $n\in\mathbb{N}^*$ first modes of the state are available to design the control. We denote P_n the projection on the approximating space. This means we consider the system (8)-(9) along with the control law

$$u^n_{BS}(t)=K_{BS}P_nz. \tag{19}$$

Theorem 3 *There exists $N\in\mathbb{N}$ such that for any $n\ge N$, for any initial condition $z_0\in\mathcal{H}^1([0,1])$, the system (8)-(9) along with the approximated control law (19) is exponentially stable at the origin, $z(t,x)\equiv 0$ in the sense of the $\mathcal{L}^2([0,1])$-norm.*

PROOF. This theorem can be proved using [31, Theorem IX.2.4] since the semigroup is analytic perturbed by a small perturbation. However, this method cannot be extended for the other examples considered in this paper, contrary to the Lyapunov-based proof used here.

The main idea of the proof consists in mapping (8)-(9) along with the control law (19) to a simpler target system with a similar structure to (17)-(18) using the transformation (16). This target system is then proved to be exponentially stable for an order of approximation n large enough. This is done by the way of a Lyapunov function. Finally, due to inequality (14), this implies the exponential stability of the original system.

Let us consider (8)-(9) along with the control law (19). Similarly to [54], differentiating (10) with respect to space, we obtain

$$w_x(t,x) = z_x(t,x) - L(x,x)z(t,x) \\ - \int_0^x R(x,\xi)z(t,\xi)d\xi. \tag{20}$$

and

$$w_{xx} = z_{xx}(t,x) - L(x,x)z_x(t,x) - R(x,x)z(t,x) \\ - \frac{d}{dx}(L(x,x))z(t,x) - \int_0^x R_x(x,\xi)z(t,\xi)d\xi.$$

Similarly, differentiating (10) with respect to time and using (8)

$$w_t(t,x) = z_t(t,x) - \int_0^x L(x,\xi)z_t(t,\xi)d\xi \\ = z_{xx}(t,x) + \lambda z(t,x) - L(x,x)z_x(t,x) + L_\xi(x,x)z(t,x) \\ - \int_0^x L_{\xi\xi}(x,\xi)z(t,\xi) + \lambda L(x,\xi)z(t,\xi)d\xi.$$

Thus, combining the two previous equations, we get

$$w_t(t,x) - w_{xx}(t,x) + cw(t,x) = z_{xx}(t,x) + \lambda z(t,x) \\ + L_\xi(x,x)z(t,x) - \int_0^x \left(L_{\xi\xi}(x,\xi)z(t,\xi) + \lambda L(x,\xi)z(t,\xi)\right) d\xi \\ - L(x,x)z_x(t,x) - z_{xx}(t,x) + L(x,x)z_x(t,x) \\ + R(x,x)z(t,x) + \frac{d}{dx}(L(x,x))z(t,x) + \int_0^x R_x(x,\xi)z(t,\xi)d\xi \\ + cz(t,x) - c\int_0^x L(x,\xi)z(t,\xi)d\xi.$$

This yields,

$$w_t(t,x) - w_{xx}(t,x) + cw(t,x) = (\lambda + c - \frac{d}{dx}(L(x,x))) \\ z(t,x) + \int_0^x \left(L_{xx}(x,\xi) - L_{\xi\xi}(x,\xi) - (\lambda+c)L(x,\xi)\right) d\xi.$$

Finally, using (12)-(13), we obtain

$$w_t(t,x) = w_{xx}(t,x) - cw(t,x). \tag{21}$$

Using (9) and (20) we obtain the Neumann boundary condition

$$w_x(t,0) = z_x(t,0) - L(0,0)z(t,0) = 0. \tag{22}$$

Using (19), we have that

$$L(1,1)z(t,1) + \int_0^1 R(1,\xi)z(t,\xi)d\xi = -u_{BS}(t).$$

Combining this result with (16), we obtain the following Neumann boundary conditions

$$w_x(t,1) = z_x(t,1) - L(1,1)z(t,1) - \int_0^1 R(1,\xi)z(t,\xi)d\xi \\ = u_{BS}^n(t) - u_{BS}(t). \tag{23}$$

Using Assumption 2 and inequality (14), we obtain

$$|K_{BS}P_nz - K_{BS}z| \leq C_nC_2||w||_{\mathcal{H}^1([0,1])}. \tag{24}$$

We now prove the stability of the system (21)-(23) with a Lyapunov analysis. Inspired by [54], let us consider the Lyapunov function candidate

$$V(t) = \int_0^1 w^2(t,x)dx. \tag{25}$$

Differentiating V with respect to time and integrating by parts yields

$$\dot{V}(t) = 2\int_0^1 w(t,x)(w_{xx}(t,x) - cw(t,x))dx \\ = -2\int_0^1 w_x^2(t,x)dx - \int_0^1 2cw^2(t,x)dx \\ + 2w(t,1)(u_{BS}^n(t) - u_{BS}(t)) \\ \leq -2\int_0^1 w_x^2(t,x)dx - \int_0^1 2cw^2(t,x)dx \\ + 2C_nC_2\alpha||w||^2_{\mathcal{H}^1([0,1])}, \tag{26}$$

where we have used (1) and (24) to obtain the last inequality. Since C_n converges to zero, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $C_n \leq \frac{\min(c,1)}{2C_2\alpha}$ This yields the existence of a constant δ such that

$$\dot{V}(t) \leq -\delta V(t) \tag{27}$$

This implies the exponential stability of the system (21)-(23) in the sense of the $\mathcal{L}^2$-norm. Due to (14), the original state z has the same properties. This concludes the proof.

Early-lumping. We now derive early-lumping control laws that are going to be compared with the late-lumping one obtained above. We start by recalling the abstract formulation of (8) in terms of operators. This abstract formulation, although it was not required for the design of the backstepping controller is useful in design of an early-lumping controller. Define $\mathcal{Z} = \mathcal{L}^2([0,1])$. We can rewrite the system in the abstract form as

$$\dot{z}(t) = \mathfrak{A}_{\text{heat}}z(t), \quad z(0) = z_0. \\ \mathfrak{B}_{\text{heat}}z(t) = u(t) \tag{28}$$

The operator $\mathfrak{A}_{\mathrm{heat}}$ is defined by

$$\begin{aligned} \mathfrak{A}_{\mathrm{heat}} : \mathcal{Z} \to \mathcal{H} = \mathcal{L}^2([0,1]) \\ z \longmapsto z_{xx} + \lambda z, \end{aligned} \quad (29)$$

with $\mathcal{Z} = \{z \in \mathcal{H}^2([0,1]) |\ z_x(0) = 0\}$, where $\mathcal{H}^2([0,1])$ indicates the Sobolev space of functions with weak second derivatives (see e.g [52]). Its domain of definition satisfies Assumption 1. We equip $\mathcal{Z}$ with the scalar product associated with the graph norm $||z||_{D(\mathfrak{A}_{\mathrm{heat}})} = ||z||_{\mathcal{L}^2[0,1]} + ||\mathfrak{A}_{\mathrm{heat}} z||_{\mathcal{L}^2[0,1]}$, which is equivalent to the $\mathcal{H}^1([0,1])$-norm. The boundary control operator $\mathfrak{B}_{\mathrm{heat}} : \mathcal{Z} \to \mathbb{R}$ is defined by

$$\mathfrak{B}_{\mathrm{heat}} z = \frac{dz}{dx}(1).$$

The eigenfunctions ϕ_i $(i = 0, ...)$ of the operator

$$Az = \mathfrak{A}_{\mathrm{heat}}, \quad D(A) = \{z \in \mathcal{Z};\ \frac{dz}{dx}(1) = 0\}$$

form a orthonormal basis for $\mathcal{L}^2(0,1)$. These eigenfunctions are (see for example, [20])

$$\phi_k(x) = \begin{cases} 1 & \text{if } k = 0 \\ \sqrt{2}\cos(k\pi x) & \text{if } k \neq 0. \end{cases} \quad (30)$$

They also form an orthogonal basis for $\mathcal{H}^1([0,1])$. Define $\chi_n = \mathrm{span}_{k=0,\cdots,n}\{\phi_k\}$ and let P_n indicate the projection onto χ_n. Then define $z^n(t,x) = P_n z(t,x) = \sum_{k=0}^{n} z_k(t)\phi_k(x)$.
Define $\mathfrak{A}_n$ by the Galerkin approximation [34,45,47]

$$\langle \mathfrak{A}_n \phi_j, \phi_k \rangle = \langle \mathfrak{A}_{\mathrm{heat}} \phi_j, \phi_k \rangle, \quad (j,k) \in [0,n]^2 \quad (31)$$

and $\mathfrak{B}_n = \mathfrak{B}_{\mathrm{heat}} P_n$.In the following we denote $\mathbf{z}^n = \Big(z_0, \cdots z_n\Big)^T$, the concatenation of different projections of z on the space χ_n. Similarly, we denote $\mathbf{z}_0^n = \Big((P_n z_0)_0, \cdots (P_n z_0)_n\Big)^T$.

The following open-loop convergence result is well-known.

Lemma 4 *[46, e.g.,Theorem 3.1] For each initial condition $z_0 \in \mathcal{Z}$, the uncontrolled approximating state $z^n(t)$, converges uniformly on bounded intervals to the exact state $z(t)$.*

Using the Galerkin approximation (31) it becomes possible to derive early-lumping controllers that can be numerically compared with the late-lumping one. Inspired by the backstepping controller, a natural way to design an early-lumping controller is to approximate the (exponentially stable) target system (17)-(18), find the eigenvalues of the resulting ODE and place the first N eigenvalues of (31) on the same location. This sequence of control laws will be denoted $u^n_{BS_{early}}$(although this is not strictly a backstepping control law). However, one must be aware that the matrices used to derive this pole placement are the ones obtained using the approximated operators $\mathfrak{A}_n$ and $\mathfrak{B}_n$.Such a finite dimensional method, that places the poles of the approximated system in the same position as the poles of the approximated target system, can be compared to the one proposed in [8] using a finite-difference discretization. This was the first attempt to use backstepping for PDEs and then the proposed finite-difference algorithm had no other claim than to be a "proof of concept" without any attempt to find an optimal numerical approximation. Consequently, it is not surprising that the Galerkin approximation we propose leads to better results.

A second method to design an early-lumping controller is linear quadratic control. Consider the quadratic functional

$$J(u^n, z_0) = \int_0^\infty \langle \mathbf{z}^n(t), \mathbf{z}^n(t) \rangle + \alpha((u^n)(t))^2 dt, \quad (32)$$

where $\alpha > 0$ is a tuning coefficient. Some convergence results can exist for parabolic equations with unbounded control operators [9,37]. The LQ controller associated with minimizing the cost (32) for the Galerkin approximation stabilizes the original PDE (8) if the number of modes n is large enough. Moreover it converges to the LQ-optimal controller for (8).

Simulation results. The following lemma is a direct consequence of Assumption 1.

Lemma 5 *The considered approximation scheme combined with the control law* (16) *satisfies Assumption 2.*

PROOF. Due to Jackson's inequality [29],[49, Exercise 1.5.14] there exists a constant $C_1 > 0$ such that for all $z \in H^1([0,1])$, if we denote z^n its projection on the basis defined by (31), we have for all $x \in [0,1]$

$$|z(x) - z^n(x)| \leq \frac{C\ln(n)}{\sqrt{n}} \omega(\frac{1}{n}, z),$$

where $\omega(\frac{1}{n}, z)$ denotes the modulus of continuity of z with the step $\frac{1}{n}$. As the function z is in $H^1([0,1])$ which is embedded in the Holder space $\mathcal{C}^{0,\frac{1}{2}}([0,1])$, using Morrey's inequality (2) we have

$$\omega(\frac{1}{n}, z) \leq 2\sup|z(x)| \leq 2||z||_{H^1}.$$

This yields the expected result, using the linearity of the control law (16).

This implies (Theorem 3) the convergence of the late-lumping backstepping controller introduced in (16).

We now compare the controller given by (19) with the two early-lumping controllers designed above. The real system is simulated using the same Galerkin approximation with the number of modes $N = 30$. The two control laws are designed using only $M < 30$ modes (different values of M will be used). We compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for the three different controllers. The parameter λ is chosen to be equal to 3. The numerical parameters used for the design of the control laws are $\alpha = 0.1$ and $c = 2$. The initial condition is $z(0, x) = 0.25$.

The simulations (see Figures 1-3) have comparable computation times (the late-lumping approach requires the computations of the kernels but this can be done once offline) and tend to show better performance for the late-lumping backstepping controller compared to the early-lumping backstepping controller when only a few number of modes is used. Regarding the control effort, the late lumping approach is more demanding at initial time. However, compared to the early-lumping backstepping, the control effort decreases faster. For a large number of modes, the behaviors are similar. These simulation results also tend to show that the early-lumping LQR controller has a better performance/control effort trade-off compared to the two other controllers. Although this could be expected when using a large number of modes, this still holds even with a few number of modes. Thus, for the heat equation the choice of the method does not seem to really matter and the LQR control law appears as the best solution. This is not surprising, as the approximations of the heat equation converge so quickly it is “almost” an ODE.

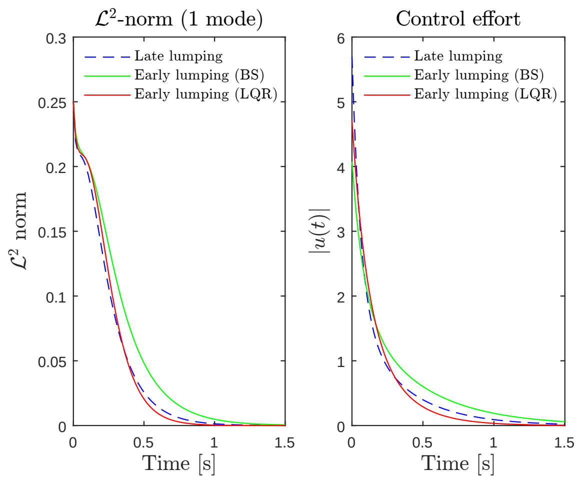

Fig. 1. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=1). The late-lumping controller and the early-lumping LQR controller have similar behavior.

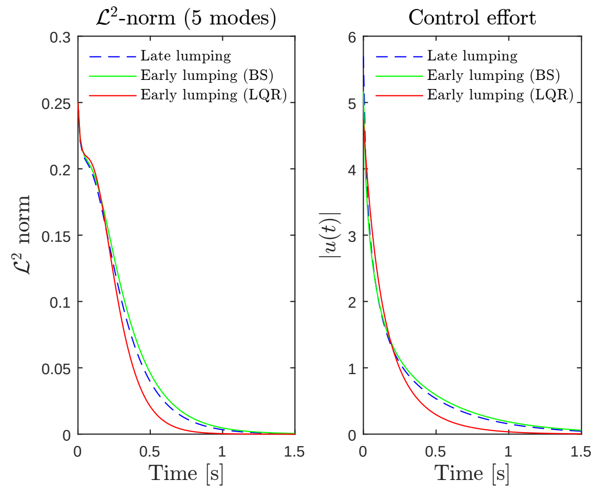

Fig. 2. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=5). The LQR controller is better in terms of performance and control effort.

4 Wave equation

A one-dimensional wave equation that is controlled from one end and contains instability at the other (free) end is considered in this section. This yields the following hyperbolic partial differential equation

$$z_{tt}(t, x) = z_{xx}(t, x), \tag{33}$$

evolving in $\{(t, x)| \quad t > 0, \ x \in [0, 1]\}$, with Neumann boundary conditions

$$z_x(t, 0) = -q z_t(t, 0), \quad z_x(t, 1) = u(t). \tag{34}$$

The parameter q is assumed different from -1 and strictly negative to avoid having an infinite number of eigenvalues in the right half plane (RHP). An infinite number of eigenvalues in the RHP would make impossible delay-robust stabilization (see [41]). The free end of the string is subject to a force proportional to the displacement, which physically may be the result of various phenomena. For instance, if the $x = 0$ end of the string is made of iron and is placed between two magnets of the same polarity, the string's end will be subject to a magnetic force which depends on its displacement. The initial condition denoted $(z^0, z_t^0) = (z(0, \cdot), z_t(0, \cdot))$ is assumed to belong to $\mathcal{H}^1([0, 1]) \times \mathcal{H}^1([0, 1])$. The system is stable but not asymptotically stable and converge to a non-zero value $(z_1, 0)$. The objective of the control design is to ensure stabilization to zero and also to improve the

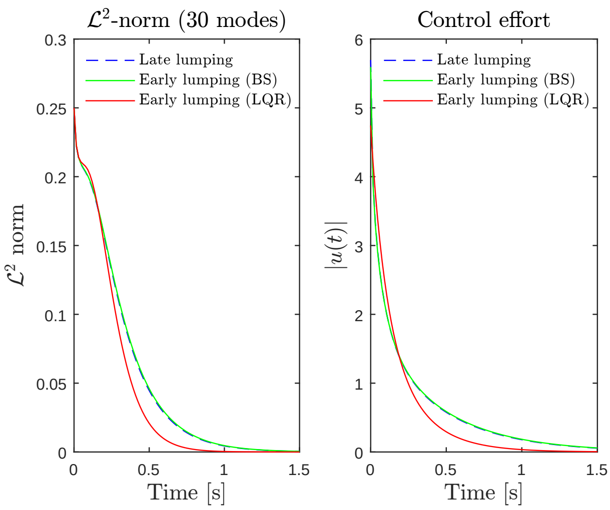

Fig. 3. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=30).The LQR controller is better in terms of performance and control effort. The two other controllers have the same behavior.

convergence rate.

Let us now give the abstract formulation of (33).

$$\frac{d}{dt}\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \mathfrak{A}_{\text{wave}}\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix}, \quad \mathfrak{B}_{\text{wave}}\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = u(t), \tag{35}$$

$$\begin{pmatrix} z(0) & \dot{z}(0) \end{pmatrix} = \begin{pmatrix} z^0 & z_t^0 \end{pmatrix}. \tag{36}$$

The operator

$$\mathfrak{A}_{\text{wave}} : \mathcal{Z} \to (\mathcal{L}^2([0,1]))^2$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_2 \\ \frac{d^2}{dx^2} z_1 \end{pmatrix}, \tag{37}$$

with

$$\mathcal{Z} = \{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1) | (z_1)_x(0) = -qz_2(0)\}$$

The operator $\mathfrak{A}_{\text{wave}}$ is densely defined. We equip $D(\mathfrak{A}_{\text{wave}})$ with the scalar product associated with the norm $\mathcal{H}^1([0,1]) \times \mathcal{H}^1([0,1])$. The operator $\mathfrak{B}_{wave}$ is defined on $\mathcal{Z}$ by

$$\mathfrak{B}_{\text{wave}}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{dz_1}{dx}(1).$$

Late-lumping controller. A late-lumping backstepping controller will be used based on that described in [55]. Consider the Volterra transformation

$$w(t,x) = -\frac{1+qc}{q^2-1}z(t,x) + \frac{q(q+c)}{q^2-1}z(t,0) - \frac{q+c}{q^2-1}\int_0^x z_t(t,\xi)d\xi, \tag{38}$$

where the constant c is an arbitrary strictly positive constant such that $c \neq 1$ and $qc \neq -1$. We have the following lemma whose proof is straightforward.

Lemma 6 *There exist constants C_1 and C_2 such that*

$$C_1(||w||_{\mathcal{H}^1([0,1])} + ||w_t||_{\mathcal{H}^1([0,1])}) \leq (||z||_{\mathcal{H}^1([0,1])} + ||z_t||_{\mathcal{H}^1([0,1])}) \leq C_2(||w||_{\mathcal{H}^1} + ||w_t||_{\mathcal{H}^1([0,1])}). \tag{39}$$

Define $K_{BS} \in \mathcal{L}(D(\mathfrak{A}_{\text{wave}}), \Re)$

$$K_{BS}z = \frac{c_0 q(q+c)}{1+qc}z(t,0) - c_0 z(t,1) - \frac{(q+c)}{1+qc}z_t(t,1) - \frac{c_0(q+c)}{1+qc}\int_0^1 z_t(t,\xi)d\xi, \tag{40}$$

$$u_{BS}(t) = K_{BS}z(t), \tag{41}$$

where c_0 is an arbitrary strictly positive coefficient (used to improve the convergence rate).

Lemma 7 *[55, Theorem 1] Transformation* (38) *along with the control law* (41) *maps the original system* (33)-(34) *to the following stable target system*

$$w_{tt}(t,x) = w_{xx}(t,x), \tag{42}$$

with Neumann boundary conditions

$$w_x(t,0) = cw_t(t,0), \quad w_x(t,1) = -c_0 w(t,1). \tag{43}$$

For any initial condition $(z(0,\cdot), z_t(0,\cdot)) \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1)$ compatible with the boundary conditions, the system (33)-(34) *along with the control law u_{BS} defined by* (41), *has a unique solution $(z, z_t) \in C([0,\infty), \mathcal{H}^1(0,1) \times \mathcal{L}^2(0,1))$ which is exponentially stable in the sense of the norm*

$$(\int_0^1 z_x(t,x)^2 dx + \int_0^1 z_t(t,x)^2 dx + z(t,1)^2)^2. \tag{44}$$

PROOF. System (42)-(43) can be obtained from (33)-(34) differentiating (38) with respect to space and time (see [55] for details). The rest of the proof is done through a Lyapunov analysis that is detailed in [55].

Consider an approximation scheme satisfying Assumption 2 and assume that only the n first modes of the state are available to design the control (where $n \in \mathbb{N}$). We denote P_n the orthogonal projection on the approximating space. This means we consider the system (33)-(34) along with the following control law

$$u^n_{BS}(t) = K_{BS} P_n z. \tag{45}$$

We then have the following theorem.

Theorem 8 *There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $(z(0,\cdot), z_t(0,\cdot)) \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1)$ compatible with the boundary conditions, the system* (33)-(34) *along with the approximated control law* (45) *is exponentially stable at the origin, $z(t,x) \equiv 0$ in the sense of the norm defined by* (44).

PROOF. This proof is similar to the one of Theorem 3. Let us consider (33)-(34) along with the control law (45). As in [55], differentiate twice (38) with respect to space to obtain

$$w_{xx}(t,x) = -\frac{1+qc}{q^2-1} z_{xx}(t,x) - \frac{q+c}{q^2-1} z_{tx}(t,x).$$

Similarly, differentiating twice (38) with respect to time and using (33), we obtain

$$\begin{aligned} w_t(t,x) &= -\frac{1}{q^2-1}(-(1+qc)z_t(t,x) + q(q+c)z_t(t,0) \\ &- (q+c)z_x(t,x) + (q+c)z_x(t,0)) \\ w_{tt}(t,x) &= -\frac{1}{q^2-1}(-(1+qc)z_{xx}(t,x) - (q+c)z_{tx}(t,0)) \end{aligned}$$

This yields the target system

$$w_{tt}(t,x) = w_{xx}(t,x), \tag{46}$$

with the following Neumann boundary conditions

$$w_x(t,0) = c w_t(t,0), \tag{47}$$

$$\begin{aligned} w_x(t,1) &= \frac{-1}{q^2-1}((1+qc)z_x(t,1) - (q+c)z_t(t,1)) \\ &= \frac{-1}{q^2-1}((1+qc)u(t) - (q+c)z_t(t,1)) \\ &= -c_0 w(t,1) + \frac{1+qc}{q^2-1}(u_{BS}(t) - u^n_{BS}(t)). \end{aligned} \tag{48}$$

Using Assumption 2 and (39), we obtain

$$|K_{BS}P^n z - K_{BS} z| \leq C_n C_2(||(w,w_t)||_{\mathcal{H}^1([0,1])}). \tag{49}$$

We now prove the stability of the system (46)-(48) with a Lyapunov analysis. Inspired by [55], let us consider the Lyapunov function candidate

$$\begin{aligned} V(t) &= \frac{1}{2}\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx + \frac{c_0}{2} w(t,1)^2 \\ &+ \delta \int_0^1 (x-2) w_x(t,x) w_t(t,x) dx \end{aligned} \tag{50}$$

Using the Cauchy Schwartz and Young's inequalities, one can show that for sufficiently small δ, there exist $m_1 > 0$ and $m_2 > 0$ such that

$$m_1 U \leq V \leq m_2 U, \tag{51}$$

where $U = ||w_x||^2 + ||w_t||^2 + w^2(1)$. In the following, we will assume that δ is small enough so that (51) is satisfied. In particular, we assume that $\delta \leq \frac{c}{1+c^2}$. For such a δ, V is positive definite. Differentiating V with respect to time and integrating by part yields

$$\begin{aligned} \dot{V}(t) &= \int_0^1 w_x(t,x) w_{tx}(t,x) + w_t(t,x) w_{xx}(t,x) dx \\ &+ \delta \int_0^1 (x-2) w_{xt} w_t + (x-2) w_x w_{xx} dx + c_0 w_t(t,1) w(t,1) \\ &= -w_t(t,1) w_x(t,1) + w_t(t,1) w_x(t,1) - w_t(t,0) w_x(t,0) \\ &+ \frac{1+qc}{q^2-1}(K_{BS} z - K_{BS} P^n z) w_t(t,1) \\ &+ \frac{\delta}{2}(-w_x^2(t,1) + 2w_x^2(t,0) - w_t^2(t,1) + 2w_t^2(t,0)) \\ &- \frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x) dx) \end{aligned}$$

Thus,

$$\begin{aligned} \dot{V} &\leq -\frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) - (c - \delta(1+c^2)) w_t^2(t,0) \\ &- c_0^2 \frac{\delta}{2} w^2(t,1) - \frac{\delta}{2}(\frac{1+qc}{q^2-1})^2 (K_{BS} z - K_{BS} P^n z)^2 + (K_{BS} z - \\ &K_{BS} P^n z) \frac{1+qc}{q^2-1} w_t(t,1) + c_0 \delta \frac{1+qc}{q^2-1} \\ &(K_{BS} z - K_{BS} P^n z) w(t,1) \\ &\leq -\frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) + c_0^2 \frac{\delta}{4} w^2(t,1) \\ &+ \frac{\delta}{4} ||w_t||^2_{\mathcal{H}^1([0,1])} + (\frac{1+qc}{q^2-1})^2 (\frac{\alpha^2}{\delta} + \frac{\delta}{2})(K_{BS} z - K_{BS} P^n z)^2, \end{aligned}$$

where we have used (2) and Young's inequality in the last line. Using (49) leads to

$$\dot{V} \leq -\frac{\delta}{4}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) - c_0^2\frac{\delta}{4}w^2(t,1)$$
$$C_n^2 C_2^2(\frac{1+qc}{q^2-1})^2(\frac{\alpha^2}{\delta}+\frac{\delta}{2})(||w||_{\mathcal{H}^1([0,1])}+||w_t||_{\mathcal{H}^1([0,1])})^2.$$

Since C_n converges to zero, using Young's and Poincarre's inequality, there exists $M > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dot{V}(t) \leq -MV(t) \tag{52}$$

This implies the exponential stability of the system (21)-(23) in the sense of the norm defined in (44). Due to (39), the original state z has the same properties. This concludes the proof.

Simulations To implement the system in simulation, and to design early-lumping controllers, a Galerkin approximation based on eigenfunctions is again used. The approximation scheme is based on a Riesz basis. Consider the family ϕ_k defined for all $k = 1, 2\ldots,$ by

$$\phi_k(x) = \begin{pmatrix}\phi_k^1(x)\\ \phi_k^2(x)\end{pmatrix} = \begin{pmatrix}\frac{1}{k\pi}\cos(k\pi x)\\ \cos(k\pi x)\end{pmatrix} \tag{53}$$

Define ϕ_0 and $\phi_{0,1}$ as

$$\phi_0(x) = \begin{pmatrix}1 & 0\end{pmatrix}^T, \quad \phi_{0,1}(x) = \begin{pmatrix}0 & 1\end{pmatrix}^T \tag{54}$$

The family $\{\phi_{0,1}, \phi_k,\ k \in \mathbb{N}\}$ forms a Riesz basis for the state space $H^1(0,1) \times L^2(0,1)$ [20]. Let us consider $n \in \mathbb{N}$, we define $\chi_n = \text{span}\{\text{span}_{i=-n,\cdots,n}\{\phi_i\}, \phi_{0,1}\}$ and denote P_n, the orthogonal projection onto χ_n. The space χ_n is equipped with the $\mathcal{H}^1$-norm.

Due to Jackson's theorem, this approximation scheme satisfies Assumption 2. This implies convergence of the late-lumping backstepping controller.

Using this approximation scheme, it is straightforward to design an early-lumping controller that places the poles at the same location as the poles of the approximated target system and an early-lumping LQR controller, following a procedure identical to the one described for the heat equation. However, since the underlying semigroup is not analytic, the convergence or performance of the controllers on the PDE is not guaranteed.

The late lumping backstepping controller (45) is compared with these two early-lumping controllers. The real system is simulated using approximation with a number of modes $N = 40$. The control laws are designed using $M < 40$ modes. We compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for the three controllers. The parameters are chosen as follow: $q = -\frac{1}{2}$, $\alpha = 0.5$, $c = 0.8$ and $c_0 = 1.05$. The choice of these parameters is motivated by an effort to have similar performance in terms of the $\mathcal{L}^2$ norm of the late-lumping backstepping controller and the early-lumping LQR controller when only one mode is used. The initial conditions are defined by $z^0(x) = 1 + \frac{1}{N\pi}\cos(N\pi x)$ and $z_t^0(x) = 1 + \cos(N\pi x)$. Comparing Figures 4-6, it is apparent that the early-lumping backstepping controller and the early-lumping LQR controller have similar behavior in this respect. However, when few modes are used, the late-lumping backstepping controller achieves similar performance but is more demanding in term of control efforts at initial time (although it requires less control effort in later times).

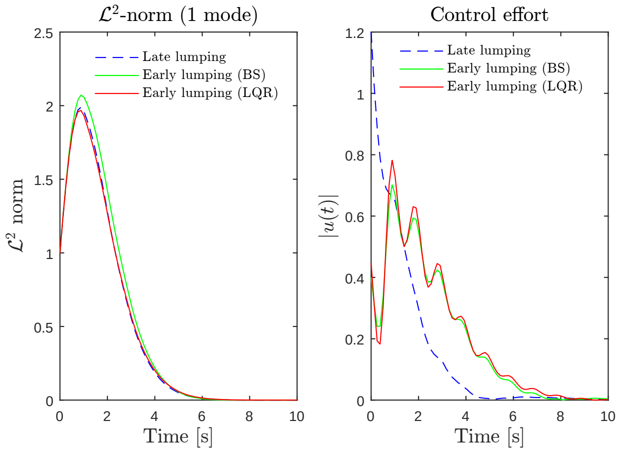

Fig. 4. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=1). The three controllers have a similar convergence rate. The late-lumping controller is better in terms of control effort. The two early-lumping controllers have similar behavior.

5 Two linear coupled hyperbolic PDEs

We consider in this section two linear first-order hyperbolic PDEs which appear for instance in Saint-Venant equations, heat exchangers equations and other linear hyperbolic balance laws (see [11]):

$$w_t(t,x) + \lambda w_x(t,x) = \sigma^{+-}z(t,x) \tag{55}$$
$$z_t(t,x) - \mu z_x(t,x) = \sigma^{-+}w(t,x), \tag{56}$$

evolving in $\{(t,x)|\quad t>0,\ x\in[0,1]\}$, with the following linear boundary conditions

$$w(t,0) = qz(t,0), \quad z(t,1) = u(t), \tag{57}$$

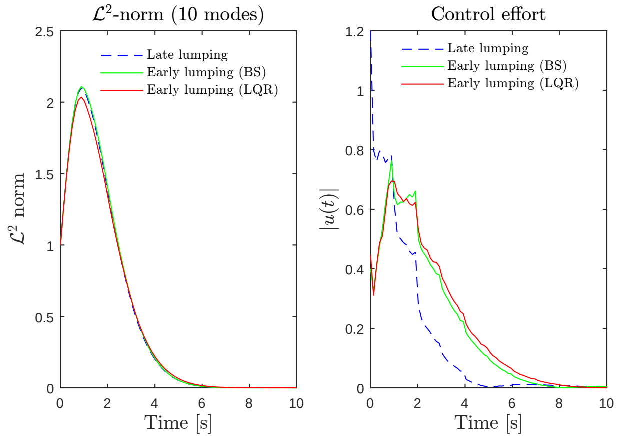

Fig. 5. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=10). The three controllers have a similar convergence rate. The late-lumping controller is better in terms of control effort. The two early-lumping controllers have similar behavior.

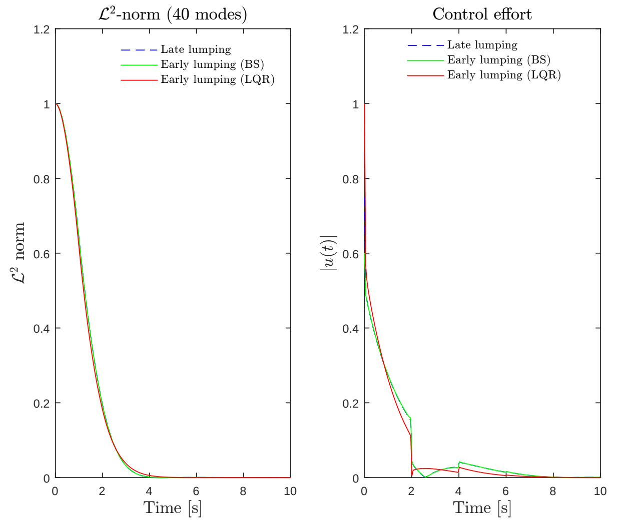

Fig. 6. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=40). The three controllers have a similar behavior

with constant coupling terms σ^{-+} and σ^{+-} and constant velocities λ and μ. The boundary coupling term q is assumed non null. Depending on the value of σ^{+-}, σ^{-+} and q, the system may be unstable [11] (the eigenvalues can curve over). The initial conditions denoted w_0 and z_0 are assumed to belong to $\mathcal{H}^1([0,1])$ and satisfy the compatibility conditions. As proved in [5], the system (55)-(57) is delay-robustly stabilizable and has a finite number of poles in the right half-plane.

Late-lumping controller. In [18] a control law that stabilizes the original infinite-dimensional system (55)-(57) using the backstepping method [33] is derived. Consider the Volterra transformation

$$\gamma(t,x) = w(t,x) - \int_0^x (K^{uu}(x,\xi)w(\xi) + K^{uv}(x,\xi)z(\xi))d\xi, \quad (58)$$

$$\beta(t,x) = z(t,x) - \int_0^x (K^{vu}(x,\xi)w(\xi) + K^{vv}(x,\xi)z(\xi))d\xi, \quad (59)$$

where the kernels $K^{uu}, K^{uv}, K^{vu}, K^{vv}$ are defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \quad \xi \leq x\}$ by a set of hyperbolic PDEs (see [18]). We have the following lemma, whose proof is straightforward.

Lemma 9 *There exist constants C_1 and C_2 such that*

$$C_1(||\gamma||_{\mathcal{H}^1([0,1])} + ||\beta||_{\mathcal{H}^1([0,1])}) \leq ||z||_{\mathcal{H}^1([0,1])} + ||w||_{\mathcal{H}^1([0,1])}, \quad (60)$$

$$(||z||_{\mathcal{H}^1([0,1])} + ||w||_{\mathcal{H}^1([0,1])}) \leq C_2(||\gamma||_{\mathcal{H}^1([0,1])} + ||\beta||_{\mathcal{H}^1([0,1])}). \quad (61)$$

Define the control law

$$u_{BS}(t) = K_{BS}\begin{pmatrix} w & z \end{pmatrix}^T, \quad (62)$$

with

$$K_{BS}\begin{pmatrix} w \\ z \end{pmatrix} = \int_0^1 K^{vu}(1,\xi)w(\xi) + K^{vv}(1,\xi)z(\xi)d\xi,$$

we have the following lemma.

Lemma 10 *[18, Theorems 3.2] Transformation (58)-(59) along with the control law (62) maps the original system (55)-(57) to the following stable target system*

$$\gamma_t(t,x) = -\lambda\gamma_x(t,x) \quad (63)$$

$$\beta_t(t,x) = \mu\beta_x(t,x) \quad (64)$$

with the following boundary conditions

$$\gamma(t,0) = q\beta(t,0), \quad \beta(t,1) = 0. \quad (65)$$

For any initial condition $(w(0,\cdot), z(0,\cdot)) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ that satisfies the compatibility conditions, the system (55)-(57) along with the control law u_{BS} defined by (62), has a unique solution $(w,z) \in C([0,\infty), \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1))$ which is exponentially stable in the sense of the $\mathcal{L}^2$-norm. As proved in [18], using the control law (62), the system actually reaches its zero equilibrium in finite time $t_f = \frac{1}{\lambda} + \frac{1}{\mu}$.

Let us consider an approximation scheme satisfying Assumption 2. Denoting by P_n the projection on the approximating space, consider the system (55)-(57) along with the control law

$$u_{BS}^n(t) = K_{BS}P_n \begin{pmatrix} w & z \end{pmatrix}^T = K_{BS}P^n \begin{pmatrix} w & z \end{pmatrix}^T. \quad (66)$$

We then have the following theorem.

Theorem 11 *There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $z_0 \in \mathcal{H}^1([0,1])$, the system* (55)-(57) *along with the approximated control law* (66) *is exponentially stable at the origin.*

PROOF. This proof is similar to that of Theorem 8. Let us consider (55)-(57) along with the control law (66). Using the results from [5], this system can be mapped to

$$\gamma_t(t,x) = -\lambda\gamma_x(t,x) \quad \beta_t(t,x) = \mu\beta_x(t,x) \quad (67)$$

$$\gamma(t,0) = q\beta(t,0), \quad \beta(t,1) = (K_{BS}P^n - K_{BS}) \begin{pmatrix} w & z \end{pmatrix}^T. \quad (68)$$

Since the approximation scheme satisfies Assumption 2, we obtain

$$|(K_{BS}P^n - K_{BS}) \begin{pmatrix} w \\ z \end{pmatrix} | \leq C_n C_2(||(\gamma,\beta)||_{\mathcal{H}^1}). \quad (69)$$

We now prove the stability of the system (67)-(68) with a Lyapunov analysis. Let us consider the Lyapunov function candidate

$$V(t) = \int_0^1 \frac{1}{\lambda} e^{-\nu x}\gamma^2(t,x) + \frac{q^2}{\mu} e^{\nu x}\beta^2(t,x)dx \quad (70)$$

where ν is a strictly positive parameters. Using the Cauchy Schwartz and Young's inequalities, one can show that there exist $m_1 > 0$ and $m_2 > 0$ such that

$$m_1(||\gamma||^2 + ||\beta||^2) \leq V \leq m_2(||\gamma||^2 + ||\beta||^2). \quad (71)$$

Differentiating V with respect to time and integrating by part yields

$$\begin{aligned} \dot{V}(t) = & -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ & + [-e^{-\nu x}\gamma^2(t,x) + q^2 e^{\nu x}\beta^2(t,x)]_0^1 \\ \leq & -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ & + q^2 e^{\nu}((K_{BS}P^n - K_{BS}) \begin{pmatrix} w \\ z \end{pmatrix})^2 \\ \leq & -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ & + C_n^2 C_2^2 q^2 e^{\nu}(||\gamma||^2 + ||\beta||^2). \end{aligned} \quad (72)$$

Since C_n converges to zero, we easily obtain using (69) that there exists $M > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dot{V}(t) \leq -MV(t) \quad (73)$$

This implies the exponential stability of the system (67)-(68). Due to (61), the original state (z,w) has the same properties. This concludes the proof.

Simulations. The real system is simulated using the Galerkin approximation with a number of modes $N = 40$. The basis we use for the approximating spaces is the same as the one introduced in the previous section (i.e the family ϕ_k defined in equation (53)-(54)).

The controller (66) is compared to two early-lumping controllers, designed similarly to those in the previous sections. The control laws are designed using only $M < 40$ modes. The system parameters are chosen as follow: $\sigma^{+-} = 0$, $\sigma^{-+} = 1$, $q = 1$. The initial conditions are defined by $w_0(x) = z_0(x) = 1$. The LQR early-lumping controller did not stabilize the system when using more than 10 modes. Therefore, in Figure 7-8, we compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for only the early-lumping backstepping controller and the late-lumping backstepping controller. The late-lumping backstepping controller still stabilizes the system in **finite-time** even with a few number of modes. The early-lumping backstepping controller also stabilizes the system (even with one mode) but the performance is not as good. However, when the number of modes increases, we obtain similar results in term of performance and control efforts for the two controllers. For this class of hyperbolic systems, it seems that the late-lumping approach allows better performance with low order controllers. Moreover, for these systems, early lumping is problematic as the convergence may require an important number of modes or may not exist (the LQR control law does not convergence for low values

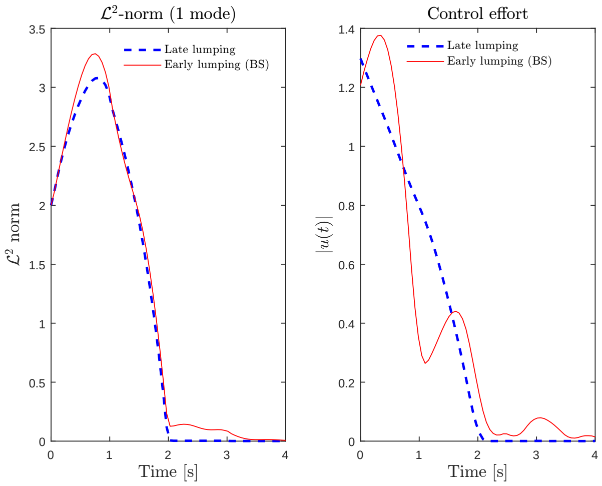

Fig. 7. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (55)-(57), N=40, M=1). The late-lumping controller has a better convergence rate and a better control effort. The early-lumping LQR controller did not converge.

of M). Of course, a more complete analysis (computing the transfer functions, analyzing the robustness margins,...) would be necessary.

Note that this Galerkin approximation may not be the best method to approximate the control law. More precisely, it has been proved in [5,11] that the considered class of hyperbolic equations can be rewritten as neutral equations with distributed delays. As multiple accurate solvers exist for such equations, it may be interesting to use them. The Galerkin approximation has been chosen here to fairly compare the early-lumping controller with the late-lumping one.

6 Remarks on observer design

All the backstepping control laws presented above are state-feedback control laws and require the value of the (approximated) state for all $x \in [0,1]$. However, the measurements, in distributed parameter systems, are rarely available across the domain. It is more common for the sensors to be placed only at the boundaries. Consequently, to envision industrial applications, for each problem presented above a state-observer has to be designed. The corresponding state estimation can then be used to derive an output-feedback control law. As these observers are usually designed as the duals to the backstepping controllers presented above, they are defined through PDEs that are similar to the ones describing the original systems. Regarding the late-lumping approximation, the solutions of these observer systems have to be approximated. Thus, the second line of Assumption 2

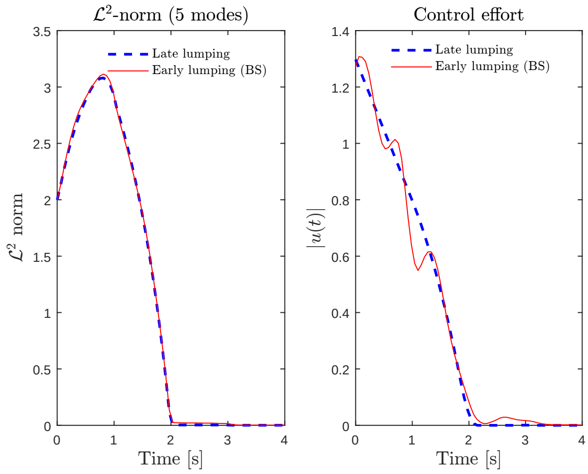

Fig. 8. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (55)-(57), N=40, M=5). The late-lumping controller has a slightly better control effort. The early-lumping LQR controller did not converge.

has to be changed by

$$\forall n \in \mathbb{N}, ||K\hat{z}_n - Kz|| \leq C_n ||z||_{(\mathcal{H}^1([0,1]))^p}, \tag{74}$$

where $\hat{z}_n$ is the approximation of order n of the observer state $\hat{z}$. Proving that backstepping observers satisfy such an assumption may not be an easy task and is out of the scope of this paper.

In this section, we just give some remarks for reflection in perspective of future work.

Let us consider the system (55) along with the boundary conditions (57). We assume that only boundary measurements at the right boundary of the spatial domain are available (i.e. we measure $w(t,1)$). For such a system, the following backstepping state observer has been designed in [61]:

$$\begin{aligned} \hat{w}_t(t,x) + \lambda \hat{w}_x(t,x) &= \sigma^{+-}\hat{z}(t,x) \\ &\quad + p_1(x)(w(t,1) - \hat{w}(t,1)), \quad (75) \\ \hat{z}_t(t,x) - \mu \hat{z}_x(t,x) &= \sigma^{+-}\hat{w}(t,x) \\ &\quad + p_2(x)(w(t,1) - \hat{w}(t,1)), \quad (76) \end{aligned}$$

with the boundary conditions

$$\hat{w}(t,0) = q\hat{z}(t,0), \quad \hat{z}(t,1) = u(t), \tag{77}$$

where the output injection gains p_1 and p_2 are continuous functions defined in [61]. It has been proved in [61] that for any control law $u(t)$, the solutions of (75)-(77) exponentially converge to the solutions of (55)-(57) (they actually converges in finite time). Moreover, combining this observer with the control law (62) leads to the design of a stabilizing output feedback law [61, Theorem 2]. The observer system (75)-(77) can be approximated using the Galerkin approximation based on the Riesz basis (53)-(54). The corresponding state is denoted $(\hat{w}_n, \hat{z}_n)$. Let us now consider the abstract formulation of (55)-(57):

$$\frac{d}{dt}\begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \mathfrak{A}\begin{pmatrix} w(t) \\ z(t) \end{pmatrix}, \quad \mathfrak{B}\begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = u(t), \tag{78}$$

where defining

$$\mathcal{Z} = \{(w, z) \in (\mathcal{H}^1([0, 1]))^2 | w(0) = qz(0)\},$$

the operator $\mathfrak{A} : \mathcal{Z} \to (\mathcal{L}^2([0, 1]))^2$ is

$$\mathfrak{A}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -\lambda \frac{d}{dx} w + \sigma^{+-} z \\ \mu \frac{d}{dx} z + \sigma^{-+} w. \end{pmatrix}.$$

The operator $\mathfrak{B} : \mathcal{Z} \to \mathbb{R}$ is defined by

$$\mathfrak{B}\begin{pmatrix} w \\ z \end{pmatrix} = z(t, 1).$$

Once projected on on the Riesz basis (53)-(54), the observer system (75)-(77) admits the following abstract representation

$$\frac{d}{dt}\begin{pmatrix} \hat{w}_n \\ \hat{z}_n \end{pmatrix} = \mathfrak{F}_n \mathfrak{C}\begin{pmatrix} w \\ z \end{pmatrix} + (\mathfrak{A}_n - \mathfrak{F}_n \mathfrak{C})\begin{pmatrix} \hat{w}_n \\ \hat{z}_n \end{pmatrix} \tag{79}$$

$$\mathfrak{B}_n \begin{pmatrix} \hat{w}_n \\ \hat{z}_n \end{pmatrix} = u(t), \tag{80}$$

where $\mathfrak{A}_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ is the Galerkin approximation of the operator $\mathfrak{A}$, where $\mathfrak{C}$ is the output operator (i.e. $\mathfrak{C}\begin{pmatrix} w \\ z \end{pmatrix} = w(1)$), and where $\mathfrak{F}$ corresponds to the projection of the output injection operator $\mathfrak{F}$ that appears in (75)-(77) and which is defined by $\mathfrak{F} : y \in \mathbb{R} \to \begin{pmatrix} p_1(x)y(t) \\ p_2(x)y(t) \end{pmatrix}$. It is straightforward to prove that $\mathfrak{F}_n$ uniformly converges to $\mathfrak{F}$. This is not however sufficient to conclude to the convergence of the late-lumping observer.

As an illustration we have pictured in Figure 9 the evolution of the state of the real system (55)-(57) and of the state of the approximated observer system (79) without any actuation for $M = 5$. The system parameters are chosen as follow: $\sigma^{+-} = 0.4$, $\sigma^{-+} = 1$, $q = 1$. The initial conditions of the real system are defined by

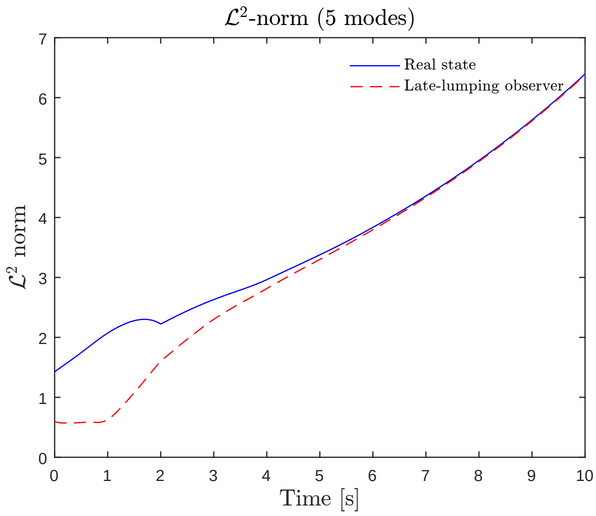

Fig. 9. Time evolution of the $\mathcal{L}^2$-norm of the state of the real system (55)-(57) and of the state of the approximated observer system (79) without any actuation (M=5).

$w_0(x) = z_0(x) = 1$, while the ones for the observer are arbitrarily chosen. It appears that, for this example, the observer state converges to the real state. This motivates further investigations. Showing convergence of the late-lumping observers, as well as the stability properties of the output feedback control law, will be the purpose of future work.

7 Concluding remarks

In this paper we have considered different systems that can be stabilized by a backstepping control law. We have proved that, under some assumptions, an finite-dimensional approximation to the controller still provides exponential stabilization. This was done through Lyapunov analysis, using the backstepping method as an analysis tool.

The late lumping backstepping controllers were compared in simulations with early-lumping controllers. Note that stability of the closed loop systems with early-lumping controllers has not been established for the two hyperbolic systems that were considered. All controllers performed well for the heat equation but for this example, a classical LQR early-lumping controller performs slightly better than the backstepping controller. This is not surprising, as the approximations of the heat equation converge so quickly that it is "almost" an ODE). For the wave equation (section 4) the late lumping controller was able to stabilize the system with a smaller number of modes than the early-lumping controllers. However, this controller appears more demanding in term of control efforts at initial time (even if the control effort decreases faster using the late lumping approach) than early-lumping controllers. For the hyperbolic system considered in section 5, performance was tuned to be similar but the late lumping controller required less control effort when a small number of modes was used.

The results presented here raise important questions about the comparison between late-lumping and early-lumping controllers. In particular, various criteria, such as transient behaviour, robustness margins, and disturbance rejection, should be considered for a fair comparison between different approaches. A current limitation to a deeper analysis is the lack of results for stability of early-lumping controller design for unbounded control operators.

This work is a first step towards practical applications of backstepping controllers. The question of the late-lumping backstepping controller-observer or the extension to systems of larger dimensions (using the results of [1,56,57]) has not been considered in this paper. As discussed in Section 6, the methods proposed in this paper might be extended to observer design as well as output feedback control. This will be the focus of future work.

References

[1] O.M Aamo, M. Krstić, and T.R Bewley. Control of mixing by boundary feedback in 2D channel flow. *Automatica*, 39(9):1597–1606, 2003.

[2] O.M Aamo, A. Smyshlyaev, and M. Krstic. Boundary control of the linearized Ginzburg–Landau model of vortex shedding. *SIAM journal on control and optimization*, 43(6):1953–1971, 2005.

[3] O.M Aamo, A Smyshlyaev, M. Krstic, and B.A Foss. Stabilization of a Ginzburg-Landau model of vortex shedding by output feedback boundary control. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 3, pages 2409–2416. IEEE, 2004.

[4] Pedro Ascencio, Alessandro Astolfi, and Thomas Parisini. Backstepping pde design: A convex optimization approach. *IEEE Transactions on Automatic Control*, 2017.

[5] J. Auriol, U. J. Aarsnes, P. Martin, and F. Di Meglio. Delay-robust control design for two heterodirectional linear coupled. *Submitted to IEEE-TAC*, 2017.

[6] J. Auriol and F. Di Meglio. Minimum time control of heterodirectional linear coupled hyperbolic PDEs. *Automatica*, 71:300–307, 2016.

[7] Antonello Baccoli, Alessandro Pisano, and Yury Orlov. Boundary control of coupled reaction–diffusion processes with constant parameters. *Automatica*, 54:80–90, 2015.

[8] A. Balogh and M. Krstic. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. *European journal of control*, 8(2):165–175, 2002.

[9] H. T. Banks and K Ito. Approximation in LQR problems for infinite dimensional systems with unbounded input operators. *Journal of Mathematical Systems Estimation and Control*, 7:119–119, 1997.

[10] H. T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM Journal on Control and Optimization*, 22(5):684–698, 1984.

[11] G. Bastin and J-M Coron. *Stability and boundary stabilization of 1-d hyperbolic systems*. Springer, 2016.

[12] D.M. Bošković, A. Balogh, and M. Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. *Mathematics of Control, Signals, and Systems (MCSS)*, 16(1):44–75, 2003.

[13] D.M Boskovic, M. Krstic, and W. Liu. Boundary control of an unstable heat equation via measurement of domain-averaged temperature. *IEEE Transactions on Automatic Control*, 46(12):2022–2028, 2001.

[14] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[15] J.A. Burns, K. Ito, and G.Propst. On non-convergence of adjoint semigroups for control systems with delays. *SIAM Jour. Control and Optim.*, 26(6):1442–1454, 1988.

[16] H.S. Carslaw. *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, volume 2. Macmillan London, 1921.

[17] A. Cheng and K. A. Morris. Well-posedness of boundary control systems. *SIAM Jour. Control & Optim.*, 42(4):1244–1265, 2003.

[18] J-M Coron, R. Vazquez, M. Krstic, and G. Bastin. Local exponential h^2 stabilization of a 2× 2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization*, 51(3):2005–2035, 2013.

[19] R. F. Curtain and D. Salamon. Finite-dimensional compensators for infinite-dimensional systems with unbounded input operators. *SIAM journal on control and optimization*, 24(4):797–816, 1986.

[20] R.F Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.

[21] Joachim Deutscher. A backstepping approach to the output regulation of boundary controlled parabolic pdes. *Automatica*, 57:56 – 64, 2015.

[22] Joachim Deutscher. Finite-time output regulation for linear 22 hyperbolic systems using backstepping. *Automatica*, 75:54 – 62, 2017.

[23] Hongyinping Feng and Bao-Zhu Guo. A new active disturbance rejection control to output feedback stabilization for a one-dimensional anti-stable wave equation with disturbance. *IEEE Transactions on Automatic Control*, 62(8):3774–3787, 2017.

[24] Hongyinping Feng and Bao-Zhu Guo. Observer design and exponential stabilization for wave equation in energy space by boundary displacement measurement only. *IEEE Transactions on Automatic Control*, 62(3):1438–1444, 2017.

[25] G. Hagen, I. Mezic, B. Bamieh, and K. Zhang. Modelling and control of axial compressors via air injection. In *American Control Conference, 1999. Proceedings of the 1999*, volume 4, pages 2703–2707. IEEE, 1999.

[26] L.F Ho and D.L. Russell. Admissible input elements for systems in Hilbert space and a carleson measure criterion. *SIAM Journal on Control and Optimization*, 21(4):614–640, 1983.

[27] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, 61(11):3301–3314, 2016.

[28] K. Ito and F. Kappel. The Trotter-Kato theorem and approximation of PDEs. *Mathematics of Computation of the American Mathematical Society*, 67(221):21–44, 1998.

[29] D. Jackson. *The theory of approximation*, volume 11. American Mathematical Soc., 1930.

[30] Lukas Jadachowski, Thomas Meurer, and Andreas Kugi. An efficient implementation of backstepping observers for time-varying parabolic pdes. *IFAC Proceedings Volumes*, 45(2):798 – 803, 2012. 7th Vienna International Conference on Mathematical Modelling.

[31] T. Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[32] M. Krstic, B-Z. Guo, A. Balogh, and A. Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica*, 44(1):63–74, 2008.

[33] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*, volume 16. Siam, 2008.

[34] I. Lasiecka. Ritz–galerkin approximation of the time optimal boundary control problem for parabolic systems with dirichlet boundary conditions. *SIAM journal on control and optimization*, 22(3):477–500, 1984.

[35] I. Lasiecka. Approximations of riccati equation for abstract boundary control problems applications to hyperbolic systems. *Numerical functional analysis and optimization*, 8(3-4):207–243, 1986.

[36] I. Lasiecka and R. Triggiani. Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations. *Journal of Differential Equations*, 47(2):246–272, 1983.

[37] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume I. Cambridge University Press, 2000.

[38] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume II. Cambridge University Press, 2000.

[39] J. L. Lions. *Optimal control of systems governed by partial differential equations problèmes aux limites*. Springer, 1971.

[40] X. Litrico and V. Fromion. Boundary control of hyperbolic conservation laws using a frequency domain approach. *Automatica*, 45(3):647–656, 2009.

[41] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM Journal on Control and Optimization*, 34(2):572–600, 1996.

[42] T. Meurer and M. Zeitz. Flatness-based feedback control of diffusion-convection-reaction systems via k-summable power series. *IFAC Proceedings Volumes*, 37(13):177–182, 2004.

[43] K. A. Morris. H_∞ output feedback control of infinite-dimensional systems via approximation. *Systems and Control Letters*, 44(3):211–217, 2001.

[44] K.A. Morris. Convergence of controllers designed using state-space methods. *IEEE Trans. Auto. Cont.*, 39(10):2100–2104, 1994.

[45] K.A. Morris. Design of finite-dimensional controllers for infinite-dimensional systems by approximation. *Jour. of Mathematical Systems, Estimation and Control*, 4(2):1–30, 1994.

[46] K.A Morris. Control of systems governed by partial differential equations. *The Control Theory Handbook*, 2010.

[47] P.A Orner and A.M Foster. A design procedure for a class of distributed parameter control systems. *Journal of Dynamic Systems, Measurement, and Control*, 93(2):86–92, 1971.

[48] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[49] M.A Pinsky. *Introduction to Fourier analysis and wavelets*, volume 102. American Mathematical Soc., 2002.

[50] C. Sagert, F. Di Meglio, M. Krstic, and P. Rouchon. Backstepping and flatness approaches for stabilization of the stick-slip phenomenon for drilling. *IFAC Proceedings Volumes*, 46(2):779–784, 2013.

[51] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of the American Mathematical Society*, 300(2):383–431, 1987.

[52] R.E. Showalter. *Hilbert space methods in partial differential equations*. Courier Corporation, 2010.

[53] A. Smyshlyaev and M. Krstic. Closed-form boundary state feedbacks for a class of 1-d partial integro-differential equations. *IEEE Transactions on Automatic Control*, 49(12):2185–2202, 2004.

[54] A. Smyshlyaev and M. Krstic. Backstepping observers for a class of parabolic PDEs. *Systems & Control Letters*, 54(7):613–625, 2005.

[55] A. Smyshlyaev and M. Krstic. Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary. *Systems & Control Letter*, 58(8):617–623, 2009.

[56] R. Vazquez and M. Krstic. Explicit integral operator feedback for local stabilization of nonlinear thermal convection loop pdes. *Systems & control letters*, 55(8):624–632, 2006.

[57] R. Vazquez and M. Krstic. A closed-form feedback controller for stabilization of the linearized 2-d navier–stokes poiseuille system. *IEEE Transactions on Automatic Control*, 52(12):2298–2312, 2007.

[58] R. Vazquez and M. Krstic. Control of 1-d parabolic PDEs with volterra nonlinearities, part i: design. *Automatica*, 44(11):2778–2790, 2008.

[59] R. Vazquez and M. Krstic. Boundary control of coupled reaction-diffusion systems with spatially-varying reaction. *IFAC-PapersOnLine*, 49(8):222–227, 2016.

[60] R. Vazquez and M. Krstic. Taking a step back: A brief history of PDE backstepping. *IFAC Proceedings Volumes*, 2017.

[61] Rafael Vazquez, Miroslav Krstic, and Jean-Michel Coron. Backstepping boundary stabilization and state estimation of a 2× 2 linear hyperbolic system. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 4937–4942. IEEE, 2011.

[62] D. Washburn. A bound on the boundary input map for parabolic equations with application to time optimal control. *SIAM Journal on Control and Optimization*, 17(5):652–671, 1979.

[63] F. Woittennek, M. Riesmeier, and S. Ecklebe. On approximation and implementation of transformation based feedback laws for distributed parameter systems. *IFAC Proceedings Volumes*, 2017.

[64] C. Xu, E. Schuster, R. Vazquez, and M. Krstic. Stabilization of linearized 2D magnetohydrodynamic channel flow by backstepping boundary control. *Systems & control letters*, 57(10):805–812, 2008.