Higher Order Random Walks, Local Spectral Expansion, and Applications

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis describes rresults from the following papers, all of which I co-authored:

1. [AJT19]: *Approximating Constraint Satisfaction Problems on High Dimensional Expanders*. Joint work with: Fernando Granha Jeronimo, Madhur Tulsiani. In, FOCS 2019.

2. [AJQ+20]: *List Decoding of Lifted Codes*. Joint work with: Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, Madhur Tulsiani. In, SODA 2020.

3. [AL20]: *Improved Analysis of Higher Order Random Walks*. Joint work with: Lap Chi Lau. In, STOC 2020.

Abstract

The study of spectral expansion of graphs and expander graphs has been an extremely fruitful line of research in Mathematics and Computer Science, with applications ranging from random walks and fast sampling to optimization. In this dissertation, we study *high dimensional local spectral expansion*, which is a generalization of the theory of spectral expansion of graphs, to simplicial complexes.

We study two random walks on simplicial complexes, which we call the *down-up walk*, which captures a wide array of natural random walks which can be used to sample random combinatorial objects via the so-called *heat-bath dynamics*, and the *swap walk*, which can be thought as a random walk on a sparse version of the Kneser graph.

First, we give a sharp bound for the spectral gap of the down-up walks in terms of the *local spectral expansion.* Using this bound, we argue that the natural Markov chains for (i) sampling a random independent of fixed size s of a graph $G = (V, E)$ is rapidly mixing, so long as $s \leq \frac{|V|}{\Delta+\eta}$ – where Δ is the maximum degree of any vertex in G, and η is the magnitude of the least eigenvalue of the adjacency matrix of G; and (ii) sampling a common independent set from two partition matroids of fixed size s is rapidly mixing, so long as $s \leq \frac{r}{3}$ – where r is the maximum size of any common independent set contained in both partition matroids.

Next, we study the spectrum of the swap walks, and show that using local spectral expansion we can relate the spectrum of the swap walk on any simplicial complex to the spectrum of the Kneser graph. We will mention applications of this result in (i) approximating constraint satisfaction problems (CSPs) on instances where the constraint hypergraph is a *high dimensional local spectral expander*; and in (ii) the construction of new families of list decodable codes based on (sparse) Ramanujan complexes of Lubotzky, Samuels, and Vishne.

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I have had the pleasure of attending the summer cluster *High Dimensional Expansion and Error Correcting Codes* in the summer of 2019, organized by Irit Dinur and Prahladh Harsha. I would like to thank Irit and Prahladh for organizing a wonderful summer cluster.

I would like to thank the many faculty members at the *Algorithms and Complexity* and *Combinatorics and Optimization* groups in the University of Waterloo, with whom I have had the pleasure of interacting. My special thanks go to Joseph Cheriyan for always showing a genuine interest in what I have to say and inviting me to the workshop on *Algebraic and Spectral Graph Theory* at the Banff International Research Station in the summer of 2016 and Gautam Kamath for his conversation and his help during the last year of my PhD. I would also like to thank to Eric Blais, Rafael Oliveira, Christopher Godsil, and Izhar Oppenheim for agreeing to be in my thesis committee.

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To my parents Suzan Alev and Samson Levi Alev, and to my grandmother Raşel Surujon, who have always been very supportive of my aspirations, I give the deepest of my thanks. Their unconditional love and support gave me the strength to carry on at the times when I was filled with self doubt.

Finally, I would like to thank the kind and merciful God who has put me in the path I am pursuing, arranged me to meet with all the wonderful people I named above, and saw it fit for me to do all the work that I did. I conclude by invoking the words of the Psalmist, "הדו לה׳ כי טוב. כי לעולם חסדו:".[1]

[1] Psalms 107:1. Praise the Lord for He is good, and his kindness is eternal.

Dedication

This thesis is dedicated to my dearly departed grandparents Hayim Levi Alev, Elvira Alev, and Boni Surujon – may they rest in peace; and to my grandmother Raşel Surujon – may she merit long and happy years.

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Chapter 1

Introduction

1.1 Higher Order Random Walks and Local Spectral Expansion: A Bird's Eye View

1.1.1 Higher Order Random Walks

Consider the following random walks [KM17, DK17, KO18, DDFH18, AJT19, DD19] defined[1] on a simplicial complex X. Initially, the random walk starts from an arbitrary face α_1 of dimension k in X.

- Down-Up Walk: In each step $t \geq 1$, we choose a uniform random element $i \in \alpha_t$ and delete i from α_t, and set α_{t+1} to be a uniform random face of dimension k in X that contains $\alpha_t \setminus \{i\}$. This is called the k-th down-up walk of X, and its transition matrix is denoted by $\mathsf{P}_k^{\triangledown}$.

- Up-Down Walk: In each step $t \geq 1$, we choose a uniform random face β of dimension $k+1$ in X that contains α_t, and choose a uniform random element $i \in \beta$ and set

[1] All the definitions in the introduction will be formally defined again in a more general setting in Chapter 2.

$\alpha_{t+1} = \beta \setminus \{i\}$. This is called the k-th up-down walk of X, and its transition matrix is denoted by $\mathsf{P}_k^{\triangle}$.

- Swap-Walk: In each step $t \geq 1$, we choose a uniform random face β of cardinality $2|\alpha_t|$ in X that contains α_t, and set $\alpha_{t+1} = \beta \setminus \alpha_t$. This is called the k-th swap walk of X, and its transition matrix is denoted by S_k.

The questions that we study in this thesis are the expansion and mixing time of these random walks, i.e. the second eigenvalue $\lambda_2(\mathsf{M})$ where $\mathsf{M} \in \left\{\mathsf{P}_k^{\triangle}, \mathsf{P}_k^{\triangledown}, \mathsf{S}_k\right\}$ and the number of steps t required for the distribution of α_t to be close to the stationary distribution of these random walks, which happens to be the uniform distribution on faces in X of dimension k.

A graph is a simplicial complex of dimension 1. The transition matrix of the lazy random walk on a graph is $\mathsf{P}_0^{\triangle}$. Similarly, for a graph the swap-walk S_0 precisely describes the non-lazy random walk.

Fundamental results in spectral graph theory state the equivalence between following results, (i) the second eigenvalue of a random walk matrix M being small, (ii) the graph described by M approximately behaving like a sparse copy of the complete graph[2], (iii) the random walk described by M rapidly mixing.[3] See [HLW06, WLP09] for surveys on this topic.

Since the theory of expander graphs has many applications, there are various motivations in generalizing these results for graphs to simplicial complexes. Several definitions of high-dimensional expanders have been studied in the literature (e.g. [LM06, Gro10, PRT16, DKW16, KM17, Opp18]), and these results have found interesting applications in discrete geometry, complexity theory, coding theory, and property testing (e.g. [LM06, MW09, FGL$^+$11, KL14, EK16, KM16, KKL16, DK17, DHK$^+$19]).

[2] In particular, the operator norm difference between the random walk matrices of a clique with self-loops and a spectral expander is small.

[3] Formally one needs the second *singular* value being small, but this is not a problem for us since in all the settings where we want to use this equivalence the second singular value will equal the second eigenvalue.

1.1.2 Local Spectral Expansion

In this thesis, we consider the definition of γ-local spectral expansion developed in [KM17, DK17, KO18, Opp18, DDFH18] for studying random walks on simplicial complexes. The local structures of a simplicial complex are described by its links. The link X_α of a face $\alpha \in X$ is defined as the simplicial complex $X_\alpha = \{\beta \setminus \alpha : \beta \in X, \beta \supset \alpha\}$. The graph $G_\alpha = (V_\alpha, E_\alpha)$ of the link X_α is defined as follows: (i) each vertex i in V_α corresponds to a singleton $\{i\}$ in X_α, (ii) two vertices $i, j \in V_\alpha$ have an edge in E_α if and only if $\{i, j\}$ is contained in some face of X_α, (iii) the weight w_{ij} of an edge $ij \in E_\alpha$ is proportional to the number of maximal faces in X_α that contains $\{i, j\}$.

Informally, a simplicial complex X is a γ-local-spectral expander if G_α is an expander graph for every $\alpha \in X$. In the following, we say X is a pure simplicial complex if every maximal face of X is of the same dimension, and we call this the dimension of X.

Definition 1.1.1 (γ-local-spectral expansion [Opp18, KO18])**.** A d-dimensional pure simplicial complex X is a γ-local-spectral expander if $\lambda_2(G_\alpha) \leq \gamma$ for every face $\alpha \in X$ of dimension up to $d-2$, where $\lambda_2(G_\alpha)$ denotes the second largest eigenvalue of the random walk matrix of G_α (where the transition probabilities are proportional to the edge weights).

Similarly, we say that X is a two-sided γ-local spectral expander if $\sigma_2(G_\alpha) \leq \gamma$ for every face α of dimension up to $d-2$, where $\sigma_2(G_\alpha)$ denotes the second largest singular value of the random walk matrix of G_α (where the transition probabilities are proportional to the edge weights).

Below, we will survey some highlights from the study of local spectral expansion that will be closely related to the results presented in this thesis,

Kaufman-Oppenheim Theorem

Kaufman and Oppenheim [KO18] proved that the k-th down-up walk and the $(k-1)$-th up-down walk have a non-trivial spectral gap as long as the simplicial complex is a γ-local-spectral expander for $\gamma < 2/k^2$.

Theorem 1.1.2 ([KO18]). *Let X be a pure d-dimensional simplicial complex. Suppose X is a γ-local-spectral expander. Then, for every $0 \le k \le d$,*

$$\lambda_2(\mathsf{P}_k^{\triangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\vartriangle}) \le 1 - \frac{1}{k+1} + \frac{k\gamma}{2},$$

Theorem 1.1.2 states that the spectral gap of $\mathsf{P}_k^{\triangledown}$ is at least $g := 1 - \lambda_2(\mathsf{P}_k^{\triangledown}) \ge \frac{1}{k+1} - \frac{k\gamma}{2}$, which implies by a standard argument (see Theorem 2.2.7) that the mixing time of these walks is at most $O(\frac{(k+1)\log(n)}{g})$ where n is the size of the ground set of X. For example, if $\gamma \le 0$, then the mixing time of $\mathsf{P}_k^{\triangledown}$ is at most $O(k^2 \log(n))$.

Oppenheim's Trickling Down Theorem

Kaufman-Oppenheim Theorem 1.1.2 provides a way to bound the mixing time of the down-up walks and up-down walks. To apply the theorem, however, one needs to check that $\lambda_2(G_\alpha) \le \gamma$ for every face $\alpha \in X$ of dimension at most $d-2$. This is not an easy task. There are too many graphs G_α to check, and these graphs are defined implicitly where computing the edge weights involve non-trivial counting problems. A very useful result by Oppenheim [Opp18] makes this task easier, by relating the second eigenvalue of the graph of a lower-dimensional link to that of a higher-dimensional link.

Theorem 1.1.3 ([Opp18]). *Let X be a pure d-dimensional simplicial complex. Suppose $\lambda_2(G_\beta) \le \gamma \le \frac{1}{2}$ for every face β of dimension k, and G_α is connected for every face α of dimension $k-1$. Then, for every face α of dimension $k-1$, it holds that*

$$\lambda_2(G_\alpha) \le \frac{\gamma}{1-\gamma}.$$

Applying this theorem inductively, we can reduce the problem of bounding $\lambda_2(G_\alpha)$ for every α to bounding $\lambda_2(G_\beta)$ for only those faces β of highest dimension.

Corollary 1.1.4 ([Opp18]). *Let X be a pure d-dimensional simplicial complex. Suppose $\lambda_2(G_\beta) \le \gamma \le \frac{1}{d}$ for every face β of dimension $d-2$, and G_α is connected for every face α. Then, for every $k \le d-2$, and for every face α of dimension k, it holds that*

$$\lambda_2(G_\alpha) \le \frac{\gamma}{1-(d-2-k)\gamma}.$$

Corollary 1.1.4 is useful for two reasons: First, note that the weight of every edge in G_β for face β of dimension $d-2$ is either zero or one, which makes the task of bounding its second eigenvalue more tractable. Second, if one can prove that $\lambda_2(G_\beta) = O(\frac{1}{d^2})$ for every face β of dimension $d-2$ and G_α is connected for every face α, then one can conclude that $\lambda_2(G_\alpha) = O(\frac{1}{d^2})$ for every face α and hence the simplicial complex is a $O(\frac{1}{d^2})$-local-spectral expander. So, the reduction of Oppenheim is basically lossless in the regime where Kaufman-Oppenheim's Theorem 1.1.2 applies.

Dikstein-Dinur-Filmus-Harsha Theorem

Dikstein, Dinur, Filmus, and Harsha [DDFH18] proved that the k-th up-down walk $\mathsf{P}_k^{\triangle}$ and the k-th down-up walk $\mathsf{P}_k^{\triangledown}$ satisfy a certain approximate commutativity relationship when X is a two-sided γ-local spectral expander. In particular, they proved

Theorem 1.1.5 ([DDFH18]). *Let X be a pure d-dimensional simplicial complex. Suppose X is a two-sided γ-local spectral expander. Then, for $0 \le k \le d-1$ there exists a vectors $\vec{\delta}, \vec{r} \in \mathbb{R}^{[0,d-1]}$ such that,*

$$\left\| \mathsf{P}_k^{\triangle} - r_k \mathsf{P}_k^{\triangledown} - \delta_k \cdot \mathsf{I} \right\|_{\text{op}} \le O(\gamma) \quad \textit{for all} \quad k = 0, \dots, d-1$$

where we have used the notation $\|\bullet\|_{\text{op}}$ for the operator norm of $\bullet$.

For the complete complex $\Delta_{n,d} = \binom{[n]}{\le d+1}$, it is a classical result ([Sta88]) that there actually does exists vectors $\vec{\mu}, \vec{s} \in \mathbb{R}^{[0,d-1]}$, such that

$$\mathsf{P}_k^{\triangle} - s_k \mathsf{P}_k^{\triangledown} - \mu_k \mathsf{I} = 0, \qquad \text{(sequential differentiability)}$$

for all $k = 0, \dots, d-1$. Indeed, the existence of the vector $\vec{\mu}$ satisfying the sequential differentiability criterion can be used to determine the entire spectrum of the operator $\mathsf{P}_k^{\triangle}$ (see, e.g. [GM15]). In this light, Theorem 1.1.5 can be seen as a way of seeing two-sided γ-local spectral expanders as an approximate version of the complete complex $\Delta_{n,d}$, quite similar to the standard results in spectral graph theory that suggest that expander graphs can be seen as approximate versions of the complete graph. Indeed, using this idea, it is shown in [DDFH18] that for small enough γ one can give an *approximate* spectral decomposition of the operator $\mathsf{P}_k^{\triangle}$ on a two-sided γ-local spectral expander X in which both the eigenvalues and the eigenvectors are *approximately* what they would be in the case of the complete complex $\Delta_{n,d}$. This is a particularly interesting connection, since the complex $\Delta_{n,d}$ contains $O(n^d)$ faces and there exists constructions of two-sided γ-local spectral expanders which can have as little as $O_\gamma(n)$ many edges based on the Ramanujan complex constructions of Lubotzky, Samuels, and Vishne ([LSV05, DK17]).

Analyzing Mixing Times of Markov Chains

Recently, Anari, Liu, Oveis Gharan, and Vinzant [ALOV19] found a striking application of Theorem 1.1.2 and Corollary 1.1.4 in proving the matroid expansion conjecture of Mihail and Vazirani [MV87], answering a long standing open question in Markov Chain Monte Carlo methods.

To illustrate their result, consider the special case of sampling a random spanning tree from a graph $G = (V, E)$. Let X be the simplicial complex where the ground set is E and each acyclic subgraph of G is a face of X. Then X is a pure d-dimensional simplicial complex, where $d = |V| - 2$ and the spanning trees of G are the maximal faces of X. Note that $\mathsf{P}_d^\triangledown$ in X is exactly the natural Markov chain on the spanning trees of G, where in each step we delete a uniformly random edge e from the current spanning tree T and add a uniformly random edge f so that $T - e + f$ is a spanning tree. So, the problem of proving the Markov chain on spanning trees is fast mixing is equivalent to upper bounding $\lambda_2(\mathsf{P}_d^\triangledown)$ of the simplicial complex X.

Using the nice structures of matroids, Anari, Liu, Oveis Gharan, and Vinzant [ALOV19] showed that the graph G_β is a complete multi-partite graph for every face β of dimension $d-2$, and this implies that $\lambda_2(G_\alpha) \leq 0$ for every face β of dimension $d-2$. Thus, it follows from Oppenheim's Corollary 1.1.4 that $\lambda_2(G_\alpha) \leq 0$ for every face α.[4] Then Kaufman-Oppenheim's Theorem 1.1.2 implies that $\lambda_2(\mathsf{P}_d^\triangledown) \leq 1 - \frac{1}{d+1}$, and thus the mixing time of the Markov chain of sampling matroid bases is at most $O(d^2 \log n)$. This provides the first FPRAS for counting the number of matroid bases, and also proves that the basis exchange graph of a matroid is an expander graph.

The proof of the matroid expansion conjecture shows that the techniques developed in higher order random walks provide a new simplicial complex approach to analyze mixing times of Markov chains. It is thus natural to investigate whether this approach can be extended to other problems. Here we would like to discuss some limitations of the current techniques. It can be shown that $\lambda_2(G_\beta) \leq 0$ only if G_β is a complete multi-partite

[4] The result that every matroid complex is a 0-local-spectral expander was also proved by Huh and Wang [HW17], using techniques from Hodge theory for matroids [AHK18] instead of Oppenheim's theorem.

graph [God] and more generally a 0-local-expander is a weighted matroid complex [BH19], and so the same analysis as in [ALOV19] only works for matroids. Note that Kaufman-Oppenheim Theorem 1.1.2 only applies when $\lambda_2(G_\alpha) \leq O(\frac{1}{d^2})$ for every face α up to dimension $d-2$. For many problems of natural interest, it does not hold that $\lambda_2(G_\beta) \leq O(\frac{1}{d^2})$ even when restricted to faces β of dimension $d-2$.

Agreement Tests and Coding Theory

In agreement testing, one is given a collection of local functions $\{\boldsymbol{f}_\alpha : \alpha \to \{-1,+1\}\}_{\alpha\in\Omega}$, where each $\alpha \in \Omega$ is a subset of $[n]$ of size k – i.e. $\Omega \subset \binom{[n]}{k}$ – and would like to decide the existence of a global function $\boldsymbol{g} : [n] \to \{-1, 1\}$ that agrees with the majority of local functions $\boldsymbol{f}_\alpha$, i.e.

$$\text{deciding whether there exists some } \boldsymbol{g} \text{ such that } \quad \Pr_\alpha[\boldsymbol{f}_\alpha \equiv \boldsymbol{g}|_\alpha] \geq 1 - \varepsilon.$$

Here the draw of the probability is over the uniform random choice of $\alpha \in \Omega$, $\boldsymbol{g}|_\alpha$[5] is the restriction of $\boldsymbol{g}$ on α, and $\varepsilon \in (0, 1)$ is a very small number. This is a very important task for PCP constructions. In [DK17], using a precursor of Theorem 1.1.2, Dinur and Kaufman showed that two-sided γ-local spectral expanders can be used to design explicit agreement testers. Though, explicit agreement tests were known, the application of [DK17] is significant since by using the *sparse* constructions of Ramanujan complexes of Lubotzky, Samuels, and Vishne [LSV05], it implies the existence of an explicit construction with *size* $O_d(n)$, whereas the previously known constructions [GS00, IKW12] required *size* $O(n^d)$. As a consequence of their result, Dinur and Kaufman already show that one can *boost* the *distance* of an error-correcting code without suffering too much in terms of *rate* by taking the *direct product* of a code-word along the faces of a two-sided local spectral expander, i.e. by re-encoding a codeword $\boldsymbol{w} \in \{-1, 1\}^n$ as

$$\{-1, 1\}^n \ni \boldsymbol{w} \mapsto (\boldsymbol{w}_\alpha)_{\alpha\in\Omega} \in \left(\{-1, +1\}^k\right)^\Omega,$$

where we wrote Ω for the collection of k-dimensional faces of X.

Their results were later improved by Dinur, Harsha, Kaufman, Navon, and Ta-Shma [DHK+19] to show that the code resulting after this re-encoding is actually also *list decodable*.

[5] $\boldsymbol{g}|_\alpha(x) = \boldsymbol{g}(x)$ for all $x \in \alpha$.

1.2 Main Results

Now, we present the main contributions of this thesis.

1.2.1 Up-Down and Down-Up Walks

Our first result, extends the local-spectral-extension approach to analyze the mixing times of more general Markov chains by bounding the second eigenvalue for higher order random walks which initially appeared in [AL20], in joint work with Lap Chi Lau,

Theorem 1.2.1 ([AL20])**.** *Let X be a pure d-dimensional simplicial complex. Define*

$$\gamma_j := \max\{\lambda_2(G_\alpha) : \alpha \in X \textit{ and } \alpha \text{ is of dimension } j\},$$

For any $0 \leq k \leq d$,

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\bigtriangleup}) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j).$$

The following are some remarks about Theorem 1.2.1.

1. A basic result is that a simplicial complex X is gallery connected (i.e. $\lambda_2(\mathsf{P}_d^{\bigtriangledown}) < 1$) if G_α is connected (i.e. $\lambda_2(G_\alpha) < 1$) for every face α of dimension up to $d-2$. Theorem 1.2.1 provides a quantitative generalization of this result.

2. A corollary of Theorem 1.2.1 is that the spectral gap $1 - \lambda_2(\mathsf{P}_k^{\bigtriangledown})$ of the k-th down-up walk is at least $\Omega(1/k)$ if X is a $O(\frac{1}{k})$-local-spectral expander. This is an improvement of Theorem 1.1.2 where it requires the simplicial complex X to be a $O(\frac{1}{k^2})$-local-spectral expander to conclude that $\mathsf{P}_k^{\bigtriangledown}$ has a non-zero spectral gap.

3. It can be shown that the spectral gap $1 - \lambda_2(\mathsf{P}_k^{\bigtriangledown})$ of the k-th down-up walk is at most $O(\frac{1}{k})$ for any simplicial complex (see Proposition 3.1.3), so Theorem 1.2.1 shows that any $O(\frac{1}{k})$-local-spectral expander has the optimal spectral gap for the k-th down-up walk up to a constant factor.

4. The refinement of having a different bound γ_j for links of different dimension is very useful for analyzing Markov chains. We will discuss some applications in Section 3.2.

5. Theorem 1.2.1 can be used to provide a tighter bound on the spectral gap of certain "longer" random walks (see Corollary 1.3.6) which were utilized by [DK17, DHK+19] for their applications in coding theory and agreement testing (see Section 1.3.5).

Combined with Oppenheim's Theorem 1.1.3, Theorem 1.2.1 provides the following bound for the second eigenvalue of higher order random walks in a black box fashion. See Section 3.1 for the proof.

Corollary 1.2.2. *Let X be a pure d-dimensional simplicial complex. For any $0 \leq k \leq d$, suppose $\gamma_{k-2} \leq \frac{1}{k+1}$ and G_α is connected for every face α up to dimension $k-2$, then*

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \leq 1 - \frac{1}{(k+1)^2}.$$

This provides a convenient way to bound the mixing time of Markov chains. Recall that the edge weights in G_β for face β of dimension $d-2$ are either zero or one, and so it is easier to bound their second eigenvalue. Corollary 1.2.2 states that as long as we can prove $\lambda_2(G_\beta) \leq 1/(d+1)$ for these unweighted graphs in the highest dimension, then we can conclude that $\mathsf{P}_d^{\bigtriangledown}$ is fast mixing.

1.2.2 Swap Walks

Our second result shows that for two-sided simplicial complexes, the swap walk matrices S_k have bounded second singular value. This result initially appeared in [AJT19], in joint work with Fernando Granha Jeronimo, and Madhur Tulsiani.

Theorem 1.2.3 ([AJT19])**.** *Let X be a pure d-dimensional simplicial complex. Define,*

$$\gamma := \max\{\sigma_2(G_\alpha) : \alpha \in X \quad \text{and } \alpha \text{ is a face of dimension up to } d-2\}.$$

If $\gamma \leq \epsilon \cdot (64 \cdot d^{d+4} 2^{3d+2})^{-1}$ where $\epsilon \in (0,1)$, then

$$\sigma_2(\mathsf{S}_k) \leq \epsilon.$$

Our proof of this result can be seen as a generalization of the previously mentioned results of [DDFH18] about the approximate spectral decomposition of $\mathsf{P}_k^{\triangle}$ using the approximate version of the sequential differentiability criterion (Theorem 1.1.5). Whereas they show that in their decomposition, the random walk $\mathsf{P}_k^{\triangle}$ over a γ-local spectral expander X has *approximate* eigenvectors and eigenvalues acting *similarly* to what they would be in the case of the complete complex $\Delta_{n,d}$, we show that S_k over a two-sided γ-local spectral expander X has an *approximate* spectral decomposition where the *approximate* eigenvectors and eigenvalues act *similarly* to how they would do on the complete complex $\Delta_{n,d}$. This result can be seen as further proof that the notion of local spectral expansion captures behaving *approximately* like the complete complex, for random walk operator not limited to the up-down walk $\mathsf{P}_k^{\triangle}$.

It is quite important that we can control the second singular value of S_k to be arbitrarily close to 0. Though, the assumption we make on the parameter γ for Theorem 1.2.3 is very demanding, this will not hinder us from using this theorem: In Section 1.3.6, we will discuss several applications of this result in approximating constraint satisfaction problems and in coding theory. In both cases, we will think of the dimension d as being a constant.

This result was improved by Dikstein and Dinur [DD19],

Theorem 1.2.4. *Let X be a pure d-dimensional simplicial complex. Define,*

$$\gamma := \max\{\sigma_2(G_\alpha) : \alpha \in X \quad \textit{and } \alpha \textit{ is a face of dimension up to } d-2\}.$$

If $\gamma \leq \epsilon \cdot k^{-2}$ where $\epsilon \in (0,1)$, then

$$\sigma_2(\mathsf{S}_k) \leq \epsilon.$$

We provide a proof sketch for this result in Appendix A.

1.3 Applications

We present several applications of Theorem 1.2.1 and Corollary 1.2.2, in analyzing mixing times of Markov chains (Section 1.3.1, Section 1.3.2, Section 1.3.3), in analyzing constructions of high-dimensional expanders (Section 1.3.4), and in analyzing longer random walks (Section 1.3.5).

We will also discuss applications of Theorem 1.2.3 in approximating constraint satisfaction problems and coding theory (Section 1.3.6)

1.3.1 Sampling Independent Sets of Fixed Size

One of the most natural simplicial complexes to consider is the independent set complex of a graph [Mes01, AB06]. Let $G = (V, E)$ be a graph. The independent set complex $I_{G,k}$ has the vertex set V as the ground set, and a subset $S \subset V$ is a face in X if and only if S is an independent set in G with $|S| \le k$.

We are interested in bounding $\lambda_2(\mathsf{P}^{\triangledown}_{k-1})$ for this simplicial complex X. The $(k-1)$-th down-up walk corresponds to a natural Markov chain on sampling independent sets of size k. Initially, the random walk starts from an arbitrary independent set S_1 of size k. In each step $t \ge 1$, we choose a uniform random vertex $u \in S_t$ and delete it from S_t, and we choose a uniform random vertex v so that $S_t - u + v$ is still an independent set of size k and set $S_{t+1} := S_t - u + v$. This Markov chain is known to mix in polynomial time for $k \le \frac{|V|}{2\Delta+1}$ where Δ is the maximum degree of G, by using the path coupling technique [BD97, MU05]. In [AL20], we proved the following more refined result,

Theorem 1.3.1. *Let $G = (V, E)$ be a graph with maximum degree Δ. Let $\mathsf{P}^{\triangledown}_{k-1}$ be the $(k-1)$-th down-up walk on the simplicial complex $I_{G,k}$. Let A_G be the adjacency matrix of G.*

$$\text{If} \quad k \le \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}, \quad \text{then} \quad \lambda_2(\mathsf{P}^{\triangledown}_{k-1}) \le 1 - \frac{1}{k^2}.$$

It is well-known that $|\lambda_{\min}(A_G)| \le \Delta$ for a graph with maximum degree Δ, and so Theorem 1.3.1 recovers the previous result that the Markov chain is fast mixing if $k \le \frac{|V|}{2\Delta}$. There are various graph classes with $|\lambda_{\min}(A_G)|$ smaller than Δ, and Theorem 1.3.1 allows us to sample larger independent sets. For example, it is known that $|\lambda_{\min}(A_G)| \le O(\sqrt{\Delta})$ for planar graphs and more generally for graphs with bounded arboricity [Hay06], and also for random graphs and more generally for two-sided expander graphs [HLW06].

1.3.2 Sampling Common Independent Sets in Two Partition Matroids

A matroid $M = (E, \mathcal{I})$ on the ground set E with the set of independent sets $\mathcal{I} \subset 2^E$ is a combinatorial object satisfying the following properties:

- (containment property) if $S \in \mathcal{I}$ and $T \subset S$, then $T \in \mathcal{I}$,

- (extension property) if $S, T \in \mathcal{I}$ such that $|S| > |T|$ then there is some $x \in S \backslash T$ such that $\{x\} \cup T \in \mathcal{I}$.

A partition matroid is the special case where the ground set E is partitioned into disjoint blocks $B_1, \ldots, B_l \subseteq E$ with parameters $0 \leq d_i \leq |B_i|$ for $1 \leq i \leq l$, and a subset S is in $\mathcal{I}$ if and only if $|S \cap B_i| \leq d_i$ for $1 \leq i \leq l$.

The intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I})$ over the same ground set E can be used to formulate various interesting combinatorial optimization problems [Sch03]. We are interested in the problem of sampling a uniform random common independent set of size k, i.e. a random subset $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|F| = k$.

Matroids naturally correspond to simplicial complexes. We let $C_{M_1,M_2,k}$ be the matroid intersection complex with ground set E, where a subset $F \subset E$ is a face in $C_{M_1,M_2,k}$ if and only if $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|F| \leq k$. The $(k-1)$-th down-up walk of this complex corresponds to a natural Markov chain on sampling common independent sets of M_1 and M_2 of size k. We show that this Markov chain is fast mixing for k up to one third the size of a maximum common independent set, when M_1 and M_2 are partition matroids and there are no two elements belonging to the same block in both matroids (i.e. there are no two elements x, y such that x and y are in the same block in M_1 and also in the same block in M_2).

In [AL20], we proved,

Theorem 1.3.2. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two given partition matroids with a common independent set of size r and no two elements belonging to the same block in both matroids. If $k \leq r/3$, then*

$$\lambda_2(\mathsf{P}^{\triangledown}_{k-1}) \leq 1 - \frac{1}{k^2},$$

where $\mathsf{P}^{\triangledown}_{k-1}$ is the $(k-1)$-th down-up walk on the matroid intersection complex $C_{M_1,M_2,k}$.

Sampling a common independent set of two partition matroids can be reduced to sampling a random matching in a bipartite graph. In this regime, stronger results are known to exist [JS89, JSV04] whose analysis is based on the canonical paths method while our analysis provides an alternative approach using a spectral method.

The proof of Theorem 1.3.2 reveals an interesting property of the links of the simplicial complex $C_{M_1,M_2,k}$. For any face β of dimension $k-3$, we show that the graph G_β is the complement of the line graph of a bipartite graph. We note that this holds for any two matroids, not just for partition matroids. By the additional assumptions that the two matroids are partition matroids and there are no two elements in the same block in both matroids, the graph G_β is the line graph of a *simple* bipartite graph. Using the fact that the adjacency matrix of the line graph of a simple graph has minimum eigenvalue at least -2, we prove that $\lambda_2(G_\beta) \leq \frac{1}{k}$ as long as $k \leq \frac{r}{3}$. We can then use Corollary 1.2.2 to conclude Theorem 1.3.2.

1.3.3 Sampling from the Gibbs Distributions of Spin Systems

Recently, Anari, Liu, and Oveis Gharan [ALO20] have used Theorem 1.2.1 to prove a strong result about sampling independent sets from the hardcore distribution. Given a graph $G = (V, E)$ and a parameter $\lambda > 0$, the problem is to sample an independent set S with probability $\frac{\lambda^{|S|}}{Z_G(\lambda)}$ where $Z_G(\lambda) := \sum_{S \subset V : S \text{ independent}} \lambda^{|S|}$ is the partition function. An important work of Weitz [Wei06] gave a deterministic fully polynomial time approximation scheme to estimate $Z_G(\lambda)$ for λ up to the "uniqueness threshold", but the exponent of the runtime depends on the maximum degree Δ of G. It is conjectured that the natural Markov chain for sampling independent sets mixes in polynomial time up to the uniqueness threshold. Anari, Liu, and Oveis Gharan prove this conjecture and obtain a polynomial time algorithm to estimate $Z_G(\lambda)$ up to the uniqueness threshold for any graph (even with unbounded maximum degree). They consider a pure n-dimensional simplicial complex for sampling independent sets, and prove that $\gamma_j = \Theta(\frac{1}{n-j})$ for $0 \leq j \leq n-2$ by using the techniques from *correlation decay*. Then it follows from Theorem 1.2.1 that the Markov chain is fast mixing. Note that it is crucial to have a different bound γ_j for links of different dimension in Theorem 1.2.1, so even when $\gamma_{n-2} = \Theta(1)$ it is still possible to conclude fast mixing.

The Markov chain considered by [ALO20], is indeed a particular case of a very general random walk model on *spin systems*: Let a graph $G = (V, E)$ and a number $q > 1$ be given. One is interested in sampling over assignments $\alpha : V \to [q]$ from the so-called Gibbs distribution Π of the system which assigns measure $\Pi(\alpha)$ to an assignment α based on the interactions between the *spins* $\alpha(u)$ and $\alpha(v)$ for all adjacent vertices $u, v \in V$. This sampling problem is also captured by the random walk $\mathsf{P}^{\triangledown}_{|V|}$ where the random walk corresponds to the so-called *heat bath dynamics*. Very recently, Chen, Liu, and Vigoda [CLV20] extended the results of [ALO20] to show, that for the case where $q = 2$, in the so-called *correlation decay* regime – where spins of vertices far from each other in the graph have little correlation – Theorem 1.2.1 can again be used to show that these more general random walks mix rapidly as well.

Inspired by these results, in independent works Chen, Galanis, Štefankovič, and Vigoda [CGSV20] and Feng, Guo, Yin, and Zhang [FGYZ20] have considered the problem of sampling uniformly random q-colourings of a graph, which is yet another sampling problem that can be tackled by the *heat bath dynamics*. They have shown that in the correlation decay regime – where colours of vertices distant from each other in the graph have little correlation – Theorem 1.2.1 again suffices to show rapid mixing.

1.3.4 Combinatorial Constructions of High Dimensional Expanders

Recently, Liu, Mohanty, and Yang [LMY19] presented an interesting combinatorial construction of a sparse simplicial complex where all higher order random walks have a constant spectral gap. Their construction is by taking a certain tensor product of a graph G on n vertices and a small H-dimensional complete simplicial complex $\Delta_{s,H}$ on s vertices.

Theorem 1.3.3 ([LMY19]). *Let G be a T-regular triangle free graph on n vertices. There is an explicit family $(X^{(s,H,G)})_{H\geq 1, s\geq H+1}$ of simplicial complexes, satisfying the following properties:*

1. *$X^{(s,H,G)}$ is a pure H-dimensional simplicial complex with $\Theta(n)$ maximal faces.*

2. *The spectral gap of the graphs of j dimensional links of the complex $X^{(s,H,G)}$ satisfies*

$$1-\gamma_j \geq \begin{cases} \frac{1}{2} & \text{if } j \in [0, H-2], \\ \left(\frac{1}{2} - \frac{1}{2(T2^H+1)}\right)(1-\sigma_2(G)) & \text{if } j = -1, \end{cases}$$

   where $\sigma_2(G)$ is the second largest eigenvalue of the normalized adjacency matrix of G.

3. *For any $-1 \leq j \leq H-2$,*

$$\lambda_2(\mathsf{P}^{\bigtriangledown}_{j+1}) = \lambda_2(\mathsf{P}^{\triangle}_{j}) \leq 1 - \Omega\left(\frac{1-\sigma_2(G)}{T^2 \cdot j^2 \cdot (s-j) \cdot 2^j}\right),$$

The main technical part of their proof is in establishing Item (3) in Theorem 1.3.3. They use the special structures of their construction and the decomposition technique from [JSTV04] to bound the spectral gap of the higher order random walks. The authors ask the question whether the spectral property in Item (2) alone is enough to prove the fast mixing result in Item (3). Note that Kaufman-Oppenheim's Theorem 1.1.2 does not apply in this regime.

Using Theorem 1.2.1, we answer their question affirmatively, by deriving Item (3) from Item (2) in a black box fashion. This slightly improves their bound and considerably simplifies their analysis.

Corollary 1.3.4 ([AL20]). *Let $X := X^{(s,H,G)}$ be a complex from Theorem 1.3.3 satisfying Item (2). For any $-1 \leq j \leq H-2$,*

$$\lambda_2(\mathsf{P}^{\triangledown}_{j+1}) = \lambda_2(\mathsf{P}^{\triangle}_{j}) \leq 1 - \Omega\left(\frac{1-\sigma_2(G)}{j \cdot 2^j}\right).$$

1.3.5 Longer Random Walks and Other Applications

Consider the following generalization of the up-down walk where we take "longer" steps. Initially, the random walk starts from an arbitrary α_1 face of dimension a in X. In each step $t \geq 1$, we sample a uniformly random face β of dimension $a+h$ that contains α_t, and set α_{t+1} to be a uniformly random subset of β of dimension a. We call this the up-down walk on $X(a)$-th with height h and write $\mathsf{P}^{(h)}_a$.

The k-th up-down walk defined before is the special case $\mathsf{P}^{(1)}_k = \mathsf{P}^{\triangle}_k$. Dinur and Kaufman [DK17] derived the following result about $\mathsf{P}^{\triangle}_{a,b}$ from the result about the ordinary up-down walks.

Corollary 1.3.5 ([DK17]). *Let X be a d-dimensional pure simplicial complex. If X is a γ-local-spectral expander, then for any $0 \leq a < b \leq d-1$,*

$$\lambda_2\big(\mathsf{P}^{(h)}_a\big) \leq \frac{a+1}{a+h+1} + O(ah \cdot \gamma).$$

Using Theorem 1.2.1, in [AL20] we proved the following result. See Section 3.1 for the proof.

Corollary 1.3.6. *Let X be a d-dimensional pure simplicial complex. If X is a γ-local-spectral expander, then for any $a, h \geq 0$ such that $a + h \leq d$ we have,*

$$\lambda_2\big(\mathsf{P}_a^{(h)}\big) \leq (1+\gamma)^h \cdot \frac{a+1}{a+h+1}.$$

In particular, if $\gamma \leq \frac{\varepsilon}{h}$ for some $0 \leq \varepsilon \leq 1$, then $\lambda_2(\mathsf{P}_a^{(h)}) \leq e^{\varepsilon} \cdot \frac{a+1}{a+h+1}$.

Whereas the bound from Corollary 1.3.5 requires $\gamma = O(\frac{1}{(a+h+1)\cdot h})$ to give a nontrivial upper bound on the second eigenvalue of $\mathsf{P}_a^{(h)}$, Corollary 1.3.6 only requires $\gamma \leq O(\frac{1}{h})$ to give a comparable bound.

Corollary 1.3.5 has found applications in agreement testing and coding theory [DK17, DHK$^+$19, AJQ$^+$20]. We believe that Corollary 1.3.6 can be of independent interest because of those applications. One potential application would be in constructing double samplers from Ramanujan complexes under a weaker expansion assumption [DK17].

1.3.6 Constraint Satisfaction Problems and Coding Theory

An important application of the expansion of the swap-walk S_k (Theorem 1.2.3) is in approximating constraint satisfaction problems. In [AJT19], in joint work with Fernando Granha Jeronimo and Madhur Tulsiani, we have shown that a k-CSP instance admits an efficient approximation algorithm based on the Sum-of-Squares hierarchy when the constraint complex is a two-sided local spectral expander. We recall: A k-CSP instance $\mathfrak{I} = (H, \mathcal{C})$ with alphabet size q, consists of a k-uniform constraint hypergraph H and a set of constraints $\mathcal{C} = \{\mathcal{C}_\eta : \eta \in H\}$ for each hyperedge $\eta \in H$ – we will call the simplicial complex X that is obtained from H by taking a downward closure, the constraint complex of $\mathfrak{I}$. The objective is to find an assignment $\boldsymbol{a} : [n] \to [q]$ to the n vertices of H, that satisfy as many of the constraints $\mathcal{C}_\eta$ as possible. It is known by the work of [BRS11, GS11] that 2-CSP instances admit an efficient approximation algorithm when their constraint graph G is an expander graph.

Sparse, explicit, and hard instances of k-CSPs are of great importance in hardness reductions. As local-spectral expanders can be sparse, a hardness result for k-CSPs would have been really exciting. In our work, we have showed that the results for approximating 2-CSPs with expanding constraint graphs, generalize to all k-CSPs (where k is a constant) whose constraint complex is a strong enough two-sided local spectral expander. The expansion of the swap-walks is a crucial ingredient in our proof. This work is also the first result in the literature that makes use of expansion in approximating k-CSPs for $k > 2$. Since high-dimensional expanders can be really sparse [LSV05, DK17], our work also is useful in the *sparse* regime where there are as few as $O(n)$ constraints. This was a limitation of previous work, which only considered dense instances.

In a follow up paper to [AJT19], together with Dylan Quintana and Shashank Srivastava [AJQ+20] we have shown that our approximation algorithm for CSPs can be used for the list decoding of *direct-sum codes* obtained from a linear code $\mathcal{C} \subset \{-1, 1\}^n$ by XOR-ing the coordinates of a codeword $\boldsymbol{w} \in \mathcal{C}$ along the faces of a high-dimensional expander X. This already suggests that an approximation routine for k-XOR is useful decoding the resulting code. It can be argued that the resulting code will have better distance than $\mathcal{C}$. Further using the Ramanujan complexes of Lubotzky, Samuels, and Vishne [LSV05] as X – which are sparse – it is possible to argue that the relative rate of the resulting code will not suffer too much. Most of our techincal work for establishing this result is in showing that it is possible to list-decode direct-sum codes efficiently by using a Sum-of-Squares SDP relaxation. Because, the Sum-of-Squares SDP hierarchy is beyond the scope of this thesis, we do not present these results here.

1.4 Related Work

Our work follows a sequence of works [KM17, DK17, Opp18, KO18, DDFH18] which use the spectral properties of the links of simplical complexes to analyze higher order random walks. Higher order random walks on simplicial complexes were first introduced by Kaufman and Mass [KM17]. They formulated related but more combinatorial notions of skeleton expansion and colorful expansion to establish fast mixing of higher order random walks. Dinur and Kaufman [DK17] introduced the definition of two-sided γ-local-spectral expanders, which is similar to Definition 1.1.1 but requires all but the first eigenvalue to have absolute value at most γ (i.e. it also controls the negative eigenvalues). They used this stronger assumption to prove a similar theorem as in Theorem 1.1.2, and applied it to construct efficient agreement tester with applications to PCP constructions. The one-sided γ-local-expander in Definition 1.1.1 was first studied by Oppenheim [Opp18], where he proved Theorem 1.1.3. Then, Kaufman and Oppenheim [KO18] strengthened the result in [DK17] and prove Theorem 1.1.2.

Dikstein, Dinur, Filmus and Harsha [DDFH18] studied an alternative definition of high dimensional expanders, based on the operator norm of the difference between the (non-lazy) up-down and down-up operators. Using this definition, they show that it is possible to approximately characterize all the eigenvalues and eigenvectors of higher order random walks. Their techniques were used in [AJT19] to analyze the "swap walks" on high dimensional expanders, with applications in designing good approximation algorithms for solving constraint satisfaction problems on high-dimensional expanders. Independently, the same "swap walks" were also studied by [DD19] under the name "complement walks", where applications in agreement testing were given.

The results in higher order random walks have also found applications in coding theory. The double samplers in [DK17] are used in [DHK+19] to design an efficient algorithm to decode direct product codes over high dimensional expanders. The swap walks in [AJT19] were also independently studied by Dikstein and Dinur in [DD19], where they proved an improvement over Theorem 1.2.3 and used this improvement to obtain sparser agreement testers.

Analyzing Mixing Times of Markov Chains

Mixing time of Markov chains is an extensively studied topic with various applications (see e.g. [WLP09, MT05]). There are several well-developed approaches to bound the mixing time of a Markov chain. Perhaps the most widely used approach is the coupling method (e.g. [Ald83, BD97]), which has applications in sampling graph colorings (e.g. [Jer95, Vig00]) and many other problems (see [WLP09]). The canonical path (or more generally multicommodity flow) method developed in [JS89, Sin92, Sin93] was used in the important problem of sampling perfect matchings in bipartite graphs [JS89, JSV04] and other problems including sampling matroid bases [FM92]. Geometric methods are used in the important problem of sampling a random point in a convex body [DFK91, LV06]. Analytical methods such as (modified) log-Sobolev inequalities and Nash inequalities [DSC+96, BT06] are useful in proving sharp bounds on mixing time, e.g. a recent paper [CGM19] used a modified log-Sobolev inequality to prove optimal mixing time of the natural Markov chain on sampling matroid bases.

The simplicial complex approach studied in this thesis is quite different from the above approaches. It is linear algebraic and designed to bound the second eigenvalue directly using ideas from simplicial complexes. On the other hand, the coupling method is probabilistic and designed to compare two random processes, while the canonical path method and the geometric method are designed to bound the underlying expansion of the graph or the geometric object. The analytical methods are more diffcult to apply and are not as widely applicable, but when they work they could be used to prove very sharp results.

1.5 Organization

We discuss the relevant definitions and mathematical foundations that will be used in this thesis in Chapter 2.

Our results about the down-up walk will be presented in Chapter 3. These results include the eigenvalue bounds Theorem 1.2.1, Corollary 1.2.2 for the up-down and down-up walks $\mathsf{P}_k^{\triangle}$, $\mathsf{P}_k^{\triangledown}$; the eigenvalue bound Corollary 1.3.6 for the longer random walks $\mathsf{P}_k^{(h)}$; and the sampling applications Theorem 1.3.1 and Theorem 1.3.2.

Our result about the swap walk (Theorem 1.2.3) will be presented in Chapter 4.

Finally, in Chapter 5 we will discuss some future directions.

Chapter 2

Preliminaries

2.1 Linear Algebra

2.1.1 Vectors and Inner-Products

Throughout this thesis, bold faces will be used for scalar functions/vectors, i.e. $\boldsymbol{f} \in \mathbb{R}^V$. For $i \in V$, the notation $\mathbf{1}_i \in \mathbb{R}^V$ will be reserved for the indicator vector of i, i.e. $\mathbf{1}_i(i) = 1$ and $\mathbf{1}_i(j) = 0$ for all $j \neq i$; and similarly for $S \subseteq V$, we will write $\mathbf{1}_S = \sum_{i \in S} \mathbf{1}_i$ for the indicator vector of S. The notation $\mathbf{1}_V$ will be used for the vector of all ones, when V is clear from context we will simply write $\mathbf{1} \in \mathbb{R}^V$ in place of $\mathbf{1}_V$.

We use $\Pi \in \mathbb{R}^V$ to denote various probability distributions, i.e. $\sum_{x \in V} \Pi(x) = 1$ and $\Pi(x) \geq 0$ for all $x \in V$ – we will adopt the convention that whenever $y \notin V$, $\Pi(y) = 0$.

Given $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$, and a distribution Π on V such that $\Pi(x) > 0$ for all $x \in V$, we use the notations $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi$ and $\|\boldsymbol{f}\|_\Pi$ to denote the inner-product and the norm with respect to the distribution Π, i.e.

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi = \boldsymbol{f}^\top \cdot \mathrm{diag}(\Pi) \cdot \boldsymbol{g} = \sum_{x \in V} \Pi(x) \boldsymbol{f}(x) \boldsymbol{g}(x) \quad \text{and} \quad \|\boldsymbol{f}\|_\Pi^2 = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi, \tag{2.1}$$

where given $\boldsymbol{v} \in \mathbb{R}^V$ the notation $\mathrm{diag}(\boldsymbol{v})$ is used for the diagonal matrix that satisfies $[\mathrm{diag}(\boldsymbol{v})](x,x) = \boldsymbol{v}(x)$ for all $x \in V$ and $[\mathrm{diag}(\boldsymbol{v})](x,y) = 0$ whenever $x \neq y$. We reserve

$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \sum_x \boldsymbol{f}(x)\boldsymbol{g}(x)$ for the standard inner-product. Given $\boldsymbol{f} \in \mathbb{R}^V$, we write $\|\boldsymbol{f}\|_{\ell_1} = \sum_{x \in V} |\boldsymbol{f}(x)|$ for its ℓ_1-norm, and $\|\boldsymbol{f}\|_{\ell_2} = (\sum_{x \in V} \boldsymbol{f}(x)^2)^{\frac{1}{2}}$ for its ℓ_2-norm.

We recall the famous Cauchy-Schwarz inequality, which allows us to bound the inner-product $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi$ of two vectors $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$ by the product of the norms $\|\boldsymbol{f}\|_\Pi$ and $\|\boldsymbol{f}\|_\Pi$.

Fact 2.1.1 (Cauchy-Schwarz Inequality)**.** *Let $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$ be vectors and suppose the vectorspace $\mathbb{R}^V$ is equipped with the inner-product defined by the distribution Π. Then, we have*

$$|\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi| \leq \|\boldsymbol{f}\|_\Pi \cdot \|\boldsymbol{g}\|_\Pi.$$

Proof. The statement is trivial when $\boldsymbol{g} = 0$, thus assume $\boldsymbol{g} \neq 0$. Write,

$$\boldsymbol{h} = \boldsymbol{f} - \frac{\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi} \cdot \boldsymbol{g}.$$

We have,

$$\begin{aligned} 0 &\leq \langle \boldsymbol{h}, \boldsymbol{h} \rangle_\Pi = \sum_x \Pi(x) \cdot \boldsymbol{h}(x)^2, \\ &= \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi + \frac{\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi^2}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi^2} \cdot \langle \boldsymbol{g}, \boldsymbol{g} \rangle^2 - \frac{2\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi} \cdot \langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi, \\ &= \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi - \frac{\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi^2}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi}. \end{aligned}$$

Rewriting the last inequality and taking square roots yields,

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi^2 \leq \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi \cdot \langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi \implies |\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi| \leq \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi} \cdot \sqrt{\langle \boldsymbol{g}, \boldsymbol{g} \rangle_\Pi}.$$

The statement follows since for any vector $\boldsymbol{a} \in \mathbb{R}^V$ one has, $\sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle_\Pi} = \|\boldsymbol{a}\|_\Pi$. $\square$

2.1.2 Matrices and Eigenvalues

In this section, we will recall some results concerning eigenvalues and eigenvectors.

We will use serif faces for matrices, i.e. $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{U \times V}$. The adjoint of the operator $\mathsf{B} \in \mathbb{R}^{U \times V}$, with respect to the inner-products defined by the distributions Π_U and Π_V on U and V, is the operator $\mathsf{B}^* \in \mathbb{R}^{V \times U}$ such that

$$\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{g} \rangle_{\Pi_U} = \langle \mathsf{B}^* \boldsymbol{f}, \boldsymbol{g} \rangle_{\Pi_V} \quad \text{for all } \boldsymbol{f} \in \mathbb{R}^U, \boldsymbol{g} \in \mathbb{R}^V.$$

If $U = V$ and $\Pi_U = \Pi_V$, the operator B is called self-adjoint if $\mathsf{B}^* = \mathsf{B}$. It is well known that the operator $\mathsf{B}^* \in \mathbb{R}^{V \times U}$ is uniquely determined by the choice of $\mathsf{B} \in \mathbb{R}^{U \times V}$ and the inner-products defined by Π_U and Π_V (see e.g. [SC97, p. 318]),

Proposition 2.1.2. *Let $\mathsf{B} \in \mathbb{R}^{U \times V}$ be arbitrary. We write B^* for the adjoint operator to B with respect to the inner-products defined by the distributions Π_U and Π_V. Then,*

$$\mathsf{B}^*(x, y) = \mathsf{B}(y, x) \cdot \frac{\Pi_V(y)}{\Pi_U(x)} \quad \textit{for all} \quad x \in U, y \in V.$$

Proof. We obtain by applying Eq. (2.1) repeatedly,

$$\mathsf{B}^*(x, y) = \mathbf{1}_x^\top \mathsf{B}^* \mathbf{1}_y = \frac{1}{\Pi_U(x)} \cdot \langle \mathbf{1}_x, \mathsf{B}^* \mathbf{1} y \rangle_{\Pi_U} = \frac{1}{\Pi_U(x)} \cdot \langle \mathsf{B}\mathbf{1}_x, \mathbf{1}_y \rangle = \frac{\Pi_V(y)}{\Pi_U(x)} \mathbf{1}_x^\top \mathsf{B}^\top \mathbf{1}_y.$$

The proposition follows by noting $\mathbf{1}_x^\top \mathsf{B}^\top \mathbf{1}_y = \mathsf{B}(y, x)$. □

When the vector spaces $\mathbb{R}^U$ and $\mathbb{R}^V$ are equipped with the counting measure, i.e. when we have the standard inner-products on $\mathbb{R}^U$ and $\mathbb{R}^V$, the adjoint matrix corresponds to the transpose, i.e. $\mathsf{B}^* = \mathsf{B}^\top$. Further, in this case when $\mathsf{B} \in \mathbb{R}^{V \times V}$ is a square matrix, it is easy to see that B is self-adjoint, i.e. $\mathsf{B} = \mathsf{B}^*$, if and only if B is symmetric, i.e. $\mathsf{B}^\top = \mathsf{B}$.

Let $\mathsf{W} \in \mathbb{R}^{V \times V}$ be a square operator. A vector $\boldsymbol{v} \in \mathbb{C}^V \setminus \{0\}$ is called an eigenvector of W if there exists a $\lambda \in \mathbb{C}$ such that, $\mathsf{W}\boldsymbol{v} = \lambda \boldsymbol{v}$. We now recall the spectral theorem (see, e.g. [HJ12])

Theorem 2.1.3 (Spectral Theorem)**.** *Let $\mathsf{W} \in \mathbb{R}^{V \times V}$ be a self-adjoint operator with respect to the inner-product defined by Π. Then, all the eigenvalues $\lambda_1(\mathsf{W}), \ldots, \lambda_{|V|}(\mathsf{W})$ are real. Further, W has an orthonormal collection of real eigenvectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{|V|} \in \mathbb{R}^V$ such that,*

$$\mathsf{W} = \sum_{i=1}^{|V|} \lambda_i(\mathsf{W}) \cdot \boldsymbol{w}_i \boldsymbol{w}_i^* \quad \textit{and} \quad \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_\Pi = 0 \quad \textit{whenever } i \neq j \quad \textit{and} \quad \|\boldsymbol{w}_i\|_\Pi = 1,$$

where $\boldsymbol{w}_i^*(v) = \boldsymbol{w}_i(v) \cdot \Pi(v)$ for all $v \in V$.

For self-adjoint operators W we adopt the convention of taking $\lambda_i(\mathsf{W})$ to be the i-th largest eigenvalue of W, i.e. we have $\lambda_1(\mathsf{W}) \geq \cdots \geq \lambda_{|V|}(\mathsf{W})$. We will also write $\lambda_{\min}(\mathsf{W})$ for the least eigenvalue $\lambda_{|V|}(\mathsf{W})$ of W. We recall the following fundamental theorem from linear algebra (see, e.g. [Bha13]),

Theorem 2.1.4 (Courant-Fischer-Weyl Minimax Principle)**.** *Let* $\mathsf{W} \in \mathbb{R}^{V\times V}$ *be a self-adjoint operator with respect to the measure* Π*. Then,*

$$\lambda_j(\mathsf{W}) = \max_{\substack{\mathcal{X}\subseteq\mathbb{R}^V,\\ \dim\mathcal{X}=j}} \min_{\substack{\boldsymbol{f}\in\mathcal{X},\\ \|\boldsymbol{f}\|_\Pi=1}} \langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle_\Pi$$

where the minimum runs over all subspaces $\mathcal{U}$ *of* $\mathbb{R}^V$ *of dimension* k*. Further, the maximizer* $\mathcal{X}$ *is spanned by the top* j *eigenvectors of* W*, i.e. there exists* $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_j$ *such that*

$$\mathsf{W} = \mathrm{span}\big\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_j\big\}$$

such that $\langle \boldsymbol{f}_k, \boldsymbol{f}_l\rangle_\Pi = 0$ *whenever* $k \neq l$*,* $\|\boldsymbol{f}\|_k = 1$ *for all* k*, and* $\mathsf{W}\boldsymbol{f}_k = \lambda_k(\mathsf{W})\boldsymbol{f}$*.*

Similarly,

$$\lambda_j(\mathsf{W}) = \min_{\substack{\mathcal{Y}\subseteq\mathbb{R}^V,\\ \dim\mathcal{Y}=n-j+1}} \max_{\substack{\boldsymbol{f}\in\mathcal{Y},\\ \|\boldsymbol{f}\|_\Pi=1}} \langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle_\Pi$$

Further, the minimizer $\mathcal{Y}$ *is the subspace that is orthogonal to the top* $j-1$ *eigenvectors of* W*, i.e. there exists* $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{j-1}$ *such that*

$$\mathcal{Y} = \mathrm{span}\big\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{j-1}\big\}^\perp = \{\boldsymbol{g} \in \mathbb{R}^V : \langle \boldsymbol{g}, \boldsymbol{f}_k\rangle_\Pi = 0 \textit{ for all } k \in [j-1]\}$$

and $\langle \boldsymbol{f}_k, \boldsymbol{f}_l\rangle_\Pi = 0$ *whenever* $k \neq l$*,* $\|\boldsymbol{f}\|_k = 1$ *for all* k*, and* $\mathsf{W}\boldsymbol{f}_k = \lambda_k(\mathsf{W})\boldsymbol{f}$*.*

A self-adjoint operator $\mathsf{A} \in \mathbb{R}^{V\times V}$ with respect to inner-product defined by Π is called positive semi-definite, denoted by $\mathsf{A} \succeq_\Pi 0$, if it satisfies $\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle_\Pi \geq 0$ for all $\boldsymbol{f} \in \mathbb{R}^V$. This condition is equivalent to the condition that $\lambda_{\min}(\mathsf{A}) \geq 0$. For self-adjoint operators $\mathsf{A} \in \mathbb{R}^{V\times V}$ and $\mathsf{B} \in \mathbb{R}^{V\times V}$ with respect to the same inner-product defined by Π, we will write $\mathsf{A} \preceq_\Pi \mathsf{B}$ if

$$\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle_\Pi \leq \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle_\Pi \quad \text{for all} \quad \boldsymbol{f} \in \mathbb{R}^V.$$

This is equivalent to $\mathsf{A} - \mathsf{B}$ being positive-semidefinite, i.e. $\mathsf{A} - \mathsf{B} \succeq_\Pi 0$. If Π is just the standard inner-product, we will drop the subscript Π.

Given an operator $\mathsf{B} \in \mathbb{R}^{V\times U}$ we will write $\sigma_i(\mathsf{B})$ for the i-th largest singular of B, i.e. we have $\sigma_i(\mathsf{B}) = \sqrt{\lambda_i(\mathsf{B}^*\mathsf{B})}$. This expression is well-defined since $\mathsf{B}^*\mathsf{B}$ is positive-semi definite,

$$\langle \boldsymbol{f}, \mathsf{B}^*\mathsf{B}\boldsymbol{f}\rangle_{\Pi_U} = \langle \mathsf{B}\boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle_{\Pi_V} = \|\mathsf{B}\boldsymbol{f}\|^2_{\Pi_V} \geq 0,$$

and therefore $\lambda_i(\mathsf{B}^*\mathsf{B}) \geq 0$. If $\mathsf{B} \in \mathbb{R}^{V\times V}$ is a self-adjoint square operator, by the Spectral Theorem 2.1.3 we can pick an orthonormal basis of eigenvectors $\{\boldsymbol{w}\}_{i=1}^{|V|}$ of B that satisfy,

$$\mathsf{B}^*\mathsf{B} = \sum_{i,j=1}^{|V|} \lambda_i(\mathsf{B})\lambda_j(\mathsf{B})\langle \boldsymbol{w}_i, \boldsymbol{w}_j\rangle \boldsymbol{w}_i\boldsymbol{w}_j^* = \sum_{j=1}^{|V|} \lambda_j(\mathsf{B})^2 \boldsymbol{w}_i\boldsymbol{w}_i^*,$$

where we have used that $\boldsymbol{w}_i^*\boldsymbol{w}_j = \langle \boldsymbol{w}_i, \boldsymbol{w}_j\rangle_\Pi = 0$ whenever $i \neq j$ and $\langle \boldsymbol{w}_i, \boldsymbol{w}_i\rangle_\Pi = 1$. Using this, observation we obtain the following simple corollary from Courant-Fischer-Weyl Theorem 2.1.4

Corollary 2.1.5. *Let $\mathsf{B} \in \mathbb{R}^{V\times V}$ be a self-adjoint operator with respect to the inner-product defined by the distribution Π. There is a bijective mapping between the eigenvalues of $\lambda_i(\mathsf{B})$ of B and the eigenvalues $\lambda_j(\mathsf{B}^*\mathsf{B})$ of $\mathsf{B}^*\mathsf{B}$, where*

$$\lambda_j(\mathsf{B}^*\mathsf{B}) = j\text{-th largest of all } |\lambda_i(\mathsf{B})|^2 \text{ for all } i = 1, \ldots, |V|.$$

In particular, since $\sigma_j(\mathsf{B}) = \sqrt{\lambda_j(\mathsf{B}^*\mathsf{B})}$ we have,

$$\sigma_j(\mathsf{B}) = j\text{-th largest of all } |\lambda_i(\mathsf{B})| \text{ for all } i = 1, \ldots, |V|.$$

We will use the following results about eigenvalues in Chapter 3; see e.g. [Bha13].

Fact 2.1.6. *Let $\mathsf{A} \in \mathbb{R}^{U\times V}$ and let A^* be the adjoint of A with respect to the inner-products on Π_U on $\mathbb{R}^U$ and Π_V on $\mathbb{R}^V$. Then, the non-zero spectrum of $\mathsf{A}\mathsf{A}^*$ coincides with that of $\mathsf{B}\mathsf{A}$ with the same multiplicity.*

Proof. Let $\boldsymbol{f} \in \mathbb{C}^U$ be an eigenvector of $\mathsf{A}\mathsf{B}$ corresponding to the eigenvalue $\lambda \neq 0$, i.e. $\mathsf{A}\mathsf{B}\boldsymbol{f} = \lambda\boldsymbol{f}$. Then, we note that $\mathsf{B}\boldsymbol{f} \in \mathbb{C}^V$ is an eigenvector of $\mathsf{B}\mathsf{A}$ of eigenvalue λ, since

$$\mathsf{B}\mathsf{A}(\mathsf{B}\boldsymbol{f}) = \mathsf{B}(\mathsf{A}\mathsf{B}\boldsymbol{f}) = \mathsf{B}(\lambda\boldsymbol{f}) = \lambda\mathsf{B}\boldsymbol{f}.$$

As we can run the same argument for eigenpairs $(\lambda, \boldsymbol{g})$ of AB to show that $(\lambda, \mathsf{A}\boldsymbol{g})$ is an eigenpair of BA, we know that the matrices AB and BA are going to have the same set of non-zero eigenvalues.

Now, we show that (geometric) multiplicities are preserved. Let $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r \in \mathbb{R}^U$ be an orthogonal collection of eigenvectors of AB corresponding to the same eigenvalue $\lambda \in \mathbb{C}\backslash\{0\}$, i.e.

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle = 0 \quad \text{and} \quad (\mathsf{AB})\boldsymbol{f}_j = \lambda \boldsymbol{f}_j \quad \text{for all } i, j = 1, \ldots, r \text{ where } i \neq j.$$

We want to show that $\boldsymbol{g}_j = \mathsf{B}\boldsymbol{f}_j$ for $j = 1, \ldots, r$ also constitute an orthogonal collection of eigenvectors of BA corresponding to the eigenvalue λ. To this end, we note $\square$

Remark 2.1.7. In the proof above we have used geometric multiplicities, i.e. the number of linearly independent eigenvectors corresponding to an eigenvalue. The same statement also holds for the algebraic multiplicities, i.e. the multiplicity of λ as a root of the characteristic polynomial. As we will only use this fact when $\mathsf{B} = \mathsf{A}^*$, we will only be dealing with self-adjoint matrices AB and BA, for which both notions of multiplicity coincide. Thus, for simplicity we omit the proof for algebraic multiplicities.

Fact 2.1.8. *Let $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{V \times V}$ be two self-adjoint matrices with respect to the inner-product defined by Π satisfying $\mathsf{A} \preceq_\Pi \mathsf{B}$. Then, $\lambda_i(\mathsf{A}) \leq \lambda_i(\mathsf{B})$ for all $1 \leq i \leq |V|$.*

Proof. Let $i \in [1, |V|]$ be arbitrary. Suppose $\mathcal{X}^\star$ is the subspace that maximizes the maximin formula in the Courant-Fischer-Weyl Theorem 2.1.4. We know since $\mathsf{A} \preceq_\Pi \mathsf{B}$, we have

$$\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle_\Pi \leq \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f} \rangle_\Pi \quad \text{for all } \boldsymbol{f} \in \mathbb{R}^V.$$

In particular,

$$\lambda_i(\mathsf{A}) = \min_{\substack{\boldsymbol{f} \in \mathcal{X}^\star, \\ \|\boldsymbol{f}\|_\Pi = 1}} \langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle_\Pi \leq \min_{\substack{\boldsymbol{f} \in \mathcal{X}^\star, \\ \|\boldsymbol{f}\|_\Pi = 1}} \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f} \rangle_\Pi.$$

And thus,

$$\lambda_i(\mathsf{A}) = \min_{\substack{\boldsymbol{f}\in\mathcal{X}^\star,\\ \|\boldsymbol{f}\|_\Pi=1}} \langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle_\Pi \leq \min_{\substack{\boldsymbol{f}\in\mathcal{X}^\star,\\ \|\boldsymbol{f}\|_\Pi=1}} \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle_\Pi \leq \max_{\substack{\mathcal{X}\subseteq\mathbb{R}^V,\\ \dim\mathcal{X}=k}} \min_{\substack{\boldsymbol{f}\in\mathcal{X},\\ \|\boldsymbol{f}\|_\Pi=1}} \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle = \lambda_i(\mathsf{B}),$$

where we have invoked the Courant-Fischer-Weyl Theorem 2.1.4. □

Theorem 2.1.9 (Cauchy Interlacing Theorem)**.** *Let* $\mathsf{A} \in \mathbb{R}^{V\times V}$ *be a symmetric matrix and* $\mathsf{B} \in \mathbb{R}^{U\times U}$ *be a principal submatrix of* A*. Let* $n = |V|$ *and* $m = |U|$*. For any* $0 \leq j \leq m$*,*

$$\lambda_j(\mathsf{A}) \geq \lambda_j(\mathsf{B}) \geq \lambda_{n-m+j}(\mathsf{A}).$$

Proof. The inequality, $\lambda_j(\mathsf{A}) \geq \lambda_j(\mathsf{B})$ follows by noticing that every subspace $\mathcal{U} \subset \mathbb{R}^U$ of dimension $\dim\mathcal{U} = j$ is also a subspace of $\mathbb{R}^V$ of dimension j. In particular,

$$\lambda_j(\mathsf{B}) = \max_{\substack{\mathcal{U}\subseteq\mathbb{R}^U,\\ \dim\mathcal{U}=j}} \min_{\substack{\boldsymbol{f}\in\mathcal{U},\\ \|\boldsymbol{f}\|=1}} \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle = \max_{\substack{\mathcal{U}\subseteq\mathbb{R}^U,\\ \dim\mathcal{U}=j}} \min_{\substack{\boldsymbol{f}\in\mathcal{U},\\ \|\boldsymbol{f}\|=1}} \langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle \leq \max_{\substack{\mathcal{X}\subseteq\mathbb{R}^V,\\ \dim\mathcal{X}=j}} \min_{\substack{\boldsymbol{f}\in\mathcal{X},\\ \|\boldsymbol{f}\|=1}} \langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle = \lambda_j(\mathsf{A}).$$

Now, from the side of the inequality that we have proven, we can infer that

$$-\lambda_{n-j+1}(\mathsf{A}) = \lambda_j(-\mathsf{A}) \geq \lambda_j(-\mathsf{B}) = -\lambda_{m-j+1}(\mathsf{B}).$$

Thus, we have $\lambda_{n-j}(\mathsf{A}) \leq \lambda_{m-j}(\mathsf{A})$, and the inequality $\lambda_{n-m+j}(\mathsf{A}) \leq \lambda_j(\mathsf{B})$ follows an index change. □

Theorem 2.1.10 (Weyl Interlacing Theorem)**.** *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{V\times V}$ *be two symmetric matrices. For any* $i, j \geq 1$ *such that* $i + j \leq |V| - 1$*,*

$$\lambda_{i+j-1}(\mathsf{A}+\mathsf{B}) \leq \lambda_i(\mathsf{A}) + \lambda_j(\mathsf{B}).$$

Proof. Let $\mathcal{Y}_1 = \operatorname{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}\}^\perp$ be the subspace that is perpendicular to orthonormal collection consisting of the first $(i-1)$ eigenvectors of A, i.e. the $\boldsymbol{a}_k$ satisfy $\|\boldsymbol{a}_k\| = 1$ for all $k \in [i-1]$, $\langle \boldsymbol{a}_k, \boldsymbol{a}_l \rangle = 0$ whenever $k \neq l$ and $\mathsf{A}\boldsymbol{a}_k = \lambda_k(\mathsf{A}) \cdot \boldsymbol{a}_k$. We know by the Courant-Fischer-Weyl Theorem 2.1.4 that,

$$\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle \leq \lambda_i(\mathsf{A}) \text{ for all } \boldsymbol{f} \in \mathcal{Y}_1 \text{ such that } \|\boldsymbol{f}\| = 1. \tag{2.2}$$

Similarly, let $\mathcal{Y}_2 = \operatorname{span}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{j-1}\}^{\perp}$ be the subspace that is perpendicular to the orthonormal collection consisting of the first $(j-1)$ eigenvectors of B, i.e. the $\boldsymbol{b}_k$ satisfy $\|\boldsymbol{b}_k\| = 1$ for all $k \in [j-1]$, $\langle \boldsymbol{b}_k, \boldsymbol{b}_l \rangle = 0$ whenever $k \neq l$ and $\mathsf{B}\boldsymbol{b}_k = \lambda_k(\mathsf{B}) \cdot \boldsymbol{b}_k$. We know by the Courant-Fischer-Weyl Theorem 2.1.4 that,

$$\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f} \rangle \leq \lambda_j(\mathsf{B}) \text{ for all } \boldsymbol{f} \in \mathcal{Y}_2 \text{ such that } \|\boldsymbol{f}\| = 1. \tag{2.3}$$

Now, note that

$$\begin{aligned} \mathcal{Y}_1 \cap \mathcal{Y}_2 &= \{\boldsymbol{f} \in \mathbb{R}^V : \langle \boldsymbol{f}, \boldsymbol{a}_k \rangle = 0 \text{ for all } k \in [i-1] \text{ and } \langle \boldsymbol{f}, \boldsymbol{b}_l \rangle = 0 \text{ for all } l \in [j-1]\}. \\ &= \operatorname{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{j-1}\}^{\perp}. \end{aligned}$$

In particular, $\dim(\mathcal{Y}_1 \cap \mathcal{Y}_2) \geq n - i - j + 2$, since $\mathcal{Y}_1 \cap \mathcal{Y}_2$ is the orthogonal subspace to a subspace of dimension at most $i + j - 2$. Now, note that

$$\langle \boldsymbol{f}, (\mathsf{A}+\mathsf{B})\boldsymbol{f} \rangle = \langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle + \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f} \rangle \leq \lambda_i(\mathsf{A}) + \lambda_j(\mathsf{B}) \text{ for all } \boldsymbol{f} \in \mathcal{Y}_1 \cap \mathcal{Y}_2 \text{ such that } \|\boldsymbol{f}\| = 1$$

where we have used $\mathcal{Y}_1 \cap \mathcal{Y}_2 \subset \mathcal{Y}_1$ and Eq. (2.2) for bounding the first term and $\mathcal{Y}_1 \cap \mathcal{Y}_2 \subset \mathcal{Y}_2$ and Eq. (2.3) for bounding the second term. Thus, using the Courant-Fischer-Weyl Theorem 2.1.4 we obtain

$$\lambda_{i+j-1}(\mathsf{A}+\mathsf{B}) = \min_{\substack{\mathcal{Y} \subseteq \mathbb{R}^V, \\ \dim \mathcal{Y} = n-i-j+2}} \max_{\substack{\boldsymbol{f} \in \mathcal{Y}, \\ \|\boldsymbol{f}\|=1}} \langle \boldsymbol{f}, (\mathsf{A}+\mathsf{B})\boldsymbol{f} \rangle \leq \max_{\substack{\boldsymbol{f} \in \mathcal{Y}_1 \cap \mathcal{Y}_2, \\ \|\boldsymbol{f}\|=1}} \langle \boldsymbol{f}, (\mathsf{A}+\mathsf{B})\boldsymbol{f} \rangle \leq \lambda_i(\mathsf{A}) + \lambda_j(\mathsf{B}),$$

where we have assumed that $\dim(\mathcal{Y}_1 \cap \mathcal{Y}_2) = n - i - j + 2$ if this is not the case, we can just take any subspace of $\mathcal{Y}_1 \cap \mathcal{Y}_2$ of dimension $n - i - j + 2$ and the theorem follows. $\square$

Let $\mathbb{R}^V, \mathbb{R}^U$ be a vector space equipped with the inner-products with respect to the measures Π_U, Π_V. We define operator norm of $\|\mathsf{A}\|_{\Pi_U \to \Pi_V}$ of a matrix $\mathsf{A} \in \mathbb{R}^{U \times V}$ as the following quantity,

$$\begin{aligned} \|\mathsf{A}\|_{\Pi_V \to \Pi_U} &= \max\left\{ \|\mathsf{A}\boldsymbol{f}\|_{\Pi_U} : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\|_{\Pi_U} = 1 \right\}, \\ &= \max\left\{ \frac{\|\mathsf{A}\boldsymbol{f}\|_{\Pi_U}}{\|\boldsymbol{f}\|_{\Pi_V}} : \boldsymbol{f} \neq 0, \boldsymbol{f} \in \mathbb{R}^V \right\}. \end{aligned} \tag{operator norm}$$

When $\mathsf{A} \in \mathbb{R}^{V \times V}$ is a square operator, we will adopt the convention $\|\mathsf{A}\|_{\Pi_U \to \Pi_V} = \|\mathsf{A}\|_{\Pi_V}$. We recall the following elementary results about the operator norm that will be of use to us in Chapter 4,

Fact 2.1.11. *Let the vector spaces $\mathbb{R}^U$ and $\mathbb{R}^V$ be equipped with inner-products with respect to the measures Π_U and Π_V and suppose $\mathsf{A} \in \mathbb{R}^{U \times V}$. Then,*

$$\|\mathsf{A}\|_{\Pi_U \to \Pi_V} = \sqrt{\lambda_1(\mathsf{A}^*\mathsf{A})} = \sigma_1(\mathsf{A}).$$

Proof. Recall that $\|\mathsf{A}\boldsymbol{f}\|_{\Pi_V}^2 = \langle \mathsf{A}\boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle_{\Pi_V}$. In particular,

$$\|\mathsf{A}\boldsymbol{f}\|_{\Pi_V}^2 = \langle \mathsf{A}\boldsymbol{f}, \mathsf{A}\boldsymbol{f} \rangle_{\Pi_V} = \langle \boldsymbol{f}, \mathsf{A}^*\mathsf{A}\boldsymbol{f} \rangle_{\Pi_U}.$$

By Courant-Fischer-Weyl Theorem 2.1.4, we know that this expression is maximized precisely when $\boldsymbol{f} \in \mathbb{R}^U$ is the top-eigenvector of $\mathsf{A}^*\mathsf{A}$ with the value $\lambda_1(\mathsf{A}^*\mathsf{A})$, which concludes the proof for the claim $\|\mathsf{A}\|_{\Pi_U \to \Pi_V} = \sqrt{\lambda_1(\mathsf{A}^*\mathsf{A})}$. The latter equality follows from Corollary 2.1.5. $\square$

Fact 2.1.12 (Submultiplicativity of the Operator Norm)**.** *Let $\mathsf{A} \in \mathbb{R}^{U \times V}$ and $\mathsf{B} \in \mathbb{R}^{V \times W}$ be given and suppose the vector spaces $\mathbb{R}^U, \mathbb{R}^V, \mathbb{R}^W$ are equipped with inner-products with respect to the measures Π_U, Π_V, and Π_W respectively. Then,*

$$\|\mathsf{AB}\|_{\Pi_W \to \Pi_U} \leq \|\mathsf{A}\|_{\Pi_V \to \Pi_U} \cdot \|\mathsf{B}\|_{\Pi_W \to \Pi_V}.$$

Proof. We have,

$$\begin{aligned}
\|\mathsf{AB}\|_{\Pi_W\to\Pi_U} &= \max\left\{\frac{\|\mathsf{AB}\boldsymbol{f}\|_{\Pi_U}}{\|\boldsymbol{f}\|_{\Pi_W}} : \boldsymbol{f}\in\mathbb{R}^W, \boldsymbol{f}\neq 0\right\},\\
&\leq \max\left\{\|\mathsf{A}\|_{\Pi_V\to\Pi_U}\cdot\frac{\|\mathsf{B}\boldsymbol{f}\|_{\Pi_V}}{\|\boldsymbol{f}\|_{\Pi_W}} : \boldsymbol{f}\in\mathbb{R}^W, \boldsymbol{f}\neq 0\right\},\\
&= \|\mathsf{A}\|_{\Pi_V\to\Pi_U}\cdot\|\mathsf{B}\|_{\Pi_W\to\Pi_V},
\end{aligned}$$

where we have used the definition of the operator norm to get the last two (in)equalities. $\square$

Fact 2.1.13 (Triangle Inequality)**.** *Let $\mathsf{A},\mathsf{B}\in\mathbb{R}^{U\times V}$ be given pair of operators, and suppose the vector spaces $\mathbb{R}^U$ and $\mathbb{R}^V$ are equipped with the inner-product with respect to the measures Π_U and Π_V. Then,*

$$\|\mathsf{A}+\mathsf{B}\|_{\Pi_V\to\Pi_U}\leq\|\mathsf{A}\|_{\Pi_V\to\Pi_U}+\|\mathsf{B}\|_{\Pi_V\to\Pi_U}.$$

Proof. We first show that for all $\boldsymbol{f},\boldsymbol{g}\in\mathbb{R}^U$ we have

$$\|\boldsymbol{f}+\boldsymbol{g}\|_{\Pi_U}\leq\|\boldsymbol{f}\|_{\Pi_U}+\|\boldsymbol{g}\|_{\Pi_U}. \tag{2.4}$$

One has,

$$\|\boldsymbol{f}+\boldsymbol{g}\|^2_{\Pi_U}=\langle\boldsymbol{f}+\boldsymbol{g},\boldsymbol{f}+\boldsymbol{g}\rangle_{\Pi_U}=\langle\boldsymbol{f},\boldsymbol{f}\rangle_{\Pi_U}+\langle\boldsymbol{g},\boldsymbol{g}\rangle_{\Pi_U}+2\langle\boldsymbol{f},\boldsymbol{g}\rangle_{\Pi_U}=\|\boldsymbol{f}\|^2_{\Pi_U}+\|\boldsymbol{g}\|^2_{\Pi_U}+2\langle\boldsymbol{f},\boldsymbol{g}\rangle_{\Pi_U}.$$

By the Cauchy-Schwarz Inequality (Fact 2.1.1) this implies,

$$\|\boldsymbol{f}+\boldsymbol{g}\|^2_{\Pi_U}\leq\|\boldsymbol{f}\|^2_{\Pi_U}+\|\boldsymbol{g}\|^2_{\Pi_U}+2\cdot\|\boldsymbol{f}\|_{\Pi_U}\|\boldsymbol{g}\|_{\Pi_U}=(\|\boldsymbol{f}\|_{\Pi_U}+\|\boldsymbol{g}\|_{\Pi_U})^2.$$

Thus, Eq. (2.4) follows by taking square root of the above inequality.

To prove Fact 2.1.13, we use the definition of the operator norm, and observe

$$\begin{aligned}
\|\mathsf{A}+\mathsf{B}\|_{\Pi_V\to\Pi_U} &= \max\left\{\|(\mathsf{A}+\mathsf{B})\boldsymbol{f}\|_{\Pi_U} : \boldsymbol{f}\in\mathbb{R}^V, \|\boldsymbol{f}\|_{\Pi_V}=1\right\},\\
&\leq \max\left\{\|\mathsf{A}\boldsymbol{f}\|_{\Pi_V\to\Pi_U}+\|\mathsf{B}\boldsymbol{f}\|_{\Pi_V\to\Pi_U} : \boldsymbol{f}\in\mathbb{R}^V, \|\boldsymbol{f}\|_{\Pi_V}=1\right\},\\
&\leq \ \|\mathsf{A}\|_{\Pi_V\to\Pi_U}+\|\mathsf{B}\|_{\Pi_V\to\Pi_U}.
\end{aligned}$$

$\square$

2.2 Graphs, Random Walks, and Eigenvalues

In this section, we will recall some basic results concerning the eigenvalues of self-adjoint row-stochastic matrices $\mathsf{M} \in \mathbb{R}^{V\times V}$ and their connection to mixing times of random walks and graph isoperimetry.

2.2.1 Random Walk Operators

An operator $\mathsf{M} \in \mathbb{R}^{U\times V}$ is called row-stochastic every non-zero entry $\mathsf{M}(i,j)$ of M is non-negative and every row-sum is 1, i.e. $\mathsf{M}\mathbf{1}$. Let $G = (V, E, w)$ be an edge-weighted undirected graph with a weight $w_e > 0$ on each edge $e \in E$. The (weighted) adjacency matrix of G is denoted by $\mathsf{A}_G \in \mathbb{R}^{V\times V}$ with $\mathsf{A}_G(u,v) = w_{uv}$ for $uv \in E$ and $\mathsf{A}_G(u,v) = 0$ for $uv \notin E$. The diagonal degree matrix of G is denoted by D_G where $\mathsf{D}_G(v,v) = \deg(v) = \sum_{u:uv\in E} w_{uv}$ for $v \in V$. The random walk matrix of G is denoted by $\mathsf{M}_G := \mathsf{D}_G^{-1}\mathsf{A}_G$. Note that M_G is a row-stochastic matrix.

If $\mathsf{M} \in \mathbb{R}^{V\times V}$ is a row-stochastic self-adjoint operator with respect to the distribution Π, then the random walk described by M is called reversible. It turns out that whenever M is self-adjoint with respect to a distribution Π, Π is the stationary measure of M (see e.g. [AF95, p. 57]),

Proposition 2.2.1. *Let $\mathsf{M} \in \mathbb{R}^{V\times V}$ be a row-stochastic matrix that is self-adjoint with respect to the distribution Π, then Π is stationary for M, i.e. $\Pi^\top \mathsf{M} = \Pi^\top$.*

We note that when $\mathsf{M} \in \mathbb{R}^{V\times V}$ describes the random walk in the undirected edge-weighted graph $G = (V, E, w)$, it is self-adjoint with respect to the distribution $\Pi = \mathsf{D}_G\mathbf{1}/2\sum_{uv\in E} w_{uv}$ as,

$$
\begin{aligned}
\left(\frac{\mathsf{D}_G\mathbf{1}}{2\sum_{uv\in E} w_{uv}}\right)^\top \mathsf{D}_G^{-1}\mathsf{A}_G &= \frac{1}{2\sum_{uv\in E} w_{uv}}\mathbf{1}^\top \mathsf{D}_G\mathsf{D}_G^{-1}\mathsf{A}_G \\
&= \frac{1}{2\sum_{uv\in E} w_{uv}}\mathbf{1}^\top \mathsf{A}_G \\
&= \frac{\mathbf{1}^\top \mathsf{D}_G}{2\sum_{uv\in E} w_v} = \left(\frac{\mathsf{D}_G\mathbf{1}}{2\sum_{uv\in E} w_{uv}}\right)^\top.
\end{aligned}
$$

Proof of Proposition 2.2.1. As M is self-adjoint, one has $\langle \mathsf{M}\boldsymbol{f}, \boldsymbol{g}\rangle_\Pi = \langle \boldsymbol{f}, \mathsf{M}\boldsymbol{g}\rangle_\Pi$ for all $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$. In particular, one has for all $i \in V$

$$[\Pi^\top \mathsf{M}](i) = \Pi^\top \mathsf{M}\mathbf{1}_i = \langle \mathbf{1}, \mathsf{M}\mathbf{1}_i\rangle_\Pi = \langle \mathsf{M}\mathbf{1}, \mathbf{1}_i\rangle_\Pi = \langle \mathbf{1}, \mathbf{1}_i\rangle_\Pi,$$

where we have used the row-stochasticity of M in the equality. Now, note that for all $i \in V$

$$\langle \mathbf{1}, \mathbf{1}_i\rangle_\Pi = 1 \cdot \Pi(i) = \Pi(i) \quad \text{and therefore} \quad [\Pi^\top \mathsf{M}](i) = \Pi(i).$$

Thus, $\Pi^\top \mathsf{M} = \Pi^\top$. □

Remark 2.2.2. From now on we will refer to self-adjoint row-stochastic matrices $\mathsf{M} \in \mathbb{R}^{V\times V}$ with respect to the inner-product defined by the distribution Π, as reversible random walks with stationary distribution Π. While there might be many distributions that are stationary for M, we will call Π *the* stationary distribution. Our abuse of nomenclature is very mild as we will later show that under the assumption of irreducibility (which we will soon discuss), Π is indeed the unique stationary distribution of M (see Corollary 2.2.5). Also, notice that in Section 2.1, we have adopted the convention of defining inner-products $\langle \bullet, \bullet\rangle_\Pi$ only for distributions $\Pi \in \mathbb{R}^V$ with $\Pi(x) > 0$ for all $x \in V$. So, whenever we say M is a reversible random walk with the stationary distribution Π, we will also assume that $\Pi(x) > 0$.

From the definition of row-stochasticity, it is clear that a row-stochastic operator $\mathsf{M} \in \mathbb{R}^{V\times V}$ has 1 as an eigenvalue, and the eigenvector corresponding to it is $\mathbf{1}$. We first show that, 1 is actually the largest eigenvalue of M in absolute value (see e.g. [WLP09, Lemma 12.1]),

Proposition 2.2.3. *Let $\mathsf{M} \in \mathbb{R}^{V\times V}$ be a reversible random walk matrix with stationary distribution Π. Then,*

$$\lambda_1(\mathsf{M}) = 1 \quad \textit{and} \quad |\lambda_{\min}(\mathsf{M})| \leq 1.$$

Proof. Let $\boldsymbol{f} \in \mathbb{R}^V \setminus \{0\}$ be an eigenvector of M corresponding to the value $\lambda \in \mathbb{R}$, i.e. $\mathsf{M}\boldsymbol{f} = \lambda \boldsymbol{f}$. We write $i^\star = \arg\max_{i\in V} |\boldsymbol{f}(i)|$. Then,

$$|\lambda| \cdot |\boldsymbol{f}(i^\star)| = |[\mathsf{M}\boldsymbol{f}](i^\star)| = \left|\sum_{j\in V} \mathsf{M}(i^\star, j) \cdot \boldsymbol{f}(j)\right| \leq \sum_{j\in V} \mathsf{M}(i^\star, j)|\boldsymbol{f}(i^\star)| \leq |\boldsymbol{f}(i^\star)|$$

where we have used that $\mathsf{M}(i^\star, \bullet)$ is a probability distribution in the last inequality. In particular, this implies $|\lambda| \leq 1$ and thus, $\lambda_1(\mathsf{M}) \leq 1$ and $|\lambda_{\min}(\mathsf{M})| \leq 1$. The statement $\lambda_1(\mathsf{M}) = 1$ follows since $\mathsf{M}\mathbf{1} = \mathbf{1}$, i.e. 1 is an eigenvalue of M. $\square$

We will call a reversible random walk matrix $\mathsf{M} \in \mathbb{R}^{V \times V}$ with stationary measure Π *irreducible* if there exists a time point $t_{ij} > 0$ for every pair of vertices $\{i, j\}$ such that $\mathsf{M}^{t_{ij}}(i, j) > 0$, i.e. in t_{ij} discrete time-steps one can reach from $i \in V$ to $j \in V$ with non-zero probability. Any reversible random walk matrix $\mathsf{M} \in \mathbb{R}^{V \times V}$ with stationary measure Π, can be thought as describing a random walk on the edge-weighted graph $G = (V, E, w)$ where

$$E = \{ij : i, j \in V \quad \text{and} \quad \mathsf{M}(i, j) > 0\}$$

and every edge $ij \in E$ is assigned the weight $w_{ij} = 2\Pi(i) \cdot \mathsf{M}(i, j)$.

Indeed, we have now

$$\mathsf{D}_G(i, i) = \sum_{ij \in E} w_{ij} = 2\Pi(i) \sum_{j \in V} \mathsf{M}(i, j) = 2\Pi(i),$$

where we have used the row-stochasticity of M in the last equality. Using this, it is easy to see that $\mathsf{M}_G(i, j) = w_{ij}/\mathsf{D}_G(i, i) = \mathsf{M}(i, j)$. Now, it is easy to see that for a row-stochastic self-adjoint matrix $\mathsf{M} \in \mathbb{R}^{V \times V}$ irreducibility is equivalent to the connectivity of the graph G, over which M represents a random walk.

Under the assumption of irreducibility, we can show that $\mathbf{1}$ is (up to scaling) the only eigenvector of M corresponding to the eigenvalue 1 (see e.g. [WLP09, Lemma 12.1]).

Theorem 2.2.4 (Perron-Frobenius)**.** *Let $\mathsf{M} \in \mathbb{R}^{V \times V}$ be a reversible random walk matrix with stationary distribution Π. M is irreducible if and only if $\lambda_2(\mathsf{M}) < 1$. Equivalently, for an edge-weighted undirected graph $G = (V, E, w)$, $\lambda_2(\mathsf{M}_G) < 1$ if and only if G is connected.*

Proof. The equivalence follows from the discussion above. Suppose and edge-weighted graph $G = (V, E, w)$ is disconnected, i.e. there exists two sets $A, B \subseteq V$ such that $A \cap B = \varnothing$, $A \sqcup B = V$, and $E(A, B) = \{ij \in E : i \in A, j \in B\} = \varnothing$. Then, since there are no edges between A and B, we note that we can write

$$\mathsf{M}_G = \begin{pmatrix} \mathsf{M}_{G[A]} & 0 \\ 0 & \mathsf{M}_{G[B]} \end{pmatrix}$$

where $G[A]$ (resp. $G[B]$) is the induced subgraph of G on A (resp. B). Now, we note that $\mathbf{1}_A$ and $\mathbf{1}_B$ are eigenvectors M_G since the row-stochasticity of M induces the row-stochasticity of $\mathsf{M}_{G[A]}$ and $\mathsf{M}_{G[B]}$. Since $\langle \mathbf{1}_A, \mathbf{1}_B \rangle_\Pi = 0$, we have by the Courant-Fischer-Weyl Theorem 2.1.4 $\lambda_2(\mathsf{M}_G) = 1$.

Suppose now that G is connected, and suppose $\boldsymbol{f} \in \mathbb{R}^V \setminus \{0\}$ is an eigenvector to 1, i.e. $\mathsf{M}_G \boldsymbol{f} = \boldsymbol{f}$. Write, $i^\star = \arg\max_{i \in V} \boldsymbol{f}(i)$. Then, note that

$$\boldsymbol{f}(i^\star) = [\mathsf{M}_G \boldsymbol{f}](i^\star) = \sum_{j : ij \in E} \mathsf{M}_G(i^\star, j) \boldsymbol{f}(j),$$

and in particular for every vertex $j \in V$ adjacent to $i^\star$, we have that $\boldsymbol{f}(j) = \boldsymbol{f}(i^\star)$, where we have used that $\mathsf{M}_G(i^\star, j) > 0$ if and only if $i^\star j \in E$ and that $\sum_{j \in V} \mathsf{M}_G(i^\star, j) = 1$ by row-stochasticity. Using the connectivity of G, we can extend this argument for every $j \in V$ by induction on distance from $i^\star$, and show that we have $\boldsymbol{f}(j) = \boldsymbol{f}(i^\star)$ for all $j \in V$. Indeed this means that $\boldsymbol{f}$ and $\mathbf{1}$ are co-linear. Since every eigenvector $\boldsymbol{f}$ to 1 is necessarily co-linear to $\mathbf{1}$ (and not orthogonal) by the Courant-Fischer-Weyl Theorem 2.1.4 we obtain that $\lambda_2(\mathsf{M}_G) < 1$. $\square$

A nice consequence of the Perron-Frobenius Theorem 2.2.4 is the uniqueness of the stationary measure for reversible Markov chains,

Corollary 2.2.5. *Let $\mathsf{M} \in \mathbb{R}^{V \times V}$ be a reversible Markov chain with stationary distribution $\Pi \in \mathbb{R}^V$. If M is irreducible, then Π is the unique stationary distribution of Π.*

Proof. The Perron-Frobenius Theorem 2.2.4 tells us that the irreducible random walk matrix M has the eigenvalue 1 with multiplicity 1. By the Spectral Theorem 2.1.3 this means that any $\boldsymbol{f} \in \mathbb{R}^V$ that satisfies $\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f} \rangle_\Pi = 1$ and $\|\boldsymbol{f}\|_\Pi = 1$ is necessarily co-linear to $\mathbf{1}$. Suppose the distribution Π_0 is stationary for M, i.e. $\Pi_0^\top \mathsf{M} = \Pi_0^\top$ and we write $\Pi_0^*(i) = \frac{\Pi_0(i)}{\Pi(i)}$. Notice that Π_0 is well-defined as $\Pi_0(i) > 0$ for all $i \in V$ as per our conventions (see Remark 2.2.2). We notice that Π_0 is co-linear to $\mathbf{1}$ if and only if $\Pi_0 = \Pi$.

We note first that,

$$\langle \Pi_0^*, \mathsf{M}\Pi_0^* \rangle_\Pi = (\Pi_0^*)^\top \cdot \mathrm{diag}(\Pi) \cdot \mathsf{M}\Pi_0^* = \Pi_0^\top \mathsf{M}\Pi_0^* = \Pi_0^\top \Pi_0^* = \sum_{j \in V} \frac{\Pi_0(j)^2}{\Pi(j)}.$$

And also,

$$\langle \Pi_0^*, \Pi_0^* \rangle_\Pi = \sum_{j \in V} \Pi(j) \cdot \frac{\Pi_0(j)^2}{\Pi(j)} = \sum_{j \in V} \frac{\Pi_0^2(j)}{\Pi(j)}.$$

Hence, $\frac{\Pi_0^*}{\|\Pi_0^*\|_\Pi}$ must be co-linear to 1, however this is only possible if $\Pi_0 = \Pi$, however this is only possible if $\Pi_0 = \Pi$. Thus, Π is the unique stationary distribution of M. $\square$

2.2.2 Graphs and Isoperimetry

Let $G = (V, E, w)$ be an edge-weighted undirected graph where $w_{ij} > 0$ for all $ij \in E$. Given a set $S \subseteq V$, the conductance $\Phi_G(S)$ of S in G is defined by,

$$\Phi_G(S) = \frac{w(E(S, \overline{S}))}{\mathrm{vol}(S)},$$

where we recall that $w(E(S,\overline{S})) = \sum_{ij \in E(S,\overline{S})} w_{ij}$ and $\mathrm{vol}(S) = \sum_{i \in S} \deg(i)$ where $\deg(i)$ is the weighted degree of the vertex S. We define the conductance of the graph G as

$$\Phi(G) = \min_{\substack{S \subset V, \\ 0 < \mathrm{vol}(S) \leq \mathrm{vol}(V)/2}} \Phi_G(S).$$

Now, writing $(X_i)_{i \geq 0}$ for the state of the random walk starting from $X_0 \in V$ we can observe

$$\Phi_G(S) = \frac{\langle \mathbf{1}_S, (\mathsf{I} - \mathsf{M}_G)\mathbf{1}_S \rangle_\Pi}{\|\mathbf{1}_S\|_\Pi^2} = \Pr[X_1 \notin S \mid X_0 \in S] \quad \text{and} \quad \Phi(G) = \min_{\substack{S \subset V, \\ 0 < \Pi(S) \leq 1/2}} \Phi_{\mathsf{M}}(S), \tag{2.5}$$

where $\Pi = \mathsf{D}_G \mathbf{1} / (2 \sum_{ij \in E} w_{ij})$ is the stationary distribution of M_G. Based on the observation above, we will write $\Phi_{\mathsf{M}}(S) := \Phi_G(S)$ and $\Phi(\mathsf{M}) := \Phi(G)$ from now on.

Now, we recall the easy direction of the Cheeger inequality, which can be used to lower bound $\lambda_2(\mathsf{M}_G)$

Theorem 2.2.6 ([AM85])**.** *Let $\mathsf{M} \in \mathbb{R}^{V \times V}$ be a row-stochastic self-adjoint matrix with stationary distribution $\Pi \in \mathbb{R}^V$. Then*

$$\frac{1 - \lambda_2(\mathsf{M})}{2} \leq \Phi(\mathsf{M}).$$

Proof. Let $S \subset V$ be the set attaining the minimum conductance, i.e. $\Phi(\mathsf{M}) = \Phi_{\mathsf{M}}(S)$ and $0 < \Pi(S) \leq 1/2$. We write $\boldsymbol{f}_S = \mathbf{1}_S - \langle \mathbf{1}, \mathbf{1}_S \rangle_\Pi \cdot \mathbf{1} = \mathbf{1}_S - \Pi(S) \cdot \mathbf{1} \in \mathbb{R}^V$ and note that $\mathbf{1}$ and $\boldsymbol{f}_S$ are orthogonal in the inner-product defined by Π, i.e.

$$\langle \mathbf{1}, \boldsymbol{f}_S \rangle_\Pi = \langle \mathbf{1}, \mathbf{1}_S - \langle \mathbf{1}, \mathbf{1}_S \rangle_\Pi \cdot \mathbf{1} \rangle_\Pi = \langle \mathbf{1}, \mathbf{1}_S \rangle_\Pi - \langle \mathbf{1}, \mathbf{1}_S \rangle_\Pi \cdot \langle \mathbf{1}, \mathbf{1} \rangle_\Pi = 0$$

where we have used the observation $\langle \mathbf{1}, \mathbf{1} \rangle_\Pi = \sum_{v \in V} \Pi(v) = 1$ since Π is a probability distribution.

Further, since $\boldsymbol{f}_S(i) = 1 - \Pi(S)$ when $i \in S$ and $\boldsymbol{f}_S(j) = -\Pi(S)$ when $j \notin S$, we have

$$\begin{aligned}
\|\boldsymbol{f}_S\|_\Pi^2 &= \langle \boldsymbol{f}_S, \boldsymbol{f}_S \rangle_\Pi, \\
&= \sum_{i \in S} (1 - \Pi(S))^2 \cdot \Pi(i) + \sum_{j \notin S} \Pi(S)^2 \cdot \Pi(j), \\
&= \Pi(S) \cdot (1 - \Pi(S))^2 + \Pi(S)^2 \Pi(\overline{S}), \\
&= \Pi(S) \cdot (1 - \Pi(S))^2 + \Pi(S)^2 \cdot (1 - \Pi(S)), \qquad (\Pi(\overline{S}) = 1 - \Pi(S)) \\
&= \Pi(S) \cdot (1 - \Pi(S)) \leq \frac{\Pi(S)}{2}, \qquad (2.6)
\end{aligned}$$

where we have used $\Pi(S) \leq 1/2$ to obtain the last inequality. Now, by the Courant-Fischer-Weyl Theorem 2.1.4 we have

$$\lambda_2(\mathsf{M}_G) \geq \min_{\substack{\boldsymbol{f} \in \mathrm{span}\{\mathbf{1}, \boldsymbol{f}_S\}, \\ \|\boldsymbol{f}\|_\Pi = 1}} \langle \boldsymbol{f}, \mathsf{M}_G \boldsymbol{f} \rangle_\Pi \geq \frac{1}{\|\boldsymbol{f}_S\|^2} \langle \boldsymbol{f}_S, \mathsf{M}_G \boldsymbol{f}_S \rangle_\Pi,$$

where we have used $\langle \boldsymbol{f}_S, \mathbf{1} \rangle_\Pi = 0$ and Proposition 2.2.3 to get the last inequality, i.e. without loss of generality we can assume that the minimizer will be colinear to $\boldsymbol{f}_S$ since the vector $\mathbf{1}$ attains the largest eigenvalue 1. Thus, we have

$$\lambda_2(\mathsf{M}_G) \cdot \|\boldsymbol{f}_S\|^2 \geq \langle \boldsymbol{f}_S, \mathsf{M}_G \boldsymbol{f}_S \rangle_\Pi \qquad (2.7)$$

Now, we observe

$$\begin{aligned}
\langle \boldsymbol{f}_S, \mathsf{M}_G \boldsymbol{f}_S \rangle_\Pi &= \langle \mathbf{1}_S - \Pi(S) \cdot \mathbf{1}, \mathsf{M}_G(\mathbf{1}_S - \Pi(S) \cdot \mathbf{1}) \rangle_\Pi, \\
&= \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi - 2\Pi(S) \cdot \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1} \rangle_\Pi + \Pi(S)^2 \cdot \langle \mathbf{1}, \mathsf{M}_G \mathbf{1} \rangle_\Pi, \\
&= \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi - 2\Pi(S) \langle \mathbf{1}_S, \mathbf{1} \rangle_\Pi + \Pi(S)^2, && (\mathsf{M}_G \mathbf{1} = \mathbf{1}) \\
&= \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi - \Pi(S)^2 && (2.8)
\end{aligned}$$

where we have used $\langle \mathbf{1}_S, \mathbf{1} \rangle_\Pi = \Pi(S)$ to obtain Eq. (2.8).

Thus, we have

$$\begin{aligned}
\|\boldsymbol{f}_S\|^2 (1 - \lambda_2(\mathsf{M}_G)) &\le \langle \boldsymbol{f}_S, (\mathsf{I} - \mathsf{M}_G) \boldsymbol{f}_S \rangle_\Pi, && \text{(by Eq. (2.7))} \\
&= \|\boldsymbol{f}_S\|^2 - \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi + \Pi(S)^2, && \text{(by Eq. (2.8))} \\
&= \Pi(S) - \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi && \text{(by Eq. (2.6))} \\
&= \langle \mathbf{1}_S, \mathbf{1}_S \rangle_\Pi - \langle \mathbf{1}_S, \mathsf{M}_G \mathbf{1}_S \rangle_\Pi && (\langle \mathbf{1}_S, \mathbf{1}_S \rangle_\Pi = \Pi(S)), \\
&= \langle \mathbf{1}_S, (\mathsf{I} - \mathsf{M}_G) \mathbf{1}_S \rangle_\Pi, \\
&= \Phi_{\mathsf{M}}(S) \cdot \|\mathbf{1}_S\|_\Pi^2 = \Pi(S) \cdot \Phi_{\mathsf{M}}(S) && \text{(by Eq. (2.5))}
\end{aligned}$$

Thus,

$$(1 - \lambda_2(\mathsf{M}_G)) \cdot \frac{\|\boldsymbol{f}_S\|_\Pi^2}{\Pi(S)} \le \Phi_{\mathsf{M}}(S).$$

The theorem follows from Eq. (2.6). □

2.2.3 Mixing Times

The mixing time $T(\varepsilon, \mathsf{P})$ of the random walk operator $\mathsf{P} \in \mathbb{R}^{V \times V}$ is defined to be the least time step where the distribution of the random walk is ε-close to the stationary distribution Π of P in the ℓ_1 distance, i.e. $\Pi^{(t)}$ for the distribution of the random walk after t-steps, we have

$$T(\varepsilon, \mathsf{P}) = \min\{t \in \mathbb{N}_{\ge 0} : \|\Pi^{(t)} - \Pi\|_{\ell_1} \le \varepsilon\}.$$

We will write $\mathsf{A}(x, \bullet)$ for the x-th row of the matrix A. Recalling that the distribution of the random walk P starting from x after t steps is given by $\Pi^{(t)} = \mathsf{P}^t(x, \bullet)$, we formally define

$$T(\varepsilon, \mathsf{P}) = \min\bigl\{t \in \mathbb{N}_{\geq 0} : \|\mathsf{P}^t(x, \bullet) - \Pi^\top\|_{\ell_1} \leq \varepsilon \text{ for all } x \in V\bigr\}. \qquad \text{(mixing time)}$$

For our applications in sampling in Section 3.2, we will use the following well known relation between the mixing time of the random walk and the spectral gap of its transition matrix. The proof we present will follow the exposition in [MT05, Proposition 1.12].

Theorem 2.2.7 (Spectral Mixing Time Bound)**.** *Let* $\mathsf{P} \in \mathbb{R}^{V \times V}$ *be a reversible random walk matrix with stationary distribution* Π*. One has,*

$$T(\varepsilon, \mathsf{P}) \leq \frac{1}{1 - \sigma_2(\mathsf{P})} \cdot \log \frac{1}{\varepsilon \cdot \sqrt{\min_{x \in V} \Pi(x)}},$$

where $\sigma_2(\mathsf{P})$ *is the second largest singular value of* P*.*

Proof. Let $x \in V$ be an arbitrary vertex. For $t \geq 0$, we define:

$$\boldsymbol{p}_x^{(t)}(y) = \frac{\mathsf{P}^t(x, y)}{\Pi(y)} - 1.$$

Note that we have,

$$\mathop{\mathbb{E}}_{y \sim \Pi} |\boldsymbol{p}_x^{(t)}(y)| = \sum_{y \sim V} \Pi(y) \cdot |\boldsymbol{p}_x^{(t)}(y)| = \sum_{y \sim V} \bigl|\mathsf{P}^t(x, y) - \Pi(y)\bigr| = \|\mathsf{P}^t(x, \bullet) - \Pi^\top\|_{\ell_1}.$$

Our plan is now bounding $\mathbb{E}_{y \sim \Pi} |\boldsymbol{p}_x^{(t)}(y)|$ for all x to get a bound for the mixing time $T(\varepsilon, \mathsf{P})$.

The function $\boldsymbol{p}_x^{(t)}$ satisfies two useful properties. Firstly, it is orthogonal to the constant function, i.e. the eigenvector corresponding to 1

$$\begin{aligned}
\langle \boldsymbol{p}_x^{(t)}, \mathbf{1} \rangle_\Pi &= \sum_{y \in V} \Pi(y) \cdot \left(\frac{\mathsf{P}^t(x, y)}{\Pi(y)} - 1\right), \\
&= \sum_{y \in V} \mathsf{P}^t(x, y) - \Pi(y), \\
&= \left(\sum_{y \in V} \mathsf{P}^t(x, y)\right) - \left(\sum_{y \in V} \Pi(y)\right) = 1 - 1 = 0,
\end{aligned} \tag{2.9}$$

since Π is a probability distribution and P^t is a row-stochastic matrix. Second, we have $\mathsf{P}\boldsymbol{p}_x^{(t-1)} = \boldsymbol{p}_x^{(t)}$ since

$$
\begin{aligned}
[\mathsf{P} \cdot \boldsymbol{p}_x^{(t-1)}](y) &= \sum_{z \in V} \mathsf{P}(y,z) \cdot \left(\frac{\mathsf{P}^{t-1}(x,z)}{\Pi(z)} - 1 \right), \\
&= \left(\sum_{z \in V} \mathsf{P}(y,z) \frac{\mathsf{P}^{t-1}(x,z)}{\Pi(z)} \right) - 1, && \text{(row-stochasticity)} \\
&= \left(\sum_{z \in V} \mathsf{P}(z,y) \cdot \frac{\Pi(z)}{\Pi(y)} \cdot \frac{\mathsf{P}^{t-1}(x,z)}{\Pi(z)} \right) - 1, && \text{(by Proposition 2.1.2 and } \mathsf{P} = \mathsf{P}^*) \\
&= \left(\frac{1}{\Pi(y)} \sum_{z \in v} \mathsf{P}(z,y) \cdot \mathsf{P}^{t-1}(x,z) \right) - 1. \\
&= \frac{\mathsf{P}^t(x,y)}{\Pi(y)} - 1 = \boldsymbol{p}_x^{(t)}(y) && (2.10)
\end{aligned}
$$

Now, the goal is to appeal to the eigenvalues of P to bound $\|\mathsf{P}^t(x,\bullet) - \Pi^\top\|_{\ell_1} = \mathbb{E}_{y\sim\Pi} |\boldsymbol{p}_x^{(t)}|$. To this end we use Jensen's inequality,

$$
\|\mathsf{P}^t(x,\bullet) - \Pi^\top\|_{\ell_1}^2 = \left(\mathop{\mathbb{E}}_{y\sim\Pi} |\boldsymbol{p}_x^{(t)}| \right)^2 \le \mathop{\mathbb{E}}_{y\sim\Pi} |\boldsymbol{p}_x^{(t)}|^2 = \|\boldsymbol{p}_x^{(t)}\|_\Pi^2.
$$

By Eq. (2.10), we can write

$$
\|\boldsymbol{p}^{(t)}\|_\Pi^2 = \|\mathsf{P} \cdot \boldsymbol{p}^{(t-1)}\|_\Pi^2 = \langle \mathsf{P}\boldsymbol{p}^{(t-1)}, \mathsf{P}\boldsymbol{p}^{(t-1)} \rangle_\Pi = \langle \boldsymbol{p}^{(t-1)}, \mathsf{P}^2 \boldsymbol{p}^{(t-1)} \rangle_\Pi.
$$

Now, note that P^2 has non-negative entries only and further, $\mathsf{P}^2\mathbf{1} = \mathsf{P} \cdot \mathsf{P} \cdot \mathbf{1} = \mathbf{1}$. Now, by Corollary 2.1.5, Courant-Fischer-Weyl Theorem 2.1.4, and Eq. (2.9) we have

$$
\|\boldsymbol{p}^{(t)}\|_\Pi^2 \le \sigma_2(\mathsf{P})^2 \cdot \|\boldsymbol{p}^{(t-1)}\|_\Pi^2 \quad \text{and inductively} \quad \|\boldsymbol{p}^{(t)}\|_\Pi^2 \le \sigma_2(\mathsf{P})^{2t} \|\boldsymbol{p}^{(0)}\|_\Pi^2 \tag{2.11}
$$

Since $\boldsymbol{p}^{(0)}(y) = \frac{\mathsf{I}(x,y)}{\Pi(y)} - 1$, we have that the y-th entry is -1 when $x \neq y$ and $1/\Pi(x) - 1$ when $x = y$. Thus,

$$
\|\boldsymbol{p}^{(0)}\|_\Pi^2 = \sum_{\substack{y \in V, \\ y \neq x}} \Pi(y) + \left(\frac{1 - \Pi(x)}{\Pi(x)} \right)^2 \cdot \Pi(x) = 1 - \Pi(x) + \frac{(1 - \Pi(x))^2}{\Pi(x)} = \frac{1}{\Pi(x)} - 1 \le \frac{1}{\Pi(x)}.
$$

Plugging this in Eq. (2.11), we have

$$\|\mathsf{P}^t(x,\bullet) - \Pi^\top\|_{\ell_1}^2 \le \|\boldsymbol{p}_x^{(t)}\|_\Pi^2 \le \sigma_2(\mathsf{P})^{2t} \cdot \frac{1}{\Pi(x)}.$$

By solving $\sigma_2(\mathsf{P})^{2t} \cdot \frac{1}{\Pi(x)} \le \varepsilon^2$, we see that

$$t \ge \frac{\log(1/(\varepsilon \cdot \sqrt{\Pi(x)})}{\log(1/\sigma_2(\mathsf{P}))} \text{ ensures } \|\mathsf{P}^t(x,\bullet) - \Pi^\top\|_{\ell_1} \le \varepsilon.$$

The expression in the Theorem is obtained (i) by taking the maximum for all starting points $x \in V$, and (ii) using the first order approximation $\log(1/\sigma_2(\mathsf{P})) \ge (1 - \sigma_2(\mathsf{P}))$. $\square$

2.3 Simplicial Complexes

A simplicial complex X is a collection of subsets that is downward closed, i.e. if $\beta \in X$ and $\alpha \subset \beta$ then $\alpha \in X$. The elements α, β in X are called faces/simplices of X. The dimension of a face α is defined as $|\alpha| - 1$, e.g. an edge is of dimension 1, a vertex/singleton is of dimension 0, the empty set is of dimension -1. The collection of faces of dimension j is denoted by $X(j)$. The dimension of a simplicial complex is defined as the maximum dimension of its faces. A d-dimensional simplicial complex is called pure if every maximal face is of dimension d. All simplicial complexes we will consider in this thesis are pure.

2.3.1 Weighted Simplicial Complexes

A simplicial complex X can be equipped with a weighted function which assigns a positive weight to each face of X. We follow the formalism of [DDFH18] where the weight function is a probability distribution Π on the faces of the same dimension. Let X be a d-dimensional simplicial complex. Given a probability distribution $\Pi := \Pi_d$ with support $X(d)$, we can inductively obtain probability distributions Π_j supported on $X(j)$ for all $j \in [-1, d-1]$ by considering the marginal distributions, i.e.

$$\Pi_j(\alpha) = \frac{1}{j+2} \sum_{\substack{\beta \in X(j+1), \\ \beta \supset \alpha}} \Pi_{j+1}(\beta). \tag{2.12}$$

Equivalently, we can understand Π_j as the probability distribution of the following random process: Sample a random face $\beta \in X(d)$ using the probability distribution Π_d, and then sample a uniformly random subset of β in $X(j)$. The pair (X, Π) will be referred as a weighted simplicial complex. We write (X, Π) simply as X when Π is the uniform distribution.

We will prove the following useful consequence of Eq. (2.12),

Proposition 2.3.1. *Let (X, Π) be a d-dimensional weighted simplicial complex. For all $-1 \leq j \leq k \leq d$, and for all $\alpha \in X(j)$ one has,*

$$\Pi_j(\alpha) = \frac{1}{\binom{k+1}{j+1}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(k)}} \Pi_k(\beta).$$

Proof. When $k = j + 1$, the proposition is true by definition (Eq. (2.12)). We proceed by induction, suppose there exists some $k \in [j, d-1]$ such that for every $\alpha \in X(j)$

$$\Pi_j(\alpha) = \frac{1}{\binom{k+1}{j+1}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(k)}} \Pi_k(\beta). \qquad \text{(induction hypothesis)}$$

With this we calulate,

$$\begin{aligned} \Pi_j(\alpha) &= \frac{1}{\binom{k+1}{j+1}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(k)}} \Pi_k(\beta), \\ &= \frac{1}{\binom{k+1}{j+1}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(k)}} \frac{1}{k+2} \sum_{\substack{\beta' \supset \beta, \\ \beta' \in X(k+1)}} \Pi_{k+1}(\beta') \qquad \text{(by Eq. (2.12))} \end{aligned}$$

Now, we note that for each $\beta' \in X(k+1)$ such that $\beta' \supset \alpha$ is summed in the RHS $|\beta'| - |\alpha| = k - j + 1$ many times, each corresponding to a unique $\beta \subset \beta'$ such that $\beta \in X(k)$ and $\beta \supset \alpha$ is obtained by removing an element in β' not contained in α.

Thus,

$$\begin{aligned} \Pi_j(\alpha) &= \frac{1}{\binom{k+1}{j+1}} \cdot \frac{k-j+1}{k+2} \cdot \sum_{\substack{\beta' \supset \alpha, \\ \beta' \in X(k+1)}} \Pi_{k+1}(\beta'), \\ &= \frac{1}{\binom{k+2}{j+1}} \sum_{\substack{\beta' \supset \alpha, \\ \beta' \in X(k+1)}} \Pi_{k+1}(\beta'). \end{aligned}$$

$\square$

Remark 2.3.2. Whenever $\beta \supsetneq \alpha$, there exists a unique $\tau \in X(|\beta| - |\alpha| - 1)$ satisfying $\beta = \alpha \sqcup \tau$. For our applications, it will often be more convenient to have the index in the summation in Eq. (2.12) and Proposition 2.3.1 to run over $\tau \in X_\alpha(|\beta| - |\alpha| - 1)$, than for $\beta \supset \alpha$.

2.3.2 Links and Link Graphs

Let (X, Π) be a pure d-dimensional weighted simplicial complex. The link X_α of a face α is the simplicial complex defined as

$$X_\alpha := \{\beta \setminus \alpha \mid \beta \in X, \beta \supset \alpha\}.$$

The probability distributions $\Pi_0, \ldots, \Pi_d$ on X can naturally be used to define the probability distributions $\Pi_0^\alpha, \ldots, \Pi_{d-|\alpha|}^\alpha$ on X_α using conditional probability. Suppose $\alpha \in X(j)$. For $\tau \in X_\alpha(l)$, we define

$$\begin{aligned}
\Pi_l^\alpha(\tau) &= \Pr_{\beta \sim \Pi_{j+1+l}}[\beta = \alpha \cup \tau \mid \beta \supset \alpha] \\
&= \frac{\Pi_{j+l+1}(\alpha \cup \tau)}{\sum_{\substack{\beta \supset \alpha, \\ \beta \in X(j+l+1)}} \Pi_{j+l+1}(\beta)}, \\
&= \frac{\Pi_{j+l+1}(\alpha \cup \tau)}{\binom{j+l+2}{l+1} \cdot \Pi_j(\alpha)} \qquad (2.13)
\end{aligned}$$

where Eq. (2.13) is obtained from Proposition 2.3.1.

Remark 2.3.3. It is a crucial observation that the measures Π_j^α satisfy Eq. (2.12) as well. For $\alpha \in X(j)$, and $\tau \in X_\alpha(l)$, one has,

$$\begin{aligned}
\Pi_l^\alpha(\tau) &= \frac{\Pi_{j+l+1}(\alpha \sqcup \tau)}{\binom{j+l+2}{l+2}} && \text{(by Eq. (2.13))}, \\
&= \frac{\frac{1}{j+l+2} \sum_{\substack{\tau' \supset \tau, \\ \tau' \in X_\alpha(l+1)}} \Pi_{j+l+2}(\alpha \sqcup \tau')}{\binom{j+l+2}{l+1} \Pi_j(\alpha)} && \text{(by Eq. (2.12) and Remark 2.3.2)}, \\
&= \frac{1}{l+2} \sum_{\substack{\tau' \supset \tau, \\ \tau' \in X_\alpha(l+1)}} \frac{\Pi_{j+l+2}(\alpha \sqcup \tau')}{\binom{j+l+3}{l+2} \Pi_j(\alpha)}
\end{aligned}$$

where in the last equality we have used the identity,

$$(j+l+2)\binom{j+l+2}{l+1} = (l+2)\binom{j+l+3}{l+2}.$$

In particular, by using Eq. (2.13) and Remark 2.3.2 again,

$$\Pi_l^\alpha(\tau) = \frac{1}{l+2} \sum_{\substack{\tau' \supset \tau, \\ \tau' \in X_\alpha(l+1)}} \Pi_{l+1}^\alpha(\tau').$$

As Proposition 2.3.1 is a simple consequence of Eq. (2.12), the conclusion of Proposition 2.3.1 is also satisfied by the measures Π_j^α for all $j = -1, \dots, d - |\alpha|$.

We will prove the following useful decomposition rule,

Proposition 2.3.4. *Let (X, Π) be a d-dimensioal weighted simplicial complex. For all $-1 \le k \le d-1$, and for all $\alpha \in X(k)$ one has*

$$\sum_{z \in X_\alpha(0)} \Pi_0(z) \cdot \Pi_k^{\{z\}}(\alpha) = \Pi_k(\alpha).$$

Proof. By Eq. (2.12) and Remark 2.3.2, we have

$$\Pi_k(\alpha) = \frac{1}{k+2} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(k+1)}} \Pi_{k+1}(\beta) = \frac{1}{k+2} \sum_{z \in X_\alpha(0)} \Pi_{k+1}(\alpha \sqcup \{z\}).$$

Using Eq. (2.13), we can write

$$\Pi_{k+1}(\alpha \sqcup \{z\}) = (k+2) \cdot \Pi_0(z) \cdot \Pi_k^{\{z\}}(\alpha).$$

The proposition follows by plugging this into the first equality. □

Remark 2.3.5. It will often be more convenient to define $\Pi_k^{\{z\}}(\alpha) = 0$ whenever $z \in \alpha$ or $\alpha \notin X_{\{z\}}(\alpha)$. Using this convention, we can write

$$\sum_{z \in X(0)} \Pi_0(z) \cdot \Pi_k^{\{z\}}(\alpha) = \Pi_k(\alpha).$$

2.3.3 Local Spectral Expansion

Given a link X_α, the graph $G_\alpha = (X_\alpha(0), X_\alpha(1), \Pi_1^\alpha)$ is defined as the 1-skeleton of X_α. More explicitly, each singleton $\{v\}$ in X_α is a vertex v in G_α, each pair $\{u, v\}$ in X_α is an edge uv in G_α, and the weight of uv in G_α is equal to $\Pi_1^\alpha(\{u, v\})$. A simple observation is that if X is a pure d-dimensional simplicial complex and Π is the uniform distribution on $X(d)$, then for any $\alpha \in X(d-2)$ the weighting Π_1^α on the edges of G_α is uniform. We will use this observation in Section 3.2.

The definition of local spectral expanders will be based on the random walk matrix of G_α. Let A_α be the adjacency matrix of G_α. Let D_α be the diagonal degree matrix where $\mathsf{D}_\alpha(x, x) = \sum_y \mathsf{A}_\alpha(x, y) = 2\Pi_0^\alpha(x)$ where the last equality is by Eq. (2.12). The random walk matrix M_α of G_α is defined as $\mathsf{M}_\alpha := \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha$, with

$$\mathsf{M}_\alpha(x, y) = \frac{\Pi_1^\alpha(x, y)}{2\Pi_0^\alpha(x)} \quad \text{for all } \{x, y\} \in X_\alpha(1) \tag{2.14}$$

The distribution Π_0^α is the stationary distribution of M_α, as

$$(\Pi_0^\alpha)^\top \mathsf{M}_\alpha = (\Pi_0^\alpha)^\top \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha = \mathbf{1}^\top \mathsf{A}_\alpha = (\Pi_0^\alpha)^\top.$$

The matrix M_α is self-adjoint with respect to the inner-product defined by Π_0^α, as

$$\langle \boldsymbol{f}, \mathsf{M}_\alpha \boldsymbol{g}\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}, \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha \boldsymbol{g}\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}, \mathsf{A}_\alpha \boldsymbol{g}\rangle = \langle \mathsf{A}_\alpha \boldsymbol{f}, \boldsymbol{g}\rangle = \langle \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha \boldsymbol{f}, \boldsymbol{g}\rangle_{\Pi_0^\alpha} = \langle \mathsf{M}_\alpha \boldsymbol{f}, \boldsymbol{g}\rangle_{\Pi_0^\alpha}.$$

So, M_α have only real eigenvalues, and an orthonormal basis of eigenvectors with respect to the inner-product defined by Π_0^α. The largest eigenvalue of M_α is 1, as $\mathsf{M}_\alpha \mathbf{1} = \mathbf{1}$ and M_α is row-stochastic.

We also note, for all $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^{X_\alpha(0)}$ we have

$$\langle \boldsymbol{f}, \mathsf{M}_\alpha \boldsymbol{g}\rangle_{\Pi_0^\alpha} = \sum_{\substack{x,y \in X_\alpha(0),\\ x \neq y}} \Pi_1^\alpha(\{x, y\}) \cdot \boldsymbol{f}(x)\boldsymbol{g}(y). \tag{2.15}$$

Given a vector $\boldsymbol{f}$, we will be interested in writing it as $\boldsymbol{f} = \boldsymbol{f}^{\mathbf{1}} + \boldsymbol{f}^{\perp \mathbf{1}}$, so that $\boldsymbol{f}^{\mathbf{1}} = c\mathbf{1}$ for some scalar c and $\langle \boldsymbol{f}^{\mathbf{1}}, \boldsymbol{f}^{\perp\mathbf{1}}\rangle_{\Pi_0^\alpha} = 0$. It follows that $c = \frac{\langle \boldsymbol{f}, \mathbf{1}\rangle_{\Pi_0^\alpha}}{\langle \mathbf{1}, \mathbf{1}\rangle_{\Pi_0^\alpha}} = \langle \boldsymbol{f}, \mathbf{1}\rangle_{\Pi_0^\alpha} = \mathbb{E}_{x\sim\Pi_0^\alpha}[\boldsymbol{f}(x)]$. We write $\mathsf{J}_\alpha = \mathbf{1}(\Pi_0^\alpha)^\top$ as the operator to map $\boldsymbol{f}$ to $\boldsymbol{f}^{\mathbf{1}}$, so that

$$\mathsf{J}_\alpha \boldsymbol{f} = (\mathbf{1}(\Pi_0^\alpha)^\top)\boldsymbol{f} = \langle \boldsymbol{f}, \Pi_0^\alpha\rangle \cdot \mathbf{1} = \mathop{\mathbb{E}}_{x\sim\Pi_0^\alpha}[\boldsymbol{f}(x)] \cdot \mathbf{1} = \boldsymbol{f}^{\mathbf{1}}. \quad \text{(projector to constant functions)}$$

Oppenheim's Theorem

Let (X, Π) be a pure d-dimensional weighted simplicial complex. For all $j = -1, \ldots, d-2$ we define,

$$\nu_j := \nu_j(X, \Pi) = \min_{\alpha \in X(j)} \lambda_{\min}(\mathsf{M}_\alpha) \quad \text{and} \quad \gamma_j := \gamma_j(X, \Pi) = \max_{\alpha \in X(j)} \lambda_2(\mathsf{M}_\alpha).$$

We say X is a γ-local-spectral expander if $\gamma_i \le \gamma$ for $-1 \le i \le d-2$. We say X is a two-sided γ-local expander if $\max\{\gamma_i, |\nu_i|\} \le \gamma$ for all $-1 \le i \le d-2$. We note that two-sided γ-local spectral expansion of the complex X implies $\sigma_2(\mathsf{M}_\alpha) \le \gamma$ for all $\alpha \in X(\le d-2)$.

We will now recall Oppenheim's Theorem [Opp18], which relates the second eigenvalue of the graph of a lower-dimensional link to that of a higher-dimensional link.

Theorem 2.3.6 (Oppenheim's Theorem)**.** *Let (X, Π) be a pure d-dimenisonal weighted simplicial complex. For any $0 \le j \le d-2$, if G_α is connected for every $\alpha \in X(j-1)$, then*

$$\nu_{j-1} \ge \nu_j + (1-\nu_j) \cdot \nu_{j-1}^2 \quad \textit{and} \quad \gamma_{j-1} \le \gamma_j + (1-\gamma_j) \cdot \gamma_{j-1}^2.$$

In particular, we have

$$\nu_{j-1} \ge \frac{\nu_j}{1-\nu_j} \quad \textit{and} \quad \gamma_{j-1} \le \frac{\gamma_j}{1-\gamma_j}.$$

Theorem 2.3.6 is proven through the so-called Garland method, which goes back to [Gar73]. The main idea is that by using Proposition 2.3.4 one can decompose an expectation over the distribution Π_0^α for $\alpha \in X(\le d-1)$ into an expectation of expectations over the distributions $\Pi_0^{\alpha \cup \{z\}}$ for $z \in X_\alpha(0)$.

Proof of Theorem 2.3.6. We first assume $j = 0$ and prove $\gamma_{-1} \le \gamma_0 + (1-\gamma_0) \cdot \gamma_{-1}^2$. We will then explain how this implies the general case.

Throughout this proof, we will write $\mathsf{M} := \mathsf{M}_\varnothing$ and for $z \in X(0)$, $\mathsf{M}_z := \mathsf{M}_{\{z\}}$ and $\mathsf{J}_z = \mathsf{J}_{\{z\}}$.

Let $\boldsymbol{f} \in \mathbb{R}^{X(1)}$ be a unit eigenvector of M to the eigenvalue $\lambda := \lambda_2(\mathsf{M})$, i.e.

$$\mathsf{M}\boldsymbol{f} = \lambda \boldsymbol{f} \quad \text{and} \quad \|\boldsymbol{f}\|_{\Pi_0}^2 = 1 \quad \text{and therefore} \quad \langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f} \rangle = \lambda. \qquad \text{(assumptions)}$$

We have by Eq. (2.15),

$$\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} = \sum_{\substack{x,y\in X(0),\\ x\neq y}} \Pi_1(\{x,y\})\cdot \boldsymbol{f}(x)\boldsymbol{f}(y). \tag{2.16}$$

We will now do calculations to show that we can think of the quadratic form over $\mathsf{M}_{\varnothing}$ as an expectation of quadratic forms over M_z for $z \in X(0)$.

Thus, we can write

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} &= \sum_{\substack{x,y\in X(0),\\ x\neq y}} \Pi_1(\{x,y\})\cdot \boldsymbol{f}(x)\boldsymbol{f}(y),\\
&= \sum_{\substack{x,y\in X(0),\\ x\neq y}} \sum_{z\in X(0)} \Pi_0(z)\cdot \Pi_1^{\{z\}}(\{x,y\})\cdot \boldsymbol{f}(x)\boldsymbol{f}(y), \quad \text{(by Proposition 2.3.4)}\\
&= \sum_{z\in X(0)} \Pi_0(z) \sum_{\substack{x,y\in X(0),\\ x\neq y}} \Pi_1^{\{z\}}(\{x,y\})\cdot \boldsymbol{f}(x)\boldsymbol{f}(y),\\
&= \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\sum_{\substack{x,y\in X(0),\\ x\neq y}} \Pi_1^{\{z\}}(\{x,y\})\cdot \boldsymbol{f}(x)\boldsymbol{f}(y)\right],\\
&= \mathop{\mathbb{E}}_{z\sim\Pi_0} \langle \boldsymbol{f}, \mathsf{M}_z\boldsymbol{f}\rangle_{\Pi_0^{\{z\}}} \quad \text{(by Eq. (2.15))}.
\end{aligned}$$

For $z \in X(0)$, we will write $V_z = X_z(0)$. We recall that M_z are row-stochastic matrices whose top eigenvector is $\mathbf{1}_{V_z} \in \mathbb{R}^{V_z}$. The projector to the top-eigenspace of M_z is given by $\mathsf{J}_z = \mathbf{1}_{V_z}(\Pi_0^{\{z\}})^\top$ for all $z \in X(0)$, and the projector to the eigenspace orthogonal to $\mathbf{1}_{V_z}$ is given by $\mathsf{I}_{V_z} - \mathsf{J}_z$.

With this we can now write,

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} &= \mathop{\mathbb{E}}_{z\sim\Pi_0} \langle \boldsymbol{f}, \mathsf{M}_z\boldsymbol{f}\rangle_{\Pi_0^{\{z\}}},\\
&= \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\langle \mathsf{J}_z\boldsymbol{f}, \mathsf{M}_z\mathsf{J}_z\boldsymbol{f}\rangle_{\Pi_0^{\{z\}}} + \langle (\mathsf{I}_{V_z}-\mathsf{J}_z)\boldsymbol{f}, \mathsf{M}_z(\mathsf{I}_{V_z}-\mathsf{J}_z)\boldsymbol{f}\rangle_{\Pi_0^{\{z\}}}\right]\\
&= \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\mathsf{J}_z\boldsymbol{f}\|^2_{\Pi_0^z} + \langle (\mathsf{I}_{V_z}-\mathsf{J}_z)\boldsymbol{f}, \mathsf{M}_z(\mathsf{I}_{V_z}-\mathsf{J}_z)\boldsymbol{f}\rangle_{\Pi_0^{\{z\}}}\right] \quad \text{(using } \mathsf{M}_z\mathsf{J}_z = \mathsf{J}_z\text{)}
\end{aligned}$$

Since $\mathsf{I}_{V_z} - \mathsf{J}_z$ is the orthogonal projection to the functions to $\mathbf{1}_{V_z}$, i.e. the top eigenvector of M_z we have,

$$\begin{aligned} \langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} &\leq \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\mathsf{J}_z\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}} + \lambda_2(\mathsf{M}_z)\cdot\|(\mathsf{I}_{V_z} - \mathsf{J}_z)\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}}\right] \\ &\leq \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\mathsf{J}_z\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}} + \gamma_0\cdot\|(\mathsf{I} - \mathsf{J}_z)\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}}\right] \end{aligned} \tag{2.17}$$

where we have used $\gamma_0 = \max_{z\sim X(0)} \lambda_2(\mathsf{M}_z)$.

We would like the analyze the terms in the expectation in Eq. (2.17). We recall that by the definition of the projector to constant functions,

$$\mathsf{J}_z\boldsymbol{f} = \left(\mathbf{1}(\Pi_0^{\{z\}})^\top\right)\boldsymbol{f} = \left(\mathop{\mathbb{E}}_{x\sim\Pi_0^{\{z\}}} \boldsymbol{f}(x)\right)\cdot\mathbf{1}$$

By Eq. (2.13) and Eq. (2.14) from the preceding section we obtain

$$\Pi_1^{\{z\}}(x) = \frac{\Pi_1(x,z)}{2\Pi_0(z)} = \mathsf{M}(z,x). \tag{2.18}$$

Thus,

$$\mathop{\mathbb{E}}_{x\sim\Pi_0^{\{z\}}}[\boldsymbol{f}(x)] = [\mathsf{M}\boldsymbol{f}](z) \quad \text{and therefore} \quad \mathop{\mathbb{E}}_{z\in X(1)}\left[\|\mathsf{J}_z\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}}\right] = \|\mathsf{M}\boldsymbol{f}\|^2_{\Pi_0}.$$

From the above (since $\mathsf{M}\boldsymbol{f} = \lambda\boldsymbol{f}$, and $\boldsymbol{f}$ is a unit eigenvector) we obtain

$$\mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\mathsf{J}_z\boldsymbol{f}\|^2_z\right] = \|\lambda\boldsymbol{f}\|^2_{\Pi_0} = \lambda^2\cdot\|\boldsymbol{f}\|^2_{\Pi_0} = \lambda^2. \tag{2.19}$$

This settles the first term in the RHS of Eq. (2.17). Now, we proceed to investigate the second term: Since J_z is an orthogonal projector, we have

$$\mathop{\mathbb{E}}_{z\sim\Pi_0}\|(\mathsf{I} - \mathsf{J}_z)\boldsymbol{f}\|_{\Pi_0^{\{z\}}} = \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}} - \|\mathsf{J}_z\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}}\right] = \mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\boldsymbol{f}\|^2_{\Pi_0^{\{z\}}}\right] - \lambda^2. \tag{2.20}$$

Now, we note

$$\begin{aligned}
\mathop{\mathbb{E}}_{z\sim\Pi_0}\left[\|\boldsymbol{f}\|_{\Pi_0^{\{z\}}}\right] &= \sum_{z\in X(0)} \Pi_0(z)\cdot \sum_{x\in X_{\{z\}}(0)} \Pi_0^{\{z\}}(x)\cdot \boldsymbol{f}(x)^2, \\
&= \sum_{z\in X(0)} \sum_{x\in X_{\{z\}}(0)} \Pi_0(z)\Pi_0^{\{z\}}(x)\cdot \boldsymbol{f}(x)^2, \\
&= \sum_{x\in X(0)} \sum_{z\in X_{\{x\}}(0)} \Pi_0(z)\cdot \Pi_0^{\{z\}}(x)\boldsymbol{f}(x)^2, \qquad (2.21)\\
&= \sum_{x\in X(0)} \Pi_0(x)\cdot \boldsymbol{f}(x)^2, \qquad \text{(by Proposition 2.3.4)}\\
&= \|\boldsymbol{f}\|^2_{\Pi_0} = 1 \qquad \text{(as } \boldsymbol{f} \text{ is a unit vector)}
\end{aligned}$$

where we have used the equivalence $x \in X_{\{z\}}(0) \iff \{x, z\} \in X(1) \iff z \in X_{\{x\}}(0)$ for Eq. (2.21). Using this Eq. (2.20) becomes

$$\mathop{\mathbb{E}}_{z\sim\Pi_0} \|(\mathsf{I}-\mathsf{J}_z)\boldsymbol{f}\|_{\Pi_0^{\{z\}}} = 1-\lambda^2.$$

We can plug this in Eq. (2.17) together with Eq. (2.19) to obtain,

$$\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} \leq \lambda^2 + \gamma_0\cdot(1-\lambda^2) = \gamma_0 + (1-\gamma_0)\cdot\lambda^2.$$

As $\boldsymbol{f}$ is a unit eigenvector of λ (see our assumptions) corresponding to the eigenvalue $\lambda = \lambda_2(\mathsf{M})$ we have

$$\langle \boldsymbol{f}, \mathsf{M}\boldsymbol{f}\rangle_{\Pi_0} = \lambda \leq \gamma_0 + (1-\gamma_0)\cdot\lambda^2.$$

Noticing that $\lambda = \lambda_2(\mathsf{M}_\varnothing) = \gamma_{-1}$ concludes, we have now proven

$$\gamma_{-1} \leq \gamma_0 + (1-\gamma_0)\cdot\gamma_{-1}^2.$$

The second expression in the theorem statement can be obtained by solving the inequality for γ_{-1} assuming $\gamma_0 < 1/2$. This assumption can be made without loss of generality as we have $\gamma_0/(1-\gamma_0) \geq 1$ when $\gamma_0 \geq 1/2$, and thus by Proposition 2.2.3, the inequality $\gamma_{-1} \leq 1 \leq \gamma_0/(1-\gamma_0)$ follows trivially.

Suppose now, $j > 0$ and $\alpha \in X(j)$. By repeating the same arguments in the link $(X_\alpha, \Pi^\alpha_{d-|\alpha|})$ we can get an upperbound on $\lambda_2(\mathsf{M}_\alpha)$ as M^X_α (the link graph of α in X) and $\mathsf{M}^{X_\alpha}_\varnothing$ (the link graph of $\varnothing$ in X_α) are identical. By taking a maximum over all α we prove the analogous statement for general γ_j.

As the analogous inequality for ν_j follows identically, we omit the proof. $\square$

An inductive argument proves the following corollary, which we will use in Section 3.2.

Corollary 2.3.7 (Oppenheim's Corollary)**.** *Let (X, Π) be a pure d-dimenisonal weighted simplicial complex. If G_α is connected for every $\alpha \in X(k)$ and every $k \leq d-2$, then*

$$\gamma_j \leq \frac{\gamma_{d-2}}{1-(d-2-j)\cdot\gamma_{d-2}}.$$

Remark 2.3.8. To get a non-trivial bound on γ_{-1} using Oppenheim's Corollary 2.3.7, i.e. to ensure $\gamma_{-1} < 1$ using this line of arguments, we need to assume $\gamma_{d-2} < 1/(d-1)$.

2.4 Higher Order Random Walks

2.4.1 Up and Down Operators

Let (X, Π) be a pure d-dimensional weighted simplicial complex. In the following definitions, $\alpha \in X(k)$, $\beta \in X(k+1)$, $\boldsymbol{f} \in \mathbb{R}^{X(j)}$, $\boldsymbol{g} \in \mathbb{R}^{X(k+1)}$, and $j \in [-1, d-1]$.

The j-th up operator $\mathsf{U}_j : \mathbb{R}^{X(j)} \to \mathbb{R}^{X(j+1)}$ is defined as

$$[\mathsf{U}_j\boldsymbol{f}](\beta) = \frac{1}{j+2}\sum_{x\in\beta}\boldsymbol{f}(\beta\backslash\{x\}) = \sum_{\substack{\alpha\subset\beta,\\ \alpha\in X(j)}}\frac{\boldsymbol{f}(\alpha)}{j+2}. \qquad \text{(up operator)}$$

The $(j+1)$-st down operator $\mathsf{D}_{j+1} : \mathbb{R}^{X(j+1)} \to \mathbb{R}^{X(j)}$ is defined as

$$[\mathsf{D}_{j+1}\boldsymbol{g}](\alpha) = \sum_{x\in X_\alpha(0)}\frac{\Pi_{j+1}(\alpha\cup\{x\})}{(j+2)\Pi_j(\alpha)}\cdot\boldsymbol{g}(\alpha\cup\{x\}) = \sum_{\substack{\beta\supset\alpha,\\ \beta\in X(j+1)}}\frac{\Pi_{j+1}(\beta)\cdot\boldsymbol{g}(\beta)}{(j+2)\Pi_j(\alpha)} \quad \text{(down operator)}$$

It is known [KO18, DDFH18] that U_j and D_{j+1} are adjoints of each other, i.e.

Proposition 2.4.1. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Suppose $j \in [-1, d-1]$. Then,*

$$\langle \boldsymbol{g}, \mathsf{U}_j \boldsymbol{f} \rangle_{\Pi_{j+1}} = \langle \mathsf{D}_{j+1} \boldsymbol{g}, \boldsymbol{f} \rangle_{\Pi_j} \quad \textit{for all } \boldsymbol{g} \in \mathbb{R}^{X(j+1)}, \boldsymbol{f} \in \mathbb{R}^{X(j)}. \tag{adjointness}$$

Equivalently, the operators U_j and D_{j+1} are adjoints of each other with respect to the inner-products over Π_j and Π_{j+1}, i.e. $\mathsf{U}_j^* = \mathsf{D}_{j+1}$ and $\mathsf{D}_{j+1}^* = \mathsf{U}_j$.

Proof. We appeal to Proposition 2.1.2. By the definition of the up operator, for $\beta \in X(j+1)$ and $\alpha \in X(j)$

$$\mathsf{U}_j(\beta, \alpha) = \mathbf{1}_\beta^\top \mathsf{U}_j \mathbf{1}_\alpha = [\mathsf{U}_j \mathbf{1}_\alpha](\beta) = \sum_{x \in \beta} \frac{\mathbf{1}_\alpha(\beta \setminus \{x\})}{j+2} = \frac{\mathbf{1}[\alpha \subset \beta]}{j+2},$$

and similarly by the definition of the down operator

$$\mathsf{D}_{j+1}(\alpha, \beta) = \mathbf{1}_\alpha^\top \mathsf{D}_{j+1} \mathbf{1}_\beta = [\mathsf{D}_{j+1} \mathbf{1}_\beta](\alpha) = \sum_{x \in X_\alpha(0)} \frac{\Pi_{j+1}(\beta) \cdot \mathbf{1}_\beta(\alpha \cup \{x\})}{(j+2) \cdot \Pi_j(\alpha)} = \frac{\Pi_{j+1}(\beta) \cdot \mathbf{1}[\beta \supset \alpha]}{(j+2) \cdot \Pi_j(\alpha)}.$$

It is clear that the equation in Proposition 2.1.2 is satisfied, and thus

$$\mathsf{D}_{j+1}^* = \mathsf{U}_j \quad \text{and} \quad \mathsf{U}_j^* = \mathsf{D}_{j+1}.$$

□

Remark 2.4.2. We have stayed consistent with the notations in [DDFH18], and named U_j and D_{j+1} up and down operators with their right-action on functions (or vectors) in mind. However, in terms of random walks, U_j describes a random down-movement from $X(j+1)$ to $X(j)$, whereas D_{j+1} describes a random up-movement from $X(j)$ to $X(j+1)$, since the action of the probability distribution is from the left.

We now remark that when viewed as random walk operators U_j and D_j respectively, compose really nicely each other: Let X be a d-dimensional simplicial complex and let $k, h \geq 0$ be such that $k + h \leq d$. For a face $\alpha \in X(k)$, we introduce the notation

$$\boldsymbol{p}_\alpha^{(h)} = (\mathsf{D}_{k+1} \cdots \mathsf{D}_{k+h})^\top \mathbf{1}_\alpha$$

for the probability vector of the random movement that starts from $\alpha \in X(k)$ and moves to $X(k+h)$ as described by the operators D_{k+j} for $j = 1, \ldots, k$. We adopt the convention that the empty product of matrices is the identity and thus, $\boldsymbol{p}_\alpha^{(0)} = \mathbf{1}_\alpha$. We prove that $\boldsymbol{p}_\alpha^{(h)}$ describes a probability distribution on the faces $\beta \in X(k+h)$ that contain α which assigns them measure proportional to $\Pi_{k+h}(\beta)$.

Proposition 2.4.3. *Let X be a d-dimensional simplicial complex and $k, h \geq 0$ such that $k + h \leq d$. For $\alpha \in X(k)$ and $\beta \in X(k+h)$ one has,*

$$\boldsymbol{p}_\alpha^{(h)}(\beta) = \mathbf{1}[\beta \supset \alpha] \cdot \frac{1}{\binom{k+h+1}{h}} \cdot \frac{\Pi_{k+h}(\beta)}{\Pi_k(\alpha)} = \Pi_{h-1}^{\alpha}(\beta \backslash \alpha).$$

Proof. Notice that for $h = 0$, the statement holds trivially. We proceed by induction: Assume that there exists some $h \geq 0$ that satisfies the induction hypothesis

$$\boldsymbol{p}_\alpha^{(h)}(\beta) = \mathbf{1}[\beta \supset \alpha] \cdot \frac{1}{\binom{k+h}{h}} \cdot \frac{\Pi_{k+h+1}(\beta)}{\Pi_k(\alpha)} \tag{induction hypothesis}$$

for all $\beta \in X(k+h)$.

For $\beta' \in X(k+(h+1))$ one has by the definition of the down operator

$$\begin{aligned}
\boldsymbol{p}_\alpha^{(h+1)}(\beta') &= [\mathsf{D}_{k+h+1}^\top \boldsymbol{p}_\alpha^{(h)}](\beta'), \\
&= \mathbf{1}_{\beta'}^\top \cdot (\mathsf{D}_{k+h+1}^\top \boldsymbol{p}_\alpha^{(h)}), \\
&= (\mathsf{D}_{k+h+1} \mathbf{1}_{\beta'})^\top \boldsymbol{p}_\alpha^{(h)}, \\
&= \frac{1}{k+h+2} \cdot \sum_{x \in \beta'} \frac{\Pi_{k+h+1}(\beta')}{\Pi_{k+h}(\beta' \backslash \{x\})} \cdot \boldsymbol{p}_\alpha^{(h)}(\beta' \backslash \{x\}).
\end{aligned}$$

Plugging in the induction hypothesis, this implies

$$\begin{aligned}
\boldsymbol{p}_\alpha^{(h+1)}(\beta') &= \frac{1}{(k+h+2)} \cdot \sum_{x \in \beta'} \frac{\Pi_{k+h+1}(\beta')}{\Pi_{k+h}(\beta' \backslash \{x\})} \cdot \left(\mathbf{1}[(\beta' \backslash \{x\}) \supset \alpha] \cdot \frac{1}{\binom{k+h+1}{h}} \cdot \frac{\Pi_{k+h}(\beta' \backslash \{x\})}{\Pi_k(\alpha)} \right), \\
&= \frac{1}{(k+h+2)} \cdot \frac{1}{\binom{k+h+1}{h}} \cdot \sum_{x \in \beta'} \mathbf{1}[\beta' \backslash \{x\} \supset \alpha] \cdot \frac{\Pi_{k+h+1}(\beta')}{\Pi_k(\alpha)}.
\end{aligned}$$

From the preceding, it is clear that $\boldsymbol{p}_\alpha^{(h)}(\beta') = 0$ if $\beta' \not\supset \alpha$. Suppose therefore $\beta' \in X(k+h+1)$ is such that $\beta' \supset \alpha$. Since there are precisely $h+1$ indices whose deletion still preserves the containment of α, we can write

$$\begin{aligned} p_\alpha^{(h+1)}(\alpha) &= \mathbf{1}[\beta' \supset \alpha] \cdot \frac{h+1}{k+h+2} \cdot \frac{1}{\binom{k+h+1}{h}} \frac{\Pi_{k+h+1}(\beta')}{\Pi_k(\alpha)}, \\ &= \mathbf{1}[\beta' \supset \alpha] \cdot \frac{1}{\binom{k+h+2}{h+1}} \cdot \frac{\Pi_{k+h+1}(\beta')}{\Pi_k(\alpha)}. \end{aligned}$$

The statement $\boldsymbol{p}_\alpha^{(h)}(\beta) = \Pi_{h-1}^{\alpha}(\beta \backslash \alpha)$ is a consequence of Eq. (2.13). □

Similarly, for $\beta \in X(k+h)$ and $\alpha \in X(k)$, we introduce the notation $\boldsymbol{q}_\beta^{(h)}$, as

$$\boldsymbol{q}_\beta^{(h)}(\alpha) = (\mathsf{U}_{k+h-1} \cdots \mathsf{U}_k)^\top \mathbf{1}_\beta,$$

for the probability vector of the random movement starting from $\beta \in X(k+h)$ and moving to $X(k)$ as described by U_{k+h-j} for $j = 1, \ldots, h$. Using Proposition 2.4.3, and the fact that $(\mathsf{U}_{k+u-1} \cdots \mathsf{U}_k)^* = \mathsf{D}_{k+u} \cdots \mathsf{D}_{k+1}$ we can see that $\boldsymbol{q}_\beta^{(h)}$ is the uniform distribution of all subsets of β contained in $X(k)$

Corollary 2.4.4. *Let $X(\leq d)$ be a simplicial complex, and $k, h \geq 0$ be parameters satisfying $k + h \leq d$. For $\beta \in X(k+h)$ and $\alpha \in X(k)$, one has*

$$\boldsymbol{q}_\beta^{(h)}(\alpha) = \frac{1}{\binom{k+h+1}{h}} \cdot \mathbf{1}[\beta \supset \alpha].$$

2.4.2 Down-Up Walk, Up-Down Walk, and Non-Lazy Up-Down Walk

We use the up and down operators to define three random walk operators on $X(j)$. The j-th down-up walk $\mathsf{P}_j^\bigtriangledown$ and the j-th up-down walk $\mathsf{P}_j^\bigtriangleup$ are defined as

$$\mathsf{P}_j^\bigtriangledown = \mathsf{U}_{j-1}\mathsf{D}_j \quad \text{and} \quad \mathsf{P}_j^\bigtriangleup = \mathsf{D}_{j+1}\mathsf{U}_j. \qquad \text{(down-up walk, up-down walk)}$$

As $\mathsf{U}_i^* = \mathsf{D}_{i+1}$, it is easy to observe that these operators are positive semi-definite. One useful property of $\mathsf{P}_j^\bigtriangleup$ and $\mathsf{P}_j^\bigtriangledown$ is that they have the same non-zero spectrum with the same multiplicity by Fact 2.1.6, and in particular $\lambda_2(\mathsf{P}_j^{\triangle}) = \lambda_2(\mathsf{P}_j^{\bigtriangledown})$. Also, we define the j-th non-lazy up-down walk as

$$\mathsf{P}_j^{\wedge} = \frac{j+2}{j+1}\left(\mathsf{P}_j^{\triangle} - \frac{1}{j+2}\mathsf{I}\right), \qquad \text{(non-lazy up-down walk)}$$

which is the up-down walk conditioned on not looping. It follows from the adjointness of U_j and D_{j+1} that all $\mathsf{P}_j^{\bigtriangledown}$, $\mathsf{P}_j^{\triangle}$, and $\mathsf{P}_j^{\wedge}$ are self-adjoint with respect to the inner-product defined by Π_j, e.g. given any $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^{X(j)}$,

$$\langle \boldsymbol{f}_1, \mathsf{P}_j^{\triangle}\boldsymbol{f}_2\rangle_{\Pi_j} = \langle \boldsymbol{f}_1, \mathsf{D}_{j+1}\mathsf{U}_j\boldsymbol{f}_2\rangle_{\Pi_j} = \langle \mathsf{U}_j\boldsymbol{f}_1, \mathsf{U}_j\boldsymbol{f}_2\rangle_{\Pi_{j+1}} = \langle \mathsf{D}_{j+1}\mathsf{U}_j\boldsymbol{f}_1, \boldsymbol{f}_2\rangle_{\Pi_j} = \langle \mathsf{P}_j^{\triangle}\boldsymbol{f}_1, \boldsymbol{f}_2\rangle_{\Pi_j}.$$

This implies that Π_j is the stationary distribution for all these random walks $\mathsf{P}_j^{\bigtriangledown}$, $\mathsf{P}_j^{\triangle}$, and $\mathsf{P}_j^{\wedge}$ by Proposition 2.2.1.

Combinatorial Interpretation: We can understand the higher order random walks as a random walk on a bipartite graph between $X(j)$ and $X(j+1)$ as explained in [ALOV19, DK17]. Consider the bipartite graph $H = (X(j), X(j+1), E)$ in which a face $\alpha \in X(j)$ and a face $\beta \in X(j+1)$ are connected if and only if $\alpha \subset \beta$. The edge $\{\alpha, \beta\} \in H$ is assigned the weight $\frac{1}{j+2} \cdot \Pi_{j+1}(\beta)$. Using Eq. (2.12), it can be seen that the weighted degree of any $\alpha \in X(j)$ is $\Pi_j(\alpha)$. And the weighted degree of any $\beta \in X(j+1)$ is exactly $\Pi_{j+1}(\beta)$. Thus, the graph H has the (weighted) random walk matrix

$$\mathsf{M}_H = \begin{pmatrix} 0 & \mathsf{U}_j \\ \mathsf{D}_{j+1} & 0 \end{pmatrix}.$$

One step of the down-up walk $\mathsf{P}_{j+1}^{\bigtriangledown}$ can be thought as a two step random walk in M_H: starting from some $\beta \in X(j+1)$, the random walk will go down from $\beta \in X(j+1)$ to $\alpha \in X(j)$ by dropping an element of β, which is chosen uniformly at random as prescribed by U_j, and then the random walk will go up from $\alpha \in X(j)$ to a random face $\beta' \in X(j+1)$ which contains α as prescribed by D_{j+1}. Similarly, one step of the up-down walk $\mathsf{P}_j^{\triangle}$ can be thought as a two step random walk in M_H starting from some $\alpha \in X(j)$. More precisely,

$$\mathsf{M}_H^2 = \begin{pmatrix} \mathsf{U}_j\mathsf{D}_{j+1} & 0 \\ 0 & \mathsf{D}_{j+1}\mathsf{U}_j & 0 \end{pmatrix} = \begin{pmatrix} \mathsf{P}_{j+1}^{\bigtriangledown} & 0 \\ 0 & \mathsf{P}_j^{\triangle} \end{pmatrix}.$$

It is instructive to check that when the distribution Π of the simplicial complex is the uniform distribution, then the down-up walks and the up-down walks are as described as in the introduction.

Remark 2.4.5 (Johson Graphs). If X is the Boolean slice of size $k+1$, i.e. $X = \binom{[n]}{\leq k}$, then the non-lazy down-up walk $\mathsf{P}_k^{\bigtriangledown}$ corresponds to a lazy random walk over the Johnson graph $J(n, k+1) = (V, E)$ where $V = \binom{[n]}{k+1}$ and there is an edge $\{\alpha, \alpha'\} \in E$ whenever $|\alpha \cap \alpha'| = |\alpha| - 2$. The random walk matrix of the Johnson graph is known to have eigenvalues bounded away from 1 [GM15]. In Chapter 3 we will prove that the down-up walks are expanding and our proof for this can be thought as a more robust version of the proof of the eigenvalue bound for the Johnson graph – indeed, when $X = \binom{[n]}{\leq k+1}$ the eigenvalue bound we give will be tight.

The down- and up- operators we have introduced can also be studied in the more general setting of partially ordered sets. We refer the interested readers to [BR98].

2.4.3 Longer Random Walks

Suppose now X is a d-dimensional simplicial complex and $a_1, a_2, h \geq 0$ are such that $a_1 \geq a_2$ and $a_1 + h \leq d$. We define the up-down walk between $X(a_1)$ and $X(a_2)$ through $X(a_1 + h)$ to be

$$\mathsf{P}_{a_1,a_2}^{(h)} = \mathsf{D}_{a_1+1} \cdots \mathsf{D}_{a_1+h} \cdot \mathsf{U}_{a_1+h-1} \cdots \mathsf{U}_{a_2}.$$

We will often refer to the parameter h as the height of the random walk. Similar to the intuition that was presented about the up-down and the down-up walks, we can think of $\mathsf{P}_{a_1,a_2}^{(h)}(\alpha_0, \alpha_1)$ as describing the following random process: Starting from some $\alpha_0 \in X(a_1)$ we first sample a face $\beta \in X(a_1 + h)$ that contains α_0 with probability $\boldsymbol{p}_{\alpha_0}^{(h)}(\beta)$, and then pick a uniformly random subset $\alpha_1 \in X(a_2)$ of β.

We will introduce the notation $\mathsf{P}_a^{(h)} := \mathsf{P}_{a,a}^{(h)}$, for the case where $a_1 = a_2 = a$.

2.4.4 Swap Operator

Let (X, Π) be a d-dimensional weighted simplicial complex and suppose $k, l, h \geq 0$ satisfy $k \geq l$ and $k + j \leq d$. We consider the following random movement from $X(k)$ to $X(l)$: Starting from from some $\alpha_0 \in X(k)$ first sample a random face $\beta \in X(k+h)$ containing α_0 with probability $\boldsymbol{p}_{\alpha_0}^{(h)}(\beta)$, and then pick a uniformly random face $\alpha_1 \in X(l)$ among all the subsets of β that satisfy $|\alpha_1 \setminus \alpha_0| = h$. We will call this random process a step of the h-swapping walk from $X(k)$ to $X(l)$,as this random walk swaps h uniformly random elements of α_0 with $\beta\backslash\alpha_0$. We will write $\mathsf{S}_{k,l}^{(h)}$ for the operator describing the random process we have written above, i.e. $\mathsf{S}_{k,l}^{(h)}(\alpha_0, \alpha_1) = \Pr[\alpha_1 \mid \alpha_0]$. We have,

Proposition 2.4.6. *Suppose X is a d-dimensional simplicial complex. Suppose, $k, l, h \geq 0$ are given such that $k \leq l$ and $k + h \leq d$. Let $\alpha_0 \in X(k)$, $\alpha_1 \in X(l)$, then*

$$\mathsf{S}_{k,l}^{(h)}(\alpha_0, \alpha_1) = \frac{\mathbf{1}[|\alpha_1\backslash\alpha_0| = h]}{\binom{k+1}{l+1-h} \cdot \binom{k+h+1}{h}} \cdot \frac{\Pi_{k+h}(\alpha_0 \cup \alpha_1)}{\Pi_k(\alpha_0)}$$

Proof. We first notice that the process we have describes above lands on $\alpha_1 \in X(l)$ from some $\beta \in X(k+h)$ containing α_0 such that $|\alpha_1\backslash\alpha_0| = h$ if and only if $\beta = \alpha_1 \cup \alpha_0$. This happens with probability $\boldsymbol{p}_{\alpha}^{(h)}(\alpha_0 \cup \alpha_1)$. Conditioned on picking $\beta = \alpha_0 \cup \alpha_1$ the probability of picking α_1 as the uniformly random set such that $|\alpha_1\backslash\alpha_0| = h$ is, $\frac{1}{\binom{k+1}{l+1-h}}$ as this corresponds to picking $l+1-h$ random elements from the set α_1 of cardinality $k+1$. By law of conditional probability,

$$\begin{aligned}
\mathsf{S}_{k,l}^{(h)}(\alpha_0, \alpha_1) &= \Pr[\alpha_1 \mid \alpha_0], \\
&= \Pr[\alpha_1 \mid \beta = \alpha_0 \cup \alpha_1] \cdot \boldsymbol{p}_{\alpha_0}^{(h)}(\alpha_0 \cup \alpha_1), \\
&= \frac{\mathbf{1}[|\alpha_1\backslash\alpha_0| = h]}{\binom{k+1}{l+1-h}} \cdot \boldsymbol{p}_{\alpha_0}^{(h)}(\alpha_0 \cup \alpha_1)
\end{aligned}$$

Using Proposition 2.4.3, we obtain

$$\mathsf{S}_{k,l}^{(h)}(\alpha_0, \alpha_1) = \frac{\mathbf{1}[|\alpha_1\backslash\alpha_0| = h]}{\binom{k+1}{l+1-h} \cdot \binom{k+h+1}{h}} \cdot \frac{\Pi_{k+h}(\alpha_0 \cup \alpha_1)}{\Pi_k(\alpha_0)},$$

which was to be demonstrated. □

We remark that $\mathsf{S}_{k,l}^{(h)}$ can be thought as the random walk $\mathsf{P}_{k,l}^{(h)}$ that is conditioned on having $|\alpha_1\backslash\alpha_0| = h$. This observation will be useful in Chapter 4. The important special cases for us will be when $h = l+1$, i.e. where $\alpha_0 \cap \alpha_1 = \varnothing$. To simplify notation, we will write $\mathsf{S}_k := \mathsf{S}_{k,k}^{(k+1)}$ and $\mathsf{S}_{k,l} := \mathsf{S}_{k,l}^{(l+1)}$.

Remark 2.4.7 (Kneser Graphs). If X is the complete complex $\Delta_{n,d}$, i.e. $X = \binom{[n]}{\leq d+1+h}$, then the swap walk S_k corresponds to the simple random walk over the Kneser graph, $K(n, k+1) = (V, E)$ where the set of vertices is $V = \binom{[n]}{k+1}$, and there is an edge between every two disjoint pair of sets in V, i.e. $E = \{\{\alpha, \alpha'\} : \alpha, \alpha' \in V, \alpha \cap \alpha' = \varnothing\}$. The random walk matrix of the Kneser graph is known to have eigenvalues that are small [GM15]. In Chapter 4 we will prove that the swap walks are expanding and our proof for this can be thought as a more robust version of the proof of the eigenvalue bound for the Kneser graphs.

Chapter 3

Down-Up Walks and Applications

3.1 Eigenvalue Bounds for Higher Order Random Walks

Our main result in this chapter is a quantitative generalization of the basic fact that a pure d-dimensional simplicial complex X is gallery connected (i.e. $\lambda_2(\mathsf{P}_d^{\triangledown}) < 1$) if and only if the graph G_α is connected for every $\alpha \in X$ up to dimension $d-2$ (i.e. $\lambda_2(\mathsf{M}_\alpha) < 1$ for all $\alpha \in X(k)$ and $-1 \leq j \leq d-2$).

For simplicity of presentation, given a d-dimensional pure simplicial complex X, we will adopt the convention of writing

$$\gamma_j := \max_{\alpha \in X(j)} \lambda_2(\mathsf{M}_\alpha) \quad \text{and} \quad \nu_j := \min_{\alpha \in X(j)} \lambda_{\min}(\mathsf{M}_\alpha),$$

where $-1 \leq j \leq d-2$.

Theorem 3.1.1. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \leq k \leq d$,*

$$\lambda_2(\mathsf{P}_k^{\triangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j).$$

Using an inductive argument as in [ALOV19, Theorem 3.3], we can prove a more general statement about the entire range of eigenvalues.

Theorem 3.1.2. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Then, for any $0 \leq k \leq d-1$ and for any $-2 \leq r \leq k$, the matrix $\mathsf{P}_k^{\triangle}$ has at most $|X(r)|$ eigenvalues with value strictly greater than*

$$1 - \frac{r+2}{k+2} \prod_{j=r}^{k-1} (1 - \gamma_j),$$

where we adopt the conventions that $X(-2) = \varnothing$ and $\prod_{j=r}^{r-1}(1-\gamma_j) = 1$.

Note that Theorem 3.1.1 is a special case of Theorem 3.1.2 where $r = -1$ (recall that $X(-1) = \{\emptyset\}$ and so $|X(-1)| = 1$). Further, Theorem 3.1.1 can only prove that $\lambda_2(\mathsf{P}_d^{\bigtriangledown}) \leq 1 - \frac{1}{d+1}$. We observe that this bound is almost tight.

Proposition 3.1.3. *Let X be a d-dimensional simplicial complex. Let $n = |X(0)|$. Suppose $2(d+1) \leq n$. Then $\lambda_2(\mathsf{P}_d^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{d-1}^{\triangle}) \geq 1 - \frac{2}{d+1}$.*

Before we prove Theorem 3.1.1 and Theorem 3.1.2, we present two corollaries of Theorem 3.1.1.

Combining with Oppenheim's Corollary 2.3.7, Theorem 3.1.1 provides a bound on the second eigenvalue of the d-th down-up walk based only on the maximum second eigenvalue of the graphs in dimension $d-2$. This will be useful in Section 3.2.

Corollary 3.1.4. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \leq k \leq d$, suppose $\gamma_k \leq \frac{1}{k+1}$ and $\gamma_j < 1$ for $-1 \leq j \leq k-2$, then*

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \leq 1 - \frac{1}{(k+1)^2}.$$

Proof. Since $\gamma_{k-2} \leq \frac{1}{k+1}$ and $\gamma_j < 1$ for $-1 \leq j \leq k-2$, it follows from Oppenheim's Corollary 2.3.7 that for any $-1 \leq j \leq k-3$,

$$\gamma_j \leq \frac{\gamma_{k-2}}{1 - (k-2-j) \cdot \gamma_{k-2}} \leq \frac{\frac{1}{k+1}}{1 - \frac{k-2-j}{k+1}} = \frac{1}{j+3}.$$

Therefore, by Theorem 3.1.1,

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} \frac{j+2}{j+3} = 1 - \frac{1}{(k+1)^2}.$$

∎

Theorem 3.1.1 implies the following result for longer random walks on local-spectral expanders.

Corollary 3.1.5. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Let $0 \leq a, h$ be such that $a + h \leq d$. If X is a γ-local-spectral expander, then*

$$\lambda_2(\mathsf{P}_a^{(h)}) \leq (1+\gamma)^h \cdot \frac{a+1}{a+h+1}.$$

The rest of this section is organized as follows. We will first prove Theorem 3.1.1 in Section 3.1.1, then Theorem 3.1.2 in Section 3.1.4, then Corollary 3.1.5 in Section 3.1.5, and finally Proposition 3.1.3 in Section 3.1.6.

3.1.1 Spectral Bound: Proof of Theorem 3.1.1

In this section, we will provide a proof for Theorem 3.1.1,

Theorem 3.1.1. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \leq k \leq d$,*

$$\lambda_2(\mathsf{P}_k^{\triangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1-\gamma_j).$$

The key lemma in proving Theorem 3.1.1 is the following result that quantifies a spectral bound on the difference of the k-th non-lazy up-down walk and the k-th down-up walk in terms of the second eigenvalue of the links at dimension $k-1$.

Lemma 3.1.6. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \leq k \leq d-1$,*

$$\nu_{k-1} \cdot \left(\mathsf{I} - \mathsf{P}_k^{\triangledown}\right) \preceq_{\Pi_k} \mathsf{P}_k^{\wedge} - \mathsf{P}_k^{\triangledown} \preceq_{\Pi_k} \gamma_{k-1} \cdot \left(\mathsf{I} - \mathsf{P}_k^{\triangledown}\right).$$

The proof of Lemma 3.1.6, will closely follow the proof of [DDFH18, Theorem 5.5], where they prove the weaker inequality

$$\nu_{k-1} \cdot \mathsf{I} \preceq_{\Pi_k} \mathsf{P}_k^{\wedge} - \mathsf{P}_k^{\triangledown} \preceq_{\Pi_k} \gamma_{k-1} \cdot \mathsf{I}. \tag{3.1}$$

We remark that a similar statement was also used in [KO18] for proving Theorem 1.1.2.

We will first show how Lemma 3.1.6 implies Theorem 3.1.1 by an inductive argument.

Proof of Theorem 3.1.1 from Lemma 3.1.6. We prove Theorem 3.1.1 by induction on k. The base case is when $k = 0$, where $\mathsf{P}_0^{\triangledown} = \mathbf{1}\Pi_0^{\top}$ is a rank one matrix and so $\lambda_2(\mathsf{P}_0^{\triangledown}) \leq 0$, and hence Theorem 3.1.1 trivially holds.

For the induction step, suppose we have

$$\lambda_2(\mathsf{P}_{j+1}^{\triangledown}) = \lambda_2(\mathsf{P}_j^{\triangle}) \leq 1 - \frac{1}{j+2}\prod_{i=-1}^{j-1}(1-\gamma_i). \qquad \text{(induction hypothesis)}$$

Since $\mathsf{P}_{j+1}^{\triangledown} = \mathsf{U}_j\mathsf{D}_{j+1}$ and $\mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j$ have the same non-zero eigenvalues with the same multiplicity by Fact 2.1.6, we only need to prove the statement for $\mathsf{P}_{j+1}^{\triangle}$. By Lemma 3.1.6,

$$\mathsf{P}_{j+1}^{\wedge} \preceq_{\Pi_{j+1}} \gamma_j \cdot \mathsf{I} + (1-\gamma_j)\mathsf{P}_{j+1}^{\triangledown}$$

It follows from Fact 2.1.8 that

$$\lambda_2(\mathsf{P}_{j+1}^{\wedge}) \leq \gamma_j + (1-\gamma_j)\cdot\lambda_2(\mathsf{P}_{j+1}^{\triangledown}) \leq 1 - \frac{1}{j+2}\prod_{i=-1}^{j}(1-\gamma_i),$$

where the last equality is by plugging in the induction hypothesis. The theorem now follows from the definition of non-lazy up-down walk, i.e.

$$\mathsf{P}_{j+1}^{\wedge} = \frac{j+3}{j+2}\left(\mathsf{P}_{j+1}^{\triangle} - \frac{1}{j+3}\mathsf{I}\right) \quad \Longleftrightarrow \quad \mathsf{P}_{j+1}^{\triangle} = \frac{j+2}{j+3}\cdot\mathsf{P}_{j+1}^{\wedge} + \frac{1}{j+3}\mathsf{I}.$$

Therefore,

$$\lambda_2(\mathsf{P}_{j+1}^{\triangle}) = \frac{j+2}{j+3}\cdot\lambda_2(\mathsf{P}_{j+1}^{\wedge}) + \frac{1}{j+3} \leq 1 - \frac{1}{j+3}\prod_{i=-1}^{j}(1-\gamma_i),$$

and this proves the induction step. □

3.1.2 Relation Between Up-Down and Down-Up Walks: Proof of Lemma 3.1.6

The proof of Lemma 3.1.6 will rest on few useful identities established in [KO18, DDFH18], which can be obtained through the "Garland Method", which decomposes the higher order random walk matrices into the random walk matrices of the links.

In the following, given $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ and $\alpha \in X(k-1)$, we use $\boldsymbol{f}_\alpha$ to denote the restriction of $\boldsymbol{f}$ to the entries in $\{\alpha \cup \{x\} \mid x \in X_\alpha(0)\}$. And recall that J_α is the projector to constant functions defined in Section 2.3.3

Lemma 3.1.7. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For all $\boldsymbol{f} \in \mathbb{R}^{X(j)}$ the following hold,*

1. $\langle \boldsymbol{f}, \mathsf{I}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}} \|\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathbb{E}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
2. $\langle \boldsymbol{f}, \mathsf{P}_j^{\triangledown}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}} \|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathbb{E}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
3. $\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \mathsf{M}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$.

We will provide a proof of Lemma 3.1.7 in Section 3.1.3 for completeness. We are ready to prove Lemma 3.1.6.

Proof of Lemma 3.1.6. Let $\boldsymbol{f} \in \mathbb{R}^{X(j)}$ be arbitrary. By Items (2) and (3) in Lemma 3.1.7, we write

$$\langle \boldsymbol{f}, (\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown})\boldsymbol{f}\rangle_{\Pi_j} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}} \left[\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right].$$

Notice that since M_α is a row-stochastic matrix (with top eigenvector $\mathbf{1}$) and the matrix J_α is the projector to its top eigenspace. Since both M_α and J_α are self-adjoint with respect to the inner-product defined by Π_0^α (see Section 2.3.3), it follows that

$$\mathsf{M}_\alpha - \mathsf{J}_\alpha \preceq_{\Pi_0^\alpha} \lambda_2(\mathsf{M}_\alpha) \cdot \mathsf{I}.$$

Moreover, since the matrix $\mathsf{M}_\alpha - \mathsf{J}_\alpha$ is only supported on the subspace perpendicular to $\mathbf{1}$, writing $\boldsymbol{f}_\alpha^{\perp\mathbf{1}}$ for the component of $\boldsymbol{f}_\alpha$ that is perpendicular to $\mathbf{1}$, we have

$$\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha^{\perp\mathbf{1}}, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha^{\perp\mathbf{1}}\rangle_{\Pi_0^\alpha}.$$

As J_α is the projector to constant functions we have, $\boldsymbol{f}_\alpha^{\perp\mathbf{1}} = (\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha$ and thus

$$\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle \;\leq\; \lambda_1(\mathsf{M}_\alpha - \mathsf{J}_\alpha) \cdot \|\boldsymbol{f}_\alpha^{\perp\mathbf{1}}\|^2_{\Pi_0^\alpha} \;\leq\; \lambda_2(\mathsf{M}_\alpha) \cdot \|(\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha}, \tag{3.2}$$

where the first inequality is by the Courant-Fischer-Weyl Theorem 2.1.4. Therefore,

$$\begin{aligned}
\langle \boldsymbol{f}, (\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\bigtriangledown})\boldsymbol{f}\rangle_{\Pi_j} &= \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right], && \text{(by Items (2) and (3) in Lemma 3.1.7)}\\
&\le \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\lambda_2(\mathsf{M}_\alpha)\cdot \|(\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|_{\Pi_0^\alpha}^2\right], && \text{(by Eq. (3.2))}\\
&\le \gamma_{j-1}\cdot \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\|(\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|_{\Pi_0^\alpha}^2\right],\\
&= \gamma_{j-1}\cdot \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\alpha, (\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right], && \text{(by } \langle \mathsf{J}_\alpha\boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\text{)}\\
&= \gamma_{j-1}\cdot\langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P}_j^{\bigtriangledown})\boldsymbol{f}\rangle_{\Pi_j} && \text{(by Items (1) and (2) in Lemma 3.1.7).}
\end{aligned}$$

This proves $\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\bigtriangledown} \preceq_{\Pi_j} \gamma_{j-1}(\mathsf{I} - \mathsf{P}_j^{\bigtriangledown})$.

To prove $\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\bigtriangledown} \succeq_{\Pi_k} \nu_{k-1}\cdot(\mathsf{I} - \mathsf{P}_k^{\bigtriangledown})$, we observe

$$\mathsf{M}_\alpha - \mathsf{J}_\alpha \ge \lambda_{\min}(\mathsf{M}_\alpha)\cdot \mathsf{I},$$

as J_α is the projector to the top-eigenspace of the matrix M_α. The proof, then follows analogously. □

3.1.3 Garland Method: Proof of Lemma 3.1.7

Here we provide a proof of Lemma 3.1.7 for completeness. These arguments are from [KO18, DDFH18].

First, we recall

Lemma 3.1.7. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For all $\boldsymbol{f} \in \mathbb{R}^{X(j)}$ the following hold,*

1. $\langle \boldsymbol{f}, \mathsf{I}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\|\boldsymbol{f}_\alpha\|_{\Pi_0^\alpha}^2 = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
2. $\langle \boldsymbol{f}, \mathsf{P}_j^{\bigtriangledown}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|_{\Pi_0^\alpha}^2 = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
3. $\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \mathsf{M}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$.

Proof. Item (1) can be proven from the identity

$$\Pi_j(\beta) = \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \frac{\Pi_j(\alpha\cup\{x\})}{k+1} = \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \Pi_{j-1}(\alpha)\cdot\Pi_0^\alpha(x),$$

where the last equality is by Eq. (2.13) that $\Pi_0^\alpha(x) = \frac{\Pi_j(\alpha\cup\{x\})}{(j+1)\cdot\Pi_{j-1}(\alpha)}$. Then,

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{I}\boldsymbol{f}\rangle_{\Pi_j} &= \sum_{\beta\in X(j)} \Pi_j(\beta)\cdot \boldsymbol{f}(\beta)^2 \\
&= \sum_{\beta\in X(j)} \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \Pi_{j-1}(\alpha)\cdot\Pi_0^\alpha(x)\cdot \boldsymbol{f}_\alpha(x)^2 \\
&= \sum_{\alpha\in X(j-1)} \Pi_{j-1}(\alpha)\cdot \sum_{x\in X_\alpha(0)} \Pi_0^\alpha(x)\cdot \boldsymbol{f}_\alpha(x)^2 \\
&= \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}.
\end{aligned}$$

Item (2) follows by appealing to the definition of the down-up walk that $\mathsf{P}_j^\triangledown = \mathsf{U}_{j-1}\mathsf{D}_j$, and so

$$\langle \boldsymbol{f}, \mathsf{P}_j^\triangledown \boldsymbol{f}\rangle_{\Pi_j} = \langle \boldsymbol{f}, \mathsf{U}_{j-1}\mathsf{D}_j\boldsymbol{f}\rangle_{\Pi_j} = \langle \mathsf{D}_j\boldsymbol{f}, \mathsf{D}_j\boldsymbol{f}\rangle_{\Pi_{j-1}}.$$

By the definition of the down operator and $\Pi_0^\alpha(x) = \frac{\Pi_j(\alpha\cup\{x\})}{(j+1)\cdot\Pi_{j-1}(\alpha)}$ from Eq. (2.13), it follows that $[\mathsf{D}_j\boldsymbol{f}](\alpha) = \sum_{x\in X_\alpha(0)} \Pi_0^\alpha(x)\cdot \boldsymbol{f}(\alpha\cup\{x\}) = \mathbb{E}_{x\sim\Pi_0^\alpha} \boldsymbol{f}_\alpha(x)$ and thus

$$\langle \boldsymbol{f}, \mathsf{P}_j^\triangledown \boldsymbol{f}\rangle_{\Pi_j} = \sum_{\alpha\in X(j-1)} \Pi_{j-1}(\alpha)\left(\mathop{\mathbb{E}}_{x\sim\Pi_0^\alpha} \boldsymbol{f}_\alpha(x)\right)^2 = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\left(\mathop{\mathbb{E}}_{x\sim\Pi_0^\alpha} \boldsymbol{f}_\alpha(x)\right)^2\right].$$

Observing that $\mathsf{J}_\alpha\boldsymbol{f}_\alpha = \mathbf{1}\cdot\mathbb{E}_{x\sim\Pi_0^\alpha} \boldsymbol{f}_\alpha(x)$ by the definition of the projector to constant functions and therefore $\|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \left(\mathbb{E}_{x\sim\Pi_0^\alpha} \boldsymbol{f}_\alpha(x)\right)^2$. Hence, Item (2) follows as

$$\langle \boldsymbol{f}, \mathsf{P}_j^\triangledown \boldsymbol{f}\rangle_{\Pi_j} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}} \|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha},$$

where we used that J_α is an orthogonal projection and so $\langle \mathsf{J}_\alpha\boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$.

For Item (3), by the definition of $\mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j$ and the definition of up operator,

$$\langle \boldsymbol{f}, \mathsf{P}_j^{\triangle}\boldsymbol{f}\rangle_{\Pi_j} \;=\; \langle \mathsf{U}_j\boldsymbol{f}, \mathsf{U}_j\boldsymbol{f}\rangle_{\Pi_{j+1}} \;=\; \sum_{\beta\in X(j+1)} \Pi_{j+1}(\beta)\cdot\sum_{x,y\in\beta}\frac{1}{|\beta|^2}\boldsymbol{f}(\beta\backslash x)\boldsymbol{f}(\beta\backslash y).$$

Now, by the definition of non-lazy up-down walk, we see that

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge}\boldsymbol{f}\rangle_{\Pi_j} &= \frac{j+2}{j+1}\cdot\langle \boldsymbol{f}, \mathsf{P}_j^{\triangle}\boldsymbol{f}\rangle_{\Pi_j} - \frac{1}{j+1}\langle \boldsymbol{f}, \boldsymbol{f}\rangle_{\Pi_j},\\
&= \sum_{\beta\in X(j+1)}\frac{\Pi_{j+1}(\beta)}{|\beta|\cdot(|\beta|-1)}\sum_{x,y\in\beta}\boldsymbol{f}(\beta\backslash x)\boldsymbol{f}(\beta\backslash y) - \frac{1}{j+1}\sum_{\alpha\in X(j)}\Pi_j(\alpha)\boldsymbol{f}(\alpha)\boldsymbol{f}(\alpha),
\end{aligned}$$

where we got the second inequality using $|\beta| = j+2$. Now, notice that sampling $\alpha\sim\Pi_j$ is the same as first sampling $\beta\sim\Pi_{j+1}$ and then sampling $x\sim\beta$ uniformly and considering $\beta\backslash x$, so we can get by Eq. (2.12) that

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge}\boldsymbol{f}\rangle_{\Pi_j} &= \sum_{\beta\in X(j+1)}\sum_{x,y\in\beta}\frac{\Pi_{j+1}(\beta)}{|\beta|\cdot(|\beta|-1)}\boldsymbol{f}(\beta\backslash x)\boldsymbol{f}(\beta\backslash y) - \frac{1}{(j+1)}\sum_{\beta\in X(j+1)}\Pi_{j+1}(\beta)\cdot\sum_{x\in\beta}\frac{\boldsymbol{f}(\beta\backslash x)\boldsymbol{f}(\beta\backslash x)}{j+2},\\
&= \sum_{\beta\in X(j+1)}\Pi_{j+1}(\beta)\sum_{\{x,y\}\in\beta}\frac{1}{\binom{|\beta|}{2}}\boldsymbol{f}(\beta\backslash x)\boldsymbol{f}(\beta\backslash y)
\end{aligned}$$

where we have obtained the last inequality by using $|\beta| = j+2$ and noticing that the sum kills the diagonal terms. Using $\tau = \beta\backslash\{x,y\}$ and the identity $\frac{\Pi_{j+1}(\beta)}{\binom{|\beta|}{2}} = \Pi_{j-1}(\tau)\cdot\Pi_1^{\tau}(\{x,y\})$ from Eq. (2.13), we can rewrite it as

$$\begin{aligned}
\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge}\boldsymbol{f}\rangle_{\Pi_j} &= \sum_{\beta\in X(j+1)}\sum_{\{x,y\}\in\beta}\Pi_1^{\tau}(\{x,y\})\cdot\Pi_{j-1}(\tau)\cdot\boldsymbol{f}(\tau\cup x)\boldsymbol{f}(\tau\cup y)\\
&= \sum_{\tau\in X(j-1)}\Pi_{j-1}(\tau)\sum_{\{x,y\}\in X_\tau(1)}\Pi_1^{\tau}(\{x,y\})\boldsymbol{f}(\tau\cup x)\boldsymbol{f}(\tau\cup y).
\end{aligned}$$

On the other hand, using the equation

$$\langle \boldsymbol{f}, \mathsf{M}_\tau\boldsymbol{f}\rangle_{\Pi_0^\tau} = \sum_{x\in X_\tau(0)}\Pi_0^\tau(x)\cdot\boldsymbol{f}(x)\cdot[\mathsf{M}_\tau\boldsymbol{f}](x) = \sum_{\{x,y\}\in X_\tau(1)}\boldsymbol{f}(x)\boldsymbol{f}(y)\cdot\Pi_1^\tau(x,y).$$

where we use $\mathsf{M}_\tau(x,y) = \frac{\Pi_1^\tau(x,y)}{2\Pi_0^\tau(x)}$ from Section 2.3.3, we can also write

$$\mathop{\mathbb{E}}_{\tau\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\tau, \mathsf{M}_\tau, \boldsymbol{f}_\tau\rangle_{\Pi_0^\tau}\right] = \sum_{\tau\in X(j-1)} \Pi_{j-1}(\tau)\cdot \sum_{\{x,y\}\in X_\tau(1)} \Pi_1^\tau(\{x,y\})\cdot \boldsymbol{f}_\tau(x)\cdot \boldsymbol{f}_\tau(y),$$

and this proves $\langle \boldsymbol{f}, \mathsf{P}_j^\wedge \boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\tau\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\tau, \mathsf{M}_\tau, \boldsymbol{f}_\tau\rangle_{\Pi_0^\tau}\right]$. □

3.1.4 Bounds for the Entire Spectrum: Proof of Theorem 3.1.2

We will prove Theorem 3.1.2 about the entire spectrum of the higher order random walks.

Theorem 3.1.2. *Let (X,Π) be a pure d-dimensional weighted simplicial complex. Then, for any $0 \le k \le d-1$ and for any $-2 \le r \le k$, the matrix $\mathsf{P}_k^\triangle$ has at most $|X(r)|$ eigenvalues with value strictly greater than*

$$1 - \frac{r+2}{k+2}\prod_{j=r}^{k-1}(1-\gamma_j),$$

where we adopt the conventions that $X(-2) = \varnothing$ and $\prod_{j=r}^{r-1}(1-\gamma_j) = 1$.

Proof. We prove by induction on k. The base case is when $k=0$, where $\mathsf{P}_0^\triangle = \frac{1}{2}\mathsf{M}_\varnothing + \frac{1}{2}\mathsf{I}$. For $r=-2$, the claim states that we have at most $|X(-2)| = 0$ eigenvalues that are strictly greater than 1, which is true since $\mathsf{P}_0^\triangle$ is a stochastic matrix. For $r=-1$, the claim tells us that that there are at most $|X(-1)| = 1$ eigenvalues which are strictly larger than $\frac{1}{2} + \frac{\gamma_{-1}}{2}$, which is true by the definition of $\gamma_{-1} = \lambda_2(\mathsf{M}_\varnothing)$ and that since $\mathsf{P}_0^\triangle$ is of rank $|X(0)|$. For $r=0$, the claim tells us that there are at most $|X(0)|$ eigenvalues that are greater than 0, which is true since this is the rank of $|X(0)|$.

For the induction step, suppose that there exists some $j \ge 1$ such that the claim of the theorem is true for all $-1 \le r \le j$. By Fact 2.1.6, $\mathsf{P}_{j+1}^\triangledown$ and $\mathsf{P}_j^\triangle$ have the same non-zero eigenvalues. By Lemma 3.1.6, $\mathsf{P}^{\wedge}_{j+1} \preceq_{\Pi_{j+1}} \gamma_j \mathsf{I} + (1-\gamma_j)\mathsf{P}^{\nabla}_j$, and thus for $-1 \le r \le j$ the matrix $\mathsf{P}^{\wedge}_{j+1}$ has at most $|X(r)|$ eigenvalues with value greater than

$$\gamma_j + (1-\gamma_j)\cdot\left(1 - \frac{r+2}{j+2}\prod_{i=r}^{j-1}(1-\gamma_i)\right) = 1 - \frac{r+2}{j+2}\prod_{i=r}^{j}(1-\gamma_i).$$

Using the definition of the non-lazy up-down walk, Eq. (non-lazy up-down walk), we have that for $-1 \le r \le j$, $\mathsf{P}^{\triangle}_{j+1}$ has at most $|X(r)|$ eigenvalues with value greater than

$$\frac{j+2}{j+3}\left(1 - \frac{r+2}{j+2}\prod_{i=r}^{j}(1-\gamma_i)\right) + \frac{1}{j+3} = 1 - \frac{r+2}{j+3}\prod_{i=r}^{j}(1-\gamma_i).$$

For $r = j+1$, the claim is trivial since the $\mathsf{P}^{\triangle}_{j+1}$ is of rank $|X(j+1)|$. $\square$

3.1.5 Longer Random Walks: Proof of Corollary 3.1.5

Corollary 3.1.5. *Let (X,Π) be a pure d-dimensional weighted simplicial complex. Let $0 \le a, h$ be such that $a+h \le d$. If X is a γ-local-spectral expander, then*

$$\lambda_2(\mathsf{P}^{(h)}_a) \le (1+\gamma)^h \cdot \frac{a+1}{a+h+1}.$$

We will use two basic facts in the proof.

Fact 3.1.8. *Let $\mathsf{M}_1 \in \mathbb{R}^{V\times U}$ and $\mathsf{M}_2 \in \mathbb{R}^{U\times W}$ be two row-stochastic matrices. Then, we have $\sigma_2(\mathsf{M}_1\cdot\mathsf{M}_2) \le \sigma_2(\mathsf{M}_1)\cdot\sigma_2(\mathsf{M}_2)$.*

Fact 3.1.9 (Bernoulli's Inequality)**.** *Let $x \ge -1$ and $r \ge 1$ be real numbers. Then, $(1+x)^r \ge 1 + r\cdot x$.*

Proof. Recall that $\mathsf{P}^{(h)}_a = \mathsf{D}_{a+1}\cdots\mathsf{D}_{a+h}\cdot\mathsf{U}_{a+h-1}\cdots\mathsf{U}_a$. As $\mathsf{U}_i^* = \mathsf{D}_{i+1}$, it can be observed that $\mathsf{P}^{(h)}_a$ is positive semi-definite and therefore, $\sigma_2(\mathsf{P}^{(h)}_a) = \lambda_2(\mathsf{P}^{(h)}_a)$. By Fact 3.1.8,

$$\lambda_2(\mathsf{P}^{(h)}_a) = \sigma_2(\mathsf{P}^{(h)}_a) \le \sigma_2(\mathsf{D}_{a+1})\cdots\sigma_2(\mathsf{D}_{a+h})\cdot\sigma_2(\mathsf{U}_{a+h-1})\cdots\sigma_2(\mathsf{U}_a).$$

Notice that as $\mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j$ and $\mathsf{D}_{j+1}^* = \mathsf{U}_j$, we have $\lambda_2(\mathsf{P}_j^{\triangle}) = \sigma_2(\mathsf{U}_j) \cdot \sigma_2(\mathsf{D}_{j+1})$. Thus, by rearranging we obtain,

$$\begin{aligned}
\lambda_2(\mathsf{P}_a^{(h)}) &\le \prod_{j=0}^{b-a-1} \lambda_2(\mathsf{P}_{a+j}^{\triangle}), \\
&\le \prod_{j=0}^{h-1}\left(1 - \frac{(1-\gamma)^{a+j+1}}{a+j+2}\right), && \text{(by Theorem 3.1.1)} \\
&\le \prod_{j=0}^{h-1}\left(1 - \frac{1-(a+j+1)\cdot\gamma}{a+j+2}\right), && \text{(by Fact 3.1.9)} \\
&= \prod_{j=0}^{h-1}\left((1+\gamma)\frac{a+j+1}{a+j+2}\right).
\end{aligned}$$

By cancellations in the telescoping product, we have

$$\lambda_2(\mathsf{P}_a^{(h)}) \le (1+\gamma)^h \cdot \frac{a+1}{a+h+1}.$$

□

3.1.6 Lower Bound for the Expansion: Proof of Proposition 3.1.3

In this section, we will prove Proposition 3.1.3, i.e.

Proposition 3.1.3. *Let X be a d-dimensional simplicial complex. Let $n = |X(0)|$. Suppose $2(d+1) \le n$. Then $\lambda_2(\mathsf{P}_d^{\triangledown}) = \lambda_2(\mathsf{P}_{d-1}^{\triangle}) \ge 1 - \frac{2}{d+1}$.*

The proof will be based on the easy side of Cheeger's inequality, Theorem 2.2.6.

Proof of Proposition 3.1.3. It is clear that there should exist a vertex $v \in X(0)$ such that $\Pi_0(v) \le \frac{1}{|X(0)|} = \frac{1}{n}$.

We consider the set $A_v \subset X(d)$ consisting of all faces in $X(d)$ containing the vertex v, i.e. $A_v = \{\beta \in X(d) : v \in \beta\}$. Note that,

$$\begin{aligned}
\Pi_d(A_v) &= (d+1) \cdot \Pi_0(v), && \text{(by using Eq. (2.12) repeatedly)}, \\
&\leq \frac{d+1}{n}, && \text{(by using } \Pi_0(v) \leq \frac{1}{n}\text{)}, \\
&\leq \frac{1}{2}. && \text{(by using } 2(d+1) \leq n\text{)}
\end{aligned}$$

By Theorem 2.2.6,

$$\frac{1-\lambda_2(\mathsf{P}_d^{\triangledown})}{2} \leq \min_{S:\Pi_d(S)\leq 1/2} \Phi(S) \leq \Phi(A_v).$$

We recall that the random-walk $\mathsf{P}_d^{\triangledown}$ starting from a face $\beta \in X(d)$ first picks an index $i \sim \beta$ uniformly at random, and then picks some face $\beta' \supset (\beta \backslash i)$ with probability proportional to $\Pi_d(\beta')$. If $\beta \in A_v$ the only way we leave A_v in a single step is when the index i we pick from β is v, which happens with probability $1/|\beta| = 1/(d+1)$.

Writing $(X_t)_{t\geq 0}$ for the state of the random walk, this means for any $\beta \in A_v$ we have

$$\Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \frac{1}{d+1}.$$

It follows that

$$\frac{1-\lambda_2(\mathsf{P}_d^{\triangledown})}{2} \leq \Phi(A_v) = \sum_{\beta\in A_v} \frac{\Pi_d(\beta)}{\Pi_d(A_v)} \cdot \Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \max_{\beta\in A_v} \Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \frac{1}{d+1}.$$

Solving the expression for $\lambda_2(\mathsf{P}_d^{\triangledown})$ proves the proposition. $\square$

3.2 Analyzing Mixing Times of Markov Chains

In this section, we will use Corollary 3.1.4 to analyze Markov chains for sampling independent sets of a graph of fixed size and sampling common independent sets of two partition matroids.

3.2.1 Sampling Independent Sets

Let $G = (V, E)$ be a graph. A subset of vertices $S \subset V$ is called an independent set if $uv \notin E$ for every pair $u, v \in S$. We are interested in the problem of sampling a uniformly random independent set of size k. We will analyze a natural Markov chain for the problem by analyzing the down-up walk of a corresponding simplicial complex.

Define the $(k-1)$-dimensional simplicial complex $I_{G,k}$ of $G = (V, E)$ as

$$I_{G,k} = \{S \subset V : |S| \le k \text{ and } S \text{ is independent}\},$$

the complex consisting of all independent sets in G of cardinality at most k. We endow $I_{G,k}$ with the uniform distribution Π_{k-1} on $I_{G,k}(k-1)$, i.e. the set of independent sets of size k. We simply write $I_{G,k}$ for the weighted simplicial complex $(I_{G,k}, \Pi_{k-1})$.

The $(k-1)$-th down-up walk $\mathsf{P}_{k-1}^{\triangledown}$ on $I_{G,k}$ corresponds to a natural Markov chain to sample independent sets of size k. It is known that this Markov chain is fast mixing when $k \le \frac{|V|}{2\Delta+1}$ using coupling techniques [BD97, MU05]. The main result in this subsection is the following improved bound using higher order random walks on simplicial complexes.

Theorem 3.2.1. *Let $G = (V, E)$ be a graph with maximum degree Δ. Let $\mathsf{P}_{k-1}^{\triangledown}$ be the $(k-1)$-th down-up walk on the simplicial complex $I_{G,k}$. Let A_G be the adjacency matrix of G.*

$$\text{If} \quad k \le \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}, \quad \text{then} \quad \lambda_2(\mathsf{P}_{k-1}^{\triangledown}) \le 1 - \frac{1}{k^2}.$$

It is well-known that $|\lambda_{\min}(\mathsf{A}_G)| \le \Delta$ for a graph with maximum degree Δ, and so Theorem 3.2.1 recovers the previous result that the Markov chain is fast mixing if $k \le \frac{|V|}{2\Delta}$. There are various graph classes with $|\lambda_{\min}(\mathsf{A}_G)|$ smaller than Δ, and Theorem 3.2.1 allows us to sample larger independent sets. For example, it is known that $|\lambda_{\min}(\mathsf{A}_G)| \le O(\sqrt{\Delta})$ for planar graphs and more generally for graphs with bounded arboricity [Hay06], and also for random graphs and more generally for two-sided expander graphs [HLW06].

Using the simple bound $\min_{S \in I_{G,k}(k-1)} \Pi_{k-1}(S) \ge n^{-k}$ as Π_{k-1} is the uniform distribution, the following mixing time result follows from Theorem 2.2.7.

Corollary 3.2.2. *Let $G = (V, E)$ be a graph with maximum degree Δ and let A_G be the adjacency matrix of G. For any $k \le n/(\Delta + |\lambda_{\min}(\mathsf{A}_G)|)$, the down-up walk $\mathsf{P}_{k-1}^{\triangledown}$ on the simplicial complex $I_{G,k}$ samples a random independent set of G of size k whose distribution* *is ε-close to the uniform distribution on all independent sets of size k in the total variation distance in*

$$T(\varepsilon, \mathsf{P}_{k-1}^{\bigtriangledown}) \leq k^2 \cdot \left(\log\left(\frac{1}{\varepsilon}\right) + k \cdot \log n\right)$$

many time steps.

This implies a polynomial time algorithm to approximately sample a uniform random independent set and also a FPRAS for approximately counting the number of independent set of size k for $k \leq \frac{n}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}$.

Proof of Theorem 3.2.1

The plan is to use Corollary 3.1.4 to prove Theorem 3.2.1. To apply Corollary 3.1.4, we need to prove that:

1. $I_{G,k}$ is a pure simplicial complex. It is a simple exercise that this complex is pure when $k \leq \frac{n}{\Delta+1}$.
2. For each $S \in I_{G,k}$ with $|S| \leq k-2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$. This is proved in Lemma 3.2.3.
3. For each $S \in I_{G,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \leq 1/k$. This is proved in Lemma 3.2.4.

Assuming the three items are proven, Theorem 3.2.1 follows immediately from Corollary 3.1.4. We will prove the second item in Section 3.2.1 and the third item in Section 3.2.1.

Proof of Lemma 3.2.3

Let $H_S = (V_S, E_S)$ be the underlying support graph of G_S of the link $(I_{G,k})_S$, i.e. G_S without edge weights. Let M_S be the random walk matrix of G_S as defined in Section 2.3.3. Note that $\lambda_2(\mathsf{M}_S) < 1$ if and only if H_S is connected.

We introduce some notation to describe H_S. We write $N_G[S]$ as the union of S and the set of vertices which are connected to a vertex in S in G, i.e.

$$N_G[S] = S \cup \{v : \text{there exists some } uv \in E(G) \text{ such that } u \in S\}.$$

For a subset of vertices $S \subset V(G)$, we write $\overline{S} = V(G) \setminus S$ for the complement of S in G, and $G[S]$ for the induced subgraph of G on S. For a graph H, we write $\overline{H}$ for the complement graph of H.

Recall that a vertex v is in V_S if and only if $S \cup \{v\}$ is an independent set in G of size $|S|+1$, and so V_S is exactly $V - N_G[S] = \overline{N_G[S]}$. Two vertices $u, v \in V_S$ have an edge in H_S if and only if $S \cup \{u, v\}$ is an independent set in G of size $|S|+2$, and so $uv \in E_S$ if and only if $uv \notin E(G)$. Therefore, we see that

$$H_S = \overline{G[V_S]} = \overline{G[\overline{N[S]}]}.$$

With the description of H_S, we are ready to prove the second item in Section 3.2.1.

Lemma 3.2.3. *Let $G = (V, E)$ be a graph with maximum degree Δ. Suppose $k \leq \frac{|V|}{\Delta+1}$. For any $S \in I_{G,k}$ with $|S| \leq k-2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$.*

Proof. Note that $\lambda_2(\mathsf{M}_S) < 1$ if and only if the underlying support graph H_S of G_S is connected, so we focus on proving the latter. To prove that H_S is connected, we prove the stronger claim that every two vertices $u, v \in H_S$ has a path of length at most two. If uv is an edge in H_S, then there is a path of length one. Suppose uv is not an edge in H_S. Then uv is an edge in G. Since G is of maximum degree Δ, it implies that $|N_G[\{u, v\}]| \leq (\deg_G(u)+1) + (\deg_G(v)+1) - 2 \leq 2\Delta$, and also

$$|V_S| = |V| - |N_G[S]| \geq |V| - |S| \cdot (\Delta+1) \geq 2\Delta + 2,$$

where we use the assumptions that $|S| \leq k-2 \leq \frac{|V|}{\Delta+1} - 2$ in the last inequality. So, there must be some vertex w such that $w \in V_S \setminus N_G[\{u, v\}]$. This implies that $wu \notin E(G)$ and $wv \notin E(G)$, and thus $wu \in E(H_S)$ and $wv \in E(H_S)$ and so there is a path of length two connecting u and v in H_S. $\square$

Proof of Lemma 3.2.4

We observe that G_S is an unweighted graph for S with $|S| = k-2$ when the distribution on $I_{G,k}(k-1)$ is the uniform distribution. Therefore, G_S is simply a scaled version of H_S, and the random walk matrix M_S of G_S is the same as the random walk matrix of H_S. To bound the second eigenvalue, we will use some simple interlacing arguments. We need the stronger assumption that $k \leq \frac{|V(G)|}{\Delta + |\lambda_{\min}(\mathsf{A}_G|)}$ in the proof of the following lemma. (Note that for any unweighted graph G, we have $|\lambda_{\min}(\mathsf{A}_G)| \geq 1$.)

Lemma 3.2.4. *Let $G = (V, E)$ be a graph with maximum degree Δ. Suppose $k \leq |V|/(\Delta + |\lambda_{\min}(\mathsf{A}_G)|)$. For any $S \in I_{G,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) \leq \frac{1}{k}$.*

Proof of Lemma 3.2.4. Recall that for S with $|S| = k-2$, the random walk matrix M_S of G_S is the same as the random walk matrix of H_S, and so we will focus on the latter. Let D_H be diagonal degree matrix of H_S. As argued above, the random walk matrix M_S of G_S is equal to $\mathsf{M}_S = \mathsf{D}_H^{-1}\mathsf{A}_H$. We can write the adjacency matrix A_H of H_S as

$$\mathsf{A}_H = \mathbf{1}\mathbf{1}^\top - \mathsf{I} - \mathsf{A}_{G[\overline{N[S]}]},$$

where $\mathsf{A}_{G[\overline{N[S]}]}$ is the adjacency matrix of $G[\overline{N[S]}]$. By Weyl's interlacing theorem,

$$\begin{aligned}
\lambda_2(\mathsf{M}_S) &\leq \lambda_2(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2}) + \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(by Theorem 2.1.10)} \\
&= \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I})\mathsf{D}_H^{-1/2}), && (\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} \text{ is of rank 1}) \\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot \lambda_1(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I}), && \text{(by Theorem 2.1.4)} \qquad (3.3) \\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_{G[\overline{N[S]}]})| - 1), && (\lambda_1(-\mathsf{A}_{G[\overline{N[S]}]}) = -\lambda_{\min}(\mathsf{A}_{G[\overline{N[S]}]})) \\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_G)| - 1), && \text{(by Theorem 2.1.9)}
\end{aligned}$$

For Eq. (3.3), we have used the consequence of the Courant-Fischer-Weyl Theorem 2.1.4 $\lambda_1(\mathsf{W}) = \max\{\langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\| = 1\}$ in the following way: Let $\mathsf{W} = -\mathsf{A}_L - \mathsf{I}$ and $\boldsymbol{g}$ be an unit top-eigenvecor of $\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}$, i.e. $\|\boldsymbol{g}\| = 1$ and $\langle \boldsymbol{g}, \mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}\boldsymbol{g}\rangle = \lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})$. Then,

$$\lambda_1(\mathsf{W}) \geq \left\langle \frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|}, \mathsf{W}\frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|} \right\rangle = \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1/2}\|^2} \geq \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1}\|}.$$

It remains to bound $\|\mathsf{D}_H^{-1}\| = (\min_v \deg_{H_S}(v))^{-1}$. As $H_S = \overline{G[\overline{N[S]}]} = \overline{G[V - N[S]]}$,

$$\deg_{H_S}(v) = |V| - |N[S]| - (\deg_{G[\overline{N[S]}]}(v) + 1) \geq |V| - (\Delta + 1)(|S| + 1),$$

where the last inequality uses that $|N[S]| \leq |S| \cdot (\Delta + 1)$ and $\deg_{G[\overline{N[S]}]}(v) \leq \deg_G(v) \leq \Delta$. Therefore, using our bound $\lambda_2(\mathsf{M}_S) \leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_G)| - 1)$, we obtain

$$\lambda_2(\mathsf{M}_S) \leq \frac{|\lambda_{\min}(\mathsf{A}_G)| - 1}{|V| - (\Delta + 1) \cdot (|S| + 1)} = \frac{|\lambda_{\min}(\mathsf{A}_G)| - 1}{|V| - (\Delta + 1) \cdot (k - 1)},$$

where we use $|S| = k - 2$. Finally, plugging in the assumption

$$k \leq \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|} \quad \Longrightarrow \quad \lambda_2(\mathsf{M}_S) \leq \frac{1}{k}.$$

□

3.2.2 Sampling Matroid Intersection

A matroid $M = (E, \mathcal{I})$ on the ground set E with the set of independent sets $\mathcal{I} \subset 2^E$ is a combinatorial object satisfying the following properties:

- (containment property) if $S \in \mathcal{I}$ and $T \subset S$, then $T \in \mathcal{I}$,
- (extension property) if $S, T \in \mathcal{I}$ such that $|S| > |T|$ then there is some $x \in S \backslash T$ such that $\{x\} \cup T \in \mathcal{I}$.

A partition matroid is the special case where the ground set E is partitioned into disjoint blocks $B_1, \ldots, B_l \subseteq E$ with parameters $0 \leq d_i \leq |B_i|$ for $1 \leq i \leq l$, and a subset S is in $\mathcal{I}$ if and only if $|S \cap B_i| \leq d_i$ for $1 \leq i \leq l$.

The intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I})$ over the same ground set E can be used to formulate various interesting combinatorial optimization problems [Sch03]. We are interested in the problem of sampling a uniform random common independent set of size k, i.e. a random subset $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|F| = k$. We will analyze a natural Markov chain for the problem by analyzing the down-up walk of a corresponding simplicial complex.

Define the $(k-1)$-dimensional matroid intersection complex $C_{M_1,M_2,k}$ of $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ as

$$C_{M_1,M_2,k} = \{S \in \mathcal{I}_1 \cap \mathcal{I}_2 : |S| \leq k\},$$

the complex consisting of all common independent sets of both matroids containing at most k elements. We endow $C_{M_1,M_2,k}(k-1)$ with the uniform distribution Π_{k-1} on the common independent sets $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|S| = k$. We write $C_{M_1,M_2,k}$ for the weighted simplicial complex $(C_{M_1,M_2,k}, \Pi_{k-1})$.

The $(k-1)$-th down-up walk $\mathsf{P}^{\triangledown}_{k-1}$ on $C_{M_1,M_2,k}$ corresponds to the following natural Markov chain to sample common independent sets of size k. Initially, the random walk starts from a common independent set S_1 of size k. In each step $t \geq 1$, we choose a uniform random element $i \in S_t$ and delete i from S_t, and set S_{t+1} to be a uniform random common independent set of size k that contains $S_t \setminus \{i\}$. The stationary distribution of $\mathsf{P}^{\triangledown}_{k-1}$ is the uniform distribution Π_{k-1}; see Section 2.4 and Proposition 2.2.1.

The main result in this subsection is the following upper bound on the second eigenvalue of $\mathsf{P}^{\triangledown}_{k-1}$.

Theorem 3.2.5. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two given partition matroids with a common independent set of size r and no two elements belonging to the same block in both matroids. If $k \leq r/3$, then*

$$\lambda_2(\mathsf{P}^{\triangledown}_{k-1}) \leq 1 - \frac{1}{k^2},$$

where $\mathsf{P}^{\triangledown}_{k-1}$ is the $(k-1)$-th down-up walk on the matroid intersection complex $C_{M_1,M_2,k}$.

Using the simple bound $\min_{S \in C_{M_1,M_2,k}(k-1)} \Pi_{k-1}(S) \geq n^{-k}$ as Π_{k-1} is the uniform distribution, the following mixing time result follows from Theorem 2.2.7.

Corollary 3.2.6. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two given partition matroids with a common independent set of size r and no two elements belonging to the same block in both matroids. For any $k \leq r/3$ the down-up walk $\mathsf{P}^{\triangledown}_{k-1}$ on the simplicial complex $C_{M_1,M_2,k}$ samples a random common independent set of M_1 and M_2 of size k whose distribution is ε-close to the uniform distribution on all common independent sets of size k in the total variation distance in*

$$T(\varepsilon, \mathsf{P}^{\triangledown}_{k-1}) \leq k^2 \cdot \left(\log\left(\frac{1}{\varepsilon}\right) + k \cdot \log n\right)$$

many time steps.

This implies a polynomial time algorithm to approximately sample a uniform random common independent set of two partition matroids M_1 and M_2 of size k and also a FPRAS for approximately counting the number of independent set of size k given $k \le \frac{r}{3}$

Proof of Theorem 3.2.5

The plan is to use Corollary 3.1.4 to prove Theorem 3.2.5. To apply Corollary 3.1.4, we need to prove that:

1. $C_{M_1,M_2,k}$ is a pure simplicial complex. This is a simple proof in Claim 3.2.7.

2. For each $S \in C_{M_1,M_2,k}$ with $|S| \le k-2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$. This is proved in Lemma 3.2.8, showing that the underlying graph of G_S is the complement of the line graph of a bipartite graph.

3. For each $S \in C_{M_1,M_2,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \le 1/k$. This is proved in Lemma 3.2.11, using the fact that the minimum eigenvalue of the adjacency matrix of the line graph of a simple graph is at least -2.

Assuming the three items are proven, Theorem 3.2.5 follows immediately from Corollary 3.1.4.

It remains to prove the three items. We will prove the second item in Section 3.2.2 and the third item in Section 3.2.2. We note that the first two items hold for any two matroids, and we only use the additional assumptions for the third item. The following is a simple proof for the first item.

Claim 3.2.7. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$. Any common independent set $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|S| < r/2$ is contained in a larger common independent set. In particular, this implies that the simplicial complex $C_{M_1,M_2,k}$ is a pure simplicial complex as long as $k \le r/2$.*

Proof. By the extension property of matroids, there is a subset $T_1 \subset T$ with $|T_1| \geq r - |S|$ such that $S \cup \{x\}$ is an independent set in $\mathcal{I}_1$ for any $x \in T_1$. Similarly, there is a subset $T_2 \subset T$ with $|T_2| \geq r - |S|$ such that $S \cup \{y\}$ is an independent set in $\mathcal{I}_2$ for any $y \in T_2$. As $|S| < r/2$, this implies that $T_1 \cap T_2 \neq \emptyset$, and $S \cup \{z\}$ is a larger independent set that contains S for any $z \in T_1 \cap T_2$. $\square$

Proof of Lemma 3.2.8

Let $H_S = (E_S, F_S)$ be the underlying support graph of G_S of the link $(C_{M_1,M_2,k})_S$, that is, H_S is G_S without edge weights. The vertex set of H_S is $E_S = \{x \in E \mid S \cup \{x\} \in \mathcal{I}_1 \cap \mathcal{I}_2\}$ and the edge set of H_S is $F_S = \{\{x, y\} \mid x, y \in E \text{ and } S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2\}$. Let M_S be the random walk matrix of G_S as defined in Section 2.3.3. It is a basic fact in spectral graph theory that $\lambda_2(\mathsf{M}_S) < 1$ if and only if H_S is connected.

We will see that H_S is the complement of the line graph of a bipartite graph B. To define the bipartite graph B, we first introduce the matroid partition property (see e.g. [ALOV19]). The matroid partition property says that there is a partition $\mathcal{P} := \{P_1, \ldots, P_p\}$ of the vertex set E_S (i.e. $\bigcup_{i=1}^{p} P_i = E_S$ and $P_i \cap P_j = \emptyset$ for $i \neq j$) with the property that for any $x, y \in E_S$,

$$S \cup \{x, y\} \notin \mathcal{I}_1 \quad \Longleftrightarrow \quad x, y \in P_i \text{ for some } 1 \leq i \leq p.$$

In words, there is a partition $\mathcal{P}$ of the vertex set E_S such that two elements x, y in E_S can be added to S to form an independent set in the first matroid M_1 if and only if x, y do not belong to the same class of the partition $\mathcal{P}$. Similarly, there is a partition $\mathcal{Q} := \{Q_1, \ldots, Q_q\}$ of the vertex set E_S such that for any two elements $x, y \in E_S$, we have $S \cup \{x, y\} \notin \mathcal{I}_2$ if and only if $x, y \in Q_i$ for some $1 \leq i \leq q$.

We use the partitions $\mathcal{P}$ and $\mathcal{Q}$ to define the bipartite graph B as follows. The vertex set of B is $P \sqcup Q$, where we create a vertex $i \in P$ in B for each P_i in $\mathcal{P}$, and we create a vertex $j \in Q$ in B for each Q_j in $\mathcal{Q}$. Each edge in B corresponds to an element in E_S. For each element $x \in E_S$, we create the edge $e_x = ij$ in B if and only if $x \in P_i$ and $x \in Q_j$. Note that the edge e_x for $x \in E_S$ is well-defined by the matroid partition property. By construction, it should be clear that the bipartite graph B satisfies the following important property:

$$e_x \text{ and } e_y \text{ do not share a vertex in } B \quad \Longleftrightarrow \quad S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2 \quad \Longleftrightarrow \quad \{x, y\} \in F_S. \tag{3.4}$$

Recall that the line graph $L(B)$ of a graph B is defined as follows: the vertex set of $L(B)$ is the edge set of B, and two vertices in $L(B)$ have an edge if and only if the corresponding edges in B share an endpoint. Let $\overline{L(B)}$ be the complement of $L(B)$ where $\overline{L(B)}$ and $L(B)$ have the same vertex set and two vertices in $\overline{L(B)}$ have an edge if and only if the corresponding vertices in $L(B)$ do not have an edge. Then, we see from Eq. (3.4) that

$$H_S = \overline{L(B)}. \tag{3.5}$$

Using the bipartite graph B, it is easy to show the second item in Section 3.2.2.

Lemma 3.2.8. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$. Suppose $k < r/2 - 1$. For any $S \in C_{M_1,M_2,k}$ with $|S| \leq k - 2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$.*

Proof. It is well known that $\lambda_2(\mathsf{M}_S) < 1$ if and only if the underlying support graph H_S of G_S is connected, so we focus on proving the latter. Since $|S| \leq k - 2 < r/2 - 3$, it follows from Claim 3.2.7 that there are four elements $a, b, c, d \in E$ such that $S \cup \{a, b, c, d\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. In the bipartite graph B in Eq. (3.5), the four elements a, b, c, d correspond to four vertex-disjoint edges e_a, e_b, e_c, e_d in B by Eq. (3.4). To prove that H_S is connected, we prove the stronger claim that every two vertices $u, v \in H_S$ has a path of length at most two. If uv is an edge in H_S, then there is a path of length one. Suppose uv is not an edge in H_S. Then e_u and e_v shares a vertex in B and so they span at most three vertices in B. This implies that $e_u \cup e_v$ cannot intersect all four (vertex-disjoint) edges e_a, e_b, e_c, e_d. So there must be an edge, say e_a, which is vertex-disjoint from both e_u and e_v. Then u-a-v is path of length two in H_S by Eq. (3.4), which completes the proof. □

Proof of Lemma 3.2.11

For the third term, we need to prove that for each $S \in C_{M_1,M_2,k}$ with $|S| = k - 2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \leq \frac{1}{k}$. We use the additional assumptions for the following property.

Claim 3.2.9. *If M_1 and M_2 are two partition matroids and there are no two elements x, y such that x, y belongs to the same block in M_1 and also the same block in M_2, then Eq. (3.5) holds with the property that the bipartite graph B is a simple graph.*

Observe that G_S is an unweighted graph for S with $|S| = k-2$ when the distribution on $C_{M_1,M_2,k}(k-1)$ is the uniform distribution (i.e. the distribution on the common independent sets of size k is the uniform distribution). This is because when $|S| = k-2$, for any $x, y \in E$, either $S \cup \{x, y\}$ is contained in exactly one or zero set of size k in $C_{M_1,M_2,k}$, and each set of size k is assigned the same weight in the uniform distribution (more formally see Eq. (2.13) for the definition of the weight). Therefore, G_S is simply a scaled version of H_S, and the random walk matrix M_S of G_S is the same as the random walk matrix of H_S.

Fact 3.2.10. *Let $G = (V, E)$ be any simple graph and $\mathsf{A}_{L(G)}$ be the adjacency matrix of the line graph of G. It holds that $\lambda_{\min}(\mathsf{A}_{L(G)}) \geq -2$.*

Proof. Define $\mathsf{B} \in \mathbb{R}^{E \times V}$ to be the edge-vertex incidence matrix of $G = (V, E)$, i.e. $\mathsf{B}(e, v) = 1[v \in e]$. Observe that

$$2\mathsf{I} + \mathsf{A}_{L(G)} = \mathsf{B}\mathsf{B}^\top \quad \Longrightarrow \quad \lambda_{\min}\left(2\mathsf{I} + \mathsf{A}_{L(G)}\right) = \lambda_{\min}\left(\mathsf{B}\mathsf{B}^\top\right) \geq 0,$$

as $\mathsf{B}\mathsf{B}^\top$ is a positive semidefinite matrix. This implies that $\lambda_{\min}(\mathsf{A}_{L(G)}) \geq -2$. □

We are ready to bound the second eigenvalue of M_S. We need the stronger assumption that $k \leq \frac{r}{3}$ in the proof of the following lemma.

Lemma 3.2.11. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two partition matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$ and there are no two elements belonging to the same block in both matroids. Suppose $k \leq r/3$. For any $S \in C_{M_1,M_2,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) \leq 1/k$.*

Proof. Recall that for S with $|S| = k-2$, the random walk matrix M_S of G_S is the same as the random walk matrix of H_S, and so we will focus on the latter. By Claim 3.2.9, H_S

is the complement of the line graph of a simple graph, and so we can write the adjacency matrix A_H of H_S as

$$\mathsf{A}_H = \mathbf{1}\mathbf{1}^\top - \mathsf{I} - \mathsf{A}_L,$$

where A_L is the adjacency matrix of the line graph $L(B)$ in Eq. (3.5). Let D_H be diagonal degree matrix of H_S. As argued above, the random walk matrix M_S of G_S is equal to $\mathsf{M}_S = \mathsf{D}_H^{-1}\mathsf{A}_H$. Using that D_H^{-1} and $\mathsf{D}_H^{-1/2}\mathsf{A}_H\mathsf{D}_H^{-1/2}$ are similar matrices and have the same spectrum, we have

$$\begin{aligned}
\lambda_2(\mathsf{M}_S) &= \lambda_2(\mathsf{D}_H^{-1/2}\mathsf{A}_H\mathsf{D}_H^{-1/2}), \\
&= \lambda_2(\mathsf{D}_H^{-1/2}(\mathbf{1}\mathbf{1}^\top - \mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}), \\
&= \lambda_2\Big(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} + \mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}\Big).
\end{aligned}$$

Using the Weyl Interlacing Theorem 2.1.10,

$$\begin{aligned}
\lambda_2(\mathsf{M}_S) &\leq \lambda_2(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2}) + \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(by Theorem 2.1.10)} \\
&= \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}), && (\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} \text{ is of rank 1}) \\
&\leq \|\mathsf{D}_H^{-1}\| \cdot \lambda_1(-\mathsf{A}_L - \mathsf{I}), && \text{(by Theorem 2.1.4)} \quad (3.6) \\
&\leq \|\mathsf{D}_H^{-1}\| \cdot (|\lambda_{\min}(\mathsf{A}_L)| - 1), && (\text{by using } \lambda_1(-\mathsf{A}_L) = -\lambda_{\min}(\mathsf{A}_L)) \\
&= \|\mathsf{D}_H^{-1}\|. && \text{(by Fact 3.2.10)} \quad (3.7)
\end{aligned}$$

For Eq. (3.6), we have used the implication of Theorem 2.1.4,

$$\lambda_1(\mathsf{W}) = \max\big\{\langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\| = 1\big\}$$

in the following way: Let $\mathsf{W} = -\mathsf{A}_L - \mathsf{I}$ and $\boldsymbol{g}$ be an unit top-eigenvector of $\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}$, i.e. $\|\boldsymbol{g}\| = 1$ and $\langle \boldsymbol{g}, \mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}\boldsymbol{g}\rangle = \lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})$. Then,

$$\lambda_1(\mathsf{W}) \geq \left\langle \frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|}, \mathsf{W}\frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|} \right\rangle = \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1/2}g\|^2} \geq \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1}\|}.$$

It remains to bound $\|\mathsf{D}_H^{-1}\| = (\min_x \deg_{H_S}(x))^{-1}$. By the definition of H_S, the degree $\deg_{H_S}(x)$ of $x \in E$ is equal to the number of elements $y \in E \setminus (S \cup \{x\})$ such that

$S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. By our assumption, there is a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size r. Since $|S \cup \{x\}| = k - 1$, by the extension property of the first matroid M_1, there are at least $r - k + 1$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_1$. Similarly, there are at least $r - k + 1$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_2$. Therefore, there are at least $r - 2k + 2$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. This implies that for any $x \in V(H_S)$,

$$\deg_{H_S}(x) \geq r - 2k + 2 \geq k \quad \implies \quad \lambda_2(\mathsf{M}_S) \leq \|\mathsf{D}_H^{-1}\| \leq \frac{1}{k},$$

where we use the assumption that $k \leq \frac{r}{3}$. $\square$

Chapter 4

Expansion Swap Walks

4.1 Statement of Results

Swap walks $\mathsf{S}_{k,l}$ arise naturally in several applications ranging from approximation algorithms for k-CSPs [AJT19], to list decoding of codes [AJQ+20], to agreement testing [DD19]. For these applications, one typically thinks of k being constant or independent of $n := |X(0)|$ – the size of the ground set. For this reason, we analyze the spectra of swap walks. We show that swap walks $\mathsf{S}_{k,l}$ over simplicial complexes (X, Π) which are two-sided γ-local spectral expanders are indeed expanding for γ sufficiently small.

Due to technical reasons, we study the case of square swap walks $\mathsf{S}_k := \mathsf{S}_{k,k}$ and rectangular swap walks $\mathsf{S}_{k,l}$ (where $k \neq l$) independently.

For the square case we prove,

Theorem 4.1.1. *Let (X, Π) be a pure d-dimensional two-sided γ local spectral expander such that $\gamma \leq \epsilon \left(64 k^{k+4} 2^{3k+2}\right)^{-1}$ where $\epsilon \in (0,1)$ and $k \geq 0$ some parameter such that $d \geq 2k+1$. Then the second largest singular value $\sigma_2(\mathsf{S}_k)$ of the swap walk S_k on $X(k)$ is*

$$\sigma_2(\mathsf{S}_k) \;\leq\; \epsilon.$$

For the rectangular case we prove,

Theorem 4.1.2. *Let (X,Π) be a pure d-dimensional two-sided γ-local spectral expander. Suppose $k,l \geq 0$ are parameters such that $d \geq k+l+1$. If $\gamma \leq \epsilon^2 \cdot (128 \cdot k^2 \cdot l^{l+4} 2^{4l+2k+6})^{-1}$ for some $\epsilon \in (0,1)$, then the second singular value of the operator $\mathsf{S}_{k,l}$ can be bounded from above*

$$\sigma_2(\mathsf{S}_{k,l}) \;\leq\; \epsilon.$$

In Appendix A, we will sketch a proof for the following improvement to Theorem 4.1.2 due to Dikstein and Dinur [DD19].

Theorem 4.1.3. *Let (X,Π) be a pure d-dimensional two-sided γ-local spectral expander. Let $k,l \geq 0$ be parameters such that $d \geq k+l+1$. Then, writing $\mathsf{S}_{k,l}$ for the swap operator on the complex (X,Π) we have*

$$\sigma_2(\mathsf{S}_{k,l}) \leq (k+1) \cdot (l+1) \cdot \gamma.$$

For simplicity, we base our exposition on the square case – the rectangular case will follow similarly (and is handled in Section 4.3.3). We prove Theorem 4.1.1 by connecting the spectral structure of S_k on a general two-sided γ-local spectral expander to the well behaved case of complete simplicial complexes, $\Delta_{n,d} = \binom{[n]}{\leq d+1}$ equipped with the uniform measure on the d-dimensional faces $\binom{[n]}{d+1}$. To distinguish these two cases we denote by $\mathsf{S}_{k,k}^{\Delta_{n,d}}$ for the swap-walk on the complete complex $\Delta_{n,d}$ and $\mathsf{S}_{k,k}$ for the swap-walk on the concrete simplicial complex (X,Π) that we care about. We recall that $\mathsf{S}_{k}^{\Delta_{n,d}}$ is the random walk operator of the well known Kneser graph $K(n,k+1)$.

Definition 4.1.4 (Kneser Graph $K(n,k)$ [GM15])**.** Let $n,k \geq 1$ be parameters such that $n \geq 2k$. The Kneser graph $K(n,k)$ is the graph $G=(V,E)$ where $V = \binom{[n]}{k}$ and $E = \{\{\alpha,\alpha'\} \mid \alpha \cap \alpha' = \varnothing\}$.

We recall that the spectrum of the Kneser graphs are well-understood.

Fact 4.1.5 (Kneser Graph [GM15])**.** *Let $n,k \geq 1$ be parameters such that $n \geq 2k$. The singular values of the unnormalized adjacency matrix of the Kneser graph $K(n,k)$ are*

$$\binom{n-k-i}{k-i},$$

for $i = 0,\ldots,k$.

This means that $\sigma_2(\mathsf{S}^{\Delta}_{k,k}) = O_k(1/n)$ as shown in Claim 4.1.6.

Claim 4.1.6. *Let $d \geq 2k+1$. The second largest singular value $\sigma_2(\mathsf{S}^{\Delta_{n,d}}_{k,k})$ of the swap operator $\mathsf{S}^{\Delta_{n,d}}_{k,k}$*

$$\sigma_2(\mathsf{S}^{\Delta_{n,d}}_{k}) \; = \; \frac{k+1}{n-k-1},$$

provided $n \geq M_k$ where $M_k \in \mathbb{N}$ only depends on k.

Proof. First note that for the complete complex $\Delta_{n,d}$, the operator $\mathsf{S}^{\Delta_{n,d}}_{k,k}$ is the random walk matrix of the Kneser graph $K(n, k+1)$. Since the degree of $K(n,k)$ is $\binom{n-k-1}{k+1}$, the result follows from Fact 4.1.5. □

Therefore, if we could claim that $\sigma_2(\mathsf{S}_k)$ of the swap walk S_k over an arbitrary two-sided γ-local spectral expander is close to $\sigma_2(\mathsf{S}^{\Delta_{n,d}}_{k})$ for small enough γ, we would conclude that the swap walks S_k have bounded second singular value $\sigma_2(\mathsf{S}_k)$ for strong two-sided local-spectral expanders. A priori there is no reason why this claim should hold since a general d-dimensional two-sided γ-local spectral expander may have much fewer faces ($O_d(n)$ versus $\sum_{j=0}^{d}\binom{n}{j+1}$ in the complete complex $\Delta_{n,d}$). Fortunately, it turns out that this claim is indeed true (up to $O_k(\gamma)$ errors).

To prove Theorem 4.1.1 we employ the machinery developed in [DDFH18]. Before we delve into the full technical analysis, it might be instructive to see how Theorem 4.1.1 is obtained from understanding the quadratic form $\langle \mathsf{S}_k \boldsymbol{f}, \boldsymbol{f}\rangle_{\Pi_k}$ for $\boldsymbol{f} \in \mathbb{R}^{X(k)}$.

First we informally recall the decomposition of $\mathbb{R}^{X(k)}$ into subspaces $\bigoplus_{i=-1}^{k} \mathcal{V}^{(k,i)} = \mathbb{R}^{X(k)}$ from [DDFH18] where the vectorspaces $\mathcal{V}^{(k;\bullet)}$ are approximately orthogonal with each other and $\mathcal{V}^{(k,i)}$ can be thought of as the subspace of approximate eigenvectors of $\mathbb{R}^{X(k)}$ (the precise definitions are deferred to Section 4.3.1). In this decomposition, $\mathcal{V}^{(k;-1)}$ is defined as the space of constant functions in $\mathbb{R}^{X(k)}$ and all the other vector spaces $\mathcal{V}^{(k;i)}$ for $i \neq -1$ are (actually) orthogonal to it. It will turn out that for small enough γ, the decomposition $\bigoplus_{i=-1}^{k} \mathcal{V}^{(k;i)}$ will be proper, i.e. every $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ can be uniquely written as $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ where $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$.

Equipped with this machinery, we prove the stronger result that for small enough γ, the swap operators S_k of any two-sided γ-local spectral expander has an *approximate spectrum* that only depends on k. Formally,

Lemma 4.1.7. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander where $\gamma \leq \epsilon \left(64k^{k+4}2^{3k+2}\right)^{-1}$ for some $\epsilon \in (0,1)$. If $d \geq 2k+1$, there exists constants $\lambda_{k,i}^{\mathsf{S}_k}$ for all $i = -1, \ldots, k$ only depending on k and i (and not on (X, Π)) such that, for any $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ with $\|\boldsymbol{f}\|_{\Pi_k} = 1$ we have*

$$\langle \mathsf{S}_k \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k} \;=\; \sum_{i=0}^{k} \lambda_{k,i}^{\mathsf{S}_k} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \epsilon.$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ is the unique decomposition satisfying $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$

Roughly, Lemma 4.1.7 suggests that the vector spaces $\mathcal{V}^{(k;i)}$ can be thought as the space of approximate eigenvectors of $\mathcal{V}^{(k;i)}$ associated with the value $\lambda_{k,i}^{\mathsf{S}_k}$. Given Lemma 4.1.7, one might be led into believing that this is all one needs to show $\sigma_2(\mathsf{S}_k)$ is indeed small, since the approximate eigenvalues $\lambda_{k,i}$ only depend on k and i. However, giving an explicit expression for these approximate eigenvalues proves to be a challenge. For this reason, we rely on the singular values of Kneser graphs, as we will elaborate later.

To aid with our proof of Lemma 4.1.7, we introduce the notion of *balanced* operators which in particular captures longer- ($\mathsf{P}_k^{(h)}$) and (as we will soon discuss) swap walks (S_k). We show that the quadratic form expression of Lemma 4.1.7 is a particular case of a general result for $\langle \mathsf{B}\boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k}$ where $\mathsf{B} \in \mathbb{R}^{X(k) \times X(k)}$ is a *balanced* operator. A *balanced* operator $\mathsf{B} \in \mathbb{R}^{X(k) \times X(k)}$ is any operator that can be obtained as linear combination of *pure balanced* operators, the later being operators that are a formal product of an equal number of up ($\mathsf{U}_\bullet$) and down ($\mathsf{D}_\bullet$) operators. The following result will be proven in Section 4.3.

Lemma 4.1.8. *Let (X, Π) be a d-dimensional two-sided γ-local spectral expander such that $\gamma \leq \epsilon \left(16k^{k+2}\ell^2 \sum_{\mathsf{W} \in \mathcal{Y}} |\alpha^{\mathsf{W}}|\right)^{-1}$, for some $\epsilon \in (0,1)$. Let $\mathcal{Y} \subseteq \{\mathsf{Y} \mid \mathsf{Y} \in \mathbb{R}^{X(k) \times X(k)}\}$ be a collection of formal operators that are product of an equal number of at most l up and down operators (i.e., pure balanced operators). Let $\mathsf{B} = \sum_{\mathsf{Y} \in \mathcal{Y}} \alpha^{\mathsf{Y}} \mathsf{Y}$ where $\alpha^{\mathsf{Y}} \in \mathbb{R}$ for all $\mathsf{Y} \in \mathcal{Y}$. Then, there exists constants $\lambda_{k,i}^{\mathsf{Y}}$ only depending on k and i (and not on (X, Π)), such that for any $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ with $\|\boldsymbol{f}\|_{\Pi_k} = 1$ we have*

$$\langle \mathsf{B}\boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k} \;=\; \sum_{i=-1}^{k} \left(\sum_{\mathsf{Y} \in \mathcal{Y}} \alpha^{\mathsf{Y}} \lambda_{k,i}^{\mathsf{Y}} \right) \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \epsilon,$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ is the unique decomposition where $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ for all $i = -1, \ldots, k$.

The above result is very useful to understand the swap-operators S_k, as we will see that they are balanced operators. We will prove the following general result in Section 4.2,

Corollary 4.1.9. *Let (X, Π) be a d-dimensional simplicial complex, and let $k, l, h \geq 0$ be parameters satisfying $k \geq l \geq h$ and, $k + h \leq d$. Then,*

$$\binom{k+1}{l+1-h} \mathsf{S}_{k,l}^{(h)} = \sum_{j=0}^{u} (-1)^{h-j} \cdot \binom{k+1+j}{l+1} \cdot \binom{h}{j} \cdot \mathsf{P}_{k,l}^{(j)}.$$

In particular, this implies that we have

$$\mathsf{S}_k = \sum_{j=0}^{k+1} (-1)^{k+1-j} \cdot \binom{k+1+j}{j} \cdot \binom{k+1}{j} \cdot \mathsf{P}_k^{(j)},$$

where we recall that the longer random walk operators $\mathsf{P}_k^{(j)} = \mathsf{D}_{k+1} \cdots \mathsf{D}_{k+j} \mathsf{U}_{k+j-1} \cdots \mathsf{U}_k$ consist of an equal number of up- and down-operators.

With all these above mentioned facts, finally we have what it takes to prove Theorem 4.1.1. For convenience we restate it below,

Theorem 4.1.1. *Let (X, Π) be a pure d-dimensional two-sided γ local spectral expander such that $\gamma \leq \epsilon \left(64k^{k+4}2^{3k+2}\right)^{-1}$ where $\epsilon \in (0,1)$ and $k \geq 0$ some parameter such that $d \geq 2k+1$. Then the second largest singular value $\sigma_2(\mathsf{S}_k)$ of the swap walk S_k on $X(k)$ is*

$$\sigma_2(\mathsf{S}_k) \;\leq\; \epsilon.$$

Proof. First we show that for $i \in [0, k]$ the approximate eigenvalue $\lambda_{k,i}^{\mathsf{S}_k}$ (from Lemma 4.1.7) of the swap operator S_k is actually zero. Note that for $i \in [0, k]$ the space $\mathcal{V}^{(k;i)}$ is a non-trivial vectorspace (i.e., $\mathcal{V}^{(k;i)}$ is not the space of constant functions). Let $\mathsf{S}_{k,k}^{\Delta_{n,d}}$ be the swap operator of the complete complex $\Delta_{n,d}$. On one hand Claim 4.1.6 gives

$$\sigma_2(\mathsf{S}_k^{\Delta_{n,d}}) \;=\; \max\left\{ \left| \langle \boldsymbol{f}, \mathsf{S}_k^{\Delta} \boldsymbol{f} \rangle_{\Pi_k} \right| : \boldsymbol{f} \in \mathbb{R}^{X(k)}, \langle \boldsymbol{f}, \mathbf{1} \rangle_{\Pi_k} = 0, \|\boldsymbol{f}\|_{\Pi_k} = 1 \right\} \;=\; O_k\left(\frac{1}{n}\right).$$

On the other hand since the complete complex $\Delta_{n,d}$ is a two-sided γ^{Δ}-local spectral expander where $\gamma^{\Delta} = O_k(1/n)$, if n is sufficiently large we have $\gamma^{\Delta} \leq \gamma$ and thus Lemma 4.1.8 and Corollary 4.1.9 can be applied to give

$$\sigma_2(\mathsf{S}_k^{\Delta_{n,d}}) \;\geq\; \max\{\langle \boldsymbol{f}_i, \mathsf{S}_k^{\Delta} \boldsymbol{f}_i \rangle_{\Pi_k} : \boldsymbol{f}_i \in \mathcal{V}^{(k;i)}, \|\boldsymbol{f}_i\|_{\Pi_k} = 1\} \;=\; \left|\lambda_{k,i}^{\mathsf{S}_k}\right| \;\pm\; O_k\left(\frac{1}{n}\right).$$

where we have also used that $\mathcal{V}^{(k;i)}$ is orthogonal to $\mathbf{1}$ for $i \geq 0$. Since n is arbitrary and $\lambda_{k,i}$ depends only on k and i, we obtain $\lambda_{k,i}^{\mathsf{S}_k} = 0$ as claimed. Now we apply Lemma 4.1.8 to the swap operator S_k of the two-sided γ-local spectral expander (X, Π). Let $\boldsymbol{f}$ be any vector that is orthogonal to $\mathbf{1}$. Since $\mathcal{V}^{(k;i)}$ are all orthogonal to the subspace spanned by $\mathbf{1}$ (i.e. $\mathcal{V}^{(k;-1)}$) except for $i = -1$, we have $\boldsymbol{f} = \sum_{i=0}^{k} \boldsymbol{f}_i$. Thus, by Lemma 4.1.7

$$\begin{aligned}
|\langle \mathsf{S}_k \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k}| \;&\leq\; \left| \sum_{i=0}^{k} \lambda_{k,i} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \epsilon \right|, \\
&=\; \epsilon. && \text{(since } \lambda_{k,i}^{\mathsf{S}_k} = 0\text{)}
\end{aligned}$$

And therefore,

$$\begin{aligned}
\sigma_2(\mathsf{S}_k) \;&= \max\{|\langle \boldsymbol{f}, \mathsf{S}_k \boldsymbol{f} \rangle_{\Pi_k}| : \boldsymbol{f} \in \mathbb{R}^{X(k)}, \langle \boldsymbol{f}, \mathbf{1} \rangle_{\Pi_k} = 0, \|\boldsymbol{f}\|_{\Pi_k} = 1\}, \\
&\leq \epsilon
\end{aligned}$$

□

4.2 Swap Walks and Longer Random Walks

The goal of this section will be to prove the following formula for the swap walks $\mathsf{S}_{k,l}^{(h)}$ in terms of the longer random walks $\mathsf{P}_{k,l}^{(j)}$ for $j = 0, \ldots, h$,

Corollary 4.1.9. *Let (X, Π) be a d-dimensional simplicial complex, and let $k, l, h \geq 0$ be parameters satisfying $k \geq l \geq h$ and, $k + h \leq d$. Then,*

$$\binom{k+1}{l+1-h} \mathsf{S}_{k,l}^{(h)} = \sum_{j=0}^{u} (-1)^{h-j} \cdot \binom{k+1+j}{l+1} \cdot \binom{h}{j} \cdot \mathsf{P}_{k,l}^{(j)}.$$

We will prove this statement in Section 4.2.1.

Our approach for proving this result will rely on first establishing a formula for the longer random walks $\mathsf{P}_{k,l}^{(h)}$ in terms of the swap walks $\mathsf{S}_{k,l}^{(j)}$,

Lemma 4.2.1. *Let $u, l, k, d \geq 0$ be given satisfying $k \geq l \geq h$, and $k + h \leq d$. Then, we have the following formula for the $\mathsf{P}_{k,l}^{(h)}$ on any d-dimensional simplicial complex X*

$$\mathsf{P}_{k,l}^{(h)} = \sum_{j=0}^{h} \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+h+1}{l+1}} \cdot \mathsf{S}_{k,l}^{(j)}.$$

We will prove this result in Section 4.2.3. The main observation the proof of Lemma 4.2.1 hinge on the following alternative characterization of the swap-walks:

We recall that $\mathsf{S}_{k,l}^{(h)}$ can be thought as describing the same process as $\mathsf{P}_{k,l}^{(h)}$, where we condition the transitions between α and α' on having a difference of fixed size: Starting from a face $\alpha \in X(k)$, first we sample a face $\beta \in X(k+h)$ from the distribution $\boldsymbol{p}_\alpha^{(h)}$ (see Section 2.4.1 and Section 2.4.4), and then we sample a uniformly random subset $\alpha' \in X(l)$ of β among all the subsets that satisfy $|\alpha' \backslash \alpha| = h$. In particular, we have $\mathsf{S}_{k,l}^{(h)}(\alpha, \alpha') = \Pr[\alpha' \mid \alpha]$.

It will be convenient for us to give an alternative description of $\mathsf{S}_{k,l}^{(h)}$: We describe the process which we will call the j-swapping walk of height h which we will represent by the stochastic operator $\mathsf{S}_{k,l}^{(h;j)}$: Starting from a face $\alpha \in X(k)$, first we sample a face $\beta \in X(k+h)$ from the distribution $\boldsymbol{p}_\alpha^{(h)}$, and then we sample a uniformly random subset $\alpha' \in X(l)$ of β among all the subsets that satisfy $|\alpha' \backslash \alpha| = j$, i.e. $\mathsf{S}_{k,l}^{(h;j)}(\alpha, \alpha') = \Pr[\alpha' \mid \alpha]$. It turns out that when $h \geq j$, the walks $\mathsf{S}_{k,l}^{(j)}$ and the walks $\mathsf{S}_{k,l}^{(h,j)}$ follow the same law. Formally,

Lemma 4.2.2. *Let (X, Π) be a d-dimensional simplicial complex for $d \geq 0$, and suppose $k, l, h, j \geq 0$ are parameters satisfying $k \geq l \geq h \geq j$, and $k + h \leq d$. Then,*

$$\mathsf{S}_{k,l}^{(h;j)} = \mathsf{S}_{k,l}^{(j)}.$$

We will prove this result in Section 4.2.2.

Finally, we will use *binomial inversion* to obtain Corollary 4.1.9 from Lemma 4.2.1. The following result is well-known (see e.g. [BS02])

Fact 4.2.3. *Let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ be arbitrary sequences. Suppose for all $n \geq 0$ we have,*

$$b_n = \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot a_j.$$

Then, we also have

$$a_n = \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot b_j.$$

For completeness, we will provide a proof of this in Section 4.2.4

4.2.1 Swap Walks in Terms of Longer Random Walks: Proof of Corollary 4.1.9

Assuming all the statements we have mentioned in the previous section, we now provide a proof for Corollary 4.1.9. We recall,

Corollary 4.1.9. *Let (X, Π) be a d-dimensional simplicial complex, and let $k, l, h \geq 0$ be parameters satisfying $k \geq l \geq h$ and, $k + h \leq d$. Then,*

$$\binom{k+1}{l+1-h} \mathsf{S}_{k,l}^{(h)} = \sum_{j=0}^{u} (-1)^{h-j} \cdot \binom{k+1+j}{l+1} \cdot \binom{h}{j} \cdot \mathsf{P}_{k,l}^{(j)}.$$

Proof. Fix faces $\alpha \in X(k)$ and $\alpha' \in X(l)$ and set for all $j = 0, \ldots, h$

$$a_j := \binom{k+1}{l+1-j} \cdot (-1)^j \cdot \mathsf{S}_{k,l}^{(j)}(\alpha, \alpha').$$

Notice that we have by Lemma 4.2.1

$$\binom{k+1+h}{l+1} \cdot \mathsf{P}_{k,l}^{(h)}(\alpha, \alpha') = \sum_{j=0}^{h} \binom{h}{j} \cdot (-1)^j \cdot a_j = \sum_{j=0}^{h} \binom{h}{j} \cdot \binom{k+1}{l+1-j} \cdot \mathsf{S}_{k,l}^{(h)}(\alpha, \alpha').$$

In particular, setting for all $j = 0, \ldots, h$

$$b_j := \binom{k+1+j}{l+1} \cdot \mathsf{P}_{k,l}^{(j)}(\alpha, \alpha'),$$

we can apply Fact 4.2.3 to obtain

$$\begin{aligned}
\binom{k+1}{l+1-h} \cdot (-1)^h \cdot \mathsf{S}_{k,l}^{(h)}(\alpha, \alpha') &= a_h, \\
&= \sum_{j=0}^{h} \binom{h}{j} \cdot (-1)^j \cdot b_j, \\
&= \sum_{j=0}^{h} \binom{h}{j} \cdot \binom{k+1+j}{l+1} \cdot (-1)^j \cdot \mathsf{P}_{k,l}^{(j)}(\alpha, \alpha').
\end{aligned}$$

Dividing both sides of this equation by $(-1)^h$ yields the desired result. $\square$

4.2.2 Swap Walks are Height Independent: Proof of Lemma 4.2.2

The main result we will prove in this section, will be that the j-swapping walks $\mathsf{S}_{k,l}^{(h;j)}$ are independent of the parameter h, so long as the parameter h is large enough, i.e. $h \geq j$,

Lemma 4.2.2. *Let (X, Π) be a d-dimensional simplicial complex for $d \geq 0$, and suppose $k, l, h, j \geq 0$ are parameters satisfying $k \geq l \geq h \geq j$, and $k + h \leq d$. Then,*

$$\mathsf{S}_{k,l}^{(h;j)} = \mathsf{S}_{k,l}^{(j)}.$$

Proof. We fix some $\alpha \in X(k)$ and $\alpha' \in X(l)$, and show that $\mathsf{S}_{k,l}^{(h;j)}(\alpha, \alpha') = \mathsf{S}_{k,l}^{(j)}(\alpha, \alpha')$. It is clear that the claim is true when $|\alpha' \backslash \alpha| \neq j$, as both quantities equal 0 in this case. Thus, we assume $|\alpha' \backslash \alpha| = j$. By Proposition 2.4.6, we know

$$\mathsf{S}_{k,l}^{(j)}(\alpha, \alpha') = \frac{1}{\binom{k+1}{l+1-j}\binom{k+j+1}{j}} \cdot \frac{\Pi_{k+j}(\alpha \cup \alpha')}{\Pi_k(\alpha)}. \tag{4.1}$$

Thus, it remains to show

$$\mathsf{S}_{k,l}^{(h;j)}(\alpha, \alpha') = \frac{1}{\binom{k+1}{l+1-j}\binom{k+j+1}{j}} \cdot \frac{\Pi_{k+j}(\alpha \cup \alpha')}{\Pi_k(\alpha)}. \tag{4.2}$$

The process $\mathsf{S}_{k,l}^{(h;j)}$ first samples a face $\beta \in X(k+h)$ from the distribution $\boldsymbol{p}_{\alpha}^{(h)}$ and then resamples a uniformly random face $\alpha^{(h;j)} \in X(l)$ among all the subsets $\bullet \in X(l)$ of β that satisfy $|\bullet\backslash\alpha| = j$. It is clear, that $\alpha^{(h;j)} = \alpha'$ is only possible when $\beta \supset \alpha \cup \alpha'$ which happens with probability,

$$\begin{aligned}
\Pr_{\beta\sim \boldsymbol{p}_\alpha^{(h)}}[\beta \supset \alpha\cup\alpha'] &= \sum_{\substack{\beta\in X(k+h),\\ \beta\supset\alpha\cup\alpha'}} \boldsymbol{p}_\alpha^{(h)}(\beta),\\
&= \sum_{\substack{\tau\in X_\alpha(h-1),\\ \tau\supset\alpha'\backslash\alpha}} \Pi_{h-1}^{\alpha}(\tau), && \text{(by Proposition 2.4.3)}\\
&= \binom{h}{j}\Pi_{j-1}^{\alpha}(\alpha'\backslash\alpha), && \text{(by Proposition 2.3.1 and } |\alpha'\backslash\alpha| = j)\\
&= \frac{\binom{h}{j}}{\binom{k+j+1}{j}}\cdot\frac{\Pi_{k+j}(\alpha\cup\alpha')}{\Pi_k(\alpha)}. && \text{(by Eq. (2.13) and } |\alpha'\backslash\alpha| = j)
\end{aligned}$$

Then, we note that conditioning on picking some $\beta \supset \alpha\cup\alpha'$, we sample α' from β with probability

$$\Pr[\alpha^{(h;j)} = \alpha' \mid \beta \supset \alpha\cup\alpha'] = \frac{1}{\binom{k+1}{l+1-j}\cdot\binom{h}{j}}$$

since $\binom{k+1}{l+1-j}\cdot\binom{h}{j}$ is the number of subsets $\bullet \in X(l)$ of β such that $|\bullet\backslash\alpha| = j$.

Thus, by law of conditional probability we have established Eq. (4.2), as

$$\begin{aligned}
\mathsf{S}_{k,l}^{(h;j)}(\alpha,\alpha') &= \Pr[\alpha^{(h;j)} = \alpha'],\\
&= \Pr[\alpha^{(h;j)} = \alpha' \mid \beta\supset\alpha\cup\alpha']\cdot \Pr_{\beta\sim\boldsymbol{p}_\alpha^{(h)}}[\beta\supset\alpha\cup\alpha'],\\
&= \frac{1}{\binom{k+j+1}{j}\binom{k+1}{l+1-j}}\frac{\Pi_{k+j}(\alpha\cup\alpha')}{\Pi_k(\alpha)}.
\end{aligned}$$

□

4.2.3 Longer Random Walks in Terms of the Swap Walks: Proof of Lemma 4.2.1

We show that the longer random walks $\mathsf{P}_{k,l}^{(h)}$ are given by an average of swap walks $\mathsf{S}_{k,l}^{(j)}$ with respect to the hypergeometric distribution.

Lemma 4.2.1. *Let $u, l, k, d \geq 0$ be given satisfying $k \geq l \geq h$, and $k + h \leq d$. Then, we have the following formula for the $\mathsf{P}_{k,l}^{(h)}$ on any d-dimensional simplicial complex X*

$$\mathsf{P}_{k,l}^{(h)} = \sum_{j=0}^{h} \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+h+1}{l+1}} \cdot \mathsf{S}_{k,l}^{(j)}.$$

Proof. Our proof will rely on the observation that starting from α the processes $\mathsf{P}_{k,l}^{(h)}$ and $\mathsf{S}_{k,l}^{(h;j)}$ can be factorized into two steps: (i) picking some $\beta \in X(k+h)$ with respect to $\boldsymbol{p}_{\alpha}^{(h)}$, (ii) picking a face $\alpha' \in X(l)$ such that $\alpha' \subset \beta$. It is only step (ii) that these processes differ. Our idea will be conditioning both processes on picking the same β, and then conditioning the first process $|\beta \backslash \alpha_1|$.

We introduce some notation: starting from α we will write $\alpha^{(h)}$ for the random face that $\mathsf{P}_{k,l}^{(h)}$ picks at step (ii), and $\alpha^{(h;j)}$ for the random face that $\mathsf{S}_{k,l}^{(h;j)}$ picks, i.e. we have $\mathsf{P}_{k,l}^{(h)}(\alpha, \alpha') = \Pr[\alpha^{(h)} = \alpha']$ and $\mathsf{S}_{k,l}^{(h;j)}(\alpha, \alpha') = \Pr[\alpha^{(h;j)} = \alpha']$. Let $\beta \in X(k)$ be an arbitrary face. We condition both processes on picking β in step (i). We write $\mathcal{E}_j(\beta)$ for the event that the uniformly random set $\alpha^{(h)} \in X(l)$ we pick satisfies $|\alpha^{(h)} \cap \beta| = j$. By elementary combinatorics,

$$\Pr_{\substack{\alpha^{(h)} \subset \beta, \\ \alpha^{(h)} \in X(l)}} [\mathcal{E}_j(\beta) \mid \beta] = \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+h+1}{l+1}}$$

where the draw of the probability is uniform over the subsets $\alpha^{(h)} \in X(l)$ of $\beta \in X(k+h)$.

By definition (Section 2.4.3 and Section 4.2.2), we have for all $\alpha' \in X(l)$,

$$\Pr[\alpha^{(h;j)} = \alpha' \mid \beta] = \Pr[\alpha^{(h)} = \alpha' \mid \beta \text{ and } \mathcal{E}_j(\beta)] \tag{4.3}$$

since $\alpha^{(h;j)}$ conditioned on β is distributed uniformly among all subsets of β in $X(l)$ that have difference of size j with α, and $\alpha^{(h)}$ conditioned on β is distributed uniformly among all the subsets of β in $X(l)$.

Now, by the law of total probability we have

$$
\begin{aligned}
\mathsf{P}_{k,l}^{(h)}(\alpha,\alpha') &= \sum_{j=0}^{h} \sum_{\beta \in X(k+h)} \boldsymbol{p}_{\alpha}^{(h)}(\beta) \cdot \Pr[\mathcal{E}_j(\beta) \mid \beta] \cdot \Pr[\alpha^{(h)} = \alpha' \mid \beta \text{ and } \mathcal{E}_j(\beta)], \\
&= \sum_{j=0}^{h} \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+1+h}{l+1}} \cdot \mathop{\mathbb{E}}_{\beta \sim \boldsymbol{p}_{\alpha}^{(h)}} \left[\Pr[\alpha^{(h)} = \alpha' \mid \beta \text{ and } \mathcal{E}_j(\beta)]\right], \\
&= \sum_{j=0}^{h} \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+h+1}{l+1}} \cdot \mathop{\mathbb{E}}_{\beta \sim \boldsymbol{p}_{\alpha}^{(h)}} \Pr\left[\alpha^{(h;j)} = \alpha' \mid \beta\right]
\end{aligned}
$$

where we used Eq. (4.3) to get the last equality. Another application of the law of total probability gives us

$$
\mathop{\mathbb{E}}_{\beta \sim \boldsymbol{p}_{\alpha}^{(h)}} \Pr\left[\alpha^{(h;j)} = \alpha' \mid \beta\right] = \Pr[\alpha^{(h;j)} = \alpha'] = \mathsf{S}_{k,l}^{(h;j)}.
$$

This allows us to write,

$$
\mathsf{P}_{k,l}^{(h)} = \sum_{j=0}^{h} \frac{\binom{h}{j}\binom{k+1}{l+1-j}}{\binom{k+h+1}{l+1}} \cdot \mathsf{S}_{k,l}^{(h;j)}(\alpha,\alpha').
$$

The statement follows using the height independence of the swap walks $\mathsf{S}_{k,l}^{(h;j)} = \mathsf{S}_{k,l}^{(j)}$ for all $j = 0, \ldots, h$, i.e. Lemma 4.2.2 $\square$

4.2.4 Binomial Inversion: Proof of Fact 4.2.3

In this section, we prove provide the proof for the *binomial inversion* result we have used to obtain Corollary 4.1.9. We recall the following well-known result (see e.g. [BS02])

Fact 4.2.3. *Let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ be arbitrary sequences. Suppose for all $n \geq 0$ we have,*

$$
b_n = \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot a_j.
$$

Then, we also have

$$
a_n = \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot b_j.
$$

Proof. Suppose we have,

$$b_n = \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot a_j \qquad \text{(assumption)}$$

for all $n \geq 0$. Then,

$$\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot b_j &= \sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot \sum_{i=0}^{j} \binom{j}{i} \cdot (-1)^i \cdot a_i, \quad \text{(by assumption)} \\
&= \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} (-1)^{j+i} \cdot a_i, \\
&= \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{n}{j} \binom{j}{i} (-1)^{(j+i)} \right) a_i, \\
&= \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{n}{i} \binom{n-i}{j-i} (-1)^{(j+i)} \right) \cdot a_i, \quad \left(\text{by } \binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}\right) \\
&= \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j+i} \right) \binom{n}{i} \cdot a_i, \\
&= \sum_{i=0}^{n} \left(\sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^{l} \right) \cdot \binom{n}{i} \cdot a_i, \qquad (l = j - i)
\end{aligned}$$

where to get the last inequality we also used $(-1)^{j-i} = (-1)^{j+i}$. Now, we conclude by the binomial theorem,

$$\sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l = (1-1)^{n-i} = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

And therefore,

$$\sum_{j=0}^{n} \binom{n}{j} \cdot (-1)^j \cdot b_j = \binom{n}{n} a_n = a_n.$$

□

4.3 Decompositions of $\mathbb{R}^{X(k)}$ and Balanced Operators

We state the definitions used in our technical proofs starting with γ-HDX from [DDFH18]. A d-dimensional simplicial complex X is called γ-HDX provided that we have

$$\|\mathsf{P}_i^{\wedge} - \mathsf{P}_i^{\triangledown}\|_{\Pi_i} \leq \gamma, \tag{γ-HDX}$$

for every $i = 0, \ldots, d$.

We observe that from Lemma 3.1.6, it trivially follows that any two-sided γ-local spectral expander is a γ-HDX.

Corollary 4.3.1. *Let (X, Π) be a d-dimensional simplicial complex. For all $j = 0, \ldots, d-1$ we have*

$$\nu_{j-1} \cdot \mathsf{I} \preceq_{\Pi_j} \mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown} \preceq_{\Pi_j} \gamma_{j-1} \cdot \mathsf{I}.$$

In particular, if (X, Π) is a pure two-sided γ-local spectral expander, i.e. $\max\{\gamma_j, |\nu_j|\} \leq \gamma$ *for all $j = -1, \cdots, d-2$ we have*

$$\left\|\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown}\right\|_{\Pi_j} \leq \gamma,$$

for all $j = 0, \ldots, d-1$ and thus (X, Π) is also a γ-HDX.

Proof. By Lemma 3.1.6, we know

$$\nu_{j-1} \cdot (\mathsf{I} - \mathsf{P}_j^{\triangledown}) \preceq_{\Pi_j} \mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown} \preceq_{\Pi_j} \gamma_{j-1} \cdot (\mathsf{I} - \mathsf{P}_j^{\triangledown})$$

since $\mathsf{P}_j^{\triangledown}$ is positive semi-definite, i.e. $\mathsf{P}_j^{\triangledown} \succeq_{\Pi_j} 0$ and $\nu_{j-1} \leq 0$.

The second claim about the operator norm, follows immediately from the definition of the positive semi-definite order $\succeq_{\Pi_j}$ and the definition of the operator norm. □

Naturally the complete complex $\Delta_{n,d}$ is a γ-HDX with as it is also a two-sided γ-local spectral expander. Moreover, in this particular case γ vanishes as n grows.

Lemma 4.3.2 (From [DDFH18])**.** *The complete complex $\Delta_{n,d}$ is a γ-HDX with $\gamma = O_d(1/n)$.*

Proof. It is easy to verify that when $\alpha \in \Delta_{n,d}(j)$ for $j = -1, \ldots, d-2$, the random walk matrix M_α of the link describes the random walk on a complete graph on $n - (j+1)$ vertices. Thus, $\sigma_2(\mathsf{M}_\alpha) \leq 1/(n-(j+1))$. $\square$

4.3.1 Decompositions of $\mathbb{R}^{X(k)}$

We first introduce some notation: Let (X, Π) be a d-dimensional simplicial complex, for $0 \leq i \leq d$ we write

$$\mathcal{H}^{(i)} := \ker \mathsf{D}_i = \left\{ \boldsymbol{f} \in \mathbb{R}^{X(i)} : \mathsf{D}_i \boldsymbol{f} = 0 \right\}$$

for the linear subspace of $\mathbb{R}^{X(i)}$ on which D_i vanishes. We set, $\mathcal{H}^{(-1)} = \mathbb{R}$. We write $\mathcal{V}^{(k;i)}$ for the linear subspace of $\mathbb{R}^{X(k)}$ that consists of lifts of functions from $\mathcal{H}^{(i)}$, i.e. for all $i > k$,

$$\mathcal{V}^{(k;i)} := \mathsf{U}_{k-1} \cdots \mathsf{U}_i \mathcal{H}^{(i)} = \left\{ \mathsf{U}_{k-1} \cdots \mathsf{U}_i \boldsymbol{h} : \boldsymbol{h} \in \mathcal{H}^{(i)} \right\}.$$

When, $i = k$ we set $\mathcal{V}^{(i;i)} = \mathcal{H}^{(i)}$. For convenience, we also introduce the shorthand notation, $\mathsf{U}_k^{(h)} \in \mathbb{R}^{X(k+h) \times X(k)}$ for the operator,

$$\mathsf{U}_k^{(h)} = \begin{cases} \mathsf{I} & \text{if } h = 0, \\ \mathsf{U}_{k+h-1} \cdots \mathsf{U}_k & \text{if } h \geq 1. \end{cases}$$

With our new notation, $\mathcal{V}^{(k;i)} = \mathsf{U}_i^{(k-i)} \ker(\mathsf{D}_i)$.

We first show that if X the least eigenvalue $\lambda_{\min}(\mathsf{M}_\alpha)$ of the links $\alpha \in X(\leq d-2)$ is large enough, then the vector spaces $\mathcal{V}^{(k;i)}$ yield a decomposition of the vector space $\mathbb{R}^{X(k)}$, i.e.

Theorem 4.3.3 ([DDFH18])**.** *Let X be a d-dimensional simplicial complex such that $|\nu_i| < \frac{1}{i+2}$ for all $= -1, \ldots, d-2$. Then, we have the decomposition,*

$$\mathbb{R}^{X(k)} = \mathcal{V}^{(k;k)} \oplus \mathcal{V}^{(k;k-1)} \oplus \cdots \oplus \mathcal{V}^{(k;-1)}$$

for all $k = 0, \ldots, d$.

In particular, every function $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ has a unique decomposition $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ such that $\boldsymbol{f}_i \in \mathcal{V}^{(i;k)}$ for all $i = -1, \ldots, k$.

Further, for all $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ and $i \geq 0$ we have $\langle \boldsymbol{f}_i, \mathbf{1} \rangle_{\Pi_k} = 0$.

Before we go on proving this theorem, we make a simple observation,

Corollary 4.3.4. *Let* (X, Π) *be a* d*-dimensional simplicial complex. For* $0 \leq l \leq d-1$ *we have,*

$$\lambda_{\min}(\mathsf{P}_l^{\triangle}) \geq \frac{1 + (l+1) \cdot \nu_{l-1}}{l+2}.$$

Proof. By Lemma 3.1.6, we know

$$\mathsf{P}_l^{\wedge} \succeq_{\Pi_l} \nu_{l-1}\mathsf{I} + (1 - \nu_{l-1})\mathsf{P}_l^{\triangledown}.$$

Using the definition of the non-lazy up-down walk $\mathsf{P}_l^{\wedge}$, this implies

$$\mathsf{P}_l^{\triangle} \succeq_{\Pi_l} \frac{1 + (l+1)\nu_{l-1}}{l+2} \cdot \mathsf{I} + \frac{l+1}{l+2} \cdot (1 - \nu_{l-1}) \cdot \mathsf{P}_i^{\triangledown} \succeq_{\Pi_l} \frac{1 + (l+1)\nu_{l-1}}{l+2} \cdot \mathsf{I},$$

where we have used $\nu_{l-1} \leq 0$ and $\mathsf{P}_l^{\triangledown} \succeq_{\Pi_l} 0$. The claim now follows from Fact 2.1.8. ☐

Proof of Theorem 4.3.3. We first prove the claim about uniqueness of the decomposition: We note that assuming the bound $|\nu_i| < \frac{1}{i+2}$, we can show that every up-down walk $\mathsf{P}_i^{\triangle}$ for $i = 0, \ldots, k-1$ on X is positive definite by Corollary 4.3.4. Since in this case,

$$\lambda_{\min}(\mathsf{P}_l^{\triangle}) = \frac{1 + (l+1) \cdot \nu_{l-1}}{l+2} > 0$$

for all $l = 0, \ldots, d-1$.

We now, prove the decomposition theorem by induction on k. The induction basis is the case when $k = -1$, where the statement is trivially true, since $\mathbb{R}^{X(-1)} = \mathbb{R}$ and $\mathcal{V}^{(-1;-1)} = \mathcal{H}^{(-1)} = \mathbb{R}$. Thus, we suppose that the statement of the theorem holds for some $k > -1$, i.e. every function from $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ has a unique decomposition $\boldsymbol{f} = \sum_{i=-1}^{k} \mathsf{U}_i^{(k-i)} \boldsymbol{h}_i$ where $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$. By applying the rank-nullity theorem to the operator D_{k+1} we can obtain that $\mathbb{R}^{X(k+1)} = \ker(\mathsf{D}_{k+1}) \oplus \mathrm{im}(\mathsf{D}_{k+1}^*) = \ker(\mathsf{D}_{k+1}) \oplus \mathrm{im}(\mathsf{U}_k)$, i.e. every $\boldsymbol{g} \in \mathbb{R}^{X(k+1)}$ can be written uniquely such that $\boldsymbol{g} = \boldsymbol{h}_{k+1} + \boldsymbol{g}'$ where $\boldsymbol{g}' \in \mathrm{im}(\mathsf{U}_k)$. In particular, there exists some $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ such that $\boldsymbol{g} = \boldsymbol{h}_{k+1} + \mathsf{U}_k \boldsymbol{f}$. As we have shown above, by our assumption

$\mathsf{P}_k^{\triangle} = \mathsf{D}_{k+1}\mathsf{U}_k$ is positive-definite, i.e. $\ker(\mathsf{P}_k^{\triangle}) = \ker(\mathsf{U}_k)$ is trivial, which means that the map U_k is injective. Thus, the unique choice of $\boldsymbol{g}'$ determines the choice of $\boldsymbol{f}$ uniquely, i.e. there exists a unique $\boldsymbol{h}_{k+1} \in \mathcal{H}^{(k+1)}$ and a unique $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ such that $\boldsymbol{g} = \boldsymbol{h}_{k+1} + \mathsf{U}_k\boldsymbol{f}$. Now, by our induction assumption we have a unique decomposition $\boldsymbol{g} = \sum_{i=-1}^{k+1} \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i$ such that $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$.

Now, we proceed to show the orthogonality claim: For $k \geq i \geq 0$, let $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ be arbitrary, i.e. $\boldsymbol{f}_i = \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i$. Then, we have

$$\langle \boldsymbol{f}_i, \mathbf{1}\rangle_{\Pi_k} = \langle \mathsf{U}_{k-1}\cdots\mathsf{U}_i\boldsymbol{h}_i, \mathbf{1}\rangle_{\Pi_k} = \langle \boldsymbol{h}_i, \mathsf{D}_{i+1}\cdots\mathsf{D}_k\mathbf{1}\rangle_{\Pi_i} = \langle \boldsymbol{h}_i, \mathbf{1}\rangle_{\Pi_i},$$

where we have used the definition of the down operator to obtain $\mathsf{D}_{i+1}\cdots\mathsf{D}_k\mathbf{1} = \mathbf{1}$. Now, the statement follows from the rank-nullity theorem: Since $\mathbf{1} \in \mathrm{im}(\mathsf{D}_i^*) = \mathrm{im}(\mathsf{U}_{i-1})$ and $\boldsymbol{h}_i \in \ker(\mathsf{D}_i)$ we have $\langle \boldsymbol{h}_i, \mathbf{1}\rangle_{\Pi_i} = 0$. $\square$

For convenience set $\vec{\delta} \in \mathbb{R}^{d-1}$ such that $\delta_i = 1/(i+2)$ for $i = 0, \ldots, d-1$. It will be convenient to work with the following equivalent definition of of γ-HDX

$$\|\mathsf{D}_{i+1}\mathsf{U}_i - (1-\delta_i)\mathsf{U}_{i-1}\mathsf{D}_i - \delta_i\mathsf{I}\|_{\Pi_i} \;\leq\; \frac{i+1}{i+2}\cdot\gamma \;\text{ for all }\; i = 0, \ldots, d-1. \qquad (\gamma\text{-HDX})$$

Towards our goal of understanding quadratic forms of swap operators we study the approximate spectrum of operators of the form $\mathsf{Y} = \mathsf{Y}_\ell\cdots\mathsf{Y}_1 \in \mathbb{R}^{X(k)\times X(k)}$ where each Y_i is either an up or down operator. We regard the expression $\mathsf{Y}_\ell\ldots\mathsf{Y}_1$ defining Y as a formal product. When we say that the spectrum of Y depends on Y we mean that it depends on k and on the formal expression $\mathsf{Y}_\ell\ldots\mathsf{Y}_1$ (i.e. the pattern of the up- and down-operators occurring in Y). By definition, the random walks $\mathsf{P}_k^{\triangle} = \mathsf{D}_{k+1}\mathsf{U}_k$ and $\mathsf{P}_k^{\triangledown} = \mathsf{U}_{k-1}\mathsf{D}_k$ are pure balanced operators where $l = 1$. Similarly, the random walk operator $\mathsf{P}_k^{(h)} = \mathsf{D}_{k+1}\cdots\mathsf{D}_{k+h}\mathsf{U}_{k+h-1}\cdots\mathsf{U}_k$ is also a pure balanced operator where $l = 2h$.

Taking linear combinations of *pure balanced* operators leads to the notion of *balanced* operators. Formally, we call $\mathsf{B} \in \mathbb{R}^{X(k)\times X(k)}$ a balanced operator provided there exists a set of pure balanced operators $\mathcal{Y}$ such that

$$\mathsf{B} \;=\; \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}}\cdot\mathsf{Y},$$

where the scalars α^{Y} for $\mathsf{Y} \in \mathcal{Y}$ satisfy $\alpha^{\mathsf{Y}} \in \mathbb{R}$.

The following observation is why we consider the notion of balanced operators,

Observation 4.3.5. *Let (X, Π) be a d-dimensional simplicial complex. Corollary 4.1.9 establishes that all swap walks $\mathsf{S}_{k,l}^{(h)}$ are balanced operators so long as they are well-defined. In particular, the operators S_k and $\mathsf{S}_{k,l}$ are balanced operators as well.*

It turns out that at a more crude level the behavior of Y is governed by how the number of up operators compares to the number of down operators. For this reason, for a pure balanced operator $\mathsf{Y} \in \mathbb{R}^{X(k)\times X(k)}$ we define the notations, define

$$\mathcal{U}(\mathsf{Y}) = \{\mathsf{Y}_i \mid \mathsf{Y}_i \text{ is an up operator}\} \quad \text{and} \quad \mathcal{D}(\mathsf{Y}) = \{\mathsf{Y}_i \mid \mathsf{Y}_i \text{ is a down operator}\}.$$

When Y is clear in the context we use $\mathcal{U} = \mathcal{U}(\mathsf{Y})$ and $\mathcal{D} = \mathcal{D}(\mathsf{Y})$.

Henceforth we adopt the convention of always assuming $\boldsymbol{h}_i \in \mathcal{H}^{(i)} = \ker(\mathsf{D}_i)$, $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ and $\boldsymbol{g} \in \mathbb{R}^{X(k)}$. This convention will make the statements of the technical results of Section 4.3.2 cleaner.

4.3.2 Quadratic Forms over Balanced Operators

Now we establish all the technical results leading to and including the analysis of quadratic forms over *balanced operators*. Our main result is Lemma 4.1.8 which can be thought of as a result about the approximate spectrum and approximate eigenvectors of balanced operators. This can be seen as a generalization of the results of [DDFH18], which proves the analogous result for the case of the up-down operators $\mathsf{P}_i^{\triangle}$. To establish our result, we need to make the error parameters in their analysis explicit. We first recall Lemma 4.1.8,

Lemma 4.1.8. *Let (X, Π) be a d-dimensional two-sided γ-local spectral expander such that $\gamma \leq \epsilon\left(16k^{k+2}\ell^2 \sum_{\mathsf{W}\in\mathcal{Y}} |\alpha^{\mathsf{W}}|\right)^{-1}$, for some $\epsilon \in (0,1)$. Let $\mathcal{Y} \subseteq \{\mathsf{Y} \mid \mathsf{Y} \in \mathbb{R}^{X(k)\times X(k)}\}$ be a collection of formal operators that are product of an equal number of at most l up and down operators (i.e., pure balanced operators). Let $\mathsf{B} = \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}}\mathsf{Y}$ where $\alpha^{\mathsf{Y}} \in \mathbb{R}$ for all $\mathsf{Y} \in \mathcal{Y}$. Then, there exists constants $\lambda_{k,i}^{\mathsf{Y}}$ only depending on k and i (and not on (X, Π)), such that for any $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ with $\|\boldsymbol{f}\|_{\Pi_k} = 1$ we have*

$$\langle \mathsf{B}\boldsymbol{f}, \boldsymbol{f}\rangle_{\Pi_k} \;=\; \sum_{i=-1}^{k}\left(\sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}}\lambda_{k,i}^{\mathsf{Y}}\right) \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} \;\pm\; \epsilon,$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ *is the unique decomposition where* $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ *for all* $i = -1, \ldots, k$.

Since swap walks are *balanced operators*, this will directly imply the following,

Lemma 4.1.7. *Let* (X, Π) *be a pure* d*-dimensional two-sided* γ*-local spectral expander where* $\gamma \leq \epsilon \left(64k^{k+4}2^{3k+2}\right)^{-1}$ *for some* $\epsilon \in (0,1)$*. If* $d \geq 2k+1$*, there exists constants* $\lambda_{k,i}^{\mathsf{S}_k}$ *for all* $i = -1, \ldots, k$ *only depending on* k *and* i *(and not on* (X, Π)*) such that, for any* $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ *with* $\|\boldsymbol{f}\|_{\Pi_k} = 1$ *we have*

$$\langle \mathsf{S}_k \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k} \;=\; \sum_{i=0}^{k} \lambda_{k,i}^{\mathsf{S}_k} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \epsilon.$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ *is the unique decomposition satisfying* $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$

The next result, Lemma 4.3.6, (implicit in [DDFH18]) will be key in establishing that the spectral structure of balanced operators on a γ-HDX is fully determined by the parameters in $\vec{\delta}$ provided γ is small enough. Note that the condition of being a γ-HDX provides a "calculus" for rearranging a single pair of up and down $\mathsf{D}_{j+1}\mathsf{U}_j$ operators. The next result treats the more general case of $\mathsf{D}_{j+1}\mathsf{U}_j \cdots \mathsf{U}_i$ where $j > i$.

Lemma 4.3.6 (Structure Lemma)**.** *Let* (X, Π) *be a pure* d*-dimensional simplicial complex that is a two-sided* γ*-local spectral expander (in particular, a* γ*-HDX). Suppose,* $\mathsf{Y} = (\mathsf{Y}_\ell \cdots \mathsf{Y}_1) \in \mathbb{R}^{X(i+\ell-2)\times X(i)}$ *is an operator such that each* Y_i *is either an up- or down-operator and* $|\mathcal{D}(\mathsf{Y})| = 1$*. Let* $\mathsf{Y}_c \in \mathcal{D}$ *be the unique down operator in* $\mathsf{Y} = \mathsf{Y}_\ell \ldots \mathsf{Y}_1$*. Then, for any operator* $\mathsf{A} \in \mathbb{R}^{X(k)\times X(i+l-2)}$ *with* $\|\mathsf{A}\|_{\Pi_{i+\ell-2}\to\Pi_k} \leq 1$ *and for all vectors* $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$ *and* $\boldsymbol{g} \in \mathbb{R}^{X(k)}$ *we have we have,*

$$\langle \mathsf{A}\mathsf{Y}_\ell \ldots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \;=\; \begin{cases} 0 & \text{if } \ell = 1 \text{ or } c = 1 \\ Q_{c,i}(\vec{\delta}) \cdot \langle \mathsf{A}\mathsf{U}_i^{(\ell-2)} \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \;\pm\; (c-1)\cdot\gamma\cdot\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k} & \text{otherwise,} \end{cases}$$

where $Q_{c,i}(\bullet)$ *is a polynomial in the variables* $\vec{\delta}$ *(from the* γ*-HDX definition) depending on* c, i *such that* $Q_{c,i}(\vec{\delta}) \leq 1$.

Proof. The statement is trivial when $\ell = 1$ or $c = 1$, since in both cases Y is of the form $\mathsf{Y} = (\mathsf{Y}_\ell \cdots \mathsf{Y}_2)\mathsf{D}_i$ and in particular, $\mathsf{Y}\boldsymbol{h}_i = 0$, since $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$. We proceed by induction on c: Assume that for any pure balanced operator $\mathsf{Y}' = \mathsf{Y}'_\ell \cdots \mathsf{Y}'_1$ such that Y'_{c-1} is the unique down operator, we have

$$\langle \mathsf{AY}'_\ell \cdots \mathsf{Y}'_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \leq Q_{c-1,i}(\delta) \cdot \langle \mathsf{AU}_i^{(l-2)} \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \pm (c-2) \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{g}\|_{\Pi_k}. \tag{IH}$$

We observe now, that since Y_c is the unique down operator in Y we must have $\mathsf{Y}_c = \mathsf{D}_{j+1}$ for $c = i + j - 2$, and in particular $\mathsf{Y}_c\mathsf{Y}_{c-1} = \mathsf{D}_{j+1}\mathsf{U}_j$. Note that since (X, Π) is a γ-HDX, we have

$$\|\mathsf{D}_{j+1}\mathsf{U}_j - (1-\delta_j) \cdot \mathsf{U}_{j-1}\mathsf{D}_j - \delta_j \mathsf{I}\|_{\Pi_j} \leq \frac{j+1}{j+2} \cdot \gamma.$$

In particular, there exists an operator $\mathsf{Q} \in \mathbb{R}^{X(j) \times X(j)}$ such that $\|\mathsf{Q}\|_{\Pi_j} \leq (j+1)/(j+2) \cdot \gamma$ that satisfies,

$$\mathsf{D}_{j+1}\mathsf{U}_j = \mathsf{Q} + (1-\delta_j) \cdot \mathsf{U}_{j-1}\mathsf{D}_j + \delta_j \mathsf{I}.$$

We substitute this, in place on $\mathsf{Y}_c\mathsf{Y}_{c-1}$,

$$\begin{aligned} \langle \mathsf{AY}\boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} &= \langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}(\mathsf{D}_{j+1}\mathsf{U}_j)\mathsf{Y}_{c-2} \cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k}, \\ &= \langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}(\mathsf{Q} + (1-\delta_j) \cdot \mathsf{U}_{j-1}\mathsf{D}_j + \delta_j \mathsf{I})\mathsf{Y}_{c-2} \cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k}. \end{aligned}$$

By expanding the above,

$$\begin{aligned} &\langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}(\mathsf{Q} + (1-\delta_j) \cdot \mathsf{U}_{j-1}\mathsf{D}_j + \delta_j \mathsf{I})\mathsf{Y}_{c-2} \cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \\ &\qquad = \langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{Q}\mathsf{Y}_{c-2} \cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \\ &\qquad\quad + (1-\delta_j) \cdot \langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{U}_{j-1}\mathsf{D}_j\mathsf{Y}_{c-2} \cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g} \rangle_{\Pi_k} \\ &\qquad\quad + \delta_j \cdot \langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{I}\mathsf{Y}_{c-2} \cdots \boldsymbol{g} \rangle_{\Pi_k} \end{aligned} \tag{4.4}$$

The first term on the RHS can be bound by the Cauchy-Schwarz-Inequality (Fact 2.1.1) and the submultiplicativity of the operator norm (Fact 2.1.12)

$$\begin{aligned}
\langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{Q}\mathsf{Y}_{c-2}\cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} &= \pm \|\mathsf{AY}_{c-2}\cdots \mathsf{Y}_1\| \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{g}\|_{\Pi_k}, \\
&= \pm \frac{j+1}{j+2} \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{g}\|_{\Pi_k}, \\
&= \pm \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{g}\|_{\Pi_k}. \qquad (4.5)
\end{aligned}$$

The second term on the RHS can be bounded by the induction hypothesis (IH) since the only down-operator on the expression $\mathsf{Y}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{U}_{j-1}\mathsf{D}_j \cdot \mathsf{Y}_{c-2}\cdots \mathsf{Y}_1$ is on the $(c-1)$-st spot, i.e.

$$\begin{aligned}
&\langle \mathsf{AY}_\ell \cdots \mathsf{Y}_{c+1}\mathsf{U}_{j-1}\mathsf{D}_j\mathsf{Y}_{c-2}\cdots \mathsf{Y}_1 \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \\
&\qquad \leq Q_{c-1,i}(\delta) \cdot \langle \mathsf{AU}_i^{(l-2)} \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \pm (c-2)\cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{g}\|_{\Pi_k} \qquad (4.6)
\end{aligned}$$

The third term can be seen to be equal to $\delta_j \cdot \langle \mathsf{AU}_i^{(\ell-2)} \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k}$ since the only down operator among Y_j is Y_c which is not present there. Plugging this, Eq. (4.5), and Eq. (4.6) into Eq. (4.4), we obtain

$$\begin{aligned}
\langle \mathsf{AY}\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \leq\ & (1-\delta_j)\cdot Q_{c-1,i}(\vec{\delta}) \cdot \langle \mathsf{AU}_i^{(l-2)} \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \\
& + \delta_j \cdot \langle \mathsf{AU}_i^{(l-2)} \boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \pm (c-2)\cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \\
& \pm \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{g}\|_{\Pi_k}.
\end{aligned}$$

Setting $Q_{c,i}(\vec{\delta}) = (1-\delta_j)Q_{c-1,i}(\vec{\delta}) + \delta_j$ yields the result. □

With Lemma 4.3.6 we are close to recover the approximate spectrum of $\mathsf{D}_{k+1}\mathsf{U}_k$ from [DDFH18]. However, in our application we will need to analyze more general operators, namely, *pure balanced* and *balanced* operators.

Lemma 4.3.7 (Refinement of [DDFH18])**.** *Let (X, Π) be a two-sided γ-local spectral expander and let $k, l, i \geq 0$ be parameters such that $d \geq k+1$ and $l \geq k \geq i$. Then for* *any* $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ *such that* $\boldsymbol{f}_i = \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i$ *and for any operator* $\mathsf{A} \in \mathbb{R}^{X(l)\times X(k)}$ *such that* $\|\mathsf{A}\|_{\Pi_k\to\Pi_l} \leq 1$, *we have*

$$\langle \mathsf{A}\mathsf{P}_k^{\triangle}\boldsymbol{f}_i, \boldsymbol{g}\rangle \;=\; \lambda_i^{\mathsf{P}_k^{\triangle}} \cdot \langle \mathsf{A}\boldsymbol{f}_i, \boldsymbol{g}\rangle \;\pm\; (k-i+1)\cdot\gamma\cdot\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_i},$$

where $\lambda_i^{\mathsf{P}_k^{\triangle}} = Q_{k-i+2,i}(\vec{\delta})$ *with* $Q_{x,y}(\bullet)$ *being defined as in Lemma 4.3.6.*

Proof. Recall that $\mathsf{P}_k^{\triangle} = \mathsf{D}_{k+1}\mathsf{U}_k$. The statement follows from Lemma 4.3.6 by setting $\mathsf{Y} = \mathsf{D}_{k+1}\mathsf{U}_i^{(k+1-i)}$. □

We also show that the powers of $\mathsf{P}_k^{\triangle} = \mathsf{D}_{k+1}\mathsf{U}_k$ behave as expected,

Lemma 4.3.8 (Exponentiation Lemma)**.** *Let* (X,Π) *be a pure d-dimensional two-sided* γ*-local spectral expander and let* $k, i \geq 0$ *be parameters such that* $d \geq k+1$ *and* $k \geq i$. *Then for any* $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ *such that* $\boldsymbol{f}_i = \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i$

$$\langle (\mathsf{D}_{k+1}\mathsf{U}_k)^s\,\boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} \;=\; \left(\lambda_i^{\mathsf{P}_k^{\triangle}}\right)^s \cdot \|\boldsymbol{f}_i\|_{\Pi_k}^2 \;\pm\; s\cdot(k-i+1)\cdot\gamma\cdot\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{f}_i\|_{\Pi_k},$$

where $\lambda_i^{\mathsf{P}_k^{\triangle}}$ *is defined as in Lemma 4.3.7.*

Proof. Follows immediately from the foregoing and the fact that $\|\mathsf{P}_k^{\triangle}\|_{\Pi_k} = 1$. □

Let again let $\mathsf{Y} = (\mathsf{Y}_\ell \cdots \mathsf{Y}_1) \in \mathbb{R}^{X(k)\times X(i)}$ be a product of up and down operators. In the case $|\mathcal{D}(\mathsf{Y})| > |\mathcal{U}(\mathsf{Y})|$, we show that Y is an operator whose kernel *approximately contains* $\mathcal{H}^{(i)} = \ker(\mathsf{D}_i)$.

Lemma 4.3.9 (Refinement of [DDFH18])**.** *Let* (X,Π) *be a pure d-dimensional simplicial complex that is a two-sided* γ*-local spectral expander (in particular, a* γ*-HDX). Suppose,* $\mathsf{Y} = (\mathsf{Y}_\ell \cdots \mathsf{Y}_1) \in \mathbb{R}^{X(j)\times X(i)}$ *is a product of up- and down- operators such that* $|\mathcal{D}(\mathsf{Y})| > |\mathcal{U}(\mathsf{Y})|$. *Then, for any operator* $\mathsf{A} \in \mathbb{R}^{X(k)\times X(j)}$ *with* $\|\mathsf{A}\|_{\Pi_j\to\Pi_k} \leq 1$ *and for all vectors* $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$ *and* $\boldsymbol{g} \in \mathbb{R}^{X(k)}$ *we have*

$$\langle \mathsf{A}\mathsf{Y}\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \;=\; \pm\,\ell^2\cdot\gamma\cdot\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k}.$$

Proof. Let $c \in [\ell]$ be the smallest index for which Y_c is a down operator. Observe that $c < \ell/2$ since $|\mathcal{D}| > |\mathcal{U}|$. We perform induction on $m = |\mathcal{D}|$. If $\ell = 1$ or $c = 1$, then the claim holds trivially since $\mathsf{Y}\boldsymbol{h}_i = (\mathsf{Y}_\ell \cdots \mathsf{Y}_2)\mathsf{D}_i\boldsymbol{h}_i = 0$. Hence, we assume $c, m > 1$ and that we have

$$\langle \mathsf{A}\mathsf{Y}'\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} = \pm\ell^2 \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i}^2 \cdot \|\boldsymbol{g}\|_{\Pi_k}^2, \tag{IH}$$

for every product $\mathsf{Y}' = \mathsf{Y}'_\ell \cdots \mathsf{Y}'_1$ of up- and down-operators satisfying the condition $m-1 \geq |\mathcal{D}(\mathsf{Y}')| > |\mathcal{U}(\mathsf{Y}')|$.

In our case, $\mathsf{W}_c\mathsf{W}_{c-1} = \mathsf{D}_{i+c}\mathsf{U}_{i+c-1}$. Thus, applying Lemma 4.3.6 we obtain

$$\begin{aligned}
\langle \mathsf{A}\mathsf{Y}\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} &= \langle (\mathsf{A}\mathsf{Y}_\ell \ldots \mathsf{Y}_{c+1})\, \mathsf{D}_{i+c}\mathsf{U}_{i+c-1}\mathsf{U}_i^{(c-1)}\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k}, \\
&= Q_{c,i}(\vec{\delta}) \cdot \langle (\mathsf{A}\mathsf{Y}_\ell \ldots \mathsf{Y}_{c+1})\, \mathsf{U}_i^{(c-1)}\boldsymbol{h}_i, \boldsymbol{g}\rangle_{\Pi_k} \;\pm\; \frac{\ell}{2} \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k}, \\
&= \pm\, Q_{c,i}(\vec{\delta}) \cdot (\ell-2)^2 \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k} \;\pm\; \frac{\ell}{2} \cdot \gamma\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k}, \qquad \text{(by IH)} \\
&= \pm\, \ell^2 \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{g}\|_{\Pi_k},
\end{aligned}$$

where in the last step we used $Q_{c,i}(\vec{\delta}) \leq 1$ and in our application of Lemma 4.3.6 we have used that $\|\mathsf{A}\mathsf{Y}_\ell \cdots \mathsf{Y}_{c+1}\|_{\Pi_{i+c} \to \Pi_k} \leq 1$. $\square$

We now consider an important particular case of product operators $\mathsf{Y} = \mathsf{Y}_\ell \cdots \mathsf{Y}_1 \in \mathbb{R}^{X(k)\times X(k)}$ where $|\mathcal{D}(\mathsf{Y})| = |\mathcal{U}(\mathsf{Y})|$, namely, the random walk operators

$$\mathsf{P}_k^{(h)} = \mathsf{D}_{k+1} \cdots \mathsf{D}_{k+h}\mathsf{U}_{k+h-1} \cdots \mathsf{U}_k.$$

We show that $\mathsf{P}_k^{(h)}$ is approximately a polynomial in the operator $\mathsf{P}_k^{\triangle} = \mathsf{D}_{k+1}\mathsf{U}_k$. As a warm up consider the case $\mathsf{P}_k^{(2)} = \mathsf{D}_{k+1}\mathsf{D}_{k+2}\mathsf{U}_{k+1}\mathsf{U}_k$. Using the inequality from the definition γ-HDX, we get

$$\begin{aligned}
\mathsf{P}_k^{(2)} &\approx (1-\delta_{k+1}) \cdot \mathsf{D}_{k+1}\mathsf{U}_k\mathsf{D}_{k+1}\mathsf{U}_k \;+\; \delta_{k+1} \cdot \mathsf{D}_{k+1}\mathsf{U}_k, \\
&= (1-\delta_{k+1}) \cdot (\mathsf{D}_{k+1}\mathsf{U}_k)^2 \;+\; \delta_{k+1} \cdot \mathsf{D}_{k+1}\mathsf{U}_k, \\
&= (1-\delta_{k+1}) \cdot \left(\mathsf{P}_k^{\triangle}\right)^2 \;+\; \delta_{k+1} \cdot \mathsf{P}_k^{\triangle}.
\end{aligned}$$

Inspecting this polynomial more carefully we see that that its coefficients form a probability distribution. This property holds in general as the following Lemma 4.3.10 shows. This gives an alternative (approximate) random walk interpretation of $\mathsf{P}_k^{(h)}$ as the walk that first selects the the length s according to the distribution encoded in the polynomial and takes s steps of the random walk $\mathsf{P}_k^{\triangle}$, as dictated by $\left(\mathsf{P}_k^{\triangle}\right)^s$.

Lemma 4.3.10 (Canonical Polynomials)**.** *Let (X, Π) be a d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). For $k, h \geq 0$ such that $d \geq k + h$ there exists a degree h univariate polynomial $F_{u,h,\vec{\delta}}$ depending only on $h, k, \vec{\delta}$ such that*

$$\left\|\mathsf{P}_k^{(h)} - F_{u,k,\vec{\delta}}\left(\mathsf{P}_k^{\triangle}\right)\right\|_{\Pi_k} \;\leq\; (h-1)^2 \cdot \gamma.$$

Moreover, the coefficients of this polynomial form a probability distribution, i.e., $F_{u,k,\vec{\delta}}(x) = \sum_{i=0}^{h} c_i x^i$ where $\sum_{i=0}^{h} c_i = 1$ and $c_i \geq 0$ for $i = 0, \ldots, h$.

Proof. For $h = 0$, $\mathsf{P}_k^{(0)} = \mathsf{I}$ and the lemma follows. Similarly, if $h = 1$, $\mathsf{P}_k^{(1)} = \mathsf{P}_k^{\triangle}$ and the lemma follows. Now suppose $h \geq 2$. Set $\mathsf{Y} = \mathsf{P}_k^{(h)}$, i.e.,

$$\mathsf{Y} \;=\; \mathsf{D}_{k+1} \ldots (\mathsf{D}_{k+h}\mathsf{U}_{k+h-1}) \ldots \mathsf{U}_k.$$

For convenience, we let $j = k + h - 1$. Using the condition for being a γ-HDX, and the submultiplicativity of the operator norm (Fact 2.1.12)we can replace $\mathsf{D}_{j+1}\mathsf{U}_j$ in Y by $(1-\delta_j)\mathsf{U}_{j-1}\mathsf{D}_j + \delta_j\mathsf{I}$ incurring an error of γ (in operator norm) and yielding

$$\left\|\mathsf{P}_k^{(h)} - \left((1-\delta_j) \cdot \mathsf{Y}' \;+\; \delta_j \cdot \mathsf{P}_k^{(h-1)}\right)\right\|_{\Pi_k} \leq \gamma,$$

where Y' was obtained from Y by moving the rightmost occurrence of a down operator (in this case D_{j+1}) one position to right. We continue this process of moving the rightmost occurrence of a down operator until the resulting operator is up to $(h-1)\cdot\gamma$ error, i.e. until we obtain

$$\left\|\mathsf{P}_k^{(h)} - \left(\alpha \cdot \underbrace{\mathsf{P}_k^{(h-1)} \cdot (\mathsf{D}_{k+1}\mathsf{U}_k)}_{=\mathsf{P}_k^{(h-1)} \cdot \mathsf{P}_k^{\triangle}} \;+\; \beta \cdot \mathsf{P}_k^{(h-1)}\right)\right\|_{\Pi_k} \leq (h-1) \cdot \gamma,$$

where $\alpha = \prod_{i=k+1}^{j}(1-\delta_i)$ and $\beta = \sum_{i=k+1}^{j} \delta_i \prod_{i=k+1}^{j}(1-\delta_i)$. Since $\delta_i = \delta_i > 0$, α, β are non-negative and satisfy $\alpha + \beta = 1$. Now the result follows from the induction hypothesis applied to $\mathsf{P}_k^{(h-1)}$, i.e. setting $F_{h,k,\vec{\delta}}(x) = \alpha \cdot x \cdot F_{h-1,k,\vec{\delta}}(x) + \beta \cdot F_{h-1,k,\vec{\delta}}(x)$. $\square$

Remark 4.3.11. Having a polynomial expression $F_{u,k,\vec{\delta}}(\mathsf{P}_k^{\triangle}) \approx \mathsf{P}_k^{(h)}$ and knowing that S_k can be written as linear combination of longer walks $\mathsf{P}_k^{(h)}$ (Corollary 4.1.9), we could deduce that S_k is also approximately a polynomial in $\mathsf{D}_{k+1}\mathsf{U}_k$. Using an error refined version of the Lemma 4.3.8 (showing that exponentiation of $\mathsf{D}_{k+1}\mathsf{U}_k$ behaves naturally), we could deduce the approximate spectrum of S_k. We avoid this approach since it analysis introduces tedious error terms and we can understand quadratic forms of *pure balanced* operators directly.

Remark 4.3.12. The canonical polynomial $F_{u,k,\vec{\delta}}(\mathsf{P}_k^{\triangle})$ will later be used in the error analysis that relates the norms $\|\boldsymbol{h}_i\|_{\Pi_i}$ and $\|\boldsymbol{f}_i\|_{\Pi_k}$ (Lemma 4.3.15).

Now we consider the case of balanced operators $\mathsf{Y} \in \mathbb{R}^{X(k)\times X(k)}$ where $|\mathcal{D}| = |\mathcal{U}|$ in full generality. We show how the quadratic form of Y behaves in terms of the approximate eigenspace decomposition $\mathbb{R}^{X(k)} = \bigoplus_{i=-1}^{k} \mathcal{V}^{(k;i)}$.

Lemma 4.3.13 (Pure Balanced Walks). *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). Suppose $\mathsf{Y} = \mathsf{Y}_\ell \dots \mathsf{Y}_1 \in \mathbb{R}^{X(k)\times X(k)}$ is a product of an equal number of up and down operators, i.e., $|\mathcal{D}(\mathsf{Y})| = |\mathcal{U}(\mathsf{Y})|$. Then for $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$*

$$\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} \;=\; \lambda_{k,i}^{\mathsf{W}} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} \;\pm\; \gamma \cdot (\ell^2 + \ell(k-i-1))\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{f}_i\|_{\Pi_k},$$

where $\lambda_{k,i}^{\mathsf{Y}}$ is a constant depending only on Y, k and i (and not (X,Π)), and $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$ is such that $\boldsymbol{f}_i = \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i$.

Proof. We perform induction on even ℓ. For $\ell = 0$, the result trivially follows so assume $\ell \geq 2$. For the induction step, we assume that for every $l' < l$ and $\widetilde{Y} = \widetilde{\mathsf{Y}}_{l'} \cdots \widetilde{\mathsf{Y}}_1$ consisting of an equal number of up- and down- operators, we have

$$\langle \widetilde{\mathsf{Y}} \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;=\; \lambda_{k,i}^{\widetilde{\mathsf{Y}}} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \gamma \cdot (\ell'^2 + \ell'(k-i-1)) \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k}, \tag{IH}$$

Let $c \in [\ell]$ be the smallest index of a down operator. Set $\mathsf{A} = \mathsf{Y}_\ell \ldots \mathsf{Y}_{c+1}$ and let $\mathsf{Y}' = \mathsf{Y}_c \ldots \mathsf{Y}_1 = \mathsf{D}_{k+c-1} \mathsf{U}_{k+c-2} \ldots \mathsf{U}_k$. Observe that

$$\langle \mathsf{Y} \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} = \langle \mathsf{A} \mathsf{Y}' \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;=\; \langle \mathsf{A} \mathsf{D}_{k+c-1} \mathsf{U}_k^{(c-1)} (\mathsf{U}_i^{(k-i)} \boldsymbol{h}_i), \boldsymbol{f}_i \rangle_{\Pi_k} = \langle \mathsf{A} \mathsf{D}_{k+c-1} \mathsf{U}_i^{(k+c-1-i)} \boldsymbol{h}_i, \boldsymbol{f}_i \rangle_{\Pi_k}.$$

Applying Lemma 4.3.6 to the RHS above gives

$$\begin{aligned}\langle \mathsf{A} \mathsf{D}_{k+c-1} \mathsf{U}_i^{(k+c-1-i)} \boldsymbol{h}_i, \boldsymbol{f}_i \rangle_{\Pi_k} &= Q_{k+c-i-1,i}(\vec{\delta}) \cdot \langle \mathsf{A} \mathsf{U}_i^{k+c-1-i} \boldsymbol{h}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \pm (c+k-i-1) \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k}, \\ &= Q_{k+c-i,i}(\vec{\delta}) \langle \mathsf{A} \mathsf{U}_k^{(c-2)} \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \pm (c+k-i-1) \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k}.\end{aligned}$$

Applying the induction hypothesis (IH) to $\mathsf{Y}'' = \mathsf{A} \mathsf{U}_k^{(c-2)}$ in the above RHS yields

$$\begin{aligned}& Q_{c-1+k-i,i}(\vec{\delta}) \cdot \lambda_{k,i}^{\mathsf{Y}''} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \\ &\pm\; Q_{c-1+k-i,i}(\vec{\delta}) \cdot \gamma \cdot ((\ell-2)^2 + (\ell-2)(k-i-1)) \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k} \\ &\pm\; (c+k-i-2) \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k} \\ &=\; \lambda_{k,i}^{\mathsf{Y}} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \gamma \cdot (\ell^2 + \ell(k-i-1)) \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{f}_i\|_{\Pi_k},\end{aligned}$$

where we have defined $\lambda_{k,i}^{\mathsf{W}} := Q_{c-1+k-i,i}(\vec{\delta}) \cdot \lambda_{k,i}^{\mathsf{W}''}$. The last inequality follows from $Q_{c-1+k-i,i}(\vec{\delta}) \leq 1$ and $c \leq \ell$. □

To understand all errors in the analysis in Lemma 4.3.13 we need to derive the approximate orthogonality of the vectors $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ and $\boldsymbol{f}_j \in \mathcal{V}^{(k;j)}$ for $i \neq j$ from [DDFH18] in more detail. We start with the following bound in terms of $\boldsymbol{h}_i, \boldsymbol{h}_j$.

Lemma 4.3.14 (Refinement of [DDFH18])**.** *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). Let k, i, j be parameters, such that $k \geq i, j$ and $i \neq j$. Then, we have*

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle_{\Pi_k} \;=\; \pm\, (2k-i-j)^2 \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \|\boldsymbol{h}_j\|_{\Pi_j}.$$

where $\boldsymbol{h}_i, \boldsymbol{h}_j$ satisfy $\mathsf{U}_i^{(k-i)} \boldsymbol{h}_i = \boldsymbol{f}_i$ and $\mathsf{U}_j^{(k-j)} \boldsymbol{h}_j = \boldsymbol{f}_j$.

Proof. Without loss of generality suppose $i > j$. We have

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle_{\Pi_k} = \langle \mathsf{U}_i^{(k-i)} \boldsymbol{h}_i, \mathsf{U}_j^{(k-j)} \boldsymbol{h}_j \rangle_{\Pi_k} \; = \; \langle \mathsf{Y} \boldsymbol{h}_i, \boldsymbol{h}_j \rangle_{\Pi_j},$$

where $\mathsf{Y} = \mathsf{D}_{j+1} \cdots \mathsf{D}_k \mathsf{U}_{k-1} \cdots \mathsf{U}_i$. Since $k - j > k - i$, i.e. $|\mathcal{D}(\mathsf{Y})| > \mathcal{U}(\mathsf{Y})|$, the statement follows from Lemma 4.3.9. $\square$

To give a bound for Lemma 4.3.14 only in terms of the *approximate eigenvector* norms $\|\boldsymbol{f}_i\|_{\Pi_k}$ and not in terms of $\|\boldsymbol{h}_i\|_{\Pi_i}$, we need to understand how the norms of $\boldsymbol{h}_i$ and $\boldsymbol{f}_i$ are related.

Lemma 4.3.15 (Refinement of [DDFH18])**.** *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). Suppose we are given parameters $k, i \geq 0$ such that $d \geq k \geq i$. We define, $\eta_{k,i} := (k-i)^2 + 1$ and let $\beta_i := \sqrt{|F_{k-i,i,\vec{\delta}}(\delta_i) \pm \gamma \cdot \eta_{k,i}|}$ where $F_{k-i,k,\vec{\delta}}$ is a canonical polynomial of degree $k - i$ from Lemma 4.3.10. Then, for all $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ we have,*

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \; = \; \beta_i^2 \cdot \langle \boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i},$$

where $\boldsymbol{h}_i \in \mathcal{H}^{(i)}$ satisfies $\boldsymbol{f}_i = \mathsf{U}_i^{(k-i)} \boldsymbol{h}_i$ and $\boldsymbol{f}_j = \mathsf{U}_j^{(k-j)} \boldsymbol{h}_j$. Furthermore, defining $\theta_{k,i} := (i+2)^{k-i}$ if $\gamma \leq 1/(2 \cdot \eta_{k,i} \cdot \theta_{k,i})$, then $\beta_i \geq \frac{1}{2\theta_{k,i}}$.

Proof. For $i = k$ the result trivially follows so we assume $k > i$. First consider the case $k = i + 1$. We have

$$\langle \mathsf{U}_i \boldsymbol{h}_i, \mathsf{U}_i \boldsymbol{h}_i \rangle_{\Pi_{i+1}} \; = \; \langle \mathsf{D}_{i+1} \mathsf{U}_i \boldsymbol{h}_i, \boldsymbol{h}_i \rangle \; = \; \delta_i \cdot \langle \boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i} \; \pm \; \gamma \cdot \langle \boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i}. \tag{4.7}$$

This follows from the condition of being γ-HDX, i.e. we have

$$\mathsf{D}_{i+1} \mathsf{U}_i = \mathsf{Q} + (1 - \delta_i) \mathsf{U}_{i-1} \mathsf{D}_i + \delta_i \cdot \mathsf{I},$$

for some operator $\mathsf{Q} \in \mathbb{R}^{X(i) \times X(i)}$ with $\|\mathsf{Q}\|_{\Pi_i} \leq \gamma$ annd thus

$$\langle \mathsf{D}_{i+1} \mathsf{U}_i \boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i} = \langle \mathsf{Q} \boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i} + \delta_i \cdot \|\boldsymbol{h}_i, \boldsymbol{h}_i\|_{\Pi_i}$$

where we have used that $\mathsf{U}_{i-1}\mathsf{D}_i\boldsymbol{h}_i = 0$. Now Eq. (4.7) follows from the Cauchy-Schwarz Inequality (Fact 2.1.1).

For general $k > i$ we have,

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} = \langle \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i, \mathsf{U}_i^{(k-i)}\boldsymbol{h}_i \rangle_{\Pi_i} \;=\; \langle \mathsf{D}_{i+1}\cdots\mathsf{D}_k\mathsf{U}_{k-1}\cdots\mathsf{U}_i\boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i} \;=\; \langle \mathsf{P}_i^{(k-i)}\boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i}.$$

Applying Lemma 4.3.10 to $\mathsf{P}_i^{(k-i)}$ together with Cauchy-Schwarz Inequality (Fact 2.1.1) yields

$$\left\langle \mathsf{P}_i^{(k-i)}\boldsymbol{h}_i, \boldsymbol{h}_i \right\rangle \;=\; \left\langle F_{k-i,i,\vec{\delta}}(\mathsf{P}_i^{\triangle}) \cdot \boldsymbol{h}_i, \boldsymbol{h}_i \right\rangle_{\Pi_i} \;\pm\; \gamma \cdot (k-i-1)^2.$$

Combining Eq. (4.7) and Lemma 4.3.8 gives

$$\langle F_{k-i,i,\vec{\delta}}(\mathsf{P}_i^{\triangle})\boldsymbol{h}_i, \boldsymbol{h}_i \rangle_{\Pi_i} \;\pm\; \gamma \cdot (k-i-1)^2 \;=\; \left\langle F_{k-i,i,\vec{\delta}}(\delta_i)\boldsymbol{h}_i, \boldsymbol{h}_i \right\rangle_{\Pi_i} \;\pm\; \gamma \cdot ((k-i)^2 + 1).$$

Since $F^N_{k-i,i,\vec{\delta}}(x) = \sum_{i=0}^{k-i} c_i x^i$ where the coefficients c_i form a probability distribution, we get

$$F^N_{k-i,i,\vec{\delta}}(\delta_i) \;\geq\; \delta_i^{k-i} \;=\; \left(\frac{1}{i+2}\right)^{k-i}.$$

□

Now, we can state the approximate orthogonality Lemma 4.3.16 in terms of the eigenvector norms.

Lemma 4.3.16 (Approximate Orthogonality (refinement of [DDFH18]))**.** *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). Suppose we are given parameters $k, i, j \geq 0$ such that $d \geq k \geq i, j$. Let $\eta_{k,s}, \theta_{k,s}, \beta_s$ for $s \in \{i, j\}$ be given as in Lemma 4.3.15. If $i \neq j$ and $\beta_i, \beta_j > 0$, then for all $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$, $\boldsymbol{f}_j \in \mathcal{V}^{(k;j)}$,*

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle_{\Pi_k} \;=\; \pm\; \frac{\gamma \cdot (2k-i-j)^2}{\beta_i \beta_j} \|\boldsymbol{f}_i\|_{\Pi_k} \|\boldsymbol{f}_j\|_{\Pi_k}.$$

Furthermore, if $\gamma \leq \min\left(1/(2 \cdot \eta_{k,i} \cdot \theta_{k,i}), 1/(2 \cdot \eta_{k,j} \cdot \theta_{k,j})\right)$, then $\beta_i, \beta_j > 0$ and

$$\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle_{\Pi_k} \;=\; \pm\, \gamma \cdot \theta_{k,i} \cdot \theta_{k,j} \cdot (2k-i-j)^2 \|\boldsymbol{f}_i\|_{\Pi_k} \|\boldsymbol{f}_j\|_{\Pi_k}.$$

Proof. The statement follows directly from Lemma 4.3.15. ◻

We will need this direct consequence of Lemma 4.3.16,

Corollary 4.3.17. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander (in particular, a γ-HDX). Suppose $k \geq 0$ is such that $d \geq k$. Let $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ be given with $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ with $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ for every $i = -1, \dots, k$. If $\gamma \leq \epsilon \cdot \left((k+2)^{2k} \cdot 4k^2\right)^{-1}$ for some $\epsilon \in (0,1)$ we have,*

$$\|\boldsymbol{f}\|_{\Pi_k}^2 = (1 \pm \epsilon) \cdot \left(\sum_{i=-1}^{k} \|\boldsymbol{f}_i\|_{\Pi_k}^2 \right).$$

Proof. Since $\|\boldsymbol{f}\|_{\Pi_k}^2 = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k}$, we have

$$\|\boldsymbol{f}\|_{\Pi_k}^2 = \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|^2 + \sum_{\substack{i,j=-1, \\ i \neq j}}^{k} \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle_{\Pi_k}.$$

By Lemma 4.3.16, we have

$$\|\boldsymbol{f}\|_{\Pi_k}^2 = \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|^2 \;\pm\; \sum_{\substack{i,j=-1, \\ i \neq j}}^{k} \gamma \cdot \theta_{k,i} \cdot \theta_{k,j} \cdot (2k)^2 \|\boldsymbol{f}_i\|_{\Pi_k} \cdot \|\boldsymbol{f}_i\|_{\Pi_k}.$$

By using the AM-GM inequality on the RHS

$$\begin{aligned}
\|\boldsymbol{f}\|_{\Pi_k}^2 &= \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|^2 \;\pm\; \sum_{i,j=-1, i \neq j}^{k} \gamma \cdot \theta_{k,i} \cdot \theta_{k,j} \cdot 2k^2 \cdot \left(\|\boldsymbol{f}_i\|_{\Pi_k}^2 + \|\boldsymbol{f}_j\|_{\Pi_k}^2 \right), \\
&= \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|^2 \;\pm\; \gamma \cdot (k+2)^{2k} \cdot 2k^2 \sum_{i,j=-1, i \neq j}^{k} \left(\|\boldsymbol{f}_i\|_{\Pi_k}^2 + \|\boldsymbol{f}_j\|_{\Pi_k}^2 \right), \qquad (\theta_{k,i}, \theta_{k,j} \leq (k+2)^k) \\
&= \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|^2 \;\pm\; \gamma \cdot (k+2)^{2k} \cdot 4k^2 \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|_{\Pi_k}^2, \\
&= \left(1 \pm \gamma \cdot (k+2)^{2k} \cdot 4k^2\right) \cdot \sum_{i=-1}^{k} \|\boldsymbol{f}_i\|_{\Pi_k}^2.
\end{aligned}$$

The statement now follows by our choice of γ. ◻

Now we finally prove Lemma 4.1.8 for bounding quadratic forms of general balanced operators. We will do this by generalizing the above analysis for pure balanced operators. We restate again for convenience,

Lemma 4.1.8. *Let (X, Π) be a d-dimensional two-sided γ-local spectral expander such that $\gamma \le \epsilon \left(16k^{k+2}\ell^2 \sum_{\mathsf{W}\in\mathcal{Y}} |\alpha^{\mathsf{W}}|\right)^{-1}$, for some $\epsilon \in (0,1)$. Let $\mathcal{Y} \subseteq \{\mathsf{Y} \mid \mathsf{Y} \in \mathbb{R}^{X(k)\times X(k)}\}$ be a collection of formal operators that are product of an equal number of at most l up and down operators (i.e., pure balanced operators). Let $\mathsf{B} = \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}}\mathsf{Y}$ where $\alpha^{\mathsf{Y}} \in \mathbb{R}$ for all $\mathsf{Y} \in \mathcal{Y}$. Then, there exists constants $\lambda_{k,i}^{\mathsf{Y}}$ only depending on k and i (and not on (X, Π)), such that for any $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ with $\|\boldsymbol{f}\|_{\Pi_k} = 1$ we have*

$$\langle \mathsf{B}\boldsymbol{f}, \boldsymbol{f}\rangle_{\Pi_k} \;=\; \sum_{i=-1}^{k} \left(\sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}}\lambda_{k,i}^{\mathsf{Y}}\right) \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} \;\pm\; \epsilon,$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ is the unique decomposition where $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$ for all $i = -1, \ldots, k$.

Proof. The statement follows from Lemma 4.3.13, the assumption on γ, and the following computation,

$$\begin{aligned}
\langle \mathsf{B}\boldsymbol{f}, \boldsymbol{f}\rangle_{\Pi_k} &= \sum_{i,j=-1}^{k} \langle \mathsf{B}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle_{\Pi_k}, \\
&= \sum_{i,j=-1}^{k} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha_{\mathsf{Y}} \cdot \langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle_{\Pi_k}, \\
&= \sum_{i=-1}^{k} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}} \left(\lambda_{k,i}^{\mathsf{Y}} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k}\right), \\
&\qquad \pm \sum_{i=-1}^{k} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}} \cdot \left(\gamma \cdot (\ell^2 + \ell \cdot k)\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{f}_i\|_{\Pi_k}\right), \\
&\qquad\quad + \sum_{\substack{i,j=-1,\\ i\ne j}}^{k} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}} \cdot \left\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_j\right\rangle_{\Pi_k}
\end{aligned}$$

where we have used Lemma 4.3.13 to obtain the expression on the last equality by rewriting the diagonal terms $\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k}$. Next, we note that for our choice of γ we have,

$$\pm \sum_{i=-1}^{k} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}} \cdot \left(\gamma \cdot (\ell^2 + \ell \cdot k)\|\boldsymbol{h}_i\|_{\Pi_i}\|\boldsymbol{f}_i\|_{\Pi_k}\right) = \pm\frac{\varepsilon}{4}\sum_{i=-1}^{k}\langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle_{\Pi_k} = \pm\frac{\varepsilon}{2},$$

where we have used Lemma 4.3.15 and Corollary 4.3.17. Next we note that we can bound the cross-terms $\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle_{\Pi_k}$. Suppose, $i > j$ then,

$$\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle_{\Pi_k} = \langle \mathsf{Y}\mathsf{U}_i^{(k-i)}\boldsymbol{h}_i, \mathsf{U}_j^{(k-j)}\boldsymbol{h}_j\rangle_{\Pi_k} = \langle \mathsf{A}\widetilde{\mathsf{Y}}\boldsymbol{h}_i, \boldsymbol{h}_j\rangle_{\Pi_j},$$

where we wrote $\widetilde{\mathsf{Y}} = \mathsf{D}_i \cdots \mathsf{D}_k\mathsf{Y}\mathsf{U}_{k-1}\cdots\mathsf{U}_i$ and $\mathsf{A} = \mathsf{D}_{j+1}\cdots\mathsf{D}_{i+1}$. Since $\widetilde{\mathsf{Y}}$ contains more down-operators than up operators and $\|\mathsf{A}\|_{\Pi_{i+1}\to\Pi_j} \le 1$ as A is row-stochastic, by Lemma 4.3.9

$$\begin{aligned}
\langle \mathsf{Y}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle_{\Pi_k} &= \pm\ (\ell + 2k + 1)^2 \cdot \gamma \cdot \|\boldsymbol{h}_i\|_{\Pi_i} \cdot \|\boldsymbol{h}_j\|_{\Pi_k}, \\
&= \pm\ (\ell + 2k + 1)^2 \cdot \gamma\Big(\|\boldsymbol{h}_i\|_{\Pi_i}^2 + \|\boldsymbol{h}_j\|_{\Pi_j}^2\Big).
\end{aligned}$$

where we have used the AM-GM inequality for the last step.

In particular, by our choice of γ,

$$\begin{aligned}
\sum_{\substack{i,j=-1,\\ i\neq j}} \sum_{\mathsf{Y}\in\mathcal{Y}} \alpha^{\mathsf{Y}} \cdot \langle \mathsf{B}\boldsymbol{f}_i, \boldsymbol{f}_j\rangle &= \pm\ (\ell + 2k + 1)^2 \cdot \gamma \cdot \left(\sum_{\mathsf{Y}\in\mathcal{Y}}\left|\alpha^{\mathsf{Y}}\right|\right) \cdot k \cdot \sum_{i=-1}^{k}\|\boldsymbol{h}_i\|_{\Pi_i}^2, \\
&= \pm\ \frac{\epsilon}{4} \cdot \sum_{i=-1}^{k}\|\boldsymbol{f}_i\|_{\Pi_k}^2, \\
&= \pm\ \frac{\epsilon}{2},
\end{aligned}$$

where we have used where we have used Lemma 4.3.15 and Corollary 4.3.17. □

We instantiate Lemma 4.3.16 for swap walks with their specific parameters. First, we introduce some notation. Using Corollary 4.1.9, (and $\mathsf{S}_k = \mathsf{S}_{k,k}^{(k+1)}$) we have

$$\mathsf{S}_k \ = \ \sum_{j=0}^{k+1}(-1)^{k+1-j} \cdot \binom{k+j+1}{k+1} \cdot \binom{k+1}{j} \cdot \mathsf{P}_k^{(j)} \ = \ \sum_{j=0}^{k}\alpha_j \cdot \mathsf{P}_k^{(j)},$$

where $\alpha_j = (-1)^{k+1-j} \cdot \binom{k+1+j}{k+1} \cdot \binom{k+1}{j}$.

Finally, we have all the pieces to prove Lemma 4.1.7 restated below.

Lemma 4.1.7. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander where $\gamma \le \epsilon \left(64k^{k+4}2^{3k+2}\right)^{-1}$ for some $\epsilon \in (0,1)$. If $d \ge 2k+1$, there exists constants $\lambda_{k,i}^{\mathsf{S}_k}$ for all $i = -1, \ldots, k$ only depending on k and i (and not on (X, Π)) such that, for any $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ with $\|\boldsymbol{f}\|_{\Pi_k} = 1$ we have*

$$\langle \mathsf{S}_k \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_k} \;=\; \sum_{i=0}^{k} \lambda_{k,i}^{\mathsf{S}_k} \cdot \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_k} \;\pm\; \epsilon.$$

where $\boldsymbol{f} = \sum_{i=-1}^{k} \boldsymbol{f}_i$ is the unique decomposition satisfying $\boldsymbol{f}_i \in \mathcal{V}^{(k;i)}$

Proof. First note that Lemma 4.3.13 establishes the existence of approximate eigenvalues $\lambda_{k,i}^{\mathsf{P}_k^{(j)}}$ of $\mathsf{P}_k^{(j)}$ corresponding to space $\mathcal{V}^{(k;i)}$ for $i = -1, \ldots, k$ such that $\lambda_{k,i}^{\mathsf{P}_k^{(j)}}$ depends only on k, i and j. We have, $\lambda_{k,i}^{\mathsf{S}_k} = \sum_{j=0}^{k+1} \alpha_j \lambda_{k,i}^{\mathsf{P}_k^{(i)}}$ where $\alpha_j = (-1)^{k+1-j} \cdot \binom{k+1+j}{k+1} \cdot \binom{k+1}{j}$. Now, to apply Lemma 4.1.8 and conclude the lemma (with Corollary 4.1.9) we only we need to bound $\sum_{j=0}^{k} |\alpha_j|$. Since

$$\sum_{j=0}^{k+1} |\alpha_j| \;=\; \sum_{j=0}^{k+1} \binom{k+j+1}{k+1} \cdot \binom{k+1}{j+1} \;\le\; 2^{k+1} \cdot \sum_{j=0}^{k} \binom{k+k+1}{j} \;=\; 2^{3k+2},$$

where we have used $\binom{k+1}{j} \le 2^{k+1}$ for eliminating the first term. $\square$

4.3.3 Rectangular Swap Walks $\mathsf{S}_{k,l}$

We turn to the spectral analysis of rectangular swap walks, i.e., $\mathsf{S}_{k,l}$ where $k \ne l$. Recall that to bound $\sigma_2(\mathsf{S}_k)$ in Theorem 4.1.1 we proved that the second sigular value of S_k on a two-sided γ-local spectral expander (X, Π) is close to the second singular value of $\mathsf{S}_k^{\Delta_{n,d}}$ using the analysis of quadratic forms over *balanced* operators from Section 4.3.2. Then we appealed to the fact that $\mathsf{S}_{k,k}^{\Delta_{n,d}}$ is expanding since it is the walk operator of the well known Kneser graph. In this rectangular case, we do not have a classical result establishing that $\mathsf{S}_{k,l}^{\Delta_{n,d}}$ is expanding, however one can still prove the following result Lemma 4.3.18.

Lemma 4.3.18 ([AJT19]). *Let parameters $d, k, l \geq 0$ be given such that $d \geq k + l + 1$. The second largest singular value $\sigma_2(\mathsf{S}^{\Delta_{n,d}}_{k+1,l+1})$ of the swap operator $\mathsf{S}^{\Delta_{n,d}}_{k,l}$ on $\Delta_{n,d}$ is*

$$\sigma_2(\mathsf{S}^{\Delta_{n,d}}_{k,l}) \;\leq\; \max\left(\frac{k+1}{n-k-1}, \frac{l+1}{n-l-1}\right),$$

provided $n \geq M_{k,l}$ where $M_{k,l} \in \mathbb{N}$ only depends on k and l.

Lemma 4.3.18 can be proven by studying a generalization of Kneser graphs which we call *bipartite Kneser* graphs. We write $K(n, k, l) = (L, R, E)$ for the bipartite graph with the bipartition (L, R) such that

$$L = \binom{[n]}{l}, \quad R = \binom{[n]}{k}, \quad \text{and} \quad E = \{\{\alpha, \beta\} : \alpha \in L, \beta \in R, \alpha \cap \beta = \varnothing\}.$$

We then note that the normalized-adjacency matrix $\mathsf{W}_{k+1,l+1}$ of the bipartite Kneser graph $K(n, k+1, l+1)$ can be written as,

$$\mathsf{W}_{k+1,l+1} = \begin{pmatrix} 0 & \mathsf{S}^{\Delta_{n,d}}_{k,l} \\ \left(\mathsf{S}^{\Delta_{n,d}}_{k,l}\right)^* & 0 \end{pmatrix},$$

where we have introduced the notation $\mathsf{S}^{\Delta_{n,d}}_{k,l}$ for the swap operator on the complete complex. We observe,

$$\mathsf{W}^2_{k+1,l+1} = \begin{pmatrix} \mathsf{S}^{\Delta_{n,d}}\left(\mathsf{S}^{\Delta_{n,d}}_{k,l}\right)^* & 0 \\ 0 & \left(\mathsf{S}^{\Delta_{n,d}}_{k,l}\right)^*\mathsf{S}^{\Delta_{n,d}}_{k,l} \end{pmatrix}$$

By Fact 2.1.6, the upper-left and bottom-right principal submatrices of $\mathsf{W}^2_{k+1,l+1}$ have the same singular values. In particular, as a consequence of the Courant-Fischer-Weyl Theorem 2.1.4 and Corollary 2.1.5 we have

$$\sigma_2\left(\mathsf{S}^{\Delta_{n,d}}_{k,l}\right) = \sqrt{\lambda_2\left(\mathsf{W}^2_{k+1,l+1}\right)} = \sigma_2(\mathsf{W}_{k+1,l+1}). \tag{4.8}$$

With this it is apparent that understanding the singular values of $\mathsf{W}_{k+1,l+1}$ is useful for understanding the singular values of $\mathsf{S}^{\Delta_{n,d}}_{k,l}$ – which is the operator which we actually care about.

It turns out, one can still establish the entire spectrum of these graphs,

Lemma 4.3.19 ([AJT19]). *The non-zero eigenvalues of the normalized walk matrix of* $\mathsf{W}_{k,l}$ *of* $K(n,k,l)$ *are*

$$\pm \frac{\binom{n-k-i}{l-i}\binom{n-l-i}{k-i}}{\binom{n-k}{l}\binom{n-l}{k}},$$

for $i = 0, \ldots, \min(k,l)$.

In particular, by Corollary 2.1.5 the singular values of $\mathsf{W}_{k,l}$ *are*

$$\frac{\binom{n-k-i}{l-i}\binom{n-l-i}{k-i}}{\binom{n-k}{l}\binom{n-l}{k}},$$

for $i = 0, \ldots, \min(k,l)$.

In [AJT19], the above lemma was proven by the observation that the precise set of eigenvectors for $\mathsf{W}_{k,l}$ is known (see, [Fil16]) as $\mathsf{W}_{k,l}$ is closely related to operators in the Johnson scheme (see, [GM15]). However, we omit this computation as it requires introducing some representation theory, which is beyond the scope of this thesis.

Now the proof follows a similar strategy to the one we have employed to prove Theorem 4.1.1, by analyzing quadratic forms over $\mathsf{S}_{k,l}^*\mathsf{S}_{k,l}$ using the results from Section 4.3.2 since by Corollary 4.1.9, the operator $\mathsf{S}_{k,l}^*\mathsf{S}_{k,l}$ balanced. Formally,

Corollary 4.3.20. *Let* (X,Π) *be a* d*-dimensional simplicial complex. Let* $k,l \geq 0$ *be parameters such that* $d \geq k \geq l$. *The operator* $\widetilde{\mathsf{S}}_{k,l} = \mathsf{S}_{k,l}^*\mathsf{S}_{k,l} \in \mathbb{R}^{X(l)\times X(l)}$ *is balanced and satisfies the following equality,*

$$\widetilde{\mathsf{S}}_{k,l} = \sum_{i,j=0}^{l+1} (-1)^{i+j}\binom{k+1+j}{l+1}\binom{k+1+i}{l+1}\binom{l+1}{i}\binom{l+1}{j}\left(\mathsf{P}_{k,l}^{(j)}\right)^*\mathsf{P}_{k,l}^{(i)}.$$

Proof. The formula follows directly from Corollary 4.1.9. We only verify that the terms $\left(\mathsf{P}_{k,l}^{(j)}\right)^*\mathsf{P}_{k,l}^{(i)}$ are pure balanced operators, i.e. contain an equal number of up- and down-operators. We have,

$$\begin{aligned}
\left(\mathsf{P}_{k,l}^{(j)}\right)^*\mathsf{P}_{k,l}^{(i)} &= (\mathsf{D}_{k+1}\cdots\mathsf{D}_{k+j}\mathsf{U}_{k+j-1}\cdots\mathsf{U}_l)^*\mathsf{D}_{k+1}\cdots\mathsf{D}_{k+i}\mathsf{U}_{k+i-1}\cdots\mathsf{U}_l, \\
&= \mathsf{D}_{l+1}\cdots\mathsf{D}_{k+j}\mathsf{U}_{k+j-1}\cdots\mathsf{U}_k\mathsf{D}_{k+1}\cdots\mathsf{D}_{k+i}\mathsf{U}_{k+i-1}\cdots\mathsf{U}_l,
\end{aligned}$$

and thus each $\left(\mathsf{P}_{k,l}^{(j)}\right)^*\mathsf{P}_{k,l}^{(i)}$ is a product of $k+j-l+i$ up- and $k+j-l+i$ down operators. □

Lemma 4.3.21. *Let (X, Π) be a pure d-dimensional two-sided γ-local expander (in particular, a γ-HDX) such that $\gamma \le \epsilon \cdot (128 \cdot \cdot k^2 \cdot l^{l+4} 2^{4l+2k+6})^{-1}$ for some $\epsilon \in (0,1)$. Let k, l be parameters such that $d \ge k + l + 1$ and $k \ge l$. For any $\boldsymbol{f} = \sum_{i=-1}^{l} \boldsymbol{f}_i \in \mathbb{R}^{X(l)}$ such that $\boldsymbol{f}_i \in \mathcal{V}^{(l;i)}$ for all $i = -1, \ldots, l$ and $\|\boldsymbol{f}\|_{\Pi_l} = 1$, we have*

$$\left\langle \widetilde{\mathsf{S}}_{k,l} \boldsymbol{f}, \boldsymbol{f} \right\rangle_{\Pi_l} = \sum_{i=-1}^{k} \lambda_{l,i}^{\widetilde{\mathsf{S}}_{k,l}} \langle \boldsymbol{f}_i, \boldsymbol{f}_i \rangle_{\Pi_l} \ \pm \ \epsilon$$

where $\lambda_{k,i}^{\widetilde{\mathsf{S}}_{k,l}}$ are constants only depending on k, l, i (and not on (X, Π))

Proof. The proof will be analogous to that of Lemma 4.1.7. By Corollary 4.3.20, we have

$$\widetilde{\mathsf{S}}_{k,l} = \sum_{i,j=0}^{l+1} \alpha_{i,j} \cdot \left(\mathsf{P}_{k,l}^{(i)}\right)^* \cdot \mathsf{P}_{k,l}^{(j)},$$

where $\alpha_{i,j} = (-1)^{i+j} \binom{k+1+j}{l+1} \binom{k+1+i}{l+1} \binom{l+1}{i} \binom{l+1}{j}$. Similarly to what we did in the proof of Lemma 4.1.7, we set set $\lambda_{l,r}^{\widetilde{\mathsf{S}}_{k,l}} = \sum_{i,j=0}^{k} \alpha_{i,j} \lambda_{l,r}^{\widetilde{\mathsf{S}}_{k,l}}$ (where $\lambda_{l,r}^{\widetilde{\mathsf{S}}_{k,l}}$ are obtained from Lemma 4.3.13). Now to apply Lemma 4.1.8, all we need to do is bound $\sum_{i,j=0}^{l+1} |\alpha_{i,j}|$.

$$\begin{aligned} \sum_{i,j=0}^{l+1} |\alpha_{i,j}| \ &= \ \sum_{i,j=0}^{l+1} \binom{k+j+1}{l+1} \cdot \binom{l+1}{j} \binom{k+i+1}{l+1} \cdot \binom{l+1}{i} \\ &\le \ 2^{2l+2} \cdot \left(\sum_{j=0}^{l+1} \binom{k+j+1}{l+1} \right) \cdot \left(\sum_{i=0}^{l+1} \binom{k+i+1}{i} \right) \ \le \ 2^{4l+2k+6}, \end{aligned}$$

we are done. $\square$

Next, we show the analogous statement to Theorem 4.1.1 for $\widetilde{\mathsf{S}}_{k,l}$ which states that the random walk $\widetilde{\mathsf{S}}_{k,l}$ on a two-sided γ-local spectral expander has very small second singular value.

Lemma 4.3.22. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander. Suppose $k, l \ge 0$ are parameters such that $d \ge k + l + 1$. If $\gamma \le \epsilon^2 \cdot (128 \cdot k^2 \cdot l^{l+4} 2^{4l+2k+6})^{-1}$*

for some $\epsilon \in (0,1)$, then the second singular value of the operator $\widetilde{\mathsf{S}}_{k,l}$ can be bounded from above

$$\sigma_2(\widetilde{\mathsf{S}}_{k,l}) \;\leq\; \epsilon^2.$$

Proof. Using Lemma 4.3.18, and the same argument as the one used to prove Theorem 4.1.1 we can conclude that $\lambda_{l,i}^{\widetilde{\mathsf{S}}_{k,l}} = 0$ for all $i \geq 0$. Then, the proof follows analogously to Theorem 4.1.1. $\square$

Now the proof of Theorem 4.1.2 follows directly, as we have by the definition of singular values

$$\sigma_2(\mathsf{S}_{k,l}) = \sqrt{\lambda_2\big(\mathsf{S}_{k,l}^*\mathsf{S}_{k,l}\big)} = \sqrt{\lambda_2(\widetilde{\mathsf{S}}_{k,l})} = \sqrt{\sigma_2(\widetilde{\mathsf{S}}_{k,l})},$$

where for the last step we have used that since $\widetilde{\mathsf{S}}_{k,l}$ is positive semi-definite, i.e. $\widetilde{\mathsf{S}}_{k,l} \succeq_{\Pi_l} 0$ we have $\sigma_2(\widetilde{\mathsf{S}}_{k,l}) = \lambda_2(\widetilde{\mathsf{S}}_{k,l})$.

Thus, we obtain Theorem 4.1.2, restated below for convenience

Theorem 4.1.2. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander. Suppose $k, l \geq 0$ are parameters such that $d \geq k + l + 1$. If $\gamma \leq \epsilon^2 \cdot (128 \cdot k^2 \cdot l^{l+4} 2^{4l+2k+6})^{-1}$ for some $\epsilon \in (0,1)$, then the second singular value of the operator $\mathsf{S}_{k,l}$ can be bounded from above*

$$\sigma_2(\mathsf{S}_{k,l}) \;\leq\; \epsilon.$$

Chapter 5

Future Directions

Finally, in this chapter we will discuss some interesting future directions.

Sampling Applications. The spectral bound from Theorem 3.1.1 for the down-up walk $\mathsf{P}_k^{\triangle}$, provides a natural framework for studying the rapid mixing of many natural chains of interest. What are other settings we can apply this machinery to establish rapid mixing? Can we use this machinery to show that the natural Markov chain for sampling a uniformly random $\Delta + o(\Delta)$ coloring of a graph of maximum degree Δ is rapidly mixing? This is a very important open question in the area of Markov chains. Currently, it is only known that this Markov chain mixes rapidly when the number of colors is at least $(11/6 - \eta)\Delta$ where η is some fixed constant [CDM+19].

Improving Oppenheim's Theorem. Oppenheim's Theorem 2.3.6 is a very powerful tool for establishing local spectral expansion, as it reduces the problem of studying the spectra of the links of any dimension, to studying the spectra of links of maximal dimension. However, a shortcoming of Oppenheim's Theorem 2.3.6 is the very strong assumption it makes on the expansion of the top-level links: In a d-dimensional complex (X, Π), to have a non-trivial implication of Theorem 2.3.6, one needs to assume $\gamma_{d-2} = O(1/d)$. This assumption is not always met, i.e. when the top level links are graphs on a constant number of vertices, in general, one cannot hope for anything better than $\gamma_{d-2} = \Omega(1)$. A natural question here is whether one can improve Theorem 2.3.6, to work with weaker priors. A very ambitious (but extremely useful) goal would be trying to understand what priors the simplicial complex (X, Π) needs to satisfy such that assuming only the connectivity of the links and $\gamma_{d-2} = \Omega(1)$ we can show $\gamma_j = O(1/(d-j))$ for $j = -1, \ldots, d-2$.

More General Random Walk Models. Though the higher order random walk framework is powerful, unfortunately it does not cover all random walks of interest. One example is the Markov chains for sampling perfect or near perfect matchings [JS89, JSV04]. Can we use local spectral techniques for establishing rapid mixing of a wider array of natural Markov chains? One particular interesting example is the Markov Chain used for perfect or near perfect matchings. This approach might also allow an angle to attack the general problem of sampling matroid intersection, of which matchings are a special case. Presently, this problem is wide open.

Modified Log-Sobolev Inequalities. Our result Theorem 3.1.1 for bounding the second eigenvalue of the down-up walk $\mathsf{P}_k^{\triangle}$ is useful in arguing rapid mixing, because it could be used to bound the spectral gap which quantifies the multiplicative decrease in the ℓ_2-distance between the distribution of the random walk and the stationary distribution after taking a step of the random walk (Theorem 2.2.7). For sharper estimates on the mixing time, one can consider the *modified log-Sobolev constant* which roughly quantifies the multiplicative decrease in the distance between the distribution of the random walk and the stationary distribution in relative entropy after taking a step of the random walk. Recently, Cryan, Guo, and Mousa [CGM19] have shown that the natural Markov chain described by $\mathsf{P}_r^{\triangledown}$ for sampling matroid bases satisfies a modified log-Sobolev inequality and established that this natural chain has optimal mixing time $O(r \log r)$. Do their result generalize to all high-dimensional expanders? More generally, is there a property of the local graphs G_α that would allow us to bound the (modified) log-Sobolev constant of the corresponding chain as opposed to the spectral gap? This would give a more systematic way of studying the (modified) log-Sobolev constants of many interesting Markov chains, which in practice is known to be a challenging task.

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Appendices

Appendix A

Improved Bound on the Second Singular Value of $\mathsf{S}_{k,l}$

A.1 Improved Bound on $\sigma_2(\mathsf{S}_{k,l})$

To avoid introducing additional notation, we only provide a sketch for the proof of the improved bound on the second singular value of the swap walk $\mathsf{S}_{k,l}$ on two-sided γ-local spectral expanders due to [DD19].

Theorem 4.1.3. *Let (X, Π) be a pure d-dimensional two-sided γ-local spectral expander. Let $k, l \geq 0$ be parameters such that $d \geq k + l + 1$. Then, writing $\mathsf{S}_{k,l}$ for the swap operator on the complex (X, Π) we have*

$$\sigma_2(\mathsf{S}_{k,l}) \leq (k+1) \cdot (l+1) \cdot \gamma.$$

To prove this result, we need to introduce the concept of partite complexes: Let (X, Π) be a d-dimensional simplicial complex. We call (X, Π) partite, if there exists a partition of $X(0) = \bigsqcup_{i=1}^{d+1} V_i$ such that $V_i \cap V_j = \varnothing$ for all $i \neq j$ and

$$|\alpha \cap V_i| = 1 \ \text{ for all } \ \alpha \in X(d), \ i = 1, \ldots, d+1.$$

We observe that a bipartite graph $G = (L, R, E)$ is a one-dimensional partite simplicial complex and thus partiteness can be seen as a generalization of bipartiteness for graphs to simplicial complexes.

We now introduce the main lemma we will use to obtain Theorem 4.1.3.

Lemma A.1.1. *Let (Y, Π) be a 2-dimensional partite complex with parts $Y(0) = V_1 \sqcup V_2 \sqcup V_3$. Let $\mathsf{M}^{1,2}, \mathsf{M}^{1,3}$, and $\mathsf{M}^{2,3}$ denote the random walk matrices of the bipartite graphs $G_{1,2} = G_\varnothing[V_1 \sqcup V_2]$, $G_{1,3} = G_\varnothing[V_1 \sqcup V_3]$ and $G_{2,3} = G_\varnothing[V_2 \sqcup V_3]$ – where we have written $G_\varnothing$ for the empty link in Y, and in every one of these random walks transitions between vertices i and j are taken with probability proportional to $\Pi_1(\{i, j\})$. If for every $v \in V_1$, the adjacency matrix of the link graph M_v satisfies $\sigma_2(\mathsf{M}_v) \le \eta$, then*

$$\sigma_2(\mathsf{M}^{2,3}) \le \eta + \sigma_2(\mathsf{M}^{1,2}) \cdot \sigma_2(\mathsf{M}^{1,3}).$$

Proof Sketch. By abuse of notation, we will take $\mathsf{M}^{i,j} \in \mathbb{R}^{V_i \times V_j}$ to be the upper right submatrix of the random walk matrix of $G_{i,j}$, i.e.

$$\mathsf{M}_{G_{i,j}} = \begin{pmatrix} 0 & \mathsf{M}^{i,j} \\ \mathsf{M}^*_{i,j} & 0 \end{pmatrix}.$$

Notice that $\sigma_2(\mathsf{M}_{G_{i,j}}) = \sigma_2(\mathsf{M}^{i,j})$.

Suppose $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$ be unit vectors such that $\boldsymbol{f}, \boldsymbol{g} \perp \mathbf{1}$.

We then note that by analogous arguments to those we have used used for proving Oppenheim's Theorem 2.3.6, one can prove for any $\boldsymbol{f} \in \mathbb{R}^{V_3}$ and $\boldsymbol{g} \in \mathbb{R}^{V_2}$, we have

$$\left\langle \mathsf{M}^{2,3}\boldsymbol{f}, \boldsymbol{g} \right\rangle = \mathop{\mathbb{E}}_{v \sim V_1}[\langle \mathsf{M}_v \boldsymbol{f}_v, \boldsymbol{g}_v \rangle], \tag{A.1}$$

where we have again abused notation to write M_v for the upper-right submatrix of the bipartite graph G_v with parts L and R confined into V_2 and V_3 (since $v \in V_1$ and Y is partite). We again follow the proof strategy of Theorem 2.3.6, and decompose $\boldsymbol{f}_v =$

$\boldsymbol{f}_v^{\mathbf{1}} + \boldsymbol{f}_v^{\perp\mathbf{1}}$ and $\boldsymbol{g}_v = \boldsymbol{g}_v^{\mathbf{1}} + \boldsymbol{g}_v^{\perp\mathbf{1}}$, where $\boldsymbol{f}_v^{\mathbf{1}}$ and $\boldsymbol{g}_v^{\mathbf{1}}$ are the parts of $\boldsymbol{f}_v$ and $\boldsymbol{g}_v$ that are parallel to the constant function $\mathbf{1}$, and $\boldsymbol{f}_v^{\perp\mathbf{1}}$ and $\boldsymbol{g}_v^{\perp\mathbf{1}}$ are the parts of $\boldsymbol{f}_v$ and $\boldsymbol{g}_v$ that are perpendicular to $\mathbf{1}$. Since M_v is row-stochastic $\mathsf{M}_v\boldsymbol{f}_v^{\mathbf{1}}$ is also parallel to $\mathbf{1}$, and $\mathsf{M}_v\boldsymbol{f}_v^{\perp\mathbf{1}}$ is also perpendicular to $\mathbf{1}$. Thus,

$$\begin{aligned}\left\langle \mathsf{M}^{2,3}\boldsymbol{f}, \boldsymbol{g}\right\rangle &= \mathop{\mathbb{E}}_{v\sim V_1}\left[\left\langle \mathsf{M}_v\boldsymbol{f}_v^{\mathbf{1}}, \boldsymbol{g}_v^{\mathbf{1}}\right\rangle + \left\langle \mathsf{M}_v\boldsymbol{f}_v^{\perp\mathbf{1}}, \boldsymbol{g}_v^{\perp\mathbf{1}}\right\rangle\right],\\ &= \mathop{\mathbb{E}}_{v\sim V_1}\left[\left\langle \mathsf{M}_v\boldsymbol{f}_v^{\mathbf{1}}, \boldsymbol{g}_v^{\mathbf{1}}\right\rangle\right] + \mathop{\mathbb{E}}_{v\sim V_1}\left[\left\langle \mathsf{M}_v\boldsymbol{f}_v^{\perp\mathbf{1}}, \boldsymbol{g}_v^{\perp\mathbf{1}}\right\rangle\right],\end{aligned}$$

For bounding the second term, we note that

$$\begin{aligned}\mathop{\mathbb{E}}_{v\sim V_1}\left[\left\langle \mathsf{M}_v\boldsymbol{f}_v^{\perp\mathbf{1}}, \boldsymbol{g}_v^{\perp\mathbf{1}}\right\rangle\right] &\leq \mathop{\mathbb{E}}_{v\sim V_1}[\sigma_2(\mathsf{M}_v)\|\boldsymbol{f}_v\|\cdot\|\boldsymbol{g}_v\|],\\ &\leq \mathop{\mathbb{E}}_{v\sim V_1}\left[\sigma_2(\mathsf{M}_v)\cdot\frac{1}{2}\cdot\left(\|\boldsymbol{f}_v\|^2+\|\boldsymbol{g}_v\|^2\right)\right], \quad \text{(by AM-GM inequality)}\\ &\leq \eta\cdot\frac{1}{2}\mathop{\mathbb{E}}_{v\sim V_1}\left[\|\boldsymbol{f}_v\|^2+\|\boldsymbol{g}_v\|^2\right],\\ &= \eta.\end{aligned}$$

Where the last step follows from a similar argument to how we have proved $\mathbb{E}_{v\sim\Pi_0}\|\boldsymbol{f}_v\|_{\Pi_v}^2 = \|\boldsymbol{f}\|_{\Pi_0}^2$ in the proof of Theorem 2.3.6 and the fact that $\boldsymbol{f}$ and $\boldsymbol{g}$ are unit vectors.

To bound the first term, $\mathbb{E}_{v\sim V_1}\langle \mathsf{M}_v\boldsymbol{f}_v^{\mathbf{1}}, \boldsymbol{g}_v^{\mathbf{1}}\rangle$, we notice that analogously to have we shown, $\mathsf{M}_v\boldsymbol{f}^{\mathbf{1}} = [\mathsf{M}_\varnothing\boldsymbol{f}](v)\cdot\mathbf{1}$ in the proof of Theorem 2.3.6, we can show,

$$\boldsymbol{g}^{\mathbf{1}} = \mathop{\mathbb{E}}_{u\sim V_3\cap Y_v(0)}\boldsymbol{g}_v(u) = [\mathsf{M}^{1,3}\boldsymbol{g}](v)\cdot\mathbf{1} \quad\text{and}\quad \boldsymbol{f}^{\mathbf{1}} = \mathop{\mathbb{E}}_{v\sim V_2\cap Y_v(0)}\boldsymbol{f}_v(u) = [\mathsf{M}^{1,2}\boldsymbol{f}](v)\cdot\mathbf{1}.$$

Now, notice that since M_v is an averaging operator, we have

$$\mathop{\mathbb{E}}_{v\in V_1}\langle \mathsf{M}_v\boldsymbol{f}_v^{\mathbf{1}}, \boldsymbol{g}_v^{\mathbf{1}}\rangle = \mathop{\mathbb{E}}_{v\in V_1}\left[[\mathsf{M}^{1,2}\boldsymbol{f}](v)\cdot[\mathsf{M}^{1,3}\boldsymbol{g}]\right] = \langle \mathsf{M}^{1,2}\boldsymbol{f}, \mathsf{M}^{1,3}\boldsymbol{g}\rangle \leq \sigma_2(\mathsf{M}^{1,2})\cdot\sigma_2(\mathsf{M}^{1,3}).$$

Then, we observe

$$\langle \mathsf{M}^{1,3}\boldsymbol{f}, \boldsymbol{g}\rangle \leq \eta + \sigma_2(\mathsf{M}^{1,2})\cdot\sigma_2(\mathsf{M}^{1,3}).$$

Since this holds for arbitrary unit vectors $\boldsymbol{f}, \boldsymbol{g}$ such that $\boldsymbol{f}, \boldsymbol{g} \perp \mathbf{1}$ the statement follows. $\square$

We can now proceed to sketch a proof for Theorem 4.1.3,

Proof Sketch for Theorem 4.1.3. We will proceed by induction on $k + l$. Suppose that $k + l = 0$, i.e. $k = l = 0$. Then, $\mathsf{S}_{k,l}$ is just the empty link $\mathsf{M}_{\varnothing}$, and by the two-sided γ-local spectral expansion criterion, the statement follows.

We suppose now that there exists an N such that for all $k, l \geq 0$ with $k + l \leq N$ we have

$$\sigma_2(\mathsf{S}_{k,l}) \leq \gamma \cdot (k+1) \cdot (l+1).$$

Suppose now, $k + l = N$. We want to show, that we have

$$\sigma_2(\mathsf{S}_{k+1,l}) \leq \gamma \cdot (k+2) \cdot (l+1).$$

To this end, we describe a 2-dimensional partite simplicial complex Y with parts,

$$Y(0) = X(k) \sqcup X(k+1) \sqcup X(l).$$

And insert a 2-dimensional face $\{\alpha_1, \alpha_2, \alpha_3\}$ into Y for $\alpha_1 \in X(k)$, $\alpha_2 \in X(k+1)$, and $\alpha_3 \in X(l)$ whenever $\alpha_1 \subset \alpha_2$, $\alpha_2 \cap \alpha_3 = \varnothing$ and $\alpha_2 \sqcup \alpha_3 \in X$. The weight associated with face $\{\alpha_1, \alpha_2, \alpha_3\}$ is proportional to $\Pi_{k+l+2}(\alpha_2 \sqcup \alpha_3)$.

We want to apply Lemma A.1.1 to conclude the theorem. We note that for all $\alpha_1 \in X(k)$, and the link graph G_{α_1} of Y_{α_1} corresponds to the swap walk $\mathsf{S}_{0,l}$ on the link of X_{α_1}, and by the inductive assumption $\sigma_2(\mathsf{M}_{\alpha_1}) \leq (l+1) \cdot \gamma$.

The bipartite graph between $X(k)$ and $X(l)$ ($G_{1,3}$ in Lemma A.1.1) is precisely the swap walk $\mathsf{S}_{k,l}$ on the complex X, and by assumption satisfies $\sigma_2(\mathsf{S}_{k,l}) \leq (k+1) \cdot (l+1) \cdot \gamma$.

And the bipartite graph between $X(k+1)$ and $X(l+1)$ ($G^{2,3}$ from Lemma A.1.1) is precisely the swap walk $\mathsf{S}_{k+1,l}$.

Thus, by Lemma A.1.1 we obtain

$$\sigma_2(\mathsf{S}_{k+1,l}) \leq (l+1)\gamma + (k+1)(l+1)\gamma \cdot \sigma_2(\mathsf{M}^{1,2}),$$

where $\mathsf{M}^{1,2}$ is the random walk matrix of the bipartite graph between $X(k)$ and $X(k+1)$ ($G_{1,2}$ in Lemma A.1.1). Since $\mathsf{M}^{1,2}$ is row-stochastic, we have $\sigma_2(\mathsf{M}^{1,2}) \leq 1$ and the statement follows. $\square$