Some Problems in Multiplicative and Additive Number Theory

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2020

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we obtain several results in number theory.

Let $k \geqslant 1$ be a natural number and $\omega_k(n)$ denote the number of distinct prime factors of a natural number n with multiplicity k. We estimate the first and the second moments of the functions ω_k, $k \geqslant 1$. Moreover, we prove that the function $\omega_1(n)$ has normal order $\log \log n$ and the functions $\omega_k(n)$ with $k \geqslant 2$ do not have normal order $F(n)$ for any nondecreasing nonnegative function F.

Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Define

$$\mathcal{M}_p(-s, \chi) := \frac{2}{p-1} \sum_{\substack{\psi \ (\mathrm{mod}\ p) \\ \psi(-1)=-1}} L(1, \psi) L(-s, \chi \overline{\psi}),$$

$$\mathcal{A}_p(\chi) := \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \sum_{\substack{1 \leqslant n_1, n_2 \leqslant N \\ \chi(n_1)=\chi(n_2)}} 1,$$

$$\Delta(s, \chi) := \sum_{n=2}^{\infty} \frac{\chi(n)\Delta(n)}{n^s}, \qquad (\Re(s) > 2)$$

where $\Delta(n)$ is the error term in the Prime Number Theorem. We investigate the mean value $\mathcal{M}_p(-s, \chi)$ for $\Re(s) > -1$, give an exact formula for the average $\mathcal{A}_p(\chi)$ and obtain the meromorphic continuation of the function $\Delta(s, \chi)$ to the region $\Re(s) > 1/2$.

Acknowledgements

Dedication

I dedicate this thesis to mathematics itself which is one of the few intellectual activities that unite not only nationalities and genders but also the short time periods we live in.

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Chapter 1

Introduction and Statements of Results

In this chapter, we introduce the topics studied in Chapters 2-4 and state the main results obtained in this thesis.

1.1 Number of Distinct Prime Factors

Let $\omega(n)$ be the number of distinct prime factors of a natural number n. The behaviour of the function $\omega(n)$ on average is understood by the estimate, [21, p. 58],

$$\sum_{n \leqslant x} \omega(n) = x \log \log x + bx + O\left(\frac{x}{\log x}\right) \tag{1.1.1}$$

where b is a constant. Thus, the behaviour of $\omega(n)$ on average is similar to $\log \log n$ and a natural question to ask is how large the deviation $|\omega(n) - \log \log n|$ on average can be. For this purpose, the concept of normal order is defined as follows, [10]. Let $f, F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be functions such that F is nondecreasing. Then $f(n)$ is said to have *normal order $F(n)$* if for any $\epsilon > 0$, the number of $n \leqslant x$ that do not satisfy the inequality

$$(1 - \epsilon)F(n) < f(n) < (1 + \epsilon)F(n)$$

is $o(x)$ as $x \to \infty$. The original definition in [10] is given for increasing F, here we extend this definition in order to include constant functions. Note that, [11, Section 22.11], the

function defined by

$$f(n) := \begin{cases} n & \text{if } n = 2^m \text{ for some } m \in \mathbb{N}, \\ 1 & \text{otherwise} \end{cases}$$

has normal order 1 since the number of $n \leqslant x$ which are of the form 2^m for some $m \in \mathbb{N}$ is $o(x)$.

In [10] (see also [11, Section 22.11]), Hardy and Ramanujan proved that $\omega(n)$ has normal order $\log \log n$. In [27], Turán showed that

$$\sum_{n \leqslant x} (\omega(n) - \log \log x)^2 \ll x \log \log x \tag{1.1.2}$$

from which it follows that the number of $n \leqslant x$ satisfying the inequality

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} > h(x) \tag{1.1.3}$$

is $o(x)$ as $x \to \infty$ for any increasing function $h(x) \to \infty$ as $x \to \infty$. Thus, the next question one may ask is whether the function on the left-hand side of (1.1.3) has a distribution. In [8], Erdös and Kac proved the remarkable result that the function on the left-hand side of (1.1.3) has normal distribution in the sense that

$$\lim_{x \to \infty} \frac{\left| \left\{ n \leqslant x : \alpha \leqslant \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leqslant \beta \right\} \right|}{x} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2//2} \, dt$$

for any $\alpha \leqslant \beta$.

In Chapter 2 of the present thesis, we consider a refined version of the function $\omega(n)$. Let $k \geqslant 1$ be a natural number and $\omega_k(n)$ denote the number of distinct prime factors of n with multiplicity k. Note that

$$\omega(n) = \sum_{k \geqslant 1} \omega_k(n)$$

for all $n \in \mathbb{N}$. In Chapter 2, we first estimate the summatory functions of $\omega_k(n)$, $k \geqslant 1$.

Theorem 1.1.1. *Let $k \geqslant 1$ be a natural number. Define*

$$c_k := \sum_{p \, prime} \frac{1}{p^k(p+1)}, \qquad b := \gamma_0 - \sum_{p \, prime} \sum_{j=2}^\infty \frac{1}{jp^j} \tag{1.1.4}$$

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where γ_0 is the Euler-Mascheroni constant. We have

$$\sum_{n \leqslant x} \omega_1(n) = x \log \log x + (b - c_1 - c_2)x + O\left(\frac{x}{\log x}\right).$$

Moreover, for $k \geqslant 2$, we have

$$\sum_{n \leqslant x} \omega_k(n) = (c_{k-1} - c_{k+1})x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x\right).$$

Here, we would like to note that all the implied constants throughout the thesis are absolute unless such dependency is indicated by a subscript in the big-oh notation and in the notation $\ll$.

Moreover, we estimate the second moments, i.e. the summatory functions of the squares, of $\omega_k(n)$, $k \geqslant 1$.

Theorem 1.1.2. *Let $k \geqslant 1$ be a natural number and c_k be defined as in (1.1.4). Define*

$$C_k := c_{k-1} - c_{k+1} + (c_{k-1} - c_{k+1})^2 - \sum_{p \text{ prime}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)^2, \qquad (k \geqslant 2).$$

We have

$$\sum_{n \leqslant x} \omega_1^2(n) = x \left(\log \log x\right)^2 + O\left(x \log \log x\right).$$

Moreover, for $k \geqslant 2$, we have

$$\sum_{n \leqslant x} \omega_k^2(n) = C_k x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x\right).$$

By Theorems 1.1.1 and 1.1.2, we deduce the following result analogous to (1.1.2).

Corollary 1.1.3. *We have*

$$\sum_{2 \leqslant n \leqslant x} \left(\omega_1(n) - \log \log n\right)^2 \ll x \log \log x.$$

Let $h(x)$ be an increasing function such that $h(x) \to \infty$ as $x \to \infty$. Then the number of natural numbers $n \leqslant x$ such that

$$\frac{|\omega_1(n) - \log \log n|}{\sqrt{\log \log n}} \geqslant h(x)$$

is $o(x)$ and thus $\omega_1(n)$ has normal order $\log \log n$.

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In contrast, we prove the following result.

Theorem 1.1.4. *Let $k \geqslant 2$ be a fixed integer. Then the function $\omega_k(n)$ does not have normal order $F(n)$ for any nondecreasing function $F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$.*

For a natural number n, let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. It is known, [11, Section 22.10], that

$$\sum_{n \leqslant x} \Omega(n) = x \log \log x + \left(b + \sum_p \frac{1}{p(p-1)} \right) x + O \left(\frac{x}{\log x} \right)$$

where b is the constant in (1.1.4) and the sum over p runs over all prime numbers as above. By using the estimate above and (1.1.1), we have

$$\sum_{n \leqslant x} (\Omega(n) - \omega(n)) = x \sum_p \frac{1}{p(p-1)} + O \left(\frac{x}{\log x} \right). \tag{1.1.5}$$

Note that

$$\sum_{n \leqslant x} (\Omega(n) - \omega(n)) = \sum_{n \leqslant x} \sum_{k \geqslant 2} (k-1)\omega_k(n). \tag{1.1.6}$$

Since the error terms for the summatory functions of $\omega_k(.)$ with $k \geqslant 2$ in our main result Theorem 1.1.1 are better than $x/\log x$, one may expect to obtain a better error term in (1.1.5) by using Theorem 1.1.1. The largest error term for the summatory functions of $\omega_k(.)$ with $k \geqslant 2$ comes from the case $k = 2$ and this error term is $x^{3/5} \log^2 x$. Thus, we do not expect to have a better error term for (1.1.5) than $x^{3/5} \log^2 x$ by using Theorem 1.1.1. However, a recent work of Hassani, [12], gives a surprisingly better error term which is $\sqrt{x}/\log x$. Although the technique we use to prove Theorem 1.1.1 is different than the ones in [12], we do not estimate the sum on the right-hand side of (1.1.6) in this thesis since Hassani's error term is much superior to the one we would have by using Theorem 1.1.1.

1.2 Discrete Mean Values of Dirichlet L-functions

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. For $\sigma > 1$, the *Riemann zeta function* $\zeta(s)$ is defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which can be written as the *Euler product*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.2.1}$$

where the product runs over all prime numbers. In 1859, Riemann, [24], (see also [25, p. 135] or [4, p. 299]), proved that the function $\zeta(s)$ has meromorphic continuation throughout the complex plane and the only pole of $\zeta(s)$ is a simple pole at $s = 1$ with residue 1. More precisely, the Riemann zeta function satisfies, [3], the functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s) \tag{1.2.2}$$

where $\Gamma(.)$ is the Gamma function.

Although the most basic link between prime numbers and the Riemann zeta function is the Euler product in (1.2.1), the connection between these two objects of arithmetic and analysis is more visible by taking the logarithmic derivatives of both sides of (1.2.1) to obtain the identity

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \qquad (\sigma > 1)$$

where the *von Mangoldt function* $\Lambda(.)$ is defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m, \ p \text{ prime}, \ m \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

Let $x \geqslant 2$ be a real number and $\pi(x)$ be the number of prime numbers less than or equal to x. In 1790's, [4, p. 2], [9], Gauss made extensive calculations to compute $\pi(x)$ for several large values of x and compared the values of $\pi(x)$ with the values of the logarithmic integral $\int_2^x \frac{du}{\log u}$ by which he suggested, without giving theoretical evidence, that

$$\pi(x) \sim \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}.$$

The asymptotic above is now called the Prime Number Theorem proved independently by Hadamard and de la Vallée Poussin in 1896, [21, p. 192]. By partial summation, the Prime Number Theorem is equivalent to

$$\sum_{n \leqslant x} \Lambda(n) \sim x.$$

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A deeper connection between prime numbers and the Riemann zeta function is seen through the fact that the error term

$$\sum_{n \leqslant x} \Lambda(n) - x \tag{1.2.3}$$

in the Prime Number Theorem is $\ll x^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$ is equivalent, [3], to the famous Riemann Hypothesis that all the zeros of the Riemann zeta function in the *critical strip* $0 < \sigma < 1$ have real part $1/2$. The best known unconditional upper bound for the size of the error term in (1.2.3) is $\ll x \exp\left(-C\left(\log x\right)^{3/5}\left(\log\log x\right)^{-1/5}\right)$ for some constant $C > 0$, first due to Vinogradov and Korobov, independently, in 1958, [13, Section 12.3], [3, p. 113].

Let $q \geqslant 2$ be a natural number. A *Dirichlet character* χ *modulo* q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying the properties

1. (*Total multiplicativity*) $\chi(nm) = \chi(n)\chi(m)$ for all $n, m \in \mathbb{Z}$,

2. (*Periodicity*) $\chi(n + q) = \chi(n)$ for all $n \in \mathbb{Z}$,

3. $\chi(n) = 0$ if and only if $(n, q) \neq 1$.

Let $\varphi(.)$ denote the Euler totient function. The set of Dirichlet characters modulo q forms a multiplicative group of order $\varphi(q)$ with the identity element χ_0, called *the principal character modulo* q, defined by $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$ with $(n, q) = 1$. For a Dirichlet character χ modulo q, let $\overline{\chi}$ be defined by $\overline{\chi}(n) = \overline{\chi(n)}$ for all $n \in \mathbb{Z}$. For $a \in \mathbb{Z}$ with $(a, q) = 1$, the Dirichlet characters modulo q satisfy the orthogonality relation

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \chi(a)\overline{\chi}(b) = \begin{cases} 1 & \text{if } a \equiv b \pmod{q}, \\ 0 & \text{otherwise} \end{cases} \tag{1.2.4}$$

where the sum runs over all Dirichlet characters modulo q. Thus, the Dirichlet characters modulo q provide a way of selecting a reduced residue class modulo q.

For a Dirichlet character χ modulo q and a complex number $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, the *Dirichlet L-function* $L(s, \chi)$ is defined by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \, \text{prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \qquad (\sigma > 1). \tag{1.2.5}$$

If χ is a *primitive Dirichlet character modulo* q, i.e. the least period of χ is q, then $L(s,\chi)$ satisfies the functional equation, [3, Chapter 9],

$$L(s,\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{q}\right)^s \frac{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)} L(1-s,\overline{\chi}) \tag{1.2.6}$$

where

$$\mathfrak{a}_\chi := \frac{1-\chi(-1)}{2} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

and the Gauss sum $\tau(\chi)$ associated with the character χ is defined by

$$\tau(\chi) := \sum_{1 \leqslant b \leqslant q-1} \chi(b) e\left(\frac{b}{q}\right), \qquad (e(x) := e^{2\pi i x},\ x \in \mathbb{C}).$$

By taking logarithmic derivatives in (1.2.5), we have

$$-\frac{L'}{L}(s,\chi) = \sum_{n=2}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}, \qquad (\sigma > 1). \tag{1.2.7}$$

The identities (1.2.4) and (1.2.7) give the link between Dirichlet L-functions and the prime numbers in arithmetic progressions. Dirichlet used the functions $L(s,\chi)$ and the orthogonality relation (1.2.4) to prove, [3, Chapters 1 and 4], that there are infinitely many prime numbers of the form $qn + a$ where $(a,q) = 1$. Due to Siegel and Walfisz, [3, Chapter 22], the strongest known form of the Prime Number Theorem in Arithmetic Progressions states that for any $A > 0$, there exists a constant $C(A) > 0$ such that if $q \leqslant (\log x)^A$, then

$$\sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \Lambda(n) = \frac{x}{\varphi(q)} + O\left(x \exp\left(-C(A)(\log x)^{1/2}\right)\right)$$

for any $a \in \mathbb{Z}$ with $(a,q) = 1$. The Generalized Riemann Hypothesis is the statement that for any Dirichlet character χ modulo q, all the the zeros of $L(s,\chi)$ in the critical strip $0 < \sigma < 1$ have real part $1/2$. On the Generalized Riemann Hypothesis, the error term the Prime Number Theorem in Arithmetic Progressions is $\ll x^{1/2} \log^2 x$, [3, p. 125].

In Chapter 3, we investigate two problems related to discrete mean values of Dirichlet L-functions. Let $p \geqslant 3$ be a prime number and χ be a nonprincipal Dirichlet character modulo p. In Chapter 3, we first consider the average

$$\mathcal{A}_p(\chi) := \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \sum_{\substack{1 \leqslant n_1, n_2 \leqslant N \\ \chi(n_1) = \chi(n_2)}} 1.$$

Our main result on the average $\mathcal{A}_p(\chi)$ is the identity given below.

Theorem 1.2.1. *[5] Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ of order $k \geqslant 2$. Then, we have*

$$\mathcal{A}_p(\chi) = \frac{p(2p-1)}{6k} + \frac{(k-1)(p+1)}{12k} + \mathfrak{a}_\chi \frac{p^2}{\pi^2 k(p-1)} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2.$$

Theorem 1.2.1 can be considered in the context of discrete mean values of Dirichlet L-functions since Theorem 1.2.1 gives a link between the mean value

$$\frac{2}{k} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2$$

and the average $\mathcal{A}_p(\chi)$ when $\chi(-1) = -1$. The proof of Theorem 1.2.1 relies on a key lemma, Lemma 3.2.9, which gives a closed formula for partial sums of a nonprincipal Dirichlet character modulo p.

In Chapter 3, we also investigate the mean value

$$\mathcal{M}_p(-s, \chi) := \frac{2}{p-1} \sum_{\substack{\psi \ (\mathrm{mod} \ p) \\ \psi(-1)=-1}} L(1, \psi) L(-s, \chi\overline{\psi})$$

where χ is a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and we ask whether the mean value $\mathcal{M}_p(-s, \chi)$ is related to $L(1-s, \chi)$ in some region for s. Our main result on the mean value $\mathcal{M}_p(-s, \chi)$ is the following theorem.

Theorem 1.2.2. *[6] Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Then, for $s = \sigma + it$ with $\sigma > -1$, $t \in \mathbb{R}$, we have*

$$\mathcal{M}_p(-s, \chi) = L(1-s, \chi) + \mathfrak{a}_\chi 2p^s L(1, \chi)\zeta(-s) + E_p(s, \chi) \tag{1.2.8}$$

where

$$E_p(s, \chi) := \frac{i^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\overline{\chi})} \left(\frac{p}{\pi}\right)^s \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} (s+1) \int_1^\infty \frac{(\lfloor x \rfloor - x + \frac{1}{2})S_{\overline{\chi}}(x)}{x^{s+2}} \, dx$$

and

$$S_{\overline{\chi}}(x) := \sum_{1 \leqslant n \leqslant x} \overline{\chi}(n).$$

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For $-1 < \sigma \leqslant 1$, we have

$$E_p(s, \chi) \ll p^{\sigma - \frac{1}{2}} \left(|t|^{\sigma + \frac{3}{2}} + \left| 1 - (\sigma - \mathfrak{a}_\chi)^2 \right| \right) \left(\frac{1 - (p^{1/2} \log p)^{-\sigma}}{\sigma(\sigma + 1)} \right).$$

In particular, if $0 < \sigma < 1/2$ is fixed and $|t| = o\left(p^{\frac{1-2\sigma}{3+2\sigma}} \right)$, then (1.2.8) holds with $E_p(s, \chi) = o(1)$ as $p \to \infty$.

The proof of Theorem 1.2.2 relies on the functional equations of Dirichlet L-functions and the Gamma function, an auxiliary result (Theorem 3.1.2) and Lemma 3.2.9.

Here we would like to note that there are some difficulties in extending the proof of Theorem 1.2.2 to primitive Dirichlet characters modulo composite numbers due to the use of the functional equations of Dirichlet L-functions associated with a product of Dirichlet characters. However, generalizing Theorem 1.2.1 is possible if one wishes to obtain an asymptotic for $\mathcal{A}_q(\chi)$ but obtaining an exact formula requires some other ideas than the ones used in the proof of Theorem 1.2.1. Further discussion in this direction is given at the end of Section 3.1.

1.3 A Dirichlet Series Related to the Error Term in the Prime Number Theorem

In this section, we state our main results given in Chapter 4. For a natural number n, let

$$\Delta(n) := \sum_{k \leqslant n} \Lambda(k) - \frac{\Lambda(n)}{2} - n$$

be the error term in the Prime Number Theorem. Here we have a modification on the nth term above and the reason for such a modification is hinted by a technique of multiplicative number theory called Perron's formula (Lemma 2.2.1 below). For a prime number $p \geqslant 3$ and a nonprincipal Dirichlet character χ modulo p, define

$$\Delta(s, \chi) := \sum_{n=2}^{\infty} \frac{\chi(n)\Delta(n)}{n^s}, \quad (\sigma > 2)$$

where $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ as usual. In Chapter 4, we investigate the meromorphic behaviour of the function $\Delta(s, \chi)$. Since $\Delta(n) = o(n)$ by the Prime Number Theorem, the series

$\Delta(s, \chi)$ is absolutely convergent for $\sigma > 2$. On the Riemann Hypothesis, $\Delta(n) \ll n^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$ and thus the series defining $\Delta(s, \chi)$ is absolutely convergent for $\sigma > 3/2$ assuming the Riemann Hypothesis. Our main result in Chapter 4, Theorem 1.3.1 below, gives the meromorphic continuation of $\Delta(s, \chi)$ in $\sigma > 1/2$ and in particular, it shows that $\Delta(s, \chi)$ is analytic in the region $\sigma > 1$ unconditionally.

Theorem 1.3.1. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. For $\sigma > 1/2$, we have*

$$\Delta(s, \chi) = \frac{\tau(\chi)\chi(-1)}{\pi i(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1, \psi)\tau\left(\overline{\chi\psi}\right) \frac{L'}{L}(s, \chi\psi) + G(s, \chi) \tag{1.3.1}$$

where

$$G(s, \chi) := L(0, \chi)\frac{L'}{L}(s, \chi_0) + \frac{L(s-1, \chi)}{s-1} + s\sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2} - s^2 \sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2(s-\rho)}$$

$$- \log(2\pi)L(s, \chi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) - \frac{1}{2}\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2}\right)$$

$$- \sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s} + \frac{\gamma_0}{2} + 1$$

and $G(s, \chi)$ is analytic in $\sigma > 1/2$. Here the sums over ρ run over the nontrivial zeros of the Riemann zeta function (the zeros ρ with $0 < \Re(\rho) < 1$) counted with multiplicity and γ_0 is the Euler-Mascheroni constant.

Theorem 1.3.1 has the following interesting corollary about exceptional zeros. It is known, [3, p. 93], that there exists a constant $c > 0$ such that if χ is a real nonprincipal character modulo q, then the real line segment

$$\mathcal{R}(q) := \left(1 - \frac{c}{\log q}, 1\right) \tag{1.3.2}$$

contains at most one zero of $L(s, \chi)$. Such a zero, if exists, is called an exceptional zero.

Corollary 1.3.2. *Let $p \geqslant 3$ be a prime number and $\mathcal{R}(p)$ be defined by (1.3.2) and $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol modulo p.*

1. *If $p \equiv 1 \pmod 4$ and $\Delta(s, \chi)$ is analytic in $\mathcal{R}(p)$ for at least one nonprincipal Dirichlet character χ modulo p with $\chi(-1) = -1$, then $L\left(s, \left(\frac{\cdot}{p}\right)\right)$ has no exceptional zeros.*

2. *If $p \equiv 3 \pmod 4$ and $\Delta(s, \chi)$ is analytic in $\mathcal{R}(p)$ for at least one nonprincipal Dirichlet character χ modulo p with $\chi(-1) = 1$, then $L\left(s, \left(\frac{\cdot}{p}\right)\right)$ has no exceptional zeros.*

We would like to note that it is possible to generalize Theorem 1.3.1 and Corollary 1.3.2 for primitive Dirichlet characters modulo q which is not necessarily a prime number. A route for such a generalization is described at the end of Section 4.1.

Chapter 2

Number of Prime Factors with a Given Multiplicity

2.1 Introduction

For a number theoretical function $f : \mathbb{N} \to \mathbb{C}$, the summatory function

$$\sum_{n \leqslant x} f(n)$$

of f is a fundamental object to study in order to understand the behaviour of f on average. In the case that $f(n) = \omega(n)$, the number of distinct prime factors of n, it is known [21, p. 58] that

$$\sum_{n \leqslant x} \omega(n) = x \log \log x + bx + O\left(\frac{x}{\log x}\right) \tag{2.1.1}$$

where

$$b := \gamma_0 - \sum_{p} \sum_{j=2}^{\infty} \frac{1}{jp^j}, \tag{2.1.2}$$

γ_0 is the Euler-Mascheroni constant and the sum $\sum_p$ ranges over all prime numbers. The second moment of ω, i.e. the summatory function of $\omega^2(.)$, satisfies, [2, Theorem 3.1.1],

$$\sum_{n \leqslant x} \omega^2(n) = x \left(\log \log x\right)^2 + O\left(x \log \log x\right) \tag{2.1.3}$$

by which we have

$$\sum_{n \leqslant x} (\omega(n) - \log \log n)^2 \ll x \log \log x.$$

Using the above estimate, one can prove that $\omega(n)$ has normal order $\log \log n$ as mentioned in Section 1.1.

In this chapter, we consider some refined versions of the $\omega(.)$ function through the following set up. For a prime number p and a natural number $n \geqslant 1$, let $\nu_p(n)$ be the multiplicity of p in the unique factorization of n, that is, $\nu_p(n)$ is the unique integer such that $p^{\nu_p(n)} \mid n$ but $p^{\nu_p(n)+1} \nmid n$. For natural numbers $k, n \geqslant 1$, define

$$\omega_k(n) := \sum_{\substack{p \mid n \\ \nu_p(n)=k}} 1$$

which counts the number of prime factors of n with multiplicity k. Note that the usual $\omega(.)$ function can be partitioned into the functions $\omega_k(.)$ with $k \geqslant 1$ as

$$\omega(n) = \sum_{k \geqslant 1} \omega_k(n)$$

for all $n \in \mathbb{N}$. We first prove the following result about the summatory functions of $\omega_k(.)$ with $k \geqslant 1$.

Theorem (Theorem 1.1.1). *Let $k \geqslant 1$ be a natural number. Define*

$$c_k := \sum_p \frac{1}{p^k(p+1)} \tag{2.1.4}$$

and let b be the constant defined by (2.1.2). We have

$$\sum_{n \leqslant x} \omega_1(n) = x \log \log x + (b - c_1 - c_2)x + O\left(\frac{x}{\log x}\right).$$

Moreover, for $k \geqslant 2$, we have

$$\sum_{n \leqslant x} \omega_k(n) = (c_{k-1} - c_{k+1})x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x\right).$$

The reason why we have a better error term above for the functions ω_k with $k \geqslant 2$ than the one for $k = 1$ is the following. For $\omega_1(.)$, we use the summatory function of $\omega(.)$ (see (2.2.8)) which gives an error term $x/\log x$ by (2.1.1). However, if $k \geqslant 2$, then the generating function corresponding to ω_k (see (2.2.4) and (2.2.8)) is analytic in a sufficiently large region (except at $s = 1$) so that we can control the error term in a better way.

The functions ω_k, $k \geqslant 1$, are neither additive nor multiplicative but the estimates given in our main results above can be put into the context of additive number theory since they are refined versions of the usual ω function which is additive. Interestingly, the proof of Theorem 1.1.1 uses a technique from multiplicative number theory.

Next, we consider the second moments of the functions ω_k, $k \geqslant 1$, and prove the following theorem.

Theorem (Theorem 1.1.2). *Let $k \geqslant 1$ be a natural number and c_k be defined as in (2.1.4). For $k \geqslant 2$, define*

$$C_k := c_{k-1} - c_{k+1} + (c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2.$$

We have

$$\sum_{n \leqslant x} \omega_1^2(n) = x \left(\log \log x \right)^2 + O \left(x \log \log x \right).$$

Moreover, for $k \geqslant 2$, we have

$$\sum_{n \leqslant x} \omega_k^2(n) = C_k x + O \left(x^{\frac{k+1}{3k-1}} \log^2 x \right).$$

Analogous to the usual $\omega(.)$ function, we have the following corollary about the function $\omega_1(.)$ and its normal order.

Corollary (Corollary 1.1.3). *We have*

$$\sum_{n \leqslant x} (\omega_1(n) - \log \log n)^2 \ll x \log \log x.$$

Let $h(x)$ be an increasing function such that $h(x) \to \infty$ as $x \to \infty$. Then the number of natural numbers $n \leqslant x$ such that

$$\frac{|\omega_1(n) - \log \log n|}{\sqrt{\log \log n}} \geqslant h(x)$$

is $o(x)$ and thus $\omega_1(n)$ has normal order $\log \log n$.

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Here we would like to note that with a bit more work, [7], we also proved that

$$\sum_{2 \leqslant n \leqslant x} (\omega_1(n) - \log\log n - C)^2 = x \log\log x + \left(2C + \frac{\pi^2}{6} - b + 2b \sum_p \frac{1}{p^3} \right) x$$
$$+ O\left(\frac{x \log\log x}{\log x} \right)$$

where $C := b - c_1 - c_2$ which appears in the first moment of the function ω_1. An analoguous estimate where ω_1 is replaced by ω and C is replaced by b can also be obtained.

Recall that the main terms for the summatory functions of ω_1 and ω_1^2 are $x \log\log x$ and $x(\log\log x)^2$, respectively. Since

$$\sum_{n \leqslant x} (\omega_1(n) - \log\log n)^2 = \sum_{n \leqslant x} \omega_1(n)^2 - 2 \sum_{n \leqslant x} \omega_1(n) \log\log n + \sum_{n \leqslant x} (\log\log n)^2, \quad (2.1.5)$$

we will see in the proof of Corollary 1.1.3 that the main terms of the three sums on the right-hand side of (2.1.5) cancel out and we obtain the first assertion of Corollary 1.1.3. However, we do not have such a cancellation for ω_k with $k \geqslant 2$. Instead, we have

$$\sum_{n \leqslant x} (\omega_k(n) - (c_{k-1} - c_{k+1}))^2 = \left(C_k - 2(c_{k-1} - c_{k+1})^2 + (c_{k-1} - c_{k+1})^2 \right) x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x \right)$$
$$= \left(C_k - (c_{k-1} - c_{k+1})^2 \right) x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x \right)$$

by Theorems 1.1.1 and 1.1.2. Since

$$C_k - (c_{k-1} - c_{k+1})^2 = c_{k-1} - c_{k+1} + (c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2 - (c_{k-1} - c_{k+1})^2$$
$$= \sum_p \left(\left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) - \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2 \right) \neq 0,$$

the analogous sum to (2.1.5) for ω_k with $k \geqslant 2$ is $\gg x$ which is of the same order of magnitude as the second moment of ω_k. This makes us wonder whether the functions $\omega_k(n)$ with $k \geqslant 2$ have normal order $F(n)$ for some nondecreasing function $F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ which is the content of the following theorem.

Theorem (Theorem 1.1.4). *Let $k \geqslant 2$ be a fixed integer. Then the function $\omega_k(n)$ does not have normal order $F(n)$ for any nondecreasing function $F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$.*

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2.2 Proof of Theorem 1.1.1

To prove Theorem 1.1.1, we need some preliminary results. First, we state a variant of Perron's formula which is used in Chapter 4 as well.

Lemma 2.2.1. *(Perron's Formula, [26, Lemma 3.12]) For $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad (\sigma > 1),$$

where $a_n = O(\Psi(n))$, $\Psi(n)$ being nondecreasing. Assume that for some $\alpha \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} = O\left(\frac{1}{(\sigma - 1)^{\alpha}}\right)$$

as $\sigma \to 1^+$. Let $T \geqslant 4$ and $T_1, T_2 \in (T/2, 2T)$. Let $c > 0$, $\sigma + c > 1$ and $x \geqslant 2$. If x is not an integer and N is the nearest integer to x, then

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT_1}^{c+iT_2} f(s+w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T(\sigma + c - 1)^{\alpha}}\right)$$
$$+ O\left(\frac{\Psi(2x)x^{1-\sigma} \log x}{T}\right) + O\left(\frac{\Psi(N)x^{1-\sigma}}{T|x - N|}\right).$$

If $x \in \mathbb{N}$, then

$$\sum_{n \leqslant x-1} \frac{a_n}{n^s} + \frac{a_x}{2x^s} = \frac{1}{2\pi i} \int_{c-iT_1}^{c+iT_2} f(s+w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T(\sigma + c - 1)^{\alpha}}\right)$$
$$+ O\left(\frac{\Psi(2x)x^{1-\sigma} \log x}{T}\right) + O\left(\frac{\Psi(x)x^{-\sigma}}{T}\right).$$

Next, we use the following upper bounds for the size of the Riemann zeta function.

Lemma 2.2.2. *[13, p. 25] Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. For $|t| \geqslant 2$, we have*

$$|\zeta(s)| \ll \begin{cases} 1 & \text{if } \sigma > 2, \\ \log|t| & \text{if } 1 \leqslant \sigma \leqslant 2, \\ |t|^{\frac{1-\sigma}{2}} \log|t| & \text{if } 0 \leqslant \sigma \leqslant 1, \\ |t|^{\frac{1}{2}-\sigma} \log|t| & \text{if } \sigma \leqslant 0. \end{cases}$$

The proof of Theorem 1.1.1 relies also on the following general result.

Proposition 2.2.3. *Let $g : \mathbb{N} \to \mathbb{C}$ be a function such that $|g(p)| \leqslant 1$ for all prime numbers p. For a fixed natural number $k \geqslant 1$, define*

$$a_{g,k}(n) := \sum_{\substack{p|n \\ \nu_p(n) \geqslant k+1}} \left(1 + g(p) + g(p)^2 + \ldots + g(p)^{\nu_p(n)-(k+1)}\right) \tag{2.2.1}$$

with the convention that empty sum is taken to be zero. Define

$$c_{g,k} := \sum_p \frac{1}{p^k(p - g(p))}.$$

Then we have

$$\sum_{n \leqslant x} a_{g,k}(n) = c_{g,k}x + O\left(x^{\frac{k+2}{3k+2}} \log^2 x\right) \tag{2.2.2}$$

where the implied constant is absolute.

Proof. Let $\sigma > 1$ and define

$$A_{g,k}(s) := \sum_{n=1}^{\infty} \frac{a_{g,k}(n)}{n^s}.$$

We have

$$\sum_p \frac{1}{p^{ks}(p^s - g(p))} = \sum_p \left(\frac{1}{p^{(k+1)s}} + \frac{g(p)}{p^{(k+2)s}} + \frac{g(p)^2}{p^{(k+3)s}} + \ldots\right) = \sum_{n=1}^{\infty} \frac{b_{g,k}(n)}{n^s} \tag{2.2.3}$$

where

$$b_{g,k}(n) := \begin{cases} g(p)^{\alpha-(k+1)} & \text{if } n = p^\alpha, \ \alpha \geqslant k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{b_{g,k}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} b_{g,k}(d)}{n^s}.$$

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We have

$$\sum_{d|n} b_{g,k}(d) = \sum_{\substack{p|n \\ \nu_p(n)\geqslant k+1}} \sum_{j=k+1}^{\nu_p(n)} g(p)^{j-(k+1)} = a_{g,k}(n).$$

Thus, we have

$$A_{g,k}(s) = \zeta(s) \sum_p \frac{1}{p^{ks}(p^s - g(p))} \tag{2.2.4}$$

for $\sigma > 1$. Since the series in (2.2.3) is absolutely convergent for $\sigma > 1/(k+1)$, the identity in (2.2.4) holds for $\sigma > 1/(k+1)$ by analytic continuation.

Note that

$$|a_{g,k}(n)| \leqslant \sum_{\substack{p|n \\ \nu_p(n)\geqslant k+1}} (1 + |g(p)| + |g(p)|^2 + \dots + |g(p)|^{\nu_p(n)-(k+1)})$$

$$\leqslant \sum_{\substack{p|n \\ \nu_p(n)\geqslant k+1}} (\nu_p(n) - k) \ll \log n \tag{2.2.5}$$

and

$$\sum_{n=1}^{\infty} \frac{|a_{g,k}(n)|}{n^\sigma} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \sum_{\substack{p|n \\ \nu_p(n)\geqslant k+1}} (\nu_p(n) - k) = \zeta(\sigma) \sum_p \frac{1}{p^{k\sigma}(p^\sigma - 1)} \ll \frac{1}{\sigma - 1}$$

as $\sigma \to 1^+$. Let $x > 2$ be half of an odd integer and let T be a real number with $2 \leqslant T \leqslant x$. By Lemma 2.2.1, we have

$$\sum_{n<x} a_{g,k}(n) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} A_{g,k}(s) \frac{x^s}{s} \, ds + O\left(\frac{x \log^2 x}{T}\right).$$

By pulling the line of integration above to the left and applying the residue theorem, we have

$$\sum_{n<x} a_{g,k}(n) = c_{g,k}x - (I_1 + I_2 + I_3) + O\left(\frac{x \log^2 x}{T}\right)$$

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where

$$c_{g,k} := \sum_p \frac{1}{p^k(p - g(p))},$$

$$I_1 := \frac{1}{2\pi i} \int_{1 + \frac{1}{\log x} + iT}^{\frac{1}{k+1} + \frac{1}{\log x} + iT} A_{g,k}(s) \frac{x^s}{s}\, ds,$$

$$I_2 := \frac{1}{2\pi i} \int_{\frac{1}{k+1} + \frac{1}{\log x} + iT}^{\frac{1}{k+1} + \frac{1}{\log x} - iT} A_{g,k}(s) \frac{x^s}{s}\, ds,$$

$$I_3 := \frac{1}{2\pi i} \int_{\frac{1}{k+1} + \frac{1}{\log x} - iT}^{1 + \frac{1}{\log x} - iT} A_{g,k}(s) \frac{x^s}{s}\, ds.$$

For $\sigma \geqslant \frac{1}{k+1} + \frac{1}{\log x}$, we have

$$\left| \sum_p \frac{1}{p^{ks}(p^s - g(p))} \right| \ll \sum_p \frac{1}{p^{(k+1)\sigma}} \leqslant \sum_p \frac{1}{p^{1 + \frac{1}{\log x}}} \leqslant \zeta\left(1 + \frac{1}{\log x}\right) \ll \log x.$$

Thus, by Lemma 2.2.2, we have

$$I_1 \ll \frac{T^{1/2}(\log x)\log T}{T} \int_{\frac{1}{k+1} + \frac{1}{\log x}}^{1} \left(\frac{x}{T^{1/2}}\right)^\sigma d\sigma + \frac{\log T}{T} \int_{1}^{1 + \frac{1}{\log x}} x^\sigma\, d\sigma$$

$$\ll \frac{x \log^2 x}{T}.$$

Similarly, we have $I_3 \ll \frac{x\log^2 x}{T}$. For I_2, we have

$$I_2 \ll x^{\frac{1}{k+1}} \log x \int_0^2 \left| \frac{1}{k+1} + \frac{1}{\log x} + it \right|^{-1} dt + x^{\frac{1}{k+1}}(\log x)(\log T) \int_2^T t^{1 - \frac{1}{k+1} - \frac{1}{\log x}}{2}} \frac{1}{t}\, dt$$

where the implied constant is absolute. Note that

$$\left| \frac{1}{k+1} + \frac{1}{\log x} + it \right|^{-1} \leqslant \left| \frac{1}{k+1} + \frac{1}{\log x} \right|^{-1} \leqslant \log x$$

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and

$$\int_{2}^{T} t^{1-\frac{1}{k+1}-\frac{1}{\log x}} \frac{1}{t}\, dt = \frac{2}{1-\frac{1}{k+1}-\frac{1}{\log x}} \left(T^{1-\frac{1}{k+1}-\frac{1}{\log x}} - 2^{1-\frac{1}{k+1}-\frac{1}{\log x}} \right)$$

$$\ll T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)}$$

where the implied constant is absolute. Thus, we have

$$I_2 \ll x^{\frac{1}{k+1}} \log^2 x + x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x$$

$$\ll x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x.$$

Thus, we have

$$\sum_{n<x} a_{g,k}(n) = c_{g,k}x + O\left(\frac{x \log^2 x}{T} \right) + O\left(x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x \right).$$

Taking $T = x^{\frac{2k}{3k+2}}$ equates the error terms above and we obtain

$$\sum_{n<x} a_{g,k}(n) = c_{g,k}x + O\left(x^{\frac{k+2}{3k+2}} \log^2 x \right)$$

where the implied constant is absolute. By (2.2.5), adding a single term $a_{g,k}(\lfloor x \rfloor + 1)$ to the left-hand side of the estimate above has contribution $\ll \log x$ and thus Proposition 2.2.3 follows. $\square$

Now, we deduce Theorem 1.1.1 from Proposition 2.2.3.

Proof of Theorem 1.1.1. Let $g(p) = -1$ for all prime numbers p. Then, with this choice of $g(.)$, we have

$$a_k(n) := a_{g,k}(n) = \sum_{\substack{p|n \\ \nu_p(n) \geq k+1}} \left(1 + g(p) + g(p)^2 + ... + g(p)^{\nu_p(n)-(k+1)} \right)$$

$$= \sum_{\substack{p|n \\ \nu_p(n) \geq k+1 \\ \nu_p(n)-k \text{ odd}}} 1 \qquad (2.2.6)$$

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which counts the number of prime factors of n whose multiplicities are of the form $k + 1$, $k + 3$, $k + 5$, ..., i.e. of the form $k + l$ for some odd natural number l. By Proposition 2.2.3, we have

$$\sum_{n \leqslant x} a_k(n) = c_k x + O_k \left(x^{\frac{k+2}{3k+2}} \log^2 x \right), \qquad (k \geqslant 1) \tag{2.2.7}$$

where, as defined in (2.1.4),

$$c_k = \sum_p \frac{1}{p^k(p+1)}.$$

Note that

$$\omega_1(n) = \omega(n) - a_1(n) - a_2(n), \tag{2.2.8}$$
$$\omega_k(n) = a_{k-1}(n) - a_{k+1}(n), \qquad (k \geqslant 2).$$

Hence, the desired result in Theorem 1.1.1 follows from (2.1.1) and (2.2.7). $\qquad\square$

2.3   The ω^e and ω^o Functions

By the estimate given in (2.2.7), we can consider the functions

$$\omega^e(n) := \sum_{\substack{p|n \\ \nu_p(n) \geqslant 2 \\ \nu_p(n) \, \text{even}}} 1,$$

$$\omega^o(n) := \sum_{\substack{p|n \\ \nu_p(n) \geqslant 3 \\ \nu_p(n) \, \text{odd}}} 1$$

which count the number of distinct prime factors of n with even multiplicities $\geqslant 2$ and with odd multiplicities $\geqslant 3$, respectively. Note that

$$\omega^e(n) = a_1(n),$$
$$\omega^o(n) = a_2(n)$$

with the notation defined in (2.2.6). Thus, we immediately have the first moments of ω^e and ω^o.

Theorem 2.3.1. *Let c_1 and c_2 be defined by (2.1.4). We have*

$$\sum_{n \leqslant x} \omega^e(n) = c_1 x + O\left(x^{\frac{3}{5}} \log^2 x\right)$$

and

$$\sum_{n \leqslant x} \omega^o(n) = c_2 x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

In order to prove Theorem 1.1.2, we need estimates for the second moments of ω^e and ω^o. We first consider the second moment of ω^e.

Theorem 2.3.2. *Let c_1 be defined by (2.1.4). Define*

$$C^e := c_1 + c_1^2 - \sum_p \frac{1}{p^2(p+1)^2}.$$

We have

$$\sum_{n \leqslant x} \omega^e(n)^2 = C^e x + O\left(x^{\frac{3}{5}} \log^2 x\right).$$

Proof. We have

$$
\begin{aligned}
\sum_{n \leqslant x} \omega^e(n)^2 &= \sum_{n \leqslant x} \left(\sum_{\substack{p|n \\ \nu_p(n) \geqslant 2,\, \text{even}}} 1 \right)^2 \\
&= \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 2,\, \text{even}}} 1 \\
&= \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 2,\, \text{even} \\ p=q}} 1 + \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 2,\, \text{even} \\ p \neq q}} 1 \\
&= \sum_{n \leqslant x} \omega^e(n) + \sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n),\nu_q(n) \geqslant 2,\, \text{even}}} 1. \qquad (2.3.1)
\end{aligned}
$$

Now, we consider

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n), \nu_q(n) \geqslant 2, \text{ even}}} 1.$$

For given primes p, q such that $pq \leqslant \sqrt{x}$ and $p \neq q$, note that the expression

$$\left\lfloor \frac{x}{p^2 q^2} \right\rfloor - \left\lfloor \frac{x}{p^2 q^3} \right\rfloor + \left\lfloor \frac{x}{p^2 q^4} \right\rfloor - \left\lfloor \frac{x}{p^2 q^5} \right\rfloor + \dots$$

counts the number of $n \leqslant x$ such that $p^2 \mid n$ and $\nu_q(n) \geqslant 2$ and $\nu_q(n)$ is even. For a natural number $\ell \geqslant 2$, define

$$f(\ell, p, q, x) := \left\lfloor \frac{x}{p^\ell q^2} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^3} \right\rfloor + \left\lfloor \frac{x}{p^\ell q^4} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^5} \right\rfloor + \dots. \qquad (2.3.2)$$

Then,

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) \geqslant 2, \text{ even} \\ \nu_q(n) \geqslant 2, \text{ even}}} 1 = \sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \left(f(2, p, q, x) - f(3, p, q, x) + f(4, p, q, x) - f(5, p, q, x) + \dots \right).$$

$$(2.3.3)$$

Now we consider $f(\ell, p, q, x)$. For a given $\ell \geqslant 2$, let $m_\ell = m_{\ell, p, q, x} \geqslant 2$ be the largest exponent such that $x/(p^\ell q^{m_\ell}) \geqslant 1$. Then $m_\ell \ll \log x$ where the implied constant is absolute. Write

$$\left\lfloor \frac{x}{p^\ell q^j} \right\rfloor = \frac{x}{p^\ell q^j} + E(\ell, j, p, q, x).$$

By using the bound $|E(\ell, j, p, q, x)| \leqslant 1$ for $j \leqslant m_\ell$ and the bound $|E(\ell, j, p, q, x)| \leqslant \frac{x}{p^\ell q^j}$ for $j \geqslant m_\ell + 1$ for which $\left\lfloor \frac{x}{p^\ell q^j} \right\rfloor = 0$, we have

$$f(\ell, p, q, x) = \sum_{j=2}^{\infty} (-1)^j \left\lfloor \frac{x}{p^\ell q^j} \right\rfloor$$

$$= \sum_{j=2}^{\infty} (-1)^j \frac{x}{p^\ell q^j} + \sum_{j=2}^{m_\ell} (-1)^j E(\ell, j, p, q, x) + \sum_{j=m_\ell+1}^{\infty} (-1)^j E(\ell, j, p, q, x)$$

$$= \frac{x}{p^\ell q^2} \frac{1}{1 + \frac{1}{q}} + O\left(\sum_{j=2}^{m_\ell} 1 + \sum_{j=m_\ell+1}^{\infty} \frac{x}{p^\ell q^j} \right)$$

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where the implied constant is absolute. For the error term above, we have

$$\sum_{j=2}^{m_\ell} 1 + \sum_{j=m_\ell+1}^{\infty} \frac{x}{p^\ell q^j} \ll \log x + \frac{x}{p^\ell q^{m_\ell+1}} \ll \log x$$

by the definition of m_ℓ. Thus, we have

$$f(\ell, p, q, x) = \frac{x}{p^\ell q(q+1)} + O\left(\log x\right). \qquad (2.3.4)$$

For given p, q with $pq \leqslant \sqrt{x}$ and $p \neq q$, let $\widetilde{m} = \widetilde{m}_{p,q,x} \geqslant 2$ be the largest exponent such that $x/(p^{\widetilde{m}}q^2) \geqslant 1$. Then $f(\ell, p, q, x) = 0$ for $\ell \geqslant \widetilde{m} + 1$ by (2.3.2) and $\widetilde{m} \ll \log x$ where the implied constant is absolute. Let

$$f(\ell, p, q, x) = \frac{x}{p^\ell q(q+1)} + \tilde{E}(\ell, p, q, x).$$

By (2.3.4), we have $|\tilde{E}(\ell, p, q, x)| \ll \log x$ for all $\ell \geqslant 2$. Since $f(\ell, p, q, x) = 0$ for $\ell \geqslant \widetilde{m}+1$, we also have $|\tilde{E}(\ell, p, q, x)| \leqslant \frac{x}{p^\ell q(q+1)}$ for $\ell \geqslant \widetilde{m} + 1$. Thus,

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \left(f(2, p, q, x) - f(3, p, q, x) + f(4, p, q, x) - f(5, p, q, x) + \dots \right)$$

$$= \sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\ell=2}^{\infty} (-1)^\ell f(\ell, p, q, x)$$

$$= \sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\ell=2}^{\infty} \left(\frac{(-1)^\ell x}{p^\ell q(q+1)} + \tilde{E}(\ell, p, q, x) \right)$$

$$= x \sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \frac{1}{p(p+1)q(q+1)} + O\left(\sum_{\substack{p,q \\ pq \leqslant \sqrt{x}}} \left(\widetilde{m} \log x + \sum_{\ell=\widetilde{m}+1}^{\infty} \frac{x}{p^\ell q(q+1)} \right) \right). \qquad (2.3.5)$$

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By the definition of $\widetilde{m}$, we have

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x}}} \left(\widetilde{m} \log x + \sum_{\ell=\widetilde{m}+1}^{\infty} \frac{x}{p^\ell q(q+1)} \right) \ll \sum_{\substack{p,q \\ pq \leqslant \sqrt{x}}} \left(\log^2 x + \frac{x}{p^{\widetilde{m}+1}q^2} \right)$$

$$\ll \log^2 x \sum_{p \leqslant \sqrt{x}} \sum_{q \leqslant \sqrt{x}/p} 1$$

$$\ll \sqrt{x} \log^2 x \sum_{p \leqslant \sqrt{x}} \frac{1}{p}$$

$$\ll \sqrt{x}(\log^2 x)(\log\log x). \qquad (2.3.6)$$

Now, we consider

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \frac{1}{p(p+1)q(q+1)}.$$

Define

$$\mathbb{1}_{\mathrm{S}}(r) := \begin{cases} r & \text{if S is true,} \\ 0 & \text{otherwise} \end{cases}$$

where S is a statement and r is a real number. Then, by the integral test, we have

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \frac{1}{p(p+1)q(q+1)} = \sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} \sum_{\substack{q \leqslant \sqrt{x}/p \\ q \neq p}} \frac{1}{q(q+1)}$$

$$= \sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} \left(c_1 - \mathbb{1}_{p \leqslant x^{1/4}} \left(\frac{1}{p(p+1)} \right) + O\left(\int_{\sqrt{x}/p}^{\infty} \frac{du}{u^2} \right) \right)$$

For the contribution of the error term above, we have

$$\sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} \int_{\sqrt{x}/p}^{\infty} \frac{du}{u^2} \ll \frac{1}{\sqrt{x}} \sum_{p \leqslant \sqrt{x}} \frac{1}{p} \ll \frac{\log\log x}{\sqrt{x}}.$$

Thus,

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \frac{1}{p(p+1)q(q+1)} = \sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} \left(c_1 - \mathbb{1}_{p \leqslant x^{1/4}} \left(\frac{1}{p(p+1)} \right) \right) + O\left(\frac{\log\log x}{\sqrt{x}} \right).$$

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Similarly, by the integral test, we have

$$\sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} \left(c_1 - \mathbb{1}_{p \leqslant x^{1/4}} \frac{1}{p(p+1)} \right) = c_1 \sum_{p \leqslant \sqrt{x}} \frac{1}{p(p+1)} - \sum_{p \leqslant x^{1/4}} \frac{1}{p^2(p+1)^2}$$

$$= c_1 \left(c_1 + O\left(\frac{1}{\sqrt{x}} \right) \right) - \sum_p \frac{1}{p^2(p+1)^2} + O\left(\frac{1}{x^{3/4}} \right)$$

$$= c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} + O\left(\frac{1}{\sqrt{x}} \right).$$

Thus, we obtain

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \frac{1}{p(p+1)q(q+1)} = c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} + O\left(\frac{\log \log x}{\sqrt{x}} \right). \qquad (2.3.7)$$

By (2.3.3), (2.3.5)-(2.3.7), we have

$$\sum_{\substack{p,q \\ pq \leqslant \sqrt{x} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n), \nu_q(n) \geqslant 2, \text{ even}}} 1 = x \left(c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} + O\left(\frac{\log \log x}{\sqrt{x}} \right) \right)$$

$$+ O\left(\sqrt{x}(\log^2 x)(\log \log x) \right)$$

$$= \left(c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} \right) x + O\left(\sqrt{x}(\log^2 x)(\log \log x) \right).$$

By (2.3.1), Theorem 1.1.1 and the above estimate, we have

$$\sum_{n \leqslant x} \omega^e(n)^2 = c_1 x + \left(c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} \right) x$$

$$+ O\left(x^{\frac{3}{5}} \log^2 x \right) + O\left(\sqrt{x}(\log^2 x)(\log \log x) \right)$$

$$= \left(c_1 + c_1^2 - \sum_p \frac{1}{p^2(p+1)^2} \right) x + O\left(x^{\frac{3}{5}} \log^2 x \right)$$

which finishes the proof. $\qquad \square$

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Now, we estimate the second moment of ω^o.

Theorem 2.3.3. *Let c_2 be defined by (2.1.4). Define*

$$C^o := c_2 + c_2^2 - \sum_p \frac{1}{p^4(p+1)^2}.$$

We have

$$\sum_{n \leqslant x} \omega^o(n)^2 = C^o x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

Proof. We have

$$\sum_{n \leqslant x} \omega^o(n)^2 = \sum_{n \leqslant x} \left(\sum_{\substack{p|n \\ \nu_p(n) \geqslant 3,\, \mathrm{odd}}} 1 \right)^2$$

$$= \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 3,\, \mathrm{odd}}} 1$$

$$= \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 3,\, \mathrm{odd} \\ p=q}} 1 + \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n),\nu_q(n) \geqslant 3,\, \mathrm{odd} \\ p \neq q}} 1$$

$$= \sum_{n \leqslant x} \omega^o(n) + \sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n),\nu_q(n) \geqslant 3,\, \mathrm{odd}}} 1. \qquad (2.3.8)$$

For a natural number $\ell \geqslant 3$ and distinct prime numbers p and q with $pq \leqslant x^{1/3}$, define

$$g(\ell, p, q, x) := \left\lfloor \frac{x}{p^\ell q^3} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^4} \right\rfloor + \left\lfloor \frac{x}{p^\ell q^5} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^6} \right\rfloor + \dots.$$

We have

$$g(\ell, p, q, x) = \sum_{j=3}^{\infty} (-1)^{j+1} \left\lfloor \frac{x}{p^\ell q^j} \right\rfloor$$

$$= \sum_{j=3}^{\infty} (-1)^{j+1} \frac{x}{p^\ell q^j} + O(\log x)$$

$$= \frac{x}{p^\ell q^2(q+1)} + O(\log x)$$

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where the implied constant is absolute. Since

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n), \nu_q(n) \geqslant 3,\, \text{odd}}} 1 = \sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\ell=3}^{\infty} (-1)^{\ell+1} g(\ell, p, q, x),$$

we have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n), \nu_q(n) \geqslant 3,\, \text{odd}}} 1 = \sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\ell=3}^{\infty} (-1)^{\ell+1} \frac{x}{p^\ell q^2(q+1)} + O\left(x^{1/3}(\log^2 x)\log\log x\right)$$

$$= x \sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \frac{1}{p^2(p+1)q^2(q+1)} + O\left(x^{1/3}(\log^2 x)\log\log x\right).$$

We also have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \frac{1}{p^2(p+1)q^2(q+1)} = \sum_{p \leqslant x^{1/3}} \frac{1}{p^2(p+1)} \left(c_2 - \mathbb{1}_{p \leqslant x^{1/6}} \left(\frac{1}{p^2(p+1)} \right) + O\left(\int_{x^{1/3}/p}^{\infty} \frac{du}{u^3} \right) \right)$$

$$= c_2^2 - \sum_{p} \frac{1}{p^4(p+1)^2} + O\left(\frac{\log\log x}{x^{2/3}} \right).$$

Thus, we obtain

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/3} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n), \nu_q(n) \geqslant 3,\, \text{odd}}} 1 = \left(c_2^2 - \sum_{p} \frac{1}{p^4(p+1)^2} \right) x + O\left(x^{1/3}(\log^2 x)\log\log x\right).$$

By (2.3.8), Theorem 1.1.1 and the above estimate, we have

$$\sum_{n \leqslant x} \omega^o(n)^2 = c_2 x + O\left(x^{\frac{1}{2}} \log^2 x\right) + \left(c_2^2 - \sum_{p} \frac{1}{p^4(p+1)^2} \right) x + O\left(x^{1/3}(\log^2 x)\log\log x\right)$$

$$= \left(c_2 + c_2^2 - \sum_{p} \frac{1}{p^4(p+1)^2} \right) x + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

2.4   Proof of Theorem 1.1.2

Now, we are ready to estimate the second moments of the functions ω_k with $k \geqslant 1$ and prove Theorem 1.1.2. First, we consider the second moment of ω_1. Note that

$$\omega_1(n) = \omega(n) - \omega^e(n) - \omega^o(n).$$

Thus,

$$\sum_{n \leqslant x} \omega_1(n)^2 = \sum_{n \leqslant x} (\omega(n) - \omega^e(n) - \omega^o(n))^2$$

$$= \sum_{n \leqslant x} \omega(n)^2 - 2 \sum_{n \leqslant x} \omega(n)(\omega^e(n) + \omega^o(n)) + \sum_{n \leqslant x} (\omega^e(n) + \omega^o(n))^2$$

$$= x(\log \log x)^2 - 2 \sum_{n \leqslant x} \omega(n)(\omega^e(n) + \omega^o(n)) + \sum_{n \leqslant x} (\omega^e(n) + \omega^o(n))^2$$

$$+ O\left(x \log \log x\right)$$

by (2.1.3). By Theorems 2.3.2, 2.3.3 and the Cauchy-Schwarz inequality, we have

$$\sum_{n \leqslant x} (\omega^e(n) + \omega^o(n))^2 \ll x.$$

By the Cauchy-Schwarz inequality, (2.1.3) and the upper bound above, we have

$$-2 \sum_{n \leqslant x} \omega(n)(\omega^e(n) + \omega^o(n)) + \sum_{n \leqslant x} (\omega^e(n) + \omega^o(n))^2 \ll \left(x \left(\log \log x\right)^2\right)^{1/2} x^{1/2} + x$$

$$\ll x \log \log x.$$

Thus, we obtain

$$\sum_{n \leqslant x} \omega_1(n)^2 = x \left(\log \log x\right)^2 + O\left(x \log \log x\right)$$

which finishes the proof of the first assertion in Theorem 1.1.2.

Let $k \geqslant 2$. We have

$$\sum_{n \leqslant x} \omega_k(n)^2 = \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n)=\nu_q(n)=k \\ p=q}} 1 + \sum_{n \leqslant x} \sum_{\substack{p,q|n \\ \nu_p(n)=\nu_q(n)=k \\ p \neq q}} 1$$

$$= \sum_{n \leqslant x} \omega_k(n) + \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n)=\nu_q(n)=k}} 1. \qquad (2.4.1)$$

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For a natural number $\ell \geqslant k$ and distinct prime numbers p and q with $pq \leqslant x^{1/k}$, define

$$h(\ell, p, q, x) := \left\lfloor \frac{x}{p^\ell q^k} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^{k+1}} \right\rfloor$$

which counts the number of $n \leqslant x$ such that $p^\ell \mid n$ and $\nu_q(n) = k$. Then

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1 = \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(h(k, p, q, x) - h(k+1, p, q, x) \right).$$

Since

$$h(\ell, p, q, x) = \frac{x}{p^\ell} \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) + O(1)$$

we have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1 = \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{x}{p^k} \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) - \frac{x}{p^{k+1}} \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) + O(1) \right)$$

$$= x \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right)$$

$$+ O\left(x^{1/k} \log\log x \right). \tag{2.4.2}$$

We have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right)$$

$$= \sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\sum_q \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) + O\left(\int_{x^{1/k}/p}^{\infty} \frac{du}{u^k} \right) \right)$$

$$= \sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\sum_q \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) + O\left(\frac{p^{k-1}}{x^{(k-1)/k}} \right) \right).$$

For the contribution of the error term above, we have

$$\sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \frac{p^{k-1}}{x^{(k-1)/k}} \ll \frac{1}{x^{(k-1)/k}} \sum_{p \leqslant x^{1/k}} \frac{1}{p} \ll \frac{\log \log x}{x^{(k-1)/k}}.$$

Since

$$c_{k-1} - c_{k+1} = \sum_p \left(\frac{1}{p^{k-1}(p+1)} - \frac{1}{p^{k+1}(p+1)} \right) = \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right),$$

we have

$$\sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\sum_q \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \right)$$

$$= \sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(c_{k-1} - c_{k+1} - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \right)$$

$$= (c_{k-1} - c_{k+1})^2 + O\left(\frac{1}{x^{(k-1)/k}} \right) - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2 + O\left(\frac{1}{x^{(2k-1)/(2k)}} \right)$$

$$= (c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2 + O\left(\frac{1}{x^{(k-1)/k}} \right).$$

Thus, we have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \left(\frac{1}{q^k} - \frac{1}{q^{k+1}} \right) = (c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2$$

$$+ O\left(\frac{\log \log x}{x^{(k-1)/k}} \right).$$

By (2.4.2) and the above estimate, we have

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1 = \left((c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right)^2 \right) x + O\left(x^{1/k} \log \log x \right).$$

$$(2.4.3)$$

By (2.4.1), (2.4.3) and Theorem 1.1.1, we obtain

$$\sum_{n \leqslant x} \omega_k(n)^2 = (c_{k-1} - c_{k+1})x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x\right)$$

$$+ \left((c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)^2\right) x + O\left(x^{1/k} \log \log x\right)$$

$$= \left(c_{k-1} - c_{k+1} + (c_{k-1} - c_{k+1})^2 - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)^2\right) x + O\left(x^{\frac{k+1}{3k-1}} \log^2 x\right)$$

which finishes the proof of Theorem 1.1.2.

2.5   Proof of Corollary 1.1.3

We have

$$\sum_{2 \leqslant n \leqslant x} (\omega_1(n) - \log \log n)^2 = \sum_{2 \leqslant n \leqslant x} \omega_1(n)^2 - 2 \sum_{2 \leqslant n \leqslant x} \omega_1(n) \log \log n + \sum_{2 \leqslant n \leqslant x} (\log \log n)^2.$$

By partial summation and Theorem 1.1.1, we have

$$\sum_{2 \leqslant n \leqslant x} \omega_1(n) \log \log n = \left(\sum_{2 \leqslant n \leqslant x} \omega_1(n)\right) \log \log x - \int_2^x \left(\sum_{n \leqslant u} \omega_1(n)\right) \frac{du}{u \log u} + O(1)$$

$$= x(\log \log x)^2 + O(x \log \log x) - \int_2^x \frac{\log \log u}{\log u} du + O\left(\int_2^x \frac{du}{\log u}\right)$$

$$= x(\log \log x)^2 + O(x \log \log x).$$

By partial summation, we have

$$\sum_{2 \leqslant n \leqslant x} (\log \log n)^2 = x(\log \log x)^2 - 2 \int_2^x \frac{\lfloor u \rfloor \log \log u}{u \log u} du + O((\log \log x)^2)$$

$$= x(\log \log x)^2 + O\left(\int_2^x \frac{\log \log u}{\log u} du\right) + O((\log \log x)^2)$$

$$= x(\log \log x)^2 + O\left(\frac{x \log \log x}{\log x}\right).$$

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Thus, we have

$$\sum_{2 \leqslant n \leqslant x} (\omega_1(n) - \log\log n)^2 = x(\log\log x)^2 + O\left(x\log\log x\right)$$

$$- 2x(\log\log x)^2 + O\left(x\log\log x\right)$$

$$+ x(\log\log x)^2 + O\left(\frac{x\log\log x}{\log x}\right)$$

$$= O\left(x\log\log x\right). \tag{2.5.1}$$

Let $h(x)$ be an increasing function such that $h(x) \to \infty$ as $x \to \infty$. Let $\mathcal{E}$ be the set of natural numbers n with $\frac{x}{\log x} \leqslant n \leqslant x$ such that

$$\frac{|\omega_1(n) - \log\log n|}{\sqrt{\log\log n}} \geqslant h(x).$$

Let $|\mathcal{E}|$ be the cardinality of $\mathcal{E}$. Then

$$\sum_{2 \leqslant n \leqslant x} (\omega_1(n) - \log\log n)^2 \geqslant \sum_{n \in \mathcal{E}} (\omega_1(n) - \log\log n)^2$$

$$\geqslant h^2(x/\log x) \sum_{n \in \mathcal{E}} \log\log n$$

$$\geqslant h^2(x/\log x) |\mathcal{E}| \log\log(x/\log x). \tag{2.5.2}$$

By (2.5.1) and (2.5.2), we have

$$\frac{|\mathcal{E}|}{x} \ll \frac{\log\log x}{h(x/\log x)\log\log(x/\log x)} \to 0$$

as $x \to \infty$ since $h(x) \to \infty$ as $x \to \infty$. This finishes the proof of the second assertion of Corollary 1.1.3 since the remaining set of natural numbers with $n < x/\log x$ is already of size $o(x)$.

2.6   Proof of Theorem 1.1.4

Now, we prove that the functions $\omega_k(n)$ with $k \geqslant 2$ do not have normal order $F(n)$ for any nondecreasing function $F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$.

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First we assume that there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$. Then $F(n) > 0$ for $n \geqslant n_0$ since F is nondecreasing. Thus,

$$\lim_{N \to \infty} \frac{|\{n \leqslant N : F(n) > 0\}|}{N} = 1.$$

For a natural number N, define

$$\mathcal{N}_0(N) := \{n \leqslant N : \omega_k(n) = 0\}.$$

Since

$$\sum_{\substack{n \leqslant N \\ n \notin \mathcal{N}_0(N)}} 1 = \sum_{p} \sum_{\substack{n \leqslant N \\ p^k \mid n \\ p^{k+1} \nmid n}} 1 \leqslant \sum_{p} \sum_{\substack{n \leqslant N \\ p^k \mid n}} 1 \leqslant N \sum_{p} \frac{1}{p^k},$$

we have

$$\frac{|\mathcal{N}_0(N)|}{N} \geqslant \frac{N - N \sum_{p} \frac{1}{p^k}}{N} = 1 - \sum_{p} \frac{1}{p^k} \geqslant 1 - \sum_{p} \frac{1}{p^2} > 1 - (\zeta(2) - 1) = 2 - \frac{\pi^2}{6} > 0.$$

Thus

$$\liminf_{N \to \infty} \left(\frac{|\{n \leqslant N : F(n) > 0\}|}{N} + \frac{|\mathcal{N}_0(N)|}{N} \right) > 1$$

and the cardinality of the set of $n \leqslant N$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(N)$. Since for such n, the inequality

$$|\omega_k(n) - F(n)| > \frac{F(n)}{2}$$

is satisfied, we deduce that $\omega_k(n)$ does not have normal order $F(n)$.

Now assume that $F(n) = 0$ for all $n \in \mathbb{N}$. Then

$$\lim_{N \to \infty} \frac{|\{n \leqslant N : F(n) = 0\}|}{N} = 1.$$

Define

$$\mathcal{N}_1(N) := \{n \leqslant N : \omega_k(n) = 1\}.$$

Since

$$|\mathcal{N}_1(N)| \geqslant \sum_{\substack{n \leqslant N \\ \nu_2(n)=k \\ \nu_p(n)<k \text{ for all } p \geqslant 3}} 1 = \sum_{\substack{n \leqslant N \\ \nu_2(n)=k}} 1 - \sum_{\substack{n \leqslant N \\ \nu_2(n)=k \\ \nu_p(n)\geqslant k \text{ for some } p \geqslant 3}} 1$$

$$= \left\lfloor \frac{N}{2^k} \right\rfloor - \left\lfloor \frac{N}{2^{k+1}} \right\rfloor - \sum_{p \geqslant 3} \sum_{\substack{n \leqslant N/2^k \\ p^k | n \\ n \text{ is odd}}} 1$$

$$\geqslant \frac{N}{2^k} - \frac{N}{2^{k+1}} - \frac{N}{2^k} \sum_{p \geqslant 3} \frac{1}{p^k} - 1,$$

we have

$$\liminf_{N \to \infty} \frac{|\mathcal{N}_1(N)|}{N} \geqslant \frac{1}{2^k} \left(\frac{1}{2} - \sum_{p \geqslant 3} \frac{1}{p^k} \right) \geqslant \frac{1}{2^k} \left(\frac{1}{2} - \sum_{p \geqslant 3} \frac{1}{p^2} \right)$$

$$> \frac{1}{2^k} \left(\frac{1}{2} - \left(\frac{\pi^2}{6} - 1 - \frac{1}{4} \right) \right)$$

$$= \frac{1}{2^k} \left(\frac{7}{4} - \frac{\pi^2}{6} \right)$$

$$> 0.$$

Thus

$$\liminf_{N \to \infty} \left(\frac{|\{n \leqslant N : F(n) = 0\}|}{N} + \frac{|\mathcal{N}_1(N)|}{N} \right) > 1$$

and the cardinality of the set of $n \leqslant N$ for which $F(n) = 0$ and $\omega_k(n) = 1$ is not $o(N)$. Since for such n, the inequality

$$|\omega_k(n) - F(n)| > \frac{F(n)}{2}$$

is satisfied, we deduce that $\omega_k(n)$ does not have normal order $F(n)$.

Chapter 3

Two Problems on Discrete Mean Values of Dirichlet L-Functions

3.1 Introduction

Let $p \geqslant 3$ be a prime number and χ be a nonprincipal Dirichlet character modulo p. The first problem we consider in this chapter is the following: What is the average of the number of solutions of $\chi(n_1) = \chi(n_2)$ with $1 \leqslant n_1, n_2 \leqslant N$ where the average is taken over N with $1 \leqslant N \leqslant p - 1$? This is measured by the quantity

$$\mathcal{A}_p(\chi) := \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \sum_{\substack{1 \leqslant n_1, n_2 \leqslant N \\ \chi(n_1) = \chi(n_2)}} 1.$$

Let us make a heuristic argument on the behaviour of $\mathcal{A}_p(\chi)$. If the order of the character χ is $k \geqslant 2$ and $p \nmid n_1$, then the value $\chi(n_1)$ is a kth root of unity. Thus, the probability of $\chi(n_2) = \chi(n_1)$ for a randomly chosen n_2 with $p \nmid n_2$ seems to be $1/k$ and one would expect that

$$\mathcal{A}_p(\chi) \sim \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \sum_{n_1 \leqslant N} \sum_{n_2 \leqslant N} \frac{1}{k} = \frac{1}{k(p-1)} \sum_{1 \leqslant N \leqslant p-1} N^2 = \frac{p(2p-1)}{6k}.$$

Thus, we may expect that $\mathcal{A}_p(\chi) \sim p^2/(3k)$. However, if the order k of the character χ is $p - 1$, then the condition $\chi(n_1) = \chi(n_2)$ is equivalent to $n_1 = n_2$ and thus $\mathcal{A}_p(\chi) = p/2$. Although this indicates that the heuristic argument given above is not valid for some large

values of k, our first main result in this chapter, Theorem 1.2.1 below, shows that the expected asymptotic $\mathcal{A}_p(\chi) \sim p^2/(3k)$ is true if k is not too large.

Theorem (Theorem 1.2.1). *[5] Let $p \geqslant 3$ be a prime number and χ be a nonprincipal Dirichlet character modulo p of order $k \geqslant 2$. Let $\mathfrak{a}_\chi = (1 - \chi(-1))/2$. Then we have*

$$\mathcal{A}_p(\chi) = \frac{p(2p-1)}{6k} + \frac{(k-1)(p+1)}{12k} + \mathfrak{a}_\chi \frac{p^2}{\pi^2 k(p-1)} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2. \qquad (3.1.1)$$

Remark 3.1.1. Theorem 1.2.1 gives an exact formula for the average $\mathcal{A}_p(\chi)$ if $\chi(-1) = 1$ since $\mathfrak{a}_\chi = 0$ in this case. If $\chi(-1) = -1$, then the problem of estimating the average $\mathcal{A}_p(\chi)$ is closely related to the discrete mean value of the Dirichlet L-functions $L(1, \chi^{2j-1})$, $1 \leqslant j \leqslant k/2$, where k is the order of the Dirichlet character χ modulo p. Moreover, the expected asymptotic behaviour $\mathcal{A}_p(\chi) \sim p^2/(3k)$ given by the heuristic argument above is true as long as

$$\frac{k}{p} + \mathfrak{a}_\chi \frac{1}{p} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2$$

tends to zero as $p \to \infty$. Since $L(1, \chi^{2j-1}) \ll \log p$ for all $1 \leqslant j \leqslant k/2$, we see that $k = o\left(\frac{p}{1 + \mathfrak{a}_\chi \log^2 p}\right)$ is a sufficient condition for $\mathcal{A}_p(\chi) \sim p^2/(3k)$ to hold. However, if $k = p - 1$, then $\mathcal{A}_p(\chi) = p/2$ and in this case, we have

$$\sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2 = \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2.$$

In [28], Walum proved that

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2 = \frac{\pi^2(p-1)(p-2)}{6p^2}. \qquad (3.1.2)$$

Thus, Theorem 1.2.1 is in accordance with the above identity since in this case the right-hand side of (3.1.1) is equal to

$$\frac{p(2p-1)}{6(p-1)} + \frac{(p-1-1)(p+1)}{12(p-1)} + \frac{p^2}{\pi^2(p-1)(p-1)} \frac{\pi^2(p-1)(p-2)}{6p^2} \frac{(p-1)}{2} = \frac{p}{2}.$$

Moreover, in [19, Theorem 1], Louboutin showed that if $p \equiv 1 \pmod 6$ is a prime number and χ is a Dirichlet character modulo p of order $k = (p-1)/3$, then

$$\frac{2}{k} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2 = \frac{\pi^2}{6} \left(1 - \frac{1}{p} \right). \tag{3.1.3}$$

Thus by Theorem 1.2.1, we have

$$\mathcal{A}_p(\chi) = \frac{p(2p-1)}{6\frac{p-1}{3}} + \frac{(\frac{p-1}{3} - 1)(p+1)}{12\frac{p-1}{3}} + \frac{p^2}{\pi^2(p-1)} \frac{\pi^2(p-1)}{12p} = \frac{7p+2}{6}.$$

In [19, Theorem 1], Louboutin also considered the case that $p \equiv 1 \pmod{10}$ is a prime number of the form $p = (a^5 - 1)/(a - 1)$ for some $a \in \mathbb{Z} \setminus \{1\}$ and $k = \frac{p-1}{5}$. In this case Louboutin proved that

$$\frac{2}{k} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2 = \frac{\pi^2}{6} \left(1 + \frac{2a(a+1)^2 - 1}{p} \right) \tag{3.1.4}$$

which in return gives

$$\begin{aligned}
\mathcal{A}_p(\chi) &= \frac{p(2p-1)}{6\frac{p-1}{5}} + \frac{(\frac{p-1}{5} - 1)(p+1)}{12\frac{p-1}{5}} + \frac{p^2}{\pi^2(p-1)} \frac{\pi^2}{12} \left(1 + \frac{2a(a+1)^2 - 1}{p} \right) \\
&= \frac{11p+3}{6} + \frac{a(a+1)^2 p}{6(p-1)}.
\end{aligned}$$

The results given in (3.1.2)-(3.1.4) are the only known cases for the mean value

$$\frac{2}{k} \sum_{j=1}^{k/2} |L(1, \chi^{2j-1})|^2 \tag{3.1.5}$$

and in general, estimating the average $\mathcal{A}_p(\chi)$ where $\chi(-1) = -1$ may lead, in the future, to a better comprehension of the mean value in (3.1.5).

Here we would like to note that one can also investigate analogous sums to $\mathcal{A}_p(\chi)$ by considering triples n_1, n_2, n_3 with $\chi(n_1) = \chi(n_2) = \chi(n_3)$ or in general by considering d-tuples $n_1, \ldots, n_d$ for $d \geqslant 3$ with common χ-values. We will consider such extensions in a future communication.

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Let q be a natural number. The discrete mean value considered in (3.1.2) is a special case of

$$\mathcal{M}(q, w, s, \epsilon; \chi) := \frac{2}{\varphi(q)} \sum_{\substack{\psi \pmod{q} \\ \psi(-1)=\epsilon}} L(w, \psi) L(s, \chi\overline{\psi}) \tag{3.1.6}$$

where $\epsilon \in \{\pm 1\}$, φ is the Euler totient function, χ is a Dirichlet character modulo a natural number (not necessarily q) and $w, s \in \mathbb{C}$ except possibly the only pole of the right-hand side of (3.1.6) at 1, if exists. As some examples of the studies on such mean values, we refer to [18] for $\mathcal{M}(q, n, n, \epsilon; \chi_0)$, to [14] and [16] for $\mathcal{M}(q, m, n, \epsilon; \chi_0)$ where $m, n \geqslant 1$ are some natural numbers and χ_0 denotes the principal Dirichlet character modulo q. For a similar mean value with complex arguments w and s but again with $\chi = \chi_0$, one may see [20] and [22]. The only work that we were able to spot in the literature where $\chi \neq \chi_0$ is [29] in which the authors consider the mean value $\mathcal{M}(p, n, 1, 1; \chi_4)$ where $p \geqslant 5$ is a prime number, $n \geqslant 2$ is an even natural number and χ_4 is the nonprincipal Dirichlet character modulo 4.

Before stating the second problem we consider in this chapter precisely, let us have a closer look at Walum's result in (3.1.2) which gives

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2 = \frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1, \psi) L(1, \overline{\psi}) \sim \frac{\pi^2}{6} = \zeta(1+1). \tag{3.1.7}$$

The reason for us to write $\zeta(1+1)$ rather than $\zeta(2)$ above is to indicate the contribution of the diagonal terms if one uses the Dirichlet series of $L(1, \psi)$ and $L(1, \overline{\psi})$. By the Pólya-Vinogradov Inequality (see Lemmata 3.2.2 and 3.2.3 below), the left-hand side of (3.1.7) can be approximated by

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \left(\sum_{a<p/2} \frac{\psi(a)}{a} + O\left(\frac{\log p}{\sqrt{p}}\right) \right) \left(\sum_{b<p/2} \frac{\overline{\psi}(b)}{b} + O\left(\frac{\log p}{\sqrt{p}}\right) \right). \tag{3.1.8}$$

By the orthogonality relation, [17, p. 191],

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \psi(a)\overline{\psi}(b) = \begin{cases} 1 & \text{if } b \equiv a \pmod{p}, \\ -1 & \text{if } b \equiv -a \pmod{p}, \\ 0 & \text{otherwise} \end{cases}$$

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for $(a, p) = 1$, the expression in (3.1.8) becomes

$$\sum_{a,b<p/2} \frac{1}{ab} \frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \psi(a)\overline{\psi}(b) + O\left(\frac{\log^2 p}{\sqrt{p}}\right) = \sum_{a<p/2} \frac{1}{a^{1+1}} + O\left(\frac{\log^2 p}{\sqrt{p}}\right)$$

$$= \zeta(1+1) + O\left(\frac{1}{p}\right) + O\left(\frac{\log^2 p}{\sqrt{p}}\right)$$

$$= \zeta(1+1) + o_{p\to\infty}(1). \qquad (3.1.9)$$

Let $p \geqslant 3$ be a prime number and χ be a nonprincipal Dirichlet character modulo p. For $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, define

$$\mathcal{M}_p(-s, \chi) := \mathcal{M}(p, 1, -s, -1; \chi) = \frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1, \psi) L(-s, \chi\overline{\psi}). \qquad (3.1.10)$$

The second problem we consider in this chapter is estimating the mean value $\mathcal{M}_p(-s, \chi)$ when $\sigma > 0$. The reason for us to consider $\mathcal{M}_p(-s, \chi)$ with $\sigma > 0$ rather than $\mathcal{M}_p(s, \chi)$ with $\sigma > 0$ is the following. For $\mathcal{M}_p(s, \chi)$ with sufficiently large $\sigma > 0$, one can effectively use the partial sums of the Dirichlet series of the functions involved (as in (3.1.8)) and observe that the resulting main term, for large p, bounded $|s|$ and for $\chi(-1) = 1$, is $L(1 + s, \chi)$ (similar to $\zeta(1 + 1)$ in (3.1.9)). Here we are curious about whether such a behaviour occurs for $\mathcal{M}_p(-s, \chi)$ with $\sigma > 0$, that is, whether $\mathcal{M}_p(-s, \chi)$ with $\sigma > 0$ approximates to $L(1 - s, \chi)$.

Our second main result in this chapter, Theorem 1.2.2 below, gives an identity for $\mathcal{M}_p(-s, \chi)$ in a larger region where $\sigma > -1$ and it shows that the behaviour explained above is still valid if $0 < \sigma < 1/2$ is fixed and $|t = \Im s| = o\left(p^{\frac{1-2\sigma}{3+2\sigma}}\right)$ as $p \to \infty$ and $\chi(-1) = 1$. Moreover, by differentiation, one can obtain some information about the derivatives $\mathcal{M}_p^{(j)}(-s, \chi)$ in $\sigma > -1$ as well.

Theorem (Theorem 1.2.2). *[6] Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and let $\mathfrak{a}_\chi = (1 - \chi(-1))/2$. Then, for $s = \sigma + it$ with $\sigma > -1$ and $t \in \mathbb{R}$, we have*

$$\mathcal{M}_p(-s, \chi) = L(1 - s, \chi) + \mathfrak{a}_\chi 2p^s L(1, \chi)\zeta(-s) + E_p(s, \chi) \qquad (3.1.11)$$

where

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$$E_p(s,\chi) := \frac{i^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\overline{\chi})}\left(\frac{p}{\pi}\right)^s \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}(s+1)\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_{\overline{\chi}}(x)}{x^{s+2}}\,dx$$

and

$$S_{\overline{\chi}}(x) := \sum_{1\leqslant n\leqslant x} \overline{\chi}(n).$$

For $-1 < \sigma \leqslant 1$, we have

$$E_p(s,\chi) \ll p^{\sigma-\frac{1}{2}}\left(|t|^{\sigma+\frac{3}{2}} + |1-(\sigma-\mathfrak{a}_\chi)^2|\right)\left(\frac{1-(p^{1/2}\log p)^{-\sigma}}{\sigma(\sigma+1)}\right).$$

In particular, if $0 < \sigma < 1/2$ is fixed and $|t| = o\left(p^{\frac{1-2\sigma}{3+2\sigma}}\right)$, then (3.1.11) holds with $E_p(s,\chi) = o(1)$ as $p \to \infty$.

In the proof of Theorem 1.2.2, we use the functional equations of the factors $L(-s,\chi\overline{\psi})$ in (3.1.10). Note that for general moduli, the product of two nonconjugate characters is not necessarily primitive even if both of them are primitive. However, the assumption that the modulus p is a prime number guarantees the fact that a nonprincipal Dirichlet character modulo p is primitive and thus one can use the functional equations corresponding to such characters. This brings us to the problem of understanding the mean value of $L(1,\psi)\tau(\chi\overline{\psi})L(s+1,\overline{\chi}\psi)$ over the characters $\psi \neq \chi$ with $\psi(-1) = -1$. In Theorem 3.1.2 below, we relate such a mean value to the function

$$\mathbf{S}(s,\chi) := \sum_{N=1}^\infty \frac{S_\chi(N)}{N^s}, \quad (\sigma > 1) \tag{3.1.12}$$

where

$$S_\chi(N) = \sum_{1\leqslant n\leqslant N} \chi(n).$$

Note that the series in (3.1.12) is absolutely convergent in $\sigma > 1$ since $|S_\chi(N)|$ is bounded in terms of p only by the Pólya-Vinogradov Inequality, Lemma 3.2.2.

Theorem 3.1.2. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$.*

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(i) *For any $s \in \mathbb{C}$, except $s = 1$ if $\chi(-1) = -1$, we have*

$$\mathbf{S}(s, \chi) = \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1 \\ \psi \neq \overline{\chi}}} L(1, \psi)\tau\left(\overline{\chi\psi}\right) L(s, \chi\psi)$$

$$+ \mathfrak{a}_\chi \frac{\tau(\chi)(p^s - 1)}{\pi i p^{s-1}(p-1)} L(1, \overline{\chi})\zeta(s) + \frac{L(s, \chi)}{2}. \tag{3.1.13}$$

Thus, the function $\mathbf{S}(s, \chi)$ is analytic everywhere on $\mathbb{C}$ if $\chi(-1) = 1$; otherwise, the only pole of $\mathbf{S}(s, \chi)$ is at $s = 1$ which is a simple pole with residue $\frac{\tau(\chi)}{\pi i}L(1, \overline{\chi})$.

(ii) *For $\sigma > 0$, except $s = 1$ if $\chi(-1) = -1$, we have*

$$\mathbf{S}(s, \chi) = \frac{L(s-1, \chi)}{s-1} + \frac{L(s, \chi)}{2} + s \int_1^\infty \frac{(\lfloor x \rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}} \, dx. \tag{3.1.14}$$

Here we would like make some remarks about extending Theorems 1.2.1 and 3.1.2 to composite moduli. Let χ be a primitive Dirichlet character modulo a composite number q. Then we first encounter with the problem of obtaining a closed formula for the partial sums $S_\chi(N)$ given by Lemma 3.2.9 below and this lemma is used in the proofs of Theorems 1.2.1 and 3.1.2. Although it is possible to have a similar identity for $S_\chi(N)$ when $(N, q) = 1$, covering the case when $(N, q) > 1$ is not straightforward. Now, consider the average

$$\mathcal{A}_q(\chi) := \frac{1}{q-1} \sum_{1 \leqslant N \leqslant q-1} \sum_{\substack{1 \leqslant n_1, n_2 \leqslant N \\ \chi(n_1)=\chi(n_2) \\ (n_1 n_2, q)=1}} 1$$

to obtain an analogue of Theorem 1.2.1. If we follow the proof of Theorem 1.2.1, we eventually come across with the sum

$$\frac{1}{q-1} \sum_{1 \leqslant N \leqslant q-1} \left(\sum_{\substack{n \leqslant N \\ (n,q)=1}} 1 \right)^2$$

which makes the problem harder if one wishes to obtain an identity. However, by estimating the sum above and using the mean square formula, given in [1],

$$\sum_{1 \leqslant N \leqslant q-1} |S_\chi(N)|^2 = \frac{q^2}{12} \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + \mathfrak{a}_\chi \frac{q^2}{\pi^2} |L(1, \chi)|^2,$$

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we can prove that

$$\mathcal{A}_q(\chi) = \left(\frac{\varphi(q)}{q}\right)^2 \frac{q(2q-1)}{6k} + \frac{(k-1)q^2}{12k(q-1)} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)$$

$$+ \mathfrak{a}_\chi \frac{q^2}{\pi^2 k(q-1)} \sum_{j=1}^{k/2} \left|L(1, \chi^{2j-1})\right|^2 + O\left(\varphi(q) 2^{\omega(q)}\right)$$

where k is the order of χ.

For a generalization of Theorem 3.1.2, one may consider the series

$$\sum_{n=1}^{\infty} \frac{\chi_0(N) S_\chi(N)}{N^s}, \quad (\sigma > 1)$$

where χ_0 is the principal Dirichlet character modulo q as an analogue of the function $\mathbf{S}(s, \chi)$ above. By having a closed formula for $S_\chi(N)$ for $(N, q) = 1$, it is possible to obtain an analogue of the first part of Theorem 3.1.2. However, for the second part of Theorem 3.1.2, in view of the identity given in (3.3.3) below, it is difficult to obtain an exact analogue of the second part of Theorem 3.1.2. Instead, one can consider the partial sums of $L(s, \chi_0)$ to proceed further.

3.2 Lemmata

Lemma 3.2.1. *[21, Corollary 4.5] Let ψ_1, ψ_2 be Dirichlet characters modulo $q \geqslant 2$. Then*

$$\frac{1}{\varphi(q)} \sum_{1 \leqslant k \leqslant q-1} \psi_1(k)\overline{\psi_2}(k) = \begin{cases} 1 & \text{if } \psi_1 = \psi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2.2. *[3, Chapter 23, Pólya-Vinogradov Inequality] Let ψ be a nonprincipal Dirichlet character modulo $q \geqslant 3$. Then*

$$\sum_{A \leqslant n \leqslant A+B} \psi(n) \ll \sqrt{q} \log q$$

for any $A, B \geqslant 1$.

Lemma 3.2.3. *Let $K \geqslant 2$ be a real number and ψ be a nonprincipal Dirichlet character modulo $q \geqslant 3$. Then*

$$\sum_{n \leqslant K} \frac{\psi(n)}{n} = L(1, \psi) + O\left(\frac{\sqrt{q} \log q}{K}\right).$$

Proof. By partial summation and Lemma 3.2.2, we have

$$\sum_{n \leqslant K} \frac{\psi(n)}{n} = \frac{1}{K} \sum_{n \leqslant K} \psi(n) + \int_1^K \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du$$

$$= \int_1^K \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du + O\left(\frac{\sqrt{q} \log q}{K}\right). \tag{3.2.1}$$

By Lemma 3.2.2, the integral in (3.2.1) can be written as

$$\int_1^\infty \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du - \int_K^\infty \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du = \int_1^\infty \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du + O\left(\sqrt{q} \log q \int_K^\infty \frac{du}{u^2}\right)$$

$$= \int_1^\infty \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du + O\left(\frac{\sqrt{q} \log q}{K}\right). \tag{3.2.2}$$

Replacing K in (3.2.1) by x and letting $x \to \infty$, we have

$$L(1, \psi) = \int_1^\infty \frac{\sum_{n \leqslant u} \psi(n)}{u^2} du. \tag{3.2.3}$$

By (3.2.1)-(3.2.3), the desired result follows. $\qquad \square$

Now, we obtain the following identity about a weighted mean value of Dirichlet L-functions.

Lemma 3.2.4. *Let $p \geqslant 3$ be a prime number and M and N be natural numbers with $M \geqslant 1$, $N \geqslant 0$ and $p \nmid M(M+N)$. Then,*

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} (\psi(M+N) - \psi(M)) \tau\left(\overline{\psi}\right) L(1, \psi) = -\frac{2\pi i N}{p} + 2\pi i \left(\left\lfloor \frac{M+N}{p} \right\rfloor - \left\lfloor \frac{M}{p} \right\rfloor\right).$$

$$\tag{3.2.4}$$

Proof. The result is clear if $N = 0$ as both sides of (3.2.4) are zero. Assume that $M, N \geqslant 1$ and $p \nmid M(M+N)$. Let $\mathfrak{C}$ be the positively oriented circular contour with center $M + \frac{N}{2}$ and radius $N/2$. Let $\mathcal{R}$ be the line segment from M to $M+N$ and $\mathfrak{C}_1, \mathfrak{C}_2$ be the parts of $\mathfrak{C}$ lying in the upper and the lower half plane, respectively. Let

$$f(w) := w/p.$$

Note that $f(w)$ is not an integer on $\mathfrak{C}$ since $p \nmid M(M+N)$. Moreover, $f(w) \in \mathbb{Z}$ for some w inside $\mathfrak{C}$ if and only if w is a multiple of p with $M < w < M + N$. Thus, the function

$$\frac{1}{e(f(w)) - 1}$$

is analytic on $\mathfrak{C}$ and the only poles of this function inside $\mathfrak{C}$ are at $w = pk$ with $M < pk < M + N$ for some integer k, which are simple poles with residue

$$\frac{1}{\frac{2\pi i}{p} e\left(\frac{pk}{p}\right)} = \frac{p}{2\pi i}.$$

Thus,

$$
\begin{aligned}
I &:= \int_{\mathfrak{C}} \frac{1}{e(f(w)) - 1}\, dw \\
&= 2\pi i \left(\frac{p}{2\pi i} \times (\text{the number of multiples of } p \text{ in the interval } (M, M+N))\right) \\
&= p\left(\left\lfloor \frac{M+N}{p} \right\rfloor - \left\lfloor \frac{M}{p} \right\rfloor\right)
\end{aligned}
\tag{3.2.5}
$$

by the residue theorem. Write

$$I = \int_{\mathfrak{C}_1} \frac{1}{e(f(w)) - 1}\, dw + \int_{\mathfrak{C}_2} \frac{1}{e(f(w)) - 1}\, dw \tag{3.2.6}$$

and consider the first term in (3.2.6). Let K be a natural number. By the identity

$$\frac{1}{z-1} = \frac{1 - z^{K+1}}{z - 1} + \frac{z^{K+1}}{z - 1}, \qquad (z \in \mathbb{C} \setminus \{1\})$$

where z is replaced by $e(f(w))$ for $w \in \mathfrak{C}_1$, we have

$$\int_{\mathfrak{C}_1} \frac{1}{e(f(w)) - 1}\, dw = -\int_{\mathfrak{C}_1} h_1(w)\, dw + \int_{\mathfrak{C}_1} \frac{e((K+1)f(w))}{e(f(w)) - 1}\, dw \tag{3.2.7}$$

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where

$$h_1(w) := \frac{1 - e((K+1)f(w))}{1 - e(f(w))}.$$

Since K is an integer, the function $h_1(w)$ is analytic on and inside $\mathfrak{C}$. Thus, by the residue theorem and the fact that $\mathfrak{C}_1 \cup \mathcal{R}$ is a positively oriented regular closed contour, the first term on the right-hand side of (3.2.7) is

$$-\int_{\mathfrak{C}_1} h_1(w)\, dw = \int_{\mathcal{R}} h_1(w)\, dw.$$

By (3.2.7) and the identity above, we have

$$\int_{\mathfrak{C}_1} \frac{1}{e(f(w)) - 1}\, dw = \int_{\mathcal{R}} h_1(w)\, dw + \int_{\mathfrak{C}_1} \frac{e((K+1)f(w))}{e(f(w)) - 1}\, dw. \qquad (3.2.8)$$

Now, we consider the second term on the right-hand side of (3.2.6). By the identity

$$\frac{1}{z - 1} = \frac{1 - z^{-K}}{z - 1} + \frac{z^{-K}}{z - 1}, \qquad (z \in \mathbb{C} \setminus \{1\})$$

where z is replaced by $e(f(w))$ for $w \in \mathfrak{C}_2$, we have

$$\int_{\mathfrak{C}_2} \frac{1}{e(f(w)) - 1}\, dw = \int_{\mathfrak{C}_2} h_2(w)\, dw + \int_{\mathfrak{C}_2} \frac{e(-Kf(w))}{e(f(w)) - 1}\, dw \qquad (3.2.9)$$

where

$$h_2(w) := \frac{1 - e(-Kf(w))}{e(f(w)) - 1}.$$

Since K is an integer, the function $h_2(w)$ is also analytic on and inside $\mathfrak{C}$. Thus, by considering the orientation of $\mathfrak{C}_2$ and by the residue theorem, we have

$$\int_{\mathfrak{C}_2} h_2(w)\, dw = \int_{\mathcal{R}} h_2(w)\, dw.$$

By (3.2.9) and the identity above, we have

$$\int_{\mathfrak{C}_2} \frac{1}{e(f(w)) - 1}\, dw = \int_{\mathcal{R}} h_2(w)\, dw + \int_{\mathfrak{C}_2} \frac{e(-Kf(w))}{e(f(w)) - 1}\, dw.$$

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By (3.2.6), (3.2.8) and the identity above, we have

$$I = \int_{\mathcal{R}} (h_1(w) + h_2(w)) \, dw + \int_{\mathfrak{C}_1} \frac{e((K+1)f(w))}{e(f(w)) - 1} \, dw + \int_{\mathfrak{C}_2} \frac{e(-Kf(w))}{e(f(w)) - 1} \, dw. \qquad (3.2.10)$$

By the definitions of $h_1(w)$ and $h_2(w)$ and the identities

$$\frac{1 - z^{K+1}}{1 - z} = \sum_{k=0}^{K} z^k, \qquad (z \in \mathbb{C} \setminus \{1\})$$

and

$$\frac{1 - z^{-K}}{z - 1} = \sum_{k=-K}^{-1} z^k, \qquad (z \in \mathbb{C} \setminus \{0, 1\}),$$

we have

$$h_1(w) + h_2(w) = \frac{1 - e((K+1)f(w))}{1 - e(f(w))} + \frac{1 - e(-Kf(w))}{e(f(w)) - 1}$$

$$= \sum_{k=-K}^{K} e(kf(w)) \qquad (3.2.11)$$

for $w \in \mathcal{R}$ since h_1 and h_2 are analytic on $\mathcal{R}$. By (3.2.5), (3.2.10) and (3.2.11), we have

$$p \left(\left\lfloor \frac{M+N}{p} \right\rfloor - \left\lfloor \frac{M}{p} \right\rfloor \right) = \sum_{k=-K}^{K} \int_{M}^{M+N} e(ku/p) \, du$$

$$+ \int_{\mathfrak{C}_1} \frac{e((K+1)w/p)}{e(w/p) - 1} \, dw + \int_{\mathfrak{C}_2} \frac{e(-Kw/p)}{e(w/p) - 1} \, dw$$

$$= T_1 + T_2 + T_3 \qquad (3.2.12)$$

where

$$T_1 := \sum_{k=-K}^{K} \int_{M}^{M+N} e(ku/p) \, du,$$

$$T_2 := \int_{\mathfrak{C}_1} \frac{e((K+1)w/p)}{e(w/p) - 1} \, dw,$$

$$T_3 := \int_{\mathfrak{C}_2} \frac{e(-Kw/p)}{e(w/p) - 1} \, dw.$$

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Since

$$\int_{M}^{M+N} e(ku/p)\, du = \begin{cases} N & \text{if } k = 0, \\ \frac{p}{2\pi i k}\left[e\left(\frac{(M+N)k}{p}\right) - e\left(\frac{Mk}{p}\right)\right] & \text{if } k \neq 0, \end{cases}$$

we have

$$T_1 = N + \frac{p}{2\pi i} \sum_{\substack{k=-K \\ k \neq 0}}^{K} \frac{e\left(\frac{(M+N)k}{p}\right) - e\left(\frac{Mk}{p}\right)}{k}$$

$$= N + \frac{p}{2\pi i} \sum_{\substack{k=-K \\ k \neq 0 \\ p \nmid k}}^{K} \frac{e\left(\frac{(M+N)k}{p}\right) - e\left(\frac{Mk}{p}\right)}{k}. \tag{3.2.13}$$

Let $c \in \mathbb{N}$ such that $p \nmid c$. Define

$$S(c) := \sum_{\substack{k=-K \\ k \neq 0 \\ p \nmid k}}^{K} \frac{e\left(\frac{ck}{p}\right)}{k}.$$

Then by (3.2.13), we have

$$T_1 = N + \frac{p}{2\pi i}\left(S(M+N) - S(M)\right). \tag{3.2.14}$$

Considering the residue classes of ck for $p \nmid ck$ in the definition of $S(c)$, we have

$$S(c) = \sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) \sum_{\substack{k=-K \\ k \neq 0 \\ p \nmid k \\ ck \equiv a \ (\mathrm{mod}\ p)}}^{K} \frac{1}{k}.$$

By the orthogonality relation (1.2.4), we have

$$\sum_{\substack{k=-K \\ k \neq 0 \\ p \nmid k \\ ck \equiv a \ (\mathrm{mod}\ p)}}^{K} \frac{1}{k} = \frac{1}{p-1} \sum_{\psi \ (\mathrm{mod}\ p)} \overline{\psi}(a)\psi(c) \sum_{\substack{k=-K \\ k \neq 0 \\ p \nmid k}}^{K} \frac{\psi(k)}{k}, \qquad (p \nmid a).$$

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Thus,

$$S(c) = \sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) \left[\frac{1}{p-1} \sum_{\psi \pmod p} \overline{\psi}(a)\psi(c) \sum_{\substack{k=-K \\ k\neq 0 \\ p\nmid k}}^{K} \frac{\psi(k)}{k} \right]$$

$$= \frac{1}{p-1} \sum_{\psi \pmod p} \psi(c) \sum_{\substack{k=-K \\ k\neq 0 \\ p\nmid k}}^{K} \frac{\psi(k)}{k} \sum_{a=1}^{p-1} \overline{\psi}(a) e\left(\frac{a}{p}\right)$$

$$= \frac{1}{p-1} \sum_{\psi \pmod p} \psi(c)\tau\left(\overline{\psi}\right) \sum_{\substack{k=-K \\ k\neq 0 \\ p\nmid k}}^{K} \frac{\psi(k)}{k}.$$

Since

$$\sum_{\substack{k=-K \\ k\neq 0 \\ p\nmid k}}^{K} \frac{\psi(k)}{k} = \begin{cases} 2\sum_{k=1}^{K} \frac{\psi(k)}{k} & \text{if } \psi(-1) = -1, \\ 0 & \text{if } \psi(-1) = 1, \end{cases}$$

we have

$$S(c) = \frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \psi(c)\tau\left(\overline{\psi}\right) \sum_{k=1}^{K} \frac{\psi(k)}{k} \tag{3.2.15}$$

for $c \in \mathbb{N}$ with $p \nmid c$. By (3.2.14), (3.2.15) and Lemma 3.2.3, we have

$$T_1 = N + \frac{p}{\pi i(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} (\psi(M+N) - \psi(M))\, \tau\left(\overline{\psi}\right) \sum_{k=1}^{K} \frac{\psi(k)}{k}$$

$$= N + \frac{p}{\pi i(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} (\psi(M+N) - \psi(M))\, \tau\left(\overline{\psi}\right) L(1,\psi) + O_p\left(\frac{1}{K}\right) \tag{3.2.16}$$

where the implied constant depends only on p.

Recall

$$T_2 = \int_{\mathfrak{C}_1} \frac{e((K+1)w/p)}{e(w/p) - 1}\, dw,$$

$$T_3 = \int_{\mathfrak{C}_2} \frac{e(-Kw/p)}{e(w/p) - 1}\, dw$$

where $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) is the part of the circle $\mathfrak{C}$ lying in the upper (resp. the lower) half plane. Now, our aim is to show that T_2 and T_3 tend to zero as $K \to \infty$. Note that for any real number A, we have

$$|e(Aw/p)| = \exp\left(-2\pi A\Im(w)/p\right). \tag{3.2.17}$$

Moreover, for any $w \in \mathfrak{C}$, we have $|e(w/p) - 1| \gg_p 1$ since $p \nmid M(M + N)$. Thus, by taking $A = K + 1$ in (3.2.17) and considering the contributions to the integral defining T_2 along which $\Im(w) > \frac{1}{\sqrt{K}}$ and $0 \leqslant \Im(w) \leqslant 1/\sqrt{K}$ separately, we have

$$T_2 = \int_{\substack{\mathfrak{C}_1 \\ \Im(w) > 1/\sqrt{K}}} \frac{e((K+1)w/p)}{e(w/p) - 1}\, dw + \int_{\substack{\mathfrak{C}_1 \\ \Im(w) \leqslant 1/\sqrt{K}}} \frac{e((K+1)w/p)}{e(w/p) - 1}\, dw$$

$$\ll_{p,M,N} \exp\left(-2\pi(K+1)/(\sqrt{K}p)\right) + \frac{1}{\sqrt{K}} \tag{3.2.18}$$

which tends to zero as $K \to \infty$. Similarly, we have

$$T_3 = \int_{\substack{\mathfrak{C}_2 \\ \Im(w) < -1/\sqrt{K}}} \frac{e(-Kw/p)}{e(w/p) - 1}\, dw + \int_{\substack{\mathfrak{C}_2 \\ \Im(w) \geqslant -1/\sqrt{K}}} \frac{e(-Kw/p)}{e(w/p) - 1}\, dw$$

$$\ll_{p,M,N} \exp\left(-2\pi(-K)/(-\sqrt{K}p)\right) + \frac{1}{\sqrt{K}} \tag{3.2.19}$$

which also tends to zero as $K \to \infty$. By (3.2.12), (3.2.16), (3.2.18), (3.2.19) and letting $K \to \infty$ among natural numbers, we have

$$p\left(\left\lfloor \frac{M+N}{p} \right\rfloor - \left\lfloor \frac{M}{p} \right\rfloor\right) = N + \frac{p}{\pi i(p-1)} \sum_{\substack{\psi \ (\mathrm{mod}\ p) \\ \psi(-1) = -1}} (\psi(M+N) - \psi(M))\, \tau\left(\overline{\psi}\right) L(1, \psi)$$

and the desired result follows by reorganizing the terms above. $\qquad\square$

Lemma 3.2.5. *Let $p \geqslant 3$ be a prime number and $k \in \mathbb{N}$. Define $\mathbb{1}_{p|k} = 1$ if $p \mid k$ and $\mathbb{1}_{p|k} = 0$ if $p \nmid k$. Then we have*

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \psi(k)\tau\left(\overline{\psi}\right) L(1,\psi) = -2\pi i \left(\frac{k}{p} - \left\lfloor \frac{k}{p} \right\rfloor\right) + \pi i \left(1 - \mathbb{1}_{p|k}\right). \tag{3.2.20}$$

Proof. Let $M = 1$ and $N = p - 2$ in Lemma 3.2.4. Since $\psi(M+N) - \psi(M) = \psi(p-1) - \psi(1) = -2$ when $\psi(-1) = -1$, we have

$$-\frac{4}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \tau\left(\overline{\psi}\right) L(1,\psi) = -\frac{2\pi i (p-2)}{p}$$

which gives

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \tau\left(\overline{\psi}\right) L(1,\psi) = \frac{\pi i (p-2)}{p}.$$

By the identity above and Lemma 3.2.4 with $M = 1$, $N \geqslant 0$ such that $p \nmid N + 1$, we have

$$-\frac{2\pi i N}{p} + 2\pi i \left\lfloor \frac{N+1}{p} \right\rfloor = \frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \left(\psi(N+1) - \psi(1)\right) \tau\left(\overline{\psi}\right) L(1,\psi)$$

$$= \frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \psi(N+1)\tau\left(\overline{\psi}\right) L(1,\psi) - \frac{\pi i (p-2)}{p}.$$

Thus,

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} \psi(N+1)\tau\left(\overline{\psi}\right) L(1,\psi) = -\frac{2\pi i N}{p} + \frac{\pi i (p-2)}{p} + 2\pi i \left\lfloor \frac{N+1}{p} \right\rfloor$$

$$= -\frac{2\pi i (N+1)}{p} + \pi i + 2\pi i \left\lfloor \frac{N+1}{p} \right\rfloor$$

$$= -2\pi i \left(\frac{N+1}{p} - \left\lfloor \frac{N+1}{p} \right\rfloor\right) + \pi i.$$

Replacing $N + 1$ by k above gives the desired result for $k \geqslant 1$ and $p \nmid k$. The case that $p \mid k$ is clear since both sides of (3.2.20) are zero. $\qquad\square$

Lemma 3.2.6. *Let $p \geqslant 3$ be a prime number and $k \in \mathbb{N}$ such that $p \nmid k$. Then we have*

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k) L(1,\psi) = \frac{\pi}{p} \cot\left(\frac{\pi k}{p}\right).$$

Proof. For $p \nmid k \in \mathbb{N}$, let $k^{-1} \in \{1,2,3,...,p-1\}$ be the inverse of k modulo p. In (3.2.20), we replace k by $k^{-1}m$ for some $m \in \mathbb{N}$. Then on multiplying both sides of (3.2.20) by $e(m/p)$ and summing over m with $1 \leqslant m \leqslant p$ and using the fact that $\tau\left(\overline{\psi}\right) = \psi(-1)\overline{\tau(\psi)} = -\overline{\tau(\psi)}$ for $\psi(-1) = -1$, the left-hand side of (3.2.20) becomes

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k) \tau\left(\overline{\psi}\right) L(1,\psi) \sum_{1 \leqslant m \leqslant p} \psi(m) e\left(\frac{m}{p}\right) = \frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k) \tau(\psi) \tau\left(\overline{\psi}\right) L(1,\psi)$$

$$= -\frac{2p}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k) L(1,\psi).$$

For the right-hand side of (3.2.20), we have

$$\sum_{1 \leqslant m \leqslant p} \left[-2\pi i \left\{ \frac{k^{-1}m}{p} \right\} + \pi i \left(1 - \mathbb{1}_{p|k^{-1}m} \right) \right] e\left(\frac{m}{p}\right) = -\pi i - 2\pi i \sum_{1 \leqslant m \leqslant p} \left\{ \frac{k^{-1}m}{p} \right\} e\left(\frac{m}{p}\right).$$

Thus, we have

$$\frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k) L(1,\chi) = \frac{\pi i}{p} \left(1 + 2 \sum_{1 \leqslant m \leqslant p} \left\{ \frac{k^{-1}m}{p} \right\} e(m/p) \right). \tag{3.2.21}$$

On writing $k^{-1}m \equiv a \pmod{p}$ where $a \in \{1,2,..,p\}$, we have

$$\sum_{1 \leqslant m \leqslant p} \left\{ \frac{k^{-1}m}{p} \right\} e(m/p) = \sum_{1 \leqslant a \leqslant p} \left\{ \frac{a}{p} \right\} e(ak/p) = \frac{1}{p} \sum_{1 \leqslant a \leqslant p-1} a \, e(ak/p). \tag{3.2.22}$$

Define

$$S(z) := \sum_{1 \leqslant a \leqslant p} a z^a, \qquad z := e(k/p).$$

Since $z \neq 1$ and $z^p = 1$, we have

$$S(z) = z\frac{d}{dz}\left(\sum_{1 \leqslant a \leqslant p} z^a\right) = z\frac{d}{dz}\left(\frac{z^{p+1}-1}{z-1}-1\right)$$

$$= z\frac{(p+1)z^p(z-1)-(z^{p+1}-1)}{(z-1)^2}$$

$$= z\frac{(p+1)(z-1)-(z-1)}{(z-1)^2}$$

$$= \frac{zp}{z-1}.$$

Thus,

$$\frac{1}{p}\sum_{1 \leqslant a \leqslant p-1} ae(ak/p) = \frac{1}{p}\left(S(z)-p\right) = \frac{1}{p}\left(\frac{zp}{z-1}-p\right) = \frac{1}{z-1}. \qquad (3.2.23)$$

By (3.2.21)-(3.2.23), we have

$$\frac{2}{p-1}\sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(k)L(1,\psi) = \frac{\pi i}{p}\left(1+\frac{2}{z-1}\right) = \frac{\pi i}{p}\frac{z+1}{z-1}.$$

Since

$$\frac{z+1}{z-1} = \frac{e^{2\pi i\frac{k}{p}}+1}{e^{2\pi i\frac{k}{p}}-1} = \frac{e^{\pi i\frac{k}{p}}\left(e^{\pi i\frac{k}{p}}+e^{-\pi i\frac{k}{p}}\right)}{e^{\pi i\frac{k}{p}}\left(e^{\pi i\frac{k}{p}}-e^{-\pi i\frac{k}{p}}\right)} = \frac{2\cos\left(\frac{\pi k}{p}\right)}{2i\sin\left(\frac{\pi k}{p}\right)} = \frac{1}{i}\cot\left(\frac{\pi k}{p}\right),$$

the desired result follows. $\qquad \square$

We continue with a special case of a result of Louboutin, [17, Proposition 1]. By using the lemmata above, we give a different proof of this special case where the modulus is a prime number.

Lemma 3.2.7. *Let* χ *be a Dirichlet character modulo a prime number* $p \geqslant 3$ *and let* $\mathfrak{a}_\chi = (1-\chi(-1))/2$. *We have*

$$\sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right) = \mathfrak{a}_\chi\frac{2p}{\pi}L(1,\overline{\chi}). \qquad (3.2.24)$$

Proof. If $\chi(-1) = 1$, then

$$\sum_{1\leqslant a\leqslant p-1} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right) = \sum_{1\leqslant a<p/2} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right) + \sum_{p/2<a\leqslant p-1} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right)$$

$$= \sum_{1\leqslant a<p/2} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right) + \sum_{1\leqslant a<p/2} \overline{\chi}(p-a)\cot\left(\frac{\pi(p-a)}{p}\right)$$

$$= \sum_{1\leqslant a<p/2} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right) - \sum_{1\leqslant a<p/2} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right)$$

$$= 0$$

since $\overline{\chi}(p-a) = \overline{\chi}(-a) = \overline{\chi}(a)$ and $\cot\left(\frac{\pi(p-a)}{p}\right) = -\cot\left(\frac{\pi a}{p}\right)$.

If $\chi(-1) = -1$, then by Lemma 3.2.6, we have

$$\sum_{1\leqslant a\leqslant p-1} \overline{\chi}(a)\left(\frac{2p}{\pi(p-1)}\sum_{\substack{\psi \ (\text{mod } p) \\ \psi(-1)=-1}} \overline{\psi}(a)L(1,\psi)\right) = \sum_{1\leqslant a\leqslant p-1} \overline{\chi}(a)\cot\left(\frac{\pi a}{p}\right).$$

By Lemma 3.2.1, the left-hand side above is equal to

$$\frac{2p}{\pi(p-1)}\sum_{\substack{\psi \ (\text{mod } p) \\ \psi(-1)=-1}} L(1,\psi)\sum_{1\leqslant a\leqslant p-1} \overline{\chi}(a)\overline{\psi}(a) = \frac{2p}{\pi}L(1,\overline{\chi})$$

Hence, the desired result follows. $\qquad\square$

Next, we state a well-known identity for Dirichlet characters (see [3, Chapter 9] or [21, Section 9.2], for example).

Lemma 3.2.8. *[3], [21] Let $p \geqslant 3$ be a prime number and $n \in \mathbb{N}$. If χ is a nonprincipal Dirichlet character modulo p, then we have*

$$\chi(n)\tau(\overline{\chi}) = \sum_{1\leqslant a\leqslant p-1} \overline{\chi}(a)e\left(\frac{an}{p}\right). \tag{3.2.25}$$

If $\chi = \chi_0$ and $p \nmid n$, then (3.2.25) still holds.

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Now, we obtain a closed formula for the partial sums

$$S_\chi(N) = \sum_{1 \leqslant n \leqslant N} \chi(n)$$

of a nonprincipal Dirichlet character χ modulo a prime number $p \geqslant 3$.

Lemma 3.2.9. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Then, for any natural number $N \geqslant 1$, we have*

$$S_\chi(N) = \frac{p\chi(N)}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)\tau\left(\overline{\chi\psi}\right)\psi(N)$$

$$+ \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1,\overline{\chi})\chi_0(N) + \frac{\chi(N)}{2}. \tag{3.2.26}$$

Proof. Since both sides of (3.2.26) are zero if $p \mid N$, we assume that $p \nmid N$. By Lemma 3.2.8, we have

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a)e\left(\frac{an}{p}\right) \tag{3.2.27}$$

for all $n \in \mathbb{N}$ since χ is nonprincipal. Then

$$S_\chi(N) = \sum_{1 \leqslant n \leqslant N} \frac{1}{\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a)e\left(\frac{an}{p}\right)$$

$$= \frac{1}{\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \sum_{1 \leqslant n \leqslant N} e\left(\frac{an}{p}\right).$$

The inner sum on the right-hand side above is equal to

$$\sum_{1 \leqslant n \leqslant N} e\left(\frac{an}{p}\right) = \frac{e\left(\frac{a}{p}\right)}{e\left(\frac{a}{p}\right)-1}\left(e\left(\frac{aN}{p}\right)-1\right), \qquad (p \nmid a).$$

Since

$$\frac{e\left(\frac{a}{p}\right)}{e\left(\frac{a}{p}\right)-1} = \frac{\cot\left(\frac{\pi a}{p}\right)}{2i} + \frac{1}{2},$$

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we have

$$S_\chi(N) = \frac{1}{\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \left(\frac{\cot\left(\frac{\pi a}{p}\right)}{2i} + \frac{1}{2} \right) \left(e\left(\frac{aN}{p}\right) - 1 \right).$$

By (3.2.27), the contribution of the term $1/2$ to the right-hand side above is

$$\frac{1}{2\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \left(e\left(\frac{aN}{p}\right) - 1 \right) = \frac{1}{2\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) e\left(\frac{aN}{p}\right) = \frac{\chi(N)}{2}.$$

Thus, we have

$$S_\chi(N) = \frac{1}{2i\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \left(e\left(\frac{aN}{p}\right) - 1 \right) \cot\left(\frac{\pi a}{p}\right) + \frac{\chi(N)}{2}$$

$$= T(\chi, N) + T(\chi) + \frac{\chi(N)}{2} \tag{3.2.28}$$

where

$$T(\chi, N) := \frac{1}{2i\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) e\left(\frac{aN}{p}\right) \cot\left(\frac{\pi a}{p}\right)$$

and

$$T(\chi) := -\frac{1}{2i\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \cot\left(\frac{\pi a}{p}\right) = -\mathfrak{a}_\chi \frac{p}{\pi i \tau(\overline{\chi})} L(1, \overline{\chi})$$

$$= \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1, \overline{\chi})$$

by Lemma 3.2.7 and $\tau(\overline{\chi}) = -\overline{\tau(\chi)}$ if $\chi(-1) = -1$.

Now, we consider $T(\chi, N)$. By Lemmata 3.2.6 and 3.2.8, we have

$$T(\chi, N) = \frac{1}{2i\tau(\overline{\chi})} \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) e\left(\frac{aN}{p}\right) \frac{2p}{\pi(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} \overline{\psi}(a) L(1, \psi)$$

$$= \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1, \psi) \sum_{1 \leqslant a \leqslant p-1} \overline{\chi}(a) \overline{\psi}(a) e\left(\frac{aN}{p}\right)$$

$$= \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1, \psi) \chi(N) \psi(N) \tau\left(\overline{\chi\psi}\right).$$

By (3.2.28) and the above identities for $T(\chi)$ and $T(\chi, N)$, the desired result follows. $\quad\square$

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3.3   Proof of Theorem 3.1.2

First, we prove Theorem 3.1.2 from which we deduce Theorem 1.2.2 in the next section. Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Recall that

$$\mathbf{S}(s, \chi) = \sum_{N=1}^{\infty} \frac{S_\chi(N)}{N^s}, \quad (\sigma > 1)$$

where

$$S_\chi(N) = \sum_{1 \leqslant n \leqslant N} \chi(n).$$

Let $\sigma > 1$. Dividing both sides of (3.2.26) by N^s and summing over $N \geqslant 1$ give

$$\mathbf{S}(s, \chi) = \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \ (\text{mod } p) \\ \psi(-1)=-1}} L(1, \psi) \tau\left(\overline{\chi\psi}\right) L(s, \chi\psi)$$

$$+ \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1, \overline{\chi}) \zeta(s) \left(1 - \frac{1}{p^s}\right) + \frac{L(s, \chi)}{2}.$$

If $\chi(-1) = -1$, then the term in the sum above with $\psi = \overline{\chi}$ contributes

$$\frac{p}{\pi i \tau(\overline{\chi})(p-1)} L(1, \overline{\chi}) \tau(\chi_0) L(s, \chi_0) = \frac{\tau(\chi)}{\pi i (p-1)} L(1, \overline{\chi}) \zeta(s) \left(1 - \frac{1}{p^s}\right).$$

By the last two identities above, we have

$$\mathbf{S}(s, \chi) = \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \ (\text{mod } p) \\ \psi(-1)=-1 \\ \psi \neq \overline{\chi}}} L(1, \psi) \tau\left(\overline{\chi\psi}\right) L(s, \chi\psi)$$

$$+ \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1, \overline{\chi}) \zeta(s) \left(1 - \frac{1}{p^s}\right) \left(1 + \frac{1}{p-1}\right) + \frac{L(s, \chi)}{2}$$

which gives the first assertion of Theorem 3.1.2 by analytic continuation.

For the second assertion of Theorem 3.1.2, we start with

$$\sum_{N \leqslant pk} \frac{S_\chi(N)}{N^s} = \sum_{N \leqslant pk} \frac{1}{N^s} \sum_{n \leqslant N} \chi(n) = \sum_{n \leqslant pk} \chi(n) \sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} \tag{3.3.1}$$

for some $k \in \mathbb{N}$ and $\sigma > 1$. Since

$$\sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} = \frac{1}{n^s} + \sum_{j \leqslant pk} \frac{1}{j^s} - \sum_{j \leqslant n} \frac{1}{j^s},$$

we have

$$\sum_{n \leqslant pk} \chi(n) \sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} = \sum_{n \leqslant pk} \chi(n) \left[\frac{1}{n^s} + \sum_{j \leqslant pk} \frac{1}{j^s} - \sum_{j \leqslant n} \frac{1}{j^s} \right]$$

$$= \sum_{n \leqslant pk} \frac{\chi(n)}{n^s} - \sum_{n \leqslant pk} \chi(n) \sum_{j \leqslant n} \frac{1}{j^s}$$

$$= S_1 - S_2, \tag{3.3.2}$$

where

$$S_1 := \sum_{n \leqslant pk} \frac{\chi(n)}{n^s},$$

$$S_2 := \sum_{n \leqslant pk} \chi(n) \sum_{j \leqslant n} \frac{1}{j^s}.$$

By [26, Equation 3.5.3], we have

$$\zeta(s) = \sum_{j \leqslant n} \frac{1}{j^s} + s \int_n^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} \, dx + \frac{n^{1-s}}{s-1} - \frac{1}{2n^s}, \quad (\sigma > 0). \tag{3.3.3}$$

Thus,

$$S_2 = \sum_{n \leqslant pk} \chi(n) \left[\zeta(s) - s \int_n^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} \, dx - \frac{n^{1-s}}{s-1} + \frac{1}{2n^s} \right]$$

$$= -\frac{1}{s-1} \sum_{n \leqslant pk} \frac{\chi(n)}{n^{s-1}} - s \sum_{n \leqslant pk} \chi(n) \int_n^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} \, dx + \frac{1}{2} \sum_{n \leqslant pk} \frac{\chi(n)}{n^s}.$$

By the definition of S_1 and the above identity for S_2, we have

$$S_1 - S_2$$

$$= \sum_{n \leqslant pk} \frac{\chi(n)}{n^s} + \frac{1}{s-1} \sum_{n \leqslant pk} \frac{\chi(n)}{n^{s-1}} + s \sum_{n \leqslant pk} \chi(n) \int_n^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} \, dx - \frac{1}{2} \sum_{n \leqslant pk} \frac{\chi(n)}{n^s}$$

$$= \frac{1}{s-1} \sum_{n \leqslant pk} \frac{\chi(n)}{n^{s-1}} + \frac{1}{2} \sum_{n \leqslant pk} \frac{\chi(n)}{n^s} + s \sum_{n \leqslant pk} \chi(n) \int_n^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} \, dx.$$

Note that

$$\sum_{n\leqslant pk} \chi(n) \int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\, dx = \int_1^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}} \left(\sum_{\substack{n\leqslant pk \\ n\leqslant x}} \chi(n) \right) dx$$

$$= \int_1^{pk} \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}}\, dx.$$

By letting $k \to \infty$ for $\sigma > 1$ and using (3.3.1), (3.3.2) and the last two identities above, we obtain

$$\mathbf{S}(s,\chi) = \frac{1}{s-1}L(s-1,\chi) + \frac{1}{2}L(s,\chi) + s\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}}\, dx.$$

Since $S_\chi(x) \ll_p 1$, the integral above is convergent for $\sigma > 0$ and hence Theorem 3.1.2 follows.

3.4   Proof of Theorem 1.2.2

Replacing s by $s+1$ in Theorem 3.1.2 and equating the expressions in (3.1.13) and (3.1.14), we have

$$T_1 + T_2 + T_3 = (s + 1) \int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+2}}\, dx \tag{3.4.1}$$

for $\sigma > -1$ where

$$T_1 := \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \ (\mathrm{mod}\ p) \\ \psi(-1)=-1 \\ \psi\neq\overline{\chi}}} L(1,\psi)\tau\left(\overline{\chi\psi}\right) L(s+1,\chi\psi),$$

$$T_2 := \mathfrak{a}_\chi \frac{\tau(\chi)(p^{s+1}-1)}{\pi i p^s(p-1)} L(1,\overline{\chi})\zeta(s+1),$$

$$T_3 := -\frac{L(s,\chi)}{s}.$$

Now, we consider T_1. Note that if $\psi(-1) = -1$ and $\psi \neq \overline{\chi}$, we have

$$\mathfrak{a}_{\chi\psi} = 1 - \mathfrak{a}_\chi$$

and

$$\tau(\overline{\chi\psi})\tau(\chi\psi) = \chi\psi(-1)\overline{\tau(\chi\psi)}\tau(\chi\psi) = -\chi(-1)p.$$

Thus, for such characters χ and ψ, we have

$$\tau\left(\overline{\chi\psi}\right)L(s+1,\chi\psi) = \tau\left(\overline{\chi\psi}\right)\frac{\tau(\chi\psi)}{i^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{p}\right)^{s+1}\frac{\Gamma\left(\frac{-s+1-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)}L(-s,\overline{\chi\psi})$$

$$= -\frac{\chi(-1)p}{i^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{p}\right)^{s+1}\frac{\Gamma\left(\frac{-s+1-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)}L(-s,\overline{\chi\psi}) \qquad (3.4.2)$$

by the functional equation (1.2.6). By (3.4.2), we have

$$T_1 = \frac{p}{\pi i \tau(\overline{\chi})(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1 \\ \psi \neq \overline{\chi}}} L(1,\psi)\left[-\frac{\chi(-1)p}{i^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{p}\right)^{s+1}\frac{\Gamma\left(\frac{-s+1-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)}L(-s,\overline{\chi\psi})\right]$$

$$= \frac{i^{\mathfrak{a}_\chi}\tau(\chi)}{\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s\frac{\Gamma\left(\frac{1-s-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)}\frac{1}{p-1}\sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1 \\ \psi \neq \overline{\chi}}} L(1,\psi)L(-s,\overline{\chi\psi}).$$

Recall that

$$\mathcal{M}_p(-s,\chi) := \frac{2}{p-1}\sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1,\psi)L(-s,\chi\overline{\psi}).$$

Since

$$\frac{1}{p-1}\sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1 \\ \psi \neq \overline{\chi}}} L(1,\psi)L(-s,\overline{\chi\psi}) = \frac{\mathcal{M}_p(-s,\overline{\chi})}{2} - \mathfrak{a}_\chi L(1,\overline{\chi})\zeta(-s)\frac{1-p^s}{p-1},$$

T_1 can be written as

$$T_1 = \frac{i^{\mathfrak{a}_\chi}\tau(\chi)}{2\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s\frac{\Gamma\left(\frac{1-s-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)}\mathcal{M}_p(-s,\overline{\chi})$$

$$+ \mathfrak{a}_\chi\frac{i\tau(\chi)}{\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s\frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\frac{p^s-1}{p-1}L(1,\overline{\chi})\zeta(-s). \qquad (3.4.3)$$

For T_2, we use the functional equation (1.2.2) of $\zeta(s)$ and write

$$T_2 = \mathfrak{a}_\chi \frac{\tau(\chi)(p^{s+1}-1)}{\pi i p^s(p-1)} L(1,\overline{\chi}) \pi^{s+\frac{1}{2}} \frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \zeta(-s)$$

$$= \mathfrak{a}_\chi \frac{i\tau(\chi)}{\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \frac{1-p^{s+1}}{p-1} L(1,\overline{\chi})\zeta(-s) \qquad (3.4.4)$$

For T_3, we have

$$T_3 = -\frac{1}{s} \frac{\tau(\chi)}{i^{\mathfrak{a}_\chi}\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)} L(1-s,\overline{\chi}) \qquad (3.4.5)$$

by the functional equation (1.2.6). Thus, by (3.4.3)-(3.4.5), we have

$$T_1 + T_2 + T_3 = \frac{i^{\mathfrak{a}_\chi}\tau(\chi)}{2\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(\frac{1-s-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)} \mathcal{M}_p(-s,\overline{\chi})$$

$$- \mathfrak{a}_\chi \frac{i\tau(\chi)}{\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} p^s L(1,\overline{\chi})\zeta(-s)$$

$$- \frac{1}{s} \frac{\tau(\chi)}{i^{\mathfrak{a}_\chi}\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)} L(1-s,\overline{\chi})$$

which is equivalent to

$$T_1 + T_2 + T_3 = \frac{1}{s} \frac{\tau(\chi)}{i^{\mathfrak{a}_\chi}\sqrt{\pi}} \left(\frac{\pi}{p}\right)^s \frac{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}$$

$$\times \left[\frac{i^{2\mathfrak{a}_\chi}}{2} \frac{\Gamma\left(\frac{1-s-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)} \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} \mathcal{M}_p(-s,\overline{\chi}) \right.$$

$$\left. - \mathfrak{a}_\chi i^{1+\mathfrak{a}_\chi} \frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} p^s L(1,\overline{\chi})\zeta(-s) - L(1-s,\overline{\chi}) \right]. \qquad (3.4.6)$$

By $s\Gamma(s) = \Gamma(s+1)$, we have

$$\frac{i^{2\mathfrak{a}_\chi}}{2} \frac{\Gamma\left(\frac{1-s-\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{s+2-\mathfrak{a}_\chi}{2}\right)} \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} = 1 \qquad (3.4.7)$$

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and

$$\mathfrak{a}_\chi i^{1+\mathfrak{a}_\chi} \frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} = 2\mathfrak{a}_\chi \frac{-\frac{s}{2}\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} = 2\mathfrak{a}_\chi. \tag{3.4.8}$$

By (3.4.6)-(3.4.8) and (3.4.1), we have

$$\mathcal{M}_p(-s,\overline{\chi}) - \mathfrak{a}_\chi 2p^s L(1,\overline{\chi})\zeta(-s) - L(1-s,\overline{\chi})$$
$$= \frac{i^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\chi)} \left(\frac{p}{\pi}\right)^s \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}(s+1) \int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+2}}\, dx \tag{3.4.9}$$

for $\sigma > -1$. Replacing χ by $\overline{\chi}$ and reorganizing the terms in (3.4.9) finish the proof of the first statement in Theorem 1.2.2.

Let

$$E_p(s,\chi) := \frac{i^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\overline{\chi})} \left(\frac{p}{\pi}\right)^s \frac{s\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)}(s+1) \int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_{\overline{\chi}}(x)}{x^{s+2}}\, dx$$

for $-1 < \sigma \leqslant 1$. By the Pólya-Vinogradov inequality, we have

$$\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_{\overline{\chi}}(x)}{x^{s+2}}\, dx \ll \int_1^A x^{-\sigma-1}\, dx + p^{1/2}\log p \int_A^\infty x^{-\sigma-2}\, dx$$
$$= \begin{cases} \log A + p^{1/2}(\log p)A^{-1} & \text{if } \sigma = 0, \\ -\frac{1}{\sigma}(A^{-\sigma} - 1) + p^{1/2}(\log p)\frac{A^{-\sigma-1}}{\sigma+1} & \text{if } \sigma \neq 0. \end{cases}$$

Taking $A = p^{1/2}\log p$ and noting that $\lim_{\sigma\to 0}(1 - A^{-\sigma})/\sigma = \log A$, we see that

$$\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_{\overline{\chi}}(x)}{x^{s+2}}\, dx \ll \frac{1 - (p^{1/2}\log p)^{-\sigma}}{(\sigma+1)\sigma}, \quad (-1 < \sigma \leqslant 1)$$

where the right-hand side above is to be interpreted as the limit $\sigma \to 0$ if $\sigma = 0$. By Stirling's formula [13, Equation A.34], we know that

$$|\Gamma(s)| = (2\pi)^{1/2}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi|t|}{2}}\left(1 + O\left(\frac{1}{|t|}\right)\right), \quad (-1 < \sigma \leqslant 1, \ |t| \geqslant 1)$$

where the implied constant is absolute. Thus,

$$\frac{s(s+1)\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} \ll |t|^{\sigma+\frac{3}{2}}, \quad (-1 < \sigma \leqslant 1, \ |t| \geqslant 1).$$

Now we consider the remaining case where $|t| < 1$. Since $\Gamma(s)$ is never zero and it has simple poles at nonpositive integers, we have

$$\frac{s(s+1)\Gamma\left(\frac{s+\mathfrak{a}_\chi}{2}\right)}{\Gamma\left(\frac{1-s+\mathfrak{a}_\chi}{2}\right)} \ll \frac{|s(s+1)(1-s+\mathfrak{a}_\chi)|}{|s+\mathfrak{a}_\chi|}, \quad (-1 < \sigma \leqslant 1, \ |t| < 1).$$

Thus,

$$E_p(s,\chi) \ll p^{\sigma-\frac{1}{2}}\left(|t|^{\sigma+\frac{3}{2}} + |(\sigma+1-\mathfrak{a}_\chi)(1-\sigma+\mathfrak{a}_\chi)|\right)\left(\frac{1-(p^{1/2}\log p)^{-\sigma}}{(\sigma+1)\sigma}\right)$$

for $-1 < \sigma \leqslant 1$ and $t \in \mathbb{R}$ which finishes the proof of Theorem 1.2.2.

3.5 Proof of Theorem 1.2.1

Define

$$\mathcal{M}(\chi) := \frac{1}{p-1}\sum_{1\leqslant N\leqslant p-1}|S_\chi(N)|^2$$

where

$$S_\chi(N) = \sum_{n\leqslant N}\chi(n).$$

First, we show that

$$\mathcal{M}(\chi) = \frac{p+1}{12} + \mathfrak{a}_\chi\frac{p^2}{\pi^2(p-1)}|L(1,\chi)|^2. \tag{3.5.1}$$

For a more general result than (3.5.1), we refer to [1]. For $1 \leqslant N \leqslant p-1$, we have

$$S_\chi(N) = T(\chi,N) + T(\chi) + \frac{\chi(N)}{2} \tag{3.5.2}$$

where

$$T(\chi,N) = \frac{p\chi(N)}{\pi i\tau(\overline{\chi})(p-1)}\sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}}L(1,\psi)\tau\left(\overline{\chi\psi}\right)\psi(N)$$

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and

$$T(\chi) = \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1, \overline{\chi})$$

by Lemma 3.2.9. By (3.5.2), we have

$$\mathcal{M}(\chi) = \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \left(|T(\chi, N)|^2 + |T(\chi)|^2 + \frac{1}{4} \right)$$

$$+ \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \left(2\Re \left(T(\chi, N) \overline{T(\chi)} \right) \right)$$

$$+ \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \left(2\Re \left(T(\chi, N) \frac{\overline{\chi(N)}}{2} \right) + 2\Re \left(T(\chi) \frac{\overline{\chi(N)}}{2} \right) \right). \tag{3.5.3}$$

By orthogonality, Lemma 3.2.1, the last sum in (3.5.3) is zero and the second sum in (3.5.3) is

$$\frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \left(2\Re \left(T(\chi, N) \overline{T(\chi)} \right) \right) = 2\Re \left(\overline{T(\chi)} \frac{p}{\pi i \tau(\overline{\chi})(p-1)} L(1, \overline{\chi}) \tau(\chi_0) \right)$$

$$= \mathfrak{a}_\chi \frac{2p}{\pi^2(p-1)} |L(1, \chi)|^2. \tag{3.5.4}$$

Again by Lemma 3.2.1, the first sum in (3.5.3) is equal to

$$\frac{p}{\pi^2(p-1)^2} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2 \left| \tau\left(\overline{\chi\psi}\right) \right|^2 + \mathfrak{a}_\chi \frac{p}{\pi^2} |L(1, \chi)|^2 + \frac{1}{4}. \tag{3.5.5}$$

By considering the cases $\chi(-1) = \pm 1$ and using (3.1.2), we have

$$\frac{p}{\pi^2(p-1)^2} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2 \left| \tau\left(\overline{\chi\psi}\right) \right|^2$$

$$= \frac{p^2}{\pi^2(p-1)^2} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} |L(1, \psi)|^2 + \mathfrak{a}_\chi \frac{p-p^2}{\pi^2(p-1)^2} |L(1, \chi)|^2$$

$$= \frac{p^2}{\pi^2(p-1)^2} \frac{\pi^2(p-1)(p-2)}{6p^2} \frac{p-1}{2} - \mathfrak{a}_\chi \frac{p}{\pi^2(p-1)} |L(1, \chi)|^2$$

$$= \frac{p-2}{12} - \mathfrak{a}_\chi \frac{p}{\pi^2(p-1)} |L(1, \chi)|^2. \tag{3.5.6}$$

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By (3.5.3)-(3.5.6), we have

$$\mathcal{M}(\chi) = \frac{p-2}{12} - \mathfrak{a}_\chi \frac{p}{\pi^2(p-1)} |L(1,\chi)|^2 + \mathfrak{a}_\chi \frac{p}{\pi^2} |L(1,\chi)|^2 + \frac{1}{4} + \mathfrak{a}_\chi \frac{2p}{\pi^2(p-1)} |L(1,\chi)|^2$$

$$= \frac{p+1}{12} + \mathfrak{a}_\chi \frac{1}{\pi^2} \left(\frac{p}{p-1} + p \right) |L(1,\chi)|^2$$

$$= \frac{p+1}{12} + \mathfrak{a}_\chi \frac{p^2}{\pi^2(p-1)} |L(1,\chi)|^2$$

which finishes the proof of (3.5.1).

Now, we deduce Theorem 1.2.1. Let $k \geqslant 2$ be the order of the Dirichlet character χ modulo p. For any integer a with $p \nmid a$, we have

$$1 + \chi(a) + \chi^2(a) + \ldots + \chi^{k-1}(a) = \begin{cases} k & \text{if } \chi(a) = 1, \\ 0 & \text{otherwise} \end{cases}$$

since $(1 - \chi(a))(1 + \chi(a) + \chi^2(a) + \ldots + \chi^{k-1}(a)) = 0$ as $p \nmid a$. Let $p \nmid n_1 n_2$ and n_2^{-1} denote the multiplicative inverse of n_2 modulo p. Then we have

$$1 + \sum_{j=1}^{k-1} \chi^j(n_1)\overline{\chi^j}(n_2) = 1 + \chi(n_1 n_2^{-1}) + \chi^2(n_1 n_2^{-1}) + \ldots + \chi^{k-1}(n_1 n_2^{-1})$$

$$= \begin{cases} k & \text{if } \chi(n_1 n_2^{-1}) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.7}$$

Let $\mathcal{E}_p(\chi)$ denote the expected behaviour of $\mathcal{A}_p(\chi)$, that is,

$$\mathcal{E}_p(\chi) := \frac{1}{k(p-1)} \sum_{1 \leqslant N \leqslant p-1} \left(\sum_{1 \leqslant n \leqslant N} 1 \right)^2 = \frac{p(2p-1)}{6k}.$$

Since the condition $\chi(n_1 n_2^{-1}) = 1$ in (3.5.7) is equivalent to $\chi(n_1) = \chi(n_2)$ for $p \nmid n_1 n_2$, we have

$$k\mathcal{E}_p(\chi) + \sum_{j=1}^{k-1} \mathcal{M}(\chi^j) = \frac{1}{p-1} \sum_{1 \leqslant N \leqslant p-1} \sum_{1 \leqslant n_1, n_2 \leqslant N} \left(1 + \sum_{j=1}^{k-1} \chi^j(n_1)\overline{\chi^j}(n_2) \right)$$

$$= k\mathcal{A}_p(\chi).$$

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Thus,

$$\mathcal{A}_p(\chi) = \mathcal{E}_p(\chi) + \frac{1}{k} \sum_{j=1}^{k-1} \mathcal{M}(\chi^j).$$

By the identity above and (3.5.1) applied to the the nonprincipal Dirichlet characters χ^j for $j = 1, 2, .., k-1$, we have

$$\mathcal{A}_p(\chi) = \mathcal{E}_p(\chi) + \frac{1}{k} \sum_{j=1}^{k-1} \left(\frac{p+1}{12} + \mathfrak{a}_{\chi^j} \frac{p^2}{\pi^2(p-1)} \left| L(1, \chi^j) \right|^2 \right)$$

$$= \mathcal{E}_p(\chi) + \frac{(k-1)(p+1)}{12k} + \frac{p^2}{\pi^2 k(p-1)} \sum_{j=1}^{k-1} \mathfrak{a}_{\chi^j} \left| L(1, \chi^j) \right|^2$$

$$= \mathcal{E}_p(\chi) + \frac{(k-1)(p+1)}{12k} + \mathfrak{a}_\chi \frac{p^2}{\pi^2 k(p-1)} \sum_{j=1}^{k/2} \left| L(1, \chi^{2j-1}) \right|^2$$

which finishes the proof of Theorem 1.2.1.

Chapter 4

A Dirichlet Series Related to the Error Term in the Prime Number Theorem

4.1 Introduction

Let $n \geqslant 2$ be a natural number and recall that

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \, p \text{ prime}, \, m \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

is the von Mangoldt function and

$$\Delta(n) = \sum_{k \leqslant n} \Lambda(k) - \frac{\Lambda(n)}{2} - n$$

is the error term in the Prime Number Theorem.

Let $p \geqslant 3$ be a prime number and χ be a nonprincipal Dirichlet character modulo p. Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ as usual. In this chapter, we investigate the meromorphic behaviour of the function

$$\Delta(s, \chi) := \sum_{n=2}^{\infty} \frac{\chi(n)\Delta(n)}{n^s}, \quad (\sigma > 2).$$

On the Riemann Hypothesis and its equivalent form that $\Delta(n) \ll n^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$, we see that the series $\Delta(s, \chi)$ is absolutely convergent in $\sigma > 3/2$. In our main result of this chapter, Theorem 1.3.1 below, we obtain the meromorphic continuation of the function $\Delta(s, \chi)$ to the region $\sigma > 1/2$ which in particular shows that $\Delta(s, \chi)$ is analytic in $\sigma > 1$.

Theorem (Theorem 1.3.1). *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. For $\sigma > 1/2$, we have*

$$\Delta(s, \chi) = \frac{\tau(\chi)\chi(-1)}{\pi i(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1, \psi)\tau\left(\overline{\chi\psi}\right) \frac{L'}{L}(s, \chi\psi) + G(s, \chi) \qquad (4.1.1)$$

where

$$G(s, \chi) := L(0, \chi)\frac{L'}{L}(s, \chi_0) + \frac{L(s-1, \chi)}{s-1} + s\sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2} - s^2 \sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2(s-\rho)}$$

$$- \log(2\pi)L(s, \chi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) - \frac{1}{2}\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right)$$

$$- \sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s} + \frac{\gamma_0}{2} + 1$$

and $G(s, \chi)$ is analytic in $\sigma > 1/2$. Here the sums over ρ run over the nontrivial zeros of the Riemann zeta function counted with multiplicity and γ_0 is the Euler-Mascheroni constant.

The classical zero-free region for Dirichlet L-functions is given by the following result, [3, p. 93]. There exists a constant $c > 0$ with the following property. If χ is a complex Dirichlet character modulo q, then $L(s, \chi)$ has no zeros in the region defined by

$$\sigma \geqslant \begin{cases} 1 - \frac{c}{\log(q|t|)} & \text{if } |t| \geqslant 1, \\ 1 - \frac{c}{\log q} & \text{if } |t| \leqslant 1. \end{cases} \qquad (4.1.2)$$

If χ is a real nonprincipal Dirichlet character modulo q, then the only possible zero of $L(s, \chi)$ in this region is a single (simple) real zero. Such a zero, if exists, is called an *exceptional zero*.

Theorem 1.3.1 gives the following corollary about exceptional zeros.

Corollary (Corollary 1.3.2). *Let $p \geqslant 3$ be a prime number and $\mathcal{R}$ be the real line segment $\left(1 - \frac{c}{\log p}, 1\right)$ where c is as in (4.1.2) and let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol modulo p.*

1. *If $p \equiv 1 \pmod 4$ and $\Delta(s, \chi)$ is analytic in $\mathcal{R}$ for at least one nonprincipal Dirichlet character χ modulo p with $\chi(-1) = -1$, then $L\left(s, \left(\frac{\cdot}{p}\right)\right)$ has no exceptional zeros.*

2. *If $p \equiv 3 \pmod 4$ and $\Delta(s, \chi)$ is analytic in $\mathcal{R}$ for at least one nonprincipal Dirichlet character χ modulo p with $\chi(-1) = 1$, then $L\left(s, \left(\frac{\cdot}{p}\right)\right)$ has no exceptional zeros.*

Remark 4.1.1. An interesting feature of Corollary 1.3.2 is that the assumption is related to the zeros of the Riemann zeta function via the error term $\Delta(.)$ and to the existence of a Dirichlet character $\chi \neq \left(\frac{\cdot}{p}\right)$ for which $\Delta(s, \chi)$ is analytic in $\mathcal{R}$, but the conclusion is about a zero of $L\left(s, \left(\frac{\cdot}{p}\right)\right)$.

Here again, we would like to note that it is possible to generalize Theorem 1.3.1 and Corollary 1.3.2 for primitive Dirichlet characters to any moduli not necessarily prime. Let χ be a primitive Dirichlet character modulo $q \geqslant 3$ and let χ_0 be the principal Dirichlet character modulo q. By using a generalization of Lemma 3.2.9 concerning the partial sums $S_\chi(N)$ where $(N, q) = 1$, one can consider the expression

$$\sum_{2 \leqslant n \leqslant qk} \chi(n) \left(\sum_{m \leqslant n} \frac{\chi_0(m)\Lambda(m)}{m^s} - \frac{\chi_0(n)\Lambda(n)}{2n^s} \right), \quad (k \in \mathbb{N})$$

in view of Propositions 4.3.1 and 4.3.2 below. Then by following the proofs of these propositions, one can obtain such a generalization of Theorem 1.3.1 and Corollary 1.3.2.

4.2   Lemmata

Lemma 4.2.1. *[21, Lemma 12.4] Let $\mathcal{A}$ denote the set of those points $s \in \mathbb{C}$ such that $\sigma \leqslant -1$ and $|s + 2k| \geqslant 1/4$ for every positive integer k. Then*

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s| + 1)$$

uniformly for $s \in \mathcal{A}$.

The following lemma is a modified version of Lemma 12.2 in [21].

Lemma 4.2.2. *[21, Lemma 12.2] Let $A > 2$ be fixed and $2 \leqslant \sigma \leqslant A$ and $t \in \mathbb{R}$ be fixed. Let $T \geqslant 4(1 + |t|)$. Then there are real numbers T_1 and T_2 with $-T - 1 \leqslant t - T_1 \leqslant -T$ and $T \leqslant t + T_2 \leqslant T + 1$ such that*

$$\frac{\zeta'}{\zeta}(\sigma + it + u - iT_1) \ll \log^2 T$$

and

$$\frac{\zeta'}{\zeta}(\sigma + it + u + iT_2) \ll \log^2 T$$

uniformly in u with $-1 - \sigma \leqslant u \leqslant A + 1 - \sigma$.

Lemma 4.2.3. *[15] Let $\epsilon > 0$ and let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Then,*

$$L(s, \chi) \ll \begin{cases} (p(|t| + 2))^{\frac{1}{2} - \sigma + \epsilon} & \text{if } \sigma \leqslant 0, \\ (p(|t| + 2))^{\frac{1 - \sigma}{2} + \epsilon} & \text{if } 0 \leqslant \sigma \leqslant 1, \\ (p(|t| + 2))^{\epsilon} & \text{if } \sigma \geqslant 1. \end{cases}$$

Lemma 4.2.4. *[21, Corollary 10.14] We have*

$$\frac{\zeta'}{\zeta}(s) = B + \frac{\log \pi}{2} - \frac{1}{s - 1} - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right)$$

where

$$B := -\sum_{\rho} \Re\frac{1}{\rho} = -\frac{\gamma_0}{2} - 1 + \frac{\log(4\pi)}{2} \tag{4.2.1}$$

and γ_0 is the Euler-Mascheroni constant.

Lemma 4.2.5. *[21, Theorem 12.5] Let $n \geqslant 2$ be a natural number and $T \geqslant 2$ be a real number. For a nontrivial zero ρ of $\zeta(s)$, let $\gamma = \Im(\rho)$. Then*

$$\sum_{k \leqslant n-1} \Lambda(k) + \frac{\Lambda(n)}{2} = n - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{n^\rho}{\rho} - \log(2\pi) - \frac{1}{2} \log\left(1 - \frac{1}{n^2}\right) + R(n, T)$$

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where

$$R(n,T) \ll (\log n) \min\left\{1, \frac{n}{T\langle n\rangle}\right\} + \frac{n\log^2(nT)}{T} \tag{4.2.2}$$

and $\langle n\rangle$ denotes the distance from n to the nearest prime power, other than n itself.

Lemma 4.2.6. *[3, Chapter 15] For $u \geqslant 2$, we have*

$$N(u) = \frac{u}{2\pi}\log\left(\frac{u}{2\pi e}\right) + O(\log u)$$

where $N(u)$ is the number of zeros ρ of the Riemann zeta function with $0 < \Im(\rho) \leqslant u$ counted with multiplicity.

4.3 Two Key Propositions

In this section, we prove two propositions that are used in the proof of the main result, Theorem 1.3.1, of this chapter.

Proposition 4.3.1. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and $\sigma > 1$. Define*

$$E(n,s) := \sum_{m\leqslant n}\frac{\Lambda(m)}{m^s} - \frac{\Lambda(n)}{2n^s}, \qquad (n \geqslant 2). \tag{4.3.1}$$

Then we have

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \sum_{2\leqslant n\leqslant pk} \chi(n)E(n,s) = \frac{\tau(\chi)\chi(-1)}{\pi i(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)\tau\left(\overline{\chi\psi}\right)\frac{L'}{L}(s,\chi\psi)$$

$$+ L(0,\chi)\frac{L'}{L}(s,\chi_0), \qquad (\sigma > 1).$$

Proof. By Lemma 3.2.9 and the functional equation (1.2.6), we have

$$S_\chi(N) = \frac{\tau(\chi)\chi(-1)\chi(N)}{\pi i(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)\tau\left(\overline{\chi\psi}\right)\psi(N)$$

$$+ L(0,\chi)\chi_0(N) + \frac{\chi(N)}{2}, \qquad (N \in \mathbb{N}). \tag{4.3.2}$$

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On multiplying both sides of (4.3.2) by $\Lambda(N)/N^s$ and summing over $N \in \mathbb{N}$ for $\sigma > 1$, we have

$$\tilde{S}(s,\chi) := \sum_{N=1}^{\infty} \frac{S_{\chi}(N)\Lambda(N)}{N^s} = \frac{\tau(\chi)\chi(-1)}{\pi i (p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)\tau\left(\overline{\chi\psi}\right)\left(-\frac{L'}{L}(s,\chi\psi)\right)$$

$$+ L(0,\chi)\left(-\frac{L'}{L}(s,\chi_0)\right) - \frac{1}{2}\frac{L'}{L}(s,\chi). \qquad (4.3.3)$$

Let $k \geqslant 1$ be a natural number. Then

$$\tilde{S}_k(s,\chi) := \sum_{N \leqslant pk} \frac{S_{\chi}(N)\Lambda(N)}{N^s} = \sum_{N \leqslant pk} \frac{\Lambda(N)}{N^s} \sum_{n \leqslant N} \chi(n)$$

$$= \sum_{n \leqslant pk} \chi(n) \sum_{n \leqslant N \leqslant pk} \frac{\Lambda(N)}{N^s}.$$

Since

$$\sum_{n \leqslant N \leqslant pk} \frac{\Lambda(N)}{N^s} = \frac{\Lambda(n)}{n^s} + \sum_{m \leqslant pk} \frac{\Lambda(m)}{m^s} - \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s},$$

and $k \in \mathbb{N}$, we have

$$\tilde{S}_k(s,\chi) = \sum_{n \leqslant pk} \chi(n)\left(\frac{\Lambda(n)}{n^s} + \sum_{m \leqslant pk} \frac{\Lambda(m)}{m^s} - \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s}\right)$$

$$= \sum_{n \leqslant pk} \frac{\chi(n)\Lambda(n)}{n^s} - \sum_{n \leqslant pk} \chi(n) \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s}.$$

Thus,

$$\tilde{S}_k(s,\chi) = \sum_{n \leqslant pk} \frac{\chi(n)\Lambda(n)}{n^s} - \sum_{2 \leqslant n \leqslant pk} \chi(n) \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s}$$

$$= \frac{1}{2}\sum_{n \leqslant pk} \frac{\chi(n)\Lambda(n)}{n^s} - \sum_{2 \leqslant n \leqslant pk} \chi(n)\left(\sum_{m \leqslant n} \frac{\Lambda(m)}{m^s} - \frac{\Lambda(n)}{2n^s}\right) \qquad (4.3.4)$$

Recall that

$$E(n,s) = \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s} - \frac{\Lambda(n)}{2n^s}, \quad (n \geqslant 2, \ \sigma > 1).$$

Note that the series $\sum_n \chi(n)E(n,s)$ is not necessarily convergent since we can find some complex number s with $\sigma > 1$ such that $E(n,s)$ does not tend to zero as $n \to \infty$. However, for $\sigma > 1$, $\tilde{S}_k(s,\chi)$ and the first term on the right-hand side of (4.3.4) converge as $k \to \infty$. Thus, the subsequence

$$\sum_{2 \leqslant n \leqslant pk} \chi(n)E(n,s)$$

of partial sums of $\sum_n \chi(n)E(n,s)$ is convergent for $\sigma > 1$ as $k \to \infty$ on natural numbers. Hence, we have

$$\tilde{S}(s,\chi) = -\frac{1}{2}\frac{L'}{L}(s,\chi) - \lim_{\substack{k \to \infty \\ k \in \mathbb{N}}} \sum_{2 \leqslant n \leqslant pk} \chi(n)E(n,s), \qquad (\sigma > 1). \qquad (4.3.5)$$

By (4.3.3) and (4.3.5), we have

$$\lim_{\substack{k \to \infty \\ k \in \mathbb{N}}} \sum_{2 \leqslant n \leqslant pk} \chi(n)E(n,s) = \frac{\tau(\chi)\chi(-1)}{\pi i(p-1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1,\psi)\tau\left(\overline{\chi\psi}\right)\frac{L'}{L}(s,\chi\psi)$$

$$+ L(0,\chi)\frac{L'}{L}(s,\chi_0), \qquad (\sigma > 1)$$

which finishes the proof of Proposition 4.3.1. $\qquad\square$

Proposition 4.3.2. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and $E(n,s)$ be defined by (4.3.1). For $\sigma > 2$, we have*

$$\lim_{\substack{k \to \infty \\ k \in \mathbb{N}}} \sum_{2 \leqslant n \leqslant pk} \chi(n)E(n,s) = \Delta(s,\chi) + \log(2\pi)L(s,\chi) + \frac{1}{2}\sum_{n=2}^{\infty}\frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right)$$

$$- \frac{\gamma_0}{2} - 1 - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{1}{1-s}L(s-1,\chi)$$

$$- s\sum_{\rho}\frac{L(s-\rho,\chi)}{\rho^2} + s^2\sum_{\rho}\frac{L(s-\rho,\chi)}{\rho^2(s-\rho)}$$

$$+ \sum_{k=1}^{\infty}\frac{L(2k+s,\chi)-1}{2k+s}$$

where the sums over ρ run over the nontrivial zeros of the Riemann zeta function counted with multiplicity and γ_0 is the Euler-Mascheroni constant.

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Proof. Let $n \geqslant 2$ be a natural number and $c := 1/\log(2n)$. Let $s = \sigma + it$ be fixed with $\sigma > 2$, $t \in \mathbb{R}$. Let $T \geqslant 4(1 + |t|)$ and T_1 and T_2 be as in Lemma 4.2.2. Note that $T_1, T_2 \in (T/2, 2T)$. Then, by Lemma 2.2.1, we have

$$E(n, s) = \sum_{m \leqslant n} \frac{\Lambda(m)}{m^s} - \frac{\Lambda(n)}{2n^s} = \frac{1}{2\pi i} \int_{c-iT_1}^{c+iT_2} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw + O\left(\frac{1}{T}\right).$$

Let $K \geqslant 5$ and assume that $\sigma - K$ is a negative odd integer. Define

$$I_1 := \frac{1}{2\pi i} \int_{c-iT_1}^{c+iT_2} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw,$$

$$I_2 := \frac{1}{2\pi i} \int_{c+iT_2}^{-K+iT_2} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw,$$

$$I_3 := \frac{1}{2\pi i} \int_{-K+iT_2}^{-K-iT_1} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw,$$

$$I_4 := \frac{1}{2\pi i} \int_{-K-iT_1}^{c-iT_1} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw.$$

Then,

$$I_1 = -\frac{\zeta'}{\zeta}(s) + \frac{n^{1-s}}{1-s} - \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^{\rho-s}}{\rho - s} - \sum_{1 \leqslant k < \frac{K-\sigma}{2}} \frac{n^{-2k-s}}{-2k - s} - (I_2 + I_3 + I_4)$$

where the sum over ρ ranges over the nontrivial zeros of $\zeta(s)$ with imaginary part γ such that $-T_1 < \gamma - t < T_2$ and such zeros are counted with multiplicity.

Now, we find upper bounds for the size of I_2, I_3 and I_4. We have

$$I_2 = \frac{1}{2\pi i} \int_{c+iT_2}^{-1-\sigma+iT_2} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw + \frac{1}{2\pi i} \int_{-1-\sigma+iT_2}^{-K+iT_2} -\frac{\zeta'}{\zeta}(s + w) \frac{n^w}{w} dw$$

$$\ll \frac{\log^2 T}{T} n^c (\sigma + 1) + \int_{-K}^{-1-\sigma} \frac{\log(|\sigma + it + u + iT_2| + 1)}{|\sigma + it + u + iT_2|} n^u \, du$$

$$\ll \frac{\log^2 T}{T} + \frac{\log T}{T} \int_{-\infty}^{-1-\sigma} n^u \, du \ll \frac{\log^2 T}{T} \tag{4.3.6}$$

by Lemmata 4.2.1 and 4.2.2. The bound in (4.3.6) holds also for I_4 by symmetry. By

Lemma 4.2.1, we have

$$I_3 \ll n^{-K} \int_{-T_1}^{T_2} \frac{\log(|\sigma + it - K + iv| + 1)}{|\sigma + it - K + iv|} \, dv$$

$$\ll n^{-K} \frac{\log(KT)}{K} T.$$

Letting $K \to \infty$ with the assumption that $\sigma - K$ is an odd integer, we obtain

$$E(n, s) = -\frac{\zeta'}{\zeta}(s) + \frac{n^{1-s}}{1-s} - \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^{\rho - s}}{\rho - s} + \sum_{k=1}^{\infty} \frac{n^{-2k-s}}{2k+s} + O\left(\frac{\log^2 T}{T}\right)$$

for $\sigma > 2$ and $n \geqslant 2$ where the implied constant depends on s but not on n. Let $x \geqslant 3$ be a multiple of p. Then,

$$\sum_{2 \leqslant n \leqslant x} \chi(n) E(n, s) = \sum_{2 \leqslant n \leqslant x} \chi(n) \left[-\frac{\zeta'}{\zeta}(s) + \frac{n^{1-s}}{1-s} - \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^{\rho - s}}{\rho - s} + \sum_{k=1}^{\infty} \frac{n^{-2k-s}}{2k+s} \right]$$

$$+ O\left(\frac{x \log^2 T}{T}\right). \tag{4.3.7}$$

Since $p \mid x$, we have

$$\sum_{2 \leqslant n \leqslant x} \chi(n) \left[-\frac{\zeta'}{\zeta}(s) + \frac{n^{1-s}}{1-s} - \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^{\rho - s}}{\rho - s} + \sum_{k=1}^{\infty} \frac{n^{-2k-s}}{2k+s} \right]$$

$$= \frac{\zeta'}{\zeta}(s) - \frac{1}{1-s} + \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{1}{\rho - s}$$

$$+ \frac{1}{1-s} \sum_{1 \leqslant n \leqslant x} \chi(n) n^{1-s} + \sum_{1 \leqslant n \leqslant x} \chi(n) n^{-s} \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^{\rho}}{s - \rho}$$

$$+ \sum_{2 \leqslant n \leqslant x} \chi(n) \sum_{k=1}^{\infty} \frac{n^{-2k-s}}{2k+s}. \tag{4.3.8}$$

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Define

$$S_1 := \frac{1}{1-s} \sum_{1 \leqslant n \leqslant x} \chi(n) n^{1-s},$$

$$S_2 := \sum_{1 \leqslant n \leqslant x} \chi(n) n^{-s} \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{n^\rho}{s - \rho},$$

$$S_3 := \sum_{2 \leqslant n \leqslant x} \chi(n) \sum_{k=1}^\infty \frac{n^{-2k-s}}{2k+s}.$$

Then by (4.3.7) and (4.3.8), we have

$$\sum_{2 \leqslant n \leqslant x} \chi(n) E(n, s) = \frac{\zeta'}{\zeta}(s) - \frac{1}{1-s} + \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{1}{\rho - s}$$

$$+ S_1 + S_2 + S_3$$

$$+ O\left(\frac{x \log^2 T}{T}\right). \tag{4.3.9}$$

For S_1, we have

$$S_1 = \frac{1}{1-s} L(s-1, \chi) + O_s\left(\int_x^\infty \left(\sum_{n \leqslant u} \chi(n)\right) u^{-\sigma} \, du\right)$$

$$= \frac{1}{1-s} L(s-1, \chi) + O_{s,p}\left(x^{1-\sigma}\right) \tag{4.3.10}$$

by partial summation and the Pólya-Vinogradov Inequality, Lemma 3.2.2.

For S_3, we have

$$S_3 = \sum_{k=1}^\infty \frac{1}{2k+s} \sum_{2 \leqslant n \leqslant x} \chi(n) n^{-2k-s} \tag{4.3.11}$$

and

$$\sum_{2 \leqslant n \leqslant x} \chi(n) n^{-2k-s} = \sum_{1 \leqslant n \leqslant x} \chi(n) n^{-2k-s} - 1$$

$$= L(2k+s, \chi) - 1 - \sum_{n \geqslant x+1} \chi(n) n^{-2k-s}. \tag{4.3.12}$$

By the integral test, we have

$$\left| \sum_{n \geqslant x+1} \chi(n) n^{-2k-s} \right| \leqslant (x+1)^{-2k-\sigma} + \int_{x+1}^{\infty} u^{-2k-\sigma} \, du$$

$$= (x+1)^{-2k-\sigma} + \frac{(x+1)^{-2k-\sigma+1}}{2k+\sigma-1} \qquad (4.3.13)$$

for $k \geqslant 1$ and $\sigma > 2$. Thus, by (4.3.11)-(4.3.13), we have

$$S_3 = \sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1 + O\left((x+1)^{-2k-\sigma} + \frac{(x+1)^{-2k-\sigma+1}}{2k+\sigma-1} \right)}{2k+s}. \qquad (4.3.14)$$

Note that the series

$$\sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s}$$

is absolutely convergent for $\sigma > -1$ since $L(2k+s, \chi) - 1 \ll 2^{-2k-\sigma}$ by (4.3.13) where $x+1$ is replaced by 2. The contribution of the error term in (4.3.14) is

$$\ll \sum_{k=1}^{\infty} \frac{x^{-2k-\sigma}}{|2k+s|} + \sum_{k=1}^{\infty} \frac{x^{-2k-\sigma+1}}{|2k+s|(2k+\sigma-1)}$$

$$\leqslant \sum_{k=1}^{\infty} x^{-2k-\sigma} + \sum_{k=1}^{\infty} \frac{x^{-1-\sigma}}{|2k+s|(2k+\sigma-1)}$$

$$\ll x^{-2-\sigma} + x^{-1-\sigma} \ll x^{-1-\sigma} \qquad (4.3.15)$$

where the implied constant is absolute. Thus,

$$S_3 = \sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s} + O\left(x^{-1-\sigma} \right). \qquad (4.3.16)$$

Now we consider S_2. By the identity,

$$\frac{1}{s-\rho} = -\frac{1}{\rho} - \frac{s}{\rho^2} + \frac{s^2}{\rho^2(s-\rho)},$$

we have

$$\sum_{1\leqslant n\leqslant x} \chi(n)n^{-s} \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{n^\rho}{s-\rho} = \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} n^\rho \left(-\frac{1}{\rho} - \frac{s}{\rho^2} + \frac{s^2}{\rho^2(s-\rho)}\right)$$

$$= \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \left(-\frac{n^\rho}{\rho}\right) - s \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{1}{\rho^2} \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^{s-\rho}}$$

$$+ s^2 \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{1}{\rho^2(s-\rho)} \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^{s-\rho}}.$$

Define

$$S_{21} := \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \left(-\frac{n^\rho}{\rho}\right),$$

$$S_{22} := -s \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{1}{\rho^2} \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^{s-\rho}},$$

$$S_{23} := s^2 \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{1}{\rho^2(s-\rho)} \sum_{1\leqslant n\leqslant x} \frac{\chi(n)}{n^{s-\rho}}.$$

Then $S_2 = S_{21} + S_{22} + S_{23}$. For S_{22}, we have

$$S_{22} = -s \sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{1}{\rho^2} \left(L(s-\rho,\chi) + O\left(\frac{1}{x^{\sigma-\beta}} + \int_x^\infty \frac{du}{u^{\sigma-\beta}}\right)\right).$$

The contribution of the error term above is

$$\ll \sum_\rho \frac{x^{\beta-\sigma+1}}{|\rho|^2(\sigma-\beta-1)} \ll x^{2-\sigma}$$

since s is fixed with $\sigma > 2$, $\Re(\rho) < 1$ for all zeros ρ and $\sum_\rho \frac{1}{\rho^2}$ is absolutely convergent. Moreover,

$$\sum_{\substack{\rho \\ -T_1<\gamma-t<T_2}} \frac{L(s-\rho,\chi)}{\rho^2} = \sum_\rho \frac{L(s-\rho,\chi)}{\rho^2} + O\left(\sum_{\substack{\rho \\ \gamma\notin(-T_1+t,T_2+t)}} \frac{|L(s-\rho,\chi)|}{|\rho|^2}\right).$$

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The error term above is

$$\sum_{\substack{\rho \\ \gamma \notin (-T_1+t, T_2+t)}} \frac{|L(s-\rho,\chi)|}{|\rho|^2} \ll \zeta(\sigma-1) \sum_{\substack{\rho \\ \gamma \notin (-T_1+t, T_2+t)}} \frac{1}{|\rho|^2}$$

$$\ll \sum_{\substack{\rho \\ \gamma \notin (-T_1+t, T_2+t)}} \frac{1}{|\rho|^2} \tag{4.3.17}$$

for $\sigma > 2$. Since t is fixed and $T_1, T_2 \in (T/2, 2T)$, we have

$$\sum_{\substack{\rho \\ \gamma \notin (-T_1+t, T_2+t)}} \frac{1}{|\rho|^2} \ll \int_T^\infty \frac{1}{u^2} d(N(u)) \tag{4.3.18}$$

where $N(u)$ is the number of zeros ρ with $0 < \gamma = \Im(\rho) \leqslant u$ counted with multiplicity. Let $R(u)$ be defined by

$$N(u) = \frac{u}{2\pi} \log \frac{u}{2\pi e} + R(u).$$

Then

$$\int_T^\infty \frac{1}{u^2} dN(u) = \int_T^\infty \frac{1}{u^2} \left(\frac{1}{2\pi} \log \left(\frac{u}{2\pi e} \right) + \frac{u}{2\pi} \frac{2\pi e}{u} \frac{1}{2\pi e} \right) du$$

$$+ \int_T^\infty \frac{1}{u^2} dR(u)$$

$$\ll \int_T^\infty \frac{\log u}{u^2} du + \left| -\frac{R(T)}{T^2} + \int_T^\infty \frac{R(u)}{u^3} du \right|$$

$$\ll \frac{1 + \log T}{T} + \frac{\log T}{T^2} \ll \frac{\log T}{T} \tag{4.3.19}$$

on integration by parts and Lemma 4.2.6. By (4.3.17)-(4.3.19), we have

$$\sum_{\substack{\rho \\ \gamma \notin (-T_1+t, T_2+t)}} \frac{|L(s-\rho,\chi)|}{|\rho|^2} \ll \frac{\log T}{T} \tag{4.3.20}$$

for $\sigma > 2$. Thus,

$$S_{22} = -s \sum_\rho \frac{L(s-\rho,\chi)}{\rho^2} + O\left(\frac{\log T}{T} \right) + O\left(x^{2-\sigma} \right) \tag{4.3.21}$$

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Similarly, we have

$$S_{23} = s^2 \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{1}{\rho^2(s-\rho)} \left(L(s-\rho, \chi) + O(x^{2-\sigma}) \right)$$

$$= s^2 \sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2(s-\rho)} + O\left(\frac{\log T}{T} \right) + O\left(x^{2-\sigma} \right) \tag{4.3.22}$$

by using the same bounds above since $|s - \rho| \geqslant \sigma - 1 > 1$.

By considering the terms with $n = 1$ and $n \geqslant 2$ in S_{21}, we have

$$S_{21} = \sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \left(-\frac{n^\rho}{\rho} \right) + \sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \left(-\frac{1}{\rho} \right) \tag{4.3.23}$$

Now, we want to replace the condition $-T_1 < \gamma - t < T_2$ by $-T \leqslant \gamma \leqslant T$ in the sums above and in the third term on the right-hand side of (4.3.9). Since T_1 and T_2 satisfy $-T - 1 \leqslant t - T_1 \leqslant -T$ and $T \leqslant t + T_2 \leqslant T + 1$, the number of zeros we include or discard by the replacement of the condition $-T_1 < \gamma - t < T_2$ by $-T \leqslant \gamma \leqslant T$ is $\ll_t \log T$ by Lemma 4.2.6. Also, for such zeros ρ, we have $\frac{1}{\rho} \ll \frac{1}{T}$ and $\frac{1}{\rho - s} \ll \frac{1}{T}$ since t is fixed and $T \geqslant 4(1 + |t|)$. Thus, for the third term on the right-hand side of (4.3.9), we have

$$\sum_{\substack{\rho \\ -T_1 < \gamma - t < T_2}} \frac{1}{\rho - s} = \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho - s} + O\left(\frac{\log T}{T} \right) \tag{4.3.24}$$

By (4.3.23) and the argument above, we have

$$S_{21} = \sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \left[\sum_{\substack{\rho \\ |\gamma| \leqslant T}} \left(-\frac{n^\rho}{\rho} \right) + O\left(\frac{n \log T}{T} \right) \right]$$

$$- \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho} + O\left(\frac{\log T}{T} \right).$$

The contributions of the error terms above are

$$\ll \frac{\log T}{T} \sum_{n \leqslant x} \frac{1}{n^{\sigma-1}} \ll \frac{\log T}{T}$$

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since $\sigma > 2$. Thus,

$$S_{21} = \sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \left(-\frac{n^\rho}{\rho} \right) - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho} + O\left(\frac{\log T}{T} \right). \tag{4.3.25}$$

By Lemma 4.2.5, we have, for $n \geqslant 2$,

$$\Delta(n) := \sum_{k \leqslant n-1} \Lambda(k) + \frac{\Lambda(n)}{2} - n = - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{n^\rho}{\rho} - \log(2\pi) - \frac{1}{2} \log\left(1 - \frac{1}{n^2} \right) + R(n,T)$$

where

$$R(n,T) \ll (\log n) \min\left\{ 1, \frac{n}{T\langle n \rangle} \right\} + \frac{n \log^2(nT)}{T} \tag{4.3.26}$$

and $\langle n \rangle$ denotes the distance from n to the nearest prime power, other than n itself. Thus,

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \left(-\frac{n^\rho}{\rho} \right) = \sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \left[\Delta(n) + \log(2\pi) + \frac{1}{2} \log\left(1 - \frac{1}{n^2} \right) - R(n,T) \right].$$

$$\tag{4.3.27}$$

Since $\Delta(n) \ll n$, we have

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)\Delta(n)}{n^s} = \sum_{n=2}^{\infty} \frac{\chi(n)\Delta(n)}{n^s} + O\left(\sum_{n > x} \frac{1}{n^{\sigma-1}} \right)$$

$$= \Delta(s,\chi) + O\left(x^{2-\sigma} \right) \tag{4.3.28}$$

and similarly

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} = L(s,\chi) - 1 + O\left(x^{1-\sigma} \right). \tag{4.3.29}$$

Since $\log\left(1 - 1/n^2 \right) \ll 1/n^2$, we have

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2} \right) = \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2} \right) + O\left(\sum_{n > x} \frac{1}{n^{\sigma+2}} \right)$$

$$= \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2} \right) + O\left(x^{-1-\sigma} \right). \tag{4.3.30}$$

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By (4.3.26), we have

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} R(n,T) \ll \sum_{2 \leqslant n \leqslant x} \frac{1}{n^\sigma} \left((\log n) \min \left\{ 1, \frac{n}{T\langle n \rangle} \right\} + \frac{n \log^2(nT)}{T} \right)$$

$$\ll \frac{\log^2 T}{T} \sum_{2 \leqslant n \leqslant x} \frac{\log^2 n}{n^{\sigma - 1}}$$

$$\ll \frac{\log^2 T}{T} \tag{4.3.31}$$

since $\sigma > 2$. By (4.3.27) and (4.3.28)-(4.3.31), we have

$$\sum_{2 \leqslant n \leqslant x} \frac{\chi(n)}{n^s} \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \left(-\frac{n^\rho}{\rho} \right) = \Delta(s,\chi) + \log(2\pi) \left(L(s,\chi) - 1 \right) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log \left(1 - \frac{1}{n^2} \right)$$

$$+ O\left(x^{2-\sigma} \right) + O\left(\frac{\log^2 T}{T} \right). \tag{4.3.32}$$

By (4.3.25) and (4.3.32), we have

$$S_{21} = \Delta(s,\chi) + \log(2\pi) \left(L(s,\chi) - 1 \right) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log \left(1 - \frac{1}{n^2} \right) - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho}$$

$$+ O\left(x^{2-\sigma} \right) + O\left(\frac{\log^2 T}{T} \right). \tag{4.3.33}$$

By (4.3.21), (4.3.22) and (4.3.33), we have

$$S_2 = \Delta(s,\chi) + \log(2\pi) \left(L(s,\chi) - 1 \right) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log \left(1 - \frac{1}{n^2} \right) - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho}$$

$$- s \sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2} + s^2 \sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2(s-\rho)}$$

$$+ O\left(x^{2-\sigma} \right) + O\left(\frac{\log^2 T}{T} \right). \tag{4.3.34}$$

By (4.3.9), (4.3.16), (4.3.10), (4.3.24) and (4.3.34), , we have

$$\sum_{2 \leqslant n \leqslant x} \chi(n)E(n,s) = \Delta(s,\chi) + \log(2\pi)\left(L(s,\chi) - 1\right) + \frac{1}{2}\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right)$$
$$+ \frac{\zeta'}{\zeta}(s) - \frac{1}{1-s} + \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho - s} - \sum_{\substack{\rho \\ |\gamma| \leqslant T}} \frac{1}{\rho}$$
$$+ \frac{1}{1-s}L(s-1,\chi) - s\sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2} + s^2\sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2(s-\rho)}$$
$$+ \sum_{k=1}^{\infty} \frac{L(2k+s,\chi) - 1}{2k+s}$$
$$+ O\left(\frac{x\log^2 T}{T}\right) + O\left(x^{2-\sigma}\right).$$

By letting $T \to \infty$ and using Lemma 4.2.4 in the form

$$\frac{\zeta'}{\zeta}(s) - \frac{1}{1-s} - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \log(2\pi) - \frac{\gamma_0}{2} - 1 - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right)$$

where γ_0 is the Euler-Mascheroni constant, we have

$$\sum_{2 \leqslant n \leqslant x} \chi(n)E(n,s) = \Delta(s,\chi) + \log(2\pi)L(s,\chi) + \frac{1}{2}\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right)$$
$$- \frac{\gamma_0}{2} - 1 - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \frac{1}{1-s}L(s-1,\chi)$$
$$- s\sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2} + s^2\sum_{\rho} \frac{L(s-\rho,\chi)}{\rho^2(s-\rho)}$$
$$+ \sum_{k=1}^{\infty} \frac{L(2k+s,\chi) - 1}{2k+s} + O\left(x^{2-\sigma}\right).$$

By taking $x = pk$ and letting $k \to \infty$ among natural numbers, we have

$$\lim_{\substack{k \to \infty \\ k \in \mathbb{N}}} \sum_{2 \leqslant n \leqslant pk} \chi(n) E(n, s) = \Delta(s, \chi) + \log(2\pi) L(s, \chi) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2}\right)$$

$$- \frac{\gamma_0}{2} - 1 - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \frac{1}{1-s} L(s-1, \chi)$$

$$- s \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2} + s^2 \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2(s - \rho)}$$

$$+ \sum_{k=1}^{\infty} \frac{L(2k + s, \chi) - 1}{2k + s}$$

for $\sigma > 2$ which finishes the proof of Proposition 4.3.2. $\qquad \square$

4.4 Proof of Theorem 1.3.1

By Propositions 4.3.1 and 4.3.2, we have

$$\Delta(s, \chi) + \log(2\pi) L(s, \chi) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2}\right)$$

$$- \frac{\gamma_0}{2} - 1 - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \frac{1}{1-s} L(s-1, \chi)$$

$$- s \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2} + s^2 \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2(s - \rho)} + \sum_{k=1}^{\infty} \frac{L(2k + s, \chi) - 1}{2k + s}$$

$$= \frac{\tau(\chi)\chi(-1)}{\pi i(p - 1)} \sum_{\substack{\psi \pmod{p} \\ \psi(-1) = -1}} L(1, \psi) \tau\left(\overline{\chi\psi}\right) \frac{L'}{L}(s, \chi\psi) + L(0, \chi) \frac{L'}{L}(s, \chi_0) \qquad (4.4.1)$$

for $\sigma > 2$. Define

$$G(s, \chi) := L(0, \chi) \frac{L'}{L}(s, \chi_0) + \frac{L(s-1, \chi)}{s - 1} + s \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2} - s^2 \sum_{\rho} \frac{L(s - \rho, \chi)}{\rho^2(s - \rho)}$$

$$- \log(2\pi) L(s, \chi) + \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right)$$

$$- \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \log\left(1 - \frac{1}{n^2}\right) - \sum_{k=1}^{\infty} \frac{L(2k + s, \chi) - 1}{2k + s} + \frac{\gamma_0}{2} + 1$$

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for $\sigma > 2$. Then by (4.4.1), we have

$$\Delta(s, \chi) = \frac{\tau(\chi)\chi(-1)}{\pi i (p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1, \psi)\tau\left(\overline{\chi\psi}\right) \frac{L'}{L}(s, \chi\psi) + G(s, \chi) \qquad (4.4.2)$$

for $\sigma > 2$. Now, we observe that the function $G(s, \chi)$ is analytic in $\sigma > 1/2$ unconditionally. The term

$$-\log(2\pi)L(s, \chi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) - \frac{1}{2}\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right) - \sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s}$$

is clearly analytic in $\sigma > 1/2$ since $L(s, \chi)$ is analytic therein and

$$\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}\log\left(1 - \frac{1}{n^2}\right) \ll \sum_{n=2}^{\infty} \frac{1}{n^{\sigma+2}},$$

and

$$\sum_{k=1}^{\infty} \frac{L(2k+s, \chi) - 1}{2k+s} \ll \sum_{k=1}^{\infty} \frac{2^{-2k-\sigma}}{|2k+s|}$$

and $\Gamma(z)$ is never zero and the poles of $\Gamma(z)$ are at $z = 0, -1, -2, \ldots$. For the term

$$L(0, \chi)\frac{L'}{L}(s, \chi_0) + \frac{L(s-1, \chi)}{s-1} + s\sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2} - s^2\sum_{\rho} \frac{L(s-\rho, \chi)}{\rho^2(s-\rho)},$$

let $s_0 = \sigma_0 + it_0$ be a complex number with $\sigma_0 := \Re(s_0) > 1/2$ such that $s_0 \neq 1$ and $\zeta(s_0) \neq 0$. Then the term

$$L(0, \chi)\frac{L'}{L}(s, \chi_0) + \frac{L(s-1, \chi)}{s-1} \qquad (4.4.3)$$

is clearly analytic at $s = s_0$. By Lemma 4.2.3 and the fact that $\Re(\rho) < 1$ for all zeros ρ of $\zeta(s)$, we have the bounds

$$L(s_0 - \rho, \chi) \ll_p \begin{cases} (|t_0 - \gamma| + 2)^{\frac{1}{2}+\epsilon} & \text{if } \sigma_0 \geq 1, \\ (|t_0 - \gamma| + 2)^{\frac{1}{2}-(\sigma_0-1)+\epsilon} & \text{if } \frac{1}{2} < \sigma_0 \leq 1 \end{cases} \qquad (4.4.4)$$

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for any $\epsilon > 0$. Note that for such a complex number s_0, we have $|s_0 - \rho| \gg 1$ for all zeros ρ and thus convergence of the sums

$$\sum_\rho \frac{L(s-\rho,\chi)}{\rho^2} \qquad \text{and} \qquad \sum_\rho \frac{L(s-\rho,\chi)}{\rho^2(s-\rho)} \tag{4.4.5}$$

when $s \to s_0$ is determined by the contributions of the zeros ρ with $|\gamma| \geqslant 2|t|$. By (4.4.4), we have

$$
\left| \sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{L(s-\rho,\chi)}{\rho^2} \right| + \left| \sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{L(s-\rho,\chi)}{\rho^2(s-\rho)} \right|
$$

$$
\ll \sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{|L(s-\rho,\chi)|}{|\rho|^2}
$$

$$
\ll_p \sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{\max\left\{ (|t_0 - \gamma| + 2)^{\frac{1}{2}+\epsilon}, (|t_0 - \gamma| + 2)^{\frac{1}{2}-(\sigma_0-1)+\epsilon} \right\}}{|\gamma|^2}
$$

$$
\ll_p \sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{\max\left\{ (|\gamma| + 2)^{\frac{1}{2}+\epsilon}, (|\gamma| + 2)^{\frac{3}{2}-\sigma_0+\epsilon} \right\}}{|\gamma|^2}. \tag{4.4.6}
$$

Since the sum $\sum_\rho \frac{1}{\gamma^{1+\epsilon}}$ is convergent for any $\epsilon > 0$, we have

$$\sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{(|\gamma| + 2)^{\frac{1}{2}+\epsilon}}{|\gamma|^2} \ll \sum_\rho \frac{1}{\gamma^{\frac{3}{2}-\epsilon}} \ll 1$$

and

$$\sum_{\substack{\rho \\ |\gamma|\geqslant 2|t|}} \frac{(|\gamma| + 2)^{\frac{3}{2}-\sigma_0+\epsilon}}{|\gamma|^2} \ll \sum_\rho \frac{1}{\gamma^{\frac{1}{2}+\sigma_0-\epsilon}} \ll 1$$

since $\sigma_0 > 1/2$ is fixed and $\epsilon > 0$ is arbitrary. Thus, the sums in (4.4.5) are absolutely convergent in this case.

If $s_0 = 1$, the the sums in (4.4.5) are absolutely convergent at $s = s_0$ as in the previous case since $|1 - \rho| \gg 1$ for all zeros ρ. Now, we consider the behaviour of the terms in

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(4.4.3) as $s \to s_0 = 1$. Since

$$
\begin{aligned}
\frac{L'}{L}(s, \chi_0) &= \frac{d}{ds} \log \left(\zeta(s) \left(1 - \frac{1}{p^s} \right) \right) \\
&= \frac{\zeta'}{\zeta}(s) + \frac{\log p}{p^s - 1} \\
&= -\frac{1}{s - 1} + f_1(s)
\end{aligned}
\tag{4.4.7}
$$

where $f_1(s)$ is analytic at $s = 1$ by Lemma 4.2.4, it is enough to consider

$$
\begin{aligned}
\lim_{s \to 1} \left(L(0, \chi) \left(-\frac{1}{s - 1} \right) + \frac{L(s - 1, \chi)}{s - 1} \right) &= \lim_{s \to 1} \frac{L(s - 1, \chi) - L(0, \chi)}{s - 1} \\
&= L'(0, \chi)
\end{aligned}
$$

and the existence of the limit above gives the analyticity of $G(s, \chi)$ at $s = 1$.

Now assume that there exists a zero ρ_0 of $\zeta(s)$ with $\beta_0 := \Re(\rho_0) > 1/2$. Let m_{ρ_0} be the multiplicity of ρ_0. By the argument in (4.4.6), the sums over zeros in the definition of $G(s, \chi)$ are absolutely convergent if s is not close to a zero ρ. Thus, for the analyticity of $G(s, \chi)$ at $s = \rho_0$ it is enough to consider the behaviour of

$$
L(0, \chi) \frac{L'}{L}(s, \chi_0) - s^2 \frac{L(s - \rho_0, \chi)}{\rho^2(s - \rho_0)} m_{\rho_0}
\tag{4.4.8}
$$

when $s \to \rho_0$ as the other terms in the definition of $G(s, \chi)$ are analytic at $s = \rho_0$. Since

$$
\frac{L'}{L}(s, \chi_0) = m_{\rho_0} \frac{1}{s - \rho_0} + f_2(s)
\tag{4.4.9}
$$

where $f_2(s)$ is analytic at $s = \rho_0$ by (4.4.7) and Lemma 4.2.4, it is enough to consider

$$
\begin{aligned}
&\lim_{s \to \rho_0} \left(L(0, \chi) m_{\rho_0} \frac{1}{s - \rho_0} - s^2 \frac{L(s - \rho_0, \chi)}{\rho^2(s - \rho_0)} m_{\rho_0} \right) \\
&= m_{\rho_0} \lim_{s \to \rho_0} \frac{L(0, \chi) - \frac{s^2}{\rho_0^2} L(s - \rho_0, \chi)}{s - \rho_0} \\
&= m_{\rho_0} \lim_{h \to 0} \frac{L(0, \chi) - \frac{h^2 + 2h\rho_0 + \rho_0^2}{\rho_0^2} L(h, \chi)}{h} \\
&= m_{\rho_0} \left(\frac{2}{\rho_0} L(0, \chi) - L'(0, \chi) \right)
\end{aligned}
$$

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and the existence of the limit above gives the analyticity of $G(s,\chi)$ at $s = \rho_0$.

Hence, $G(s,\chi)$ is analytic in $\sigma > 1/2$ unconditionally and this finishes the proof of Theorem 1.3.1 by analytic continuation and (4.4.2).

4.5   Proof of Corollary 1.3.2

By Euler's criterion, [23, Corollary 2.38], we have

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}.$$

Thus, if $p \equiv 1 \pmod 4$, then the Legendre symbol $\left(\frac{\cdot}{p}\right)$ is an even Dirichlet character, i.e. $\left(\frac{-1}{p}\right) = 1$. If χ is an odd Dirichlet character, i.e. $\chi(-1) = -1$, then the Dirichlet characters $\chi\psi$ in the first term on the right-hand side of (4.1.1) range over all even Dirichlet characters and thus one of the terms in this sum is

$$L\left(1, \overline{\chi}\left(\frac{\cdot}{p}\right)\right) \tau\left(\left(\frac{\cdot}{p}\right)\right) \frac{L'}{L}\left(s, \left(\frac{\cdot}{p}\right)\right).$$

If $\rho \in \mathcal{R}$ is a zero of $L\left(s, \left(\frac{\cdot}{p}\right)\right)$, then the pole of the above function at $s = \rho$ can not be canceled by another term in the sum on the right-hand side of (4.1.1). This gives the desired contradiction with the assumption that $\Delta(s,\chi)$ is analytic in $\mathcal{R}$. By a similar argument, the second assertion of Corollary 1.3.2 follows and this finishes the proof.

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