Counting Pentagons in Triangle-free Binary Matroids

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A rank-n binary matroid is a spanning subset E of $\mathbb{F}_2^n \setminus \{0\}$, a triangle is a set of three elements from E which sum to zero, and the density of a rank-n binary matroid is $|E|/2^n$. We begin by giving a new exposition of a result due to Davydov and Tombak, which states that if E is a rank-n triangle-free matroid of density greater than $1/4$, then there is a dimension-$(n-2)$ subspace of $\mathbb{F}_2^n$ which is disjoint from E. With this as a starting point, we provide a recursive structural decomposition for all maximal triangle-free binary matroids of density greater than $1/4$. A key component of this decomposition is an analogous characterization of matroids which are maximal with respect to containing exactly one triangle.

A pentagon in a binary matroid E is a set of 5 elements which sum to zero. We conjecture that if E is a rank-n triangle-free binary matroid, then E contains at most 2^{4n-16} pentagons, and provide a potential extremal example. We first resolve this conjecture when E has density at most $\sqrt[4]{120}/16 \approx 0.20568$. Thereafter, we use our structural decomposition to show that the conjecture holds for matroids with density greater than $1/4$. This leaves the interval $\left(\sqrt[4]{120}/16, 1/4\right]$, where the conjecture remains unresolved.

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Chapter 1

Introduction

This thesis provides a recursive structural decomposition for simple binary matroids which have a large number of elements and are triangle-free. We then use this decomposition to bound the number of pentagons in large triangle-free binary matroids. The material touches matroid theory, coding theory, and additive combinatorics; we will use terminology that fits most closely with additive combinatorics.

A *simple binary matroid* (hereafter a *matroid*) is a subset E of the binary vector space $\mathbb{F}_2^n$ which does not contain the zero vector. This definition deviates significantly from the standard definition of an abstract binary matroid. The *rank* of a matroid $E \subseteq \mathbb{F}_2^n$ is the dimension of the subspace which it spans. If E spans all of $\mathbb{F}_2^n$ we say that E is *full-rank*. It is often convenient to assume that E is full-rank by restricting to the span and making a linear change of coordinates. Two matroids in dimension n are *isomorphic* if there is a linear isomorphism which takes one to the other.

A *hyperplane* is a dimension-$(n-1)$ subspace of $\mathbb{F}_2^n$. A *rank-n affine geometry* is a set of the form $E = \mathbb{F}_2^n \setminus H$ where H is a hyperplane. All rank-n affine geometries are isomorphic. When the dimension is obvious we will drop the prefix "rank-n" and simply refer to such sets as *affine geometries*.

A *triangle* in a matroid E is a set $\{x_1, x_2, x_3\} \subseteq E$ of three elements of E such that $x_1 + x_2 + x_3 = 0$. Since matroids do not contain the zero vector, any triple $(x_1, x_2, x_3) \in E^3$ whose sum is zero will be a triangle when treated as a set. Note also that if we restrict to the subspace spanned by x_1, x_2, x_3, the matroid is isomorphic to the set of columns of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We say a matroid is *triangle-free* if it contains no triangles. A triangle-free matroid is *maximal* if it is not a proper subset of another triangle-free matroid. In the language of additive combinatorics, a rank-n matroid E is triangle-free if E and $E+E$ are disjoint, and maximal if E and $E+E$ cover all of $\mathbb{F}_2^n$ (i.e. $E \cup (E+E) = \mathbb{F}_2^n$). Rank-$n$ triangle-free matroids can contain at most $\frac{1}{2}(2^n)$ points and equality holds only for rank-n *affine geometries* $E = \mathbb{F}_2^n \setminus H$ for a hyperplane H [4].

A *pentagon* is a matroid E is a set $\{x_1, \dots, x_5\} \subseteq E$ of five elements of E such that $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ and no proper subset sums to zero. A matroid E could contain a tuple $(x_1, \dots, x_5) \in E^5$ which sums to zero, where $x_1 + x_2 + x_3 = 0$ and $x_4 = x_5$; we do not consider this a pentagon. Note that when E is triangle-free, every 5-tuple in E^5 whose sum is zero forms a pentagon as a set. If we restrict to the subspace spanned by a pentagon and just the elements of the pentagon, the matroid is isomorphic to the set of columns of the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We refer to the isomorphism class of the above matroid as *the* pentagon, with a definite article. The pentagon is a 4-dimensional simple binary matroid.

The *density* of a full-rank matroid $E \subseteq \mathbb{F}_2^n$ is the fraction $|E|/|\mathbb{F}_2^n| = |E|/2^n$. As an example, the pentagon has density $5/16$, and every affine geometry has density $1/2$. Since matroids never contain the zero vector it may seem reasonable to compare the density to $2^n - 1$ instead of the full $|\mathbb{F}_2^n|$. Our choice is similar to how the degree of a vertex in an n vertex graph is often compared to n instead of $n-1$. Our choice also makes it much easier to compare densities of matroids of different rank, and is more consistent with the additive combinatorics literature.

For a matroid $E \subseteq \mathbb{F}_2^n$, we call $E \times \mathbb{F}_2 \subseteq \mathbb{F}_2^{n+1}$ the *doubling of* E. Note that E and $E \times \mathbb{F}_2$ will have the same density in their ambient spaces. We can also repeat this process k times to obtain the matroid $E \times \mathbb{F}_2^k$, which we call the *k-th doubling of* E. In this context we can also say that the rank-n affine geometry is the $(n-1)$-th doubling of the rank-1 matroid $\{1\}$. If E is triangle-free, then it is easy to check that its doubling will be triangle-free as well; if $(x_1, \delta_1), (x_2, \delta_2), (x_3, \delta_3)$ were to form a triangle in $E \times \mathbb{F}_2$, then so would x_2, x_2, x_3 form a triangle in E (Note that since $0 \notin E$ we know that $(0, 1) \notin E \times \mathbb{F}_2$). Moreover, if E is maximal triangle-free then $E \times \mathbb{F}_2$ will be also be maximal.

The purpose of Chapter 2 is to give a new exposition of the following result. In its original form, this result (and its proof) were coding theoretic; here we rephrase it in terms of matroids.

Theorem 1.0.1 (Davydov and Tombak, [5])**.** *If $E \subseteq \mathbb{F}_2^n$ is triangle-free and the density of E is greater than $1/4$, then there is a dimension-$(n-2)$ subspace of $\mathbb{F}_2^n$ that is disjoint from E.*

A codimension-2 subspace is a highly structured set which contains a quarter of the total ambient space; One would not expect a random matroid to have a large disjoint subspace. This condition imposes a lot of structure on triangle-free sets, and in Chapter 3 we obtain a recursive structural decomposition for all triangle-free matroids with density greater than $1/4$. In the final chapter, we propose the following conjecture on the number of pentagons in triangle-free sets, and use this structural decomposition to resolve it in the case of triangle-free sets of density greater than $1/4$.

Conjecture 1.0.2. *Every triangle-free matroid E with rank n contains at most 2^{4n-16} pentagons. Moreover, equality holds only for the $(n-4)$-th doubling of the pentagon.*

Earlier we mentioned that our definition of a simple binary matroid differs somewhat from the standard definition; we now reconcile the definitions. Following Oxley [17], an abstract matroid is a pair (X, r) where X is a finite set, called the *ground set* and r is a function $r : 2^X \to \mathbb{N}$ called the *rank function* such that

(R1) $r(A) \leq |A|$ for any subset $A \subseteq X$,

(R2) if $A \subseteq B \subseteq X$ then $r(A) \leq r(B)$, and

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for any two subsets A, B of X.

An abstract matroid is *simple* if every singleton has rank 1, and every pair of elements from X has rank 2. A simple abstract matroid is *binary* if there exists a set of vectors $E \subseteq \mathbb{F}_2^n$ and a bijection $\tau : X \to E$ such that for every subset $A \subset X$ its rank $r(A)$ is equal to the rank of the matrix with columns $\tau(A)$. This definition can be extended for matroids representable over other fields.

Given a binary abstract matroid there is a unique matroid which represents it up to linear isomorphism. The set $E \subseteq \mathbb{F}_2^n$, along with a labeling of the elements, contains all the information needed to reconstruct the rank function of the matroid it represents. It is safe to identify a binary matroid with a set of vectors in $\mathbb{F}_2^n$, as binary representations are unique after a choice of basis [17]. The same is not always true for larger fields.

We will often consider the matroid obtained from E by restricting its ambient space $\mathbb{F}_2^n$ to some proper k-dimensional subspace W. To do this carefully, we must first make a linear change of coordinates so that W is the subspace whose first $n-k$ entries are zero. This change can be any linear isomorphism which takes a basis of the subspace W to the final k standard basis vectors of $\mathbb{F}_2^n$. We then take the set of vectors $W \cap E$, and consider them as a subset of $\mathbb{F}_2^k$ by restricting our attention to the final k entries. We will most often restrict to hyperplanes, in which case our change of coordinates must only make the first coordinate zero.

1.1 Graph Theory

Both main theorems of this thesis can be seen as analogues of results in graph theory. Several classical theorems about binary matroids can also be cast in this way, and a few recent papers [See, [1], [13], [16]] have also extended results about graphs with the subgraph and induced-subgraph orders to the setting of binary matroids. To properly tell this story it helps to begin with a simple example.

Recall that a rank-n affine geometry is the triangle-free matroid A with $\frac{1}{2}(2^n)$ points such that $A = \mathbb{F}_2^n \setminus H$ for a hyperplane H. We say that a binary matroid E is *affine* if it is contained in an affine geometry, or equivalently if there is a hyperplane H of $\mathbb{F}_2^n$ which is disjoint from E. Since every affine matroid is contained in an affine geometry, they will all be triangle-free; more than that, affine matroids contain no odd circuit. In fact, this gives an alternate characterization of affine matroids, they are precisely those sets which contain no odd circuits.

Proposition 1.1.1 ([17])**.** *$E \subseteq \mathbb{F}_2^n$ is affine if and only if it contains no odd circuit.*

Proof. Suppose that E is affine, and let H be the hyperplane which is disjoint from E. Let $\xi \in \mathbb{F}_2^n$ be the vector orthogonal to every element of H. Then $\langle \xi, x \rangle = 1$ for all $x \in E$. Suppose E did contain an odd circuit $\{x_1, \ldots, x_{2k+1}\}$ such that $x_1 + \cdot + x_{2k+1} = 0$. Then

$$1 = \sum_{i=1}^{2k+1} 1 = \sum_{i=1}^{2k+1} \langle \xi, x_i \rangle = \langle \xi, 0 \rangle = 0.$$

For the converse, suppose that E contains no odd circuits. Select a basis $\{x_1, \ldots, x_n\} \subseteq E$. Then every element of E must be written as a sum of an odd number of the basis elements. Let H be the hyperplane of elements of $\mathbb{F}_2^n$ which are the sum of an even number of basis elements. Then H is disjoint from E, and E is an affine set. $\square$

In this way, affine matroids can be compared to bipartite graphs, and the characterization above is analogous to the basic result that a graph is bipartite if and only if it contains no odd cycles. The analogy extends to a version of chromatic number for binary matroids. Define the *critical number* $\chi(E)$ of a binary matroid $E \subseteq \mathbb{F}_2^n$ to be the minimum co-dimension of a subspace of $\mathbb{F}_2^n$ which is disjoint from E (see Oxley [17], Chapter 9 for more details). The connection between chromatic number of graphs and critical number can actually be made explicit.

Proposition 1.1.2 ([17])**.** *If G is a graph with n vertices we can represent it by the set $M(G) \subset \mathbb{F}_2^n$ of its vertex-edge incidence vectors. Then $\chi(M(G)) = \lceil \log_2(\chi(G)) \rceil$, where $\chi(G)$ is the chromatic number of G*

Proof. Let G be a graph and let $M(G) \subset \mathbb{F}_2^n$ be the set of columns of its vertex-edge incidence matrix. First we show that G is 2-colourable if and only if $\chi(M(G)) = 1$. The graph G is 2-colourable if and only if G contains no odd cycles. Every odd cycle of G is an odd cycle of $M(G)$ so G contains no odd cycles if and only if $M(G)$ contains no odd cycles. By the previous Proposition, $M(G)$ contains no odd cycles if and only $\chi(G) = 1$.

Now suppose that $\chi(M(G)) \leq k$, by definition this implies that there exists a codimension-k subspace in the complement of $M(G)$. This codimension-k subspace is the intersection of k-hyperplanes, so $\chi(M(G) \leq k$ if and only if $M(G)$ is the union of k affine sets. By the previous paragraph this is equivalent to saying that the edge set of G is the union of k bipartite subgraphs. Finally, G is the union of k bipartite subgraphs if and only if G is 2^k-colourable, by assigning to each vertex the set of all its colours in each of the k bipartite subgraphs. Hence, $\chi(M(G)) \leq k$ if and only if $\chi(G) \leq k$, and the result follows by considering the minimum. $\square$

With this new notation we can restate the theorem of Davydov and Tombak; if $E \subseteq \mathbb{F}_2^n$ is triangle-free and the density of E is greater than $1/4$, then $\chi(E) \leq 2$. Geelen and Nelson [6] give a construction showing that no such result holds for density below $1/4$. That is, they show that for each $\epsilon > 0$ and each integer $c \geq 1$, there is a simple triangle-free binary matroid E with density greater than $1/4 - \epsilon$ such that $\chi(E) \geq c$.

Together, these results are analogous to the solution to the Erdös-Simonovits Problem by Brandt and Thomassé [2]. They showed that every triangle-free graph with minimum degree greater than $|V|/3$ is 4-colourable, where it was known that no such result holds for minimum degree below $|V|/3$. For every $\epsilon > 0$, Hajnal [see[2]], provided graphs with minimum degree $(1/3-\epsilon)n$ and arbitrarily high chromatic number. In our setting triangle-free sets take the place of triangle-free graphs, and density greater than $1/4$ takes the place of minimum degree greater than $|V|/3$. According to the exponential relationship between critical number and chromatic number, the conclusion that all such graphs satisfy $\chi(G) \leq 4$ translates to the conclusion that all our matroids have $\chi(E) \leq \lceil \log_2(4) \rceil = 2$. It is not the case that all such translations preserve truth, but they can at least lead to reasonable conjectures. To help develop a sense for the connection between the two conclusions we can compare a pair of easier results.

Theorem 1.1.3 (Govaerts and Storme, [7])**.** *If E is a triangle-free matroid with density greater than $5/16$, then there is a hyperplane that is disjoint from E ($\chi(E) = 1$).*

This result matches with an easy case of the Anrásfai-Erdös-Sós Theorem, which states that if a triangle free graph has minimum degree greater than $2|V|/5$ then it is 2-colourable. Here the direct translation of the conclusion works again; the conclusion that all such graphs have $\chi(G) \leq 2$ becomes the conclusion that all such matroids have $\chi(E) \leq \lceil \log_2(2) \rceil = 1$. We should also note the similarities in the conditions of the two theorems. The result of Govaerts and Storme does not hold for any density at most $5/16$. The pentagon and its repeated doublings are triangle-free sets with density exactly $5/16$, but which have critical number two (They contain pentagons, so cannot be affine). The obstruction to improving the graph theoretic analogue is strikingly similar; the 5-cycle and its balanced blowups are triangle-free graph with minimum degree $2|V|/5$, but which require at least three colours.

The analogy between blowups for graphs and doublings for binary matroids does not end here. Consider Turán's Theorem, which states that if an n-vertex graph is K_{r+1}-free, then it has at most $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ edges and, moreover, that equality hold only for the Turán graph $\mathrm{T}(n, r)$. The Turán graph is the (almost) balanced blowup of K_r. The analogous result for binary matroids is the Bose-Burton theorem. It states that if a rank-n binary matroid does not contain a subset isomorphic to $\mathbb{F}_2^{r+1} \setminus \{0\}$ then it contains at most $\left(1 - \frac{1}{2^r}\right) 2^n$ points. In this setting equality holds only for the Bose-Burton geometry $\mathrm{BB}(n, r)$, which is obtained from the full space $\mathbb{F}_2^n$ by deleting a codimension-r subspace [4]. While this classical construction of $\mathrm{BB}(n, r)$ does not mention doublings, we can also say that

$$\mathrm{BB}(n, r) = (\mathbb{F}_2^r \setminus \{0\}) \times \mathbb{F}_2^{n-r},$$

where the codimension-r subspace is now $\{0\} \times \mathbb{F}_2^{n-r}$. The largest possible binary matroids, $\mathbb{F}_2^r \setminus \{0\}$, are taking the same role as the largest possible simple graphs, K_r, and repeated doublings are taking the role of balanced blowups.

While Brandt and Thomassé obtained their result by first giving a structural characterization of triangle-free graphs with large minimum degree, here we take the opposite approach. We can prove the result of Davydov and Tombak without completely characterizing the structure of triangle-free sets with density greater than $1/4$. In Chapter 3 we then use their result to deduce a recursive decomposition. We then extend this decomposition to sets which contain exactly one triangle, and $\frac{1}{4}(2^n) + 2$ elements.

In Chapter 4 we attempt to extend another result from graphs to binary matroids. The motivation is a result of Grzesik [9], and independently by Hatami, Hladký, Král, Norine, and Razborov [10], resolving a question of Erdös about the number of pentagons in triangle-free graphs.

Theorem 1.1.4. *Every triangle-free graph with n vertices contains at most $(n/5)^5$ cycles of length five. Moreover, when n is divisible by 5 equality is obtained only by the balanced blow-up of the 5-cycle.*

Both groups proved this result using the technique of flag algebras, which does not apply readily to our geometric setting. We formulate an analogous conjecture using our established translation techniques. Triangle-free sets will replace triangle-free graphs, and the repeated doubling of the pentagon will replace the balanced blowup of the 5-cycle.

Conjecture 1.1.5. *Every triangle-free matroid E with rank n contains at most 2^{4n-16} circuits of length five. Moreover, equality holds only when E is a repeated doubling of the pentagon.*

1.2 Additive Combinatorics

Because it is easy to describe the condition that a binary matroid is triangle-free in terms of the additive structure of the set, we find that questions about triangle-free matroids are amenable to the techniques of additive combinatorics. Recall that a matroid $E \subseteq \mathbb{F}_2^n \setminus \{0\}$ is triangle-free if E and $E + E$ are disjoint, and moreover is maximal if E and $E + E$ cover all of $\mathbb{F}_2^n$. These will both be useful restatements when we want to use results from additive group theory and discrete Fourier analysis.

While our current definition of doubling is quite convenient when we start with a matroid and add extra dimensions, it becomes a little awkward to determine whether a binary matroid is a doubling of a smaller matroid once we have to apply an isomorphism. Luckily, we can detect whether a matroid is a doubling using results from additive combinatorics.

Let G be an additive group and $E \subseteq G$, then define the *stabiliser subgroup*

$$\mathrm{Stab}(E) = \{g \in G : g + E = E\}.$$

Note that if $E \subseteq \mathbb{F}_2^n$ and $v \in \text{Stab}(E) \setminus \{0\}$ then E can be partitioned into pairs of the form $\{x, x+v\}$. These pairs are cosets of the subspace $\{0, v\}$; let σ be the corresponding quotient map. Then E is the doubling of $\sigma(E)$. Conversely, note that the vector $(0, \ldots, 0, 1) \in \mathbb{F}_2^n \times \mathbb{F}_2$ will be in $\text{Stab}(E \times \mathbb{F}_2)$.

Kneser's Theorem gives control over the size of $\text{Stab}(A+B)$ in terms of the sizes of A, B and $A+B$.

Theorem 1.2.1 (Kneser, [12])**.** *Let G be an abelian group, and A and B be subsets of G. If $|A| + |B| \leq |G|$ then*

$$|A+B| \geq |A| + |B| - |\operatorname{Stab}(A+B)|.$$

In particular, $|\operatorname{Stab}(A+B)| \geq |A| + |B| - |A+B|$.

We will also use some notation from Fourier analysis; for a more general treatment see the book of Tao and Vu [18]. For a function $f : \mathbb{F}_2^n \to \mathbb{C}$ we define its *Fourier transform* to be the function $\hat{f} : \mathbb{F}_2^n \to \mathbb{C}$ so that

$$\hat{f}(\xi) = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} f(x)(-1)^{\langle \xi, x \rangle}.$$

Here, $\langle \xi, x \rangle$ denotes $\xi^T x$. Formally, in place of $(-1)^{\langle \xi, x \rangle}$ we should use the function $e_\xi(x)$ which takes value 1 if $\langle \xi, x \rangle = 0$, and -1 if $\langle \xi, x \rangle = 1$, but the above notation is so evocative that we choose to abuse notation for the sake of clarity. The functions e_ξ for $\xi \in \mathbb{F}_2^n$ are called the *characters* and form a basis for the 2^n-dimensional complex vector space of functions $f : \mathbb{F}_2^n \to \mathbb{C}$. The values $\hat{f}(\xi)$ are referred to as the *Fourier coefficients*. When we are given a function $f : \mathbb{F}_2^n \to \mathbb{C}$ it is usually represented in the standard basis of indicator function for singletons. The Fourier transform is a unitary change of basis from the standard basis to the basis of characters, where $\hat{f}(\xi)$ is the coefficient of e_ξ in the decomposition. This leads to the Fourier inversion formula, which allows us to reconstruct the original function from the Fourier coefficients:

$$f(x) = \sum_{\xi \in \mathbb{F}_2^n} \hat{f}(\xi) e_\xi(x).$$

For a binary matroid E we will slightly abuse notation and also use E to denote its characteristic function $E : \mathbb{F}_2^n \to \{0, 1\}$. In this case $\widehat{E}(0)$ is equal to the density of the matroid E, an important reason we defined density as we did. Moreover, if ξ is non-zero and H is the hyperplane of points such that $\langle \xi, x\rangle = 0$, then

$$\begin{aligned}
\widehat{E}(\xi) &= \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} E(x)(-1)^{\langle \xi, x\rangle} \\
&= \frac{1}{2^n}\sum_{x\in E}(-1)^{\langle \xi, x\rangle} \\
&= \frac{1}{2^n}\left(|E\cap H| - |E\setminus H|\right) \\
&= \frac{2|E\cap H| - |E|}{2^n} = \left(\frac{|E\cap H|}{|H|} - \frac{|E|}{2^n}\right);
\end{aligned}$$

the Fourier coefficients measure how a matroid is distributed on hyperplanes. If $\widehat{E}(\xi) = 0$ then the points of E are evenly divided between the hyperplane H and its complement, while if $\widehat{E}(\xi) = |E|/2^n$ then $E \subseteq H$.

Proposition 1.2.2. *For a binary matroid $E \subseteq \mathbb{F}_2^n$ and $\xi \in \mathbb{F}_2^n \setminus \{0\}$ then $\widehat{E}(\xi) = \left(\frac{|E\cap H|}{|H|} - \frac{|E|}{2^n}\right)$, where H is the hyperplane of points orthogonal to ξ.*

If f and g are two functions from $\mathbb{F}_2^n$ to $\mathbb{C}$ then we define their *convolution*, $f * g : \mathbb{F}_2^n \to \mathbb{C}$, by

$$f * g(x) = \frac{1}{2^n}\sum_{y\in\mathbb{F}_2^n} f(y)g(x-y) = \frac{1}{2^n}\sum_{y\in\mathbb{F}_2^n} f(x-y)g(y).$$

When A and B are subsets of $\mathbb{F}_2^n$ their convolution gives us information about the additive structure of the sum set $A + B$;

$$A * B(x) = \frac{1}{2^n}\#\{(a,b)\in A\times B : a + b = x\}.$$

The Fourier transform and convolution are related by the equation $\widehat{f * g} = \hat{f}\cdot\hat{g}$. Combining this fact with the Fourier inversion formula can give important statistics about the additive structure of a binary matroid.

Since a binary matroid E does not contain the zero vector, the Fourier inversion formula implies that

$$\sum_{\xi\in\mathbb{F}_2^n}\widehat{E}(\xi) = \sum_{\xi\in\mathbb{F}_2^n}\widehat{E}(\xi)e_\xi(0) = E(0) = 0.$$

Moreover, by considering the repeated convolutions of the characteristic function we can use the Fourier coefficients to count the number of triangles and pentagons.

$$\frac{1}{2^{2n}}\#\left\{(x_1,x_2,x_3)\in E^3 : x_1+x_2+x_3=0\right\} = \sum_{\xi\in\mathbb{F}_2^n}\widehat{E}(\xi)^3, \text{ and}$$
$$\frac{1}{2^{4n}}\#\left\{(x_1,\dots,x_5)\in E^5 : x_1+\dots+x_5=0\right\} = \sum_{\xi\in\mathbb{F}_2^n}\widehat{E}(\xi)^5.$$

If E is triangle-free, then every tuple in $\left\{(x_1,\dots,x_5)\in E^5 : x_1+\dots+x_5=0\right\}$ is a pentagon, whereas, in general, a tuple could be a triangle and a duplicated element of E. These formulas do not count pentagons and triangles directly, since they care about the order of elements in the tuple. This is easy to fix, however, as we can say that the number of pentagons in a triangle-free binary matroid E will be

$$\frac{2^{4n}}{5!}\sum_{\xi\in\mathbb{F}_2^n}\widehat{E}(\xi)^5.$$

To illustrate some subtle strangeness about this formula, consider applying it to the pentagon itself. As a subset of $\mathbb{F}_2^4$ there are 15 hyperplanes which must be considered. Recall that the pentagon can be viewed as the set of standard basis vectors along with their sum. Ten of the hyperplanes of $\mathbb{F}_2^n$ intersect the pentagon in three elements, and the remaining five each intersect in a single element. We can calculate the number of pentagons using the above formula and Proposition 1.2.2:

$$\begin{aligned}
\frac{2^{4n}}{5!}\sum_{\xi\in\mathbb{F}_2^4}\widehat{P}(\xi)^5. &= \frac{2^{16}}{5!}\left(\left(\frac{5}{16}\right)^5 + 10\cdot\left(\frac{3}{8}-\frac{5}{16}\right)^5 + 5\cdot\left(\frac{1}{8}-\frac{5}{16}\right)^5\right) \\
&= \frac{2^{16}}{5!}\left(\left(\frac{5}{16}\right)^5 + 10\cdot\left(\frac{1}{16}\right)^5 + 5\cdot\left(\frac{-3}{16}\right)^5\right) \\
&= \frac{1}{16\cdot 5!}\left(5^5 + 10\cdot 1^5 + 5\cdot(-3)^5\right) \\
&= \frac{1}{1920}\left(3125 + 10 - 5\cdot 243\right) = 1,
\end{aligned}$$

which is a somewhat inefficient way to show that the pentagon contains exactly one pentagon.

Chapter 2

Davydov and Tombak

Recall that a binary matroid is a set $E \subseteq \mathbb{F}_2^n \setminus \{0\}$. A triangle in a binary matroid is a triple $(x, y, z) \in E^3$ such that $x + y + z = 0$. We say that a matroid is triangle-free if it contains no triangles. The purpose of this chapter is to give an updated exposition of the following result.

Theorem 2.0.1 (Davydov and Tombak, [5])**.** *If $E \subseteq \mathbb{F}_2^n$ is triangle-free and $|E| > \frac{1}{4}(2^n)$ then there is a dimension-$(n-2)$ subspace that is disjoint from E.*

This result is not explicitly stated in [5], so we hope to give a presentation of the paper which exposes this result more clearly. Moreover, the original paper is written in Russian and in the context of coding theory. Our exposition will more closely align with matroid theory and additive combinatorics, and will be entirely in English.

The organization will be a bit backwards. We will begin in Section 2.1 with some consequences of the above result. The importance of presenting these results first is that they will be part of the inductive step when we finally attack the proof of Theorem 2.0.1 in Section 2.3. Between these two sections we will also give some new proofs of other required structural facts which appear in the Russian coding theory literature [14] [5].

Throughout this section we make a distinction between the use of parentheses and brackets for matrices. When we use parentheses it refers to the matrix itself, while square brackets refer to the set of columns of the matrix. This distinction is important, but largely intuitive.

2.1 Consequences

All of these results appear in some form in [5]. The geometric presentation of the proofs is due to Bruen and Wehlau [3], while we provide only some modifications to the statements and proofs, plus a few additional corollaries. The goal of this section is to see what we can deduce about a large triangle-free matroid E, once we know it has a subspace of dimension $n-2$ in its complement.

We begin with some simple observations.

Lemma 2.1.1. *If E is a triangle-free matroid that is disjoint from some hyperplane H and $|E| > \frac{1}{4}(2^n)$, then the only maximal triangle-free matroid containing E is the affine geometry $\mathbb{F}_2^n \setminus H$.*

Proof. Consider a point $v \in H$. The point v induces a partition of $\mathbb{F}_2^n \setminus \{0, v\}$ into $\frac{1}{2}(2^n)$ pairs of the form $\{x, x+v\}$. Half of these pairs will be contained in H, and the remaining $\frac{1}{4}(2^n)$ pairs are a partition of $\mathbb{F}_2^n \setminus H$. One of these parts of $\mathbb{F}_2^n \setminus H$ must have both its elements in E, since $E > \frac{1}{4}(2^n)$. This would create a triangle if we were to add v to E, so E can only be extended by elements which are not in H. Clearly all points in $\mathbb{F}_2^n \setminus H$ extend E without creating triangles, and $\mathbb{F}_2^n \setminus H$ is an affine geometry. □

Lemma 2.1.2. *If E is a maximal triangle-free matroid and $|E| < \frac{1}{2}(2^n)$ then E intersects every hyperplane of $\mathbb{F}_2^n$.*

Proof. Suppose E is maximal triangle-free and disjoint from a hyperplane H. Then E is contained in the affine geometry $\mathbb{F}_2^n \setminus H$, so by maximality $E = \mathbb{F}_2^n \setminus H$ and $|E| = 2^{n-1}$. □

Definition 2.1.3. Let E be a triangle-free matroid which is disjoint from some dimension-$(n-2)$ subspace W. Let H be a hyperplane which contains W. We say that E is (H, W)-*maximal* if every point in $H \setminus W$ is either in E or $E+E$. We say that E is W-*quasi-maximal* if every point in $\mathbb{F}_2^n \setminus W$ is either in E or $E+E$. Note that E is W-quasi-maximal if and only if it is (H, W)-maximal for all three hyperplanes containing W.

For a triangle-free matroid E in $\mathbb{F}_2^n$ which is quasi-maximal with respect to an $(n-2)$-dimensional subspace W, we let H_A, H_B, H_C be the three hyperplanes which contain W. Then, let $A = H_A \cap E, B = H_B \cap E, C = H_C \cap E$ so that A, B, C is a partition of E. We refer to the tuple $(W; H_A, H_B, H_C; A, B, C)$ as the *frame* of E (See Figure 2.1).

Lemma 2.1.4. *Let E be a triangle-free matroid disjoint from a codimension-2 subspace W with frame $(W; H_A, H_B, H_C; A, B, C)$. Then $|A + B| \leq |A| + |B| - 1$ with equality if and only if E is (W, H_C)-maximal and satisfies $|E| = \frac{1}{4}(2^n) + 1$.*

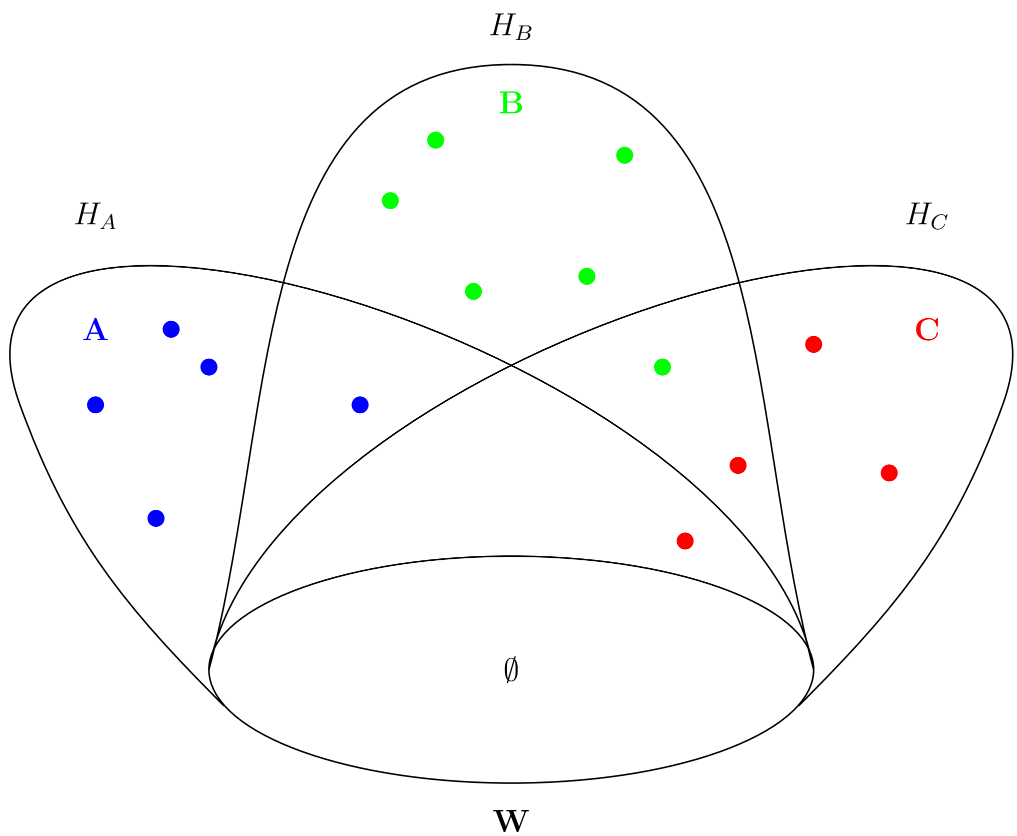


Figure 2.1: A Matroid in its Frame

Proof. Since E is triangle-free, $A+B$ is a subset of $(H \setminus W) \setminus C$. Therefore

$$\begin{aligned}|A+B| &\leq \frac{1}{4}(2^n) - |C| \\ &= \frac{1}{4}(2^n) - (|E| - |A| - |B|) \\ &\leq \frac{1}{4}(2^n) - \left(\frac{1}{4}(2^n) + 1\right) + |A| + |B| \\ &= |A| + |B| - 1,\end{aligned}$$

with equality if and only if $|E| = \frac{1}{4}(2^n) + 1$ and $A + B = W_C \setminus C$. $\square$

Lemma 2.1.5. *Let E be a triangle-free matroid which is W-quasi-maximal. If H is a hyperplane containing W and $E \cap H$ is a doubling, then E is itself a doubling.*

Proof. To use the same notation as earlier we suppose that $H = H_C$. Then $C = C + v$ for some $v \in W$ (if v were outside W then $H_C \setminus W$ would be sent to another coset of W). Now assume that E is not a doubling. Then there is some point $x \in E$ such that $x' = x + v$ is not in E. By quasi-maximality x' must be in $E+E$; there are $y, z \in E$ such that $x' = y+z$. Then without loss of generality $z \in C$ and $z+v \in E$ as well. Thus $(x, y, z+v)$ is a triangle of $\mathbb{F}_2^n$ fully contained in E, a contradiction. $\square$

Recall that for an additive group G and $E \subseteq G$ we define

$$\mathrm{Stab}(E) = \{v \in G : v + E = E\}.$$

Moreover, if $E \subseteq \mathbb{F}_2^n$ and $v \in \mathrm{Stab}(E) \setminus \{0\}$ then E can be partitioned into pairs of the form $\{x, x+v\}$. These pairs are cosets of the subspace $\{0, v\}$ so let σ be the corresponding quotient map. Then E is the doubling of $\sigma(E)$.

Kneser's Theorem gives control over the size of $\mathrm{Stab}(A + B)$ in terms of the sizes of A, B and $A + B$.

Theorem 2.1.6 (Kneser, [12])**.** *Let G be an abelian group, and A and B be subsets of G. If $|A| + |B| \leq |G|$ then*

$$|A + B| \geq |A| + |B| - |\mathrm{Stab}(A + B)|.$$

In particular, $|\mathrm{Stab}(A + B)| \geq |A| + |B| - |A + B|$.

The following is an easy combination of all the previous results and Kneser's Theorem.

Theorem 2.1.7. *Let E be a triangle-free matroid that is W-quasi-maximal. If $|E| \geq \frac{1}{4}(2^n) + 2$, then E is a doubling.*

Proof. By Lemma 2.1.4, $|A+B| \leq |A|+|B|-2$. Kneser's theorem then implies that $A+B$ has non-trivial stabilizer, so $A+B$ is a doubling. Since E is (H_C, W)-maximal, we have $C = W_C \setminus (A+B)$, and C is a doubling. Finally, by Lemma 2.1.5, E is a doubling. $\square$

Theorem 2.0.1 implies that every maximal triangle-free matroid E with $|E| > \frac{1}{4}(2^n)$ is quasi-maximal with respect to some codimension-2 subspace W. If we then apply Theorem 2.1.7 we conclude that if E is maximal triangle-free and $|E| \geq \frac{1}{4}(2^n) + 2$, then E is a doubling. Now, assuming Theorem 2.0.1, we can use the above consequences to obtain some interesting corollaries.

Corollary 2.1.8. *If E is a maximal triangle-free matroid with $|E| > \frac{1}{4}(2^n)$, then $|E| = \frac{1}{4}(2^n) + 2^k$ for some $k \in \{0, \dots, n-4\}$ or $k = n-2$.*

Proof. First note that the result is obvious when $|E| = \frac{1}{4}(2^n) + 1$ and $n \neq 3$, so we may assume that $|E| > \frac{1}{4}(2^n) + 1$.

Now assume that E is a counterexample with n minimum. When $n = 3$, there is no maximal triangle-free matroid in $\mathbb{F}_2^3$ with exactly $\frac{1}{4}(2^3)+1 = 3$ elements; any three elements would form a basis and thus be contained in an affine geometry. This explains why we never need $k = n-3$ in the conclusion of the theorem. Now assume that $n \neq 3$

Since $|E| \geq \frac{1}{4}(2^n) + 2$ we know that E is a doubling of a set $E' \subseteq \mathbb{F}_2^{n-1}$, and since $|E'| < |E|$ we know that $|E'| = \frac{1}{4}(2^{n-1}) + 2^k$ for some $k \in \{0, \dots, (n-1)-4\}$ or $k = (n-1)-2$. Since $|E| = 2|E'|$ we conclude that $|E| = \frac{1}{4}(2^n)+2^k$ for some $k \in \{1, \dots, n-4\}$ or $k = n-2$. $\square$

Corollary 2.1.9. *If E is a triangle-free matroid with density at least $\frac{1}{4} + \frac{1}{2^{d+1}}$, then E is contained in a doubling of a triangle-free matroid of rank at most d or is a doubling of a triangle-free matroid of rank $d+1$.*

Proof. We proceed by induction on the dimension n. We may assume that $n > d+1$ as otherwise the result is trivial. If $E \subseteq \mathbb{F}_2^n$ with density at least $\frac{1}{4} + \frac{1}{2^{d+1}}$ then E has cardinality at least $2^{n-2} + 2^{n-d-1} \geq \frac{1}{4}(2^n) + 2$. Hence, E is the doubling of a matroid $E' \subseteq \mathbb{F}_2^{n-1}$. Since E' has the same density as E the induction hypothesis implies that E' is contained in a doubling of a triangle-free matroid of rank at most d or is a doubling of a triangle-free matroid of rank $d+1$. Since doubling is associative the same conclusion holds for E. $\square$

Corollary 2.1.10 (Govaerts and Storme, [7]). *If E is a triangle-free matroid with density at least $5/16$, then E is either affine or a repeated doubling of the pentagon.*

Proof. By the previous corollary E is contained in a doubling of a triangle-free matroid of rank at most 3, or is the doubling of a rank-4 triangle-free matroid. It suffices to check that every triangle-free matroid of rank at most 4 is either affine or the pentagon. Every triangle-free matroid of rank at most 3 is affine. Since any maximal triangle-free matroid which is not affine must contain an odd cycle, and a pentagon is maximally triangle-free, the only remaining triangle-free set in rank 4 is the pentagon. □

2.2 Coding Theory Bounds

In the coding-theoretic approach used by Davydov and Tombak they require bounds on the size of a "minimum weight code word". In the language of matroid theory this corresponds to finding a hyperplane of $\mathbb{F}_2^n$ whose intersection with E is maximum. Using the matroid theoretic interpretation, we will present a few important bounds related to this quantity.

Proposition 2.2.1. *If $E \subseteq \mathbb{F}_2^n \setminus \{0\}$ then there exists a hyperplane H of $\mathbb{F}_2^n$ so that $|H \cap E| < \frac{1}{2}|E|$.*

Proof. While an elementary double-counting argument can obtain this result, we present a proof (which is not essentially different) using Fourier analysis. Let $\hat{E}$ be the Fourier transform of the characteristic function for the set $E \subseteq \mathbb{F}_2^n \setminus \{0\}$. Since 0 is not in E we know by Fourier inversion that

$$\sum_{\xi \in \mathbb{F}_2^n} \hat{E}(\xi) = 0.$$

Since $\hat{E}(0) = |E|/2^n$ is positive, there must be some ξ so that $\hat{E}(\xi)$ is negative. Let H be the corresponding hyperplane. Then, using the formula for the Fourier coefficients from Proposition 1.2.2,

$$\hat{E}(\xi) = \left(\frac{|E \cap H|}{|H|} - \frac{|E|}{2^n} \right) < 0,$$

and we can conclude that $|H \cap E| < \frac{1}{2}|E|$. □

We can use this result to obtain an opposing bound for triangle-free sets.

Proposition 2.2.2. *If $E \subseteq \mathbb{F}_2^n$ is triangle-free and not an affine geometry then there exists a hyperplane H of $\mathbb{F}_2^n$ such that $|H \cap E| > \frac{1}{2}|E|$.*

Here we give two proofs, the first will again be Fourier-analytic, while the second will be combinatorial and inductive. This time the two methods do seem fundamentally different.

Proof 1 (Fourier-Analytic). As in the proof of the previous proposition we have by Fourier inversion that

$$\sum_{\xi \in \mathbb{F}_2^n} \hat{E}(\xi) = 0.$$

Moreover, since E is triangle free, we know that

$$\sum_{\xi \in \mathbb{F}_2^n} \hat{E}(\xi)^3 = 0.$$

Now suppose that no such hyperplane H exists. This means that $\hat{E}(\xi) < 0$ for every $\xi \neq 0$. But then

$$\hat{E}(0) = \sum_{\xi \neq 0} |\hat{E}(\xi)|, \text{ and } \hat{E}(0)^3 = \sum_{\xi \neq 0} |\hat{E}(\xi)|^3.$$

This is only possible if $\hat{E}(\xi) = 0$ for all but one non-zero ξ. But then Fourier inversion implies that E is an affine geometry. □

Proof 2 (Combinatorial). The proof is by induction on n. The base case will be in $\mathbb{F}_2^2$ where the only non-affine triangle-free matroid consists of a single point.

Since E is triangle-free and not an affine geometry we know that $|E| < 2^{n-1}$. Let H' be a hyperplane so that $|H' \cap E| < \frac{1}{2}|E|$ using the previous proposition. Then $E \cap H'$ is also too small to be an affine geometry when restricted to H' and the restriction is triangle-free.

We may inductively assume that there exists a hyperplane W of H' so that $|W \cap E| > \frac{1}{2}|H' \cap E|$. The two other hyperplanes which contain W will induce a partition of $\mathbb{F}_2^n \setminus H'$, and in particular one must contain at least half the points from $E \setminus H'$. Let this hyperplane be H. Then it contains at least

$$|W \cap E| + \frac{1}{2}|E \setminus H'| > \frac{1}{2}|H' \cap E| + \frac{1}{2}|E \setminus H'| = \frac{1}{2}|E|$$

points, as desired. □

This proposition is also a consequence of Griesmer's bound for error correcting codes [[15], Section 17.5]. The next result, due to Logachev, shows that for certain examples the Griesmer bound can be improved [14]. Here we give a sketch of the proof.

Proposition 2.2.3. *If $E \subseteq \mathbb{F}_2^n$ is triangle-free, $n \geq 5$, and $|E| = \frac{1}{4}(2^n) + 1$ then there is a hyperplane of H of $\mathbb{F}_2^n$ such that $|H \cap E| \geq \frac{1}{8}(2^n) + 2$.*

Proof sketch. The proof is by induction on n. For the base case note that the pentagon would be a counterexample in $\mathbb{F}_2^4$, since no hyperplane contains more than 3 of its elements, so the requirement that $n \geq 5$ is necessary. The result is true, however, in $\mathbb{F}_2^5$, and can be seen by checking all the required examples.

Note that applying the previous proposition would only guarantee $|H \cap E| \geq \frac{1}{8}(2^n) + 1$. We follow the same structure as the proof of the previous proposition. Let H' be a hyperplane of $\mathbb{F}_2^n$ so that $|H' \cap E| > \frac{1}{2}|E|$. Applying the previous proposition, we suppose that $|H' \cap E| = \frac{1}{8}(2^n) + 1$. Then the restriction to H' will satisfy conditions of the inductive hypothesis. Hence, there exists a hyperplane W of H' so that $|W \cap E| = \frac{1}{16}(2^n) + 2$. The two other hyperplanes which contain W partition the elements of $E \setminus H'$, so one must contain at least half the elements of $E \setminus H'$. Let H be such a hyperplane. Then

$$|H \cap E| \geq |W \cap E| + \frac{1}{2}|E \setminus H'| \geq \frac{1}{16}(2^n) + 2 + \frac{1}{16}(2^n) = \frac{1}{8}(2^n) + 2.$$

□

Finally, we have a bound which relates the maximum density of a hyperplane, to the maximum density of a dimension-$(n-2)$ subspace contained in that hyperplane.

Proposition 2.2.4. *Let $E \subseteq \mathbb{F}_2^n$ and H be a hyperplane of $\mathbb{F}_2^n$ with $|E \cap H|$ maximum. If W is an $(n-2)$-dimensional subspace of H, then $|E \setminus H| \leq 2|(H \cap E) \setminus W|$.*

Proof. The two other hyperplanes which contain W partition $E \setminus H$ into two sets. Since H has maximum intersection with E, each of these parts must be no bigger than $|(H \cap E) \setminus W|$. Summing the two inequalities we get the desired bound. □

This proposition is almost too easy to be worth citing elsewhere, but its proof captures the primary way we use maximality of hyperplanes.

2.3 Their Main Result

Now we have enough to reduce the main problem to its most difficult lemma. We will prove the following, which is enough to deduce the main result.

Theorem 2.3.1. *Let $E \subseteq \mathbb{F}_2^n$ be a triangle-free matroid with $|E| \geq \frac{1}{4}(2^n) + 2$. Then there exists a codimension-2 subspace which is disjoint from E.*

First we will give the simple argument that shows this is all we need.

Proof of Theorem 2.0.1. Let E be a matroid with $|E| = \frac{1}{4}(2^n) + 1$. Then consider the doubling $E \times \mathbb{F}_2^n$. This will have $\frac{1}{4}(2^{n+1}) + 2$ elements, and so satisfies the hypothesis of Theorem 2.3.1. There is a codimension-2 subspace W in the complement of $E \times \mathbb{F}_2$, and the intersection of W with the subspace $\mathbb{F}_2^n \times \{0\}$ will be a codimension of codimension at most 2 which is disjoint from E. $\square$

The proof of the main theorem requires the following difficult lemma.

Lemma 2.3.2. *Let $E \subseteq \mathbb{F}_2^n$ be triangle free with $|E| \geq \frac{1}{4}(2^n)+2$ and let H be the hyperplane with $|H \cap E|$ maximum. If the restriction $E \cap H$ is contained in a repeated doubling of the pentagon, then there is a codimension-2 subspace which is disjoint from E*

We will postpone the proof of this lemma to a later section. We can now give a proof of Theorem 2.3.1.

Proof of Theorem 2.3.1 from Lemma 2.3.2. The proof is by induction on n. For $n = 4$ the result is easy to prove using the preliminary results in Section 2.1.

Let $E \subseteq \mathbb{F}_2^n$ be triangle free with $|E| \geq \frac{1}{4}(2^n) + 2$ and let H be its densest hyperplane. If E is affine then we are already done, so we assume this is not the case. By Proposition 2.2.2 we know that $|H \cap E| > \frac{1}{2}|E|$. Since we know the size of $|E|$ this implies that

$$|E \cap H| \geq \frac{1}{4}(2^{n-1}) + 2.$$

Now we can apply induction to the restriction $E \cap H$, and use our results from Section 2.1 to obtain a little more structure. In particular, using Corollary 2.1.8, we know that $E \cap H$ is contained in a doubling of a rank-d triangle-free matroid X with $|X| = \frac{1}{4}(2^d) + 1$ for some $d \geq 0$. If $d \leq 3$ then we know that X is affine, and moreover that $E \cap H$ is affine as well.

If $d \geq 5$ then by Proposition 2.2.3 there is a hyperplane W' of X so that $|X \setminus W'| \leq \frac{1}{8}(2^d) - 1$. This doubles to a hyperplane W of $E \cap H$ with

$$|(E \cap H) \setminus W| \leq 2^{(n-1)-d}\left(\frac{1}{8}(2^d) - 1\right) = \frac{1}{8}(2^{n-1}) - 2^{(n-1)-d}.$$

Proposition 2.2.4 now implies that

$$|E \setminus H| \leq 2|(E \cap H) \setminus W| \leq \frac{1}{8}(2^n) - 2^{n-d}.$$

Meanwhile, since $E \cap H$ is contained in a doubling of X we know that $|E \cap H| \leq \frac{1}{4}(2^{n-1}) + 2^{(n-1)-d}$. Putting these together we conclude that

$$\begin{aligned}
|E| &= |E \setminus H| + |E \cap H| \\
&\leq \left(\frac{1}{8}(2^n) - 2^{n-d}\right) + \left(\frac{1}{8}(2^n) + 2^{(n-1)-d}\right) \\
&= \frac{1}{4}(2^n) - 2^{(n-1)-d} < \frac{1}{4}(2^n).
\end{aligned}$$

This is a contradiction, so the case $d \geq 5$ is ruled out. If $d = 4$ then $E \cap H$ is contained in a doubling of the pentagon, and so the result follows from Lemma 2.3.2.

□

2.4 Difficult Lemma

We will restate the difficult Lemma 2.3.2 and provide a proof. The proof will follow the same structure as the original in [5] with only some modifications to the presentation.

Lemma 2.4.1 (Lemma 2.3.2)**.** *Let $E \subseteq \mathbb{F}_2^n$ be triangle free with $|E| \geq \frac{1}{4}(2^n) + 2$ and let H be its densest hyperplane. If the restriction $E \cap H$ is contained in a repeated doubling of the pentagon, then there is a codimension-2 subspace which is disjoint from E*

Suppose that $|E| = \frac{1}{4}(2^n) + \beta$ where $\beta \geq 2$. We can represent E in matrix form M as follows:

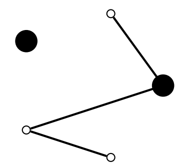
$$M = \left(\begin{array}{c} 0\ldots01\ldots1 \\ \hline R_0 \\ \hline R_1 \end{array}\right) = \left(\begin{array}{c|c} 0\ldots0 & 1\ldots1 \\ \hline A & B \\ \hline C & D \end{array}\right)$$

where we have now identified the hyperplane H with the set of vectors with 0 as the first entry. The submatrix R_0 will be the top four rows of M, and R_1 will be the remaining $n-5$ rows. The matrices A and C will have exactly $|E \cap H|$ columns, while B and D have $|E \setminus H|$ columns. Both A and B have 4 rows, while C and D have $n-5$ rows. Now if there is a codimension-2 subspace of $\mathbb{F}_2^5$ which is disjoint from

$$\left[\begin{array}{c} 0\ldots01\ldots1 \\ \hline R_0 \end{array}\right]$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(a) The set of disjoint columns of B



(b) The constructed set X

Figure 2.2: Example of Graph Construction

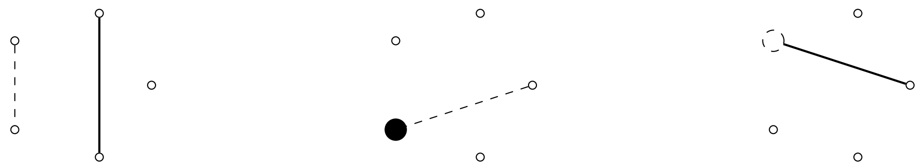
this will extend to a codimension-2 subspace disjoint from the columns of M, and hence give a codimension-2 subspace disjoint from E. For this reason we chose to investigate the smaller subset of rows.

We can perform row operations so that the columns of A will all be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

since $H \cap E$ is contained in a doubling of the pentagon. The columns of B, on the other hand, can be any element of $\mathbb{F}_2^4$, including the zero vector. To clarify our case analysis, we will work with subsets of the vertices and edges of a graph on 5 vertices. For any matroid E satisfying the assumptions of Lemma 2.3.2, we begin by considering the complete graph K_5, where we identify the 5 vertices with the 5 different columns of A, and the 10 edges with their pairwise sums (Note: the pairwise sums are distinct). From here we construct a set $X(E) \subseteq V(K_5) \cup E(K_5)$ where X contains the vertex v if v is a column of B, and contains the edge e if e is a column of B. Every symmetry of K_5 is also a symmetry of the columns of A; any relabeling of the vertices can be achieved by row operations inside A. Note that this construction cannot remember whether the zero vector is a column of B, so we will have to treat this case separately.

We now describe a labeling game on these graphs. Given a matroid E satisfying the conditions of Lemma 2.3.2 we begin by constructing the set $X(E)$. We are then allowed to label the vertices and edges of the underlying K_5 by columns of A. We can assign label a to vertex v if $a + v \in [B]$, or label a to edge e if $a + e \in [B]$ (Note: this is equivalent to the existence of $b \in B$ so that $a + b = e$). These rules can be interpreted in terms of the graph and set X (see Figure 2.3: the black components are elements of X and the dashed components can be labeled using the first vertex).



(a) Rule 1 (b) Rule 2 (c) Rule 3

Figure 2.3: Labeling Rules

The goal of this game is to label every element of $V(K_5) \cup E(K_5)$ (or equivalently the set $\mathbb{F}_2^4 \setminus \{0\}$), without using any label more than 4 times. If we can accomplish our goal then we can derive a contradiction and so ignore the matroid E which we started with.

Lemma 2.4.2. *Let $E \subseteq \mathbb{F}_2^n$ be triangle-free with $|E| \geq \frac{1}{4}(2^n) + 2$ and let H be its densest hyperplane. If the restriction $E \cap H$ is contained in a repeated doubling of the pentagon, then there exists no labeling of $V(K_5) \cup E(K_5)$ using $X(E)$ and Rules $1-3$ which uses every label at most 4 times.*

Proof. Suppose that $|E| = \frac{1}{4}(2^n) + \beta$ where $\beta \geq 2$, and represent E in matrix form M as above:

$$M = \left(\begin{array}{c} 0\dots01\dots1 \\ \hline R_0 \\ \hline R_1 \end{array}\right) = \left(\begin{array}{c|c} 0\dots0 & 1\dots1 \\ \hline A & B \\ \hline C & D \end{array}\right).$$

Let $A_{1000}, A_{0100}, A_{0010}, A_{0001}$, and A_{1111} form a partition of the columns of

$$\left(\begin{array}{c} A \\ \hline C \end{array}\right)$$

according to the corresponding entries of A. Every element of A_{1000} will have the same top four entries $(1,0,0,0)$, every element of A_{0100} will have top four entries $(0,1,0,0)$, and so on. Thus, the matrix

$$\left(\begin{array}{c|c|c|c|c} 0\dots0 & 0\dots0 & 0\dots0 & 0\dots0 & 0\dots0 \\ \hline A_{1000} & A_{0100} & A_{0010} & A_{0001} & A_{1111} \end{array}\right)$$

contains all the columns of M whose first entry is zero. Let $a_v = \#[A_v]/2^{n-5}$, then the $[A_v]$ is a set of $a_v \cdot 2^{n-5}$ vectors in $\mathbb{F}_2^{n-1}$. Note that $|E \cap H| = 2^{n-5} \sum_{v \in [A]} a_v$.

Let Ψ_x denote the set of points in $(E+E) \setminus H$ where the top entry is one and the next four entries are the vector $x \in \mathbb{F}_2^4$. For clarity, note that

$$(E+E) \setminus H = \bigcup_{x \in \mathbb{F}_2^4} \Psi_x, \text{ and } |(E+E) \setminus H| = \sum_{x \in \mathbb{F}_2^4} |\Psi_x|.$$

Suppose that there was a labeling of $V(K_5) \cup E(K_5)$ using $X(E)$ and Rules $1-3$ which used every label at most 4 times. Let $\phi : \mathbb{F}_2^4 \setminus 0 \to [A]$ be the corresponding labeling of $\mathbb{F}_2^4 \setminus \{0\}$, where we consider edges as the sum of their two endpoints. The Rules $1-3$ imply that for every $x \in \mathbb{F}_2^4$ there exists a vector $b \in B$ such that $\phi(x) + b = x$. Hence, there exists a vector $y \in E \setminus H$ such that $A_{\phi(x)} + y \subseteq \Psi_x$. Moreover, we can use the inequality $|A_\phi(x)| \leq |\Psi_x|$ to bound the size of $(E + E) \setminus H$:

$$\begin{aligned}
|(E+E) \setminus H| &= \sum_{x \in \mathbb{F}_2^4} |\Psi_x| \\
&\geq |\Psi_0| + \sum_{x \in \mathbb{F}_2^4 \setminus \{0\}} |A_{\phi(x)}| \\
&\geq \sum_{x \in \mathbb{F}_2^4 \setminus \{0\}} a_{\phi(x)} \cdot 2^{n-5} \\
&\geq 2^{n-5} \sum_{x \in \mathbb{F}_2^4 \setminus \{0\}} 1 + 2^{n-5} \sum_{x \in \mathbb{F}_2^4 \setminus \{0\}} (1 - a_{\phi(x)}) \\
&\geq 15 \cdot 2^{n-5} + 4 \cdot 2^{n-5} \sum_{v \in [A]} (1 - a_v) \\
&= 15 \cdot 2^{n-5} - 20 \cdot 2^{n-5} + 4 \cdot 2^{n-5} \cdot \sum_{v \in [A]} a_v \\
&= 4 \cdot 2^{n-5} \cdot \sum_{v \in [A]} a_v - 5 \cdot 2^{n-5},
\end{aligned}$$

Where the first inequality follows from our argument above, the second follows from the definition of a_v, and the fourth inequality follows from the fact that each element of $[A]$ is used as an label for at most 4 vectors in $\mathbb{F}_2^4 \setminus \{0\}$.

Since E is triangle free the sets $(E + E) \setminus H$ and $E \setminus H$ are disjoint. Hence,

$$\begin{aligned}
|\mathbb{F}_2^n \setminus H| &\geq |E \setminus H| + |(E+E) \setminus H| \\
&= |E| - |E \cap H| + |(E+E) \setminus H| \\
&\geq 8 \cdot 2^{n-5} - 2^{n-5} \cdot \sum_{v \in [A]} a_v + 4 \cdot 2^{n-5} \cdot \sum_{v \in [A]} a_v - 5 \cdot 2^{n-5} \\
&\geq 3 \cdot 2^{n-5} + 3 \cdot 2^{n-5} \sum_{v \in [A]} a_v.
\end{aligned}$$

Since $|\mathbb{F}_2^n \setminus H| = 2^{n-1}$ the above inequality implies that $\sum_{v \in [A]} a_v \leq 13/3$.

Now, there must be some i, j, k such that $a_i + a_j + a_k \geq \frac{3}{5}\sum_{v\in[A]} a_v$, by majority. Let W be the hyperplane of H which contains the three sets A_i, A_j, and A_k. By the maximality of $|H \cap E|$ we know that the other two hyperplanes of $\mathbb{F}_2^n$ containing W each contain at most $2^{n-5} \cdot \frac{2}{5}\sum_{v\in[A]} a_v$ points from E (See Proposition 2.2.4). Thus, finally, we can bound the number of points in E:

$$\begin{aligned}
|E| &\leq |E \cap H| + 2 \cdot \frac{2}{5} \sum_{v\in[A]} a_v \cdot 2^{n-5} \\
&\leq \frac{13}{3} \cdot 2^{n-5} + 2 \cdot \frac{2}{5} \cdot \frac{13}{3} \cdot 2^{n-5} \\
&= \frac{117}{15} \cdot 2^{n-5} < 2^{n-2}.
\end{aligned}$$

This contradiction completes the proof; no such labeling can exist. □

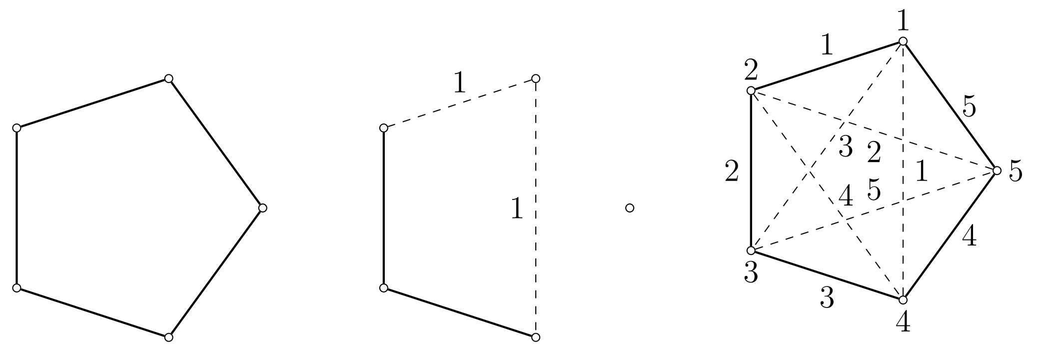
Remark that if we assume B has no zero column then there exists no labeling of $V(K_5) \cup E(K_5)$ using $X(E)$ and Rules $1-3$ which uses every label at most 4 times, even if we allow one element of $V(K_5) \cup E(K_5)$ to go unlabelled. The only modification necessary to the proof is in the lower bound on $|(E + E) \setminus H|$, where we we can only use the weak estimate $|\Psi_x| \geq 0$ for the unlabelled element x. Luckily, we can compensate by using the bound $|\Psi_0| = 2^{n-5}$, since we know B has no zero column.

Now let us work through one example of how to construct a good labeling using Rules $1-3$. Suppose that X contains the edges of a 5-cycle. Our goal is to assign labels to all the vertices and edges of K_5. There is only one way to label vertices (Rule 3, Figure 2.3c); we label the vertices by their neighbours cyclically around the 5 cycle. Now we use Rule 1 (Figure 2.3a) two different ways starting from each vertex.

Figure 2.4 shows how we can use Rule 1 to label two edges using vertex 1, and how we can repeat this process for each vertex to get the completed labeling. The labeling shown uses each label at most 3 times, which is enough to apply Lemma 2.4.2. Hence, if E satisfies the conditions of Lemma 2.3.2 we know that $X(E)$ cannot contain a 5-cycle.

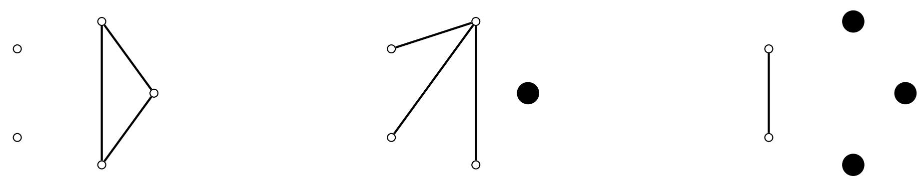
As another example, suppose that $[B]$ contained all of $\mathbb{F}_2^4$. Then $X(E) = V(K_5) \cup E(K_5)$, and so X contains the edges of a 5-cycle. Our previous argument implies this situation cannot occur; from now on we may assume that $[B]$ is missing some column. By row reducing using the top row (which is all 1s above B), we may in fact assume that B has no zero column.

Now we need a way to use the graph representation $X(E)$ to find a codimension-2 subspace disjoint from E.



(a) The 5-cycle (b) Using Rule 1 two ways (c) The completed labeling

Figure 2.4: Constructing a Labeling When X contains a 5-cycle



(a) Triangle (b) Vertex plus star (c) Edge plus three vertices

Figure 2.5: Subgraphs Which Give Codimension-2 Subspaces

Lemma 2.4.3. *If $X(E)$ does not intersect a subgraph of the form [Fig 2.5a], [Fig 2.5b], or [Fig 2.5c], then there is a codimension-2 subspace of $\mathbb{F}_2^n$ which is disjoint from E.*

Proof. We will show the subspace for each subgraph separately. In each case we may assume, by the symmetry of the representation, that the missing subgraph is exactly the one indicated in Figure 2.5.

Suppose that there is a subgraph of the form [Fig 2.5a] which is disjoint from $X(E)$. Then the subspace of vectors from $\mathbb{F}_2^n$ whose first five entries form a column from

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right],$$

is a codimension-2 subspace which is disjoint from E.

Suppose that there is a subgraph of the form [Fig 2.5b] which is disjoint from $X(E)$. Then the subspace of vectors from $\mathbb{F}_2^n$ whose first five entries form a column from

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right],$$

is a codimension-2 subspace which is disjoint from E.

Suppose that there is a subgraph of the form [Fig 2.5a] which is disjoint from $X(E)$. Then the subspace of vectors from $\mathbb{F}_2^n$ whose first five entries form a column from

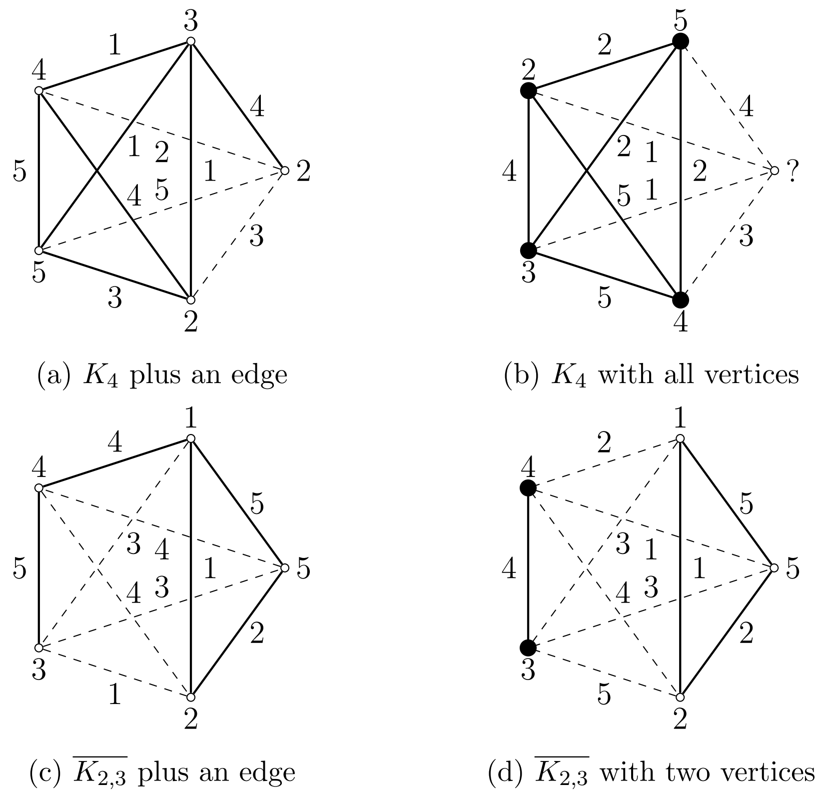
$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right],$$

is a codimension-2 subspace which is disjoint from E. $\square$

Now we can move on to the final case analysis.

Proof of Lemma 2.3.2. First, let us reiterate the conditions of Lemma 2.3.2. Let $E \subseteq \mathbb{F}_2^n$ be triangle free with $|E| \geq \frac{1}{4}(2^n) + 2$ and let H be its densest hyperplane. Suppose the restriction $E \cap H$ is contained in a repeated doubling of the pentagon. Our goal is to show that there is a codimension-2 subspace of $\mathbb{F}_2^n$ which is disjoint from E.

Suppose that no such subspace exists. Then Lemma 2.4.3 implies that $X(E)$ must intersect every subgraph of the form [Fig 2.5a], [Fig 2.5b], and [Fig 2.5c]. In particular, if we consider only the edges of $X(E)$, the complement of the graph $X(E)$ is triangle-free. This implies that the complement of $X(E)$ is contained in a maximal triangle-free graph. There are three such graphs on 5 vertices: the 5-cycle, the star $K_{1,4}$ and the complete bipartite graph $K_{2,3}$. Hence, $X(E)$ must contain all the complementary edges of one of these three graphs. The complement of a 5-cycle is another 5-cycle, the complement of the star is K_4, and the complement of $K_{2,3}$ is a disjoint triangle and edge. We have already constructed a labeling for the 5-cycle (Figure 2.4). If $X(E)$ contains K_4 then it must either contain an additional edge, or it must contain every vertex of the K_4 since $X(E)$ intersects every subgraph of the from [Fig 2.5b]. If $X(E)$ contains $\overline{K_{2,3}}$, then it must either contain



(a) K_4 plus an edge

(b) K_4 with all vertices

(c) $\overline{K_{2,3}}$ plus an edge

(d) $\overline{K_{2,3}}$ with two vertices

Figure 2.6: Labelings For the Remaining Cases

an additional edge, or it must contain both vertices of the single edge. We give a labeling respecting Rules $1-3$ (Figure 2.3) for each of these four cases in Figure 2.6. Using Lemma 2.4.2 this leads to a contradiction, so there must exist a subspace of $\mathbb{F}_2^n$ which is disjoint from E. $\square$

Chapter 3

Large Triangle-free Sets

This chapter provides a recursive construction for all triangle-free binary matroids with greater than 1/4 density. In a subsequent chapter we use this characterization to bound the number of pentagons in dense triangle-free matroids. The notation and terminology are borrowed heavily from both matroid theory and additive combinatorics.

Recall that a simple binary matroid is a subset E of the binary vector space $\mathbb{F}_2^n$ which does not contain the zero vector. Such a set is triangle-free if it contains no three points x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = 0$. A triangle-free matroid is maximal if it is not a proper subset of another triangle-free set. In the language of additive combinatorics, a rank-n matroid E is triangle-free if E and $E+E$ are disjoint, and maximal if E and $E+E$ cover all of $\mathbb{F}_2^n$. Let E be a triangle-free set which is disjoint from some co-dimension-2 subspace W. Recall that E is W-*quasi-maximal* if every point in $\mathbb{F}_2^n \setminus W$ is either in E or $E+E$. If H is a hyperplane which contains W we say that E is (H,W)-maximal if $H \setminus W$ is covered by the sets E and $E+E$.

Recall that the density of a rank-n binary matroid E is the fraction $|E|/|\mathbb{F}_2^n| = |E|/2^n$. In this section we will be investigating the structure of triangle-free sets with density greater than 1/4.

For a set $E \subseteq \mathbb{F}_2^n$ we construct the set $E \times \mathbb{F}_2$ which we call the *doubling of* E. Note that E and $E \times \mathbb{F}_2$ will have the same density in their ambient spaces. If E is triangle-free, then it is easy to check that its doubling will be triangle-free as well. A rank-n set E is a doubling if and only if there exists a point $v \in \mathbb{F}_2^n$ such that $E + v = E$.

Recall Davydov and Tombak's Theorem implies that ever large maximal triangle-free matroid is W-quasimaximal for some codimension-2 subspace W. Theorem 2.1.7 then implies that every maximal triangle-free set with more than $\frac{1}{4}(2^n) + 1$ points is a doubling.

Thus, to understand the structure of triangle-free matroids with density greater than $1/4$, it suffices to understand those maximal triangle-free matroids with exactly $\frac{1}{4}(2^n)+1$ points. We refer to such sets as *irreducible* triangle-free. Let E be a triangle-free set in $\mathbb{F}_2^n$ which is W-quasi-maximal for some codimension-2 subspace W (such a subspace exists for irreducible matroids by Theorem 2.0.1). Let H_A, H_B, H_C be the three hyperplanes which contain W, and $A = H_A \cap E, B = H_B \cap E, C = H_C \cap E$ so that A, B, C is a partition of E. We refer to the tuple $(W; H_A, H_B, H_C; A, B, C)$ as the *frame* of E (See Figure 2.1).

In Section 3.1 we provide two constructions of irreducible triangle-free matroids. The first appears as-is in [3], while the second significantly strengthens their construction of so-called fractal caps. The decomposition in Section 3.2 is also given in [1], but now matches our strengthened construction to provide a complete recursive characterization.

To do this, we rely on two older results: Kemperman's Theorem for sumsets in abelian groups, and the result Davydov and Tombak from the previous chapter. We will use give the relevant pieces of Kemperman's theorem as they are needed, but repeat the result of Davydov and Tombak here.

Theorem 3.0.1 (Davydov and Tombak, [5]). *If $E \subseteq \mathbb{F}_2^n$ is triangle-free and $|E| > \frac{1}{4}(2^n)$ then there is a co-dimension-2 subspace that is disjoint from E.*

The construction in Section 3.1 requires sets with $\frac{1}{4}(2^n)+2$ points which contain exactly one triangle. To fully understand the structure of these sets more fully, we give a matching recursive characterization of these in Sections 3.3 and 3.4 .

3.1 Constructing Irreducible Triangle-free Sets

Here we describe two constructions for large maximal triangle-free sets which are not doublings. By Theorem 2.1.7 it suffices to construct maximal triangle-free sets with exactly $\frac{1}{4}(2^n) + 1$ elements. The first construction was given in [3] and is slightly rephrased here. The second is based on the construction of fractal caps in [3], but has been strengthened here with a new proof.

The first construction begins with an $(n-2)$-dimensional subspace W. We consider its three affine cosets W_A, W_B and W_C and select a point $z \in W_C$ and a subset A of W_A.

We construct a triangle-free set $\mathrm{NA}(W, z, A)$ of size $\frac{1}{4}(2^n) + 1$ as follows. Let $B = W_B \setminus (A + z)$. We define $\mathrm{NA}(W, z, A) := A \cup B \cup \{z\}$. Note that the hyperplane $W \cup W_C$ meets $\mathrm{NA}(W, z, A)$ only in the point z. This motivates us to call sets constructed in this way *nearly-affine*.

Proposition 3.1.1. *A set $R = \mathrm{NA}(W, z, A)$ constructed as above is triangle-free with exactly $\frac{1}{4}(2^n) + 1$ elements. Moreover,*

(1) every point in $W_A \cup W_B$ is in $(R + z) \cup R$,

(2) if A is not a doubling then R is W-quasi-maximal, and

(3) if $|A| \neq \frac{1}{8}(2^n)$ then every point of W is in $R + R$.

Proof. It is clear that R is a triangle-free set of size $\frac{1}{4}(2^n) + 1$. To see (1) note that $W_B \setminus B = z + A$ and $W_A \setminus A = z + B$.

For (2) we suppose that R is not quasi-maximal. Then there is $w \in W_C$ with $w \neq z$ such that $w \notin A + B$. Now let $v = z + w$. Then $v + A = z + (w + A) = z + W_B \setminus B = z + z + A = A$, so B is a doubling.

For (3) suppose that $|A| \neq \frac{1}{8}(2^n)$. Then by the symmetry of A and B we may assume that $|A| > \frac{1}{8}(2^n)$. Now for every point w of W there are $\frac{1}{8}(2^n)$ lines that pass through W_B, and so one of them must contain two points of R, by majority. □

While the technical definition of nearly-affine sets given above is convenient for stating the conditions that give maximality, it does obfuscate the simplicity of their structure. A simpler construction is to begin with a hyperplane H, and the affine geometry $E = \mathbb{F}_2^n \setminus H$. Then add an additional point $z \in H$ to the set. This will create many triangles which pair off the elements of E. To create a nearly-affine set we remove exactly one element from each of these pairs.

If W is a codimension-2 subspace which is a subset of H not containing the point z, then we get a frame $(W; H_A, H_B, H; A, B, C)$ where $C = \{z\}$ and for each pair of points from E exactly one of A and B contain a representative from that pair. The set constructed here is precisely the nearly affine set $\mathrm{NA}(W, z, A)$.

Our next construction is slightly more complicated. We first pick an integer $t \leq n - 1$ and a rank-$(n-t+2)$ matroid $S \subseteq \mathbb{F}_2^{n-t+2} \setminus \{0\}$ which contains exactly one triangle $\bar{\Lambda}$. Next, we need a rank-t set $T \subseteq \bar{\Lambda} \times \mathbb{F}_2^{t-2}$ and let F be the subspace $\{0\} \times \mathbb{F}_2^{t-2} \subseteq \mathbb{F}_2^{n-t+2} \times \mathbb{F}_2^{t-2}$. Note that, inside the subspace $\Lambda = (\bar{\Lambda} \cup \{0\}) \times \mathbb{F}_2^{t-2}$, F is a codimension-2 subspace which is disjoint from T.

We combine these two sets S and T to create a set $\mathrm{ET}(S, T) \subseteq \mathbb{F}_2^n$ as follows. First, we perform $t - 2$ doublings of the set $S \setminus \bar{\Lambda}$ to obtain a set $(S \setminus \bar{\Lambda}) \times \mathbb{F}_2^{t-2} \subseteq \mathbb{F}_2^n$. We complete the construction by adding the set T into $\bar{\Lambda} \times \mathbb{F}_2^{t-2}$, which is currently empty after the doubling. Hence, $\mathrm{ET}(S, T) = (S \setminus \bar{\Lambda}) \times \mathbb{F}_2^{t-2} \cup T$.

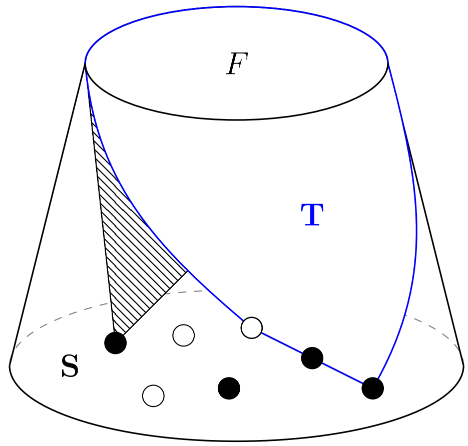


Figure 3.1: Structure of an Exploded-triangle Set

Note that the image of $\mathrm{ET}(S,T)$ after under the quotient by $F = \{0\} \times \mathbb{F}_2^{t-2}$ is S. If we were to allow S to be a triangle in $\mathbb{F}_2^2$, then $\mathrm{ET}(S,T)$ would isomorphic to T. We have required that $t \leq n-1$ to avoid this problem. A set constructed in this way is referred to as an *exploded-triangle set*.

Proposition 3.1.2. *A set $R = \mathrm{ET}(S,T)$ has cardinality $|T| + 2^{t-2}(|S|-3)$, and every triangle of R is contained in T.*

Proof. It is easy to calculate the size of R:

$$|R| = |T| + |(S \setminus \bar{\Lambda})| = |T| + 2^{t-2}(|S|-3)$$

Suppose that R does contain a triangle $\{x, y, z\}$. Let $\overline{x}, \overline{y}, \overline{z}$ denote the images of x, y, z respectively under the quotient by F. The images will be non-zero since F is disjoint from R. Moreover, it must be that $\overline{x}, \overline{y}$, and $\overline{z}$ are all elements of S. If they were all distinct then they would have to form a triangle in S. Since the only such triangle is $\overline{\Lambda}$, the triangle $\{x, y, z\}$ would be a subset of T. Thus, we may assume without loss of generality that $\overline{x} = \overline{y}$. But then $\overline{z} = \overline{x} + \overline{y} = 0$, a contradiction. □

We will use this construction to build maximal triangle-free sets with cardinality $\frac{1}{4}(2^n)+1$ by adding a few extra conditions on the sets S and T. If we require that $|S| = \frac{1}{4}(2^s)+2$, and $|T| = \frac{1}{4}(2^t)+1$, and that T is triangle-free, then Proposition 3.1.2 implies that $\mathrm{ET}(S,T)$ will be triangle-free with cardinality $\frac{1}{4}(2^n)+1$. We capture maximality in the following Lemma.

Lemma 3.1.3. *If S is maximal with respect to containing exactly one triangle Λ, and T is F-quasi-maximal in $\Lambda = (\bar{\Lambda} \cup \{0\}) \times \mathbb{F}_2^{t-2}$, then $R = \mathrm{ET}(S, T)$ is maximal triangle-free.*

Proof. We show that every point $w \notin R$ is in $R + R$. First, suppose that $w \in \Lambda \setminus F$. Then the quasi-maximality of T within the subspace Λ is enough to guarantee that w is in $T + T \subseteq R + R$. If $w \in F$ then $S' + w = S'$ and thus we know that $w \in R + R$.

Now suppose that $w \notin \Lambda$. Then the image $\overline{w}$ of w under the quotient by F is non-zero and $\overline{w} \notin S$. Since S is maximal, $\overline{w} = \overline{x} + \overline{y}$ for some $\overline{x}, \overline{y} \in S$. Let W, X, Y denote the cosets of F in R which contain the points $\overline{w}$, $\overline{x}$, and $\overline{y}$. Since $\overline{\Lambda}$ is the unique triangle of S we may assume without loss of generality that $\overline{x} \notin \overline{\Lambda}$. Then $X \subseteq R$ and by Lemma 2.1.2 $Y \cap R \neq \emptyset$. We conclude that every point of W is in $R + R$, and in particular the point w. Thus, R is maximal triangle-free. $\square$

We will address how to construct sets with exactly one triangle in Section 3.3. For now, we turn to a decomposition theorem for triangle-free sets of cardinality $\frac{1}{4}(2^n)+1$, and show that all such sets are either nearly-affine or exploded-triangle.

3.2 Decomposing Large Triangle-free Sets

Since maximal triangle-free matroids with more than $\frac{1}{4}(2^n) + 1$ elements are always doublings, we will only be concerned with decomposing irreducible triangle-free matroids (maximal triangle-free sets with exactly $\frac{1}{4}(2^n) + 1$ elements).

As in [3] we will introduce some definitions from Kemperman [11].

Let X and Y be non-empty subsets of an abelian group G.

Definition 3.2.1. If $g \in G$ then $\nu_g(X, Y)$ denotes the number of representations of g as $g = x + y$ with $x \in X$ and $y \in Y$. For a subset C of G denote by $\mathrm{Stab}(C)$ the subgroup $\mathrm{Stab}(C) = \{g \in G : g + C = C\}$.

Definition 3.2.2. We say that C is *periodic* if $\mathrm{Stab}(C) \neq \{0\}$. Note that when $G = \mathbb{F}_2^n$ this definition coincides with that of the set C being a doubling.

Definition 3.2.3. $P_1(X, Y)$ denotes the collection of pairs (F, D) such that

(1) F is a finite subgroup of G with $|F| \geq 2$;

(2) D is the intersection of $X+Y$ with some F-coset, moreover $(X+Y) \setminus D$ is a union of one or more F-cosets;

(3) if $X+Y$ is periodic then D is an F-coset and there is at least one $d \in D$ with $\nu_d(X,Y) = 1$;

(4) if $\sigma : G \to G/F$ is the quotient map, then $\sigma(D)$ has exactly one representation $\sigma(D) = \bar{x} + \bar{y}$ where $\bar{x} \in \sigma(X)$ and $\bar{y} \in \sigma(Y)$.

Let E be an irreducible triangle-free set with frame $(W; H_A, H_B, H_C; A, B, C)$. In order to understand the structure of these matroids, we wish to apply the results from [11] to the subsets A, B and C.

Definition 3.2.4. If E is W-quasi-maximal and $P_1(A,B) \cup P_1(A,C) \cup P_1(B,C)$ is non-empty, then we say that E is P_1-*decomposable* (with respect to W). If the pair (F,D) is in $P_1(A,B) \cup P_1(A,C) \cup P_1(B,C)$ then we say that F is a P_1-*decomposing subgroup* for E.

Note that an exploded-triangle set is P_1-decomposable with $F \cup \{0\}$ as a P_1-decomposing subgroup. (This does rely on the fact that every set S which contains exactly one triangle and $\frac{1}{4}(2^s)+2$ elements has a codimension-2 subspace in the complement, which we will see in a subsequent section).

The following is the definition of an *elementary pair* given in [11] specialized to the case where $G = \mathbb{F}_2^n$.

Definition 3.2.5. Let X and Y be non-empty subsets of $G = \mathbb{F}_2^n$. Then (X,Y) is and *elementary* if at least one of the following conditions holds:

(i) either $|X| = 1$ or $|Y| = 1$;

(ii) for some non-trivial subgroup F of G, each of X and Y is contained in an F-coset and $X+Y$ is itself an F-coset, moreover precisely one element g of G satisfies $\nu_g(X,Y) = 1$;

(iii) X is a non-periodic subset of some F-coset $x+F$ and Y is of the form $Y = g_0 + ((x+F) \setminus X)$, moreover $X+Y$ is obtained from an F-coset by deleting a single element from that coset and *no* element g of G satisfies $\nu_g(X,Y) = 1$.

Kemperman's original definition in [11] was for general abelian groups and included a fourth condition, but this reduces to condition (i) when $G = \mathbb{F}_2^n$. We now state the theorem that connects elementary pairs to P_1-decomposable pairs. This is Lemma 5.2 of [11] which will then specialize to our setting.

Theorem 3.2.6 ([11]). *Let X and Y be non-empty subsets of an additive group G. If $|X+Y| = |X| + |Y| - 1$, and either $X+Y$ is non-periodic or there exists at least one g such that $\nu_g(X,Y) = 1$, then either (X,Y) is an elementary pair or $P_1(A,B)$ is non-empty.*

Kneser's Theorem (Theorem 2.1.6) guarantees that $|X+Y| \geq |X| + |Y| - 1$, whenever $X+Y$ is non periodic. Kemperman's Theorem essentially gives a structural characterization of when equality holds. The statement of the above theorem is needlessly technical for our purposes. The following corollary simplifies the preconditions for our particular setting.

Corollary 3.2.7. *Let E be an irreducible triangle-free matroid which is W-quasi-maximal, and $(W; H_A, H_B, H_C; A, B, C)$ be its frame. Then either (A,B) is an elementary pair, $P_1(A,B)$ is non-empty.*

Proof. From Lemma 2.1.2 we know that A and B are non-empty. If $A+B$ were periodic that would imply that $C = (H_C \backslash W) \backslash (A+B)$ is periodic as well, which finally would imply, by Lemma 2.1.5, that E as a whole were periodic. This is impossible since $|E|$ is odd, so we know that $A+B$ is non-periodic.Theorem 2.1.4 implies that the $|A+B| = |A|+|B|-1$, since $|E| = \frac{1}{4}(2^n) + 1$ and E is H_C-maximal. Finally, by Theorem 3.2.6, we conclude that either (A,B) is an elementary pair or that $P_1(A,B)$ is empty. $\square$

We now work towards connecting this result to our two constructions of triangle-free sets in the previous section. This next structural result appears in [3] as Theorem 12.1; we repeat the proof here for completeness.

Lemma 3.2.8. *Let E be an irreducible triangle-free matroid which is W-quasi-maximal, and $(W; H_A, H_B, H_C; A, B, C)$ be its frame. If E is not P_1-decomposable with respect to W then one of the three hyperplanes containing W meets E in a unique point (i.e. E is a nearly-affine set).*

Proof. Since E is not P_1-decomposable, Corollary 3.2.7 implies that $(A,B), (A,C)$ and (B,C) are elementary pairs. Without loss of generality we first consider only the pair (A,B)

From the definition of elementary pairs see that in case (i) either H_A or H_B will meet E in a single point. Case (ii) implies that $A+B$ is periodic which we already saw was impossible.

Assume case (iii) occurs. Then there exists a subgroup F of $\mathbb{F}_2^n$ such that $A+B$ is obtained from an F-coset by deleting a single element from that coset. Let $m(C)$ be the rank of this subgroup F. Now $|C| = 2^{n-2} - 2^{m(C)} + 1$. If $m = n-2$ then $|C| = 1$ and H_C intersects E in a single point.

Therefore we may assume that $m(C) \leq n-3$. Now assume we have reached the same case for the remaining pairs (A, C) and (B, C). By the symmetry of the roles of A, B and C we may further assume that $m(A) \leq n-3$ and $m(B) \leq n-3$. But then

$$\begin{aligned} |E| &= |A| + |B| + |C| = 3(2^{n-2}+1) - (2^{m(A)} + 2^{m(B)} + 2^{m(C)}) \\ &\geq 3(2^{n-2}+1) - 3(2^{n-3}) \\ &= 3 \cdot 2^{n-3} + 3 > 2^{n-2} + 1 = |E|. \end{aligned}$$

This is a contradiction and so it must be that one of A, B, C contains only a single point. $\square$

The knowledge that either $P_1(A, B)$ is non-empty or (A, B) does not inherently respect the symmetry of a matroid E's frame $(W; H_A, H_B, H_C; A, B, C)$. In the above proof we were able to take advantage of the additional symmetry to handle the case where all three pairs $(A, B), (A, C)$, and (B, C) are elementary. Next, we show that we can also recover the symmetry in the case that E is P_1-decomposable.

Lemma 3.2.9. *Let E be an irreducible triangle-free matroid which is W-quasi-maximal, and $(W; H_A, H_B, H_C; A, B, C)$ be its frame. If F is a P_1-decomposing subgroup for E then there exist $a_0 \in A$, $b_0 \in B$, and $c_0 \in C$ such that:*

(1) $A = A_0 \cup A'$, *where* $\emptyset \neq A_0 \subsetneq a_0 + F$, *and* $\operatorname{Stab}(A') = F$;

(2) $B = B_0 \cup B'$, *where* $\emptyset \neq B_0 \subsetneq b_0 + F$, *and* $\operatorname{Stab}(B') = F$;

(3) $C = C_0 \cup C'$, *where* $\emptyset \neq C_0 \subsetneq c_0 + F$, *and* $\operatorname{Stab}(C') = F$; *and*

(4) $a_0 + b_0 + c_0 = 0$.

Proof. Without loss of generality we may assume that $(F, D) \in P_1(A, B)$. From the definition of $P_1(A, B)$ we know that $A + B = D \cup D'$ where D' is a non-empty union of F-cosets, and $D \subseteq F + d_0$ with $d_0 \in D$. Since $A + B$ is non-periodic, we know that D is a proper non-empty subset of $d_0 + F$. Moreover, if $\sigma : \mathbb{F}_2^n \to \mathbb{F}_2^n / F$ is the quotient map, $\sigma(D)$ has exactly one representation $\sigma(D) = \bar{a_0} + \bar{b_0}$ with $\bar{a_0} \in \sigma(A)$, and $\bar{b_0} \in \sigma(B)$. Pick a_0, b_0 from A and B respectively so that $\sigma(a_0) = \bar{a_0}$ and $\sigma(b_0) = \bar{b_0}$, and $a_0 + b_0 = d_0$.

Consider an F-coset $a + F$ which is contained in $H_A \setminus W$ but is different from $a_0 + F$. Suppose that $a + F$ is not fully contained in E. Then select a point $a + f_1 \in (a + F) \setminus E$.

By the quasi-maximality of E there exist points $b + f_2 \in B$ and $c + f_3 \in C$ such that $f_1 + a = (f_2 + b) + (f_3 + c)$. Because the decomposition of $\sigma(d_0) == \bar{a_0} + \bar{b_0}$ is unique, $c + F \neq d_0 + F$, and so $c + F \subseteq E$. Since $(b + f_2) + (c + F) = F + a$, we see that $F + a$ is disjoint from E. Symmetrically, we can apply the same argument to each coset contained in W_B and conclude that every F-coset other than $F + a_0, F + b_0$ and $F + d_0$ is either fully contained in E or is disjoint from E. So we may write

(1) $A = A_0 \cup A'$, where $\emptyset \neq A_0 \subseteq a_0 + F$, and $\mathrm{Stab}(A') = F$;

(2) $B = B_0 \cup B'$, where $\emptyset \neq B_0 \subseteq b_0 + F$, and $\mathrm{Stab}(B') = F$; and

(3′) $A + B = (A_0 + B_0) \cup D'$, with $\emptyset \neq (A_0 + B_0) \subsetneq d_0 + F$.

Moreover, it cannot be that $A_0 = a_0 + F$, as $D = (a_0 + F) + B_0$ would be the full coset $d_0 + F$; similarly $B_0 \subsetneq b_0 + F$. Because E is quasi-maximal we can see that $D \cup D'$ is the complement of C inside $H_C \setminus W$. Thus, C is comprised of $C_0 = (F + d_0) \setminus (A_0 + B_0)$ and a (possibly empty) union of F-cosets. Setting $c_0 = d_0$ we see that $a_0 + b_0 = c_0$, which is equivalent to the desired $a_0 + b_0 + c_0 = 0$. □

Now we know that P_1-decomposable sets are quite structured. They are nearly periodic, except for three deficient cosets. Moreover, the three interesting cosets lie on a triangle in the quotient. This already feels familiar from our construction of exploded-triangle matroids, but first we have a few technical conditions to verify.

The following lemma and most of the previous proof appear in [3].

Lemma 3.2.10. *Let E be an irreducible triangle-free matroid which is W-quasi-maximal, and let $(W; H_A, H_B, H_C; A, B, C)$ be its frame. If F is a P_1-decomposing subgroup for E with respect to W, then*

(1) $F \subseteq W$, and

(2) $|F| = 2^k$ for $k \leq n - 3$.

Proof. Assume without loss of generality that $(F, D) \in P_1(A, B)$. Let $d + F$ be a coset of F which is contained fully in $A + B$. Such a coset exists by the definition of $P_1(A, B)$ Consider $f \in F \setminus \{0\}$. Then $d = d + 0 \in d + F$ and also $d + f \in d + F$. Thus d and $d + f$ are two distinct points in $H_C \setminus W$ and so $f = (d + f) + d \in W$. This proves (1).

Statement (1) implies that $k \leq n - 2$. From the definition of $P_1(A, B)$, the set $(A + B)$ contains D along with the full F-coset $d + F$ which is disjoint from D. Thus $k \leq n - 3$. □

We can now investigate the structure of the quotient. Let $S(E) = \sigma(A) \sqcup \sigma(B) \sqcup \sigma(C)$ be the image of E under the quotient map $\sigma : \mathbb{F}_2^n \to \mathbb{F}_2^n / F$. Then (note that $s \geq 3$) so that $S \subseteq \mathbb{F}_2^s$.

Lemma 3.2.11. *Let $E \subseteq \mathbb{F}_2^n$ be an irreducible triangle-free matroid which is W-quasi-maximal, and let $(W; H_A, H_B, H_C; A, B, C)$ be its frame. If F is a P_1-decomposing subgroup for E then the image, S, of E under the quotient by F has rank s at least three, contains a unique triangle, and satisfies $|S| = \frac{1}{4}(2^s) + 2$.*

Proof. Lemma 3.2.10 implies that $|F| = 2^k$ for $k \geq n - 3$. Thus the quotient, S, will have dimension $s = n - k \geq 3$.

Let $A_0 \cup A', B_0 \cup B', C_0 \cup C'$ and a_0, b_0, c_0 be decomposition and cosets representatives obtained from Lemma 3.2.9. Let $(A_0 + B_0) \cup D'$ be the corresponding decomposition of $A + B$. Since E is quasi-maximal C is the complement of $A + B$ in $H_C \setminus W$. Hence,

$$|C'| = |H_C \setminus W| - |c_0 + F| - |D'| = 2^{n-2} - 2^k - |D'|.$$

Lemma 2.1.4 implies that $|A| + |B| = |A + B| + 1$, since E is irreducible . If we consider the equation $|A| + |B| = |A + B| + 1$ and reduce mod $|F|$ all the full cosets vanish, leaving $|A_0| + |B_0| - |A_0 + B_0| \in \{1, 1 + |F|\}$. In the case of the latter, Kneser's Theorem [12] would imply that $\mathrm{Stab}(A_0 + B_0) > |F|$, but since we know that $A_0 + B_0 \subseteq c_0 + F$ this is impossible. We conclude that $|A_0| + |B_0| = |A_0 + B_0| + 1$. Now from the decomposition into cosets,

$$\begin{aligned}
|S| &= 3 + \frac{1}{|F|} (|A'| + |B'| + |C'|) \\
&= 3 + \frac{1}{2^k} \left(|A'| + |B'| + 2^{n-2} - 2^k - |D'|\right) \\
&= 2 + \frac{1}{2^k} \left(|A'| + |B'| - |D'| + 2^{n-2}\right) \\
&= 2 + 2^{n-k} = \frac{1}{4}(2^s) + 2.
\end{aligned}$$

We know that $a_0 + F, b_0 + F$ and $c_0 + F$ form a triangle in S by Lemma 3.2.9(4). We denote this triangle by $\bar{\Lambda}$. To see that it is the only one, let $v, w \in S$ such that $w \notin \bar{\Lambda}$. Now we know that the preimage of w is a fully contained in E and the preimage of v intersects E. It follows that the preimage of $v + w$ is disjoint from E. Hence, $\bar{\Lambda}$ is the only triangle in S. $\square$

All that remains is to understand the structure of the three degenerate cosets. Let $\Lambda = (a_0 + F) \sqcup (b_0 + F) \sqcup (c_0 + F) \sqcup F$ be the preimage of the unique triangle $\bar{\Lambda}$ in S, extended to a subspace by including F. Let $t = k+2$ so that Λ is a t-dimensional subspace. Let $T = \Lambda \cap E$. Then $T = A_0 \sqcup B_0 \sqcup C_0$ and is triangle-free.

Lemma 3.2.12. *Let $E \subseteq \mathbb{F}_2^n$ be an irreducible triangle-free matroid which is W-quasi-maximal, and let $(W; H_A, H_B, H_C; A, B, C)$ be its frame. If F be a P_1-decomposing subgroup for E and Λ is the preimage of the unique triangle in E/F, then $T = \Lambda \cap E$ is quasi-maximal with respect to F when restricted to Λ. Moreover, for $w \in \Lambda \setminus F$ if $w = u + v$ with $u, v \in E$ then in fact u and v are in T.*

Proof. It suffices to prove the second half of the statement since we assumed E to be W-quasi-maximal and $W \cap \Lambda = F$.

To this end take $w \in (\Lambda \setminus F) \setminus T$ and suppose that there are $u, v \in E$ so that $w = u + v$. Assume that $u \notin \Lambda$. Then $v \notin \Lambda$ as well, since Λ is a subspace. Now the F-cosets $F + u$ and $F + v$ are both fully contained in E. This would imply that $F + w$ is disjoint from E but this is a contradiction since A_0, B_0 and C_0 are non-empty. ☐

Finally, the sets S and T constructed above show that $E = \mathrm{ET}(S, T)$. Therefore, every P_1-decomposable quasi-maximal triangle-free set is an *exploded-triangle* set, and is maximal triangle-free. Combined with Lemma 3.2.8 we obtain the following result.

Theorem 3.2.13. *If E is a rank-n maximal triangle-free set with $|E| = \frac{1}{4}(2^n) + 1$ then E is either a nearly-affine set, or E is an exploded-triangle set.*

3.3 Constructing Maximum One-triangle Sets

One ingredient for our construction of exploded-triangle sets is a rank-s set S with $|S| = \frac{1}{4}(2^s) + 2$ and exactly one triangle. For this reason we look to understand the structure of such sets. First, we note that every such set is maximal with respect to having a single triangle.

Proposition 3.3.1. *If $E \subseteq \mathbb{F}_2^n$ contains exactly one triangle then $|E| \leq \frac{1}{4}(2^n) + 2$.*

Proof. Let E be a set with a single triangle Δ. For each point v in $\mathbb{F}_2^n \setminus \Delta$ we form a rank-3 subspace F_v spanned by Δ and the point v. Then the sets $F_v \setminus \Delta$ form a partition of $\mathbb{F}_2^n \setminus \Delta$. Now, each such $F_v \setminus \Delta$ contains at most one point of E, as otherwise we are guaranteed an additional triangle. Summing over the partition and Δ we conclude that $|E| \leq \frac{1}{4}(2^n) + 2$. ☐

From now on we will refer to such sets as *max one-triangle sets*. Using the results of the previous section we give an simple way to construct max one-triangle sets. The following result appears as Theorem 13.8 in [3].

Theorem 3.3.2. *If E is W-quasi-maximal with $|E| = \frac{1}{4}(2^n) + 1$, then there exists a point x in $\mathbb{F}_2^n \setminus E$ such that x has a unique representation $x = y + z$ for $y, z \in E$ and $x \notin W$.*

Proof. We proceed by induction with respect the decomposition of exploded triangle sets. Our base case will be the nearly-affine sets.

If E is nearly-affine, then there is hyperplane H containing W which meets E in a unique point z. This point partitions $\mathbb{F}_2^n \setminus H$ into $\frac{1}{4}(2^n)$ pairs whose sum is z. Pick such a pair $\{x, y\}$ so that $y \in E$. Then $x = y + z$ is a unique representation for the point $x \in \mathbb{F}_2^n \setminus E$, and it is disjoint from H.

Now let $E = \mathrm{ET}(S, T)$ be an exploded-triangle set which decomposes into a max one-triangle set S, and a maximal triangle-free set T, where T is contained in the subspace Λ. By the induction hypothesis, there is a point $x \in \Lambda \setminus T$ which has a unique representation $x = y + z$ for $y, z \in T$. Lemma 3.2.12 implies that every triangle of $\mathrm{ET}(S, T \cup \{x\})$ is a triangle in $T \cup \{x\}$, so there are no other representations for x. $\square$

We can see from this proof that we may slightly modify the use of the exploded-triangle construction in Section 3.1, by allowing the set T to be one-triangle instead of triangle-free. If S and T are both max one-triangle then Proposition 3.1.2 implies that $\mathrm{ET}(S, T)$ will contain exactly one triangle, and will have cardinality $\frac{1}{4}(2^n) + 2$.

In the next section we argue that all max one-triangle sets, except for one small exception, are constructed using this modified exploded-triangle construction, or are obtained from a nearly-affine set by adding a single point.

3.4 Decomposing Maximum One-triangle Sets

We start with a max one-triangle set E with triangle $\Delta = \{x, y, z\}$ and consider what happens when we delete the element x. The set $E \setminus x$ will be triangle-free and have $\frac{1}{4}(2^n)$ elements, but may not be maximal triangle-free. There will be one small exceptional example given by the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which we refer to as $U_{2,3} \oplus U_{1,1}$. The following lemma identifies this as an important example.

Lemma 3.4.1. *Let E be a max one-triangle set with triangle Δ. If there is a hyperplane H of $\mathbb{F}_2^n$ which meets E in a single point then either there is $x \in \Delta$ such that $E \setminus x$ is nearly-affine, or $E \cong U_{2,3} \oplus U_{1,1}$.*

Proof. Let z be the point at the intersection of H and E. Let W be a codimension-2 subspace of $\mathbb{F}_2^n$ contained in H and disjoint from z. Let H_A and H_B be the remaining two hyperplanes containing W and let $H_C := H$. Let x, y be the remaining two points of Δ. Then without loss of generality we may assume that $x \in H_A$ and that $|E \cap H_A|$ is even. Let $A = (E \cap H_A) \setminus \{x\}$; we claim that $E \setminus x = \mathrm{NA}(W, z, A)$ and is nearly-affine. To see this let $B = H_B \cap E$, and $A = W_A \setminus (z + B)$, then note that $E \setminus x = A \cup B \cup \{z\}$. Since $|A|$ is odd $E \setminus x$ satisfies condition (2) of Proposition 3.1.1. When $n > 3$ the set $E \setminus x$ also satisfies condition (3) and thus is nearly-affine. When $n = 3$ the only one-triangle set is $U_{2,3} \oplus U_{1,1}$. □

Now we move on to the case that $E \setminus x$ is not maximal.

Lemma 3.4.2. *Let E be a max one-triangle set with triangle Δ. If $E \setminus x$ is not maximal triangle-free, then either E is obtained from a nearly-affine set by adding an extra point, or by the modified exploded-triangle construction*

Proof. Since $E \setminus x$ is not maximal triangle-free we may extend it to a maximal set R. By Corollary 2.1.8 it must be that $|R| = 2^{n-2} + 2^d$ for some $d \geq 1$. We also know that it will be a doubling of a maximal triangle-free set $\overline{R}$ of rank $n - d$.

Suppose that $R + x = R$. Then we partition R into $2^{n-3} + 2^{d-1}$ pairs of points whose sum is x. We must delete a point from all but one of these pairs to get a set which is one-triangle. Hence,

$$|E \setminus x| \leq 2^{n-2} + 2^d - (2^{n-3} + 2^{d-1} - 1) = 2^{n-3} + 2^{d-1} + 1.$$

Since we know the cardinality of $E \setminus x$ this implies that $d \geq n - 2$. Thus R is a doubling of a triangle-free set of rank at most 2. In this case R must be affine and we conclude by Lemma 3.4.1 that E is obtained from a nearly-affine set by adding an additional point.

Now suppose that $R + x \neq R$ and let $\sigma : \mathbb{F}_2^n \to \mathbb{F}_2^{n-d}$ be the quotient map taking R to $\overline{R}$. Let $\Delta = \{x, y, z\}$, and $\overline{x}$,$\overline{y}$, $\overline{z}$ be the images of x, y, z. If $|\sigma^{-1}(\overline{y}) \cap E| + |\sigma^{-1}(\overline{z}) \cap E| \geq 2^d + 2$ Then x is contained in at least two triangles. So it must be that

$$|\sigma^{-1}(\overline{y}) \cap E| + |\sigma^{-1}(\overline{z}) \cap E| \leq 2^d + 1.$$

Since $|\sigma^{-1}(\overline{y}) \cup \sigma^{-1}(\overline{z})| = 2^{d+1}$ and $|R| = |E \setminus x| + 2^d - 1$, we conclude that $|\sigma^{-1}(v)| = 2^d$ for every other point $v \in \overline{R}$. It follows that $\overline{R} \cup \{\overline{x}\}$ contains only a single triangle $\overline{x}, \overline{y}, \overline{z}$. Moreover, Lemma 3.4.1 implies that $T := \sigma^{-1}(\overline{x}, \overline{y}, \overline{z}) \cap E$ is obtained from a nearly-affine set by adding an additional point, or $T \cong U_{2,3} \oplus U_{1,1}$. If $\overline{R}$ has rank 2 then $E = T$, and otherwise $E = \mathrm{ET}(\overline{R} \cup \{\overline{x}\}, T)$. □

Now all that remains is the case that $E \setminus x$ is maximal triangle-free. We know from Theorem 3.2.13 that $E \setminus x$ is either nearly-affine, or an exploded-triangle set. If $E \setminus x$ is nearly-affine then this already fits our hypothesis. When $E \setminus x$ is an exploded-triangle set with parts S and T we must check that x lies in T to confirm that E is obtained from the modified exploded-triangle construction.

Lemma 3.4.3. *Let E be a max one-triangle set with triangle Δ such that $E \setminus x = \mathrm{ET}(S, T)$ is maximal triangle-free. If x does not lie in the span of T then there is another point $y \in \Delta$ such that $E \setminus y$ is not maximal.*

Proof. Let $\sigma : \mathbb{F}_2^n \to \mathbb{F}_2^s$ be the quotient map which sends the set $E \setminus x$ to S. Since T is triangle-free, we must have performed at least two doublings in the exploded-triangle construction, and know that $\sigma(x)$ is non-zero. Otherwise, x creates more than just one triangle.

Let $\overline{\Lambda}$ be the triangle of S. Let $\overline{y}$ and $\overline{z}$ be the points in S such that $\overline{y} + \overline{z} = \sigma(x)$. Since we are assuming that $\sigma(x) \notin \overline{\Lambda}$ we may assume without loss of generality that $\overline{z} \notin \overline{\Lambda}$. Now we know that $\sigma^{-1}(\overline{z}) \subseteq E$, and so it must be that $\sigma^{-1}(\overline{y})$ intersects E in a single point. If the intersection were larger then we would have more than one triangle through x. Let y be the lonely point in $\sigma^{-1}(\overline{y}) \cap E$. Then $E \setminus y$ is triangle-free and is contained in a repeated doubling of $S \setminus \overline{y}$, so we know that $E \setminus y$ is not maximal. □

Combining the above results we come to the following conclusion.

Theorem 3.4.4. *If E is a max one-triangle set then either $E \cong U_{2,3} \oplus U_{1,1}$, E is obtained from a nearly-affine set by adding a single point, or E is obtained from the modified exploded-triangle construction.*

Chapter 4

Counting Pentagons

Recall the graph-theoretic result of Grzesik [9], and independently by Hatami, Hladký, Král, Norine, and Razborov [10], about the number of pentagons in triangle-free graphs.

Theorem 4.0.1. *Every triangle-free graph with n vertices contains at most $(n/5)^5$ cycles of length five. Moreover, when n is divisible by 5 equality is obtained only by the balanced blow-up of the pentagon.*

We wish to extend this result to binary matroids, and state a conjecture which we believe is the correct analogue is this setting. As evidence for this conjecture, we resolve it in the case of matroids with density greater than $1/4$.

Recall that a *pentagon* in a simple binary matroid E is a set $x_1, \dots, x_5 \in E$ of five elements of E such that $x_1+x_2+x_3+x_4+x_5 = 0$ and no proper subset sums to zero. Note that when E is triangle-free every set of five elements whose sum is zero forms a pentagon. We will use $\#P(E)$ to denote the number of pentagons in a set E. Before stating our main conjecture we must get some preliminary definitions and propositions out of the way.

Proposition 4.0.2. *If $E = X \times \mathbb{F}_2$ for a triangle free set $X \subseteq \mathbb{F}_2^{n-1}$, then $\#P(E) = 16 \cdot \#P(X)$.*

Proof. We consider E to be the set of vectors in $\mathbb{F}_2^n$ of the form (x, δ) where $x \in X$ and $\delta \in \{0, 1\}$.

Now suppose that $\{(x_i, \delta_i)\}_{i=1}^5$ is a pentagon in E. Then $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Since X is triangle-free and does not contain the zero vector, $\{x_i\}_{i=1}^5$ is a pentagon in X. We also know that $\delta_5 = \delta_1 + \delta_2 + \delta_3 + \delta_4$, and moreover every choice of $\delta_1, \delta_2, \delta_3, \delta_4$ gives another pentagon in E. $\square$

The $(n-4)$-th doubling of the pentagon will be a subset of $\mathbb{F}_2^n$ and, by the above proposition, will contain $16^{n-4} = 2^{4n-16}$ pentagons. We conjecture that this is an upper bound on the number of pentagons in each triangle-free binary matroid.

Conjecture 4.0.3. *Every triangle-free binary matroid E with rank n contains at most 2^{4n-16} pentagons. Moreover, equality holds only when E is a repeated doubling of the pentagon.*

4.1 Counting Pentagons in Low Density Sets

We can provide an upper bound on the number of pentagons in a binary matroid in terms of only the density. This bound ignores all of the structure of the set, but still gets within a constant factor of our conjectured bound for modest densities.

Proposition 4.1.1. *If $E \subseteq \mathbb{F}_2^n$ is a binary matroid with density α then E contains at most $\frac{2^{16}\cdot\alpha^4}{5!} \cdot 2^{4n-16}$ pentagons.*

Proof. The last element of a pentagon can always be determined by the first 4 elements, so we can represent a pentagon by a 4-tuple of elements from E. For each pentagon there will be 5! different 4-tuples in E^4 which determine that pentagon. Hence, the number of pentagons in E is at most

$$\frac{|E^4|}{5!} \leq \frac{(\alpha \cdot 2^n)^4}{5!} = \frac{2^{16} \cdot \alpha^4}{5!} \cdot 2^{4n-16}.$$

$\square$

Note that this does not require that E is a triangle-free set. Now we can apply this naive bound to our setting. The Theorem of Govaerts and Storme implies that a triangle-free set with density greater than 5/16 is affine, and so contains no pentagons. If we apply the above bound to sets with density at most 5/16 we find that triangle-free sets contain at most $\frac{625}{120} \cdot 2^{4n-16}$ pentagons. In the next section we will confirm the conjecture for triangle-free sets with density greater than 1/4. Now if we apply the proposition to sets with density at most 1/4 we would find that triangle-free sets contain at most $\frac{256}{120} \cdot 2^{4n-16}$ pentagons, which is slightly more than double the bound we want. Finally, any set with density at most $\frac{\sqrt[4]{5!}}{16} \approx 0.20568$ contains at most 2^{4n-16} pentagons, leaving a small gap where we have been unable to verify the conjecture.

Theorem 4.1.2. *Conjecture 4.0.3 holds for every triangle-free binary matroid E with density at most $\frac{\sqrt[4]{5!}}{16} \approx 0.20568$.*

Corollary 4.1.3. *If Conjecture 4.0.3 holds for matroids of density greater that $1/4$ then every triangle-free binary matroid E with rank n contains at most $\frac{256}{120}2^{4n-16}$ pentagons.*

4.2 Counting Pentagons in Large Triangle-free Sets

Proposition 4.0.2 implies that it suffices to verify this conjecture for sets which are not doublings. In the case where E has density greater than $1/4$ we can combine Theorems 2.1.7 and 3.2.13 to see that is suffices to check the nearly-affine and exploded-triangle sets. We prove the following result which is slightly stronger but allows for an easier induction.

Theorem 4.2.1. *If $E \subseteq \mathbb{F}_2^n$ is max one-triangle, then the number of pentagons in E is at most 2^{4n-16}.*

This result combined with Theorem 3.3.2 implies Conjecture 4.0.3 for sets with density greater than $1/4$, since every large triangle-free set is contained in a max one-triangle set. We consider the cases of Theorem 3.4.4 separately, first the nearly affine sets and then the modified exploded-triangle sets, before combining them to complete the result.

We begin with the small-rank cases. The following bound, which refers to the computational results described in the appendix, is tight for M_9.

Lemma 4.2.2. *If $E \subseteq \mathbb{F}_2^n$ for $n \in \{5, 6\}$ is max one-triangle then $\#P(E) \leq \frac{11}{16}2^{4n-16}$.*

Note that the case $n = 4$ is the pentagon with an extra point, which exactly meets the bound in Conjecture 4.0.3. For $n = 3$, there is only the set $U_{2,3} \oplus U_{1,1}$ which contains no pentagons. For larger examples we begin with the one-triangle sets obtained by adding a single point to a nearly affine set.

Lemma 4.2.3. *If $E \subseteq \mathbb{F}_2^n$ for $n \geq 7$ and is obtained from a nearly affine set by adding a single point then E has at most $\frac{2}{3}2^{4n-16}$ pentagons.*

Proof. Let $x, y \in E$ be such that $E \setminus x$ is nearly-affine and $E \setminus \{x, y\}$ is affine, and let H be the hyperplane so that $H \cap (E \setminus x) = \{y\}$. Every pentagon of E must contain either x or y. The number of pentagons containing y in $E \setminus x$ is at most

$$\frac{1}{4!}(2^{n-2})^3 = \frac{1}{3}2^{3n-9},$$

since each pentagon is determined by any three of its points other than y. If $x \in H$ the same argument gives that the number of pentagons containing x in E is at most $\frac{1}{3}2^{3n-9}$, since no pentagon can contain both x and y. If $x \notin H$ then every pentagon of E contains y, and x is contained in at most $\frac{1}{3}2^{2n-4} \leq \frac{1}{3}2^{3n-9}$ pentagons.

In either case the total number of pentagons is at most $\frac{2}{3}2^{3n-9}$ which, since $n \geq 7$ is at most $\frac{2}{3}2^{4n-16}$. $\square$

Now we must consider the one-triangle sets obtained from the modified exploded-triangle construction. Let $E \subseteq \mathbb{F}_2^n$ be such a one-triangle set composed of one-triangle parts $S \subseteq \mathbb{F}_2^s$ and $T \subseteq \mathbb{F}_2^t$. Let $\Lambda = \{x, y, z\}$ be the triangle in S. We will write every point in E in the form (w, v) where $w \in S$ and $v \in V(t-2, 2)$. Let $\{(w_i, v_i)\}_{i=1}^5$ be a pentagon in E. We break into three cases:

(1) $\{w_i\}_{i=1}^5$ is a pentagon in S,

(2) $\{(w_i, v_i)\}_{i=1}^5$ is a pentagon in T, or

(3) $\{(w_i, v_i)\}_{i=1}^5 = \{(x, v_1), (y,v_2), (z, v_3), (w_0, v_4)(w_0, v_5)\}$ for some $w_0 \in S \setminus \Lambda$.

Every pentagon of E must fall into one of these three cases, which can be seen using the ideas from the proof of Proposition 4.0.2. We can give an upper bound on the number of pentagons of type (3) immediately, without any knowledge of the precise structure of S and T. Since $|T| = 2^{t-2} + 2$ there will be at most

$$(2^{s-2} - 1)\frac{2^{t-2}}{2}\left(\frac{2^{t-2} + 2}{3}\right)^3$$

pentagons of type (3), as we must select $w_0 \in S \setminus \Lambda$, the pair $\{(w_0, v_4), (w_0, v_5)\}$ and the points v_1, v_2, v_3. Moreover, when $t \geq 4$ the above quantity is at most

$$(2^{t-3})^4(2^{s-2} - 1).$$

When $t = 3$ structural conditions guarantee that the number of pentagons of type (3) is exactly $(2^{s-2} - 1) = (2^{t-3})^4(2^{s-2} - 1)$, since $\{(x, v_1), (y,v_2), (z, v_3)\}$ must not be a triangle in $T \cong U_{2,3} \oplus U_{1,1}$.

Now we are ready to count pentagons in sets obtained from the modified exploded triangle construction.

Lemma 4.2.4. *If $E \subseteq \mathbb{F}_2^n$ is obtained from the modified exploded triangle construction and $n \geq 5$ then the number of pentagons in E is at most $\left(\frac{11}{16} + \frac{1}{2^2 \cdot 7}\right) \cdot 2^{4n-16}$.*

Proof. Define a function $f : \mathbb{N} \to [0,1]$ as follows. Let

$$f(n) = \frac{11}{16} + \frac{1}{2^5} \sum_{i=0}^{n-6} 2^{-3i}.$$

Note that

$$\frac{11}{16} = f(5) \leq f(n) \leq \frac{11}{16} + \frac{1}{2^2 \cdot 7},$$

and that

$$f(n) + \frac{1}{2^{3n-10}} = f(n+1).$$

We will show that E has at most $f(n) \cdot 2^{4n-16}$ pentagons.

We proceed by induction on n. Our base cases are those sets obtained from nearly affine sets by adding a single point. Lemmas 4.2.2 and 4.2.3 imply these examples have at most $\frac{11}{16} \cdot 2^{4n-16}$ pentagons.

Suppose that $n \geq 7$ and the result holds for all smaller n. Now let $S \subseteq \mathbb{F}_2^s$ and $T \subseteq \mathbb{F}_2^t$ be so that $E = \mathrm{ET}(S, T)$. We break into cases based on the rank of the set S.

Case 1: $s = 3$

When $s = 3$ we know that $S \cong U_{2,3} \oplus U_{1,1}$. We count the pentagons according to the three types laid out above. In this case there are no pentagons of type (1) since S contains no pentagons. By the induction hypothesis, and considering the small cases, there will be no more than 2^{4t-16} pentagons of type (2). Since $|T| = 2^{t-2} + 2$ there will be at most

$$\frac{2^{t-2}}{2} \binom{2^{t-2}+2}{3}^3$$

pentagons of type (3), as we must select the points v_1, v_2, v_3 and the pair $\{(w_0, v_4)(w_0, v_5)\}$. Thus, the set E has at most

$$\frac{2^{t-2}}{2} \binom{2^{t-2}+2}{3}^3 + 2^{4t-16}$$

pentagons, which is at most $\frac{11}{16} \cdot 2^{4n-16}$ for $n \geq 6$.

Case 2: $s = 4$

When $s = 4$ we know that S is the pentagon with an additional point. Let $\Lambda = \{x, y, z\}$ be the triangle in S. Now S has exactly one pentagon, and it intersect Λ in the points x and y. Thus there are at most

$$(2^{t-2})^2 \binom{2^{t-2}+2}{2}$$

pentagons of type (1), since $(x, w_1), (y, w_2) \in T$. Again we know there will be at most 2^{4t-16} pentagons of type (2). Finally, there will be at most $(2^{t-3})^4(2^2 - 1)$ pentagons of type (3). We conclude that there are at most

$$2^{4t-16} + (2^{t-4})^4(2^6 - 16) \leq 2^6 \cdot (2^{t-4})^4$$

pentagons of either type (2) or (3). Now E has at most

$$(2^{t-2})^2 \binom{2^{t-2}+2}{2} + 2^6 \cdot (2^{t-4})^4$$

pentagons, which is at most $\frac{11}{16} \cdot 2^{4n-16}$ for $n \geq 7$.

Case 3: $s \geq 5$

When $s \geq 5$ we know by the induction hypothesis that S contains at most $f(s) \cdot 2^{4s-16}$ pentagons. Thus, by Lemma 4.0.2, there are at most

$$(2^{t-2})^4 \cdot f(s) \cdot 2^{4s-16} = f(s) \cdot 2^{4s+4t-24} = f(s) \cdot 2^{4n-16}$$

pentagons of type (1). Again, we know that there are at most 2^{4t-16} pentagons of type (2). Finally, there will be at most $(2^{t-3})^4(2^{s-2} - 1)$ pentagons of type (3). Combining the pentagons of types (2) and (3) we find that there are at most

$$\begin{aligned} 2^{4t-16} + (2^{t-3})^4(2^{s-2} - 1) &= 2^{4t-16} + (2^{t-4})^4(2^{s+2} - 16) \\ &\leq (2^{t-4})^4 \cdot 2^{s+2} \\ &= 2^{(4s+4t-24)-3s+10} = \frac{2^{4n-16}}{2^{3s-10}} \end{aligned}$$

pentagons which are not of type (1). Finally we conclude that there are at most

$$f(s) \cdot 2^{4n-16} + \frac{1}{2^{3s-10}} \cdot 2^{4n-16} = f(s+1) \cdot 2^{4n-16} \leq f(n) \cdot 2^{4n-16}$$

pentagons in E.

□

By combining Lemmas 4.2.2, 4.2.3, and 4.2.4 we conclude that every max one-triangle set with rank at least 5 has at most $\frac{81}{112}2^{4n-16}$ pentagons. Theorem 3.3.2 implies that every triangle-free set with $\frac{1}{4}(2^n)+1$ points is contained in a max one-triangle set, so the same bound applies. We have therefore obtained the following strengthening of Conjecture 4.0.3 for large triangle-free sets.

Theorem 4.2.5. *If E is a large triangle-free set with more than $\frac{81}{112}2^{4n-16}$ pentagons, then E is contained in a repeated doubling of the pentagon.*

4.3 The Possibility of a Regularity Approach

Our goal is to use Green's Regularity Lemma for abelian groups in the setting of binary matroids to allow us to bound the number of pentagons in triangle free binary sets. Let $G = \mathbb{F}_2^n$ and $E \subseteq G$. We say that E is *ϵ-uniform in G* if for every hyperplane H of G the estimate

$$\left| \frac{|E \cap H|}{|H|} - \frac{|E|}{|G|} \right| \le \epsilon$$

holds. Let F be a subspace of G and E a subset of G. For each $g \in G$ we define $F^g(E) = \{h \in F : h + g \in X\}$. Note that $F^g(E) = g + (X \cap (g + F))$, so it will be helpful to think of $F^g(E)$ as a 'shifted' representative of the intersection of E with the coset $g+F$. We say that F is *ϵ-regular with respect to E* if $F^g(E)$ is ϵ-uniform in F for all but $\epsilon|G|$ values of $g \in G$.

The following result of Green [8] guarantees the existence of a uniform subspace of bounded codimension depending on ϵ. Here $W(t)$ represents a tower of twos of height $\lceil t \rceil$.

Theorem 4.3.1 (Green, [8])**.** *Let $G = \mathbb{F}_2^n$ and $E \subseteq G$ and let $\epsilon > 0$ be a real number. Then there is a subspace F of G that is ϵ-regular with respect to E and G and has codimension at most $W(\epsilon^{-3})$ in G.*

We will also make use of a counting lemma for triangles in a regular partition.

Lemma 4.3.2. *Let F be a subgroup of G and let $g_1, g_2, g_3 \in G$. If A is a subset of G where $|F^{g_i}(A)| = \alpha_i|F|$ and $F^{g_1}(E)$ is ϵ-uniform, then $T(g_1.g_2.g_3)$ (the number of triples (x_1, x_2, x_3) such that $x_i \in F^{g_i}(E)$ and $x_1 + x_2 + x_3 = 0$) satisfies*

$$T(g_1, g_2, g_3) \ge (\alpha_1\alpha_2\alpha_3 - \epsilon)|F|^2.$$

Now we define a reduced set that allows us to convert a triangle-free set into a triangle-free partition into ϵ-uniform parts.

Definition 4.3.3. Let $A \subseteq G$ and let F be an ϵ-regular subgroup for A. Define the reduced set A' as follows. For each $g \in G$ determine if either:

(i) $F^g(A)$ is not ϵ-uniform, or

(ii) $|F^g(A)| \leq \epsilon^{1/3}|F|$.

If either (i) or (ii) hold for $g \in G$ then we delete the entire coset $g + F$ from A.

Note that both conditions depend only on the coset $g+F$ and not the representative g itself. In this process we delete at most $\epsilon|G|$ points which satisfy condition (i) and at most $\epsilon^{1/3}|G|$ points which satisfy condition (ii). We now argue that if A was triangle free, then the image of A under the quotient by F is also triangle free.

Suppose that A' had a triangle in its quotient, and let $g_1, g_2, g_3 \in G$ be such that the cosets $g_i + F$ are distinct and $g_1 + g_2 + g_3 = 0$. Then

$$T(g_1, g_2, g_3) \geq (\alpha_1\alpha_2\alpha_3 - \epsilon)|F|^2 > (\epsilon^{1/3}\epsilon^{1/3}\epsilon^{1/3} - \epsilon)|F|^2 = 0$$

by the triangle counting lemma. Thus there exists a triple (x_1, x_2, x_3) such that $x_i \in F^{g_i}(E)$ and $x_1 + x_2 + x_3 = 0$. But then $x_1 + g_1, x_2 + g_2, x_3 + g_3$ form a triangle in A', and also a triangle of A.

Now each point we deleted to create A' was contained in at most $|E|^3/4!$ pentagons, which is at most $\frac{1}{3}2^{3n-4}$ for $|E| \leq \frac{1}{4}|G|$. Thus A contains at most

$$\frac{2^7(\epsilon + \epsilon^{1/3})}{3} \cdot 2^{4n-16} \leq \frac{2^8\epsilon^{1/3}}{3} \cdot 2^{4n-16}$$

pentagons in addition to those contained in A'.

For fixed ϵ the quotient of A' by F will have bounded rank. If we could show that all such triangle free sets had at most $d \cdot 2^{4n-16}$ pentagons then we could deduce that every triangle free set has at most

$$\left(\frac{2^8\epsilon^{1/3}}{3} + d\right) \cdot 2^{4n-16}$$

pentagons.

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APPENDICES

Appendix A

Computational Results in Low Rank

A.1 Small Examples

This section comprises the examples of maximal triangle-free subsets of $\mathbb{F}_2^n$ with $\frac{1}{4}(2^n)+1$ points for $n \leq 6$. Every max one-triangle set of rank at most 6 is a one-element deletion away from one of the 7 sets listed below. There far too many max one-triangle sets to list here, but they can be easily generated if we start with the following list.

For each example we will also give the number of pentagons, and the maximum number of pentagons among its one-triangle extensions.

$$U_{4,5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The set $U_{4,5}$ is the pentagon itself. All its single-element extensions are isomorphic, and none contain any additional pentagons.

$$M_9 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The set M_9 is nearly-affine with $|A| = 1$. It contains 7 pentagons. It has two non-isomorphic one-triangle extensions, one with the same 7 pentagons, and another with 11 pentagons.

$$M_{17}^1 = \begin{bmatrix} 1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&1&0&1&1&0&0&1&1&0&1&1&0&0 \\ 0&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0 \\ 0&0&0&1&1&0&0&1&1&0&0&1&1&1&0&0&1 \\ 0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$M_{17}^2 = \begin{bmatrix} 1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1 \\ 0&1&0&0&1&0&1&1&0&0&1&1&0&0&1&0&1 \\ 0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&1 \\ 0&0&0&1&1&0&0&1&1&0&0&1&1&1&0&0&1 \\ 0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$M_{17}^3 = \begin{bmatrix} 1&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&1 \\ 0&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1 \\ 0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$M_{17}^4 = \begin{bmatrix} 1&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&0 \\ 0&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&0 \\ 0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0 \\ 0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$M_{17}^5 = \begin{bmatrix} 1&0&0&0&1&0&0&0&1&0&0&0&1&1&1&1&0 \\ 0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0&0 \\ 0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0&0 \\ 0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

All the rank-6 examples are also nearly-affine. M^1_{17} contains 35 pentagons and its extensions contain at most 63. M^2_{17} contains 67 pentagons and its extensions contain at most 127. M^3_{17} contains 72 pentagons and its extensions contain at most 91. M^4_{17} contains 75 pentagons and its extensions contain at most 95. M^5_{17} contains 80 pentagons and its extensions contain at most 95.

A.2 Python Code

The following python code was used to perform the calculations needed in the previous section.

```
m9 = map_vec_to_int( [
      '001111',
      '010111',
      '011011',
      '011101',
      '011110',
      '100000',
      '111000',
      '110100',
      '101100'
])

m117 = map_vec_to_int( [
      '001111',
      '100111',
      '101011',
      '101101',
      '101110',
      '100000',
      '101100',
      '101010',
      '100110',
      '110000',
      '111100',
      '111010',
      '110110',
```

```
    '110001',
    '110111',
    '111011',
    '111101',
])

def pentagon_count(m):
    n = len(m)
    count = 0
    for i0 in range(n):
        for i1 in range(i0+1, n):
            for i2 in range(i1+1, n):
                for i3 in range(i2+1, n):
                    for i4 in range(i3+1, n):
                        if m[i0]^m[i1]^m[i2]^m[i3]^m[i4] == 0:
                            count += 1

    return(count)


def pentagon_per_point(m):
    n = len(m)
    count = 0
    per_point = [0 for i in range(n)]
    for i0 in range(n):
        for i1 in range(i0+1, n):
            for i2 in range(i1+1, n):
                for i3 in range(i2+1, n):
                    for i4 in range(i3+1, n):
                        if m[i0]^m[i1]^m[i2]^m[i3]^m[i4] == 0:
                            count += 1
                            per_point[i0] += 1
                            per_point[i1] += 1
                            per_point[i2] += 1
                            per_point[i3] += 1
                            per_point[i4] += 1

    print(per_point)
```

```
def single_triangle_examples(m):
    n = len(m)
    triangles = {}
    for i0 in range(n):
        for i1 in range(i0+1,n):
            t = m[i0] ^ m[i1]
            if t not in triangles:
                triangles[t] = 0
            triangles[t] += 1

    print(len(triangles)+n)
    for t in list(triangles.keys()):
        if triangles[t] > 1:
            del triangles[t]
        else:
            triangles[t] = pentagon_count(m)

    for t in triangles.keys():
        for j1 in range(n):
            for j2 in range(j1+1, n):
                for j3 in range(j2+1, n):
                    for j4 in range(j3+1, n):
                        if t^m[j1]^m[j2]^m[j3]^m[j4] == 0:
                            triangles[t] += 1
    print(triangles.values())

print('Rank 5 Examples (max 16):')
print(pentagon_count(m9))

print('Rank 6 Examples (max 256):')
print(pentagon_count(m117))
```