Sparsity in Critical Graphs with Small Clique Number

by

Matthew Eliot Kroeker

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In 1998, Reed conjectured that for every graph G, $\chi(G) \le \lceil \frac{1}{2}(\Delta(G)+1+\omega(G)) \rceil$, and proved that there exists $\varepsilon > 0$ such that $\chi(G) \le \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$ for every graph G. Recently, much effort has been made to prove this result for increasingly large values of ε in graphs with sufficiently large maximum degree. One of the main lemmas used in deriving these bounds states that graphs which are list-critical are sparse. This result generally follows by applying a sufficient condition for list colouring complete multipartite graphs with parts of bounded size, and until recently a theorem of Erdős, Rubin and Taylor for list colouring complete multipartite graphs with parts of size at most two was used. The current bottleneck in bounding $\chi(G)$ for an improved value of ε is the case of small clique number. We derive new density lemmas exploiting this case by showing that our graph is contained in a complete multipartite graph with many parts of size three. In order to list colour in this setting, we apply a theorem of Noel, West, Wu and Zhu, as well as our own unbalanced variant of this result.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The study of graph colouring is one of the oldest and most foundational disciplines in graph theory. A colouring of a graph is simply a labeling of its vertices such that adjacent vertices do not receive the same label. More formally, for a positive integer k, a *k-colouring* of a graph G is a map $\phi : V(G) \longrightarrow \{1, ..., k\}$ such that for all $uv \in E(G)$, $\phi(u) \neq \phi(v)$. If G has a k-colouring, then it is said to be *k-colourable*. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k for which G is k-colourable.

The motivation for our interest in the chromatic number is very natural. Indeed, to colour some graph G with no restriction on the number of colours is a triviality: putting the vertices into bijection with $\{1, ..., |V(G)|\}$, we obtain a proper colouring. It follows intuitively that, when attempting to colour some graph in actual practice, one is typically interested in using as few colours as possible. For instance, graph colouring frequently emerges in problems in computer science, and in this setting, colouring typically becomes a problem of cost minimization.

Graph theorists' interest in studying the chromatic number is further motivated by the fact that to find colourings of graphs with a minimum number of colours is a hard problem. In fact it is well known that, for a graph G, to determine if $\chi(G) \leq k$, for fixed $k \geq 3$, is NP-complete in general. Thus considerable attention has been devoted to finding upper bounds on the chromatic number for various classes of graphs. The chromatic number has two well-known trivial bounds, each given by a fundamental graph parameter. For a graph G, we denote the clique number of G

by $\omega(G)$ and its maximum degree by $\Delta(G)$. Clearly $\chi(G) \geq \omega(G)$; and it is easily verified that $\chi(G) \leq \Delta(G) + 1$ by colouring the vertices of G in a greedy manner.

Questions surrounding these two trivial bounds have also garnered lots of attention and have given rise to some beautiful structural theory. For instance, the lower bound $\omega(G)$ is central to the study of perfect graphs. (A graph is *perfect* if, in all of its induced subgraphs, the chromatic number is equal to the clique number). With regard to the trivial upper bound, the following characterization of Brooks is foundational in the theory of graph colouring.

Theorem 1.1.1 ([4]). *Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.*

In 1998, Reed [16] conjectured a very nice upper bound on the chromatic number in terms of both of these parameters. Essentially, he suggests that the chromatic number of any graph is bounded above by the average of the two.

Conjecture 1.1.2 ([16]). *For any graph G,*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

While Conjecture 1.1.2 remains open, recent partial results take the form of an "epsilon version," stemming from the following result of Reed, which he proved in the same paper as evidence for the conjecture.

Theorem 1.1.3 ([16]). *There exists $\varepsilon > 0$ such that, for any graph G,*

$$\chi(G) \leq \lceil (1 - \varepsilon)(\Delta(G) + 1) + \varepsilon\omega(G) \rceil.$$

Theorem 1.1.3 offers a starting point for progress towards proving Conjecture 1.1.2; to prove the conjecture in full is equivalent to proving Theorem 1.1.3 for $\varepsilon = \frac{1}{2}$.

In [12], King and Reed presented a much shorter proof of Theorem 1.1.3. While Reed's original proof was rather complicated, their new technique reduced the problem to a simpler argument by applying a theorem of King [11] to show that a minimum counterexample must have small clique number. We will outline this proof technique in more detail in the next section, but it basically consists of three steps. We apply King's theorem, and then the result follows from two main lemmas, which we refer to as the *density lemma* and the *sparsity lemma*.

In this setting, the term *density* refers to the quantity of edges in the neighbourhoods of a graph. A graph is called dense if it has some neighbourhood with a lot of edges (we will define this more formally in the next section); otherwise it is called sparse. The density lemma essentially tells us that our graph is sparse by induction, and the sparsity lemma tells us that we can colour sparse graphs. This is the basic idea behind the proof technique of King and Reed.

Implicit in King and Reed's proof of Theorem 1.1.3 is the following result, which says that, for graphs with sufficiently large maximum degree, the bound holds for $\varepsilon = \frac{1}{320e^6}$.

Theorem 1.1.4 ([12]). *For $\varepsilon = \frac{1}{320e^6}$, there exists $\Delta(\varepsilon)$ such that if G is a graph with $\Delta(G) > \Delta(\varepsilon)$, then*

$$\chi(G) \leq \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil.$$

The main point of King and Reed's paper [12] was not to optimize the bound on ε in Theorem 1.1.4, but rather to present a shorter proof of Theorem 1.1.3. In the interest of improving the bound on ε, Bonamy, Perrett and Postle presented enhanced versions of the sparsity and density lemmas in [3]. Their main result was an improvement of the bound in Theorem 1.1.4 to $\varepsilon \leq \frac{1}{26}$. In [5], Delcourt and Postle improved this further to $\varepsilon \leq \frac{1}{13}$ via a new density lemma.

The purpose of this thesis is to pursue further improvement to the density lemma, specifically in the special case of graphs with very small clique number. In the section which follows, we will elaborate on the overall proof technique and give a detailed exposition of the sparsity and density lemmas and how they are applied. In the next chapter we will focus on list-colouring, which is in fact the setting in which the density lemma is proved. We will derive an "unbalanced" sufficient condition for list-colouring graphs with colour classes of size at most three, generalizing a result of Delcourt and Postle central to the proof of their density lemma. In Chapter 3, we show that list-critical graphs with small clique number have a large number of disjoint triangles in their complement. Combining this with our list colouring condition, we derive two different density lemmas.

1.2 The Sparsity-Density Paradigm

When one is interested in proving a conjecture, it is natural to try to prove a partial or weaker version of the statement. Since Conjecture 1.1.2 proposes that the chromatic number is bounded above by an average of the maximum degree plus one and the clique number, a natural weakening is to bound the chromatic number by a non-trivial convex combination of these two parameters. So the natural question that arises is the following.

Question 1.2.1. *For which* $\varepsilon \in [0, \frac{1}{2}]$ *is it true that, for any graph* G*,*

$$\chi(G) \leq \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil?$$

Observe that this bound is a strengthening of the trivial bound $\chi(G) \leq \Delta(G)+1$. Thus by answering Question 1.2.1 in the affirmative for increasingly large values of ε, we are dragging the chromatic number away from its trivial upper bound and closer to its trivial lower bound $\omega(G)$.

In this section we provide an outline of the proof technique introduced in [12] by King and Reed. The technique was refined by Bonamy, Perrett and Postle in [3], whose main result answered Question 1.2.1 as follows for graphs with sufficiently large maximum degree.

Theorem 1.2.2 ([3])**.** *There exists* $\Delta > 0$ *such that, for any graph* G *with* $\Delta(G) > \Delta$*,*

$$\chi(G) \leq \left\lceil \frac{25}{26}(\Delta(G)+1) + \frac{1}{26}\omega(G) \right\rceil.$$

This result was further improved upon by Delcourt and Postle in [5], who showed that the bound in Question 1.2.1 holds for $\varepsilon = \frac{1}{13}$, provided that the maximum degree is sufficiently large (we will defer our discussion of this result until the end of this section, since it requires techniques beyond the scope of this thesis). Furthermore, their result was proved in the setting of list colouring, a generalization of (standard) colouring which we now introduce.

A *list assignment* for a graph G is a function which maps each vertex v of G to a set $L(v)$ of colours. Given a list assignment L for G, an *L-colouring* of G is a function $\phi : V(G) \longrightarrow \bigcup_{v \in V(G)} L(v)$ such that for each $v \in V(G)$, $\phi(v) \in L(v)$, and for every edge $uv \in E(G)$, $\phi(u) \neq \phi(v)$. We say that G is *L-colourable* if there exists an L-colouring for G.

For a positive integer k, we call L a *k-list-assignment* for G if $|L(v)| \geq k$ for every $v \in V(G)$. We call G *k-list-colourable* if G has an L-colouring for every k-list-assignment L for G. Finally, the *list-chromatic number*, which we denote $\chi_\ell(G)$, is the smallest integer k for which G is k-list-colourable. Note that, while $\chi(G) \leq \chi_\ell(G)$, it is well known that the reverse inequality does not hold in general. The question of when the chromatic number and this list chromatic number are equal is itself a well studied topic, and will be discussed further in Chapter 2.

Before going further, we require the notion of *criticality* in colouring and list colouring. For a positive integer k, a graph G is *k-critical* if G is not k-colourable but every proper subgraph of G is. In this way, criticality describes those graphs which show up in inductive colouring arguments. For a list assignment L of G, we say that G is *L-critical* if G is not L-colourable but every proper subgraph of G is.

We are now in a position to outline the main results we require, namely the sparsity and density lemmas, both of which are proved in the setting of list colouring. We begin by formalizing what we mean by sparsity. Note that, for a graph G and $X \subseteq V(G)$, $G[X]$ denotes the *subgraph induced by X*, that is, the maximal subgraph of G whose vertex set is X. For $v \in V(G)$, the *neighbourhood of v in G* is the set of all vertices adjacent to v in G, and is denoted by $N(v)$.

Definition 1 ([3])**.** *Let $\sigma \in [0,1]$. We call a graph G σ-sparse if for each $v \in V(G)$*

$$|E(G[N(v)])| \leq (1-\sigma)\binom{\Delta(G)}{2}.$$

The density lemma states that graphs which are critical with respect to some list assignment must be sparse. The following statement is a general framework for how the density lemma is formulated; it states that a critical graph is $\sigma(\varepsilon, x)$-sparse for some desired $\varepsilon > 0$, where $x \in (0,1)$ is the proportion of the size of the clique number relative to the maximum degree. The actual purpose of the density lemma is to specify such a function $\sigma(\varepsilon, x)$, which is ideally as large as possible. We leave it unspecified here for the sake of generality.

Statement 1.2.3 (Density Lemma)**.** *Let $\varepsilon, x \in (0,1)$ and let G be a graph with maximum degree at most Δ and clique number at most $x(\Delta+1)$. Let L be a $\lceil (1-\varepsilon(1-x))(\Delta+1)\rceil$-list-assignment for G. If G is L-critical, then for every $v \in V(G)$,*

$$|E(G[N(v)])| \leq (1-\sigma(\varepsilon, x))\binom{\Delta}{2}.$$

Note that the term $\lceil (1-\varepsilon(1-x))(\Delta+1)\rceil$ is just our desired bound on the chromatic number. In particular, if G has clique number $\omega = x(\Delta+1)$ for some

$x \in (0,1)$, then

$$\begin{aligned}\lceil (1-\varepsilon(1-x))(\Delta+1) \rceil &= \lceil (\Delta+1) - \varepsilon(\Delta+1) + \varepsilon x(\Delta+1) \rceil \\ &= \lceil (1-\varepsilon)(\Delta+1) + \varepsilon\omega \rceil.\end{aligned}$$

We denote the bound in this way because we generally consider the clique number as a proportion of the maximum degree.

The density lemma naturally provides a first step in our proof: when trying to colour our graph, we may assume that it is critical by induction. Bonamy, Perrett and Postle attained the following.

Theorem 1.2.4 ([3])**.** *For $\varepsilon, x \in (0,1)$ with $\varepsilon < \frac{1-x}{2}$, Statement 1.2.3 holds with*

$$\sigma(\varepsilon, x) = \frac{(1-x-2\varepsilon)^2}{2}.$$

The next step in the proof is to show that we can colour sparse graphs. This is the role of the sparsity lemma.

While we presented the density lemma in terms of a more general framework (that is, we did not specify the function $\sigma(\varepsilon, x)$ in Statement 1.2.3), we will state the actual sparsity lemma of Bonamy, Perrett and Postle given in [3]. The reason for this is that their sparsity lemma is state of the art and will be taken for granted in this thesis, while our focus will be on seeking improvements to the density lemma.

Theorem 1.2.5 (Sparsity Lemma [3])**.** *Let $\sigma \in [0, 0.9]$ and let $\varepsilon = 0.3012\sigma - 0.1283\sigma^{\frac{3}{2}}$. There exists $\Delta_1(\sigma)$ such that if G is a σ-sparse graph with $\Delta(G) > \Delta_1(\sigma)$, then*

$$\chi_\ell(G) \leq (1-\varepsilon)(\Delta(G)+1).$$

The sparsity and density lemmas can be combined to prove a colouring result of the following form. (This is a general framework of Lemma 5.2 in [3]).

Statement 1.2.6 (Main Colouring Lemma [3])**.** *For each $\varepsilon, x \in [0,1]$, there exists $\Delta_2(\varepsilon, x)$ such that if G is a graph with maximum degree at most $\Delta > \Delta_2(\varepsilon, x)$ and clique number at most $\omega = x(\Delta+1)$, then*

$$\chi(G) \leq \lceil (1-\varepsilon)(\Delta+1) + \varepsilon\omega \rceil,$$

provided that, for $\sigma(\varepsilon, x)$ for which Statement 1.2.3 is satisfied,

$$\varepsilon(1-x) \leq 0.3012\sigma(\varepsilon, x) - 0.1283\sigma(\varepsilon, x)^{\frac{3}{2}}. \tag{1.1}$$

Proof. Set $\Delta_2(\varepsilon, x) = \Delta_1(\sigma(\varepsilon, x))$. Suppose to the contrary, that there exists $\Delta > \Delta_2(\varepsilon, x)$ for which there exists a graph G with maximum degree at most Δ and clique number at most $\omega = x(\Delta + 1)$ such that G is not k-colourable, where $k = \lceil (1-\varepsilon)(\Delta+1)+\varepsilon\omega \rceil$. Choose this graph G with $|V(G)|+|E(G)|$ as small as possible, so that G is k-critical. Note that

$$\begin{aligned} k &= \lceil (1-\varepsilon)(\Delta+1) + \varepsilon x(\Delta+1) \rceil \\ &= \lceil (1-\varepsilon(1-x))(\Delta+1) \rceil. \end{aligned}$$

By Statement 1.2.3, for each $v \in V(G)$,

$$|E(G[N(v)])| \leq (1-\sigma(\varepsilon, x))\binom{\Delta}{2}. \tag{1.2}$$

Now let G' be obtained from G by adding, if necessary, an independent set to the neighbourhood of some vertex in G, so that $\Delta(G') = \Delta$. Then the condition given by (1.2) holds for G', that is, G' is $\sigma(\varepsilon, x)$-sparse. Since

$$\varepsilon(1-x) \leq 0.3012\sigma(\varepsilon, x) - 0.1283\sigma(\varepsilon, x)^{\frac{3}{2}},$$

it now follows by Theorem 1.2.5, that G is k-colourable. □

The bound on $\varepsilon(1-x)$ given in (1.1), which entirely depends on the sparsity and density lemmas, essentially determines the value of ε we can achieve in answering Question 1.2.1. Thus we see that achieving greater levels of sparsity in Statement 1.2.3, or increasing the bound on ε in Theorem 1.2.5, leads to improvement. To prove the main result (that is, a result such as Theorem 1.2.2), the other main ingredient we require is the following theorem of King.

Theorem 1.2.7 ([11]). *If G is a graph with $\omega(G) > \frac{2}{3}(\Delta(G)+1)$, then G has an independent set I such that, for every maximum clique X of G, $I \cap X \neq \emptyset$.*

We now give a brief description of how Statement 1.2.6 and Theorem 1.2.7 combine to prove the main result. For the sake of concreteness, we will specifically consider the proof of Theorem 1.2.2. By iteratively applying Theorem 1.2.7 to a graph G and removing the independent sets, we obtain a graph G' with clique number at most $\frac{2}{3}(\Delta(G')+1)$. Hence we can apply Statement 1.2.6 to G' with $x \in [0, \frac{2}{3}]$, from which it follows that inequality (1.1), with the function $\sigma(\varepsilon, x)$ given by Theorem 1.2.4, is satisfied for $\varepsilon = \frac{1}{26}$. Colouring each of the deleted independent sets a distinct new colour, we are able to obtain our desired colouring. (For a full proof, see [3]).

As mentioned above, Theorem 1.2.2 was improved upon by Delcourt and Postle in [5]. Not only did they obtain a twofold improvement on the value of ε, but their main result holds in the setting of list colouring.

Theorem 1.2.8 ([5])**.** *There exists $\Delta > 0$ such that, for any graph G with $\Delta(G) > \Delta$,*

$$\chi_\ell(G) \leq \left\lceil \frac{12}{13}(\Delta(G)+1) + \frac{1}{13}\omega(G) \right\rceil.$$

The improved upper bound in Theorem 1.2.8 follows from Delcourt and Postle's enhanced density lemma.

Theorem 1.2.9 ([5])**.** *For $\varepsilon, x \in (0,1)$, Statement 1.2.3 holds with*

$$\sigma_1(\varepsilon, x) = \left(1 - \frac{1-x}{2} - \varepsilon\right)(1 - x - 2\varepsilon).$$

While the technique outlined in this section is done mostly in the setting of list colouring, there is one key obstruction to proving the full result for the list chromatic number, namely the application of Theorem 1.2.7. Indeed, when we remove from G an independent set hitting every maximum clique, we may not be able to colour every vertex in this set with the same colour, as the lists of the vertices may have empty intersection. Delcourt and Postle were able to overcome this issue using their improved version of the density lemma, and by modifying their technique for colouring sparse graphs in the special case of very low sparsity [5].

1.3 Deriving the Density Lemma

The focus of this thesis is on improving the density lemma. In this section, we outline how the density lemma is derived in general, and then describe the techniques we will use to seek improvement. While our definition of sparsity in Section 1.2 is in terms of neighbourhoods, we actually derive the density lemma by showing that, if G is L-critical, then *all* of its induced subgraphs have a bounded number of edges. Indeed, the density lemma from [5] is essentially a consequence[1] of a more general result from

[1] This result does not quite imply Theorem 1.2.9, whose proof has minute differences by virtue of the setting in which it is applied. This yields a negligible ε^2-term (Postle, personal communication).

[9], which yields a lower bound on $|E(\overline{H})|$ (where $\overline{H}$ denotes the complement graph) for each induced subgraph H of an L-critical graph G. Before we state this result, we require the following definition, which will be used throughout the thesis.

Definition 2 ([9]). *Let G be a graph with list assignment L. For each $v \in V(G)$, define*

$$\mathrm{Save}_L(v) = d(v) + 1 - |L(v)|.$$

Theorem 1.3.1 ([9]). *Suppose that G is L-critical with respect to a list assignment L, and let H be a non-empty induced subgraph of G. If M is a matching in $\overline{H}$, then*

$$|E(\overline{H})| \geq |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u).$$

The general technique used to prove this theorem (as well as the density lemma in [3]) will be used throughout this thesis, and is not difficult to describe. If H is a non-empty induced subgraph of G, then, by criticality, $G - V(H)$ has an L-colouring ϕ. Let L' be the list-assignment for H given by

$$L'(v) = L(v) \setminus \{\phi(u) : u \in N(v) \setminus V(H)\}.$$

Then H is not L'-colourable (for otherwise there exists an L-colouring of G). So there must exist some vertex $v \in V(H)$ with $|L'(v)|$ small, and therefore having many of its neighbours outside of H. This yields a lower bound on $d_{\overline{H}}(v)$. We then remove v from H and repeat the process.

Clearly, the effectiveness of this technique depends on the upper bound for $|L'(v)|$. But bounding $|L'(v)|$ naturally requires some sufficient condition for list colourability. Namely, if we know that $\chi_\ell(H) \leq k$, then it follows that $|L'(v)| < k$. Prior to Delcourt and Postle's paper [5], the density lemma was derived using the following classical result of Erdős, Rubin and Taylor.

Theorem 1.3.2 ([6]). *If G is a complete k-partite graph in which every part has size at most two, then $\chi_\ell(G) \leq k$.*

In applying this theorem to our induced subgraph H, we are interested in making k as small as possible. To do this, we take a large matching in the complement of H. In particular, if we take a matching M in $\overline{H}$, then it follows that, for some $v \in V(H)$, $|L'(v)| < |V(H)| - |M|$.

Delcourt and Postle achieved a significant improvement to the density lemma by using the above technique together with the following "unbalanced" version of Theorem 1.3.2. Note that, in a complete multipartite graph, we may refer to a part with i vertices as an *i-part*.

Theorem 1.3.3 ([5])**.** *Let G be a complete k-partite graph on n vertices in which every part has size at most two. If G has k_1 1-parts and k_2 2-parts, and L is a list-assignment for G such that*

(a) $|L(v)| \geq k$ *if v is in a 1-part,*

(b) $|L(v)| \geq k_2$ *if v is in a 2-part, and*

(c) $|L(u)| + |L(v)| \geq n$ *if $\{u, v\}$ is a 2-part,*

then G is L-colourable.

Deriving the bound on $|E(\overline{H})|$ (where H is again an induced subgraph of some L-critical graph G) follows as before, but this time there are cases to consider. In particular, one of (a), (b) or (c) is not satisfied for H and L'. So, given some matching M in $\overline{H}$, we use this theorem to derive lower bounds on $d_{\overline{H}}(v)$ for some $v \in V(H)$ or $|\delta_{\overline{H}}(\{u, v\})|$ for some $uv \in M$, where $\delta_{\overline{H}}(\{u, v\})$ denotes the cut in $\overline{H}$ induced by $\{u, v\}$. Then we repeat the procedure for either $H - v$ or $H - \{u, v\}$ respectively. Note that this technique derives its power from the list colouring condition (namely Theorems 1.3.2 and 1.3.3) and the size of the matching M in the complement of the induced subgraph H.

Interestingly, the bound $\chi_\ell(G) \leq \lceil (1-\varepsilon)(\Delta+1)+\varepsilon\omega \rceil$ in [5] was found to become worse as the clique number tends to zero, that is, as the clique number becomes smaller, so does the value of ε for which inequality (1.1) is satisfied. So in order to improve on Theorem 1.2.8, we are interested in the case where $\omega = x(\Delta + 1)$, where x is small. This case offers more structure to be exploited for the density lemma. In particular, we will show in Chapter 3 that, in an induced subgraph H of an L-critical graph G with small clique number, we can find not only a large matching in $\overline{H}$, but a large collection of disjoint triangles in $\overline{H}$. As this forces the vertices of H into a smaller number of colour classes, this holds potential to improve Theorem 1.3.1. The other key ingredient we require is a list colouring condition for complete multipartite graphs with parts of size at most three. For this, we will use the following theorem of Noel, West, Wu and Zhu.

Theorem 1.3.4 ([14]). *For any graph G,*

$$\chi_\ell(G) \leq \max\left\{\chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil\right\}.$$

An outline for the remainder of the thesis is as follows. In Chapter 2 we will take a closer look at list colouring and derive an unbalanced version of Theorem 1.3.4. Then in Chapter 3 we will show that in the case of small clique number, we can find a large collection of disjoint triangles in the complement of our induced subgraph H. We will use this fact to derive two different density lemmas for the case of small clique number. The first follows from Theorem 1.3.4.

Theorem 1.3.5. *Let G be an L-critical graph, where L is a k-list-assignment, with maximum degree at most Δ and clique number at most ω. Let H be an induced subgraph of G, and define*

$$t := \sqrt{(\Delta + 1 - k)^2 + (\Delta + 1 - k)\omega + \frac{\omega^2}{2}}$$

and

$$\ell_1 := |V(H)| - 2(\Delta + 1 - k) - \omega - \frac{t}{2}.$$

Then

$$\begin{aligned} |E(\overline{H})| \geq\ & \frac{5|V(H)|^2}{18} - \frac{|V(H)|(\Delta + 1 - k)}{9} - \frac{8(\Delta + 1 - k)^2}{9} - \frac{2|V(H)|\omega}{9} \\ & - \frac{|V(H)|t}{9} - \frac{5(\Delta + 1 - k)\omega}{9} - \frac{5(\Delta + 1 - k)t}{18} - \frac{5\omega^2}{9} - \frac{5\omega t}{9} \\ & - \frac{5t^2}{36} - \frac{41|V(H)|}{18} + \frac{32\omega}{9} + \frac{25(\Delta + 1 - k)}{9} + \frac{16t}{9} - \frac{14}{3} \\ & - (\Delta + 1 - k)(\ell_1 + 1), \end{aligned}$$

provided that $\ell_0 := |V(H)| - 2(\Delta + 1 - k) - 4\omega - 2t + 6 \geq 0$.

Next we will apply our unbalanced version of Theorem 1.3.4 to derive the following.

Theorem 1.3.6. *Let G be an L-critical graph, and let H be a non-empty induced subgraph of G. If H is a spanning subgraph of a complete k-partite graph with k_i parts of size i for each $i \in \{1, 2, 3\}$ and $k = k_1 + k_2 + k_3$, then*

$$|E(\overline{H})| \geq k_3\left(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\right) + k_2(k_1 + k_2 + 3k_3 - 1) - \sum_{u \in V(H)} \mathrm{Save}_L(u).$$

1.4 Reed's Conjecture

As explained in the previous sections, prior density lemmas were derived using a sufficient condition for list colouring complete multipartite graphs with parts of size at most two, and we have seen that, the stronger this condition is, the more sparsity we can extract. Our aim is to improve the density lemma for the case of small clique number.

Since the list colouring conditions used in previous research are in terms of the number of parts in the complete multipartite graph, it is intuitively desirable for the number of parts in the graph to be small. Hence if we can prove that graphs with small clique number, in addition to having large matchings in their complements, have large collections of disjoint triangles in their complements, then we can derive more sparsity, provided that we have a suitable list colouring condition. Theorem 1.3.4, as well as the unbalanced version we derive in Chapter 2, will play this role.

In this section, we examine the overall effectiveness of our density lemmas, namely Theorems 1.3.5 and 1.3.6, in the setting of the sparsity-density paradigm. In particular, for the sake of comparison to previous work on Reed's conjecture, we derive bounds of the form given by Statement 1.2.3 from these results.

In Chapter 3, we prove that, if H is a non-empty induced subgraph of an L-critical graph G with maximum degree at most Δ and clique number at most $x(\Delta+1)$, where L is a $\lceil (1-\varepsilon(1-x))(\Delta+1)\rceil$-list-assignment, then there exists a collection T of disjoint triangles in $\overline{H}$ with

$$\begin{aligned} |T| \geq \frac{|V(H)|}{3} - \frac{2\varepsilon(1-x)(\Delta+1)}{3} - \frac{x(\Delta+1)}{3} \\ - \frac{2(\Delta+1)}{3}\sqrt{(1-x)^2\varepsilon^2 + x(1-x)\varepsilon + \frac{x^2}{2}} \end{aligned} \tag{1.3}$$

(see Theorem 3.1.3). Note that the expression t in Theorem 1.3.5 comes from the square root term in (1.3). For convenience, we use the following shorthand definition throughout this section: for $\varepsilon, x \in [0,1]$, let

$$\gamma(\varepsilon, x) = \sqrt{(1-x)^2\varepsilon^2 + x(1-x)\varepsilon + \frac{x^2}{2}}.$$

The result we obtain from Theorem 1.3.5 is as follows.

Corollary 1.4.1. *Let $\varepsilon, x \in [0,1]$ such that*

$$1 - 3\varepsilon(1-x) - 4x - 2\gamma(\varepsilon, x) \geq 0$$

is satisfied. There exists $\Delta_3(\varepsilon, x)$ such that, if G is an L-critical graph with maximum degree at most $\Delta > \Delta(\varepsilon, x)$ and clique number at most $x(\Delta+1)$, where L is a $\lceil(1-\varepsilon(1-x))(\Delta+1)\rceil$-list-assignment, then for every $v \in V(G)$,

$$|E(G[N(v)])| \leq (1 - \sigma_2(\varepsilon, x))\binom{\Delta}{2},$$

where

$$\begin{aligned}\sigma_2(\varepsilon, x) = \frac{5}{9} - \frac{20\varepsilon(1-x)}{9} + \frac{17\varepsilon^2(1-x)^2}{9} - \frac{4x}{9} + \frac{11\varepsilon x(1-x)}{18} - \frac{5x^2}{4} \\ - 2\gamma(\varepsilon, x)\left(\frac{1}{9} + \frac{5x}{9} - \frac{2\varepsilon(1-x)}{9}\right).\end{aligned}$$

The proofs of Corollary 1.4.1 and the analogue corresponding to Theorem 1.3.6 are both straightforward but computationally longwinded (mostly because of inequality (1.3)), so we will instead simply outline each.

To prove Corollary 1.4.1, we apply Theorem 1.3.5 with $H = G[N(v)]$ for some vertex $v \in V(G)$, $k = \lceil(1-\varepsilon(1-x))(\Delta+1)\rceil$ and $\omega = x(\Delta+1)$. Note that, since G is critical, $d(v) \geq \lceil(1-\varepsilon(1-x))(\Delta+1)\rceil$, for otherwise we could obtain an L-colouring of G by deleting v. Therefore

$$\begin{aligned}\ell_0 &\geq (1-\varepsilon(1-x))(\Delta+1) + 2\varepsilon(1-x)(\Delta+1) - 4x(\Delta+1) \\ &\quad - 2\gamma(\varepsilon, x)(\Delta+1) + 6 \\ &\geq (\Delta+1)(1 - \varepsilon(1-x) + 2\varepsilon(1-x) - 4x - 2\gamma(\varepsilon, x)) \\ &\geq 0,\end{aligned}$$

by assumption. Then we apply the lower bound we obtain for $|E(\overline{H})|$ and the fact that $|E(\overline{H})| = \binom{d(v)}{2} - |E(H)|$ to obtain an upper bound on $|E(H)|$. The desired bound then follows by applying the bound $d(v) \leq \Delta$ and taking Δ sufficiently large to ignore the linear terms in Δ.

Corresponding to Theorem 1.3.6, we have the following result.

Corollary 1.4.2. *Let $\varepsilon, x \in [0,1]$. There exists $\Delta_4(\varepsilon, x)$ such that, if G is an L-critical graph with maximum degree at most $\Delta > \Delta_4(\varepsilon, x)$ and clique number at* *most $x(\Delta+1)$, where L is a $\lceil(1-\varepsilon(1-x))(\Delta+1)\rceil$-list-assignment, then for every $v \in V(G)$,*

$$|E(G[N(v)])| \leq (1-\sigma_3(\varepsilon,x))\binom{\Delta}{2},$$

where

$$\begin{aligned}\sigma_3(\varepsilon,x) = {} & \frac{14}{27}+\frac{4\varepsilon^2x^2}{27}+\frac{26\varepsilon x}{9}-\frac{22\varepsilon x^2}{27}-\frac{8\varepsilon^2 x}{27}-\frac{56\varepsilon}{27}-\frac{10x}{27}+\frac{4\varepsilon^2}{27}-\frac{x^2}{9}\\ & +\gamma(\varepsilon,x)\left(\frac{4\varepsilon}{27}-\frac{4\varepsilon x}{27}-\frac{2}{27}+\frac{20x}{27}\right).\end{aligned}$$

We obtain this result using a similar technique as before, this time applying Theorem 1.3.6. We let $k_3 = |T|$, where T is a maximum collection of disjoint triangles in $\overline{H}$, and let k_2 be the size of a maximum matching in $\overline{H} - V(T)$ (where $V(T)$ denotes the set of vertices contained in the triangles of T). Using the fact that $k_1 = d(v) - 2k_2 - 3k_3$, the bound from Theorem 1.3.6 yields

$$|E(\overline{H})| \geq k_3\left(d(v)-2k_2-\frac{2k_3}{3}-\frac{8}{3}\right)+k_2(d(v)-k_2-1)-\sum_{u\in V(H)}\mathrm{Save}_L(u).$$

It follows by an easy maximality argument (see Section 3.1) that

$$k_2 \geq \frac{|V(H)\setminus V(T)|-\omega(H)}{2} \geq \frac{(d(v)-3k_3)-x(\Delta+1)}{2}.$$

Applying this to the above bound and simplifying, we have

$$\begin{aligned}|E(\overline{H})| \geq {} & k_3\left(\frac{k_3}{12}-\frac{x(\Delta+1)}{2}\right)+\frac{d(v)^2}{4}-\frac{x^2(\Delta+1)^2}{4}-\sum_{u\in V(H)}\mathrm{Save}_L(u)\\ & -\left(\frac{d(v)-3k_3-x(\Delta+1)}{2}\right)-\frac{8k_3}{3}.\end{aligned} \tag{1.4}$$

Finally, we apply the lower bound on k_3 from (1.3) and perform a similar analysis as before to obtain Corollary 1.4.2. The following table compares Corollaries 1.4.1 and 1.4.2 with Delcourt and Postle's density lemma, Theorem 1.2.9, by showing various values of ε for which inequality (1.1) is satisfied given various values of x.

$\sigma_1(\varepsilon, x)$		$\sigma_2(\varepsilon, x)$		$\sigma_3(\varepsilon, x)$	
x	ε	x	ε	x	ε
0	0.0795	0	0.0825	0	0.0789
0.01	0.0801	0.01	0.0825	0.01	0.0794
0.02	0.0807	0.02	0.0825	0.02	0.0798
0.03	0.0813	0.03	0.0824	0.03	0.0802
0.04	0.0819	0.04	0.0822	0.04	0.0807
0.05	0.0824	0.05	0.0820	0.05	0.0812

Table 1.1: Values of x and ε satisfying (1.1).

We see that $\sigma_2(\varepsilon, x)$ beats $\sigma_1(\varepsilon, x)$ up to $x = 0.05$. One can check that, at $x = 0.045$, $\sigma_1(\varepsilon, x)$ and $\sigma_2(\varepsilon, x)$ both achieve $\varepsilon = 0.0821$ (roughly 1/12.17), yielding marginal progress towards Reed's conjecture.

Finally, we note that, unfortunately, $\sigma_3(\varepsilon, x)$ does not even beat $\sigma_1(\varepsilon, x)$ in the case of small clique number. One can check that, when the clique number is small, it does indeed beat the sparsity level corresponding to Theorem 1.3.1, but only by a small margin. (As noted in the footnote at the beginning of Section 1.3, Theorem 1.2.9 was derived in such a way that a negligible ε^2-term appears, but even this overpowers $\sigma_3(\varepsilon, x)$).

In order to understand the comparison between Theorem 1.3.1 and Corollary 1.4.2, consider the inequality (1.4). Applying Theorem 1.3.1 to $H = G[N(v)]$, we have that

$$|E(\overline{H})| \geq \frac{d(v)^2}{4} - \frac{x^2(\Delta+1)^2}{4} - \sum_{u \in V(H)} \mathrm{Save}_L(u).$$

Hence the term $k_3(\frac{1}{12}k_3 - \frac{1}{2}x(\Delta+1))$ in (1.4) essentially tells us the benefit we derive from having antitriangles in our graph. Given (1.3), one can estimate that this is indeed insignificant.

Chapter 2

Unbalanced List Colouring

Recall that the goal of the density lemma is to prove that if G is L-critical with respect to some list assignment L, then all of its induced subgraphs have a bounded number of edges. Naturally, a crucial step in proving a theorem of this form is to apply some sufficient condition for list-colourability. Recall that we denote the *list chromatic number* for a graph G by $\chi_\ell(G)$, and that for any graph G, $\chi(G) \leq \chi_\ell(G)$. At first glance, one might expect the reverse inequality to hold, however it was observed in both [6] and [18] that this is not true in general. In fact, Erdős, Rubin and Taylor showed that there exist bipartite graphs with arbitrarily large list chromatic number [6]. Alon further proved that the list chromatic number of a graph is bounded below in terms of its average degree (see [1] and [2]).

List colouring was introduced independently by Erdős, Rubin and Taylor [6] and by Vizing [18], and has been a very active research area ever since. Much research has been motivated by the problem of characterizing those graphs G for which $\chi_\ell(G) = \chi(G)$. For instance, the famous *list colouring conjecture*, independently posed by several researchers throughout the 1970s and 80s (see [8]), asserts that this equality holds if G is a line graph. Theorem 1.3.2, which was proved by Erdős, Rubin and Taylor in their original paper, gives one class of graphs for which the two chromatic numbers are equal, namely complete multipartite graphs with parts of size at most two. Over twenty years later, Ohba [15] conjectured the broad generalization that if G is a graph with $|V(G)| \leq 2\chi(G) + 1$, then $\chi_\ell(G) = \chi(G)$. In this way, Ohba suggested that the property of having equal chromatic and list-chromatic numbers extended to all graphs in which the chromatic number was large relative to the number of vertices.

Ohba achieved some partial progress in this direction, showing that $|V(G)| \leq$

$\chi(G) + \sqrt{2\chi(G)}$ was sufficient [15]. Reed and Sudakov [17] later improved this to $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$. The full conjecture was finally proved in [13] by Noel, Reed and Wu.

Theorem 2.0.1 ([13])**.** *For any graph G, if $|V(G)| \leq 2\chi(G)+1$, then $\chi_\ell(G) = \chi(G)$.*

In [14], Theorem 2.0.1 was extended and used to prove Theorem 1.3.4, one of the main tools we use in this thesis. The main result of this chapter is the following unbalanced version of Theorem 1.3.4.

Theorem 2.0.2. *Let G be a complete k-partite graph on n vertices such that every part has size at most three. If G has k_i i-parts for each $i \in \{1, 2, 3\}$, and L is a list assignment for G such that*

(a) $|L(u)| \geq k$ *if u is in a 1-part,*

(b) $|L(u)| \geq k_2 + k_3$ *if u is in a 2-part,*

(c) $|L(u)| \geq \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3$ *if u is in a 3-part,*

(d) $|L(u)| + |L(v)| \geq n$ *if $\{u, v\}$ is a 2-part, and*

(e) $|L(u)| + |L(v)| + |L(w)| \geq 2k_1 + 3k_2 + 4k_3$ *if $\{u, v, w\}$ is a 3-part,*

then G is L-colourable.

Our proof of Theorem 2.0.2 uses the general technique introduced in [14]. The central idea is the notion of *merging* vertices, which is defined as follows.

Definition 3 ([14])**.** *Let G be a graph and let L be a list-assignment for G. To* ***merge*** *non-adjacent vertices u and v in G is to replace them with a single vertex w whose neighbourhood is $N(u) \cup N(v)$, and set $L(w) = L(u) \cap L(v)$.*

Observe that, if we find an L-colouring of G *after* a series of merges, then we have found an L-colouring of G, because only non-adjacent vertices can be merged. We will soon illustrate the usefulness of merging, but we must first introduce the technique of using Hall's Theorem for finding systems of distinct representatives in the setting of list colouring.

Given a collection $\{X_1, ..., X_n\}$ of finite non-empty sets (n a positive integer), we call a collection $\{x_1, ..., x_n\}$ of elements a *system of distinct representatives* (SDR) for $\{X_1, ..., X_n\}$ if $x_i \in X_i$ for each $i \in \{1, ..., n\}$ and $x_i \neq x_j$ for each $i \neq j$. Hall's Theorem characterizes those collections of such sets for which there exists an SDR.

Theorem 2.0.3 (Hall's Theorem, [7]). *Let $\{X_1, ..., X_n\}$ be a collection of finite non-empty sets. Then $\{X_1, ..., X_n\}$ has a system of distinct representatives if and only if for any $S \subseteq \{1, ..., n\}$, $|\bigcup_{i \in S} X_i| \geq |S|$.*

Hall's Theorem has become a standard tool for finding list colourings in graphs, as any SDR for a collection of lists induces a proper list colouring. To illustrate how we might apply Hall's Theorem in actual practice, consider, as an example, the situation of G being a complete k-partite graph with parts of size at most three. For a set S of vertices in G, we first consider the case where S contains, say, a pair of vertices from the same 3-part. Let $L(S)$ denote the union of lists of vertices in S. If we can show that $|L(S)| \geq |S|$ in this case, then we may assume that $|S| \leq k_1 + 2k_2 + k_3$. Continuing to consider various cases, we gradually reduce S until we have shown that Hall's inequality holds in general.

Note that Hall's Theorem can also be viewed in terms of matchings in bipartite graphs. In particular, given a bipartite graph B with bipartition (X, Y), there exists a matching in B covering all the vertices of X if and only if for every $S \subseteq X$, $|N_B(S)| \geq |S|$ (where $N_B(S) = \bigcup_{v \in S} N_B(v)$). We may opt to use this graph theoretic version of Hall's Theorem in attempting to find an L-colouring of a graph G; we simply consider a bipartite graph B with bipartition $(V(G), C)$, where $C = \bigcup_{v \in V(G)} L(v)$ and $vc \in E(B)$ if and only if $c \in L(v)$.

Now, why is the notion of merging useful when applying Hall's Theorem? The short answer is, it gives us more control when trying to bound $|L(S)|$. Again, take as an example the case where S contains two vertices $\{x, y\}$ from an (unmerged) 3-part in G. Then

$$|L(S)| \geq |L(x)| + |L(y)| - |L(x) \cap L(y)|.$$

Now if $|L(x) \cap L(y)|$ is large, then we are at a disadvantage here; but if we choose to merge such pairs having large intersection, then we obtain a better bound in the above case. Moreover, when we consider the possibility of S containing a merged 3-part $\{x, w\}$ (where w is merged and x is not), we have that $|L(S)| \geq |L(x)| + |L(w)|$ (we will prove this later), and we have that $|L(w)|$ is large by construction, which is again to our advantage.

Now let us proceed to prove Theorem 2.0.2. Our first step is to use induction to derive some properties of a minimum counterexample.

A very useful and standard lemma in list colouring is the so-called "Small Pot Lemma," proved in [10] and [17], which states that if $\chi_\ell(G) > k$, then there exists a k-list-assignment L for G such that G is not L-colourable and $|\bigcup_{v \in V(G)} L(v)| < |V(G)|$.

Since our result does not concern the list-chromatic number, but rather an unbalanced sufficient condition for list colouring, we prove the following generalization.

Lemma 2.0.4. *If G is not L-colourable for some list-assignment L, then there exists a list-assignment L' for G such that $|L'(v)| = |L(v)|$ for each $v \in V(G)$, $|\bigcup_{v\in V(G)} L'(v)| < |V(G)|$, and G is not L'-colourable.*

Proof. Let (G, L) be a counterexample on a minimum number of vertices. So G is not L-colourable and no such list-assignment L' exists. We may further choose our minimum counterexample such that $|\bigcup_{v\in V(G)} L(v)|$ is as small as possible.

Let $C = \bigcup_{v\in V(G)} L(v)$. Let B denote the bipartite graph with bipartition $(V(G), C)$, where, for each $v \in V(G)$ and $c \in C$, $vc \in E(B)$ if and only if $c \in L(v)$. Now, since (G, L) is a counterexample, $|C| \geq |V(G)|$. Thus, since G is not L-colourable, there is no matching in B saturating C. Therefore, by Hall's Theorem, there exists $S \subseteq C$ such that $|S| > |N_B(S)|$. Choose such a set S which is minimal with respect to this property. Take $T \subsetneq S$ such that $|T| = |S| - 1$. Then by minimality of S, Hall's condition holds for T. So there exists a matching M in B saturating T. Moreover,

$$|N_B(S)| \geq |N_B(T)| \geq |T| \geq |S| - 1 \geq |N_B(S)|.$$

Hence $|T| = |N_B(T)|$; so the matching M saturates both T and $N_B(T)$.

Observe that, by definition, $L(v) \cap T = \emptyset$ for each $v \in V(G) \setminus N_B(T)$. Thus

$$\big(\cup_{v\in V(G-N_B(T))} L(v)\big) \cap \big(\cup_{v\in N_B(T)} L(v)\big) = \emptyset. \tag{2.1}$$

Therefore $G - N_B(T)$ has no L-colouring; for such an L-colouring together with the bijection defined by M would yield an L-colouring of G by (2.1), a contradiction.

Clearly $|S| \geq 2$, for otherwise $N_B(S) = \emptyset$. Hence $T \neq \emptyset$, and therefore $N_B(T) \neq \emptyset$. By minimality of our counterexample, there exists a list-assignment L' for $G - N_B(T)$ such that $G - N_B(T)$ is not L'-colourable, $|L'(v)| = |L(v)|$ for each $v \in V(G - N_B(T))$, and $|\bigcup_{v\in V(G-N_B(T))} L'(v)| < |V(G)| - |N_B(T)|$.

Claim: $|\bigcup_{v\in V(G-N_B(T))} L(v)| \geq |V(G)| - |N_B(T)|$.

Proof. Recall that, since T and $N_B(T)$ are in bijection, it follows by definition that $|\bigcup_{v\in N_B(T)} L(v)| = |T| = |N_B(T)|$. Hence, by (2.1), we have

$$\begin{aligned}
|\cup_{v\in V(G-N_B(T))} L(v)| &= |(\cup_{v\in V(G)} L(v)) \setminus (\cup_{v\in N_B(T)} L(v))| \\
&= |\cup_{v\in V(G)} L(v)| - |\cup_{v\in N_B(T)} L(v)| \\
&= |\cup_{v\in V(G)} L(v)| - |N_B(T)| \\
&\geq |V(G)| - |N_B(T)|,
\end{aligned}$$

proving the claim. $\square$

It now follows by the claim, that

$$|\cup_{v\in V(G-N_B(T))} L(v)| \geq |V(G)| - |N_B(T)| > |\cup_{v\in V(G-N_B(T))} L'(v)|. \tag{2.2}$$

Also, we may clearly assume that

$$(\cup_{v\in V(G-N_B(T))} L'(v)) \cap (\cup_{v\in N_B(T)} L(v)) = \emptyset.$$

Now define a list-assignment L'' for G by

$$L''(v) = \begin{cases} L(v) & if\ v \in N_B(T) \\ L'(v) & if\ v \in V(G) \setminus N_B(T). \end{cases}$$

Then, because $G - N_B(T)$ is not L'-colourable, G is not L''-colourable. Moreover $|L''(v)| = |L(v)|$ for each $v \in V(G)$. But by construction and equations (2.1) and (2.2), we have that

$$\begin{aligned} |\cup_{v\in V(G)} L''(v)| &= |\cup_{v\in V(G-N_B(T))} L'(v)| + |\cup_{v\in N_B(T)} L(v)| \\ &< |\cup_{v\in V(G-N_B(T))} L(v)| + |\cup_{v\in N_B(T)} L(v)| \\ &= |\cup_{v\in V(G)} L(v)|. \end{aligned}$$

But then, by choice of our minimum counterexample, it now follows that (G, L'') is not a counterexample. So there exists a list-assignment $L^{(3)}$ for G such that G is not $L^{(3)}$-colourable, $|L^{(3)}(v)| = |L''(v)|$ for each $v \in V(G)$, and $|\bigcup_{v\in V(G)} L^{(3)}(v)| < |V(G)|$. But then $|L^{(3)}(v)| = |L(v)|$ for each $v \in V(G)$ by construction, contrary to the assumption the (G, L) is a counterexample. $\square$

For the purpose of dealing with unions of lists when applying Hall's Theorem, we now use induction to prove that 3-parts have empty list intersection in a minimum counterexample.

Proposition 2.0.5. *Let (G, L) be a counterexample to Theorem 2.0.2 on a minimum number of vertices. If A is a 3-part of G, then $\bigcap_{v\in A} L(v) = \emptyset$.*

Proof. Suppose to the contrary that there exists a 3-part A in G such that there exists $c \in \bigcap_{v\in A} L(v)$. Let $G' = G - A$, and let

$$L'(x) = L(x) \setminus \{c\}$$

for each $x \in V(G')$. Now let k_i' denote the number of i-parts in G' for each $i \in \{1,2,3\}$. We show that conditions (a) to (e) from Theorem 2.0.2 hold for G' and L'. Suppose first that u belongs to a 1-part of G'. Then

$$k_1' + k_2' + k_3' = k_1 + k_2 + k_3 - 1 \le |L(u)| - 1 \le |L'(u)|.$$

Now suppose that u belongs to a 2-part of G'. Then

$$k_2' + k_3' = k_2 + k_3 - 1 \le |L(u)| - 1 \le |L'(u)|.$$

If u belongs to a 3-part of G', then

$$\frac{2}{3}k_1' + \frac{2}{3}k_2' + \frac{4}{3}k_3' = \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3 - \frac{4}{3} \le |L(u)| - 1 \le |L'(u)|.$$

So (a), (b) and (c) hold for G' and L'. Now let $\{u, v\}$ be a 2-part of G'. Then

$$k_1' + 2k_2' + 3k_3' = k_1 + 2k_2 + 3k_3 - 3 \le |L(u)| + |L(v)| - 2 \le |L'(u)| + |L'(v)|.$$

Finally, if $\{u, v, w\}$ is a 3-part of G', then

$$\begin{aligned} 2k_1' + 3k_2' + 4k_3' &= 2k_1 + 3k_2 + 4k_3 - 4 \\ &\le |L(u)| + |L(v)| + |L(w)| - 3 \\ &\le |L'(u)| + |L'(v)| + |L'(w)|. \end{aligned}$$

Hence the conditions of Theorem 2.0.2 are satisfied for G' and L'. By minimality of (G, L), there exists an L'-colouring of G'. Colouring the vertices of A with c, we obtain an L-colouring of G. □

We are now in a position to prove Theorem 2.0.2. We shall consider a minimum counterexample (G, L). Throughout, let P_i denote the set of i-parts in G for each $i \in \{1,2,3\}$. Following the notation of [14], for $A \in P_3$, define

$$\ell(A) = \max\{|L(u) \cap L(v)| : \{u, v\} \subseteq A\}.$$

Call a pair $\{u, v\} \subseteq A$ achieving $\ell(A)$ a *maximum pair* in A. For $S \subseteq V(G)$, let $L(S) = \bigcup_{v \in S} L(v)$.

Before we prove Theorem 2.0.2, we require a notion of a "good" merge, that is, a merge obtained from a pair of vertices with sufficiently large list intersection. For

$A \in P_3$, call a pair $\{u,v\} \subseteq A$ *good* if $|L(u) \cap L(v)| \geq \frac{1}{3}k_1 + \frac{1}{3}k_2$. Again following the notation of [14], for each $A \in P_3$ we define

$$L_A = \{c : c \in L(u) \cap L(v) \text{ for some good pair } \{u,v\} \subseteq A\}.$$

As outlined earlier in this chapter, the technique we will use to prove Theorem 2.0.2 largely consists of merging vertices, and then finding an SDR using Hall's Theorem. One of the clever tricks used in [14], which we will also employ here, is to apply Hall's Theorem to a subset of parts in the graph, and *then* merge according to the resulting SDR (see Claim 3 in the proof). This is the main purpose of good pairs.

Before proving Theorem 2.0.2, we will show that every 3-part in our minimum counterexample indeed has a good pair. First we require the following.

Proposition 2.0.6. *Let A be a 3-part in a counterexample (G,L) to Theorem 2.0.2 on a minimum number of vertices. Then*

$$\sum_{\{u,v\} \subseteq A} |L(u) \cap L(v)| \geq k.$$

Proof. Note that, for our minimum counterexample (G,L), we may assume that $|\bigcup_{u \in V(G)} L(u)| \leq n-1$. Thus, by (e) and Proposition 2.0.5,

$$\begin{aligned}\sum_{\{u,v\} \subseteq A} |L(u) \cap L(v)| &= \sum_{u \in A} |L(u)| - |L(A)| \\ &\geq 2k_1 + 3k_2 + 4k_3 - (n-1) \\ &= k_1 + k_2 + k_3,\end{aligned}$$

and the result follows. □

Lemma 2.0.7. *Let (G,L) be a counterexample to Theorem 2.0.2 on a minimum number of vertices. If A is a 3-part of G, then A has a good pair.*

Proof. Suppose that A does not have a good pair. Then

$$\sum_{\{u,v\} \subseteq A} |L(u) \cap L(v)| < 3\Big(\frac{1}{3}k_1 + \frac{1}{3}k_2\Big) \leq k,$$

contrary to Proposition 2.0.6. □

We are now ready to prove the main theorem.

Proof of Theorem 2.0.2. Let (G, L) be a counterexample to Theorem 2.0.2 on a minimum number of vertices. By Lemma 2.0.4, we may assume that $|\bigcup_{v\in V(G)} L(v)| \leq n-1$.

Fix a set $Z_3' \subseteq P_3$ with $|Z_3'| = \lceil \frac{1}{3}k_3 \rceil$. Let t_3 be the largest integer such that there exists $Z_3 \subseteq P_3 \setminus Z_3'$ with the properties that $|Z_3| = t_3 - \lceil \frac{1}{3}k_3 \rceil$ and

$$\ell(A) \geq t_3 - \left\lceil \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil$$

for all $A \in Z_3$. Choose such a set Z_3 with $\sum_{A\in Z_3} \ell(A)$ as large as possible.

Claim 1: For every $A \in (P_3 \setminus Z_3') \setminus Z_3$, $\ell(A) \leq t_3 - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$.

Proof. Suppose to the contrary that there exists $A' \in (P_3 \setminus Z_3') \setminus Z_3$ such that $\ell(A') > t_3 - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$. Suppose first that for every $A \in Z_3, \ell(A) > t_3 - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$. Let $Z_3'' = Z_3 \cup \{A'\}$. Then $Z_3'' \subseteq P_3 \setminus Z_3'$ is a set of size $(t_3 + 1) - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$ such that

$$\ell(A) \geq (t_3 + 1) - \left\lceil \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil$$

for every $A \in Z_3''$, contrary to our choice of t_3.

Suppose now that there exists a set $A \in Z_3$ with $\ell(A) = t_3 - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$. Let $Z_3'' = (Z_3 \setminus \{A\}) \cup \{A'\}$. Then $|Z_3''| = t_3 - \lceil \frac{1}{3}k_3 \rceil$, and $\ell(A) \geq t_3 - \lceil \frac{1}{3}(k_3 - k_1 - k_2) \rceil$ for each $A \in Z_3''$. However,

$$\sum_{A\in Z_3''} \ell(A) > \sum_{A\in Z_3} \ell(A),$$

contrary to our choice of Z_3. $\square$

Now let t_2 be the largest integer such that there exists $Z_2 \subseteq P_2$ with the properties that $|Z_2| = t_2$ and

$$|L(u) \cap L(v)| \geq t_2 + k_3$$

for every $\{u, v\} \in Z_2$. Choose such a set Z_2 with $\sum_{\{u,v\}\in Z_2} |L(u) \cap L(v)|$ as large as possible. The following Claim follows by an argument nearly identical to the proof of Claim 1.

Claim 2: For every $\{u, v\} \in P_2 \setminus Z_2$, $|L(u) \cap L(v)| \leq t_2 + k_3$.

Now for each $A \in P_3 \setminus Z'_3$, merge a maximum pair in A if and only if $A \in Z_3$. Similarly, for every $\{u, v\} \in P_2$, merge u and v if and only if $\{u, v\} \in Z_2$. Let T_3 and T_2 denote the respective resulting sets of merged vertices.

Now we define a merge in each part of Z'_3. For each $A \in Z'_3$, let $B(A)$ denote the set of pairs in A which are not good. Then $|B(A)| \leq 2$ by Lemma 2.0.7. It now follows by Proposition 2.0.6 that for each $A \in Z'_3$,

$$\begin{aligned}
|L_A| &= \sum_{\{u,v\} \subseteq A} |L(u) \cap L(v)| - \sum_{\{u,v\} \in B(A)} |L(u) \cap L(v)| \\
&\geq k - 2(\frac{1}{3}k_1 + \frac{1}{3}k_2) \\
&\geq \frac{1}{3}k_1 + \frac{1}{3}k_2 + k_3.
\end{aligned}$$

Let $X = \{L(w) : w \in T_3\} \cup \{L_A : A \in Z'_3\}$.

Claim 3: X has an SDR.

Proof. We show that Hall's Inequality holds for $S \subseteq X$. Note that $|S| \leq t_3$. For notational convenience, let $L(S)$ denote the union of sets contained in S. If $L_A \in S$ for some $A \in Z'_3$, then $|L(S)| \geq |L_A| \geq k_3 \geq |S|$. So the inequality holds in this case and we may now assume that $|S| \leq t_3 - \lceil \frac{1}{3}k_3 \rceil$. If $L(w) \in S$ for some $w \in T_3$, then

$$|L(S)| \geq |L(w)| \geq t_3 - \left\lceil \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil \geq t_3 - \left\lceil \frac{1}{3}k_3 \right\rceil \geq |S|,$$

and the claim follows. □

Now for each $A \in Z'_3$, merge a pair of vertices in A according to the SDR for X given by the claim. That is, for each $A \in Z'_3$, if c is the colour representing L_A in the SDR, then merge the unique pair $\{u, v\} \subseteq A$ for which $c \in L(u) \cap L(v)$. Together with the merges resulting from Z_3 and Z_2, we have now defined precisely $t_2 + t_3$ merges in G. We now use Hall's Theorem to prove that the collection of lists of vertices (after merges) has an SDR, and thereby obtain an L-colouring of G.

Let S be a set of vertices in G after merges. Then $|S| \leq n - t_2 - t_3$. Suppose first that S contains an unmerged 3-part A. Then by construction and Claim 1,

$$\begin{aligned}
|L(S)| \geq |L(A)| &\geq \sum_{u \in A} |L(u)| - \sum_{\{u,v\} \subseteq A} |L(u) \cap L(v)| \\
&\geq 2k_1 + 3k_2 + 4k_3 - 3\Big(t_3 - \Big\lceil \frac{1}{3}(k_3 - k_1 - k_2) \Big\rceil \Big) \\
&\geq 2k_1 + 3k_2 + 4k_3 - 3t_3 + (k_3 - k_1 - k_2) \\
&= k_1 + 2k_2 + 5k_3 - 3t_3 \\
&\geq k_1 + 2k_2 + 3k_3 - t_3 \\
&\geq |S|.
\end{aligned}$$

So $|S| \leq k_1 + 2k_2 + 2k_3 - t_2$. Now suppose that S contains an unmerged 2-part $\{x, y\}$. Then by construction and Claim 2,

$$\begin{aligned}
|L(S)| &\geq |L(x)| + |L(y)| - |L(x) \cap L(y)| \\
&\geq k_1 + 2k_2 + 3k_3 - (t_2 + k_3) \\
&= k_1 + 2k_2 + 2k_3 - t_2 \\
&\geq |S|.
\end{aligned}$$

Hence $|S| \leq k_1 + k_2 + 2k_3$. Now suppose that S intersects an unmerged 3-part at exactly two points, say $\{x, y\}$. Then, again by Claim 1 and construction,

$$\begin{aligned}
|L(S)| &\geq |L(x)| + |L(y)| - |L(x) \cap L(y)| \\
&\geq 2\Big(\frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3\Big) - \Big(t_3 - \Big\lceil \frac{1}{3}(k_3 - k_1 - k_2) \Big\rceil \Big) \\
&\geq \frac{4}{3}k_1 + \frac{4}{3}k_2 + \frac{8}{3}k_3 - t_3 + \frac{1}{3}(k_3 - k_1 - k_2) \\
&= k_1 + k_2 + 3k_3 - t_3 \\
&\geq |S|.
\end{aligned}$$

So $|S| \leq k_1 + k_2 + k_3 + t_3$. Suppose now that S contains a merged 3-part resulting from Z_3, say $\{x, w\}$, where w is merged and x is not. Then by Proposition 2.0.5 and construction,

$$\begin{aligned}
|L(S)| &\geq |L(x)| + |L(w)| \\
&\geq \left\lceil \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3 \right\rceil + t_3 - \left\lceil \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil \\
&= \left\lceil k + \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil + t_3 - \left\lceil \frac{1}{3}(k_3 - k_1 - k_2) \right\rceil \\
&\geq k + t_3 \\
&\geq |S|.
\end{aligned}$$

Thus $|S| \leq k + \lceil \frac{1}{3}k_3 \rceil$. Now suppose that S contains a merged 3-part $\{x, w\}$ resulting from Z_3', again where w is merged and x is not. Then, since w was obtained through a good merge,

$$\begin{aligned}
|L(S)| &\geq |L(x)| + |L(w)| \\
&\geq \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3 + \frac{1}{3}k_1 + \frac{1}{3}k_2 \\
&= k + \frac{1}{3}k_3.
\end{aligned}$$

So $|L(S)| \geq |S|$ in this case. Thus $|S| \leq k$. If S contains a vertex from a 1-part, then $|L(S)| \geq k \geq |S|$. So $|S| \leq k_2 + k_3$. If S contains a vertex from an unmerged 2-part, then $|L(S)| \geq k_2 + k_3 \geq |S|$. Hence $|S| \leq t_2 + k_3$. Now if S contains a vertex from a merged 2-part, then, by construction, $|L(S)| \geq t_2 + k_3 \geq |S|$. Hence $|S| \leq k_3$, and clearly if S contains an unmerged vertex from a 3-part, then $|L(S)| \geq |S|$. Hence S contains only merged vertices resulting from 3-parts. But the collection of lists for these remaining vertices has an SDR by Claim 3 and construction. So $|L(S)| \geq |S|$ by Hall's Theorem. Therefore, by Hall's Theorem, the collection of lists of vertices after merges has an SDR. So G has an L-colouring. $\square$

Chapter 3

Sparsity and Antitriangles

In this chapter we set out to achieve our main goal: to derive density lemmas tailored to the case of small clique number. Theorems 1.3.4 and 2.0.2 give us list colouring conditions which take into account the existence of triangles in the complement graph (which we may informally refer to as *antitriangles*), so our first task in this chapter is to show that list-critical graphs with small clique number indeed have sizeable collections of such triangles. We may refer to this result as the *antitriangle lemma*. After we have done this, we will derive density lemmas in two different ways. First we will prove Theorem 1.3.5, and then prove Theorem 1.3.6 in the following section.

While the proofs of both density lemmas follow the technique described in Chapter 1, the key difference between them is in whether or not the triangle-matching structure is preserved after each iteration. Recall that, to derive a bound on $|E(\overline{H})|$, we show that H is not colourable with respect to a particular list assignment and we thereby obtain a lower bound on $d_{\overline{H}}(v)$ for some $v \in V(H)$. We then repeat this procedure for $H - v$.

Since we are list-colouring the induced subgraph H with respect to a partitioning of $V(H)$ into 1-, 2- and 3-parts, we may view H as a spanning subgraph of $K_{|V(H)|} - T - M$, where T is a collection of disjoint triangles, and M is a matching in $K_{|V(H)|} - V(T)$. But when we remove the vertex v from H this structure is modified, depending on whether v is in a triangle or a matched edge or otherwise. This is to say, the triangle-matching structure that $H - v$ inherits from H depends on the choice of v.

It is not necessarily the case, however, that $H - v$ must inherit this structure. Indeed, we could take a new collection of triangles and a new matching after each iteration. In fact, this is what we will do to prove Theorem 1.3.5. As antitriangles are advantageous to us, it is in our interest to have as many as possible in each iteration. It is feasible to do this in proving Theorem 1.3.5, because the list colouring condition we will use is Theorem 1.3.4, and consequently the lower bound on $d_{\overline{H}}(v)$ is the same regardless of what part of the structure v comes from. This is not the case, however, in the proof of Theorem 1.3.6, in which we instead apply our unbalanced list colouring condition, Theorem 2.0.2.

If we renew the triangle-matching structure after every iteration when using an unbalanced list colouring condition, then it is possible that, in every iteration, the vertex v for which we obtain a lower bound on $d_{\overline{H}}(v)$ belongs to a "worst case" part of the structure, that is, a part of the structure for which the list colouring condition yields the worst possible lower bound on $d_{\overline{H}}(v)$. So in this setting, it is most sensible to allow $H - v$ to inherit the structure of H. This was also done in [9] to prove Theorem 1.3.1.

3.1 Finding Triangles in the Complement

In order to find a collection of antitriangles, we begin with the following existence condition.

Lemma 3.1.1. *Let H be a graph. If $\overline{H}$ has no triangles, then*

$$|E(\overline{H})| \leq \frac{\omega(H)|V(H)|}{2}.$$

Proof. Suppose that $\overline{H}$ has no triangles. Then for each $u \in V(H)$, $N_{\overline{H}}(u)$ is a clique in H. Hence

$$|E(\overline{H})| = \frac{1}{2} \sum_{u \in V(H)} d_{\overline{H}}(u) \leq \frac{1}{2} \sum_{u \in V(H)} \omega(H) = \frac{\omega(H)|V(H)|}{2},$$

proving the result. □

Since our ultimate objective is to derive a density lemma, we want to find a collection of disjoint antitriangles in an induced subgraph H of an L-critical graph G. In this setting, we can use a known density lemma, namely Theorem 1.3.1, to obtain a lower bound on $|E(\overline{H})|$. Combining this with Lemma 3.1.1, we can prove the existence of an antitriangle in H. Iterating this procedure, we obtain a collection of disjoint antitriangles.

Since Theorem 1.3.1 is in terms of a matching in $\overline{H}$, it is desirable to find a sizeable such matching. The following standard result will be helpful.

Proposition 3.1.2. *If G is a graph, then $\overline{G}$ has a matching of size at least $\lceil \frac{1}{2}(|V(G)| - \omega(G))\rceil$.*

Proof. Let M be a maximal matching in $\overline{G}$. Then $V(\overline{G})\backslash V(M)$ forms an independent set in $\overline{G}$, and therefore forms a clique in G. Hence

$$|V(G)| - 2|M| = |V(\overline{G}) \setminus V(M)| \leq \omega(G),$$

and the result follows. $\square$

Now we are ready to prove the antitriangle lemma.

Lemma 3.1.3 (Antitriangle Lemma)**.** *Let G be an L-critical graph with maximum degree at most Δ and clique number at most ω, where L is a k-list-assignment. Let H be a non-empty induced subgraph of G, and let T be a maximal collection of disjoint triangles in $\overline{H}$. Then*

$$|T| \geq \frac{|V(H)|}{3} - \frac{2(\Delta+1-k)}{3} - \frac{\omega}{3} - \frac{2}{3}\sqrt{(\Delta+1-k)^2 + (\Delta+1-k)\omega + \frac{\omega^2}{2}}.$$

Proof. Since T is maximal, $\overline{H - V(T)}$ is triangle-free. Therefore, by Lemma 3.1.1, we have

$$|E(\overline{H - V(T)})| \leq \frac{\omega(H - V(T))(|V(H)| - 3|T|)}{2} \leq \frac{\omega(H)(|V(H)| - 3|T|)}{2}. \quad (3.1)$$

Let M be a maximum matching in $\overline{H - V(T)}$. By applying Theorem 1.3.1 to $H - V(T)$ and M, we get

$$|M|(|V(H)| - 3|T| - |M|) - \sum_{u \in V(H - V(T))} \mathrm{Save}_L(u) \leq \frac{\omega(H)(|V(H)| - 3|T|)}{2}.$$

Also, by definition, $\mathrm{Save}_L(v) \leq \Delta + 1 - k$ for each $v \in V(G)$. Hence

$$|M|(|V(H)| - 3|T| - |M|) - (\Delta+1-k)(|V(H)| - 3|T|) \leq \frac{\omega(H)(|V(H)| - 3|T|)}{2}. \quad (3.2)$$

Since M is a maximum matching in $\overline{H - V(T)}$, it follows by Proposition 3.1.2, that

$$|M| \geq \frac{|V(H)| - 3|T| - \omega(H - V(T))}{2} \geq \frac{|V(H)| - 3|T| - \omega(H)}{2}.$$

Hence

$$|M|(|V(H)| - 3|T| - |M|) \geq \left(\frac{|V(H)| - 3|T| - \omega(H)}{2}\right)\left(\frac{|V(H)| - 3|T| + \omega(H)}{2}\right).$$

Now applying this bound to (3.2), multiplying both sides by 4 and simplifying, we have

$$\begin{aligned} |V(H)|^2 - 6|T||V(H)| + 9|T|^2 - \omega(H)^2 - 4(\Delta + 1 - k)(|V(H)| - 3|T|) \\ \leq 2\omega(H)(|V(H)| - 3|T|). \end{aligned}$$

Equivalently,

$$\begin{aligned} 9|T|^2 + \big(- 6|V(H)| + 12(\Delta + 1 - k) + 6\omega(H)\big)|T| \\ + |V(H)|^2 - \omega(H)^2 - 4(\Delta + 1 - k)|V(H)| - 2\omega(H)|V(H)| \leq 0. \end{aligned} \tag{3.3}$$

Observe that the lefthand side of (3.3) is a quadratic function in $|T|$. Hence we may apply the quadratic formula to find its roots and thereby obtain a lower bound on $|T|$. Let us denote the lefthand side of (3.3) by $f(|T|)$. Setting $f(|T|) = 0$ and simplifying, we obtain

$$\begin{aligned} |T| = {} & \frac{6|V(H)| - 12(\Delta + 1 - k) - 6\omega(H)}{18} \\ & \pm \frac{1}{18}\sqrt{144(\Delta + 1 - k)^2 + 144\omega(H)(\Delta + 1 - k) + 72\omega(H)^2}. \end{aligned}$$

Hence, since the coefficient of $|T|^2$ in (3.3) is positive, if $f(|T|) \leq 0$, then

$$|T| \geq \frac{|V(H)|}{3} - \frac{2(\Delta + 1 - k)}{3} - \frac{\omega(H)}{3} - \frac{2}{3}\sqrt{(\Delta + 1 - k)^2 + (\Delta + 1 - k)\omega(H) + \frac{\omega(H)^2}{2}},$$

completing the proof. □

3.2 Sparsity via Renewed Structure

In this section we use Theorem 1.3.4 to derive the density lemma Theorem 1.3.5. We will consider an induced subgraph H of an L-critical graph G, and throughout will view H in terms of its triangle-matching structure. For convenience, we define the following functions gleaned from Lemma 3.1.3.

Definition 4. *Let G be an L-critical graph, where L is a k-list-assignment, with maximum degree at most Δ and clique number at most ω, and let H be an induced subgraph of G. We define*

$$f(|V(H)|) = \frac{|V(H)|}{3} - \frac{2(\Delta+1-k)}{3} - \frac{\omega}{3} - \frac{2}{3}t,$$

where

$$t := \sqrt{(\Delta+1-k)^2 + (\Delta+1-k)\omega + \frac{\omega^2}{2}}.$$

Now we adapt Theorem 1.3.4 to the triangle-matching framework as follows.

Lemma 3.2.1. *Let G be an L-critical graph with maximum degree at most Δ and clique number at most ω, where L is a k-list-assignment. Let H be an induced subgraph of G. Then*

$$\chi_\ell(H) \le \max\left\{\frac{|V(H)| - f(|V(H)|) + \omega}{2}, \frac{|V(H)|}{2} - \frac{f(|V(H)|)}{6} + \frac{\omega}{6} + \frac{2}{3}\right\}.$$

Proof. By Lemma 3.1.3, there exists a collection T of disjoint triangles in $\overline{H}$ such that $|T| \ge f(|V(H)|)$. By Proposition 3.1.2, there exists a matching M in $\overline{H - V(T)}$ such that

$$|M| \ge \frac{|V(H)| - 3|T| - \omega(H - V(T))}{2} \ge \frac{|V(H)| - 3|T| - \omega}{2}.$$

Observe that, since

$$\chi(H) \le |T| + |M| + (|V(H)| - 3|T| - 2|M|) = |V(H)| - 2|T| - |M|,$$

it follows by Theorem 1.3.4, that

$$\chi_\ell(H) \le \max\left\{|V(H)| - 2|T| - |M|, \left\lceil \frac{2|V(H)| - 2|T| - |M| - 1}{3} \right\rceil\right\}.$$

Now we apply our bound on $|M|$ and then our bound on $|T|$ to each of these quantities. Observe that

$$|V(H)| - 2|T| - |M| \le |V(H)| - 2|T| - \left(\frac{|V(H)| - 3|T| - \omega}{2}\right) = \frac{|V(H)| - |T| + \omega}{2}.$$

Also

$$\left\lceil \frac{2|V(H)| - 2|T| - |M| - 1}{3} \right\rceil \leq \frac{2|V(H)|}{3} - \frac{2|T|}{3} - \frac{|M|}{3} + \frac{2}{3}$$
$$\leq \frac{2|V(H)|}{3} - \frac{2|T|}{3} - \frac{1}{3}\left(\frac{|V(H)| - 3|T| - \omega}{2}\right) + \frac{2}{3}$$
$$= \frac{|V(H)|}{2} - \frac{|T|}{6} + \frac{\omega}{6} + \frac{2}{3}.$$

The result now follows by applying the bound $|T| \geq f(|V(H)|)$ to each of these bounds. $\square$

Now that we have a condition for list-colouring which accounts for the triangle-matching structure of the graph, we can obtain a lower bound on the degree of some vertex in $\overline{H}$. In order to do this, we require a succinct way of dealing with the maximizer in Lemma 3.2.1. In particular, given parameters as in Lemma 3.2.1, the outcome of the maximizer depends only on $|V(H)|$. Thus we define the following.

Definition 5. *Let G be an L-critical graph with maximum degree at most Δ and clique number at most ω, where L is a k-list-assignment. For an induced subgraph H of G, define*

$$M(H) := \max\left\{\frac{|V(H)| - f(|V(H)|) + \omega}{2}, \frac{|V(H)|}{2} - \frac{f(|V(H)|)}{6} + \frac{\omega}{6} + \frac{2}{3}\right\}.$$

The following lemma will serve as the main driving engine in proving Theorem 1.3.5.

Lemma 3.2.2. *Let G be an L-critical graph with maximum degree at most Δ and clique number at most ω, where L is a k-list-assignment. If H is a non-empty induced subgraph of G, then there exists $a \in V(H)$ such that*

$$d_{\overline{H}}(a) \geq |V(H)| - M(H) - Save_L(a).$$

Proof. Since H is non-empty and G is L-critical, there exists an L-colouring ϕ of $G - V(H)$. For each $v \in V(H)$, let

$$L'(v) = L(v) \setminus \{\phi(u) : u \in N(v) \setminus V(H)\}$$

Now H is not L'-colourable, since otherwise there exists an L-colouring of G. Hence, by Lemma 3.2.1, there exists $a \in V(H)$ such that $|L'(a)| < M(H)$. Now

$$|L'(a)| \geq |L(a)| - (d_G(a) - d_H(a)) = d_H(a) + 1 - Save_L(a).$$

Therefore $d_H(a) \leq M(H) - 1 + Save_L(a)$. Hence

$$\begin{aligned} d_{\overline{H}}(a) &= (|V(H)| - 1) - d_H(a) \\ &\geq |V(H)| - 1 - M(H) + 1 - Save_L(a) \\ &= |V(H)| - M(H) - Save_L(a), \end{aligned}$$

as desired. □

Now that we have obtained a lower bound on $d_{\overline{H}}(a)$ for some vertex a in H, where H is any non-empty induced subgraph of an L-critical graph G, we can derive a lower bound on $|E(\overline{H})|$. We simply apply the lower bound on $d_{\overline{H}}(a)$ given by Lemma 3.2.2, and then re-apply Lemma 3.2.2 to the induced subgraph $H - a$. Repeating this process, we obtain a lower bound on $|E(\overline{H})|$.

Observe that, by virtue of the $f(|V(H)|)$-terms implicit in Lemma 3.2.2, we are lower-bounding $d_{\overline{H}}(a)$ in terms of a maximal collection of antitriangles in the induced subgraph H in each iteration of this procedure. In this sense, we renew the triangle-matching structure in each iteration. The following easy technical lemma provides a framework for this process.

Lemma 3.2.3. *Let G be an L-critical graph with maximum degree at most Δ and clique number at most ω, where L is a k-list-assignment. If H is a non-empty induced subgraph of G, then there exists a sequence of pairs (H_i, a_i), $i \in \{0, ..., |V(H)| - 1\}$, such that*

(a) *H_i is a graph with $a_i \in V(H_i)$,*

(b) *$H_0 = H$ and $H_i = H_{i-1} - a_{i-1}$ for each $i \in \{1, ..., |V(H)| - 1|\}$, and*

(c) *for each $i \in \{0, ..., |V(H)| - 1\}$,*

$$d_{\overline{H_i}}(a_i) \geq (|V(H)| - i) - M(H_i) - Save_L(a_i).$$

Proof. We proceed by induction on $|V(H)|$. If $|V(H)| = \{a\}$ for some $a \in V(G)$, then by Lemma 3.2.2, $d_{\overline{H}}(a) \geq |V(H)| - M(H) - Save_L(a)$. Now the result holds for the sequence consisting only of (H_0, a_0), where $H = H_0$ and $a = a_0$.

Now suppose that $|V(H)| \geq 2$, and that the result holds for all non-empty induced subgraphs of G on fewer than $|V(H)|$ vertices. Again, by Lemma 3.2.2, there exists $a \in V(H)$ such that $d_{\overline{H}}(a) \geq |V(H)| - M(H) - Save_L(a)$. By induction, there exists a sequence of pairs (H_i, a_i), $i \in \{1, ..., |V(H)| - 1\}$, where $H_1 = H - a$, $a_i \in V(H_i)$ for each $i \in \{1, ..., |V(H)| - 1\}$ and $H_i = H_{i-1} - a_{i-1}$ for each $i \in \{2, ..., |V(H)| - 1\}$. Appending the pair (H_0, a_0), where $H_0 = H$ and $a_0 = a$, then gives the desired sequence. $\square$

Let us now prove the main result of this section.

Proof of Theorem 1.3.5. First, let us fix a sequence (H_i, a_i), $i \in \{0, ..., |V(H)| - 1\}$, as given by Lemma 3.2.3. Now the number ℓ_0 gives us a threshold for determining the value of $M(H_i)$ for each $i \in \{0, ..., |V(H)| - 1\}$. In particular,

$$\frac{|V(H_i)|}{2} - \frac{f(|V(H_i)|)}{6} + \frac{\omega}{6} + \frac{2}{3} \geq \frac{|V(H_i)| - f(|V(H_i)|) + \omega}{2}$$

if and only if $i \leq \ell_0$.

By Lemma 3.2.2, for each $i \leq \lfloor \ell_0 \rfloor$, we have

$$\begin{aligned} d_{\overline{H_i}}(a_i) &\geq (|V(H)| - i) - \left(\frac{|V(H_i)|}{2} - \frac{f(|V(H)| - i)}{6} + \frac{\omega}{6} + \frac{2}{3}\right) - Save_L(a_i) \\ &\geq \frac{|V(H)|}{2} - \frac{i}{2} + \frac{1}{6}\left(\frac{|V(H)| - i}{3} - \frac{2(\Delta + 1 - k)}{3} - \frac{\omega}{3} - \frac{2t}{3}\right) - \frac{\omega}{6} \\ &\quad - \frac{2}{3} - (\Delta + 1 - k) \\ &= \frac{10|V(H)|}{18} - \frac{10i}{18} - \frac{2(\Delta + 1 - k)}{18} - \frac{4\omega}{18} - \frac{2t}{18} - \frac{2}{3} - (\Delta + 1 - k). \end{aligned} \tag{3.4}$$

Similarly, for each $i \geq \lfloor \ell_0 \rfloor + 1$,

$$\begin{aligned} d_{\overline{H_i}}(a_i) &\geq (|V(H)| - i) - \left(\frac{|V(H)| - i}{2} - \frac{f(|V(H)| - i)}{2} + \frac{\omega}{2}\right) - Save_L(a_i) \\ &\geq \frac{|V(H)|}{2} - \frac{i}{2} + \frac{1}{2}\left(\frac{|V(H)| - i}{3} - \frac{2(\Delta + 1 - k)}{3} - \frac{\omega}{3} - \frac{2t}{3}\right) \\ &\quad - \frac{\omega}{2} - (\Delta + 1 - k) \\ &\geq \frac{2|V(H)|}{3} - \frac{2i}{3} - \frac{(\Delta + 1 - k)}{3} - \frac{2\omega}{3} - \frac{t}{3} - (\Delta + 1 - k). \end{aligned} \tag{3.5}$$

Now $|E(\overline{H})| = \sum_{i=0}^{|V(H)|-1} d_{\overline{H_i}}(a_i)$. We apply the lower bounds given by (3.4) and (3.5) to obtain our desired result. However, in order to ensure that we do not include negative terms in this sum, we need a stronger upper bound on the indices we include. In particular, observe that the lower bound given by (3.5) is non-negative if and only if $i \leq \ell_1$. Hence

$$\begin{aligned}
|E(\overline{H})| &= \sum_{i=0}^{|V(H)|-1} d_{\overline{H_i}}(a_i) \\
&= \sum_{i=0}^{\lfloor \ell_0 \rfloor} d_{\overline{H_i}}(a_i) + \sum_{i=\lfloor \ell_0 \rfloor+1}^{|V(H)|-1} d_{\overline{H_i}}(a_i) \\
&\geq \sum_{i=0}^{\lfloor \ell_0 \rfloor} \left(\frac{10|V(H)|}{18} - \frac{10i}{18} - \frac{2(\Delta+1-k)}{18} - \frac{4\omega}{18} - \frac{2t}{18} - \frac{2}{3} \right) \\
&\quad + \sum_{i=\lfloor \ell_0 \rfloor+1}^{\lfloor \ell_1 \rfloor} \left(\frac{2|V(H)|}{3} - \frac{2i}{3} - \frac{(\Delta+1-k)}{3} - \frac{2\omega}{3} - \frac{t}{3} \right) \\
&\quad - (\Delta+1-k)(\lfloor \ell_1 \rfloor + 1).
\end{aligned} \tag{3.6}$$

In the lower bound given by (3.6), let us denote the first sum by S_1 and the second by S_2. We now simplify S_1 and S_2 respectively. Observe that

$$\begin{aligned}
S_1 &= \frac{10|V(H)|(\lfloor \ell_0 \rfloor + 1)}{18} - \frac{10}{18}\left(\frac{\lfloor \ell_0 \rfloor(\lfloor \ell_0 \rfloor + 1)}{2} \right) - \frac{2(\Delta+1-k)(\lfloor \ell_0 \rfloor + 1)}{18} \\
&\quad - \frac{4\omega(\lfloor \ell_0 \rfloor + 1)}{18} - \frac{2t(\lfloor \ell_0 \rfloor + 1)}{18} - \frac{2(\lfloor \ell_0 \rfloor + 1)}{3} \\
&\geq \frac{10|V(H)|(\ell_0+1)}{18} - \frac{10|V(H)|}{18} - \frac{10}{18}\left(\frac{\ell_0(\ell_0+1)}{2} \right) - \frac{2(\Delta+1-k)(\ell_0+1)}{18} \\
&\quad - \frac{4\omega(\ell_0+1)}{18} - \frac{2t(\ell_0+1)}{18} - \frac{2(\ell_0+1)}{3} \\
&= (\ell_0+1)\left(\frac{10|V(H)|}{18} - \frac{5\ell_0}{18} - \frac{2(\Delta+1-k)}{18} - \frac{4\omega}{18} - \frac{2t}{18} - \frac{2}{3} \right) - \frac{10|V(H)|}{18}.
\end{aligned}$$

Substituting the value of ℓ_0, expanding and simplifying, we obtain

$$\begin{aligned}S_1 \geq\ & \frac{5|V(H)|^2}{18} - \frac{|V(H)|(\Delta+1-k)}{9} - \frac{8(\Delta+1-k)^2}{9} - \frac{2|V(H)|\omega}{9} - \frac{|V(H)|t}{9} \\ & - \frac{32(\Delta+1-k)\omega}{9} - \frac{16(\Delta+1-k)t}{9} - \frac{32\omega^2}{9} - \frac{32\omega t}{9} - \frac{8t^2}{9} \\ & - \frac{17|V(H)|}{18} + \frac{140\omega}{9} + \frac{70(\Delta+1-k)}{9} + \frac{70t}{9} - \frac{49}{3}.\end{aligned}$$

Now let us consider S_2. Observe that

$$\begin{aligned}S_2 =\ & \frac{2|V(H)|(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} - \frac{2}{3}\sum_{i=\lfloor\ell_0\rfloor+1}^{\lfloor\ell_1\rfloor} i - \frac{(\Delta+1-k)(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} \\ & - \frac{2\omega(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} - \frac{t(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3}.\end{aligned}$$

Now

$$\sum_{i=\lfloor\ell_0\rfloor+1}^{\lfloor\ell_1\rfloor} i = \sum_{i=0}^{\lfloor\ell_1\rfloor-\lfloor\ell_0\rfloor-1} (i+\lfloor\ell_0\rfloor+1) = (\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)\left(\frac{\lfloor\ell_1\rfloor + \lfloor\ell_0\rfloor + 1}{2}\right).$$

Hence

$$\begin{aligned}S_2 =\ & \frac{2|V(H)|(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} - \frac{2(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3}\left(\frac{\lfloor\ell_1\rfloor + \lfloor\ell_0\rfloor + 1}{2}\right) \\ & - \frac{(\Delta+1-k)(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} - \frac{2\omega(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} - \frac{t(\lfloor\ell_1\rfloor - \lfloor\ell_0\rfloor)}{3} \\ \geq\ & \frac{2|V(H)|(\ell_1 - \ell_0 - 1)}{3} - \frac{2(\ell_1 - \ell_0 + 1)}{3}\left(\frac{\ell_1 + \ell_0 + 1}{2}\right) \\ & - \frac{(\Delta+1-k)(\ell_1 - \ell_0 + 1)}{3} - \frac{2\omega(\ell_1 - \ell_0 + 1)}{3} - \frac{t(\ell_1 - \ell_0 + 1)}{3} \\ =\ & (\ell_1 - \ell_0 + 1)\left(\frac{2|V(H)|}{3} - \frac{(\ell_1 + \ell_0 + 1)}{3} - \frac{(\Delta+1-k)}{3} - \frac{2\omega}{3} - \frac{t}{3}\right) - \frac{4|V(H)|}{3}.\end{aligned}$$

Now $\ell_1 - \ell_0 + 1 = 3\omega + \frac{3t}{2} - 5$, and

$$\ell_1 + \ell_0 + 1 = 2|V(H)| - 4(\Delta+1-k) - 5\omega - \frac{5t}{2} + 7.$$

Substituting these values into our lower bound on S_2 and expanding, we obtain

$$S_2 \geq 3(\Delta+1-k)\omega + 3\omega^2 + 3\omega t + \frac{3(\Delta+1-k)t}{2} + \frac{3t^2}{4} - 12\omega - 6t - 5(\Delta+1-k) - \frac{4|V(H)|}{3} + \frac{35}{3}.$$

Substituting the lower bounds on S_1 and S_2 into (3.6) gives the desired result. $\square$

3.3 Sparsity via Inherited Structure

The main result of this section is Theorem 1.3.6, which we prove using Theorem 2.0.2, our unbalanced list colouring condition from Chapter 2. Like in the previous section, the first step is to find cuts in $\overline{H}$ whose sizes can be bounded below. Because we are using an unbalanced list colouring condition, there are several cases to consider. The following lemma yields the various possibilities.

Lemma 3.3.1. *Let G be an L-critical graph, and let H be an non-empty induced subgraph of G. If H is a spanning subgraph of a complete k-partite graph with k_i parts of size i for $i \in \{1,2,3\}$ and $k_1+k_2+k_3 = k$, then either there exists a vertex $v \in V(H)$ such that either*

(a) *v is in a 1-part and $d_{\overline{H}}(v) > k_2 + 2k_3 - \mathrm{Save}_L(v)$;*

(b) *v is in a 2-part and $d_{\overline{H}}(v) > k_1 + k_2 + 2k_3 - \mathrm{Save}_L(v)$;*

(c) *v is in a 3-part and $d_{\overline{H}}(v) > \frac{1}{3}k_1 + \frac{4}{3}k_2 + \frac{5}{3}k_3 - \mathrm{Save}_L(v)$;*

or there exists a 2-part $\{u,v\}$ such that

(d) *$|\delta_{\overline{H}}(\{u,v\})| > k_1 + 2k_2 + 3k_3 - 2 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v)$,*

or there exists a 3-part $\{u,v,w\}$ such that

(e) *$|\delta_{\overline{H}}(\{u,v,w\})| > k_1 + 3k_2 + 5k_3 - 6 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w)$.*

Proof. Since G is L-critical and H is non-empty, there exists an L-colouring ϕ of $G - V(H)$. For each $v \in V(H)$, define

$$L'(v) = L(v) \setminus \{\phi(u) : u \in N(v) \setminus V(H)\}.$$

Then H is not L'-colourable, for otherwise G has an L-colouring. Therefore, by Theorem 2.0.2, either there exists a vertex $v \in V(H)$ such that

- v is in a 1-part and $|L'(v)| < k$;
- v is in a 2-part and $|L'(v)| < k_2 + k_3$;
- v is in a 3-part and $|L'(v)| < \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3$;

or there exists a 2-part $\{u, v\}$ of H such that $|L'(u)| + |L'(v)| < k_1 + 2k_2 + 3k_3$, or there exists a 3-part $\{u, v, w\}$ such that $|L'(u)| + |L'(v)| + |L'(w)| < 2k_1 + 3k_2 + 4k_3$. Before considering each case, note that for each $v \in V(H)$,

$$\begin{aligned} |L'(v)| &\geq |L(v)| - (d_G(v) - d_H(v)) \\ &= d_H(v) + 1 - \mathrm{Save}_L(v). \end{aligned} \tag{3.7}$$

Suppose first that there exists a vertex $v \in V(H)$ such that v is in a 1-part and $|L'(v)| < k$. Then, by (3.7),

$$d_H(v) < k - 1 + \mathrm{Save}_L(v).$$

Hence

$$\begin{aligned} d_{\overline{H}}(v) &= (|V(H)| - 1) - d_H(v) \\ &> (|V(H)| - 1) - (k - 1 + \mathrm{Save}_L(v)) \\ &= k_2 + 2k_3 - \mathrm{Save}_L(v). \end{aligned}$$

So (a) holds in this case. Now suppose that there exists $v \in V(H)$ such that v is in a 2-part and $|L'(v)| < k_2 + k_3$. Again, by (3.7),

$$d_H(v) < k_2 + k_3 - 1 + \mathrm{Save}_L(v).$$

Thus

$$\begin{aligned} d_{\overline{H}}(v) &> (|V(H)| - 1) - (k_2 + k_3 - 1 + \mathrm{Save}_L(v)) \\ &= k_1 + k_2 + 2k_3 - \mathrm{Save}_L(v). \end{aligned}$$

So (b) holds in this case. Now suppose that there exists $v \in V(H)$ such that v is in a 3-part and $|L'(v)| < \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3$. So by (3.7),

$$d_H(v) < \frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3 - 1 + \mathrm{Save}_L(v).$$

Hence

$$\begin{aligned}d_{\overline{H}}(v) &= (|V(H)|-1) - \left(\frac{2}{3}k_1 + \frac{2}{3}k_2 + \frac{4}{3}k_3 - 1 + \mathrm{Save}_L(v)\right)\\ &= \frac{1}{3}k_1 + \frac{4}{3}k_2 + \frac{5}{3}k_3 - \mathrm{Save}_L(v).\end{aligned}$$

So (c) holds in this case. Suppose now that there exists a 2-part $\{u, v\}$ such that

$$|L'(u)| + |L'(v)| < k_1 + 2k_2 + 3k_3.$$

Then, by (3.7),

$$d_H(u) + d_H(v) < k_1 + 2k_2 + 3k_3 - 2 + \mathrm{Save}_L(u) + \mathrm{Save}_L(v).$$

It now follows that

$$\begin{aligned}|\delta_{\overline{H}}(\{u, v\})| &= 2(|V(H)| - 2) - (d_H(u) + d_H(v))\\ &> 2|V(H)| - 4 - k_1 - 2k_2 - 3k_3 + 2 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v)\\ &= k_1 + 2k_2 + 3k_3 - 2 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v).\end{aligned}$$

So (d) holds in this case. Finally, suppose that there exists a 3-part $\{u, v, w\}$ such that

$$|L'(u)| + |L'(v)| + |L'(w)| < 2k_1 + 3k_2 + 4k_3.$$

Then, by (3.7),

$$d_H(u) + d_H(v) + d_H(w) < 2k_1 + 3k_2 + 4k_3 - 3 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w).$$

Therefore

$$\begin{aligned}|\delta_{\overline{H}}(\{u, v, w\})| &= 3(|V(H)| - 3) - d_H(u) - d_H(v) - d_H(w)\\ &> 3|V(H)| - 9 - 2k_1 - 3k_2 - 4k_3 + 3 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w)\\ &= k_1 + 3k_2 + 5k_3 - 6 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w).\end{aligned}$$

So (e) holds in this case, and the result follows. $\square$

Now let us prove the main theorem. Unlike in Section 3.2, here we do not take a new collection of antitriangles after each iteration; rather the structure of the graph can be modified by the choice of vertex.

Proof of Theorem 1.3.6. We proceed by induction on $|V(H)|$. By hypothesis, one of (a), (b), (c), (d) or (e) from Lemma 3.3.1 holds. Suppose first that there exists $v \in V(H)$ such that v is in a 1-part and $d_{\overline{H}}(v) \geq k_2 + 2k_3 + 1 - \mathrm{Save}_L(v)$. Then, by induction,

$$\begin{aligned}
|E(\overline{H})| &= d_{\overline{H}}(v) + |E(\overline{H - v})| \\
&\geq (k_2 + 2k_3 + 1 - \mathrm{Save}_L(v)) + k_3\Big((k_1 - 1) + \frac{7}{3}k_3 - \frac{8}{3}\Big) \\
&\quad + k_2((k_1 - 1) + k_2 + 3k_3 - 1) - \sum_{u \in V(H-v)} \mathrm{Save}_L(u) \\
&\geq k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) + k_2(k_1 + k_2 + 3k_3 - 1) - \sum_{u \in V(H)} \mathrm{Save}_L(u).
\end{aligned}$$

Now suppose that there exists $v \in V(H)$ such that v is in a 2-part and $d_{\overline{H}}(v) \geq k_1 + k_2 + 2k_3 + 1 - \mathrm{Save}_L(v)$. By induction,

$$\begin{aligned}
|E(\overline{H})| &= d_{\overline{H}}(v) + |E(\overline{H - v})| \\
&\geq (k_1 + k_2 + 2k_3 + 1 - \mathrm{Save}_L(v)) + k_3\Big((k_1 + 1) + \frac{7}{3}k_3 - \frac{8}{3}\Big) \\
&\quad + (k_2 - 1)((k_1 + 1) + (k_2 - 1) + 3k_3 - 1) - \sum_{u \in V(H-v)} \mathrm{Save}_L(u) \\
&= k_1 + k_2 + 2k_3 + 1 + k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) + k_3 \\
&\quad + k_2(k_1 + k_2 + 3k_3 - 1) - k_1 - k_2 - 3k_3 + 1 - \sum_{u \in V(H)} \mathrm{Save}_L(u).
\end{aligned}$$

So the result holds in this case. Suppose now that there exists $v \in V(H)$ such that

v is in a 3-part and $d_{\overline{H}}(v) > \frac{1}{3}k_1 + \frac{4}{3}k_2 + \frac{5}{3}k_3 - \mathrm{Save}_L(v)$. By induction,

$$
\begin{aligned}
|E(\overline{H})| &= d_{\overline{H}}(v) + |E(\overline{H-v})| \\
&> \Big(\frac{1}{3}k_1 + \frac{4}{3}k_2 + \frac{5}{3}k_3 - \mathrm{Save}_L(v)\Big) + (k_3 - 1)\Big(k_1 + \frac{7}{3}(k_3 - 1) - \frac{8}{3}\Big) \\
&\quad + (k_2 + 1)(k_1 + (k_2 + 1) + 3(k_3 - 1) - 1) - \sum_{u \in V(H-v)} \mathrm{Save}_L(u) \\
&= \frac{1}{3}k_1 + \frac{4}{3}k_2 + \frac{5}{3}k_3 + k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) - \frac{7}{3}k_3 - k_1 - \frac{7}{3}k_3 + 5 \\
&\quad + k_2(k_1 + k_2 + 3k_3 - 1) - 2k_2 + k_1 + k_2 + 3k_3 - 3 - \sum_{u \in V(H)} \mathrm{Save}_L(u) \\
&\geq k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) + k_2(k_1 + k_2 + 3k_3 - 1) - \sum_{u \in V(H)} \mathrm{Save}_L(u).
\end{aligned}
$$

Now suppose that there exists a 2-part $\{u, v\}$ of H such that

$$
|\delta_{\overline{H}}(\{u, v\})| \geq k_1 + 2k_2 + 3k_3 - 1 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v).
$$

Let $H' = H - \{u, v\}$. Then, by induction,

$$
\begin{aligned}
|E(\overline{H})| &= |\delta_{\overline{H}}(\{u, v\})| + |E(\overline{H'})| \\
&\geq (k_1 + 2k_2 + 3k_3 - 1 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v)) + k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) \\
&\quad + (k_2 - 1)(k_1 + (k_2 - 1) + 3k_3 - 1) - \sum_{u \in V(H')} \mathrm{Save}_L(u) \\
&= k_1 + 2k_2 + 3k_3 - 1 + k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) \\
&\quad + k_2(k_1 + k_2 + 3k_3 - 1) - k_2 - k_1 - k_2 - 3k_3 + 2 - \sum_{u \in V(H)} \mathrm{Save}_L(u).
\end{aligned}
$$

Thus the result holds in this case. Finally, suppose that there exists a 3-part $\{u, v, w\}$ of H such that

$$
|\delta_{\overline{H}}(\{u, v, w\})| \geq k_1 + 3k_2 + 5k_3 - 5 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w).
$$

Again, let $H' = H - \{u, v, w\}$. Then, by induction,

$$\begin{aligned}
|E(\overline{H})| &= |\delta_{\overline{H}}(\{u, v, w\})| + |E(\overline{H'})| \\
&\geq k_1 + 3k_2 + 5k_3 - 5 - \mathrm{Save}_L(u) - \mathrm{Save}_L(v) - \mathrm{Save}_L(w) \\
&\quad + (k_3 - 1)\Big(k_1 + \frac{7}{3}(k_3 - 1) - \frac{8}{3}\Big) + k_2(k_1 + k_2 + 3(k_3 - 1) - 1) \\
&\quad - \sum_{u \in V(H')} \mathrm{Save}_L(u) \\
&= k_1 + 3k_2 + 5k_3 - 5 + k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) - \frac{7}{3}k_3 - k_1 - \frac{7}{3}k_3 + 5 \\
&\quad + k_2(k_1 + k_2 + 3k_3 - 1) - 3k_2 - \sum_{u \in V(H)} \mathrm{Save}_L(u) \\
&\geq k_3\Big(k_1 + \frac{7}{3}k_3 - \frac{8}{3}\Big) + k_2(k_1 + k_2 + 3k_3 - 1) - \sum_{u \in V(H)} \mathrm{Save}_L(u),
\end{aligned}$$

and the result now follows. $\square$

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