Cost-Efficient Contingent Claims with Choquet Pricing

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We examine a problem of the Neyman-Pearson type [22], in which an investor seeks the cheapest contingent claim that achieves a minimum performance subject to a maximum allowed risk exposure. Specifically, our problem minimizes a non-linear cost functional, subject to both a minimum performance measure and a maximum risk measure, where all expectations are taken in the sense of Choquet. Solutions to our problem are called *cost-efficient claims*, and possess a desirable monotonicity property as shown by Ghossoub [14]; the claims are anti-comonotonic with respect to the underlying asset, and therefore a hedge against its risk. By viewing our problem in the context of convex optimization, we apply a Karush-Kuhn-Tucker theorem to give necessary and sufficient conditions for cost efficiency. Such conditions also hold when the distortion functions are assumed to be absolutely continuous, but not necessarily continuously differentiable. This allows us to consider a broader set of risk measures, including the popular conditional value at risk (a.k.a. the expected shortfall). Under some additional assumptions, we explicitly characterize cost-efficient claims in closed-form, thereby extending the results of [16]. Finally, a numerical example is provided to illustrate our results in full detail.

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Chapter 1

Introduction

In a seminal paper, Schied [22] examines problems in which an investor seeks to raise an amount of capital $P_0 \geq 0$ by issuing a contingent claim with a fixed maturity. While there are many ways to construct such a claim, the investor also desires to achieve this amount of capital at a minimal amount of risk. The pool of available claims is represented by a collection $\mathcal{X}$ of uniformly bounded random variables on a given non-atomic probability space $(\Omega, \mathcal{E}, \mathbb{P})$, and the risk associated with a contingent claim $Y \in \mathcal{X}$ is determined by a risk measure $\rho : \mathcal{X} \to \mathbb{R}$. Furthermore, the amount of capital raised by such a claim Y is given by a pricing functional $\mathcal{P} : \mathcal{X} \to \mathbb{R}$ defined by $\mathcal{P}(Y) = \mathbb{E}[\xi Y] = \int \xi \cdot Y \, d\mathbb{P}$, where ξ is a state price density (i.e., a strictly positive random variable with $\mathbb{E}[\xi] = 1$). When the risk measure ρ is simply the expectation with respect to $\mathbb{P}$, this problem reduces to the classical Neyman-Pearson problem for randomized tests - as such, Schied calls this problem a *Neyman-Pearson problem* for the risk measure ρ. Such problems arise naturally in portfolio choice theory and risk management; indeed, since the classical mean-variance portfolio theory of Markowitz, investors' portfolios are constructed to meet a desirable average return while minimizing the variance, interpreted as the risk exposure. Schied [22] shows the existence of optimal solutions to the Neyman-Pearson problem when ρ satisfies certain properties: namely, monotonicity, convexity, and continuity from above. Moreover, he gives a closed-form characterization of the solution when ρ is a quantile-based risk measure, while retaining the assumption of linear pricing.

Ghossoub [16] interprets the aforementioned problem as one of seeking a contingent claim with the minimal price, among those that satisfy a minimum desired level of performance. In his case, the problem is formulated as

$$\inf_{Y\in\mathcal{X}}\left\{\mathcal{C}(Y): 0\leq Y\leq N,\, \mathcal{P}(Y)\geq P_0\right\}, \tag{1.1}$$

where $\mathcal{C}:\mathcal{X}\to\mathbb{R}$ is a pricing rule and $\mathcal{P}:\mathcal{X}\to\mathbb{R}$ is a performance measure. While Schied [22] assumes the existence of a state-price density and a linear pricing rule $\mathcal{C}(Y)=\int\xi\cdot Y\,d\mathbb{P}$, linearity is generally not exhibited by securities markets with imperfections (e.g., the existence of bid-ask spreads). For example, such inefficiencies can be captured by a sublinear cost functional (i.e., positive homogeneous and subadditive), which can be represented as the maximum of a collection $\mathcal{L}$ of linear positive pricing rules (e.g., Jouini and Kallal [18]). In some cases, such a maximum can be represented as a Choquet pricing rule with respect to a submodular capacity, as examined by Chateauneuf et al. [8]. A Choquet pricing rule is a nonlinear pricing rule of the form

$$\mathcal{C}(Y)=\int Y\,d\nu,$$

where ν is a non-additive measure (a capacity), and integration is in the sense of Choquet. We refer to Appendix A.2.1 for more on non-additive measures and Choquet integration. Araujo et al. [2], Chateauneuf and Cornet [7], and Cerreia-Vioglio et al. [6] also provide characterizations of Choquet pricing rules.

In an earlier work, Ghossoub [14] proves existence of an optimal solution to the Neyman-Pearson problem 1.1, if the capacity ν is continuous and strongly diffuse, and if $\mathcal{C}$ and $\mathcal{P}$ preserve uniformly bounded pointwise convergence. His results are also extended to a similar problem with an additional risk constraint which a claim must satisfy. In particular, the risk of a claim is given by the functional $\mathcal{R}:\mathcal{X}\to\mathbb{R}$, also assumed to preserve uniformly bounded pointwise convergence. Given a maximum risk tolerance level R_0, the problem is therefore

$$\inf_{Y\in\mathcal{X}}\left\{\mathcal{C}(Y): 0\leq Y\leq N,\, \mathcal{P}(Y)\geq P_0,\, \mathcal{R}(Y)\leq R_0\right\}, \tag{1.2}$$

and existence of an optimal solution is guaranteed. Such a problem arises naturally in risk management, as financial institutions often encounter regulatory restrictions on their risk exposure. We note that this setup differs from that of Schied [22] in that the objective is not to minimize the risk of a claim, but its cost, over a set of claims meeting a performance threshold and not exceeding a risk threshold.

Ghossoub [16] fully characterizes optimal solutions to Problem 1.1, in the case when the Choquet pricing rule and performance functional are given by expectations with respect to a distorted probability measure: that is, a capacity of the form $\nu = T \circ \mathbb{P}$, where $\mathbb{P}$ is a probability measure and T is an increasing and continuous function with $T(0) = 0$ and $T(1) = 1$. However, the results obtained therein do not directly extend to optimal solutions to Problem 1.2. For example, while a Lagrange multiplier method was applied to Problem 1.1, such a method relies on the fact that the performance constraint will always be attained at an optimum - this is not the case for Problem 1.2.

In this thesis, we address Problem 1.2 and the characterization of its optimal solutions. Specifically, we consider a scenario where $\mathcal{C}, \mathcal{P}$,and $\mathcal{R}$ are Choquet integrals with respect to possibly different distortions. That is,

$$\begin{aligned}
\mathcal{C}(Y) &= \int Y \, dT_1 \circ \mathbb{P}\,, \\
\mathcal{P}(Y) &= \int p(Y) \, dT_2 \circ \mathbb{P}\,, \\
\mathcal{R}(Y) &= \int r(Y) \, dT_3 \circ \mathbb{P}\,,
\end{aligned}$$

where T_1, T_2, and T_3 are given distortion functions, p is a concave and increasing real-valued function, and r is a convex and increasing real-valued function. Applying similar methods to those in [16], we can reformulate this problem using quantile functions by a change of measure. We then show that under certain reasonable conditions, Problem 1.2 can be understood as a convex optimization problem, and therefore a Karush-Kuhn-Tucker (KKT) theorem can be applied to obtain necessary and sufficient conditions for optimality. These conditions are also shown to hold under the less restrictive assumption that T_2 and T_3 are non-decreasing and absolutely continuous; this differs from the setting in [16] in that these distortions no longer need to be strictly increasing and continuously differentiable. This allows us to consider risk measures such as the expected shortfall, which can be represented as a distorted expectation with an absolutely continuous distortion function

[10]. Finally, under a few special conditions, similar to those imposed by Ghossoub [14], an optimal solution can be explicitly characterized in closed form.

The rest of this thesis is organized as follows. Chapter 2 introduces the main problem in detail and includes some definitions. Chapter 3 frames the problem in the context of convex optimization, and introduces a KKT theorem to obtain conditions for optimality. Chapter 4 gives two special cases in which explicit characterization is possible. Also in Chapter 4, we provide a numerical example in which a full explicit characterization of an optimal solution is given. Some related background and results about rearrangements, non-additive measures, and Choquet integration are given in the Appendices.

Chapter 2

Problem Setup and Formulation

2.1 Contingent Claims

In a given financial market, an investor wishes to hedge the risk of a security's random payoff, by purchasing a contingent claim (a derivative instrument). The random payoff of such an asset depends on a collection S of states of the world, equipped with a σ-algebra $\mathcal{E}$ of events. The payoff is represented by a random variable X on the measurable space $(S, \mathcal{E})$.

Let $\Sigma = \sigma\{X\}$ be the σ-algebra on S generated by X. We further assume that the space (S, Σ) is equipped with a probability measure $\mathbb{P}$, with the following additional assumptions on X:

1. $X \in L^\infty(S, \Sigma, \mathbb{P})$; and,

2. X is a continuous random variable with respect to $\mathbb{P}$. That is, the Borel probability measure $\mathbb{P} \circ X^{-1}$ is non-atomic.

Let $B(\Sigma)$ denote the linear space of all bounded, real-valued, and Σ-measurable functions on (S, Σ), and $B^+(\Sigma)$ its positive cone. When equipped with the supnorm[1], $B(\Sigma)$ is

[1]For all $Y \in B(\Sigma)$, the *supnorm* of Y is defined by $||Y||_{sup} = \sup\{|Y(s)| : s \in S\} < +\infty$.

a Banach space [1, Theorem 14.2]. By Doob's measurability theorem [1, Theorem 4.41], for any $Y \in B(\Sigma)$, there exists a bounded, Borel-measurable map $I : \mathbb{R} \to \mathbb{R}$ such that $Y = I \circ X$, and $Y \in B^+(\Sigma)$ if and only if the function I is nonnegative. We can then identify the collection of random payoffs of all contingent claims on X with $B^+(\Sigma)$.

2.2 Pricing rule

The market prices contingent claims through a cost functional $\mathcal{C} : B(\Sigma) \to \mathbb{R}$, assumed to be non-linear due to market frictions. Specifically, we consider a Choquet pricing rule, with respect to the distortion of the measure $\mathbb{P}$. Appendix A.2.1 provides background on non-additive measures and Choquet integration.

Assumption 2.2.1. (Choquet Pricing) There exists a distortion function $T_1 : [0, 1] \to [0, 1]$ such that:

1. $T_1(0) = 0$ and $T_1(1) = 1$;
2. T_1 is strictly increasing and continuously differentiable on $[0, 1]$;
3. $\mathcal{C}(Y) = \int Y \, dT_1 \circ \mathbb{P}$.

Here, we assume neither convexity nor concavity of the distortion function T_1.

2.3 Performance and Risk Measurement

Assumption 2.3.1. There exists distortion functions T_2, T_3, a function p, and a function r such that:

1. $T_2(0) = T_3(0) = 0$ and $T_2(1) = T_3(1) = 1$;
2. T_2 and T_3 are non-decreasing and absolutely continuous on $[0, 1]$;
3. p is strictly increasing and strictly concave;

4. r is strictly increasing and strictly convex;

5. $\mathcal{P}(Y) = \int p(Y)\, dT_2 \circ \mathbb{P}$;

6. $\mathcal{R}(Y) = \int r(Y)\, dT_3 \circ \mathbb{P}$.

One could interpret the function p as a utility function for the investor. Therefore, concavity of p agrees with the notion of diminishing marginal utility. We could similarly interpret the convexity of r through the notion of disutility, using a symmetric argument.

2.4 The Investor's Problem

The investor desires a minimum performance $P_0 \in \mathbb{R}^+$, subject to a maximum risk tolerance level $R_0 \in \mathbb{R}^+$. We also assume that the payoffs of all contingent claims available to the investor are bounded by some sufficiently large $N \in (0, +\infty)$. Hence, the investor's problem can be summarized as

$$\inf_{Y \in B^+(\Sigma)} \left\{ \int Y\, dT_1 \circ \mathbb{P} : 0 \leq Y \leq N, \int p(Y)\, dT_2 \circ \mathbb{P} \geq P_0, \int r(Y)\, dT_3 \circ \mathbb{P} \leq R_0 \right\}. \quad (2.1)$$

The set of contingent claims available to the investor is called the feasibility set, defined as

$$\mathcal{F} := \left\{ Y \in B^+(\Sigma) \,\middle|\, Y \leq N, \int p(Y)\, dT_2 \circ \mathbb{P} \geq P_0, \int r(Y)\, dT_3 \circ \mathbb{P} \leq R_0 \right\}.$$

It is possible that $\mathcal{F} = \varnothing$, e.g., if the performance constraint or the risk measure constraint cannot be met. For example, if $P_0 > p(N) = \int p(N)\, dT_2 \circ \mathbb{P}$, the performance constraint can never be met. To rule out trivial situations, we further assume the following:

Assumption 2.4.1. $0 \leq P_0 \leq p(N)$.

Assumption 2.4.2. $0 \leq R_0 \leq r(N)$.

However, it is important to note that these assumptions do not cover all possible scenarios that would admit an empty feasibility set. For a given performance measure P_0, it is possible that the risk tolerance level R_0 is so low that every contingent claim with sufficient performance is too risky for the investor. To avoid this, we impose the following additional assumption:

Assumption 2.4.3. There exists a $Y_0 \in B^+(\Sigma)$ such that

$$\int p(Y_0)\, dT_2 \circ \mathbb{P} > P_0, \int r(Y_0)\, dT_3 \circ \mathbb{P} < R_0.$$

This ensures that the performance level P_0 and risk tolerance level R_0 are selected so that there is still at least one contingent claim that is feasible for the investor. The strictness of the inequality is also important, as will become clear when trying to characterize the optimal claim.

2.5 Cost-Efficient Claims

Definition 2.5.1. A contingent claim $Y \in B^+(\Sigma)$ is *cost-efficient* if

1. $Y \in \mathcal{F}$, i.e., it is feasible for the investor; and,
2. $\mathcal{C}(Y) \leq \mathcal{C}(Z), \quad \forall Z \in \mathcal{F}$.

A contingent claim Y is *strictly cost-efficient* if it is cost-efficient, and there does not exist $Z \in \mathcal{F}$ such that

(i) $\mathbb{P}(Y \neq Z) > 0$, and,

(ii) $\mathcal{C}(Z) = \mathcal{C}(Y)$.

That is, for all $Z \in \mathcal{F}$ such that $\mathbb{P}(Y \neq Z) > 0$, we have $\mathcal{C}(Y) < \mathcal{C}(Z)$.

Since the cost functional, the performance measure, and the risk measure are all law-invariant (i.e., they depend only on the distribution of the contingent claim Y), Proposition 2.5.5 below gives a characterization of the shape of cost-efficient claims. The following arguments are adapted from Ghossoub [14, 16], and are summarized here.

Definition 2.5.2. Two functions $Y_1, Y_2 \in B(\mathcal{E})$ are said to be *comonotonic* if

$$[Y_1(s) - Y_1(s')]\,[Y_2(s) - Y_2(s')] \geq 0, \quad \text{for all } s, s' \in S.$$

Similarly, the functions Y_1, Y_2 are said to be *anti-comonotonic* or *countermonotonic* if

$$[Y_1(s) - Y_1(s')]\,[Y_2(s) - Y_2(s')] \leq 0, \quad \text{for all } s, s' \in S.$$

Lemma 2.5.3. *Let $\mathcal{F}^{\downarrow} \subseteq \mathcal{F}$ denote all the elements of $\mathcal{F}$ that are anti-comonotonic with X. Then for each $Y \in \mathcal{F}$, there exists a $\tilde{Y} \in \mathcal{F}^{\downarrow}$ such that $\mathcal{C}(Y) = \mathcal{C}(\tilde{Y})$, $\mathcal{P}(Y) = \mathcal{P}(\tilde{Y})$, and $\mathcal{R}(Y) = \mathcal{R}(\tilde{Y})$.*

Proof. Define $\tilde{Y}$ to be the non-increasing $\mathbb{P}$-upper-equimeasurable rearrangement of Y with respect to X, as defined in Appendix A.2.2. Then $\tilde{Y} \leq N$, and

$$\begin{aligned}
\mathcal{C}(Y) &= \int Y\, dT_1 \circ \mathbb{P} \\
&= \int_0^{\infty} T_1(\mathbb{P}(\{s \in S : Y(s) > t\}))\, dt \\
&= \int_0^{\infty} T_1(\mathbb{P}(\{s \in S : \tilde{Y}(s) > t\}))\, dt \\
&= \int \tilde{Y}\, dT_1 \circ \mathbb{P} \\
&= \mathcal{C}(\tilde{Y})\,.
\end{aligned}$$

Similarly, one can show that $\mathcal{P}(Y) = \mathcal{P}(\tilde{Y})$ and $\mathcal{R}(Y) = \mathcal{R}(\tilde{Y})$. □

Lemma 2.5.4. *(Helly's Compactness Theorem) If $(f_n)_n$ is a uniformly bounded sequence of non-increasing real-valued functions on a closed interval $\mathcal{I}$ in $\mathbb{R}$ with bound N, then there exists a non-increasing real-valued bounded function f^* on $\mathcal{I}$, also with bound N, and a subsequence of $(f_n)_n$ that converges pointwise to f^* on $\mathcal{I}$.*

Proof. See [11, pp. 165-166]. □

Proposition 2.5.5. *(Ghossoub [14]) Assuming that Problem 2.1 has a non-empty feasibility set, it admits a solution which is anti-comonotonic with X. Moreover, any strictly cost-efficient claim is necessarily anti-comonotonic with X.*

Proof. By Lemma 2.5.3, we see that if $\mathcal{F} \neq \varnothing$, then $\mathcal{F}^{\downarrow} \neq \varnothing$. Also, we can choose a minimizing sequence $\{Y_n\}_n$ in $\mathcal{F}^{\downarrow}$, that is,

$$\lim_{n\to\infty} \mathcal{C}(Y_n) = H := \inf_{Y\in\mathcal{F}} \mathcal{C}(Y)\,.$$

Since $0 \leq Y_n \leq N$ for all n, the sequence $\{Y_n\}_n$ is uniformly bounded. Also, for each n we have $Y_n = I_n \circ X$, and consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of non-decreasing Borel-measurable functions. Therefore by Lemma 2.5.4, there exists a non-decreasing function I^* and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ converging pointwise to I^*. Since I^* is also Borel-measurable, $Y^* := I^* \circ X \in B^+(\Sigma)$ and $0 \leq Y^* \leq N$, and Y^* is anti-comonotonic with X. Moreover, the sequence $\{Y_m\}_m$ defined by $Y_m := I_m \circ X$ converges pointwise to Y^*.

Recall that the distortion functions T_1, T_2, and T_3 were assumed to be continuous. Since $\mathbb{P}$ is a probability measure, it is a continuous capacity [15], and hence so are $T_1 \circ \mathbb{P}$, $T_2 \circ \mathbb{P}$, and $T_3 \circ \mathbb{P}$. Since p and r are continuous and non-decreasing, they are also Borel-measurable and bounded on any closed and bounded subset of $\mathbb{R}$. Since the Choquet integral with respect to a continuous capacity is an operator on $B^+(\Sigma)$ which preserves uniformly bounded convergence, it follows that $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{R}$ all preserve uniformly bounded pointwise convergence. Therefore it follows that $Y^* \in \mathcal{F}^{\downarrow}$. Also,

$$\mathcal{C}(Y^*) = \lim_{m\to\infty} \mathcal{C}(Y_m) = \lim_{n\to\infty} (Y_n) = H\,.$$

Hence, Y^* solves Problem 2.1, and is anti-comonotonic with X.

Finally, suppose for the sake of contradiction that there exists a strictly cost-efficient claim Y_0 not anti-comonotonic with X. Then taking $\tilde{Y}_0$ in the sense of Lemma 2.5.3, we have $\mathcal{C}(Y_0) = \mathcal{C}(\tilde{Y}_0)$, and $\tilde{Y}_0 \in \mathcal{F}$, contradicting the strict cost-efficiency of Y_0. □

Chapter 3

Towards a Characterization of Optimal Solutions

3.1 Quantile Formulation and Change of Variables

For each $Y \in B^+(\Sigma)$, let $F_Y(t) := \mathbb{P}(\{s \in S : Y(s) \leq t\})$ be the cumulative distribution function (cdf) of Y with respect to the probability measure $\mathbb{P}$. Let $F_Y^{-1}(t)$ denote the left-continuous inverse of the cdf F_Y (i.e., a *quantile* of Y), defined as

$$F_Y^{-1}(t) := \inf\left\{z \in \mathbb{R}^+ \middle| F_Y(z) \geq t\right\}, \quad \forall t \in [0,1].$$

Let $\mathcal{Q}$ denote the collection of all quantile functions, and let $\mathcal{Q}^*$ denote the collection of all quantile functions f which satisfy $0 \leq f(t) \leq N$ for all $t \in (0,1)$. Then

$$\mathcal{Q} = \left\{f : (0,1) \to \mathbb{R} \,\middle|\, f \text{ is non-decreasing and left-continuous}\right\},$$

and

$$\mathcal{Q}^* = \{f \in \mathcal{Q} : 0 \leq f(t) \leq N, \quad \forall\, 0 < t < 1\}.$$

Then the following uniform transform is instrumental in reformulating our problem in terms of quantiles.

Lemma 3.1.1. *The following conditions hold:*

i) $U := F_X(X)$ *is a random variable on* $(S, \Sigma, \mathbb{P})$ *with a uniform distribution on* $(0, 1)$*,*

ii) $X = F_X^{-1}(U)$ *almost surely (with respect to* $\mathbb{P}$*); and,*

iii) for each $Y \in \mathcal{F}$*, the function* $Y^* := F_Y^{-1}(1 - F_X(X)) = F_Y^{-1}(1 - U)$ *satisfies:*

  (a) $Y^* \in \mathcal{F}^{\downarrow}$*;*

  (b) Y *and* Y^* *have the same distribution with respect to* $\mathbb{P}$*; and,*

  (c) $\mathcal{C}(Y) = \mathcal{C}(Y^*), \mathcal{P}(Y) = \mathcal{P}(Y^*)$*, and* $\mathcal{R}(Y) = \mathcal{R}(Y^*)$*.*

Proof. See Lemma 9.3 of [14]. □

Now consider the problem:

$$\inf_{f \in \mathcal{Q}^*} \left\{ \int_0^1 T_1'(t) f(1-t)\, dt : \int_0^1 T_2'(t) p(f(1-t))\, dt \geq P_0, \int_0^1 T_3'(t) r(f(1-t))\, dt \leq R_0 \right\}. \tag{3.1}$$

Lemma 3.1.2. *If* f^* *is optimal for Problem 3.1, then* $Y^* := f^*(1 - F_X(X))$ *is optimal for Problem 2.1 and anti-comonotonic with* X*.*

Proof. Since each distortion is absolutely continuous, by Lemma 3.1.1, we have

$$\begin{aligned}
\mathcal{C}(Y) &= \int Y\, dT_1 \circ \mathbb{P} \\
&= \int T_1'(1-U) F_Y^{-1}(U)\, d\mathbb{P} \\
&= \int_0^1 T_1'(1-t) F_Y^{-1}(t)\, dt \\
&= \int_0^1 T_1'(t) F_Y^{-1}(1-t)\, dt\,.
\end{aligned}$$

A similar calculation for T_2 and T_3 yields

$$\mathcal{P}(Y) = \int_0^1 T_2'(t) p\left(F_Y^{-1}\right)(1-t)\, dt\,,$$

and

$$\mathcal{R}(Y) = \int_0^1 T_3'(t) r \left(F_Y^{-1}\right)(1-t)\, dt\,.$$

Now, let f^* be optimal for Problem 3.1. Then since $\mathcal{Q}^*$ is a collection of quantile functions, then there exists $Z^* \in B^+(\Sigma)$ such that f^* is the quantile of Z^*. Hence $0 \leq Z^* \leq N$, and the feasibility of f^* gives

$$\begin{aligned}
\mathcal{P}(Z^*) &= \int_0^1 T_2'(t) p \left(F_{Z^*}^{-1}\right)(1-t)\, dt \\
&= \int_0^1 T_2'(t) p(f^*(1-t))\, dt \\
&\geq P_0\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}(Z^*) &= \int_0^1 T_2'(t) r \left(F_{Z^*}^{-1}\right)(1-t)\, dt \\
&= \int_0^1 T_2'(t) r(f^*(1-t))\, dt \\
&\geq R_0\,.
\end{aligned}$$

Therefore Z^* is feasible for Problem 2.1. Hence, by Lemma 3.1.1, defining $Y^* := f^*(1 - F_X(X))$ is feasible for Problem 2.1, anti-comonotonic with X, and such that $\mathcal{C}(Z^*) = \mathcal{C}(Y^*), \mathcal{P}(Z^*) = \mathcal{P}(Y^*)$, and $\mathcal{R}(Z^*) = \mathcal{R}(Y^*)$.

It remains to show optimality of Y^*. If Y is any other feasible claim for Problem 2.1, then by Lemma 3.1.1, the function $Z = F_Y^{-1}(1 - F_X(X))$ is feasible for Problem 2.1 and anti-comonotonic with X. Moreover, $F_Z = F_Y$. Let $f = F_Y^{-1}$ so that $Z = f(1-U)$. Then we have

$$\mathcal{P}(Z) = \int p(Z)\, dT_2 \circ \mathbb{P} = \int_0^1 T_2'(t) p(f(1-t))\, dt \geq P_0\,,$$

and

$$\mathcal{R}(Z) = \int r(Z)\, dT_3 \circ \mathbb{P} = \int_0^1 T_3'(t) r(f(1-t))\, dt \leq R_0\,.$$

Therefore f is feasible for Problem 3.1. Since f^* is optimal for Problem 3.1, then

$$\begin{aligned}\int Z\, dT_1 \circ \mathbb{P} &= \int_0^1 T_1'(t) f(1-t)\, dt \\ &\geq \int_0^1 T_1'(t) f^*(1-t)\, dt \\ &= \int Z^*\, dT_1 \circ \mathbb{P}\,,\end{aligned}$$

showing optimality of Y^*.

□

By Lemma 3.1.2, if we can solve the quantile reformulation problem 3.1, then we can recover a cost-efficient claim for the original problem 2.1.

Using the substitution $v(t) = 1 - T_1^{-1}(1-t)$ and $z = v^{-1}(t)$, the objective can be rewritten as

$$\begin{aligned}\int_0^1 T_1'(t) f(1-t)\, dt &= \int_0^1 T_1'(1-t) f(t)\, dt \\ &= \int_0^1 f(t)\, d(1 - T_1(1-t)) \\ &= \int_0^1 f(t)\, dv^{-1}(t) \\ &= \int_0^1 f(v(z))\, dz \\ &= \int_0^1 q(z)\, dz,\end{aligned}$$

where $q = f \circ v$ is increasing and can be viewed as a quantile function. The performance constraint can also be rewritten, as

$$\begin{aligned}\int_0^1 T_2'(t)p(f(1-t))\,dt &= \int_0^1 T_2'(1-t)p(f(t))\,dt \\ &= \int_0^1 T_2'(1-v(z))p(f(v(z)))\,dv(z) \\ &= \int_0^1 T_2'(1-v(z))p(q(z))v'(z)\,dz \\ &= \int_0^1 T_2'(T_1^{-1}(1-t))p(q(t))(T_1^{-1})'(1-t)\,dt \\ &= \int_0^1 p(q(t))\frac{T_2'(T_1^{-1}(1-t))}{T_1'(T_1^{-1}(1-t))}\,dt \\ &= \int_0^1 p(q(t))\psi_2'(t)\,dt,\end{aligned}$$

where we define $\psi_2(t) := 1 - T_2(T_1^{-1}(1-t))$. Similarly, the risk constraint can be rewritten as

$$\int_0^1 T_3'(t)r(f(1-t))\,dt = \int_0^1 r(q(t))\psi_3'(t)\,dt,$$

with $\psi_3(t) := 1 - T_3(T_1^{-1}(1-t))$.

Now, define the set $\mathcal{Q}^{**}$ by

$$\mathcal{Q}^{**} := \Big\{ q : (0,1) \to \mathbb{R} \,\Big|\, q \text{ is non-decreasing and left-continuous, and} \\ 0 \le q(t) \le N, \text{ for each } 0 < t < 1 \Big\}. \quad (3.2)$$

Consider the following problem:

$$\inf_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\,dt : \int_0^1 p(q(t))\psi_2'(t)\,dt \ge P_0, \int_0^1 r(q(t))\psi_3'(t)\,dt \le R_0 \right\}. \quad (3.3)$$

Lemma 3.1.3. *If q^* is optimal for Problem 3.3, then the function f^* defined by $f^*(t) = q^*(1 - T_1(1-t))$ is optimal for Problem 3.1. Furthermore, $Y^* := f^*(1 - F_X(X)) = q^*(1 - T_1(F_X(X)))$ is optimal for Problem 2.1 and anti-comonotonic with X.*

Proof. Suppose q^* is optimal for Problem 3.3, and let $f^* = q^* \circ T_1$. Then $q^*(t) = f^*(v(t))$ for all $t \in [0,1]$. Since q^* is feasible for Problem 3.3, we have that q^* is non-decreasing and left-continuous, and therefore so is f^*: i.e., $f^* \in \mathcal{Q}$. Also, since $0 \le q^* \le N$, we also have $0 \le f^* \le N$, so $f^* \in \mathcal{Q}^*$. Then by the above,

$$\int_0^1 T_2'(t)p(f^*(1-t))\,dt = \int_0^1 p(q^*(t))\psi_2'(t)\,dt \ge P_0,$$

and

$$\int_0^1 T_3'(t)r(f^*(1-t))\,dt = \int_0^1 r(q^*(t))\psi_3'(t)\,dt \le R_0,$$

showing feasibility of f^* for Problem 3.1.

To show optimality of f^* for Problem 3.1, let f be any other feasible solution, and define $q := f \circ v$. Then by the above, we have

$$\int_0^1 T_1'(t)f(1-t)\,dt = \int_0^1 q(z)\,dz.$$

Feasiblity of q for Problem 3.3 follows similarly as the above. Since f is feasible for Problem 3.1, then it is non-decreasing and left-continuous, and therefore so is q. Also, since $0 \le f \le N$, we also have $0 \le q \le N$. Finally, by feasibility of f, we also have

$$\int_0^1 T_2'(t)p(f(1-t))\,dt = \int_0^1 p(q(t))\psi_2'(t)\,dt \ge P_0,$$

and

$$\int_0^1 T_3'(t)r(f(1-t))\,dt = \int_0^1 r(q(t))\psi_3'(t)\,dt \le R_0.$$

This guarantees feasibility of q. Therefore

$$\begin{aligned}\int_0^1 T_1'(t)f(1-t)\,dt &= \int_0^1 q(z)\,dz \\ &\ge \int_0^1 q^*(z)\,dz \\ &= \int_0^1 T_1'(t)f^*(1-t)\,dt,\end{aligned}$$

showing optimality of f^* for Problem 3.1. The remainder follows from Lemma 3.1.2. □

3.2 A Convex Programming Approach

The set $\mathcal{Q}^{**}$ is a set of quantile functions - in this case, a set of left-continuous functions $q : [0,1] \to \mathbb{R}$. Since each $q \in \mathcal{Q}^{**}$ is bounded by N, we can identify $\mathcal{Q}^{**}$ as a subset of $L^\infty([0,1])$. When equipped with the essential supnorm[1], $L^\infty([0,1])$ is a Banach space [1, Theorem 13.5], so we can appeal to the theory of convex optimization to solve Problem 3.3, provided that the objective function and the constraints satisfy certain regularity conditions.

Definition 3.2.1. (Convex functions) Let X be a real linear space, and $f : X \to \mathbb{R}$ a real-valued function on X. Then f is *convex* if

$$f(t_1x_1 + t_2x_2) \le t_1f(x_1) + t_2f(x_2)\,,$$

for all $x_1, x_2 \in X$ and $t_1, t_2 \ge 0$ with $t_1 + t_2 = 1$.

Theorem 3.2.2. *(Barbu & Precupanu [3]) Let X be a real linear space and let $h : X \to \bar{\mathbb{R}}$ be a given convex function. Define the feasible set $\mathcal{F}_X$ by*

$$\mathcal{F}_X = \left\{ x \in X \;\middle|\; g_i(x) \le 0 \;\; \forall i = 1, \ldots, n, \quad r_j(x) = 0 \;\; \forall j = 1, \ldots, m \right\},$$

where g_i and r_j are extended real-valued functions on X, convex and affine respectively. Consider the optimization problem

$$\min \left\{ h(x) \;\middle|\; x \in \mathcal{F}_X \right\},$$

and suppose further that this $\mathcal{F}_X$ is nonempty. If $x^* \in \mathcal{F}_X$ is an optimal solution to this optimization problem, that is,

$$h(x^*) = \min \left\{ h(x) \;\middle|\; x \in \mathcal{F}_X \right\},$$

then there exist $n + m + 1$ real numbers $\theta^*, \lambda_1^*, \ldots, \lambda_n^*, \mu_1^*, \ldots, \mu_m^*$ satisfying:

(i) $\theta^* \ge 0$;

[1] For all $q \in L^\infty([0,1])$, $||q||_{esssup} = \inf\{M > 0 : |q(x)| \le M \text{ for } \mathbb{P}\text{-a.e. } x\}$.

(ii) $\lambda_i^*, \mu_j^* \geq 0$ *for all* $i = 1, \ldots, n$, $j = 1, \ldots, m$*;*

(iii) $\theta^* h(x^*) \leq \theta^* h(x) + \sum_{i=1}^n \lambda_i^* g_i(x) + \sum_{j=1}^m \mu_j^* r_j(x)$ *for all* $x \in X$*;*

(iv) $\lambda_i^* g_i(x^*) = 0$ *for all* $i = 1, \ldots, n$*.*

Proof. See Theorem 3.1 from [3] and references therein. □

Note that if all g_i are identically zero, then this reduces to a classical optimization problem that can be solved by the Lagrangian multiplier method. Therefore this theorem can be seen as an extension of the Lagrangian method for inequality constraints, and the real numbers λ_i^* and μ_j^* can be interpreted as Lagrangian multipliers, as shown by the following definition.

Definition 3.2.3. (Lagrangian function) For a convex optimization problem as above, define its *Lagrangian function* as

$$\mathcal{L}(x, \lambda, \mu) = h(x) + \sum_{i=1}^{n} \lambda_i g_i(x) + \sum_{j=1}^{m} \mu_j r_j(x), \tag{3.4}$$

for all $(x, \lambda, \mu) \in X \times \mathbb{R}_+^n + \mathbb{R}^m$.

We see that Theorem 3.2.2 gives necessary conditions for an optimal solution; under some additional assumptions, we can show that these conditions are sufficient as well.

Definition 3.2.4. (Slater condition) An optimization problem satisfies the *Slater condition* if there exists a point $x_0 \in \mathcal{F}_X$ such that

$$g_i(x_0) < 0 \quad \forall i = 1, \ldots, n\,.$$

Theorem 3.2.5. *Under the hypotheses of Theorem 3.2.2, if we further assume that the Slater condition holds, then the necessary conditions of Theorem 3.2.2 are sufficient for optimality, and equivalently,* $\theta > 0$*. Furthermore, it suffices to optimize the Lagrangian function* $\mathcal{L}(x, \lambda^*, \mu^*)$ *as a function of* x *over* X*.*

Proof. See Theorems 3.7-3.9 from [3]. □

To apply the aforementioned convex programming technique, we can rewrite Problem 3.3 as follows:

$$\inf_{q\in\mathcal{Q}^{**}}\left\{\int_0^1 q(t)\,dt : \int_0^1 -p(q(t))\psi_2'(t)\,dt + P_0 \le 0, \int_0^1 r(q(t))\psi_3'(t)\,dt - R_0 \le 0\right\}. \quad (3.5)$$

We have two inequality constraints and no equality constraints. Define

$$g_1(q) = \int_0^1 -p(q(t))\psi_2'(t)\,dt + P_0\,;$$
$$g_2(q) = \int_0^1 r(q(t))\psi_3'(t)\,dt - R_0\,.$$

Then the following results show that Problem 3.5 satisfies the conditions of Theorems 3.2.2 and 3.2.5.

Proposition 3.2.6. *The objective cost functional* $\int_0^1 q(t)\,dt$ *is convex as a function of* q.

Proof. This follows trivially by linearity of the integral. □

One small detail to note is that Problem 3.5 is an optimization problem over the domain $\mathcal{Q}^{**}$, whereas the statement in Theorem 3.2.2 is a problem over a general linear space X. This inconsistency can be resolved by noting that the set $\mathcal{Q}^{**}$ is itself a convex set in the larger space $L^\infty([0,1])$. Define the objective $h(q)$ over $L^\infty([0,1])$ by

$$h(q) = \begin{cases} \int_0^1 q(t)\,dt, & q \in \mathcal{Q}^{**}\,; \\ +\infty, & q \notin \mathcal{Q}^{**}\,. \end{cases}$$

Then it is straightforward to verify that in light of Proposition 3.2.6, the extension h is also convex as a function of q. We can therefore optimize over convex domains in the same manner as we do over real linear spaces [4].

Proposition 3.2.7. $g_2(q)$ *is a convex function of* q.

Proof. By Assumption 2.3.1, r is convex. Let $0 \leq t_1, t_2 \in \mathbb{R}$ such that $t_1 + t_2 = 1$. Then

$$\begin{aligned} g_2(t_1 q_1 + t_2 q_2) &= \int_0^1 r(t_1 q_1(t) + t_2 q_2(t))\psi_3'(t)\, dt - R_0 \\ &\leq \int_0^1 \left[t_1 r(q_1(t)) + t_2 r(q_2(t))\right] \psi_3'(t)\, dt - R_0 \\ &= t_1 \left(\int_0^1 r(q_1(t))\psi_3'(t)\, dt - R_0\right) + t_2 \left(\int_0^1 r(q_2(t))\psi_3'(t)\, dt - R_0\right) \\ &= t_1 g_2(q_1) + t_2 g_2(q_2)\,. \end{aligned}$$

□

Proposition 3.2.8. *$g_1(q)$ is a convex function of q.*

Proof. Similar. □

Proposition 3.2.9. *Problem 3.5 satisfies the Slater condition.*

Proof. Recall from Assumption 2.4.3 the existence of $Y_0 \in B^+(\Sigma)$ such that

$$\int p(Y_0)\, dT_2 \circ \mathbb{P} > P_0, \int r(Y_0)\, dT_3 \circ \mathbb{P} < R_0\,.$$

Then defining $q_0(t) = F_{Y_0}^{-1}(1 - T_1^{-1}(t))$ gives

$$\int p(q_0(t))\psi_2'(t)\, dt = \int p(Y_0)\, dT_2 \circ \mathbb{P} > P_0\,;$$

and

$$\int r(q_0(t))\psi_3'(t)\, dt = \int r(Y_0)\, dT_3 \circ \mathbb{P} < R_0\,.$$

by the quantile reformulation arguments from the previous section. Furthermore, $q_0 \in \mathcal{Q}^{**}$ by definition of $\mathcal{Q}^{**}$. Then q_0 satisfies the Slater condition for Problem 3.5, as desired. □

This shows that f, g_1, g_2 satisfy the regularity properties of Theorem 3.2.2. Applying this theorem therefore gives the existence of $\lambda \in \mathbb{R}^+, \mu \in \mathbb{R}^*$, such that for any optimal solution $\bar{q}$, we have

$$\int_0^1 \bar{q}(t)\, dt \leq \int_0^1 q(t)\, dt + \lambda \left(\int_0^1 -p(q(t))\psi_2'(t)\, dt + P_0\right) + \mu \left(\int_0^1 r(q(t))\psi_3'(t)\, dt - R_0\right), \tag{3.6}$$

for all $q \in \mathcal{Q}^{**}$. Also, by part (iv) of Theorem 3.2.2, we have

$$\int_0^1 -p(\bar{q}(t))\psi_2'(t)\,dt + P_0 = \int_0^1 r(\bar{q}(t))\psi_3'(t)\,dt - R_0 = 0.$$

Adding these terms to the left-hand side of Equation 3.6 and cancelling out the P_0 and R_0 terms gives

$$\begin{aligned}\int_0^1 \bar{q}(t)\,dt - \lambda \int_0^1 p(\bar{q}(t))\psi_2'(t)\,dt + \mu \int_0^1 r(\bar{q}(t))\psi_3'(t)\,dt \\ \leq \int_0^1 q(t)\,dt - \lambda \int_0^1 p(q(t))\psi_2'(t)\,dt + \mu \int_0^1 r(q(t))\psi_3'(t)\,dt, \quad \forall q \in \mathcal{Q}^{**}.\end{aligned}$$

Therefore, $\bar{q}$ must also be an optimal solution to the problem

$$\min_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\,dt - \lambda \int_0^1 p(q(t))\psi_2'(t)\,dt + \mu \int_0^1 r(q(t))\psi_3'(t)\,dt \right\}. \tag{3.7}$$

Conversely, Theorem 3.2.5 shows that an optimal solution q^* for Problem 3.7 is also optimal for Problem 3.3. For the remainder of this chapter, we focus on solving Problem 3.7.

3.2.1 Non-negativity of Multipliers

In the previous section, we proved the existence of the multipliers $\lambda, \mu \geq 0$. The special case when these multipliers are actually equal to zero warrants some attention. If a multiplier is zero, then consider a similar optimization problem, but without the multiplier's corresponding constraint. Then by applying the methodology from the previous section, we find that these optimization problems must have the same solution, since their solutions are characterized by the same conditions. This implies that imposing this condition was redundant, since it does not affect the optimum.

In our case, the performance constraint is never redundant - without the performance constraint, we could take a contingent claim identically equal to zero, which would minimize the cost. This is clearly a trivial situation, so we can safely assume $\lambda > 0$. The fact that the performance constraint is tight at optimum then follows from Theorem 3.2.2. Alternatively, we can show this constructively, by only relying on the monotonicity properties of our performance and risk measures, as shown below.

Proposition 3.2.10. *Suppose q^* is optimal for Problem 3.3. Then*

$$\int_0^1 p(q^*(t))\psi_2'(t)\,dt = P_0.$$

Proof. Suppose, for the sake of contradiction, that $\int_0^1 p(q^*(t))\psi_2'(t)\,dt > P_0$. Then by monotonicity of p and continuity of the integral, there exists $\varepsilon > 0$ such that

$$\int_0^1 p(q^*(t) - \varepsilon)\psi_2'(t)\,dt = P_0.$$

Let $\bar{q} := \max\{0, q^* - \varepsilon\}$. Then it is clear that $0 \leq \bar{q} \leq q^* \leq N$, and $\bar{q}$ is increasing and left continuous, implying that $q^* \in \mathcal{Q}^{**}$. Now note that

$$\{t \in [0,1] : q^*(t) = 0\} = \{t \in [0,1] : q^*(t) = \bar{q}(t)\}.$$

To show $(\subseteq)$, if $q^*(t) = 0$, then $\bar{q}(t) = \max\{0, 0 - \varepsilon\} = 0 = q^*(t)$. For the reverse direction $(\supseteq)$, suppose $q^*(t) = \bar{q}(t)$ but $q^*(t) > 0$. Then

$$0 < \bar{q}(t) = q^*(t) - \varepsilon < q^*(t),$$

which is a contradiction. Therefore if we define the set $\mathcal{A} := \{t \in [0,1] : \bar{q}(t) < q^*(t)\}$, we have

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}(q^* > 0) > 0.$$

Then by checking the cost functional applied to $\bar{q}$, we have

$$\begin{aligned}
\int_0^1 \bar{q}(t)\,dt &= \int_{\mathcal{A}} \bar{q}(t)\,dt + \int_{[0,1]\setminus\mathcal{A}} \bar{q}(t)\,dt \\
&= \int_{\mathcal{A}} \bar{q}(t)\,dt + \int_{[0,1]\setminus\mathcal{A}} q^*(t)\,dt \\
&< \int_{\mathcal{A}} q^*(t)\,dt + \int_{[0,1]\setminus\mathcal{A}} q^*(t)\,dt \\
&= \int_0^1 q^*(t)\,dt.
\end{aligned}$$

Therefore, $\bar{q}$ has a strictly lower cost than q^*. Also, since r is increasing, it follows that

$$\int_0^1 r(\bar{q}(t))\psi_3'(t)\,dt \le \int_0^1 r(q^*(t))\psi_3'(t)\,dt \le R_0\,.$$

Finally, define the set $\mathcal{B} := \{t \in [0,1] : q^*(t) - \varepsilon > 0\}$. Then

$$\begin{aligned}
\int_0^1 p(\bar{q}(t))\psi_2'(t)\,dt &= \int_{\mathcal{C}} p(\bar{q}(t))\psi_2'(t)\,dt + \int_{[0,1]\setminus\mathcal{B}} p(\bar{q}(t))\psi_2'(t)\,dt \\
&= \int_{\mathcal{B}} p(q^*(t) - \varepsilon)\psi_2'(t)\,dt + 0 \\
&= \int_0^1 p(q^*(t) - \varepsilon)\psi_2'(t)\,dt - \int_{[0,1]\setminus\mathcal{B}} p(q^*(t) - \varepsilon)\psi_2'(t)\,dt \\
&= P_0 - \int_{[0,1]\setminus\mathcal{B}} p(\underbrace{q^*(t) - \varepsilon}_{\le 0})\psi_2'(t)\,dt \\
&\ge P_0\,.
\end{aligned}$$

Therefore $\bar{q}$ is feasible for Problem 3.3, but has a strictly lower cost than q^*. This contradicts the optimality of q^*. $\square$

We have shown constructively that the performance constraint is always tight at optimum, and it is never redundant. However, a similar construction cannot be applied to the risk constraint, so the redundancy of the risk constraint requires separate consideration. In fact, the risk constraint in our case can be redundant, in the case that R_0 is set too high, as shown by the following argument.

Suppose we take a similar problem as Problem 2.1, but without the risk constraint:

$$\inf_{Y \in B^+(\Sigma)} \left\{ \int Y\,dT_1 \circ \mathbb{P} : 0 \le Y \le N,\ \int p(Y)\,dT_2 \circ \mathbb{P} \ge P_0 \right\}. \tag{3.8}$$

Then this is also a convex programming problem, and has a fully characterized optimal solution, as shown in [16]. Suppose Y^* is optimal for Problem 3.8. We can compute the risk measure of this claim, as

$$R^* := \int r(Y^*)\,dT_3 \circ \mathbb{P}.$$

If our risk tolerance level R_0 exceeds the value of R^*, then Y^* would be optimal for Problem 2.1 as well, which would imply that the risk level R_0 was redundant. This would correspond to a vanishing multiplier $\mu = 0$. Therefore for the remainder of this analysis, we assume

Assumption 3.2.11. $R_0 \leq R^*$.

That is, the risk tolerance is low enough as to effect the choice of the investor. This reflects that in the market, there are claims with higher cost for the same performance, but at lower risk. The mathematical implication is that both multipliers λ, μ are strictly positive, which will be important in the following section.

3.3 An Envelope Relaxation Problem

Recall Problem 3.7:

$$\min_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\psi_2'(t)\, dt + \mu \int_0^1 r(q(t))\psi_3'(t)\, dt \right\}.$$

Suppose we fix a value of t in $[0, 1]$, and consider the integrand:

$$y - \lambda p(y)\psi_2'(t) + \mu r(y)\psi_3'(t)\,. \tag{3.9}$$

Then viewing this expression as a function of y, we can define $\bar{y}_t$ to be the argument of the minimum

$$\bar{y}_t := \arg\min\{y - \lambda p(y)\psi_2'(t) + \mu r(y)\psi_3'(t)\}\,.$$

Then by defining $\bar{q}(t) = \bar{y}_t$, it follows that this $\bar{q}$ would minimize the integral pointwise, and therefore admit a possible solution to Problem 3.7. However, in order for $\bar{q}$ to be feasible, we must have $\bar{q} \in \mathcal{Q}^{**}$; that is, $\bar{q}$ would need to be bounded by N, non-negative, and non-decreasing. These conditions are not necessarily met by the minimum of Equation 3.9. To guarantee monotonicity, we adopt a convex/concave envelope approach.

3.3.1 Convex and Concave Envelopes

Definition 3.3.1. (Convex Envelope) For a real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, the *convex envelope* of f on the interval $[0,1]$ is the real-valued function g, which is the greatest convex function that is pointwise dominated by f.

Proposition 3.3.2. *Rockafellar and Wets [21, Prop. 2.31] show that for all $x \in \mathbb{R}$,*

$$g(x) = \inf \left\{ \sum_{i=1}^{n} \lambda_i f(x_i) : \sum_{i=1}^{n} \lambda_i x_i = x, \ \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geq 0 \ \text{ for } i = 1, \cdots, n \right\}. \quad (3.10)$$

It is straightforward to verify that the convex envelope satisfies the following properties.

Proposition 3.3.3. *If g is the convex envelope of f, then*

1. *g is continuous and convex on $[0,1]$;*
2. *$g(0) = f(0)$ and $g(1) = f(1)$;*
3. *for all $x \in [0,1]$, $g(x) \leq f(x)$;*
4. *g is affine on $\{x \in [0,1] : g(x) < f(x)\}$;*
5. *if f is non-decreasing, then so is g;*
6. *if f is strictly increasing, then so is g;*
7. *if f is continuously differentiable on $(0,1)$, then so is g.*

Proof. See [17]. □

Definition 3.3.4. (Concave Envelope) For a real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, the *concave envelope* of f on the interval $[0,1]$ is the real-valued function g, which is the least concave function that pointwise dominates f.

The concave envelope also satisfies the following properties:

Proposition 3.3.5. *If h is the concave envelope of f, then*

1. *h is continuous and concave on $[0, 1]$,*

2. *$h(0) = f(0)$ and $h(1) = f(1)$,*

3. *for all $x \in [0, 1]$, $h(x) \geq f(x)$,*

4. *h is affine on $\{x \in [0, 1] : h(x) > f(x)\}$,*

5. *if f is non-decreasing, then so is h,*

6. *if f is strictly increasing, then so is h,*

7. *if f is continuously differentiable on $(0, 1)$, then so is h.*

Proof. Similar to Proposition 3.3.3. □

Now let δ_2 be the convex envelope of ψ_2, and let δ_3 be the concave envelope of ψ_3. Consider the related problem

$$\inf_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt : \int_0^1 p(q(t))\delta_2'(t)\, dt \geq P_0, \int_0^1 r(q(t))\delta_3'(t)\, dt \leq R_0 \right\}, \tag{3.11}$$

and its associated Lagrangian minimization problem

$$\inf_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\, dt + \mu \int_0^1 r(q(t))\delta_3'(t)\, dt \right\}. \tag{3.12}$$

As in Equation 3.9, we can consider the integrand of Problem 3.12 as a function of a variable y,

$$\Delta_t(y) := y - \lambda p(y)\delta_2'(t) + \mu r(y)\delta_3'(t)\,. \tag{3.13}$$

3.3.2 Solving the Envelope Relaxation Problem - A Special Case

Now our goal is to characterize the minimum of $\Delta_t(y)$ for a fixed t. For mathematical convenience, we assume for the time being that following additional conditions hold.

Assumption 3.3.6. The distortions T_2 and T_3 are strictly increasing.

Assumption 3.3.7. The distortions T_2 and T_3 are twice continuously differentiable.

As a consequence of Assumption 3.3.7 and Propositions 3.3.3 and 3.3.5, we can write $\delta_2''(t)$ and $\delta_3''(t)$ to be the continuous second derivatives of $\delta_2(t)$ and $\delta_3(t)$ respectively.

Proposition 3.3.8. *The function $\Delta_t(y)$ is convex for all $t \in [0,1]$. Under Assumption 3.3.6, it is strictly convex.*

Proof. By taking the second derivative, we have

$$\Delta_t''(y) = -\lambda p''(y)\delta_2'(t) + \mu r''(y)\delta_3'(t).$$

Note that since p is strictly concave and r is strictly convex, we have $p''(y) < 0$ and $r''(y) > 0$. From the previous section, we know that $\mu, \lambda > 0$. Also, since ψ_2, ψ_3 are non-decreasing, then δ_2, δ_3 are non-decreasing by Propositions 3.3.3 and 3.3.5, so $\delta_2'(t) \geq 0$ and $\delta_3'(t) \geq 0$. Therefore

$$-\lambda p''(y)\delta_2'(t) + \mu r''(y)\delta_3'(t) \geq 0,$$

showing convexity of $\Delta_t(y)$. Under Assumption 3.3.6, we have $\delta_2'(t) > 0$ and $\delta_3'(t) > 0$, and

$$-\lambda p''(y)\delta_2'(t) + \mu r''(y)\delta_3'(t) > 0,$$

so $\Delta_t(y)$ is strictly convex. □

Recall that for a strictly convex function, checking the first order condition is enough to guarantee a unique global minimum. Taking the first derivative of $\Delta_t(y)$ gives

$$\Delta_t'(y) = 1 - \lambda p'(y)\delta_2'(t) + \mu r'(y)\delta_3'(t).$$

Setting $\Delta'_t(y) = 0$ and rearranging gives

$$1 = \lambda p'(y)\delta'_2(t) - \mu r'(y)\delta'_3(t). \tag{3.14}$$

Since $\Delta'_t(y)$ is strictly increasing and continuous, it is possible to define $\bar{q}(t)$ such that $y = \bar{q}(t)$ satisfies equation 3.14. That is, for all $t \in [0, 1]$,

$$\bar{q}(t) = \arg\min_{y} \Delta_t(y),$$

from which it is easy to see that

$$\bar{q} \in \operatorname*{arg\,min}_{q \in L^\infty([0,1])} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta'_2(t)\, dt + \mu \int_0^1 r(q(t))\delta'_3(t)\, dt \right\}.$$

Furthermore, by strict convexity of the integrand, any function minimizing the above integral coincides with $\bar{q}$ almost everywhere, as shown in the following result:

Proposition 3.3.9. *If $\tilde{q}$ is an element of*

$$\operatorname*{arg\,min}_{q \in L^\infty([0,1])} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta'_2(t)\, dt + \mu \int_0^1 r(q(t))\delta'_3(t)\, dt \right\}, \tag{3.15}$$

then $\tilde{q}(t) = \bar{q}(t)$ for almost all $t \in [0, 1]$.

Proof. Define

$$\mathcal{L}(q) = \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta'_2(t)\, dt + \mu \int_0^1 r(q(t))\delta'_3(t)\, dt,$$

and suppose $\tilde{q}$ minimizes $\mathcal{L}(q)$. Define $V = \mathcal{L}(\tilde{q}) = \mathcal{L}(\bar{q})$ as the optimum value. Suppose for the sake of contradiction that $\tilde{q}$ and $\bar{q}$ are not equal almost everywhere - that is, they differ on a set of strictly positive measure.

By strict convexity of r and strict concavity of p, $\mathcal{L}$ is strictly convex. Furthermore, for $\alpha \in (0, 1)$, the convex combination $\alpha\tilde{q} + (1-\alpha)\bar{q}$ is a strict convex combination. Therefore

$$\begin{aligned} \mathcal{L}(\alpha\tilde{q} + (1-\alpha)\bar{q}) &< \alpha\mathcal{L}(\tilde{q}) + (1-\alpha)\mathcal{L}(\bar{q}) \\ &= \alpha V + (1-\alpha)V \\ &= V\,. \end{aligned}$$

Therefore the function $\alpha\tilde{q} + (1-\alpha)\bar{q}$ is a strict improvement over both $\tilde{q}$ and $\bar{q}$ - a contradiction. □

In order for $\bar{q}$ to also solve Problem 3.12, we need to show that $\bar{q}$ is in $\mathcal{Q}^{**}$. Therefore $\bar{q}$ needs to satisfy two additional properties; it must be bounded between 0 and N, and it must be non-decreasing. For the first property, since $\bar{q}$ does not necessarily take values in between 0 and N, we must restrict its range. If we can also confirm that $\bar{q}$ is non-decreasing, then we recover an optimal solution to Problem 3.12, as shown by Proposition 3.3.10 below.

Proposition 3.3.10. *Let $\mu, \lambda > 0$ be strictly positive constants, and let $\delta_2, \delta_3 : [0,1] \to [0,1]$ be strictly increasing, twice continuously differentiable functions, convex and concave respectively. Let $\bar{q}(t)$ be as defined above, and define $q^*(t) := \max\{0, \min\{N, \bar{q}(t)\}\}$. Then*

$$q^* \in \operatorname*{arg\,min}_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\, dt + \mu \int_0^1 r(q(t))\delta_3'(t)\, dt \right\},$$

and any other minimizer q is equivalent to q^* almost everywhere.

Proof. We begin by noticing that by definition, $0 \le q^*(t) \le N$ for all $t \in [0,1]$. Also, if $\bar{q}(t) \neq q^*(t)$, then $\bar{q}(t) < 0$ or $\bar{q}(t) > N$, and we can consider these cases separately.

If $\bar{q}(t) < 0$, then this means that the strictly convex function $\Delta_t(y)$ obtained its minimum at a value of $y < 0$. Since $\Delta_t(y)$ is convex, it is increasing after its minimum, and so it is increasing on $[0, N]$. It follows that the value of $y \in [0, N]$ to minimize $\Delta_t(y)$ on $[0, N]$ is $y = 0$. Hence, in this case, $q^*(t) = \max\{0, \min\{N, \bar{q}(t)\}\} = 0$ obtains the minimum.

Similarly, if $\bar{q}(t) > N$, then $\Delta_t(y)$ is decreasing on $[0, N]$. In this case, $q^*(t) = \max\{0, \min\{N, \bar{q}(t)\}\} = N$ obtains the minimum. This justifies the choice of $q^*(t)$ as the argument to minimize $\Delta_t(y)$, given the restriction that it must be bounded from below by 0 and from above by N.

It remains to show that $q^*(t)$ is left-continuous and non-decreasing. It suffices to show these properties for $\bar{q}$. By implicitly differentiating Equation 3.14, we obtain

$$\begin{aligned} 0 &= \lambda p'(y)\delta_2''(t) + \lambda p''(y)\delta_2'(t)\frac{dy}{dt} - \mu r'(y)\delta_3''(t) - \mu r''(y)\delta_3'(t)\frac{dy}{dt}; \\ \frac{dy}{dt} &= \frac{\mu r'(y)\delta_3''(t) - \lambda p'(y)\delta_2''(t)}{\lambda p''(y)\delta_2'(t) - \mu r''(y)\delta_3'(t)}. \end{aligned}$$

Recall that p is strictly increasing and concave, and r is strictly increasing and convex. Furthermore, δ_2 is strictly increasing and convex, and δ_3 is strictly increasing and concave. This implies

$$\mu r'(y)\delta_3''(t) - \lambda p'(y)\delta_2''(t) \leq 0,$$

and

$$\lambda p''(y)\delta_2'(t) - \mu r''(y)\delta_3'(t) < 0,$$

which shows that

$$\frac{dy}{dt} \geq 0\,.$$

So $\bar{q}$ is monotone and continuous (and therefore left-continuous), and hence so is q^*. Therefore $q^* \in \mathcal{Q}^{**}$, and we have

$$q^* \in \operatorname*{arg\,min}_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\,dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\,dt + \mu \int_0^1 r(q(t))\delta_3'(t)\,dt \right\}\,,$$

as desired. Finally, any other minimizer is equivalent to q^* almost everywhere by the argument of Proposition 3.3.9. $\square$

The later steps in this proof show the necessity of taking the envelopes δ_2 and δ_3 instead of ψ_2 and ψ_3: the second order conditions are essential for ensuring monotonicity of q^*.

3.3.3 The Envelope Relaxation Problem - The General Case

We turn our attention to removing Assumptions 3.3.6 and 3.3.7. These lead to a few technical difficulties, but the underlying structure of the solution remains the same.

Non-decreasing Distortions

The first assumption was that T_2 and T_3 were strictly increasing. This was important in guaranteeing $\delta_2'(t) > 0$ and $\delta_3'(t) > 0$ for all $t \in [0, 1]$. If we relax this assumption and allow T_2 and T_3 to be non-decreasing, then we have the possibility of $\delta_2'(t) = \delta_3'(t) = 0$, which would give a non-strict convex $\Delta_t(y)$ as in Proposition 3.3.8.

If $\delta_2'(t) = \delta_3'(t) = 0$, then the implication is that $\Delta_t(y) = y$, which is minimized as y tends to $-\infty$. Restricting the domain of $\Delta_t(y)$ to $[0, N]$ shows that defining $q^*(t) = 0$ in this case would be the correct choice to minimize $\Delta_t(y)$. It is also necessary to verify that q^* defined in this way is still monotone.

Note that since δ_2 is convex, the only way it can be flat (i.e., $\delta_2'(t) = 0$) is if it was flat all the way from zero: that is,

$$\delta_2'(t) = 0 \implies \delta_2'(s) = 0 \;\; \forall s \in [0, t].$$

So for all $s \in [0, t]$, it follows that $\Delta_s(y) = s + \mu r(y)\delta_3'(s)$, and since r is increasing, this function obtains its minimum at $y = 0$, so whenever we have $t \in [0, 1]$ such that $\delta_2'(t) = \delta_3'(t) = 0$, we define $q^*(s) = 0$ for all $0 \leq s \leq t$. This is the unique choice to minimize $\Delta_s(y)$, so we see that our construction of $q^*(t)$ as the minimizer of $\Delta_t(y)$ is still unique in this case. Furthermore, this is consistent with the condition that q^* must be monotone.

Absolutely Continuous Distortions

The second assumption was that T_2 and T_3 were twice continuously differentiable. If instead, T_2 and T_3 are assumed to just be absolutely continuous, then although δ_2 and δ_3 are differentiable almost everywhere, they are no longer necessarily continuously differentiable. In particular, we can no longer use the implicit differentiation method from Proposition 3.3.10, since the derivative $\frac{dy}{dx}$ does not necessarily exist for all $t \in [0, 1]$.

Note that by convexity and concavity of the absolutely continuous functions $\delta_2(t)$ and $\delta_3(t)$ respectively, we can still define $\delta_2'(t)$ and $\delta_3'(t)$ as their non-decreasing and non-increasing first derivatives almost everywhere. Given this, we can still prove that $\bar{q}(t)$ is monotone.

Proposition 3.3.11. *Suppose δ_2 and δ_3 are absolutely continuous, with derivatives $\delta_2'(t)$ and $\delta_3'(t)$ defined almost everywhere on $[0, 1]$. Define $\bar{q}$ on the subset $\mathcal{D}$ of $[0, 1]$ where the aforementioned derivatives exist, such that $\bar{q}(t)$ satisfies*

$$1 = \lambda p'(\bar{q}(t))\delta_2'(t) - \mu r'(\bar{q}(t))\delta_3'(t), \quad \forall t \in \mathcal{D}\,. \tag{3.16}$$

Then $\bar{q}(t)$ is non-decreasing.

Proof. Let $t_1, t_2 \in \mathcal{D}$ with $t_1 < t_2$. Then by Equation 3.16,

$$\lambda p'(\bar{q}(t_1))\delta_2'(t_1) - \mu r'(\bar{q}(t_1))\delta_3'(t_1) = \lambda p'(\bar{q}(t_2))\delta_2'(t_2) - \mu r'(\bar{q}(t_2))\delta_3'(t_2).$$

Rearranging yields

$$\lambda p'(\bar{q}(t_1))\delta_2'(t_1) - \lambda p'(\bar{q}(t_2))\delta_2'(t_2) = \mu r'(\bar{q}(t_1))\delta_3'(t_1) - \mu r'(\bar{q}(t_2))\delta_3'(t_2).$$

Looking at the left-hand side, we can write the expression as

$$\begin{aligned}
&\lambda p'(\bar{q}(t_1))\delta_2'(t_1) - \lambda p'(\bar{q}(t_2))\delta_2'(t_2) \\
&= \lambda p'(\bar{q}(t_1))\delta_2'(t_1) - \lambda p'(\bar{q}(t_2))\delta_2'(t_1) + \lambda p'(\bar{q}(t_2))\delta_2'(t_1) - \lambda p'(\bar{q}(t_2))\delta_2'(t_2) \\
&= \lambda\delta_2'(t_1)\left[p'(\bar{q}(t_1)) - p'(\bar{q}(t_2))\right] + \lambda p'(\bar{q}(t_2))\left[\delta_2'(t_1) - \delta_2'(t_2)\right].
\end{aligned}$$

Similarly, the right hand side can be rewritten as

$$\mu\delta_3'(t_1)[r'(\bar{q}(t_1) - r'(\bar{q}(t_2))] + \mu r'(\bar{q}(t_2))[\delta_3'(t_1) - \delta_3'(t_2)]\,.$$

Therefore we have

$$\begin{aligned}
\lambda\delta_2'(t_1)[p'(\bar{q}(t_1)) - &p'(\bar{q}(t_2))] + \lambda p'(\bar{q}(t_2))[\delta_2'(t_1) - \delta_2'(t_2)] \\
&= \mu\delta_3'(t_1)[r'(\bar{q}(t_1)) - r'(\bar{q}(t_2))] + \mu r'(\bar{q}(t_2))[\delta_3'(t_1) - \delta_3'(t_2)]; \\
\lambda\delta_2'(t_1)[p'(\bar{q}(t_1)) - &p'(\bar{q}(t_2))] + \mu\delta_3'(t_1)[r'(\bar{q}(t_2)) - r'(\bar{q}(t_1))] \\
&= \lambda p'(\bar{q}(t_2))[\delta_2'(t_2) - \delta_2'(t_1)] + \mu r'(\bar{q}(t_2))[\delta_3'(t_1) - \delta_3'(t_2)].
\end{aligned}$$

Since p and r are increasing, $p'(\bar{q}(t_2)), r'(\bar{q}(t_2)) \geq 0$. Also, since δ_2' is non-decreasing and δ_3' is non-increasing, we have

$$\lambda p'(\bar{q}(t_2))[\delta_2'(t_2) - \delta_2'(t_1)] + \mu r'(\bar{q}(t_2))[\delta_3'(t_1) - \delta_3'(t_2)] \geq 0,$$

which implies

$$\lambda\delta_2'(t_1)[p'(\bar{q}(t_1)) - p'(\bar{q}(t_2))] + \mu\delta_3'(t_1)[r'(\bar{q}(t_2)) - r'(\bar{q}(t_1))] \geq 0.$$

From this, we see that at least one of $p'(\bar{q}(t_1)) - p'(\bar{q}(t_2))$ and $r'(\bar{q}(t_2)) - r'(\bar{q}(t_1))$ is no less than zero. In either case, since p' is strictly decreasing and r' is strictly increasing, we have $\bar{q}(t_1) \leq \bar{q}(t_2)$, and so $\bar{q}(t)$ is non-decreasing as a function of t. $\square$

We summarize these results in the following theorem.

Theorem 3.3.12. *Let $\mu, \lambda > 0$ be strictly positive constants, and let $\delta_2, \delta_3 : [0,1] \to [0,1]$ be non-decreasing absolutely continuous functions, convex and concave respectively. Denote by δ_2' and δ_3' the derivatives of δ_2 and δ_3 respectively, and denote by $\mathcal{D}$ the set on which both these derivatives exist. Suppose that $\bar{q}$ satisfies*

$$1 = \lambda p'(\bar{q}(t))\delta_2'(t) - \mu r'(\bar{q}(t))\delta_3'(t) \quad \forall t \in \mathcal{D},$$

and define $q^*(t) := \max\{0, \min\{N, \bar{q}(t)\}\}$ for all $t \in \mathcal{D}$. Then

$$q^* \in \operatorname*{arg\,min}_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\, dt + \mu \int_0^1 r(q(t))\delta_3'(t)\, dt \right\},$$

and any other minimizer q is equivalent to q^* almost everywhere.

Proof. Follows from the above arguments as well as the proof of Proposition 3.3.10. $\square$

Chapter 4

Closing the Optimization Gap

While Theorem 3.3.12 gives an explicit characterization of the optimal solution for the envelope relaxation problem, this does not immediately imply a solution to Problem 3.3. Equivalently, this does not characterize the solution to the original Problem 2.1. However, in two special cases, it can be shown that the solution to the envelope relaxation problem is also optimal for the original problem.

Throughout this section, define $\mathcal{D}$ to be the set of all $t \in [0, 1]$ on which the derivatives of $\psi_2, \delta_2, \psi_3, \delta_3$ exist. Then since T_2, T_3 are assumed to be absolutely continuous, these derivatives exist almost everywhere; hence, $\mathcal{D}$ differs from $[0, 1]$ by a set of measure zero.

4.1 A Special Case: An Ambiguity Averse Risk Measure

The first case is when the envelope relaxation, Problem 3.12, is the same as Problem 3.7 - that is, we have $\psi_2'(t) = \delta_2'(t)$ and $\psi_3'(t) = \delta_3'(t)$. This situation can occur when the investor is ambiguity averse in their risk evaluation, as outlined below.

Definition 4.1.1. (Ambiguity Aversion Index) For a given distortion function T assumed to be twice differentiable, define the *ambiguity aversion index* of T as the ratio

$$AA_T := \frac{T''}{T'}\,.$$

Given two twice differentiable distortion functions T_1 and T_2, T_2 is said to be less ambiguity averse than T_1 if $AA_{T_2} \leq AA_{T_1}$.

The index AA_T was introduced by Carlier and Dana [5], by comparison with the classical Arrow-Pratt index of risk aversion.

The assumption that the distortion T is twice differentiable can also be relaxed to an absolutely continuous T. In this case, as in the previous section, the proofs become more technical, but the underlying intuition and methodology remain the same. In the following discussion, for simplicity, we assume the existence of a continuous second derivative, but as in Proposition 3.3.11, the proofs can be extended; the details are not given here.

Lemma 4.1.2. *Suppose that T_2 is less ambiguity averse than T_1. Then $\psi_2(t) = 1 - T_2(T_1^{-1}(1-t))$ is concave as a function of t.*

Proof. By proposition, we have for all t,

$$\frac{T_2''(1-t)}{T_2'(1-t)} \leq \frac{T_1''(1-t)}{T_1'(1-t)},$$

from which it follows that

$$\begin{aligned} 0 &\leq \frac{T_2'(1-t)T_1''(1-t) - T_1'(1-t)T_2''(1-t)}{[T_1'(1-t)]^2} \\ 0 &\leq \frac{d}{dt}\left(\frac{T_2'(1-t)}{T_1'(1-t)}\right)\,. \end{aligned}$$

Therefore the function $t \mapsto \frac{T_2'(1-t)}{T_1'(1-t)}$ is a non-decreasing function of t. Since $T_1^{-1}(1-t)$ decreases as $1-t$ decreases, it follows that $t \mapsto \frac{T_2'(T_1^{-1}(1-t))}{T_1'(T_1^{-1}(1-t))}$ is also a non-decreasing function of t. That is, $\psi_2'(t)$ is non-decreasing, and so $\psi_2(t)$ is convex. $\square$

Corollary 4.1.3. *Suppose that T_3 is more ambiguity averse than T_1. Then $\psi_3(t) = 1 - T_3(T_1^{-1}(1-t))$ is concave as a function of t.*

Proof. Similar. □

Then in this case, we have the following result, which consolidates all the arguments from this section so far.

Theorem 4.1.4. *Suppose that T_2 is less ambiguity averse than T_1, and T_3 is more ambiguity averse than T_1. Then an optimal contingent claim Y^* to minimize Problem 2.1 is*

$$Y^* := \max\{0, \min\{N, \bar{q}(1 - F_X(X))\}\},$$

where $\bar{q}$ satisfies

$$1 = \lambda p'(\bar{q}(t))\psi_2'(t) - \mu r'(\bar{q}(t))\psi_3'(t),$$

with

$$\psi_i(t) = 1 - T_i(T_1^{-1}(1-t)), \quad i = 2, 3,$$

and λ, μ are the Lagrange multipliers satisfying

$$\int_0^1 p(\bar{q}(t))\psi_2'(t)\, dt = P_0$$

and

$$\int_0^1 r(\bar{q}(t))\psi_3'(t)\, dt = R_0.$$

Proof. By the quantile reformulation shown by Problem 3.3, we can solve the related problem

$$\inf_{f \in \mathcal{Q}^*} \left\{ \int_0^1 T_1'(t) f(1-t)\, dt : \int_0^1 T_2'(t) p(f(1-t))\, dt \geq P_0, \int_0^1 T_3'(t) r(f(1-t))\, dt \leq R_0 \right\}.$$

This is a convex programming problem, so by applying Theorem 3.2.2, we guarantee the existence of $\lambda, \mu > 0$ such that the solution of the related Problem 3.7

$$\min_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\, dt - \lambda \int_0^1 p(q(t))\psi_2'(t)\, dt + \mu \int_0^1 r(q(t))\psi_3'(t)\, dt \right\}$$

coincides with the solution of Problem 3.3. Now note that by Lemma 4.1.2, since T_2 is less ambiguity averse than T_1, then ψ_2 is convex. Similarly, by Corollary 4.1.3, ψ_3 is concave. Therefore, it follows that

$$\begin{aligned}\delta_2(t) &= \psi_2(t); \text{ and,}\\ \delta_3(t) &= \psi_3(t) \text{ for all } t \in \mathcal{D}.\end{aligned}$$

Therefore we can directly apply Proposition 3.3.10. Hence, defining $\bar{q}$ such that

$$1 = \lambda p'(\bar{q}(t))\psi_2'(t) - \mu r'(\bar{q}(t))\psi_3'(t) \quad \forall t \in \mathcal{D},$$

it follows that $q^* := \max\{0, \min\{N, \bar{q}\}\}$ is optimal for Problem 3.7. By Lemma 3.1.3, the claim $Y^* := q^*(1 - F_X(X)) = \max\{0, \min\{N, \bar{q}(1 - F_X(X))\}\}$ is optimal for Problem 2.1, as desired.

Finally, by optimality of q^* and assertion (iv) of Theorem 3.7, it follows that

$$\int_0^1 p(q^*(t))\psi_2'(t)\, dt = P_0,$$

and

$$\int_0^1 r(q^*(t))\psi_3'(t)\, dt = R_0.$$

□

In the case when the cost function exhibits ambiguity neutrality, we can simplify the statement of Theorem 4.1.4 even further. Note that ambiguity neutrality of the cost function means that the distortion T is the identity function. Then we have the following characterization:

Proposition 4.1.5. *Let $T_1 : [0, 1] \to [0, 1]$ be the identity function - i.e., $T_1(t) = t$. Then a distortion function $T_2 : [0, 1] \to [0, 1]$ is less ambiguity averse then T_1 if and only if T_2 is concave.*

Proof. By definition, we have

$$\frac{T_2''(t)}{T_2'(t)} \leq \frac{T_1''(t)}{T_1'(t)} = 0 \quad \forall t \in \mathcal{D},$$

which happens if and only if $T_2''(t) \leq 0$ for all $t \in \mathcal{D}$. □

Corollary 4.1.6. *Let $T_1 : [0,1] \to [0,1]$ be the identity function - i.e., $T_1(t) = t$. Then a distortion function $T_3 : [0,1] \to [0,1]$ is more ambiguity averse then T_1 if and only if T_3 is convex.*

Proof. Similar. □

The following results are direct corollaries of the above.

Corollary 4.1.7. *Let T_1 be the identity, and T_2 concave. Then $\psi_2(t)$ is convex.*

Proof. Direct consequence of Proposition 4.1.5 and Proposition 4.1.3. □

Corollary 4.1.8. *Let T_1 be the identity, and T_3 convex. Then $\psi_3(t)$ is concave.*

Proof. Similar. □

Then we have the following statement, a simple case of Theorem 4.1.4:

Corollary 4.1.9. *Suppose that T_1 is the identity, T_2 is concave, and T_3 is convex. Then an optimal contingent claim Y^* for Problem 2.1 is*

$$Y^* := \max\{0, \min\{N, \bar{q}(1 - F_X(X))\}\},$$

where $\bar{q}$ satisfies

$$1 = \lambda p'(\bar{q}(t))\psi_2'(t) - \mu r'(\bar{q}(t))\psi_3'(t) \quad \forall t \in \mathcal{D},$$

with

$$\psi_i(t) = 1 - T_i(1 - t), \quad \forall t \in \mathcal{D} \text{ and } i = 2, 3,$$

and λ, μ are the Lagrange multipliers satisfying

$$\int_0^1 p(\bar{q}(t))\psi_2'(t)\, dt = P_0,$$

and

$$\int_0^1 r(\bar{q}(t))\psi_3'(t)\, dt = R_0.$$

Proof. A consequence of Theorem 4.1.4, after applying Corollaries 4.1.7 and 4.1.8. □

4.1.1 A Numerical Example

We now consider a simple numerical example to illustrate the result of Theorem 4.1.4. Suppose that the distortion function T_1 is given by an inverse S-shaped distortion function, such as the one used in Cumulative Prospect Theory [19, 23]. That is, for all $t \in [0, 1]$,

$$T_1(t) = \frac{t^\gamma}{(t^\gamma + (1-t)^\gamma)^{\frac{1}{\gamma}}} .$$

We take $\gamma = \frac{3}{4}$, so that for all $t \in [0, 1]$,

$$T_1(t) = \frac{t^{\frac{3}{4}}}{(t^{\frac{3}{4}} + (1-t)^{\frac{3}{4}})^{\frac{4}{3}}} .$$

Then it is straightforward to verify that T_1 is continuous and strictly increasing on $[0, 1]$.

Furthermore, suppose that the performance functional is a proportional hazard risk measure, as in [24]. That is, for all $t \in [0, 1]$,

$$T_2(t) = t^\alpha ,$$

for $0 < \alpha \leq 1$. Here, let $\alpha = \frac{1}{2}$, so that $T_2(t) = \sqrt{t}$ for all $t \in [0, 1]$. Finally, take T_3 to be the conjugate of T_2, that is, $T_3(t) = 1 - \sqrt{1-t}$. The three distortions are shown in Figure 4.1.

Then by taking derivatives, it can be shown that these distortions satisfy the conditions of Theorem 4.1.4. In particular, T_2 is less ambiguity averse than T_1, and T_3 is more ambiguity averse than T_1. The ambiguity aversion indices of each distortion are shown in Figure 4.2.

Next, suppose that the functions p and r have an exponential form:

$$u(t) = \begin{cases} (1 - e^{-at})/a, & a \neq 0 , \\ t, & a = 0 , \end{cases}$$

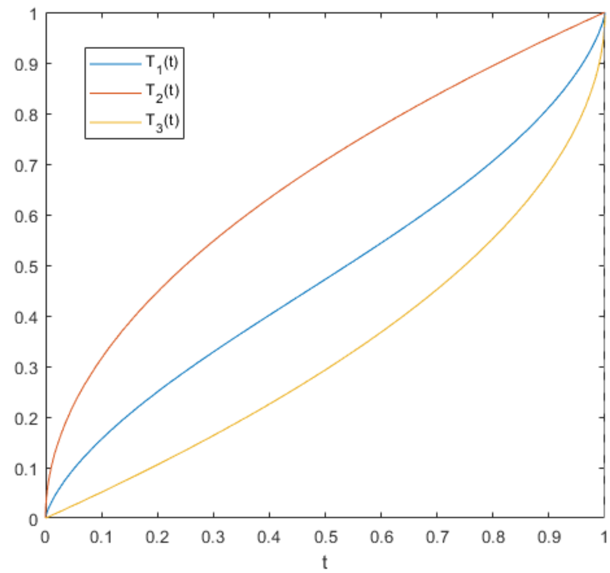

Figure 4.1: Distortions

for some $a \in \mathbb{R}$. For p, take $a = 1$, and for r, take $a = -1$. For clarity, we scale both functions by a factor of 100, so that for all $t \in \mathbb{R}$, we have

$$\begin{aligned} p(t) &= 100(1 - e^{-t}) \\ r(t) &= 100(e^{t} - 1)\,. \end{aligned}$$

In the context of decision-making under uncertainty, the class of exponential utilities is popular because it displays constant absolute risk aversion (CARA).

Then it is straightforward to verify that p is concave and r is convex. Take $N = 100, P_0 = 37.55$, and consider the problem without a risk constraint:

$$\inf_{Y \in B^+(\Sigma)} \left\{ \int Y \, dT_1 \circ \mathbb{P} : 0 \leq Y \leq 100, \int p(Y) \, dT_2 \circ \mathbb{P} \geq 37.55 \right\} .$$

Through the results of Section 3.1, this can be rewritten as

$$\inf_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t) \, dt : \int_0^1 p(q(t)) \psi_2'(t) \, dt \geq 37.55 \right\} . \tag{4.1}$$

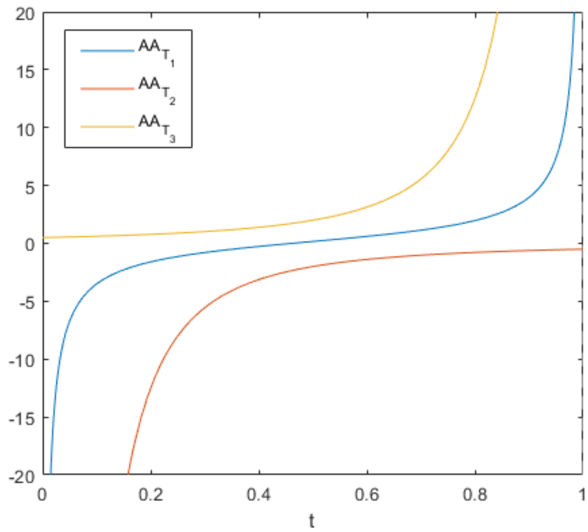

Figure 4.2: Ambiguity Aversion Indices

Then by methods from [16], it can be show that Problem 4.1 admits a solution $q_1^*(t)$ with the value for the Lagrange multiplier $\lambda \approx 0.0129$. The same result can be obtained from the results of Section 3.3, by taking a large value for R_0, forcing the multiplier μ to vanish. Checking both performance and risk functions for $q_1^*(t)$, we obtain

$$\int_0^1 p(q_1^*(t))\psi_2'(t)\,dt = 37.55\,,$$
$$\int_0^1 r(q_1^*(t))\psi_3'(t)\,dt = 22.44\,.$$

Now suppose we take $R_0 = 21.75$, and consider the problem

$$\inf_{Y\in B^+(\Sigma)} \left\{ \int Y\,dT_1 \circ \mathbb{P} : 0 \le Y \le 100, \int p(Y)\,dT_2 \circ \mathbb{P} \ge 37.55, \int r(Y)\,dT_3 \circ \mathbb{P} \le 21.75 \right\}. \tag{4.2}$$

Then it is clear that q_1^* is no longer feasible for Problem 4.2, since its risk is too high. By the result of Theorem 4.1.4, it follows that Problem 4.2 admits a solution q_2^* with

$(\lambda, \mu) \approx (11.50, 8.26)$. Comparing the objective for both q_1^* and q_2^*, we obtain

$$\int_0^1 q_1^*(t)\, dt = 0.3075\,,$$
$$\int_0^1 q_2^*(t)\, dt = 0.3114\,.$$

Notice that q_2^* has a higher cost than q_1^*, while attaining the same performance at a lower level of risk. This is consistent with the intuition that in order to attain the same performance at lower risk, a premium needs to be paid, i.e., the price of a hedge. The plots for q_1^* and q_2^* are shown in Figure 4.3.

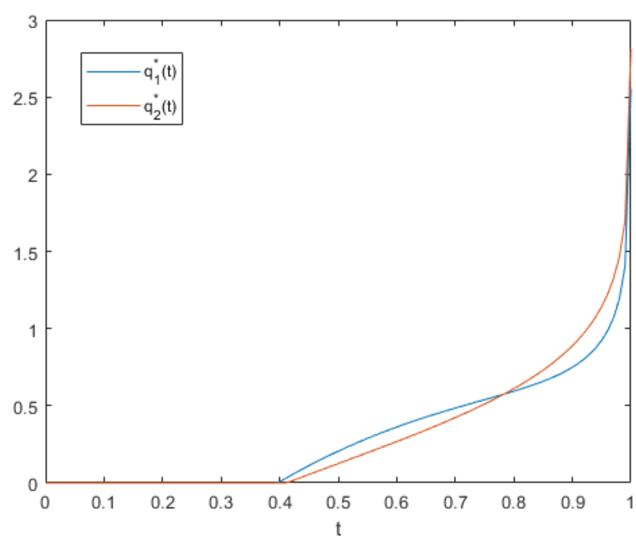

Figure 4.3: Two Optimal Claims

4.2 Another Special Case: An Ambiguity Seeking Risk Measure

Now suppose the opposite scenario: suppose T_2 is more ambiguity averse than T_1, and T_3 is less ambiguity averse than T_1. By Corollary 4.1.3 and Proposition 4.1.2, ψ_2 is concave and ψ_3 is convex.

Note that since $\psi_2(0) = 0$, $\psi_2(1) = 1$, and ψ_2 is concave on $[0, 1]$, then its convex envelope is the line segment connecting its endpoints. In particular, this implies δ_2 is the identity, i.e., $\delta_2(t) = t$. Similarly, we have $\delta_3(t) = t$. So in this case, we have $\psi_2 \neq \delta_2$ and $\psi_3 \neq \delta_3$, but we are still able to characterize the optimal solution of Problem 2.1. We start with the following result.

Lemma 4.2.1. *Let $q \in \mathcal{Q}$, $q \geq 0$ be a non-negative quantile function. Then*

$$\int_0^1 q(t)\psi_2'(t)\, dt \leq \int_0^1 q(t)\delta_2'(t)\, dt.$$

Proof. Consider the following integral:

$$\int_0^1 [\psi_2(t) - \delta_2(t)]\, dq(t)\,.$$

By applying Fubini's theorem, we can rewrite this expression as follows:

$$\begin{aligned}
\int_0^1 [\delta_2(t) - \psi_2(t)]\, dq(t) &= \int_0^1 [(\psi_2(1) - \delta_2(1)) - (\psi_2(t) - \delta_2(t))]\, dq(t) \\
&= \int_0^1 \int_t^1 [\psi_2'(s) - \delta_2'(s)]\, ds\, dq(t) \\
&= \int_0^1 \int_0^s [\psi_2'(s) - \delta_2'(s)]\, dq(t)\, ds \\
&= \int_0^1 \left[\int_0^s dq(t)\right] [\psi_2'(s) - \delta_2'(s)]\, ds \\
&= \int_0^1 q(s)[\psi_2'(s) - \delta_2'(s)]\, ds \\
&= \int_0^1 q(t)[\psi_2'(t) - \delta_2'(t)]\, dt\,. \qquad (4.3)
\end{aligned}$$

Equation 4.3 is particularly useful, and will appear several times in the proceeding analysis.

Now note that since δ_2 is the convex envelope of ψ_2, it follows that

$$\delta_2(t) \leq \psi_2(t), \forall t \in [0,1],$$

implying that

$$0 \geq \int_0^1 [\delta(t) - \psi_2(t)]\, dq(t) = \int_0^1 q(t)[\psi_2'(t) - \delta_2'(t)]\, dt\,.$$

Therefore,

$$\int_0^1 q(t)\delta_2'(t)\, dt \geq \int_0^1 q(t)\psi_2'(t)\, dt,$$

as desired. □

In the previous proof, it is interesting to note that the convexity of δ_2 is never applied. In fact, we only required that δ_2 and ψ_2 be equal at their endpoints, and that δ_2 is dominated by ψ_2. From this, we can conclude a similar result for δ_3 and ψ_3, noting that ψ_3 dominates δ_3, therefore switching the two functions' roles in the proof.

Corollary 4.2.2. *Let $q \in \mathcal{Q}$, $q \geq 0$ be a non-negative quantile function. Then*

$$\int_0^1 q(t)\psi_3'(t)\, dt \geq \int_0^1 q(t)\delta_3'(t)\, dt.$$

Proof. Similar to Lemma 4.2.1, by the preceding discussion. □

Lemma 4.2.3. *Suppose T_2 is more ambiguity averse than T_1, and T_3 is less ambiguity averse than T_1. Then if q^* is an optimal solution of Problem 3.11, then*

$$\int_0^1 \delta_2'(t)p(q^*(t))\, dt = \int_0^1 \psi_2'(t)p(q^*(t))\, dt,$$

and

$$\int_0^1 \delta_3'(t)r(q^*(t))\, dt = \int_0^1 \psi_3'(t)r(q^*(t))\, dt.$$

Proof. We show the first equality, as the second is similar. Note that since p is continuous and increasing, we have $p \circ q^*$ is non-decreasing and left-continuous, so it can be viewed as a quantile function. Therefore applying Equation 4.3 gives

$$\int_0^1 [\delta_2(t) - \psi_2(t)]\, dp(q^*(t)) = \int_0^1 p(q^*(t))[\psi_2'(t) - \delta_2'(t)]\, dt\,.$$

Let $\bar{q}$ and q^* be defined as in the proof of Proposition 3.3.10. Since q^* is flat at 0 or N when $q^* \neq \bar{q}$, we see that

$$dq^*(t) = \begin{cases} 0, & q^*(t) \neq \bar{q}(t)\,; \\ d\bar{q}(t), & q^*(t) = \bar{q}(t)\,. \end{cases}$$

Recall that the expression for the derivative $d\bar{q}(t)$ is

$$d\bar{q}(t) = \frac{\mu r'(\bar{q}(t))\delta_3''(t) - \lambda p'(\bar{q}(t))\delta_2''(t)}{\lambda p''(\bar{q}(t))\delta_2'(t) - \mu r''(\bar{q}(t))\delta_3'(t)}, \quad \forall t \in \mathcal{D}.$$

However, since T_2 is more ambiguity averse than T_1, by Corollary 4.1.3, ψ_2 is concave and δ_2 is the identity, i.e., $\delta_2(t) = t$, and $\delta_2''(t) = 0$. Similarly, $\delta_3''(t) = 0$, implying that $d\bar{q}(t) = 0$ for all t. Therefore

$$0 = \int_0^1 [\delta_2(t) - \psi_2(t)]\, dp(q^*(t)) = \int_0^1 p(q^*(t))[\psi_2'(t) - \delta_2'(t)]\, dt\,.$$

Hence,

$$\int_0^1 \delta_2'(t)p(q^*(t))\, dt = \int_0^1 \psi_2'(t)p(q^*(t))\, dt\,,$$

as desired. The other equation follows similarly.

Note that instead of using the argument of Proposition 3.3.10, we could have also used the more general Theorem 3.3.12 and reached the same conclusion. The details are more complicated, but as usual, the underlying method is the same. □

Theorem 4.2.4. *Suppose that T_2 is more ambiguity averse than T_1, and T_3 is less ambiguity averse than T_1. Then an optimal contingent claim Y^* to minimize Problem 2.1 is a constant, satisfying*

$$Y^* := \max\{0, \min\{N, \bar{q}\}\},$$

where $\bar{q}$ satisfies

$$1 = \lambda p'(\bar{q}) - \mu r'(\bar{q}),$$

and λ, μ are the Lagrange multipliers satisfying

$$p(Y^*) = P_0\,,$$

and

$$r(Y^*) = R_0\,.$$

Proof. First, recall Problem 3.11

$$\inf_{q \in \mathcal{Q}^{**}} \left\{ \int_0^1 q(t)\,dt : \int_0^1 p(q(t))\delta_2'(t)\,dt \geq P_0, \int_0^1 r(q(t))\delta_3'(t)\,dt \leq R_0 \right\}\,,$$

and its associated Lagrangian

$$\int_0^1 q(t)\,dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\,dt + \mu \int_0^1 r(q(t))\delta_3'(t)\,dt\,.$$

Suppose q is feasible for Problem 3.3: that is, the problem before taking the envelope. Then by Proposition 4.2.1 and Corollary 4.2.2, it follows that q is feasible for Problem 3.11 as well. This guarantees that the feasibility set of Problem 3.11 is non-empty; in fact, it contains the feasibility set of Problem 3.3.

Note now that the arguments of subsection 3.2.1 apply to Problem 3.11 as well. Firstly, the performance constraint is always tight at optimum, implying $\lambda > 0$. Furthermore, the results of [16] show that in the absence of a risk constraint, Problem 3.3 and Problem 3.11 have equivalent solutions. That is, if the multiplier μ of Problem 3.3 is zero, then equivalently, so is the multiplier for Problem 3.11. It then follows from Assumption 3.2.11 that when solving Problem 3.11, we will have strictly positive multipliers.

By the above, since the feasibility set of Problem 3.3 is non-empty, a solution must exist; denote this solution by $\tilde{q}$. By Theorem 3.2.2 and the arguments of Section 3.2.1, we guarantee a choice of $\lambda, \mu > 0$ such that

(i)

$$\int_0^1 p(\tilde{q}(t))\delta_2'(t)\,dt = P_0,$$

(ii)

$$\int_0^1 r(\tilde{q}(t))\delta_3'(t)\,dt = R_0,$$

(iii) $\tilde{q}$ minimizes the Lagrangian

$$\int_0^1 q(t)\,dt - \lambda \int_0^1 p(q(t))\delta_2'(t)\,dt + \mu \int_0^1 r(q(t))\delta_3'(t)\,dt.$$

However, by Theorem 3.3.12, the unique minimizer of the Lagrangian, for a given λ, μ, takes the form $q^*(t) := \max\{0, \min\{N, \bar{q}(t)\}\}$, with $\bar{q}(t)$ defined to satisfy

$$1 = \lambda p'(\bar{q}(t))\delta_2'(t) - \mu r'(\bar{q}(t))\delta_3'(t)\,.$$

By uniqueness of this minimizer, $q^* = \tilde{q}$ almost everywhere, so it follows that q^* is optimal and

(i)

$$\int_0^1 p(q^*(t))\delta_2'(t)\,dt = \int_0^1 p(\tilde{q}(t))\delta_2'(t)\,dt = P_0\,,$$

(ii)

$$\int_0^1 r(q^*(t))\delta_3'(t)\,dt = \int_0^1 r(\tilde{q}(t))\delta_3'(t)\,dt = R_0\,.$$

Now recall that T_2 was assumed to be less ambiguity averse than T_1, and T_3 more ambiguity averse. Then $\delta_2(t) = \delta_3(t) = t$, so we can rewrite the above equations as

(i)

$$\int_0^1 p(q^*(t))\,dt = \int_0^1 p(\tilde{q}(t))\,dt = P_0\,,$$

(ii)

$$\int_0^1 r(q^*(t))\,dt = \int_0^1 r(\tilde{q}(t))\,dt = R_0\,.$$

But also, from the proof of Lemma 4.2.3, it followed that $q^*(t)$ was flat everywhere - i.e., it can be written as a constant $y^* := q^*(t)$. It remains to show optimality of y^* for Problem 3.3. For this, we invoke Proposition 3.2.5, by showing that this choice of y^* and λ, μ satisfy all the sufficient conditions.

Firstly, by the above and Lemma 4.2.3, since y^* is optimal for Problem 3.11,

(i)
$$\int_0^1 p(y^*)\,dt = \int_0^1 p(y^*)\psi_2'(t)\,dt = P_0\,,$$

(ii)
$$\int_0^1 r(y^*)\,dt = \int_0^1 r(y^*)\psi_3'(t)\,dt = R_0\,,$$

showing that the choice of y^* binds both performance and risk constraints. Furthermore, by Proposition 4.2.1, Corollary 4.2.2, and Lemma 4.2.3, we have

$$\begin{aligned}
&\int_0^1 q(t)\,dt - \lambda\int_0^1 p(q(t))\psi_2'(t)\,dt + \mu\int_0^1 r(q(t))\psi_3'(t)\,dt \\
\geq &\int_0^1 q(t)\,dt - \lambda\int_0^1 p(q(t))\delta_2'(t)\,dt + \mu\int_0^1 r(q(t))\delta_3'(t)\,dt \\
\geq &\int_0^1 y^*\,dt - \lambda\int_0^1 p(y^*)\delta_2'(t)\,dt + \mu\int_0^1 r(y^*)\delta_3'(t)\,dt \\
= &\int_0^1 y^*\,dt - \lambda\int_0^1 p(y^*)\psi_2'(t)\,dt + \mu\int_0^1 r(y^*)\psi_3'(t)\,dt\,,
\end{aligned}$$

showing that the Lagrange condition is satisfied as well. Therefore y^* is optimal for Problem 3.3, and so the constant claim $Y^* := y^*$ is optimal for Problem 2.1 by Lemma 3.1.3. □

The statement of this theorem requires some further interpretation. Since we obtain $p(Y^*) = P_0$ and $r(Y^*) = R_0$, this implies that $R_0 = r(p^{-1}(P_0))$. This value also only depends on the fixed functions p, r and performance level P_0. Therefore, there is only one possible value for R_0 for which the risk constraint is not redundant in the sense of Assumption 3.2.11, and such that the problem is feasible. However, this value must coincide with the value R^*. Therefore in this special case, the risk constraint never affects the optimal contingent claim, except by possibly making the problem infeasible.

The intuition behind this is as follows. Suppose that T_1, T_2, and T_3 satisfy the conditions of Theorem 4.2.4, and suppose we solve the problem without including a risk constraint. Then by results from [16], we see that the optimal contingent claim is a constant. Upon introduction of a risk constraint, we might be forced to obtain another claim attaining the same performance, but is less risky. We can intuitively understand a claim's risk to be how "flat" it is. Since we already have a constant claim, this claim is "flat" enough that it is not possible to reduce the risk while maintaining the same performance. Of course, since we are dealing with distorted probability measures, the result of Theorem 4.2.4 is not obvious.

Corollary 4.2.5. *Suppose that T_1 is the identity, T_2 is convex, and T_3 is concave. Then an optimal contingent claim Y^* to minimize Problem 2.1 is a constant claim, satisfying*

$$Y^* := \max\{0, \min\{N, \bar{q}\}\},$$

where $\bar{q}$ satisfies

$$1 = \lambda p'(\bar{q}) - \mu r'(\bar{q}),$$

and λ, μ are the Lagrange multipliers satisfying

$$p(Y^*) = P_0\,,$$

and

$$r(Y^*) = R_0.$$

Proof. Direct consequence of Theorem 4.2.4 and Corollaries 4.1.7 and 4.1.8. □

As an example of a concave distortion function, for a fixed $p \in [0, 1)$, consider

$$T_3(t) := \min\left\{\frac{t}{1-p}, 1\right\}, \quad t \in [0,1]\,.$$

Then the corresponding distorted expectation is the expected shortfall [10]:

$$\int_0^1 r(Y)\, dT_3 \circ \mathbb{P} = \mathrm{ES}_p[r(Y)] = \frac{1}{1-p}\int_p^1 F^{-1}_{r(Y)}(t)\, dt\,.$$

Corollary 4.2.6. *Suppose the investor has the following problem:*

$$\inf_{Y \in B^+(\Sigma)} \left\{ \int Y \, d\mathbb{P} : \;\; 0 \le Y \le N, \;\; \int p(Y) \, dT_2 \circ \mathbb{P} \ge P_0, \;\; \mathrm{ES}_p[r(Y)] \le R_0 \right\},$$

where $p \in [0,1)$ *is a given constant, and* T_2 *is convex, indicating ambiguity aversion when calculating the performance measure. Then* Y^* *is optimal, where*

$$Y^* := \max\{0, \min\{N, \bar{q}\}\},$$

where $\bar{q}$ *satisfies*

$$1 = \lambda p'(\bar{q}) - \mu r'(\bar{q}),$$

and λ, μ *are the Lagrange multipliers satisfying*

$$p(y^*) = P_0 \, ,$$

and

$$r(y^*) = R_0.$$

That is, a constant claim is optimal.

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APPENDICES

Appendix A

Equimeasurable Rearrangements and Supermodularity

All of the results in this Appendix are taken from Ghossoub [14] and references therein, to which we refer for proofs, additional results, and additional references.

A.1 Rearrangement with Respect to a Probability Measure

Let $(S, \mathcal{G}, \mu)$ be a probability space and let $V \in L^\infty(S, \mathcal{G}, \mu)$ be a continuous random variable (i.e., $\mu \circ V^{-1}$ is nonatomic) with range $V(S) \subset \mathbb{R}^+$.

For each $Z \in L^\infty(S, \mathcal{G}, \mu)$, let $F_{Z,\mu}(t) = \mu\big(\{s \in S : Z(s) \leq t\}\big)$ denote the cumulative distribution function of Z with respect to the probability measure μ, and let $F^{-1}_{Z,\mu}(t)$ be the left-continuous inverse of the distribution function $F_{Z,\mu}$ (that is, the quantile function of Z w.r.t. μ), defined by

$$F^{-1}_{Z,\mu}(t) = \inf\left\{z \in \mathbb{R}^+ : F_{Z,\mu}(z) \geq t\right\}, \ \forall t \in [0,1]\,. \tag{A.1}$$

Proposition A.1.1. *For any $Y \in L^\infty(S, \mathcal{G}, \mu)$, define $\tilde{Y}_\mu$ and $\overline{Y}_\mu$ as follows:*

$$\overline{Y}_\mu = F^{-1}_{Y,\mu}(F_{V,\mu}(V)) \quad \text{and} \quad \tilde{Y}_\mu = F^{-1}_{Y,\mu}(1 - F_{V,\mu}(V)).$$

Then,

(i) Y*,* $\tilde{Y}_\mu$*, and* $\overline{Y}_\mu$ *have the same distribution under* μ*.*

(ii) $\overline{Y}_\mu$ *is comonotonic with* V*.*

(iii) $\tilde{Y}_\mu$ *is anti-comonotonic with* V*.*

(iv) For each $L \in \mathbb{R}$*, if* $0 \leq Y \leq L$*, then* $0 \leq \tilde{Y}_\mu \leq L$*, and* $0 \leq \overline{Y}_\mu \leq L$*.*

(v) For each $Z \in L^\infty(S, \mathcal{G}, \mu)$*, If* $0 \leq Y \leq Z$*, then* $0 \leq \tilde{Y}_\mu \leq \tilde{Z}_\mu$*, and* $0 \leq \overline{Y}_\mu \leq \overline{Z}_\mu$*.*

(vi) If Z^* *is any other element of* $L^\infty(S, \mathcal{G}, \mu)$ *that has the same distribution as* Y *under* μ *and that is comonotonic with* V*, then* $Z^* = \overline{Y}_\mu$*,* μ*-a.s.*

(vii) If Z^{**} *is any other element of* $L^\infty(S, \mathcal{G}, \mu)$ *that has the same distribution as* Y *under* μ *and that is anti-comonotonic with* V*, then* $Z^{**} = \tilde{Y}_\mu$*,* μ*-a.s.*

$\tilde{Y}_\mu$ is called the nonincreasing μ-rearrangement of Y with respect to V, and $\overline{Y}_\mu$ is called the non-decreasing μ-rearrangement of Y with respect to V.

Since $\mu \circ V^{-1}$ is nonatomic, it follows that $F_{V,\mu}(V)$ has a uniform distribution over $(0,1)$ [13, Lemma A.25]. Letting $U := F_{V,\mu}(V)$, it follows that U is a random variable on the probability space (S, Σ, μ) with a uniform distribution on $(0,1)$ and that $V = F^{-1}_{V,\mu}(U)$, μ-a.s., that is, $\overline{V}_\mu = V$, μ-a.s.

A.2 Rearrangement with Respect to a Capacity

A.2.1 Non-Additive Measures and Choquet Integration

Definition A.2.1. (Capacities) A (normalized) *capacity* on a measurable space (S, Σ) is a set function $\upsilon : \Sigma \to [0,1]$ such that

1. $\upsilon(\varnothing) = 0$;
2. $\upsilon(S) = 1$; and,

3. υ is monotone: for any $A, B \in \Sigma,\ A \subseteq B \Rightarrow \upsilon(A) \leq \upsilon(B)$.

The capacity υ is said to be:

- *supermodular* (or convex) if $\upsilon(A \cup B) + \upsilon(A \cap B) \geq \upsilon(A) + \upsilon(B)$, for all $A, B \in \Sigma$; and,
- *submodular* (or concave) if $\upsilon(A \cup B) + \upsilon(A \cap B) \leq \upsilon(A) + \upsilon(B)$, for all $A, B \in \Sigma$.

For instance, if $(S, \Sigma, \mathbb{P})$ is a probability space and $T : [0,1] \to [0,1]$ is an increasing function, such that $T(0) = 0$ and $T(1) = 1$, then the set function $\upsilon := T \circ \mathbb{P}$ is a capacity on (S, Σ) called a *distorted probability measure.* The function T is usually called a *probability distortion.* If, moreover, the distortion function T is convex (resp. concave), then the capacity $\upsilon = T \circ \mathbb{P}$ is supermodular (resp. submodular) [9, Ex. 2.1].

Definition A.2.2. (Choquet Integral) Let υ be a capacity on (S, Σ). The *Choquet integral* of $Y \in B(\Sigma)$ with respect to υ is defined by

$$\int Y \, d\upsilon := \int_0^{+\infty} \upsilon(\{s \in S : Y(s) \geq t\}) \, dt + \int_{-\infty}^0 [\upsilon(\{s \in S : Y(s) \geq t\}) - 1] \, dt,$$

where the integrals are taken in the sense of Riemann.

Remark A.2.3. *The Choquet integral with respect to a measure is simply the usual Lebesgue integral with respect to that measure [20, p. 59].*

The following proposition gives some additional properties of the Choquet integral.

Proposition A.2.4. *Let ν be a capacity on $(S, \mathcal{G})$.*

1. *If $\phi_1, \phi_2 \in B(\mathcal{G})$ are comonotonic, then $\int (\phi_1 + \phi_2) \, d\nu = \int \phi_1 \, d\nu + \int \phi_2 \, d\nu$.*
2. *If $\phi \in B(\mathcal{G})$ and $c \in \mathbb{R}$, then $\int (\phi + c) \, d\nu = \int \phi \, d\nu + c$.*
3. *If $A \in \mathcal{G}$ then $\int \mathbf{1}_A \, d\nu = \nu(A)$.*
4. *If $\phi \in B(\mathcal{G})$ and $a \geq 0$, then $\int a \, \phi \, d\nu = a \int \phi \, d\nu$.*
5. *If $\phi_1, \phi_2 \in B(\mathcal{G})$ are such that $\phi_1 \leq \phi_2$, then $\int \phi_1 \, d\nu \leq \int \phi_2 \, d\nu$.*
6. *If ν is submodular, then for any $\phi_1, \phi_2 \in B(\mathcal{G})$, $\int (\phi_1 + \phi_2) \, d\nu \leq \int \phi_1 \, d\nu + \int \phi_2 \, d\nu$.*

A.2.2 Rearrangements with Respect to a Capacity

Definition A.2.5. The capacity $\nu \circ X^{-1}$ is said to be *diffuse* if for any $t \in \mathbb{R}$, we have $\nu \circ X^{-1}(\{t\}) = 0$.

Definition A.2.6 (Ghossoub [15])**.** The capacity ν is said to be *strongly diffuse with respect to* X if for any $a, b \in \mathbb{R}$ with $a \leq b$,

$$\nu \circ X^{-1}\Big((a,b)\Big) = \nu \circ X^{-1}\Big(\,[a,b]\,\Big).$$

When ν is strongly diffuse with respect to X, the capacity $\nu \circ X^{-1}$ will be called *strongly diffuse*. Strong diffuseness implies diffuseness. For capacities that are distortions of a probability measure, we have the following stronger result.

Proposition A.2.7 (Ghossoub [15])**.** *Let ν be a capacity on (S, Σ) and let X be a random variable on (S, Σ), and suppose that ν is a distorted probability measure of the form $\nu = T \circ \mathbb{P}$, for some probability measure $\mathbb{P}$ on (S, Σ) and some distortion function $T : [0,1] \to [0,1]$, strictly increasing with $T(0) = 0$ and $T(1) = 1$. Then the following are equivalent.*

1. *$\nu \circ X^{-1}$ is strongly diffuse;*
2. *$\nu \circ X^{-1}$ is diffuse; and,*
3. *$\mathbb{P} \circ X^{-1}$ is diffuse (i.e., nonatomic).*

Definition A.2.8. (Upper distribution) Let ν be a capacity on the measurable space (S, Σ) and let $\phi \in B(\Sigma)$. Define the upper-distribution of ϕ with respect to ν as the function

$$\begin{aligned} G_{\nu,\phi} &: \mathbb{R} \to [0,1] \\ &\qquad t \mapsto G_{\nu,\phi}(t) := \nu\big(\{s \in S : \phi(s) > t\}\big). \end{aligned}$$

If $\phi_1, \phi_2 \in B(\Sigma)$, we write $\phi_1 \overset{\nu}{\sim} \phi_2$ to mean that ϕ_1 and ϕ_2 have the same upper-distribution with respect to ν. Then a mapping $V : B(\Sigma) \to \mathbb{R}$ is said to be ν-upper-law-invariant if for any $\phi_1, \phi_2 \in B(\Sigma)$,

$$\phi_1 \overset{\nu}{\sim} \phi_2 \implies V(\phi_1) = V(\phi_2).$$

The Choquet integral is an example of a ν-upper-law-invariant functional on $B(\Sigma)$. Note that $G_{\nu,\psi}$ is nonincreasing, and if ν is continuous from below, then $G_{\nu,\psi}$ is right-continuous [9, p. 46]. Moreover, if $\nu = T \circ \mathbb{P}$, for some probability measure $\mathbb{P}$ on (S, Σ) and some distortion function $T : [0,1] \to [0,1]$, then for any $\phi_1, \phi_2 \in B(\Sigma)$, if ϕ_1 and ϕ_2 are identically distributed[1] according to $\mathbb{P}$, then they have the same upper-distribution with respect to ν. Finally, if ν is a *bone fide* additive measure, then two functions have the same upper-distribution with respect to ν if and only if they are identically distributed according to ν.

Remark A.2.9. *In particular, if $\phi = I \circ X$, the previous definition is equivalent to the map*

$$\begin{aligned} G_{\nu,X,I} &: \mathbb{R} \to [0,1] \\ &t \mapsto G_{\nu,X,I}(t) := \nu \circ X^{-1}(z \in [0,M] : I(z) > t)\,. \end{aligned}$$

Definition A.2.10. (Non-Decreasing Upper-Equimeasurable Rearrangement) Define the function $\tilde{I} : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\tilde{I}(t) := \inf \left\{ z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leq \nu \circ X^{-1}([0,t]) \right\},$$

and for each $Y = I \circ X$, define the function $\tilde{Y} := \tilde{I} \circ X$. Then $\tilde{Y}$ is called the *non-decreasing ν-upper-equimeasurable rearrangement of Y with respect to X.*

Proposition A.2.11. *If ν is continuous and strongly diffuse with respect to X, the following hold:*

1. *$\tilde{I}$ is nonincreasing and Borel-measurable.*
2. *$\tilde{I}$ is right-continuous.*
3. *For all $t \in \mathbb{R}^+$, $G_{\nu,X,I}(\tilde{I}(t)) \leq \nu \circ X^{-1}\big([0,t]\big)$.*
4. *If $I_1, I_2 : [0,M] \to \mathbb{R}^+$ are such that $I_1 \leq I_2$, then $\tilde{I}_1 \leq \tilde{I}_2$.*
5. *I and $\tilde{I}$ have the same upper-distribution with respect to $\nu \circ X^{-1}$.*

[1] That is, $\mathbb{P} \circ \phi_1^{-1}(B) = \mathbb{P} \circ \phi_2^{-1}(B)$, for any Borel set B.

6. *If* $\|I\|_{sup} = N$ $(< +\infty)$, *then* $\|\tilde{I}\|_{sup} \leq N$.

7. *If* $\{I_n\}_n$ *is a sequence of bounded Borel-measurable functions from* $[0, M]$ *into* $\mathbb{R}^+$ *such that* $I_n \uparrow I$, *for some bounded Borel-measurable function* $I : [0, M] \to \mathbb{R}^+$, *then* $\tilde{I}_n \uparrow \tilde{I}$.