Coefficient spaces arising from locally compact groups

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

I am the sole author of Chapters 1 and 2. Chapter 3 is joint work with Jason Crann and Chapter 4 is joint work with Brian Forrest and Matthew Wiersma.

Abstract

This thesis studies two disjoint topics involving coefficient spaces and algebras associated to locally compact groups. First, Chapter 3 investigates the connection between amenability and compactness conditions on locally compact groups and the homology of the Fourier algebra when viewed as a completely contractive Banach algebra. We provide characterizations of relative 1-projectivity, 1-flatness, and 1-biflatness of the Fourier algebra. These allow us to deduce a new hereditary property for an amenability condition, namely that inner amenability passes to closed subgroups. Our techniques also allow us to show that inner amenability coincides with Property (W) and to settle a conjecture regarding the cb-multiplier completion of the Fourier algebra. Our second theme is coefficient spaces arising from L^p-representations of locally compact groups. Chapter 4 is motivated by a question of Kaliszewski, Landstad, and Quigg regarding whether two coefficient space constructions coincide. We are able to provide a positive answer in special cases, in particular for the group $SL(2,\mathbb{R})$. We establish several results regarding the non-separability of algebras related to the L^p-Fourier algebras, and characterize when these algebras have a bounded approximate identity.

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Chapter 1

Introduction

The field of abstract harmonic analysis applies the tools of Banach algebra and operator algebra theory to study locally compact groups with the objective of characterizing the underlying structure of the group in terms of the associated algebras. The representation theory of such groups naturally produces algebras of operators on Hilbert space and also leads to the study of algebras of coefficient functions, which have historically been an essential tool in analyzing the structure of abelian and compact locally compact groups. The property of amenability for a general locally compact groups affords much of the tight correspondence between properties of the group and the associated algebras that is present in the classical compact and abelian settings. In the setting of coefficient algebras, seminal work of Ruan [74] shows that amenability of a locally compact group G translates into operator amenability of its Fourier algebra $A(G)$, and unpublished work of Ruan shows that amenability also corresponds to the coincidence of the Fourier-Stieltjes algebra $B(G)$ and the algebra of completely bounded multipliers of the Fourier algebra $M_{cb}A(G)$. These themes have been pursued by several authors [3, 76, 79], showing that amenability properties of G correspond to homological properties of $A(G)$ and to the identity of various coefficient algebras associated to G. In the context of operator algebras, amenability implies the injectivity of the group von Neumann algebra $L(G)$, while a result of Lau and Paterson [58] states that the converse holds only when the group is inner amenable. This weaker amenability condition is closely connected to the structure of the algebras associated to G and has recently appeared in several disjoint contexts [2, 5, 61].

Since the work of Ruan established completely contractive Banach algebras as the correct category in which to study the Fourier algebra, a rich correspondence between homological properties of $A(G)$ and amenability properties of the locally compact group G has been found [24, 76]. In this thesis we use operator algebraic techniques inspired by recent work in locally compact quantum groups to study this correspondence, providing homological characterizations for inner amenability and several related amenability and compactness conditions. The operator amenability of a completely contractive Banach algebra is equivalent to it being operator biflat and having a bounded approximate identity, and for the Fourier algebra a classical result of Leptin [56] states that amenability of G is already equivalent to $A(G)$ having a bounded approximate identity. The remaining condition, operator biflatness of $A(G)$, has been investigated by Aristov, Runde, and Spronk [4], who show that G being a quasi-SIN group is sufficient to guarantee operator biflatness of $A(G)$. A main result in this thesis provides a partial converse. In addition, we provide the first examples of locally compact groups G for which $A(G)$ fails to be operator biflat, proving a conjecture of [4]. We moreover give a dual characterization of inner amenability, answering a question of Lau and Paterson [59, Example 5], and show that inner amenability coincides with Property (W), answering a question of Anantharaman-Delaroche [2, Problem 9.1].

There is a comprehensive literature studying coefficient spaces of locally compact groups, beginning with the foundational work of Eymard [23], that continues to produce new insights into operator algebras and abstract harmonic analysis [19, 36]. With an interest in producing exotic group C^*-algebras, Brown and Guentner recently introduced the notion of L^p- representations of locally compact groups [9]. These give rise to new families of coefficient spaces, namely the L^p-Fourier and Fourier-Stieltjes algebras $A_{L^p,BG}(G)$ and $B_{L^p,BG}(G)$. The study of these spaces is the second major theme of this thesis. We consider a related construction introduced by Kaliszewski, Landstad, and Quigg [45] and investigate when the associated coefficient spaces coincide with those defined by Brown and Guentner, showing that this holds for $SL(2,\mathbb{R})$ and in certain cases for free groups. We provide a concise proof of a result of Okayasu [66] that the spaces $B_{\ell^p,BG}(\mathbb{F}_d)$ are distinct for distinct values of p, where $\mathbb{F}_d$ is the free group on $d \geq 2$ generators, and elaborate several results of Wiersma [86] regarding whether the spaces $A_{L^p,BG}(G)$ are distinct for distinct values of $p \in [2, \infty)$.

This thesis is organized as follows. Chapter 2 defines the basic concepts in the homology theory of completely contractive Banach algebras, outlines the basic theory of Fourier and Fourier-Stieltjes algebras, discusses amenability of locally compact groups, and defines the L^p-coefficient spaces. Chapter 3 studies the relationship between the homology of the Fourier algebra and amenability and compactness conditions. In Section 3.1 new homological and operator algebraic characterizations of inner amenability are developed. These allow us to show that inner amenability passes to closed subgroups, a hereditary property that has not appeared in the literature. We end this section by showing that inner amenability coincides with Anantharaman-Delaroche's Property (W) and giving a homological characterization of IN groups. Section 3.2 characterizes relative biflatness of the Fourier algebra $A(G)$ in terms of the existence of approximate indicators for the diagonal subgroup of $G \times G$, providing a converse to a result of Aristov, Runde, and Spronk [4]. We also confirm a conjecture of these authors in the case of totally disconnected groups, showing that relative biflatness of $A(G)$ is equivalent to the QSIN condition for these groups. In section 3.3 we study actions of discrete groups on compact groups to produce examples of groups with Fourier algebras that fail to be relatively biflat. The chapter ends with Section 3.4, where we provide many examples of weakly amenable groups for which $A_{cb}(G)$, the closure of $A(G)$ in $M_{cb}A(G)$, fails to be operator amenable, answering a question of Forrest, Runde, and Spronk [26]. Chapter 4 is concerned with coefficient spaces arising from L^p-representations. We begin in Section 4.1 by collecting preliminary results on the various coefficient spaces we investigate. Section 4.2 provides a simplified proof of a result of Okayasu and shows that a conjecture of Kaliszewski, Landstad, and Quigg holds for free groups in certain cases. In Sections 4.3 and 4.4 we consider abelian groups and IN groups, respectively, strengthening results of Wiersma and Taylor [82, 86]. Section 4.5 extends characterizations of amenability provided by Kaniuth and Ülger and by Chu and Xu to several coefficient algebras. Finally, Section 4.6 studies the group $SL(2, \mathbb{R})$, showing in particular that a conjecture of Kaliszewski, Landstad, and Quigg holds for this group.

Chapter 2

Preliminaries

The application of Banach algebra homology to the study of amenability of locally compact groups began with work of Barry Johnson [43] who showed that a locally compact group G is amenable exactly when the convolution algebra $L^1(G)$ is amenable as a Banach algebra. This foundational result motivated the definition of amenability for Banach algebras and led to the development of a homology theory for Banach algebras [40]. The work of Ruan [74] showed that to find a homology theory for the Fourier algebra $A(G)$ that successfully captures amenability properties of the group G, it is necessary to incorporate the operator space structure of $A(G)$ by working in the category of completely contractive Banach algebras.

In the setting of discrete groups, the amenability of a group G was discovered to correspond to finite dimensional approximation properties of operator algebras associated to G. Namely, Lance [54] showed that for discrete groups the reduced group C^*-algebra $C^*_\lambda(G)$ is nuclear exactly when G is amenable, and a result of Connes shows that the group von Neumann algebra $L(G)$ is injective precisely when G is amenable. These results were subsequently extended to general locally compact groups by Lau and Paterson [58] and Paterson [68], who found that it was necessary to impose an additional condition, inner amenability of the group, to recover the characterizations of amenability.

We begin by outlining the basic definitions in the homology theory of completely contractive Banach algebras and introducing the algebras of functions and operators that will be our concern throughout this thesis. We then introduce the amenability conditions for locally compact groups that we will study. The chapter ends by introducing the L^p-representations and associated spaces of functions that are discussed in Chapter 4. We assume that the reader has familiarity with the fundamentals of abstract harmonic analysis on locally compact groups and with the theory of Banach algebras and operator spaces.

2.1 Fourier algebras and homology of completely contractive Banach algebras

In this section we introduce the objects and properties that are the focus of Chapter 3. Our main purpose here is to establish notational conventions; we do not attempt to provide a comprehensive introduction to these topics and refer the reader to [22, 69] for an introduction to operator space theory and to [23] or the recent book [47] for the theory of Fourier and Fourier-Stieltjes algebras.

2.1.1 Completely contractive Banach algebras and their modules

Let A be a **completely contractive Banach algebra**, that is a Banach algebra for which the product map $A \otimes A \to A$ extends to a complete contraction on the operator space projective tensor product $A \widehat{\otimes} A$. An operator space E is a **right A-module** if E carries a right module action of A and the action map $E \otimes A \to E$ extends to a complete contraction $m_E : E \widehat{\otimes} A \to E$. The category of right A-modules with completely bounded right A-module homomorphisms as morphisms is denoted $\boldsymbol{mod}{-}A$. We similarly define the category of left A-modules and A-bimodules, writing $A{-}\boldsymbol{mod}$ and $A\ {-}\boldsymbol{mod}{-}\ A$ for these, respectively. The algebra A is always an A-bimodule when equipped with left and right multiplication.

Given $E \in \boldsymbol{mod}{-}A$ the dual E^* carries a natural left A-module structure via

$$\langle a \cdot \varphi, e \rangle = \langle \varphi, e \cdot a \rangle, \quad \varphi \in E^*, a \in A, e \in E.$$

The analogous definition yields a right A-module structure on E^* for $E \in A{-}\boldsymbol{mod}$.

The **balanced** or **A-module tensor product** of $E \in \boldsymbol{mod}{-}A$ and $F \in A{-}\boldsymbol{mod}$ is

$$E \widehat{\otimes}_A F = E \widehat{\otimes} F / N, \quad N = \langle e \cdot a \otimes f - e \otimes a \cdot f : a \in A, e \in E, f \in F \rangle,$$

where $\langle \cdot \rangle$ denotes closed linear span. In this setting hom-tensor duality identifies the space $CB_A(E, F^*)$ of completely bounded right A-module homomorphisms from E to F^* with $\left(E \widehat{\otimes}_A F\right)^*$ via

$$\langle \Psi, e \otimes f \rangle = \langle \Psi(e), f \rangle, \quad \Psi \in CB_A(E, F^*), e \in E, f \in F.$$

Define $E \in \boldsymbol{mod}{-}A$ to be

1. **faithful** if for $e \in E$ nonzero there is $a \in A$ with $e \cdot a \neq 0$,
2. **essential** if $\langle E \cdot A \rangle = E$,
3. **induced** if the induced map $\tilde{m}_E : E \widehat{\otimes}_A A \to E$ is a completely isometric isomorphism (note that $N \subset \ker(m_E)$ always holds).

Finally, we call A **self-induced** if $\tilde{m}_A : A \widehat{\otimes}_A A \to A$ is a completely isometric isomorphism.

2.1.2 Projectivity and injectivity of modules

Given a completely contractive Banach algebra A, any $E \in \boldsymbol{mod}{-}A$ can be made into a right module over the forced unitization $A^+ := A \oplus_1 \mathbb{C}$ by defining

$$e \cdot (a + \lambda) = e \cdot a + \lambda e, \quad e \in E, a \in A, \lambda \in \mathbb{C}.$$

Given $C \geq 1$ we call E **relatively C-projective** if there is a right inverse $\Phi^+ : E \to E \widehat{\otimes} A^+$ to $m_E^+ : E \widehat{\otimes} A^+ \to E$ that is a morphism in $\boldsymbol{mod}{-}A$ and satisfies $\|\Phi^+\|_{cb} \leq C$. When E is essential we may omit the unitizations throughout in this definition [20, Proposition 1.2].

The space $CB_A(A^+, E)$ is canonically a right A-module with action

$$(\Psi \cdot a)(b) = \Psi(ab), \quad \Psi \in CB_A\left(A^+, E\right), a \in A, b \in A^+$$

and we have a natural completely contractive morphism $\Delta_E^+ : E \to CB(A^+, E)$ defined by

$$\Delta_E^+(e)(a) = e \cdot a, \quad e \in E, a \in A^+.$$

For $C \geq 1$ we say E is **relatively C-injective** if there is a left inverse $\Phi^+ : CB(A^+, E) \to E$ that is a morphism in $\boldsymbol{mod}{-}A$ with $\|\Phi^+\|_{cb} \leq C$. If E is faithful then unitizations may be omitted throughout this definition [20, Proposition 1.7]. The right A-module E is C**-injective** if for any $F, G \in$ $\boldsymbol{mod}\text{—}A$, completely isometric morphism $\iota : F \to G$, and morphism $\Psi : F \to E$, there is a morphism $\Psi' : G \to E$ such that $\Psi'\iota = \Psi$ and $\|\Psi'\|_{cb} \leq C\|\Psi\|_{cb}$. Note that relative C-injectivity admits an analogous characterization, namely asserting that, for $F, G \in \boldsymbol{mod}\text{—}A$, completely isometric morphism $\iota : F \to G$ that has completely bounded inverse, and morphism $\Psi : F \to E$, there is a morphism $\Psi' : G \to E$ such that $\Psi'\iota = \Psi$ and $\|\Psi'\|_{cb} \leq C\|\Psi\|_{cb}\|\iota^{-1}\|_{cb}$. Thus relative injectivity asks that we be able to extend a smaller family of morphisms, exactly those for which there is no operator space obstruction to carrying out the extension. In this way relative injectivity focuses on the A-module structure and in a sense disregards the operator space structure.

Example 2.1.1. (Injective von Neumann algebras) A von Neumann algebra M is trivially a left $\mathbb{C}$-module and 1-injectivity of M in $\mathbb{C}$-mod is exactly the assertion that M is an injective operator space.

Analogous definitions are made for left A-modules. We call $E \in \boldsymbol{mod}\text{—}A$ (**relatively**) C**-flat** if E^* is (relatively) C-injective. When $E \in A - \boldsymbol{mod}\text{—}A$ we call E (**relatively**) C**-biflat** when E^* is (relatively) C-injective in $A - \boldsymbol{mod}\text{—}A$.

The operator space projective tensor product $A\widehat{\otimes}A$ is an A-bimodule via

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca, \quad a, b, c \in A.$$

The algebra A is **operator amenable** if it has a **bounded approximate diagonal**, that is a net (u_α) in $A\widehat{\otimes}A$ satisfying

$$a \cdot u_\alpha - u_\alpha \cdot a \to 0 \text{ and } m_A(u_\alpha) \cdot a \to a, \quad a \in A.$$

This is equivalent to A being relatively C-biflat for some $C \geq 1$ and having a bounded approximate identity [74, Proposition 2.4].

2.1.3 The group von Neumann algebra and Fourier algebra

The **left regular representation** of a locally compact group G is the (strongly continuous unitary) representation on $L^2(G)$ given by

$$\lambda_s \xi(t) = \xi\left(s^{-1}t\right), \quad \xi \in L^2(G), s, t \in G.$$

Note that throughout $L^p(G)$ denotes the p-integrable functions on G with respect to a fixed Haar measure on G. The von Neumann algebra generated in $B(L^2(G))$ by the operators $\{\lambda_s : s \in G\}$ is the **group von Neumann algebra** of G, denoted $L(G)$. The **right regular representation** of G on $L^2(G)$ is given by

$$\rho_s \xi(t) = \xi(ts)\Delta(s)^{1/2}, \quad \xi \in L^2(G), s,t \in G,$$

where Δ denotes the modular function of G. The commutant of $L(G)$ is the von Neumann algebra $R(G)$ generated in $B(L^2(G))$ by $\{\rho_s : s \in G\}$.

The group von Neumann algebra is in standard form on $L^2(G)$ (see [35]), from which it follows that every normal state on $L(G)$ is a vector state, meaning of the form $\omega_\xi(x) := \langle x\xi|\xi\rangle$ for a unique vector ξ in the positive cone $\mathcal{P} := \overline{\{\eta * J\eta : \eta \in \mathcal{C}_c(G)\}}$. Here $\mathcal{C}_c(G)$ is the space of continuous compactly supported functions on G and J is the modular conjugation given by

$$J\xi(s) = \overline{\xi(s^{-1})}\Delta(s)^{-1/2}, \quad \xi \in L^2(G), s \in G.$$

The predual of $L(G)$ may be identified with the space of coefficient functions of λ,

$$\left\{\lambda_{\xi,\eta} : \xi, \eta \in L^2(G)\right\},$$

where $\lambda_{\xi,\eta}$ is the function on G defined by $\lambda_{\xi,\eta}(s) = \langle \lambda_s \xi|\eta\rangle$ and the duality is given by

$$\langle x, \lambda_{\xi,\eta}\rangle = \langle x\xi|\eta\rangle, \quad x \in L(G), \xi, \eta \in L^2(G).$$

This space of coefficient functions is closed under pointwise multiplication, making it a completely contractive Banach algebra called the **Fourier algebra** and denoted $A(G)$. The norm on $A(G)$ is given by

$$\|u\|_{A(G)} = \inf\left\{\|\xi\|_{L^2(G)}\|\eta\|_{L^2(G)} : u = \lambda_{\xi,\eta}, \xi, \eta \in L^2(G)\right\}.$$

For $H \leq G$ a closed subgroup, the closed ideal

$$I(H) = \{u \in A(G) : u|_H = 0\}$$

is essential by the argument of [4, Proposition 1.7]. Herz' restriction theorem [41] asserts that the restriction map $r : A(G) \to A(H)$ is a complete quotient map with kernel $I(H)$.

The product map $A(G)\hat{\otimes}A(G)\to A(G)$, where $\hat{\otimes}$ denotes the operator space projective tensor product, has adjoint $\Gamma : L(G)\to L(G\times G)$ that defines a coassociative comultiplication on $L(G)$. Here $\overline{\otimes}$ is the von Neumann algebra tensor product and we've made use of the identities

$$\left(A(G)\hat{\otimes}A(G)\right)^{*}=L(G)\overline{\otimes}L(G)=L(G\times G) \tag{2.1.1}$$

shown in [22, Theorem 7.2.4]. This comultiplication can be expressed as $\Gamma(x)=V(x\otimes 1)V^{*}$ where $x\in L(G)$ and V is the unitary in $L^{\infty}(G)\overline{\otimes}L(G)$ given by

$$V\xi(s,t)=\xi\left(s,s^{-1}t\right),\quad \xi\in L^{2}(G\times G),s,t\in G.$$

Note that we have $\Gamma(\lambda_s)=\lambda_s\otimes\lambda_s$ for $s\in G$, and that the pentagonal relation

$$V_{12}V_{13}V_{23}=V_{23}V_{12} \tag{2.1.2}$$

holds for V, where $V_{12}=V\otimes 1$, $V_{23}=1\otimes V$, $V_{13}=(\sigma\otimes 1)V_{23}(\sigma\otimes 1)$, and σ is the flip map on $L^{2}(G\times G)$.

The natural dual action of $A(G)$ on $L(G)$ can be written in terms of the comultiplication as

$$u\cdot x=x\cdot u=(\mathrm{id}\otimes u)\Gamma(x)=(u\otimes \mathrm{id})\Gamma(x),\quad u\in A(G),x\in L(G),$$

where we have used the fact that $A(G)$ is commutative, hence the left and right dual actions coincide. The module $L(G)$ is faithful as a left, right, or bimodule over $A(G)$. Hom-tensor duality and the identities (2.1.1) allow us to identify $CB(A(G),L(G))$ with $L(G\times G)$ and under this identification $\Delta_{L(G)}=\Gamma$.

2.1.4 Amenability type properties for locally compact groups

Recall that a locally compact group G is called **amenable** if there is a left translation invariant state on $L^{\infty}(G)$, meaning a state m that is invariant under the induced left translation action of G on functions in $L^{\infty}(G)$:

$$m(s\cdot\phi)=m(\phi),\text{ where }(s\cdot\phi)(t)=\phi\left(s^{-1}t\right),\quad \phi\in L^{\infty}(G),s,t\in G.$$

In Chapter 3 we study similar properties that involve the conjugation action of G in place of left translation. For $p \in [1, \infty)$ this action induces a representation of G on $L^p(G)$ via

$$(\beta_p(s) f)(t) = f\left(s^{-1}ts\right) \Delta(s)^{1/p}, \quad f \in L^p(G), s, t \in G.$$

In the case $p = 2$ we obtain a strongly continuous unitary representation of G, the **conjugation representation**. Conjugation also defines a representation on $L^\infty(G)$ by

$$(\beta_\infty(s) \phi)(t) = \phi\left(s^{-1}ts\right), \quad \phi \in L^\infty(G), s, t \in G.$$

We call G **inner amenable** when there is a state m on $L^\infty(G)$ satisfying $m(\beta_\infty(s)\phi) = m(\phi)$ for $\phi \in L^\infty(G), s \in G$. Any amenable group is inner amenable [30, p.29].

Example 2.1.2. (Discrete, compact, and abelian groups) Inner amenability is only relevant for locally compact groups that are not discrete in the sense that discrete groups are trivially inner amenable: the point mass at the identity $m(\phi) = \phi(e)$ defines a conjugation invariant state on $\ell^\infty(G)$ when G is discrete. It's clear that abelian groups are inner amenable since any state is conjugation invariant. Compact groups are also inner amenable because in this case $1_G \in L^1(G)$ defines a conjugation invariant state on $L^\infty(G)$.

A representation $\pi : G \to B(H)$ induces a right G-action on $B(H)$ via

$$T \cdot_\pi s = \pi(s)^* T \pi(s), \quad T \in B(H), s \in G.$$

In [6], Bekka defines π to be **amenable** if there is a G-invariant state m on $B(H)$ for this action, meaning that

$$\langle m, T \cdot_\pi s \rangle = \langle m, T \rangle, \quad T \in B(H), s \in G,$$

and showed the following:

Proposition 2.1.3. *We have the following:*

1. *G is amenable if and only if λ is amenable.*

2. *G is inner amenable if and only if β_2 is amenable.*

Example 2.1.4. Another class of inner amenable groups is the **IN groups**, those locally compact groups G which have a compact neighborhood of the identity K that is conjugation invariant, $s^{-1}Ks = K$ for all $s \in G$. Such a group carries the conjugation invariant mean $|K|^{-1} \chi_K \in L^1(G)$, where $|K|$ is the Haar measure of K, so is indeed inner amenable. **SIN groups** satisfy a stronger conjugation invariance property, namely having a neighborhood base at the identity consisting of compact sets that are invariant under conjugation. It can be shown that the classes in Example 2.1.2 are SIN [67, Proposition 12.1.9]. The group $\mathbb{T}^2 \rtimes SL(2, \mathbb{Z})$ is IN while failing to be SIN [60].

Using results of Mosak [64] and Stokke [80], these conjugation invariance properties of a locally compact group may be restated in terms of centrality in the convolution algebra $L^1(G)$.

Proposition 2.1.5. *For a locally compact group G, we have the following*

1. *G is SIN if and only if $L^1(G)$ has a bounded approximate identity in its center,*
2. *G is IN if and only if $L^1(G)$ has nontrivial center,*
3. *G is inner amenable if and only if $L^1(G)$ has an asymptotically central net of states, i.e. there are positive, norm one (f_α) in $L^1(G)$ satisfying $\|\beta_1(s) f_\alpha - f_\alpha\|_{L^1(G)} \to 0$ uniformly on compact subsets of G.*

Motivated by these characterizations, we call a locally compact group G **quasi-SIN** (**QSIN**) if $L^1(G)$ has an asymptotically central bounded approximate identity. It's clear that SIN groups are QSIN and that QSIN groups are inner amenable, so QSIN can be thought of as a condition between inner amenability and SIN. The class of QSIN groups was introduced in [60], where the following are shown.

Proposition 2.1.6. *For a locally compact group G, we have the following*

1. *G is QSIN if and only if $L^\infty(G)$ has a conjugation invariant state m with the additional property that $m(f) = f(e)$ for $f \in \mathcal{C}_b(G)$, the continuous bounded functions on G.*
2. *If G is amenable, then G is QSIN.*

In summary, the following relations hold between the properties we've introduced.

$$\begin{array}{ccccc} & & \text{discrete} & & \\ & & \Downarrow & & \\ \text{compact, abelian} & \Rightarrow & \text{SIN} & & \\ \Downarrow & & \Downarrow & & \\ \text{amenable} & \Rightarrow & \text{QSIN} & \Rightarrow & \text{inner amenable} \end{array}$$

2.1.5 Completely bounded multipliers of the Fourier algebra

As we did above for the left regular representation, given any representation $\pi : G \to B(H)$ of a locally compact group G, we let $\pi_{\xi,\eta}(s) = \langle \pi(s)\xi|\eta\rangle$ for $s \in G$. The collection $B(G)$ of all such coefficient functions is the **Fourier-Stieltjes algebra** of G, a completely contractive Banach algebra under pointwise multiplication and the norm

$$\|u\|_{B(G)} = \inf\{\|\xi\|_H \|\eta\|_H : u = \pi_{\xi,\eta} \text{ for some rep } \pi : G \to B(H), \xi, \eta \in H\}.$$

It's clear that $A(G)$ is contained in $B(G)$, but it is moreover a closed ideal and coincides with the closure of the compactly supported functions in $B(G)$ [23]. We write

$$P(G) = \{\pi_{\xi,\xi} : \pi : G \to B(H) \text{ is a representation and } \xi \in H\}$$

for the cone of continuous **positive definite functions** on G and $P_1(G)$ for the convex set of continuous positive definite functions of norm one.

The **completely bounded multipliers of the Fourier algebra** is the space $M_{cb}A(G)$ of functions $m : G \to \mathbb{C}$ with $mA(G) \subset A(G)$ and for which the product map $u \mapsto mu$ is completely bounded on $A(G)$. The space $M_{cb}A(G)$ is a completely contractive Banach algebra when normed by

$$\|m\|_{M_{cb}A(G)} = \|u \mapsto mu\|_{cb}$$

and contains $B(G)$. We let $A_{cb}(G)$ denote the closure of $A(G)$ in the norm on $M_{cb}A(G)$.

The algebra $M_{cb}A(G)$ is indeed the algebra of completely bounded multipliers of $A(G)$ as a completely contractive Banach algebra and it is known that $M_{cb}A(G)$ coincides with $B(G)$ exactly when G is amenable. One implication is due to Losert [57], the other appears in unpublished work of Ruan and also in [55].

Leptin's result shows that amenability of G is equivalent to $A(G)$ having a bounded approximate identity, and motivates calling G **weakly amenable** when $A(G)$ has an approximate identity bounded in the smaller norm of $M_{cb}A(G)$. It was shown in [24] that this is equivalent to the algebra $A_{cb}(G)$ having a bounded approximate identity.

2.2 Coefficient spaces and L^p-representations

This section outlines the preliminaries needed in Chapter 4.

2.2.1 Group C^*-algebras and coefficient spaces

Let G be a locally compact group and let $\pi : G \to B(H)$ be a representation of G. Recall that the (strongly continuous unitary) representations of G are in one-to-one correspondence with nondegenerate $*$-representations of the convolution algebra $L^1(G)$, and that the $*$-representation of $L^1(G)$ on $B(H)$ determined by π is given by

$$\langle \pi(f)\xi|\eta\rangle := \int_G f(s)\langle \pi(s)\xi|\eta\rangle\, ds, \quad f \in L^1(G).$$

The C^*-algebra generated in $B(H)$ by $\pi(L^1(G))$ is the **group C^*-algebra of π**, denoted $C^*_\pi(G)$. We call the group C^*-algebra of the left regular representation λ the **reduced group C^*-algebra** of G and the C^*-algebra of the **universal representation** $\pi_u : G \to B(H_u)$ the **universal group C^*-algebra**, writing $C^*(G)$ for the latter. The dual of $C^*(G)$ is identified with the Fourier-Stieltjes algebra $B(G)$ via the duality

$$\langle u, \pi_u(f)\rangle = \int_G uf, \quad u \in B(G), f \in L^1(G). \tag{2.2.1}$$

In particular $B(G)$ is a dual space and carries a weak* topology. Let

$$A_\pi(G) = \langle \pi_{\xi,\eta} : \xi, \eta \in H\rangle,$$

where again $\langle\cdot\rangle$ indicates the closed linear span, and let $B_\pi(G)$ be the closure of $A_\pi(G)$ in the weak*-topology on $B(G)$. These are the **coefficient spaces** associated to the representation π. The dual of $C^*_\pi(G)$ can be identified with $B_\pi(G)$ with duality given by a formula analogous to equation 2.2.1.

2.2.2 L^p-representations and the L^p-Fourier and Fourier-Stieltjes algebras

Let G be a locally compact group and fix $p \in [1,\infty]$. We call a representation $\pi : G \to B(H)$ an L^p**-representation** if there is a dense subspace $H_0 \subset H$ such that $\pi_{\xi,\xi} \in L^p(G)$ for all $\xi \in H_0$. This notion was introduced by Brown and Guentner [9] for the purpose of constructing exotic group C^*-algebras. A related notion was introduced by Kaliszewski, Landstad, and Quigg [45]: we call π a $KLQ - L^p$**-representation** if there are dense subspaces $H_1, H_2 \subset H$ such that $\pi_{\xi,\eta} \in L^p(G)$ for all $\xi \in H_1$ and $\eta \in H_2$. The polarization identity

$$\pi_{\xi,\eta} = \sum_{k=0}^{3} i^k \pi_{\xi+i^k\eta,\xi+i^k\eta}, \quad \xi,\eta \in H$$

makes it clear that an L^p-representation is a $KLQ - L^p$-representation.

Example 2.2.1. (Left regular representation is an L^p-representation) Since $\mathcal{C}_c(G)$ is dense in $L^2(G)$ and $\lambda_{\xi,\xi} \in \mathcal{C}_c(G)$ when $\xi \in \mathcal{C}_c(G)$, the left regular representation is an L^p-representation for all $p \in [2,\infty]$.

Set

$$A_{L^p,BG}(G) = \langle \pi_{\xi,\eta} : \pi \text{ is an } L^p\text{- representation on } H \text{ and } \xi,\eta \in H\rangle,$$
$$A_{L^p,KLQ}(G) = \langle \pi_{\xi,\eta} : \pi \text{ is a } KLQ - L^p\text{- representation on } H \text{ and } \xi,\eta \in H\rangle.$$

The space $A_{L^p,BG}(G)$ is the L^p**-Fourier algebra** of G and its weak* closure in $B(G)$ is the L^p**-Fourier-Stieltjes algebra** of G, denoted $B_{L^p,BG}(G)$. We have $A_{L^p,BG}(G) \subset A_{L^p,KLQ}(G)$ and in Chapter 4 we study when these spaces or their weak* closures coincide.

Proposition 2.2.2. *We have*

$$A_{L^p,BG}(G) = \langle P(G) \cap L^p(G)\rangle \quad \textit{and}$$
$$A_{L^p,KLQ}(G) = \langle B(G) \cap L^p(G)\rangle,$$

where the span closures are taken in the norm on $B(G)$.

Proof. The first equality is established in [86], we prove the second. Suppose $\pi_{\xi,\eta} \in A_{L^p,KLQ}(G)$ for some $KLQ - L^p$- representation π on H and $\xi, \eta \in H$. Let H_1 and H_2 be as in the definition of a $KLQ - L^p$-representation. Then there are $\xi_\alpha \in H_1$ and $\eta_\alpha \in H_2$ converging to ξ and η respectively and we have $\pi_{\xi_\alpha,\eta_\alpha} \in L^p(G)$. Since

$$\begin{aligned}\|\pi_{\xi,\eta} - \pi_{\xi_\alpha,\eta_\alpha}\|_{B(G)} &\leq \|\pi_{\xi,\eta} - \pi_{\xi_\alpha,\eta}\|_{B(G)} + \|\pi_{\xi_\alpha,\eta} - \pi_{\xi_\alpha,\eta_\alpha}\|_{B(G)} \\ &= \|\pi_{\xi-\xi_\alpha,\eta}\|_{B(G)} + \|\pi_{\xi_\alpha,\eta-\eta_\alpha}\|_{B(G)} \\ &\leq \|\xi - \xi_\alpha\|_H \|\eta\|_H + \|\xi_\alpha\|_H \|\eta - \eta_\alpha\|_H \\ &\to 0\end{aligned}$$

it follows that $\pi_{\xi,\eta}$ is in the $B(G)$-norm closure of $L^p(G)$, as required.

Let $\pi_{\xi,\eta} \in B(G) \cap L^p(G)$ for some representation π on H and $\xi, \eta \in H$. Let H' and H''denote the G-invariant norm-closed subspaces of H generated by ξ and η, respectively. Let P' denote the orthogonal projection from H onto H'. Replacing H'' with $P'H''$ and η with $P'\eta$, we may assume that H'' is contained in H'. Now let P'' denote the orthogonal projection onto $H'' \subset H'$. Replacing H' with H'' and ξ with $P''\xi$, we may also assume that $H' = H''$. Finally, replace H with H' so that if H_1 and H_2 denote the G-invariant subspaces of H generated by ξ and η, respectively, then H_1 and H_2 are each dense in H. It is clear that $\pi_{\xi',\eta'} \in L^p(G)$ for $\xi' \in H_1$ and $\eta' \in H_2$ since $L^p(G)$ is invariant under both left and right translation by G. It follows that π is a $KLQ - L^p$-representation of G. □

2.2.3 $\mathcal{C}_0$-representations and the Rajchman algebra

Motivated to produce a C^*-algebraic characterization of the Haagerup property, in [9] Brown and Guentner call a representation $\pi : G \to B(H)$ a **$\mathcal{C}_0$-representation** if there is a dense subspace $H_0 \subset H$ such that $\pi_{\xi,\xi} \in \mathcal{C}_0(G)$ for all $\xi \in H_0$. The span closure in $B(G)$ of coefficient functions of $\mathcal{C}_0$-representations

$$\langle \pi_{\xi,\eta} : \pi \text{ is a } \mathcal{C}_0\text{- representation on } H \text{ and } \xi, \eta \in H \rangle$$

is called the **Rajchman algebra** and coincides with $B(G) \cap \mathcal{C}_0(G)$. This algebra, historically denoted by $B_0(G)$, contains the Fourier algebra $A(G)$ because $\lambda_{\xi,\xi} \in \mathcal{C}_0(G)$ for all $\xi \in \mathcal{C}_c(G)$, meaning the left regular representation is always a $\mathcal{C}_0$-representation.

Chapter 3

Homology of the Fourier algebra

The celebrated result of Ruan characterizing amenability of a locally compact group G in terms of operator amenability of its Fourier algebra $A(G)$ is dual to the classical result of Johnson that amenability of G corresponds to amenability of $L^1(G)$. Since the work of Johnson developing a homology theory for $L^1(G)$ [43], effort has been made to find a dual homological theory for the Fourier algebra. In this direction, Spronk and Samei independently showed that $A(G)$ is always operator weakly amenable [79, 77] and Aristov showed that operator biprojectivity of $A(G)$ corresponds to discreteness of G [3], which are in analogy to results of Johnson and Helemskii for $L^1(G)$ [44, 39]. This work was continued by Ruan and Xu [76] who showed that $A(G)$ is relatively operator 1-projective when G is an IN group and that $A(G)$ is relatively operator 1-flat when G is inner amenable. In this section we establish the converse of both of these results. Aristov, Runde, and Spronk showed that the QSIN condition on G guarantees the existence of a bounded approximate indicator for the diagonal subgroup G_Δ in $G \times G$ and that this in turn implies the relative operator biflatness of $A(G)$. We establish a converse to the first implication and show that all three conditions coincide in many special cases.

By the work of Leptin [56] and Ruan [74] the algebra $A(G)$ is operator amenable exactly when it has a bounded approximate identity and these conditions are each equivalent to amenability of G. In analogy with Leptin's result, Forrest [24] showed that G is weakly amenable exactly when $A_{cb}(G)$ has a bounded approximate identity, which led to the natural conjecture of [26] that $A_{cb}(G)$ is operator amenable precisely when G is weakly amenable. Using the results of this chapter we are able to provide a large family of counter examples.

This chapter is based on joint work with Jason Crann [18].

3.1 Relative flatness and inner amenability

In this section we study the relationship between inner amenability of a locally compact group G and relative flatness of its Fourier algebra $A(G)$ as a module over itself. Inner amenability is defined in terms of $L^{\infty}(G)$, our first result provides a dual characterization in terms of the group von Neumann algebra $L(G)$ and answers a question of Lau and Paterson [59, Example 5]. Recall from Subsection 2.1.4 that the left regular representation induces a G-action on $B(L^2(G))$ and therefore also on $L(G)$ via

$$x \cdot_{\lambda} s = \lambda_s^* x \lambda_s, \quad x \in L(G), s \in G.$$

On $L(G)$ this action coincides with that induced by the conjugation representation β_2 since $\beta_2(s) = \lambda_s \rho_s$ for $s \in G$ and the operators ρ_s commute with operators in $L(G)$.

Proposition 3.1.1. *A locally compact group G is inner amenable if and only if there exists a G-invariant state on $L(G)$.*

Proof. If G is inner amenable, then by [6, Theorem 2.4] there exists a β_2-invariant state m on $B(L^2(G))$. The restriction of m to $L(G)$ is necessarily G-invariant because

$$\langle m, x \cdot_{\pi} s\rangle = \langle m, \lambda_s^* \rho_s^* x \rho_s \lambda_s\rangle = \langle m, \beta_2(s)^* x \beta_2(s)\rangle = \langle m, x\rangle$$

for all $x \in L(G)$, $s \in G$.

Conversely, suppose m is a G-invariant state on $L(G)$. Since $L(G)$ is standardly represented on $L^2(G)$, there exists a net of unit vectors (ξ_{α}) in $\mathcal{P}$ such that $(\omega_{\xi_{\alpha}})$ converges to m in the weak* topology of $L(G)^*$. By G-invariance, it follows that

$$\beta_2(s) \cdot \omega_{\xi_{\alpha}} \cdot \beta_2(s)^* - \omega_{\xi_{\alpha}} = \omega_{\beta_2(s)\xi_{\alpha}} - \omega_{\xi_{\alpha}} \to 0$$

weakly in $A(G) = L(G)_*$ for all $s \in G$. Passing to convex combinations, we obtain a net of unit vectors (η_γ) in $\mathcal{P}$ satisfying

$$\left\|\beta_2(s)\cdot\omega_{\eta_\gamma}\cdot\beta_2(s)^*-\omega_{\eta_\gamma}\right\|_{A(G)}=\left\|\omega_{\beta_2(s)\eta_\gamma}-\omega_{\eta_\gamma}\right\|_{A(G)}\to 0,\quad s\in G.$$

However, since $\beta_2(s)=\lambda_s\rho_s=\lambda_s J\lambda_s J$ we have $\beta_2(s)\mathcal{P}\subseteq\mathcal{P}$ for any $s\in G$ by [35, Theorem 1.1]. Then [35, Lemma 2.10] entails

$$\|\beta_2(s)\eta_\gamma-\eta_\gamma\|^2_{L^2(G)}\leq\left\|\omega_{\beta_2(s)\eta_\gamma}-\omega_{\eta_\gamma}\right\|_{A(G)}\to 0,\quad s\in G.$$

Letting $f_\gamma:=|\eta_\gamma|^2$, we obtain a net of states in $L^1(G)$ satisfying

$$\|\beta_1(s)f_\gamma-f_\gamma\|_{L^1(G)}=\left\|\omega_{\beta_2(s)\eta_\gamma}-\omega_{\eta_\gamma}\right\|_{L^1(G)}\leq 2\,\|\beta_2(s)\eta_\gamma-\eta_\gamma\|_{L^2(G)}\to 0, s\in G.$$

Any weak* cluster point of (f_γ) in $L^\infty(G)^*$ will therefore be conjugation invariant, hence G is inner amenable. □

This characterization yields the following new hereditary property of inner amenability.

Corollary 3.1.2. *Let G be a locally compact group and let H be a closed subgroup of G. If G is inner amenable, then H is inner amenable.*

Proof. Let $L_H(G)=\{\lambda_G(s):s\in H\}''\subseteq L(G)$ and let $r_H:A(G)\to A(H)$ be the restriction map. It's not hard to verify that its adjoint $r_H^*:L(H)\to L(G)$ satisfies

$$r_H^*(\lambda_H(s))=\lambda_G(s),\quad s\in H,$$

and consequently determines a $*$-homomorphism on $\mathrm{span}\lambda_H(H)$, hence also on $L(H)$ because r_H^* is weak* continuous. Since r_H surjects by Herz' restriction theorem, its adjoint is injective with closed rage and therefore an isometric $*$-isomorphism onto $\mathrm{ran}(r_H^*)=\ker(r_H)^\perp=I(H)^\perp=L_H(G)$. Thus, if m is a G-invariant state on $L(G)$ then $m_H:=m|_{L_H(G)}\circ r_H^*\in L(H)^*$ is an H-invariant state on $L(H)$, so H is inner amenable by Proposition 3.1.1. □

In [58, Corollary 3.2], Lau and Paterson showed that

G is amenable $\iff$ G is inner amenable and $L(G)$ is an injective operator space providing a key connection between amenability and inner amenability in the setting of locally compact groups. We now establish another characterization of inner amenability that allows us to write this equivalence in purely homological terms.

Theorem 3.1.3. *A locally compact group G is inner amenable if and only if $A(G)$ is relatively 1-flat in* $\mathbf{mod} - A(G)$.

Proof. If G is inner amenable, then by Proposition 2.1.5 there exists a net of states (f_α) in $L^1(G)$ satisfying

$$\|\beta_1(s)f_\alpha - f_\alpha\|_{L^1(G)} \to 0, \quad s \in G,$$

uniformly on compact sets. The square roots $\xi_\alpha := f_\alpha^{1/2} \in L^2(G)$ then satisfy

$$\|\beta_2(s)\xi_\alpha - \xi_\alpha\|^2_{L^2(G)} \leq \|\beta_1(s)f_\alpha - f_\alpha\|_{L^1(G)} \to 0, \quad s \in G,$$

uniformly on compact sets. Thus, combining [76, Lemma 3.1, Lemma 4.1], it follows that $\Gamma : L(G) \to L(G \times G)$ has a completely contractive left inverse Φ which is a left $A(G)$-module map. Since $L(G)$ is faithful in $A(G) - \mathbf{mod}$, this entails the relative 1-injectivity of $L(G)$ in $A(G) - \mathbf{mod}$, and hence, the relative 1-flatness of $A(G)$ in $\mathbf{mod} - A(G)$.

Conversely, relative 1-flatness of $A(G)$ in $\mathbf{mod} - A(G)$ implies the existence of a completely contractive morphism $\Phi : L(G\times G) \to L(G)$ satisfying $\Phi\circ\Gamma = \mathrm{id}_{L(G)}$. It follows that $\Gamma \circ \Phi : L(G \times G) \to L(G \times G)$ is a projection of norm one onto the image of Γ. Thus, by [85], $\Gamma \circ \Phi$ is a $\Gamma(L(G))$-bimodule map, which by injectivity of Γ yields the identity

$$x\Phi(T)y = \Phi(\Gamma(x)T\Gamma(y)) \tag{3.1.1}$$

for all $x, y \in L(G)$ and $T \in L(G \times G)$.

For $x \in L(G)$, the module property of Φ implies $u \cdot \Phi(x \otimes 1) = \Phi(x \otimes u \cdot 1) = u(e)\Phi(x \otimes 1)$ for all $u \in A(G)$. The standard argument then shows $\Phi(x \otimes 1) \in \mathbb{C}1$, so that $m : L(G) \to \mathbb{C}$ defined by

$$\langle m, x\rangle = \Phi(x \otimes 1), \quad x \in L(G)$$

yields a state on $L(G)$. Moreover, by equation (3.1.1) we obtain

$$\begin{aligned}\langle m, \lambda_s x \lambda_s^* \rangle &= \Phi(\lambda_s x \lambda_s^* \otimes 1) = \Phi((\lambda_s \otimes \lambda_s)(x \otimes 1)(\lambda_s^* \otimes \lambda_s^*)) \\ &= \Phi(\Gamma(\lambda_s)(x \otimes 1)\Gamma(\lambda_s^*)) = \lambda_s \Phi(x \otimes 1)\lambda_s^* = \Phi(x \otimes 1) \\ &= \langle m, x \rangle\end{aligned}$$

for any $x \in L(G)$ and $s \in G$. Thus, m is a G-invariant state on $L(G)$, which by Proposition 3.1.1 implies that G is inner amenable. □

With this result and [16, Corollary 5.3], and recalling from Subsection 2.1.2 that relative 1-flatness of $A(G)$ in **mod** $- A(G)$ is exactly relative 1-injectivity of its dual $L(G)$ in $A(G) -$ **mod**, we may now rephrase the result of Lau and Paterson in homological terms:

$$L(G) \text{ is 1-injective in } A(G)\textbf{-mod} \iff \begin{array}{l} L(G) \text{ is relatively 1-injective in} \\ A(G)\textbf{-mod} \text{ and 1-injective in} \\ \mathbb{C}\textbf{-mod} \end{array}$$

In [2], given a locally compact group G, Anantharaman-Delaroche calls $u \in B(G \times G)$ **properly supported** if $\operatorname{supp}(u) \cap G \times K$ and $\operatorname{supp}(u) \cap K \times G$ are compact for every compact set $K \subset G$ and defines G to have **property** (W) if

> for every $\epsilon > 0$ and compact set $K \subset G$ there is $u \in P(G \times G)$ properly supported with $|u(s,s) - 1| < \epsilon$ for all $s \in K$.

It was shown in [2, Proposition 4.6] that inner amenable groups have property (W) but the converse was left open [2, Problem 9.1]. We are able to establish the equivalence of these two properties using Theorem 3.1.3.

Theorem 3.1.4. *A locally compact group G is inner amenable if and only if it has Property (W).*

Proof. Suppose G has Property (W), witnessed by a net (u_α) of properly supported positive definite functions in $B(G \times G)$ satisfying

$$|u_\alpha(s,s) - 1| \to 0, \quad s \in G$$

uniformly on compact sets. Without loss of generality we may assume that $u_\alpha(e,e) = 1$ for all α. By Nielson's lemma [65, Lemma 10.3] (see also [19, Proposition 5.1]) it follows that

$$u_\alpha|_{G_\Delta} \cdot v \to v, \quad v \in A(G).$$

Moreover, since u_α is properly supported, for any $v \in A(G)$ with compact support, the function $u_\alpha \cdot (1 \otimes v) \in B(G \times G)$ is compactly supported, and hence lies in $A(G \times G)$. Thus, $u_\alpha \cdot (1 \otimes v) \in A(G \times G)$ for all $v \in A(G)$, and

$$\begin{aligned}\|[u_\alpha \cdot (1 \otimes v_{ij})]\|_{M_n(A(G\times G))} &= \|[u_\alpha \cdot (1 \otimes v_{ij})]\|_{M_n(B(G\times G))} \\ &\leq \|u_\alpha\|_{B(G\times G)} \|[v_{ij}]\|_{M_n(A(G))},\end{aligned}$$

so that $\|u_\alpha\|_{CB(A(G),A(G\times G))} \leq \|u_\alpha\|_{B(G\times G)} = 1$. Define maps $\Phi_\alpha : L(G\times G) \to L(G)$ by

$$\langle \Phi_\alpha(X), v\rangle = \langle u_\alpha \cdot (1 \otimes v), X\rangle, \quad X \in L(G \times G), v \in A(G).$$

Then $\|\Phi_\alpha\|_{cb} \leq \|u_\alpha\|_{CB(A(G),A(G\times G))} \leq 1$, and

$$\begin{aligned}\langle \Phi_\alpha(u \cdot X), v\rangle &= \langle u_\alpha \cdot (1 \otimes v), u \cdot X\rangle \\ &= \langle u_\alpha \cdot (1 \otimes vu), X\rangle \\ &= \langle \Phi_\alpha(X), vu\rangle \\ &= \langle u \cdot \Phi_\alpha(X), v\rangle\end{aligned}$$

for all $X \in L(G\times G)$ and $u, v \in A(G)$. Passing to a subnet if necessary, we may assume that (Φ_α) converges weak* to $\Phi \in CB\left(L(G \times G), L(G)\right) = (L(G \times G)\widehat{\otimes}A(G))^*$. Then, with $\Gamma : L\left(G\right) \to L\left(G \times G\right)$ denoting the comultiplication as defined in Subsection 2.1.3,

$$\langle \Phi(\Gamma(x)), v\rangle = \lim_\alpha \langle u_\alpha \cdot (1 \otimes v), \Gamma(x)\rangle = \lim_\alpha \left\langle u_\alpha|_{G_\Delta} \cdot v, x\right\rangle = \lim_\alpha \langle v, x\rangle = \langle x, v\rangle$$

for all $x \in L(G)$ and $v \in A(G)$. Hence, $\Phi : L(G \times G) \to L(G)$ is a completely contractive left $A(G)$-module inverse to Γ, entailing the relative 1-flatness of $A(G)$ in $\mathbf{mod} - A(G)$, and therefore the inner amenability of G by Theorem 3.1.3. □

We conjecture that inner amenability of G is equivalent to relative C-

flatness of $A(G)$ in $\boldsymbol{mod}{-}A(G)$ for any $C>1$. The following proposition will allow us to produce examples supporting this conjecture.

Proposition 3.1.5. *Let G be a locally compact group and let H be a closed subgroup. If $L(G)$ is C-injective in $A(G)-\mathbf{mod}$ for $C\geq 1$, then $L(H)$ is C-injective in $A(H)-\mathbf{mod}$.*

Proof. Let $r: A(G)\twoheadrightarrow A(H)$ be the complete quotient map given by restriction (see Subsection 2.1.3). Then $B(L^2(H))$ becomes a left $A(G)$-module via

$$u\cdot T=(\mathrm{id}\otimes r(u))\,\Gamma^r(T),\quad u\in A(G), T\in B\left(L^2(H)\right),$$

where $\Gamma^r: B(L^2(H))\to B(L^2(H))\overline{\otimes}L(H)$ is the canonical lifting of the co-multiplication on $L(H)$ given by

$$\Gamma^r(T)=V(T\otimes 1)V^*,\quad T\in B\left(L^2(H)\right).$$

Clearly, $L(H)$ is a closed $A(G)$-submodule of $B(L^2(H))$. Hence, the inclusion $L(H)\hookrightarrow L(G)$ extends to a morphism $E: B(L^2(H))\to L(G)$ with $\|E\|_{cb}\leq C$. We show that $E(B(L^2(H)))=L(H)$. To this end, fix $T\in B(L^2(H))$. Then for $u\in A(G)$ and $v\in I(H)$, we have

$$\langle E(T),u\cdot v\rangle=\langle v\cdot E(T),u\rangle=\langle E(v\cdot T),u\rangle=0$$

as $r(v)=0$. Since $I(H)$ is essential it follows that $E(T)\in I(H)^{\perp}=L(H)$. Thus, $E: B(L^2(H))\to L(H)$ is a completely bounded $A(H)$-module projection with $\|E\|_{cb}\leq C$. By [71], there is an $A(H)$-invariant state m on $L(H)$ satisfying

$$\langle m,u\cdot x\rangle=u(e)\langle m,x\rangle,\quad u\in A(H), x\in L(H),$$

it follows that $L(H)$ is an amenable quantum group, and the proof of [17, Theorem 5.5] implies that $B(L^2(H))$ is 1-injective in $A(H)-\mathbf{mod}$. Thus, $L(H)$ is C-injective in $A(H)-\mathbf{mod}$. □

Corollary 3.1.6. *Let G be a locally compact group such that $L(G)$ is C-injective in $A(G)-\mathbf{mod}$ for some $C\geq 1$. Then every closed inner amenable subgroup of G is amenable.*

Proof. By Proposition 3.1.5 we know that $L(H)$ is C-injective in $A(H)-\mathbf{mod}$ for any closed subgroup H. Hence, there exists a completely bounded projection $E : B(L^2(H)) \to L(H)$, which, by [12, Theoerem 3.1] (see also [70]) implies that $L(H)$ is an injective von Neumann algebra. If H is inner amenable, then by [58, Corollary 3.2] it is necessarily amenable. $\square$

Corollary 3.1.7. *Let G be a locally compact group containing $\mathbb{F}_2$ as a closed subgroup and for which $L(G)$ is 1-injective in $\mathbb{C} - \mathbf{mod}$. Then $L(G)$ is not relatively C-injective in $A(G) - \mathbf{mod}$ for any $C \geq 1$.*

Proof. If $L(G)$ were relatively C-injective in $A(G) - \mathbf{mod}$, then it would be C-injective in $A(G) - \mathbf{mod}$ by [16, Proposition 2.3]. Since $\mathbb{F}_2$ is inner amenable, Corollary 3.1.6 would imply that it is amenable, a contradiction. $\square$

For an almost connected locally compact group G it is known that $L(G)$ is injective [68] and that G is amenable exactly when it fails to contain $\mathbb{F}_2$ as a closed subgroup [73]. Thus Corollary 3.1.7 implies that almost connected nonamenable groups have Fourier algebras that fail to be relatively C-flat for all $C \geq 1$. For $n \geq 2$, the groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, and $SO(1, n)$ fall into this class, so in particular their Fourier algebras fail to be relatively biflat, confirming a conjecture posed in [4].

We close this section by establishing the converse of [76, Lemma 3.2], providing a characterization of relative projectivity of $A(G)$.

Proposition 3.1.8. *Let G be a locally compact group. Then $A(G)$ is relatively 1-projective in $\mathbf{mod} - A(G)$ if and only if G is an IN group.*

Proof. Assuming relative 1-projectivity of $A(G)$ in $\mathbf{mod} - A(G)$, there exists a normal completely contractive left $A(G)$-module map $\Phi : L(G \times G) \to L(G)$ such that $\Phi \circ \Gamma = \mathrm{id}_{L(G)}$. By the proof of Theorem 3.1.3 we obtain a normal G-invariant state on $L(G)$, which, by [83, Proposition 4.2] implies that G is IN. The converse follows from [76, Lemma 3.2]. $\square$

3.2 Relative biflatness

Let G be a locally compact group. Given a closed subgroup H, a bounded net (m_α) in $B(G)$ is called an **bounded approximate indicator** for H if

$$\lim_\alpha m_\alpha|_H u = u \text{ and } \lim_\alpha m_\alpha v = 0 \text{ for all } u \in A(H), v \in I(H).$$

This definition was introduced by Aristov, Runde, and Spronk in [4], where it is shown that $A(G)$ is relatively C-biflat whenever the diagonal subgroup $G_\Delta \leq G \times G$ has an approximate indicator with bound C. Recalling that relative biflatness is weaker than operator amenability, this result is a natural since $A(G)$ is operator amenable exactly when G_Δ has an approximate indicator in the smaller algebra $A(G \times G)$. We prove the converse to the result of Aristov, Runde, and Spronk when $C = 1$.

Theorem 3.2.1. *Let G be a locally compact group. Then $A(G)$ is relatively 1-biflat if and only if G_Δ has a contractive approximate indicator.*

Proof. We need only establish necessity. Consider the right $L^1(G)$-action on $L(G)$ given by

$$x \lhd f = \int_G \lambda_s^* x \lambda_s f(s) ds, \quad x \in L(G), f \in L^1(G).$$

For $f \in L^1(G)$, we let

$$\begin{aligned}
&\hat{\Theta}(f) : L(G) \to L(G) : x \mapsto x \lhd f, \\
&\hat{\theta}_f : L(G \times G) \to L(G \times G) : X \mapsto \int_G (\lambda_s^* \otimes \lambda_s^*) X (\lambda_s \otimes \lambda_s) f(s) ds.
\end{aligned}$$

Relative 1-biflatness of $A(G)$ implies the existence of a completely contractive $A(G)$-bimodule left inverse $\Phi : L(G \times G) \to L(G)$ to Γ. It follows as in Theorem 3.1.3 that $\Gamma \circ \Phi$ is a $\Gamma(L(G))$-bimodule map. By Wittstock's bimodule extension theorem [87], this map extends to an $\Gamma(L(G))$-bimodule map $\Psi : B(L^2(G \times G)) \to B(L^2(G \times G))$. Moreover, [63, Lemma 2.3] allows us to approximate Ψ in the point weak* topology by a net (Ψ_α) of normal completely bounded $\Gamma(L(G))$-bimodule maps. Thus, for any $X \in L(G \times G)$, we have

$$\begin{aligned}
\Gamma\circ\Phi(\hat\theta_f(X)) &= \Psi(\hat\theta_f(X)) = \Psi\left(\int_G(\lambda_s^*\otimes\lambda_s^*)\,X\,(\lambda_s\otimes\lambda_s)\,f(s)ds\right)\\
&= \lim_\alpha \Psi_\alpha\left(\int_G(\lambda_s^*\otimes\lambda_s^*)\,X\,(\lambda_s\otimes\lambda_s)\,f(s)ds\right)\\
&= \lim_\alpha\left(\int_G \Psi_\alpha\left((\lambda_s^*\otimes\lambda_s^*)\,X\,(\lambda_s\otimes\lambda_s)\right)f(s)ds\right)\\
&= \lim_\alpha\left(\int_G \Psi_\alpha\left(\Gamma\left(\lambda_s^*\right)X\Gamma\left(\lambda_s\right)\right)f(s)ds\right)\\
&= \lim_\alpha\left(\int_G \Gamma(\lambda_s^*)\Psi_\alpha(X)\Gamma(\lambda_s)f(s)ds\right)\\
&= \lim_\alpha \hat\theta_f\left(\Psi_\alpha(X)\right) = \hat\theta_f\left(\Psi(X)\right) = \hat\theta_f\left(\Gamma\circ\Phi(X)\right),
\end{aligned}$$

where we used normality of Ψ_α and $\hat\theta_f$ in the fourth and eighth equality, respectively. By definition of $\hat\theta_f$, we have $\hat\theta_f\circ\Gamma = \Gamma\circ\hat\Theta(f)$, so the above calculation entails $\Gamma\circ\Phi\circ\hat\theta_f = \Gamma\circ\hat\Theta(f)\circ\Phi$, which, by injectivity of Γ, implies $\Phi\circ\hat\theta_f = \hat\Theta(f)\circ\Phi$.

As in the proof of Theorem 3.1.3 , the restriction $\Phi|_{L(G)\otimes 1}$ defines a state $m\in L\left(G\right)^*$. The bimodule property of Φ ensures that m is invariant for the $A(G)$-action on $L\left(G\right)$, that is,

$$\langle m, u\cdot x\rangle = u(e)\left\langle m,x\right\rangle, \quad x\in L\left(G\right), u\in A(G).$$

Moreover, for $f\in L^1\left(G\right)$ and $x\in L\left(G\right)$ we have

$$\begin{aligned}
\langle m, x\lhd f\rangle &= \Phi\left(\int_G(\lambda_s^*\otimes\lambda_s^*)\,(x\otimes 1)\,(\lambda_s\otimes\lambda_s)\,f(s)ds\right)\\
&= \Phi\left(\hat\theta_f(x\otimes 1)\right)\\
&= \hat\Theta(f)\left(\Phi(x\otimes 1)\right)\\
&= \langle f,1\rangle\left\langle m,x\right\rangle.
\end{aligned}$$

Approximating $m\in L(G)^*$ in the weak* topology by a net of states (u_β) in $A(G)$, it follows that

$$u_\beta\cdot v - v(e)u_\beta \to 0 \text{ and } f\lhd u_\beta - \langle f,1\rangle\, u_\beta\to 0$$

weakly in $A(G)$ for all $v\in A(G)$ and $f\in L^1\left(G\right)$, where $f\lhd u_\beta = \left(\hat\Theta(f)\right)_*(u_\beta)$. Passing to convex combinations, we obtain a net of states (u_γ) in $A(G)$ satisfying

$$\|u_\gamma \cdot v - v(e)u_\gamma\|_{A(G)}, \|f \lhd u_\gamma - \langle f, 1\rangle u_\gamma\|_{A(G)} \to 0, \quad v \in A(G), f \in L^1(G). \tag{3.2.1}$$

For $s \in G$ and $v \in A(G)$ we define $s \lhd v \in A(G)$ by $s \lhd v(t) = v(s^{-1}ts)$, $s, t \in G$. Then by left invariance of the Haar measure it follows that

$$s \lhd (f \lhd v) = (l_s f) \lhd v, \quad s \in G, f \in L^1(G), v \in A(G), \tag{3.2.2}$$

where $l_s f(t) = f(st)$ for $s, t \in G$. Fix a state $f_0 \in L^1(G)$, and consider the net $(f_0 \lhd u_\gamma)$. For $\varepsilon > 0$, take a neighborhood U of the identity $e \in G$ such that

$$\|l_s f_0 - f_0\|_{L^1(G)} < \frac{\varepsilon}{2}, \quad s \in U.$$

Then for any compact set $K \subseteq G$, there exist $s_1, ..., s_n \in K$ such that $K \subseteq \cup_{i=1}^n U s_i$. Take γ_ε such that for $\gamma \geq \gamma_\varepsilon$

$$\|(l_{s_i} f_0) \lhd u_\gamma - u_\gamma\|_{A(G)} < \frac{\varepsilon}{4}, \quad 1 \leq i \leq n.$$

Applying (3.2.2) together with the $L^1(G)$-invariance in (3.2.1), it follows by the standard argument (see [75, Lemma 7.1.1]) that

$$\|k \lhd (f_0 \lhd u_\gamma) - f_0 \lhd u_\gamma\|_{A(G)} < \varepsilon, \quad k \in K.$$

Hence, the net $(f_0 \lhd \psi_\gamma)$ satisfies

$$\|s \lhd (f_0 \lhd u_\gamma) - f_0 \lhd u_\gamma\|_{A(G)} \to 0, \quad s \in G,$$

uniformly on compact sets. Using both the $A(G)$ and $L^1(G)$-invariance from equation (3.2.1), a 3ε-argument also shows that

$$\|(f_0 \lhd u_\gamma) \cdot v - v(e) f_0 \lhd u_\gamma\|_{A(G)} \to 0, \quad v \in A(G).$$

Forming $|f_0 \lhd u_\gamma|^2$, we may further assume $f_0 \lhd u_\gamma(s) \geq 0$ for all $s \in G$, as one may easily verify using boundedness and multiplicativity of the G-action that

$$\left\|u \cdot |f_0 \lhd u_\gamma|^2 - u(e)\, |f_0 \lhd u_\gamma|^2\right\|_{A(G)}, \left\|s \lhd |f_0 \lhd u_\gamma|^2 - |f_0 \lhd u_\gamma|^2\right\|_{A(G)} \to 0$$

for all $u \in A(G)$ and for all $s \in G$, uniformly on compact sets.

Now, since $L(G)$ is standardly represented on $L^2(G)$, there exist unit vectors $\xi_\gamma \in \mathcal{P}$ satisfying

$$\omega_{\xi_\gamma}\Big|_{L(G)} = f_0 \lhd u_\gamma.$$

Note that $J\xi_\gamma = \xi_\gamma$ and that ξ_γ is necessarily real-valued by uniqueness. For any $s \in G$ we have $s \lhd \omega_{\xi_\gamma} = \omega_{\beta_2(s)\xi_\gamma}$ and $\beta_2(s)\mathcal{P} \subseteq \mathcal{P}$. Thus [35, Lemma 2.10] implies

$$\|\beta_2(s)\xi_\gamma - \xi_\gamma\|^2_{L(G)} \leq \left\|\omega_{\beta_2(s)\xi_\gamma} - \omega_{\xi_\gamma}\right\|_{A(G)} = \|s \lhd u_\gamma - u_\gamma\|_{A(G)} \to 0 \quad (3.2.3)$$

for all $s \in G$, uniformly on compact sets.

Define the function $\varphi_\gamma \in P_1(G \times G) \subseteq B(G \times G)$ by

$$\varphi_\gamma(s,t) = \langle \lambda_s \rho_t \xi_\gamma | \xi_\gamma \rangle, \quad s,t \in G,$$

and consider the associated normal completely positive map $\Theta(\varphi_\gamma)$ in $CB_{A(G\times G)}(L(G \times G))$ given by

$$\Theta(\varphi_\gamma)(\lambda_s \otimes \lambda_t) = \varphi_\gamma(s,t)\lambda_s \otimes \lambda_t, \quad s,t \in G.$$

We claim that the bounded net $(\Theta(\varphi_\gamma))$ clusters to a completely positive $A(G \times G)$-module projection $L(G \times G) \to L(G_\Delta)$.

To verify the claim, first consider the net $\left(\omega_{\xi_\gamma}\right)$ in $T(L^2(G)) = B(L^2(G))_*$. By passing to a subnet we may assume that $\left(\omega_{\xi_\gamma}\right)$ converges weak* to a state $M \in B(L^2(G))^*$. For each γ define the unital completely positive map $\Phi_\gamma : L(G \times G) \to L(G)$ by

$$\Phi_\gamma(X) = \left(\mathrm{id} \otimes \omega_{\xi_\gamma}\right) V(1 \otimes U) X (1 \otimes U) V^*, \quad X \in L(G \times G),$$

where U is the self-adjoint unitary given by $U = \hat{J}J$, and $\hat{J}$ is complex conjugation on $L^2(G)$. Since $\Gamma(x) = V(x \otimes 1)V^*$, $x \in L(G)$, and $UL(G)U = L(G)'$, one easily sees that the range of Φ_γ is indeed contained in $L(G)$.

For every γ and $s, t \in G$, we have

$$\begin{aligned}
&\Theta(\varphi_\gamma)(\lambda_s \otimes \lambda_t) \\
&= \langle \lambda_s \rho_t \xi_\gamma, \xi_\gamma \rangle \lambda_s \otimes \lambda_t \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes \lambda_t \otimes \lambda_s \rho_t) \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes 1 \otimes \lambda_s)(1 \otimes \lambda_t \otimes \rho_t) \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes 1 \otimes \lambda_s)(1 \otimes (1 \otimes U)V(\lambda_t \otimes 1)V^*(1 \otimes U)) \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes 1 \otimes 1)(1 \otimes (1 \otimes U)V(\lambda_t \otimes \rho_s)V^*(1 \otimes U)) \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes 1 \otimes 1)(1 \otimes V(\lambda_t \otimes \rho_s)V^*) \quad (\text{as } U\xi_\gamma = \xi_\gamma) \\
&= \left(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi_\gamma}\right)(\lambda_s \otimes 1 \otimes 1)(1 \otimes V((1 \otimes U)(\lambda_t \otimes \rho_s)(1 \otimes U))V^*) \\
&= \lambda_s \otimes \Phi_\gamma(\lambda_t \otimes \lambda_s) \\
&= \lambda_s \otimes \Phi_\gamma(\Sigma(\lambda_s \otimes \lambda_t)) \\
&= (\mathrm{id} \otimes \Phi_\gamma \circ \Sigma)(\lambda_s \otimes \lambda_s \otimes \lambda_t) \\
&= (\mathrm{id} \otimes \Phi_\gamma \circ \Sigma)(\Gamma \otimes \mathrm{id})(\lambda_s \otimes \lambda_t)
\end{aligned}$$

By normality we see that $\Theta(\varphi_\gamma) = (\mathrm{id} \otimes \Phi_\gamma \circ \Sigma)(\Gamma \otimes \mathrm{id})$. Since (Φ_γ) is bounded, it follows that (Φ_γ) converges in the stable point weak* topology to the map $\Phi_M \in CB(L(G \times G), L(G))$ given by

$$\Phi_M(X) = (\mathrm{id} \otimes M)V(1 \otimes U)X(1 \otimes U)V^*, \quad X \in L(G \times G).$$

Hence, the net $(\Theta(\varphi_\gamma))$ converges weak* to a map $\Theta \in CB(L(G \times G))$ satisfying

$$\Theta = (\mathrm{id} \otimes \Phi_M \circ \Sigma)(\Gamma \otimes \mathrm{id}).$$

If Φ_M were a left $A(G)$-module left inverse to Γ, it would follow that $\Theta = \Gamma \circ \Phi_M \circ \Sigma$, hence the claim. We therefore turn to the required properties of Φ_M.

First, let $\hat{V}$ be the unitary in $L(G)' \otimes L^\infty(G)$ given by

$$\hat{V}\zeta(s,t) = \zeta(st,t)\Delta(t)^{1/2}, \quad \zeta \in L^2(G \times G), s,t \in G.$$

Then, for $\eta \in L^2(G)$, the compact convergence (3.2.3) entails

$$\left\| V\sigma\hat{V}\sigma\eta \otimes \xi_\gamma - \eta \otimes \xi_\gamma \right\|^2_{L^2(G\times G)} = \int_G \int_G |\eta(s)|^2 \, |\beta_2(s)\xi_\gamma(t) - \xi_\gamma(t)|^2 \, ds dt \to 0.$$

Noting that $\hat{V} = \sigma(1 \otimes U)V(1 \otimes U)\sigma$, for $X \in L(G \times G)$ we therefore have

$$\begin{aligned}
\langle \Phi_M(X), \omega_\eta \rangle &= \lim_\gamma \langle V(1 \otimes U)X(1 \otimes U)V^* \eta \otimes \xi_\gamma, \eta \otimes \xi_\gamma \rangle \\
&= \lim_\gamma \langle (1 \otimes U)V(1 \otimes U)X(1 \otimes U)V^*(1 \otimes U)\eta \otimes \xi_\gamma, \eta \otimes \xi_\gamma \rangle \\
&= \lim_\gamma \left\langle \sigma \hat{V} \sigma X \sigma \hat{V}^* \sigma \eta \otimes \xi_\gamma, \eta \otimes \xi_\gamma \right\rangle \\
&= \lim_\gamma \langle V^* X V \eta \otimes \xi_\gamma, \eta \otimes \xi_\gamma \rangle \\
&= \langle (\mathrm{id} \otimes M) V^* X V, \omega_\eta \rangle .
\end{aligned}$$

Since $\eta \in L^2(G)$ was arbitrary, by linearity we obtain

$$\Phi_M(X) = (\mathrm{id} \otimes M)(V^* X V), \quad X \in L(G \times G),$$

from which it follows that $\Phi_M \circ \Gamma = \mathrm{id}_{L(G)}$. We now show the $A(G)$-module property. For $X \in L(G \times G)$ and $u \in A(G)$, we have

$$\begin{aligned}
\Phi_M(u \cdot X) &= \Phi_M((\mathrm{id} \otimes \mathrm{id} \otimes u)(V_{23} X_{12} V_{23}^*)) \\
&= (\mathrm{id} \otimes M)(\mathrm{id} \otimes \mathrm{id} \otimes u)(V_{12}^* V_{23} X_{12} V_{23}^* V_{12}) \\
&= (\mathrm{id} \otimes M)(\mathrm{id} \otimes \mathrm{id} \otimes u)(V_{13} V_{23} V_{12}^* X_{12} V_{12} V_{23}^* V_{13}^*) \quad \text{(by (2.1.2))} \\
&= (\mathrm{id} \otimes u)(V(\mathrm{id} \otimes M \otimes \mathrm{id})(V_{23} V_{12}^* X_{12} V_{12} V_{23}^*) V^*) .
\end{aligned}$$

Denoting by $\pi : T(L^2(G)) \twoheadrightarrow A(G)$ the canonical restriction map, and recalling that $M|_{L(G)}$ is $A(G)$-invariant, for $\tau, \omega \in T(L^2(G))$, we have

$$\begin{aligned}
&\langle (\mathrm{id} \otimes M \otimes \mathrm{id})(V_{23} V_{12}^* X_{12} V_{12} V_{23}^*), \tau \otimes \omega \rangle \\
&= \langle (M \otimes \mathrm{id}) V((\tau \otimes \mathrm{id})(V^* X V) \otimes 1) V^*, \omega \rangle \\
&= \langle M, \pi(\omega) \cdot ((\tau \otimes \mathrm{id}) V^* X V) \rangle \\
&= \langle \omega, 1 \rangle \langle M, (\tau \otimes \mathrm{id}) V^* X V \rangle \\
&= \langle M \otimes \omega, (\tau \otimes \mathrm{id})(V^* X V) \otimes 1 \rangle \\
&= \langle (\mathrm{id} \otimes M \otimes \mathrm{id})(V^* X V \otimes 1), \tau \otimes \omega \rangle \\
&= \langle \Phi_M(X) \otimes 1, \tau \otimes \omega \rangle .
\end{aligned}$$

Since τ and Ω in $T\left(L^2\left(G\right)\right)$ were arbitrary, it follows that

$$\begin{aligned}\Phi_M(u\cdot X) &= (\mathrm{id}\otimes u)(V(\mathrm{id}\otimes M\otimes \mathrm{id})\left(V_{23}V_{12}^*A_{12}V_{12}V_{23}^*\right)V^*)\\ &= (\mathrm{id}\otimes u)\left(V(\Phi_M(X)\otimes 1)V^*\right)\\ &= u\cdot\Phi_M(X).\end{aligned}$$

Our original claim is therefore established, and $\Theta(\varphi_\gamma)$ converges weak* in $CB\left(L(G\times G)\right)$ to an $A(G\times G)$-module projection Θ from $L(G\times G)$ onto $L\left(G_\Delta\right)=\Gamma\left(L(G)\right)$. Then

$$\varphi_\gamma|_{G_\Delta}\cdot u-u\to 0$$

weakly for $u\in A\left(G_\Delta\right)$, and using the fact from [21, Theorem 6.5] that

$$\Gamma\left(L(G)\right)=\left\{X\in L(G\times G):(\Gamma\otimes\mathrm{id})(X)=(\mathrm{id}\otimes\Gamma)(X)\right\},$$

together with the essentiality $I\left(G_\Delta\right)=\langle I(G_\Delta)\cdot A(G\times G)\rangle$, we also have

$$\varphi_\gamma\cdot v\to 0$$

weakly for $v\in I\left(G_\Delta\right)$. Passing to convex combinations, and noting that $(\varphi_\gamma)\subseteq P_1(G\times G)$, we obtain a contractive approximate indicator for G_Δ in $P_1(G\times G)$. □

We conjecture that a locally compact group G has relatively 1-biflat Fourier algebra exactly when G is QSIN. The characterization of amenability due to Lau and Paterson discussed in Section 3.1 and Theorem 3.1.3 together assert that, when $L\left(G\right)$ is 1-injective in $\mathbb{C}$-**mod**, relative 1-biflatness of $A\left(G\right)$ implies amenability of G and hence that G is QSIN, by Proposition 2.1.6. It follows that our conjecture is valid for locally compact groups with injective von Neumann algebras, which includes type I groups and almost connected groups. We are able to verify the conjecture also for totally disconnected groups.

Proposition 3.2.2. *Let G be a totally disconnected locally compact group. Then $A(G)$ is relatively 1-biflat if and only if G is QSIN.*

Proof. Sufficiency follows from [4, Theorem 2.4], so suppose that $A(G)$ is relatively 1-biflat. Proceeding as in the proof of Theorem 3.2.1, we obtain a net of states (u_γ) in $A(G)$ (appearing as $|f_0 \lhd u_\gamma|^2$in the proof of Theorem 3.2.1) satisfying

$$\|v \cdot u_\gamma - v(e)u_\gamma\|_{A(G)}, \|s \lhd u_\gamma - u_\gamma\|_{A(G)} \to 0$$

for all $v \in A(G)$ and for all $s \in G$, where $v \cdot u_\gamma$ denotes the product in $A(G)$.

Now, because G is totally disconnected there is a neighborhood basis $\mathcal{H}$ of the identity consisting of compact open subgroups. By [23, Lemme 4.13] for each $H \in \mathcal{H}$ there exists a state $\varphi_H \in A(G)$ satisfying $\operatorname{supp}(\varphi_H) \subseteq H^2 \subseteq H$ and

$$\|\varphi_H \cdot v - v(e)\varphi_H\|_{A(G)} \to 0, \quad v \in A(G).$$

For each $H \in \mathcal{H}$, a standard 3ε-argument shows

$$\|s \lhd (\varphi_H \cdot u_\gamma) - \varphi_H \cdot u_\gamma\|_{A(G)} \to 0, \quad s \in G.$$

Denoting the index set of (u_γ) by $\mathcal{C}$, we form the product $\mathcal{I} := \mathcal{H} \times \mathcal{C}^{\mathcal{H}}$. For each $\alpha = \left(H, (\gamma_H)_{H\in\mathcal{H}}\right) \in \mathcal{I}$, letting $u_\alpha := \varphi_H \cdot u_{\gamma(H)}$, we obtain a net of states in $A(G)$ satisfying the iterated convergence

$$\lim_{\alpha\in I} \|s \lhd u_\alpha - u_\alpha\|_{A(G)} = \lim_{H\in\mathcal{H}} \lim_{\gamma\in v} \|s \lhd \varphi_H \cdot u_\gamma - \varphi_H \cdot u_\gamma\|_{A(G)} = 0$$

for all $s \in G$ by [51, pg. 69]. Moreover, $\operatorname{supp}(u_\alpha) \to \{e\}$, in the sense that for every neighborhood U of the identity, there exists α_U such that $\operatorname{supp}(u_\alpha) \subseteq U$ for $\alpha \geq \alpha_U$.

Let (ξ_α) be the unique representing vectors from $\mathcal{P}$ for the net (u_α). For each $\alpha = \left(H, (\gamma_H)_{H\in\mathcal{H}}\right)$, u_α is supported in the open subgroup H, i.e., $u_\alpha \in A(H) \subseteq A(G)$. Under the canonical subspace inclusion $L^2(H) \hookrightarrow L^2(G)$ we have $\mathcal{P}_H = \overline{\{f * Jf \mid f \in C_c(H)\}} \subseteq \mathcal{P}_G$, so by uniqueness of representing vectors [35, Lemma 2.10], we may assume $\operatorname{supp}(\xi_\alpha) \subseteq H$.

Applying Haagerup's Powers–Størmer inequality [35, Lemma 2.10] once again, we obtain

$$\left\|\omega_{\beta_2(s)\xi_\alpha} - \omega_{\xi_\alpha}\right\|_{L^1(G)}^2 \leq 4\|\beta_2(s)\xi_\alpha - \xi_\alpha\|_{L^2(G)}^2 \leq 4\|s \lhd u_\alpha - u_\alpha\|_{A(G)} \to 0, s \in G.$$

Letting $f_\alpha := |\xi_\alpha|^2$, we obtain a net of states in $L^1(G)$ satisfying

$$\|\beta_1(s)f_\alpha - f_\alpha\|_{L^1(G)} = \left\|\omega_{\beta_2(s)\xi_\alpha} - \omega_{\xi_\alpha}\right\|_{L^1(G)} \to 0, \quad s \in G,$$

and $\mathrm{supp}(f_\alpha) \to \{e\}$. Since the latter implies (f_α) is a bounded approximate identity for $L^1(G)$, it follows that G is QSIN. $\square$

3.3 Examples arising from actions of discrete groups on compact groups

In this section we produce additional examples of locally compact groups with Fourier algebras that fail to be relatively 1-biflat. Let $K \rtimes H$ be the semidirect product of a discrete group H and an infinite compact group K. Let

$$L_0^2(K) = \left\{ \xi \in L^2(K) : \int_K \xi = 0 \right\}$$

and let $\pi_K : H \to B(L_0^2(K))$ denote the representation given by

$$\pi_K(h)\xi = h \cdot \xi, \quad h \in H, \xi \in L_0^2(K),$$

where $(h \cdot \xi)(k) = \xi(h^{-1}kh)$ for $h \in H$, $k \in K$, and $h^{-1}kh$ is the product in $K \rtimes H$. Recall that the action of H on K is **ergodic** if for any Borel set $E \subset K$ such that $E \triangle h \cdot E$ is a null set for all $h \in H$, it must be that E is either a null set or a co-null set. This is equivalent to the assertion that 1_K is the unique normal H-invariant mean on $L^\infty(K)$.

Proposition 3.3.1. *Let $K \rtimes H$ be the semidirect product of an infinite compact group K by a discrete group H. If $A(K \rtimes H)$ is relatively 1-biflat, then π_K weakly contains the trivial representation.*

Proof. Let G denote $K \rtimes H$. As in the proof of Theorem 3.2.1, relative 1-biflatness of $A(G)$ yields a net of states (ω_{ξ_α}) in $A(G)$ with $\xi_\alpha \in \mathcal{P}_G$ satisfying

$$\|v \cdot \omega_{\xi_\alpha} - v(e)\omega_{\xi_\alpha}\|_{A(G)}, \ \|s \lhd \omega_{\xi_\alpha} - \omega_{\xi_\alpha}\|_{A(G)} \to 0, \quad v \in A(G), s \in G.$$

Arguing as in the proof of Proposition 3.2.2, we may assume $\mathrm{supp}(\omega_{\xi_\alpha}) \to \{e\}$ and, since K is an open subgroup of G, we may identify $A(K)$ with a subspace of $A(G)$ and further assume that $\mathrm{supp}(\xi_\alpha) \subseteq K$. Viewing $L^2(K)$ as a subspace of $L^2(G)$ via extension by zero, we have $\beta_2^G(h)\xi = h \cdot \xi$ for $\xi \in L^2(K)$ and $h \in H$ by unimodularity of G, and, noting once again that $\beta_2^G(G)\mathcal{P}_G \subseteq \mathcal{P}_G$, [35, Lemma 2.10] implies

$$\begin{aligned}\|h \cdot \xi_\alpha - \xi_\alpha\|^2_{L^2(K)} &= \left\|\beta_2^G(h)\xi_\alpha - \xi_\alpha\right\|^2_{L^2(G)} \\ &\le \left\|\omega_{\beta_2^G(h)\xi_\alpha} - \omega_{\xi_\alpha}\right\|_{A(G)} \\ &= \|h \lhd \omega_{\xi_\alpha} - \omega_{\xi_\alpha}\|_{A(G)} \\ &\to 0\end{aligned}$$

for all $h \in H$. Let $\xi_\alpha = \xi_\alpha^0 + c_\alpha 1_K$ correspond to the decomposition $L^2(K) = L^2_0(K) \oplus_2 \mathbb{C}1_K$, so that $1 = \|\xi^0_\alpha\|^2_{L^2_0(K)} + |c_\alpha|^2$ and $\|h \cdot \xi^0_\alpha - \xi^0_\alpha\|_{L^2_0(K)} \to 0$ for all $h \in H$. Fix a neighborhood U of the identity in K with $|K \setminus U| > 0$. If it were the case that $\|\xi^0_\alpha\|_{L^2_0(K)} \to 0$, then, for α large enough that $|c_\alpha|^2 > \frac{1}{2}$ and $\operatorname{supp}(\omega_{\xi_\alpha}) \subseteq U$, we have for $k \in K \setminus U$ that

$$0 = \omega_{\xi_\alpha}(k) = \omega_{\xi^0_\alpha}(k) + \omega_{\xi^0_\alpha, c_\alpha 1_K}(k) + \omega_{c_\alpha 1_K, \xi^0_\alpha}(k) + |c_\alpha|^2 = \omega_{\xi^0_\alpha}(k) + |c_\alpha|^2$$

because $\xi^0_\alpha \in L^2_0(K)$, whence

$$-\frac{1}{2}|K \setminus U| > \int_{K\setminus U} \omega_{\xi^0_\alpha}(k)dk = \langle \omega_{\xi^0_\alpha}, \lambda_K(1_{K\setminus U})\rangle_{A(K),L(K)} \to 0,$$

a contradiction. Therefore, passing to a subnet if necessary, we may assume $\|\xi^0_\alpha\|_{L^2_0(K)}$ is bounded away from zero, in which case the vectors ξ^0_α may be normalized while retaining the property that $\|h \cdot \xi^0_\alpha - \xi^0_\alpha\|_{L^2_0(K)} \to 0$ for all $h \in H$. Thus π_K weakly contains the trivial representation. ☐

A locally compact group G has Kazhdan's **property (T)** if any representation of G that weakly contains the trivial representation must in fact contain the trivial representation.

Corollary 3.3.2. *Let $K \rtimes H$ be the semidirect product of an infinite compact group K by a discrete group H such that the action of H on K is ergodic. If $A(K \rtimes H)$ is relatively 1-biflat, then H does not have Kazhdan's property (T).*

Proof. If H had Kazhdan's property (T), then π_K would contain the trivial representation and we would obtain a nonzero vector $\xi \in L^2_0(K)$ such that $h \cdot \xi = \xi$ for all $h \in H$, contradicting the ergodicity of the H-action on K. ☐

This shows, for example, that if K is an infinite compact group with an ergodic action of $SL(n, \mathbb{Z})$ by automorphisms and $n \geq 3$, then the Fourier algebra of $K \rtimes SL(n, \mathbb{Z})$ is not relatively 1-biflat.

The QSIN condition on a locally compact group G is equivalent to the existence of a conjugation invariant mean on $L^\infty(G)$ extending evaluation at the identity on $\mathcal{C}_0(G)$. In [60] it is shown that for $n \geq 2$ the group $\mathbb{T}^n \rtimes SL(n, \mathbb{Z})$ fails to be QSIN by appealing to the fact that the Haar integral on $\mathbb{T}^n$ is the unique mean on $L^\infty(\mathbb{T}^n)$ that is invariant under the $SL(n,\mathbb{Z})$-action. Indeed, the restriction to $L^\infty(\mathbb{T}^n)$ of any conjugation invariant mean on $L^\infty(\mathbb{T}^n \rtimes SL(n,\mathbb{Z}))$ is clearly invariant under the action of $SL(n,\mathbb{Z})$. For semidirect products associated to ergodic actions as above, we have the following.

Corollary 3.3.3. *Let $K \rtimes H$ be the semidirect product of an infinite compact group K by a discrete group H such that the action of H on K is ergodic. If $A(K \rtimes H)$ is relatively 1-biflat, then there is an H-invariant mean on $L^\infty(K)$ distinct from the Haar integral on K.*

Proof. By [28, Theorem 1.6], $L^\infty(K)$ admits an H-invariant mean distinct from the Haar measure when π_K, considered as a representation on $L^2_0(K,\mathbb{R})$, weakly contains the trivial representation. We may assure that the almost invariant vectors for π_K produced in Proposition 3.3.1 are real valued by replacing the states ω_{ξ_α} with $\omega_{\xi_\alpha}\overline{\omega_{\xi_\alpha}}$, in which case we have $\omega_{\xi_\alpha}\overline{\omega_{\xi_\alpha}} = \omega_{\xi'_\alpha}$ for $\xi'_\alpha \in \mathcal{P}_{K\rtimes H}$ that are then real-valued by uniqueness. □

Since the $SL(2,\mathbb{Z})$-action on $\mathbb{T}^2$ is ergodic, this confirms that the Fourier algebra of $\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})$ fails to be relatively 1-biflat. Note, however, that $\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})$ is an IN group and hence $A(\mathbb{T}^2 \rtimes SL(2,\mathbb{Z}))$ is relatively 1-flat by Theorem 3.1.3. More examples of groups H and K and conditions on these pairs for which there is a unique H-invariant mean on $L^\infty(K)$ may be found in [7] and [28].

3.4 Operator amenability of $A_{cb}(G)$

Given a closed subgroup H of a locally compact group G, we may consider approximate indicators for H consisting of completely bounded multipliers by replacing $B(G)$ with $M_{cb}A(G)$ in the definition of Section 3.2. The existence of an approximate indicator for G_Δ in the larger algebra $M_{cb}A(G \times G)$ still yields relative biflatness of $A(G)$, the proof of [4, Proposition 2.3] carrying over mutatis mutandis.

Recall that $A_{cb}(G)$ denotes the closure of $A(G)$ in the norm on $M_{cb}A(G)$. For the algebra $A_{cb}(G)$, the existence of a bounded approximate identity is equivalent to weak amenability of G [24]. This is in close analogy to Leptin's theorem [56], which asserts that $A(G)$ has a bounded approximate identity exactly when G is amenable. By a theorem of Ruan [74], amenability of G is also equivalent to the operator amenability of $A(G)$, and it was suggested in [26] that $A_{cb}(G)$ may be operator amenable exactly when G is weakly amenable. The following proposition, in combination with Corollary 3.1.7, yields a large class of counter-examples.

Theorem 3.4.1. *Let G be a locally compact group such that $A_{cb}(G)$ is operator amenable. Then G_Δ has a bounded approximate indicator in $A_{cb}(G \times G)$.*

Proof. Write

$$\Delta : A_{cb}(G)\widehat{\otimes}A_{cb}(G) \to A_{cb}(G)$$

for the product map,

$$r : A_{cb}(G \times G) \to A_{cb}(G)$$

for restriction to the diagonal G_Δ in $G \times G$, and

$$\Lambda : A_{cb}(G)\widehat{\otimes}A_{cb}(G) \to A_{cb}(G \times G)$$

for the complete contraction defined on elementary tensors by $\Lambda(u \otimes v) = u \times v$, so that $\Delta = r\Lambda$. Let (X_α) be an approximate diagonal for $A_{cb}(G)$ of bound C and set $m_\alpha = \Lambda(X_\alpha)$. We show that the net (m_α) is an approximate indicator for G_Δ. Let $u \in A(G)$ have compact support and choose $v \in A(G)$ with $v \equiv 1$ on $\operatorname{supp}(u)$ [23, Lemme 3.2], so that $u = uv$ and

$$\|ur(m_\alpha) - u\|_{A(G)} = \|u\Delta(X_\alpha) - u\|_{A(G)} \leq \|u\|_{A(G)}\|v\Delta(X_\alpha) - v\|_{A_{cb}(G)} \to 0.$$

As $A(G)$ is Tauberian and the net $(r(m_\alpha))$ is bounded in $\|\cdot\|_{A_{cb}(G)}$, a routine estimate shows that the above holds for all $u \in A(G)$.

We claim that the functions in $I(G_\Delta)$ of the form $(a \times 1_G - 1_G \times a)\,v$ for $a \in A(G)$ and $v \in A(G \times G)$ have dense span. Recall that $A(G)$ is self-induced [21], in particular

$$\ker \Delta_{A(G)} = \langle ab \otimes c - a \otimes bc : a, b, c \in A(G) \rangle,$$

and that the map $a \otimes b \mapsto a \times b$ induces a completely isometric isomorphism

$A(G)\widehat{\otimes}A(G) \to A(G \times G)$ taking $\ker \Delta_{A(G)}$ onto $I(G_\Delta)$, from which it follows that

$$I(G_\Delta) = \langle ab \times c - a \times bc : a, b, c \in A(G) \rangle .$$

Since $\{a \times c : a, c \in A(G)\}$ has dense span in $A(G \times G)$,

$$\begin{aligned} I(G_\Delta) &= \langle b \cdot (a \times c) - (a \times c) \cdot b : a, b, c \in A(G) \rangle \\ &= \langle b \cdot v - v \cdot b : b \in A(G) \text{ and } v \in A(G \times G) \rangle \\ &= \langle (b \times 1_G - 1_G \times b)\, v : b \in A(G) \text{ and } v \in A(G \times G) \rangle . \end{aligned}$$

For such elements of $I(G_\Delta)$,

$$\begin{aligned} \| (b \times 1_G - 1_G \times b)\, v m_\alpha \|_{A(G\times G)} &\leq \|v\|_{A(G\times G)} \|b \cdot m_\alpha - m_\alpha \cdot b\|_{A_{cb}(G\times G)} \\ &\leq \|v\|_{A(G\times G)} \|b \cdot X_\alpha - X_\alpha \cdot b\|_{A_{cb}(G)\widehat{\otimes}A_{cb}(G)} \\ &\to 0, \end{aligned}$$

where the second inequality uses that Λ is a contractive $A(G)$-bimodule map. The density claim above and the boundedness of (m_α) imply that $\|um_\alpha\|_{A(G\times G)}$ converges to zero for all $u \in I(G_\Delta)$. □

Corollary 3.4.2. *Let G be a locally compact group containing $\mathbb{F}_2$ as a closed subgroup and for which $L(G)$ is 1-injective in $\mathbb{C} - \mathbf{mod}$. Then $A_{cb}(G)$ is not operator amenable.*

Proof. If $A_{cb}(G)$ were operator amenable then an approximate indicator for G_Δ would exist, implying that $L(G)$ is relatively C-injective in $A(G) - \mathbf{mod}$ for some $C \geq 1$ by the completely bounded multiplier analogue of [4, Proposition 2.3], in contradiction to Corollary 3.1.7. □

Any weakly amenable, nonamenable, almost connected group G satisfies the hypotheses of Corollary 3.4.1 by [68] and [73, Theorem 5.5]. Since weak amenability is preserved under compact extensions [14, Proposition 1.3] and almost connected groups have injective group von Neumann algebras, if K is any compact group with an action of G by automorphisms, then $K \rtimes G$ is weakly amenable and $A_{cb}(K \rtimes G)$ fails to be operator amenable.

Chapter 4

Coefficient spaces arising from L^p-representations

In this chapter we study coefficient spaces arising from L^p-representations of locally compact groups. Following Brown and Guentner's introduction of L^p-representations for the purpose of producing exotic group C^*-algebras, it became natural to investigate the dual spaces $B_{L^p,BG}(G)$ of these C^*-algebras as well as other related spaces of coefficient functions. Wiersma and others [86, 46] carried out this work, focusing primarily on the L^p-Fourier and Fourier-Stieltjes algebras $A_{L^p,BG}(G)$ and $B_{L^p,BG}(G)$. These algebras coincide with the norm and weak* closures of the span of $P(G) \cap L^p(G)$ respectively, where $P(G)$ is the cone of positive definite functions on G, and both arise naturally from operator algebras associated to L^p-representations. However, the norm and weak* closures of $B(G) \cap L^p(G)$ are more suitable objects of study from the perspective of harmonic analysis. Motivated by this, Kaliszewski, Landstad, and Quigg [45] studied these latter algebras and conjectured that they coincide with the L^p-Fourier and Fourier-Stieltjes algebras.

This chapter is based on joint work with Matthew Wiersma and Brian Forrest [27].

4.1 Preliminaries

Our primary motivation is the following conjecture due to Kaliszewski, Landstad, and Quigg [46].

Conjecture 4.1.1. *$B_{L^p,BG}(G) = B_{L^p,KLQ}(G)$ for every locally compact group G and $1 \leq p \leq \infty$.*

With the aim of developing some context in which to understand this conjecture, we will study several related spaces of coefficient functions and focus on establishing when they coincide or fail to. In the latter case we establish qualitative results about their relative sizes.

Recall that $B_{L^p,BG}(G)$ and $B_{L^p,KLQ}(G)$ are the weak* closures of $A_{L^p,BG}(G)$ and $A_{L^p,KLQ}(G)$ in $B(G)$, respectively. We believe the above conjecture is true and moreover conjecture the following.

Conjecture 4.1.2. *$A_{L^p,BG}(G) = A_{L^p,KLQ}(G)$ for every locally compact group G and $1 \leq p \leq \infty$.*

We are not aware of any locally compact groups for which either conjecture was known to hold, aside from cases in which the conjectures become trivial, for example when the group is amenable in the case of Conjecture 4.1.1 and when the group is compact in the case of Conjecture 4.1.2. We will show that Conjecture 4.1.1 holds for the group $SL(2,\mathbb{R})$.

When G is a non-unimodular locally compact group and $1 \leq p < \infty$, Conjecture 4.1.2 implies that $A_{L^p,KLQ}(G)$ is closed under the check operation $u \mapsto \check{u}$, where $\check{u}(s) = u(s^{-1})$ for $s \in G$. It is not known if this holds.

With the aim of potentially distinguishing these spaces, we introduce several auxiliary spaces of coefficient functions. Let

$$\begin{aligned}
\mathrm{A}_{\mathrm{L}^{p+},\mathrm{BG}}(G) &:= \bigcap_{\epsilon>0} \mathrm{A}_{\mathrm{L}^{p+\epsilon},\mathrm{BG}} = \overline{\operatorname{span} \mathrm{P}(G) \cap \bigcap_{\epsilon>0} \mathrm{L}^{p+\epsilon}(G)}^{\|\cdot\|_{B(G)}}, \\
\mathrm{A}_{\mathrm{L}^{p-},\mathrm{BG}}(G) &:= \overline{\bigcup_{\epsilon>0} \mathrm{A}_{\mathrm{L}^{p-\epsilon},\mathrm{BG}}}^{\|\cdot\|_{B(G)}} = \overline{\operatorname{span} \mathrm{P}(G) \cap \bigcup_{\epsilon>0} \mathrm{L}^{p-\epsilon}(G)}^{\|\cdot\|_{B(G)}}, \\
\mathrm{A}_{\bigcup \mathrm{L}^{p},\mathrm{BG}}(G) &:= \overline{\bigcup_{1\leq p<\infty} \mathrm{A}_{\mathrm{L}^{p},\mathrm{BG}}(G)}^{\|\cdot\|_{B(G)}} = \overline{\operatorname{span} \mathrm{P}(G) \cap \bigcup_{1\leq p<\infty} \mathrm{L}^{p}(G)}^{\|\cdot\|_{B(G)}}, \\
\mathrm{A}_{\mathrm{L}^{p+},\mathrm{KLQ}}(G) &:= \bigcap_{\epsilon>0} \mathrm{A}_{\mathrm{L}^{p+\epsilon},\mathrm{KLQ}} = \overline{\mathrm{B}(G) \cap \bigcap_{\epsilon>0} \mathrm{L}^{p+\epsilon}(G)}^{\|\cdot\|_{B(G)}}, \\
\mathrm{A}_{\mathrm{L}^{p-},\mathrm{KLQ}}(G) &:= \overline{\bigcup_{\epsilon>0} \mathrm{A}_{\mathrm{L}^{p-\epsilon},\mathrm{KLQ}}}^{\|\cdot\|_{B(G)}} = \overline{\mathrm{B}(G) \cap \bigcup_{\epsilon>0} \mathrm{L}^{p-\epsilon}(G)}^{\|\cdot\|_{B(G)}}, \\
\mathrm{A}_{\bigcup \mathrm{L}^{p},\mathrm{KLQ}}(G) &:= \overline{\bigcup_{1\leq p<\infty} \mathrm{A}_{\mathrm{L}^{p},\mathrm{KLQ}}(G)}^{\|\cdot\|_{B(G)}} = \overline{\mathrm{B}(G) \cap \bigcup_{1\leq p<\infty} \mathrm{L}^{p}(G)}^{\|\cdot\|_{B(G)}}.
\end{aligned}$$

Since functions in $B(G)$ are left uniformly continuous, it follows that the spaces discussed so far all lie in the Rajchman algebra $B_0(G)$. The weak* closures of these spaces in $B(G)$ are labeled as follows.

$$\begin{aligned}&\mathrm{B}_{\mathrm{L}^p,\mathrm{BG}}(G), \quad \mathrm{B}_{\mathrm{L}^{p+},\mathrm{BG}}(G), \quad \mathrm{B}_{\mathrm{L}^{p-},\mathrm{BG}}(G), \quad \mathrm{B}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G),\\ &\mathrm{B}_{\mathrm{L}^p,\mathrm{KLQ}}(G), \quad \mathrm{B}_{\mathrm{L}^{p+},\mathrm{KLQ}}(G), \quad \mathrm{B}_{\mathrm{L}^{p-},\mathrm{KLQ}}(G), \quad \mathrm{B}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G).\end{aligned}$$

We will frequently drop the subscripts BG and KLQ when making statements about these spaces that hold for both the BG and KLQ variants. In these contexts, when multiple spaces are being referred to, they are to be understood as all being of BG or KLQ type.

We will make frequent use of the following Hölder-type result.

Lemma 4.1.3. *Let G be a locally compact group. If $1 \leq p, q, r < \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then*

$$uv \in \mathrm{A}_{\mathrm{L}^r}(G)$$

for all $u \in \mathrm{A}_{\mathrm{L}^p}(G)$, $v \in \mathrm{A}_{\mathrm{L}^q}(G)$ and

$$uv \in \mathrm{B}_{\mathrm{L}^r}(G)$$

for all $u \in \mathrm{B}_{\mathrm{L}^p}(G)$, $v \in \mathrm{B}_{\mathrm{L}^q}(G)$.

Proof. This is proved for the BG-case in [86, Proposition 4.5]. The KLQ case is similar but easier. We include it for the sake of completeness.

Suppose $u \in \mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G)$ and $v \in \mathrm{A}_{\mathrm{L}^q,\mathrm{KLQ}}(G)$. Then we can find sequences $(u_n) \subset \mathrm{B}(G) \cap \mathrm{L}^p(G)$ and $(v_n) \subset \mathrm{B}(G) \cap \mathrm{L}^q(G)$ converging to u and v, respectively. Then $(u_n v_n) \subset \mathrm{B}(G) \cap \mathrm{L}^r(G)$ converges to uv, hence $uv \in \mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G)$. It is similarly shown that

$$uv \in \mathrm{B}_{\mathrm{L}^r}(G)$$

for all $u \in \mathrm{B}_{\mathrm{L}^p}(G)$ and $v \in \mathrm{B}_{\mathrm{L}^q}(G)$ since multiplication is jointly weak* continuous on bounded subsets of $B(G)$ and u, v can be approximated by bounded nets in $\mathrm{A}_{\mathrm{L}^p}(G)$ and $v \in \mathrm{A}_{\mathrm{L}^q}(G)$, respectively. ☐

Let G be a locally compact group and let $2 < p < \infty$. For $u \in A_{\mathrm{L}^p}(G)$ we have $u^n \in A_{\mathrm{L}^{\frac{p}{n}}}(G)$ for all $n \geq 1$ by Lemma 4.1.3. It follows that if $n \geq \frac{p}{2}$ then $u^n \in A_{\mathrm{L}^2}(G) = A(G)$. Since $A(G)$ is a closed ideal in each of the algebras we are considering, this tells us that if $\mathcal{A}$ is any of these algebras, then $\mathcal{A}/A(G)$ is a commutative radical Banach algebra.

Lemma 4.1.4. *Let G be a locally compact group and let $\mathcal{A}$ denote any of the algebras*

$$\begin{array}{llll} \mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G), & \mathrm{A}_{\mathrm{L}^{p+},\mathrm{BG}}(G), & \mathrm{A}_{\mathrm{L}^{p-},\mathrm{BG}}(G), & \mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G), \\ \mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G), & \mathrm{A}_{\mathrm{L}^{p+},\mathrm{KLQ}}(G), & \mathrm{A}_{\mathrm{L}^{p-},\mathrm{KLQ}}(G), & \mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G). \end{array}$$

Then the Gelfand spectrum $\Delta(\mathcal{A})$ is G under the identification $s \mapsto \varphi_s$, where $\varphi_s(u) = u(s)$ for $s \in G$.

Proof. If $1 \le p \le 2$ then $\mathcal{A} = A(G)$ (see Lemma 4.1.7 below) and it is well known that $\Delta(A(G)) = G$. Thus assume that $2 < p < \infty$. It is also known that $\Delta(\mathcal{A}) = G$ when $\mathcal{A} = \mathrm{A}_{\mathrm{L}^q,\mathrm{BG}}(G)$ [86, Proposition 6.1]. Given Lemma 4.1.3, the argument used in [86, Proposition 6.1] is easily modified to show that $\Delta(\mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G)) = A(G)$. In fact, this can also be done for the algebras $\mathrm{A}_{\mathrm{L}^{p+},\mathrm{BG}}(G)$, $\mathrm{A}_{\mathrm{L}^{p-},\mathrm{BG}}(G)$, $\mathrm{A}_{\mathrm{L}^{p+},\mathrm{KLQ}}(G)$ and $\mathrm{A}_{\mathrm{L}^{p-},\mathrm{KLQ}}(G)$.

Now let $\mathcal{A} = \mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ and $\varphi \in \Delta(\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G))$. Since $\varphi \neq 0$, there exist $1 < p < \infty$ such that $\varphi|_{\mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G)} \neq 0$. But then there exists an $s \in G$ such that $\varphi|_{\mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G)} = \varphi_s$ and Lemma 4.1.3 then implies that $\varphi_{|_{\mathrm{A}_{\mathrm{L}^q,\mathrm{BG}}(G)}} = \varphi_s$ for all $1 < q < \infty$. Since convergence in $B(G)$ implies uniform convergence, it follows that $\varphi = \varphi_s$.

The proof that $\Delta(\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)) = G$ is similar to this last case. □

We record the following result for future reference.

Theorem 4.1.5. ([48]) *For an non-compact abelian locally compact group G we have $G \subsetneq \Delta(B_0(G))$.*

It follows that both $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ and $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$ are properly contained in $B_0(G)$ for non-compact abelian G. We later establish this result for a larger class of groups.

We will frequently make use of the following lemmas.

Lemma 4.1.6. (*[86, Proposition 4.6]*) *If H is an open subgroup of a locally compact group G and $1 \le p \le \infty$, then*

$$\mathrm{A}_{\mathrm{L}^p}(G)|_H = \mathrm{A}_{\mathrm{L}^p}(H) \text{ and } \mathrm{B}_{\mathrm{L}^p}(G)|_H = \mathrm{B}_{\mathrm{L}^p}(H).$$

Lemma 4.1.7. (*[23, Proposition 3.4]*) *If G is a locally compact group and $1 \le p \le 2$, then $\mathrm{A}_{\mathrm{L}^p}(G) = \mathrm{A}(G)$.*

4.2 Free groups

It was shown by Okayasu in [66] that $B_{L^p,\mathrm{BG}}(\mathbb{F}_d) \neq B_{L^q,\mathrm{BG}}(\mathbb{F}_d)$ for integers $d \geq 2$ and $2 \leq p < q \leq \infty$. In this subsection we apply a result of Haagerup to provide a new and shorter proof of Okayasu's result. For $0 < \alpha \leq 1$ let $\phi_\alpha : \mathbb{F}_d \to \mathbb{C}$ denote the positive definite function $\phi_\alpha(s) = \alpha^{|s|}$, where $|s|$ denotes the length of the reduced word $s \in \mathbb{F}_d$.

Lemma 4.2.1. ([15]) *Let $2 \leq d < \infty$. The positive definite function ϕ_α extends to a positive linear functional on $\mathrm{C}^*_r(\mathbb{F}_d)$ if and only if $\alpha \leq (2d-1)^{-1/2}$. Equivalently, this states that $\phi_\alpha \in \mathrm{B}_r(\mathbb{F}_d)$ if and only if $\alpha \leq (2d-1)^{-1/2}$.*

Theorem 4.2.2. *Let $2 \leq p \leq \infty$. Then $\phi_\alpha \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ if and only if $\alpha \leq \alpha_p := (2d-1)^{-1/p}$.*

Proof. When $p = \infty$ we have $\mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d) = B(\mathbb{F}_d)$ and the claim is trivial, so assume p is finite. Observe that ϕ_α is ℓ^p-summable for each $\alpha < (2d-1)^{-1/p} = \alpha_p$ and hence $\phi_\alpha \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ for each $\alpha \leq \alpha_p$. Suppose towards a contradiction that $\phi_\alpha \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ for some $\alpha > \alpha_p$ and choose q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then

$$\phi_\alpha \cdot \phi_{\alpha_q} = \phi_{\alpha \cdot \alpha_q} \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d) \cdot \mathrm{B}_{\ell^q,\mathrm{BG}}(\mathbb{F}_d) \subset \mathrm{B}_{\ell^2,\mathrm{BG}}(\mathbb{F}_d) = \mathrm{B}_r(\mathbb{F}_d).$$

This contradicts Haagerup's result since $\alpha \cdot \alpha_q > \alpha_p \cdot \alpha_q = (2d-1)^{-1/2}$. Thus $\phi_\alpha \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ if and only if $\alpha \leq \alpha_p$. □

It follows from the above result that the algebras $\mathrm{C}^*_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ are pairwise distinct for $2 \leq p \leq \infty$. Okayasu deduced this result from the following characterization.

Theorem 4.2.3. *Let $d \geq 2$ be an integer, $2 \leq p \leq \infty$, and $u \in P(\mathbb{F}_d)$. The following are equivalent:*

1. *$u \in \mathrm{B}_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$.*
2. *$\sup_{k\geq 0} u\chi_k(1+k)^{-1} \in \ell^p(\mathbb{F}_d)$ where $\chi_k : \mathbb{F}_d \to \{0,1\}$ denotes the characteristic function for words of length k.*
3. *The function $s \mapsto u(s)(1+|s|)^{-1-\frac{2}{p}}$ on $\mathbb{F}_d$ belongs to $\ell^p(\mathbb{F}_d)$.*
4. *$u\phi_\alpha \in \ell^p(\mathbb{F}_d)$ for every $0 < \alpha < 1$.*

We conjecture that the following KLQ-variant holds.

Conjecture 4.2.4. *Let $d \geq 2$ be an integer, $2 \leq p \leq \infty$, and $u \in B(\mathbb{F}_d)$. The following are equivalent:*

1. $u \in \mathrm{B}_{\ell^p,\mathrm{KLQ}}(\mathbb{F}_d)$.

2. $\sup_{k\geq 0} u\chi_k(1+k)^{-1} \in \ell^p(\mathbb{F}_d)$.

3. *The function* $s \mapsto u(s)(1+|s|)^{-1-\frac{2}{p}}$ *on* $\mathbb{F}_d$ *belongs to* $\ell^p(\mathbb{F}_d)$.

4. $u\phi_\alpha \in \ell^p(\mathbb{F}_d)$ *for every* $0 < \alpha < 1$.

Since $B_{\ell^p,\mathrm{BG}}(\mathbb{F}_d)$ and $B_{\ell^p,\mathrm{KLQ}}(\mathbb{F}_d)$ are spanned by positive definite functions, Conjecture 4.2.4 holds if and only if Conjecture 4.1.1 holds for free groups. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) above are straightforward, but the implication (1) $\Rightarrow$ (2) appears to be more difficult. Lemma 4.1.3 allows us to establish Conjecture 4.2.4 when p is an even integer.

Theorem 4.2.5. *Conjecture 4.2.4 holds when $p = 2n$ for $n \geq 1$.*

Proof. It suffices to show (1) $\Rightarrow$ (4) when u is positive definite by Theorem 4.2.3. If $u \in P(\mathbb{F}_d) \cap B_{\ell^{2n},\mathrm{KLQ}}(\mathbb{F}_d)$ then $u^n \in \mathrm{B}_r(\mathbb{F}_d)$ by Lemma 4.1.3 and so $u^n\phi_\alpha \in \ell^2(\mathbb{F}_d)$ for every $\alpha \in (0,1)$ by Theorem 4.2.3. Therefore $u\phi_\alpha \in \ell^{2n}(\mathbb{F}_d)$ for all $\alpha \in (0,1)$. □

4.3 Non-separability of $A_{L^p}(G)$ for abelian groups

In [86] it was shown that for non-compact, abelian, locally compact groups G the spaces $\mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G)$ are pairwise distinct for $2 \leq p < \infty$. In this section we prove that for both $\mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G)$ and $\mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G)$, not only are these spaces distinct for distinct $2 \leq p < \infty$, but for $2 \leq p < q < \infty$ the quotient $A_{\mathrm{L}^q}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$ is non-separable. In particular, $\mathrm{A}_{\mathrm{L}^p}(G)$ is non-separable for $2 < p < \infty$. In contrast, $A_{\mathrm{L}^2}(G) = A(G)$ is known to be separable when G is second countable.

We will need the theory of Riesz product constructions, which may be found in [34]. Let Γ be an infinite discrete abelian group and G be its compact Pontryagin dual. Then Γ admits an infinite **dissociate** subset, meaning a subset $\Theta \subset \Gamma$ so that every $\gamma \in \Gamma$ can be written in the form

$$\gamma = \prod_{\theta\in\Theta} \theta^{\epsilon_\theta}$$

for $\epsilon_\theta \in \{0, 1, -1\}$, $\epsilon_\theta \subset \{0, 1\}$, if $\theta^2 = 1$, $\epsilon_\theta = 0$ for all but finitely many $\theta \in \Theta$, and the choices of ϵ_θ are unique. If Θ is a dissociate subset of Γ and $a : \Theta \to \mathbb{C}$ is any function such that $|a(\theta)| \leq \frac{1}{2}$ for $\theta \in \Theta$ satisfying $\theta^2 \neq 1$ and $a(\theta) \in (-1, 1)$ when $\theta^2 = 1$, then there exists a probability measure P on G that has Fourier-Stieltjes transform

$$\widehat{P}(\gamma) = \begin{cases} \prod_{\theta \in \Theta} a(\theta)^{(\epsilon_\theta)}, & \text{if } \gamma = \prod_{\theta \in \Theta} \theta^{\epsilon_\theta},\ \epsilon_\theta \in \{0, 1, -1\} \\ & \text{is 0 for all but finitely many } \theta \\ 0, & \text{otherwise} \end{cases}$$

where

$$a^{(\epsilon)} := \begin{cases} 1 & \text{if } \epsilon = 0 \\ a & \text{if } \epsilon = 1 \\ \overline{a} & \text{if } \epsilon = -1 \end{cases}$$

for $a \in \mathbb{C}$ and $\epsilon \in \{-1, 0, 1\}$. This construction is called the **Riesz product construction** and the probability measure P is said to be based on Θ and a. The set of all Riesz product constructions is denoted $R(G)$. By work of Hewitt and Zuckerman, $\widehat{P} \in \mathrm{A}(\Gamma)$ if and only if $a \in \ell^2(\Theta)$.

The following result was proved by Wiersma for $\mathrm{A}_{\mathrm{L}^q,\mathrm{BG}}(G)$, but the proof for $\mathrm{A}_{\mathrm{L}^p,\mathrm{KLQ}}(G)$ is identical, see [86, Theorem 5.1].

Lemma 4.3.1. *Let G be a compact abelian group with dual group Γ. Suppose that $\mu \in R(G)$ is based on Θ and a. Then $\widehat{\mu} \in \mathrm{A}_{\mathrm{L}^p}(\Gamma)$ if and only if $a \in \ell^p(\Theta)$.*

We will also need the following two results.

Lemma 4.3.2. *(Zygmund* [88]*) Let P and P' be Riesz product measures based on Θ and a and Θ' and a', respectively. If*

$$\sum_{\theta \in \Theta \cap \Theta'} |a(\theta) - a'(\theta)|^2 = \infty,$$

then $P \perp P'$.

Lemma 4.3.3. *(Hewitt-Zuckerman [38]) Let P be a Riesz product measure based on Θ and a. If $a \notin \ell^2(\Theta)$, then $P \perp Q$ for every $Q \in L^1(G)$.*

From this, we deduce the following.

Lemma 4.3.4. *Let G be a compact abelian group with dual group Γ. If P is a Riesz product measure based on Θ and a such that $a \notin \ell^p(\Theta)$ for some*

$2 \le p < \infty$, *then*

$$P \perp \widehat{u}$$

for every $u \in \mathrm{A}_{\ell^p}(\Gamma)$.

Proof. Let $2 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Choose a pointwise non-negative function $b \in \ell^\infty(\Theta)$ with $\|b\|_\infty \le 1$ so that $ab \in \ell^q(\Theta)$ but $ab^{\frac{2}{q}} \notin \ell^p(\Theta)$, which is possible by [86, Lemma 5.1]. We let P' be the Riesz product measure based on Θ and $c := \frac{1}{2}|ab|^{\frac{p}{q}}$. Then $\widehat{P'} \in \mathrm{A}_{\ell^q}(\Gamma)$ by Lemma 4.3.1. Let $u \in \mathrm{A}_{\ell^p}(\Gamma)$ be a normalized positive definite function. Then

$$u\widehat{P'} \in \mathrm{A}_{\ell^2}(\Gamma) = \mathrm{A}(\Gamma)$$

by Lemma 4.1.3, implying $\widehat{u} * P' \in \mathrm{L}^1(G)$. Next observe that $P * P'$ is the Riesz product measure based on Θ and ac. Observe that

$$\sum_{\theta \in \Theta} |a(\theta)c(\theta)|^2 = \frac{1}{4} \sum_{\theta \in \Theta} |a(\theta)b(\theta)^{\frac{2}{q}}|^p = \infty.$$

It follows that $P * P' \perp \mathrm{L}^1(G)$ by Lemma 4.3.3. Therefore

$$\|P - \widehat{u}\| \ge \|P * P' - \widehat{u} * P'\| = 2,$$

implying that $P \perp \widehat{u}$. ☐

Theorem 4.3.5. *Let* Γ *be an infinite discrete abelian group. Then*

$$\mathrm{A}_{\ell^{p+}}(\Gamma)/\mathrm{A}_{\ell^p}(\Gamma)$$

is non-separable for $2 \le p < \infty$ *and*

$$A_{\ell^p}(\Gamma)/\mathrm{A}_{\ell^{p-}}(\Gamma)$$

is non-separable for $2 < p < \infty$.

Proof. Let $2 \le p < \infty$ and $q : \mathrm{A}_{\ell^{p+}}(\Gamma) \to \mathrm{A}_{\ell^{p+}}(\Gamma)/\mathrm{A}_{\ell^p}(\Gamma)$ be the quotient map. Choose an infinite dissociate set $\Theta \subset \Gamma$ and $a \in \bigcap_{\epsilon > 0} \ell^{p+\epsilon}(\Theta) \backslash \ell^p(\Theta)$ with $\|a\|_\infty \le \frac{1}{2}$. For each $t \in (0,1)$, let P_t be the Riesz product measure based on Θ and ta. Then $\widehat{P_t} \in \mathrm{A}_{\ell^{p+}}(\Gamma)$ and

$$\left\| q(\widehat{P_t}) - q(\widehat{P_{t'}}) \right\| = \|P_t - P_{t'}\|$$

for $t, t' \in (0,1)$ by Lemma 4.3.1. Further, by Lemma 4.3.2 we have $P_t \perp P_{t'}$ for $t \neq t'$ since

$$\sum_{\theta \in \Theta} |ta(\theta) - t'a(\theta)|^2 = |t - t'| \sum_{\theta \in \Theta} |a(\theta)|^2 = \infty.$$

It follows that $\left\| q(\widehat{P_t}) - q(\widehat{P_{t'}}) \right\| = 2$ for all distinct $t, t' \in (0,1)$ and hence $\mathrm{A}_{\ell^{p+}}(\Gamma)/\ \mathrm{A}_{\ell^p}(\Gamma)$ is non-separable for $2 \leq p < \infty$. A similar proof shows that $\mathrm{A}_{\ell^p}(\Gamma)/\ \mathrm{A}_{\ell^{p-}}(\Gamma)$ is non-separable for $2 < p < \infty$. ☐

Lemma 4.3.6. *Suppose G is a locally compact group containing a compact normal subgroup K. Let $q : \mathrm{B}(G) \to \mathrm{B}(G/K)$ denote the quotient map given by*

$$q(u)(sK) = \int_K u(sk)\, ds.$$

Then $q\left(\mathrm{A}_{\mathrm{L}^p}(G)\right) = \mathrm{A}_{\mathrm{L}^p}(G/K)$.

Proof. It is a simple exercise to check that q maps $\mathrm{A}_{\mathrm{L}^p}(G)$ into $\mathrm{A}_{\mathrm{L}^p}(G/K)$ since q maps $\mathrm{L}^p(G)$ into $L^p(G/K)$ and $P(G)$ into $P(G/K)$. By [86, Lemma 5.6] (and its analog for the KLQ case, which has an identical proof), we have

$$\mathrm{A}_{L^p}(G/K) = \mathrm{A}_{\mathrm{L}^p}(G : K) := \{u \in \mathrm{A}_{\mathrm{L}^p}(G) : u \text{ is constant on cosets of } K\}$$

and hence $q\left(A_{L^p}\left(G\right)\right) = A_{L^p}\left(G/K\right)$. ☐

Corollary 4.3.7. *Let G be a non-compact abelian locally compact group containing an open compact subgroup K. Then*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ and

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$.

Proof. This follows from Theorem 4.3.5 and Lemma 4.3.6. ☐

To generalize this result to all non-compact abelian locally compact groups, we need the following extension results.

Theorem 4.3.8. *([34, Theorem A.7.1]) Let* Γ *be a lattice in an abelian locally compact group* G*. If* $u \in \mathrm{A}(G)$ *is a normalized positive definite function with* $\Gamma \cap \operatorname{supp}(u) = \{e\}$ *and* $\Gamma \cap (s + \operatorname{supp}(u))$ *finite for every* $s \in G$*, then the map* $J_u : \mathrm{B}(\Gamma) \to \mathrm{B}(G)$ *defined by*

$$J_u v(s) = \sum_{\gamma \in \Gamma} u(s-\gamma) v(\gamma)$$

is an isometry mapping $P(\Gamma)$ *into* $P(G)$ *and* $\mathrm{A}(\Gamma)$ *into* $\mathrm{A}(G)$*. Further* $J_u v|_\Gamma = v$ *for every* $v \in \mathrm{B}(\Gamma)$*.*

Lemma 4.3.9. *([86, Lemma 5.4]) Let* $G = \mathbb{R}^n \times K$ *for some compact abelian group* K *and* $n \geq 1$*. Choose* $u \in P(G) \cap \mathrm{A}(G)$ *normalized with* $\operatorname{supp}(u) \subset [1/3, 1/3]^n \times K$ *and suppose* P *is a Riesz product measure of* $\mathbb{Z}^n$ *based on* Θ *and* a*. Then* $J_u \widehat{P} \in \mathrm{A}_{\mathrm{L}^p}(G)$ *if and only if* $a \in \ell^p(\Theta)$*.*

Lemma 4.3.10. *Let* $G = \mathbb{R}^n \times K$ *for some compact abelian group* K *and* $n \geq 1$*. Then*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ *and*

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$*.*

Proof. We use arguments similar to those establishing Lemma 4.3.4 and Theorem 4.3.5. Fix $u \in P(G) \cap \mathrm{A}(G)$ so that $\operatorname{supp}(u) \subset [1/3, 1/3]^n \times K$. Let Θ be an infinite dissociate subset of $\mathbb{Z}^n$ and choose $a \in \bigcap_{q>p} \ell^q(\Theta) \backslash \ell^p(\Theta)$ with $\|a\|_\infty \leq \frac{1}{2}$. Let P_t denote the Reisz product measure of $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$ based on Θ and ta for $t \in (0,1)$. Choose q satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. By the proof of Lemma 4.3.4, there exists $c \in \ell^q(\Theta)$ with $\|c\|_\infty \leq \frac{1}{2}$ such that $ac \notin \ell^2(\Theta)$. Let P' be the Riesz product measure of $\mathbb{T}^n$ based on Θ and c. Fix $v \in \mathrm{A}_{\mathrm{L}^p}(G)$. We have $J_u \widehat{P'} \cdot v \in \mathrm{A}(G)$ by Lemmas 4.1.3 and 4.3.9. For $t \neq t'$,

$$\begin{aligned} \left\| \left(J_u \widehat{P_t} - J_u \widehat{P_{t'}} \right) - v \right\| &\geq \left\| \left(J_u \widehat{P'} \cdot J_u \widehat{P_t} - J_u \widehat{P'} \cdot J_u \widehat{P_{t'}} \right) - J_u \widehat{P'} \cdot v \right\| \\ &\geq \left\| \widehat{P'} \cdot \widehat{P_t} - \widehat{P'} \cdot \widehat{P_{t'}} - \widehat{P'} \cdot (v|_{\mathbb{Z}^n}) \right\| \\ &= \left\| P' * P_t - P' * P_{t'} - P' * \widehat{v|_{\mathbb{Z}^n}} \right\|. \end{aligned}$$

By the Herz restriction theorem $J_u \widehat{P'} \cdot v \in \mathrm{A}(G)$ implies $\widehat{P'} \cdot v \in \mathrm{A}(\mathbb{Z}^n)$, hence

$P' * \widehat{v|_{\mathbb{Z}^n}} \in L^1(\mathbb{T}^n)$. Further, $P' * P_t$ and $P' * P_{t'}$ are the Riesz product measures based on Θ and tac and $t'ac$, respectively. Hence $P' * P_t \perp P' * P_{t'}$ because $(t-t')ac \notin \ell^2(\Theta)$ by Lemma 4.3.4. Thus

$$\left\| P' * P_t - P' * P_{t'} - P' * \widehat{v|_{\mathbb{Z}^n}} \right\| \geq 2,$$

showing that

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$. The result for $\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$ is proven similarly. □

Theorem 4.3.11. *Let G be a non-compact abelian group. Then*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ and

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$.

Proof. By the structure theorem for abelian locally compact groups, G admits an open subgroup of the form $K \times \mathbb{R}^n$ for some $n \geq 0$ and compact abelian group K. If $n \geq 1$, then the result follows from Lemma 4.3.10 and Lemma 4.1.6, otherwise $n = 0$ and the result follows from Corollary 4.3.7. □

4.4 Non-separability of $A_{L^p}(G)$ for IN groups

In [86, Theorem 5.8] it is shown that $\mathrm{A}_{\mathrm{L}^p,\mathrm{BG}}(G) \subsetneq A_{\mathrm{L}^q,BG}(G)$ when G is an almost connected SIN group and $2 \leq p < q < \infty$. In this section we extend this result to larger classes of groups and show that many of the results in the previous section extend to not only almost connected $[SIN]$-groups but also a large class of IN groups, including many classes of discrete groups.

Theorem 4.4.1. *Let G be a non-compact almost connected IN group. Then*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ *and*

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$*.*

Proof. Assume towards a contradiction that $\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$ is separable for some $2 \leq p < \infty$. By the structure theorem for almost connected locally compact groups [67], G admits a compact normal subgroup K such that G/K is a Lie group. By Lemma 4.3.6, it follows that $\mathrm{A}_{\mathrm{L}^{p+}}(G/K)/\mathrm{A}_{\mathrm{L}^p}(G/K)$ is separable. As such we may assume that G is itself a Lie group. In this case, the connected component of the identity G_e is open in G. It follows from Lemma 4.1.6 that $\mathrm{A}_{\mathrm{L}^{p+}}(G_e)/\mathrm{A}_{\mathrm{L}^p}(G_e)$ is separable. Finally, since G is non-compact and almost connected, G_e is itself a non-compact connected IN group. This means that there exists a compact normal subgroup K_0 of G_e such that $V = G_e/K_0$ is a vector group. However, a second application of Lemma 4.3.6 implies that $\mathrm{A}_{\mathrm{L}^{p+}}(V)/\mathrm{A}_{\mathrm{L}^p}(V)$ is also separable, in contradiction to Theorem 4.3.11.

The proof that $\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$ is non-separable for $2 < p < \infty$ is similar. □

It remains unclear how to remove the condition that the IN group G be almost connected. However, we can show for IN groups that if

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G) \text{ or } \mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is separable for some $2 < p < \infty$, then G must have compact connected component of the identity. This holds for another important class of locally compact groups, the maximally almost periodic groups. Recall that a locally compact group is said to be **maximally almost periodic** (**MAP**) if the finite dimensional representations of G separate points of G.

Corollary 4.4.2. *If G is an IN group or a MAP group for which either*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G) \text{ or } \mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is separable for some $2 < p < \infty$*, then* G_e *is compact. In particular,* G *has an open compact subgroup.*

Proof. Every locally compact group has an open almost connected subgroup G_0. However, if G is either an IN group or a MAP group, then G_0 is an almost connected IN group. It now follows from Theorem 4.4.1 that G_0 is open and compact. This implies that G_e is compact. □

We will now turn our attention to discrete groups. The key observation in this case is as follows:

Lemma 4.4.3. *Let G be a discrete group with an infinite abelian subgroup. Then*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^{p}}(G)$$

is non-separable for $2 \leq p < \infty$ and

$$\mathrm{A}_{\mathrm{L}^{p}}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$.

Proof. This follows immediately from Theorem 4.3.11 since the infinite abelian subgroup is open. □

Adian and Novikov showed that there are infinite discrete groups without infinite abelian subgroups [1]. Nevertheless, there are a number of important classes of groups G for which we know that the conclusion above holds. Recall that a discrete group G is called:

- **periodic** if each of its element has finite order,
- **locally finite** if each of its finite subsets generate a finite subgroup,
- **elementary amenable** if G belongs to the smallest class of groups which contains all finite groups and all abelian groups and is closed under taking subgroups, quotients, extensions, and direct unions,
- **linear** if G is a subgroup of $GL(n, F)$ for some field F.

It's clear that every locally finite group is periodic and that the converse is false. Hall and Kulatilaka ([37]) and Kargarpolov ([50]) independently established that infinite locally finite groups contain infinite abelian subgroups.

Linear groups have been shown to satisfy the Tits Alternative [84], meaning that if G is an infinite finitely generated linear group then either G contains a copy of $\mathbb{F}_2$ or G is virtually solvable. Since virtually solvable groups are elementary amenable, it follows that every infinite linear group also contains an infinite abelian subgroup. It is known that infinite finitely generated groups that are hyperbolic in the sense of Gromov satisfy the Tits Alternative [32], hence also contain infinite abelian subgroups. There are other important classes of geometric groups that are known to satisfy the Tits Alternative: mapping class groups [42], [62], $Out(F_n)$ [8], various groups of birational transformations of algebraic surfaces [11], and some large subclasses of CAT(0) groups [78]. As such these classes of geometric groups have the property that if G is infinite, then the conclusion of Theorem 4.4.4 holds.

Theorem 4.4.4. *Let G be an infinite discrete group which is in one of the following classes:*

1. *locally finite,*
2. *elementary amenable,*
3. *linear,*
4. *polynomial growth, meaning that for each compact neighborhood K of the identity there is an integer p such that $|K^n| = O(n^p)$.*

Then

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ and

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$.

Proof. We need only prove prove (4). Thus assume G has polynomial growth and that either $\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$ or $\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$ is separable for some $2 \leq p < \infty$ or $2 < p < \infty$, respectively. Let H be a finitely generated subgroup of G. Then H also has polynomial growth. It follows from a result of Gromov ([31]) that H has a nilpotent subgroup N of finite index in H. If N were infinite, it would contain an infinite abelian subgroup H_1. In this case we would have that either $\mathrm{A}_{\mathrm{L}^{p+}}(H_1)/\mathrm{A}_{\mathrm{L}^p}(H_1)$ or $\mathrm{A}_{\mathrm{L}^p}(H_1)/\mathrm{A}_{\mathrm{L}^{p-}}(H_1)$ is separable for some $2 \leq p < \infty$ or $2 < p < \infty$, respectively. But this is impossible unless

H_1 is finite. Consequently we get that G is locally finite, hence that G is finite by (1). $\square$

Hewitt and Zuckerman have shown that $A(G) \subsetneq B_0(G)$ when G is a non-compact abelian group [38] and Taylor showed that the same holds for non-compact second countable IN groups [82]. We now show that Taylor's result holds without the second countability assumption.

Theorem 4.4.5. *If G is a non-compact IN group, then $A(G) \subsetneq B_0(G)$.*

Proof. Assume first that G is discrete with $A(G) = B_0(G)$. Let H be a finitely generated subgroup of G. Then $A(H) = B_0(H)$ so H is finite by Taylor's result. It follows that G is locally finite. But then if G is infinite it must contain an infinite abelian subgroup H_1 with $A(H_1) = B_0(H_1)$. This contradicts the result of Hewitt and Zuckerman, hence G must be finite.

Next assume that G is an IN group with $A(G) = B_0(G)$. It follows that the intersection of all the compact conjugation invariant neighborhoods of the identity in G is a compact normal subgroup K with G/K a SIN group [33, Theorem 2.5]. Since $A(G/K) = B_0(G/K)$, we may assume without loss of generality that G is itself a SIN group. Then G has a compact normal subgroup K_1 with G/K_1 a Lie group [33, Corollary 2.17]. Thus we can assume without loss of generality that G is a SIN Lie group. Let G_e be the connected component of the identity in G. Since G is a Lie group, G_e is open. As such we have that $A(G_e) = B_0(G_e)$ for the connected SIN group G_e. By Corollary 4.4.2, we get that G_e is compact. However, since G_e is compact, normal, and open, the discrete group G/G_e satisfies $A(G/G_e) = B_0(G/G_e)$. Then G/G_e must be finite and ultimately G is compact. $\square$

We can improve the previous theorem in the class of groups of polynomial growth.

Corollary 4.4.6. *Let G be an IN group with polynomial growth. Let $\mathcal{A}$ be one of $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$. If $\mathcal{A} = B_0(G)$, then G is compact.*

Proof. Since polynomial growth is inherited by open subgroups and quotients by compact normal subgroups, arguing as in the proof of Theorem 4.4.5 allows us to assume that G is a SIN Lie group. This means that $G_e = K \times \mathbb{R}^n$ for some compact group K and $n \geq 0$ (see [67]). Moreover, we get that $\mathcal{A}(G_e) = B_0(G_e)$. If $n \geq 1$, we would conclude that $\mathcal{A}(\mathbb{R}^n) = B_0(\mathbb{R}^n)$, which is false by Theorem 4.1.5. Thus G_e is a compact, open, normal subgroup of G. But then G/G_e is a discrete group of polynomial growth with $\mathcal{A}(G/G_e) = B_0(G/G_e)$. If G/G_e is infinite then there is an infinite abelian subgroup H with $\mathcal{A}(H) = B_0(H)$, again a contradiction. Thus G/G_e must be finite and hence G is compact. □

We strongly suspect that the previous Corollary holds without the assumption that G is of polynomial growth. In fact, we make the following conjecture:

Conjecture 4.4.7. *Let G be a non-compact IN group. Then $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G) \subsetneq B_0(G)$ and $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G) \subsetneq B_0(G)$. Moreover,*

$$\mathrm{A}_{\mathrm{L}^{p+}}(G)/\mathrm{A}_{\mathrm{L}^p}(G)$$

is non-separable for $2 \leq p < \infty$ and

$$\mathrm{A}_{\mathrm{L}^p}(G)/\mathrm{A}_{\mathrm{L}^{p-}}(G)$$

is non-separable for $2 < p < \infty$.

It is reasonable to ask whether the previous conjecture holds for every non-compact locally compact group. This is not the case: it is well known that the Fourier and Rajchman algebras coincide for the $ax + b$ group [52].

4.5 Δ-weak approximate identities and the BSE condition

Leptin's theorem asserts that the Fourier algebra of a locally compact group G has a bounded approximate identity if and only if G is amenable [56]. We show that for $2 < p < \infty$ the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$ have bounded approximate identity only under rather restrictive conditions. In fact, the next result shows that this can only happen if G is amenable and if $\mathrm{A}_{\mathrm{L}^p}(G) = A(G)$.

Proposition 4.5.1. *Let G be a locally compact group. Assume that $\mathcal{A}$ is one of the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$, $A_{\mathrm{L}^{p+}}(G)$, or $A_{\mathrm{L}^{p-}}(G)$ and that $\mathcal{A}$ has a bounded approximate identity for some $2 \leq p < \infty$. Then G is amenable and $\mathcal{A} = A(G)$.*

Proof. If (u_α) is a bounded approximate identity for $\mathcal{A}$, then it is a routine exercise to show that (u_α^n) is also a bounded approximate identity for $\mathcal{A}$ for $n \geq 1$. By Lemma 4.1.3 we can find $n \in \mathbb{N}$ large enough so that $(u_\alpha^n) \subset A(G)$. Given $u \in \mathcal{A}$, we have

$$\lim_\alpha \|u_\alpha^n u - u\|_{B(G)} = 0.$$

Since $u_\alpha^n u \in A(G)$ and $A(G)$ is closed in $\mathcal{A}$, this tells us that $u \in A(G)$ and hence that $\mathcal{A} = A(G)$. The amenability of G follows from Leptin's Theorem. □

It is clear that the above argument will not work for the algebras $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ and $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$. However, we can say the following.

Proposition 4.5.2. *Let G be a locally compact group for which $A_{\mathrm{L}^{p-}}(G) \subsetneq A_{\mathrm{L}^p}(G)$ for some $2 < p < \infty$. If either $\mathcal{A} = \mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathcal{A} = \mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$, then $\mathcal{A}$ does not have a bounded approximate identity.*

Proof. Assume that (u_α) is a bounded approximate identity for $\mathcal{A}$. Then by the density of $\bigcup_{2\leq p<\infty} \mathrm{A}_{\mathrm{L}^p}(G)$ in $\mathcal{A}$, we may assume that $(u_\alpha) \subset \bigcup_{2\leq p<\infty} \mathrm{A}_{\mathrm{L}^p}(G)$. If $u \in \mathrm{A}_{\mathrm{L}^p}(G)$ then $(uu_\alpha) \subset A_{\mathrm{L}^{p-}}(G)$, but $\|uu_\alpha - u\|_{B(G)} \to 0$ implies that $u \in A_{\mathrm{L}^{p-}}(G)$ as well. This contradicts our assumption on G. □

We have seen that we cannot expect $\mathrm{A}_{\mathrm{L}^p}(G)$ to have a bounded approximate identity unless $\mathrm{A}_{\mathrm{L}^p}(G) = A(G)$. However, there is a natural weaker notion of approximate identity that may be more appropriate for these algebras.

Definition 4.5.3. Let $\mathcal{A}$ be a commutative Banach algebra. A bounded net (u_α) in $\mathcal{A}$ is called a **Δ-weak bounded approximate identity** if

$$\lim_\alpha \varphi(u_\alpha) = 1$$

for every $\varphi \in \Delta(\mathcal{A})$.

Kaniuth and Ülger have shown that amenability of G is equivalent to $A(G)$ having a Δ-weak bounded approximate identity [49], and in related work Chu and Xu showed that this is equivalent to $\lambda(G)$ being weakly closed in $L(G)$ [13, Corollary 2.8]. We establish these equivalences below for some of the algebras we're concerned with. Recall from Lemma 4.1.4 that these algebras have spectrum G and that, for $s \in G$, we let φ_s denotes the corresponding evaluation functional.

Theorem 4.5.4. *Let G be a locally compact group and let $\mathcal{A}$ be one of the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$, $A_{\mathrm{L}^{p+}}(G)$, or $A_{\mathrm{L}^{p-}}(G)$, where $2 \le p < \infty$. The following are equivalent:*

1. *G is amenable,*

2. *$\mathcal{A}$ has a Δ-weak bounded approximate identity,*

3. *$G = \Delta(\mathcal{A})$ is weakly closed in $\mathcal{A}^*$.*

Proof. To see that (1) implies (2), assume G is amenable. Leptin's Theorem asserts that $A(G)$ has a bounded approximate identity (u_α) and it is easy to see that $u_\alpha(s) \to 1$ for each $s \in G$. Now since $A(G) \subset \mathrm{A}_{\mathrm{L}^p}(G)$ and $\Delta(\mathcal{A}) = G$, it follows immediately that (u_α) is a Δ-weak bounded approximate identity for $\mathcal{A}$. This establishes (2).

Next we assume that $\mathcal{A}$ has Δ-weak bounded approximate identity (u_α). Let Γ be any weak* cluster point of (u_α) in $\mathcal{A}^{**}$. Then it follows that $\Gamma(\varphi_s) = 1$ for each $s \in G$. In particular, this shows that 0 cannot be in the closure of $G = \Delta(\mathcal{A})$. Hence G is weakly closed, so (2) implies (3).

It remains to show that (3) implies (1). Suppose (1) fails, so that G is nonamenable. By [13, Corollary 2.8], $\Delta\left(A\left(G\right)\right)$ is not weakly closed. Then 0 is in the weak closure of $\Delta\left(A\left(G\right)\right)$ and there is a net (φ_{s_α}) in $\Delta\left(A\left(G\right)\right) = G$ converge weakly to 0 in $L\left(G\right)$. Given $\Gamma \in \mathcal{A}^{**}$ and a bounded net (u_β) in $\mathcal{A}$ converging weak* to Γ, Lemma 4.1.3 asserts that there is $n \ge 1$ with $u_\beta^n \in A\left(G\right)$ for every β. Passing to a subnet if necessary, we may assume that u_β^n has a weak* limit Γ' in $A\left(G\right)^{**}$. Since

$$\Gamma\left(\varphi_{s_\alpha}\right)^n = \lim_\beta \varphi_{s_\alpha}\left(u_\beta\right)^n = \lim_\beta \varphi_{s_\alpha}\left(u_\beta^n\right) = \Gamma'\left(\varphi_{s_\alpha}\right) \to 0$$

it follows that $\Gamma\left(\varphi_{s_\alpha}\right)$ converges to 0. Thus (φ_{s_α}) converges weakly to 0 in $\mathcal{A}^*$ and (3) fails to hold. □

Definition 4.5.5. Let $\mathcal{A}$ be a commutative Banach algebra with $\Delta(\mathcal{A}) = X$. Let $\mathcal{C}_{BSE}(X)$ denote the space of all continuous bounded functions u on X which satisfy the following: there exists a positive constant β such that for every finite collection $c_1, c_2, \ldots, c_n$ of complex numbers and $x_1, x_2, \ldots, x_n \in X$ we have

$$\left|\sum_{i=1}^n c_i u(x_i)\right| \le \beta \left\|\sum_{i=1}^n c_i \varphi_{x_i}\right\|.$$

We let $\mathcal{M}(A)$ denote the multiplier algebra of $\mathcal{A}$. We say that $\mathcal{A}$ is a **BSE-algebra** if the Gelfand transform of $\mathcal{M}(A)$ restricted to X is exactly $C_{BSE}(X)$.

The previous definition was introduced by Takahasi and Hatori and was motivated by a theorem of Bochner, Schoenberg, and Eberlein showing that if G is an abelian group, then $A(G)$ is a BSE-algebra (see [81, p.149]). Characterizing when commutative Banach algebras are BSE-algebras has been studied by several authors and recent attention has focused specifically on subalgebras of the Fourier-Stieltjes algebra. In [49, Theorem 5.1] Kaniuth and Ülger showed that $A(G)$ is BSE if and only if G is amenable. Even more recently, Kaniuth, Lau, and Ülger looked at the nature of BSE-ideals in the Rajchman algebra $B_0(G)$. Using their work and the well known fact that all BSE-algebras have Δ-weak bounded approximate identities [81, Corollary 5], we are now in a position to use Theorem 4.5.4 to extend Kaniuth and Ülger's result to the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$.

Theorem 4.5.6. *Let G be a locally compact group and let $\mathcal{A}$ be one of the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$, $A_{L^{p+}}(G)$, or $A_{L^{p-}}(G)$, where $2 \le p < \infty$. The following are equivalent:*

1. *G is amenable.*
2. *$\mathcal{A}$ is a BSE-algebra.*

Proof. Assuming G is amenable, Theorem 4.5.4 asserts that $\mathcal{A}$ has a Δ-weak bounded approximate identity. Since $\Delta(\mathcal{A}) = G$ by Lemma 4.1.4, it follows from [48, Theorem 3.7] that $\mathcal{A}$ is a BSE-algebra.

Now assume instead that $\mathcal{A}$ is a BSE-algebra. By [81, Corollary 5], $\mathcal{A}$ has a Δ-weak bounded approximate identity and Theorem 4.5.4 asserts that G is amenable. □

The reader might notice that in both Theorem 4.5.4 and Theorem 4.5.6 we omit the case where the algebra under consideration is either $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$.

Proposition 4.5.7. *Let G be an amenable locally compact group and let $\mathcal{A}$ be either of the algebras $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$. Then $\mathcal{A}$ has a Δ-weak bounded approximate identity and thus is a BSE-algebra.*

Proof. The first statement follows immediately from Kaniuth and Ülger's result that $A(G)$ has a Δ-weak bounded approximate identity, since $A(G) \subset \mathcal{A}$ and $\Delta(A(G)) = G = \Delta(\mathcal{A})$. The second statement now follows from [48, Theorem 3.7]. $\square$

It is clear that the argument of Theorem 4.5.4 establishing that if $\mathcal{A}$ has a Δ-weak bounded approximate identity then so does $A(G)$ will not work when $\mathcal{A}$ is either $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$. However, in [48, Theorem 3.2] the authors claim that for a closed translation invariant ideal I in $B_0(G)$ that contains $A(G)$, I has a Δ-weak bounded approximate identity if and only if G is amenable and $\Delta(I) = G$. This would indeed imply that if $\mathcal{A}$ is either $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(G)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(G)$ and if $\mathcal{A}$ has a Δ-weak bounded approximate identity, then G is amenable, since $\mathcal{A}$ satisfies all of the conditions of [48, Theorem 3.2]. Unfortunately, there is an error in the proof of this result. In fact, the proof cannot be repaired.

Theorem 4.5.8. *Let $\mathcal{A}$ be one of the algebras $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(\mathbb{F}_2)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(\mathbb{F}_2)$. Then $\mathcal{A}$ has a Δ-weak bounded approximate identity and thus is a BSE-algebra.*

Proof. As in Section 4.2, for $0 < \alpha \leq 1$ let $\phi_\alpha : \mathbb{F}_2 \to \mathbb{C}$ denote the positive definite function $\phi_\alpha(s) = \alpha^{|s|}$. It follows from Theorem 4.2.2 that $\phi_\alpha \in \mathcal{A}$ for each $0 < \alpha \leq 1$. Moreover, it is clear that

$$\lim_{\alpha \to 0^+} \phi_\alpha(s) = \lim_{\alpha \to 0^+} |s|^\alpha = 1, \quad s \in \mathbb{F}_2.$$

Thus $(\phi_\alpha)_{0<\alpha\leq 1}$ is a Δ-weak bounded approximate identity for $\mathcal{A}$ and [48, Theorem 3.7] implies that $\mathcal{A}$ is a BSE-algebra. $\square$

We have seen that if $\mathcal{A}$ is one of the algebras $\mathrm{A}_{\mathrm{L}^p}(G)$, $A_{\mathrm{L}^{p+}}(G)$, or $A_{\mathrm{L}^{p-}}(G)$, where $2 \leq p < \infty$, then $G = \Delta(\mathcal{A})$ is weakly closed in $\mathcal{A}^*$ if and only if G is amenable. We now show that this fails for $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(\mathbb{F}_2)$ and $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(\mathbb{F}_2)$.

Corollary 4.5.9. *Let $\mathcal{A}$ be either $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{BG}}(\mathbb{F}_2)$ or $\mathrm{A}_{\bigcup \mathrm{L}^p,\mathrm{KLQ}}(\mathbb{F}_2)$. Then $\mathbb{F}_2 = \Delta(\mathcal{A})$ is weakly closed.*

Proof. By Theorem 4.5.8 $\mathcal{A}$ has a Δ-weak bounded approximate identity $(\phi_\alpha)_{0<\alpha\leq 1}$. If Γ is any weak* cluster point of $(\phi_\alpha)_{0<\alpha\leq 1}$ in $\mathcal{A}^{**}$, then $\Gamma(\varphi_s) = 1$ for each $s \in \mathbb{F}_2$ and 0 cannot be in the closure of $\Delta(\mathcal{A})$. Hence $\Delta(\mathcal{A})$ is weakly closed. $\square$

4.6 The group $SL(2, \mathbb{R})$

We now focus our attention on the special linear group $SL(2,\mathbb{R})$. We have seen examples for which $A_{\bigcup L^p}(G)$ coincides with the Rajchman algebra $B_0(G)$ because $A_{L^p}(G) = B_0(G)$ for some $p \in [2,\infty)$. We show that $A_{\bigcup L^p}(SL(2,\mathbb{R})) = B_0(G)$ despite the fact that the spaces $B_{L^p}(SL(2,\mathbb{R}))$ are distinct for distinct values of $2 \le p < \infty$ [86, Corollary 7.2].

Proposition 4.6.1. *Let* $G = \mathrm{SL}(2,\mathbb{R})$. *For* $2 \le p < \infty$ *we have* $\mathrm{B}_{\mathrm{L}^p,\mathrm{BG}}(G) = \mathrm{B}_{\mathrm{L}^p,\mathrm{KLQ}}(G)$.

Proof. We need only show $\mathrm{B}_{\mathrm{L}^p,\mathrm{KLQ}}(G) \subset \mathrm{B}_{\mathrm{L}^p,\mathrm{BG}}(G)$. Let π be a KLQ-L^p-representation of G. Then the direct integral decomposition of π does not include the complementary representation π_r for $1 < r < \frac{2}{p} - 1$ nor the trivial representation except on a null set by [72, Theorem 9.1]. Hence $\mathrm{A}_\pi(G) \subset \mathrm{B}_{\mathrm{L}^p,\mathrm{BG}}(G)$ by [86, Lemma 7.1]. The conclusion follows. $\square$

Theorem 4.6.2. *Let* $G = \mathrm{SL}(2,\mathbb{R})$. *Then* $\mathrm{A}_{\bigcup \mathrm{L}^p}(G) = \mathrm{B}_0(G)$.

Proof. Let σ be a $\mathcal{C}_0$-representation of G and write $\sigma = \int^{\oplus} \pi \, d\mu(\pi)$ as the direct integral of irreducible representations. Then the singleton $\{1_G\}$ is μ-null since otherwise σ would contain 1_G as a subrepresentation. Let $\xi = (\xi_\pi) \in H_\sigma \subset \prod_{\pi \in \widehat{G}} H_\pi$. For every $n \ge 1$ define $\xi_n = (\xi_{n,\pi})$ where $\xi_{n,\pi} = 0$ if π is the complimentary representation π_r for some $-1 < r < -1 + \frac{1}{n}$, and $\xi_{n,\pi} = \xi_\pi$ otherwise. Note that ξ_n is measurable since we are restricting to a closed subset of $\widehat{G}$. Then $\xi_n \to \xi$ in norm. By [86, Lemma 7.1], $\pi_{\xi_n,\xi_n} \in \mathrm{B}_{\pi_{2n}}(G)$ and so $(\pi^n_{\xi_n,\xi_n}) \subset \mathrm{B}_r(G)$. It follows that $(\pi^n_{\xi_n,\xi_n}) \subset \mathrm{L}^{2+\epsilon}(G)$ for every $\epsilon > 0$ by [53, Theorem 11]. In particular, $\pi_{\xi_n,\xi_n} \in \mathrm{A}_{\bigcup \mathrm{L}^p}(G)$ for every n and hence $\pi_{\xi,\xi} \in \mathrm{A}_{\bigcup \mathrm{L}^p}$. We have shown that $\mathrm{B}_0(G) \subset \mathrm{A}_{\bigcup \mathrm{L}^p}(G)$, while the reverse containment always holds. $\square$

In [29], Ghandehari proves the non-existence of an approximate identity in $B_0(SL(2,\mathbb{R}))$ by first showing that this algebra is not square dense, requiring an extensive argument. Using the preceding theorem we are able to give a more concise argument.

Proposition 4.6.3. *Let* $G = \mathrm{SL}(2,\mathbb{R})$. *Then* $\mathrm{B}_0(G)$ *does not have an approximate identity.*

Proof. Suppose that $\mathrm{B}_0(G) = \mathrm{A}_{\bigcup \mathrm{L}^p}(G)$ has approximate identity (e_α). By density, we may assume that $(e_\alpha) \subset \bigcup_{p<\infty} \mathrm{A}_{\mathrm{L}^p}(G)$. Let π_r, $-1 < r < 0$, be a complementary representation of G. Then $\mathrm{A}_{\pi_r}(G) \subset \mathrm{L}^{p+\epsilon}(G)$ for every $\epsilon > 0$ by [53, Corollary on p.58], where $p := \frac{2}{1+r}$. Thus if $\xi \in H_{\pi_r}$ and $u := (\pi_r)_{\xi,\xi}$, then $e_\alpha u \in \bigcup_{q<p} \mathrm{A}_{\mathrm{L}^q_+}(G)$, implying that $u \in \overline{\bigcup_{q<p} \mathrm{A}_{\mathrm{L}^q_+}}$. Since π_r is an irreducible representation, it follows that $\mathrm{A}_{\pi_r}(G) \subset \mathrm{A}_{\mathrm{L}^q_+}(G)$ for some $q < p$. This contradicts [86, Lemma 7.1] and so we conclude that $\mathrm{B}_0(G)$ does not have an approximate identity. □

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