The S-Matrix of Gauge and Gravity Theories and The Two-Black Hole Problem

by

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Alfredo Guevara was the sole author for Sections 1.1, 1.4, 2.1, 2.7 and 5.1, together with the general Introduction section, which were not written for publication. This thesis consists in part of five manuscripts written for publication. Research presented in chapter 3 corresponds to the single-author paper:

- A. Guevara, *Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering*, JHEP04(2019) 033 [hep-th:1706.02314].

Exceptions to sole authorship of material, detailing the contribution from the thesis author, are presented as follows:

Research presented in Chapter 1: A.G. proposed the computations that were carried out together with coauthor F. Bautista, A.G. provided the explanation and interpretation of results given in this chapter.

- Citation: Y. F. Bautista and A. Guevara, *From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems*, [hep-th:1903.12419].

Research presented in Chapter 2: A.G. carried out several checks of the computations previously proposed by coauthors A. Ochirov and J. Vines. A.G. contributed to the writing of the manuscript.

- Citation: A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, [1906.10071].

Research presented in Chapter 4: A.G. proposed the exponential structure of the scattering amplitudes and its relation to the soft expansion. He further obtained the Leading Singularity to finally match a scattering angle conjectured by coauthor J. Vines from a classical perspective. In addition, A.G. constructed the sections of the manuscript and carried out the computations which were later cross-checked with coauthor A. Ochirov.

- Citation: A. Guevara, A. Ochirov and J. Vines, *Scattering of Spinning Black Holes from Exponentiated Soft Factors*, JHEP09(2019) 056 [hep-th:1812.06895].

Research presented in Chapters 5-10: A.G. constructed the rational map formula for odd number of particles, which is the central result of part II of the thesis. He first introduced the T-symmetry and used it to obtain the corresponding integration measure in terms of rational maps. He further implemented the soft limit construction to obtain the corresponding integrand, matching a previous conjecture checked numerically by coauthor C. Wen. Different extensions presented 8 and 9 were derived jointly with supervisor F. Cachazo and coauthors M. Heydenman, J.H. Schwarz, S. Mizera and C. Wen. A.G. collaborated actively on the writing of the manuscript.

- Citation: F. Cachazo, A. Guevara, M. Heydeman, S. Mizera, J. H. Schwarz and C. Wen, *The S Matrix of 6D Super Yang-Mills and Maximal Supergravity from Rational Maps*, JHEP 09 (2018) 125, [hep-th:1805.11111].

Abstract

This thesis is devoted to diverse aspects of scattering amplitudes in gauge theory and gravity including interactions with matter particles. In Part I we focus on the applications of massive scattering amplitudes in gravity to the Black Hole two-body problem. For this we construct a classical limit putting especial emphasis on the multipole expansion of certain massive amplitudes, which we will use to model spinning black holes in a large distance effective regime or particle approximation. In Part II we study scattering amplitudes in six dimensions, and construct a compact formula analogous to the four-dimensional Witten-RSV/rational maps formulation. This provides a supersymmetric extension of moduli space localization formulae such as the CHY integral. We explore the cases of Super Yang-Mills and Maximal Supergravity theories, among others.

Acknowledgements

I would like to express my deep gratitude to my supervisor Freddy Cachazo for giving me the chance to learn so much from him.

It is also a pleasure to thank all the professors, colleagues and friends who have encompassed and guided me during my academic carreer, especially Jorge Zanelli, Fernando Izaurieta, Sebastian Mizera and Fabian Bautista. Also a big part of my development as a scientist is due to fruitful and enjoyable collaborations with N. Early, D. García-Sepúlveda, M. Heydenman, B. Gimenez-Umbert, A. Ochirov, P. Pais, J.H. Schwarz, J. Vines, C. Wen and Y. Zhang. I also wish to thank all the community of Perimeter Institute for providing an amazing environment for academic life in general.

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Dedication

A Perlita.

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Introduction

Quantum Field Theory (QFT) provides a strong framework to describe a wide range of natural phenomena, underlying our current understanding of interacting systems, providing a unification of Quantum Mechanics and Special Relativity which has led to spectacularly successful results. For instance, the anomalous magnetic moment of the electron as predicted by Quantum Electrodynamics has been found in agreement with experiments up to ten parts in a billion [200], making it one of the most succesful predictions in the history of physics.

Scattering Amplitudes are one of the keys in the predictive power of QFT, as they correspond to the main physical observables provided by the framework. In 2003, Witten revolutionized the computation of Amplitudes by introducing a twistor-string description of the S-matrix in $\mathcal{N}=4$ Super Yang-Mills Theory ($\mathcal{N}=4$ SYM) [251]. Despite the fact that his description focused on such theory it was soon realized that the novel methods that followed his construction were indeed more general. For instance, three powerful tools that have emerged as a byproduct are:

- *Recursion relations* such as the Britto-Cachazo-Feng-Witten (BCFW) recursion [57, 56]: Tree-level amplitudes written in spinor-helicity representation can be bootstrapped from their analytic structure in gauge and gravity theories, among other various examples.

- *One-shell techniques for loop amplitudes*: Exploting unitarity at loop level leads to the computation of singular parts of the amplitude based solely on tree-level building blocks. In certain cases, 1-loop amplitudes are determined exactly by their leading singularities! [12].

- *Reformulation of the S-matrix via localized integrals*: One can construct field theory scattering amplitudes via integrals defined in a certain moduli space, in a similar fashion to string theory. After the seminal work of Witten for $\mathcal{N}=4$ SYM, this picture has been extended to a wide variety of theories, the main example being the Cachazo-He-Yuan (CHY) formulation [65, 66, 64, 59].

In this thesis we show how the aforementioned on-shell techniques can be used to deepen our understanding of perturbative gravity and how they intertwine with its gauge theory analogues. Our main object of study are tree and 1-loop amplitudes with matter particles interacting with gravitons or gluons.

The first part is devoted to the application of such framework to the study of the two Black Hole (2BH) problem in General Relativity (GR). Using scattering amplitudes for massive particles, and through a classical limit, we gain new insight into the relevant classical quantities: Effective 2BH potentials, scattering angles and Gravitational Wave radiation. We focus on the phenomenologically relevant case of rotating Black Holes which will be in striking correspondence with quantum particles with spin. Along the way we examine the gauge theory analogue of such objects using QED scattering amplitudes and find a close relation between the two theories induced by the so-called double copy relations.

In the second part of this thesis we explore the perturbative S-matrix of a special class of supersymmetric gauge and gravity theories living in six dimensions. The purpose of this endeavour is to unveil new structures underlying the corresponding QFT amplitudes inspired by Witten's construction. Indeed, we will construct an all multiplicity formula for these amplitudes à la Witten, i.e. employing localized integrals over a certain moduli space. The novel formulation of Supersymmetric Yang-Mills theory provides new building blocks that can be then used for a wide range of theories, including Supergravity. The arena of six dimensions is particularly interesting since it is tailor made for phenomenological applications involving massive particles in four dimensions.

Part I: From Scattering Amplitudes to Rotating Black Holes

Since the early days of QFT, the use of effective methods to describe low energy regimes of more fundamental theories has proven extremely successful. One of the most notable applications consists in obtaining an effective classical description of a certain macroscopic system that could be eventually tested via experiment. That this effective description is obtained as $\hbar \to 0$ (which we will refer to as *classical limit*) was already elaborated by Bohr in his correspondence principle [50], referring to the fact that expectation values could be used to reproduce classical physics in the regime of large quantum numbers.

In a modern setup, Effective Field Theories (EFT) provide a framework for reducing the degrees of freedom of an underlying QFT, being also well suited to perform separation of relevant scales. This can be done even if the underlying high-energy (quantum) theory is unknown. In this direction, the problem of General Relativity as an EFT has been studied as a tool for obtaining predictions whenever the relevant energy scales are much smaller than M_{Planck} [215, 101]. For this regime the methods of QFT can be safely applied to compute long range observables. In a classical ($\hbar \to 0$) and a long distance ($r \to \infty$) regime we can model coupled interacting systems such as Black Holes with simple perturbative computations in an effective QFT framework. In the context of this thesis, the motivation for these problems stems from the always increasing interest in the measurement of (classical, long-range) gravitational waves (GW) as definitive tests of the classical aspects of gravity. Indeed, this interest led to the acclaimed first experimental detection of the GW150914 signal by LIGO in 2015 [1, 252], which was preceded by a huge development in diverse theoretical aspects.

Specifically, the binary inspiral stage, defined by the characteristic scale $v^2 \sim Gm/r$, has been the subject of extensive research since it can be addressed with analytical methods. For rotating Black Holes, these also incorporate an expansion in powers of the spin vector or angular momentum, namely $S^\mu = ma^\mu \sim v$, which we will refer to as the *multipole expansion* [49, 119, 224].

The key object in the study of the binary inspiral problem is the effective potential associated to a two-body system. This potential admits a non-relativistic expansion in powers of $v^2 \sim Gm/r$, known as the post-Newtonian (PN) expansion. Pioneered by the seminal work of Einstein-Infeld-Hoffman long ago [110], several attempts have been made to evaluate the potential at higher PN orders. In addition, the electromagnetic analog of the effective potential has been also discussed in [113, 154, 151, 161] in the context of classical corrections to Coulomb scattering. As expected the long range behavior of this potential, i.e. the $\frac{1}{r^n}$ falloff, is identical to the gravitational case. The computations are simpler in general and thus it also serves as a toy model for the PN problem.

The EFT approach is based on using Feynman diagrammatic techniques and treating the PN expansion as a perturbative loop expansion [106, 100, 195, 5]. In this thesis we will employ a setup consisting on the $2 \to 2$ scattering amplitude of massive objects m_a and m_b, interacting through the exchange of multiple gravitons (Fig. 4.1a). The fact that this amplitude develops a classical piece (even at loop orders) stems from the form of the massive propagator, which for e.g. scalar fields can be normalized as [153]

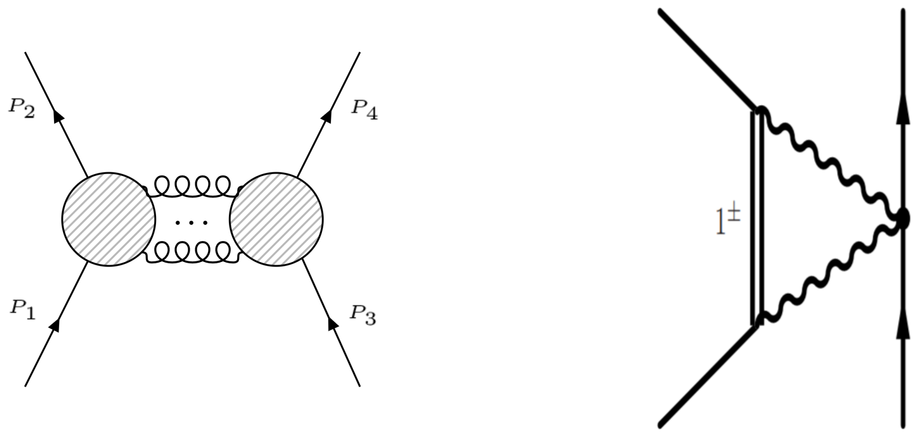


Figure 1: Left: A typical scattering process contributing to the effective potential. Two massive (spinning or spinless) particles exchange several gravitons with $t = (P_1 - P_2)^2$. For a single graviton we have $V \sim \frac{G m_a m_b}{t}$, leading to the Newton potential. Right: A 1-loop diagram contributing to such process. In the classical/long-range regime $\hbar \to 0$ and $t \to 0$, the diagram scales as $G^2\hbar \times \frac{m}{\hbar\sqrt{-t}} = \mathcal{O}(\hbar^0)$.

$$\langle \phi_1(0)\phi_2(x) \rangle = \hbar \int \frac{d^4p}{(2\pi)^4} e^{ipx} \frac{1}{p^2 - m^2/\hbar - i\epsilon} \,. \tag{1}$$

The appearance of $\hbar$ in the denominator spoils the naive homogeneous power counting of loop diagrams and can lead to cancellations as $\hbar \to 0$. In fact, the interplay between this massive propagator and the exchange of massless bosons with low momentum transfer ($t \to 0$) will lead to classical combinations as described in (Fig. 4.1a). In this case the classical potential can be obtained from the long range behavior of the amplitude after implementing the well known Born approximation [113, 154, 237]. This classical piece can be extracted for instance by setting the COM (Center of Mass) frame, in which the momentum transfer reads $|\vec{q}| = \sqrt{-t}$ and corresponds to the Fourier conjugate of the large distance r separating the two massive objects.

When the compact objects are rotating, calculation of these quantities in classical GR has proved extremely long and tedious, even though there have been remarkable simplifications in the context of non-relativistic approaches [132, 224, 126, 212, 115]. In this part of the thesis we will derive and extend all the previous results by exploiting and substantiating the following remarkable fact: *The BH multipole structure up to the*

Limit	Perturbation theory
Newtonian gravity $c \to \infty$	post-Newtonian $\frac{m_1}{m_2} \sim 1, \quad \frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1$
special relativity $G \to 0$	post-Minkowskian $\frac{m_1}{m_2} \sim 1, \quad \frac{Gm}{rc^2} \ll \frac{v^2}{c^2} \sim 1$

Figure 2: Comparison between different regimes.

2^{2s}*-pole level is reproduced by considering spin-s particles which are minimally coupled to gravity.* This means we will incorporate spin in the massive legs of 4.1a. This was first hinted in [236] up to the spin-2 level for the leading-PN-order corrections to the two-body interaction potential, following work along similar lines in [154, 155]. We will argue that this correspondence holds in the limit of large spin quantum number. We will further set the grounds to extend the results to all orders in spin by generalizing the multipole expansion of QFT amplitudes and contrasting it to its classical part appearing in the PN framework.

On the QFT side, a generalization of minimal-coupling amplitudes to arbitrary spins s has been proposed recently in [13] using a new massive spinor-helicity formalism. They provide an effective, bottom-up approach to the description of higher spin particles, which are directly related to higher orders in our multipole expansion. We find perfect agreement between this effective theory of higher spins and our effective description of rotating BHs.

A particular focus of more recent interest has been the use of scattering amplitudes to produce explicit classical results in the post-Minkowskian (PM) approximation, which resums the expansion in small speeds while still expanding in weak coupling, see Figure 2. This framework is well known for the studying gravitational radiation from compact sources but has also recently appeared in classical scattering of two Black Holes seen as an unbounded system [197, 95]. For Black Holes which are rotating, the first results for spin-orbit coupling in the PM scheme have been computed only recently, from purely classical considerations, at 1PM and 2PM orders [35, 36]. Going beyond the pole-dipole level, higher-multipole couplings specific to black holes (BHs) were treated at 1PM order in [238], by means of a classical effective action approach [176, 214] matched to the linearized Kerr solution, to all orders in the spin-induced multipole expansion. We will find a consistent picture of how these amplitudes encode the complete tower of spinning-BH multipole moments, at least at 1PM order. We will also treat radiative effects in our multipole expansion at this order, and finally provide new conjectural results for conservatinve scattering at 2PM order for high orders in spin.

This part of the thesis is organized as follows. In Chapter 1 we introduce the multipole expansion in arbitrary dimensions for spinning particles, together with the concepts of double copy and soft expansion. As an application we compute leading PM radiation for a system of two spinning sources. In Chapter 2 we specialize to $D = 4$ and the Kerr Black-Hole. We compute the momentum and spin deflection of two such objects at leading PM order. In Chapter 3 we begin the study of the effective 2-body potential for such objects, and we extend the previous computation up to 1-loop introducing new methods. Finally, in Chapter 4 we use these methods to compute the 1-loop correction (2PM) to the scattering angle in an aligned-spin case.

Part II: Scattering Amplitudes in Six Dimensional SYM & Maximal Supergravity from Rational Maps

Novel formulations of scattering amplitudes have been the subject of great interest especially since the introduction of Witten's twistor string theory in 2003 [251]. Witten proposed a formulation of the S-matrix of four-dimensional $\mathcal{N} = 4$ super Yang–Mills theory (SYM) based on correlations functions of a certain string theory, the topological B-model. Its correlations functions are computed by integrating over the moduli space of maps from n-punctured Riemman spheres into the twistor space constructed long ago by Penrose, see Figure 3. Shortly after Witten's conjecture, Roiban, Spradlin, and Volovich (RSV) gave evidence that by integrating over only maps to *connected* curves in twistor space the complete tree-level S matrix could be recovered [222]. The corresponding formula is now known as the Witten–RSV formula and led to deep insights into the analytic structure of the S-matrix of gauge theories, including pure Yang-Mills. It was both originally formulated in twistor space and in the more familiar momentum space suitable to study scattering processes.

Extending such worldsheet formulations to other theories then became a natural open problem. In the case of perturbative gravity at tree-level, formulas based on rational maps into twistor space for 4D $\mathcal{N} = 8$ supergravity (SUGRA) were developed in 2012 by Cachazo, Geyer, Skinner and Mason [61, 71, 70]. These developments gave more impetus to the search for similar phenomena in other theories and perhaps other spacetime dimensions of interest. One of the main obstacles to extending the formalism to higher

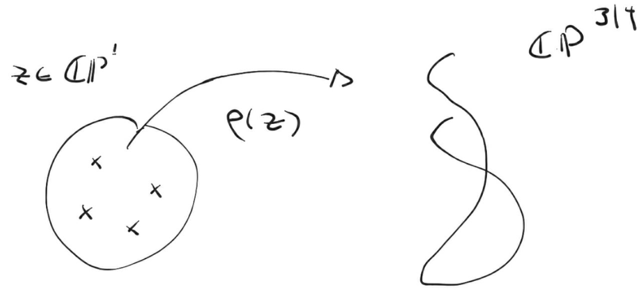


Figure 3: The moduli space of Witten's string theory consists of maps $\rho(z)$ from punctured spheres to twistor space $\mathbb{CP}^{3|4}$. The image in $\mathbb{CP}^{3|4}$ is a one (complex) parameter surface.

dimensions was the heavy use of *spinor-helicity variables* in 4D. They provide a natural decomposition of a massless particle's momentum p_μ into chiral and antichiral components:

$$p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \quad , \alpha, \dot{\alpha} = 1, 2 \tag{2}$$

where $\sigma^\mu_{\alpha\dot{\alpha}}$ are the extended Pauli matrices in 4D. This obstruction for higher dimensions was partially removed in 2009 when Cheung and O'Connell introduced the 6D spinor-helicity formalism [84]. However, straightforward extensions of connected formulas were not found, hinting that new ingredients were needed in 6D. Obtaining such formulation in 6D is very interesting for a variety of reasons. On a practical side, besides the interest in their own right, computing 6D SYM formulas would allow, for instance, via dimensional reduction, for 1) a unification of 4D helicity sectors for massless gauge and gravity amplitudes and 2) obtaining amplitudes along the Coulomb branch of $\mathcal{N} = 4$ SYM, which contains massive particles such as W bosons [25].

On a separate front, more recently connected formulas have been found for the effective theories living on a D5-brane and a M5-brane in 10D and 11D Minkowski spacetime, respectively [149]. These are 6D theories with $\mathcal{N} = (1, 1)$ and $(2, 0)$ supersymmetry, respectively. The former is the supersymmetric version of Born–Infeld theory, and the latter describes analogous interactions for a supermultiplet containing an Abelian self-dual tensor.

It is well known that scattering amplitudes can make symmetries manifest that the corresponding Lagrangian does not. A striking and unexpected example is dual superconformal invariance of $\mathcal{N} = 4$ SYM, which combines with the standard super-conformal invariance to generate an infinite-dimensional structure known as the Yangian of PSU$(2,2|4)$ [103]. In 6D, the M5-brane theory provides an even more fundamental example. The self-dual condition on the three-form field strength causes difficulties in writing down a manifestly Lorentz invariant action for the two-form gauge field [205, 2]. In contrast, the formulas found in [149] for the complete tree-level S-matrix are manifestly Lorentz invariant. These examples highlight the importance of finding explicit formulas for the complete tree-level S-matrix, as they can provide new insights into known symmetries of theories or even the discovery of unexpected ones.

This part of the thesis is dedicated to the exploration of worldsheet formulas in 6D. We will first conjecture explicit such formulas for the complete tree-level S matrix of $\mathcal{N}=(1,1)$ SYM with U(N) gauge group, the effective theory on N coincident D5-branes, and provide strong evidence for its validity. The S-matrix of this theory has been studied previously in [99, 25, 158, 98, 55, 94, 206]. We will use the building block arising in this case to further construct the amplitudes of maximal SUGRA.

One of the main results of this thesis is to uncover a fascinating structure that appears in the definition of the maps for an odd number of particles, which is in striking contrast with the simple and almost trivial ingredients emerging at even multiplicity. We derive an explicit integrand for the $\mathcal{N}=(1,1)$ SYM odd-multiplicity amplitudes, which enters the moduli space integration. This provides a new building block that can be used to construct a wide range of theories, including gravity, by mixing it with other building blocks. Remarkably, the connected formula for odd multiplicity is derived using a general statement of QFT known as the Weinberg's soft theorem, which we apply to the corresponding even-multiplicity result. Furthermore we recast our maps in the form of the so-called Veronese maps, and provide a new relation to a mathematical structure known as the symplectic (or Lagrangian) Grassmannian.

Having explicit integrands for the complete 6D $\mathcal{N}=(1,1)$ SYM tree amplitudes allows the construction of the 6D $\mathcal{N}=(2,2)$ SUGRA S-matrix by a certain squaring of the gauge theory integrand, which contains the necessary new supersymmetric information. We end with various applications to other theories in four, five, and six dimensions. These include mixed superamplitudes of 6D $\mathcal{N}=(1,1)$ SYM coupled to a single D5-brane, 5D SYM and SUGRA, and also 4D scattering amplitudes involving massive particles of $\mathcal{N}=4$ SYM on the Coulomb branch of its moduli space. The formulas for 5D theories take forms very similar to those of 6D, but with additional constraints on the rational maps to incorporate 5D massless kinematics. In order to describe the massive amplitudes of $\mathcal{N}=4$ SYM on the Coulomb branch, we utilize the spinor-helicity formalism recently developed for massive particles in 4D [13], which in fact can naturally be viewed as a dimensional reduction of 6D massless helicity spinors. We also would like to emphasize that, although it is a straightforward reduction of our 6D formula, this is the first time that a connected formula has been proposed for 4D $\mathcal{N}=4$ SYM away from the massless point of the moduli space.

This part of the thesis is organized as follows. In Chapter 5, we review the general construction of rational maps from $\mathbb{CP}^1$ to the null cone in general spacetime dimensions. We also review 4D constructions and then 6D maps for an even number of particles. Chapters 6 and 7 are devoted to $\mathcal{N}=(1,1)$ SYM amplitudes in 6D. Chapter 6 deals with an even number of particles while Chapter 7 provides the main results of this thesis by presenting formulas and consistency checks for odd multiplicity. In Chapter 8 we discuss a linear form of the scattering maps in 6D and its relationship to the symplectic Grassmannian. Extensions and applications are presented in Chapter 9. We conclude and give a discussion of future directions in Chapter 10. In Appendix I we present the algebra of the new T-shift, and in Appendix I we give details of the soft-limit calculations.

Part I

From Scattering Amplitudes to Rotating Black Holes

Chapter 1

The Multipole Expansion and the Classical Limit

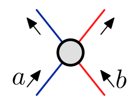
1.1 Introduction

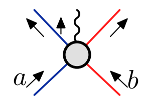
Long time ago, with the advent of QFT, it was observed that dynamics of classical massive objects subject to long-range forces could be described from the classical limit of Scattering Amplitudes [160, 107, 52, 19, 136, 100, 153, 154]. This picture has seen renewed interest with the aim of providing analytic templates in early stages of events where Gravitational Waves (GW) are generated. Such interest has led to remarkable results in the so-called Post-Minkowskian (PM) framework, originally devised as an appropriate formalism to treat radiation in asymptotic regions (far from the events) [95, 134, 39, 96, 86, 240, 135, 90, 28, 7]. To introduce the PM expansion in a nutshell, we note that the classical limit of an n-point amplitude can be expanded perturbatively

$$\langle M_n \rangle := \lim_{\hbar \to 0} M_n = G^m \times \left(M_n^{(1)}(GMq) + M_n^{(2)}(GMq)^2 + \ldots \right) \tag{1.1}$$

where M and q are some characteristic mass of the particles and momentum transfer respectively. The power $(Gmq)^{L+1}$ arises from an L-loop correction to the amplitude, after factors of $\hbar$ are cancelled as illustrated in the introduction chapter. Note that there is no non-relativistic expansion in characteristic velocities of the objects, in contrast to e.g. the Post-Newtonian expansion. Hence the PM framework is preferred from a QFT viewpoint as it is fully relativistic.

In this thesis we will focus on the amplitudes with four matter particles: two incoming and two outgoing. After the classical limit, a matter line in the amplitude can be thought as some classical trajectory for our compact objects. Two key instances are the following:

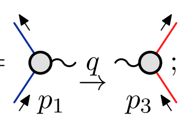
$M_4 =$ 

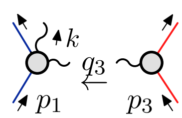
, $\quad M_5 =$ 

, (1.2)

The bodies a and b carrying internal structure are here understood as point particles with spin, which we will use in the following sections for our effective description of rotating Black Holes. While M_4 has been studied to high PM/loop orders and carries conservative information, M_5 involves an external graviton carrying radiation and it is much more subtle to study. It has only recently been introduced in this context by O'Connell et al. in the spinless case [170, 190]. Even though these objects control universal effects such as the Coulombian/Newtonian potentials, both M_4 and M_5 strongly depend on the matter content a priori.

In this chapter we consider the tree-level versions of the above amplitudes. We will construct their classical piece, $\langle M_n \rangle$, and argue that the reason for it to be universal (i.e. independent of matter-matter interactions) is that it is precisely identified with their decomposition into fundamental *amplitudes* that only depend on the coupling of matter to a massless carrier. The main example we provide are two factorizations occuring in the classical limit at tree level. Diagramatically, they can be presented as

$\langle M_4 \rangle =$ 

; $\langle M_5 \rangle =$ 

$+ \ (1 \leftrightarrow 3)\,.$ (1.3)

One may naively suspect that the gravitational case is far more complicated than e.g. the QED case. This is not true when we factor the classical piece in the above fashion, and in fact it is instructive to study QED at the same time. In this chapter we denote by $A_n^{h,s}$ the transition amplitudes of a massive spin-s state emitting $n-2$ massless particles of helicity $h = 1, 2$. For instance, the previous equations read, schematically,

$$\langle M_4^{h,s_1,s_3} \rangle = A_3^{h,s_1}(p_1) P^h(q) A_3^{h,s_3}(p_3)\,, \tag{1.4}$$
$$\langle M_5^{h,s_1,s_3} \rangle = A_4^{h,s_1}(p_1) P^h(q_3) A_3^{h,s_3}(p_3) + A_3^{h,s_1}(p_1) P^h(q_1) A_4^{h,s_3}(p_3) \tag{1.5}$$

where $P^h(q)$ is the propagator of a massless helicity h particle with momenta q. The case $h = 1$, i.e. photon emission was computed very early in the history of QED, first using the so-called old-fashioned perturbation theory [244, 225]. We start by reconsidering the objects $A_n^{h,s}$ in light of recent developments and unveil several new structures hidden in them. As an introductory example, one can study the so-called *soft expansion* and *double copy* properties of A_3^h and A_4^h for a spinless particle. From string theory arguments a relation between gravity and EM amplitudes in any dimension is known, including the cases of spinless states. At low multiplicity, it can be realized in the following formula [167, 24]:

$$A_n^{\mathrm{ph},0} \times A_n^{\mathrm{ph},0} = K_n A_n^{\mathrm{gr},0}, \quad n = 3, 4\,. \tag{1.6}$$

with $K_3{=}1$ and $K_4{=}\frac{1}{2}\frac{k_1\cdot k_2}{p_1\cdot k_1\, p_1\cdot k_2}$, [1] where $p_1{+}k_1{=}p_2{+}k_2$ (we denote by p_i massive momenta and by k_i massless momenta). In the LHS photons have polarizations ϵ_i^μ and in the RHS gravitons have polarization $\epsilon_i^{\mu\nu} = \epsilon_i^\mu \epsilon_i^\nu$ and the same momenta.

We now discuss the other key concept: The soft expansion. First note that while A_3 corresponds to a classical on-shell current and can be used to evaluate conservative effects, it is not enough for the computation of radiative effects even at Leading Order (LO) in the coupling [228, 127]. This is a reflection of the fact that it does not posses orbit multipoles, in contrast with A_4. Let us define orbit multipoles as each of the terms appearing in the soft-expansion of A_n, namely the $\omega \to 0$ expansion after scaling $k = \omega \hat{k}$, for some external photon/graviton momentum. Such expansion is trivial for A_3. For A_4, the expansion truncates at subleading order for photons [184, 121]. It follows from (1.6) that it truncates at subsubleading order for gravitons. As a consequence, both amplitudes can be fully constructed from the lowest orders in their soft expansion. To see this we just need to show how such orders are constrained by unitarity in QFT, a fact which is known as the Soft Theorem. For instance, for A_4 the only seed required is the amplitude $A_3^h(p_1, k_1) = (\epsilon\cdot p_1)^h$ which is fixed up to a constant from gauge invariance in the spinless case. The soft theorem with respect to $k_2 = \omega \hat{k}_2$ can be written compactly as

$$A_4^{\mathrm{ph}} = \frac{1}{2} \sum_{a=1,2} \frac{\epsilon_2\cdot p_a}{k_2\cdot p_a} e^{\frac{2F_2\cdot J_a}{\epsilon_2\cdot p_a}} A_3^{\mathrm{ph}} {=} \frac{1}{2}\left[\frac{p_1\cdot\epsilon_1 F_k}{p_1\cdot k_2\, p_2\cdot k_2} {-} \frac{F_\epsilon}{p_1\cdot k_2}\right], \tag{1.7}$$

where $F_2\cdot J_a = F_2^{\mu\nu} J_{a\mu\nu}$, with

$$F_2^{\mu\nu} = 2k_2^{[\mu}\epsilon_2^{\nu]} \propto \omega\,, \tag{1.8}$$

[1]We restore units in the final results and redefine $-iJ_{\mathrm{CS}} \to J_{\mathrm{here}}$ with respect to [72]. We work in mostly minus signature.

is the field strength and

$$J_a^{\mu\nu} = p_a^\mu \frac{\partial}{\partial p_{a,\nu}} - p_a^\nu \frac{\partial}{\partial p_{a,\mu}} \,, \tag{1.9}$$

is the angular momentum operator on a spinless massive particle. We define the shorthand notation $F_k = p_1{\cdot}F_2{\cdot}k_1$, $F_\epsilon = p_1{\cdot}F_2{\cdot}\epsilon_1$. Analogously,

$$\begin{aligned} A_4^{\rm gr} = & \sum_{p_a=p_1,p_2,k_1} \frac{1}{2}\frac{(\epsilon_2\cdot p_a)^2}{k_2\cdot p_a} e^{\frac{2F_2\cdot J_a}{\epsilon_2\cdot p_a}} A_3^{\rm gr} = \frac{1}{2k_1{\cdot}k_2}\times \\ & \left[\frac{(p_1{\cdot}\epsilon_1)^2}{p_1{\cdot}k_2 p_2{\cdot}k_2}F_k^2 - 2\frac{p_1{\cdot}\epsilon_1}{p_1{\cdot}k_2}F_k F_\epsilon + \frac{p_2{\cdot}k_2}{p_1{\cdot}k_2}F_\epsilon^2\right]. \end{aligned} \tag{1.10}$$

Given that $F_2{\cdot}J_a$ truncates when acting on A_3, the exponential has been inserted to get the soft-expansion at the desired order. The result not only manifests the double copy (1.6) but, as we will show, it generates the frequency expansion of classical radiation in these theories. The first term of the soft expansion therefore determines the dipole radiation formula in EM and the Einstein's quadrupole radiation in GR, whereas the subleading orders contribute to electric/magnetic higher multipoles [163]. For bodies with long range interactions as in (1.3), this corresponds to an expansion in powers of their orbital angular momentum, hence the name orbit multipole.

1.2 Spin-Multipoles in Arbitrary Dimension

Our goal is to promote the above discussion for the case of spinning sources, which introduces a rich new set of structures. In fact, the seed $A_3^{h,s}$ is not unique, neither its soft expansion is trivial in contrast to the spinless case $A_3^{h,0}$. As first pointed out by Weinberg, the expansion encodes corrections to $A_3^{h,0}$ [244, 184, 186, 135]. Here we will construct the multipole expansion in arbitrary dimensions, the reader interested in $D=4$ should skip to the next chapter. To write the expansion in modern language, note that as the spin is the only quantum number available for the massive state, for any n we can write [2]

$$A_n^{h,s}(J) \;\; = \;\; \mathcal{H}_n \times \langle p_1, \epsilon_1 | \sum_{j=0}^{\infty} \omega^{(2j)}_{\mu_1\cdots\mu_{2j}} J^{\mu_1\mu_2}\cdots J^{\mu_{2j-1}\mu_{2j}} | p_2, \epsilon_2\rangle$$

[2] Formally, this can be argued via the generalized Wigner-Eckart theorem of e.g. [3], even if the group is non-compact.

$$= \mathcal{H}_n \langle p_1, \epsilon_1 | (\omega^0 \mathbb{I} + \omega^{(2)}_{\mu\nu} J^{\mu\nu} + \ldots) | p_2, \epsilon_2 \rangle \,, \tag{1.11}$$

where $J^{\mu\nu}$ is the (intrinsic) Lorentz generator and now acts on spin-s states which we label $|p_i, \epsilon_i\rangle$, but which will be hereafter omitted. Products of $J^{\mu\nu}$ are symmetrized since $[J, J] \sim J$ can be put in terms of lower multipoles. The sum is then guaranteed to truncate due to the Cayley-Hamilton theorem as the dimension of spin states is finite. We encode the helicities of the photons/gravitons in the prefactor $\mathcal{H}_n$. Importantly this expansion in terms of multipoles agrees with the soft expansion as we will illustrate.

To begin, let us consider photon emission for low spins $s \in \{\frac{1}{2}, 1\}$. We can construct the multipole expansion for a gravitational amplitude ($h = 2$) if we first write down (1.11) for QED ($h = 1$) and then perfom a double copy similar to (1.6). Indeed, from two multipole operators X and X' acting on spin-s states, we introduce an operator $X \odot X'$ acting on spin-$2s$ as

$$X \odot X' = \begin{cases} 2^{-\lfloor D/2 \rfloor} \mathrm{tr}(X \not{\varepsilon}_1 \bar{X}' \not{\varepsilon}_2) \,, & 2s = 1, \\ \phi_{1\mu_1\nu_1} \left(X^{\mu_1}_{\mu_2} X'^{\nu_1}_{\nu_2} \right) \phi_2^{\mu_2\nu_2} \,, & 2s = 2, \end{cases} \tag{1.12}$$

where ε and ϕ are the respective massive polarizations and $\bar{X}$ denotes charge conjugation. We will show that these operations can be used to obtain scattering amplitudes in a gravity theory of a massive spin-$2s$ field [21, 208]. Here we will only need the following extension of (1.6):

$$A_n^{\mathrm{ph},s} \odot A_n^{\mathrm{ph},\tilde{s}} = K_n A_n^{\mathrm{gr},s+\tilde{s}} \,, \quad n = 3, 4 \,. \tag{1.13}$$

The case $s = 0$, $\tilde{s} \neq 0$ was introduced by Holstein et al. [150, 43]. It was used to argue that the gyromagnetic ratios of both $A_n^{\mathrm{ph},1}$ and $A_n^{\mathrm{gr},1}$ must coincide, setting $g = 2$ as a natural value [150, 90]. We introduce the case $s, \tilde{s} \neq 0$ as a further universality condition, and find it imposes strong restrictions on $A_n^{h,s}$ for higher spins. More importantly, it can be used to directly obtain multipoles in the classical gravitational theory.

For (1.13) to hold we need to put $A_n^{h,s}$ into the form (1.11). The coefficients $\omega^{(2j)}$ are universal once we consider minimal-coupling amplitudes, which are obtained from QED at $s = \frac{1}{2}$ and from the $W^{\pm}$-boson model at $s = 1$ [150]. The 3-pt. seeds in any dimension can be put as

$$A_3^{s,\mathrm{ph}} = \epsilon \cdot p_1 \left(\mathbb{I} + J \right) \,, \quad J = \frac{\epsilon_\mu q_\nu J^{\mu\nu}}{\epsilon \cdot p_1} \,, \tag{1.14}$$

for $q = p_1 - p_2$. Denoting each operator by the corresponding $\mathrm{SO}(D-1,1)$ Young diagram, i.e. $1 = \mathbb{I}$ and $\ydiagram{1,1} = J^{\mu\nu}$, the operation (1.12) gives the rules

$$1_s \odot 1_s = 1_{2s}\,, \quad 1_s \odot \ydiagram{1,1}_{\,s} = \frac{1}{2}\ydiagram{1,1}_{\,2s}\,, \tag{1.15}$$

$$\ydiagram{1,1}_{\,s} \odot \ydiagram{1,1}_{\,s} = \ydiagram{2,2}_{\,2s} + \ydiagram{2}_{\,2s} + \hat{1}_{2s}\,, \tag{1.16}$$

which are a subset of the irreducible representations allowed by the Clebsch-Gordan decomposition. Rule (1.16) is explained in (1.25) below. The first term we denote by $\Sigma^{\mu\nu\rho\sigma}$ and has the symmetries of a Weyl tensor, i.e. it is the traceless part of $\{J^{\mu\nu}, J^{\rho\sigma}\}$. For instance, the $s = 2$ amplitude as obtained from (1.13) is

$$A_3^{\mathrm{gr},2} = (\epsilon{\cdot}p_1)^2\, \phi_2{\cdot} \left(\mathbb{I} + \frac{\epsilon_\mu q_\nu J^{\mu\nu}}{\epsilon{\cdot}p_1} + \frac{W_{\mu\nu\alpha\beta}}{4(\epsilon{\cdot}p_1)^2}\Sigma^{\mu\nu\alpha\beta}\right){\cdot}\phi_1\,, \tag{1.17}$$

where $W_{\mu\nu\alpha\beta} := q_{[\mu}\epsilon_{\nu]}q_{[\alpha}\epsilon_{\beta]}$ is the Weyl tensor of the graviton, reproducing the expected Weyl-quadrupole coupling [132, 212, 216, 176, 90], as shown in Appendix A.

To deeper understand these results, let us demand $A_3^{\mathrm{gr},s}$ to be constructible for *any* spin from the following formula:

$$A_3^{\mathrm{gr},s+\tilde{s}}(J^{\mu\nu} \oplus \tilde{J}^{\mu\nu}) = A_3^{\mathrm{ph},s}(J^{\mu\nu}) \odot A_3^{\mathrm{ph},\tilde{s}}(\tilde{J}^{\mu\nu})\,, \tag{1.18}$$

Note that this is an extension of the double copy formula (1.13), relating gravitational amplitudes to those of QED. Here $J^{\mu\nu} \oplus \tilde{J}^{\mu\nu}$ is the generator acting on a spin $s + \tilde{s}$ representation. This relation yields the condition $A_3^{1,s}A_3^{1,\tilde{s}} = A_3^{1,s+\tilde{s}}A_3^{1,0}$ on the $J^{\mu\nu}$ operators. Using that $[J, \tilde{J}] = 0$ and assuming the coefficients in (1.11) to be independent of the spin leads to

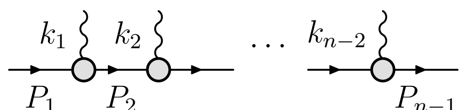
$$A_3^{h,s}(J) = (\epsilon{\cdot}p_1)^h \times e^{\omega_{\mu\nu}J^{\mu\nu}}\,, \quad h = 1, 2 \tag{1.19}$$

with $\omega_{\mu\nu} = \frac{k_{[\mu}\epsilon_{\nu]}}{\epsilon\cdot p_1}$ and $\mathcal{H}_3 = (\epsilon{\cdot}p_1)^h$ fixed by the previous examples. This easily recovers such cases and matches the Lagrangian derivation [236] for $s \in \{\frac{1}{2}, 1, 2\}$ in any dimension D. After some algebra, (1.19) leads to the D=4 photon-current derived in [183, 182] for *arbitrary* spin via completely different arguments. On the gravity side, it matches the Kerr stress-energy tensor derived in [238] together with its spinor-helicity form recently found in [135], as we show in Appendix B. For $s > h$ and $D > 4$, (1.19) contains a pole in $\epsilon{\cdot}p$ which reflects the well-known fact that such interactions cannot involve only elementary (point-like) particles [13]. In Appendix A we show such pole cancels for the multipoles that we are interested in the classical theory and provide a local form of (1.19).

What is the meaning of the exponential e^J? It corresponds to a finite Lorentz transformation induced by the massless emission. That is, it is easy to check that

$$p_2^\mu = [e^J]^\mu{}_\nu p_1^\nu\,, \tag{1.20}$$

hence for generic spin it maps the state $|p_1,\varepsilon_1\rangle$ into $|p_2,\tilde{\varepsilon}_2\rangle$, where $\tilde{\varepsilon}_2 \neq \varepsilon_2$ is another polarization for p_2. This means e^J is composed both of a boost and a SO$(D-1)$ Wigner rotation. The boost can be removed in order to match SO$(D-1)$ multipoles in the classical theory, see Appendix A. Also, as e^J is a Lorentz transformation, $|\varepsilon_2\rangle$ must live in the same irrep as $|\varepsilon_1\rangle$. This means that a projector is not needed when these objects are glued. A corollary of this is a simple formula for the full factorizations of $A_n^{h,s}$, e.g.



$$= \prod_i (P_i\cdot\epsilon_i)^h \langle\varepsilon_2|e^{J_{n-2}}\cdots e^{J_1}|\varepsilon_1\rangle = \prod_i (P_i\cdot\epsilon_i)^h \langle\varepsilon_2|\tilde{\varepsilon}_2\rangle, \tag{1.21}$$

where $P_i = p_1 + k_1 + \ldots + k_{i-1}$ and $J_i = \frac{k_{i\mu}\epsilon_{i\nu}J^{\mu\nu}}{\epsilon_i\cdot P_i}$. Each 3-pt. amplitude here maps P_i to P_{i+1} and their composition maps p_1 to p_2. The state $|\tilde{\varepsilon}_2\rangle$ depends on all $\{k_i,\epsilon_i\}_{i=1}^n$ as well as their ordering. This factorization is enough to obtain the classical spin-multipoles of M_5 at least up to the quadrupole order we are interested in. To see this, we use the Baker-Campbell-Hausdorff formula in (1.21) and get the form

$$\begin{aligned} A_4^{\mathrm{ph},s} = &\frac{1}{2}\left[\frac{p_1\cdot\epsilon_1 p_2\cdot\epsilon_2}{p_1\cdot k_1}\langle\varepsilon_2|e^{J_1+J_2-\frac{1}{2}[J_1,J_2]+\cdots}|\varepsilon_1\rangle + \right.\\ &\left.\frac{p_2\cdot\epsilon_1 p_1\cdot\epsilon_2}{p_2\cdot k_1}\langle\varepsilon_2|e^{J_1'+J_2'+\frac{1}{2}[J_1',J_2']+\cdots}|\varepsilon_1\rangle + \mathrm{c.t.}\right]. \end{aligned} \tag{1.22}$$

This is the spin analog of (1.7), where the exponential tracks the desired order. Setting $\mathcal{H}_4{=}\frac{1}{2}\frac{1}{p_1\cdot k_1 p_1\cdot k_2}$ in (1.11), this gives for $s{\leq}1$

$$\begin{aligned} \omega_{(2)}^{\mu\nu} &= -\frac{p_1\cdot F_1\cdot p_2}{2}F_2^{\mu\nu} - \frac{p_1\cdot F_2\cdot p_2}{2}F_1^{\mu\nu} - \frac{p_1\cdot(k_1+k_2)}{4}[F_1,F_2]^{\mu\nu}\,,\\ \omega_{(4)}^{\mu\nu\rho\sigma} &= \frac{k_1\cdot k_2}{16}\left(F_1^{\mu\nu}F_2^{\rho\sigma} + F_2^{\mu\nu}F_1^{\rho\sigma}\right)\,. \end{aligned} \tag{1.23}$$

The role of the contact term 'c.t.' in (1.22) is to restore gauge invariance. Here it is only needed for $\omega^{(0)}$, thus by comparison with (1.7) one finds c.t. $=\epsilon_1{\cdot}\epsilon_2$ and $\omega^{(0)}=p_1{\cdot}F_1{\cdot}F_2{\cdot}p_1$. Already for spin-$\frac{1}{2}$ it is clear that this decomposition of the Compton amplitude is not evident at all from a Feynman-diagram computation [43, 199], whereas here it is direct. A key point of this splitting is that under the *double* soft deformation $k_3=\tau\hat{k}_3, k_4=\tau\hat{k}_4$, the multipole $\omega^{(2j)}$ is $\mathcal{O}(\tau^j)$, whose leading order will be the classical contribution. It is now instructive to further decompose $A^{\text{ph,s}}$ into irreps., which follows from

$$\omega^{(4)}_{\mu\nu\rho\sigma}\{J^{\mu\nu},J^{\rho\sigma}\}=\begin{cases}\hat{1}[\omega^{(4)}]+[\omega^{(4)}]_{\mu\nu}Q^{\mu\nu}, & s=1,\\ \hat{1}[\omega^{(4)}]+[\omega^{(4)}]_{\mu\nu\rho\sigma}\ell^{\mu\nu\rho\sigma}, & s=\frac{1}{2},\end{cases}$$

where $\ell^{\mu\nu\rho\sigma}=\{J^{[\mu\nu},J^{\rho\sigma]}\}=\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and

$$\hat{1}=\frac{J_{\mu\nu}J^{\mu\nu}}{2},\;\; Q^{\mu\nu}=\square\!\square=\{J^{\mu\rho},J_{\rho}{}^{\nu}\}+\frac{4}{D}\eta^{\mu\nu}\hat{1}. \tag{1.24}$$

The notation $[\omega^{(4)}]$ denotes the corresponding projections. Among them we will be interested in the quadrupole operator $Q^{\mu\nu}$, only present for $s\geq 1$.

Finally, $A_4^{\text{gr,s}}$ is found from (1.13) and matches the Lagrangian result for $s\leq 2$. We have used that (1.16) reads

$$\begin{aligned}J_s^{\mu\nu}\odot J_s^{\rho\sigma}=&\frac{1}{4}\Sigma_{2s}^{\mu\nu\rho\sigma}+\frac{\alpha_D}{D-2}\eta^{[\sigma[\nu}Q_{2s}^{\mu]\rho]}\\ &+\frac{\beta_D}{2D(D-1)}\eta^{\sigma[\nu}\eta^{\mu]\rho}\hat{1}_{2s}\,.\end{aligned} \tag{1.25}$$

The normalizations α_D,β_D depend solely on D. However, it cancels out in the full computation and hence we set $\alpha_D{=}\beta_D{=}1$ hereafter. Similarly, the condition $A_4^{\text{ph},\frac{1}{2}}A_4^{\text{ph},\frac{1}{2}}{=}A_4^{\text{ph},0}A_4^{\text{ph},1}$, as implied by (1.13), can be traced at this order to $[\omega^{(2)}\omega^{(2)}]_{\mu\nu}=[\omega^{(4)}]_{\mu\nu}\omega^{(0)}$, which holds up to terms subleading in the double soft limit.

1.3 An application: Radiation in EM and Gravity

Very recently, Kosower et al. [170] have provided a QFT derivation of the following formulae

$$\Delta p^{\mu}{=}\int\frac{d^Dq}{(2\pi)^{D-2}}\delta(2q{\cdot}p_1)\delta(2q{\cdot}p_3)q^{\mu}e^{iq\cdot b}\langle M_4^h\rangle\,, \tag{1.26}$$

$$\mathcal{J}_h(k) = \int \frac{d^D q_1}{(2\pi)^{D-2}} \delta(2\, q_1 \cdot p_1) \delta(2\, q_3 \cdot p_3) e^{iq_1 b_1} e^{iq_3 b_3} \langle M_5^h \rangle, \tag{1.27}$$

encoding classical observables at LO in the coupling [129]. Here $\Delta p^\mu = \frac{\partial \chi}{\partial b_\mu}$ is the (conservative) momentum deflection of a massive body in a classical scattering setup, where the function χ is the scattering angle [39]. The current $\mathcal{J}_h(k)$ reads $\epsilon_\mu J^\mu$ $(h = 1)$ and $\epsilon_{\mu\nu} T^{\mu\nu}$ $(h = 2)$ and corresponds to the field radiated at $r \to \infty$. Even though these were proven for $D = 4$, matching with classical results shows that they hold in any D.

Classical Limits

The classical limit $\langle M \rangle$ is obtained by rescaling $q_i \to \hbar q_i$ which also implies $k \to \hbar k$. The justification for this comes from equations (1.26)-(1.27), where q^μ appears in the Fourier transform as a wavenumber (the conjugate to the distance b) rather than a momenta hq^μ. We refer the reader to [170] for futher details.

We now extend this rule to include spin. Following [19] we recall that in e.g. $D = 4$ the spin operators satisfy

$$[S_i, S_j] = \frac{i}{\hbar} \epsilon_{ijk} S^k \tag{1.28}$$

This means we can recover the classical operator by scaling $J \to \frac{1}{\hbar} J$, namely by inverting the Dirac quantization procedure. Hence the classical limit corresponds to a large angular momentum regime as in e.g. [135], see also [86]. Nevertheless, observe that the combination $qJ \sim \hbar^0$ is finite and will correspond to our (classical) multipole expansion.

The $\hbar \to 0$ limit is captured by the factorizations of M_4 and M_5 given in (1.3). For M_4, this was argued in [62], where the classical piece was identified as most singular part in q^2 up to 1-loop, see also [114]. For M_5, the key point is to introduce the average momentum transfer $q = \frac{q_1 - q_3}{2}$, after which we expect the same criteria to apply. In fact, noting that $d^D q_1 = d^D q$ in (1.27) already shows, after some algebra, that contact terms in q^2 appearing in $\langle M_5^h \rangle$ will lead to local quantum contributions.

Let us now apply equation (1.3) explicitly for the scalar case (the spin case will be covered below using the double copy formula). To start with, consider $\langle M_4^h \rangle = \frac{n_h}{q^2}$ where n_h a local numerator, e.g. it does not have poles. Its scalar parts are $n_{\rm ph} = p_1 \cdot p_3$ and

$$\langle M_4^{\rm gr} \rangle = \frac{n_{\rm gr}}{q^2} = \frac{\sqrt{32\pi G}}{q^2} \left[(p_1 \cdot p_3)^2 - \frac{m_a^2 m_b^2}{D-2} \right], \tag{1.29}$$

where the factor of $D-2$ arises from the graviton propagator. In $D=4$ we can evaluate (1.26) to recover the 1PM scattering angle as in [39], first derived in the classical context by Portilla [210, 211]. Moving to $\langle M_5^h \rangle$, the factorization of (1.3) together with the classical limit imply the form

$$\langle M_5^h \rangle = \frac{1}{(q\cdot k)^{h-1}} \left[\frac{n_h^{(a)}}{(q^2 - q\cdot k)(p_1\cdot k)^2} \pm \frac{n_h^{(b)}}{(q^2 + q\cdot k)(p_3\cdot k)^2} \right] , \tag{1.30}$$

where we pick $(-)$ for $h=2$. The spurious pole $q\cdot k$ arises from the t-channel of $A_4^{\mathrm{gr},s}$, and its cancellation provides a nice check of our formula. This further shows that the classical limits of M_4 and M_5 are universal and do not depend on the spin of the massive particle. This was emphasized in [43] at four points and it is the first example of such universality at five points.

Classical Soft Theorem: Gravitational Memory Effect

As an application of orbit multipoles let us study $\langle M_5^{\mathrm{gr}} \rangle$ for scalars. The numerators $n^{(a)}$ can be read off directly from (1.10): Replacing ϵ_1 by p_3, powers of the orbit multipole F_ϵ translate to powers of $F_p = p_1\cdot F\cdot p_3$, whereas F_k now becomes $F_{iq} = \eta_i (p_i \cdot F\cdot q)$, with $\eta_1 = -1, \eta_3 = 1$. The soft expansion (1.10) with respect to $k_2 = k$ becomes

$$n_{\mathrm{gr}}^{(a)} = \frac{F_{1q}^2}{2} e^{-\frac{F_p}{F_{1q}}(p_1\cdot k)\frac{\partial}{\partial(p_1\cdot p_3)}} \left[(p_1\cdot p_3)^2 - \frac{m_b^2 m_a^2}{D-2} \right] . \tag{1.31}$$

Further writing $\frac{1}{q^2 \pm q\cdot k} = e^{\pm q\cdot k \frac{\partial}{\partial q^2}} \frac{1}{q^2}$ turns (1.30) into

$$\boxed{\langle M_5^{\mathrm{gr}} \rangle = \sum_{i=1,3} \mathcal{S}_i e^{\eta_i \left(F_p \frac{p_i\cdot k}{F_{iq}} \frac{\partial}{\partial(p_1\cdot p_3)} + q\cdot k \frac{\partial}{\partial q^2} \right)} \langle M_4^{\mathrm{gr}} \rangle} \tag{1.32}$$

where $\mathcal{S}_i = \frac{\eta_i}{2} \frac{F_{iq}^2}{(p_i\cdot k)^2 q\cdot k}$ (for photons we find $\mathcal{S}_i = \frac{F_{iq}}{2(p_i\cdot k)^2}$). This expression can be used to obtain $\langle M_5^{\mathrm{gr}} \rangle$ from $\langle M_4^{\mathrm{gr}} \rangle$ as an expansion in the graviton momenta k^μ to any desired order at leading order in the coupling. Recall that $\langle M_4 \rangle$ is associated to conservative effects (as will be further studied in the following chapters) whereas $\langle M_5 \rangle$ contains a radiation kernel according to (1.27). The spurious pole in $\mathcal{S}_i$ cancels out and one can check explicitly that $\mathcal{S}_1 + \mathcal{S}_3$ corresponds to the $\hbar \to 0$ limit of the Weinberg Soft Factor for the full M_5 [243]. The first order of the exponential analogously corresponds to the

$\hbar \to 0$ limit of the subleading soft factor of Low [184, 186].

Let us focus for simplicity on the leading order of (1.32). By considering bounded orbits with $\omega \sim \frac{v}{r}$ the GW frequency expansion becomes a non-relativistic expansion [130]. It can be checked that the LO in fact leads to Einstein's Quadrupole Formula, see discussion below. For classical scattering we can use the LO to obtain the Memory Effect as $r{\to}\infty$. Plugging (1.32) into (1.27) we get

$$\int \frac{d^D q}{(2\pi)^{D-2}} \delta(2q\cdot p_1)\delta(2q\cdot p_3) e^{iq\cdot(b_1-b_3)} \left(\sum_{i=1,3} \mathcal{S}_i\right) \langle M_4^{\mathrm{gr}}\rangle$$

as $k \to 0$. Evaluating the sum and using (1.26) as a definition of $\Delta p_1 = -\Delta p_3$ we obtain

$$\epsilon_{\mu\nu}T^{\mu\nu} = \frac{F_p/2}{p_1{\cdot}k p_3{\cdot}k}\left(\frac{p_1}{p_1{\cdot}k}{+}\frac{p_3}{p_3{\cdot}k}\right){\cdot}F{\cdot}\Delta p{+}\mathcal{O}(k^0), \tag{1.33}$$

which at leading order in Δp (or G, if restored) becomes

$$T^{\mu\nu}(k) = \sqrt{8\pi G} \times \Delta\left[\frac{p_1^\mu p_1^\nu}{p_1{\cdot}k} + \frac{p_3^\mu p_3^\nu}{p_3{\cdot}k}\right]^{\mathrm{TT}}. \tag{1.34}$$

In position space this gives the burst memory wave derived by Braginsky and Thorne [54] in $D = 4$ (a $\frac{1}{4\pi r}$ factor arises from the ret. propagator as $r{\to}\infty$ [129, 138]), see also [203, 192, 226] for $D > 4$. Here we have provided a direct connection with the Soft Theorem (1.32), alternative to the expectation-value argument [234, 233]. This can also be seen as the Black Hole Bremsstrahlung of [189, 191] generalized to consistently include the dynamics of the sources.

Classical Double Copy: Results for Radiation with Spin

As the numerators in eqs. (1.29) and (1.30) correspond to $A_n^{h,s}$ amplitudes, the multipole double copy can be directly promoted to $\langle M_4\rangle$ and $\langle M_5\rangle$. From a classical perspective, the factorization of (1.3) implies that the photon numerators can always be written as $n_{\mathrm{ph}} = t_{a\mu}t_b^\mu$ where t_a and t_b *only* depend on particle 1 and 3 respectively. The simplest example is the scalar piece in $\langle M_4^{\mathrm{ph}}\rangle$, where $t_a = p_1$ and $t_b = p_3$. The KLT formula (1.13) translates to

$$\boxed{n_{\mathrm{gr}} = n_{\mathrm{ph}} \odot n_{\mathrm{ph}} - \mathrm{tr}(n_{\mathrm{ph}} \odot n_{\mathrm{ph}})} \tag{1.35}$$

where we defined the trace operation as $\mathrm{tr}(n \odot n) = \frac{(t_{a\mu}\odot t_a^\mu)(t_{b\mu}\odot t_b^\mu)}{D-2}$. By combining (1.35) with eqs. (1.29) and (1.30), this establishes a classical double-copy formula that can be directly proved from QFT via a classical limit.

Let us start with $\langle M_4\rangle$ as example. To keep notation simple consider only particle a to have spin. From (1.14) we find that at the dipole level (e.g. linear in J) the numerator for $\langle M_4^{\rm ph}\rangle$ is $n_{\frac{1}{2}}^{\rm ph} = n_0^{\rm ph} + p_3{\cdot}J_a{\cdot}q$. The gravity result follows from (1.35) by dropping contact terms in q^2. The rules (1.15) readily give the scalar and dipole parts, including (1.29). For the quadrupole part, rule (1.25) gives

$$\frac{(p_3{\cdot}J_a{\cdot}q)\odot(p_3{\cdot}J_a{\cdot}q) - \mathrm{tr}(\cdots)}{q^2} = \frac{1}{4}\frac{p_{3\mu}q_\nu p_{3\alpha}q_\beta \Sigma_a^{\mu\nu\alpha\beta}}{q^2}\,, \tag{1.36}$$

Using (A.7), the SO(D − 1) quadrupole [212, 175, 176] reads

$$\frac{1}{4}\frac{p_{3\mu}q_\nu p_{3\alpha}q_\beta \Sigma_a^{\mu\nu\alpha\beta}}{q^2} \to \left((p_1{\cdot}p_3)^2 - \frac{m_a^2 m_b^2}{D-2}\right)\frac{q\cdot\bar{Q}_a\cdot q}{2(D-3)q^2 m_a^2}. \tag{1.37}$$

Up to this order this agrees with the $D = 4$ computation [238, 135, 194]. Agreement to all orders in spin is obtained from the formula (A.12) in Appendix A.

Moving to $\langle M_5\rangle$, in the examples that follow the numerators $n_{\rm ph}$ can be read either from classical results up to dipole order [190, 170, 127, 180], from QED Bremsstrahlung, or from (1.7), (1.14) and (1.23). They are all in agreement [3]. For photons, the scalar part is

$$n_0^{(a)}{=}4e^3 p_1{\cdot}R_3{\cdot}F{\cdot}p_1, \quad n_0^{(b)}{=}4e^3 p_3{\cdot}R_1{\cdot}F{\cdot}p_3, \tag{1.38}$$

where $R_i^{\mu\nu}{=}p_i^{[\mu}(\eta_i 2q{-}k)^{\nu]}$. For the spin part we have

$$\begin{aligned} n_{\frac{1}{2}}^{(a)} &{=} n_0^{(a)} - 2e^3\left[p_1{\cdot}R_3{\cdot}kF{\cdot}J_a - F_{1q}R_3{\cdot}J_a + p_1{\cdot}k\,[F,R_3]{\cdot}J_a\right],\\ n_{\frac{1}{2}}^{(b)} &{=} n_0^{(b)} + 2e^3 p_3{\cdot}F{\cdot}\hat{R}_a{\cdot}p_3, \end{aligned} \tag{1.39}$$

with $\hat{R}_a^{\mu\nu}{=}\,(2q{+}k)^{[\mu}\,J_a^{\nu]\alpha}(2q{+}k)_\alpha$. Recall these numerators live in the support of $\delta(p_i{\cdot}q_i)$ in (1.27). Writing them as $n_{\frac{1}{2}}{=}t_a{\cdot}t_b$ one finds $t_b^{(a)}{=}p_3$ and $t_a^{(b)} = p_1{+}J_a{\cdot}(2q{+}k)$. The scalar and dipole pieces obtained from (1.35) then recover the results of [190, 127, 180] for Pure and Fat Gravity (we obtain the latter as the limit $D\to\infty$). This provides a

[3]Ref. [180] may contain a typo. Reproducing the computation leads to a relative (−) sign between eqs. 26 and 27.

strong cross-check of our method. Using (1.25) we can also compute the quadrupole order. For instance, the $Q^{\mu\nu}$ piece reads

$$\begin{aligned}\frac{n^{(a)}|_Q}{q\cdot k}=&\frac{(32\pi G)^{\frac{3}{2}}}{8(D-2)}\Bigg[(p_1\cdot p_3 F_{1q}-p_1\cdot k F_p)\,\{R_3,F\}\cdot Q_a+\\&\frac{m_b^2}{(D-2)}\,(F_{1q}\{F,Y\}\cdot Q_a-2p_1\cdot k\,p_1\cdot F\cdot Q_a\cdot F\cdot q)\Bigg],\end{aligned}$$

with $Y^{\mu\nu}=p_1^{[\mu}(2q-k)^{\nu]}$, whereas $n^{(b)}|_Q=0$. As before, we have dropped contact terms in q^2 and used the support of $\delta(p_i\cdot q_i)$. This result can be shown to agree with a much more lengthy computation of the full $M_5^{\rm gr}$ using Feynman diagrams. At this order, $M_5^{\rm gr}$ contains classical quadrupole pieces and quantum scalar and dipole pieces. Interestingly, while the scalar part is trivial to identify, we have found that the dipole part can be cancelled by adding the spin-1 spin-0 interaction $(B_\mu\partial^\mu\phi)^2$ to the Lagrangian, which signals its quantum nature.

1.4 Outlook of the Chapter

In this chapter we have shown that key techniques of Scattering Amplitudes such as soft theorems and double copy can be promoted directly to study classical phenomena associated to spinnining objects. These techniques drastically streamline the computation of radiation and spin effects; both are phenomenologically important for Black Holes, which are believed to be extremely spinning in nature [220, 219]. Indeed, in the following chapters we will further explore the connection of our amplitudes with the problem of two spinning Black Holes in $D=4$, in the conservative sector. We now outline some other directions:

The $A_n^{h,s}$ series: The amplitudes for two massive particles emitting photons/gravitons constitute building blocks of classical pieces even at loop orders, as illustrated in the Introduction of this thesis. Here we have studied the case $s\leq 2$ for $n=4$. For $s>2$ the amplitudes $A_4^{h,s}$ were studied in [90] in the context of the $\mathcal{O}(G^2)$ potential and were found to contain polynomial ambiguities. We expect our construction, including soft expansion and double copy, to be a criteria for resolving such ambiguities and lead to further classical predictions. In the scalar setup, we expect $A_n^{\rm gr,0}$ to be relevant even for $n>4$. In fact, $A_5^{\rm gr,0}$ as a double copy has been recently pointed out as a key ingredient in the computation of the $\mathcal{O}(G^3)$ potential by Bern et al. [28]. All these results made strong use of the $D=4$ spinor-helicty formalism. Specializing our treatment of radiation to $D=4$ is also a natural future direction in the hunt of simplifications even at loop orders, as in [134, 62].

Soft Theorem/Memory Effect: It would be interesting to understand the meaning of the higher orders of (1.32), considering for instance the Spin Memory Effect [202, 198]. Motivated by the infinite soft theorems of [137, 78] one could expect the corrections are related to a hierarchy of symmetries. One may also incorporate spin contributions and study their interplay with such orders [138]. In the applications side, it is desirable to further investigate (1.32) at loop level [30, 141], which could lead to a simple way of obtaining $\langle M_5 \rangle$ from $\langle M_4 \rangle$ and relate conservative and non-conservative phenomena at higher orders in G.

Generic Orbits: In this section we have considered scattering of two classical objects a and b. This is an unbounded process but it may be related to bounded (circular or quasicircular) orbits as the dynamics of both are controlled by the same equations of motion. Now, for orbits more general than scattering the radiation kernel $\mathcal{J}(k)$ does not have the support of $\delta(2p_i \cdot q_i)$, which arises only for asymptotic trajectories [130, 228]. In fact, for bounded orbits direct computation shows $\mathcal{J}(k)$ contains the subleading terms $p_i \cdot q_i \sim \omega$. Very nicely, we have checked such terms indeed can be recovered as they match with eqs. (1.38),(1.39), which in turn arise from the form in (1.23) via a natural "$F \to R$ replacement". We do not have an understanding of why this is the case. One could also try to explore the gravity case to see if a similar replacement works.

Chapter 2

The Multipole Expansion in Four Dimensions and the Kerr Black Hole

2.1 Introduction

In the last chapter we have studied the multipole decomposition of certain graviton/photon emission processes. Even though we have performed this computation in arbitrary dimension, our final aim is to improve understanding of the dynamics of four dimensional (spinning) black holes. Thus, in this chapter we consider the multipole expansion (1.11) especialized to four dimensions. We shall see that in four dimensions several simplifications arise and we can effectively obtain observables at all orders in spin, namely, considering a particle of infinite spin label s. Even though we compute these for scattering events, the matching to classical results allow us to gain confidence in our effective description so that we can further push it to obtain an effective potential for bounded systems. We will take a first step in this direction in the next chapter.

The main simplification in the $D = 4$ case comes from the fact that spin is described by the Pauli-Lubanski pseudovector

$$S_{\lambda} := ma_{\lambda} = \frac{1}{2m}\epsilon_{\lambda\mu\nu\rho}J^{\mu\nu}p^{\rho}, \tag{2.1}$$

such that the spin multipoles appearing in (1.11) can now be written as powers (i.e. tensor products) of this operator. For instance, we find identities such as

$$J^{\mu\alpha}J_{\alpha}^{\ \nu} = S^{\mu}S^{\nu} + \dots \tag{2.2}$$

where ... depend on the boost as detailed on appendix A. Now, rather than applying this identity repeatedly we will instead rederive the amplitudes precisely in terms of S^{μ}. In this chapter we will only need to do so at three points. We anticipate that the spin vector effectively reproduces the internal structure of our particles and its norm $\frac{|S|}{m} = |a|$ will correspond, in a classical limit, to the radius of the disc singularity in Kerr, hence $|a| < m$.

We will use this expansion to provide a new form of the scattering angle $\chi = \frac{\Delta \mathbf{p}}{|\mathbf{p}|}$ introduced in the previous chapter. Furthermore, we will consider general spin directions at full 1PM order and compute the spin deflection Δa^{μ} arising from the scattering. In other words, in this chapter we aim to fully reproduce the seminal results of Vines [238] to all orders in spin, computed from a purely classical perspective using linearized form of Einstein's equations (for two Black Holes). Moreover, in the forthcoming chapters we will provide new results by extending this to 2PM in an aligned-spin setup. This means that the two-body scattering is confined to a single constant orbital plane, and the spin vectors are conserved, pointing orthogonally to that plane. In the general case, both the orbital plane and the spin vectors are rotated in the course of the interaction.

The starting point is the tree-level amplitude for one-graviton exchange between two massive spin-s particles, which we previosly labeled as $\langle M_4 \rangle$, especialized to four dimensions and to all orders in spin. We will glue two of the minimal-coupling three-point amplitudes as depicted in figure 2.1. We streamline the treatment of the spin-exponentiated structure by incorporating the additional Lorentz boosts of the previous chapter into the spin exponentials. Finally, we adapt to our needs a general formalism [170, 193] for extracting gauge-invariant classical observables from amplitudes. This has been used in [193] to compute the net changes in the momenta and spins for two-body scattering at 1PM order, reproducing the results for BHs up to quadrupolar order from minimally coupled spin-1 particles. Here we extend such calculation to arbitrary spins s, and in the limit $s \to \infty$ obtain all orders in the BH multipole expansions at 1PM order [238].

2.2 Minimal coupling and the No-Hair Theorem

In this section we revisit the angular-momentum exponentiation (1.19) that is inherent to the gravitational coupling of spinning black holes and the corresponding amplitudes, this time in terms of the Pauli-Lubanski/spin vector.

At the linearized-gravity level, we can define a classical stress-energy tensor serving as an effective source for a single Kerr black hole. This means that in the harmonic gauge, the linearized form of the Kerr solution can be read from $\partial^2 h^{\mu nu} = T^{\mu\nu}_{\rm BH}$, where

$$T^{\mu\nu}_{\rm BH}(x) = \frac{1}{m}\int d\tau\, p^{(\mu} \exp(a*\partial)^{\nu)}{}_{\rho} p^{\rho} \delta^4(x-u\tau), \tag{2.3}$$

where we have used the shorthand notation

$$(a*b)^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} a_\alpha b_\beta. \tag{2.4}$$

and denoted the mass by m, classical momentum by $p^\mu = mu^\mu$ and spin by $S^\mu = ma^\mu$. The spin transversality condition $p \cdot a = 0$ is also imposed. Note that the case $a = 0$ corresponds to a point particle coupled to gravity, it will be associated to the Schwarzschild metric. This effective source can be used to analyze the dynamics of e.g. geodesics in Kerr background at the linearized order. See [238] for further details.

The corresponding coupling of the BH to gravity is

$$\begin{aligned} S_{\rm BH} &= -\frac{\kappa}{2}\int \hat{d}^4k\, h_{\mu\nu}(k) T^{\mu\nu}_{\rm BH}(-k) \\ &= -\kappa \int \hat{d}^4k\, \hat{\delta}(2p\cdot k) p^{(\mu} \exp(a*ik)^{\nu)}{}_{\rho} p^{\rho} h_{\mu\nu}(k), \end{aligned} \tag{2.5}$$

where the coupling constant is $\kappa = \sqrt{32\pi G}$. Here and below the hats over the delta functions and measures encode appropriate positive or negative powers of 2π, respectively. Putting the graviton on shell, $h_{\mu\nu}(k) \to \hat{\delta}(k^2)\varepsilon_\mu\varepsilon_\nu$, we can rewrite the characteristic angular-momentum exponential in another form

$$h_{\mu\nu}(k) T^{\mu\nu}_{\rm BH}(-k) = \hat{\delta}(k^2)\hat{\delta}(p\cdot k)(p\cdot\varepsilon)^2 \exp\left(-i\frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}\right), \tag{2.6}$$

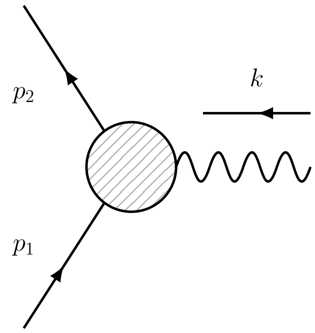


Figure 2.1: Three-point amplitude

now involving a transverse spin tensor $S^{\mu\nu}$

$$S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} p_\rho a_\sigma \qquad \Rightarrow \qquad p_\mu S^{\mu\nu} = 0. \tag{2.7}$$

More explicitly, the above transition relies on the equality

$$(p \cdot \varepsilon)^{j-1} \varepsilon_\mu \big[(a * ik)^j\big]^\mu{}_\nu p^\nu = \big(-ik_\mu \varepsilon_\nu S^{\mu\nu}\big)^j, \tag{2.8}$$

that is easiest verified in the frame and gauge where $k = (k^0, 0, 0, k^0)$, $\varepsilon = (0, \varepsilon^1, \pm i\varepsilon^1, 0)$ and $p = (p^0, p^1, 0, p^0)$.

The fact the the energy-momentum tensor (i.e. the linearized metric) of the Kerr Black Hole only depends on its momentum p^μ and spin a^μ, instead of free parameters arising for each of the infinite multipoles, is a manifestation of the no-hair theorem. We have already encountered such exponential structure in our three-point amplitudes. Here we will find the same exponential in the minimal-coupling amplitudes of the previous chapter, now written in the form originally proposed by Arkani-Hamed, Huang and Huang [13]. In appendix B we outline their representation, which is based in *spinor-helicity* variables for both massive and massless momenta. In this language the amplitudes read (when especializing to $D = 4$ we change notation $A_3^{\pm 2, s} \to \mathcal{M}_3$)

$$\mathcal{M}_3(p_1^{\{a\}}, -p_2^{\{b\}}, k^+) = -\frac{\kappa}{2} \frac{\langle 1^a 2^b \rangle^{\odot 2s}}{m^{2s-2}} x^2, \tag{2.9a}$$

$$\mathcal{M}_3(p_1^{\{a\}}, -p_2^{\{b\}}, k^-) = -\frac{\kappa}{2} \frac{[1^a 2^b]^{\odot 2s}}{m^{2s-2}} x^{-2}. \tag{2.9b}$$

The arguments of scattering amplitudes are treated as incoming, so the present choice corresponds to the momentum configuration shown in figure 2.1. The key point is that the little-group dependence on the massive particles 1 and 2 correspond to the su(2) $\approx$ so(3) indices $\{a_1, \dots, a_{2s}\}$ and $\{b_1, \dots, b_{2s}\}$ respectively. The symbol $\odot$ denotes tensor product symmetrized over each massive particle's indices. Furthermore, x is the positive-helicity factor

$$x = \frac{[k|p_1|r\rangle}{m\langle k\,r\rangle} = -\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^+) = \left[\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^-)\right]^{-1}, \tag{2.10}$$

that is dimensionless and independent of the reference momentum r on the on-shell three-point kinematics [13].

Now one can start by noticing that the amplitudes $\mathcal{M}_3^{(0)} = -\kappa(p_1 \cdot \varepsilon^\pm)^2$, given by the scalar case of eq. (2.9), correspond precisely to the $S^{\mu\nu} = 0$ case of the vertex (2.6). Moreover, we will recast the spin structure of the amplitudes (2.9) in an exponential form,

$$\langle 2^b 1^a\rangle^{\odot 2s} = [2^b|^{\odot 2s} \exp\!\left(-i\frac{k_\mu \varepsilon_\nu^+ \bar{\sigma}^{\mu\nu}}{p_1 \cdot \varepsilon^+}\right)|1^a]^{\odot 2s}, \tag{2.11a}$$

$$[2^b 1^a]^{\odot 2s} = \langle 2^b|^{\odot 2s} \exp\!\left(-i\frac{k_\mu \varepsilon_\nu^- \sigma^{\mu\nu}}{p_1 \cdot \varepsilon^-}\right)|1^a\rangle^{\odot 2s}, \tag{2.11b}$$

featuring a tensor-product version of the chiral and antichiral spinorial generators

$$\sigma^{\mu\nu} = \frac{i}{4}[\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu], \qquad \bar{\sigma}^{\mu\nu} = \frac{i}{4}[\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu]. \tag{2.12}$$

In the following we will translate between the spin-operator exponentials (2.11) and the classical-spin exponential (2.6) and thus identify the minimal-coupling amplitudes (2.9) with Kerr black holes. (A complementary identification was done in [90] by matching to the Wilson coefficients in the one-body effective field theory of a Kerr black hole [176, 173].) Such a translation involved sticking to either chiral or antichiral representation, so that one of the amplitudes in eq. (2.9) contains no apparent dependence on the spin operator.

2.3 General integer-spin setup

Let us provide some details on the action on the spin operator on our $su(2)$ states (this section can be skipped if the reader is familiar with such action). Although the minimal-coupling amplitudes (2.9) of [13] are valid for both integer and half-integer spins, for simplicity we will concentrate on the former case. Spin-s polarization tensors with little group indices $\{a_1 \dots a_{2s}\}$ are constructed as [135, 90]

$$\varepsilon_{p\mu_1\dots\mu_s}^{a_1\dots a_{2s}} = \varepsilon_{p\mu_1}^{(a_1 a_2} \dots \varepsilon_{p\mu_s}^{a_{2s-1}a_{2s})}, \qquad \varepsilon_{p\mu}^{ab} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}. \tag{2.13}$$

We adopt the spinor-helicity conventions of [199], so the spin-1 polarization vectors are spacelike and obey the standard properties that are expected from them:

$$p \cdot \varepsilon_p^{ab} = 0, \tag{2.14a}$$

$$\varepsilon_{p\mu}^{ab} \varepsilon_{p\nu ab} = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}, \tag{2.14b}$$

$$\varepsilon_{p11} \cdot \varepsilon_p^{11} = \varepsilon_{p22} \cdot \varepsilon_p^{22} = 2\varepsilon_{p12} \cdot \varepsilon_p^{12} = -1, \tag{2.14c}$$

$$(\varepsilon_{p\mu}^{ab})^* = \varepsilon_{p\mu ab} = \epsilon_{ac}\epsilon_{bd}\varepsilon_{p\mu}^{cd}. \tag{2.14d}$$

In particular, the last line follows from the conjugation rule

$$(\lambda_{p\alpha}^{\;\;a})^* = \operatorname{sgn}(p^0)\tilde{\lambda}_{p\dot{\alpha}a} \quad \Leftrightarrow \quad (\tilde{\lambda}_{p\dot{\alpha}}^{\;\;a})^* = -\operatorname{sgn}(p^0)\lambda_{p\alpha a}, \tag{2.15}$$

which implements the fact that in the little group SU(2) upper and lower indices are related by complex conjugation.

Since the polarization tensors are essentially symmetrized tensor products $\varepsilon_p^{\odot s}$, the action of the Lorentz generators is trivially induced by the vector representation

$$\Sigma^{\mu\nu,\sigma}{}_{\tau} = i[\eta^{\mu\sigma}\delta_\tau^\nu - \eta^{\nu\sigma}\delta_\tau^\mu], \tag{2.16}$$

namely,

$$\begin{aligned}(\Sigma^{\mu\nu})^{\sigma_1\dots\sigma_s}{}_{\tau_1\dots\tau_s} = \Sigma^{\mu\nu,\sigma_1}{}_{\tau_1}\delta_{\tau_2}^{\sigma_2}\dots\delta_{\tau_s}^{\sigma_s} \\ + \dots + \delta_{\tau_1}^{\sigma_1}\dots\delta_{\tau_{s-1}}^{\sigma_{s-1}}\Sigma^{\mu\nu,\sigma_s}{}_{\tau_s}.\end{aligned} \tag{2.17}$$

These matrices realize the Lorentz algebra on the one-particle states of spin s, which are represented by the polarization tensors (2.13).

A more convenient spin quantity to deal with is the Pauli-Lubanski vector given in (2.1). Here $S^{\mu\nu}$ is the spin tensor, the transverse part of which can be reconstructed from the vector as

$$S_\perp^{\mu\nu} = \frac{1}{m}\epsilon^{\mu\nu\rho\sigma}p_\rho S_\sigma \quad \Rightarrow \quad p_\mu S_\perp^{\mu\nu} = 0. \tag{2.18}$$

Understanding eq. (2.1) in the operator sense, we can derive the general form of one-particle matrix element of the Pauli-Lubanski spin operator, here denoted by Σ^μ,

$$\begin{aligned} \varepsilon_p^{\{a\}}\cdot \Sigma^\mu \cdot \varepsilon_p^{\{b\}} = {} & \frac{s(-1)^{s-1}}{2m} \\ & \times\big\{\langle p^{(a_1}|\sigma^\mu|p^{(b_1}] + [p^{(a_1}|\bar{\sigma}^\mu|p^{(b_1}\rangle\big\}\epsilon^{a_2b_2}\dots\epsilon^{a_{2s})b_{2s})}. \end{aligned} \tag{2.19}$$

One way to give meaning to this formula is to lower one set of indices and set it equal to the other: it then produces an expectation value

$$\frac{\varepsilon_{p\{a\}}\cdot \Sigma^\mu \cdot \varepsilon_p^{\{a\}}}{\varepsilon_{p\{a\}}\cdot \varepsilon_p^{\{a\}}} = \begin{cases} s s_p^\mu, & a_1 = \ldots = a_{2s} = 1, \\ (s-1)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+1, \\ (s-2)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+2, \\ \quad\dots \\ -s s_p^\mu, & a_1 = \ldots = a_{2s} = 2, \end{cases} \tag{2.20}$$

where we have also accounted for the nontrivial normalization of the tensors. This shows that spin is quantized in terms of the unit-spin vector

$$s_p^\mu = -\frac{1}{2m}\big\{\langle p_1|\sigma^\mu|p^1] + [p_1|\bar{\sigma}^\mu|p^1\rangle\big\} \tag{2.21}$$

that is transverse and spacelike, $p\cdot s_p = 0$, $s_p^2 = -1$. This vector is familiar from textbook discussions of the Dirac spin, in which context it may be written as $\frac{1}{2m}\bar{u}_{p1}\gamma^\mu\gamma^5 u_p^1 = -\frac{1}{2m}\bar{u}_{p2}\gamma^\mu\gamma^5 u_p^2$. According to eq. (2.20), this vector corresponds to the spin quantization axis and identifies the $(2s+1)$ distinct wavefunctions $\varepsilon_p^{1\ldots12\ldots2}$ with states of definite spin projection.

Moving on towards the scattering context, let us consider a three-point kinematics $p_1 + k = p_2$ shown in figure 2.1. A naive extension of the spin matrix element (2.19), now between states with different momenta of mass m, is

$$\begin{aligned} \varepsilon_2^{\{b\}}\cdot \Sigma^\mu[p_\text{a}]\cdot \varepsilon_1^{\{a\}} = {} & \frac{s}{4m^{2s}}\big\{\langle 1^a|\sigma^\mu|2^b] + [1^a|\bar{\sigma}^\mu|2^b\rangle\big\} \\ & \odot\big\{\langle 1^a2^b\rangle - [1^a2^b]\big\}\odot\langle 1^a2^b\rangle^{\odot(s-1)}\odot[1^a2^b]^{\odot(s-1)}. \end{aligned} \tag{2.22}$$

Here we have encoded the symmetrization of the little-group indices into the modified tensor-product symbol $\odot$, and the indices on the right-hand side should be regarded as abstract placeholders. It is important to stress that the symmetrization encoded in the symbol $\odot$ only acts inside the two SU(2)-index sets $\{a\}$ and $\{b\}$ separately, as symmetrizing a little-group index of momentum p_1 with that of p_2 would be mathematically inconsistent.

Notice that in eq. (2.22) we must specify that the Pauli-Lubanski operator is defined with respect to the average momentum $p_{\rm a} = (p_1 + p_2)/2$. It is this momentum that we will associate with the classical momentum $p_{\rm a}^\mu = m u_{\rm a}^\mu$ of one of the incoming black holes, so it makes sense to define a spin vector to be orthogonal to it. Eq. (2.22) treats the chiral and antichiral spinors on an equal footing. However, even in the spin-1 case the angular-momentum exponentiation (2.11), present in the exclusively chiral and antichiral spinorial representations, is opaque at the level of such a matrix element. The reason for that is physically important. As discussed in [176], a consistent picture of spin-induced multipoles of a pointlike particle must be formulated in the particle's rest frame, in which the spin does not precess [245]. Therefore, the formula (2.22) is too naive, as it involves a spin operator defined for momentum $p_{\rm a}$ but acts with it on the states with momenta $p_{\rm a} \pm k/2$. The cure for that is to take into account additional Lorentz boosts, which we will now proceed to do.

2.4 Angular-momentum exponentiation

In the previous chapter we have seen that all spin multipoles of the amplitude can be extracted through a finite Lorentz boost. This boost is needed to bridge the gap between two states with different momenta. In this way, the quantum picture is made consistent with the classical notion of spin-induced multipoles of a pointlike object on a worldline [176]. Here we introduce such a construction in terms of the spinor-helicity variables. Our main result is the form (2.35) below, from which we will extract all powers of spin. Its equivalence to the covariant formalism of the previous chapter is explained in the Appendix B.

To start, we again note that any two four-vectors p_1 and p_2 of equal mass m may be related by

$$p_2^\rho = \exp\!\big(i\mu_{12} p_1^\mu p_2^\nu \Sigma_{\mu\nu}\big)^\rho{}_\sigma p_1^\sigma, \tag{2.23}$$

where we have used the generators (2.16). The numeric prefactor in the exponent is explicitly

$$\mu_{12} = \frac{\log\!\big[\frac{1}{m^2}\big(p_1\cdot p_2+\sqrt{(p_1\cdot p_2)^2-m^2}\big)\big]}{\sqrt{(p_1\cdot p_2)^2-m^2}} = \frac{1}{m^2}+\mathcal{O}(k^2). \tag{2.24}$$

Here we are only interested in the strictly on-shell setup, for which $k^2=(p_2-p_1)^2=0$. The corresponding spinorial transformations are

$$|2^b\rangle = U_{12}{}^b{}_a \exp\!\bigg(\frac{i}{m^2}p_1^\mu k^\nu \sigma_{\mu\nu}\bigg)|1^a\rangle, \tag{2.25a}$$

$$|2^b] = U_{12}{}^b{}_a \exp\!\bigg(\frac{i}{m^2}p_1^\mu k^\nu \bar{\sigma}_{\mu\nu}\bigg)|1^a], \tag{2.25b}$$

where $U_{12}\in \mathrm{SU}(2)$ is a little-group transformation that depends on the specifics of the massive-spinor realization. The duality properties of the spinorial generators (2.12) allow us to easily rewrite the above exponents as

$$\frac{i}{m^2}p_1^\mu k^\nu \sigma_{\mu\nu,\alpha}{}^\beta = k\cdot a_\alpha{}^\beta, \qquad \frac{i}{m^2}p_1^\mu k^\nu \bar{\sigma}_{\mu\nu,}{}^{\dot\alpha}{}_{\dot\beta} = -k\cdot a^{\dot\alpha}{}_{\dot\beta}, \tag{2.26}$$

where we have defined chiral representations for the Pauli-Lubanski operators

$$a^{\mu,}{}_\alpha{}^\beta = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma}p_{\mathrm{a}\nu}\sigma_{\rho\sigma,\alpha}{}^\beta, \tag{2.27a}$$

$$a^{\mu,\dot\alpha}{}_{\dot\beta} = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma}p_{\mathrm{a}\nu}\bar{\sigma}_{\rho\sigma,}{}^{\dot\alpha}{}_{\dot\beta}. \tag{2.27b}$$

(Note that the product $k\cdot a$ is insensitive to the difference between p_1 and $p_{\mathrm{a}}=p_1+k/2$ in the above definitions, so we could pick the latter for further convenience.) Extension to the higher-spin states represented by tensor products of massive spinors is analogous to eq. (2.17), e.g.

$$\begin{aligned}(a^\mu)_{\alpha_1\ldots\alpha_{2s}}{}^{\beta_1\ldots\beta_{2s}} &= a^{\mu,}{}_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\ldots\delta_{\alpha_{2s}}^{\beta_{2s}} \\ &\quad +\ldots+\delta_{\alpha_1}^{\beta_1}\ldots\delta_{\alpha_{2s-1}}^{\beta_{2s-1}}a^{\mu,}{}_{\alpha_{2s}}{}^{\beta_{2s}},\end{aligned} \tag{2.28}$$

so we have

$$\begin{aligned} |2\rangle^{\odot 2s} &= e^{k\cdot a}\big\{U_{12}|1\rangle\big\}^{\odot 2s}, & |2]^{\odot 2s} &= e^{-k\cdot a}\big\{U_{12}|1]\big\}^{\odot 2s}, \\ \langle 2|^{\odot 2s} &= \big\{U_{12}\langle 1|\big\}^{\odot 2s}e^{-k\cdot a}, & [2|^{\odot 2s} &= \big\{U_{12}[1|\big\}^{\odot 2s}e^{k\cdot a}, \end{aligned} \tag{2.29}$$

where the second line follows from the antisymmetry of $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ in the sense of $\epsilon^{\alpha\beta}\sigma^{\mu\nu,}{}_{\beta}{}^{\gamma}\epsilon_{\gamma\delta} = -\sigma^{\mu\nu,}{}_{\delta}{}^{\alpha}$.

Let us now inspect the spin dependence of the three-point amplitudes. In [135] we used their representation (2.11) for that. In terms of the same Pauli-Lubanski operators (2.27), they can be rewritten in a simpler form:

$$\mathcal{M}_3^{(s)}(k^+) = -\frac{\kappa x^2}{2m^{2s-2}}[2|^{\odot 2s} e^{-2k\cdot a}|1]^{\odot 2s}, \tag{2.30a}$$

$$\mathcal{M}_3^{(s)}(k^-) = -\frac{\kappa x^{-2}}{2m^{2s-2}}\langle 2|^{\odot 2s} e^{2k\cdot a}|1\rangle^{\odot 2s}, \tag{2.30b}$$

which can be derived from eq. (2.9) using the identities

$$[1^a k] = x\langle 1^a k\rangle, \qquad [2^b k] = x\langle 2^b k\rangle, \tag{2.31a}$$

$$[1^a 2^b] = -\langle 1^a 2^b\rangle + \frac{x}{m}\langle 1^a k\rangle\langle k\, 2^b\rangle. \tag{2.31b}$$

The apparent spin dependence in the amplitude formulae above is of the form $e^{\mp 2k\cdot a}$, whereas there seems to be no such dependence in the original formulae (2.9). This apparent contradiction is resolved by taking into account the transformations (2.25): the true angular-momentum dependence inherent to the minimal-coupling amplitudes is independent of the spinorial basis. (Indeed, it must also match the covariant formula (E.3).) For example, the plus-helicity amplitude (2.9a) involves $\langle 12\rangle^{\odot 2s}$, which in the chiral representation is simply

$$\langle 21\rangle^{\odot 2s} = \left\{U_{12}\langle 1|\right\}^{\odot 2s} e^{-k\cdot a}|1\rangle^{\odot 2s}, \tag{2.32}$$

whereas in the antichiral representation it is

$$[2|^{\odot 2s} e^{-2k\cdot a}|1]^{\odot 2s} = \left\{U_{12}[1|\right\}^{\odot 2s} e^{-k\cdot a}|1]^{\odot 2s}. \tag{2.33}$$

As pointed out in the Appendix, appendix E it is now natural to strip the spin-states to cleanly obtain the spin dependence. Alternatively, in the classical (and arbitrary-spin) limit we should treat the operator in-between as a C-number. In that case, both expressions above become unambiguously

$$\lim_{s\to\infty} m^{2s}(U_{12})^{\odot 2s} e^{-k\cdot a}. \tag{2.34}$$

The factor of m^{2s} cancels in the actual amplitudes:

$$\mathcal{M}_3^{(\infty)}(k^+) \approx -\frac{\kappa}{2} m^2 x^2 e^{-k\cdot a} \lim_{s\to\infty} (U_{12})^{\odot 2s}, \tag{2.35a}$$

$$\mathcal{M}_3^{(\infty)}(k^-) \approx -\frac{\kappa}{2} m^2 x^{-2} e^{k\cdot a} \lim_{s\to\infty} (U_{12})^{\odot 2s}. \tag{2.35b}$$

The remaining unitary factor of $(U_{12})^{\odot 2s}$ parametrizes an arbitrary little-group transformation that corresponds to the choice of the spin quantization axis (2.21). As such, it is inherently quantum-mechanical and therefore should be removed in the classical limit. Indeed, it also appears in the simple product of polarization tensors

$$\begin{aligned}
\lim_{s\to\infty} \varepsilon_2 \cdot \varepsilon_1 &= \lim_{s\to\infty} \frac{1}{m^{2s}} \langle 21\rangle^{\odot s} \odot [21]^{\odot s} \\
&= \lim_{s\to\infty} \frac{1}{m^{2s}} \big\{U_{12}\langle 1|\big\}^{\odot s} e^{-k\cdot a} |1\rangle^{\odot s} \\
&\qquad \times \big\{U_{12}[1|\big\}^{\odot s} e^{k\cdot a} |1]^{\odot s} = \lim_{s\to\infty} (U_{12})^{\odot 2s},
\end{aligned} \tag{2.36}$$

where a is defined by eq. (2.28) but with half as many slots. So we interpret the factor of $(U_{12})^{\odot 2s}$ as the state normalization in accord with the notion of GEV, to be introduced in the following chapter.

2.5 Four-point amplitude

Finally, we are now ready to compute the four-point amplitude that contains the complete information about classical 1PM scattering of two spinning black holes, with masses m_a and m_b. To introduce the spin deflection Δa^μ in an analogous footing to Δp^μ, we first recast here formula (1.26) in the notation of [170],

$$\Delta p_\text{a}^\mu = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_\text{a}\cdot k) \hat{\delta}(2p_\text{b}\cdot k) k^\mu e^{-ik\cdot b/\hbar} i\mathcal{M}_4(k) \right\rangle\!\!\right\rangle. \tag{2.37}$$

The angle-bracket notation involves a careful analysis of suitable wavefunctions $\psi_{\text{a,b}}$ and powers of $\hbar$, and as we argued in the last chapter amounts to setting the momenta to their classical values as follows

$$k^\mu = \hbar \bar{k}^\mu \to 0, \qquad p_1^\mu, p_2^\mu \to m_\text{a} u_\text{a}^\mu, \qquad p_3^\mu, p_4^\mu \to m_\text{b} u_\text{b}^\mu. \tag{2.38}$$

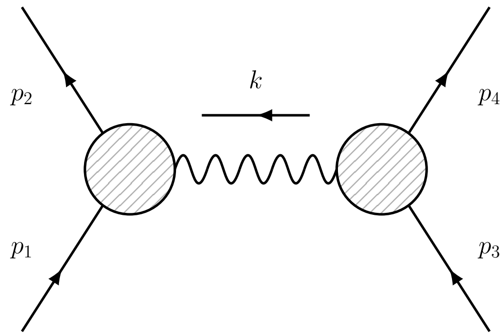


Figure 2.2: Four-point amplitude for elastic scattering of two distinct massive particles

First of all, we note that in the quantum-mechanical setting of [170] both p_1 and p_2 are associated with the momentum of the first incoming black hole. This is consistent with the equitable identification

$$p_\mathrm{a} = (p_1 + p_2)/2, \qquad p_\mathrm{b} = (p_3 + p_4)/2, \tag{2.39}$$

that we will follow. Moreover, the classical limit (2.38) prescribes inspecting soft-graviton exchange in the $t = k^2$ channel, in which the graviton's momentum is taken to zero uniformly. Here, however, we are going to adhere to an alternative strategy we will name *Holomorphic Classical Limit* (further detailed in the next chapter): We compute the residue of the scattering amplitude on the pole at $t = 0$ on finite complex kinematics and analytically continue the result to real kinematics at a later stage. As shown in figure 2.2, the four-point amplitude then conveniently factorizes into two three-point ones:

$$\begin{aligned}
&\mathcal{M}_4^{(s_\mathrm{a},s_\mathrm{b})}(p_1,-p_2,p_3,-p_4) \\
&= \frac{-1}{t}\sum_\pm \mathcal{M}_3^{(s_\mathrm{a})}(p_1,-p_2,k^\pm)\mathcal{M}_3^{(s_\mathrm{b})}(p_3,-p_4,-k^\mp) + \mathcal{O}(t^0) \\
&= \frac{-(\kappa/2)^2}{m_\mathrm{a}^{2s_\mathrm{a}-2}m_\mathrm{b}^{2s_\mathrm{b}-2}t}\Big(x_\mathrm{a}^2 x_\mathrm{b}^{-2}\langle 21\rangle^{\odot 2s_\mathrm{a}}[43]^{\odot 2s_\mathrm{b}} \\
&\qquad\qquad + x_\mathrm{a}^{-2}x_\mathrm{b}^2[21]^{\odot 2s_\mathrm{a}}\langle 43\rangle^{\odot 2s_\mathrm{b}}\Big) + \mathcal{O}(t^0),
\end{aligned} \tag{2.40}$$

of which we now have complete understanding.

The helicity factors enter the above amplitude in simple combinations evaluated on the pole kinematics as

$$x_{\rm a}/x_{\rm b} = \gamma(1-v), \qquad x_{\rm b}/x_{\rm a} = \gamma(1+v), \tag{2.41}$$

where we have introduced the following interchangeable variables that describe the total energy of the black-hole scattering process:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{p_{\rm a}\cdot p_{\rm b}}{m_{\rm a}m_{\rm b}} = u_{\rm a}\cdot u_{\rm b}. \tag{2.42}$$

Evaluating the spin-dependent terms using eq. (2.29) and taking into account the direction of k^μ, we get

$$\begin{aligned}
\mathcal{M}_4^{(s_{\rm a},s_{\rm b})} =& \frac{-(\kappa/2)^2\gamma^2}{m_{\rm a}^{2s_{\rm a}-2}m_{\rm b}^{2s_{\rm b}-2}t} \\
\times\Big(&(1-v)^2\big\{U_{12}\langle 1|\big\}^{\odot 2s_{\rm a}}\exp(-k\cdot a_{\rm a})|1\rangle^{\odot 2s_{\rm a}} \\
&\qquad\times\big\{U_{34}[3|\big\}^{\odot 2s_{\rm b}}\exp(-k\cdot a_{\rm b})|3]^{\odot 2s_{\rm b}} \\
&+(1+v)^2\big\{U_{12}[1|\big\}^{\odot 2s_{\rm a}}\exp(k\cdot a_{\rm a})|1]^{\odot 2s_{\rm a}} \\
&\qquad\times\big\{U_{34}\langle 3|\big\}^{\odot 2s_{\rm b}}\exp(k\cdot a_{\rm b})|3\rangle^{\odot 2s_{\rm b}}\Big) + \mathcal{O}(t^0).
\end{aligned} \tag{2.43}$$

It is straightforward to check that the same result is obtained if we choose to Lorentz-transform the states symmetrically to their averages: $p_1, p_2 \to p_{\rm a}$ and $p_3, p_4 \to p_{\rm b}$.

Before we take the classical limit, we should note that the above contractions of the Pauli-Lubanski pseudovector are parity-odd. To obtain a parity even expression, we observe that on the pole kinematics $k^2 = 0$ the Levi-Civita spin contractions satisfy

$$i\epsilon_{\mu\nu\rho\sigma}p_{\rm a}^\mu p_{\rm b}^\nu k^\rho a_{\rm a}^\sigma = m_{\rm a}m_{\rm b}\gamma v(k\cdot a_{\rm a}), \tag{2.44a}$$
$$i\epsilon_{\mu\nu\rho\sigma}p_{\rm a}^\mu p_{\rm b}^\nu k^\rho a_{\rm b}^\sigma = m_{\rm a}m_{\rm b}\gamma v(k\cdot a_{\rm b}). \tag{2.44b}$$

These equalities can be derived by squaring the left-hand sides and computing the resulting Gram determinants using that $k^2 = p_{\rm a}\cdot k = p_{\rm b}\cdot k = p_{\rm a}\cdot a_{\rm a} = p_{\rm b}\cdot a_{\rm b} = 0$. Therefore, introducing a two-form constructed from two initial BH momenta

$$w^{\mu\nu} = \frac{2p_{\rm a}^{[\mu}p_{\rm b}^{\nu]}}{m_{\rm a}m_{\rm b}\gamma v}, \qquad [w*a]_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}w^{\alpha\beta}a^\nu, \tag{2.45}$$

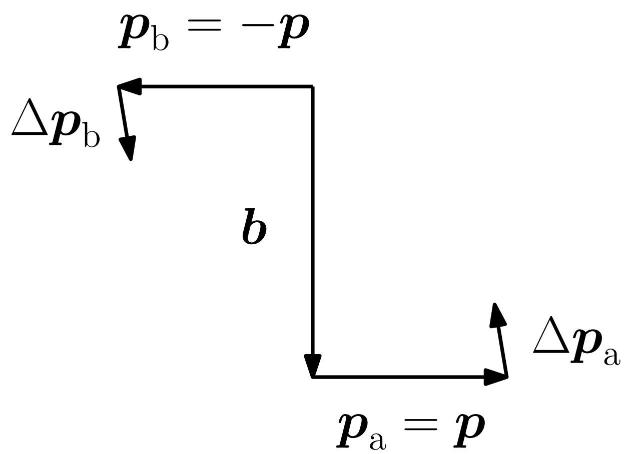


Figure 2.3: The BH three-momenta in the center-of-mass frame and the impact parameter between them.

and stripping the unitary transition factors $U_{12}^{\odot 2s_{\mathrm{a}}}$ and $U_{34}^{\odot 2s_{\mathrm{b}}}$ we obtain the classical limit of the scattering amplitude (2.43) as

$$\langle \mathcal{M}_4(k) \rangle = -\Big(\frac{\kappa}{2}\Big)^2 \frac{m_{\mathrm{a}}^2 m_{\mathrm{b}}^2}{k^2} \gamma^2 \sum_{\pm} (1 \pm v)^2 \exp[\pm i(k \cdot w * a_0)], \tag{2.46}$$

where $a_0^\mu = a_{\mathrm{a}}^\mu + a_{\mathrm{b}}^\mu$ is the total spin pseudovector. Note that from now on we consider the above expression to be valid for any values of transfer momentum momentum k.

As suggested by eq. (2.37) and the scattering-angle formula [39], in the classical picture we consider the transfer momentum k as a Fourier variable dual to the impact parameter b, which is a spacelike vector orthogonal to both of initial momenta, $b \cdot p_{\mathrm{a}} = b \cdot p_{\mathrm{b}} = 0$. Therefore, we define the scattering function

$$\langle \mathcal{M}_4(b) \rangle = \int \hat{d}^4 k \, \hat{\delta}(2p_{\mathrm{a}} \cdot k) \hat{\delta}(2p_{\mathrm{b}} \cdot k) e^{-ik\cdot b} \langle \mathcal{M}_4(k) \rangle. \tag{2.47}$$

The above Fourier transform is easiest performed in the center-of-mass (COM) frame, where $p_{\mathrm{a}} = (E_{\mathrm{a}}, \boldsymbol{p})$ and $p_{\mathrm{b}} = (E_{\mathrm{b}}, -\boldsymbol{p})$, see figure 2.3. In this frame the eikonal integration measure [39] becomes explicitly

$$\int \hat{d}^4 k \, \hat{\delta}(2p_{\mathrm{a}} \cdot k) \hat{\delta}(2p_{\mathrm{b}} \cdot k) e^{-ik\cdot b} \tag{2.48}$$

$$\overset{\text{COM frame}}{=} \frac{1}{4(E_{\rm a}+E_{\rm b})|\boldsymbol{p}|} \int \hat{d}^2\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{b}} \bigg|_{k^0=\boldsymbol{p}\cdot\boldsymbol{k}=0} \tag{2.49}$$

In other words, the integration is strictly spacelike and restricted by $\boldsymbol{p}\cdot\boldsymbol{k}=0$ to the two-dimensional subspace orthogonal to the initial momenta, which is the same subspace where the impact parameter is defined. Using $(E_{\rm a}+E_{\rm b})|\boldsymbol{p}| = m_{\rm a}m_{\rm b}\gamma v$, we compute

$$\begin{aligned} \langle \mathcal{M}_4(\boldsymbol{b})\rangle &= \frac{\kappa^2 m_{\rm a}m_{\rm b}\gamma}{16v} \sum_{\pm}(1\pm v)^2 \\ &\times \int \frac{d^2\boldsymbol{k}}{(2\pi)^2\boldsymbol{k}^2} \exp\!\big[i\boldsymbol{k}\cdot\big(\boldsymbol{b}\mp[w*a_0]_{2d}\big)\big]\Big|_{\boldsymbol{p}\cdot\boldsymbol{k}=0} \\ &= -\frac{\kappa^2 m_{\rm a}m_{\rm b}\gamma}{32\pi v} \sum_{\pm}(1\pm v)^2 \log\big|\boldsymbol{b}\mp[w*a_0]_{2d}\big|. \end{aligned} \tag{2.50}$$

Here by $[w*a_0]_{2d}$ we have denoted the appropriate spacelike projection of the four-vector $w*a_0$. However, recall that

$$[w*a_0]^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}a_{0\nu}p_{{\rm a}\rho}p_{{\rm b}\sigma}}{m_{\rm a}m_{\rm b}\gamma v}, \tag{2.51}$$

i.e. the vector $w*a_0$ is transverse to $p_{\rm a}$ and $p_{\rm b}$ and hence lies in the same plane as $\boldsymbol{k}$ and $\boldsymbol{b}$. Therefore, no information is lost in the two-dimensional projection above, so we can safely uplift the scattering function (2.50) to its Lorentz-invariant form

$$\langle \mathcal{M}_4(b)\rangle = -Gm_{\rm a}m_{\rm b}\frac{\gamma}{v}\sum_{\pm}(1\pm v)^2 \log\sqrt{-(b\mp w*a_0)^2}. \tag{2.52}$$

2.6 Linear and angular impulses

In this section we relate the scattering function (2.52) in the impact-parameter space to the classical changes in linear and angular momentum of a BH after gravitational scattering off another BH. This problem was treated to all orders in spins at 1PM order in [238], producing the result (rewritten in the mostly-minus metric convention)

$$\Delta p_{\rm a}^\mu = Gm_{\rm a}m_{\rm b}\Re Z^\mu, \tag{2.53a}$$

$$\Delta a_{\rm a}^\mu = -\frac{Gm_{\rm b}}{m_{\rm a}}\big[p_{\rm a}^\mu(a_{\rm a}\cdot\Re Z) + \epsilon^{\mu\nu\rho\sigma}(\Im Z_\nu)p_{{\rm a}\rho}a_{{\rm a}\sigma}\big], \tag{2.53b}$$

in terms of an auxiliary complex vector

$$Z^\mu = \frac{\gamma}{v} \sum_{\pm} \big(1 \pm v\big)^2 [\eta^{\mu\nu} \mp i(*w)^{\mu\nu}] \frac{(b \mp w * a_0)_\nu}{(b \mp w * a_0)^2}. \tag{2.54}$$

Now we can observe that differentiating the scattering function (2.52) automatically produces its real and imaginary parts:

$$\frac{\partial}{\partial b^\mu} \langle \mathcal{M}_4(b) \rangle = -G m_{\rm a} m_{\rm b} \Re Z_\mu, \tag{2.55a}$$
$$\frac{\partial}{\partial a^\mu} \langle \mathcal{M}_4(b) \rangle = G m_{\rm a} m_{\rm b} \Im Z_\mu. \tag{2.55b}$$

Using the known solution (2.53), we can identify

$$\begin{aligned} \Delta p_{\rm a}^\mu &= -\frac{\partial}{\partial b_\mu} \langle \mathcal{M}_4(b) \rangle, \\ \Delta a_{\rm a}^\mu &= \frac{1}{m_{\rm a}^2} \left[p_{\rm a}^\mu a_{\rm a}^\nu \frac{\partial}{\partial b^\nu} - \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\nu} a_{{\rm a}\rho} \frac{\partial}{\partial a_{\rm a}^\sigma} \right] \langle \mathcal{M}_4(b) \rangle. \end{aligned} \tag{2.56}$$

At this point, we have merely matched the derivatives of our scattering function (2.52) to the known result (2.53). Let us now promote this empirical matching to a derivation, under the assumption that our approach is consistent with that of [170].

Indeed, the first line of eq. (2.56) gives precisely the impulse formula (2.37) from [170]. So let us focus on the second line. In [193], Maybee, O'Connell and Vines have extended the classical-limit approach of [170] to include corrections in spin. Their starting point for (the expectation value of) the lowest-order angular impulse is

$$\begin{aligned} \Delta S_{\rm a}^\mu = \Bigg\langle\!\!\Bigg\langle \int \hat{d}^4 k \, \hat{\delta}(2p_{\rm a} \cdot k) \hat{\delta}(2p_{\rm b} \cdot k) e^{-ik\cdot b} \\ \times \left(-\frac{i}{m_{\rm a}^2} p_{\rm a}^\mu S_{\rm a}^\nu k_\nu \mathcal{M}_4(k) + \big[S_{\rm a}^\mu, i\mathcal{M}_4(k)\big] \right) \Bigg\rangle\!\!\Bigg\rangle. \end{aligned} \tag{2.57}$$

Here the amplitude is considered to be a function of a one-particle spin vector acting on the space of physical spin degrees of freedom, i.e. the little-group indices. Therefore, we interpret it as the matrix element

$$(S_{\rm a}^\mu)_{\{a\}}{}^{\{b\}} = (-1)^s \varepsilon_{{\rm a}\{a_1 \dots a_{2s}\}} \cdot \Sigma^\mu \cdot \varepsilon_{\rm a}^{\{b_1 \dots b_{2s}\}}, \tag{2.58}$$

where the prefactor of $(-1)^s$ is due to the spacelike normalization of the polarization vectors in eq. (2.14c). The explicit form for such a spin vector at finite spin is given in eq. (2.19). It corresponds to the generator of the little-group transformations: as an operator it satisfies the so(3) algebra in the rest frame of $p_{\rm a}$. This can also be stated covariantly as

$$[S_{\rm a}^{\mu}, S_{\rm a}^{\nu}] = \frac{i}{m_{\rm a}}\epsilon^{\mu\nu\rho\sigma}p_{{\rm a}\rho}S_{{\rm a}\sigma}. \tag{2.59}$$

As $\mathcal{M}_4$ is a function of $S_{\rm a}^{\mu}$, these so(3) rotations act as

$$[S_{\rm a}^{\mu}, \mathcal{M}_4] = \frac{i}{m_{\rm a}}\epsilon^{\mu\nu\rho\sigma}p_{{\rm a}\nu}S_{{\rm a}\rho}\frac{\partial \mathcal{M}_4}{\partial S_{\rm a}^{\sigma}}. \tag{2.60}$$

Therefore, we obtain the formula for the change in rescaled spin

$$\begin{aligned}\Delta a_{\rm a}^{\mu} = \frac{1}{m_{\rm a}^2}\bigg\langle\!\!\bigg\langle \int \hat{d}^4k\,\hat{\delta}(2p_{\rm a}\cdot k)\hat{\delta}(2p_{\rm b}\cdot k)e^{-ik\cdot b} \\ \times\bigg[p_{\rm a}^{\mu}a_{\rm a}^{\nu}(-ik_{\nu}) - \epsilon^{\mu\nu\rho\sigma}p_{\nu}a_{{\rm a}\rho}\frac{\partial}{\partial a_{\rm a}^{\sigma}}\bigg]\mathcal{M}_4(k)\bigg\rangle\!\!\bigg\rangle,\end{aligned} \tag{2.61}$$

which maps directly to the second line of eq. (2.56). Now that the impulse formulae (2.56) have meaning on their own, we see that plugging the scattering function (2.52) gives a novel derivation for the complete 1PM solution (2.53).

2.7 Summary and Discussion

We have obtained the dynamically complete solution to the (net) problem of conservative spinning black-hole scattering at 1PM order as given in [238], using minimal-coupling scattering amplitudes with arbitrarily large quantum spin [13]. We have rederived the spin-exponentiated structure of such three-point amplitudes in four dimensions in a way that takes into account the Lorentz boost between the incoming and outgoing momenta as in the previous chapter. Here we have shown that considering this boost in terms of spinor-helicity variables streamlines the discussion of the spin exponentiation, as well as allows for a cleaner connection to the classical notion of spin in general relativity [176].

We have computed the change of the momentum and spin of the scattered black holes at 1PM order using a four-point one-graviton-exchange amplitude, which in our holomorphic classical limit $t \to 0$ is factorized into two three-point minimal-coupling amplitudes. We have adopted the formulae of [170, 193] in a way that allowed us to extract the full spin dependence of the linear and angular impulses of the black holes. In this way, we obtained a complete match to the known solution 1PM solution [238], which allows for spins of the incoming black holes in arbitrary directions. It is promising that our calculation displayed a sufficiently uniform level of complexity all the way between the starting point and the final result, even despite the more complex nature of the quantum degrees of freedom. This is thanks to the remarkable fact that the spin multipoles of a black hole exponentiate [139, 238], which we could exploit and thus avoid explicit multipole expansions.

There are several interesting future directions to the treatment presented in this chapter. One of the most relevant ones is the extension to higher loop, or PM, orders which may require to include radiative corrections as in the previous chapter, or also and finite-size effects [37, 175, 82, 76]. It is also interesting to explore the test-body limit [238, 240] to improve our understanding of the effective potential in the sense of [7]. Furthermore, as elucidated on the previous chapter, it may prove beneficial to gain insight from the double-copy framework.[1]

We have paved the way to higher-order calculations for the spin effects in classical black-hole scattering using the modern amplitude techniques, namely, using an on-shell and gauge-invariant framework. In the next chapters we will mainly address the 1-loop or 2PM extension of this setup.

[1] Recent progress in the double-copy construction applied to the binary-inspiral problem can be found in [127, 130, 180, 228, 209, 20, 207]

Chapter 3

The Effective Two-Body Potential

3.1 Introduction

So far we have obtained 1PM classical observables to all orders in spin. This recovers previous results from GR and provides a justification for our effective description of spinning black holes. However, we would like to push the approach further and obtain new results that would otherwise be difficult to derive from purely classical methods. In this direction, we now initiate the treatment of loop corrections in the classical limit of the amplitudes for spinning black holes. To introduce this aspect we will first study the conservative effective potential between the two bodies. We then come back to the problem of the (2PM) scattering angle in the next section.

The effective potential has been long studied in the Post-Newtonian (PN) framework, which is obtained from our Post-Minkowskian approach by further expanding in the average velocity as explained in table 2, i.e. while assuming the Virial constraint $v \sim Gm/r$. As we have mentioned, even though the potential is computed in a scattering setup it can be applied to bounded (circular or quasicircular) systems since the equations of motion are the same. The condition $v \sim Gm/r$ can be safely applied to the early stages of the inspiral orbit [49], hence our computation could eventually lead to better analytic understanding of such stage.

For spinning objects, one of the distinctive characteristics of the PN expansion is the treatment of the binary system as localized sources endowed with a tower of multipole moments. The evaluation of higher multipole moments starting at 1.5PN requires to incorporate spin into the massive particles involved in the scattering process

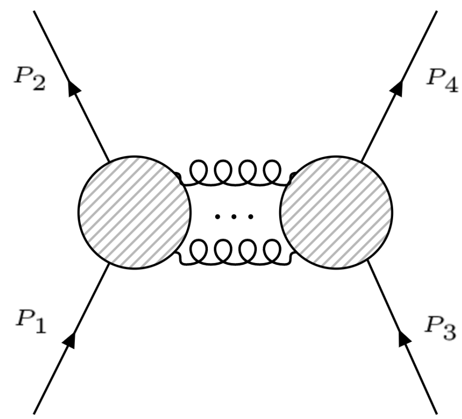


Figure 3.1: Two massive particles represented by $P_1^2 = P_2^2 = m_a^2$ and $P_3^2 = P_4^2 = m_b^2$ exchange several gravitons. The momentum transfer is given by $K = P_1 - P_2 = (0, \vec{q})$ in the COM frame.

[212, 213, 237], along with radiative corrections. These spin contributions account for the internal angular momentum of the objects in the macroscopic setting [19, 237]. The universality of the gravitational coupling implies that it is enough to consider massive particles of spin S to evaluate the spin multipole effects up to order $2S$ in the spin vector $|\vec{S}|$. The computation of the potential was first done up to 1-loop by Holstein and Ross in [156] and then by Vaidya in [237], leading to $|\vec{S}|^2$ and $|\vec{S}|^4$ results, respectively. The electromagnetic counterpart has also been discussed up to $|\vec{S}|^2$ [154]. As we have seen, higher spin multipole moments are characterized by containing higher powers of the momentum transfer $|\vec{q}|$ and $|\vec{S}|$, arising in the classical combination $|\vec{q}||\vec{S}|$. Thus, in order to evaluate classical spin effects an expansion of the amplitude to arbitrarily subleading orders in $|\vec{q}| = \sqrt{-t}$ is required. This, together with the natural increase in difficulty for manipulating higher spin degrees of freedom in loop QFT processes [156], renders the computation virtually doable only within the framework of intrinsic non-relativistic approaches along with the aid of a computer for higher PN orders [217, 169, 174, 179].

In this chapter we find that the combination of several new methods can bypass some of the aforementioned difficulties. We provide fully relativistic formulas leading to the classical potential of the amplitude valid for any spin at both tree and 1-loop level. The difficulty in extracting arbitrarily subleading momentum powers is avoided by noting that the $t \to 0$ and $|\vec{q}| \to 0$ expansions can be disentangled outside the COM frame. That is, we evaluate the classical piece in a covariant way by selecting the *leading* order in the limit $t \to 0$, which we approach by using complexified momenta. We find that the multipole terms are fully visible at leading order, and propose Lorentz covariant expressions for them in terms of the momentum transfer K^μ. These expressions can then be analytically extended to the COM frame by putting $K = (0, \vec{q})$. This is what we call the *holomorphic classical limit* (HCL).

To bypass the intrinsic complications due to the evaluation of higher spin loop processes we draw upon a battery of modern techniques based on the analytic structure of scattering amplitudes. In fact, techniques such as spinor helicity formalism, on-shell recursion relations (BCFW), and unitarity cuts have proven extremely fruitful for both computations of gravity and gauge theory amplitudes [32, 31, 251, 73, 26, 196]. In this context, several simplifications in the computation of the 1-loop potential have already been found for scalar particles in [197, 44, 151]. Pioneered by the work of Bjerrum-Bohr et al. [41] these methods were applied to the light-bending case [45, 42, 15, 152, 58], where one of the external particle carries helicity $|h| \in \{0, \frac{1}{2}, 1\}$, and universality with respect to $|h|$ was found. Here we extend these approaches by considering two more techniques, both very recently developed as a natural evolution of the previously mentioned. The first one appeared in [63], where Cachazo and the author proposed to use a generalized form of unitarity cuts, known as the Leading Singularity (LS), in order to extract the classical part of gravitational amplitudes leading to the effective potential. It was shown that while at tree level this simply corresponds to computing the t channel residue, at 1-loop the LS associated to the triangle diagram leads to a fully relativistic form containing the 1PN correction for scalar particles, through a multidispersive treatment in the t channel. The second technique is the multipole expansion developed in the previous chapters, which gives a representation and gives a representation for *massive* states of arbitrary spin completely built from spinor helicity variables. Hence we use such construction to compute the LS associated to both the gravitational and electromagnetic triangle diagram as well as the respective tree level residues, this time including higher spin in the external particles. The combination of these techniques with the HCL leads to a direct evaluation of the 1-loop correction to the classical piece. The result is expressed in a compact and covariant manner in terms of spinor helicity operators, which are then matched to the standard spin operators of the EFT. As a crosscheck we recover the results for both gravity and EM presented in [156, 154, 197, 237] for $S \leq 1$. As a bypass, by suitably defining the massless limit, we are also able to address the light-like scattering situation and check the proposed universality of light bending phenomena.

This chapter is organized as follows. In section 3.2 we review the kinematics and spin considerations associated to the $2 \to 2$ process, which motivates the holomorphic classical limit. We then proceed to give a short overview of the notation and conventions used along the work, specifically those regarding manipulations of spinor helicity variables. Next, in section 3.3 we review scalar scattering and implement the HCL to extract the electromagnetic and gravitational classical part from leading singularities at tree and 1-loop level, including the light bending case. Next, in section 3.4 we introduce the new spinor helicity representation for massive kinematics, leaving the details to Appendix C, and use it to extend the previous computations to spinning particles. In section 3.5 we discuss the applications of these results as well as possible future directions. Finally, in Appendix D we provide a prescription to match our results to the standard form of EFT operators appearing in the effective potential for the cases $S = \frac{1}{2}, 1$.

3.2 Preliminaries

3.2.1 Kinematical Considerations and the HCL

In the EFT framework, the off-shell effective potential can be extracted from the S-matrix element associated to the process depicted in Fig. 3.1, see e.g. [197]. The standard kinematical setup for this computation is given by the Center of Mass (COM) coordinates, which are defined by $\vec{P}_1 + \vec{P}_3 = 0$. We can check that 4-particle kinematics for this setup imply

$$(P_1 + P_3) \cdot (P_1 - P_2) = 0 \,, \tag{3.1}$$

which means that the momentum transfer vector $K := (P_1 - P_2)$ has the form

$$K = (0, \vec{q}) \,, \quad t = K^2 = -\vec{q}^{\,2} \,, \tag{3.2}$$

in the COM frame. For completeness, we also define here the average momentum $\vec{p}$ as

$$\frac{P_1 + P_2}{2} = (E_a, \vec{p}) \,, \quad \frac{P_3 + P_4}{2} = (E_b, -\vec{p}) \,, \tag{3.3}$$

where E_a, E_b are the respective energies for the COM frame, while $\vec{p}^{\,2} \propto v^2$ gives the characteristic velocity of the problem. From these definitions we can solve for the explicit form of the momenta P_i, $i \in \{1, 2, 3, 4\}$, and also easily check the transverse condition $\vec{p} \cdot \vec{q} = 0$. In the non-relativistic limit $\frac{\sqrt{-t}}{m} = \frac{|\vec{q}|}{m} \to 0$, the center of mass energy $\sqrt{s}$ can be parametrized as a function of $\vec{p}^2$. In fact,

$$
\begin{aligned}
s &= (P_1 + P_3)^2\,, \\
&= (E_a + E_b)^2 \\
&= (m_a + m_b)^2 \left(1 + \frac{\vec{p}^2}{m_a m_b} + O(\vec{p}^4)\right) + O(\vec{q}^2)
\end{aligned}
\tag{3.4}
$$

Note that the remaining kinematic invariant may be obtained as $u = 2(m_a^2 + m_b^2) - t - s$.

In practice, we regard the amplitude for Fig. 3.1 as a function $M(t, s)$, which may contain poles and branch cuts in both variables. At this point we can also introduce the spin vector S^μ, which will be in general constructed from polarization tensors associated to the spinning particles, see e.g. [237]. Suppose for instance that the particle m_b carries spin, then the spin vector satisfies the transversal condition

$$
S^\mu (P_3 + P_4)_\mu = 0\,,
\tag{3.5}
$$

implying that in the non-relativistic regime $\vec{p} \to 0$ the 4-vector becomes purely spatial, i.e. $S^\mu \to (0, \vec{S})$.

The PN expansion and the corresponding EM analog then proceed by extracting the classical (i.e. $\hbar$-independent) part of the scattering amplitude $M(t, s)$ expressed in these coordinates. This is done by selecting the lowest order in $|\vec{q}|$ for fixed powers of G, spin $|\vec{S}|$ and $\vec{p}^2$ [197, 237]. This claim is argued by dimensional analysis, where it is clear that for a given order in G each power of $|\vec{q}|$ carries a power of $\hbar$ unless a spin factor $|\vec{S}|$ is attached [153, 212, 197]. Here G is equivalent to 1PN order and acts as a loop counting parameter, while the latter quantities can be counted as 1PN corrections each [237]. For a given number of loops and fixed value of s, the expansion around $t = -\vec{q}^2 = 0$ used to select the classical pieces coincides with the non-relativistic limit $\frac{\vec{q}}{m} \to 0$. Additionally, in the COM frame the 2^{2n}-pole and 2^{2n-1}-pole interactions due to spin emerge in the form [212, 154, 237]

$$
V_S = c_1(|\vec{p}|) S_1^{i_1 \ldots i_{2n}} q_{i_1} \ldots q_{i_{2n}} + c_2(|\vec{p}|) S_2^{i_1 \ldots i_{2n}} q_{i_1} \ldots q_{i_{2n-1}} p_{i_{2n}} = O(|\vec{q}|^{2n-1})\,,
\tag{3.6}
$$

where $S_j^{i_1 \ldots i_{2n}}$, $j = 1, 2$, are constructed from polarization tensors of the scattered particles in such a way that the powers of $|\vec{S}|$ exactly match the powers of $|\vec{q}|$ in V_S. They are, in consequence, classical contributions and correspond to the so-called mass ($j = 1$) and current ($j = 2$) multipoles [239]. These terms arise in the scattering amplitude when one of the external particles, for instance the one with mass m_a, carries spin $S_a \geq n$. Note that in order to evaluate spin effects a non-relativistic expansion to arbitrary high orders in $|\vec{q}|$ is required. To deal with this difficulty we note that (3.6) is obtained, through the non-relativistic expansion, from the generic covariant form

$$S^{\mu_1 \cdots \mu_m} K_{\mu_1} \cdots K_{\mu_k} (P_{a_{k+1}})_{\mu_{k+1}} \cdots (P_{a_m})_{\mu_m}\,, \quad a_i \in \{1, 3\}. \tag{3.7}$$

where $k = 2n$ for mass multipoles and $k = 2n + 1$ for current multipoles. These spin forms are characteristic of the multipole interactions in the sense that they are partly determined by general constraints[1] and they emerge already in the tree level amplitude, being consistently reproduced at the loop level [156]. We give explicit examples of these for $S = \frac{1}{2}, 1$ in Appendix D. Once the non-relativistic limit is taken by expanding (3.7) with respect to $\vec{q}$ and $\vec{p}$, these terms lead to the structures present in V_S, i.e. they capture the complete spin-dependent couplings, together with some higher powers of $|\vec{q}|$ which are quantum in nature. The advantage of writing the multipole terms in the covariant form is that these are completely visible once the limit $t = K^2 \to 0$ is taken, that is, at leading order in the t expansion. All the neglected pieces, i.e. subleading orders in t, which are not captured by these multipole forms simply correspond to quantum corrections. Thus our strategy is to compute the coefficients associated to these EFT operators[2] in the $t \to 0$ limit. This is done by examining the leading order of an arbitrary linear combination of them and performing the match with the classical piece of the amplitude, obtained by computing the leading singularity [63]. The explicit matching procedure is demonstrated in Appendix D, where we use spinor helicity variables to write the multipole terms. The idea is that at $t = 0$ the expression (3.6) is not well defined but (3.7) is. This means that we can write our answer for the EFT potential in terms of (3.7) and then proceed to analytically continue it to the region $t \neq 0$, which is easily achieved by putting $K = (0, \vec{q})$ and the corresponding expressions for P_i. The evaluation of the classical piece near $t = 0$ is the holomorphic classical limit (HCL).

A few final remarks regarding the HCL are in order. First, as anticipated the term holomorphic stems from the on-shell condition $P_i \cdot K = \pm K^2$, $i \in \{1, ..., 4\}$, which for $t \to 0$ yields $P_i \cdot K \to 0$. In turn, this implies that the external momenta P_i must be complexified. Hence, in order to reach the $t = 0$ configuration we must consider an

[1] For instance, they vanish whenever the momentum transfer K is orthogonal to the polarization tensors $K_{\mu_1} \epsilon^{\mu_1 \cdots \mu_S} = 0$ as can be checked in [237], or equivalently, when it is aligned with the spin vector.

[2] Hereafter we may refer to the multipole terms (3.6), (3.7) as EFT operators indistinctly. This is in order to contrast them with the spinor operators to be defined in section 3.4, which will be then matched to EFT operators.

analytic trajectory in the kinematic space, which we can parametrize in terms of a complex variable β. We introduce such trajectory explicitly in section 3.3.2, where we also evaluate the amplitude as $\beta \to 1$. Second, we stress that just the HCL is enough to recover the classical potential with arbitrary multipole corrections. The complete non-relativistic limit can be further obtained by expanding around $s \to (m_a + m_b)^2$, i.e. expanding in $\vec{p}^2$ for a given power of $|\vec{q}|$. These corrections in $\vec{p}^2$ account for higher PN corrections when implemented through the Born approximation, which at 1-loop also requires to subtract the iterated tree level potential. We perform the procedure only at the level of the amplitude and refer to [113, 156, 237, 197] for details on iterating higher PN corrections. As the expressions we provide for the classical piece correspond to all the orders in $\vec{p}^2$ encoded in a covariant way, we regard the HCL output as a fully relativistic form of the classical potential. In fact, the construction is covariant since it is based on the null condition for K, which will also prove useful when defining the massless limit of external particles for addressing light-like scattering. Finally, the soft behavior of the momentum transfer $K \to 0$, which is the equivalent of $\frac{\vec{q}}{m} \to 0$ for COM coordinates, is not needed and we find that it does not lead to further insights on the behavior of the potential.

3.3 Scalar Scattering

In this section we compute the Leading Singularity for gravitational scattering of both tree and 1-loop level amplitudes for the no spinning case, as first presented in [63]. This time we embed the computation into the framework of the HCL, which will lead directly to the classical contribution. We also present, without additional effort, the analogous results for the EM case. Along the way we introduce new variables which will prove helpful for the next sections. Recall the form of the three-point amplitudes presented in the previous chapter (2.9). For $s = 0$ we have

$$A^{(+h)}_{scalar} = \alpha(mx)^h \,, \qquad A^{(-h)}_{scalar} = \alpha(m\bar{x})^h \,. \tag{3.8}$$

The (minimal) coupling constant α has to be chosen according to the theory under consideration, determined once the helicity $|h|$ is given, i.e. $h = \pm 1$ for EM and $h = \pm 2$ for gravity. Regarding the gravitational interaction, its universal character allows us to fix the coupling by $\alpha = \frac{\kappa}{2} = \sqrt{8\pi G}$ irrespective of the particle type, whereas for EM it will depend on the electric charge carried by such particle.

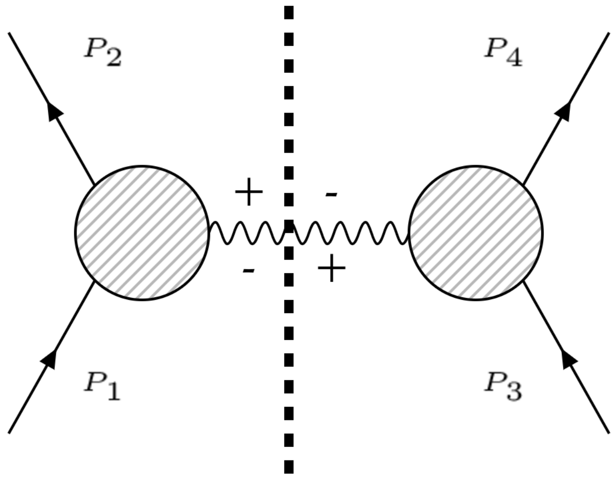


Figure 3.2: A one photon/graviton exchange process. In the HCL the internal particle is on-shell and the two polarizations need to be considered.

3.3.1 Tree Amplitude

Let us start by computing the tree level contributions to the classical potential. As explained in [63], these can be directly obtained from the Leading Singularity, which for tree amplitudes is simply the residue at $t = 0$. Here, it is transparent that the analytic expansion around such pole will yield subleading terms t^n, $n \geq 0$, which are ultralocal (e.g. quantum) once the Fourier transform is implemented in COM coordinates $t = -\vec{q}^2$ [156]. Furthermore, by unitarity this residue precisely corresponds to the product of on-shell 3pt amplitudes (see Fig. 3.2), that is to say, we can use the leading term in the HCL to evaluate the classical potential. As we have seen, even though there exist different couplings contributing to the s and u channel, these correspond to contact interactions between the different particles and do not lead to a long-range potential [63].

With these considerations we proceed to compute the leading contribution to the Coulomb potential by considering the one-photon exchange diagram. Summing over both helicities and using (3.8) we find

$$M^{(0)}_{(0,0,1)} = \frac{1}{t}\left(A_3^{(+1)}(P_1)A_3^{(-1)}(P_3) + A_3^{(-1)}(P_1)A_3^{(+1)}(P_3)\right) = \alpha^2\frac{m_a m_b}{t}\left(x_1\bar{x}_3 + \bar{x}_1 x_3\right) . \quad (3.9)$$

Here we have used $M^{(0)}_{(S_a,S_b,|h|)}$ to denote the classical piece of the $2 \to 2$ amplitude, as opposed to the notation $A_n(P_i)$ which we reserve for the n pt amplitudes used as building blocks. The index (0) indicates leading order (tree level), which will be equivalent to 0PN for the gravitational case. Recall the subindex $(S_a, S_b, |h|) = (0, 0, 1)$ denotes spinless particles exchanging a photon.

At this stage we introduce the kinematic variables

$$u := m_a m_b x_1 \bar{x}_3 \,, \quad v := m_a m_b \bar{x}_1 x_3. \tag{3.10}$$

Note that these variables are defined only in the HCL, i.e. for $t = 0$. Each of these carries no helicity, i.e. it is invariant under little group transformations of the internal particle. However, they represent the contribution from the two polarizations in the exchange of Fig. 3.2, and as such they are swapped under parity. In Appendix D we give explicit expressions for u and v in terms of their parity even and odd parts. Nevertheless, we stress that for this and the remaining sections the only identities which are needed can be stated as

$$uv = m_a^2 m_b^2 \,, \quad u + v = 2P_1 \cdot P_3 \,, \tag{3.11}$$

We then regard the new variables as a (parity sensitive) parametrization of the s channel emerging in the HCL. Further expanding in the non-relativistic limit yields $u, v \to m_a m_b$.

With these definitions, we can now proceed to write the result in a parity invariant form as

$$M^{(0)}_{(0,0,1)} = \alpha^2 \frac{u+v}{t} = \alpha^2 \frac{s - m_a^2 - m_b^2}{t} \,. \tag{3.12}$$

After implementing COM coordinates and including the proper relativistic normalization, this leads to the Coulomb potential in Fourier space, which can be expanded in the limit $s \to (m_a + m_b)^2$. In fact, assuming both particles carry the same electric charge $e = \frac{\alpha}{\sqrt{2}}$, we can use (3.2), (3.4) to write

$$\frac{M^{(0)}_{(0,0,1)}}{4E_a E_b} = -\frac{e^2}{\vec{q}^2} \left(1 + \frac{\vec{p}^2}{m_a m_b} + \ldots \right) \,. \tag{3.13}$$

Of course, we can also easily compute the one-graviton exchange diagram. The answer is again given by the parity invariant expression

$$M^{(0)}_{(0,0,2)} = \alpha^2 \frac{u^2+v^2}{t} = \frac{\kappa^2}{4}\frac{(s-m_a^2-m_b^2)^2 - 2m_a^2 m_b^2}{t}\,. \tag{3.14}$$

Again, this leads to a relativistic expression for the Newtonian potential, and can be put into the standard form by using the dictionary provided in subsection 3.2.1

$$\frac{M^{(0)}_{(0,0,2)}}{4E_aE_b} = 4\pi G \frac{m_a m_b}{\vec{q}^2}\left(1 + \frac{(3m_a^2 + 8m_am_b + 3m_b^2)}{2m_a^2m_b^2}\vec{p}^2 + ...\right), \tag{3.15}$$

in agreement with the computations in [106, 40, 197, 63].

Two final remarks are in order. First, it is interesting that the gravitational result can be directly obtained by squaring the u, v variables, i.e. squaring both contributions from the EM case. This will be a general property that we will encounter again for the discussion of the Compton amplitude in the next section, as was already pointed out in the first chapter [44] in relation with the double-copy construction. Second, it is worth noting that up to this point no parametrization of the external momenta was needed. In other words, the tree level computation was done solely in terms of (pseudo)scalar variables. As we will see now, the 1-loop case can be addressed with the help of an external parametrization specifically designed for the HCL. This parametrization will provide an extension of the variables u and v in a sense that will become clear.

3.3.2 1-Loop Amplitude: Triangle Leading Singularity

Here we proceed to compute the triangle LS [63] in order to obtain the first classical correction to the potential. This leading singularity is associated to the 1-loop diagram arising from two photons/gravitons exchange, Fig. 3.3. As explained in [63], the LS of the triangle diagram captures the second discontinuity of the amplitude as a function of t, which is precisely associated to the non-analytic behavior $\frac{1}{\sqrt{-t}} = \frac{1}{|\vec{q}|}$. In the gravitational case this accounts for G^2 corrections or equivalently 1PN. In order to track exclusively this contribution we proceed to discard higher (analytic and non-analytic) powers of t by appealing to the HCL. This can be implemented to any order in t by means of the following parametrization of the external kinematics

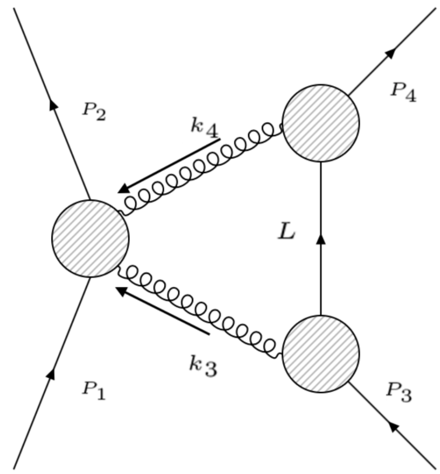


Figure 3.3: The triangle diagram used to compute the leading singularity, corresponding to the b- topology. The a-topology is obtained by reflection, i.e. by appropriately exchanging the external particles.

$$
\begin{aligned}
P_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta| \,, \\
P_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda| \,, \\
\frac{t}{m_b^2} &= \frac{(\beta-1)^2}{\beta} \,, \\
\langle\lambda\eta\rangle &= [\lambda\eta] = m_b \,.
\end{aligned}
\tag{3.16}
$$

The parametrization is constructed by first defining a complex null vector $K = |\lambda]\langle\lambda|$ orthogonal to P_3 and P_4. Then the bispinors $(P_3)_{\alpha\dot{\alpha}}$ and $(P_4)_{\alpha\dot{\alpha}}$ are expanded in a suitably constructed basis, which also provides the definition of $|\eta]_{\dot{\alpha}}$ and $\langle\eta|_\alpha$ up to a scale which is fixed by the fourth condition. As explained in Appendix C (following the lines of [14]) this basis also provides a representation for the little group associated to massive states. The dimensionless parameter β was called x in [63] and was introduced as a natural description of the t channel. In this sense, this parametrization should be regarded as an extension of the one presented there, which can be recovered for $\beta^2 \neq 1$ by means of the shift

$$|\eta] \to |\eta] + \frac{\beta}{1-\beta^2}|\lambda]\,, \quad \langle\eta| \to \langle\eta| - \frac{\beta}{1-\beta^2}\langle\lambda|\,. \tag{3.17}$$

However, in this case we are precisely interested in the degenerate point $\beta = 1$, i.e. $t = 0$, in order to define the HCL. For this point we have $P_4 - P_3 = K = |\lambda]\langle\lambda|$ as the null momentum transfer. As opposed to the tree level case, such momentum is not associated to any particle in the exchange of Fig. (3.3), but distributed between the internal photons/gravitons. In general for $\beta \neq 1$, K is just an auxiliary vector and thus we need not to consider little group transformations for $|\lambda]$, $\langle\lambda|$, i.e. these are fixed spinors. Finally, we also provide a parametrization for the s channel by extending the definitions (3.10) for $t \neq 0$

$$u = [\lambda|P_1|\eta\rangle\,, \quad v = [\eta|P_1|\lambda\rangle\,, \tag{3.18}$$

such that $u + v = 2P_1 \cdot P_3$ and $uv \to m_a^2 m_b^2 \quad \text{as} \quad \beta \to 1$.

We are now well equipped to evaluate the triangle Leading Singularity. Here we sketch the computation of the contour integral and refer to [63] for further details. It is given by

$$\begin{aligned} M^{(1,b)}_{(0,0,|h|)} = \frac{1}{4} \sum_{h_3,h_4=\pm|h|} \int_{\Gamma_{\text{LS}}} & d^4L\, \delta(L^2 - m_b^2)\, \delta(k_3^2)\, \delta(k_4^2) \\ & \times A_4(P_1, -P_2, k_3^{h_3}, k_4^{h_4}) \times A_3(P_3, -L, -k_3^{-h_3}) \times A_3(-P_4, L, -k_4^{-h_4})\,, \end{aligned} \tag{3.19}$$

where the superscript $(1, b)$ denotes the (1-loop) triangle b-topology depicted in Fig. 3.3. The a-topology is simply obtained by exchanging particles m_a and m_b: We leave the explicit procedure for the Appendix and in the following we deal only with $M^{(1,b)}_{(0,0,|h|)}$. In (3.19) we denote by A_3 and A_4 to the respective tree level amplitudes entering the diagram (note the minus sign for *outgoing* momenta), and

$$k_3 = -L + P_3\,, \quad k_4 = L - P_4\,. \tag{3.20}$$

The sum is performed over propagating internal states and enforces matching polarizations between the 3pt and 4pt amplitudes. Γ_{LS} is a complex contour defined to enclose the emerging pole in (3.19). This pole will be explicit after a parametrization for the loop momenta L is implemented and the triple-cut corresponding to the three delta functions is performed. This will leave only a 1-dimensional contour integral for a suitably defined $z \in \mathbb{C}$, where $L = L(z)$. We now use the previously defined basis of spinors to parametrize

$$\begin{aligned} L(z) &= z\ell + \omega K\,, \\ \ell &= A|\eta]\langle\lambda| + B|\lambda]\langle\eta| + AB|\lambda]\langle\lambda| + |\eta]\langle\eta|\,, \end{aligned} \tag{3.21}$$

where one scale in ℓ has been absorbed into z and we have further imposed the condition $\ell^2 = 0$. Using Eqs. (3.16), we find that implementing the triple-cut in (3.19) fixes $\omega(z) = -\frac{1}{z}$, while $A(z), B(z)$ become simple rational functions of z and β. The integral then takes the form

$$\begin{aligned} M^{(1,b)}_{(0,0,|h|)} = \sum_{h_3,h_4} \frac{\beta}{16(\beta^2-1)m_b^2} \int_{\Gamma_{\mathrm{LS}}} \frac{dy}{y} A_4(P_1, -P_2, k_3^{h_3}(y), k_4^{h_4}(y)) \\ \times A_3(P_3, -L(y), -k_3^{-h_3}(y)) \times A_3(-P_4, L(y), -k_4^{-h_4}(y))\,, \end{aligned} \tag{3.22}$$

where $y := -\frac{z}{(\beta-1)^2}$ and we now define the contour to enclose the emergent pole at $y = \infty$, i.e. $\Gamma_{\mathrm{LS}} = S^1_\infty$.[3] The internal massless momenta are given by

$$\begin{aligned} k_3(y) &= \underbrace{\frac{1}{\beta+1}\left(|\eta](\beta^2-1)y + |\lambda](1+\beta y)\right)}_{|k_3]} \underbrace{\frac{1}{\beta+1}\left(\langle\eta|(\beta^2-1) - \frac{1}{y}\langle\lambda|(1+\beta y)\right)}_{\langle k_3|}, \\ k_4(y) &= \underbrace{\frac{1}{\beta+1}\left(-\beta|\eta](\beta^2-1)y + |\lambda](1-\beta^2 y)\right)}_{|k_4]} \underbrace{\frac{1}{\beta+1}\left(\frac{1}{\beta}\langle\eta|(\beta^2-1) + \frac{1}{y}\langle\lambda|(1-y)\right)}_{\langle k_4|}\,. \end{aligned} \tag{3.23}$$

As $\frac{\beta}{\beta^2-1} \to \frac{m_b}{2\sqrt{-t}}$ for the HCL, we find that the expression (3.22) already contains the required classical correction when the leading term of the integrand, around $\beta = 1$, is extracted. We can straightforwardly evaluate the 3pt amplitudes at $\beta = 1$, giving finite contributions. This simplification will indeed prove extremely useful for the $S > 0$ cases in section 3.4. On the other hand, for the 4pt amplitude the limit $\beta \to 1$ is needed to

[3] Also the choice $y = 0$ is permitted for the contour, i.e. $\Gamma_{\mathrm{LS}} = S^1_0$. This choice does not matter in the HCL since the leading piece in (3.22) is invariant under the inversion of the contour [63].

obtain a finite answer, since it contains a pole in the t channel.

Explicitly, at $\beta = 1$ the internal momenta are given by

$$\begin{aligned} k_3^0(y) &= \underbrace{\frac{1}{2}|\lambda](1+y)}_{|k_3]}\ \underbrace{\frac{-1}{2y}\langle\lambda|(1+y)}_{\langle k_3|}\,, \\ k_4^0(y) &= \underbrace{\frac{1}{2}|\lambda](1-y)}_{|k_4]}\ \underbrace{\frac{1}{2y}\langle\lambda|(1-y)}_{\langle k_4|}\,. \end{aligned} \tag{3.24}$$

We thus note that in the HCL both internal momenta are collinear and aligned with the momentum transfer K. For the standard non-relativistic limit defined in the COM frame the condition $\beta \to 1$ certainly implies the soft limit $K \to 0$ and in general leads to vanishing momenta for the gravitons and vanishing 3pt amplitudes at $\beta = 1$.

Now, using the expression (3.16) for the momenta P_3 and (outgoing) P_4, we readily find

$$x_3 = x_4 = -y\,, \tag{3.25}$$

such that the 3pt amplitudes are given (at $\beta = 1$) by

$$\begin{aligned} A_3(P_3, -L(y), -k_3^{+|h|}(y))A_3(-P_4, L(y), -k_4^{-|h|}(y))\Big|_{\beta=1} &= \alpha^2 m_b^2 \\ A_3(P_3, -L(y), -k_3^{+|h|}(y))A_3(-P_4, L(y), -k_4^{+|h|}(y))\Big|_{\beta=1} &= \alpha^2 m_b^2 (y^2)^{|h|}\,. \end{aligned} \tag{3.26}$$

We note that for $h_3 = -h_4$ the contribution from the 3pt amplitudes is invariant under conjugation. In fact, as can be already checked from (3.24) the conjugation is induced by $y \to -y$. Even though the full contribution from the triangle leading singularity requires to sum over internal helicities, in the HCL $\beta \to 1$ the conjugate configuration $h_3 = -h_4 = -|h|$ yields the same residue, while the configurations $h_3 = h_4$ yield none as we explain below. This means that the full result can be obtained by evaluating the case $h_3 = -h_4 = +|h|$ and inserting a factor of 2. Returning to the computation, (3.22) now reads

$$M^{(1,b)}_{(0,0,|h|)} = \frac{\alpha^2}{16}\left(\frac{m_b}{\sqrt{-t}}\right)\int_{\infty}\frac{dy}{y}A^{(-+)}_{(4,|h|)}(\beta \to 1)\,, \tag{3.27}$$

where $A^{(-+)}_{(4,|h|)}(\beta \to 1)$ is the leading order of the 4pt. Compton-like amplitude, given by

$$A^{(-+)}_{(4,|h|)} = \alpha^2 \begin{cases} \dfrac{\langle k_3|P_1|k_4]^2}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 1 \\ \\ \frac{1}{t} \times \dfrac{\langle k_3|P_1|k_4]^4}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 2 \end{cases} \tag{3.28}$$

We note that the stripped Compton amplitudes (3.28) exhibit the double-copy factorization $A_{(4,2)} = 4\frac{(k_3\cdot P_1)(k_3\cdot P_2)}{t}(A_{(4,1)})^2$ as explained in [44]. We will come back at this point in section 3.4. By considering the definitions (3.18), and using (3.23) together with momentum conservation constraints, we find the HCL expansions

$$\begin{aligned} \langle k_3|P_1|k_4] &= (\beta-1)\left(u\frac{1-y}{2} + v\frac{1+y}{2} + \frac{(v-u)(1-y^2)}{4y}\right) + O(\beta-1)^2\,, \\ \langle k_3|P_1|k_3] &= \langle k_3|P_2|k_3] + O(\beta-1)^2 = (\beta-1)\frac{(v-u)(1-y^2)}{4y} + O(\beta-1)^2\,. \end{aligned} \tag{3.29}$$

where it is understood that u, v are evaluated at $\beta = 1$. We note that the conjugation $y \to -y$ is equivalent to change $u \leftrightarrow v$, as expected. Also, we can now argue why the Compton amplitude gives a finite answer in the limit $\beta \to 1$. Consider for instance the gravitational case. By unitarity, such limit induces a t channel factorization into a 3-graviton amplitude and a scalar-scalar-graviton amplitude A_3. Because of the collinear configuration (3.24) at $\beta = 1$, the 3-graviton amplitude vanishes at the same rate as the t channel propagator $\sim (\beta-1)^2$, yielding a finite result. Note that, for this factorization, regular terms in t will contribute to the result and hence these 3pt factors are not enough to compute the HCL answer.

At this stage we exhibit for completeness the expressions for the Compton amplitude in the case of same helicities. It is given by

$$A^{(4,|h|)}_{(++)} = \alpha^2 \begin{cases} \dfrac{[k_3 k_4]^2}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 1 \\ \\ \frac{1}{t} \times \dfrac{[k_3 k_4]^4}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 2 \end{cases} \tag{3.30}$$

By expanding $[k_3 k_4]$ in an analogous form to (3.29) and, together with (3.26), inserting it back into (3.22) we easily find that this gives indeed vanishing residue. In fact, this can also be checked to any order in $(\beta - 1)$, i.e. with no expansion at all [63]. As anticipated, the configurations $h_3 = h_4$ simply do not lead to a classical potential.

Finally, by inserting (3.29) into (3.27) we find that the residue is trivial ($\text{Res}_\infty = 1$) for $|h| = 1$, while for $|h| = 2$ we have

$$M^{(1,b)}_{(0,0,2)} = \frac{3\alpha^4 m_b}{2^7\sqrt{-t}}(5u^2 + 6uv + 5v^2)\,. \tag{3.31}$$

The expression is indeed symmetric in u, v, as expected by parity invariance. By using (3.11) we can write (3.32) in an analogous form to its tree level counterpart (3.14)

$$M^{(1,b)}_{(0,0,2)} = G^2\pi^2 \frac{3m_b}{2\sqrt{-t}} \left(5(s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2\right)\,. \tag{3.32}$$

The contribution $M^{(1,a)}_{(0,0,2)}$ is obtained by exchanging $m_a \leftrightarrow m_b$. After implementing the Born approximation as explained in [113, 156], this indeed recovers the 1PN form of the effective potential including the corrections in $\vec{p}^2$ [40, 156, 63, 197, 237].

3.3.3 Massless Probe Particle

Here we show that the massless case $m_a = 0$ can be regarded as a smooth limit defined in the variables u, v. In this case such limit is natural to define since both massless and massive scalar fields contain the same number of degrees of freedom. In Appendix (C) we show, however, how to extend this construction to representations with nonzero spin. In the following we focus for simplicity in the gravitational case, the electromagnetic analog being straightforward. Moreover, the gravitational case is motivated by the study of light bending phenomena within the framework of EFT, see [41, 58].

In order to discuss the massless limit, it is convenient to absorb the mass into the definition of $x, \bar{x}$ given in (2.10), i.e. these quantities now carry units of energy. Then, the massless condition $P_3\bar{P}_3 = 0$ is equivalent to $x_3\bar{x}_3 = 0$, thus one of the helicity configurations in (3.8) must vanish at $\beta = 1$. This choice corresponds to selecting one of the graviton polarizations to give vanishing contribution, that is either $u = 0$ or $v = 0$.

Due to parity invariance the election is not relevant, hence we put $v = 0$ and find from (3.11)

$$u = s - m_b^2\,, \tag{3.33}$$

which in turn yields

$$\begin{aligned} M^{(0)}_{m_a=0} &= \alpha^2 \frac{u^2}{t} \\ &= \alpha^2 \frac{(s-m_b^2)^2}{t}\,. \end{aligned} \tag{3.34}$$

Analogously, for the 1-loop correction (3.31) we find

$$\begin{aligned} M^{(1,b)}_{m_a=0} &= \frac{3\alpha^4 m_b (5u^2)}{2^7\sqrt{-t}} \\ &= \frac{15\alpha^4}{2^7} \times \frac{m_b(s-m_b^2)^2}{\sqrt{-t}}\,. \end{aligned} \tag{3.35}$$

After including the normalization factor $(4E_aE_b)^{-1} \approx (4E_am_b)^{-1}$ we find that this recovers the 1PN correction of the effective potential for a massless probe particle [152, 41]. It is important to note that in this result only the $b-$topology LS contributes, i.e. no symmetrization is needed. This is because the triangle LS scales with the mass, i.e. for the $a-$topology is proportional to $\frac{m_a}{\sqrt{-t}}$ and thus vanishes in this case. In fact, classical effects require at least one massive propagator entering the loop diagram [153], see also discussion. We will again resort to this fact in section 3.4.3, where we construct the massless limit for spinning particles.

3.4 HCL for Spinning Particles

In this section we proceed to consider the case of particles with nonzero spin. That is, we extend the computation of the triangle leading singularity presented in section 3.3 but this time for external particles with masses m_a, m_b and spins S_a, S_b respectively. By using the Born approximation, the LS leads to the 1-loop effective potential arising in gravitational or electromagnetic scattering of spinning objects, already computed in [156] for $S_a, S_b \in \{0, \frac{1}{2}, 1\}$. Here we provide an explicit expression for the tree level LS and a contour integral representation for the 1-loop correction, both valid for any spin. We explicitly expand the contour integral for $S_a \leq 1$, S_b arbitrary. In Appendix D we explain how to recover the results of [156] by projecting our corresponding expression in the HCL to the standard EFT operators.

We start by combining our parametrization of the HCL with the massive spinor variables we have studied previously. Recall that the spin-S space is spanned by $2S+1$ polarization states, corresponding to the spin S representation of $SU(2)$. Following the lines of section 3.3 we will focus on the 3pt. amplitudes $A_3(P_3, P_4, K)$ as operators acting on in this space, which will then serve as building blocks for the leading singularities. In our case, it will be natural to take advantage of the parametrization of the previous section,

$$\begin{aligned} P_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta|\,, \\ P_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda|\,, \end{aligned} \tag{3.36}$$

to construct the little group representation for momenta P_3 and P_4 (carrying the same spin S) in a simultaneous fashion. We will denote the respective $2S+1$ dimensional Hilbert spaces by V_3^S and $\bar{V}_4^S$. In appendix C we explicitly construct $V_3^{\frac{1}{2}}$ and $\bar{V}_4^{\frac{1}{2}}$ starting from the well known Dirac spinor representation. For general spin, a basis for these spaces is given by the $2S$-th rank tensors [4]

$$\begin{aligned} |m\rangle &= \frac{1}{[\lambda\eta]^S}\underbrace{|\lambda]\odot\ldots\odot|\lambda]}_{m}\odot\underbrace{|\eta]\odot\ldots\odot|\eta]}_{2S-m}\in V_3^S\,, \\ \langle n| &= \frac{1}{[\lambda\eta]^S}\underbrace{[\lambda|\odot\ldots\odot[\lambda|}_{n}\odot\underbrace{[\eta|\odot\ldots\odot[\eta|}_{2S-n}\in \bar{V}_4^S\,, \end{aligned} \tag{3.37}$$

with $m, n = 0, \ldots, 2S$. Here the symbol $\odot$ denotes the symmetrized tensor product. The normalization is chosen for latter convenience, i.e.

$$\eta_{\dot\alpha}\odot\lambda_{\dot\beta} = \frac{\eta_{\dot\alpha}\lambda_{\dot\beta}+\eta_{\dot\beta}\lambda_{\dot\alpha}}{\sqrt{2}}\,, \tag{3.38}$$

[4] The notation $|m\rangle$ for the states may seem unfortunate since it is similar to the one for angle (chiral) spinors. However, as we will be mostly using the anti-chiral representation for spinors, the risk of confusion is low.

$$\eta_{\dot\alpha}\odot\lambda_{\dot\beta}\odot\lambda_{\dot\gamma} \;=\; \frac{\eta_{\dot\alpha}\lambda_{\dot\beta}\lambda_{\dot\gamma}+\eta_{\dot\beta}\lambda_{\dot\alpha}\lambda_{\dot\gamma}+\eta_{\dot\gamma}\lambda_{\dot\alpha}\lambda_{\dot\beta}}{\sqrt{3}}\,,$$

etc. As we explicitly show below, in this framework we regard the 3pt amplitudes as operators $A_S : \bar{V}_4^S \otimes V_3^S \to \mathbb{C}$, that is, they are to be contracted with the states given in (3.37) for both particles. The representation is symmetric and anti-chiral in the sense that it is spanned by symmetrizations of the anti-chiral spinors $|\lambda]$, $|\eta]$. Further details on the choice of basis and the chirality are given in Appendix C (see also [14]).

Consider then the 3pt amplitudes for two particles of momenta P_3, P_4 and spin S interacting with a massless particle of momenta $K = P_4 - P_3$ and helicity $\pm h$. From (3.36) we see that the on-shell condition $K^2 = 0$ sets $\beta = 1$, i.e. $K = |\lambda]\langle\lambda|$. For the massless particle, we choose the standard representation in terms of the spinors $\langle k| = \frac{\langle\lambda|}{\sqrt{x}}$ and $[k| = \sqrt{x}[\lambda|$, where x carries helicity weight $+1$ and agrees with the definition (2.10) for our parametrization. Note that $[\lambda|$ and $\langle\lambda|$ remain fixed under little group transformations. With these conventions the minimally coupled 3pt amplitudes are given by the operators

$$\begin{aligned} A_S^{(+h)} &= \alpha(mx)^h\left(1-\frac{|\lambda][\lambda|}{m}\right)^{\otimes 2S} = \alpha(mx)^h\left(1-\frac{|\lambda][\lambda|}{m}\right)\otimes\ldots\otimes\left(1-\frac{|\lambda][\lambda|}{m}\right)\,, \\ A_S^{(-h)} &= \alpha(m\bar{x})^h = \alpha\left(\frac{m}{x}\right)^h\,. \end{aligned} \tag{3.39}$$

These expressions represent extensions of the ones given in (3.8). Note that we have omitted trivial tensor structures (i.e. the identity operator) in (3.39). For example, in the second line the explicit index structure is

$$\left(A_S^{(-h)}\right)^{\dot\alpha_1\ldots\dot\alpha_{2S}}_{\dot\beta_1\ldots\dot\beta_{2S}} = \alpha\left(\frac{m}{x}\right)^h\left(\mathbb{I}^{\otimes 2S}\right)^{\dot\alpha_1\ldots\dot\alpha_{2S}}_{\dot\beta_1\ldots\dot\beta_{2S}} = \alpha\left(\frac{m}{x}\right)^h\delta^{\dot\alpha_1}_{\dot\beta_1}\ldots\delta^{\dot\alpha_{2S}}_{\dot\beta_{2S}}\,. \tag{3.40}$$

The value for the amplitude is now obtained as the matrix element $\langle n|A_S^{(\pm h)}|m\rangle$. This contraction is naturally induced by the bilinear product $[\,,\,]$ of spinors. For instance, consider the matrix element associated to the transition of particle of momenta P_3 and polarization $|m\rangle$ to momenta P_4 and polarization $|n\rangle$, while absorbing a graviton:

$$A^{m+(-h)\to n} = \langle n|A_S^{(-h)}|m\rangle = \alpha\left(\frac{m_b}{x}\right)^h\langle n|m\rangle\,, \tag{3.41}$$

where the contraction

$$\langle n|m\rangle = (-1)^m \delta_{m+n,2S} \tag{3.42}$$

is induced by (3.37). The relation of this contraction with the inner product, and the corresponding normalizations, are discussed in Appendix C. We note further that for helicity $-h$ the only non trivial amplitudes are of the form $\langle n|A_S^{(-h)}|2S-n\rangle$ and correspond to the scalar amplitude. This is a consequence of choosing the anti-chiral basis. For $+h$ helicity this is not the case, but the fact that $A_S^{(+h)}$ is to be contracted with totally symmetric states of V_3^S and $\bar{V}_4^S$ allows us to commute any two factors in the tensor product of (3.39). That is, we can expand without ambiguity

$$\begin{aligned} A_S^{(+h)} &= \alpha(mx)^h \left(1 - \frac{|\lambda][\lambda|}{m}\right)^{\otimes 2S} \\ &= \alpha(mx)^h \left(1 - 2S\frac{|\lambda][\lambda|}{m} + \binom{2S}{2}\frac{|\lambda][\lambda| \otimes |\lambda][\lambda|}{m^2} + \dots\right), \end{aligned} \tag{3.43}$$

where we again omitted the trivial operators in the tensor product. As we explain in Appendix C, $|\lambda][\lambda|$ is proportional to the spin vector, hence we call it *spin operator* hereafter (see also [14]). Here we can see that in general the contraction $\langle 0|A_S|2S\rangle$ projects out the spin operator, again recovering the scalar amplitude.

3.4.1 Tree Amplitudes

We follow the lines of section 3.3 and evaluate the $2 \to 2$ t channel residue. This time we assign spins S_a, S_b to the particles of mass m_a, m_b, respectively. However, in order to construct the corresponding $SU(2)$ representation (3.37) for the momenta P_1, P_2, we need to repeat the parametrization for P_3 and P_4 given in (3.36). This time we have

$$\begin{aligned} P_1 &= |\hat{\eta}]\langle\hat{\lambda}| + |\hat{\lambda}]\langle\hat{\eta}|\,, \\ P_2 &= \beta|\hat{\eta}]\langle\hat{\lambda}| + \frac{1}{\beta}|\hat{\lambda}]\langle\hat{\eta}| + |\hat{\lambda}]\langle\hat{\lambda}|\,, \end{aligned} \tag{3.44}$$

together with the normalization $[\hat{\lambda}\hat{\eta}] = m_a$. Both parametrizations can be matched in the HCL, effectively reducing the apparent degrees of freedom. In fact, $\beta \to 1$ yields

$|\lambda]\langle\lambda| \to -|\hat{\lambda}]\langle\hat{\lambda}|$. Recall that at $\beta = 1$ the tree level process of Fig. 3.2 consists of a photon/graviton exchange, with corresponding momentum $K = |\lambda]\langle\lambda|$. For this internal particle we choose the spinors

$$|K] = |\hat{\lambda}] = \frac{|\lambda]}{\gamma}\,,\; |K\rangle = |\hat{\lambda}\rangle = -\gamma|\lambda\rangle\,, \tag{3.45}$$

for some $\gamma \in \mathbb{C}$. By using the definitions (3.16) for both P_1 and P_3 we find $x_1 = 1\,, \quad \bar{x}_3 = -\gamma^2\,,$ Using (3.10) we can then solve for γ, completely determining $|\hat{\lambda}]$ and $\langle\hat{\lambda}|$:

$$\gamma^2 = -\frac{u}{m_a m_b} = -\frac{m_a m_b}{v}. \tag{3.46}$$

After this detour, we are ready to compute the tree level residue. The $2 \to 2$ amplitude is here regarded as the operator $M^{(0)}_{(S_a,S_b,|h|)} : V_1^{S_a} \otimes \bar{V}_2^{S_a} \otimes V_3^{S_b} \otimes \bar{V}_4^{S_b} \to \mathbb{C}$, where $V_1^{S_a}, \bar{V}_2^{S_a}$ are constructed in analogous manner to (3.37). Using the expansion (3.43) we find our first main result

$$\begin{aligned}
M^{(0)}_{(S_a,S_b,|h|)} &= \alpha^2 \frac{(m_a m_b)^h}{t}\left((x_1\bar{x}_3)^h\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{2S_a} + (\bar{x}_1 x_3)^h\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}\right) \\
&= \frac{\alpha^2}{t}\left(u^h\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{2S_a} + v^h\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}\right) \\
&= \frac{\alpha^2}{t}\left(u^h - 2u^h S_a \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \otimes \mathbb{I}_b + S_a(2S_a - 1)\frac{|\tilde{\lambda}]|\hat{\lambda}][\hat{\lambda}|[\tilde{\lambda}|}{m_a^2} \otimes \mathbb{I}_b\right. \\
&\quad \left. + v^h - 2v^h S_b\, \mathbb{I}_a \otimes \frac{|\lambda][\lambda|}{m_b} + \dots\right)\,,
\end{aligned} \tag{3.47}$$

where $h = 1$ for Electromagnetism and $h = 2$ for Gravity. In the third and fourth line we exhibited explicitly the identity operators for both representations to emphasize that the spin operators act on different spaces and hence cannot be summed. In Appendix C it is argued, by examining the $S = \frac{1}{2}$ and $S = 1$ case, that the binomial expansion is in direct correspondence with the expansion in multipoles moments and hence to the PN expansion for the gravitational case. That is to say we can match the operators $|\hat{\lambda}][\hat{\lambda}|^{\otimes 2n}$, $|\hat{\lambda}][\hat{\lambda}|^{\otimes 2n-1}$ to the spin operators (3.7) in the HCL and compute the respective coefficients in the EFT expression, as we demonstrate in Appendix D for the cases in the literature, i.e. $S \leq 1$. Note further that we can easily identify universal multipole interactions as predicted by [156, 44] for the minimal coupling, the leading one corresponding to scalar (orbital) interaction. Here we emphasize again that all these multipole interactions can be easily seen at $\beta = 1$, in contrast with the COM frame limit.

Finally, note that the parametrization that we introduced did not seem relevant in order to obtain (3.47). However, it is indeed implicit in the choice of basis of states needed to project the operator $M^{(0)}_{(S_a,S_b,h)}$ into a particular matrix element. Next we compute the 1-loop correction for this process, which requires extensive use of the parametrization.

3.4.2 1-Loop Amplitude

We now compute the triangle LS (3.19) for the case in which the external particles carry spin. We explicitly expand the contour integral in the HCL for the case $S_a \leq 1$ and S_b arbitrary. The limitation for S_a simply comes from the fact that for $S_a \leq 1$ the four point Compton amplitude has a well known compact form [44] both for EM and gravity. Let us remark that the expression for higher spins is also known in terms of the new spinor helicity formulation [14], but we will leave the explicit treatment for future work. Additionally, the case $S_a \leq 1$ is enough to recover all the 1-loop results for the scattering amplitude in the literature [154, 156], and suffices here to demonstrate the effectiveness of the method (see Appendix D). Note that the final result is obtained by considering the two triangle topologies for the leading singularity, which can be obtained by symmetrization as we explain below.

In the following we regard the 3pt and 4pt amplitudes entering the integrand (3.19) as 2×2 operators equipped with the natural multiplication. Analogous to the scalar case, only the opposite helicities contribute to the residue and both configurations give the same contribution, hence we focus only on $(+-)$. Furthermore, the 3pt amplitudes can also be readily obtained at $\beta = 1$, by using (3.24) into (3.39). They give

$$
\begin{aligned}
A_3(P_3, -L(y), k_3^{+i}(y))A_3(-P_4, L(y), k_4^{-i}(y))\Big|_{\beta=1} &= \alpha^2 m_b^2 \left(1 - \frac{|k_3][k_3|}{y m_b}\right)^{2S_b}, \\
&= \alpha^2 m_b^2 \left(1 - \frac{(1+y)^2}{4y}\frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}.
\end{aligned}
\tag{3.48}
$$

This time note that the y variable carries helicity weight $+1$, as can be seen from plugging k_3 and P_3 in (3.16). This means that we needed to restore the helicity factor y in the first line in order to account for little group transformations of k_3. As in the tree level case, eq. (3.48) corresponds to an expansion in terms of spin structures that "survive" the limit $\beta = 1$.

We now proceed to compute the 4pt Compton amplitude in the limit $\beta \to 1$. For this, consider

$$A^{(S_a)}_{(4,|h|)} = \alpha^2 \begin{cases} \Gamma^{\otimes 2S_a} \dfrac{\langle k_3|P_1|k_4]^{2-2S_a}}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 1, \\ \Gamma^{\otimes 2S_a} \frac{1}{t} \times \dfrac{\langle k_3|P_1|k_4]^{4-2S_a}}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 2, \end{cases} \tag{3.49}$$

for $S_a \in \{0, \frac{1}{2}, 1\}$. Here we have defined the 2×2 matrix [14]

$$\Gamma = |k_4]\langle k_3|P_1 + P_2|k_3\rangle[k_4| \, . \tag{3.50}$$

As anticipated, the 4pt. amplitude takes a compact form for $S_a \leq 1$, and exhibits remarkable factorizations relating EM and gravity [44]. Furthermore, we have already computed the expansions (3.29), hence we only need to compute the leading term in Γ! Using the parametrizations (3.16), (3.44), (3.45) together with (3.23), we find

$$\Gamma = (\beta - 1)\left(\hat{u}\frac{(1-y)}{2} + v\frac{(1+y)}{2} + (v - \hat{u})\frac{1-y^2}{4y}\right) + O(\beta - 1)^2 \, , \tag{3.51}$$

where

$$\hat{u} = u\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right) . \tag{3.52}$$

We see that the expansion effectively attaches a "spin factor" $\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)$ to u in the expression (3.29). This is expected since the $A^{(S_a)}_{(4,i)}$ is built from the 3pt amplitudes (3.39), which can be obtained from the scalar case by promoting $x_1^h \to x_1^h \left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{S_a}$ while $\bar{x}_1$

remains the same. Consequently, the expression (3.51) precisely reduces to its scalar counterpart once the spin operator is projected out: Comparing both expansions we find

$$\mathrm{Tr}(\Gamma) = 2\langle k_3|P_1|k_4], \tag{3.53}$$

as required by (3.50). The conjugation $y \to -y$ in Γ effectively swaps $\tilde{u} \leftrightarrow v$. This time this transformation also modifies the contribution from the 3pt amplitudes (3.48), but once the residue is computed the leading singularity is still invariant (in the HCL).

Finally, considering the contribution $h_3 = -h_4 = -2$ in eq. (3.22):

$$\begin{aligned} M^{(1,b)}_{(S_a,S_b,2)} = \frac{\beta}{8(\beta^2-1)m_b^2} \int_{\Gamma_{\mathrm{LS}}} \frac{dy}{y} & A_4(P_1, -P_2, k_3^{-2}(y), k_4^{+2}(y)) \\ & \otimes A_3(P_3, -L(y), -k_3^{+2}(y)) A_3(-P_4, L(y), -k_4^{-2}(y)), \end{aligned} \tag{3.54}$$

and inserting (3.51), (3.29), (3.49) together with (3.48), we find our second main result for the classical piece associated to spinning particles

$$\begin{aligned} M^{(1,b)}_{(S_a,S_b,2)} = \frac{\alpha^4}{16} \frac{m_b}{\sqrt{-t}(v-u)^2} \int_\infty \frac{dy}{y^3(1-y^2)^2} \left(\hat{u}y(1-y) + vy(1+y) + (v-\hat{u})\,\frac{1-y^2}{2}\right)^{\otimes 2S_a} \\ \times \left(uy(1-y) + vy(1+y) + \frac{(v-u)(1-y^2)}{2}\right)^{4-2S_a} \otimes \left(1 - \frac{(1+y)^2}{4y}\frac{|\lambda][\lambda|}{m_b}\right)^{\otimes S_b} \end{aligned} \tag{3.55}$$

together with the analogous expression for $|h| = 1$. The residue can then be computed and expanded as a polynomial in spin operators. Evidently, the factor $\Gamma^{\otimes 2S_a}$ is responsible for these higher multipole interactions, together with the spin operators coming from the 3pt amplitudes (3.48). Finally, symmetrization is needed in order to derive the classical potential. This means that we need to consider the triangle topology obtained by exchanging particles m_a and m_b. This can be easily done since our expressions are general as long as S_a, $S_b \leq 1$. In appendix D we explicitly show how to construct the full answer for $S_a = S_b = \frac{1}{2}$ in terms of the standard EFT operators, and find full agreement with the results in [156]. This time it can be readily checked that the Electromagnetic case also leads to analogous spin corrections, which coincide with those given in [154].

3.4.3 Light Bending for Arbitrary Spin

We will now implement the construction of Appendix C to obtain the massless limit in a similar fashion as we did for the scalar case in sec. 3.3.3. We will again focus on the gravitational case since it is of interest for studying light bending phenomena, addressed in detail in [42, 15] for particles with non trivial helicity.

Let us then proceed to take the massless limit of the parametrization (3.44) (at $\beta = 1$) corresponding to $\tau|\hat{\eta}] \to 0$. This yields $x_1 \to 0$, which is in turn equivalent to $u \to 0$. We get from (3.47), using (3.11)

$$
\begin{aligned}
M^{(0)}_{(h_a,S_b,2)} &= \alpha^2 \frac{v^2}{t}\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b} \\
&= \alpha^2 \frac{(s-m_b^2)^2}{t}\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b} ,
\end{aligned}
\tag{3.56}
$$

where $S_a = h_a$ now corresponds to the helicity of particle a. This operator is to be contracted with the states $|0\rangle$, $|2h_a\rangle$ associated to momenta P_3 and the corresponding ones for P_4, which carry the information of the polarizations. It is however trivial in the sense that it is proportional to the identity for such states, in particular being independent of h_a. In the non-relativistic limit we find $s - m_b^2 \to 2m_b E$, with $E \ll m_b$ corresponding to the energy of the massless particle. This shows how the low energy effective potential obtained from (3.56) is independent of the type of massless particle, as long as it is minimally coupled to gravity. This is the universality of the light bending phenomena previously proposed in [42]. It may seem that this claim depends on the choice $u = 0$ or $v = 0$ for defining the massless limit, since for $v = 0$ the operator $\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{[\hat{\lambda}\hat{\eta}]}\right)^{2h_a}$ would certainly show up in the result. However, as argued in the Appendix C, the choice $v = 0$ is supplemented by the choice of a different basis of states for the massless representation, such that this operator is again proportional to the identity and hence independent of h_a.

To argue for the universality at the 1-loop level, we consider the massless limit of (3.51), given by

$$
\Gamma \to (\beta - 1)\left(v(1+y) + v\frac{1-y^2}{2y}\right) , \tag{3.57}
$$

which is precisely the massless limit of $\langle k_3|P_1|k_4]$, i.e. the corresponding factor for the scalar case. The conclusion is that the behavior of $A^{(S_a)}_{(4,2)}$ is again independent of $S_a = h_a$, hence showing the universality. The LS for gravity now reads

$$M^{(1,b)}_{(h_a,S_b,2)} = \left(\frac{\alpha^4}{2^8}\right) \frac{(s-m_b)^2 m_b}{\sqrt{-t}} \int_\infty \frac{dy\,(1+y)^6}{y^3(1-y)^2} \left(1 - \frac{(1+y)^2}{4y} \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b} . \tag{3.58}$$

This leading singularity is all what is needed to compute the classical potential for the massless case, since as explained in subsection 3.3.3 the $a-$topology has vanishing LS. Thus, we note that because there is no need to symmetrize there is no restriction on S_a at all. This means that, up to 1-loop, we have access to all orders of spin corrections for a massless particle interacting with a rotating point-like source. The expression can be used in principle to recover the multipole expansion of the Kerr black hole solution up to order G^2, see discussion.

3.5 Discussion

In this chapter we have proposed the implementation of a new technique, the Leading Singularity, in order to extract in a direct manner the classical behavior of a variety of scattering amplitudes, including arbitrarily high order spin effects. This classical piece can then be used to construct an effective field theory for long range gravitational or electromagnetic interactions. It was shown in [63] that for the gravitational case the 1-loop correction to such interaction is completely encoded into the triangle leading singularity. In this chapter we have reproduced this result and extended the argument to the electromagnetic case in a trivial fashion. The reason this is possible is because the triangle LS captures the precise non-analytic dependence of the form $t^{-\frac{1}{2}}$, which carries the subleading contribution to the potential. As explained in the Introduction, this structure arises from the interplay between massive and massless propagators entering the loop diagrams. This is the case whenever massive particles exchange multiple massless particles which mediate long range forces, such as photons or gravitons.

We have also included the tree level residues for both cases, which correspond to the leading Newtonian and Coulombian potentials. In this case, both computations were completely analogous and the gravitational contribution could be derived by “squaring” the electromagnetic one. This is reminiscent of the double copy construction. At 1-loop level, such construction is most explicitly realized in the factorization properties of the Compton amplitude. In the overall picture, this set of relations between gravity and EM amplitudes renders the computations completely equivalent. Even though the latter carries phenomenological interest by itself, it can also be regarded as a model for understanding long range effects arising in higher PN corrections, including higher loop and spin orders.

The HCL was designed as a suitable limit to extract the relevant orders in t from the complete classical leading singularities introduced in [63]. When embedded in this framework, the computation of the triangle LS proves not only simpler but also leads directly to $t^{-\frac{1}{2}}$ contribution including all the spin interactions. As explained in section 3.2.1 and explicitly shown in Appendix D, the covariant form of these interactions allows us to discriminate them from the purely quantum higher powers of t, which appear merged in the COM frame. In order to distinguish them we resorted to the following criteria: For a given power of G, a subleading order in $|\vec{q}|$ can be classical if it appears multiplied by the appropriate power of the spin vector $|\vec{S}|$. In the HCL framework this is easily implemented since the combination $|\vec{q}||\vec{S}|$ will always emerge from a covariant form which does not vanish for $t \to 0$. For instance, for $S = \frac{1}{2}$, the spin-orbit interaction only arises from $\epsilon_{\alpha\beta\gamma\delta} P_1^{\alpha} P_3^{\beta} K^{\gamma} S^{\delta}$ and can be tracked directly at leading order.

In striking contrast with previous approaches, the evaluation of spin effects at 1-loop does not involve increased difficulty with respect to the scalar case and can be put on equal footing. This is a direct consequence of implementing the massive representation with spinor helicity variables, which certainly bypasses all the technical difficulties associated to the manipulation of polarization tensors. As an important outcome we have proved that the forms of the higher multipole interactions are independent of the spin we assign to the scattered particles. This is a consequence of the equivalence principle, which we have implemented by assuming minimally coupled amplitudes. The expressions have been explicitly shown to agree with the previous results in the literature for the lowest spin orders, corresponding to $S = 1$ and $S = \frac{1}{2}$, yielding spin-orbit, quadrupole and spin-spin interactions. We emphasize, however, that the proposed expressions correspond to a relativistic completion of these results, in the sense that they contain the full $\vec{p}^2$ expansion.

At this point one could argue that the former difficulty of the diagrammatic computations has been transferred here to the difficulty in performing the matching to the EFT operators. In fact, in order to obtain the effective potential (in terms of vector fields) it is certainly necessary to translate the spinor helicity operators to their standard forms, as was done in Appendix C for $S = \frac{1}{2}$ and $S = 1$. We do not think that this should be regarded as a complication. First, as a consequence of the universality we have found, it is clear that we only need to perform the translation once and for a particle of a given

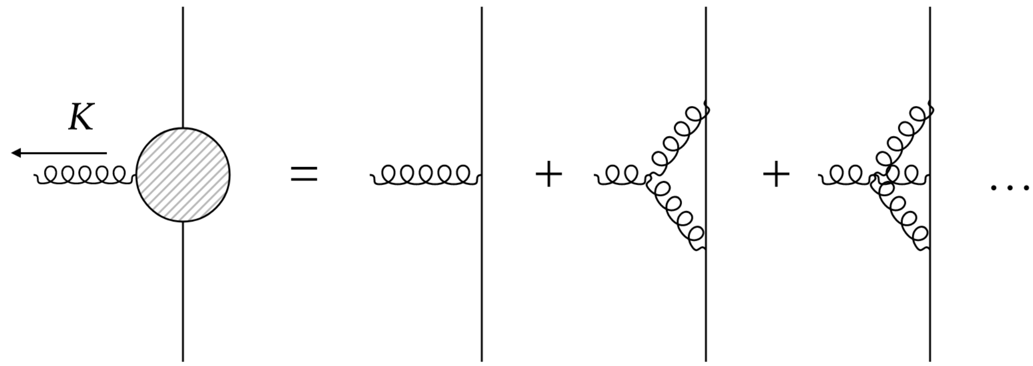


Figure 3.4: The matrix element of the stress-energy tensor $\langle T_{\mu\nu}(K)\rangle$ corresponds to the 3 point function associated to a pair of massive particles and an external off-shell graviton. The coupling to internal gravitons emanating from the massive source yields, in the long-range behavior, higher corrections in G.

spin S, as high as the order of multipole corrections we require. Second and more important, we think that this work along with e.g. [197, 44, 41, 151, 58, 63] will serve as a further motivation towards a complete reformulation of the EFT framework which naturally integrates recent developments in scattering amplitudes. For instance, one could aim for a reformulation of the effective potential, or even better, its replacement by a gauge invariant observable, solely in terms of spinor helicity variables so that no translation is needed to address the dynamics of astrophysical objects. Next we give some proposals for future work along these lines.

The most pressing future direction is the extension of the leading singularity techniques in the context of classical corrections at higher loops [63]. This is supported by the fact that higher orders in the PN expansion are associated to characteristic non-analytic structures in the t channel [197], which are precisely what the LS captures. By consistency with the PN expansion such higher orders would require to include spin multipole corrections, so that both the HCL and the new spin representation emerge as promising additional tools for such construction. One could hope that with these methods the scalar and the spinning case will be again on equal footing. Additionally, the PN expansion also requires to incorporate radiative corrections and finite-size effects. The latter may be included within the spin representation presented here by introducing non-minimal couplings, see e.g. [176].

The first consistency check for higher loop classical corrections is to reproduce known solutions to Einstein equations. In the spirit of [106, 197] and the more modern implementations [191, 128] we could argue that this work indeed represent progress towards the derivation of classical spacetimes from scattering amplitudes. As argued by Donoghue [100, 102] a way to obtain the spacetime metric is to compute the long-range behavior of the off-shell expectation value $\langle T_{\mu\nu}(K)\rangle$ illustrated in Fig. 3.4, which yields the Schwarzschild/Kerr solutions through Einstein equations. At first glance it would seem that is not possible to compute this matrix element using the on-shell methods here exposed. However, this is simply analogous to the fact that we require an off-shell two-body potential for the PN problem. The solution is, of course, to attach another external particle to turn Fig. 3.4 into the scattering process of Fig. 3.1. In this way we can get information about off-shell subprocesses by examining the $2 \to 2$ amplitude.

A simple way to proceed in that direction is to incorporate probe particles whose backreaction can be neglected. In fact, the massless case explored in subsections 3.3.3 and 3.4.3 can be regarded as a probe particle choice. The lack of backreaction is realized in the fact that only one triangle topology is needed for obtaining the classical piece of the amplitude, which in turn can be thought of containing the process of Fig. 3.4. Furthermore, this piece has no restriction in the spin S of the massive particle, i.e. we can compute both the tree level and 1-loop potential to arbitrarily high multipole terms. By extracting the matrix element $\langle T_{\mu\nu}(K)\rangle$ we could recover both leading and subleading orders in G to arbitrary order in angular momentum of the Kerr solution, see also [237]. In fact, it was recently proposed [239] that by examining a probe particle in the Kerr background the generic form of the multipole terms entering the 2-body Hamiltonian can be extracted at leading order in G and arbitrary order in spin.

Of course, it is also tempting to explore the opposite direction, outside the probe particle limit. One could try to obtain an expression for the effective (i.e. long-range) vertex of Fig. 3.4, including higher couplings with spin, expressed in terms of spinor variables. Then an effective potential could be constructed in terms of several copies of these vertices, for instance to address the n-body problem in GR.

Chapter 4

The Aligned-spin Scattering Angle at 2PM

4.1 Introduction

One of the main results of Chapter 2 was a 1PM expression for the momentum deflection Δp of black holes scattering, to all orders in spin. Now, in an aligned-spin case all the three momenta $\vec{a}_1, \vec{a}_2$ and $\vec{L}$ (the orbital angular momentum of the system) are parallel and normal to the scattering plane of the two black holes. Hence the scattering angle $\theta(b) = \frac{|\Delta \mathbf{p}|}{|\mathbf{p}|}$ is well defined in this plane, where $\mathbf{p}$ is the three momentum that lies on it. Such observable is easily deduced from the formulae (2.56), leading to the eikonal formula [164, 4, 39],

$$2 \sin \frac{\theta}{2} = -\frac{E}{(2m_a m_b \gamma v)^2} \frac{\partial}{\partial b} \int \frac{d^2 \boldsymbol{k}}{(2\pi)^2} \, e^{i\boldsymbol{k}\cdot\boldsymbol{b}} \langle \mathcal{M}_4^{(s_a,s_b)} \rangle, \tag{4.1}$$

In this chapter we will pursue this framework and extend it to 2PM order, providing such new results for the first time. We will do so armed with our new analytic tool for the classical computation, the Leading Singularity technique at 1-loop. Also, as it turns out, formula (4.1) can be extended to 2PM in an almost trivial fashion, thereby bypassing the complications we encountered in the previous chapter when aiming for the effective potential. Note that in the most general non-aligned case, the general deflection Δp must be computed instead, as we also did for 1PM order.

We will reformulate the 1-loop computation of the potential done in the previous section to adapt it to the scattering angle $\theta(b)$. A key aspect of this Post-Minkowskian computation will be the soft expansion. To see why, let us recap and further ellaborate on the soft theorems briefly discussed in Chapter 1. In 2014, Cachazo and Strominger [72] showed that the soft limit of tree-level gravity amplitudes is controlled by the action of the angular momentum operator $J^{\mu\nu}$, i.e.

$$\mathcal{M}_{n+1} = \sum_{i=1}^{n} \left[\frac{(p_i \cdot \varepsilon)^2}{p_i \cdot k} + i \frac{(p_i \cdot \varepsilon)(k_\mu \varepsilon_\nu J_i^{\mu\nu})}{p_i \cdot k} - \frac{1}{2} \frac{(k_\mu \varepsilon_\nu J_i^{\mu\nu})^2}{p_i \cdot k} \right] \mathcal{M}_n + \mathcal{O}(k^2), \tag{4.2}$$

up to sub-subleading order. Here the soft momentum k corresponds to the external soft graviton, and we have constructed its polarization tensor as $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$. The sum is over the remaining external particles with momenta p_i^μ, and the operators $J_i^{\mu\nu}$ acting on them include both orbital and spin parts of the angular momentum. The first term is simply the standard Weinberg soft factor [243], whose universality is associated to the equivalence principle. Following the QED results of Low [185, 187], the subleading behaviour of gravity amplitudes was first studied long ago by Gross and Jackiw [133, 162]. Indeed, it was already observed in [133, 162] that the subleading soft theorem follows from gauge invariance (see [248, 29] for a modern perspective), and because of this, it also adopts a universal form up to subleading order. Starting at sub-subleading order the soft expansion can depend on the matter content and EFT operators present in the theory [171, 227, 34], although it is known that gauge invariance still provides partial information at all orders [137, 181]. On a different front, the realization that soft theorems correspond to Ward identities for asymptotic symmetries at null infinity [232] has led to impressive and wide-reaching developments [144, 72, 74, 166, 29, 109, 77], see [233] for a recent review. Following such correspondence, an infinite tower of Ward identities has indeed been proposed to follow from all orders in the soft expansion [78].

Recently, a classical version of the soft theorem up to sub-subleading order has been used by Laddha and Sen [172] to derive the spectrum of the radiated power in black-hole scattering with external soft graviton insertions. This relies on the remarkable fact that conservative and non-conservative long-range effects of interacting black holes can be computed from the scattering of massive point-like sources [108, 48, 197, 39]. Indeed, as we have seen in Chapter 1, rotating black holes can be treated via a spin-multipole expansion, the order $2s$ of which can be reproduced by scattering spin-s minimally coupled particles exchanging gravitons [236].

Here we present a complementary picture to the one of Chapter 1 by employing the soft theorem in the conservative sector (i.e. no external gravitons), focusing on rotating black

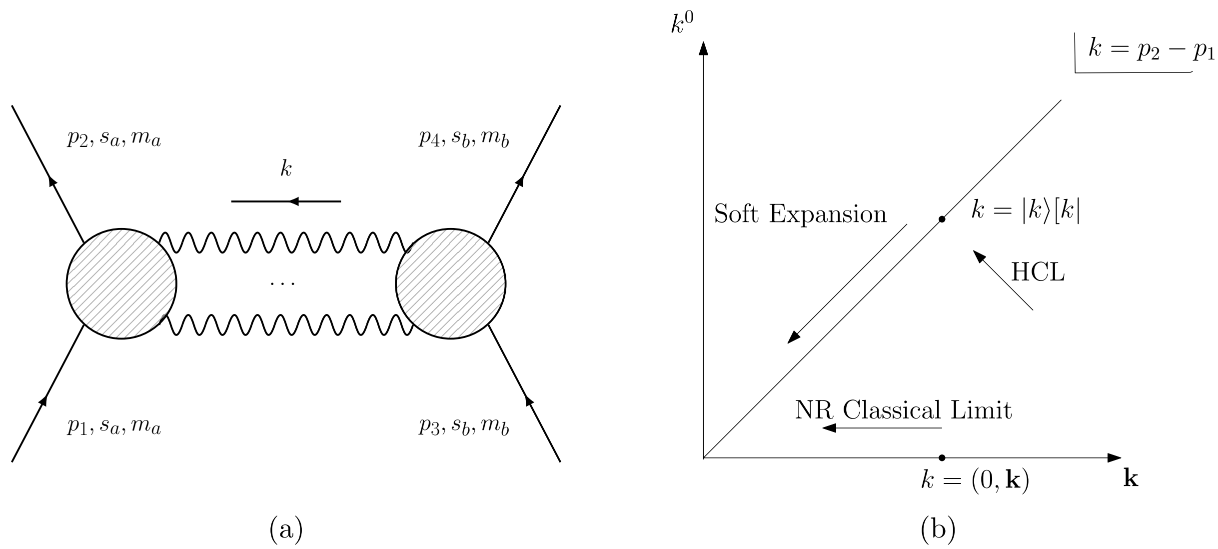


Figure 4.1: **(a)** Four-point amplitude involving the exchange of soft gravitons, which leads to classical observables. The external massive states are interpreted as two black-hole sources.
(b) Comparison between the HCL and the non-relativistic limit in the COM frame [155, 154, 236]. Spin effects require subleading orders in the nonrelativistic (NR) classical limit, but can be fully determined at the leading order in HCL through the soft expansion.

holes and at the same time extending the soft factor in (4.2) to higher orders in the soft expansion. This is achieved in the following way: We use the Holomorphic Classical Limit (HCL) from the previous chapter, which sets the external kinematics such that the momentum transfer k between the massive sources is null. On the support of the leading-singularity construction, which drops $\mathcal{O}(\hbar)$ parts, the condition $k^2 = 0$ reduces the amplitude to a purely classical expansion in spin multipoles of the form $\sim k^n S^n$, where S carries the intrinsic angular momentum of the black hole (see figure 4.1b). This precisely matches the soft expansion once the momentum transfer is recognized as the graviton momentum and the classical spin vector S is identified with the angular momentum J_i of the matter particles.

Even though the fact that classical gravitational quantities can be reproduced from QFT computations has been known for a long time, the precise conceptual foundations of the matching are still lacking.[1] The goal of one of the previous section was simply to show the agreement of the LS method with the previous computations of [154, 155, 236]. It is

[1]Very recent progress on relating classical observables to quantum amplitudes has been made in [170].

only after computing the effective potential from this amplitude that one matches the post-Newtonian potential of general relativity.

In the previous chapter the computation of the classical piece of the amplitude was made direct, through the leading singularity, for arbitrary spin and all orders in the center-of-mass energy E. Both the tree-level and one-loop versions of this computation correspond to a single order in the post-Minkowskian (PM) expansion (see e.g. recent discussion in [95, 35, 238, 97, 36, 39, 86, 240] and many more references therein), i.e. at a fixed power of G. However, the explicit match to the standard QFT amplitude was only performed up to spin-1 and leading order in E (which corresponds to the standard PN expansion). Moreover, the computation of the PN effective potential through the Born approximation suffers some complications [155, 197]. Such potential is not gauge-invariant, i.e. not an observable, and can undergo canonical and non-canonical transformations that become cumbersome when spin is considered as part of the phase space. Moreover, at one loop the Born approximation itself requires the subtraction of tree-level pieces and suffers from some (apparent) inconsistencies already at spin-1 [154]. For these reasons a more direct conversion from the LS into a gravitational observable is evidently needed. Very recently, a direct approach was proposed in the amplitudes setup to evaluate the scattering angle of classical general relativity [39], i.e. the deflection angle of two massive particles in the large-impact-parameter regime. It was demonstrated that for scalar particles the scattering angle computed by Westphal [247] can be obtained via a simple 2D Fourier transform of the classical limit of the amplitude.

Here we will show that the natural extension of 2PM the scattering angle for aligned spins (4.1) can be computed with spinning particles directly from the LS. The building blocks needed for this computation are the three-point amplitude and the Compton amplitude for massive spinning particles interacting with soft gravitons. We will use the soft expansion with respect to the internal gravitons to write the building blocks in an exponentiated form, which fits naturally into the Fourier transform leading to the first and second post-Minkowskian (1PM and 2PM) scattering angles in a resummed form.

We find the following expression for the aligned-spin scattering angle θ as a function of the masses m_a and m_b, the rescaled spins (ring radii, intrinsic angular momenta per mass) a_a and a_b, the relative velocity at infinity v, and the proper impact parameter b (the impact parameter separating the zeroth-order/asymptotic worldlines defined by each black hole's Tulczyjew spin supplementary condition [235]):

$$\theta = \frac{GE}{v^2}\left[\frac{(1+v)^2}{b+a_a+a_b} + \frac{(1-v)^2}{b-a_a-a_b}\right] - \pi G^2 E \frac{\partial}{\partial b}\left[m_b f(a_a, a_b) + m_a f(a_b, a_a)\right] + \mathcal{O}(G^3), \tag{4.3a}$$

where $E=\sqrt{m_a^2+m_b^2+2m_am_b\gamma}$ with $\gamma=(1-v^2)^{-1/2}$, and

$$f(\sigma,a)=\frac{1}{2a^2}\left(-b+\frac{(\jmath+\varkappa-2a)^5}{4v\varkappa\big[(\jmath+\varkappa)^2-(2va)^2\big]^{3/2}}\right)+\mathcal{O}(\sigma^5), \tag{4.3b}$$

with

$$\jmath=vb+\sigma+a, \qquad \varkappa=\sqrt{\jmath^2-4va(b+v\sigma)}. \tag{4.3c}$$

This agrees with previous classical computations to all orders in spin at tree level (at linear order in G) [35, 238] and through linear order in spin at one loop (at order G^2) [36], as well as with the conjectural one-loop quadratic-in-spin expression presented in [240]. Moreover, eq. (4.3) resums those contributions in a compact form, including higher orders in spin. We have indicated that the expression (4.3b) is valid up to quartic order in one of the spins (but to all orders in the other spin) according to the minimally coupled higher-spin amplitudes.

4.2 Multipole expansion of three- and four-point amplitudes

4.2.1 Massive spin-1 matter and the Generalized Expectation Value (GEV)

We start our discussion of the multipole expansion by dissecting the case of graviton emission by two massive vector (spin-1) fields. The reader should note that this greatly resembles the discussion on chapter 1 with the difference that here we start from the amplitude derived in Appendix F from Feynman rules, and we replace the notion of boosts by a normalization we call the Generalized Expectation Value (GEV).

The corresponding three-particle amplitude obtained in Appendix F reads[2]

$$\mathcal{M}_3(p_1,p_2,k)=-2(p\cdot\varepsilon)\big[(p\cdot\varepsilon)(\varepsilon_1\cdot\varepsilon_2)-2k_\mu\varepsilon_\nu\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}\big], \qquad p=\frac{1}{2}(p_1-p_2), \tag{4.4}$$

where p is the average momentum of the spin-1 particle before and after the graviton

[2] We omit the constant-coupling prefactors $-(\kappa/2)^{n-2}$ in front of tree-level amplitudes, we use $\kappa=\sqrt{32\pi G}$. Also note that we work in the mostly-minus metric signature.

emission and the polarization tensor of the graviton $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$ (with momentum $k = -p_1 - p_2$) is split into two massless polarization vectors. The derivation of eq. (4.4) from the Proca action is detailed in Appendex F, which also motivates that the term involving $\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}$ can be thought of as an angular-momentum contribution to the scattering. In other words, we are tempted to interpret the combination $\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}$ as being (proportional to) the classical spin tensor.

However, we now face our first challenge: as explained in [154, 155, 236], the spin-1 amplitude contains up to quadrupole interactions, i.e. quadratic in spin, whereas only the linear piece is apparent in eq. (4.4). To rewrite this contribution in terms of multipoles, we can use a redefined spin tensor

$$S^{\mu\nu} = -\frac{i}{\varepsilon_1 \cdot \varepsilon_2} \left\{ 2\varepsilon_1^{[\mu}\varepsilon_2^{\nu]} - \frac{1}{m^2} p^{[\mu} \big((k\cdot\varepsilon_2)\varepsilon_1 + (k\cdot\varepsilon_1)\varepsilon_2 \big)^{\nu]} \right\}. \tag{4.5}$$

It is introduced in Appendex G via a two-particle expectation value/matrix element, which we call the generalized expectation value (GEV)

$$S^{\mu\nu} = \frac{\varepsilon_{2\sigma} \hat{\Sigma}^{\mu\nu,\sigma}{}_\tau \varepsilon_1^\tau}{\varepsilon_{2\sigma}\varepsilon_1^\sigma}. \tag{4.6}$$

Here $\hat{\Sigma}^{\mu\nu}$ is constructed as an angular-momentum operator shifted in such a way that its GEV satisfies the Fokker-Tulczyjew covariant spin supplementary condition (SSC) [116, 235]

$$p_\mu S^{\mu\nu} = 0. \tag{4.7}$$

In this chapter we find this condition to be crucial for the matching to the rotating-black-hole computation of [238], as the classical spin tensor $S^{\mu\nu}$ satisfies the above SSC by definition. The purpose of this SSC is to constrain the mass-dipole components S^{0i} of the spin tensor of an object to vanish in its rest frame. In a classical setting it puts the reference point for the intrinsic spin of a spatially extended object at its rest-frame center of mass.

Inserting this spin tensor in eq. (4.8), we rewrite the above amplitude as

$$\mathcal{M}_3(p_1, p_2, k) = -m^2 x^2 (\varepsilon_1 \cdot \varepsilon_2) \left[1 - \frac{i\sqrt{2}}{mx} k_\mu \varepsilon_\nu S^{\mu\nu} + \frac{(k\cdot\varepsilon_1)(k\cdot\varepsilon_2)}{m^2(\varepsilon_1\cdot\varepsilon_2)} \right], \tag{4.8}$$

where for further convenience we also expressed the scalar products $p \cdot \varepsilon$ using a helicity variable x first introduced in [12]

$$x = \sqrt{2}\,\frac{p\cdot\varepsilon}{m} \tag{4.9}$$

(at higher points it becomes gauge-dependent but can still be used as a shorthand). Now, in the GEV of the amplitude,

$$\langle\mathcal{M}_3\rangle = \frac{\varepsilon_{2\sigma}\mathcal{M}_3^{\sigma\tau}\varepsilon_{1,\tau}}{\varepsilon_{2\sigma}\varepsilon_1^{\sigma}} = -m^2x^2\left[1 - i\frac{k_\mu\varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon} + \frac{(k\cdot\varepsilon_1)(k\cdot\varepsilon_2)}{m^2(\varepsilon_1\cdot\varepsilon_2)}\right], \tag{4.10}$$

we recognize the dipole coupling as the term linear in both k and S. Indeed, particles with spin couple naturally to the field-strength tensor of the graviton $F_{\mu\nu} = 2k_{[\mu}\varepsilon_{\nu]}$, analogously to the magnetic dipole moment $F_{\mu\nu}S^{\mu\nu}$.[3] Following the non-relativistic limit, the third term was identified in [154, 155, 236, 134] to be the quadrupole interaction $\propto \big(F_{\mu\nu}S^{\mu\nu}\big)^2$ for spin-1. It may seem a priori puzzling that we wish to regard the interaction $(k\cdot\varepsilon_1)(k\cdot\varepsilon_2)$ as the square of $F_{\mu\nu}S^{\mu\nu}$. This is because the statement is true at the levels of spin operators, but not at the level of (generalized) expectation values, i.e. $\langle F_{\mu\nu}\hat{\Sigma}^{\mu\nu}\rangle^2 \neq \langle\big(F_{\mu\nu}\hat{\Sigma}^{\mu\nu}\big)^2\rangle$. In order to expose the exponential structure described in the introduction and construct such spin operators at any order, we are going to recast the multipole expansion in terms of spinor-helicity variables.

Spin-1 amplitude in spinor-helicity variables

Based on the massive spinor formulae reviewed in Appendix C, we can now write down concrete spinor-helicity expressions for the amplitude (4.4). Choosing the polarization of the graviton to be negative, we have

$$\varepsilon_1^{a_1a_2}\cdot\varepsilon_2^{b_1b_2} = -\frac{1}{m^2}\langle 1^{(a_1}2^{(b_1}\rangle\left[\langle 1^{a_2)}2^{b_2)}\rangle - \frac{1}{mx}\langle 1^{a_2)}k\rangle\langle k\,2^{b_2)}\rangle\right], \tag{4.11a}$$

$$\big[(\varepsilon_1\cdot\varepsilon_2)k_\mu\varepsilon_\nu^- S^{\mu\nu}\big]^{a_1a_2b_1b_2} = \frac{i}{\sqrt{2}m^2}\langle 1^{(a_1}k\rangle\left[\langle 1^{a_2)}2^{(b_1}\rangle - \frac{1}{2mx}\langle 1^{a_2)}k\rangle\langle k\,2^{(b_1}\rangle\right]\langle k\,2^{b_2)}\rangle, \tag{4.11b}$$

$$(k\cdot\varepsilon_1^{a_1a_2})(k\cdot\varepsilon_2^{b_1b_2}) = -\frac{1}{2m^2x^2}\langle 1^{(a_1}k\rangle\langle 1^{a_2)}k\rangle\langle k\,2^{(b_1}\rangle\langle k\,2^{b_2)}\rangle, \tag{4.11c}$$

[3] We thank Yu-tin Huang for emphasizing to us the analogy to the electromagnetic Zeeman coupling, see e.g. [150, 127]. Indeed, in a non-covariant form, this was already related to the soft expansion long ago [244].

where we have reduced all $[1^a|$ and $|2^b]$ to the chiral spinor basis of $\langle 1^a|$ and $|2^b\rangle$ using the following identities for the three-point kinematics,[4]

$$[1^a k] = x^{-1}\langle 1^a k\rangle, \qquad [2^b k] = -x^{-1}\langle 2^b k\rangle, \qquad [1^a 2^b] = \langle 1^a 2^b\rangle - \frac{1}{mx}\langle 1^a k\rangle\langle k\, 2^b\rangle. \tag{4.12}$$

We also use x for x_- henceforth, i.e. it carries helicity -1 unless stated otherwise. From eq. (4.11) we can see that going to the chiral spinor basis has both an advantage and a disadvantage. On the one hand, the multipole expansion becomes transparent in the sense that the spin order of a term is identified by the leading power of $|k\rangle\langle k|$. On the other hand, the exponential structure of the vector basis is spoiled by a shift by higher multipole terms. However, this is just an artifact of the chiral basis, and we should see that the answer obtained from the generalized expectation value is the same.

The main advantage of the spinor-helicity variables for what we wish to achieve in this thesis is that now we can switch to spinor tensors $\langle 1^{(a_1}| \otimes \langle 1^{a_2)}|$ and $|2^{(b_1}\rangle \otimes |2^{b_2)}\rangle$, as representations of the massive-particle states 1 and 2. Introducing the symbol $\odot$ for the symmetrized tensor product, we can rewrite eq. (4.11a) as

$$\varepsilon_1\!\cdot\!\varepsilon_2 = -\frac{1}{m^2}\langle 1|^{\odot 2}\bigg[\mathbb{I}\odot\mathbb{I} - \frac{1}{mx}\mathbb{I}\odot|k\rangle\langle k|\bigg]|2\rangle^{\odot 2} = -\frac{1}{m^2}\bigg[\langle 12\rangle^{\odot 2} - \frac{1}{mx}\langle 12\rangle\odot\langle 1k\rangle\langle k2\rangle\bigg]. \tag{4.13}$$

Here the operators have their lower indices symmetrized, i.e. $(A\odot B)_{\alpha_1\beta_1}^{\alpha_2\beta_2} = A_{(\alpha_1}^{\beta_1} B_{\alpha_2)}^{\beta_2}$, and the notation assumes that the reader keeps in mind the spins associated with each momentum. Combining all the terms in eq. (4.11) into the amplitude, we obtain

$$\mathcal{M}_3(p_1,p_2,k^-) = x^2\bigg[\langle 12\rangle^{\odot 2} - \frac{2}{mx}\langle 12\rangle\odot\langle 1k\rangle\langle k2\rangle + \frac{1}{m^2x^2}\langle 1k\rangle^{\odot 2}\langle k2\rangle^{\odot 2}\bigg]. \tag{4.14}$$

Now in the multipole expansion of the Kerr stress-energy tensor (2.6), the quadrupole operator is of the simple form $(k_\mu\varepsilon_\nu S^{\mu\nu})^2$, whereas in our amplitude (4.8) it has the form $(k\cdot\varepsilon_1)(k\cdot\varepsilon_2) \propto \langle 1k\rangle^{\odot 2}\langle k2\rangle^{\odot 2}$. One then could wonder if in some sense the latter is the square of $(k_\mu\varepsilon_\nu S^{\mu\nu})$. We now show that this is precisely the case if the angular momentum is realized as a differential operator.

In appendix H we construct the differential form of the angular-momentum operator in momentum space, which involves the standard orbital piece and the "intrinsic" contribution dependent on spin. This operator admits a much simpler realization in

[4]The transition between the chiral spinors $|p^a\rangle$ and the antichiral ones $|p^a]$ is always possible [13] via the Dirac equations $p^{\dot\alpha\beta}|p^a\rangle_\beta = m|p^a]^{\dot\alpha}$ and $p_{\alpha\dot\beta}|p^a]^{\dot\beta} = m|p^a\rangle_\alpha$.

terms of spinor variables, similar to the one derived in [251] for the massless case. For a massive particle of momentum $p_{\alpha\dot{\beta}} = \lambda_{p\alpha}{}^{a}\tilde{\lambda}_{p\dot{\beta}a}$ we find that the differential operator for the total angular momentum is given by

$$J_{\alpha\dot{\alpha},\beta\dot{\beta}} = 2i\left[\lambda_{p(\alpha}{}^{a}\frac{\partial}{\partial\lambda_p^{\beta)a}}\epsilon_{\dot{\alpha}\dot{\beta}} + \epsilon_{\alpha\beta}\tilde{\lambda}_{p(\dot{\alpha}}{}^{a}\frac{\partial}{\partial\tilde{\lambda}_p^{\dot{\beta})a}}\right]. \tag{4.15}$$

We can now act with the operator $k_\mu\varepsilon_\nu J^{\mu\nu}$ on the product state $|p^a\rangle^{\odot 2} = |p^{a_1}\rangle \otimes |p^{a_2}\rangle$. For the negative helicity of the graviton, we have

$$k_\mu\varepsilon_\nu^- J^{\mu\nu} = \frac{1}{4\sqrt{2}}\lambda^\alpha\lambda^\beta\epsilon^{\dot{\alpha}\dot{\beta}}J_{\alpha\dot{\alpha},\beta\dot{\beta}} = -\frac{i}{\sqrt{2}}\langle k\,p^a\rangle\langle k\,\frac{\partial}{\partial\lambda_p^a}\rangle, \qquad \langle k\,\frac{\partial}{\partial\lambda_p^b}\rangle|p^a\rangle = |k\rangle\delta_b^a. \tag{4.16}$$

Applying the spinor differential operator above, we find[5]

$$\left(\frac{ik_\mu\varepsilon_\nu^- J^{\mu\nu}}{p\cdot\varepsilon^-}\right)|p\rangle^2 = \frac{2}{mx}|k\rangle\langle k\,p\rangle|p\rangle, \tag{4.17a}$$

$$\left(\frac{ik_\mu\varepsilon_\nu^- J^{\mu\nu}}{p\cdot\varepsilon^-}\right)^2|p\rangle^2 = \frac{2}{m^2x^2}|k\rangle^2\langle k\,p\rangle^2, \tag{4.17b}$$

$$\left(\frac{ik_\mu\varepsilon_\nu^- J^{\mu\nu}}{p\cdot\varepsilon^-}\right)^j|p\rangle^2 = 0, \qquad j \geq 3. \tag{4.17c}$$

Although it is the differential operator that realizes the soft theorem, its algebraic form is easy to obtain on three-particle kinematics. Indeed, if we take a tensor-product version $-(\sigma^{\mu\nu}\otimes\mathbb{I} + \mathbb{I}\otimes\sigma^{\mu\nu})$ of the standard $\mathrm{SL}(2,\mathbb{C})$ chiral generator $\sigma^{\mu\nu} = i\sigma^{[\mu}\bar{\sigma}^{\nu]}/2$ and use it as an algebraic realization of $J^{\mu\nu}$, it is direct to check that it acts in the same way as the differential operator above:

$$\frac{ik_\mu\varepsilon_\nu^- J^{\mu\nu}}{p\cdot\varepsilon^-} = \frac{|k\rangle\langle k|}{mx}\otimes\mathbb{I} + \mathbb{I}\otimes\frac{|k\rangle\langle k|}{mx}. \tag{4.18}$$

[5] More explicitly, we have

$$\begin{aligned} i\sqrt{2}(k_\mu\varepsilon_\nu^- J^{\mu\nu})|p^a\rangle^{\odot 2} &= \langle k\,p^b\rangle\left\{\left[\langle k\,\frac{\partial}{\partial\lambda_p^b}\rangle|p^{a_1}\rangle\right]\otimes|p^{a_2}\rangle + |p^{a_1}\rangle\otimes\left[\langle k\,\frac{\partial}{\partial\lambda_p^b}\rangle|p^{a_2}\rangle\right]\right\} \\ &= |k\rangle\langle k\,p^{a_1}\rangle\otimes|p^{a_2}\rangle + |p^{a_1}\rangle\otimes|k\rangle\langle k\,p^{a_2}\rangle = 2|k\rangle\langle k\,p^a\rangle\odot|p^a\rangle, \end{aligned}$$

with similar manipulations for higher powers.

These identities allow us to reinterpret the last two terms in the amplitude formula (4.14) as the non-zero powers of this dipole operator acting on the state $|1\rangle^2$:

$$-\frac{2}{mx}\langle 12\rangle\langle 1k\rangle\langle k2\rangle = \langle 2|^2\left(\frac{ik_\mu\varepsilon^-_\nu J_1^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)|1\rangle^2, \qquad \frac{1}{m^2x^2}\langle 1k\rangle^2\langle k2\rangle^2 = \frac{1}{2}\langle 2|^2\left(\frac{ik_\mu\varepsilon^-_\nu J_1^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)^2|1\rangle^2, \tag{4.19}$$

and rewrite the amplitude as

$$\mathcal{M}_3(p_1,p_2,k^-) = x^2\langle 2|^2\left\{1+i\left(\frac{k_\mu\varepsilon^-_\nu J_1^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)-\frac{1}{2}\left(\frac{k_\mu\varepsilon^-_\nu J_1^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)^2\right\}|1\rangle^2. \tag{4.20}$$

It is now clear that these terms

- match the differential operators of the soft expansion (4.2);
- correspond to the scalar, spin dipole and quadrupole interactions in the expansion of the Kerr energy momentum tensor (2.6) and its spin-1 amplitude representation (4.10). Note that the sign flip in the dipole term comes from the sign difference between the algebraic and differential Lorentz generators, as pointed out in the beginning of appendix H.

In this way, we interpret the three terms in the amplitude (4.14) as the multipole contributions with respect to the chiral spinor basis, despite the fact that they do not equal the multipoles in eq. (4.8) individually. Furthermore, as the operator $(k_\mu\varepsilon^-_\nu J^{\mu\nu})^j$ annihilates the spin-1 state for $j\geq 3$, the three terms can be obtained from an exponential

$$\mathcal{M}_3(p_1,p_2,k^-) = x^2\langle 2|^2\exp\left(i\frac{k_\mu\varepsilon^-_\nu J^{\mu\nu}}{p\cdot\varepsilon^-}\right)|1\rangle^2. \tag{4.21}$$

It can be checked explicitly that acting with the operator on the state $\langle 2|^2$ yields the same result, i.e. in this sense the operator $k_\mu\varepsilon_\nu J^{\mu\nu}/(p\cdot\varepsilon)$ is self-adjoint.[6] On the other hand, choosing the other helicity of the graviton will yield the parity conjugated version of eq. (4.21):

$$\mathcal{M}_3(p_1,p_2,k^+) = \frac{1}{x^2}[2|^2\exp\left(i\frac{k_\mu\varepsilon^+_\nu J^{\mu\nu}}{p\cdot\varepsilon^+}\right)|1]^2. \tag{4.22}$$

[6]The division by $p\cdot\varepsilon$ implicitly relies on the fact that the action of $k_\mu\varepsilon_\nu J^{\mu\nu}$ on the helicity variable x vanishes. Note also that $k_\mu\varepsilon_\nu J^{\mu\nu}/(p\cdot\varepsilon)$ should become $k_\mu\varepsilon_\nu J_2^{\mu\nu}/(p_2\cdot\varepsilon)$ when acting on $|2\rangle^2$.

In the next section we extend this procedure to arbitrary spin. Let us point out that the explicit amplitude can be brought into a compact form by changing the spinor basis. In fact, the three-point identities (4.12) imply that the amplitude formula (4.14) collapses into

$$\mathcal{M}_3(p_1, p_2, k^-) = [12]^2 x^2. \tag{4.23}$$

However, let us stress that this form completely hides the spin structure that was already explicit in the vector form (4.8). The purpose of the insertion of the differential operators is precisely to extract the spin-dependent pieces from the minimal coupling (4.23), which will then be matched to the Kerr black hole.

4.2.2 Exponential form of gravitational Compton amplitude

As we have already obtained an exponential form of the three point amplitude, the task of this section is to extend the construction presented to the Compton amplitude, without the support of three-particle kinematics.[7] In particular, we will show that for the cases of interest the following holds

$$\hat{\mathcal{M}}_4^{(s)}(p_1, p_2, k_3^+, k_4^-) = \mathcal{M}_4^{(0)} \exp\left(i \frac{k_\mu \varepsilon_\nu J^{\mu\nu}}{p \cdot \varepsilon} \right). \tag{4.24}$$

Here the linear and angular momentum p and $J^{\mu\nu}$ in the exponential operator may act either on massive state 1 or 2. Moreover, the momentum k and the polarization vector ε can be associated to either of the two gravitons. Explicitly, we have

$$[2|^{2s} \exp\left(i \frac{k_{3\mu} \varepsilon_{3\nu}^+ J^{\mu\nu}}{p \cdot \varepsilon_3^+} \right) |1]^{2s} = \langle 2|^{2s} \exp\left(i \frac{k_{4\mu} \varepsilon_{4\nu}^- J^{\mu\nu}}{p \cdot \varepsilon_4^-} \right) |1\rangle^{2s}. \tag{4.25}$$

The importance of this amplitude (as opposed to the same-helicity case) is that it controls the classical contribution at order G^2, as was shown directly in [134, 39]. in [134] the classical piece was argued to lead to the correct 2PN potential after a Fourier transform. In the new approach of [39] the classical contribution in the spinless case was identified by computing the scattering angle. In section 4.3 we will use the Compton amplitude as an input for computing the scattering angle with spin up to order S^4,

[7] Historically, the Compton amplitude was the prototype in the discovery of subleading soft theorems [185, 133, 162]. The construction provided in section 4.2.3 is in a sense reminiscent of Low's original derivation of the subleading factor in QED [185].

agreeing with previously known results at order S^2. We will see that this exponential form is extremely suitable for the computation of the latter as a Fourier transform.

Our strategy is the following: we first consider the action of the exponentiated soft factor acting on the three-point amplitude, as an all-order extension of the Cachazo-Strominger soft theorem. We have checked that this agrees with the known versions of the Compton amplitude [13, 46] for $s \leq 2$. We leave the problem of obtaining the case $s \geq 2$ for future investigation, but we will comment on it at the end of section 4.2.3.

To obtain eq. (4.24) we first propose an all-order extension of the soft expansion (4.2) with respect to the graviton $k_4 = |4\rangle[4|$:

$$\left[\frac{(p_1 \cdot \varepsilon_4)^2}{p_1 \cdot k_4} \exp\left(i \frac{k_{4\mu}\varepsilon_{4\nu}J_1^{\mu\nu}}{p_1 \cdot \varepsilon_4}\right) + \frac{(p_2 \cdot \varepsilon_4)^2}{p_2 \cdot k_4} \exp\left(i \frac{k_{4\mu}\varepsilon_{4\nu}J_2^{\mu\nu}}{p_2 \cdot \varepsilon_4}\right) \right.$$
$$\left. + \frac{(k_3 \cdot \varepsilon_4)^2}{k_3 \cdot k_4} \exp\left(i \frac{k_{4\mu}\varepsilon_{4\nu}J_3^{\mu\nu}}{k_3 \cdot \varepsilon_4}\right)\right] \mathcal{M}_3^{(s)}(p_1, p_2, k_3^+). \tag{4.26}$$

As stated in the introduction, two main problems arise when trying to interpret eq. (4.2) as an exponential acting on the lower-point amplitude. The first is that gauge invariance of the denominator $p_i \cdot \varepsilon_4$ is not guaranteed. Here we simply fix $\varepsilon_4^- = \sqrt{2}|4\rangle[3|/[43]$, so the last term in eq. (4.26) vanishes, as we will show in a moment. The second problem is that one still has to sum over two exponentials, which would spoil the factorization of eq. (4.24). The solution is that in this case both exponentials give the exact same contribution. In the language of the previous section, this is the fact that one can act with the operator $k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}/(p \cdot \varepsilon_4)$ either on $\langle 2|^{2s}$ or $|1\rangle^{2s}$, giving the same result.

Let us first inspect the three-point amplitude entering eq. (4.26),

$$\mathcal{M}_3^{(s)}(p_1, p_2, k_3^+) = \mathcal{M}_3^{(0)} \frac{\langle 12\rangle^{2s}}{m^{2s}}, \qquad \mathcal{M}_3^{(0)} = m^2 x_3^2 = \frac{\langle 4|1|3]^2}{\langle 34\rangle^2} = \frac{\langle 4|1|2|4\rangle^2}{\langle 34\rangle^4}, \tag{4.27}$$

where we used $\varepsilon_3^+ = \sqrt{2}|4\rangle[3|/\langle 43\rangle$. As explained in [72], in order for the action of the differential operator to be well defined, we need to solve momentum conservation and express $\mathcal{M}_3^{(0)}$ in terms of independent variables. Solving for $|3]$ and $|4]$ yields the last expression in eq. (4.27). Now to evaluate the third term in eq. (4.26), we recall from appendix H

$$J^{\text{self-dual}}_{3\,\alpha\dot{\alpha},\beta\dot{\beta}} = 2i\lambda_{3(\alpha}\frac{\partial}{\partial\lambda_3^{\beta)}}\epsilon_{\dot{\alpha}\dot{\beta}} \qquad \Rightarrow \qquad k_{4\mu}\varepsilon_{4\nu}J_3^{\mu\nu} = -\frac{i}{\sqrt{2}}\langle 43\rangle\langle 4\frac{\partial}{\partial\lambda_3}\rangle. \tag{4.28}$$

As the only place where $\langle 3|$ appears in eq. (4.27) is in the contraction with $|4\rangle$, we see that the above differential operator annihilates the scalar three-point amplitude $\mathcal{M}_3^{(0)}$. Moreover, since the prefactor $\langle 12\rangle^{2s}$ in the spin-s amplitude $\mathcal{M}_3^{(s)}$ does not depend on $|3\rangle$, we conclude that the exponential operator in the third term of (4.26) acts always trivially. The zeroth-order of the soft theorem $\propto (k_3 \cdot \varepsilon_4)^2$ then vanishes by going to the chosen gauge, hence the last term drops as promised.

Let us now look at the angular momenta of the massive particles. A similar inspection of $\langle 4|1|2|4\rangle = \langle 4\,1^a\rangle[1_a 2_b]\langle 2^b 4\rangle$ shows that the scalar piece $\mathcal{M}_0^{(3)}$ is in the kernel of the operators

$$k_{4\mu}\varepsilon_{4\nu}J_1^{\mu\nu} = -\frac{i}{\sqrt{2}}\langle 4\,1^a\rangle\langle 4\,\frac{\partial}{\partial\lambda_1^a}\rangle, \qquad k_{4\mu}\varepsilon_{4\nu}J_2^{\mu\nu} = -\frac{i}{\sqrt{2}}\langle 4\,2^a\rangle\langle 4\,\frac{\partial}{\partial\lambda_2^a}\rangle. \tag{4.29}$$

Therefore, eq. (4.26) is simplified to

$$\mathcal{M}_3^{(0)}\left[\frac{(p_1\cdot\varepsilon_4)^2}{p_1\cdot k_4}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J_1^{\mu\nu}}{p_1\cdot\varepsilon_4}\right)+\frac{(p_2\cdot\varepsilon_4)^2}{p_2\cdot k_4}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J_2^{\mu\nu}}{p_2\cdot\varepsilon_4}\right)\right]\frac{\langle 12\rangle^{2s}}{m^{2s}}. \tag{4.30}$$

Moreover, our choice of the reference spinor for ε_4 implies $p_1\cdot\varepsilon_4 = -p_2\cdot\varepsilon_4 = p\cdot\varepsilon$, where $p = (p_1 - p_2)/2$ is the average momentum of the massive particle before and after Compton scattering.

From the discussion of the previous section on the action of the angular-momentum operator on $\langle 2|^{2s}$ and $|1\rangle^{2s}$, we also have

$$\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J_1^{\mu\nu}}{p_1\cdot\varepsilon_4}\right)\langle 12\rangle^{2s} = \exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J_2^{\mu\nu}}{p_2\cdot\varepsilon_4}\right)\langle 12\rangle^{2s} = \langle 2|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s}. \tag{4.31}$$

Hence we obtain

$$\frac{1}{m^{2s}}\mathcal{M}_3^{(0)}\left[\frac{(p_1\cdot\varepsilon_4)^2}{p_1\cdot k_4}+\frac{(p_2\cdot\varepsilon_4)^2}{p_2\cdot k_4}\right]\langle 2|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s}, \tag{4.32}$$

where we recognize the scalar Weinberg soft factor. Recall that in this gauge $k_3\cdot\varepsilon_4 = 0$, so there is no contribution from the other graviton. As an easy check, we observe that the scalar Compton amplitude, written e.g. in [13, 46], can be constructed solely from this soft factor:

$$\mathcal{M}_4^{(0)} = \mathcal{M}_3^{(0)}\left[\frac{(p_1\cdot\varepsilon_4)^2}{p_1\cdot k_4}+\frac{(p_2\cdot\varepsilon_4)^2}{p_2\cdot k_4}\right] = -\frac{\langle 4|1|3]^4}{(2p_1\cdot k_4)(2p_2\cdot k_4)(2k_3\cdot k_4)}. \tag{4.33}$$

This proves that eq. (4.24) can be obtained from the all-order extension of the soft theorem (4.26). Finally, the property (4.25) is checked by repeating the computation for the opposite-helicity graviton k_3.

4.2.3 Factorization and soft theorems

In view of the exponentiation formulas, we now show how factorization is realized in this operator framework. For the pole $(k_3+k_4)^2 \to 0$ it is evident, so we will focus on the pole $(p_1 \cdot k_4) \to 0$. In that limit the scalar part factors as $\mathcal{M}_4^{(0)} \to \mathcal{M}_{3,\mathrm{L}}^{(0)}\mathcal{M}_{3,\mathrm{R}}^{(0)}/(2p_1\cdot k_4)$ corresponding to the product of the respective three-point amplitudes. Let us denote the internal momentum by $p_I = p_1 + k_4$. Unitarity demands that the operator piece in (4.24) behaves as

$$\langle 2|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s} \to \frac{1}{m^{2s}}[2|^{2s}\exp\left(i\frac{k_{3\mu}\varepsilon_{3\nu}J^{\mu\nu}}{p\cdot\varepsilon_3}\right)|I_a]^{2s}\langle I^a|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s}. \tag{4.34}$$

Here the insertion of $p_I = |I_a]\langle I^a|$ is needed since the exponential operators act on different bases. In order to show the above property, it is enough to write the left factor in the chiral basis, which is possible on the three-particle kinematics of the factorization channel:

$$\begin{aligned}\frac{1}{m^{2s}}[2|^{2s}\exp\left(i\frac{k_{3\mu}\varepsilon_{3\nu}J^{\mu\nu}}{p\cdot\varepsilon_3}\right)|I_a]^{2s}\langle I^a|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s} \\ = \frac{1}{m^{2s}}\langle 2\, I_a\rangle^{2s}\langle I^a|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s} = \langle 2|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s}.\end{aligned} \tag{4.35}$$

On the other hand, we could have inserted the resolution of the identity in the right factor

$$\begin{aligned}\frac{1}{m^{2s}}[2|^{2s}\exp\left(i\frac{k_{3\mu}\varepsilon_{3\nu}J^{\mu\nu}}{p\cdot\varepsilon_3}\right)|I_a]^{2s}\langle I^a|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s} \\ = \frac{1}{m^{2s}}[2|^{2s}\exp\left(i\frac{k_{3\mu}\varepsilon_{3\nu}J^{\mu\nu}}{p\cdot\varepsilon_3}\right)|I_a]^{2s}[I^a 1]^{2s} = [2|^{2s}\exp\left(i\frac{k_{3\mu}\varepsilon_{3\nu}J^{\mu\nu}}{p\cdot\varepsilon_3}\right)|1]^{2s}.\end{aligned} \tag{4.36}$$

Putting this together with the scalar piece we can write, for instance,

$$\mathcal{M}_4^{(s)} \xrightarrow[p_1\cdot k_4\to 0]{} \frac{\mathcal{M}_{3,\mathrm{L}}^{(0)}\mathcal{M}_{3,\mathrm{R}}^{(0)}}{2p_1\cdot k_4}\frac{1}{m^{2s}}\langle 2|^{2s}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}J^{\mu\nu}}{p\cdot\varepsilon_4}\right)|1\rangle^{2s} \tag{4.37}$$

$$= \frac{\mathcal{M}_{3,\mathrm{R}}^{(0)}}{2p_1 \cdot k_4} \exp\left(i \frac{k_{4\mu} \varepsilon_{4\nu} J_1^{\mu\nu}}{p_1 \cdot \varepsilon_4} \right) \mathcal{M}_{3,\mathrm{L}}^{(0)} \frac{\langle 12 \rangle^{2s}}{m^{2s}} = \frac{(p_1 \cdot \varepsilon_4)^2}{p_1 \cdot k_4} \exp\left(i \frac{k_{4\mu} \varepsilon_{4\nu} J_1^{\mu\nu}}{p_1 \cdot \varepsilon_4} \right) \mathcal{M}_{3,\mathrm{L}}^{(s)}.$$

Here, using $\mathcal{M}_{3,\mathrm{R}}^{(0)} = \mathcal{M}_3^{(0)}(p_1, p_I, k_4^-) = 2(p_1 \cdot \varepsilon_4^-)^2$, we have recovered the extension of the soft theorem (4.26), that we used as a starting point of this section, in the limit $p_1 \cdot k_4 \to 0$. The origin of the exponential soft factor in this case is nothing but the three-point amplitude of spin-s particles, written as a series in the angular momentum. Therefore, in our case the statement of the subsubleading soft theorem (4.2) follows from factorization of amplitudes of massive particles with spin.

Let us remark that, in analogy to the three-point case, the exponential factor can be brought into a compact form. For example, one can check that

$$\langle 2|^{2s} \exp\left(i \frac{k_{4\mu} \varepsilon_{4\nu} J_1^{\mu\nu}}{p_1 \cdot \varepsilon_4} \right) |1\rangle^{2s} = \left[\langle 21 \rangle + \frac{[43]}{\langle 4|1|3]} \langle 24 \rangle \langle 41 \rangle \right]^{2s} = m^{2s} \left(\frac{[13]\langle 42 \rangle + \langle 14 \rangle [32]}{\langle 4|1|3]} \right)^{2s}, \tag{4.38}$$

which converts the Compton amplitude into the form

$$\mathcal{M}_4^{(s)} = -\frac{\langle 4|1|3]^{4-2s}}{(2p_1 \cdot k_4)(2p_2 \cdot k_4)(2k_3 \cdot k_4)} \big([13]\langle 42 \rangle + \langle 14 \rangle [32] \big)^{2s} \tag{4.39}$$

that is given in [13]. We remark, however, that this expression completely hides the spin dependence that we need here for the classical computation.

It was pointed out in [13] that the formula (4.39) is only valid up to $s \leq 2$. For higher spins, one has to eliminate the spurious pole $\langle 4|1|3]$ that appears at the fifth order by adding contact terms. From our perspective, this spurious pole corresponds precisely to the contribution from $p_1 \cdot \varepsilon_4$ appearing at higher orders in the soft expansion (4.38). Let us remark, however, that our result (4.24) non-trivially extends the Cachazo-Strominger soft theorem in the case of the Compton amplitude for minimally coupled spinning particles. This is because for $s = 2$ the exponential is truncated only at the fourth order in the angular momentum, whereas only the second order was guaranteed by the soft theorem. This extension is what enables us in section 4.3 to obtain the scattering angle at order S^4, by means of a Fourier transform acting directly on the exponential. We leave the study of the contributions from contact terms at higher spin orders for future work.

4.3 Scattering angle as Leading Singularity

4.3.1 Linearized stress-energy tensor of Kerr Solution

In section 4.2 we have shown that the three-point and Compton amplitudes can be written in an exponential form. We have also motivated the definition of a generalized expectation value of an operator $\mathcal{O}$ acting on two massive states, represented by their polarization tensors,

$$\langle \mathcal{O} \rangle = \frac{\varepsilon_{2,\mu_1 \ldots \mu_s} \mathcal{O}^{\mu_1 \ldots \mu_s, \nu_1 \ldots \nu_s} \varepsilon_{1,\nu_1 \ldots \nu_s}}{\varepsilon_{2,\mu_1 \ldots \mu_s} \varepsilon_1^{\mu_1 \ldots \mu_s}}. \tag{4.40}$$

Let us first show how to apply this definition to match the form of the stress-energy tensor of a single Kerr black hole that we derived in the introduction:

$$h_{\mu\nu}(k) T^{\mu\nu}(-k) = (2\pi)^2 \delta(k^2) \delta(p \cdot k) (p \cdot \varepsilon)^2 \exp\left(-i \frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p \cdot \varepsilon}\right), \tag{4.41}$$

There is a subtle but important point already present in this classical matching that will guide us in the following subsection on a path to the classical scattering angle. The crucial difference between the angular momentum operator $J^{\mu\nu}$ appearing in the soft theorem and the classical spin $S^{\mu\nu}$ appearing in the expansion of $T^{\mu\nu}$ is that the latter satisfies the SSC (4.7). Moreover, there is an obvious sign flip in the respective exponents, due to the sign difference between the differential and algebraic generators, as mentioned in section 4.2.1 and appendix H. Therefore, following section 4.2.1 (see also appendix G) we relate the two by

$$J^{\mu\nu} = -S^{\mu\nu} + \frac{1}{m^2} p^\mu p_\alpha J^{\alpha\nu} - \frac{1}{m^2} p^\nu p_\alpha J^{\alpha\mu}, \tag{4.42}$$

which implies that the soft operator reads, at $p \cdot k = 0$,

$$\frac{k_\mu \varepsilon_\nu J^{\mu\nu}}{p \cdot \varepsilon} = -\frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p \cdot \varepsilon} + \frac{1}{m^2} k_\mu p_\nu J^{\mu\nu}. \tag{4.43}$$

The key observation is that this operator acts on a chiral representation. That is, for negative helicity, if the states are built from the spinors $|1\rangle^{2s}$ and $|2\rangle^{2s}$ then the operator is algebraically realized by $J^{\mu\nu} = -\sigma^{\mu\nu} = -i\sigma^{[\mu}\bar{\sigma}^{\nu]}/2$, which is self-dual. This means that

$$\frac{1}{m^2}k_\mu p_\nu J^{\mu\nu} = \frac{i}{2m^2}\epsilon^{\mu\nu\rho\sigma}k_\mu p_\nu J_{\rho\sigma} = -\frac{i}{2m^2}\epsilon^{\mu\nu\rho\sigma}k_\mu p_\nu S_{\rho\sigma} = -ia\cdot k. \tag{4.44}$$

On the three-point kinematics, one can show that

$$a\cdot k = \pm i\frac{k_\mu\varepsilon^\pm_\nu S^{\mu\nu}}{p\cdot\varepsilon^\pm}, \tag{4.45}$$

so eq. (4.43) becomes

$$\frac{k_\mu\varepsilon_\nu J^{\mu\nu}}{p\cdot\varepsilon} = -2\frac{k_\mu\varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}. \tag{4.46}$$

It can be checked that this factor-of-two relation is independent of the helicity of the graviton. To compute the generalized expectation value, we will also need to consider the product $\varepsilon_1^{(s)}\cdot\varepsilon_2^{(s)}$. To that end we use the following representation of polarization tensors, obtained as tensor products of the spin-1 polarization vectors (C.3)

$$\varepsilon_1^{(s)} = \varepsilon_1^{\otimes s} = \frac{2^{s/2}}{m^s}\big(|1\rangle[1|\big)^{\odot s}, \qquad \varepsilon_2^{(s)} = \varepsilon_2^{\otimes s} = \frac{2^{s/2}}{m^s}\big(|2\rangle[2|\big)^{\odot s}, \tag{4.47}$$

where we now take p_2 to be outgoing, so $|2\rangle$ is minus that of section 4.2. This leads to

$$\begin{aligned}
\lim_{s\to\infty} m^{2s}\varepsilon_{2,\mu_1\ldots\mu_s}\varepsilon_1^{\mu_1\ldots\mu_s} &= \lim_{s\to\infty}\langle 21\rangle^s[12]^s = \lim_{s\to\infty}\langle 2|^{2s}\left(1+\frac{|k\rangle\langle k|}{mx}\right)^s|1\rangle^{2s} \\
&= \lim_{s\to\infty}\langle 2|^{2s}\left[\sum_{j=0}^{s}\binom{s}{j}\left(\frac{|k\rangle\langle k|}{mx}\right)^j\right]|1\rangle^{2s} = \lim_{s\to\infty}\langle 2|^{2s}\left[\sum_{j=0}^{s}\binom{2s}{j}\left(\frac{|k\rangle\langle k|}{2mx}\right)^j\right]|1\rangle^{2s} \\
&= \lim_{s\to\infty}\langle 2|^{2s}\exp\!\left(\frac{i}{2}\frac{k_\mu\varepsilon_\nu J^{\mu\nu}}{p\cdot\varepsilon}\right)|1\rangle^{2s} = \lim_{s\to\infty}\exp\!\left(-i\frac{k_\mu\varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}\right)\langle 21\rangle^{2s},
\end{aligned} \tag{4.48}$$

where we have used the $s\to\infty$ limit and in the last line we extracted the operator as a GEV. The same manipulation can be done for the three-point minus-helicity amplitude:

$$\lim_{s\to\infty} m^{2s}\varepsilon_{2,\mu_1\ldots\mu_s}\mathcal{M}_3^{(s),\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}\varepsilon_{1,\nu_1\ldots\nu_s} = m^2x^2\lim_{s\to\infty}\exp\!\left(-2i\frac{k_\mu\varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}\right)\langle 21\rangle^{2s}. \tag{4.49}$$

Here we would like to emphasize a key point. Even though the exponential operator is always present at finite spin, it is only in the infinite-spin limit that the expansion does not truncate. This leads to

$$\lim_{s\to\infty}\langle\mathcal{M}_3^{(s)}\rangle = 2(p\cdot\varepsilon)^2\exp\!\left(-i\frac{k_\mu\varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}\right), \tag{4.50}$$

which recovers the Kerr gravitational coupling (4.41), as promised in (2.6) — this time with the SSC condition incorporated. The plus-helicity graviton gives the same GEV. One can also keep the minus helicity and redo the computation in the antichiral basis:

$$\lim_{s\to\infty} m^{2s}\varepsilon_{2,\mu_1\ldots\mu_s}\mathcal{M}_3^{(s),\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}\varepsilon_{1,\nu_1\ldots\nu_s} = m^2x^2\lim_{s\to\infty}[21]^{2s}, \tag{4.51a}$$

$$\lim_{s\to\infty} m^{2s}\varepsilon_{2,\mu_1\ldots\mu_s}\varepsilon_1^{\mu_1\ldots\mu_s} = \lim_{s\to\infty}\exp\!\left(-i\frac{k_\mu\varepsilon_\nu^+ S^{\mu\nu}}{p\cdot\varepsilon^+}\right)[21]^{2s} = \lim_{s\to\infty}\exp\!\left(i\frac{k_\mu\varepsilon_\nu^- S^{\mu\nu}}{p\cdot\varepsilon^-}\right)[21]^{2s}. \tag{4.51b}$$

Therefore, the GEV (4.50) is invariant with respect to the choice of the spinor basis as well.

Finally, we notice that the self-dual condition is natural when considering a definite-helicity coupling, e.g. $k_\mu\varepsilon_\nu^- J^{\mu\nu}$ projects out the anti-self-dual piece. However, we should keep in mind that this is just an artifact of our choice of chiral spinor basis to describe that coupling. It would be interesting to find a non-chiral form, analogous to the vector parametrization of section 4.2.1, in such a way that the amplitude already contains the covariant-SSC spin tensor built in.

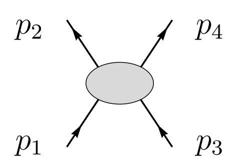
4.3.2 Kinematics and scattering angle for aligned spins

We now consider scattering of two massive spinning particles, one with mass m_a, spin (quantum number) s_a, initial momentum p_1, and final momentum p_2, and the other with mass m_b, spin s_b, initial momentum p_3, and final momentum p_4,

$$p_1^2 = p_2^2 = m_a^2, \qquad p_3^2 = p_4^2 = m_b^2, \tag{4.52}$$

following here the conventions of [134]. The total amplitude

$$\mathcal{M}_4^{(s_a,s_b)} \quad = \tag{4.53}$$



is a function of the external momenta and the external spin states (polarization tensors). We define as usual

$$s = p_{\rm tot}^2, \qquad t = k^2, \tag{4.54}$$

where $p_{\rm tot}$ is the total momentum, and k is the momentum transfer,

$$p_{\rm tot} = p_1 + p_3 = p_2 + p_4, \qquad k = p_2 - p_1 = p_3 - p_4. \tag{4.55}$$

The Mandelstam variable s, the total center-of-mass-frame energy E, the relative velocity v (between the inertial frames attached to the incoming momenta p_1 and p_3, with $v > 0$), and the corresponding relative Lorentz factor γ — each of which determines all the others, given fixed rest masses m_a and m_b — are related by

$$s = E^2 = m_a^2 + m_b^2 + 2m_a m_b \gamma, \qquad \frac{p_1 \cdot p_3}{m_a m_b} = \gamma = \frac{1}{\sqrt{1-v^2}}. \tag{4.56}$$

At $t = 0$, it is convenient to fix the little-group scaling of the internal graviton (for tree-level one-graviton exchange). Following [134], we can choose it as

$$x_b = \sqrt{2}\,\frac{p_b \cdot \varepsilon^-(-k)}{m_b} = -\sqrt{2}\,\frac{p_b \cdot \varepsilon^-}{m_b} = 1. \tag{4.57}$$

This implies

$$x_a^{-1} = -\sqrt{2}\,\frac{p_a \cdot \varepsilon^+}{m_a} = -\frac{\langle r|p_a|k]}{m_a\langle r\,k\rangle} = \gamma(1-v), \qquad x_a = \sqrt{2}\,\frac{p_a \cdot \varepsilon^-}{m_a} = -\frac{[r|p_a|k\rangle}{m_a[r\,k]} = \gamma(1+v). \tag{4.58}$$

We consider the case, in the classical limit, in which the two particles' rescaled spin vectors

$$a_a^\mu = \frac{1}{2m_a^2}\varepsilon^\mu{}_{\nu\rho\sigma}p_a^\nu S_a^{\rho\sigma}, \qquad a_b^\mu = \frac{1}{2m_b^2}\varepsilon^\mu{}_{\nu\rho\sigma}p_b^\nu S_b^{\rho\sigma}, \tag{4.59}$$

are aligned with the system's total angular momentum. They are orthogonal to the constant scattering plane, and are conserved. The scattering plane is defined containing all the momenta, see e.g. [238]. Here p_a is the average momentum $p_a = (p_1 + p_2)/2 = p_1 + \mathcal{O}(k) = p_2 + O(k)$, similarly for p_b. In this "aligned-spin case", up to order G^2, we will find that the classical scattering angle θ by which both bodies are scattered in the center-of-mass frame, is given by the same relation as for the spinless case [164, 4, 39],

$$\theta + \mathcal{O}(\theta^3) = 2\sin\frac{\theta}{2} = -\frac{E}{(2m_a m_b \gamma v)^2}\frac{\partial}{\partial b}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{b}} \lim_{s_a, s_b \to \infty} \langle \mathcal{M}_4^{(s_a,s_b)} \rangle + \mathcal{O}(G^3), \quad (4.60)$$

where $\langle \mathcal{M}_4^{(s_a,s_b)} \rangle$ is the generalized expectation value of the amplitude (4.53), the momentum transfer $\boldsymbol{k}$ is integrated over the 2D scattering plane, and $\boldsymbol{b}$ is the vectorial impact parameter with magnitude b, counted from the second particle to the first as in [238]. Compared to the nonspinning/scalar case, this version of (4.60) differs only in that the aligned spin components a_a and a_b, the magnitudes of the vectors in (4.59), will appear as scalar parameters in the amplitude. While we do not claim to provide a first-principles derivation of the applicability of (4.60) to the spinning case with aligned spins, we find that its use here produces results which are (quite nontrivially) fully consistent with the results of [35, 238, 36, 240] for aligned-spin scattering angles for binary black holes.

4.3.3 Second post-Minkowskian order

In this section we derive a compact form for the 2PM (or $\mathcal{O}(G^2)$) aligned-spin scattering angle. It is obtained from the one-loop version of the four-point amplitude (4.53) through the triangle leading singularity proposed in [134] for computing its classical piece. The LS is now given by a contour integral for a single complex variable y that remains in the loop integration after cutting the three propagators of figure 4.2:

$$\ell^2(y) = m_b^2, \qquad (p_3 - \ell(y))^2 = 0, \qquad (p_4 - \ell(y))^2 = 0. \quad (4.61)$$

It was argued in [43, 62, 134] that for the spinless case the Compton amplitude for identical helicities leads to no classical contribution. This fact is also true for arbitrary spin, as will be proven somewhere else. This implies that only the opposite-helicity case treated in section 4.2.2 is needed, together with three-point interactions. The derivation is thus valid (to describe minimally coupled elementary particles) at least up to $\mathcal{O}(a_a^4)$ and to all orders in a_b, where a_a is the rescaled spin of the particle that appears in the Compton amplitude, and a_b is the spin of other particle. As explained already in [13, 134] and emphasized in section 4.2.2 the Compton amplitude needs the introduction of contact terms for $s_a > 2$. Nevertheless, the exponential structure found already for $s_a \leq 2$ fits very nicely into the Fourier transform and leads to a compact formula for the scattering function, which can be computed directly once the multipole operators have been identified. The final formula resums all orders in both spins, but is not justified

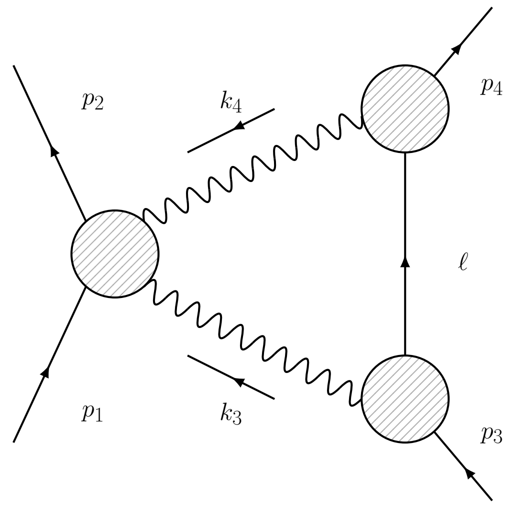


Figure 4.2: Triangle leading-singularity configuration

starting at $\mathcal{O}(a_a^5)$. We finally expand in spins and find perfect agreement with the linear- and quadratic-order-in-spin results of [36] and [240]. The computation of the possible contributions to the LS from contact terms arising in the higher-spin Compton amplitude is left for future work.

Our strategy is to identify the spin-multipole-coupling operators $\boldsymbol{k} \times \hat{\boldsymbol{p}} \cdot \boldsymbol{a}_a$ and $\boldsymbol{k} \times \hat{\boldsymbol{p}} \cdot \boldsymbol{a}_b$ in the exponential form of the three and four point amplitudes entering the triangle leading singularity, see figure 4.2. This is done on the support of the Holomorphic Classical Limit,[8] which accounts for a null momentum transfer $k^2 = 0$ and recovers the three-point kinematics studied in section 4.2. The soft expansion in k accounts for a simultaneous expansion in both powers of spin.

Let us first recap the triangle leading singularity, also introducing a more economic formulation of it. It consists of a contour integral obtained by gluing three-point amplitudes with the Compton amplitude. Our starting point is the expression

$$\frac{i(\kappa/2)^4}{8m_b\sqrt{-t}}\int_{\Gamma_{\text{LS}}}\frac{dy}{2\pi y}\hat{\mathcal{M}}_4^{(s_a)}(p_1,-p_2,k_3^+,k_4^-)\otimes\hat{\mathcal{M}}_3^{(s_b)}(p_3,-\ell,-k_3^-)\frac{|\ell\rangle^{2s}\langle\ell|^{2s}}{m_b^{2s}}\hat{\mathcal{M}}_3^{(s_b)}(-p_4,\ell,-k_4^+), \tag{4.62}$$

where we have inserted the operator $|\ell\rangle\langle\ell|$ in-between the three-point amplitudes to denote operator multiplication, in the same sense as in section 4.2.1. Here Γ_{LS} is the leading-singularity contour that can be obtained at either $|y| = \epsilon$ or $|y| \to \infty$. The loop momenta, together with their corresponding spinors, are functions of y given by

[8] The name "Holomorphic Classical Limit" is due to the external momenta being complex at that point.

eq. (3.17) of [134]. Here we will only need the following limits:

$$
\begin{aligned}
|k_3] &= \frac{1}{2}|k](1+y) + \mathcal{O}\left(\frac{\sqrt{-t}}{m_b}\right), & \langle k_3| &= \frac{1}{2y}\langle k|(1+y) + \mathcal{O}\left(\frac{\sqrt{-t}}{m_b}\right),\\
|k_4] &= \frac{1}{2}|k](1-y) + \mathcal{O}\left(\frac{\sqrt{-t}}{m_b}\right), & \langle k_4| &= -\frac{1}{2y}\langle k|(1-y) + \mathcal{O}\left(\frac{\sqrt{-t}}{m_b}\right),\\
\langle k_3 k_4\rangle &= \frac{\sqrt{-t}}{y} + \mathcal{O}\left(\frac{t}{m_b^2}\right), & \langle k_4|p_1|k_3] &= \frac{m_a\gamma}{2y}[2y - v(1+y^2)]\sqrt{-t} + \mathcal{O}\left(\frac{t}{m_b^2}\right).
\end{aligned}
\tag{4.63}
$$

Recall that at $t=0$ the momentum transfer reads $k = |k]\langle k|$ and the scaling of the spinors $|k]$, $\langle k|$ is fixed by the condition (4.57). In turn, this fixes the little-group scaling of both internal gravitons k_3 and k_4. We can now insert the exponential expressions (for $s_a \leq 2$) and evaluate the scalar pieces, obtaining

$$
\frac{i(\kappa/2)^4}{8m_b\sqrt{-t}}\int_{\Gamma_{\mathrm{LS}}}\frac{dy}{2\pi y}\mathcal{M}_4^{(0)}(p_1,-p_2,k_3^+,k_4^-)\mathcal{M}_3^{(0)}(p_3,-\ell,-k_3^-)\mathcal{M}_3^{(0)}(-p_4,\ell,-k_4^+) \tag{4.64}
$$

$$
\times \exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}^- J_a^{\mu\nu}}{p_1\cdot\varepsilon_4^-}\right)\otimes\exp\left(-i\frac{k_{3\mu}\varepsilon_{3\nu}^- J_b^{\mu\nu}}{p_3\cdot\varepsilon_3^-}\right) \tag{4.65}
$$

$$
= -\frac{i\kappa^4 m_a^2 m_b^3\gamma^2}{2^9 v^2\sqrt{-t}}\int_{\Gamma_{\mathrm{LS}}}\frac{dy\,[2y-v(1+y^2)]^4}{2\pi y^3(1-y^2)^2}\exp\left(i\frac{k_{4\mu}\varepsilon_{4\nu}^- J_a^{\mu\nu}}{p_1\cdot\varepsilon_4^-}\right)\otimes\exp\left(-i\frac{k_{3\mu}\varepsilon_{3\nu}^- J_b^{\mu\nu}}{p_3\cdot\varepsilon_3^-}\right), \tag{4.66}
$$

to leading orders in t.

Before proceeding to compute the GEV, let us clarify an important point. Recall that in the tree-level case the exponential operator was truncated at order $2s$ in the expansion. The infinite spin limit did not alter the lower orders in the exponential but simply accounted for promoting such finite number of terms to a full series. We assume such condition still holds for the Compton amplitude, that is, the first five orders reproducing the exponential expansion are not spoiled in the infinite spin limit. The reason is that at arbitrary spin, the introduction of contact terms is only needed to cancel the spurious pole coming from the exponent, which appears as a pole in the amplitude only at fifth order.

With the previous consideration, the above operator formula in the infinite spin limit is fourth-order exact in the expansion of the left exponential and fully exact in the expansion of the right exponential. Let us now proceed to evaluate the exponents of both. The exponential factor on the right can be obtained straight at $t=0$ kinematics.

In fact, using

$$k_3 = \frac{(1+y)^2}{4y} k, \tag{4.67}$$

we find

$$\exp\left(-i\frac{k_{3\mu}\varepsilon^{-}_{3\nu}J_b^{\mu\nu}}{p_3\cdot\varepsilon^{-}_3}\right) = \exp\left(-i\frac{(1+y)^2}{4y}\frac{k_\mu\varepsilon^{-}_\nu J_b^{\mu\nu}}{p_3\cdot\varepsilon^{-}}\right) = \exp\left(-i\frac{(1+y)^2}{2y}\boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_b\right), \tag{4.68}$$

where the polarization vector ε_3^- for k_3 can be taken as the vector ε^- for k, up to a scale that cancels. We have again identified $k_\mu \varepsilon_\nu J_b^{\mu\nu}/(p_3\cdot\varepsilon_3) = 2\boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_b$ as the classical operator that will enter the GEV, whereas the y dependence contributes to the contour integral.

Now, recall that the left exponential corresponds to the Compton amplitude and was fixed in section 4.2.2 using $k_3\cdot\varepsilon_4 = 0$, i.e.

$$\varepsilon_4^- = -\sqrt{2}\frac{|k_3]\langle k_4|}{[k_3 k_4]}, \tag{4.69}$$

which is singular at $t=0$. In order to evaluate, it we will need the following trick. First note that at $t\neq 0$ the numerator is gauge invariant, hence we can write

$$k_{4\mu}\varepsilon^{-}_{4\nu}J_a^{\mu\nu} = k_{4\mu}\hat{\varepsilon}^{-}_{4\nu}J_a^{\mu\nu}, \tag{4.70}$$

where

$$\hat{\varepsilon}_4^- = -\sqrt{2}\frac{|r]\langle k_4|}{[r\,k_4]} \tag{4.71}$$

and $|r]$ is some reference spinor such that $[r\,k_4]\neq 0$. This means that in the limit we have

$$\begin{aligned}\lim_{t\to 0}\frac{k_{4\mu}\varepsilon^{-}_{4\nu}J_a^{\mu\nu}}{p_1\cdot\varepsilon_4^-} &= \left(k_{4\mu}\hat{\varepsilon}^{-}_{4\nu}J_a^{\mu\nu}\right)_{t=0}\lim_{t\to 0}(p_1\cdot\varepsilon_4^-)^{-1} \\ &= \left(\frac{k_{4\mu}\hat{\varepsilon}^{-}_{4\nu}J_a^{\mu\nu}}{p_1\cdot\hat{\varepsilon}_4^-}\right)_{t=0}\times (p_1\cdot\hat{\varepsilon}_4^-)\big|_{t=0}\lim_{t\to 0}(p_1\cdot\varepsilon_4^-)^{-1}\,. \end{aligned}\tag{4.72}$$

The limit can be evaluated directly using eq. (4.63). We find

$$\lim_{t\to 0}(p_1 \cdot \varepsilon_4^-) = -\frac{\gamma m_a}{2\sqrt{2}y^2}\big[2y - v(1+y^2)\big]. \tag{4.73}$$

On the other hand, recall that at $t=0$ we recover three-particle kinematics for p_1, p_2 and k. This means that the combination

$$(p_1 \cdot \hat{\varepsilon}_4^-)\big|_{t=0} = -\frac{[r|p_1|k_4\rangle}{\sqrt{2}[r\,k_4]}\bigg|_{t=0} = +\frac{1}{y}\frac{[r|p_1|k\rangle}{\sqrt{2}[r\,k]} \tag{4.74}$$

is independent of the choice of r. Using eq. (4.58) we can identify this factor with

$$-\frac{1}{y}(p_1 \cdot \varepsilon^-) = -\frac{\gamma m_a}{\sqrt{2}y}(1+v), \tag{4.75}$$

Putting all together in (4.72) and using $k_4 = -\frac{(1-y)^2}{4y}k$, we have

$$\begin{aligned}\lim_{t\to 0}\frac{k_{4\mu}\varepsilon_{4\nu}^- J_a^{\mu\nu}}{p_1\cdot\varepsilon_4^-} &= \left(\frac{k_{4\mu}\hat{\varepsilon}_{4\nu}^- J_a^{\mu\nu}}{p_1\cdot\hat{\varepsilon}_4^-}\right)_{t=0} \times \frac{1}{y}(p_1\cdot\varepsilon^-) \times \frac{2\sqrt{2}y^2}{\gamma m_a}\big[2y - v(1+y^2)\big]^{-1} \\ &= -\frac{(1-y)^2(1+v)}{4y - 2v(1+y^2)}\left(\frac{k_\mu \varepsilon_\nu^- J_a^{\mu\nu}}{p_1\cdot\varepsilon^-}\right) = -\frac{(1-y)^2(1+v)}{2y - v(1+y^2)}\boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_a.\end{aligned} \tag{4.76}$$

Attaching the same normalization as in the previous section in order to compute the GEV, we write the leading order (i.e. dropping $\mathcal{O}(t^0)$ terms) of our contour integral as

$$-\frac{i\kappa^4 m_a^2 m_b^3 \gamma^2}{2^9 v^2\sqrt{-t}}\int_{\Gamma_{\text{LS}}}\frac{dy\,[2y - v(1+y^2)]^4}{2\pi y^3(1-y^2)^2}\exp\!\left(-i\frac{1+y^2-2vy}{2y - v(1+y^2)}\boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_a - i\frac{1+y^2}{2y}\boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_b\right). \tag{4.77}$$

As already explained, Γ_{LS} can be chosen as a contour around zero or infinity. This inversion accounts for a parity conjugation of the amplitude, and the equivalence follows from parity invariance of the triangle diagram [62]. Here let us unify both descriptions by means of the change of variables

$$z = \frac{1+y^2}{2y}. \tag{4.78}$$

Both contours around $y=\infty$ and $y=0$ are mapped to $z=\infty$. At the same time the polynomial structure gets reduced to at most quadratic, at the cost of introducing a branch cut in the integral. We now have the one-loop triangle contribution as

$$\langle \mathcal{M}_\triangleleft \rangle = -4\pi^2 \frac{G^2 m_a^2 m_b^3}{\sqrt{-t}} \int_{\Gamma_{\rm LS}} \frac{dz}{2\pi i} \frac{\gamma^2 (1-vz)^4}{v^2 (z^2-1)^{3/2}} \exp\left(-i \frac{z-v}{1-vz} \boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_a - iz \boldsymbol{k}\times\hat{\boldsymbol{p}}\cdot\boldsymbol{a}_b \right), \quad (4.79)$$

which now incorporates the second helicity assignment for the exchanged gravitons. We have also inserted a factor of -4 to account for the HCL difference between a triangle integral and its leading singularity. Note that the branch cut singularity is induced by the massive propagators inside the Compton amplitude and does not lead to classical contributions. The essential singularity at $z = 1/v$ is induced by the unphysical pole $p_1 \cdot \varepsilon_4$ in the exponential expansion. We take the contour around infinity to be $\Gamma_{\rm LS} = \{|z| = R\}$ for some large but finite radius, $R > 1/v$, for reasons we will explain in a moment. Then the contribution to the scattering angle (4.60) reads

$$\begin{aligned} \theta_\triangleleft &= \pi G^2 E \frac{m_b}{2v^4} \frac{\partial}{\partial b} \int_{\Gamma_{\rm LS}} \frac{dz}{2\pi i} \frac{(1-vz)^4}{(z^2-1)^{3/2}} \int \frac{d^2\boldsymbol{k}}{2\pi |\boldsymbol{k}|} \exp\left(i\boldsymbol{k} \cdot \left[\boldsymbol{b} - z \hat{\boldsymbol{p}} \times \boldsymbol{a}_b - \frac{z-v}{1-vz} \hat{\boldsymbol{p}} \times \boldsymbol{a}_a \right] \right) \\ &= \pi G^2 E \frac{m_b}{2v^4} \frac{\partial}{\partial b} \int_{\Gamma_{\rm LS}} \frac{dz}{2\pi i} \frac{(1-vz)^4}{(z^2-1)^{3/2}} \left| b - z a_b - \frac{z-v}{1-vz} a_a \right|^{-1}, \end{aligned} \quad (4.80)$$

where we have specialized to aligned spins. The total one-loop contribution to the scattering angle is $\theta_\triangleleft + \theta_\triangleright$, where $\theta_\triangleright$ is obtained by exchanging $m_a \leftrightarrow m_b$ and $a_a \leftrightarrow a_b$.

Let us now discuss the choice of contour $\Gamma_{\rm LS}$ in

$$\int_{\Gamma_{\rm LS}} \frac{dz}{2\pi i} \frac{(1-vz)^4}{(z^2-1)^{3/2}} \left| b - z a_b - \frac{z-v}{1-vz} a_a \right|^{-1} = \frac{1}{va} \int_{\Gamma_{\rm LS}} \frac{dz}{2\pi i} \frac{(vz-1)^5}{(z^2-1)^{3/2} (z-z_+)(z-z_-)}, \quad (4.81a)$$

where

$$z_+ + z_- = \frac{bv + a_a + a_b}{v a_b}, \qquad z_+ z_- = \frac{b + v a_a}{v a_b}. \quad (4.81b)$$

The root z_+ is distinguished from z_- by demanding $z_+ \to \infty$ as $a_b \to 0$. We now show that the appropriate leading singularity in the contour integral is given by the residues at z_+ and ∞, by ensuring the consistency of the small-spin expansion. If we were to take an expansion around $a_a, a_b \to 0$ the poles at z_+ and z_- would disappear at every order, leaving poles only at $z = \infty$ and $z = 1/v$ together with the branch cut at $z \in (-1, 1)$. In that case, the leading-singularity prescription in the integral (4.79) simply grabs the pole at $z = \infty$ and drops the branch cut contribution together with the pole at $z = 1/v$. The non-expanded expression (4.81a) resumes part of the contributions from both $z = \infty$ and

$z = 1/v$ into poles located at z_+ and z_-, respectively. This can be seen by noticing that $z_+ \to \infty$ and $z_- \to 1/v$ as $a_a, a_b \to 0$. This is the reason we consider a contour at finite radius $R > 1/v$ in eq. (4.79), so that, as long as $R < z_+$ as well, the contour integral can be evaluated from the poles at $z = \infty$ and $z = z_+$.

With this contour prescription, evaluating the integral in eq. (4.80) yields the explicit results given by eq. (4.3) in the introductory summary. Let us stress that the formulas (4.80) and (4.3) can only be expected to be valid up to fourth order in a_a. Nevertheless, they condense non-trivial information for the scattering angle up to that order into a simple contour integral.

4.3.4 Checks and Ending Remarks

We have checked that the 2PM scattering angle presented in this section precisely matches the one-loop linear-in-spin classical computation of [36], as well as the conjectural one-loop quadratic-in-spin expression given in [240], based on results from the exact quadrupolar test-black-hole limit [38] expanded to order G^2 and on next-to-next-to-leading-order post-Newtonian results [177, 178]. Let us remark that our results provides a very simple conjecture for higher orders in spin. On other hand, many open questions and future directions are outlined in Chapter 10.

Part II

Scattering Amplitudes in Six Dimensional SYM & Maximal Supergravity from Rational Maps

Chapter 5

Rational Maps and Connected Formulas

5.1 Introduction

So far we have been focused on the study of massive amplitudes interacting with gluons and gravitons. The multipole expansion studied in Part I revealed that these amplitudes contain certain freedom associated to different effective couplings. For instance, an effective theory for a vector boson field W and its conjugate $\bar{W}$ contains a gyromagnetic ratio g associated to the coupling [150],

$$\mathcal{L}_{\text{int}} = (g-1)F^{a}_{\mu\nu}T_a^{I\bar{J}}W_I^{\mu}\bar{W}_{\bar{J}}^{\nu}\,, \tag{5.1}$$

whereas we have found that only the value $g=2$ is consistent with double copy and black hole dynamics. It then seems that an enhancement of symmetry is required in order to fix such ambiguities associated massive particles that appear even at $n=3$ points.

In this part of the thesis we will study scattering amplitudes in maximally supersymmetric theories. Supersymmetry provides a powerful yet simple framework that uniquely determines effective couplings from an underlying symmetry. In fact, in the Coulomb branch of $\mathcal{N}=4$ Super Yang-Mills (SYM) theory the gyromagnetic coupling (5.1) is precisely fixed as $g=2$. This is expected since supersymmetry is also deeply intertwined with double copy, which was indeed first introduced for supersymmetric theories (see for instance the pioneering work [26]).

As reviewed in the Introduction, in 2003 Witten introduced a rational map picture for $\mathcal{N}=4$ SYM theory [251], which led to a line of research that unveiled fascinating structures present in massless amplitudes. The extension of such structures for the massive case has been since then an important problem, with promising developments such as those explored in Part I of this thesis. Furthermore, the (massive) Coulomb branch of SYM is one of the natural candidates for extending the modern techniques of massless scattering, see e.g. [93].

As it is well known, natural bridge between massless and massive scattering amplitudes is provided by dimensional or KK reduction. This motivates us to study massless amplitudes in dimensions $D>4$. In this part of the thesis we will take a first step in extending Witten's construction to higher dimensions, specifically to $D=6$. We will use supersymmetry as a guiding tool for constructing a moduli space integral, analogous to the Witten-RSV formula, in the case of maximally supersymmetric Yang-Mills and Gravity theories. As anticipated, both such theories will be related by a double copy procedure, this time performed (and trivialized) at the level of our integrand.

Applications of our construction not only include $\mathcal{N}=4$ SYM amplitudes, but also scattering amplitudes for D5-branes, supersymmetric theories in $D=5$, a non-abelian $(2,0)$ theory, and a striking connection to a geometrical structure known as the Symplectic Grassmannian.

In order to introduce the formulation, we begin in this chapter by reviewing the rational map picture for an arbitrary space-time dimension. As anticipated in the introductory chapter, this picture is deeply connected with the CHY formulation of massless (bosonic) amplitudes. We then discuss the specialization to 4D and include supersymmetry by studying the Witten–RSV formulas for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA. Finally, we give a short preview of the six dimensional extension introduced in this thesis: We provide the form of the even-n rational maps in 6D, whose generalizations will be the subject of later chapters.

5.2 Review: Rational Maps and the CHY integral

Let us first consider scattering of n massless particles in an arbitrary space-time dimension. Even though we focus here only on bosonic degrees of freedom, this will set the ground to extend the Witten-RSV construction to $D=6$ in the next chapter.

To each particle, labeled by the index i, we associate a puncture at $z=\sigma_i$ on the Riemann sphere, $\mathbb{CP}^1$, whose local coordinate is z. We then introduce polynomial maps,

$p^\mu(z)$, of degree $n-2$. They are constructed such that the momentum p_i^μ associated to the ith particle is given by:

$$p_i^\mu = \frac{1}{2\pi i}\oint_{|z-\sigma_i|=\varepsilon} \frac{p^\mu(z)}{\prod_{j=1}^n (z-\sigma_j)}\, dz, \tag{5.2}$$

which means that $p^\mu(z)$ can be written as a polynomial in z:

$$p^\mu(z) = \sum_{i=1}^n p_i^\mu \prod_{j\neq i}(z-\sigma_j). \tag{5.3}$$

Here we take all momenta to be incoming, so that momentum conservation is given by $\sum_{i=1}^n p_i^\mu = 0$. We call $p^\mu(z)$ the *scattering map*.

In order to relate the positions of the punctures σ_i to the kinematics, the additional condition that the scattering map is null, i.e., $p^2(z)=0$ for all z, is imposed. Since $p^2(z)$ is of degree $2n-4$ and it is already required to vanish at n points, σ_i, requiring $p^\mu(z)$ to be null gives $n-3$ additional constraints. Using (5.3) these constraints can be identified by considering the combination

$$\frac{p^2(z)}{\prod_{i=1}^n (z-\sigma_i)^2} = \sum_{i,j=1}^n \frac{p_i \cdot p_j}{(z-\sigma_i)(z-\sigma_j)} = 0. \tag{5.4}$$

The expression (5.4) does not have any double poles, since the punctures are distinct and all of the momenta are null, $p_i^2 = 0$. Requiring that residues on all the poles vanish implies:

$$E_i := \sum_{j\neq i} \frac{p_i \cdot p_j}{\sigma_{ij}} = 0 \qquad \text{for all } i, \tag{5.5}$$

where $\sigma_{ij} = \sigma_i - \sigma_j$. These are the so-called *scattering equations* [65]. Due to the above counting, only $n-3$ of them are independent. In fact, $\sum_i \sigma_i^\ell E_i$ automatically vanishes for $\ell = 0,1,2$ as a consequence of the mass-shell and momentum-conservation conditions.

Using the $\mathrm{SL}(2,\mathbb{C})$ symmetry of the scattering equations to fix three of the σ_i coordinates, there are $(n-3)!$ solutions of the scattering equations for the remaining σ_i's for generic kinematics [65].

The scattering equations connect the moduli space of n-punctured Riemann spheres to the external kinematic data. Tree-level n-particle scattering amplitudes of massless theories can be computed using the Cachazo–He–Yuan (CHY) formula, which takes the form [66]:

$$\mathcal{A}_n^{\text{theory}} = \int d\mu_n \, \mathcal{I}_L^{\text{theory}} \, \mathcal{I}_R^{\text{theory}}. \tag{5.6}$$

$\mathcal{I}_L^{\text{theory}}$ and $\mathcal{I}_R^{\text{theory}}$ are left- and right-integrand factors, respectively, and they depend on the theory under consideration. Their precise form is not important for now, other than that they carry weight -2 under an $\mathrm{SL}(2,\mathbb{C})$ transformation for each puncture, i.e., $\mathcal{I}_{L/R}^{\text{theory}} \to \prod_{i=1}^n (C\sigma_i + D)^2 \, \mathcal{I}_{L/R}^{\text{theory}}$ when $\sigma_i \to (A\sigma_i + B)/(C\sigma_i + D)$ and $AD - BC = 1$. Correctly identifying the separation into left- and right-integrands is important for making the double-copy properties of amplitudes manifest.

Let us now review the CHY measure:

$$d\mu_n = \delta^D\big(\textstyle\sum_{i=1}^n p_i^\mu\big) \left(\displaystyle\prod_{i=1}^n \delta(p_i^2) \right) \frac{\prod_{i=1}^n d\sigma_i}{\mathrm{vol}\, \mathrm{SL}(2,\mathbb{C})} {\prod_i}' \delta(E_i). \tag{5.7}$$

This is a distribution involving momentum conservation and null conditions for the external momenta. The factor $\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})$ denotes the fact that it is necessary to quotient by the $\mathrm{SL}(2,\mathbb{C})$ redundancy on the Riemann surface by fixing the positions of three of the punctures, specifically $i = p, q, r$. Similarly, the prime means that the corresponding three scattering equations are redundant and should be removed. Fixing these redundancies leads to

$$\begin{aligned} \int d\mu_n &= \delta^D\big(\textstyle\sum_{i=1}^n p_i^\mu\big) \left(\displaystyle\prod_{i=1}^n \delta(p_i^2) \right) \int (\sigma_{pq}\sigma_{qr}\sigma_{rp})^2 \prod_{i \neq p,q,r} \big(d\sigma_i \, \delta(E_i)\big) \\ &= \delta^D\big(\textstyle\sum_{i=1}^n p_i^\mu\big) \left(\displaystyle\prod_{i=1}^n \delta(p_i^2) \right) \sum_{s=1}^{(n-3)!} \left. \frac{(\sigma_{pq}\sigma_{qr}\sigma_{rp})^2}{\det \left[\frac{\partial E_i}{\partial \sigma_j} \right]} \right|_{\sigma_i = \sigma_i^{(s)}}, \end{aligned} \tag{5.8}$$

which can be shown to be independent of the choice of labels p, q, r. The delta functions fully localize the measure on the $(n-3)!$ solutions $\{\sigma_i^{(s)}\}$ of the scattering equations. The measure transforms with $\mathrm{SL}(2,\mathbb{C})$-weight 4 in each puncture, so that the CHY integral (5.6) is $\mathrm{SL}(2,\mathbb{C})$-invariant.

Finally, one of the advantages of the CHY formulation is that soft limits can be derived from a simple application of the residue theorem [65]. Under the soft limit of an $(n+1)$-point amplitude with the last particle soft, i.e., $\tau \to 0$ where $p_{n+1} = \tau \hat{p}_{n+1}$, the measure behaves as

$$\int d\mu_{n+1} = \delta(p_{n+1}^2) \int d\mu_n \, \frac{1}{2\pi i} \oint_{|\hat{E}_{n+1}|=\varepsilon} \frac{d\sigma_{n+1}}{E_{n+1}} + O(\tau^0). \tag{5.9}$$

Here we have rewritten the scattering equation $\hat{E}_{n+1} = 0$ as a residue integral. Note that $E_{n+1} = \tau \hat{E}_{n+1}$ is proportional to τ, and thus the displayed term is dominant. Therefore the scattering equation associated to the last particle completely decouples in the limit $\tau \to 0$. For each of the $(n-3)!$ solutions of the remaining scattering equations, the contour $\{|\hat{E}_{n+1}| = \varepsilon\}$ localizes on $n-2$ solutions [65].

5.2.1 Four Dimensions and Witten-RSV formula

Since the scattering equations are valid in an arbitrary dimension, they do not capture aspects specific to certain dimensions, such as fermions or supersymmetry. In order to do so, it is convenient to express the scattering maps using the spinor-helicity variables appropriate to a given dimension. We start with the well-understood case of 4D. Various aspects of specifying CHY formulations to 4D have also been discussed in [142, 253].

The momentum four-vector of a massless particle in 4D Lorentzian spacetime can be written in terms of a pair of two-component bosonic spinors, l^α and $\tilde{l}^{\dot\alpha}$, which transform as $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations of the $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(3,1)$ Lorentz group

$$p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha} p^\mu = l^\alpha \tilde{l}^{\dot\alpha} \qquad \alpha = 1,2, \quad \dot\alpha = \dot1, \dot2. \tag{5.10}$$

For physical momenta, l and $\pm\tilde{l}$ are complex conjugates. However, when considering analytic continuations, it is convenient to treat them as independent. The little group for a massless particle[1] in 4D is U(1). Its complexification is $\mathrm{GL}(1,\mathbb{C})$. l and $\tilde{l}$ transform oppositely under this group so that the momentum is invariant. In discussing n-particle scattering amplitudes, we label the particles by an index $i = 1, 2, \ldots, n$. It is important to understand that there is a distinct little group associated to each of the n particles. Thus, the little group $\mathrm{GL}(1,\mathbb{C})$ transforms the spinors as $\lambda_i \to t_i \lambda_i$ and $\tilde\lambda_i \to t_i^{-1}\tilde\lambda_i$, leaving only three independent degrees of freedom for the momentum. Lorentz-invariant spinor products are given by: $\langle \lambda_i \lambda_j \rangle = \varepsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$ and $[\tilde\lambda_i \tilde\lambda_j] = \varepsilon_{\dot\alpha\dot\beta} \tilde\lambda_i^{\dot\alpha} \tilde\lambda_j^{\dot\beta}$. It is sometimes convenient to simplify further and write $\langle ij \rangle$ or $[ij]$. Given a scattering amplitude, expressed in terms of spinor-helicity variables, one can deduce the helicity of the ith

[1] In this thesis we only consider massless particles that transform trivially under translations of the full little group of Euclidean motions in $D-2$ dimensions.

particle by determining the power of t_i by which the amplitude transforms. For example, the most general Parke–Taylor (PT) formula for maximally helicity violating (MHV) amplitudes in 4D YM theory is as follows [201]: if gluons i and j have negative helicity, while the other $n-2$ gluons have positive helicity, then the (color-stripped) amplitude is

$$A_n^{\rm YM}(1^+2^+\cdots i^-\cdots j^-\cdots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{5.11}$$

Since the scattering map $p^\mu(z)$ in (5.3) is required to be null for all z, it can also be expressed in a factorized form in terms of spinors:

$$p^{\alpha\dot\alpha}(z) = \rho^\alpha(z)\tilde\rho^{\dot\alpha}(z). \tag{5.12}$$

The roots of $p^{\alpha\dot\alpha}(z)$ can be distributed among the polynomials $\rho(z)$, $\tilde\rho(z)$ in different ways, such that their degrees add up to $n-2$. When $\deg\rho(z) = d$ and $\deg\tilde\rho(z) = \tilde d = n-d-2$, the maps are said to belong to the dth sector. We parametrize the polynomials as:

$$\rho^\alpha(z) = \sum_{k=0}^{d}\rho_k^\alpha z^k, \qquad \tilde\rho^{\dot\alpha}(z) = \sum_{k=0}^{\tilde d}\tilde\rho_k^{\dot\alpha} z^k. \tag{5.13}$$

The spinorial maps (5.12) carry the same information as the scattering equations, and therefore they can be used to redefine the measure. Here it is natural to introduce a measure for each sector as:

$$\int d\mu_{n,d}^{\rm 4D} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^d d^2\rho_k \prod_{k=0}^{\tilde d} d^2\tilde\rho_k}{\text{vol SL}(2,\mathbb{C})\times\text{GL}(1,\mathbb{C})}\frac{1}{R(\rho)R(\tilde\rho)}\prod_{i=1}^n \delta^4\left(p_i^{\alpha\dot\alpha} - \frac{\rho^\alpha(\sigma_i)\tilde\rho^{\dot\alpha}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right). \tag{5.14}$$

These measures contain an extra $\text{GL}(1,\mathbb{C})$ redundancy, analogous to the little group symmetries of the momenta, which allows fixing one coefficient of $\rho(z)$ or $\tilde\rho(z)$. $R(\rho)$ denotes the resultant $R(\rho^1(z),\rho^2(z),z)$ and similarly for $R(\tilde\rho)$ [120, 59]. The physical reason resultants appear in the denominator can be understood by finding the points in the moduli space of maps where they vanish. A resultant of any two polynomials, say $\rho_1(z)$ and $\rho_2(z)$, vanishes if and only if the two polynomials have a common root z^*. If such a z^* exists then the map takes it to the tip of the momentum-space null cone, i.e., to the strict soft-momentum region. This is a reflection of the fact that in four (and lower) dimensions IR divergences are important in theories of massless particles. The measure is giving the baseline for the IR behavior while integrands can change it depending on the theory. As reviewed below, the gauge theory and gravity integrands contain $(R(\rho)R(\tilde{\rho}))^{\mathtt{s}}$, where $\mathtt{s} = 1$ for YM and $\mathtt{s} = 2$ for gravity, which coincides with the spins of the particles. Combined with the factor in the measure one has $(R(\rho)R(\tilde{\rho}))^{\mathtt{s}-1}$, which indicates that the IR behavior improves as one goes from a scalar theory, with $\mathtt{s} = 0$, to gravity [75].

Summing over all sectors gives the original CHY measure:

$$\int d\mu_n = \sum_{d=1}^{n-3} \int d\mu_{n,d}^{\rm 4D}. \tag{5.15}$$

This separation works straightforwardly for theories where the integrand only depends on σ_i's and not on the maps. One such theory is the bi-adjoint scalar whose amplitudes are given by

$$m(\alpha|\beta) = \int d\mu_n \, \mathrm{PT}(\alpha) \, \mathrm{PT}(\beta) = \sum_{d=1}^{n-3} m_{n,d}(\alpha|\beta), \tag{5.16}$$

where $\mathrm{PT}(\alpha)$ is the Parke–Taylor factor. The definition for the identity permutation is

$$\mathrm{PT}(12\cdots n) = \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}}. \tag{5.17}$$

In general α denotes a permutation of the indices $1, 2, \ldots, n$. The quantities $m_{n,d}(\alpha|\beta)$ are the "scalar blocks" defined in [75]. In the dth sector the number of solutions is given by the Eulerian number $\left\langle {n-3 \atop d-1} \right\rangle$, as conjectured in [229] and proved in [64]. Upon summation (5.15) gives all $\sum_{d=1}^{n-3} \left\langle {n-3 \atop d-1} \right\rangle = (n-3)!$ solutions of the scattering equations. Note that momentum conservation and the factorization conditions that ensure masslessness are built into the measure (5.14).

An alternative version of the above constraints, which is closer to the original Witten–RSV formulas, can be obtained by integrating-in auxiliary variables t_i and $\tilde{t}_i$

$$\begin{aligned} \delta^4\left(p_i^{\alpha\dot{\alpha}} - \frac{\rho^\alpha(\sigma_i)\tilde{\rho}^{\dot{\alpha}}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right) = \delta(p_i^2) \int dt_i \, d\tilde{t}_i \, \delta\left(t_i\tilde{t}_i - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right) \\ \times \delta^2\big(\lambda_i^\alpha - t_i\,\rho^\alpha(\sigma_i)\big)\delta^2\big(\tilde{\lambda}_i^{\dot{\alpha}} - \tilde{t}_i\,\tilde{\rho}^{\dot{\alpha}}(\sigma_i)\big). \end{aligned} \tag{5.18}$$

This formulation helps to linearize the constraints and make the little-group properties of theories with spin, such as Yang–Mills theory, more manifest.

The on-shell tree amplitudes of $\mathcal{N} = 4$ SYM theory in 4D are usually written as a sum over sectors

$$\mathcal{A}_n^{\mathcal{N}=4\ \mathrm{SYM}} = \sum_{d=1}^{n-3} \mathcal{A}_{n,d}^{\mathcal{N}=4\ \mathrm{SYM}}. \tag{5.19}$$

The dth sector has $n-2-2d$ units of "helicity violation": $d \to n-2-d$ corresponds to reversing the helicities. Partial amplitudes in each sector are given by

$$\mathcal{A}_{n,d}^{\mathcal{N}=4\ \mathrm{SYM}}(\alpha) = \int d\mu_{n,d}^{\mathrm{4D}}\, \mathrm{PT}(\alpha) \left(R(\rho) R(\tilde{\rho}) \int d\Omega_{\mathrm{F},d}^{(4)} \right), \tag{5.20}$$

where $d\Omega_{\mathrm{F},d}^{(4)}$ denotes integrations over fermionic analogs of the maps $\rho(z)$ and $\tilde{\rho}(z)$ implementing the $\mathcal{N}=4$ supersymmetry, whose precise form can be found in [64].

Due to the fact that the little group is Spin(4) in 6D, it is expected that the SYM amplitudes in 6D should not separate into helicity sectors. Dimensional reduction to 4D would naturally lead to a formulation with unification of sectors. This may appear somewhat puzzling as (5.19) and (5.20) seem to combine the measure in a given sector with an integrand that is specific to that sector. This puzzle is resolved by noticing that

$$R(\rho) = \det{}' \Phi_d, \qquad R(\tilde{\rho}) = \det{}' \tilde{\Phi}_{\tilde{d}}, \tag{5.21}$$

where $[\Phi_d]_{ij} := \langle ij \rangle / (t_i t_j \sigma_{ij})$ and $[\tilde{\Phi}_{\tilde{d}}]_{ij} := [ij]/(\tilde{t}_i \tilde{t}_j \sigma_{ij})$ for $i \neq j$. The diagonal components are more complicated and depend on d and $\tilde{d}$ [71, 59]. The corresponding reduced determinants are computed using submatrices of size $d \times d$ and $\tilde{d} \times \tilde{d}$, respectively. One of the main properties of these reduced determinants is that they vanish when evaluated on solutions in sectors that differ from their defining degree, i.e.,

$$\int d\mu_{n,d}^{\mathrm{4D}} \det{}' \Phi_{d'} \det{}' \tilde{\Phi}_{\tilde{d}'} = \delta_{d,d'} \int d\mu_{n,d}^{\mathrm{4D}} \det{}' \Phi_d \det{}' \tilde{\Phi}_{\tilde{d}}. \tag{5.22}$$

Using this it is possible to write the complete amplitude in terms of factors that can be uplifted to 6D and unified!

$$\mathcal{A}_n^{\mathcal{N}=4\ \mathrm{SYM}}(\alpha) = \int \left(\sum_{d=1}^{n-3} d\mu_{n,d}^{\mathrm{4D}} \right) \mathrm{PT}(\alpha) \left(\sum_{d'=1}^{n-3} \det{}' \Phi_{d'} \det{}' \tilde{\Phi}_{\tilde{d}'} \int d\Omega_{\mathrm{F},d'}^{(4)} \right). \tag{5.23}$$

Finally, it is worth mentioning that (5.22) can be used to write unified 4D $\mathcal{N}=8$

SUGRA amplitudes, via the double copy, as

$$\mathcal{M}_n^{\mathcal{N}=8\text{ SUGRA}} = \int \left(\sum_{d=1}^{n-3} d\mu_{n,d}^{\text{4D}}\right)\left(\sum_{d'=1}^{n-3} \det{}' \Phi_{d'} \det{}' \tilde{\Phi}_{\tilde{d}'} \int d\Omega_{\text{F},d'}^{(4)}\right)\left(\sum_{d'=1}^{n-3} \det{}' \Phi_{d'} \det{}' \tilde{\Phi}_{\tilde{d}'} \int d\hat{\Omega}_{\text{F},d'}^{(4)}\right).$$

5.3 Six Dimensions: Even Multiplicity

We are finally in position to study scattering maps in 6D. It turns out that the 6D spinor-helicity formalism requires separate treatments for amplitudes with an even and an odd number of particles. In this section we review the construction for an even number of particles, as was recently introduced in the context of M5- and D5-brane scattering amplitudes [149]. (These theories only have non-vanishing amplitudes for n even.) A formula for odd multiplicity, which is required for Yang–Mills theories, is one of the main results of this thesis and it is given in Chapter 7.

The little group for massless particles in 6D is $\text{Spin}(4) \sim \text{SU}(2) \times \text{SU}(2)$. We use indices without hats when referring to representations of the first SU(2) or its $\text{SL}(2,\mathbb{C})$ complexification and ones with hats when referring to the second SU(2) or its $\text{SL}(2,\mathbb{C})$ complexification. Momenta of massless particles are parametrized in terms of 6D spinor-helicity variables $l_i^{A,a}$ by [84]:

$$p_i^{AB} = \sigma_\mu^{AB} p_i^\mu = \langle \lambda_i^A \lambda_i^B \rangle, \qquad A, B = 1, 2, 3, 4, \tag{5.24}$$

where σ_μ^{AB} are six antisymmetric 4×4 matrices, which form an invariant tensor of $\text{Spin}(5,1)$. The angle bracket denotes a contraction of the little-group indices:

$$\langle \lambda_i^A \lambda_i^B \rangle = \epsilon_{ab} \lambda_i^{A,a} \lambda_i^{B,b} = \lambda_i^{A+} \lambda_i^{B-} - \lambda_i^{A-} \lambda_i^{B+}, \qquad a, b = +, -. \tag{5.25}$$

ϵ_{ab} is an invariant tensor of the SU(2) little group, as well as its $\text{SL}(2,\mathbb{C})$ complexification. The on-shell condition, $p_i^2 = 0$, is equivalent to the vanishing of the Pfaffian of p_i^{AB}. The little group transforms the spinors as $\lambda_i^{A,a} \to (L_i)^a_b \lambda_i^{A,b}$, where $L_i \in \text{SL}(2,\mathbb{C})$, leaving only five independent degrees of freedom for the spinors, appropriate for a massless particle in six dimensions. The momenta can be equally well described by conjugate spinors $\tilde{\lambda}_{i,A,\hat{a}}$:

$$p_{i,AB} = \frac{1}{2} \epsilon_{ABCD}\, p_i^{CD} = [\tilde{\lambda}_{i,A} \tilde{\lambda}_{i,B}], \tag{5.26}$$

where

$$[\tilde{\lambda}_{i,A}\tilde{\lambda}_{i,B}] = \epsilon^{\hat{a}\hat{b}}\tilde{\lambda}_{i,A,\hat{a}}\tilde{\lambda}_{i,B,\hat{b}} = \tilde{\lambda}_{i,A,\hat{+}}\tilde{\lambda}_{i,B,\hat{-}} - \tilde{\lambda}_{i,A,\hat{-}}\tilde{\lambda}_{i,B,\hat{+}}, \qquad \hat{a},\hat{b} = \hat{+},\hat{-}. \tag{5.27}$$

These conjugate spinors belong to the second (inequivalent) four-dimensional representation of the Spin(5, 1) $\sim$ SU*(4) Lorentz group, and they transform under the right-handed little group. Using the invariant tensors of SU*(4), Lorentz invariants can be constructed as follows:

$$\langle \lambda_i^a \lambda_j^b \lambda_k^c \lambda_l^d \rangle = \epsilon_{ABCD}\lambda_i^{A,a}\lambda_j^{B,b}\lambda_k^{C,c}\lambda_l^{D,d}, \tag{5.28}$$

$$[\tilde{\lambda}_{i,\hat{a}}\tilde{\lambda}_{j,\hat{b}}\tilde{\lambda}_{k,\hat{c}}\tilde{\lambda}_{l,\hat{d}}] = \epsilon^{ABCD}\tilde{\lambda}_{i,A,\hat{a}}\tilde{\lambda}_{j,B,\hat{b}}\tilde{\lambda}_{k,C,\hat{c}}\tilde{\lambda}_{l,D,\hat{d}}, \tag{5.29}$$

$$\langle \lambda_i^a | \tilde{\lambda}_{j,\hat{b}}] = \lambda_i^{A,a}\tilde{\lambda}_{j,A,\hat{b}} = [\tilde{\lambda}_{j,\hat{b}} | \lambda_i^a \rangle. \tag{5.30}$$

The l and $\tilde{l}$ variables are not independent. They are related by the condition

$$\langle \lambda_i^a | \tilde{\lambda}_{i,\hat{a}}] = 0, \tag{5.31}$$

for all a and $\hat{a}$. We also have

$$\epsilon_{ABCD}\, p_i^{AB} p_j^{CD} = 2\, p_{i,AB}\, p_j^{AB} = 8\, p_i \cdot p_j. \tag{5.32}$$

Using the notation given above, the scattering maps can be written in terms of 6D spinor-helicity variables:

$$p^{AB}(z) = \langle \rho^A(z)\rho^B(z)\rangle. \tag{5.33}$$

In the following we take the spinorial maps $\rho^{A,a}(z)$, for $a \in \{+,-\}$, to be polynomials of the same degree. In contrast to 4D, we can also consider non-polynomial forms of the maps (such that (5.33) is still a polynomial), see discussion at the end of Section 7.1.2. Note that this choice is consistent with the action of the group denoted SL$(2,\mathbb{C})_\rho$. This is the same abstract group as the little group, but it does not refer to a specific particle.

Let us now focus on the construction for n even. In this case the degree of the polynomials is $m = \frac{n}{2} - 1$. Thus they can be expanded as:

$$\rho^{A,a}(z) = \sum_{k=0}^{m} \rho_k^{A,a} z^k. \tag{5.34}$$

With these maps the polynomial constructed in (5.33) is null and has the correct degree $n-2$. By the arguments reviewed in Chapter **??** we conclude that the equations constructed from $\rho^{A,a}(z)$,

$$p_i^{AB} = \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\,, \tag{5.35}$$

imply the scattering equations for $\{\sigma_i\}$. However, the converse, i.e., that any solution of the scattering equations is a solution to (5.35) is not guaranteed. This was checked numerically in [149] for even multiplicity up to $n=8$ particles. In this part of the thesis we give an inductive proof of this fact in Appendix J, obtained by considering consecutive soft limits of the maps. Using this fact together with the counting of delta functions we then argue that the following measure

$$\int d\mu_{n\ \text{even}}^{6\text{D}} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)} \frac{1}{V_n^2} \prod_{i=1}^n \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \tag{5.36}$$

is equivalent to the CHY measure given in (5.7), after integrating out the ρ moduli. Also, it has momentum conservation and null conditions built-in. The formula contains the Vandermonde factor

$$V_n = \prod_{1\leq i<j\leq n} \sigma_{ij}. \tag{5.37}$$

which is needed to match the $\text{SL}(2,\mathbb{C})_\sigma$ weight of (5.7). In order to avoid confusion, we use the notation $\text{SL}(2,\mathbb{C})_\sigma$ for the Möbius group acting on the Riemann sphere. Just as the $\text{SL}(2,\mathbb{C})_\sigma$ symmetry can be used to fix three of the σ coordinates, the $\text{SL}(2,\mathbb{C})_\rho$ symmetry can be used to fix three of the coefficients of the polynomial maps $\rho^{A,a}(z)$. This form of the measure imposes $6n$ constraints on $5n-6$ integration variables, leaving a total of $n+6$ delta functions which account for the n on-shell conditions and the six momentum conservation conditions. Fixing the values of σ_1, σ_2, σ_3 and of $\rho_0^{1,+}$, $\rho_0^{1,-}$, $\rho_0^{2,+}$, the gauge-fixed form of the measure becomes:

$$\int d\mu_{n\ \text{even}}^{6\text{D}} = \int \frac{J_\rho\, J_\sigma}{V_n^2} \left(\prod_{i=4}^n d\sigma_i\right) d\rho_0^{2,-} d^2\rho_0^3 d^2\rho_0^4 \left(\prod_{k=1}^m d^8\rho_k\right) \prod_{i=1}^n \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right),$$

where the Jacobians are[2]

$$J_\sigma = \sigma_{12}\sigma_{23}\sigma_{31}, \qquad\qquad J_\rho = \rho_0^{1,+}\langle\rho_0^1\,\rho_0^2\rangle. \tag{5.38}$$

It is convenient to use a short-hand notation for the bosonic delta functions:

$$\Delta_B = \prod_{i=1}^{n}\delta^6\left(p_i^{AB} - \frac{\langle\rho^A(\sigma_i)\,\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) = \delta^6(\textstyle\sum_{i=1}^n p_i^{AB})\left(\displaystyle\prod_{i=1}^{n}\delta(p_i^2)\right)\hat{\Delta}_B, \tag{5.39}$$

where $\hat{\Delta}_B$ is

$$\hat{\Delta}_B = \delta^4\left(p_n^{AB} - \frac{\langle\rho^A(\sigma_n)\,\rho^B(\sigma_n)\rangle}{\prod_{i\neq n}\sigma_{ni}}\right)\prod_{i=1}^{n-2}\delta^5\left(p_i^{AB} - \frac{\langle\rho^A(\sigma_i)\,\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right)\prod_{i=1}^{n}p_i^{12}\left(\frac{p_{n-1}^{24}}{p_{n-1}^{12}} - \frac{p_n^{24}}{p_n^{12}}\right). \tag{5.40}$$

Here the five dimensional delta functions are chosen such that $\{A,B\}\neq\{3,4\}$, whereas $\{A,B\}\neq\{3,4\},\{1,3\}$ for the four dimensional ones, and the additional factors are the Jacobian of taking out the momentum conservation and on-shell conditions [149]. Alternatively, a covariant extraction of the on-shell delta functions can be obtained by introducing auxiliary variables M_i that linearize the constraints, analogous to the ones given in (5.18), as follows:

$$\delta^6\left(p_i^{AB} - \frac{\langle\rho^A(\sigma_i)\,\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) = \delta(p_i^2)\int d^4M_i\,|M_i|^3\,\delta\left(|M_i| - \prod_{j\neq i}\sigma_{ij}\right) \times \delta^8\left(\rho^{A,a}(\sigma_i) - (M_i)_b^a\,\lambda_i^{A,b}\right), \tag{5.41}$$

where $|M_i|$ denotes the determinant of the matrix M_i, and for some purpose it is more convenient to use this version of constraints. This form connects the maps directly to the external 6D spinors, and is a 6D version of the Witten–RSV constraints, which we explore in Chapter 8.

[2]This Jacobian can be derived from the identity $\int d^6p_0\,\delta(p_0^2) = \int J_\rho\, d\rho_0^{2,-}d\rho_0^{3,+}\,d\rho_0^{3,-}\,d\rho_0^{4,+}\,d\rho_0^{4,-}$, since the map component $p_0^{AB} = \langle\rho_0^A\,\rho_0^B\rangle$ is a null vector.

Chapter 6

$\mathcal{N} = (1, 1)$ Super Yang–Mills: Even Multiplicity

In the remnant of this thesis we will propose a formula based on rational maps for the tree amplitudes of 6D maximal SYM theory, which has $\mathcal{N} = (1, 1)$ non-chiral supersymmetry. This theory describes the non-abelian interactions of a vector, four scalars, and four spinors all of which are massless and belong to the adjoint representation of the gauge group. As usual, we will generally consider color-stripped SYM amplitudes. Some properties of these amplitudes have been discussed in [99, 25, 55, 206] using 6D $\mathcal{N} = (1, 1)$ superspace.

In addition to the usual spacetime and gauge symmetries of Yang–Mills theory, the $\mathcal{N} = (1, 1)$ theory has a Spin(4) $\sim$ SU(2) $\times$ SU(2) R symmetry group. The intuitive way to understand this is to note that this theory arises from dimensional reduction of 10D SYM theory, and the R symmetry corresponds to rotations in the four transverse directions. This group happens to be the same as the little group, which is just a peculiarity of this particular theory. From these and other considerations, one may argue that 6D $\mathcal{N} = (1, 1)$ SYM with U(N) gauge symmetry (in the perturbative regime with no theta term) describes the IR dynamics of N coincident D5-branes in type IIB superstring theory [249]. In contrast to 4D $\mathcal{N} = 4$ SYM, the gauge coupling in six dimensions has inverse mass dimension, so this theory is non-renormalizable and not conformal. This is not an issue for the tree amplitudes that we consider in this thesis. Further dimensional reduction on a T^2 leads to 4D $\mathcal{N} = 4$ SYM, and this provides a consistency check of the results.

Six-dimensional $\mathcal{N} = (1, 1)$ SYM is a theory with 16 supercharges. Its physical degrees of freedom form a 6D $\mathcal{N} = (1,1)$ supermultiplet consising of eight on-shell bosons and eight on-shell fermions. These may be organized according to their quantum numbers under the four SU(2)'s of the little group and R symmetry group. For example, the vectors belong to the representation $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$, which means that they are doublets of each of the little-group SU(2)'s and singlets of each of the R symmetry SU(2)'s. In this notation, the fermions belong to the representation $(\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2})$, and the scalars belong to the representation $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$. (Whether one writes $(\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ is a matter of convention.)

It is convenient to package all 16 of these particles into a single on-shell "superparticle", by introducing four Grassmann numbers (per superparticle),

$$\Phi(\eta) = \phi^{1\hat{1}} + \eta_a \psi^{a\hat{1}} + \tilde{\eta}_{\hat{a}} \hat{\psi}^{\hat{a}1} + \eta_a \tilde{\eta}_{\hat{a}} A^{a\hat{a}} + (\eta)^2 \phi^{2\hat{1}} + (\tilde{\eta})^2 \phi^{1\hat{2}} + \cdots + (\eta)^2 (\tilde{\eta})^2 \phi^{2\hat{2}} . \quad (6.1)$$

Here η_a and $\tilde{\eta}_{\hat{a}}$ are the four Grassmann numbers, and the SU(2) indices a and $\hat{a}$ are little-group indices as before. The explicit 1's and 2's in the spectrum described above are R symmetry indices. Since the superfield transforms as a little-group scalar, this formulation makes the little-group properties manifest, but it obscures the R symmetry. By means of an appropriate Grassmann Fourier transform one could make the R symmetry manifest, but then the little-group properties would be obscured as explained in [149]. The choice that has been made here turns out to be the more convenient one for the study of superamplitudes.

When discussing an n-particle amplitude the Grassmann coordinates carry an additional index i, labeling the n particles, just like the spinor-helicity coordinates. Thus, the complete color-stripped on-shell n-particle tree amplitude will be a cyclically symmetric function of the l_i's and the η_i's. The various component amplitudes correspond to the terms with the appropriate dependence on the Grassmann coordinates. Thus, the superamplitude is like a generating function in which the Grassmann coordinates play the role of fugacities. This is an on-shell analog of the use of superfields in the construction of Lagrangians. Fortunately, it exists in cases where the latter does not exist.

Often we will refer to the momenta p_i^{AB} and supercharges q_i^A, $\tilde{q}_{iA}$ of the on-shell states. For $(1,1)$ supersymmetry, they can be expressed in terms of the Grassmann coordinates:

$$q_i^A = \epsilon^{ab} \lambda_{ia}^A \eta_{ib} = \langle \lambda_i^A \, \eta_i \rangle, \quad \tilde{q}_{iA} = \epsilon^{\hat{a}\hat{b}} \tilde{\lambda}_{iA\hat{a}} \tilde{\eta}_{i\hat{b}} = [\tilde{l}_{iA} \, \tilde{\eta}_i], \quad (6.2)$$

and the superamplitudes should be annihilated by the supercharges $Q^A = \sum_{i=1}^n q_i^A$ and $\tilde{Q}_A = \sum_{i=1}^n \tilde{q}_{iA}$. These symmetries will be manifest in the formulas that follow. However, there are eight more supercharges, involving derivatives with respect to the η coordinates, which should also be conserved. Once one establishes the first eight supersymmetries and the R symmetry, these supersymmetries automatically follow. The explicit form of the derivatively realized supercharges is:

$$\bar{q}_i^A = \lambda_{ia}^A \frac{\partial}{\partial \eta_{ia}}, \quad \tilde{\bar{q}}_{iA} = \tilde{l}_{iA\hat{a}} \frac{\partial}{\partial \tilde{\eta}_{i\hat{a}}}, \tag{6.3}$$

In terms of these Grassmann variables, one may also write the generators of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ R symmetry group. One first notes that they obey the anti-commutation relations:

$$\left\{\eta_a, \frac{\partial}{\partial \eta^b}\right\} = \epsilon_{ab}, \quad \left\{\tilde{\eta}_{\hat{a}}, \frac{\partial}{\partial \tilde{\eta}^{\hat{b}}}\right\} = \epsilon_{\hat{a}\hat{b}}\,. \tag{6.4}$$

In terms of these, the six generators of the R symmetry group may be defined as

$$R^+ = \eta_a \eta^a, \quad R^- = \frac{\partial}{\partial \eta^a}\frac{\partial}{\partial \eta_a}, \quad R = \eta_a \frac{\partial}{\partial \eta_a} - 1, \tag{6.5}$$

$$\widetilde{R}^+ = \tilde{\eta}_{\hat{a}} \tilde{\eta}^{\hat{a}}, \quad \widetilde{R}^- = \frac{\partial}{\partial \tilde{\eta}^{\hat{a}}}\frac{\partial}{\partial \tilde{\eta}_{\hat{a}}}, \quad \widetilde{R} = \tilde{\eta}_{\hat{a}} \frac{\partial}{\partial \tilde{\eta}_{\hat{a}}} - 1\,, \tag{6.6}$$

which have the standard raising and lowering commutation relations. These generate a global symmetry of $\mathcal{N} = (1, 1)$ SYM. It is easy to see that linear generators R and $\widetilde{R}$ annihilate amplitudes since they are homogeneous polynomials of degree n in both η and $\tilde{\eta}$. The non-linearly realized ones become more transparent in an alternative form of the constraints that we will discuss in Chapter 8. As explained earlier, this is due to the choice of parametrization of the non-chiral on-shell superspace.

As discussed in previous literature for tree-level amplitudes of 6D $\mathcal{N} = (1, 1)$ SYM, the four-particle partial amplitude is particularly simple when expressed in terms of the supercharges:

$$\mathcal{A}_4^{\mathcal{N}=(1,1)\mathrm{SYM}}(1234) = \delta^6\left(\sum_{i=1}^{4} p_i^{AB}\right) \frac{\delta^4\left(\sum_{i=1}^4 q_i^A\right)\delta^4\left(\sum_{i=1}^4 \tilde{q}_{i,A}\right)}{s_{12}\, s_{23}}. \tag{6.7}$$

Here and throughout this thesis one should view this expression as a superamplitude; the component amplitudes may be extracted by Grassmann integration. For example, in terms of the Lorentz invariant brackets the four-gluon amplitude is:

$$\mathcal{A}_4(A_{a\hat{a}}A_{b\hat{b}}A_{c\hat{c}}A_{d\hat{d}}) = \delta^6\left(\sum_{i=1}^{4} p_i^{AB}\right)\frac{\langle 1_a 2_b 3_c 4_d\rangle[1_{\hat{a}}2_{\hat{b}}3_{\hat{c}}4_{\hat{d}}]}{s_{12}\, s_{23}}. \tag{6.8}$$

Using the formalism of rational maps for the 6D spinor-helicity variables, the main technical result of this Chapter is a formula for the n-point generalization of the superamplitude when n is even. The formula for odd n will be given in Chapter 7.

6.1 Connected Formula

We propose that the connected formula for even-multiplicity 6D $\mathcal{N}=(1,1)$ SYM amplitudes is given by

$$\boxed{\mathcal{A}_{n\text{ even}}^{\mathcal{N}=(1,1)\text{ SYM}}(\alpha) = \int d\mu_{n\text{ even}}^{\text{6D}}\, \text{PT}(\alpha)\left(\text{Pf}'A_n \int d\Omega_F^{(1,1)}\right)}\,, \tag{6.9}$$

where $d\mu_{n\text{ even}}^{\text{6D}}$ is the measure given in (5.36), and we will shortly explain other ingredients that enter this formula. This formula is inspired by the D5-brane effective field theory scattering amplitudes written as a connected formula [149], where the factor of $(\text{Pf}'A_n)^2$ has been replaced with $\text{PT}(\alpha)$ given in (5.17). This is a standard substitution in the CHY formalism for passing from a probe D-brane theory to a Yang–Mills theory. Since the only non-vanishing amplitudes of the D5-brane theory have even n, this only works for the even-point amplitudes of SYM.

As indicated explicitly in the expression (6.9), the integrand of (6.9) factorizes into two half-integrands. Such a factorization of the integrand will be important later when we deduce the formulas for 6D SUGRA with $\mathcal{N}=(2,2)$ supersymmetry. The left half-integrand $\text{PT}(\alpha)$ is the Parke–Taylor factor, where α is a permutation that denotes the color ordering of Yang–Mills partial amplitudes. The right half-integrand further splits into two quarter-integrands. The first of these is the reduced Pfaffian of the antisymmetric matrix A_n, whose entries are given by:

$$[A_n]_{ij} = \begin{cases} \dfrac{p_i\cdot p_j}{\sigma_{ij}} & \text{if} \quad i\neq j, \\ 0 & \text{if} \quad i=j, \end{cases} \qquad \text{for} \qquad i,j=1,2,\ldots,n. \tag{6.10}$$

Since this matrix has co-rank 2, its Pfaffian vanishes. Instead, one defines the reduced Pfaffian:

$$\mathrm{Pf}'A_n = \frac{(-1)^{p+q}}{\sigma_{pq}} \mathrm{Pf}\, A_n^{[pq]}, \tag{6.11}$$

where we have removed two rows and columns labeled by p and q, and denoted the resulting reduced matrix by $A_n^{[pq]}$. The reduced Pfaffian is independent of the choice of p and q [68] and transforms under $\mathrm{SL}(2,\mathbb{C})_\sigma$ in an appropriate way.

The remaining quarter integrand is the fermionic integration measure responsible for implementing the 6D $\mathcal{N} = (1,1)$ supersymmetry [149], which we will review here. The formula is

$$d\Omega_F^{(1,1)} = V_n \left(\prod_{k=0}^{m} d^2\chi_k \, d^2\tilde{\chi}_k \right) \Delta_F \widetilde{\Delta}_F, \tag{6.12}$$

where $m = \frac{n}{2} - 1$, as before. This measure contains the Vandermonde determinant V_n, as well as a fermionic measure and fermionic delta functions. The integration variables arise as the coefficients of the fermionic rational maps, which are defined by

$$\chi^a(z) = \sum_{k=0}^{m} \chi_k^a \, z^k, \qquad \tilde{\chi}^{\hat{a}}(z) = \sum_{k=0}^{m} \tilde{\chi}_k^{\hat{a}} \, z^k, \tag{6.13}$$

where χ_k^a and $\tilde{\chi}_k^{\hat{a}}$ are Grassmann variables. The fermionic delta functions, Δ_F and $\widetilde{\Delta}_F$ are given by:

$$\Delta_F = \prod_{i=1}^{n} \delta^4 \left(q_i^A - \frac{\langle \rho^A(\sigma_i)\, \chi(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right), \tag{6.14}$$

$$\widetilde{\Delta}_F = \prod_{i=1}^{n} \delta^4 \left(\tilde{q}_{i,A} - \frac{[\tilde{\rho}_A(\sigma_i)\, \tilde{\chi}(\sigma_i)]}{\prod_{j \neq i} \sigma_{ij}} \right). \tag{6.15}$$

These delta functions are built from the external chiral and anti-chiral supercharges of each particle and are responsible for the $(1,1)$ supersymmetry in this formalism. Conservation of half of the 16 supercharges is made manifest by this expression. As in (6.7), the component amplitudes can be extracted by Grassmann integration of the appropriate η_a's and $\tilde{\eta}_{\hat{a}}$'s, which enter via the supercharges.

Even though the maps $\tilde{\rho}_{A\hat{a}}(z)$ appear explicitly in $\widetilde{\Delta}_F$, just as in the construction of

D5-brane amplitudes [149], the integration measure does not include additional integrations associated to the maps $\tilde{\rho}_{A\hat{a}}(z)$. If it did, the formula, for instance, would have the wrong mass dimension to describe SYM amplitudes in 6D. Instead, the $\tilde{\rho}$ coefficients are fixed by the conjugate set of rational constraints

$$p_{i,AB} - \frac{[\tilde{\rho}_A(\sigma_i)\,\tilde{\rho}_B(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}} = 0\,, \tag{6.16}$$

for all $i = 1, 2, \ldots, n$. These equations are not enough to determine all of the $\tilde{\rho}^{\hat{a}}_{A,k}$'s. One needs to utilize $\mathrm{SL}(2,\mathbb{C})_{\tilde{\rho}}$ to fix the remaining ones. The resulting amplitude is independent of choices that are made for the $\mathrm{SL}(2,\mathbb{C})_{\tilde{\rho}}$ fixing because $\tilde{\rho}_{A,\hat{a}}(\sigma_i)\tilde{\chi}^{\hat{a}}(\sigma_i)$ and the fermionic measure $d^2\tilde{\chi}_k$ are $\mathrm{SL}(2,\mathbb{C})_{\tilde{\rho}}$ invariant. The usual scattering amplitudes A_n are obtained by removing the bosonic and fermionic on-shell conditions ("wave functions"), which appear as delta functions, namely,

$$\mathcal{A}_n^{\mathcal{N}=(1,1)\text{ SYM}} = \delta^6\left(\sum_{i=1}^n p_i\right)\left(\prod_{i=1}^n \delta(p_i^2)\delta^2(\tilde{\lambda}_{i,A,\hat{a}}\,q_i^A)\delta^2(\lambda^B_{i,b}\,\tilde{q}_{i,B})\right)A_n^{\mathcal{N}=(1,1)\text{ SYM}}. \tag{6.17}$$

It is straightforward to show that this formula produces the correct four-point superamplitude of 6D $\mathcal{N} = (1,1)$ SYM, expressed in (6.7). A quick way to see it is to utilize the relation between the D5-brane amplitudes and the amplitudes of 6D $\mathcal{N} = (1,1)$ SYM. As we discussed previously, they are related by the exchange of $(\mathrm{Pf}'A_n)^2$ with the Parke–Taylor factor $\mathrm{PT}(\alpha)$. The four-point superamplitude for the D5-brane theory is given by [149]

$$A_4^{\text{D5-brane}} = \delta^4\left(\sum_{i=1}^4 q_i^A\right)\delta^4\left(\sum_{i=1}^4 \tilde{q}_{i,A}\right). \tag{6.18}$$

From the explicit solution of the four-point scattering equations for the σ_i's, it is easy to check that the effect of changing from $(\mathrm{Pf}'A_4)^2$ to $\mathrm{PT}(1234)$, defined in (5.17), is to introduce an additional factor of $1/(s_{12}\,s_{23})$. Namely, on the support of the scattering equations, we have the following identity for the $\mathrm{SL}(2,\mathbb{C})_\sigma$-invariant ratio,

$$\frac{\mathrm{PT}(1234)}{(\mathrm{Pf}'A_4)^2} = \frac{1}{s_{12}\,s_{23}}\,. \tag{6.19}$$

Thus, combining this identity and the D5-brane formula (6.18), we arrive at the result of the four-point of 6D $\mathcal{N} = (1,1)$ SYM (6.7). We have further checked numerically that the above formula reproduces the component amplitudes of scalars and gluons for $n = 6, 8$, obtained from Feynman diagram computations.

6.2 Comparison with CHY

This section presents a consistency check of the integrand by comparing a special bosonic sector of the theory with a CHY formula of YM amplitudes valid in arbitrary spacetime dimensions. This comparison actually also gives a derivation of the integrand in (6.9).

We begin with the general form of the superamplitude,

$$\mathcal{A}_{n\ \text{even}}^{\mathcal{N}=(1,1)\ \text{SYM}}(\alpha) = \int d\mu_{n\ \text{even}}^{\text{6D}} \int d\Omega_F^{(1,1)} \times \mathcal{J}_{n\,\text{even}}\,, \tag{6.20}$$

where the measures $d\mu_{n\ \text{even}}^{\text{6D}}$ and $d\Omega_F^{(1,1)}$ take care of 6D massless kinematics and 6D $\mathcal{N} = (1,1)$ supersymmetry, respectively. The goal is then to determine the integrand $\mathcal{J}_{n\,\text{even}}$. The strategy is to consider a particular component amplitude by performing fermionic integrations of the superamplitude $\mathcal{A}_{n\ \text{even}}^{\mathcal{N}=(1,1)\ \text{SYM}}(\alpha)$ such that our formula can be directly compared to the known CHY integrand, thereby determining $\mathcal{J}_{n\,\text{even}}$.

To make the fermionic integration as simple as possible, it is convenient to consider a specific all-scalar amplitude, for instance,

$$\mathcal{A}_n(\phi_1^{1\hat{1}}, \ldots, \phi_{\frac{n}{2}}^{1\hat{1}}, \phi_{\frac{n}{2}+1}^{2\hat{2}}, \ldots, \phi_n^{2\hat{2}}). \tag{6.21}$$

Half of the particles have been chosen to be the scalar of the top component of the superfield, while the other half are the scalar of the bottom component of the superfield. This equal division is required to obtain a non-zero amplitude, because the superamplitude is homogeneous of degree n both in the η and $\tilde{\eta}$ coordinates. Due to this convenient choice of the component amplitude, the fermionic integral over χ's and $\tilde{\chi}$'s can be done straightforwardly. Explicitly, for the component amplitude we are interested in,

$$\int d\Omega_F^{(1,1)} \implies V_n\, J_{\text{w}} \int \prod_{k=0}^{m} d^2\chi_k\, d^2\tilde{\chi}_k\, \Delta_F^{\text{proj}} \widetilde{\Delta}_F^{\text{proj}}, \tag{6.22}$$

where we have taken out the fermionic wave functions as in (6.17), which results in a Jacobian $J_{\text{w}} = \prod_{i=1}^{n} \frac{1}{(p_i^{13})^2}$ in the above expression. Furthermore, the fermionic delta functions are projected to the component amplitude of interest,

$$\Delta_F^{\text{proj}} = \prod_{i\in Y}\prod_{A=1,3}\delta\left(\frac{\langle\rho^A(\sigma_i)\,\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right)\prod_{i\in\overline{Y}}p_i^{13}, \tag{6.23}$$

$$\widetilde{\Delta}_F^{\text{proj}} = \prod_{i\in Y}\prod_{A=2,4}\delta\left(\frac{[\tilde\rho_A(\sigma_i)\,\tilde\chi(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right)\prod_{i\in\overline{Y}}p_i^{13}. \tag{6.24}$$

Here Y labels all the scalars $\phi^{1\hat{1}}$, namely $Y := \{1,\ldots,\frac{n}{2}\}$, and $\overline{Y}$ labels the other type of scalars $\phi^{2\hat{2}}$, so $\overline{Y} := \{\frac{n}{2}+1,\ldots,n\}$.

Carrying out the integrations over $d^2\chi_k$ and $d^2\tilde\chi_k$, we see that the maps $\rho_a^A(\sigma_i)$ combine nicely into $\langle\rho^A(\sigma_i)\,\rho^B(\sigma_i)\rangle$, which on the support of the rational map constraints becomes $p_i^{AB}\prod_{j\neq i}\sigma_{ij}$. Concretely, we have,

$$\int\prod_{k=0}^{m}d^2\chi_k\,d^2\tilde\chi_k\prod_{i\in Y}\prod_{A=1,3}\delta\left(\frac{\langle\rho^A(\sigma_i)\,\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right)\prod_{B=2,4}\delta\left(\frac{[\tilde\rho_B(\sigma_i)\,\tilde\chi(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right)$$
$$=\prod_{i\in Y}p_i^{13}p_{i,24}\times\prod_{i\in Y\,J\in\overline{Y}}\frac{1}{\sigma_{iJ}^2}\,. \tag{6.25}$$

Collecting terms, we find that the wave-function Jacobian $J_{\rm w}$ cancels out completely, and we obtain the final result

$$V_nJ_{\rm w}\int\prod_{k=0}^{m}d^2\chi_k\,d^2\tilde\chi_k\,\Delta_F^{\text{proj}}\widetilde{\Delta}_F^{\text{proj}} = V_n\prod_{i\in Y\,J\in\overline{Y}}\frac{1}{\sigma_{iJ}^2} := J_F\,, \tag{6.26}$$

where we have defined the final result to be J_F. Therefore we have,

$$\mathcal{A}_n(\phi_1^{1\hat{1}},\ldots,\phi_{\frac{n}{2}}^{1\hat{1}},\phi_{\frac{n}{2}+1}^{2\hat{2}},\ldots,\phi_n^{2\hat{2}}) = \int d\mu_{n\text{ even}}^{\text{6D}}\ (J_F\times\mathcal{J}_{n\,\text{even}}). \tag{6.27}$$

We are now ready to compare this result directly with CHY amplitude, which is given by

$$\mathcal{A}_n(\phi_1^{1\hat{1}},\ldots,\phi_{\frac{n}{2}}^{1\hat{1}},\phi_{\frac{n}{2}+1}^{2\hat{2}},\ldots,\phi_n^{2\hat{2}}) = \int d\mu_n\ \text{PT}(\alpha)\,\text{Pf}'\Psi_n\big|_{\text{project}}\,, \tag{6.28}$$

and $d\mu_n = d\mu_{n\text{ even}}^{\text{6D}}$ if we restrict the CHY formula to 6D.

The notation $\Psi_n\big|_{\text{project}}$ denotes projection of the matrix Ψ_n of the CHY formulation to the specific scalar component amplitude we want via dimensional reduction. In particular, the "polarization vectors" should satisfy $\varepsilon_i \cdot \varepsilon_I = 1$ if $i \in Y$ and $I \in \overline{Y}$ or vice versa. If they belong to the same set, then we have $\varepsilon_i \cdot \varepsilon_j = \varepsilon_I \cdot \varepsilon_J = 0$. Furthermore, $p_i \cdot \varepsilon_j = 0$ for all i and j, i.e., both sets, since all of the vectors are dimensionally reduced to scalars. Let us now recall the definition of the matrix Ψ_n that enters the CHY construction of YM amplitudes. It can be expressed as

$$\Psi_n = \begin{pmatrix} A_n & -C_n^{\intercal} \\ C_n & B_n \end{pmatrix}, \tag{6.29}$$

where A_n is given in (6.10), and B_n and C_n are $n \times n$ matrices defined as

$$[B_n]_{ij} = \begin{cases} \dfrac{\varepsilon_i \cdot \varepsilon_j}{\sigma_{ij}} & \text{if} \quad i \neq j, \\ 0 & \text{if} \quad i = j. \end{cases} \qquad [C_n]_{ij} = \begin{cases} \dfrac{p_j \cdot \varepsilon_i}{\sigma_{ij}} & \text{if} \quad i \neq j, \\ -\displaystyle\sum_{l \neq i} \dfrac{p_l \cdot \varepsilon_i}{\sigma_{il}} & \text{if} \quad i = j. \end{cases} \tag{6.30}$$

Like A_n, the matrix Ψ_n is also an antisymmetric matrix of co-rank 2. Its non-vanishing reduced Pfaffian is defined as

$$\mathrm{Pf}'\Psi_n = \frac{(-1)^{p+q}}{\sigma_{pq}} \mathrm{Pf}\Psi_n^{[pq]}, \tag{6.31}$$

where $\Psi_n^{[pq]}$ denotes the matrix Ψ_n with rows p, q and columns p, q removed. These should be chosen from the first n rows and columns. Otherwise, the result is independent of the choice of p, q.

For the specific choice of the component amplitude described above, $C_n = 0$ and the reduced Pfaffian $\mathrm{Pf}'\Psi_n$ becomes

$$\mathrm{Pf}'\Psi_n\big|_{\text{project}} = \mathrm{Pf}'A_n \times \mathrm{Pf}\, B_n\big|_{\text{scalar}}, \tag{6.32}$$

where the "projected" matrix B_n is

$$[B_n\big|_{\text{scalar}}]_{iJ} = \begin{cases} \dfrac{1}{\sigma_{iJ}} & \text{if} \quad i \in Y, \quad J \in \overline{Y}, \\ 0 & \quad\text{otherwise}. \end{cases} \tag{6.33}$$

Using the above result, we find

$$\mathrm{Pf}\, B_n\big|_{\text{scalar}} = \det\left(\frac{1}{\sigma_{iJ}}\right) \qquad \text{where} \quad i \in Y, \quad J \in \overline{Y}. \tag{6.34}$$

Comparing (6.27) with (6.28), we deduce that the even-point integrand should be given by

$$\mathcal{J}_{n\,\text{even}}(\alpha) = \mathrm{PT}(\alpha)\,\mathrm{Pf}' A_n \frac{\mathrm{Pf}\, B_n\big|_{\text{scalar}}}{J_F}\,. \tag{6.35}$$

It is easy to prove that $\mathrm{Pf}\, B_n\big|_{\text{scalar}}$ and J_F are actually identical. In particular, one can see that they, as rational functions, have the same zeros and poles. So we obtain the desired result, $\mathcal{J}_{n\,\text{even}}(\alpha) = \mathrm{PT}(\alpha)\,\mathrm{Pf}' A_n$.

Chapter 7

$\mathcal{N} = (1, 1)$ Super Yang–Mills: Odd Multiplicity

This chapter presents the formula for $\mathcal{N} = (1, 1)$ SYM amplitudes with odd multiplicity. This case is considerably subtler than the case of even n. It is perhaps the most novel aspect of this part of the thesis. Nevertheless, we will show that it can be written in a form entirely analogous to the even-point case:

$$\boxed{\mathcal{A}_{n\text{ odd}}^{\mathcal{N}=(1,1)\text{ SYM}}(\alpha) = \int d\mu_{n\text{ odd}}^{6\text{D}}\,\text{PT}(\alpha)\left(\text{Pf}'\widehat{A}_n \int d\widehat{\Omega}_F^{(1,1)}\right).} \tag{7.1}$$

The following sections describe the different ingredients in this expression.

Section 7.1 starts by presenting the form of the rational maps for n odd and studying the corresponding redundancies that enter in the integration measure. The explicit form of $d\mu_{n\text{ odd}}^{6\text{D}}$, given in (7.20), is deduced by considering a soft limit of an amplitude with n even. In particular, we deduce the existence of an emergent shift invariance acting on the rational maps. The discussion of how this new invariance interacts with the groups $\text{SL}(2, \mathbb{C})_\sigma$ and $\text{SL}(2, \mathbb{C})_\rho$ is relegated to Appendix I. Appendix I presents the detailed derivation of the form of the maps, as well as the measure, from the soft limit of the even-point formula (6.9).

Section 7.2 discusses the form of the integrand for odd n, which can also be derived by carefully examining the soft limit. The fermionic integration measure $d\widehat{\Omega}_F^{(1,1)}$ is given explicitly in (7.39). We show that the odd-n analog of the A_n matrix is an antisymmetric $(n + 1) \times (n + 1)$ matrix, which is denoted $\widehat{A}_n$. It is constructed from $(n + 1)$ momenta:

the original n momenta of external particles and an additional special null vector, $p_\star^{AB}$, defined through an arbitrarily chosen puncture $\sigma_\star$. The formula for the matrix $\widehat{A}_n$ is given in (7.75), and $p_\star^{AB}$ in (7.76).

In Section 7.3 we describe consistency checks of (7.1). This includes a comparison with the CHY formula in the bosonic sector, as was done for n even in 6.2. We also present a computation of the three-point superamplitude [99] directly from the connected formula.

7.1 Rational Maps and the Measure

Let us consider the definition of the scattering maps in 6D for the odd-point case $n = 2m + 1$:

$$p_i^{AB} = \frac{p^{AB}(\sigma_i)}{\prod_{j\neq i} \sigma_{ij}}. \tag{7.2}$$

This formula implies the scattering equations if $p^{AB}(z)$ is a polynomial of degree $n - 2 = 2m - 1$ such that the vector $p^{AB}(z)$ is massless for any value of z. The latter is realized by requiring

$$p^{AB}(z) = \langle \rho^A(z)\rho^B(z)\rangle = \rho^{A,+}(z)\rho^{B,-}(z) - \rho^{A,-}(z)\rho^{B,+}(z), \tag{7.3}$$

as in the case of even n. The polynomials $\rho^{A,+}(z)$ and $\rho^{A,-}(z)$ should have the same degree, since we want to maintain $\mathrm{SL}(2,\mathbb{C})_\rho$ symmetry. This is achieved by choosing $\deg \rho^{A,a}(z) = m$. However, this produces an undesired term of degree $2m = n - 1$ in $p^{AB}(z)$. This term can be made to vanish by requiring that the coefficient of z^m in $\rho^{A,a}(z)$ takes the special form

$$\rho_m^{A,a} = \omega^A \xi^a, \tag{7.4}$$

since then $\langle \rho_m^A \, \rho_m^B\rangle = 0$. This is the first new feature we encounter for odd n. The maps for $n = 2m + 1$ then become

$$\rho^{A,a}(z) = \sum_{k=0}^{m-1} \rho_k^{A,a} \, z^k + \omega^A \xi^a \, z^m, \tag{7.5}$$

$$\tilde{\rho}_A^{\hat{a}}(z) = \sum_{k=0}^{m-1} \tilde{\rho}_{A\,k}^{\hat{a}} \, z^k + \tilde{\omega}_A \tilde{\xi}^{\hat{a}} \, z^m. \tag{7.6}$$

Note that the spinor ξ^a, which we also write as $|\xi\rangle$, involves a projective scale that can be absorbed into ω^A, which is invariant under $\mathrm{SL}(2,\mathbb{C})_\rho$. In other words, ξ^a are homogeneous coordinates on $\mathbb{CP}^1$. For instance, this freedom can be used to set

$$|\xi\rangle = \begin{pmatrix} 1 \\ \xi \end{pmatrix}. \tag{7.7}$$

In the following we use the symbol ξ to denote both the two-component spinor and its only independent component.

After plugging this form of the (chiral) maps into equations (7.2) we find the expected $(n-3)!$ solutions. This is consistent since, as we show in Appendix I, this version of the maps can be obtained directly from a soft limit of the even multiplicity ones. However, a counting argument quickly leads to the fact that we must fix an extra component of the maps when solving the equations: There are $5n-6$ independent equations for $5n+1$ variables, which implies the existence of seven redundancies. Six of them are of course the $\mathrm{SL}(2,\mathbb{C})$'s present in the even case, but there is an emergent redundancy that we call *T-shift* symmetry. It is the subject of the next section.

7.1.1 Action of the T Shift

Consider the following transformation on the polynomials

$$\rho^A(z) \;\to\; \hat{\rho}^A(z) \;=\; (\mathbb{I} + z\,T)\rho^A(z). \tag{7.8}$$

Here T is a 2×2 matrix labeled by little-group indices. In order to preserve the bosonic delta functions, Δ_B, we require that for any value of z and for any polynomial $\rho^A(z)$:

$$\begin{aligned} p^{AB}(z) &= \hat{p}^{AB}(z) \\ &= \langle (\mathbb{I}+zT)\rho^A(z)\ (\mathbb{I}+zT)\rho^B(z)\rangle \\ &= \langle \rho^A(z)\,\rho^B(z)\rangle + z\left(\langle T\rho^A(z)\ \rho^B(z)\rangle + \langle \rho^A(z)\ T\rho^B(z)\rangle\right) + z^2\langle T\rho^A(z)\ T\rho^B(z)\rangle. \end{aligned} \tag{7.9}$$

Thus we obtain the following conditions

$$T^{\intercal}\epsilon + \epsilon T = 0, \qquad\qquad T^{\intercal}\epsilon T = 0, \tag{7.10}$$

where $T^\intercal$ is the transpose of T and ϵ is the 2×2 antisymmetric matrix. The first condition is equivalent to

$$\operatorname{Tr} T = 0, \tag{7.11}$$

which implies that $T^2 \propto \mathbb{I}$. The second condition then fixes

$$T^2 = 0. \tag{7.12}$$

What is the meaning of the conditions (7.11) and (7.12)? They guarantee that the transformation (7.8) is a z-dependent $\mathrm{SL}(2,\mathbb{C})_\rho$ transformation, hence preserving the polynomial map $p^{AB}(z)$. In other words, (7.11) and (7.12) are equivalent to

$$\det(\mathbb{I} + z\,T) = 1 \qquad \text{for any } z. \tag{7.13}$$

We discuss such transformations in more generality in the next chapters. For now, let us further impose that T preserves the degree of the maps, i.e., $\deg\, \hat{\rho}^{A,a}(z) = \deg\, \rho^{A,a}(z) = m$, that is

$$T^a_b\, \rho_m^{A,b} = 0, \tag{7.14}$$

where $\rho_m^{A,a}$ is the top coefficient. This means that the kernel of T consists of the four spinors $\rho_m^{A,a}$ with $A = 1,2,3,4$. In general this would force the 2×2 matrix T to vanish. However, for an odd number of particles

$$\rho_m^{A,a} = \omega^A \xi^a \qquad \Longrightarrow \qquad T^a_b\, \xi^b = 0. \tag{7.15}$$

These two equations, together with condition (7.11), fix three of the four components of T. It is easy to see that the solution is

$$T = \alpha |\xi\rangle\langle\xi|, \tag{7.16}$$

where $\alpha \in \mathbb{C}$ is a complex scale. Therefore we have found a redundancy on the coefficients of the maps given by the transformation (7.8). This is an inherent consequence of the description of the moduli space in terms of the polynomials (7.5). In fact, in Appendix I we show how T is necessary from a purely group-theoretic point of view, when regarding the equivalent maps as representations of a bigger group, identified as $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{C}^2$. Finally, in Appendix I we show how the soft limit of the even-multiplicity maps gives another interpretation of T that is reminiscent of the little group of the soft particle.

Let us close this part of the section by noting that T produces the following shift on the top component of the polynomial:

$$\hat{\rho}_m^{A,a} = \rho_m^{A,a} + T\rho_{m-1}^{A,a} = \omega^A \xi^a + \alpha \langle \xi \, \rho_{m-1}^A \rangle \xi^a, \tag{7.17}$$

or equivalently,

$$\hat{\omega}^A = \omega^A + \alpha \langle \xi \, \rho_{m-1}^A \rangle, \tag{7.18}$$

which will be useful in the next section.

7.1.2 Measure

Let us introduce the measure for $n = 2m+1$, which can be obtained from the soft limit of the measure for $n = 2m+2$. This leads to the correct choice of integration variables, and the integral localizes on the solutions of the scattering equations. Specifically, we consider an amplitude for $n+1 = 2m+2$ particles, the last one of which is chosen to be a gluon. In the soft limit of the gluon momentum, i.e., $p_{2m+2} = \tau \hat{p}_{2m+2}$ and $\tau \to 0$, the even-point measure takes the form

$$\int d\mu_{2m+2}^{\text{6D}} = \delta(p_{n+1}^2) \int d\mu_{2m+1}^{\text{6D}} \frac{1}{2\pi i} \oint_{|\hat{E}_{n+1}|=\varepsilon} \frac{d\sigma_{n+1}}{E_{n+1}} + O(\tau^0), \tag{7.19}$$

where the odd-point measure is given by:

$$\boxed{d\mu_{2m+1}^{\text{6D}} = \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right) \; d^4\omega \, \langle \xi d\xi \rangle}{\text{vol } (\text{SL}(2,\mathbb{C})_\sigma, \text{SL}(2,\mathbb{C})_\rho, \text{T})} \frac{1}{V_n^2} \prod_{i=1}^n \delta^6 \left(p_i^{AB} - \frac{p^{AB}(\sigma_i)}{\prod_{j \neq i} \sigma_{ij}} \right).} \tag{7.20}$$

This is derived in detail in Appendix J. The volume factor here implies modding out by the action of the two $\text{SL}(2,\mathbb{C})$ groups, as well as the T-shift. Furthermore,

$$E_{n+1} = \tau \hat{E}_{n+1} = \tau \sum_{i=1}^n \frac{\hat{p}_{n+1} \cdot p_i}{\sigma_{n+1,i}} = 0 \tag{7.21}$$

corresponds to the scattering equation for the soft particle. The factor of τ in E_{n+1} makes the first term in the expansion of the $(n+1)$-particle measure singular as $\tau \to 0$. As we explain below, the measure given here for $n = 2m+1$ has the correct $\text{SL}(2,\mathbb{C})_\sigma$ scaling, which is degree $4n$.

Let us now proceed to the explicit computation of the measure. Note that the redundancies can be used to fix seven of the $5n+1$ variables, leaving $5n-6$ integrations. This precisely matches the number of bosonic delta functions, which can be counted in the same way as in the even-point case. Therefore, as before, all the integration variables are localized by the delta functions. In order to carry out the computations, one needs use the seven symmetry generators to fix seven coordinates and obtain the corresponding Jacobian. The order in which this is done is also important, since T does not commute with the other generators. In order to make contact with the even-point counterpart, let us first fix the T-shift symmetry. Because T merely generates a shift in the coefficients of the polynomial, it can be seen that the measure in (7.20) is invariant. Now, let us regard the symmetry parameter α as one of the integration variables in favor of fixing one of the four components ω^A. For instance, one can choose ω^1 as fixed, and then integrate over the parameters $\{\alpha, \omega^2, \omega^3, \omega^4\}$. It can be checked from (7.18) that this change of variables induces the Jacobian

$$d^4\hat{\omega} = \langle \xi \, \rho^1_{m-1} \rangle \, d\alpha \, d\omega^2 d\omega^3 d\omega^4. \tag{7.22}$$

The other ingredients in the measure are invariant under this transformation, i.e.,

$$\begin{aligned} \Delta_B(\hat{\rho}, \sigma) &= \Delta_B(\rho, \sigma), & (7.23) \\ \prod_{k=0}^{m-1} d^8\hat{\rho}_k &= \prod_{k=0}^{m-1} d^8\rho_k. & (7.24) \end{aligned}$$

The dependence on α can then be dropped, with the corresponding integration formally canceling the volume factor for the T-shift in the denominator of (7.20). The measure in this partially-fixed form is now

$$d\mu^{\text{6D}}_{2m+1} = \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right) \, d^3\omega \, \langle \xi \, \rho^1_{m-1} \rangle \langle \xi d\xi \rangle}{\text{vol } (\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)} \, \frac{1}{V_n^2} \prod_{i=1}^n \delta^6 \left(p_i^{AB} - \frac{p^{AB}(\sigma_i)}{\prod_{j\neq i} \sigma_{ij}} \right). \tag{7.25}$$

Note that the factor $d^3\omega \langle \xi \rho^1_{m-1} \rangle \langle \xi d\xi \rangle$ is invariant under the projective scaling of ξ_α. By construction, it is also invariant under the action of the T shift, implying that ω^1 may be set to any value. However, after making these choices Lorentz invariance is no longer manifest.

Let us show explicitly how this measure has the correct $\text{SL}(2,\mathbb{C})_\sigma$-scaling under the transformation $\sigma \to t\sigma$ together with the scaling of the coefficients in the maps,

$$\rho_k^{Aa} \to t^{m-k} \rho_k^{Aa}. \tag{7.26}$$

In particular, this implies that $\rho_m^{Aa} = \omega^A \xi^a$ is invariant. As is apparent from (7.8), the parameter α carries $\mathrm{SL}(2,\mathbb{C})_\sigma$-scaling -1, as does the T volume $\langle \xi \rho_{m-1}^1 \rangle$ using (7.26). Since the projective scaling of ξ is completely independent from the $\mathrm{SL}(2,\mathbb{C})_\sigma$ transformation, none of the components ξ^a and ω^A transform. Now, we find

$$\prod_{k=0}^{m-1} d^8 \rho_k \quad \to \quad t^{n^2-1} \prod_{k=0}^{m-1} d^8 \rho_k, \tag{7.27}$$

$$\frac{1}{V_n^2} \prod_{i=1}^{n} d\sigma_i \quad \to \quad t^{4n-n^2} \frac{1}{V_n^2} \prod_{i=1}^{n} d\sigma_i, \tag{7.28}$$

$$\langle \xi \, \rho_{m-1}^1 \rangle \langle \xi d\xi \rangle d^3\omega \quad \to \quad t \, \langle \xi \, \rho_{m-1}^1 \rangle \langle \xi d\xi \rangle d^3\omega, \tag{7.29}$$

leading to the scaling weight of $4n$ for the full measure as required.

Having carried out these checks, we are now in position to give the final form of the measure for $n = 2m+1$ in the same way as explained earlier for even n. For this, we eliminate the remaining $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$ symmetry by performing the standard fixing of $\sigma_1, \sigma_2, \sigma_3$ and $\rho_0^{1,+}, \rho_0^{1,-}, \rho_0^{2,+}$. Note that we fixed the lowest coefficients ρ_0^{Aa}, because they are not affected by the T-shift. Otherwise, this would interfere with the T-shift. Finally, we extract the mass shell and momentum conservation delta functions as in (5.39). This leads to

$$d\mu_{2m+1}^{\mathrm{6D}} = \frac{J_\rho \, J_\sigma}{V_n^2} \left(\prod_{i=4}^{n} d\sigma_i \right) d\rho_0^{2,-} d^2\rho_0^3 \, d^2\rho_0^4 \left(\prod_{k=1}^{m-1} d^8 \rho_k \right) d^3\omega \, d\xi \, \langle \xi \, \rho_{d-1}^1 \rangle \, \hat{\Delta}_B, \tag{7.30}$$

where the Jacobians are given in (5.38), and $\hat{\Delta}_B$ is given in (5.40).

7.1.3 Transformations of the Maps

Having checked the scaling of the measure, here we consider other $\mathrm{SL}(2,\mathbb{C})_\sigma$ transformations, as we will see that they lead to other interesting new features of the odd-point rational maps. In particular, let us consider the inversion $\sigma_i \to -1/\sigma_i$.[1] Under

[1] The minus sign is for convenience only. Sign reversal is already established as a consequence of the scaling symmetry.

this inversion, the rational map transforms as,

$$\frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \to \frac{\langle \rho'^A(\sigma_i)\, \rho'^B(\sigma_i)\rangle}{(\prod_{j=1}^{n}\sigma_j^{-1})(\prod_{j\neq i}\sigma_{ij})}\,, \tag{7.31}$$

and the new object $\rho'^A(\sigma_i)$ entering the map is given by

$$\rho'^A(\sigma_i) = (-1)^m \sum_{k=0}^{m} (-1)^k \rho^A_{k,a}\, \sigma_i^{m-k-\frac{1}{2}}\,. \tag{7.32}$$

Note that this is actually not a polynomial due to the fact that n is odd. To keep the rational-map constraints unchanged, we require that the coefficients transform as

$$\rho^A_{k,a} \to \rho'^A_{k,a} = (-1)^k\, \rho^A_{m-k,a}. \tag{7.33}$$

Then, up to an overall factor, the transformation exchanges the degree-k coefficient with the degree $m-k$ coefficient just like in the case of even n. What is different from the even-point case is the non-polynomial property of $\rho'^A(\sigma_i)$. Therefore, inversion turns the polynomial map into a non-polynomial one of the following form

$$\rho'^A_a(z) = \sum_{k=0}^{m} \rho'^A_{k,a}\, z^{k-\frac{1}{2}}\,. \tag{7.34}$$

Now the lowest-degree coefficient, $\rho'^A_{0,a}$, which is proportional to the highest coefficient of the original map, has the special factorized form

$$\rho'^A_{0,a} = \omega^A\, \xi_a, \tag{7.35}$$

where we have used (7.4). Therefore the product $p^{AB}(z) = \langle \rho^A(z)\ \rho^B(z)\rangle$ remains a degree-$(n-2)$ polynomial.

Although we only use polynomial maps throughout this thesis, it is worth mentioning that the above non-polynomial form of the maps could be used equally well. This discussion makes it is clear that for odd multiplicity a general $\mathrm{SL}(2,\mathbb{C})_\sigma$ transformation can take the original polynomial maps to a more complicated-looking, but equivalent, version of maps. This is a consequence of the fact that the seven generators of $\mathrm{SL}(2,\mathbb{C})_\sigma$, $\mathrm{SL}(2,\mathbb{C})_\rho$, T do not close into a group, as can already be seen from the fact that

$$\text{vol}\ (\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, T) \tag{7.36}$$

carries weight -1 under scaling of $\mathrm{SL}(2,\mathbb{C})_\sigma$. Compositions of the action of these seven generators on the maps lead to the general transformations of the form

$$\rho^{A,a}(z) \to (e^{\mathcal{T}(z)})^a_b\, \rho'^{A,b}(z), \tag{7.37}$$

where $\mathcal{T}^a_b(z)$ is a traceless 2×2 matrix depending on z.

It is interesting to study the subalgebra that preserves the form of the polynomial maps, which for even multiplicity is just $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$. In Appendix I we obtain the corresponding algebra for odd multiplicity: We first show that the generators of $\mathrm{SL}(2,\mathbb{C})_\sigma$ and $\mathrm{SL}(2,\mathbb{C})_\rho$ do not commute in general and that a subset of these recombines into the algebra $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{C}^2$. This includes inversion and T-shift, but it requires a partially fixed $\mathrm{SL}(2,\mathbb{C})_\rho$ frame.

7.2 Integrand from Soft Limits

Here we apply the soft limit to the even-point integrand in order to obtain the odd-point version, with the soft factor included. The answer is composed of two pieces:

$$\mathcal{I}^{\mathcal{N}=(1,1)\ \mathrm{SYM}}_{\mathrm{odd}} = \int d\widehat{\Omega}^{(1,1)}_F \times \mathcal{J}_{\mathrm{odd}}. \tag{7.38}$$

The fermionic measure $d\widehat{\Omega}^{(1,1)}_F$ can be obtained in a way similar to the bosonic one, and we relegate its derivation to Appendix J. The result is

$$\boxed{d\widehat{\Omega}^{(1,1)}_F = V_n\, dg\, d\tilde{g} \prod_{k=0}^{m-1} d^2\chi_k\, d^2\tilde{\chi}_k \prod_{i=1}^{n} \delta^4\!\left(q^A_i - \frac{\langle \rho^A(\sigma_i)\, \chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \delta^4\!\left(\tilde{q}_{i,A} - \frac{[\tilde{\rho}_A(\sigma_i)\, \tilde{\chi}(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right).} \tag{7.39}$$

The fermionic maps are constructed such that $\langle \rho(z)\, \chi(z)\rangle$ and its conjugate are polynomials of degree $n-2$, and take a form similar to the bosonic map in (7.5). Specifically,

$$\chi^a(z) = \sum_{k=0}^{m-1} \chi^a_k\, z^k + g\xi^a z^m, \tag{7.40}$$

$$\tilde{\chi}^{\hat{a}}(z) = \sum_{k=0}^{m-1} \tilde{\chi}^{\hat{a}}_k\, z^k + \tilde{g}\tilde{\xi}^{\hat{a}} z^m, \tag{7.41}$$

where the χ's and $\tilde{\chi}$'s, as well as g and $\tilde{g}$, are Grassmann coefficients. Note that the same spinors ξ^a and $\tilde{\xi}^{\hat{a}}$ appear in the coefficients of z^m for both the bosonic maps and the fermionic maps. This form ensures the coefficient of z^{2m} in the product of maps vanishes, so that the product has the desired degree, $2m-1=n-2$, for an odd number of particles.

With this parametrization of the maps, the first check is to show that the construction has the right Grassmann degree. As in the case of the even n, we need to remove the fermionic "wave functions" $\prod_{i=1}^n \delta^2(\tilde{\lambda}_{i,A,\hat{a}}\, q_i^A)\delta^2(\lambda_{i,b}^B \tilde{q}_B)$. This leaves an integrand with Grassmann degree of $4n$, as required. Having established the Grassmann degree of the integrand, let us next count the number of fermionic integrations. There are $4m$ χ and $\tilde{\chi}$ integrals and two g and $\tilde{g}$ integrals, giving a total of $4m+2=2n$ integrations. The final amplitude thus has Grassmann degree $2n$. More precisely, just as for even n, it has degree n in both the η's and the $\tilde{\eta}$'s, which is what we expect for the superamplitudes of 6D $\mathcal{N}=(1,1)$ SYM in the representation (6.1).

The factor $\mathcal{J}_{\text{odd}}$, which is purely bosonic, contains a contour integral in σ_{n+1} that emerges from the soft limit of the measure (7.19). Therefore, it encodes all of the dependence on the soft particle. Using the identity permutation $\mathbb{I}_n$ and setting $\sigma_{n+1}=z$ for convenience, we show in Appendix J that for a soft gluon

$$S^{a\hat{a}}\mathcal{J}_{\text{odd}} = \text{PT}(\mathbb{I}_n)\,\frac{\sigma_{1n}}{2\pi i}\oint_{|\hat{\mathcal{E}}_{n+1}|=\varepsilon}\frac{dz}{\mathcal{E}_{n+1}}\times\frac{\text{Pf}' A_{n+1}}{(z-\sigma_1)(z-\sigma_n)}\,\frac{x^a}{\langle \xi\,\Xi\rangle}\,\frac{\tilde{x}^{\hat{a}}}{[\tilde{\xi}\,\tilde{\Xi}]}. \tag{7.42}$$

Let us explain the various terms appearing in this formula. First, the vanishing of $\mathcal{E}_{n+1}=\tau\hat{\mathcal{E}}_{n+1}=p(z)\cdot p_{n+1}$ is the rescaled scattering equation for the soft $(n{+}1)$th particle (on the support of the hard scattering equations), such that $\mathcal{E}_{n+1}=E_{n+1}\prod_{i=1}^n(z-\sigma_i)$. In terms of the 6D spinor-helicity formalism, Weinberg's soft factor for a gluon is given by

$$S^{a\hat{a}}=\frac{[\tilde{\lambda}^{\hat{a}}_{n+1}|p_1\tilde{p}_n|\lambda^a_{n+1}\rangle}{s_{n+1,1}s_{n,n+1}}=\frac{\tilde{\lambda}^{\hat{a}}_{n+1,A}\,p_1^{AB}\,\tilde{p}_{n,BC}\,\lambda^{a,C}_{n+1}}{s_{n+1,1}s_{n,n+1}}. \tag{7.43}$$

The reduced Pfaffian can be expanded as

$$\text{Pf}' A_{n+1}=\frac{(-1)^{n+1}}{\sigma_{1n}}\sum_{i=2}^{n-1}(-1)^i\frac{p_{n+1}\cdot p_i}{z-\sigma_i}\,\text{Pf}\,A^{[1,i,n,n+1]}_{n+1}, \tag{7.44}$$

where $A^{[1,i,n,n+1]}_{n+1}$ denotes the matrix A_{n+1} with rows and columns $1,i,n,n+1$ removed. This odd-point integrand by construction does not depend on τ, the scaling parameter introduced to define the soft limit. It is also independent of the choice of polarization $(a, \hat{a})$ and the direction of the soft momentum $p_{n+1} = \tau\, \hat{p}_{n+1}$. Recall that $\xi^a = (1, \xi)$ is determined from the hard scattering maps, while $\Xi^a = (\Xi^+, \Xi^-)$ and $x^a = (x, -1)$ are given by the following linear equations (for $z = \sigma_{n+1}$):

$$\langle \Xi\, \rho^A(z) \rangle = \langle x\, \lambda^A_{n+1} \rangle, \qquad\qquad [\tilde{\Xi}\, \tilde{\rho}^A(z)] = [\tilde{x}\, \tilde{\lambda}^A_{n+1}]. \tag{7.45}$$

Introducing a reference spinor r^A and contracting the first of the preceding two equations with $\epsilon_{ABCD}\lambda^{B,a}_{n+1}\rho^{C,b}(z)r^D$, we obtain

$$x^a \langle \rho^b(z)|\tilde{p}_{n+1}|r\rangle = \Xi^b \langle \lambda^a_{n+1}|\tilde{p}(z)|r\rangle. \tag{7.46}$$

This can be used to make the z-dependence explicit in the integrand. Contracting with ξ_b and repeating these steps for the anti-chiral piece gives

$$\frac{x^a}{\langle \xi\, \Xi\rangle}\,\frac{\tilde{x}^{\hat{a}}}{[\tilde{\xi}\,\tilde{\Xi}]} = \frac{\langle \lambda^a_{n+1}|\tilde{p}(z)|r\rangle\, [\tilde{r}|p(z)|\tilde{\lambda}^{\hat{a}}_{n+1}]}{\xi^b\langle \rho_b(z)|\tilde{p}_{n+1}|r\rangle\, [\tilde{r}|p_{n+1}|\tilde{\rho}_{\hat{b}}(z)]\tilde{\xi}^{\hat{b}}}, \tag{7.47}$$

where $|r\rangle$ and $[\tilde{r}|$ are independent reference spinors. Hence

$$S^{a\hat{a}}\mathcal{J}_{\text{odd}} = \text{PT}(\mathbb{I}_n)\,\frac{\sigma_{1n}}{2\pi i}\oint_{|\hat{\mathcal{E}}_{n+1}|=\varepsilon}\frac{dz}{\mathcal{E}_{n+1}} \times \frac{\text{Pf}' A_{n+1}}{(z-\sigma_1)(z-\sigma_n)}\,\frac{\langle \lambda^a_{n+1}|\tilde{p}(z)|r\rangle[\tilde{r}|p(z)|\tilde{\lambda}^{\hat{a}}_{n+1}]}{\xi^b\langle \rho_b(z)|\tilde{p}_{n+1}|r\rangle[\tilde{r}|p_{n+1}|\tilde{\rho}_{\hat{b}}(z)]\tilde{\xi}^{\hat{b}}}. \tag{7.48}$$

In Section 7.2.1 we evaluate this integral via contour deformation. However, let us point out here the difficulties arising when trying to evaluate this integral. For a given solution of the hard punctures $\{\sigma_i\}_{i=1}^n$ the scattering equation $\hat{\mathcal{E}}_{n+1} = 0$ is a polynomial equation of degree $n-2$ in z, which in general does not have closed-form solutions. In the CHY formalism the soft limit can be evaluated by deforming the contour and enclosing instead the hard punctures at $z = \sigma_i$. This is because the CHY integrand can be decomposed into Parke–Taylor factors, which altogether yield $1/z^2$ as the fall off at infinity. The argument can be straightforwardly repeated for the Witten–RSV formula in four dimensions, as we outline in Appendix I. In the case of (7.48) we find the leading behavior at infinity to be exactly $1/z^2$. However, the new contour will also enclose the poles associated to the brackets in the denominator, which are given by the solutions of a polynomial equation of degree $(n-3)/2$. Since these contributions to the integral also turn out to be cumbersome to evaluate, in the next section we introduce a novel contour deformation that allows us to evaluate the integral without the need to compute these individual contributions.

7.2.1 Contour Deformation

The soft factor $S^{a\hat{a}}$, given by (7.43), is still contained in the integrand $\mathcal{J}_{\text{odd}}$ and introduces an apparent dependence on the soft momentum. In order to extract it and evaluate the contour integral at the same time, we perform a complex shift of the soft momentum p_{n+1}. More specifically, for a given solution of the hard data $\{\sigma_i, \rho, \tilde{\rho}\}$, we perform a holomorphic shift in $|\lambda_{n+1}\rangle$ and use it to extract the odd-point integrand as a residue.

First, consider a reference null six-vector $Q = |q_a\rangle\langle q^a|$. (The Lorentz indices are implicit.) Using the little-group symmetry, the spinors can be adjusted such that

$$\langle \rho_a(\sigma_n)|\tilde{q}_b] = m\, \epsilon_{ab}, \tag{7.49}$$

together with $\langle q_a|\tilde{q}_b] = 0$. Here $m^2 = 2\, p(\sigma_n)\cdot Q$ is a mass scale that drops out at the end of the computation, so we set $m = 1$ for convenience. Note that $\tilde{q}_{b,A}$ transforms under the antifundamental representation of the Lorentz group, $\mathrm{SU}^*(4)$, but under the *chiral* $\mathrm{SL}(2,\mathbb{C})_\rho$. Now consider a shift described by a complex variable w:

$$\begin{aligned} |\lambda^a_{n+1}\rangle \quad &\rightarrow \quad |\lambda^a_w\rangle = |\rho^a_n\rangle + w\,|q^a\rangle & (7.50)\\ |\tilde{\lambda}^{\hat{+}}_{n+1}] \quad &\rightarrow \quad |\tilde{\lambda}^{\hat{+}}_w] = |\tilde{\rho}_n] + w\, C^a|\tilde{q}_a], & (7.51)\end{aligned}$$

where $|\rho^a_n\rangle^A$ is shorthand for $\rho^{A,a}(\sigma_n)$, while

$$|\tilde{\rho}_n]_A = [\tilde{\xi}\, \tilde{\rho}_A(\sigma_n)], \tag{7.52}$$

and the index A has been suppressed in the preceding equations. Without loss of generality, we may make the deformation for a specific choice of the polarization, which we have chosen to be $\hat{a} = \hat{+}$ in the second line. The only requirement for $|\tilde{\lambda}^{\hat{+}}_{n+1}]$ is that

$$0 = \langle \lambda^a_{n+1}|\tilde{\lambda}^{\hat{+}}_{n+1}] = \langle \lambda^a_w|\tilde{\lambda}^{\hat{+}}_w] = \langle \rho^a_n|\tilde{q}_b]C^b + \langle q^a|\tilde{\rho}_n], \qquad a = +,-\,, \tag{7.53}$$

and using (7.49) this implies $C^a = \langle q^a|\tilde{\rho}_n]$.

The shifted soft factor that we utilize is

$$S^{a\hat{+}}_w = \frac{\langle \lambda^a_w|\tilde{p}_n p_1|\tilde{\lambda}^{\hat{+}}_w]}{s_{w,1}s_{w,n}}, \tag{7.54}$$

which has a simple pole at $w = 0$. After a short computation one can show that

$$\frac{1}{2\pi i}\oint_{|w|=\varepsilon} dw\, S_w^{a\hat{+}} = \frac{C^a}{2\pi i}\oint_{|w|=\varepsilon}\frac{dw}{w} = C^a. \tag{7.55}$$

Thus, the odd-point integrand can be recast in the form

$$\mathcal{J}_{\text{odd}} = \text{PT}(\mathbb{I}_n) \times \frac{1}{2\pi i\, C^a}\oint_{|w|=\varepsilon} dw\, I_w^a, \tag{7.56}$$

with

$$I_w^a = \frac{1}{2\pi i}\oint_{|p(z)\cdot p_w|=\varepsilon}\frac{dz}{p(z)\cdot p_w} \times \frac{\sigma_{1n}\,\text{Pf}'A_{n+1}}{(z-\sigma_1)(z-\sigma_n)}\,\frac{x^a}{\langle\xi\,\Xi\rangle}\,\frac{\tilde{x}^{\hat{+}}}{[\tilde{\xi}\,\tilde{\Xi}]}\,. \tag{7.57}$$

As $w \to 0$, the soft momentum $p_w \to p(\sigma_n)$, and hence we expect $z \to \sigma_n$. In fact, we claim that this solution is the only one contributing to the singularity in w. Therefore we may redefine the contour as enclosing only the pole at σ_n, and

$$\begin{aligned} I_w^a &= \frac{1}{2\pi i}\oint_{|z-\sigma_n|=\varepsilon}\frac{dz}{p(z)\cdot p_w} \times \frac{\sigma_{1n}\,\text{Pf}'A_{n+1}}{(z-\sigma_1)(z-\sigma_n)}\,\frac{x^a}{\langle\xi\,\Xi\rangle}\,\frac{\tilde{x}^{\hat{+}}}{[\tilde{\xi}\,\tilde{\Xi}]} &(7.58)\\ &= \frac{\text{Pf}'A_{n+1}|_{z=\sigma_n}}{p(\sigma_n)\cdot p_\omega}\,\frac{\langle\lambda_w^a|\tilde{p}(\sigma_n)|r\rangle[\tilde{r}|p(\sigma_n)|\tilde{\lambda}_w]}{\langle\rho_n|\tilde{p}_w|r\rangle[\tilde{r}|p_w|\tilde{\rho}_n]}. &(7.59)\end{aligned}$$

One can show that:

$$p(\sigma_n)\cdot p_w = \frac{w^2}{2}, \tag{7.60}$$

$$\text{Pf}'A_{n+1}|_{z=\sigma_n} = \frac{\omega}{2}\frac{1}{\sigma_{n1}}\sum_{i=2}^{n-1}(-1)^i\frac{\langle q^a|\tilde{p}_i|\rho_{n,a}\rangle}{\sigma_{ni}}\text{Pf}\,A_{n+1}^{[1,i,n,n+1]} + \mathcal{O}(w^2), \tag{7.61}$$

where we have used the identity

$$\sum_{i=2}^{n-1}(-1)^i\frac{p_n\cdot p_i}{\sigma_{ni}}\text{Pf}\,A_{n+1}^{[1,i,n,n+1]} = 0. \tag{7.62}$$

We also have that

$$\frac{[\tilde{r}|p(\sigma_n)|\tilde{\lambda}_w]}{[\tilde{r}|p_w|\tilde{\rho}_n]} = \frac{w\,[\tilde{r}|\rho_{n,a}\rangle\langle\rho_n^a|\tilde{q}_b]C^b}{w\,[\tilde{r}|\rho_{n,b}\rangle\,C^b} + \mathcal{O}(w) = 1 + \mathcal{O}(w). \tag{7.63}$$

Note that for the chiral piece we can set $|r\rangle$ such that $\tilde{p}(\sigma_n)|r\rangle = |\tilde{\rho}_n]$. Then

$$\frac{\langle\lambda_w^a|\tilde{p}(\sigma_n)|r\rangle}{\langle\rho_n|\tilde{p}_w|r\rangle} = \frac{w\,\langle q^a|\tilde{\rho}_n]}{w\,\epsilon_{ABCD}\,\xi^c\,\rho_{n,c}^A\,\rho_{n,b}^B\,q^{C,b}\,r^D} + \mathcal{O}(w), \tag{7.64}$$

where the contraction in the denominator evaluates to

$$\epsilon_{ABCD}\,\xi^c\,\rho_{n,c}^A\,\rho_{n,b}^B\,q^{C,b}\,r^D = \langle q|\tilde{p}(\sigma_n)|r\rangle = \langle q|\tilde{\rho}_n], \tag{7.65}$$

with $|q\rangle^A := \xi^a q_a^A$. Hence we obtain

$$\lim_{w\to 0}\frac{\langle\lambda_w^a|\tilde{p}(\sigma_n)|r\rangle}{\langle\rho_n|\tilde{p}_w|r\rangle} = \frac{C^a}{\langle q|\tilde{\rho}_n]}. \tag{7.66}$$

Putting everything together we find

$$\begin{aligned}\mathcal{J}_{\text{odd}} &= \mathrm{PT}(\mathbb{I}_n)\times\frac{1}{2\pi i\,C^a}\oint_{|w|=\varepsilon}\frac{dw}{w}\,\frac{C^a}{\sigma_{n1}}\sum_{i=2}^{n-1}(-1)^i\frac{\langle q^a|\tilde{p}_i|\rho_{n,a}\rangle}{\sigma_{ni}\langle q|\tilde{\rho}_n]}\mathrm{Pf}\,A_{n+1}^{[1,i,n,n+1]}\\ &= \mathrm{PT}(\mathbb{I}_n)\times\frac{\langle q^a|\tilde{X}_{(1,n)}|\rho_{n,a}\rangle}{\langle q|\tilde{\rho}_n]},\end{aligned} \tag{7.67}$$

where for convenience we have defined the null vector

$$X_{(1,n)}^{AB} := \frac{1}{\sigma_{n1}}\sum_{i=2}^{n-1}(-1)^i\frac{p_i^{AB}}{\sigma_{ni}}\mathrm{Pf}\,A_{n+1}^{[1,i,n,n+1]} = \frac{1}{2}\epsilon^{ABCD}\tilde{X}_{(1,n),CD}. \tag{7.68}$$

Despite using the notation $\mathrm{Pf}\,A_{n+1}^{[1,i,n,n+1]}$, this Pfaffian is completely independent of the soft momentum and the associated puncture. As anticipated, the expression is independent of the scale of q, so we can remove the normalization condition $2\,p(\sigma_n)\cdot Q = 1$, turning $|q^a\rangle$ into a completely arbitrary spinor. Expanding the numerator of (7.67) in a basis given by $\{\xi,\zeta\}$, where ζ is a reference spinor such that $\langle\xi\zeta\rangle = 1$, we find:

$$\mathcal{J}_{\rm odd} = {\rm PT}(\mathbb{I}_n) \times \frac{\langle q|\tilde{X}_{(1,n)}|\pi_n\rangle - \langle w|\tilde{X}_{(1,n)}|\rho_n\rangle}{\langle q|\tilde{\rho}_n]}, \tag{7.69}$$

where

$$|\pi_n\rangle^A = \langle \zeta\, \rho^A(\sigma_n)\rangle \tag{7.70}$$

is the conjugate component of $|\rho_n\rangle$. Also,

$$|w\rangle^A = \langle \zeta\, q^A\rangle. \tag{7.71}$$

In particular, the fact that the integrand is independent of w implies the non-trivial identity:

$$\tilde{X}_{(1,n)}|\rho_n\rangle = 0, \tag{7.72}$$

which yields the following form of the integrand

$$\mathcal{J}_{\rm odd} = {\rm PT}(\mathbb{I}_n) \times \frac{\langle q|\tilde{X}_{(1,n)}|\pi_n\rangle}{\langle q|\tilde{\rho}_n]}. \tag{7.73}$$

Using (7.72) this expression can be recast in a non-chiral form. Let us introduce another reference spinor $|\tilde{q}]$ and consider

$$\mathcal{J}_{\rm odd} = {\rm PT}(\mathbb{I}_n) \times \frac{\langle q|\tilde{X}_{(1,n)}\, p(\sigma_n)|\tilde{q}]}{\langle q|\tilde{\rho}_n]\langle \rho_n|\tilde{q}]}. \tag{7.74}$$

Note that $\tilde{p}(\sigma_n)_{AB} X^{BC}_{(1,n)} = -\tilde{X}_{(1,n)\,,AB}\, p(\sigma_n)^{BC}$. Finally, using the definition of $X_{(1,n)}$ in (7.68) we recognize that the second factor in (7.74) is in fact a reduced Pfaffian of an antisymmetric $(n+1)\times(n+1)$ matrix constructed out of A_n with an additional column and row (labeled by $\star$) attached. We call this matrix $\widehat{A}_n$. Restoring the original integration variables, its entries are given by:

$$\boxed{[\widehat{A}_n]_{ij} = \begin{cases} \dfrac{p_i\cdot p_j}{\sigma_{ij}} & \text{if} \quad i\neq j, \\ 0 & \text{if} \quad i=j, \end{cases} \qquad \text{for} \qquad i,j=1,2,\ldots,n,\star,} \tag{7.75}$$

where

$$p_\star^{AB} = \frac{2\, q^{[A} p^{B]C}(\sigma_\star)\tilde{q}_C}{q^D[\tilde{\rho}_D(\sigma_\star)\,\tilde{\xi}]\langle\rho^E(\sigma_\star)\,\xi\rangle\tilde{q}_E} \tag{7.76}$$

is a reference null vector entering the final row and column, q and $\tilde{q}$ are arbitrary spinors, and $\sigma_\star$ is a reference puncture that can be set to one of the punctures associated to removed rows and columns. In fact, we have numerical evidence that $\sigma_\star$ can be chosen completely arbitrarily without changing the result. Here, $q^{[A}p^{B]C}$ denotes the antisymmetrization $q^A p^{BC} - q^B p^{AC}$. The reduced Pfaffian is then defined analogously to (6.11), with the restriction that the starred column and row are not removed. Independence of the choice of removed columns and rows follows from the analogous statement for n even. It is straightforward to confirm that $\text{Pf}\,'\widehat{A}_n$ transforms as a quarter-integrand in the $\text{SL}(2,\mathbb{C})_\rho$-frame studied in Appendix I, and that its mass dimension is $n-2$, as required. This completes the derivation of the odd-point formula (7.1). The reasoning was complicated, but the result is as simple as could be hoped for.

7.3 Consistency Checks

We have checked numerically that the new formula (7.1) correctly reproduces the 6D SYM amplitudes of gluons and scalars directly computed from Feynman diagrams, up to $n = 7$. In this section we perform additional consistency checks of the formula. We begin by re-deriving the odd-point integrand $\mathcal{I}_{\text{odd}}$ by comparing it with the corresponding CHY expression for a particular bosonic sector following a similar argument used earlier for the case of even n. We will then show analytically that the formula leads to the correct three-point super-amplitude of 6D SYM. It is worth noting that the three-point amplitudes in 6D YM are rather subtle due to the special kinematics first explained in [84]. As we will see, our formula gives a natural parametrization of the special three-point kinematics.

7.3.1 Comparison with CHY

This section presents an alternative derivation of the integrand of the odd-point amplitudes. The method we will use here is similar to the one for the even-point case given in section 6.2. It is based on comparison to known results of the CHY formulation of YM amplitudes in general spacetime dimensions. This method of derivation is independent of and very different from the soft-theorem derivation presented in the previous sections. Therefore it constitutes an additional consistency check.

Let us begin with the general form of the odd-point amplitudes of 6D $\mathcal{N}=(1,1)$ SYM,

$$\mathcal{A}_n^{\mathcal{N}=(1,1)\,\mathrm{SYM}} = \int d\mu_n^{\mathrm{6D}}\, d\widehat{\Omega}_F^{(1,1)} \times \mathcal{J}_{n\,\mathrm{odd}}, \tag{7.77}$$

for $n=2m+1$. Recall that the bosonic measure $d\mu_n^{\mathrm{6D}}$ is defined in (7.20), and the fermionic measure $d\widehat{\Omega}_F^{(1,1)}$ is given in (7.39), which is the part that is more relevant to the discussion here. The goal is to determine the integrand $\mathcal{J}_{n\,\mathrm{odd}}$. As mentioned above, we will follow the same procedure as in the case of even n, namely comparison of our formula with the CHY formulation of amplitudes for adjoint scalars and gluons. To do so, we consider a particular component of the amplitude. Due to the fact that n is odd and the scalars have to appear in pairs, it is not possible to choose all the particles to be scalars. The most convenient choice of the component amplitudes one with $n-1$ scalars and one gluon. Concretely, in the same notation as before, we choose to consider

$$\mathcal{A}_n(\phi_1^{1\hat{1}}, \ldots, \phi_m^{1\hat{1}}, \phi_{m+1}^{2\hat{2}}, \ldots, \phi_{2m}^{2\hat{2}}, A_n^{a\hat{a}})\,, \tag{7.78}$$

where $A_n^{a\hat{a}}$ is a gluon.

As in Section 6.2, we integrate out the fermionic variables so as to extract the desired component amplitude. The computation is similar to the one for even n, but slightly more complicated due to the appearance of $A_n^{a\hat{a}}$ in the middle term of the superfield. Projecting to this component amplitude, we obtain

$$\int d\Omega_F^{(1,1)} \implies V_n J_{\mathrm{w}} \int \prod_{k=0}^{m-1} d^2\chi_k\, d^2\tilde{\chi}_k\, dg\, d\tilde{g}\, d\eta_n^a d\tilde{\eta}_n^{\hat{a}}\;\; \Delta_F^{\mathrm{proj}} \widetilde{\Delta}_F^{\mathrm{proj}}. \tag{7.79}$$

The factor $J_{\mathrm{w}} = \prod_{i=1}^n \frac{1}{(p_i^{13})^2}$ arises from extracting the fermionic wave functions. The fermionic delta functions are given by

$$\Delta_F^{\mathrm{proj}} = \prod_{A=1,3} \delta\left(q_n^A - \frac{\langle \rho^A(\sigma_n)\,\chi(\sigma_n)\rangle}{\prod_{j\neq n}\sigma_{nj}}\right) \prod_{i\in Y} \delta\left(\frac{\langle \rho^A(\sigma_i)\,\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \prod_{i\in \overline{Y}} p_i^{13}, \tag{7.80}$$

$$\widetilde{\Delta}_F^{\mathrm{proj}} = \prod_{A=2,4} \delta\left(\tilde{q}_{nA} - \frac{[\tilde{\rho}_A(\sigma_n)\,\tilde{\chi}(\sigma_n)]}{\prod_{j\neq n}\sigma_{nj}}\right) \prod_{i\in Y} \delta\left(\frac{[\tilde{\rho}_A(\sigma_i)\,\tilde{\chi}(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right) \prod_{i\in \overline{Y}} p_i^{13}, \tag{7.81}$$

with $Y := \{1, \ldots, m\}$ and $\overline{Y} := \{m+1, \ldots, n-1\}$. Compared to the even-particle case, we have an additional contribution coming from the gluon $A_n^{a\hat{a}}$. Performing the fermionic integrations leads to the final result,

$$d\widehat{\Omega}_F^{(1,1)} \implies (J_F)_{a\hat{a}} = \frac{l_{n,a}^{[A}\langle\rho^{B]}(\sigma_n)\ \xi\rangle}{p_n^{AB}} \frac{\tilde{l}_{n,\hat{a},[C}[\tilde{\rho}_{D]}(\sigma_n)\ \tilde{\xi}]}{p_{n,CD}} \frac{V_n}{\prod_{j\neq n}\sigma_{nj}^2} \prod_{i\in Y, J\in\overline{Y}} \frac{1}{\sigma_{iJ}^2}, \tag{7.82}$$

where the square brackets denote anti-symmetrization on indices A, B and C, D. Note that although the formula for $(J_F)_{a\hat{a}}$ exhibits explicit Lorentz indices A, B and C, D, it is actually independent of the choice of these indices. Therefore, we have only made the dependence on the little-group indices a and $\hat{a}$ explicit in $(J_F)_{a\hat{a}}$. They appear because the component amplitude contains a gluon $A_n^{a\hat{a}}$.

Having extracted the component amplitude that we want, we can compare it to the corresponding result from the CHY formulation. From the comparison, we find that the odd-point integrand is given by

$$\mathcal{J}_{n\,\mathrm{odd}}(\alpha) = \frac{\mathrm{Pf}'(\Psi_{\mathrm{project}})_{a\hat{a}}}{(J_F)_{a\hat{a}}} \times \mathrm{PT}(\alpha)\,. \tag{7.83}$$

This ratio should be scalar, independent of the choice of the little-group indices a and $\hat{a}$. As in the case of n even, $\mathrm{Pf}'\Psi_{\mathrm{project}}$ is defined by the usual $\mathrm{Pf}'\Psi$, projected to the component amplitude under consideration. In the present case this means that the dot products of a pair of polarization vectors for scalars particles are the same as before, namely $\varepsilon_i \cdot \varepsilon_I = 1$ if $i \in Y$ and $I \in \overline{Y}$, and otherwise they vanish. Furthermore $\varepsilon_i \cdot \varepsilon_n = 0$, and $p_i \cdot \varepsilon_j = 0$ if $j \neq n$. Using these rules, the original reduced Pfaffian $\mathrm{Pf}'\Psi$ simplifies to

$$\mathrm{Pf}'(\Psi_{\mathrm{project}})_{a\hat{a}} = \det(\Delta_m) \sum_{i=1}^{n-2} (-1)^i \frac{p_i \cdot (\varepsilon_n)_{a\hat{a}}}{\sigma_{in}} \mathrm{Pf}A_n^{[i,n-1,n]}\,, \tag{7.84}$$

where the $m \times m$ matrix Δ_m has entries given by $\frac{1}{\sigma_{iI}}$ for $i \in Y$ and $I \in \overline{Y}$.

The ratio entering the integrand $\mathcal{J}_{n\,\mathrm{odd}}$ in (7.83) can be dramatically simplified. To demonstrate this, note that as $\mathcal{J}_{n\,\mathrm{odd}}(\alpha)$ is a scalar, the following two tensors are proportional,

$$l_{n,a}^{[A}\langle\rho^{B]}(\sigma_n)\ \xi\rangle \tilde{l}_{n,\hat{a},[C}[\tilde{\rho}_{D]}(\sigma_n)\ \tilde{\xi}] \times R = p_n^{AB}\, p_{n,CD} \sum_{i=1}^{n-2} (-1)^i \frac{p_i \cdot (\varepsilon_n)_{a\hat{a}}}{\sigma_{in}} \mathrm{Pf}A_n^{[i,n-1,n]}, \tag{7.85}$$

where the proportionality factor R is a scalar. After multiplying both sides of this equation with $l_n^{A,a}\tilde{l}^{\hat{a}}_{n,C}$ and contracting indices a and $\hat{a}$, we obtain

$$\begin{aligned}\langle\rho^A(\sigma_n)\,\xi\rangle[\tilde{\rho}_C(\sigma_n)\,\tilde{\xi}]\times R &= \sum_{i=1}^{n-2}(-1)^i\,\frac{l_n^{A,a}\,p_i\cdot(\varepsilon_n)_{a\hat{a}}\,\tilde{l}^{\hat{a}}_{n,C}}{\sigma_{in}}\,\mathrm{Pf}A_n^{[i,n-1,n]}\\ &= \sum_{i=1}^{n-2}\frac{(-1)^i}{\sigma_{in}}\frac{p_n^{AB}\,p_{i,BD}\,\varrho^{DE}\,p_{n,EC}}{\varrho\cdot p_n}\mathrm{Pf}A_n^{[i,n-1,n]},\end{aligned} \tag{7.86}$$

where in the last line we used the spinor form of the polarization vector $(\varepsilon_n)_{a\hat{a}}$ [84], with ϱ a reference vector. Collecting everything and plugging R back into the integrand, we arrive at:

$$\mathcal{J}_{n\,\mathrm{odd}}(\alpha) = \frac{\mathrm{PT}(\alpha)}{\sigma_{n-1,\,n}}\sum_{i=1}^{n-2}\frac{(-1)^i}{\sigma_{in}}\,\frac{p^{AB}(\sigma_n)\,p_{i,BD}\,\varrho^{DE}\,p_{n,EC}}{\varrho\cdot p_n\,\rho_n^A\,\tilde{\rho}_{n,C}}\;\mathrm{Pf}A_n^{[i,n-1,n]}, \tag{7.87}$$

where we have also simplified the σ-dependent part, and defined

$$\rho_n^A := \langle\rho^A(\sigma_n)\,\xi\rangle\,,\quad \tilde{\rho}_{n,C} := [\tilde{\rho}_C(\sigma_n)\,\tilde{\xi}]\,, \tag{7.88}$$

as in the previous section. Furthermore, using the identity

$$\sum_{i=1}^{n-2}(-1)^i\,\frac{p_i\cdot p_n}{\sigma_{in}}\,\mathrm{Pf}A_n^{[i,n-1,n]} = 0\,, \tag{7.89}$$

the summation in the expression of $\mathcal{J}_{n\,\mathrm{odd}}(\alpha)$ can be further simplified, leading to the final form of the integrand:

$$\mathcal{J}_{n\,\mathrm{odd}}(\alpha) = \frac{\mathrm{PT}(\alpha)}{\sigma_{n-1,\,n}}\sum_{i=1}^{n-2}(-1)^i\,\frac{p^{AB}(\sigma_n)\,p_{i,BC}}{\sigma_{in}\,\rho_n^A\,\tilde{\rho}_{n,C}}\,\mathrm{Pf}A_n^{[i,n-1,n]}\,. \tag{7.90}$$

This result is actually a Lorentz scalar, as it should be, even though it appears to depend on the explicit Lorentz spinor indices A and C. The above expression agrees with (7.74) after contraction with reference spinors in the numerator and denominator and choosing $\sigma_\star = \sigma_n$. In the derivation here, we have chosen particles n as well as $n{-}1$ to be special. However, the final result should be independent of such a choice, and therefore we have a complete agreement with (7.74), the result obtained by using the soft theorem.

7.3.2 Three-point Amplitude

Here we derive analytically the three-point amplitude from our odd-n formula. As explained in [84], the three-point amplitude requires additional considerations such as an adequate parametrization of its special kinematics. Here we find that our formula naturally leads to such a parametrization together with the correct supersymmetric expression. Since the result, which is quite subtle, exists in the literature [99], it is nice to see that our formula reproduces the known result. It turns out that it is more convenient to use the linearized constraints introduced in (5.41). So we start with the following integral representation of the superamplitude:

$$\mathcal{A}_3^{\mathcal{N}=(1,1)\ \mathrm{SYM}}(123) = \int d\mu_3^{\mathrm{6D}} \frac{\mathcal{J}_3}{(V_3)^3} \int d^2\chi_0\, d^2\tilde{\chi}_0\, dg\, d\tilde{g} \prod_{i=1}^{3} \delta^2(\eta_i^b M_{i,b}^a - \chi^a(\sigma_i))\delta^2(\tilde{\eta}_i^{\hat{b}} \widetilde{M}_{i,\hat{b}}^{\hat{a}} - \tilde{\chi}^{\hat{a}}(\sigma_i)). \tag{7.91}$$

The fermionic delta functions in the above formula are the fermionic versions of the linear constraints, and we will discuss the n-point version of these constraints in Chapter 8. For now we take this as a given, and write the degree 1 three-point maps as:

$$\begin{aligned} \rho^{A,a}(z) &= \rho_0^{A,a} + \omega^A \xi^a\, z, \\ \chi^a(z) &= \chi_0^a + g\, \xi^a\, z, \end{aligned} \tag{7.92}$$

together with their conjugates $\tilde{\rho}_{A\hat{a}}(z)$ and $\tilde{\chi}_{\hat{a}}(z)$. Imposing the orthogonality condition $\rho^{A,a}(z)\tilde{\rho}_{A,\hat{a}}(z) = 0$ we find:

$$\begin{aligned} \rho_0^{A,a}\, \tilde{\rho}_{0,A,\hat{a}} &= 0, \\ \rho_0^{A,a}\, \tilde{\omega}_A\, \tilde{\xi}_{\hat{a}} + \xi^a\, \omega^A\, \tilde{\rho}_{0,A,\hat{a}} &= 0, \\ \omega^A\, \tilde{\omega}_A &= 0. \end{aligned} \tag{7.93}$$

The solution to the middle constraint is given by

$$\begin{aligned} \rho_0^{A,a}\, \tilde{\omega}_A &= t\, \xi^a, \\ \omega^A\, \tilde{\rho}_{0,A,\hat{a}} &= -t\, \tilde{\xi}_{\hat{a}}, \end{aligned} \tag{7.94}$$

for some scale t. Recall that the top component of each map, i.e., $\xi^a\omega^A$ and its conjugate, carries a $\mathrm{GL}(1,\mathbb{C})$ freedom which we previously used to fix $\xi^+ = 1$. For reasons that will become apparent soon, here it is more convenient to use this scaling to fix $t = V_3$. Using this and the previous equations we find the following relation:

$$\rho^{A,a}(\sigma_i)\tilde{\rho}_{A,\hat{a}}(\sigma_j) = V_3\,\xi^a\tilde{\xi}_{\hat{a}}\,\sigma_{ij}. \tag{7.95}$$

Let us now evaluate the integrand in the representation of (7.73) and (7.68):

$$\begin{aligned}\mathcal{J}_3 &= \frac{1}{(V_3)^2\sigma_{13}}\times\frac{\langle q|\tilde{p}(\sigma_1)|\pi_3\rangle}{\langle q|\tilde{\rho}_3]}\\ &= \frac{1}{(V_3)^2\sigma_{13}}\frac{q^A\tilde{\rho}_{A,\hat{a}}(\sigma_1)\tilde{\rho}^{\hat{a}}_B(\sigma_1)\rho^B_a(\sigma_3)\zeta^a}{q^A\tilde{\rho}^{\hat{a}}_{0,A}\tilde{\xi}_{\hat{a}}}\\ &= 1/V_3, \end{aligned} \tag{7.96}$$

where we used $\tilde{X}_{(1,3)} = -\frac{\tilde{p}(\sigma_1)}{V_3\sigma_{13}}$, $\mathrm{Pf}\,A_4^{[1,2,3,4]} = 1$, $\langle\xi\zeta\rangle = 1$ and $q^A\tilde{\rho}^{\hat{a}}_A(\sigma)\tilde{\xi}_{\hat{a}} = q^A\tilde{\rho}^{\hat{a}}_{0,A}\tilde{\xi}_{\hat{a}}$. For three points, the $\mathrm{SL}(2,\mathbb{C})_\sigma$ symmetry completely fixes all three σ's, and we have,

$$\int d\mu_3^{\mathrm{CHY}} = (V_3)^2. \tag{7.97}$$

Plugging this into (7.91) we are left with

$$\mathcal{A}_3^{\mathcal{N}=(1,1)\ \mathrm{SYM}}(123) = F_3^{(1,0)}F_3^{(0,1)}\,, \qquad F_3^{(1,0)} = \frac{1}{V_3}\int d\chi_0^+ d\chi_0^-\, dg\prod_{i=1}^{3}\delta^2(\eta_i^b M^a_{i,b} - \chi^a(\sigma_i)), \tag{7.98}$$

together with its conjugate $F_3^{(0,1)}$. We find that now the three-point amplitude only involves fermionic integrals and factorizes into chiral and antichiral pieces. However, this form is not completely satisfactory as it still carries redundancies. In order to match this expression with the known ones [84, 94], we note that (7.95) can be inverted as follows: Pick three labels $\{i,j,k\} = \{1,2,3\}$ for the external particles, then

$$\langle\lambda_i^a|\tilde{\lambda}_j^{\hat{a}}] = V_3\,\frac{M^a_{i,b}\xi^b\widetilde{M}^{\hat{a}}_{j\hat{b}}\tilde{\xi}^{\hat{b}}}{|M_i|\,|\widetilde{M}_j|}\sigma_{ij} = \epsilon_{ijk}\left(M^a_{i,b}\xi^b\right)\left(\widetilde{M}^{\hat{a}}_{j,\hat{b}}\tilde{\xi}^{\hat{b}}\right)\,, \tag{7.99}$$

where ϵ_{ijk} is the sign of the permutation (ijk), as usual. This allows us to read off the variables defined in [84] for the special case of three-point kinematics. Since $\det\langle\lambda_i^a|\tilde{\lambda}_j^{\hat{a}}] = 0$,

$$u_i^a = M^a_{i,b}\,\xi^b\,, \qquad \tilde{u}_i^{\hat{a}} = \widetilde{M}^{\hat{a}}_{i,\hat{b}}\,\tilde{\xi}^{\hat{b}}. \tag{7.100}$$

It is easy to check that they satisfy $u_i^a \lambda_{i,a}^A = u_j^a \lambda_{j,a}^A$ for any i, j. Their duals, defined as

$$w_i^a = \frac{M_{i,b}^a \zeta^b}{\sigma_{ij}\sigma_{ik}}\,, \qquad \tilde{w}_i^{\hat{a}} = \frac{\widetilde{M}_{i,\hat{b}}^{\hat{a}} \tilde{\zeta}^{\hat{b}}}{\sigma_{ij}\sigma_{ik}}\,, \tag{7.101}$$

satisfy $\langle u_i\, w_i\rangle = [\tilde{u}_i\, \tilde{w}_i] = 1$. Since the maps are constructed such that momentum conservation is guaranteed, the condition imposed in [84],

$$\sum_{i=1}^{3} \omega_i^a \lambda_{i,a}^A = \zeta^a \sum_{i=1}^{3} \frac{\rho_a^A(\sigma_i)}{|M_i|} = 0, \tag{7.102}$$

is also satisfied by virtue of the residue theorem. Furthermore, note that there are scaling and shifting redundancies in the definition of $u_i, \tilde{u}_i, w_i, \tilde{w}_i$ [84]. In particular, these variables are defined up to a rescaling,

$$u_i \to \alpha u_i\,, \quad \tilde{u}_i \to \alpha^{-1}\tilde{u}_i\,, \quad w_i \to \alpha^{-1} w_i\,, \quad \tilde{w}_i \to \alpha \tilde{w}_i\,, \tag{7.103}$$

which is a reflection of scaling redundancy of ξ and ζ. Additionally, there is a shift redundancy in w_i,

$$w_i \to w_i + b_i u_i\,, \tag{7.104}$$

with $\sum_{i=1}^{3} b_i = 0$ corresponds to the redundancy $\zeta \to \zeta + b\,\xi$ in the defining condition $\langle \zeta\xi\rangle = 1$. Let us now fix this SL$(2,\mathbb{C})$ redundancy by setting $\xi = (1,0)$ and $\zeta = (0,1)$. Then

$$M_i = \begin{pmatrix} u_i^+ & u_i^- \\ \sigma_{ij}\sigma_{ik} w_i^+ & \sigma_{ij}\sigma_{ik} w_i^- \end{pmatrix}, \tag{7.105}$$

and similarly for the conjugate. We will now focus on the chiral piece F_3. Following [99], we define $\mathbf{w}_i = w_i^a \eta_{i,a}$ and $\mathbf{u}_i = u_i^a \eta_{i,a}$. Then we evaluate the fermionic integrals as follows

$$\begin{aligned} F_3^{(1,0)} &= \frac{1}{V_3} \int d\chi_0^+ d\chi_0^- dg \prod_{i=1}^{3} \delta(\sigma_{ij}\sigma_{ik}\mathbf{w}_i - \chi_0^+ - g\,\sigma_i)\delta(\mathbf{u}_i - \chi_0^-) \\ &= \frac{1}{V_3}(\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_3) \int d\chi_0^+ dg \prod_{i=1}^{3} \delta(\sigma_{i\,i+1}\sigma_{i\,i+2}\mathbf{w}_i - \chi_0^+ - g\,\sigma_i) \\ &= (\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2\mathbf{u}_3 + \mathbf{u}_3\mathbf{u}_1)(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3), \end{aligned} \tag{7.106}$$

where we have omitted the notation "δ" for fermionic delta functions. The final result is in precise agreement with the three-point superamplitude given, e.g., in [94]. For example, the three-gluon amplitude is:

$$\mathcal{A}_3(A_1^{a\hat{a}}, A_2^{b\hat{b}}, A_3^{c\hat{c}}) = \left(u_1^a u_2^b w_3^c + u_1^a w_2^b u_3^c + w_1^a u_2^b u_3^c\right)\left(\tilde{u}_1^{\hat{a}} \tilde{u}_2^{\hat{b}} \tilde{w}_3^{\hat{c}} + \tilde{u}_1^{\hat{a}} \tilde{w}_2^{\hat{b}} \tilde{u}_3^{\hat{c}} + \tilde{w}_1^{\hat{a}} \tilde{u}_2^{\hat{b}} \tilde{u}_3^{\hat{c}}\right) . \quad (7.107)$$

Chapter 8

Linear Form of the Maps

In this chapter we present an alternative version of the connected formula for tree-level scattering amplitudes in 6D $\mathcal{N} = (1, 1)$ SYM. We make use of "linear" constraints involving λ_a^A and η_a directly, instead of the quadratic combinations $p^{AB} = \langle\lambda^A\lambda^B\rangle$ and $q^A = \langle\lambda^A\eta\rangle$. This form of the constraints is a natural generalization of the 4D Witten–RSV formula, in the form of (5.18). We have previously presented the linear constraints in (5.41). However, our conventions in this chapter differ from the previous formula by the change of variables $W_i = M_i^{-1}$. Since the M_i's are 2×2 matrices, the two formulations differ by where W_i appears in the constraints as well as an overall Jacobian. For certain computations such as the soft limits, it may be preferable to use the previous version of the constraints.

One way in which the linear constraints differ from the quadratic constraints is that the on-shell conditions are no longer built in. Instead, they are enforced by the introduction of spinor-helicity variables. Another feature of the linear form is that it makes manifest more of the symmetries, including the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ R symmetry. We will also give evidence that this representation may be a step towards a Grassmannian formulation of 6D theories [9].

As in the previous formulation of 6D theories, there are additional subtleties when the number of particles n is odd. As before, the maps appropriate for odd n require the T symmetry, which acts as a redundancy of these maps. SYM amplitudes follow by pairing these constraints with the integrands found previously.

Using the linear constraints for even- and odd-point SYM amplitudes, in Section 8.3 we obtain a version of these constraints that is even closer to the original Witten–RSV form. In the case of 4D, this version is sometimes known as the *Veronese embedding* [8]. This is achieved by evaluating the integral over the original rational maps $\rho_a^A(z)$, $\chi_a(z)$, $\tilde{\chi}_{\hat{a}}(z)$, leaving an integral over only the punctures and the W_i variables. This allows one to view the linear constraints as those for a symplectic (or Lagrangian) Grassmannian acting on a vector built from the external kinematic data.

As an application of this formulation, we also present an alternative version of the tree-level amplitudes of the Abelian $(2,0)$ M5-brane theory. Since this theory does not have odd-point amplitudes, it is not a focus of the present work. Still, the linear version of the tree amplitudes of this theory have some advantages compared to the formula presented in [149].

8.1 Linear Even-Point Measure

The linear form of the 6D even-point measure is obtained by introducing an integration over GL(2) matrices $(W_i)_a^b$ associated to each particle (or puncture):

$$\begin{aligned}\int d\mu_{n\text{ even}}^{\text{6D}} &= \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k}{\text{vol}\left(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho\right)} \frac{1}{V_n^2} \prod_{i=1}^n \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \\ &= \left(\prod_{i=1}^n \delta(p_i^2)\right) \int \frac{\prod_{i=1}^n \prod_{k=0}^m d^8\rho_k}{\text{vol}\left(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho\right)} \mathcal{W}(\lambda,\rho,\sigma), \qquad (8.1)\end{aligned}$$

where

$$\mathcal{W}(\lambda,\rho,\sigma) = \prod_{i=1}^n \int d^4W_i\, \delta^8\big(\lambda_{ia}^A - (W_i)_a^b \rho_b^A(\sigma_i)\big)\, \delta\left(|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right) \qquad (8.2)$$

and $|W_i| = \det W_i$. The total number of delta functions exceeds the number of integrations by $n+6$, accounting for the mass-shell and momentum-conservation delta functions. This step introduces $4n$ integrals in addition to the previous $5n-6$ that were previously present after accounting for the $\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho$ symmetry. It allowed us to extract the n mass-shell constraints $\delta(p_i^2)$.

Before proceeding, let us comment on the $\text{SL}(2,\mathbb{C})$ indices of the matrix $(W_i)_a^b$. Throughout this thesis, we have used the Latin indices $a=+,-$ to denote both the "global" $\text{SL}(2,\mathbb{C})_\rho$ indices as well as the little-group indices of the external particles. The latter was not visible when all the external data entered the formulas through the little-group invariant combinations p_i^{AB}, q_i^A, and $\tilde{q}_{iA}$. In passing to the linear form, we have introduced one matrix $(W_i)_a^b$ *per particle*. We should view the upper index as global, because it contracts with the maps, whereas the lower index must transform under the little group of the ith external particle in order for the delta functions to be little-group invariant. So each W_i transforms as a bi-fundamental under the global $\mathrm{SL}(2,\mathbb{C})_\rho$ and the ith $\mathrm{SL}(2,\mathbb{C})$ little group. (The corresponding feature was also present in 4D when the t_i and $\tilde{t}_i$ variables were introduced.) More explicitly, it is sometimes useful to solve for them in favor of the maps as follows: If we pick $\{A,B\} \subset \{1,2,3,4\}$, then

$$p_i^{AB} W_{i,b}^a = \frac{\rho^{[A,a}(\sigma_i)\,\lambda_{i,b}^{B]}}{\prod_{j\neq i}\sigma_{ij}}, \tag{8.3}$$

the above solution also makes clear the difference between these two $\mathrm{SL}(2,\mathbb{C})$ indices. Despite this subtlety, we have elected not to use different notations for the different kinds of the $\mathrm{SL}(2,\mathbb{C})$ indices, though it is always easy to distinguish them based on the context.

The passage to linear constraints works analogously for the fermionic delta functions. The relevant identity is now:

$$\Delta_F = \prod_{i=1}^n \delta^4\left(q_i^A - \frac{\langle\rho^A(\sigma_i)\,\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) = \prod_{i=1}^n \delta^2\left(\tilde{l}_{iA\hat{a}}q_i^A\right)\delta^2\left(\eta_i^a - (W_i)_b^a\chi^b(\sigma_i)\right), \tag{8.4}$$

$$\widetilde{\Delta}_F = \prod_{i=1}^n \delta^4\left(\tilde{q}_{i,A} - \frac{[\tilde{\rho}_A(\sigma_i)\,\tilde{\chi}(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right) = \prod_{i=1}^n \delta^2\left(\lambda_{ia}^A\tilde{q}_{iA}\right)\delta^2\left(\tilde{\eta}_i^{\hat{a}} - (\widetilde{W}_i)_{\hat{b}}^{\hat{a}}\tilde{\chi}^{\hat{b}}(\sigma_i)\right). \tag{8.5}$$

These formulas are only valid on the support of the bosonic delta functions. Just like $\tilde{\rho}_k$, the conjugate set of matrices, $\widetilde{W}_i$, are not integrated over. Rather, they are solved for by the conjugate set of constraints, as in (6.16). As before, this form allows us to explicitly extract the super-wave-function factors leaving linear fermionic delta functions in the η and $\tilde{\eta}$ variables.

For the case of 6D $\mathcal{N}=(1,1)$ SYM with n even, the right-hand integrand, which is the Parke–Taylor factor, does not depend on this change of variables. So we can now assemble the even-point integrand, which in terms of the usual maps,

$$\chi^a(z) = \sum_{k=0}^m \chi_k^a\, z^k, \qquad \tilde{\chi}^{\hat{a}}(z) = \sum_{k=0}^m \tilde{\chi}_k^{\hat{a}}\, z^k, \tag{8.6}$$

is given by

$$\mathcal{I}_{n\ \text{even}}^{\mathcal{N}=(1,1)\,\text{SYM}} = \text{PT}(\alpha)\left(V_n\,\text{Pf}'A_n \int \left(\prod_{k=0}^{m} d^2\chi_k\, d^2\tilde{\chi}_k\right)\Delta_F\widetilde{\Delta}_F,\right). \tag{8.7}$$

Removing the mass-shell delta functions, the explicit formula for the linear form of the even-point scattering amplitudes of 6D $\mathcal{N}=(1,1)$ SYM is

$$\boxed{\begin{aligned}\mathcal{A}_{n\ \text{even}}^{\mathcal{N}=(1,1)\ \text{SYM}}(\alpha) = &\int \frac{\prod_{i=1}^n d\sigma_i\, \prod_{k=0}^m d^8\rho_k\, d^2\chi_k\, d^2\tilde{\chi}_k}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)}\ \text{PT}(\alpha)\ \mathcal{W}(\lambda,\rho,\sigma)\\ &\times V_n\,\text{Pf}'A_n \prod_{i=1}^{n} \delta^2\big(\eta_i^a - (W_i)^a_b\chi^b(\sigma_i)\big)\,\delta^2\Big(\tilde{\eta}_i^{\hat{a}} - (\widetilde{W}_i)^{\hat{a}}_{\hat{b}}\tilde{\chi}^{\hat{b}}(\sigma_i)\Big)\,.\end{aligned}} \tag{8.8}$$

So far we have used the fact that the kinematic data associated to a given particle in 6D can be encoded in two pairs of spinors, λ^A_{ia} and $\tilde{\lambda}^{i\hat{a}}_A$. However, using the overall scaling it is also possible to associate the chiral part, λ^A_{ia}, with a line in $\mathbb{CP}^3$ and two points on it, where the two components, $a=\pm$, label the points. The linear formula implements the transformation from one description to the other. $\rho^A_a(\sigma_i)$ can be taken to define a line in $\mathbb{CP}^3$, while each row of the 2×2 matrix W_i can be interpreted as defining the homogeneous coordinates for two points on this line. We believe that this new viewpoint would be useful in writing formulas in the 6D version of twistor space.

An added benefit of the linear form is that it makes parts of the non-linearly realized R symmetry generators manifest, as we mentioned previously. Recalling (6.5), the generators are quadratic in the η_i, $\tilde{\eta}_i$ variables and their derivatives. In particular, let us consider the generators $R^+ = \sum_{i=1}^n \eta_{i,a}\eta_i^a$ and $\widetilde{R}^+ = \sum_{i=1}^n \tilde{\eta}_{i,\hat{a}}\tilde{\eta}_i^{\hat{a}}$. One may verify that these are symmetry generators by first noting that under the support of the delta functions

$$\eta_{ia} = (W_i)_{ab}\chi^b(\sigma_i), \quad \tilde{\eta}_{i\hat{a}} = (\widetilde{W}_i)_{\hat{a}\hat{b}}\tilde{\chi}^{\hat{b}}(\sigma_i)\,. \tag{8.9}$$

Similar to how one constructs the momenta p_i^{AB} from antisymmetric combinations of the analogous bosonic delta functions for λ^A_{ia}, we can construct the combinations:

$$R^+ = \sum_{i=1}^{n}\langle\eta_i\eta_i\rangle = \sum_{i=1}^{n}(W_i)_{ab}(W_i)^a_c\chi^b(\sigma_i)\chi^c(\sigma_i) = \sum_{i=1}^{n}|W_i|\,\chi_b(\sigma_i)\chi^b(\sigma_i)\,, \tag{8.10}$$

and similarly for $\widetilde{R}^+$. Under the support of the bosonic delta functions the determinant $|W_i|$ can be replaced by $(\prod_{j\neq i}\sigma_{ij})^{-1}$, whereas $\chi_b(\sigma_i)\chi^b(\sigma_i)$ is a polynomial of degree $n-2$ in σ_i. Using the identity

$$\sum_{i=1}^{n} \frac{\sigma_i^k}{\prod_{j\neq i}\sigma_{ij}} = 0\,, \qquad \text{for} \qquad k = 0, 1, \ldots, n-2\,, \tag{8.11}$$

which can be understood as a consequence of a residue theorem, we find that $R^+ = 0$. This means that the amplitude is supported on configurations such that $\sum_i \eta_{ia}\eta_i^a = 0$ and $\sum_i \tilde{\eta}_{i\hat{a}}\tilde{\eta}_i^{\hat{a}} = 0$, which proves the conservation of this R charge. The vanishing of the final R symmetry generators, R^- and $\tilde{R}^-$, which are second derivative operators, is still not made manifest in this formulation, but it is not hard to prove. For example, a Grassmann Fourier transform interchanges the role of η and $\partial/\partial\eta$.

As a final application, we apply the formalism of linear constraints to the tree amplitudes of a single M5-brane in 11D Minkowski spacetime. This provides an example of a 6D theory with $(2,0)$ supersymmetry; the amplitudes in the rational maps formalism are given by [149]:

$$\mathcal{A}_n^{\text{M5-brane}} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k\, d^4\chi_k}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)} \Delta_B\, \Delta_F\, \frac{(\text{Pf}' A_n)^3}{V_n}\,, \tag{8.12}$$

where

$$\Delta_B = \prod_{i=1}^{n} \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{8.13}$$

$$\Delta_F = \prod_{i=1}^{n} \delta^8\left(q_i^{AI} - \frac{\langle \rho^A(\sigma_i)\,\chi^I(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{8.14}$$

and $I = 1, 2$ denotes the two chiral supercharges.

Since this theory has only even-point amplitudes, we do not need the machinery of odd-point rational maps in this case. Introducing the W_i variables, the bosonic measure is identical to that of SYM. The fermionic delta functions with $\mathcal{N} = (2,0)$ supersymmetry become:

$$\Delta_F = \prod_{i=1}^{n} \delta^4\left(\tilde{l}_{iA\hat{a}} q_i^{AI}\right) \delta^4\left(\eta_i^{aI} - (W_i)^a_b \chi^{bI}(\sigma_i)\right), \tag{8.15}$$

so the amplitudes have the representation:

$$\boxed{\begin{aligned}\mathcal{A}_n^{\text{M5-brane}} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k\, d^4\chi_k}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)}\, \mathcal{W}(\lambda,\rho,\sigma) \\ \times\, (\text{Pf}' A_n)^3\, V_n \prod_{i=1}^n \delta^4\big(\eta_i^{aI} - (W_i)^a_b \chi^{bI}(\sigma_i)\big)\ .\end{aligned}} \tag{8.16}$$

It is worth noting that for chiral $\mathcal{N} = (2,0)$ supersymmetry there is no need to introduce $\tilde{\rho}$, $\widetilde{W}_i$, or $\tilde{\chi}$. In some sense, the 6D chiral theories appear more natural than their non-chiral counterparts. This theory has USp(4) R symmetry, which can be verified in the linear formulation by the technique described above.

By the same reasoning, the D5-brane formula [149], which has $\mathcal{N} = (1,1)$ supersymmetry, can be recast in a similar form with the same fermionic delta functions as in (8.8)

8.2 Linear Odd-Point Measure

To complete the discussion for the $\mathcal{N} = (1,1)$ SYM odd-point measure and integrand in this formalism, we introduce the parametrization of the odd-point maps described in Section 7.1. As before, we define

$$\rho_a^A(z) \;=\; \sum_{k=0}^{m-1} \rho_{a,k}^A\, z^k + \omega^A \xi_a\, z^m, \tag{8.17}$$

$$\tilde{\rho}_A^{\hat{a}}(z) \;=\; \sum_{k=0}^{m-1} \tilde{\rho}_{A\,k}^{\hat{a}}\, z^k + \tilde{\omega}_A \tilde{\xi}^{\hat{a}}\, z^m, \tag{8.18}$$

and similarly for the fermionic partners, where $m = (n-1)/2$. In the case where we used constraints for $p^{AB}(z)$, the introduction of this parametrization of the maps included a new redundancy. This was because the polynomial $\langle \rho^A(z) \rho^B(z)\rangle$ has a shift symmetry of the form $\rho^A(z) \;\rightarrow\; \hat{\rho}^A(z) = (\mathbb{I} + z\,T)\rho^A(z)$, where $T^a_b = \alpha \xi^a \xi_b$ and α is a parameter. The invariance of the product is still required in the linear formalism, and there must be a redundancy that reduces the number of components of ω^A and ξ_a. As before, the integrations over the moduli and the Riemann sphere are completely localized by the bosonic delta functions, which requires five independent components.

We will find that the appropriate choice is to keep the general action of the T-shift on

$\rho_a^A(\sigma_i)$ but now allowing the W_i to transform at the same time. The linear constraint $\delta^8\left(\lambda_{ia}^A - (W_i)_a^b \rho_b^A(\sigma_i)\right)$ is left invariant under the T-shift, which now explicitly depends on each puncture σ_i:

$$\rho_b^A(\sigma_i) \to \rho_b^A(\sigma_i) + \alpha\sigma_i \xi^c \xi_b \rho_c^A(\sigma_i) \tag{8.19}$$

$$(W_i)_a^b \to (W_i)_a^b - \alpha\sigma_i \xi^b \xi_c (W_i)_a^c\,, \tag{8.20}$$

or more abstractly,

$$\rho^A(\sigma_i) \to (\mathbb{I} + \sigma_i\, T)\, \rho^A(\sigma_i) \tag{8.21}$$

$$(W_i)_a \to (\mathbb{I} - \sigma_i\, T^{\intercal})\, (W_i)_a\,. \tag{8.22}$$

These transformations leave the product invariant by virtue of (7.10). Recall that the lower index a on $(W_i)_a^b$ is the little-group index for the ith particle, and it does not participate in the shift.

With the maps and redundancy more or less the same as in chapter 7, we may now write down the measure associated to the linear constraints for odd n, which takes a similar form as the even-point one:

$$\begin{aligned}\int d\mu_{n\,\mathrm{odd}}^{\mathrm{6D}} &= \int \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right)\, d^4\omega\, \langle \xi d\xi \rangle}{\mathrm{vol}\,\left(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T}\right)}\, \frac{1}{V_n^2} \prod_{i=1}^n \delta^6\left(p_i^{AB} - \frac{p^{AB}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right) \\ &= \left(\prod_{i=1}^n \delta(p_i^2)\right) \int \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right)\, d^4\omega\, \langle \xi d\xi \rangle}{\mathrm{vol}\,\left(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T}\right)}\, \mathcal{W}(\lambda,\rho,\sigma).\end{aligned} \tag{8.23}$$

We are free to fix the scaling and T-shift symmetry of this measure exactly as before, so all Jacobians will be the same as in previous chapters. Therefore in terms of the linear maps, the superamplitudes of 6D $\mathcal{N}=(1,1)$ SYM can be expressed as

$$\boxed{\begin{aligned}\mathcal{A}_{n\ \mathrm{odd}}^{\mathcal{N}=(1,1)\ \mathrm{SYM}}(\alpha) &= \int \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right)\, d^4\omega\, \langle \xi d\xi \rangle}{\mathrm{vol}\,\left(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T}\right)}\, \mathcal{W}(\lambda,\rho,\sigma)\ \mathrm{PT}(\alpha)\ \mathrm{Pf}'\widehat{A}_n \\ &\times V_n \int \prod_{k=0}^{m-1} d^2\chi_k d^2\tilde{\chi}_k\, dg\, d\tilde{g} \prod_{i=1}^n \delta^2\left(\eta_i^a - (W_i)_b^a \chi^b(\sigma_i)\right) \delta^2\left(\tilde{\eta}_i^{\hat{a}} - (\widetilde{W}_i)_{\hat{b}}^{\hat{a}} \tilde{\chi}^{\hat{b}}(\sigma_i)\right),\end{aligned}} \tag{8.24}$$

where, as before, the fermionic maps for n odd are defined to be

$$\chi^a(z) = \sum_{k=0}^{m-1} \chi_k^a\, z^k + g\xi^a z^m, \tag{8.25}$$

$$\tilde{\chi}^{\hat{a}}(z) = \sum_{k=0}^{m-1} \tilde{\chi}_k^{\hat{a}}\, z^k + \tilde{g}\tilde{\xi}^{\hat{a}} z^m. \tag{8.26}$$

8.3 Veronese Maps and Symplectic Grassmannian

The preceding results can be brought even closer to the original Witten–RSV formulation by integrating out the moduli $\rho_{a,k}^A$ of the maps, which leaves an integral over only the σ_i and the W_i. This will allow us to show that these constraints apply to the elements of a *symplectic* Grassmannian. Let us begin with the even-n case and recast the bosonic measure:

$$\begin{aligned}\int d\mu_{n\ \text{even}}^{6\text{D}} &= \int \frac{\left(\prod_{i=1}^n d\sigma_i\, d^4W_i\right)\prod_{k=0}^m d^8\rho_k}{\text{vol}\left(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho\right)} \prod_{i=1}^n \delta^8\left(\lambda_{ia}^A - (W_i)_a^b \rho_b^A(\sigma_i)\right)\delta\left(|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right)\\ &= \int \frac{\prod_{i=1}^n d\sigma_i\, d^4W_i}{\text{vol}\left(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_W\right)} \prod_{k=0}^m \delta^8\left(\sum_{i=1}^n (W_i)_a^b \sigma_i^k \lambda_{ib}^A\right)\prod_{i=1}^n \delta\left(|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right).\end{aligned} \tag{8.27}$$

This result can be obtained by using the following identity for each of the eight components separately [222, 250, 59],

$$\prod_{k=0}^m \delta\left(\sum_{i=1}^n \sigma_i^k X_i\right) = V_n \int \left(\prod_{k=0}^m d\rho_k\right)\prod_{i=1}^n \delta\left(\rho(\sigma_i) - X_i \prod_{j\neq i}\sigma_{ij}\right), \tag{8.28}$$

where $\rho(z) = \sum_{k=0}^m \rho_k\, z^k$ denotes any component of the polynomial map. Starting with (8.4), one can obtain a similar result for the fermions. Specifically,

$$\int \left(\prod_{k=0}^m d^2\chi_k\right)\prod_{i=1}^n \delta^2\left(\eta_{i,a} - (W_i)_a^b \chi_b(\sigma_i)\right) = \prod_{k=0}^m \delta^2\left(\sum_{i=1}^n (W_i)_a^b \sigma_i^k \eta_{i,b}\right). \tag{8.29}$$

We now note that $(W_i)_a^b \sigma_i^k$ forms an $n \times 2n$ matrix:

$$C_{a,k;i,b} = (W_i)_a^b \, \sigma_i^k \,, \tag{8.30}$$

where we group the index k with the global $\mathrm{SL}(2,\mathbb{C})$ index a and the index i with the ith little-group $\mathrm{SL}(2,\mathbb{C})$ index b. Interestingly, under the constraints $|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}} = 0$, the matrix C formed in this way is symplectic satisfying

$$C \cdot \Omega \cdot C^T = 0 \,, \tag{8.31}$$

which follows from the application of the identity (8.11) to each block matrix of the product. Here Ω is a symplectic metric: an anti-symmetric $2n \times 2n$ matrix with non-zero entries at $\Omega_{i,i+1} = -\Omega_{i+1,i} = 1$. Therefore C is a symplectic Grassmannian, which was mentioned in [9] for its possible applications to scattering amplitudes. Here we construct the sympletic Grassmannian explicitly in the spirit of the Veronese maps as discussed in [8] to relate Witten–RSV formulas with Grassmannian formulations for 4D $\mathcal{N}=4$ SYM [53]. Using the $n \times 2n$ matrix C, one may rewrite the constraints nicely as

$$\sum_{i=1}^{n} (W_i)_a^b \sigma_i^k \lambda_{ib}^A := (C \cdot \Omega \cdot \Lambda)_a{}^A = 0 \,, \tag{8.32}$$

where $\Lambda^A = \lambda_{i,b}^A$ is a $2n$-dimensional vector. The fermionic constraints take a similar form with the same Grassmannian. Geometrically, this is a 6D version of the orthogonality conditions of the 4D Grassmannian described in [11].

Similarly, when $n = 2m+1$ is odd, the identity (8.28) leads to

$$\begin{aligned}\int d\mu_{n\ \mathrm{odd}}^{\mathrm{6D}} =& \int \frac{\prod_{i=1}^n d\sigma_i d^4 W_i \, (\prod_{k=0}^{m-1} d^8\rho_k) d^4\omega \langle \xi d\xi\rangle}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma \,, \mathrm{SL}(2,\mathbb{C})_\rho \,, T)} \prod_{i=1}^n \delta^8\big(\lambda_{ia}^A - (W_i)_a^b \rho_b^A(\sigma_i)\big)\, \delta\left(|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right) \\ =& \int \frac{(\prod_{i=1}^n d\sigma_i \, d^4 W_i) \, \langle \xi d\xi\rangle}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma \,, \mathrm{SL}(2,\mathbb{C})_W \,, T)} \prod_{k=0}^{m-1} \delta^8\left(\sum_{i=1}^n (W_i)_a^b \sigma_i^k \lambda_{ib}^A\right) \\ & \times \, \delta^4\left(\sum_{i=1}^n \xi^a (W_i)_a^b \sigma_i^m \lambda_{ib}^A\right) \prod_{i=1}^n \delta\left(|W_i| - \frac{1}{\prod_{j\neq i}\sigma_{ij}}\right) .\end{aligned} \tag{8.33}$$

For odd n, the form of the Grassmannian constraints is unmodified except for the highest degree, σ_i^m. The highest-degree terms must be modified so that the number of constraints decreases by 5 when passing from even to odd, which is the case for this expression. Note that we have integrated out the Lorentz spinor ω^A but not the global little-group spinor ξ^a. One of the $\mathrm{SL}(2,\mathbb{C})_\rho$ generators can be used to fix the only independent component in ξ^a, making it effectively arbitrary. This nontrivial relation leaves only four independent constraints for the highest-degree part of the Grassmannian.

For odd n, this Veronese form also has the T-shift symmetry inherited from that of the W_i's, as shown in (8.20). The T-shift acts on the Grassmannian as

$$(W_i)_a^b\,\sigma_i^k = C_{a,k;i,b} \to C_{a,k;i,b} - \alpha\xi_a\xi^c C_{c,k+1;i,b}\,,\quad k=0,\ldots,m-1\,, \tag{8.34}$$

$$\xi^a(W_i)_a^b\,\sigma_i^m = \xi^a C_{a,m;i,b} \to \xi^a C_{a,m;i,b}\,. \tag{8.35}$$

The term of highest degree is invariant under the shift due to $\langle\xi\,\xi\rangle = 0$. This shift can be interpreted as a special kind of row operation on the Grassmannian in which the rows of degree k are translated by the rows of degree $k+1$ with the exception of the highest-degree rows.

One must now fix the various redundancies of this description. In the end the number of integrals should equal the number of constraints after gauge fixing. There are $5n$ integrals before fixing the two $\mathrm{SL}(2,\mathbb{C})$'s and $5n-6$ after fixing them. These choices can be used to fix three of the punctures σ_i as well as two values of a W_i and one component of ξ_a. Finally, the T-shift can be used to fix the last value of the chosen W_i.

The fermionic delta functions satisfy a similar identity,

$$\int dg \prod_{k=0}^{m-1} d^2\chi_k \prod_{i=1}^{n} \delta^2\left(\eta_{i,a} - (W_i)_a^b\chi_b(\sigma_i)\right) = \delta\left(\sum_{i=1}^{n}\xi^a(W_i)_a^b\sigma_i^m\eta_{i,b}\right)\prod_{k=0}^{m-1}\delta^2\left(\sum_{i=1}^{n}(W_i)_a^b\sigma_i^k\eta_{i,b}\right). \tag{8.36}$$

Now $(W_i)_a^b\sigma_i^k$ with $k=0,1,\ldots,m-1$ combines with $\xi^a(W_i)_a^b\sigma_i^m$ to form an $n\times 2n$ symplectic matrix acting on the vector of external Grassmann variables, entirely analogous to the constraints for the external spinors.

Using these relations, it is then straightforward to rewrite all of the superamplitudes given in previous chapters in terms of the Veronese maps. In the case of 6D $\mathcal{N}=(1,1)$ SYM and odd n, in the integrand, the term $\mathrm{Pf}'\widehat{A}_n$ contains a special "momentum" vector, which we recall here:

$$p_\star^{AB} = \frac{2\,q^{[A}p^{B]C}(\sigma_\star)\tilde{q}_C}{q^D[\tilde{\rho}_D(\sigma_\star)\,\tilde{\xi}]\langle\rho^E(\sigma_\star)\,\xi\rangle\tilde{q}_E}\,. \tag{8.37}$$

This shows that $p_\star^{AB}$ is in general a function of the moduli $\rho^A_{a,k}$ and $\tilde{\rho}^{\hat{a}}_{A,k}$. Therefore, when we integrate out the moduli and express the amplitudes in the Veronese form, we should solve for $\rho^A_{a,k}$ and $\tilde{\rho}^{\hat{a}}_{A,k}$ in terms of the W_i's and $\widetilde{W}_i$'s, as well as the σ_i's. If we choose $\sigma_\star$ to be one of the σ_i's, then it is trivial to express $p_\star^{AB}$ in terms of W_i and σ_i by using the relation,

$$\rho^A_a(\sigma_i) = (M_i)_a^b \, \lambda^A_{i,b} \,, \tag{8.38}$$

and a similar relation for $\tilde{\rho}^{\hat{a}}_A(\sigma_i)$, and recalling that $M_i = W_i^{-1}$. If, instead, we choose $\sigma_\star$ to be arbitrary, $\rho^A_a(\sigma_\star)$ can also be determined in terms of M_i and σ_i using the above relation (8.38), since $\rho^A_a(z)$ is a degree $m = \frac{n-1}{2}$ polynomial, and there are n such relations.

Chapter 9

Various Theories in D$\leq$ 6 and $\mathcal{N}=4$ SYM on the Coulomb Branch

This chapter describes some interesting applications and consistency checks of the 6D SYM formulas that we have obtained. In particular we will see how the formalism is versatile and allow us to derive new formulas for other S-matrices. We start by writing down a formula for 6D $\mathcal{N}=(2,2)$ supergravity amplitudes in Section 9.1, which in a nutshell follows from putting together two copies of the $\mathcal{N}=(1,1)$ SYM formula studied in previous chapters. Then in Section 9.2 we consider mixed amplitudes by coupling the 6D $\mathcal{N}=(1,1)$ SYM with a single D5-brane.

We will also study the dimensional reduction of these theories. This aspect is interesting since provides a way of obtaining massive particles also encoded in an (on-shell) supersymmetric multiplet. We begin with the reduction to five dimensions in Section 9.3, followed by $\mathcal{N}=4$ SYM on the Coulomb branch in Section 9.4. We obtain new connected formulas for all tree-level scattering amplitudes of these theories, including for instance W-bosons interacting with gluons. In Section 9.5 we study dimensional reduction to 4D $\mathcal{N}=4$ SYM at the origin of the moduli space and comment on certain subtleties of the case.

We close the chapter with a discussion on a possible formulation of the intriguing $\mathcal{N}=(2,0)$ six dimensional gauge theory.

9.1 $\mathcal{N} = (2,2)$ Supergravity in Six Dimensions

In this section we consider the tree amplitudes of $\mathcal{N} = (2,2)$ supergravity in 6D. Even though 6D $\mathcal{N} = (2,2)$ SUGRA is nonrenormalizable, it has a well-known UV completion. This completion is given by Type IIB superstring theory compactified on T^4 or (equivalently) M theory compactified on T^5. In either case, the theory has an $\mathrm{E}_{5,5}(\mathbb{Z}) = \mathrm{SO}(5,5;\mathbb{Z})$ U-duality group. This is a discrete global symmetry. (It is believed that string theory does not give rise to continuous global symmetries [18].) The low-energy effective description of this theory, which is the 6D supergravity theory under consideration here, extends this symmetry to the continuous non-compact global symmetry $\mathrm{Spin}(5,5)$. However, much of this symmetry is non-perturbative, and only the compact subgroup $\mathrm{Spin}(5) \times \mathrm{Spin}(5)$ is realized as a symmetry of the supergravity tree-level scattering amplitudes. (Recall that $\mathrm{Spin}(5) = \mathrm{USp}(4)$.) This symmetry is the relevant R symmetry group. This is called an R symmetry group because particles with different spins belong to different representations of this group even though they form an irreducible supermultiplet. The UV complete theory and its low-energy supergravity effective description are both maximally supersymmetric. This means that there are 32 local supersymmetries, gauged by the gravitino fields. It also implies that the supergravity theory has a 6D Minkowski-space solution that has 32 unbroken global supersymmetries. When we discuss scattering amplitudes, this is the background geometry under consideration. If we further reduce to four dimensions, we get $\mathcal{N} = 8$ supergravity, which has nonperturbative $\mathrm{E}_{7(7)}$ symmetry. Again, only the compact subgroup, which is $\mathrm{SU}(8)$ in this case, is the R symmetry of the tree amplitudes.

The 6D $\mathcal{N} = (2,2)$ supergravity multiplet contains 128 bosonic and 128 fermionic degrees of freedom, which can be elegantly combined into a scalar superparticle by introducing eight Grassmann coordinates in a manner that will be described below. This multiplet contains six different spins, i.e., little-group representations, which we will now enumerate. They are characterized by their $\mathrm{SU}(2) \times \mathrm{SU}(2)$ little-group representations and their $\mathrm{USp}(4) \times \mathrm{USp}(4)$ R symmetry representations. The graviton transforms as $(\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ under these four groups. Similarly, the eight gravitinos belong to $(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}, \mathbf{3}; \mathbf{4}, \mathbf{1})$. Also, the ten two-form particles belong to $(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{5}) + (\mathbf{1}, \mathbf{3}; \mathbf{5}, \mathbf{1})$. The 16 vector particles belong to $(\mathbf{2}, \mathbf{2}; \mathbf{4}, \mathbf{4})$, the spinors belong to $(\mathbf{2}, \mathbf{1}; \mathbf{4}, \mathbf{5}) + (\mathbf{1}, \mathbf{2}; \mathbf{5}, \mathbf{4})$, and the scalars belong $(\mathbf{1}, \mathbf{1}; \mathbf{5}, \mathbf{5})$. As in the case of the SYM theory, the amplitudes will be presented in a form that makes the helicity properties of the particles straightforward to read off, but only a subgroup of the R symmetry will be manifest. With some effort, one can prove that the entire $\mathrm{USp}(4) \times \mathrm{USp}(4)$ R symmetry is actually realized. Even though this is a non-chiral (left-right symmetric) theory, corresponding left- and right-handed particles have their R symmetry factors interchanged. So this interchange should be understood to be part of the definition of the reflection symmetry.

The on-shell superfield description of the supergravity multiplet, analogous to the one for the SYM multiplet in (6.1), utilizes eight Grassmann coordinates denoted $\eta^{I,a}$ and $\tilde{\eta}^{\hat{I},\hat{a}}$. It contains 128 bosonic and 128 fermionic modes with the spectrum enumerated above. It has the schematic form

$$\Phi(\eta) = \phi + \ldots + \eta_a^I \, \eta_{b,I} \, \tilde{\eta}_{\hat{a}}^{\hat{I}} \, \tilde{\eta}_{\hat{b},\hat{I}} \, G^{ab;\hat{a}\hat{b}} + \ldots + (\eta)^4 (\tilde{\eta})^4 \bar{\phi} \,. \tag{9.1}$$

Note that $I = 1, 2$ and $\hat{I} = \hat{1}, \hat{2}$ label components of an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subgroup of the R symmetry group. Only this subgroup of the $\mathrm{USp}(4) \times \mathrm{USp}(4)$ R symmetry is manifest in this formulation. The on-shell field $G^{ab;\hat{a}\hat{b}}$ in the middle of the on-shell superfield is the 6D graviton. We have only displayed this field and two of the 25 scalar fields.

The supergravity superamplitudes have total symmetry in the n scattered particles. This is to be contrasted with the cyclic symmetry of the color-stripped SYM amplitudes. For instance, the four-point superamplitude is given by

$$M_4^{\mathcal{N}=(2,2)\ \mathrm{SUGRA}} \;=\; \delta^6\left(\sum_{i=1}^{4} p_i^{AB}\right) \frac{\delta^8\left(\sum_{i=1}^4 q_i^{A,I}\right) \delta^8\left(\sum_{i=1}^4 \tilde{q}_{i,\hat{A}}^{\hat{I}}\right)}{s_{12}\, s_{23}\, s_{13}} \,, \tag{9.2}$$

which has manifest permutation symmetry. Here the supercharges are defined as $q_i^{A,I} = \lambda_{i,a}^A \eta_i^{I,a}$, and $\tilde{q}_{i,\hat{A}}^{\hat{I}} = \tilde{\lambda}_{i,\hat{A},\hat{a}} \tilde{\eta}_i^{\hat{I},\hat{a}}$. As in the case of the SYM theory, these are half of the supercharges, and the other half involve η derivatives. Conservation of these additional supercharges automatically follows from the first set together with the R symmetry.

Thanks to the separation of the $\mathcal{N} = (1,1)$ SYM formulas into the measure, left- and right-integrands, the formulas for $\mathcal{N} = (2,2)$ SUGRA amplitudes follow from the standard KLT argument [168] in the context of CHY formulations [67]. One replaces the Parke–Taylor factor with a second copy of the remaining half-integrand. The resulting connected formula for amplitudes of all multiplicities can be written in a compact form:

$$\boxed{\mathcal{M}_n^{\mathcal{N}=(2,2)\ \mathrm{SUGRA}} = \int d\mu_n^{\mathrm{6D}} \left(\mathrm{Pf}' A_n\right)^2 \int d\Omega_F^{(2,2)}.} \tag{9.3}$$

Here the fermionic measure $d\Omega_F^{(2,2)}$ that implements the 6D $\mathcal{N} = (2,2)$ supersymmetry is the double copy of the $\mathcal{N}=(1,1)$ version $d\Omega_F^{(1,1)}$, with

$$\chi_k^a \to \chi_k^{I\,a}, \quad \tilde{\chi}_k^{\hat{a}} \to \tilde{\chi}_k^{\hat{I}\,\hat{a}}, \quad g \to g^I, \quad \tilde{g} \to \tilde{g}^{\hat{I}}. \tag{9.4}$$

Here $I=1,2$ and $\hat{I}=1,2$ are the SU(2) $\times$ SU(2) R symmetry indices. Explicitly, the measure is defined as

$$d\Omega_F^{(2,2)} = V_n^2 \left(\prod_{k=0}^{m} d^4\chi_k \, d^4\tilde{\chi}_k\right) \Delta_F^{(8)} \, \widetilde{\Delta}_F^{(8)} \,, \tag{9.5}$$

for even $n=2m+2$, and

$$d\widehat{\Omega}_F^{(2,2)} = V_n^2 \, d^2g \, d^2\tilde{g} \left(\prod_{k=0}^{m-1} d^4\chi_k \, d^4\tilde{\chi}_k\right) \Delta_F^{(8)} \, \widetilde{\Delta}_F^{(8)} \,, \tag{9.6}$$

for odd $n=2m+1$. The fermionic delta functions are also a double copy of the $\mathcal{N}=(1,1)$ ones, and they are given by

$$\Delta_F^{(8)} = \prod_{i=1}^{n} \delta^8 \left(q_i^{I,A} - \frac{\rho_a^A(\sigma_i)\chi^{I,a}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{9.7}$$

$$\widetilde{\Delta}_F^{(8)} = \prod_{i=1}^{n} \delta^8 \left(\tilde{q}_{i,A}^{\hat{I}} - \frac{\tilde{\rho}_{A,\hat{a}}(\sigma_i)\tilde{\chi}^{\hat{I},\hat{a}}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right). \tag{9.8}$$

Finally, it is understood that the reduced Pfaffian in the integrand refers to the matrix A_n in (6.10) for the even-point case and the hatted matrix $\widehat{A}_n$ in (7.75) for the odd-point case.

9.2 $\mathcal{N}=(1,1)$ Super Yang–Mills Coupled to D5-branes

Since we now have connected formulas for the scattering amplitudes in the effective field theories of the D5-brane and $\mathcal{N}=(1,1)$ SYM in 6D, we can consider mixed amplitudes involving both kinds of particles. It was proposed in [60] that these types of amplitudes admit a simple CHY formula, which interpolates between the Parke–Taylor factor PT(α) for the non-Abelian theory and $(\mathrm{Pf}'A_n)^2$ for the Abelian one. Such a construction was used in [60] to write down amplitudes coupling Non-linear Sigma Model (NLSM) pions to bi-adjoint scalars, as well as their supersymmetrization in 4D involving Volkov–Akulov theory (effective theory on a D3-brane) [241, 165] and $\mathcal{N}=4$ SYM. Related models were later written down in the context of string-theory amplitudes [81]. These mixed amplitudes are also parts of the unifying relations for scattering amplitudes [88, 85]. In all of the above cases, the connected formula selects preferred couplings between the two theories. They were identified in [87, 188] in the case of the NLSM coupled to bi-adjoint scalar theory.

Following the same approach allows us to write down a formula coupling the D5-brane effective theory to 6D SYM:

$$\boxed{\mathcal{A}_n^{\text{D5-brane} \,\oplus\, \text{SYM}}(\alpha) = \int d\mu_n^{\text{6D}} \left(\text{PT}(\alpha)\ (\text{Pf}A_{\overline{\alpha}})^2\right)\left(\text{Pf}'A_n \int d\Omega_{\text{F}}^{(1,1)}\right),} \tag{9.9}$$

where α represents the states of SYM, which are color ordered, and the complement, $\overline{\alpha}$, represents states of the Abelian D5-brane theory. We have also used the fact that the D5-brane theory and 6D SYM have identical supermultiplets and the same supersymmetry. Here the right-integrand, which is common between the two theories, remains unchanged. Of course, whenever the total number of particles n is odd, one should make use of the odd-multiplicity counterparts of the reduced Pfaffian and the bosonic and fermionic measures. In the left-integrand we have a Parke–Taylor factor constructed out of only the SYM states that enter the color ordering α. The D5-brane states belonging to $\overline{\alpha}$ do not have color labels, and hence they appear in the formula through the permutation-invariant Pfaffian. The matrix $A_{\overline{\alpha}}$ is an $|\overline{\alpha}| \times |\overline{\alpha}|$ minor of $\left[\frac{p_i \cdot p_j}{\sigma_{ij}}\right]$ with columns and rows labeled by the D5-brane states. This implies that the above amplitude in non-vanishing only if the number of D5-brane particles $|\overline{\alpha}|$ is even.

Note that whenever $|\alpha| = 2$, i.e., only two states are SYM particles, the left integrand reduces to the square of a reduced Pfaffian, and the amplitude is equal to the D5-brane amplitude, though two particles carry color labels. Hence the first non-trivial amplitude in this mixed theory arises for $n=5$:

$$\mathcal{A}_5^{\text{D5-brane} \,\oplus\, \text{SYM}}(345) = \frac{1}{4} s_{12} \left(s_{23}\, \mathcal{A}_5^{\mathcal{N}=(1,1)\ \text{SYM}}(12345) - s_{24}\, \mathcal{A}_5^{\mathcal{N}=(1,1)\ \text{SYM}}(12435)\right). \tag{9.10}$$

Here we used KLT to rewrite (9.9) in terms of the NLSM $\oplus\ \phi^3$ amplitudes from [60] and 6D $\mathcal{N}=(1,1)$ SYM ones, and presented the final result in terms of the SYM amplitudes. Symmetry in labels $1,2$ and antisymmetry with respect to $3,4,5$ of the right-hand side follows from the BCJ relations [24]. Expressions for 5-point SYM amplitudes can be found in [99].

The construction of these mixed amplitudes uniquely defines nontrivial interactions between the two sectors, as the amplitude given above illustrates. It is a curious fact that these interactions have not yet been explored from a Lagrangian point of view. There are indications that the interactions implied by these amplitude constructions may have better soft behavior than any other possible interactions. This warrants further exploration.

9.3 5D SYM and SUGRA

Let us now consider 5D SYM and SUGRA with maximal supersymmetry. The spin of a massless particle in 5D is given by a Spin(3) = SU(2) little-group representation. The appropriate spinor-helicity formalism can be conveniently obtained from the 6D one, with additional constraints, see for instance [242]. Concretely, a 5D massless momentum can be expressed

$$p^{AB} = \lambda_a^A \lambda_b^B \epsilon^{ab} \,. \tag{9.11}$$

This is identical to the 6D formula, but now there is only one kind of λ_a^A due to the fact that the little-group consists of a single SU(2), which can be identified with the diagonal subgroup of the SU(2) × SU(2) little group in 6D. Of course, one still has to impose a further condition to restrict the momentum to 5D. The additional constraint that achieves this is

$$\Omega_{AB} \lambda_a^A \lambda_b^B \epsilon^{ab} = 0 \,. \tag{9.12}$$

Here Ω_{AB} is the anti-symmetric invariant tensor of Spin(4, 1), which is a non-compact version of USp(4). Here we choose $\Omega_{13} = \Omega_{24} = 1$, and the other components of Ω_{AB} vanish for $A < B$. Note that the antisymmetry of Ω_{AB} implies that $\Omega_{AB}\lambda_a^A\lambda_b^B = c\,\epsilon_{ab}$. Therefore (9.12) actually implies that $\Omega_{AB}\lambda_a^A\lambda_b^B = 0$ for all $a, b = 1, 2$. This fact will be useful later.

Having set up the kinematics, we are now ready to present the formulas for the scattering amplitudes of 5D theories. Let us begin with 5D maximal SYM theory. This theory has Spin(5) = USp(4) R symmetry. The spectrum of an on-shell supermultiplet consists of a vector that transforms as $(\mathbf{3}, \mathbf{1})$, spinors $(\mathbf{2}, \mathbf{4})$, and scalars $(\mathbf{1}, \mathbf{5})$. The bold-face integers label little-group and R symmetry representations. The on-shell superfield of the theory can be expressed,

$$\Phi(\eta) = \phi + \eta_a^I \psi_I^a + \epsilon_{IJ}\eta_a^I \eta_b^J A^{ab} + \epsilon^{ab}\eta_a^I \eta_b^J \phi_{IJ} + (\eta^3)_a^I (\bar{\psi})_I^a + (\eta^4)\bar{\phi}\,. \tag{9.13}$$

The index $I = 1, 2$ labels a doublet of an SU(2) subgroup of the R symmetry group, whereas the entire little-group properties are manifest. This superfield is the dimensional reduction of the 6D on-shell superfield (6.1) obtained by removing all hats from 6D little-group indices. This works because the 5D SU(2) little group corresponds to the diagonal subgroup of the 6D SU(2) × SU(2) little group. One consequence of this is that the 6D gluon reduces to the 5D gluon with three degrees of freedom and a scalar. Similarly, 5D amplitudes can be obtained directly from the 6D ones by making the substitution

$$\tilde{l}_A^{\hat{a}} \to \Omega_{AB} l^{a\,B}\,. \tag{9.14}$$

A 6D Lorentz contraction, such as $V^A \tilde{V}_A$, now is realized by the use of Ω_{AB}, namely $V^A \tilde{V}_A \to \Omega_{AB} V^A V^B$. For instance, the four-gluon amplitude is given by

$$\mathcal{A}_4(A_{a_1b_1}, A_{a_2b_2}, A_{a_3b_3}, A_{a_4b_4}) = \delta^5\left(\sum_{i=1}^4 p_i^{AB}\right) \frac{\langle 1_{a_1} 2_{b_1} 3_{c_1} 4_{d_1}\rangle \langle 1_{a_2} 2_{b_2} 3_{c_2} 4_{d_2}\rangle}{s_{12}\, s_{23}} + \text{sym.}\,, \tag{9.15}$$

where the symmetrization is over the little-group indices of each gluon.

This procedure gives the following color-ordered tree-level superamplitudes for 5D maximal SYM:

$$\boxed{\mathcal{A}_n^{\text{5D SYM}}(\alpha) = \int d\mu_n^{\text{5D}}\ \text{PT}(\alpha)\ \left(\text{Pf}' A_n \int d\Omega_F^{(8)}\right).} \tag{9.16}$$

Here the 5D measure is defined as

$$\int d\mu_{n\,\text{even}}^{\text{5D}} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)} \frac{1}{V_n^2}\ \Delta_B^{5D}, \tag{9.17}$$

for even n, and

$$\int d\mu_{n\,\text{odd}}^{\text{5D}} = \int \frac{\left(\prod_{i=1}^n d\sigma_i \prod_{k=0}^{m-1} d^8\rho_k\right)\ d^4\omega\ \langle \xi d\xi\rangle}{\text{vol}\ (\text{SL}(2,\mathbb{C})_\sigma, \text{SL}(2,\mathbb{C})_\rho, \text{T})}\ \frac{1}{V_n^2}\ \Delta_B^{5D}\,, \tag{9.18}$$

for odd n. The 5D delta-function constraints $\Delta_B^{\rm 5D}$ will be defined later. We see that the integration variables and symmetry groups are identical to those of 6D, and the same for the maps,

$$\rho_a^A(z) = \sum_{k=0}^{m} \rho_{a\,k}^A\, z^k\,, \qquad \chi_a(z) = \sum_{k=0}^{m} \chi_{a\,k}\, z^k \tag{9.19}$$

and similarly for conjugate ones. Here $m = \frac{n}{2} - 1$ or $m = \frac{n-1}{2}$ depending on whether n is even or odd, and the highest coefficients factorize if n is odd, namely $\rho_{a\,m}^A = \omega^A \xi_a, \chi_{a\,m} = g\,\xi_a$ for $n = 2m+1$.

Let us now examine the 5D delta-function constraints $\Delta_B^{\rm 5D}$. We propose that the 5D conditions for the rational maps are given by

$$\Delta_B^{\rm 5D} = \prod_{i=1}^{n} \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \prod_{j=1}^{n-1} \delta\left(\frac{\Omega_{AB}\langle \rho^A(\sigma_j)\rho^B(\sigma_j)\rangle}{\prod_{l\neq j}\sigma_{jl}}\right)\,, \tag{9.20}$$

where the first part is identical to the 6D version, and the second part imposes additional constraints to incorporate the 5D kinematic constraints (9.12). The constraints should only be imposed for $(n-1)$ particles, because the remaining one is then automatically satisfied due to momentum conservation. As in the case of 6D, momentum conservation and on-shell conditions are built into (9.20), so to compute the usual scattering amplitudes we should pull out the corresponding delta functions,

$$\Delta_B^{\rm 5D} = \delta^5\Big(\sum_{i=1}^{n} p_i^{AB}\Big) \left(\prod_{i=1}^{n} \delta(p_i^2)\delta(\Omega_{AB} p_i^{AB})\right) \hat{\Delta}_B^{5D}\,. \tag{9.21}$$

Note that besides the usual on-shell conditions $p_i^2 = 0$, there are additional conditions $\Omega_{AB}\, p_i^{AB} = 0$ that one has to extract. 5D momentum conservation is now implemented by restricting, for instance, the Lorentz indices in the δ^5-function to be $\{A, B\} \neq \{2, 4\}$. Then the remaining constraints $\hat{\Delta}_B^{5D}$ are given by

$$\begin{aligned} \hat{\Delta}_B^{\rm 5D} \;&=\; \prod_{j=1}^{n-1} \delta\left(\frac{\Omega_{AB}\langle \rho^A(\sigma_j)\rho^B(\sigma_j)\rangle}{\prod_{l\neq j}\sigma_{jl}}\right) \prod_{i=1}^{n-2} \delta^4\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\,\rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \\ &\times\; \delta^3\left(p_n^{AB} - \frac{\langle \rho^A(\sigma_n)\,\rho^B(\sigma_i)\rangle}{\prod_{j\neq n}\sigma_{nj}}\right) \prod_{i=1}^{n} p_i^{12} \left(\frac{p_{n-1}^{14}}{p_{n-1}^{12}} - \frac{p_n^{14}}{p_n^{12}}\right)\,, \end{aligned} \tag{9.22}$$

where the δ^4-function has $\{A, B\} = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}$, and the δ^3-function has $\{A, B\} = \{1, 2\}, \{1, 3\}, \{1, 4\}$. Of course, the final result is independent of the choices we make here. Altogether the number of independent of delta functions is $5n - 6$, which matches with the number of integration variables (after modding out the symmetry factors). It is also straightforward to check that the formula has the correct power counting for the scattering amplitudes of 5D SYM. Finally, we remark that just like the rational maps in 6D, the 5D rational constraints also incorporate all $(n - 3)!$ solutions because of the non-trivial summation over the little-group indices.

The reduction of supersymmetry to lower dimensions is straightforward, and therefore the 5D fermionic measure, $d\Omega_F^{(8)}$, is almost identical to the 6D version, except that the fermionic maps $\chi^a(\sigma_i)$ and $\tilde{\chi}^{\hat{a}}(\sigma_i)$ now combine into $\chi^{Ia}(\sigma_i)$ (with $I = 1, 2$), just as the η's and $\tilde{\eta}$'s combined to give η^I, as we discussed previously. The corresponding 5D fermionic delta functions are therefore given by

$$\Delta_F^{(8)} = \prod_{i=1}^{n} \delta^8 \left(q_i^{AI} - \frac{\langle \rho^A(\sigma_i) \chi^I(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right) , \tag{9.23}$$

whereas the fermionic on-shell conditions that have to be taken out for computing scattering amplitudes become,

$$\prod_{i=1}^{n} \delta^4 \left(\Omega_{AB} \lambda_{a,i}^A q_i^{BI} \right) . \tag{9.24}$$

As usual, these constraints allow one to introduce the Grassmann coordinates η_{ia}^I by writing the supercharges in the form $\lambda\eta$. Furthermore, the meaning of the factor denoted $\mathrm{Pf}' A_n$ in (9.16) takes a different form depending on whether the number of particles is even or odd. Recall that if n is even, $\mathrm{Pf}' A_n$ is defined in (6.11), whereas for odd n, it is given in (7.75). For both cases, the reduction to 5D is straightforward using (9.14) and the discussion following it. We have carried out various checks of this formula by comparing it with explicit component amplitudes from Feynman diagrams; these analytically agree for $n = 3, 4$, and numerically agree up to $n = 8$.

Next we present the formula for the tree-level amplitudes of 5D maximal supergravity, which can be obtained either by a double copy of the 5D SYM formula or by a direct reduction of the 6D SUGRA formula. Either procedure gives the result

$$\boxed{\mathcal{M}_n^{\text{5D SUGRA}} = \int d\mu_n^{\text{5D}} \, (\mathrm{Pf}' A_n)^2 \int d\Omega_F^{(16)},} \tag{9.25}$$

with the fermionic measures and delta-functions all doubled up,

$$\Delta_F^{(16)} = \prod_{i=1}^{n} \delta^{16}\left(q_i^{AI} - \frac{\langle \rho^A(\sigma_i)\, \chi^I(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{9.26}$$

where now $I = 1,2,3,4$. This makes an SU(4) subgroup of the USp(8) R symmetry manifest. Again, the details of the formula depend on whether n is even or odd.

Finally, it is worth mentioning that there are analogous formulas for the superamplitudes of the world-volume theory of a D4-brane, which are nonzero only when n is even. These can be obtained either as the dimensional reduction of a D5-brane world-volume theory or of an M5-brane world-volume theory. Using the 5D measures, the probe D4-brane amplitudes can be expressed as

$$\mathcal{A}_n^{\text{D4-brane}} = \int d\mu_n^{\text{5D}}\, (\text{Pf}' A_n)^2 \left(\text{Pf}' A_n \int d\Omega_F^{(8)}\right), \tag{9.27}$$

where the number of particles, n, is always taken to be even.

9.4 $\mathcal{N} = 4$ SYM on the Coulomb Branch

A further application of our 6D formulas involves the embedding of 4D *massive* kinematics into the 6D massless kinematics. In this approach, we view some components of the 6D spinors as 4D masses [25, 157]. In the case of 6D $\mathcal{N} = (1,1)$ SYM, this procedure allows us to obtain amplitudes for 4D $\mathcal{N} = 4$ SYM on the Coulomb branch.

9.4.1 4D Massive Amplitudes from 6D Massless Ones

Four-dimensional $\mathcal{N} = 4$ SYM on the Coulomb branch can be achieved by giving vevs to scalar fields of the theory. For instance, in the simplest case,

$$\langle (\phi^{12})^I_J \rangle = \langle (\phi^{34})^I_J \rangle = v\, \delta^I_J\,, \tag{9.28}$$

other scalars have zero vev. Here "12" and "34" are $SU(4)$ R symmetry indices, whereas I, J are color indices for the gauge group U(M). So the vev spontaneously breaks the gauge group from U($M{+}N$) to U(N) $\times$ U(M), and the off-diagonal gauge bosons, which are bifundamentals of U(N) $\times$ U(M), denoted W and $\overline{W}$, gain mass. In the simple example, given above, all of the masses are equal, with $m = g_{\rm YM}\, v$. One can consider more general situations with different masses, as our formulas will describe. There have been many interesting studies of $\mathcal{N} = 4$ SYM on the Coulomb branch in the context of scattering amplitudes. For instance, the masses introduced by moving onto the Coulomb branch can be used as IR regulators [6, 145, 146]; one can also study the low-energy effective action by integrating out the masses, which has led to interesting supersymmetric non-renormalization theorems [83, 33]. The subject we are interested in here is to study the tree-level massive amplitudes of $\mathcal{N} = 4$ SYM on the Coulomb branch [93].

One can obtain 4D massive amplitudes from 6D massless ones via dimensional reduction. As discussed in [25], 4D massive kinematics can be parametrized by choosing the 6D spinor-helicity coordinates to take the special form

$$l_a^A = \begin{pmatrix} -\kappa\mu_\alpha & l_\alpha \\ \tilde{l}^{\dot\alpha} & \tilde\kappa\tilde\mu^{\dot\alpha} \end{pmatrix}\,, \qquad \tilde{l}_{A\hat a} = \begin{pmatrix} \kappa'\mu^\alpha & l^\alpha \\ -\tilde{l}_{\dot\alpha} & \tilde\kappa'\tilde\mu_{\dot\alpha} \end{pmatrix}\,, \tag{9.29}$$

where

$$\kappa = \frac{M}{\langle l\mu\rangle}\,, \quad \tilde\kappa = \frac{\widetilde{M}}{[l\mu]}\,, \quad \kappa' = \frac{\widetilde{M}}{\langle l\mu\rangle}\,, \quad \tilde\kappa' = \frac{M}{[l\mu]}\,, \tag{9.30}$$

and $M\widetilde{M} = m^2$ is the mass squared. As usual, the indices α and $\dot\alpha$ label spinor representations of the 4D Lorentz group $\mathrm{SL}(2,\mathbb{C})$. With this setup, a 4D massive momentum is given by

$$p_{\alpha\dot\alpha} = l_\alpha \tilde{l}_{\dot\alpha} + \rho\, \mu_\alpha\tilde\mu_{\dot\alpha}\,, \tag{9.31}$$

with $\rho = \kappa\tilde\kappa = \kappa'\tilde\kappa'$. We have decomposed a massive momentum into two light-like momenta, where $\mu_\alpha\tilde\mu_{\dot\alpha}$ can be considered a reference momentum.

$\mathcal{N} = 4$ SYM on the Coulomb branch can be viewed as a dimensional reduction of 6D $\mathcal{N} = (1,1)$ SYM with massless particles. For instance, the four-point amplitude involving two massive conjugate W bosons and two massless gluons, $A(W_1^+, \overline{W}_2^-, g_3^-, g_4^-)$ can be obtained from the 6D pure gluon amplitude,

$$A_4^{\rm 6D\ YM}(A_1^{+\hat{+}}, A_2^{-\hat{-}}, A_3^{-\hat{-}}, A_4^{-\hat{-}}) = \frac{\langle 1^+ 2^- 3^- 4^-\rangle [1^{\hat+} 2^{\hat-} 3^{\hat-} 4^{\hat-}]}{s_{12}\, s_{23}}\,. \tag{9.32}$$

Plugging in the massive spinors (9.29), and using the identity,

$$\langle 1_{+} 2_{-} 3_{-} 4_{-} \rangle = -\tilde{\kappa}_2 [1\mu] \langle 34 \rangle \,, \quad [1_{\hat{+}} 2_{\hat{-}} 3_{\hat{-}} 4_{\hat{-}}] = -\tilde{\kappa}'_2 [1\mu] \langle 34 \rangle \,, \tag{9.33}$$

as well as the definition of κ in (9.30), the result can be expressed as,

$$A_4^{\text{6D}}(W_1^+, \overline{W}_2^-, g_3^-, g_4^-) = \frac{m^2 [1\mu]^2 \langle 34 \rangle^2}{[2\mu]^2 s_{12} (s_{23} - m^2)} \,, \tag{9.34}$$

which agrees with the result in [93].

Alternatively, one can choose a different way of parameterizing 4D massive kinematics,

$$l_a^A = \begin{pmatrix} l_{\alpha,1} & l_{\alpha,2} \\ \tilde{l}_1^{\dot{\alpha}} & \tilde{l}_2^{\dot{\alpha}} \end{pmatrix} \,, \qquad \tilde{l}_{A\hat{a}} = \begin{pmatrix} l_1^{\alpha} & l_2^{\alpha} \\ \tilde{l}_{\dot{\alpha},1} & \tilde{l}_{\dot{\alpha},2} \end{pmatrix} \,, \tag{9.35}$$

where we split the Lorentz indices $A \Rightarrow \{\alpha, \dot{\alpha}\}$, and 1 and 2 are little-group indices of massive particles in 4D. The momentum and mass are given by

$$p_{\alpha,\dot{\alpha}} = l_{\alpha,a} \tilde{l}_{\dot{\alpha},b} \epsilon^{ab} \,, \quad l_{\alpha,a} l_{\beta,b} \epsilon^{ab} = M \epsilon_{\alpha\beta} \,, \quad \tilde{l}_{\dot{\alpha},a} \tilde{l}_{\dot{\beta},b} \epsilon^{ab} = M \epsilon_{\dot{\alpha}\dot{\beta}} \,, \tag{9.36}$$

with $M^2 = m^2$. The advantage of this setup is that it makes the massive 4D little group Spin(3) = SU(2) manifest. In fact, it actually leads to the massive spinor-helicity formalism of the recent work [13], which one can refer to for further details. In this formalism, for instance,

$$A_4^{\text{6D}}(W_1^{ab}, \overline{W}_2^{cd}, g_3^-, g_4^-) = \frac{([1_a 2_c][1_b 2_d]) \langle 34 \rangle^2}{s_{12}\,(s_{23} - m^2)} + \text{sym} \,, \tag{9.37}$$

and

$$A_4^{\text{6D}}(W_1^{ab}, \overline{W}_2^{cd}, g_3^+, g_4^-) = \frac{(\langle 1_a 4 \rangle [2_c 3] - \langle 2_c 4 \rangle [1_a 3])(\langle 1_b 4 \rangle [2_d 3] - \langle 2_d 4 \rangle [1_b 3])}{s_{12}\,(s_{23} - m^2)} + \text{sym} \,, \tag{9.38}$$

where a, b and c, d are SU(2) little-group indices of the massive particles $W_1^{ab} = W_1^{ba}$ and $\overline{W}_2^{cd} = \overline{W}_2^{dc}$, respectively. The notation "+ sym" means that one should symmetrize on the little-group indices of each massive W boson. Here we have also defined

$$[1_a 2_b] = \tilde{l}_{1,\dot{\alpha},a} \tilde{l}_{2,\dot{\beta},b} \epsilon^{\dot{\alpha}\dot{\beta}} \,, \quad \langle 1_a 2_b \rangle = l_{1,\alpha,a} l_{2,\beta,b} \epsilon^{\alpha\beta} \,, \tag{9.39}$$

for massive spinors. Note if $a \neq b$, they vanish in the massless limit which sets

$l_{\dot\alpha,+} = \tilde{l}_{\dot\alpha,-} = 0$. While if $a = b$, they reduce to the usual spinor brackets for 4D massless particles. Clearly, this formalism is very convenient for massive amplitudes, as was emphasized in [13].

9.4.2 Massive SUSY

Amplitudes for 4D $\mathcal{N} = 4$ SYM on the Coulomb branch can be constructed using the massive spinor-helicity formalism. Recall that the 16 supercharges of a particle in 6D $(1,1)$ SYM can be expressed in the form

$$q^A = l^A_a \eta^a \,, \quad \overline{q}^A = l^A_a \frac{\partial}{\partial \eta_a} \,, \tag{9.40}$$

$$\tilde{q}_A = \tilde{l}_{A\hat{a}} \tilde{\eta}^{\hat{a}} \,, \quad \overline{\tilde{q}}_A = \tilde{l}_{A\hat{a}} \frac{\partial}{\partial \tilde{\eta}_{\hat{a}}} . \tag{9.41}$$

The reduction to the supercharges of a 4D massive particle can be obtained using (9.35),

$$q^{I\alpha} = l^\alpha_- \eta^I_+ - l^\alpha_+ \eta^I_- \,, \quad \overline{q}^{I\dot\alpha} = \tilde{l}^{\dot\alpha}_+ \frac{\partial}{\partial \eta^I_+} + \tilde{l}^{\dot\alpha}_- \frac{\partial}{\partial \eta^I_-} \,, \tag{9.42}$$

$$\tilde{q}^I_\alpha = l_{\alpha-} \frac{\partial}{\partial \eta^I_-} + l_{\alpha+} \frac{\partial}{\partial \eta^I_+} \,, \quad \overline{\tilde{q}}_{\dot\alpha} = \tilde{l}_{\dot\alpha+} \eta^I_- - \tilde{l}_{\dot\alpha-} \eta^I_+ \,, \tag{9.43}$$

where we have identified $\{\eta, \tilde{\eta}\}$ as η^I with $I = 1, 2$. Their anti-commutation relations are

$$\{q^{I\alpha}, \tilde{q}^{J\beta}\} = M\epsilon^{IJ}\epsilon^{\alpha\beta} \,, \quad \{\overline{q}^{I\dot\alpha}, \overline{\tilde{q}}^{J\dot\beta}\} = M\epsilon^{IJ}\epsilon^{\alpha\beta} , \tag{9.44}$$

$$\{q^{I\alpha}, \overline{q}^{J\dot\alpha}\} = \epsilon^{IJ} p^{\alpha\dot\alpha} \,, \quad \{\tilde{q}^{I\alpha}, \overline{\tilde{q}}^{J\dot\beta}\} = -\epsilon^{IJ} p^{\alpha\dot\alpha} . \tag{9.45}$$

The central charge Z satisfies $Z^2 = M^2 = m^2$, which reflects the fact that the W's of $\mathcal{N} = 4$ SYM on the Coulomb branch are BPS. To reduce to the massless case, one sets $l^\alpha_+ = \tilde{l}_{\dot\alpha-} = 0$ and identifies $l^\alpha_+ = l^\alpha$ and $\tilde{l}_{\dot\alpha-} = \tilde{l}_{\dot\alpha}$. That is, of course, the familiar (super) spinor-helicity formalism for $\mathcal{N} = 4$ SYM at the origin of moduli space. With the introduction of supersymmetry, a massive supermultiplet of $\mathcal{N} = 4$ SYM on the Coulomb branch can be neatly written as

$$\Phi(\eta) = \phi + \eta^I_a \psi^a_I + \epsilon_{ab} \eta^{Ia} \eta^{Jb} \phi_{IJ} + \epsilon_{IJ} \eta^I_a \eta^J_b A^{ab} + (\eta)^2 \eta^J_a \bar{\psi}^a_J + (\eta)^4 \bar{\phi} \,, \tag{9.46}$$

which contains one vector, four fermions, and five scalars. One scalar has been eaten by the vector, compared to the massless case with six scalars.

We can also express the massive amplitudes supersymmetrically. For instance, the superamplitude for the four-point amplitude with a pair of conjugate W-bosons, considered previously, can be written as

$$A_4 = \frac{\delta_F^4\,\tilde{\delta}_F^4}{s_{12}(s_{23}-m^2)}\,, \tag{9.47}$$

with the fermionic delta-functions given by

$$\begin{aligned}
\delta_F^4 &= \delta^4(l_{1a}^{\alpha}\eta_1^{Ia} + l_{2a}^{\alpha}\eta_2^{Ia} + l_3^{\alpha}\eta_3^{I,-} + l_4^{\alpha}\eta_4^{I,-})\,, &(9.48)\\
\tilde{\delta}_F^4 &= \delta^4(\tilde{l}_{1a}^{\dot{\alpha}}\eta_1^{Ia} + \tilde{l}_{2a}^{\dot{\alpha}}\eta_2^{Ia} + \tilde{l}_3^{\dot{\alpha}}\eta_3^{I,+} + \tilde{l}_4^{\dot{\alpha}}\eta_4^{I,+})\,. &(9.49)
\end{aligned}$$

These delta functions make the conservation of eight supercharges manifest.

9.4.3 Massive Amplitudes on the Coulomb Branch of $\mathcal{N} = 4$ SYM

Having set up the 4D massive kinematics and supersymmetry, we are ready to write down a general Witten–RSV formula for 4D scattering amplitudes of $\mathcal{N} = 4$ SYM on the Coulomb branch by a simple dimensional reduction of 6D massless $\mathcal{N} = (1,1)$ SYM. The formula is

$$\boxed{\mathcal{A}_n^{\mathcal{N}=4\text{ SYM CB}}(\alpha) = \int d\mu_n^{\text{CB}}\, \text{PT}(\alpha) \left(\text{Pf}' A_n \int d\Omega_F^{(4),\text{CB}} \right).} \tag{9.50}$$

The measure $d\mu_n^{\text{CB}}$ is obtained directly from the 6D massless one with the following replacement of the bosonic delta functions:

$$\Delta_B \to \prod_{i=1}^{n} \delta^4\left(p_i^{\alpha\dot{\alpha}} - \frac{\langle \rho^{\alpha}(\sigma_i)\,\tilde{\rho}^{\dot{\alpha}}(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \right) \delta\left(M_i - \frac{\langle \rho^1(\sigma_i)\rho^2(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \right) \delta\left(\widetilde{M}_i - \frac{\langle \tilde{\rho}^{\dot{1}}(\sigma_i)\tilde{\rho}^{\dot{2}}(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \right), \tag{9.51}$$

and using massive kinematics of (9.35), we set $\widetilde{M}_i = M_i$ for $i = 1, 2, \ldots, n-1$, where $M_i^2 = m_i^2$ is the mass squared of the ith particle ($\widetilde{M}_n = M_n$ is a consequence of 6D momentum conservation). The mass m_i^2, is m_W^2 or 0, as appropriate, for the simple symmetry breaking pattern described previously. Similarly, for the fermionic part

$$\Delta_F \to \prod_{i=1}^{n} \delta^4 \left(q_i^{\alpha,I} - \frac{\langle \rho^{\alpha}(\sigma_i)\, \chi^I(\sigma_i)\rangle}{\prod_{j\neq i} \sigma_{ij}} \right) , \quad \tilde{\Delta}_F \to \prod_{i=1}^{n} \delta^4 \left(\tilde{q}_i^{\dot{\alpha},I} - \frac{\langle \tilde{\rho}^{\dot{\alpha}}(\sigma_i)\, \chi^I(\sigma_i)\rangle}{\prod_{j\neq i} \sigma_{ij}} \right) , \tag{9.52}$$

where the supercharges $q_i^{\alpha,I}$ and $\tilde{q}_i^{\dot{\alpha},I}$ are defined in the previous section.

The polynomial maps are defined as usual,

$$\rho_a^{\alpha}(z) = \sum_{k=0}^{m} \rho_{k,a}^{\alpha}\, z^k , \quad \tilde{\rho}_a^{\dot{\alpha}}(z) = \sum_{k=0}^{m} \tilde{\rho}_{k,a}^{\dot{\alpha}}\, z^k , \quad \chi_a^I(z) = \sum_{k=0}^{m} \chi_{k,a}^I\, z^k . \tag{9.53}$$

They can be understood as a reduction from the 6D maps,

$$\rho_a^A(z) = \begin{pmatrix} \rho_{\alpha,1}(z) & \rho_{\alpha,2}(z) \\ \tilde{\rho}_1^{\dot{\alpha}}(z) & \tilde{\rho}_2^{\dot{\alpha}}(z) \end{pmatrix} , \qquad \tilde{\rho}_{A\hat{a}}(z) = \begin{pmatrix} \rho_1^{\alpha}(z) & \rho_2^{\alpha}(z) \\ \tilde{\rho}_{\dot{\alpha},1}(z) & \tilde{\rho}_{\dot{\alpha},2}(z) \end{pmatrix} . \tag{9.54}$$

Again, we have to treat amplitudes with n even and n odd differently. So $n = 2m + 2$ or $n = 2m + 1$ if n is even or odd, and the highest coefficients in the maps take the factorized form if n is odd.

The factor $\text{Pf}' A_n$ in the integrand is defined differently depending on whether n is even or odd, but they are straightforward reductions from 6D ones. For instance, we find that the odd-point Pfaffian can be constructed with the additional vector

$$p_*^{\alpha\dot{\alpha}} = \frac{2\, r^{\alpha}\, \tilde{\rho}_a^{\dot{\alpha}}(\sigma_*) \langle \rho^a(\sigma_*), r\rangle}{(\xi_b \langle \rho^b(\sigma_*), r\rangle)^2} , \qquad m_* = 0 . \tag{9.55}$$

This is obtained from (7.76) by splitting the 6D spinor index A into 4D ones $\alpha, \dot{\alpha}$ according to (9.54), and choosing the reference spinors as $q^A = (r_{\alpha}; 0)$, $\tilde{q}_A = (r^{\alpha}; 0)$. The same manipulations are required for the description of the n scattered particles, according to (9.35).

If the amplitudes involve massless external particles, we set $m_i = 0$ for them. The massive particle masses should satisfy the conservation constraint $\sum_i m_i = 0$, which is imposed by the rational maps automatically. Note that it is necessary to keep track of the signs of masses, even though the inertial mass is always $|m|$. This would be the only condition for a general 4D theory obtained by dimensional reduction. Specifying the particular Coulomb branch of $\mathcal{N} = 4$ SYM requires that we impose further conditions. In the simplest cases, where all of the massive particles have the same mass, we have

$m_i^2 = m_W^2$ for all i, but all the W bosons have mass m_W, whereas all the $\overline{W}$'s have mass $-m_W$. More generally, different masses can be assigned to different massive particles, but if we assign m as the mass of a W boson, then we should then assign $-m$ to the corresponding conjugate $\overline{\mathrm{W}}$ boson. Therefore $\sum_i m_i = 0$ is satisfied in pairs. Finally, due to the color structure, a W boson and its conjugate $\overline{\mathrm{W}}$ boson should appear in adjacent pairs with gluons sandwiched in between. For instance, there are nontrivial amplitudes of the type $A_n(W_1, g_2, \ldots, g_{i-1}, \overline{W}_i, \tilde{g}_{i+1}, \ldots, g_n)$, with gluons g and $\tilde{g}$ belonging to the gauge groups U(N) and U(M), respectively.

We checked that the formula produces correct four-point amplitudes in previous section. It also gives correct five- and six-point ones such as

$$\begin{aligned}
A_5(g_1^+, g_2^+, g_3^+, W_4^{ab}, \overline{W}_5^{cd}) &= \frac{\langle 4_a 5_c\rangle\langle 4_b 5_d\rangle[1|p_5(p_1+p_2)|3]}{\langle 12\rangle\langle 23\rangle(s_{51}-m^2)(s_{34}-m^2)} + \mathrm{sym}\,, &(9.56)\\
A_6(g_1^+, g_2^+, g_3^+, g_4^+, W_5^{ab}, \overline{W}_6^{cd}) &= \frac{\langle 5_a 6_c\rangle\langle 5_b 6_d\rangle[1|p_6(p_1+p_2)(p_3+p_4)p_5|4]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle(s_{61}-m^2)(s_{612}-m^2)(s_{45}-m^2)} + \mathrm{sym}\,,
\end{aligned}$$

or SU(4) R symmetry-violating amplitudes that vanish in the massless limit, such as

$$\begin{aligned}
A_5(\phi_1^{34}, \phi_2^{34}, \phi_3^{34}, W_4^{ab}, \overline{W}_5^{cd}) &= -\frac{m\,\langle 4_a 5_c\rangle[4_b 5_d]}{(s_{51}-m^2)(s_{34}-m^2)} + \mathrm{sym}\,, &(9.57)\\
A_6(\phi_1^{34}, \phi_2^{34}, \phi_3^{34}, \phi_4^{34}, W_5^{ab}, \overline{W}_6^{cd}) &= -\frac{m^2\langle 5_a 6_c\rangle[5_b 6_d]}{(s_{61}-m^2)(s_{612}-m^2)(s_{45}-m^2)} + \mathrm{sym}\,. &(9.58)
\end{aligned}$$

When restricted to W bosons with helicity ± 1, they are all in agreement with the results in [93], but now in a form with manifest SU(2) little-group symmetry for the massive particles. One can also consider cases in which the massive particles are not adjacent, for instance

$$A_4(W_1^{ab}, g_2^+, \overline{W}_3^{cd}, \tilde{g}_4^+) = \frac{\langle 1_a 3_c\rangle\langle 1_b 3_d\rangle[24]^2}{(s_{12}-m^2)(s_{23}-m^2)} + \mathrm{sym}\,, \tag{9.59}$$

$$A_4(W_1^{ab}, g_2^-, \overline{W}_3^{cd}, \tilde{g}_4^+) = \frac{(\langle 1_a 2\rangle[3_c 4] - \langle 3_c 2\rangle[1_a 4])(\langle 1_b 2\rangle[3_d 4] - \langle 3_d 2\rangle[1_b 4])}{(s_{12}-m^2)\,(s_{23}-m^2)} + \mathrm{sym}\,. \tag{9.60}$$

9.5 Reduction to Four Dimensions: Special Sectors

One can further reduce our 6D formulas down to 4D massless kinematics. It is interesting that 4D kinematics induces a separation into sectors, as reviewed in Section 5.2.1, whereas there is no natural separation into sectors in higher dimensions. In fact, one of the motivations for developing formulas in 6D is to unify all of the 4D sectors. Here we will explain how to naturally obtain the integrand of 4D theories from 6D via dimensional reduction in the middle ($d = \tilde{d}$) and "next to middle" ($d = \tilde{d} \pm 1$) sectors for even and odd multiplicity, respectively. However, the emergence of the other sectors is more difficult to see via dimensional reduction, even though all sectors are present. We will comment on this at the end of this subsection.

For the first case, it was already argued in [149] that the 6D measure for rational maps reduces to the corresponding 4D measure provided the maps behave regularly under the dimensional reduction, i.e., they reduce to the ones appearing in the Witten–RSV formula. After reviewing the reduction for n even, we will generalize the argument to odd n for the near-to-middle sectors, i.e., $d = \tilde{d} \pm 1$.[1]

Let us first consider the even-point case, $n = 2m + 2$. For the solutions corresponding to the middle sector $d = \tilde{d} = m$, the maps behave as follows [149]:

$$\rho_a^A(z) \to \begin{pmatrix} 0 & \rho_\alpha(z) \\ \tilde{\rho}^{\dot{\alpha}}(z) & 0 \end{pmatrix}, \qquad \tilde{\rho}_{A\hat{a}}(z) \to \begin{pmatrix} 0 & \rho^\alpha(z) \\ \tilde{\rho}_{\dot{\alpha}}(z) & 0 \end{pmatrix}, \tag{9.61}$$

where $\deg \rho_\alpha(z) = \deg \tilde{\rho}_{\dot{\alpha}}(z) = d$. Here we have used the 4D embedding described in [84], with the analogous behaviour for the kinematic data λ_a^A and $\tilde{\lambda}_{A\hat{a}}$. This corresponds to setting $p_i^{AB} = 0$ for $\{A, B\} = \{1, 2\}, \{3, 4\}$. Note further that the action of the subgroup GL(1,$\mathbb{C}$) $\subset$ SL(2,$\mathbb{C}$) $\times$ SL(2,$\mathbb{C}$) is manifest and given by $\rho_-^A \to \ell \rho_-^A$, $\rho_+^A \to \frac{1}{\ell}\rho_+^A$, etc.

Consider now the fermionic piece of the SYM integrand in (6.9):

$$V_n \int \prod_{k=0}^{d} d^2\chi_k d^2\tilde{\chi}_k \prod_{i=1}^{n} \delta^4\left(q_i^A - \frac{\rho_a^A(\sigma_i)\chi^a(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right) \delta^4\left(\tilde{q}_{i,A} - \frac{\tilde{\rho}_{A,\hat{a}}(\sigma_i)\tilde{\chi}^{\hat{a}}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right). \tag{9.62}$$

Under the embedding (9.61) this becomes

$$V_n \int \prod_{k=0}^{d} d\chi_k^+ d\chi_k^- d\tilde{\chi}_{k-} d\tilde{\chi}_{k+} \times \prod_{i=1}^{n} \delta^2\left(q_{i\alpha}^1 - \frac{\rho_\alpha(\sigma_i)\chi^-(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right) \delta^2\left(\tilde{q}_i^{\dot{\alpha}1} - \frac{\tilde{\rho}^{\dot{\alpha}}(\sigma_i)\chi^+(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right)$$

[1] One can input kinematics $\{p_i^\mu\}$ in $D = 4 + \epsilon$ dimensions and study the behaviour of the 6D maps as $\epsilon \to 0$. We find that when the solution corresponds to the aforementioned sectors the maps are regular. This implies that the measure is finite and reproduces the CHY measure of Section 5, valid for both 6D and 4D. For other sectors the maps become divergent and additional care is needed to define the limit of the measure.

$$\times\, \delta^2\left(\tilde{q}^2_{i\dot\alpha} - \frac{\tilde\rho_{\dot\alpha}(\sigma_i)\tilde\chi_-(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right)\delta^2\left(q_i^{\alpha 2} - \frac{\rho^\alpha(\sigma_i)\tilde\chi_+(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{9.63}$$

where we have labeled $q^A = (q^1_\alpha; \tilde{q}^{\dot\alpha 1})$ and $\tilde{q}_A = (\tilde{q}^2_{\dot\alpha}; q^{\alpha 2})$. We can now identify the 4D fermionic degrees of freedom as

$$\tilde\chi^{\hat I} = (\chi^-, \tilde\chi_+)\,, \qquad \chi^I = (\chi^+, \tilde\chi_-), \tag{9.64}$$

with $I = 1,2$ and $\hat I = 1,2$ transforming under the manifest $\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{SU}(4)$ R symmetry group in 4D. Hence, the fermionic piece is

$$\int d\Omega_F = V_n \int \prod_{k=0}^{d} d^2\chi_k^I d^2\tilde\chi_k^{\hat I} \times \prod_{i=1}^{n}\delta^4\left(q_i^{\alpha\hat I} - \frac{\rho^\alpha(\sigma_i)\tilde\chi^{\hat I}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right)\delta^4\left(\tilde{q}_i^{\dot\alpha I} - \frac{\tilde\rho^{\dot\alpha}(\sigma_i)\chi^I(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right). \tag{9.65}$$

The remaining part of the even-multiplicity integrand is trivially reduced to four dimensions, since the matrix $[A_n]_{ij} = \frac{p_i\cdot p_j}{\sigma_{ij}}$ is not sensitive to any specific dimension. Alternatively, it can be seen that under the embedding (9.61) and the support of the bosonic delta functions [149]

$$V_n\,\mathrm{Pf}' A_n \to R^d(\rho)\, R^{\tilde d}(\tilde\rho)\,. \tag{9.66}$$

Let us now derive the analogous statement for $n = 2m+1$. We assume $\tilde d = d-1$ (with the case $\tilde d = d+1$ being completely analogous). The embedding (9.61) can then be obtained by fixing the components $\xi = \tilde\xi = (1,0)$ and $\zeta = \tilde\zeta = (0,1)$ for the odd-point maps (recall that we defined $\{\xi,\zeta\}$ as an $\mathrm{SL}(2,\mathbb{C})_\rho$ basis). For the fermionic part we again introduce two polynomials $\chi^I(\sigma)$ and $\tilde\chi^{\hat I}(\sigma)$ of degrees d and $\tilde d$. The top components of the polynomial χ^I can be identified as

$$(\chi^1_d, \chi^2_d) = (\chi^+_d, \tilde\chi_-) = (g, \tilde g) \tag{9.67}$$

according to (7.40) and (7.41). The bosonic part of the integrand becomes

$$\mathrm{Pf}'\widehat{A}_n = \frac{1}{\sigma_{n1}}\sum_{i=2}^{n-1}\frac{(-1)^i}{\sigma_{ni}}\frac{[q|P(\sigma_i)|\tilde\rho_{\hat a}(\sigma_n)]\zeta^{\delta\hat a}}{[q|\rho_a(\sigma_n)\rangle\xi^a}\mathrm{Pf}\,A^{[1in]} \tag{9.68}$$

$$= \frac{1}{\sigma_{n1}} \sum_{i=2}^{n-1} \frac{(-1)^i}{\sigma_{ni}} \frac{[qi]\langle i\rho(\sigma_n)\rangle}{[q\tilde{\rho}(\sigma_n)]} \mathrm{Pf}\, A^{[1,i,n]}. \tag{9.69}$$

We have checked numerically up to $n = 7$ that this expression coincides with $V_n^{-1} R^d(\rho) R^{\tilde{d}}(\tilde{\rho})$ for $\tilde{d} = d - 1$ on the support of the 4D equations (5.12). Hence for this sector ($d = \tilde{d}$ for even n or $d = \tilde{d} \pm 1$ for odd n) the integrand can be recast into the non-chiral form of the Witten–RSV formula, and the amplitude is given by [64]:

$$\begin{aligned} \mathcal{A}_{n,d}^{\mathcal{N}=4\ \mathrm{SYM}} &= \int \mu_{n,d}^{\mathrm{4D}}\, R^d(\rho) R^{\tilde{d}}(\tilde{\rho}) \int \prod_{k=0}^{d} d^2\chi_k^I \prod_{k=0}^{\tilde{d}} d^2\tilde{\chi}_k^{\hat{I}} \\ &\quad \times \prod_{i=1}^{n} \delta^4\left(q_i^{\alpha\hat{I}} - \frac{\rho^\alpha(\sigma_i)\tilde{\chi}^{\hat{I}}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}} \right) \delta^4\left(\tilde{q}_i^{\dot{\alpha}I} - \frac{\tilde{\rho}^{\dot{\alpha}}(\sigma_i)\chi^I(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}} \right). \end{aligned} \tag{9.70}$$

Let us finally comment on other sectors. First of all, given the fact that the 6D rational maps contain all $(n-3)!$ solutions, it is clear that all the sectors are there. One can see it by considering completely integrating out all the moduli ρ's, then reducing the 6D formulas to 4D will not be different from the dimensional reduction of the original CHY formulations. However, from the procedure outlined above, it is subtle to see how other sectors emerge directly by dimensional reduction. As we will discuss in Section 9.5, this subtlety is closely related to the fact that both $\mathrm{Pf}' A_n$ (for even n) and $\mathrm{Pf}' \widehat{A}_n$ (for odd n) vanish for the kinematics of the non-middle sectors (for even n) and the non next-to-middle sectors (for odd n).

Degenerate Kinematics in 6D

The topic we now address has to do with a very important assumption made in the construction of our formulas. Up to this point we have been using maps of degree $n-2$ from $\mathbb{CP}^1$ into the null cone defined by

$$p^{AB}(z) = \langle \rho^A(z)\, \rho^B(z)\rangle = \rho^{A,+}(z)\rho^{B,-}(z) - \rho^{A,-}(z)\rho^{B,+}(z), \tag{9.71}$$

with $\rho^{A,+}(z)$ and $\rho^{A,-}(z)$ both polynomials of degree $(n-2)/2$ for even n and $(n-1)/2$ for odd n. The assumption made so far is that these maps are sufficient to cover the entire relevant moduli space for the computation of Yang–Mills amplitudes. In particular, a natural question is what happens when $d_+ = \deg\, \rho^{A,+}(z)$ and $d_- = \deg\, \rho^{A,-}(z)$ are allowed to be distinct and whether such maps are needed to cover regions of the moduli space when the external kinematics takes special values.

Let us start the discussion with n even. Considering maps of general degrees d_+ and d_-, subject to the constraint $d_+ + d_- = n-2$, we may require $\Delta := d_+ - d_- \geq 0$ without loss of generality. While for generic kinematics $\Delta = 0$ maps exist for all $(n-3)!$ solutions of the scattering equations, we find that there are codimension one or higher subspaces for which some solutions escape the "coordinate patch" covered by $\Delta = 0$ maps.

There are three matrices that control all connected formulas presented in this work. They are K_n, A_n and Φ_n. The first and the last one only appeared implicitly. For reader's convenience we list below the definition of all three even though A_n has been previously defined:

$$(K_n)_{ij} = \begin{cases} p_i \cdot p_j & i \neq j, \\ 0 & i = j, \end{cases} \quad (A_n)_{ij} = \begin{cases} \frac{p_i \cdot p_j}{\sigma_{ij}} & i \neq j, \\ 0 & i = j, \end{cases} \quad (\Phi_n)_{ij} = \begin{cases} \frac{p_i \cdot p_j}{\sigma_{ij}^2}, & i \neq j, \\ -\sum_{k \neq i} \frac{p_i \cdot p_k}{\sigma_{ik}^2} & i = j. \end{cases}$$

The physical meaning of the first one is clear: It is the matrix of kinematic invariants. The second is the familiar A_n matrix whose reduced Pfaffian enters in all the formulas we have presented. Finally, Φ_n is the Jacobian matrix of the scattering equations.

In dimensions $D \geq n-1$ the number of independent Mandelstam invariants is $n(n-3)/2$, and therefore the matrix K_n has rank $n-1$. When $D < n-1$ the space of Mandelstam invariants has the lower dimension $(D-1)n - D(D+1)/2$, and therefore the matrix K_n has a lower rank. This is easy to understand as the momentum vectors in $p_i \cdot p_j$ start to satisfy linear dependencies. In general, if the dimension is D, then so is the rank of the K_n matrix. The rank of K_n is therefore a measure of the minimal spacetime dimension where a given set of kinematic invariants $p_i \cdot p_j$ can be realized as physical momentum vectors. By contrast, in general the matrices A_n and Φ_n have ranks $n-2$ and $n-3$, respectively, for any spacetime dimension D.

At this point we have numerical evidence up to $n = 10$ to support the following picture: There exist subspaces in the space of 6D kinematic invariants where some solutions to the scattering equations lower the rank of A_n while keeping the rank of K_n and Φ_n the same as is expected for generic kinematics.

Since A_n is antisymmetric, its rank decreases by multiples of two. Moreover, we find that when the rank has decreased by $2r$, i.e., its new rank is $n - 2(r+1)$, the maps that cover such solutions of the scattering equations are those for which $\Delta = 2r$. From the definition $\Delta = d_+ - d_-$ it is clear that the maps needed to cover these new regions are of degree $d_+ = n/2 + (r-1)$ and $d_- = n/2 - (r+1)$.

The extreme case $r = n/2 - 2$, i.e., when $d_- = 1$, is never reached while keeping the rank of K_n equal to six. In fact, it is only when the rank of K_n becomes four that such maps are needed. Note that decreasing the rank of K_n to four implies that such kinematic points can be realized by momenta embedded in 4D spacetime. In 4D it is well-known that the solutions to the scattering equations split into sectors, as discussed in Section 5, and maps of different degrees are needed to cover all solutions.

For odd n the preceding statement needs to be refined. To see why, recall that in Section 7.1.3 we introduced the notion of z-dependent $\mathrm{SL}(2,\mathbb{C})$ transformations (7.37). In particular, for $d_+ = \Delta + d_-$ one has the following redundancy of the maps:

$$\rho^{A,+}(z) \to \rho^{A,+}(z) + u(z)\rho^{A,-}(z)\,, \tag{9.72}$$

where $\deg u(z) = \Delta$. This is an intrinsic redundancy of each sector, since the maps (and hence the matrix A_{ij}) are invariant under such transformation. However, when $d_+ < \Delta + d_-$ this transformation will "shift" between sectors, leading to maps that satisfy $d_+ = \Delta + d_-$, but are equivalent to those with lower degree. We see that these points of the moduli space should be modded out in order to define sector decomposition. The way to recognize them is to notice that when $d_+ < \Delta + d_-$ the transformation (9.72) will determine coefficients of $\rho^+(z)$ in terms of those of $\rho^-(z)$. Hence, for n even, the natural way of modding out such cases is to consider the moduli space with completely independent coefficients of $\rho^+(z)$ and $\rho^-(z)$, which is what we did so far. For n odd, this is not the case since in general the top coefficients of $\rho^+(z)$ and $\rho^-(z)$ are related, i.e., $\rho^+_{d_+} = \xi\rho^-_{d_-}$. A natural choice is then to set $\xi = 0$, which effectively removes linear dependences within coefficients. Hence we define the $\Delta = 1$ sector as the one with $d_+ = d_- + 1$ and all the coefficients being independent. The transformations (9.72) are now the standard $\mathrm{SL}(2,\mathbb{C})_\rho$ shift and the T shift, which we leave as the redundancies of the sector. We further note that since the degree of the map $p^{AB}(z) = \rho_+^{[A}(z)\rho_-^{B]}(z)$ is odd, for even $\Delta = d_+ - d_-$ there will be trivial linear relations among coefficients. This motivates us to label the sectors as

$$\Delta = d_+ - d_- = 2r + 1. \tag{9.73}$$

The maps that we have used so far correspond to $r = 0$. We find that for $r > 0$ it is the odd-point analog of the reduced Pfaffian, $\mathrm{Pf}'\widehat{A}_n$, defined in Section 7, that vanishes for the troublesome solutions supported by these maps.

Finally, we comment that, even though integrands of all the theories we consider in this thesis contain $\mathrm{Pf}A_n$ or $\mathrm{Pf}'\widehat{A}_n$, the full integrand does not necessarily vanish for the missing solutions in the degenerate kinematics sector (as we approach it via analytic continuation). In fact, depending on the projected components of the supermultiplet, the fermionic integrations may generate singularities for these solutions such that they contribute finitely. This can happen in all the theories considered so far except in the case of M5 brane and D-branes, where there are enough powers of $\mathrm{Pf}\, A_n$ to generate a zero for the degenerate solutions. These facts can also be seen by considering purely bosonic amplitudes and directly using CHY formulas. This means that at degenerate kinematic points there are solutions to the scattering equations that require maps with $|\Delta| > 0$. However, this is of course not a problem for our formulas: As we mentioned earlier the degenerate regions of kinematic space are of codimension one or higher, so we can define the amplitudes by analytic continuation of the $\Delta = 0$ formulas. In practice, the integral over the maps moduli space can be first performed in a generic configuration close to the degenerate kinematics, after which the degenerate configuration can be easily reached. We leave a complete exploration of the moduli space of maps for all values of Δ, together with the related topic of 4D dimensional reduction, for future research.

9.6 Non-abelian $\mathcal{N} = (2, 0)$ Formula

As discussed in Section 6.1, the 6D $\mathcal{N} = (1, 1)$ non-abelian SYM amplitudes for even n can be obtained from those of the D5-brane theory by replacing $(\mathrm{Pf}' A_n)^2$ with the Parke–Taylor factor $\mathrm{PT}(\alpha)$. It is natural to ask what happens if we apply the same replacement to the M5-brane formula [149], at least for an even number n of particles. This procedure leads to a formula with a non-abelian structure and $\mathcal{N} = (2, 0)$ supersymmetry,

$$\mathcal{A}_n^{\mathcal{N}=(2,0)}(\alpha) = \int \frac{\prod_{i=1}^n d\sigma_i \, \prod_{k=0}^m d^8\rho_k \, d^4\chi_k}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho)} \prod_{i=1}^n \delta^6\left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right)$$
$$\times\ \delta^8\left(q_i^{AI} - \frac{\langle \rho^A(\sigma_i)\, \chi^I(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right) \frac{\mathrm{Pf}' A_n}{V_n}\, \mathrm{PT}(\alpha)\,. \qquad (9.74)$$

One would be tempted to speculate that these new formulas compute some observable in the mysterious $\mathcal{N} = (2, 0)$ theory that arises in the world-volume of multiple coincident M5-branes. Of course, this would be too naive based on what it is currently known about the $\mathcal{N} = (2, 0)$ theory; simple dimensional arguments suggest that the $\mathcal{N} = (2, 0)$ theory does not have a perturbative parameter and hence a perturbative S matrix. Moreover, explicit no-go results have been obtained preventing the existence of three-particle amplitudes with all the necessary symmetries [158, 94].

Here we take the viewpoint that since (9.74) is well defined as an integral, i.e., it has all correct redundancies, $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$, it is worth exploring in its own right. Moreover, the new non-abelian $\mathcal{N}=(2,0)$ formulas can be combined with non-abelian $\mathcal{N}=(0,2)$ formulas using the KLT procedure in order to compute $\mathcal{N}=(2,2)$ supergravity amplitudes. Given that the non-abelian $\mathcal{N}=(2,0)$ formulas are purely chiral, they have some computational advantages over their $\mathcal{N}=(1,1)$ Yang–Mills cousins, which are traditionally used in KLT.

A natural step in the study of any connected formula based on rational maps is to consider its behavior under factorization. Any physical amplitude must satisfy locality and unitarity: a tree-level amplitude should only have simple poles when non-overlapping Mandelstam variables approach to zero, and the corresponding residues should be products of lower-point ones.

Let us investigate these physical properties of the non-abelian $\mathcal{N}=(2,0)$ formula. Already for $n=4$ we find a peculiar behavior under factorization. As we discussed in Section 6.1, the net effect of changing from $(\mathrm{Pf}' A_4)^2$ to the Parke–Taylor factor $\mathrm{PT}(1234)$ is to introduce an additional factor of $1/(s_{12}\, s_{23})$. Therefore, for $n=4$ the non-abelian $(2,0)$ formula gives [246]:

$$\mathcal{A}_4^{\mathcal{N}=(2,0)}(1234) = \delta^6\left(\sum_{i=1}^{4} p^{AB}\right) \frac{\delta^8(\sum_{i=1}^4 q_i^{A,I})}{s_{12}\, s_{23}}\,, \tag{9.75}$$

which is related to that of 6D $\mathcal{N}=(1,1)$ SYM by a simple change to the fermionic delta functions. Comparing with the four-point amplitude of the theory of a probe M5-brane, the new feature is that $\mathcal{A}_4^{\mathcal{N}=(2,0)}(1234)$ has simple poles at $s_{12}\to 0$ and $s_{23}\to 0$, and the question is what the corresponding residues are. In order to explore the singularity in the s_{12}-channel, let us define the following two objects at $s_{12}=0$:

$$x_{23} = w_2^a \langle 2_a | 3_{\hat{b}}] \tilde{u}_3^{\hat{b}}\,, \qquad \tilde{x}_{23} = \tilde{w}_2^{\hat{a}} [2_{\hat{a}} | 3_b\rangle u_3^b. \tag{9.76}$$

It is easy to check that $s_{23} = x_{23}\tilde{x}_{23}$. One can then show that the residue is given by

$$\lim_{s_{12}\to 0} s_{12}\, \mathcal{A}_4^{\mathcal{N}=(2,0)}(1234) = \delta^6\left(\sum_{i=1}^{4} p_i^{AB}\right) \frac{x_{23}^2}{s_{23}} \int d^4\eta_P^I\, F_3^{(2,0)}(1,2,P) F_3^{(2,0)}(-P,3,4), \tag{9.77}$$

where $F_3^{(2,0)}$ is obtained from $\mathcal{A}_3^{\mathcal{N}=(1,1)\,\mathrm{SYM}}$ by the replacement of fermionic delta functions (7.106) to make it $\mathcal{N}=(2,0)$ supersymmetric. Note that the left-hand side still diverges as $s_{23}\to 0$. These three-point objects, $F_3^{(2,0)}$, are ambiguous since they are not invariant under α-scaling of (7.103) as we discussed in Section 7.3.2, which is a redundancy of the three-particle special kinematics [158, 94]. However, equation (9.77) is well-defined, because the prefactor on the right-hand side precisely cancels out the ambiguity. Moreover, the scaling acts by sending $x_{23} \to \alpha x_{23}$, $\tilde{x}_{23} \to \alpha^{-1}\tilde{x}_{23}$, so it is clear that there is a choice of $\alpha = \alpha(w_2, \tilde{u}_3)$ that sets $x_{23}^2 = s_{23}$. For this choice the four-particle residue can in fact be written as a product of the three-point objects $F_3^{(2,0)}$ summed over internal states. Note, however, that the two $F_3^{(2,0)}$ factors cannot be regarded as independent amplitudes, i.e., they are non-local, since they are defined only in the frame $\frac{x_{23}^2}{s_{23}} = 1$, which in turn depends on all four particles involved. A similar decomposition can be achieved by implementing an unfixed α-scale, but using the shift redundancy (7.104), $w_i \to w_i + b_i u_i$, to set

$$w_2^a \langle 2_a | 3_{\hat{b}}] \tilde{w}_3^{\hat{b}} + w_1^a \langle 1_a | 3_{\hat{b}}] \tilde{w}_3^{\hat{b}} = 0. \tag{9.78}$$

In this frame we find $\frac{x_{23}^2}{s_{23}} = [\tilde{u}_P \tilde{u}_{-P}]\langle w_P w_{-P}\rangle$, and we can write[2]

$$\lim_{s_{12}\to 0} s_{12}\, \mathcal{A}_4^{\mathcal{N}=(2,0)}(1234) = \delta^6\left(\sum_{i=1}^{4} p_i^{AB}\right) \int d^4\eta_P^I\, F_3^{a\hat{a}}(1,2,P) F_{3,a\hat{a}}(-P,3,4)\,, \tag{9.79}$$

with

$$F_3^{a\hat{a}}(1,2,P) := F_3^{(2,0)}(1,2,P) w_P^a \tilde{u}_P^{\hat{a}}\,, \tag{9.80}$$

which now resembles the three-particle amplitude of higher spin states with $\mathcal{N} = (2,0)$ supersymmetry, as described in [94]. Non-locality is now present because the objects are not b-shift invariant. In fact, the defining frame given by (9.78) again depends on the kinematics of all the particles involved. We hence recognize two different "frames" in which the residue of $\mathcal{A}_4^{\mathcal{N}=(2,0)}(1234)$ is given by a sum over exchanges between three-point $\mathcal{N} = (2,0)$ objects.

Since the residue is not given by local quantities we expect that the non-abelian $\mathcal{N} = (2,0)$ formulas give rise to a generalization of physical scattering amplitudes whose meaning might be interesting to explore. Note that the same computation for the 6D $\mathcal{N} = (1,1)$ SYM theory yields no prefactor, and therefore the residue of a four-point amplitude is precisely a product of two three-point amplitudes summed over the exchange of all allowed on-shell states in the theory, as required by unitarity.

[2] We thank Yu-tin Huang for this observation.

We have further checked that the naive non-abelian $(2,0)$ integral formula for odd multiplicity does not have the required $(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, T)$ redundancies anymore, i.e., it depends on the "fixing" of σ's and ρ's. In the case of three particles this is a reflection of the α-scaling ambiguity and is again in agreement with the analysis of [158, 94].

Along the same line of thought, one may further construct 6D $\mathcal{N}=(4,0)$ "supergravity" formulas by the double copy of two non-abelian $\mathcal{N}=(2,0)$ formulas discussed previously and $\mathcal{N}=(3,1)$ "supergravity" formulas by the double copy of the non-abelian $\mathcal{N}=(2,0)$ formulas with $\mathcal{N}=(1,1)$ SYM. The possible existence of a 6D $\mathcal{N}=(4,0)$ theory and its relation to supergravity theories have been discussed in [159]; also see the recent works [147, 148] on constructing the actions of 6D free theories with $\mathcal{N}=(4,0)$ or $\mathcal{N}=(3,1)$ supersymmetry.[3] These constructions clearly will lead to well-defined integral formulas as far as the $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$ redundancies are concerned. For instance, the four-point formulas should be given by (9.2), but with a change of the fermionic delta functions in the numerator such that they implement $\mathcal{N}=(4,0)$ or $\mathcal{N}=(3,1)$ supersymmetry. However, as we can see already at four points, the formulas contain kinematics poles, and the residues do not have clear physical interpretations, just like the $\mathcal{N}=(2,0)$ non-abelian formulas above.

It is worth mentioning that, even though all these formulas are pathological in 6D, upon dimensional reduction to lower dimensions the non-abelian $\mathcal{N}=(2,0)$ formulas or the $\mathcal{N}=(4,0)$ and $\mathcal{N}=(3,1)$ "supergravity" formulas actually behave as well as 6D $\mathcal{N}=(1,1)$ SYM or 6D $\mathcal{N}=(2,2)$ supergravity. In fact, they give the same results. This phenomenon has already been observed for branes, where the $\mathcal{N}=(2,0)$ M5-brane formulas and the $\mathcal{N}=(1,1)$ D5-brane formulas both reduce to the same D4-brane amplitudes in 5D.

[3] The double copy of the $(2,0)$ spectrum to produce the $(4,0)$ one was discussed in [89], and more recently in [51].

Chapter 10

General Discussion

Part I

In this thesis we have presented a new connection between scattering amplitudes and conservative and non-conservative classical gravitational observables, in particular for scattering of spinning black holes. This approach constructs such quantities in an economic way through leading singularities and classical limits. It also complements the general picture regarding the extraction of classical results from on-shell methods, provided e.g. in [47, 197, 86].

It was already pointed out in [236] that amplitudes for massive spin-s particles lead to a classical potential for bodies with spin-induced multipoles such as black holes or neutron stars. The amplitudes match the classical potential up to the 2^{2s}-pole level, or up to order S^{2s}, where S is the body's intrinsic angular momentum:

- a scalar particle corresponds to a monopole (with no higher multipoles);
- a spin-1/2 particle adds only a dipole $\propto S$, yielding the $\mathcal{O}(S^1)$ spin-orbit effects which are universal (body-independent) in gravity;
- a spin-1 particle further adds a spin-induced quadrupole $\propto S^2$, specifically matching the quadrupole of a spinning black hole when constructed with minimal coupling. Note that the quadrupole level corresponds to the order at which the soft theorem stops being universal.
- a spin-3/2 particle adds a black-hole octupole $\propto S^3$, etc.

The complete spin-multipole series of a black hole is seemingly obtained by taking the limit $s \to \infty$ for a massive spin-s particle minimally coupled to gravity. This correlation was shown by Vaidya [236] with explicit calculations at leading post-Newtonian orders, corresponding to the nonrelativistic limits of tree-level amplitudes, up to the spin-2 or S^4 level. In this thesis, we have provided further evidence that this correspondence holds, fully relativistically, to all orders in spin at tree level, and for at least the first few orders in spin at one-loop order. It is, however, not yet clear why we should expect this correspondence between classical black holes and minimally coupled quantum particles with $s \to \infty$ and $\hbar \to 0$, and to what extent we should expect it to hold.

As we have emphasized throughout the text, the natural desired extension of the leading-singularity method is the extraction of higher orders, both in loops (i.e. Post-Minkowskian) and in powers of spin. Examples of higher-loop leading singularities were computed for gravitational theories in [62], so it would be interesting to see if these can be also applied to compute classical observables. On the other hand, extending the range of validity in powers of spin is now clearly related to the problem of understanding deeper orders in the soft expansion. More precisely, it is known that these orders depend both on the matter content and the coupling to gravity [171, 227], hence one could hope that such problem is tractable at least for matter minimally coupled to gravity [13], thus describing black holes. Our methodology clearly resembles a soft bootstrap approach [221], and it would be desirable to formally implement it via recursion relations [111, 79].

It was found in [141], by means of a BCFW argument, that in the MHV sector of gravity amplitudes there is also a natural exponential completion of the soft theorem. A general statement for gravity amplitudes is however still missing. There are a few evident problems for the naive extrapolation of the formula (4.2) to higher orders. As we have seen, increasing the powers of angular momentum, encoded in the gauge-invariant combination $(k_\mu \varepsilon_\nu J_i^{\mu\nu})$, requires decreasing the powers of the numerator $(p \cdot \varepsilon_i)$, which generates unphysical poles. This is precisely what prevents from obtaining a natural candidate for the gravitational Compton amplitude at J^5, namely spin $S = 5/2$. As a consequence of this the results we have derived for the 2PM scattering angle and the 1PN two-body potential should only be trusted up to S^4 order. The extension of this to higher orders in spin may require a careful understanding of higher-spin massive particles in the classical limit.

Finally, another obvious question which arises from our construction based on soft theorems is whether it is possible to establish a link between BMS symmetries studied at null/spatial infinity [232, 144, 166, 109, 77, 233] (or at the black hole horizon [140, 204]) and classical observables arising from massive amplitudes. The natural candidate for such a connection is radiative effects [214, 129, 128, 190, 228], as explored in [172] from the point of view of soft theorems. Finally, it would be also interesting to see a link between the exponentiation presented here and the exponentiation of IR divergences [243, 117, 27, 91, 233] that has been known in QED for a long time. The latter one has recently appeared in the computation of tail effects from the EFT perspective [131, 214, 173].

Part II

In this part of the thesis we presented new connected formulas for tree-level scattering amplitudes of 6D $\mathcal{N}=(1,1)$ SYM theory as well as for $\mathcal{N}=(2,2)$ SUGRA via the double-copy procedure. Due to the peculiar properties of 6D spinor-helicity variables, scattering amplitudes of even and odd number of particles must be treated differently. In the case of even multiplicity, our formulas are direct extensions of the results for the world-volume theory of a probe D5-brane [149]. By considering a soft limit of the even-point formulas we obtained the rational maps and the integrands for odd multiplicity, with many interesting features and novelties. In particular, a new redundancy, which we call T-shift symmetry, emerges for the odd-point worldsheet formulas. Interestingly, the T shift intertwines with the original Möbius $\mathrm{SL}(2,\mathbb{C})_\sigma$ and $\mathrm{SL}(2,\mathbb{C})_\rho$ redundancies. Another new feature is the generalized Pfaffian $\mathrm{Pf}'\widehat{A}_n$ in the integrand. Besides the original n punctures, it contains an additional reference puncture, which can be set to an arbitrary value. Associated to the new puncture there is a special "momentum" vector. The special vector is used to increase the size of the original matrix A_n to $(n{+}1)\times(n{+}1)$ such that it has a non-vanishing reduced Pfaffian for odd n. Moreover, since the special null vector $p_\star$ has zero mass dimension, $\mathrm{Pf}'\widehat{A}_n$ has the correct mass dimension for Yang–Mills amplitudes. It would be of great interest to better understand the physical origin of the additional puncture and the additional special vector. One clear future direction is to obtain an ambitwistor model that realizes all of these new features of the odd-multiplicity connected formulas.

We also presented the 6D formulas in alternative forms, with constraints linearly in terms of the 6D external helicity spinors. They are a direct analog of the Witten–RSV formulations for 4D $\mathcal{N}=4$ SYM. By integrating out the moduli of maps, the linear maps can be further recast into a form with a symplectic Grassmannian structure. The symplectic Grassmannian is realized in terms of 6D version of the Veronese maps.

Having obtained formulas for 6D theories, we also considered their dimensional reduction to 5D and 4D leading to various new connected formulas. By reducing to 5D for massless kinematics and utilizing the 5D spinor-helicity formalism, we obtained new formulas for 5D SYM and SUGRA theories. Reduction to 4D reproduced the original Witten–RSV formula for $\mathcal{N}=4$ SYM in 4D for the middle helicity sector for even n and the next-to-middle sector for odd n. The appearance of other disconnected sectors for 4D kinematics is rather subtle, and we leave it for future investigations. On the other hand, it is very nice that reduction to 4D massive kinematics turns out to be more straightforward without such subtleties. By doing so, we deduced a connected formula for massive amplitudes of 4D $\mathcal{N}=4$ SYM on the Coulomb branch.

Another natural future application of our 6D formulas would be to use the procedure of forward limits in [143, 69] to obtain the one-loop integrand of 4D $\mathcal{N}=4$ SYM. Since now we have manifestly supersymmetric formulas for amplitudes in 5D and 6D, we are in a good position to apply the forward limit procedure of [69] supersymmetrically. This procedure might lead to an analog of the Witten–RSV formulas at loop level, which might be genuinely different from previous formulations [122, 123, 69, 124, 125]. We leave this as a future research direction.

Even though 6D $\mathcal{N}=(1,1)$ SYM is not a conformal theory, its planar scattering amplitudes enjoy a dual conformal symmetry [98] just like $\mathcal{N}=4$ SYM in 4D [105, 104]. Such hidden symmetries are often obscured in traditional ways of representing the amplitudes (such as Feynman diagrams), and become more transparent in modern formulations, such as the Grassmannian [11, 9], as shown in [10]. It would be of interest to investigate whether our 6D $\mathcal{N}=(1,1)$ SYM formulas, especially the version in terms of the Veronese maps or its ultimate symplectic Grassmannian form, can make dual conformal symmetry manifest.

Having succeeded in using the spinor-helicity formalism to study supersymmetric theories in 6D, it is tempting to try to carry out analogous constructions in even higher dimensions where supersymmetric theories still exist, such as ten or eleven. The main challenge is that in $D>6$ one has to impose non-linear constraints on the spinors. Not long after the 6D spinor-helicity formalism was developed, a proposal for a 10D version was introduced [80], also see recent work [16, 17] for 10D and 11D theories. It would be interesting to pursue this line of research further.

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Appendix A

From $\mathrm{SO(D-1,1)}$ to $\mathrm{SO(D-1)}$ multipoles

In order to compare with classical results for spinning bodies it is sometimes necessary to choose a frame through the Spin Supplementary Condition (SSC). Let us show how this arises from our setup.

We have shown that the spin multipoles correspond to finite $\mathrm{SO(D-1,1)}$ transformations which map $p_1 \to p_2$. Such Lorentz transformations are composed of both a boost and a $\mathrm{SO(D-1)}$ Wigner rotation. Spin multipoles of a massive spinning body are defined with respect to a reference time-like direction and form irreps. of $\mathrm{SO(D-1)}$ acting on the transverse directions [176, 173]. Hence, it is natural to identify such action with Wigner rotations of the massive states entering our amplitude. A simple choice for the time-like direction is the average momentum $u = \frac{p}{m} = \frac{p_1+p_2}{2m}$. In this frame boosts are obtained as $K^\nu = u_\nu J^{\mu\nu}$ whereas Wigner rotations read $S^{\mu\nu} = J^{\mu\nu} - 2u^{[\mu}K^{\nu]}$. Adopting $S^{\mu\nu}$ as classical spin tensor then corresponds to the *covariant* SSC, i.e. $u_\nu S^{\nu\mu} = 0$ [218, 217, 238] (other choice was used in [90, 155]). The momenta p_1 and p_2 can be aligned canonically to p through the boost,

$$p_1 = e^{\frac{q}{2m}\cdot K} p \,, \quad p_2 = e^{-\frac{q}{2m}\cdot K} p \,, \tag{A.1}$$

which defines canonical polarization vectors ε, $\tilde{\varepsilon}$ for p through (recall p_2 is outgoing):

$$\varepsilon_1 = e^{\frac{q}{2m}\cdot K} \varepsilon \,, \quad \varepsilon_2 = \tilde{\varepsilon}\, e^{\frac{q}{2m}\cdot K} \,. \tag{A.2}$$

This replacement can then be applied to the multipole expansion (1.11), yielding an extra power of q for each power of J, hence preserving the $\hbar$-scaling. We find

$$\begin{aligned}
\varepsilon_1{\cdot}\varepsilon_2 &= \varepsilon{\cdot}\tilde{\varepsilon}+\frac{1}{m}q_\mu\varepsilon K^\mu\tilde{\varepsilon}+\mathcal{O}(K^2)\,, &\text{(A.3)}\\
\varepsilon_1 J^{\mu\nu}\varepsilon_2 &= \varepsilon S^{\mu\nu}\tilde{\varepsilon}+2u^{[\mu}\varepsilon K^{\nu]}\tilde{\varepsilon}+ \\
&\quad \frac{q_\alpha}{m}\varepsilon\{K^\alpha,S^{\mu\nu}\}\tilde{\varepsilon}+\mathcal{O}(K^2)\,, &\text{(A.4)}\\
\varepsilon_1\{J^{\mu\nu},J^{\rho\sigma}\}\varepsilon_2 &= \varepsilon\{S^{\mu\nu},S^{\rho\sigma}\}\tilde{\varepsilon}+\mathcal{O}(K)\,, &\text{(A.5)}
\end{aligned}$$

(for generic spin K and S are independent). In terms of irreducible representations this decomposition can be thought of as branching SO(D − 1, 1) into SO(D − 1) [22]. For instance, the dipole branches as $\ydiagram{1,1} \to \ydiagram{1,1} + \ydiagram{1}$, which is a transverse dipole plus a transverse vector irrep, K^μ. In the same way, in general the $\ydiagram{2,2}$ irrep. of SO(D − 1, 1) also contains a $\ydiagram{2}$ piece for SO(D − 1). This is the reason we can extract a quadrupole from Weyl piece in (1.37), namely by combining (A.5) with the replacement rule

$$\{S^{\mu\nu},S^{\rho\sigma}\}=\frac{2}{D{-}3}\left(\bar{\eta}^{\sigma[\mu}\bar{Q}^{\nu]\rho}{-}\bar{\eta}^{\rho[\mu}\bar{Q}^{\nu]\sigma}\right)+\text{other irreps} \tag{A.6}$$

where $\bar{\eta}^{\mu\nu}=\eta^{\mu\nu}-u^\mu u^\nu$. Thus we have the identity (c.f. [231, 82])

$$\begin{aligned}
\omega_{\mu\nu\rho\sigma}\Sigma^{\mu\nu\rho\sigma} &= [\omega]^{\ydiagram{2,2}}_{\mu\nu\rho\sigma}\langle\varepsilon_1|\{J^{\mu\nu},J^{\rho\sigma}\}|\varepsilon_2\rangle\,,\\
&= \frac{4}{D-3}[\omega]^{\ydiagram{2,2}}_{\mu\nu\rho\sigma}u^\mu\bar{Q}^{\nu\rho}u^\sigma+O(K)\,.
\end{aligned} \tag{A.7}$$

For instance, we extract a quadrupole contribution from $A_3^{h,s}$ in (1.17):

$$A_3^{h,s}|_{\bar{Q}}{=}\frac{1}{4}\left(\epsilon\cdot p_1\right)^h\frac{q\cdot\bar{Q}\cdot q}{D-3}\,. \tag{A.8}$$

Of course, the SO(D − 1, 1) quadrupole present in $A_4^{h,s}$ also contains a SO(D − 1) quadrupole. It follows from (A.5) that it can be read through

$$Q^{\mu\sigma}{=}\bar{Q}^{\mu\sigma}{-}\frac{4}{D(D-1)}\bar{\eta}^{\mu\sigma}S^2{+}\mathcal{O}(K)\,. \tag{A.9}$$

In general the SO(D − 1) multipoles defined through the covariant SSC are given directly from the SO(D − 1, 1) ones, up to $O(K)$ terms. Due to unitarity, one expects the latter to drop from the amplitude, at least for A_3. Let us show explicitly how this happens.

Note that 3-pt. kinematics implies $[q\cdot K, q\cdot J\cdot\epsilon]=0$ and hence the spin piece of the 3-pt. amplitude (1.19) reads

$$\begin{aligned}\varepsilon_1 e^{\frac{q\cdot J\cdot\epsilon}{\epsilon\cdot p}}\varepsilon_2 = \tilde{\varepsilon}\exp\left(\frac{q_\mu\epsilon_\nu J^{\mu\nu}}{\epsilon\cdot p}+\frac{q_\mu K^\mu}{m}\right)\varepsilon &= \tilde{\varepsilon}e^{\mathcal{S}}\varepsilon \\ = \sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\varepsilon}\left(\frac{q_\mu\epsilon_\nu S^{\mu\nu}}{\epsilon\cdot p}\right)^n\varepsilon\,,\end{aligned} \tag{A.10}$$

where one can check that the sum truncates at order $2s$. Thus the boost (A.1) is effectively subtracted from the finite Lorentz transformation leading to the interpretation of the 3-pt. formula as a little-group rotation induced via photon/graviton emission. We end with a comment on the case $s>h$ and $D>4$: Note that the pole $\epsilon\cdot p$ cancels in (A.8) for any dimension. This means we can provide a local form of the 3-pt. amplitude which contains the same multipoles as the exponential. For instance,

$$\begin{aligned}\bar{A}_3^{\mathrm{ph},2} = (\epsilon\cdot p)\,\phi_2\cdot\Bigg(\mathbb{I}+\frac{\epsilon_\mu q_\nu J^{\mu\nu}}{\epsilon\cdot p}+\frac{q_\mu q_\rho}{4m^2\,\epsilon\cdot p}\times \\ \left[\epsilon_\nu p_\sigma+\epsilon_\sigma p_\nu-\frac{\eta_{\nu\sigma}\,(\epsilon\cdot p)}{D-3}\right]\{J^{\mu\nu},J^{\rho\sigma}\}\Bigg)\cdot\phi_1\,,\end{aligned} \tag{A.11}$$

also yields (A.8) and reduces to (1.19) in $D=4$. In general the 2^n-poles [173, 238] of (A.10) are obtained by performing $\left\lfloor\frac{n}{2}\right\rfloor$ traces with the spatial metric $\bar{\eta}^{\alpha\beta}$ appearing in (A.6). The result takes the local form

$$\begin{aligned}A_3^{h,s}\Big|_{2^n-\text{poles}} = (\epsilon\cdot p)^h\sum_{n=0}^{\infty}\left(\alpha_n+\beta_n\frac{q_\mu\epsilon_\nu S^{\mu\nu}}{\epsilon\cdot p}\right) \\ \times\,\bar{Q}^{(n)}_{\mu_1\cdots\mu_{2n}}q^{\mu_1}\cdots q^{\mu_{2n}}\,,\end{aligned} \tag{A.12}$$

where α_n, β_n depend on the dimension D, and $\bar{Q}^{(n)}$ are the transverse multipoles. In four dimensions we find $\bar{Q}^{(n)}$ to be a tensor product of the Pauli-Lubanski vector S^μ [173, 90], and $\alpha_n=\frac{m^{-2n}}{(2n)!}$, $\beta_n=\frac{m^{-2n}}{(2n+1)!}$.

Appendix B

From Polarization Vectors to Spinor-Helicity Multipoles

Here we show the exponential forms presented here for spin-multipoles contain as particular cases the ones of [135], which implemented massive spinor-helicity variables in $D=4$ [13]. Consider first $A_3^{\text{gr},s}$: For plus helicity of the graviton, the expression derived in [135] reads

$$A_{3,+}^{\text{gr},s} = \frac{(p\cdot\epsilon)^2}{m^{2s}}\langle 2|^{2s} e^{\frac{k_\mu \epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}} |1\rangle^{2s} \,, \tag{B.1}$$

where $\epsilon{=}\epsilon^+$ carries the graviton helicity and $|\lambda\rangle^{2s}$ stands for the product $|\lambda^{(a_1}\rangle_{\alpha_1}\cdots|\lambda^{a_{2s})}\rangle_{\alpha_{2s}}$ of $\text{SL}(2,\mathbb{C})$ spinors associated to each massive particle. The generator $J^{\mu\nu}$ in the exponent acts on such chiral representation. The labels a_i are completely symmetrized little-group indices. The explicit construction of the massive spinors is not needed here (c.f. [13]), but solely the fact that spin-s polarization tensors can be expressed compactly as

$$\varepsilon_1{=}\frac{1}{m^s}|1\rangle^s|1]^s \,, \quad \varepsilon_2{=}\frac{1}{m^s}|2\rangle^s|2]^s \,, \tag{B.2}$$

where $|1^a]_{\dot\alpha}$ and $|2^a]_{\dot\alpha}$ live in the antichiral representation of $\text{SL}(2,\mathbb{C})$. Inserting them into (1.19) we obtain

$$\langle\varepsilon_2|A_3^{\text{gr},s}|\varepsilon_1\rangle{=}\frac{(p\cdot\epsilon)^2}{m^{2s}}\langle 2|^s e^{\frac{k_\mu\epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}}|1\rangle^s[2|^s e^{\frac{k_\mu\epsilon_\nu \bar{J}^{\mu\nu}}{p\cdot\epsilon}}|1]^s \,, \tag{B.3}$$

where $J^{\mu\nu}$ and $\tilde{J}^{\mu\nu}$ are given by

$$J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}\otimes\mathbb{I}^{\otimes(s-1)}+\mathbb{I}\otimes\frac{1}{2}\sigma^{\mu\nu}\otimes\mathbb{I}^{\otimes(s-2)}+\cdots, \tag{B.4}$$

$$\tilde{J}^{\mu\nu} = \frac{1}{2}\tilde{\sigma}^{\mu\nu}\otimes\mathbb{I}^{\otimes(s-1)}+\mathbb{I}\otimes\frac{1}{2}\tilde{\sigma}^{\mu\nu}\otimes\mathbb{I}^{\otimes(s-2)}+\cdots, \tag{B.5}$$

with $\sigma^{\mu\nu}=\sigma^{[\mu}\tilde{\sigma}^{\nu]}$ and $\tilde{\sigma}^{\mu\nu}=\tilde{\sigma}^{[\mu}\sigma^{\nu]}$. They satisfy the self-duality conditions

$$J^{\mu\nu}=\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}J_{\rho\sigma}\,, \quad \tilde{J}^{\mu\nu}=-\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{J}_{\rho\sigma}\,. \tag{B.6}$$

As it is well known, choosing the graviton to have plus helicity leads to a self-dual field strength tensor, which in turn implies that $k_{[\mu}\epsilon^{+}_{\nu]}\tilde{J}^{\mu\nu}=0$. Then (B.3) reads

$$\langle\varepsilon_2|A_3^{\mathrm{gr},s}|\varepsilon_1\rangle=\frac{(p\cdot\epsilon)^2}{m^{2s}}\langle 2|^s e^{\frac{k_\mu\epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}}|1\rangle^s[21]^s\,. \tag{B.7}$$

We can now plug the identity $[21]^s{=}\langle 2|^s e^{\frac{k_\mu\epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}}|1\rangle^s$ from [135] to obtain:

$$\langle\varepsilon_2|A_3^{\mathrm{gr},s}|\varepsilon_1\rangle{=}\frac{(p{\cdot}\epsilon)^2}{m^{2s}}\langle 2|^s e^{\frac{k_\mu\epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}}|1\rangle^s\langle 2|^s e^{\frac{k_\mu\epsilon_\nu J^{\mu\nu}}{p\cdot\epsilon}}|1\rangle^s. \tag{B.8}$$

which has the structure of our formula (1.18), now in "spinor space". Extending the generators $J^{\mu\nu}$ to act on $2s$ slots, i.e. $J^{\mu\nu}\otimes\mathbb{I}^s+\mathbb{I}^s\otimes J^{\mu\nu}\to J^{\mu\nu}$, then recovers (B.1). Consider now $A_{4,+-}^{\mathrm{gr},s}$ for $s\leq 2$ as given in [135], where $(+-)$ denotes the helicity of the gravitons $k_1=|\hat{1}]\langle\hat{1}|$ and $k_2=|\hat{2}]\langle\hat{2}|$,

$$A_{4,+-}^{\mathrm{gr},s}=\frac{\langle\hat{1}|P_1|\hat{2}]^4 m^{-2s}}{p_1{\cdot}k_1\,p_1{\cdot}k_2\,k_1{\cdot}k_2}\langle 2|^{2s}e^{\frac{k_{1\mu}\epsilon_{1\nu}J^{\mu\nu}}{p\cdot\epsilon_1}}|1\rangle^{2s}\,. \tag{B.9}$$

In order to match this we double copy our formula (1.22). The sum in (1.22) exponentiates if we impose $[J_1,J_2]=0$, which in turn is only possible if the polarizations are aligned, i.e. $\epsilon_1\propto\epsilon_2$. When the states have opposite helicity this can be achieved via a gauge choice. This yields

$$\frac{k_{1\mu}\epsilon_{1\nu}J^{\mu\nu}}{p_1\cdot\epsilon_1}+\frac{k_{2\mu}\epsilon_{2\nu}J^{\mu\nu}}{p_2\cdot\epsilon_2}=\frac{k_\mu\epsilon_{1\nu}J^{\mu\nu}}{p\cdot\epsilon_1}, \tag{B.10}$$

where $k = k_1 + k_2$. Expression (1.22) thus becomes

$$A_4^{\mathrm{ph},s}\Big|_{\epsilon_1 \propto \epsilon_2} = \frac{p_1{\cdot}\epsilon_1\, p_2{\cdot}\epsilon_2\, k_1{\cdot}k_2}{p_1{\cdot}k_1 p_1{\cdot}k_2} \langle \varepsilon_1 | e^{\frac{k_\mu \epsilon_{1\nu} J^{\mu\nu}}{p\cdot\epsilon_1}} | \varepsilon_2 \rangle\,. \tag{B.11}$$

(note that $\mathrm{ct} = \epsilon_1 \cdot \epsilon_2$ drops out). The formula (1.13) gives

$$A_4^{\mathrm{gr},s}|_{\epsilon_1 \propto \epsilon_2} = \frac{(p_1{\cdot}\epsilon_1)^2 (p_2{\cdot}\epsilon_2)^2\, k_1{\cdot}k_2}{p_1{\cdot}k_1\, p_1{\cdot}k_2} \langle \varepsilon_1 | e^{\frac{k_\mu \epsilon_{1\nu} J^{\mu\nu}}{p\cdot\epsilon_1}} | \varepsilon_2 \rangle\,, \tag{B.12}$$

for $s \leq 2$. This can be shown to match (B.9) following the same derivation as before and fixing $\epsilon_1 = \frac{|\hat{1}\rangle[\hat{2}|}{[\hat{1}\hat{2}]}$, $\epsilon_2 = \frac{|\hat{1}\rangle[\hat{2}|}{\langle\hat{1}\hat{2}\rangle}$. Note finally that, even though in any dimension D there is an helicity choice such that (1.22) becomes (B.12), the factorization of (1.3) requires to sum over all helicities of internal gravitons.

Appendix C

Spinor Helicity Variables for Massive Kinematics

Here we construct the $SU(2)$ states (3.37) and their respective operators written in terms of anti-chiral spinors, first proposed in [14] as a presentation of the massive little group. Let us first introduce a covariant formulation. Then we present a fix-$SU(2)$ redundancy form of the massive spinors in which we use in the text to define the Holomorphic Classical Limit.

The massive spinor-helicity formalism is well suited to describe scattering amplitudes for massive particles with spin. Much like its massless counterpart, this formalism allows to construct all of the scattering kinematics from basic $\mathrm{SL}(2,\mathbb{C})$ spinors that transform covariantly with respect to the little group of the associated particle. The massive little group is $\mathrm{SU}(2)$, so the Pauli-matrix map from two-spinors to momenta

$$p_{\alpha\dot{\beta}} = p_\mu \sigma^\mu_{\alpha\dot{\beta}} = \epsilon_{ab} |p^a\rangle_\alpha [p^b|_{\dot{\beta}} = |p^a\rangle_\alpha [p_a|_{\dot{\beta}} = \lambda_\alpha{}^a \tilde{\lambda}_{\dot{\beta}a}, \tag{C.1}$$

involves a contraction of the $\mathrm{SU}(2)$ indices $a, b, \ldots = 1, 2$ (not to be confused with the spinorial $\mathrm{SL}(2,\mathbb{C})$ indices $\alpha, \beta, \ldots = 1, 2$ and $\dot{\alpha}, \dot{\beta}, \ldots = 1, 2$). This is in contrast to the massless case, where the little group is $\mathrm{U}(1)$, so its index is naturally hidden inside the complex nature of massless two-spinors

$$k_{\alpha\dot{\beta}} = k_\mu \sigma^\mu_{\alpha\dot{\beta}} = |k\rangle_\alpha [k|_{\dot{\beta}} = \lambda_\alpha \tilde{\lambda}_{\dot{\beta}}. \tag{C.2}$$

Now just as λ_α and $\tilde{\lambda}_{\dot{\beta}}$ are convenient to built massless polarization vectors (C.4), we can use the massive spinors $\lambda_\alpha{}^a$ and $\tilde{\lambda}_{\dot{\beta}}{}^b$ to construct spin-S external wavefunctions. For instance, massive polarization vectors are explicitly

$$\varepsilon_{p\mu}^{ab} = \frac{\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m} \qquad \Rightarrow \qquad \begin{cases} p\cdot\varepsilon_p^{ab} = 0, \\ \varepsilon_{p\mu}^{ab}\varepsilon_{p\nu ab} = \eta_{\mu\nu} - \dfrac{p_\mu p_\nu}{m^2} \\ \varepsilon_{p11}\cdot\varepsilon_p^{11} = \varepsilon_{p22}\cdot\varepsilon_p^{22} = 2\varepsilon_{p12}\cdot\varepsilon_p^{12} = 1, \end{cases} \tag{C.3}$$

where the symmetrized little-group indices (ab) represent the physical spin-projection numbers $1, 0, -1$ with respect to a spin quantization axis, as chosen by the massive spinor basis. Note that the vector indices, as well as their dotted and undotted spinorial counterparts, must always be contracted and do not represent a physical quantum number.

Let us also point out here that the massless polarization vectors and hence the associated helicity variable (4.9) can be written in terms of massless spinors as

$$\varepsilon_+^\mu = \frac{\langle r|\sigma^\mu|k]}{\sqrt{2}\langle r\,k\rangle}, \qquad \varepsilon_-^\mu = -\frac{[r|\bar{\sigma}^\mu|k\rangle}{\sqrt{2}[r\,k]} \qquad \Rightarrow \qquad x_+ = \frac{\langle r|p|k]}{m\langle r\,k\rangle}, \qquad x_- = -\frac{[r|p|k\rangle}{m[r\,k]} = -\frac{1}{x_+}, \tag{C.4}$$

where x is independent of the reference momentum r on the three-point on-shell kinematics.

In (3.37) we considered two massive particles (with same mass m_b and spin S) and constructed the spaces V_3^S, $\bar{V}_4^S$ associated to their respective states. We also introduced the contraction between these states which will naturally occur in the matrix element of the scattering processes:

$$\langle n|m\rangle = (-1)^m\delta_{m+n,2S}\,.$$

This follows from the normalization explained in (3.38). It is also natural to define an inner product for each space if we identify $\bar{V}_4^S = \left(V_3^S\right)^*$, i.e. as providing a dual basis for V_3^S [1]. With these conventions, we can expand any operator $O \in \left(V_3^S\right)^* \otimes \left(\bar{V}_4^S\right)^*$ as

[1]The contraction $\langle n|m\rangle$, as defined, is antisymmetric for fermions. This is reminiscent of the spin-statistics theorem, as such form is proportional to the minimally coupled 3pt amplitude. On the other hand, in order to interpret this contraction as an inner product it is necessary to introduce the dual map $\zeta : V^S \to \left(V^S\right)^*$. For instance, defining $\zeta : |n\rangle \mapsto (-1)^{2s}\langle n|$ leads to a symmetric expression.

$$O = \sum_{n,m \leq 2S+1} (-1)^{n+m-2S} |\bar{n}\rangle\langle\bar{m}| \, \langle n|O|m\rangle \,, \tag{C.5}$$

where $\bar{m} = 2S - m, \bar{n} = 2S - n$. Of course, this expansion is general for any choice of basis as long as $|\bar{n}\rangle, \langle\bar{m}|$ are defined as the duals. It is even possible to use different states for V_3^S and V_4^S, spanned by different spinors $\{|\lambda], |\eta]\}$ and $\{|\bar{\lambda}], |\bar{\eta}]\}$. However, it is natural to use the basis (3.37) as it coincides for both massive particles entering the 3pt amplitude, and also coincides with the dual basis up to relabelling. Next we explicitly illustrate the natural map between the states (3.37) and the well known Dirac spinor representation for $S = \frac{1}{2}$, focusing on the parametrization (3.16) which is suitable for the Holomorphic Classical Limit. We also show how to construct the chiral presentation in terms of angle spinors, in which the basis for both particles turn out to be different.

The basis of solutions for the (momentum space) Dirac equation are given in terms of the spinors

$$\begin{aligned} u_3^+ &= \begin{pmatrix} \langle\lambda| \\ [\lambda| \end{pmatrix} , \qquad u_3^- = \begin{pmatrix} -\langle\eta| \\ [\eta| \end{pmatrix} , \\ \bar{u}_4^+ &= (-\beta|\lambda\rangle \;\; |\lambda]) \,, \qquad \bar{u}_4^- = (\frac{|\eta\rangle}{\beta} + |\lambda\rangle \;\; |\eta]) \,. \end{aligned} \tag{C.6}$$

Thus it is now natural to use $|\eta]$ and $|\lambda]$ to construct the $S = \frac{1}{2}$ representation for the (outgoing) particle P_4, and similarly for P_3. This yields an anti-chiral representation of $SU(2)$. From the definition (3.37) we find (slightly abusing the notation)

$$|+\rangle = \frac{|\lambda]}{\sqrt{m_b}} \,, \quad |-\rangle = \frac{|\eta]}{\sqrt{m_b}} \;\; \in V_3^{\frac{1}{2}} \,. \tag{C.7}$$

and analogously for $\langle\pm| \in \bar{V}_4^{\frac{1}{2}}$. The expansion (C.5) leads to the 2×2 operator

$$O = \frac{1}{m_b} \left(-|\lambda][\lambda| \, O_{(--)} + |\lambda][\eta| \, O_{(-+)} + |\eta][\lambda| \, O_{(+-)} - |\eta][\eta| \, O_{(++)} \right) \,. \tag{C.8}$$

Had we used the chiral part, we would have selected a different basis for each of the massive particles. In fact, the chiral part is obtained by acting with P_3, P_4 on the anti-chiral states, respectively. This means that the change of basis (for $S = \frac{1}{2}$) is given by

$$\bar{O} = \frac{\bar{P}_3 O P_4}{m^2}\,, \tag{C.9}$$

where we have used matrix multiplication, with the extension to higher values of spin being straightforward.

For completeness we present here some useful expressions obtained at $\beta = 1$, even though they can easily be computed in general

$$\begin{aligned}
m^2 \bar{u}_4 \gamma_\mu u_3 \to m^2 \gamma_\mu &= 2(P_4)_\mu |\eta][\lambda| - 2(P_3)_\mu |\lambda][\eta| - 2v_\mu |\lambda][\lambda| \\
&= 2m(P_3)_\mu + 2K_\mu |\eta][\lambda| - 2v_\mu |\lambda][\lambda|\,, \\
\bar{u}_4 u_3 \to \mathbb{I}_{2\times 2} &= \frac{(P_3)^\mu}{m}\gamma_\mu = 2 - \frac{|\lambda][\lambda|}{m}\,, \\
\frac{m^2}{2}\bar{u}_4 \gamma_5 \gamma_\mu u_3 \to m^2 S_\mu &= 2K_\mu |\eta][\eta| - 2(R_\mu + \frac{1}{2}v_\mu)|\lambda][\lambda| \\
&\quad + 2(u_\mu - v_\mu + K_\mu)|\eta][\lambda| + 2(u_\mu - v_\mu)|\lambda][\eta|\,,
\end{aligned} \tag{C.10}$$

where

$$\begin{aligned}
2v_\mu &= [\eta|\sigma_\mu|\lambda\rangle\,, \quad 2u_\mu = [\lambda|\sigma_\mu|\eta\rangle\,, \\
v_\mu + u_\mu &= (P_3)_\mu\,, \quad 2R_\mu = [\eta|\sigma_\mu|\eta\rangle\,.
\end{aligned} \tag{C.11}$$

Here $\mathbb{I}_{2\times 2}$ is the identity operator for Dirac spinors, projected into the two-dimensional subspaces spanned by the wavefunctions $u^{\pm}$. On the other hand, in the second line we used the identity

$$1 = \frac{|\eta][\lambda| - |\lambda][\eta|}{[\lambda\eta]}\,. \tag{C.12}$$

From the fourth line of (C.10), using $2q \cdot K = -m^2$ we find in the HCL

$$S_\mu K^\mu = |\lambda][\lambda|\,. \tag{C.13}$$

This is the reason we call $|\lambda][\lambda|$ a spin operator. One may wonder why the spin operator appears in the expansion of $\mathbb{I}_{2\times 2}$, which contains the scalar contribution. Even though $\mathbb{I}$ and $\gamma_5\gamma_\mu$ are orthogonal as Dirac matrices, this does not hold once they are projected into the 2D subspace of physical states. This is consistent with the non-relativistic expansions of [156], where the form $\bar{u}_4 u_3$ indeed contributes to the spin interaction. In fact, this is also true for higher spin generalizations as we now show.

Motivated by the manifest universality found in section 3.4, i.e. expression (3.47), we consider the following extensions for arbitrary spin S_b (not to be confused with the spin vector S_μ)

$$\begin{aligned} S_\mu K^\mu &= 2S_b|\lambda][\lambda|\,, \\ \mathbb{I}_{(2S_b+1)} &= 2\left(1 - S_b\frac{|\lambda][\lambda|}{m}\right)\,, \end{aligned} \tag{C.14}$$

As explained in the discussion after Eq. (3.39), we omit the trivial part of the operators on the RHS. This allows to keep the expressions compact and makes the universality manifest. Let us briefly perform a nontrivial check of equations (C.14) for higher spins. We do so by examining the representation for $S_b = 1$, which in the standard framework is given by polarization vectors satisfying $\epsilon^{(i)}\cdot P = 0$, $i = 1,2,3$, for a given momentum $P^2 = m_b^2$. In terms of spinor helicity variables the polarization vectors ϵ_3 and ϵ_4 are represented as operators acting on V_3^1 and $\bar{V}_4^1$ respectively. Explicitly, we can choose [2]

$$\begin{aligned} \frac{m_b^2(\epsilon_3)_\mu}{2} &\to [\lambda|[\lambda|R_\mu - [\lambda|[\eta|(u-v+K)_\mu - [\eta|[\eta|K_\mu\,, \\ \frac{m_b^2(\epsilon_4)_\mu}{2} &\to |\lambda]|\lambda](R+\frac{1}{2}P_3)_\mu - |\lambda]|\eta](u-v-K)_\mu - |\eta]|\eta]K_\mu\,, \end{aligned}$$

Using this expression it is easy to check the validity of (C.14) for $S_b = 1$, with [156]

$$\begin{aligned} \epsilon_3\cdot\epsilon_4 &\to \mathbb{I}_3\,, \\ \frac{1}{2m_b}\epsilon_{\mu\alpha\beta\gamma}\epsilon_4^\alpha\epsilon_3^\beta(P_3+P_4)^\gamma &\to S_\mu\,. \end{aligned} \tag{C.15}$$

[2] Here we use the notation $[\lambda|[\eta|$ to account for the standard tensor product, i.e. not symmetrized. Of course, we can replace $[\lambda|[\eta| \to \frac{1}{\sqrt{2}}[\lambda|\odot[\eta|$, where $\odot$ involves the normalization (3.38).

Also, we can now derive the form of the quadrupole interaction. It is given by

$$(\epsilon_4 \cdot K)(\epsilon_3 \cdot K) = |\lambda]|\lambda] \otimes [\lambda|[\lambda| \,. \tag{C.16}$$

We will use this expression in appendix D to translate the leading singularity into standard EFT operators.

For illustration purposes, let us close this section by constructing the representation of the 3pt amplitudes for $S = \frac{1}{2}$ massive fields with a graviton. Let the polarization of the massless particle be described by $|\bar{\lambda}] = \sqrt{x}|\lambda]$, $\langle\bar{\lambda}| = \frac{\langle\lambda|}{\sqrt{x}}$, where x carries helicity 1 (recall $|\lambda]$ is fixed). The 3pt amplitude is given by [237]

$$\begin{aligned} A^{(+2)}_{\frac{1}{2}} &= \frac{\alpha m}{2}\gamma_\mu \frac{[\bar{\lambda}|\sigma^\mu|\eta\rangle[\bar{\lambda}|P_3|\eta\rangle}{\langle\eta\bar{\lambda}\rangle^2}\,, \\ A^{(-2)}_{\frac{1}{2}} &= \frac{\alpha m}{2}\gamma_\mu \frac{[\eta|\sigma^\mu|\bar{\lambda}\rangle[\eta|P_3|\bar{\lambda}\rangle}{[\eta\bar{\lambda}]^2}\,. \end{aligned} \tag{C.17}$$

Here we have fixed the reference spinor entering in the 3pt. amplitudes to be η. Using (C.10) together with (3.16) we find

$$\begin{aligned} A^{(+2)}_{\frac{1}{2}} &= \alpha(mx)^2\left(1 - \frac{|\lambda][\lambda|}{m}\right)\,, \\ A^{(-2)}_{\frac{1}{2}} &= \alpha\left(\frac{m}{x}\right)^2\,, \end{aligned} \tag{C.18}$$

precisely agreeing with (3.39) for $|h| = 2$. Furthermore, in the chiral representation we find, using (C.9)

$$\begin{aligned} \bar{A}^{(+2)}_{\frac{1}{2}} &= \alpha\left(\frac{m}{\bar{x}}\right)^2\,, \\ \bar{A}^{(-2)}_{\frac{1}{2}} &= \alpha(m\bar{x})^2\left(1 - \frac{|\lambda\rangle\langle\lambda|}{m}\right)\,. \end{aligned} \tag{C.19}$$

where $\bar{x}$ is defined as the conjugate of x, see e.g. eq (2.10).

Massless Representation

We can extend the treatment described in section 3.3.3 in order to construct the states for massless particles. The idea is to use the two highest weight states $|0\rangle$, $|2S\rangle$ of the massive representation as the physical polarizations of the massless one, after the limit is taken. The massless case can be formally defined by introducing a variable τ in the parametrization (3.16), i.e. by putting either $|\eta] \mapsto \tau|\eta]$ or $|\eta\rangle \mapsto \tau|\eta\rangle$ and then proceed to take the limit $\tau \to 0$. This parametrizes the mass of both $P_3(\tau)$ and $P_4(\tau)$ as $m^2(\tau) = \tau m^2$. Next we proceed to sketch the procedure leading to the massless 3pt. amplitudes[3] and study both choices $\tau|\eta] \to 0$ and $\tau|\eta\rangle \to 0$. As these amplitudes correspond to the building blocks for both the tree level residue and the triangle LS in section 3.4, showing that they can be recovered from our expressions (3.39) proves the equivalence with the standard spinor helicity approach for massless particles. This approach was recently implemented in [42].

In the following we will consider $\beta = 1$. Indeed, the massless deformation of the momenta is only consistent in the HCL since $t = \tau\frac{(\beta-1)^2}{\beta}m_b^2 \to 0$ as $\tau \to 0$. This is enough for our purposes in section 3.4 since we evaluate both the tree level residue and triangle LS by neglecting subleading contributions in t. For the choice $|\eta] \mapsto \tau|\eta]$ we thus have

$$\begin{aligned} P_3 &= \tau|\eta]\langle\lambda| + |\lambda]\langle\eta| \longrightarrow |\lambda]\langle\eta|\,, \\ P_4 &= \tau|\eta]\langle\lambda| + |\lambda](\langle\eta| + \langle\lambda|) \longrightarrow |\lambda](\langle\eta| + \langle\lambda|)\,, \\ K &= |\lambda]\langle\lambda|\,. \end{aligned} \tag{C.20}$$

In the following we choose $|\lambda]$,$\langle\lambda|$ to represent the polarizations of the particle K. As explained in section (3.3.3), it is convenient to reabsorb the mass into the definition of x (2.10), thus we have

$$x = \tau[\lambda\eta] = \tau m\,, \quad \bar{x} = \langle\lambda\eta\rangle = m\,. \tag{C.21}$$

This means $\tau|\eta] \to 0$ is equivalent to the limit $x \to 0$, keeping $\bar{x}$ fixed. As the reference spinor $|\eta]$ is also fixed, we can assume that the neither the basis (3.37) nor the operators (3.39) depend on τ in any other way that is not through x. With these considerations, we

[3] At this level we keep the discussion general for S and h. Of course, (interacting) massless higher spin particles are known to be inconsistent by very fundamental principles, thus effectively restricting our choices to $S, h \leq 2$.

find for the massless limit

$$A_S^{(h)} = 0\,, \quad A_S^{(-h)} = \alpha \bar{x}^h\,, \tag{C.22}$$

where at this stage $\bar{x} = \langle \lambda\eta \rangle$ is not restricted since the original mass m is not relevant after the limit is taken. We then note that all the positive helicity amplitudes vanish. In fact, these amplitudes can be described in terms of square brackets, thus it is expected that they vanish for the $\tau = 0$ limit in (C.20). Now, the negative helicity amplitudes in the standard spinor helicity notation read [112]

$$\begin{aligned} A(3^{+S}, 4^{-S}, K^{-h}) &= \alpha \frac{\langle K3 \rangle^{h-2S} \langle K4 \rangle^{h+2S}}{\langle 43 \rangle^h} \\ &= \alpha \bar{x}^h\,. \end{aligned} \tag{C.23}$$

Note that this derivation is also valid for $A(3^{-S}, 4^{+S}, K^{-h})$ up to a possible sign. Also, the configuration $A(3^{+S}, 4^{+S}, K^{-h})$ together with its conjugate do not correspond to the minimal coupling and thus vanish. In order to interpret these amplitudes as matrix elements of (C.22), we need to specify the basis of states for the massless particles. It turns out that just the highest weight states in (3.37) are enough for this purpose. That is, we find

$$\begin{aligned} A(3^{+S}, 4^{-S}, K^{-h}) = \langle 2S | A_S | 0 \rangle \quad &, \quad A(3^{-S}, 4^{+S}, K^{-h}) = \langle 0 | A_S | 2S \rangle\,, \\ A(3^{+S}, 4^{+S}, K^{-h}) = \langle 2S | A_S | 2S \rangle \quad &, \quad A(3^{+S}, 4^{+S}, K^{-h}) = \langle 2S | A_S | 2S \rangle\,, \end{aligned} \tag{C.24}$$

therefore showing the equivalence of both approaches for massless particles. Here we note that a somehow more straightforward approach is to define the massless limit directly in the expectation values (C.24), following [14]. Instead, we have opted for constructing the corresponding operators (C.22), since our integral expressions in section 3.4 are given in terms of them. These operators are defined for the basis built from the fixed spinors $|\lambda]$ and $|\eta]$, which are reminiscent of the massive representation in (C.20).

The choice $|\eta\rangle \mapsto \tau|\eta\rangle$ is completely analogous and yields

$$A_S^{(h)} = \alpha x^h \left(1 - \frac{|\lambda][\lambda|}{[\lambda\eta]} \right)^S \,, \quad A_S^{(-h)} = 0\,, \tag{C.25}$$

i.e. vanishing negative helicity amplitudes. This is expected since we have

$$
\begin{aligned}
P_3 &= |\eta]\langle\lambda| + \tau|\lambda]\langle\eta| \;\longrightarrow |\eta]\langle\lambda|\,, \\
P_4 &= |\eta]\langle\lambda| + \tau|\lambda]\langle\eta| + |\lambda]\langle\lambda| \;\longrightarrow (|\lambda] + |\eta])\langle\lambda|\,.
\end{aligned}
\tag{C.26}
$$

However, this time we note that the natural basis of spinors for P_4 is given by $|\bar{\eta}] := |\lambda] + |\eta]$ and $|\lambda]$. When expressed in terms of this basis, the expression (C.25) takes a form analogous to (C.22). Hence we construct the states $\langle 0|$,$\langle 2S|$ in $\bar{V}_4^S$ in terms of these spinors.

Appendix D

Matching the Spin Operators

Here we explain how to recover the standard form of the potential in terms of generic spin operators (3.7), starting from the results of section 3.4. As usual throughout this work, we focus on the gravitational case since it presents greater difficulty in the standard approaches. We give two examples which illustrate how the procedure works. First, we present the tree level result for the case $S_a = 0$, $S_b = 1$, which yields both a spin-orbit and a quadrupole interaction. Second, we discuss the matching at 1-loop level for the case $S_a = S_b = \frac{1}{2}$. Both computations were done in [156] using standard Feynman diagrammatic techniques, which lead to notably increased difficulty with respect to the scalar case. Here we find that the computations are straightforward using the techniques introduced throughout this work. In fact, all the computations in [156] can be redone in a direct way and we leave them as an exercise for the reader. The same can be done for the EM case in order to recover the results previously presented in [154].

The starting point for both cases are the explicit expressions for the variables u, v that we used to construct the amplitudes. We can easily solve them from Eq. (3.11). We find

$$\begin{aligned} 2u &= s - m_a^2 - m_b^2 + \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4m_a^2 m_b^2}\,, \\ 2v &= s - m_a^2 - m_b^2 - \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4m_a^2 m_b^2}\,, \end{aligned} \tag{D.1}$$

where the square root corresponds to the parity odd piece. From the definition (3.10) it is clear that under the exchange $P_1 \leftrightarrow P_3$ (which we perform below), u and v must also be exchanged. Now, to keep the notation compact, let us write

$$P_1 \cdot P_3 = r m_a m_b\,, \quad r > 1.$$

Note that in the non-relativistic regime we have $r \to 1$. Now we can write D.1 as

$$u = m_a m_b \left(r + \sqrt{r^2-1}\right) , \qquad v = m_a m_b \left(r - \sqrt{r^2-1}\right) . \tag{D.2}$$

Consider now the case $S_a = 0$, $S_b = 1$. Let us construct a linear combination of the EFT operators associated to scalar, spin-orbit, and quadrupole interaction, that is [156, 237]

$$\bar{M}^{(1)}_{(0,1,2)} = \alpha^2 \frac{(m_a m_b)^2}{t} \left(c_1(r)\epsilon_3 \cdot \epsilon_4 + c_2(r) \frac{\epsilon_{\mu\alpha\beta\gamma} K^\mu P_1^\alpha P_3^\beta S^\gamma}{m_a m_b^2} + c_3(r) \frac{(\epsilon_4 \cdot K)(\epsilon_3 \cdot K)}{m_b^2} \right) . \tag{D.3}$$

The reason we call $\epsilon_3 \cdot \epsilon_4$ a scalar interaction is because, as will be evident in a moment, it is the only piece surviving the contraction $\langle 0|\bar{M}^{(1)}_{(0,1,2)}|2\rangle$, which we identified as the scalar amplitude (see discussion below Eq. (3.43)).

Note that we have not assumed the non-relativistic limit in the u, v variables, only the HCL $t = 0$ which selects the classical contribution. The operators can now be expanded using (C.14), (C.16). For this, it is enough to note that in the HCL the spin-orbit piece takes the form

$$\epsilon_{\mu\alpha\beta\gamma} K^\mu P_1^\alpha P_3^\beta S^\gamma = -\frac{K \cdot S}{2} \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4 m_a^2 m_b^2} = m_a m_b (K \cdot S) \sqrt{r^2-1} \, . \tag{D.4}$$

We then find

$$\bar{M}^{(0)}_{(0,1,2)} = \alpha^2 \frac{(m_a m_b)^2}{t} \left(2c_1 - 2\frac{|\lambda][\lambda|}{m_b}(c_1 - c_2\sqrt{r^2-1}) + c_3 \frac{|\lambda]|\lambda\rangle \otimes [\lambda|[\lambda|}{m_b^2} \right) .$$

Comparing now with the expression (3.47), which in this case reads

$$\begin{aligned} M^{(0)}_{(0,1,2)} &= \frac{\alpha^2}{t} \left(u^2 + v^2 \left(1 - \frac{|\lambda][\lambda|}{m_b} \right)^2 \right) \\ &= \frac{\alpha^2}{t} \left(u^2 + v^2 - 2v^2 \frac{|\lambda][\lambda|}{m_b} + v^2 \frac{|\lambda]|\lambda] \otimes [\lambda|[\lambda|}{m_b^2} \right) \end{aligned}$$

$$
\begin{aligned}
&= \alpha^2 \frac{(m_a m_b)^2}{t}\left((4r^2-2) - 2\left(2r^2-1-2r\sqrt{r^2-1}\right)\frac{|\lambda][\lambda|}{m_b}\right. \\
&\quad \left. + \left(2r^2-1-2r\sqrt{r^2-1}\right)\frac{|\lambda]|\lambda]\otimes[\lambda|[\lambda|}{m_b^2}\right),
\end{aligned}
$$

we find

$$
\begin{aligned}
c_1 &= 2r^2-1\,, \\
c_2 &= 2r\,, \\
c_3 &= 2r^2-1-2r\sqrt{r^2-1}\,.
\end{aligned}
$$

The result in [156] can then be recovered by imposing the non-relativistic limit $s \to s_0$, which in this case reads $r \to 1$[1]. Even though we computed the residue in (D.3) at $t=0$, it is evident that this expression can be analytically extended to the region $t \neq 0$ in which the COM frame can be imposed, as described in (3.2.1). This is precisely done in [156] where the effective potential is obtained from this expression after implementing the Born approximation.

The 1-loop result for $S_a = 0$, $S_b = 1$ can be computed in the same fashion, by using the expressions provided in section (3.4.2). Expectedly, the EFT operators are exactly the same that appeared at tree level, but the behavior of the coefficients c_1, c_2 and c_3 as functions of r differs in the sense that it can involve poles at $r=1$. We now illustrate all this by considering the more complex case also addressed in [156], namely $S_a = S_b = \frac{1}{2}$.

For $S = \frac{1}{2}$ the multipole operators are restricted to the scalar and spin-orbit interaction. They read [156]

$$
\mathcal{U} = \bar{u}_4 u_3 \quad , \quad \mathcal{E} = \epsilon_{\alpha\beta\gamma\delta} P_1^\alpha P_3^\beta K^\gamma S^\delta\,.
$$

In our case we will consider two copies of these operators, one for each particle. That is to say we propose the following form for the 1-loop leading singularity

$$
\bar{M}^{(1)}_{(\frac{1}{2},\frac{1}{2},2)} = \left(\frac{\alpha^4}{16}\right)\frac{(m_a m_b)^2}{\sqrt{-t}}\left(c_{11}\mathcal{U}_a\mathcal{U}_b + c_{12}\frac{\mathcal{U}_a\mathcal{E}_b}{m_b^2 m_a} + c_{21}\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b} + c_{22}\frac{\mathcal{E}_a\mathcal{E}_b}{m_b^3 m_a^3}\right) \tag{D.5}
$$

[1]There are, however, some discrepancies in conventions which may be fixed by replacing $-\epsilon_f^{b*} \to \epsilon_4$, $iS_b \to S_b$ in [156]. We find our conventions more appropriated since the sign in the scalar interaction is the same for any spin.

$$
\begin{aligned}
= \quad & \alpha^4 \frac{(m_a m_b)^2}{4\sqrt{-t}} \left(c_{11} - \frac{c_{11} - c_{12}\sqrt{r^2-1}}{2} \left(\frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \right) - \frac{c_{11} + c_{21}\sqrt{r^2-1}}{2} \left(\frac{|\lambda][\lambda|}{m_b} \right) \right. \\
& \left. + \frac{\left(c_{11} - (c_{12} - c_{21})\sqrt{r^2-1} - c_{22}(r^2-1)\right)}{4} \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \otimes \frac{|\lambda][\lambda|}{m_b} \right) .
\end{aligned}
$$

Here we have used (D.4),(C.13) and (C.10). A minus sign was introduced when implementing (C.13) for particle m_a, which arises from the mismatch between both parametrizations in the HCL, i.e. $K = |\lambda]\langle\lambda| = -|\hat{\lambda}][\hat{\lambda}|$. We proceed to compare this with the sum of the two triangle leading singularities given by (3.22), using the results of section 3.4.2. The result can be written

$$
M^{(1,\text{full})}_{(\frac{1}{2},\frac{1}{2},2)} = M^{(1,b)}_{(\frac{1}{2},\frac{1}{2},2)} + M^{(1,a)}_{(\frac{1}{2},\frac{1}{2},2)} ,
$$

where $M^{(1,a)}_{(\frac{1}{2},\frac{1}{2},2)}$ is obtained by exchanging $m_a \leftrightarrow m_b$, $|\hat{\lambda}][\hat{\lambda}| \leftrightarrow |\lambda][\lambda|$ and $u \leftrightarrow v$ in

$$
\begin{aligned}
M^{(1,b)}_{(\frac{1}{2},\frac{1}{2},2)} = & \left(\frac{\alpha^4}{16} \right) \frac{m_b}{\sqrt{-t}(v-u)^2} \int_{\infty} \frac{dy}{y^3(1-y^2)^2} \left(\hat{u}y(1-y) + vy(1+y) + (v - \hat{u}) \frac{1-y^2}{2} \right) \\
& \times \left(uy(1-y) + vy(1+y) + \frac{(v-u)(1-y^2)}{2} \right)^3 \otimes \left(1 - \frac{(1+y)^2}{4y} \frac{|\lambda][\lambda|}{m_b} \right) ,
\end{aligned}
\tag{D.6}
$$

with $\hat{u} = u \left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \right)$. After computing the contour integral, we can easily solve for the coefficients c_{ij}, $i, j \in \{1, 2\}$. In order to compare with the results in the literature, we first perform the non-relativistic expansion

$$
\begin{aligned}
c_{11} &= 6(m_a + m_b) + \ldots \\
c_{12} &= \frac{4m_a + 3m_b}{2(r-1)} + 11 \left(m_a + \frac{3}{4} m_b \right) + \ldots \\
c_{21} &= \frac{3m_a + 4m_b}{2(r-1)} + 11 \left(\frac{3}{4} m_a + m_b \right) + \ldots \\
c_{22} &= \frac{m_a + m_b}{4(r-1)^2} + \frac{9(m_a + m_b)}{2(r-1)} + \ldots
\end{aligned}
\tag{D.7}
$$

Note that even though the coefficients present poles, they are parity invariant in the sense that they do not contain square roots. To put the result in the same form as [156], we need to further extract the standard spin-spin interaction term from our operator $\mathcal{E}_a\mathcal{E}_b$. This accounts for extracting the classical piece, which can be obtained by returning to the physical region $t = K^2 \neq 0$. Using (D.4) we find

$$\mathcal{E}_a\mathcal{E}_b = m_a m_b(r^2-1)\left((S_a\cdot K)(S_b\cdot K) - K^2(S_a\cdot S_b)\right) + rK^2(P_1\cdot S_b)(P_3\cdot S_a) + O(K^3)\,,$$

where $O(K^3) = O(|\vec{q}|^3)$ denotes quantum contributions, i.e. higher orders in $|\vec{q}|$ for a fixed power of spin $|\vec{S}|$. However, we note the presence of the couplings $P_i\cdot S \sim \vec{v}\cdot\vec{S}$ which certainly do not appear in the effective potential [19, 156, 237]. In fact, in the standard EFT framework these couplings are dropped by the so-called Frenkel-Pirani conditions or Tulczyjew conditions [118][2]. In our case they have emerged due to our bad choice of ansatz (D.5). In fact, the right choice is now clearly obtained by replacing

$$\mathcal{E}_a\mathcal{E}_b \to m_a m_b(r^2-1)\left((S_a\cdot K)(S_b\cdot K) - K^2(S_a\cdot S_b)\right)\,,$$

corresponding to the correct spin-spin interaction term [212], which is already visible at tree level [154, 156, 237]. Note, however, that this does not modify the HCL of this operator, which comes solely from the first term. Consequently, our results (D.7) are still valid and indeed they agree with the ones in the literature [156]. They can be regarded as a fully relativistic completion leading to the effective potential up to order G^2.

[2]They can arise, however, when including non-minimal couplings corresponding to higher dimensional operators, see e.g. [176]

Appendix E

Spin multipoles from boosts

Here we provide the link between the construction of Chapter 1 applied to the three-point amplitudes and show how it simplifies in the spinor-helicity formalism of Chapter 2. Consider the three-point amplitude in the covariant form as given there:

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \varepsilon_2 \cdot \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \cdot \varepsilon_1, \tag{E.1}$$

where ε_1 and ε_2 are spin-s polarization tensors, the generators $\Sigma^{\mu\nu}$ are given in eq. (2.17), and $\mathcal{M}_3^{(0)} = -\kappa(p_1 \cdot \varepsilon)^2$ corresponds to the gravitational interaction of a scalar particle. It was proposed that in order to extract classical multipoles (forming representations of the little group in the sense of Ref. [176]) the spin states must be evaluated at the same momenta. On the three-point kinematics, the polarization states for p_1 and p_2 are related via

$$\varepsilon_2 = \exp\left(\frac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\right) \tilde{\varepsilon}_1, \qquad \tilde{\varepsilon}_1 = U_{12}^{(s)} \varepsilon_1, \tag{E.2}$$

where $U_{12}^{(s)}$ is a tensor representation of an SO(3) little-group transformation. Note that in the rest frame of particle 1 the Lorentz transformation $p_1^\mu k^\nu \Sigma_{\mu\nu} = m k^i \Sigma^{0i}$ is nothing but the canonical choice for the boost needed to shift p_1 to $p_1 + k$. One can show that the two exponents commute on the three-point kinematics, so

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \tilde{\varepsilon}_1 \exp\left(-\frac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\right) \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \varepsilon_1$$

$$= \mathcal{M}_3^{(0)} \tilde{\varepsilon}_1 \exp\!\left(-i\frac{k_\mu \varepsilon_\nu \Sigma_\perp^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \varepsilon_1, \tag{E.3}$$

where we have defined

$$\Sigma_\perp^{\mu\nu} = \Sigma^{\mu\nu} + \frac{2}{m^2} p_1^{[\mu} \Sigma^{\nu]\rho} p_{1\rho}, \qquad p_{1\mu} \Sigma_\perp^{\mu\nu} = 0, \tag{E.4}$$

as the operator that corresponds to the transverse spin tensor (2.18). Being a transverse tensor, it can be used to construct representations of the little group. The first such j spin multipoles of e.g. Ref. [176] are recovered by expanding the exponential to order j and stripping ε_1 and $\tilde{\varepsilon}_1$. For finite spin s, it was observed that this exponential truncates at order $2s$, whereas eq. (E.1) truncates at order s.

Let us now apply the spinor-helicity formalism to the above argument. Picking for concreteness the negative helicity, it was shown in [20] that eq. (E.1) can be rewritten as our amplitude:

$$\mathcal{M}_3^{(s)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} \langle 2|^{\odot 2s} \exp\!\left(-i\frac{k_\mu \varepsilon_\nu^- \sigma^{\mu\nu}}{p_1 \cdot \varepsilon^-}\right) |1\rangle^{\odot 2s}. \tag{E.5}$$

On the other hand, in Chapter 2 we noted that in the chiral spinor-variable basis self-duality of $\sigma^{\mu\nu}$ implies

$$-i\frac{k_\mu \varepsilon_\nu^- \sigma^{\mu\nu}}{p_1 \cdot \varepsilon^-} = -2i\frac{k_\mu \varepsilon_\nu^- \sigma_\perp^{\mu\nu}}{p_1 \cdot \varepsilon^-} = 2k \cdot a, \tag{E.6}$$

where a^μ is given by (2.28) and $\sigma_\perp^{\mu\nu}$ is the transverse projection of $\sigma^{\mu\nu}$, as in eq. (E.4). The crucial factor of two arises here because in the spinor variables we cannot distinguish between the orbital or intrinsic pieces of the angular momentum. Indeed, the $p_1 \to p_2$ boost considered in eq. (E.2) acts on the chiral basis as

$$\frac{i}{m^2} p_1^\mu k^\nu \sigma_{\mu\nu} = k \cdot a \quad \Rightarrow \quad |2\rangle^{\odot 2s} = e^{k\cdot a} \big\{ U_{12} |1\rangle \big\}^{\odot 2s} \tag{E.7}$$

in accord with eqs. (2.26) and (2.29) in the main text. This boost compensates the factor of two in eq. (E.6), so

$$\begin{aligned}\mathcal{M}_3^{(s)} &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\big\{U_{12}\langle 1|\big\}^{\odot 2s} e^{-k\cdot a} e^{2k\cdot a}|1\rangle^{\odot 2s} \\ &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\big\{U_{12}\langle 1|\big\}^{\odot 2s} e^{k\cdot a}|1\rangle^{\odot 2s}.\end{aligned} \tag{E.8}$$

Now compare this to eq. (E.3), where two distinct exponentials combined into an exponential of a SO(3) rotation (E.4). We see that in the four-dimensional chiral spinor basis it trivialized down to two exponentials, identical up to a numerical prefactor.

One might see an apparent contradiction in eq. (E.7). Namely, that the right-hand side involves the little-group rotation $k \cdot a$ preserving p_1, whereas the left-hand side corresponds to a boost $p_1 \to p_1 + k$. The reason that this is consistent is because eq. (E.7) is a chirality-dependent statement. In fact, the corresponding relation for the antichiral spinors involves a sign flip:

$$\frac{i}{m^2} p_1^\mu k^\nu \bar{\sigma}_{\mu\nu} = -k \cdot a, \tag{E.9}$$

as given in eq. (2.26). More concretely, consider the following relations

$$\big[e^{k\cdot a}\big]_\alpha{}^\beta p_{1\beta\dot\beta}\big[e^{-k\cdot a}\big]^{\dot\beta}{}_{\dot\alpha} = p_{1\alpha\dot\alpha}, \tag{E.10a}$$

$$\big[e^{k\cdot a}\big]_\alpha{}^\beta p_{1\beta\dot\beta}\big[e^{k\cdot a}\big]^{\dot\beta}{}_{\dot\alpha} = p_{2\alpha\dot\alpha}, \tag{E.10b}$$

where $p_{i\alpha\dot\alpha} = |i^a\rangle_\alpha [i_a|_{\dot\alpha}$ as usual. The first relation is simply the statement that the Pauli-Lubanski operator generates little-group rotations, whereas the second relation shows that thanks to the sign flip $k \cdot a$ can effectively act as a boost. Therefore, eq. (E.7) and eq. (E.9) contain no real contradiction and reflect the 'square root' nature of the spinor-helicity representation.

Appendix F

Three-point amplitude with spin-1 matter

Here we compute the three-point amplitude (4.4) starting from the massive spin-1 Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \tag{F.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In order to compute the minimal cubic vertex to gravity, one needs to the extract the energy-momentum tensor sourced by this field. In principle, this can be done by covariantizing this action, i.e. by promoting $\partial_\mu \to \nabla_\mu$, and then inspecting the metric variation, $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}$. Let us, however, take an alternative route of computing the energy-momentum tensor directly in flat space. The reason is that this procedure will explicitly identify the contribution of the intrinsic angular momentum of the particle.

A textbook application of Noether's theorem for translations yields the following tensor

$$T_N^{\mu\nu} = -F^{\mu\sigma}\partial^\nu A_\sigma - \eta^{\mu\nu}\mathcal{L} \qquad \Rightarrow \qquad \partial_\mu T_N^{\mu\nu} = 0. \tag{F.2}$$

Its contraction with an on-shell graviton, $\varepsilon_{\mu\nu}T_N^{\mu\nu}$, fails to give the correct three-point amplitude, as opposed to the one obtained from covariantization. The reason is that $T_N^{\mu\nu}$ lacks symmetry in its indices (notice e.g. $\partial_\nu T_N^{\mu\nu} \neq 0$), therefore its orbital angular momentum $L^{\lambda\,\mu\nu} = x^\mu T_N^{\lambda\nu} - x^\nu T_N^{\lambda\mu}$ is not conserved. Let us fix that by generalizing $T_N^{\mu\nu}$

to a larger class of tensors that are all conserved due to eq. (F.2):

$$T^{\mu\nu} = T_N^{\mu\nu} + \partial_\lambda B^{\lambda\mu\,\nu}, \qquad B^{\lambda\mu\,\nu} = -B^{\mu\lambda\,\nu} \qquad \Rightarrow \qquad \partial_\mu T^{\mu\nu} = 0, \tag{F.3}$$

where the Belinfante tensor $B^{\mu\nu\rho}$ [23, 223] may be adjusted to yield a symmetric energy-momentum tensor matching the gravitational one. To do that, we apply Noether's theorem to Lorentz transformations. The conservation of the total angular momentum $L^{\lambda\,\mu\nu} + S^{\lambda\,\mu\nu}$ then implies

$$T_N^{\mu\nu} - T_N^{\nu\mu} = -\partial_\lambda S^{\lambda\,\mu\nu}, \qquad S^{\lambda\,\mu\nu} = -i\frac{\partial\mathcal{L}}{\partial(\partial_\lambda A^\sigma)}\Sigma^{\mu\nu,\sigma}{}_\tau A^\tau = iF^{\lambda\sigma}\Sigma^{\mu\nu}_{\sigma\tau}A^\tau. \tag{F.4}$$

Here $\Sigma_{\mu\nu}$ are the Lorentz generators $\Sigma^{\mu\nu,\sigma}{}_\tau = i[\eta^{\mu\sigma}\delta^\nu_\tau - \eta^{\nu\sigma}\delta^\mu_\tau]$ that will help us identify the spin contribution inside the three-point amplitude. Imposing that the corrected tensor $T^{\mu\nu}$ be symmetric now yields the condition $\partial_\lambda B^{\lambda[\mu\,\nu]} = \frac{1}{2}\partial_\lambda S^{\lambda\,\mu\nu}$, which is solved by

$$B^{\lambda\mu\,\nu} = \frac{1}{2}\big[S^{\lambda\,\mu\nu} + S^{\mu\,\nu\lambda} - S^{\nu\,\lambda\mu}\big]. \tag{F.5}$$

Contracting the resulting energy-momentum tensor with a traceless symmetric graviton $h_{\mu\nu}$ and integrating by parts, we obtain the gravitational interaction vertex

$$-h_{\mu\nu}T^{\mu\nu} = h_{\mu\nu}F^{\mu\sigma}\partial^\nu A_\sigma - i(\partial_\lambda h_{\mu\nu})F^{\nu\sigma}\Sigma^{\lambda\mu}_{\sigma\tau}A^\tau, \tag{F.6}$$

where we suppress the coupling-constant factor $\kappa/2$. Its momentum-space version in the scattering amplitude gives the following contributions:

$$h_{\mu\nu}F^{\mu\sigma}\partial^\nu A_\sigma \;\to\; -(p_2\cdot\varepsilon_3)\big[(p_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2) - (p_1\cdot\varepsilon_2)(\varepsilon_1\cdot\varepsilon_3)\big] + (1\leftrightarrow 2), \tag{F.7a}$$

$$-i(\partial_\mu h_{\nu\rho})F^{\rho\sigma}\Sigma^{\mu\nu}_{\sigma\tau}A^\tau \;\to\; ip_{3\mu}\varepsilon_{3\nu}\big[(p_1\cdot\varepsilon_3)(\varepsilon_1\cdot\Sigma^{\mu\nu}\cdot\varepsilon_2) - (\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\Sigma^{\mu\nu}\cdot\varepsilon_2)\big] + (1\leftrightarrow 2). \tag{F.7b}$$

where the transverse polarization vectors ε_1 and ε_2 correspond to the massive spin-1 matter and two copies of ε_3 belong to the massless graviton. Putting the above terms together and using the three-point on-shell kinematic conditions $p_1\cdot p_3 = p_2\cdot p_3 = 0$, we obtain the amplitude

$$\mathcal{M}_3 = 2(p_1\cdot\varepsilon)\big[(p_1\cdot\varepsilon)(\varepsilon_1\cdot\varepsilon_2) - 2p_{3\mu}\varepsilon_{3\nu}\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}\big], \tag{F.8}$$

The second term in eq. (F.8) comes from $\varepsilon_1\cdot\Sigma^{\mu\nu}\cdot\varepsilon_2^\tau = 2i\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}$, which in appendix G we interpret as a spin expectation value, so it can be regarded as the spin contribution to the gravitational interaction.

Appendix G

Spin tensor for spin-1 matter

Here we construct the spin tensor for a massive spin-1 particle for the three-particle kinematics of section 4.2.1. The starting point is the one-particle expectation value of the angular-momentum operator in the quantum-mechanical sense:

$$S_p^{\mu\nu} = \frac{\langle p|\Sigma^{\mu\nu}|p\rangle}{\langle p|p\rangle} = \frac{\varepsilon^*_{p\sigma}\Sigma^{\mu\nu,\sigma}{}_\tau \varepsilon_p^\tau}{\varepsilon_p^* \cdot \varepsilon_p} = 2i\varepsilon_p^{*[\mu}\varepsilon_p^{\nu]}, \qquad \Sigma^{\mu\nu,\sigma}{}_\tau = i[\eta^{\mu\sigma}\delta^\nu_\tau - \eta^{\nu\sigma}\delta^\mu_\tau], \tag{G.1}$$

where for now we suppress the spin-projection/little-group labels of the states. We also used the Lorentz generators $\Sigma^{\mu\nu}$ in the vector representation. Due to the transversality of the both massive polarization vectors, $p \cdot \varepsilon_p = 0$, this spin tensor immediately satisfies the SSC (4.7).

Now a natural way to extend eq. (G.1) to the case of two different states (one incoming with momentum p_1 and one outgoing with p_2) is to introduce a generalized expectation value such that it gives one for a unit operator:

$$S_{12}^{\mu\nu} = \frac{\langle 2|\Sigma^{\mu\nu}|1\rangle}{\langle 2|1\rangle} = \frac{\varepsilon^*_{2\sigma}\Sigma^{\mu\nu,\sigma}{}_\tau \varepsilon_1^\tau}{\varepsilon_2^* \cdot \varepsilon_1} = \frac{2i\varepsilon_2^{*[\mu}\varepsilon_1^{\nu]}}{\varepsilon_2^* \cdot \varepsilon_1}. \tag{G.2}$$

Since in section 4.2 we consider all momenta incoming, we suppress the conjugation sign[1]

[1]The conjugation rule between the incoming and outgoing states in the massive spinor-helicity formalism amounts to lowering and raising the little-group indices, as indicated by the completeness relation in eq. (C.3). For instance, in the helicity basis [13, 199] of spinors for a massive momentum $p^\mu = (E, \vec{p}) =$

and rewrite the above as

$$S_{12}^{\mu\nu} = -2i\varepsilon_1^{[\mu}\varepsilon_2^{\nu]}/(\varepsilon_1\cdot\varepsilon_2), \tag{G.4}$$

which is the (normalized) angular momentum contribution obtained in appendix F from Noether's theorem. Now in a classical computation [238] it is desirable to consider a spin tensor that satisfies the spin supplementary condition (4.7). Although eq. (G.4) is a legitimate definition, it does not satisy the covariant SSC (4.7) with respect to the average momentum $p = (p_1 - p_2)/2$ of the massive particle before and after graviton emission:

$$p_\mu S_{12}^{\mu\nu} = -\frac{i}{2}\big((k\cdot\varepsilon_2)\varepsilon_1^\nu + (k\cdot\varepsilon_1)\varepsilon_2^\nu\big)/(\varepsilon_1\cdot\varepsilon_2) \neq 0, \tag{G.5}$$

where $k = -p_1 - p_2$ is the momentum transfer. However, the spin tensor is intrinsically ambiguous, as the separation between the orbital and intrinsic pieces of the total angular momentum is relativistically frame-dependent. In a classical setting, for instance, the reference point for the intrinsic angular momentum of a spatially extended body (as opposed to its overall orbital momentum about the origin) is at its center of mass, but it gets shifted by a frame change (see e.g. [230]). This ambiguity allows the spin tensor to be transformed as $S^{\mu\nu} \to S^{\mu\nu} + p^{[\mu}r^{\nu]}$, where the difference $p^{[\mu}r^{\nu]}$ for some vector r^ν accounts for the relative shift between $S^{\mu\nu}$ and $L^{\mu\nu} \sim p^{[\mu}\partial/\partial p_{\nu]}$. Adjusting r^ν to accommodate for the SSC (4.7), we obtain

$$S^{\mu\nu} = S_{12}^{\mu\nu} + \frac{2}{m^2}p_\lambda S_{12}^{\lambda[\mu}p^{\nu]} = -\frac{i}{\varepsilon_1\cdot\varepsilon_2}\bigg\{2\varepsilon_1^{[\mu}\varepsilon_2^{\nu]} - \frac{1}{m^2}p^{[\mu}\big((k\cdot\varepsilon_2)\varepsilon_1 + (k\cdot\varepsilon_1)\varepsilon_2\big)^{\nu]}\bigg\}, \tag{G.6}$$

where we have used that $p^2 = m^2$ for a null momentum transfer k. Finally, we note that in the classical limit $k \to 0$ we retrieve the spin tensor (G.4) as the covariant-SSC one.

$(E, P\hat{p})$, the one-particle spin quantization is explicitly

$$m\langle a^\mu\rangle_p^{ab} = \frac{1}{2m}\epsilon^{\mu\nu\lambda\rho}(\varepsilon_{pab}\cdot\Sigma_{\nu\lambda}\cdot\varepsilon_p^{ab})p_\rho = \begin{cases} s_p^\mu, & a=b=1, \\ 0, & a+b=3, \\ -s_p^\mu, & a=b=2, \end{cases} \qquad \begin{aligned} p^\mu &= (E,\vec{p}) = (E, P\hat{p}), \\ s_p^\mu &= \frac{1}{m}(P, E\hat{p}). \end{aligned} \tag{G.3}$$

Appendix H

Angular-momentum operator

Here we consider the total angular momentum

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \qquad L^{\text{pos.}}_{\mu\nu} = 2ix_{[\mu}\frac{\partial}{\partial x^{\nu]}} \tag{H.1}$$

in terms of the spinor-helicity variables. The starting point is the momentum-space form of the orbital piece

$$L_{\mu\nu} = 2ip_{[\mu}\frac{\partial}{\partial p^{\nu]}} = p_\sigma \Sigma_{\mu\nu,}{}^{\sigma}{}_{\tau}\frac{\partial}{\partial p_\tau}, \tag{H.2}$$

in which we encounter the Lorentz generators $\Sigma^{\mu\nu}$ again. Since $\Sigma_{\mu\nu,\sigma\tau}$ is antisymmetric in both pairs of indices, we notice the subtle difference in signs between the actions of the differential and algebraic operators, $L_{\mu\nu}p^\rho = -\Sigma_{\mu\nu,}{}^{\rho}{}_{\sigma}p^\sigma$, also valid for $J_{\mu\nu}$ below.

Massless Case

Let us warm up with the case of a massless $k^\mu = \langle k|\sigma^\mu|k]/2$. The spinorial version of the angular momentum (H.2) is [251]

$$J^{\mu\nu} = \left[\lambda^\alpha \sigma^{\mu\nu,}{}_{\alpha}{}^{\beta}\frac{\partial}{\partial\lambda^\beta} + \tilde{\lambda}_{\dot\alpha}\bar{\sigma}^{\mu\nu,\dot\alpha}{}_{\dot\beta}\frac{\partial}{\partial\tilde{\lambda}_{\dot\beta}}\right], \tag{H.3}$$

where the matrices

$$\sigma^{\mu\nu,}{}_{\alpha}{}^{\beta} = \frac{i}{4}\big(\sigma^{\mu}_{\alpha\dot\gamma}\bar\sigma^{\nu,\dot\gamma\beta} - \sigma^{\nu}_{\alpha\dot\gamma}\bar\sigma^{\mu,\dot\gamma\beta}\big), \qquad \bar\sigma^{\mu\nu,\dot\alpha}{}_{\dot\beta} = \frac{i}{4}\big(\bar\sigma^{\mu,\dot\alpha\gamma}\sigma^{\nu}_{\gamma\dot\beta} - \bar\sigma^{\nu,\dot\alpha\gamma}\sigma^{\mu}_{\gamma\dot\beta}\big) \tag{H.4}$$

are the left-handed and right-handed representations of the Lorentz-group algebra. Note that the spinor map $\{\lambda_\alpha, \tilde\lambda_{\dot\alpha}\} \to k^\mu$ is not invertible for massless particles, but we can still use the chain rule

$$\frac{\partial}{\partial\lambda^\alpha} = \frac{\partial k^\mu}{\partial\lambda^\alpha}\frac{\partial}{\partial k^\mu} = \frac{1}{2}\sigma^{\mu}_{\alpha\dot\beta}\tilde\lambda^{\dot\beta}\frac{\partial}{\partial k^\mu}, \qquad \frac{\partial}{\partial\tilde\lambda_{\dot\alpha}} = \frac{1}{2}\bar\sigma^{\mu,\dot\alpha\beta}\lambda_\beta\frac{\partial}{\partial k^\mu} \tag{H.5}$$

to check the consistency between eqs. (H.2) and (H.3). Namely, the action of spinorial generator on a function of momentum k^μ coincides with that of the vectorial one.

The generator (H.3), which can be more concisely written in spinor indices as

$$J_{\alpha\dot\alpha,\beta\dot\beta} = \sigma^{\mu}_{\alpha\dot\alpha}\sigma^{\nu}_{\beta\dot\beta}J_{\mu\nu} = 2i\left[\lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}}\epsilon_{\dot\alpha\dot\beta} + \epsilon_{\alpha\beta}\tilde\lambda_{(\dot\alpha}\frac{\partial}{\partial\tilde\lambda^{\dot\beta)}}\right], \tag{H.6}$$

has more information than its momentum-space counterpart, as it cares about the helicity of the massless particle. For instance, when we write the polarization tensors in terms of spinor-helicity variables,

$$\varepsilon^{+}_{\alpha\dot\alpha} = \sqrt{2}\,\frac{|r\rangle_\alpha[k|_{\dot\alpha}}{\langle r\,k\rangle}, \qquad \varepsilon^{-}_{\alpha\dot\alpha} = -\sqrt{2}\,\frac{|k\rangle_\alpha[r|_{\dot\alpha}}{[r\,k]}, \tag{H.7}$$

we do not regard them as functions of k^μ but rather of its spinors λ_α and $\tilde\lambda_{\dot\alpha}$. Of course, an integer spin should not by itself depend on the auxiliary spinors. Fortunately, we can show that the action of the differential operator (H.6) is precisely that of the algebraic generator $\Sigma_{\mu\nu}$, which constitutes the intrinsic angular momentum

$$(\varepsilon S^{\mu\nu})_\tau = \varepsilon_\sigma\Sigma^{\mu\nu,\sigma}{}_{\tau} = 2i\varepsilon^{[\mu}\delta^{\nu]}_{\tau} \qquad \Rightarrow \qquad (\varepsilon S_{\alpha\dot\alpha,\beta\dot\beta})_{\gamma\dot\gamma} = 2i[\varepsilon_{\alpha\dot\alpha}\epsilon_{\beta\gamma}\epsilon_{\dot\beta\dot\gamma} - \epsilon_{\alpha\gamma}\epsilon_{\dot\alpha\dot\gamma}\varepsilon_{\beta\dot\beta}]. \tag{H.8}$$

Specializing to the negative-helicity case for concreteness, we indeed find

$$J_{\alpha\dot\alpha,\beta\dot\beta}\varepsilon^{-}_{\gamma\dot\gamma} = (\varepsilon^{-}S_{\alpha\dot\alpha,\beta\dot\beta})_{\gamma\dot\gamma} + \frac{2\sqrt{2}i}{[q\,k]^2}\epsilon_{\alpha\beta}[q|_{\dot\alpha}[q|_{\dot\beta}|k\rangle_\gamma[k|_{\dot\gamma}. \tag{H.9}$$

Here the last term is a gauge contribution explicitly proportional to $k_{\gamma\dot\gamma}$, so it can be discarded in a physical amplitude.

Therefore, we conclude that the spinorial differential operator (H.6) incorporates both the orbital and intrinsic contributions, so it is the total angular-momentum operator.

Massive Case

It is direct to generalize the above discussion to massive momenta $p^\mu = \langle p^a|\sigma^\mu|p_a]/2$ [92]. The angular-momentum operator in the space of massive spinors $\{\lambda^a_\alpha, \tilde{\lambda}^b_{\dot\beta}\}$ is given by

$$J^{\mu\nu} = \left[\lambda^{\alpha a}\sigma^{\mu\nu,}{}_{\alpha}{}^{\beta}\frac{\partial}{\partial\lambda^{\beta a}} + \tilde{\lambda}^a_{\dot\alpha}\bar{\sigma}^{\mu\nu,\dot\alpha}{}_{\dot\beta}\frac{\partial}{\partial\tilde{\lambda}^a_{\dot\beta}}\right], \qquad J_{\alpha\dot\alpha,\beta\dot\beta} = 2i\left[\lambda^{\ a}_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)a}}\epsilon_{\dot\alpha\dot\beta} + \epsilon_{\alpha\beta}\tilde{\lambda}^{\ a}_{(\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}^{\dot\beta)a}}\right]. \tag{H.10}$$

This operator is by construction invariant under the little group SU(2). Using the chain rule

$$\frac{\partial}{\partial\lambda^{\alpha a}} = \frac{\partial p^\mu}{\partial\lambda^{\alpha a}}\frac{\partial}{\partial p^\mu} = \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}\tilde{\lambda}^{\dot\beta}_a\frac{\partial}{\partial p^\mu}, \qquad \frac{\partial}{\partial\tilde{\lambda}^a_{\dot\alpha}} = -\frac{1}{2}\bar{\sigma}^{\mu,\dot\alpha\beta}\lambda_{\beta a}\frac{\partial}{\partial p^\mu}, \tag{H.11}$$

it is again easy to check that the action on a function of $p_{\alpha\dot\beta} = \lambda^a_\alpha\epsilon_{ab}\tilde{\lambda}^b_{\dot\beta}$ is the same as that of eq. (H.2). Finally, the action on polarization tensors can be tested to be a Lorentz transformation. The spin-s tensors are parametrized in terms of massive spinor-helicity variables as

$$\varepsilon^{a_1\ldots a_{2s}}_{\alpha_1\dot\alpha_1\ldots\alpha_s\dot\alpha_s} = \frac{2^{s/2}}{m^s}\lambda^{(a_1}_{\alpha_1}\tilde{\lambda}^{a_2}_{\dot\alpha_1}\cdots\lambda^{a_{2s-1}}_{\alpha_s}\tilde{\lambda}^{a_{2s})}_{\dot\alpha_s}. \tag{H.12}$$

with an obvious extension by an additional factor of Dirac spinor [13, 199] for half-integer spins. Indeed, since $J^{\mu\nu}$ is a first-order differential operator, it distributes when acting on $\varepsilon^{a_1\cdots a_{2s}}$ and naturally expands into the left- and right-handed Lorentz generators:

$$\begin{aligned} J^{\mu\nu}\varepsilon^{a_1\ldots a_{2s}}_{\alpha_1\dot\alpha_1\ldots\alpha_s\dot\alpha_s} = \frac{2^{s/2}}{m^s}\bigg\{ & \left[\epsilon_{\alpha_1\beta}\left(\lambda^{\alpha(a_1}\sigma^{\mu\nu,}{}_{\alpha}{}^{\beta}\right)\right]\tilde{\lambda}^{a_2}_{\dot\alpha_1}\cdots\lambda^{a_{2s-1}}_{\alpha_s}\tilde{\lambda}^{a_{2s})}_{\dot\alpha_s} \\ & + \lambda^{(a_1}_{\alpha_1}\left[\tilde{\lambda}^{a_2}_{\dot\alpha}\bar{\sigma}^{\mu\nu,\dot\alpha}{}_{\dot\alpha_2}\right]\cdots\lambda^{a_{2s-1}}_{\alpha_s}\tilde{\lambda}^{a_{2s})}_{\dot\alpha_s} + \ldots\bigg\}. \end{aligned} \tag{H.13}$$

Appendix I

Symmetry Algebra

This appendix examines the group of redundancies of the odd-point scattering maps that preserves their polynomial form. This consists of a five-dimensional subalgebra of the full Lie algebra. We will examine this five-dimensional algebra now, and leave the analysis of the full algebra for the future. More concretely, we first fix two generators of $\mathrm{SL}(2,\mathbb{C})_\rho$ corresponding to dilations and special conformal transformations in a suitable way, and then show that the algebra of residual symmetries corresponds to the semidirect product $\mathrm{SL}(2,\mathbb{C})_\sigma \ltimes \mathbb{C}^2$.

It is instructive to start by analyzing the even-point symmetry group $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$ in this setup. For $n = 2m+2$ let us call the polynomials $\rho^{A,+}(z) = \varpi^A(z)$ and $\rho^{A,-}(z) = \vartheta^A(z)$, both of degree m. We consider transformations $(z, \sigma_i, \rho^{A,a}) \to (\hat{z}, \hat{\sigma}_i, \hat{\rho}^{A,a})$ such that

$$\frac{\hat{\varpi}^{[A}(\hat{z})\hat{\vartheta}^{B]}(\hat{z})}{\prod_{i=1}^{n}(\hat{z}-\hat{\sigma}_i)}d\hat{z} \;=\; \frac{\varpi^{[A}(z)\vartheta^{B]}(z)}{\prod_{i=1}^{n}(z-\sigma_i)}dz. \tag{I.1}$$

This contains the $\mathrm{SL}(2,\mathbb{C})_\rho$ transformations, which can be defined as the stability subgroup satisfying $\hat{z} = z$. Among these, let us consider only the shift:[1]

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})_\rho\,, \qquad e^{\alpha J}: \quad \hat{\varpi}(z) = \varpi(z) + \alpha\vartheta(z)\,, \qquad \hat{\vartheta}(z) = \vartheta(z). \tag{I.2}$$

The other two generators should be thought as fixed. For instance, consider the $\mathrm{SL}(2,\mathbb{C})_\sigma$

[1] In this section we will mostly suppress the $\mathrm{SU}^*(4)$ index, since it is not relevant to what follows.

scaling $\hat{z} = e^{\alpha} z$. This induces the following transformation on the polynomials:

$$e^{\alpha \ell_0}: \qquad \hat{\varpi}(z) = e^{p\alpha} \varpi(e^{-\alpha} z)\,, \qquad \hat{\vartheta}(z) = e^{q\alpha} \vartheta(e^{-\alpha} z), \tag{I.3}$$

with $p + q = n - 1$. Since the generator of $\mathrm{SL}(2, \mathbb{C})_\rho$ scaling is fixed, so are the values of p and q, which will be determined below. Similarly, for the shift $\hat{z} = z + \beta$ we find

$$e^{\beta \ell_{-1}}: \qquad \hat{\varpi}(z) = \varpi(z - \beta)\,, \qquad \hat{\vartheta}(z) = \vartheta(z - \beta). \tag{I.4}$$

The last generator is defined by $\ell_{+1} = \mathcal{I}\,\ell_{-1}\,\mathcal{I}$, where inversion $\mathcal{I}$ acts in the following way. Consider the transformation $\hat{z} = -1/z$. The polynomials should then transform as

$$\mathcal{I}: \qquad \hat{\varpi}(z) = z^m\, Y^{\frac{1}{2}}\, \varpi\left(-1/z\right)\,, \qquad \hat{\vartheta}(z) = z^m\, Y^{\frac{1}{2}}\, \vartheta\left(-1/z\right), \tag{I.5}$$

where $Y = \prod_{i=1}^{n} \sigma_i$. It is straightforward to check that $\mathcal{I}^2 = (-1)^m \mathbb{1}$. The minus sign can be neglected since we are only interested in a representation of $\mathrm{PSL}(2, \mathbb{C})_\sigma$, which corresponds to the Möbius transformations acting on the punctures, for which we have the $\mathbb{Z}_2$ identification $-\mathbb{1} \cong \mathbb{1}$. Let us consider the action of the following composition

$$\begin{aligned}
\mathcal{I}\, e^{\alpha \ell_0}\, \mathcal{I}(\varpi(z)) &= \mathcal{I}\, e^{\alpha \ell_0} \left(z^m\, Y^{\frac{1}{2}}(\sigma_i)\, \varpi\left(-1/z\right) \right) \\
&= \mathcal{I} \left(e^{p\alpha} e^{-\alpha m} e^{\frac{-\alpha n}{2}}\, z^m\, Y^{\frac{1}{2}}(\sigma_i)\, \varpi\left(-e^{\alpha}/z\right) \right) \\
&\cong e^{p\alpha} e^{-\alpha m} e^{\frac{-\alpha n}{2}}\, \varpi(e^{\alpha} z),
\end{aligned} \tag{I.6}$$

where the symbol $\cong$ indicates we have used the $\mathbb{Z}_2$ identification. Imposing $\mathcal{I}\,\ell_0\,\mathcal{I} = -\ell_0$ we find:

$$-p + m + \frac{2m + 2}{2} = p \qquad \Longrightarrow \qquad p = q = m + \frac{1}{2}, \tag{I.7}$$

which coincides with the choice of [149]. The analysis for $\vartheta(z)$ is identical. It then follows that

$$e^{\alpha J} e^{\beta \ell_0} \begin{pmatrix} \varpi(z) \\ \vartheta(z) \end{pmatrix} = e^{\beta \ell_0} e^{\alpha J} \begin{pmatrix} \varpi(z) \\ \vartheta(z) \end{pmatrix}, \tag{I.8}$$

or equivalently, $[J, \ell_0] = 0$. We also have

$$\mathcal{I}\, e^{\alpha J}\, \mathcal{I}(\varpi(z)) = \mathcal{I} \left[z^m\, Y^{\frac{1}{2}} \left(\varpi\left(-1/z\right) + \alpha \vartheta\left(-1/z\right) \right) \right]$$

$$
\begin{aligned}
&\cong \varpi(z) + \alpha\vartheta(z) \\
&= e^{\alpha J}\varpi(z),
\end{aligned} \tag{I.9}
$$

which gives $\mathcal{I}\, J\, \mathcal{I} = J$ or $[\mathcal{I}, J] = 0$. This analysis is consistent with the fact that we are considering the subalgebra $\mathrm{SL}(2,\mathbb{C})_\sigma \times J$ of the direct product $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho$ and $\mathcal{I}$ belongs to the first group.

Let us now examine how this situation changes when considering the odd-point maps with $n = 2m+1$. Now, we fix the generators of $\mathrm{SL}(2,\mathbb{C})_\rho$ such that $\deg \varpi(z) = m$ and $\deg \vartheta(z) = m-1$. Note that this is consistent with the fact that J is a residual symmetry. In fact, the actions of J, ℓ_0 and ℓ_{-1} are not modified, even though the values of p, q differ, as we will show below. The inversion $\mathcal{I}$ now acts as

$$
\mathcal{I}: \qquad \hat{\varpi}(z) = z^m\, Y^{\frac{1}{2}}\, \varpi\left(-1/z\right), \qquad \hat{\vartheta}(z) = z^{m-1} Y^{\frac{1}{2}}\, \vartheta\left(-1/z\right). \tag{I.10}
$$

Repeating the computation in (I.6) we find that

$$
-p + m + \frac{2m+1}{2} = p \qquad \Longrightarrow \qquad \begin{cases} p = m + \frac{1}{4}, \\ q = m - \frac{1}{4}. \end{cases} \tag{I.11}
$$

Furthermore, we have:

$$
\begin{aligned}
e^{\alpha J} e^{\beta \ell_0} \begin{pmatrix} \varpi(z) \\ \vartheta(z) \end{pmatrix} &= \begin{pmatrix} e^{(m+1/4)\beta}\varpi(e^{-\beta}z) + \alpha e^{(m-1/4)\beta}\vartheta(e^{-\beta}z) \\ e^{(m-1/4)\beta}\vartheta(e^{-\beta}z) \end{pmatrix} \\
&= \begin{pmatrix} e^{(m+1/4)\beta}\left(\varpi(e^{-\beta}z) + \tilde{\alpha}\vartheta(e^{-\beta}z)\right) \\ e^{(m-1/4)\beta}\vartheta(e^{-\beta}z) \end{pmatrix} \\
&= e^{\beta\ell_0} e^{\tilde{\alpha} J} \begin{pmatrix} \varpi(z) \\ \vartheta(z) \end{pmatrix},
\end{aligned} \tag{I.12}
$$

where $\tilde{\alpha} := \alpha e^{-\beta/2}$. This means that $[J, \ell_0] = -\frac{1}{2}J$.

In contrast to the case of even n, we have shown that for odd n the group structure is a semidirect extension of $\mathrm{SL}(2,\mathbb{C})_\sigma$ by an (Abelian) shift factor J. In other words, the Riemann sphere symmetry group $\mathrm{SL}(2,\mathbb{C})_\sigma$ and the group $\mathrm{SL}(2,\mathbb{C})_\rho$ are intertwined. Moreover, we will now show that the J extension of $\mathrm{SL}(2,\mathbb{C})_\sigma$ is not enough to achieve closure of the group. In fact, consider

$$
\mathcal{I}\, e^{\alpha J}\, \mathcal{I}(\varpi(z)) = \mathcal{I}\left[z^m\, Y^{\frac{1}{2}}\, \varpi\left(-1/z\right) + \alpha z^{m-1} Y^{\frac{1}{2}} \vartheta\left(-1/z\right)\right]
$$

$$
\begin{aligned}
&= \mathcal{I}\left[z^m\, Y^{\frac{1}{2}}\, \varpi_{(\alpha)}(-1/z)\right] \\
&\cong \varpi_{(\alpha)}(z),
\end{aligned} \tag{I.13}
$$

where we have defined the polynomial

$$
\varpi_{(\alpha)}(z) := \varpi(z) - \alpha z \vartheta(z) = e^{\alpha T}\varpi(z). \tag{I.14}
$$

This shows that conjugating the shift J by an inversion leads to a new shift symmetry not present in the even-n case: $\mathcal{I}J\mathcal{I} = -T$. This precisely corresponds to the T-shift symmetry, introduced previously, acting on the fixed frame with $\xi = (1, 0)$. Conjugating the equation $[J, \ell_0] = -\frac{1}{2}J$ we find:

$$
[T, \ell_0] = \frac{1}{2}T. \tag{I.15}
$$

Because J and T are Abelian shifts it follows that $[J, T] = 0$, i.e., they generate the translation group $\mathbb{C}^2$ and transform as a doublet under $\mathrm{SL}(2, \mathbb{C})_\sigma$. The rest of the $\mathrm{SL}(2, \mathbb{C})_\sigma \ltimes \mathbb{C}^2$ algebra is

$$
[\ell_1, T] = [\ell_{-1}, J] = 0, \qquad [\ell_{-1}, T] = -J, \qquad [\ell_1, J] = T, \qquad [\ell_i, \ell_j] = (i-j)\ell_{i+j}. \tag{I.16}
$$

More succinctly, if we define $(J, T) = (T_{-1/2}, T_{1/2})$, then we have $[T_r, T_s] = 0$ and

$$
[\ell_i, T_r] = \left(\frac{i}{2} - r\right) T_{i+r}, \quad i = -1, 0, 1 \quad r = \pm 1/2, \tag{I.17}
$$

as well as

$$
\mathcal{I}\, l_i\, \mathcal{I}^{-1} = -l_{-i} \quad i = -1, 0, 1 \quad \text{and} \quad \mathcal{I}\, T_r\, \mathcal{I}^{-1} = -T_{-r} \quad r = \pm 1/2. \tag{I.18}
$$

Finally, one can directly check that the remaining generators of $\mathrm{SL}(2, \mathbb{C})_\rho$ do not preserve the polynomial form of the maps. Hence we claim that this is the maximal subalgebra associated to polynomial maps.

Details of the Soft-Limit Calculations

In this appendix we analyze the soft limit of the connected formula, treating the measure and the integrand separately. Because of its simplicity, we start in I with the soft limit of the CHY measure and the deformation of the maps in 4D. In J we turn to the even-n measure for 6D, where several new technical ingredients appear due to the $\mathrm{SL}(2,\mathbb{C})$ little-group structure. This analysis allows us to recover the form of the odd-point maps and measure as well as the emergent symmetry T discussed in appendix A. In J the odd-point integrand is derived from the even-point one for the case of $\mathcal{N}=(1,1)$ SYM. Finally, in J we obtain the even-n measure from the odd-n one, and use it to prove that the constraints have $(n-3)!$ solutions.

Four Dimensions

Let us illustrate the use of the soft limit by considering the simpler 4D case first. Here we will focus on the CHY measure in the Witten–RSV form and show that it has the form given in (5.9). In particular, we consider the measure associated to the dth sector and show that in the soft limit, where $p_{n+1}=\tau\,\hat{p}_{n+1}$ with $\tau\to 0$, we have

$$\int d\mu^{\text{4D}}_{n+1,d}=\delta(p_{n+1}^2)\int d\mu^{\text{4D}}_{n,d-1}\frac{1}{2\pi i}\oint_{[\tilde{\lambda}_{n+1},\tilde{\rho}(\sigma_{n+1})]=0}\frac{d\sigma_{n+1}}{E_{n+1}}+\text{conj.}+\mathcal{O}(\tau^0), \tag{I.19}$$

where the scattering equation for the soft particle takes the form

$$E_{n+1}=\sum_{i=1}^{n}\frac{p_{n+1}\cdot p_i}{\sigma_{n+1,i}}=\frac{\langle\lambda_{n+1}\,\rho(\sigma_{n+1})\rangle[\tilde{\lambda}_{n+1}\,\tilde{\rho}(\sigma_{n+1})]}{\prod_{i=1}^{n}\sigma_{n+1,i}} \tag{I.20}$$

In (I.19) "conj." means to consider the first term with the conjugated contour $[\tilde{\lambda}_{n+1}\,\tilde{\rho}(\sigma_{n+1})]\to\langle\lambda_{n+1}\,\rho(\sigma_{n+1})\rangle$ as well as conjugated sector $d\to\tilde{d}=n-2-d$. By summing (I.19) over all sectors we obtain (5.9).

Let us now consider the soft limit of

$$\int d\mu^{\text{4D}}_{n+1,d}=\int\frac{\prod_{i=1}^{n+1}d\sigma_i\prod_{k=0}^{d}d^2\rho_k\prod_{k=0}^{\tilde{d}}d^2\tilde{\rho}_k}{\text{vol }\mathrm{SL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})}\frac{1}{R^d(\rho)R^{\tilde{d}}(\tilde{\rho})}\prod_{i=1}^{n+1}\delta^4\left(p_i^{\alpha\dot{\alpha}}-\frac{\rho^{\alpha}(\sigma_i)\tilde{\rho}^{\dot{\alpha}}(\sigma_i)}{\prod_{j\neq i}^{n+1}\sigma_{ij}}\right). \tag{I.21}$$

The strategy is to first isolate the leading $1/\tau$ factor, which in this case comes from the resultants. As we will show, in the soft limit $R^d(\rho)R^{\tilde{d}}(\tilde{\rho})\sim\tau$, which allows us to evaluate the rest of the measure for $\tau=0$ (except for the factor $\delta(p_{n+1}^2)$). What makes the case of 4D simple is that $p_{n+1}\to 0$ has only two solutions: $\lambda_{n+1}\to 0$ or $\tilde{\lambda}_{n+1}\to 0$, which account for the two terms in (I.19). Choosing $\lambda_{n+1}\to 0$, the delta function for the last particle in

(I.21) takes the form:

$$\delta^4\left(p^{\alpha\dot\alpha}_{n+1}-\frac{\rho^\alpha(\sigma_{n+1})\tilde\rho^{\dot\alpha}(\sigma_{n+1})}{\prod_{i=1}^n\sigma_{n+1,i}}\right)\to\int dt\,d\tilde t\,\delta^2(\tilde\lambda_{n+1}-\tilde t\tilde\rho(\sigma_{n+1}))\,\delta^2(t\rho(\sigma_{n+1}))\,\delta\left(t\,\tilde t-\frac{1}{\prod_{i=1}^n\sigma_{n+1,i}}\right)$$
$$=\left(\prod_{i=1}^n\sigma_{n+1,i}\right)^2\int\tilde t d\tilde t\,\delta^2(\tilde\lambda_{n+1}-\tilde t\tilde\rho(\sigma_{n+1}))\delta^2(\rho(\sigma_{n+1})),\qquad\text{(I.22)}$$

where we have used (5.18) and dropped the factor $\delta(p^2_{n+1})$. If we now introduce a reference spinor $|q]$, we can recast the result in the form

$$\left(\prod_{i=1}^n\sigma_{n+1,i}\right)^2\int\tilde t d\tilde t\,\delta\left(\tilde t-\frac{[\tilde\lambda_{n+1}\,q]}{[\tilde\rho(\sigma_{n+1})\,q]}\right)\delta\left([\tilde\lambda_{n+1}\,\tilde\rho(\sigma_{n+1})]\right)\delta^2(\rho(\sigma_{n+1}))$$
$$=\left(\prod_{i=1}^n\sigma_{n+1,i}\right)\frac{1}{t}\,\delta\left([\tilde\lambda_{n+1}\,\tilde\rho(\sigma_{n+1})]\right)\delta^2(\rho(\sigma_{n+1})),\qquad\text{(I.23)}$$

where now

$$t=\frac{1}{\prod_{i=1}^n\sigma_{n+1,i}}\frac{[\tilde\rho(\sigma_{n+1})\,q]}{[\tilde\lambda_{n+1}\,q]}.\qquad\text{(I.24)}$$

The first constraint is a polynomial equation of degree $n-d$ in σ_{n+1}, which we used for the contour in (I.19). To manipulate the second constraint let us reparametrize the polynomial as

$$\rho^\alpha(z)=\hat\rho^\alpha(z)(z-\sigma_{n+1})+r^\alpha.\qquad\text{(I.25)}$$

Here $\hat\rho^\alpha(z)=\sum_{k=0}^{d-1}\hat\rho^\alpha_k z^k$ is a polynomial of degree $d-1$, whose coefficients are shifted from those of $\rho^\alpha(z)$. Therefore the Jacobian is one, i.e.,

$$\prod_{k=0}^d d^2\rho_k=d^2r\prod_{k=0}^{d-1}d^2\hat\rho_k\qquad\text{(I.26)}$$

Integration over r^α eliminates the second delta function in (I.23), since

$$\int d^2r\delta^2(\rho(\sigma_{n+1}))=1\,,\qquad\text{(I.27)}$$

setting $r = 0$, i.e., $\rho^\alpha(z) = \hat{\rho}^\alpha(z)(z - \sigma_{n+1})$. Putting everything together, (I.21) becomes

$$\int \frac{\prod_i^n d\sigma_i \prod_{k=0}^{d-1} d^2\hat{\rho}_k \prod_{k=0}^{\tilde{d}} d^2\tilde{\rho}_k}{\text{vol SL}(2,\mathbb{C}) \times \text{GL}(1,\mathbb{C})} \prod_{i=1}^{n} \delta^4 \left(p_i^{\alpha\dot{\alpha}} - \frac{\hat{\rho}^\alpha(\sigma_i)\, \tilde{\rho}^{\dot{\alpha}}(\sigma_i)}{\prod_{j\neq i}^n \sigma_{ij}} \right)$$
$$\times \frac{1}{2\pi i} \oint_{[\tilde{\lambda}_{n+1}\ \tilde{\rho}(\sigma_{n+1})]=0} \frac{d\sigma_{n+1}}{\left(\frac{[\tilde{\lambda}_{n+1}\ \tilde{\rho}(\sigma_{n+1})]}{\prod_{i=1}^n \sigma_{n+1,i}} \right)} \left(\frac{1}{t\, R^d(\rho) R^{\tilde{d}}(\tilde{\rho})} \right) + \mathcal{O}(\tau^0). \tag{I.28}$$

Note that in the bosonic delta functions the puncture σ_{n+1} has completely dropped thanks to the definition of $\hat{\rho}$. We will not prove it here, but using the definition (5.21) in terms of the matrices Φ_d and $\tilde{\Phi}_{\tilde{d}}$ one can show that in the soft limit the resultants behave as

$$t\, R^d(\rho) R^{\tilde{d}}(\tilde{\rho}) = \langle \lambda_{n+1}\, \rho(\sigma_{n+1}) \rangle R^{d-1}(\hat{\rho}) R^{\tilde{d}}(\tilde{\rho}) + \mathcal{O}(\tau^2) \tag{I.29}$$

where $\lambda_{n+1} = \mathcal{O}(\tau)$ is responsible for the leading behaviour, as anticipated. This concludes the proof of (I.19). The extension of this procedure to the integrand in (5.20), including the redefinition of the fermionic maps, is straightforward in 4D, but we do not present it here. After including the integrand one can deform the contour for σ_{n+1} such that it encloses two of the other punctures, i.e., at $\sigma_{n+1} = \sigma_i$. This leads to the soft limit of the $\mathcal{N} = 4$ SYM amplitude.

Appendix J

From Even to Odd Multiplicity in 6D

Let us now consider the case of n odd in 6D. We show that the expression (7.20) can be obtained from the soft limit of the $n+1 = 2m+2$ measure after extracting the corresponding wave function and scattering equation. That is,

$$\int d\mu_{2m+2}^{\text{6D}} = \delta(p_{n+1}^2) \int d\mu_{2m+1}^{\text{6D}} \frac{1}{2\pi i} \oint_{|\hat{E}_{n+1}|=\varepsilon} \frac{d\sigma_{n+1}}{E_{n+1}} + \mathcal{O}(\tau^0), \tag{J.1}$$

where

$$d\mu_{2m+1}^{\text{6D}} = \frac{\left(\prod_{i=1}^{n} d\sigma_i\right) \left(\prod_{k=0}^{m-1} d^8\rho_k\right) \, d^4\omega \, \langle \xi d\xi \rangle}{\text{vol } (\text{SL}(2,\mathbb{C})_\sigma, \text{SL}(2,\mathbb{C})_\rho, \text{T})} \frac{1}{V_n^2} \Delta_B. \tag{J.2}$$

The maps entering the bosonic delta functions Δ_B are defined in (5.39). As in 4D, the strategy is to first isolate the τ^{-1} piece and then manipulate the delta function for particle $n+1$ to get the corresponding scattering equation. In Section J we achieve the first goal by proving that if $\hat{p}_{n+1}^{AB} = v^{[A}q^{B]}$ is the direction of the soft momentum, where $p_{n+1} = \tau \, \hat{p}_{n+1}$, then

$$\begin{aligned} \int d\mu_{2m+2}^{\text{6D}} = \frac{1}{\tau} \delta(p_{n+1}^2) \int \frac{d\sigma_{n+1} \, \left(\prod_{i=1}^{n} d\sigma_i\right) \left(\prod_{k=0}^{m} d^8\rho_k\right)}{\text{vol}\,(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_\rho)\, V_n^2} \Delta_B^{(n)} \\ \times \left(\prod_{i=1}^{n} \sigma_{n+1,i} \right) \int dx \, d^2\Xi \; \delta^8 \left(\rho_a^A(\sigma_{n+1}) - \Xi_a(q^A + x\, v^A) \right) \; + \mathcal{O}(\tau^0). \end{aligned} \tag{J.3}$$

Here $\Delta_B^{(n)}$ contains the bosonic delta functions for the n hard particles, but still depends on σ_{n+1} and the even-multiplicity maps. Since the leading power of τ has been extracted in this expression, the integral can be evaluated for $\tau = 0$. Note that this expression is invariant under little-group transformations of the soft particle. In fact, the $\mathrm{SL}(2,\mathbb{C})_\rho$ transformation

$$q \;\to\; Dq + Bv \tag{J.4}$$
$$v \;\to\; Cq + Av \tag{J.5}$$

with $AD - BC = 1$ is equivalent to the following change of variables

$$x \;\to\; \hat{x} \;=\; \frac{Ax+B}{Cx+D}, \tag{J.6}$$
$$\Xi_a \;\to\; \hat{\Xi}_a \;=\; \Xi_a(Cx+D), \tag{J.7}$$

which leaves the measure invariant, i.e., $dx\, d^2\Xi = d\hat{x}\, d^2\hat{\Xi}$. The reason for introducing the variables x and Ξ will become clear in the following section. In Section J we redefine the maps and isolate the scattering equation as a contour prescription for the puncture σ_{n+1} associated to the soft particle, leading to (J.1).

Derivation of (J.3)

We start with the following identity

$$\Delta_B^{(n+1)} = \Delta_B^{(n)}\,\delta(p_{n+1}^2)\int d^4M|M|^3\delta^8\left(\rho^{Aa}(\sigma_{n+1}) - M_b^a\lambda_n^{Ab}\right)\delta\!\left(|M| - \prod_{i=1}^{n}\sigma_{n+1,i}\right), \tag{J.8}$$

where we have utilized the linear constraints in (5.41), and denoted $M = M_{n+1}$. Now, up to linear order in τ, the most general form of the soft momenta can be written as

$$\lambda_{n+1}^{Aa} = \beta^a v^A + \tau q^{Aa} \tag{J.9}$$

which gives $p_{n+1}^{AB} = \tau v^{[A}q^{B]} + \mathcal{O}(\tau^2)$, once we set $q^A := \beta_a q^{Aa}$. Unlike 4D, where the soft condition $p_{n+1} \to 0$ has only two branches (the holomorphic and antiholomorphic soft limits,) here we have a family of solutions due to the less trivial $\mathrm{SL}(2,\mathbb{C})$ structure. Let us now assume that as $\tau \to 0$ all the components of the maps $\rho^{Aa}(z)$ and the σ_i's stay finite, as determined by the delta functions Δ_B, since they should be localized by the equations of the hard particles.

In the limit $\tau \to 0$, the matrix M has a singular piece:

$$M = \frac{\overline{M}}{\tau} + M_0 + \mathcal{O}(\tau^1). \tag{J.10}$$

The strategy is to input this ansatz into the delta functions and evaluate the result power by power in τ leaving only four components of M to be integrated. That is, impose

$$\rho^{Ab}(\sigma_{n+1}) = \left(\frac{\overline{M}_a^b}{\tau} + M_{0,a}^b\right)\left(\beta^a v^A + \tau q^{Aa}\right) \tag{J.11}$$

$$= \frac{\overline{M}_a^b \beta^a}{\tau} v^A + M_{0,a}^b \beta^a v^A + \overline{M}_a^b q^{Aa} + \mathcal{O}(\tau^1)\,, \tag{J.12}$$

$$\prod_{i=1}^{n} \sigma_{n+1,i} = |M| = \frac{1}{\tau^2}|\overline{M}| + \frac{\langle \overline{M}^+\, M_0^- \rangle - \langle \overline{M}^-\, M_0^+ \rangle}{\tau} + |M_0|\,. \tag{J.13}$$

Here $\overline{M}^+, \overline{M}^-, M_0^+, M_0^-$ denote the respective columns of the matrices $\overline{M}$ and M_0. From the finiteness of the LHS of (J.12) and (J.13), we see that $\overline{M}$ is degenerate and β is a null eigenvector, that is

$$\overline{M}_a^b = \Xi^b\, \beta_a\,. \tag{J.14}$$

Equating terms at order τ^{-1},

$$0 = \langle \overline{M}^+\, M_0^- \rangle - \langle \overline{M}^-\, M_0^+ \rangle = \langle \beta\, \Xi_a M_0^a \rangle \Longrightarrow \Xi_a M_{0,b}^a \beta^b = 0\,. \tag{J.15}$$

This result allows to introduce variables x and $\overline{x}$ defined by

$$M_{0,b}^a\, \beta^b = x\, \Xi^a, \qquad \Xi_a\, M_{0,b}^a = \overline{x}\, \beta_b\,. \tag{J.16}$$

The general solution of these equations for M_0 can be expressed in the basis of spinors β and Ξ as

$$M_{0,b}^a = \frac{\overline{x}\, \beta^a \beta_b + x\, \Xi^a \Xi_b}{\langle \Xi \beta \rangle} + \gamma\, \Xi^a \beta_b, \tag{J.17}$$

and thus

$$M_b^a = \frac{\overline{x}\, \beta^a \beta_b + x\, \Xi^a \Xi_b}{\langle \Xi \beta \rangle} + \left(\gamma + \frac{1}{\tau}\right) \Xi^a \beta_b\,. \tag{J.18}$$

The component γ is a fixed constant, which can only be determined by considering subleading orders in τ. This is consistent since it only contributes to the result at order $O(\tau^1)$. In fact, choosing the change of variables $\{M^a_b\} \to \{x, \overline{x}, \Xi^+, \Xi^-\}$, we find

$$d^4M = x\left(\frac{1+\gamma\tau}{\tau}\right) dx\, d\overline{x}\, d^2\Xi \sim \frac{x}{\tau} dx\, d\overline{x}\, d^2\Xi. \tag{J.19}$$

Having identified the singular dependence on τ, we can now select the leading pieces of the arguments inside the delta functions, yielding

$$\delta\left(|M| - \prod_{i=1}^{n} \sigma_{n+1,i}\right) = \delta\left(x\overline{x} - \prod_{i=1}^{n} \sigma_{n+1,i}\right), \tag{J.20}$$

$$\delta^8\left(\rho^{Ab}(\sigma_n) - M^b_a \lambda^{Aa}_n\right) = \delta^8\left(\rho^{Ab}(\sigma_n) - \Xi^b(x\, v^A + q^A)\right). \tag{J.21}$$

Integrating out $\overline{x}$, writing $V^2_{n+1} = V^2_n \prod_{i=1}^n \sigma^2_{i,\,n+1}$, and substituting in the identity (J.8), we finally arrive at the desired result (J.3).

Derivation of (J.1)

In this section we consider the expression (J.3) without the integration over Ξ^a, i.e., taking Ξ^a to be a fixed spinor. We will also introduce an auxiliary spinor ξ such that $\langle \Xi\, \xi \rangle = 1$. Note that ξ still has one free component, which we choose to be $\xi^+ = 1$. The integration over Ξ^a will be restored later.

We start by expanding the polynomial maps in basis vectors as

$$\rho^{A,a}(z) = \Xi^a\, \omega^A(z) + \xi^a\, \pi^A(z), \tag{J.22}$$

the delta functions of (J.3) as

$$\delta^8\left(\rho^{Ab}(\sigma_{n+1}) - \Xi^b(x\, v^A + q^A)\right) = \delta^4\left(\pi^A(\sigma_{n+1})\right) \delta^4\left(\omega^A(\sigma_{n+1}) - x\, v^A - q^A\right), \tag{J.23}$$

$$\Delta^{(n)}_B = \prod_{i=1}^{n} \delta^6\left(p^{AB}_i - \frac{\omega^{[A}(\sigma_i)\pi^{B]}(\sigma_i)}{\prod_{j\neq i}^{n+1} \sigma_{ji}}\right), \tag{J.24}$$

and the integration measure as

$$\prod_{k=0}^{m} d^8\rho_k = \prod_{k=0}^{m} d^4\omega_k\, d^4\pi_k\,. \tag{J.25}$$

As in 4D, we now parametrize $\pi^A(z) = (z - \sigma_{n+1})\hat{\pi}^A(z) + r^A$, so that the first term vanishes at the last puncture. This change of variables gives,

$$\prod_{k=0}^{m} d^4\pi_k = d^4 r \prod_{k=0}^{m-1} d^4\hat{\pi}_k\,, \tag{J.26}$$

$$\delta^4(\pi^A(\sigma_{n+1})) = \delta^4(r^A)\,. \tag{J.27}$$

On the support of the first delta function,

$$\Delta_B^{(n)}\Big|_{r^A=0} = \prod_{i=1}^{n} \delta^6\left(p_i^{AB} - \frac{\omega^{[A}(\sigma_i)\hat{\pi}^{B]}(\sigma_i)}{\prod_{j\neq i}^{n}\sigma_{ij}}\right) =: \Delta_B^{(n)}(\omega,\hat{\pi}). \tag{J.28}$$

Note that this result does not depend on σ_{n+1}.

The leading-order term in (J.3) can be rewritten in the form

$$\begin{aligned}\frac{\delta(p_{n+1}^2)}{\tau}\int d^2\Xi \int \frac{\prod_{k=0}^{m} d^4\omega_k \prod_{k=0}^{m-1} d^4\hat{\pi}_k \prod_{i=1}^{n} d\sigma_i}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho)\, V_n^{\,2}} \times \Delta_B^{(n)}(\omega,\hat{\pi}) \\ \times \int d\sigma_{n+1}\, dx\, \delta^4\left(\omega^A(\sigma_{n+1}) - x\, v^A - q^A\right)\left(\prod_{i=1}^{n}\sigma_{n+1,i}\right).\end{aligned} \tag{J.29}$$

The integration over $\int d^2\Xi$ has effectively dropped out of the integral. In principle we could use it to cancel two of the integrations over $\mathrm{SL}(2,\mathbb{C})^2$ in the denominator. However, this would fix part of the $\mathrm{SL}(2,\mathbb{C})^2$ invariance, which we want to be present in the odd version of the measure. Instead, let us reintroduce the integration to get a manifestly symmetric answer. To achieve this we revert to the change of basis (J.22), i.e., for fixed $\{\Xi,\xi\}$ we define

$$\hat{\rho}^{A,a}(z) = \xi^a\,\omega^A(z) - \Xi^a\,\hat{\pi}^A(z). \tag{J.30}$$

This transformation is defined coefficient by coefficient as an $\mathrm{SL}(2,\mathbb{C})$ transformation except for the top one, which is not invertible. In fact,

$$d^4\omega_m \prod_{k=0}^{m-1} d^4\omega_k\, d^4\hat{\pi}_k = d^4\omega_m \prod_{k=0}^{m-1} d^8\hat{\rho}_k \tag{J.31}$$

and

$$\Delta_B^{(n)}(\omega,\hat{\pi}) = \Delta_B^{(n)}(\hat{\rho}) = \prod_{i=1}^{n} \delta^6\left(p_i^{AB} - \frac{\langle \hat{\rho}^A(\sigma_i)\,\hat{\rho}^B(\sigma_i)\rangle}{\prod_{j\neq i}^{n}\sigma_{ij}}\right)\,, \tag{J.32}$$

where the highest coefficient of the map is given by

$$\hat{\rho}_m^{A,a} = \xi^a\,\omega_m^A = \begin{pmatrix} 1 \\ \xi \end{pmatrix}\omega_m^A, \tag{J.33}$$

with $\Xi^+\xi - \Xi^- = 1$. Noting that

$$\omega^A(\sigma_{n+1}) = \langle \Xi\,\hat{\rho}^A(\sigma_{n+1})\rangle = \Xi^+\hat{\rho}^{A,-}(\sigma_{n+1}) - \Xi^-\,\hat{\rho}^{A,+}(\sigma_{n+1})\,, \tag{J.34}$$

the integral becomes

$$\frac{\delta(p_{n+1}^2)}{\tau}\int \frac{d^4\omega_m\prod_{k=0}^{m-1} d^8\hat{\rho}_k\prod_{i=1}^{n} d\sigma_i}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma\times\mathrm{SL}(2,\mathbb{C})_\rho)\,V_n^2}\Delta_B^{(n)}(\hat{\rho})\int d\sigma_{n+1}\,d^2\Xi\,dx\,\delta^4\left(D^A\right)\left(\prod_{i=1}^{n}\sigma_{n+1,i}\right). \tag{J.35}$$

where

$$D^A = \Xi^+\hat{\rho}^{A,-}(\sigma_{n+1}) - \Xi^-\hat{\rho}^{A,+}(\sigma_{n+1}) - x\,v^A - q^A. \tag{J.36}$$

Now we note that

$$\left(\prod_{i=1}^{n}\sigma_{n+1,\,i}\right)\int d^2\Xi dx\,\delta^4\left(D^A\right) = \left(\prod_{i=1}^{n}\sigma_{n+1,\,i}\right)\,\delta\left(\langle\hat{\rho}^+(\sigma_{n+1})\hat{\rho}^-(\sigma_{n+1})\,v\,q\rangle\right) = \delta(\hat{E}_{n+1}). \tag{J.37}$$

In the last line we recognize the scattering equation for the soft particle (in a form analogous to (I.20)), which we now implement as a contour integral for σ_{n+1}. This gives

$$\frac{\delta(p_{n+1}^2)}{\tau}\int \frac{d^4\omega_m\prod_{k=0}^{m-1} d^8\hat{\rho}_k\prod_{i=1}^{n} d\sigma_i}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma\times\mathrm{SL}(2,\mathbb{C})_\rho)\;V_n^2}\,\frac{1}{2\pi i}\oint_{|\hat{E}_{n+1}|=\varepsilon}\frac{d\sigma_{n+1}}{\hat{E}_{n+1}}\,\Delta_B^{(n)}(\hat{\rho}). \tag{J.38}$$

We have arrived at a compact expression. However, there is subtle but essential caveat. Recall that $\Delta_B^{(n)}(\hat{\rho})$ contains the variable $\xi = \frac{1+\Xi^-}{\Xi^+}$ in the top component of the polynomial, $\hat{\rho}_m$. This variable still depends on the soft puncture σ_{n+1}. In fact, it is implicitly defined through the relation

$$\langle \Xi\, \hat{\rho}^A(\sigma_{n+1}) \rangle = x\, v^A + q^A\,. \tag{J.39}$$

In order to decouple ξ from this soft equation, we introduce a new redundancy that will enable us to turn ξ into an integration variable (which will be fixed by the hard data). Since v^A and q^A are only defined through $\hat{p}_{n+1}^{AB} = v^{[A}q^{B]}$, the formula must be invariant under $v \to \frac{v}{\alpha}$, $q \to \alpha q$. According to (J.39), such a transformation can be absorbed into a transformation of (Ξ^a, x, ξ) as follows:

$$x \to \frac{x}{\alpha^2}\,, \qquad \Xi^a \to \frac{\Xi^b}{\alpha}\,, \qquad \xi \to \xi + \frac{\alpha - 1}{\Xi^+} = \frac{\alpha + \Xi^-}{\Xi^+}\,. \tag{J.40}$$

Since α is arbitrary, we add an additional integration in the form

$$1 = \frac{\int \frac{d\alpha}{\Xi^+}}{\mathrm{vol(T)}} = \frac{\int d\xi}{\mathrm{vol(T)}}\,, \tag{J.41}$$

which should be regarded as a formal definition of the T-shift measure. Note that this is not $\mathrm{SL}(2,\mathbb{C})^2$ covariant, signaling that the Jacobian is sensitive to the $\mathrm{SL}(2,\mathbb{C})^2$ frame. Using this, we recast the formula as promised

$$\int d\mu_{2m+2}^{\mathrm{CHY}} \to \delta(p_{n+1}^2) \int \frac{d\xi d^4\omega \prod_{k=0}^{m-1} d^8\hat{\rho}_k \prod_{i=1}^{n} d\sigma_i}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T})} \frac{\Delta_B^{(n)}(\hat{\rho})}{V_n^2} \frac{1}{2\pi i} \oint_{|\hat{E}_{n+1}|=\varepsilon} \frac{d\sigma_{n+1}}{E_{n+1}}. \tag{J.42}$$

Some comments are in order. We have used the little-group scaling of the soft particle to introduce a new redundancy in the hard equations. As the notation makes clear, this redundancy can be identified with the shift transformation explored in Section 7. Note that this symmetry was absent in (J.38), which can be regarded as a T-fixed version of the final measure. The reason is that while $\Delta_B^{(n)}(\hat{\rho})$ is invariant under the shift $\hat{\rho}(z) \to \hat{\rho}(z) + z\beta\xi\langle\xi, \hat{\rho}\rangle$, equation (J.39) is not, meaning that the shift parameter β can be determined in terms of v and q. By averaging over the little group, i.e., over different choices of v and q, we unfix this redundancy.

Integrand of $\mathcal{N} = (1,1)$ SYM for Odd Multiplicity

Let us now apply the prescription given in the previous section, this time at the level of the $\mathcal{N} = (1,1)$ integrand. For $n+1 = 2m+2$, this integrand can be broken down as follows:

$$\mathcal{I}_{2m+2} = \mathrm{PT}(\mathbb{I}_{n+1})\, \mathrm{Pf}' A_{n+1}\, V_{n+1} \int \prod_{k=0}^{m} d^2\chi_k d^2\tilde{\chi}_k\, \Delta_F^{(n+1)} \widetilde{\Delta}_F^{(n+1)}$$
$$= \delta^2\Big(Q^A_{n+1}\tilde{\lambda}_{n+1,A,\hat{a}}\Big)\, \delta^2\Big(\lambda^A_{n+1,a}\tilde{Q}_{n+1,A}\Big)\ V_n\, \mathrm{PT}(\mathbb{I}_n)\, \mathrm{Pf}' A_{n+1} \left(\prod_{i=1}^{n} \sigma_{n+1,i}\right) \frac{\sigma_{1n}}{\sigma_{1,n+1}\sigma_{n+1,n}}$$
$$\times \int \prod_{k=0}^{m} d^2\chi_k d^2\tilde{\chi}_k\, \Delta_F^{(n)} \widetilde{\Delta}_F^{(n)} \delta^2\big(\eta^a_{n+1} - W^a_b \chi^b(\sigma_{n+1})\big)\, \delta^2\Big(\tilde{\eta}^{\hat{a}}_{n+1} - \widetilde{W}^{\hat{a}}_{\hat{b}} \tilde{\chi}^{\hat{b}}(\sigma_{n+1})\Big). \quad (\text{J.43})$$

Here $W = W_{n+1} = M^{-1}_{n+1}$, as defined in Section 8. The fermionic delta functions are defined in (6.14), from which we have extracted the on-shell conditions of the soft particle (recall that $Q^A = \lambda^A_a \eta^a$, etc.). We will first project out the $(n+1)$th gluon and then take the corresponding momentum to be soft. For a given polarization this will generate Weinberg's soft factor for the even point amplitude. In Section 7.2.1 we extract it to obtain the odd-point integrand.

A simple choice of polarization is $(a, \hat{a}) = (+, \hat{+})$, where the spinor in (J.9) and its conjugate are set to

$$\beta = \tilde{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{J.44})$$

We will proceed with this special choice, but the answer for a general polarization $(a, \hat{a})$ will be deduced at the end. For now, note that the soft factor (7.43) for this choice is

$$S^{+\hat{+}} = \frac{\tau^2\, [\tilde{q}|p_1 \tilde{p}_n|q\rangle}{\tau^2\, \hat{s}_{n+1,1} \hat{s}_{n+1,n}}, \quad (\text{J.45})$$

where we have explicitly exhibited the powers of τ. Since they cancel, and the measure in (J.1) contributes a power of τ^{-1}, we expect the integrand to be of order τ^1. In fact, the factor of τ comes from the expansion of the Pfaffian, i.e., $\mathrm{Pf}' A_{n+1} = \tau \widehat{\mathrm{Pf}'} A_{n+1}$. Now, to extract the aforementioned polarization from the amplitude we perform the following fermionic integration

$$\mathcal{I}^{+\hat{+}}_{2m+1} := \int d^4\eta_{n+1} d^4\tilde{\eta}_{n+1}\, \eta^1_{n+1} \tilde{\eta}^1_{n+1}\, \widehat{\mathcal{I}}_{2m+2}$$

$$= \tau\, V_n\, \mathrm{PT}(\mathbb{I}_n)\, \frac{\widehat{\mathrm{Pf}'} A_{n+1}}{\prod_{i=1}^n \sigma_{n+1,i}} \frac{\sigma_{1n}}{\sigma_{1,n+1}\sigma_{n+1,n}}$$
$$\times \int \prod_{k=0}^{m} d^2\chi_k d^2\tilde{\chi}_k \Delta_F^{(n)} \widetilde{\Delta}_F^{(n)} \delta\left(\overline{W}_a^{+} \chi^a(\sigma_{n+1})\right) \delta\left(\widetilde{\overline{W}}_{\hat{a}}^{\hat{+}} \tilde{\chi}^{\hat{a}}(\sigma_{n+1})\right), \qquad \text{(J.46)}$$

where $\widehat{\mathcal{I}}_{2m+2}$ corresponds to $\mathcal{I}_{2m+2}$ stripped of its on-shell delta functions. We also have $W = \overline{W}/(\prod_{i=1}^n \sigma_{n+1,i})$ with

$$\overline{W}_b^a = \epsilon^{ac}\epsilon_{bd} M_c^d = \frac{\overline{x}\, \beta^a \beta_b + x\, \Xi^a \Xi_b}{\langle \Xi \beta \rangle} + \left(\gamma + \frac{1}{\tau}\right) \beta^a \Xi_b \qquad \text{(J.47)}$$
$$\Rightarrow \overline{W}_a^{+} = \frac{x\, \Xi^+ \Xi_a}{\langle \Xi \beta \rangle} = x\, \Xi_a, \qquad \text{(J.48)}$$

using (J.18). Here we have implicitly followed all of the steps that were used in Section J to simplify the form of the W variables in the soft limit. The antichiral piece works in the same way. Even though $\widetilde{M}$ is not integrated, its behaviour in the soft limit allows us to define the antichiral counterparts $\tilde{\Xi}$ and $\tilde{x}$:

$$\widetilde{\overline{W}}_{\hat{a}}^{\hat{+}} = \epsilon_{\hat{a}\hat{b}}\ \widetilde{M}_{\hat{-}}^{\hat{b}} = \tilde{x}\, \tilde{\Xi}_{\hat{a}}. \qquad \text{(J.49)}$$

In direct correspondence to the bosonic case of Section J, we have managed to make explicit the τ dependence in the integrand, and therefore we can evaluate the delta functions $\Delta_F^{(n)} \widetilde{\Delta}_F^{(n)}$ for $\tau = 0$.

We follow now Section J, in which the basis element ξ was defined such that $\langle \xi \Xi \rangle = 1$ for a given Ξ^a. Then the polynomials are expanded as

$$\chi^a(z) = \xi^a l(z) + \Xi^a r(z) \qquad \text{(J.50)}$$
$$\tilde{\chi}^{\hat{a}}(z) = \tilde{\xi}^{\hat{a}} \tilde{l}(z) + \tilde{\Xi}^{\hat{a}} \tilde{r}(z), \qquad \text{(J.51)}$$

where $l(z)$ and $r(z)$ are degree-m polynomials with Grassmann coefficients. Dropping the powers of τ, we obtain

$$\mathcal{I}_{2m+1}^{+\hat{+}} = V_n\, \mathrm{PT}(\mathbb{I}_n)\, \frac{\widehat{\mathrm{Pf}'} A}{\prod_{i=1}^n \sigma_{n+1,i}} \frac{\sigma_{1n}}{\sigma_{1,n+1}\sigma_{n+1,n}}$$
$$\times \int \prod_{k=0}^{m} dl_k\, dr_k\, d\tilde{l}_k\, d\tilde{r}_k\, \Delta_F^{(n)} \widetilde{\Delta}_F^{(n)} \delta\left(l(\sigma_{n+1})\right) \delta\left(\tilde{l}(\sigma_{n+1})\right)\ x\, \tilde{x}\,. \qquad \text{(J.52)}$$

All of the following expressions for the integrand should be thought as multiplied by the measure, as we continue to parallel the manipulations of Section J. Now we put $l(z) = (z - \sigma_{n+1})\hat{l}(z) + b$, and we note that the fermionic delta functions fix $b = 0$ in the same way as the bosonic delta functions fixed $r^A = 0$ in (J.27). Using (J.22) we have

$$\begin{aligned}\Delta_F^{(n)} &= \prod_{i=1}^{n} \delta^4\left(Q_i^A - \frac{\omega^A(\sigma_i)l(\sigma_i) - \pi^A(\sigma_i)r(\sigma_i)}{\prod_{j\neq i}^{n+1}\sigma_{ij}}\right)\\ &= \prod_{i=1}^{n} \delta^4\left(Q_i^A - \frac{\omega^A(\sigma_i)\hat{l}(\sigma_i) - \hat{\pi}^A(\sigma_i)r(\sigma_i)}{\prod_{j\neq i}^{n}\sigma_{ij}}\right) = \prod_{i=1}^{n} \delta^4\left(Q_i^A - \frac{\langle\hat{\rho}^A(\sigma_i)\,\hat{\chi}(\sigma_i)\rangle}{\prod_{j\neq i}^{n}\sigma_{ij}}\right),\end{aligned} \tag{J.53}$$

where we have defined

$$\hat{\chi}^a(z) = \xi^a r(z) - \Xi^a \hat{l}(z), \tag{J.54}$$

and $\hat{\rho}^A(\sigma)$ is given by (J.30). The top component of this fermionic map is given by $\hat{\chi}^a_m = \xi^a r_m$. We identify $r_m = g$, hence agreeing with the fermionic maps introduced in Section 7. We now have

$$\mathcal{I}^{++\hat{+}}_{2m+1} = V_n\,\mathrm{PT}(\mathbb{I}_n)\,\frac{\widehat{\mathrm{Pf}'}A}{\prod_{i=1}^{n}\sigma_{n+1,i}}\frac{\sigma_{1n}}{\sigma_{1,n+1}\sigma_{n+1,n}}x\,\tilde{x}\int dg\,d\tilde{g}\prod_{k=0}^{m-1} d^2\chi_k^a d^2\tilde{\chi}_k^{\hat{a}}\,\Delta_F^{(n)}\widetilde{\Delta}_F^{(n)}. \tag{J.55}$$

Recall that at this stage the map component $\xi = \frac{1+\Xi^-}{\Xi^+}$ is determined implicitly by (J.39), which in turn depends on σ_{n+1}. Therefore the σ_{n+1} dependence cannot be isolated yet.

The final step is to turn ξ into an extra variable, which is equivalent to unfixing the T-shift symmetry, as explained at the end of Section J. This is done by performing the transformation (J.40). However, as $\mathcal{I}^{++\hat{+}}_{2m+1}$ will be divided by $S^{++\hat{+}}$, given in (J.45), we also need to consider the scaling of the soft spinors $q \to q/\alpha$. Doing the corresponding scaling for the antichiral piece, $\tilde{q} \to \tilde{q}/\tilde{\alpha}$, we effectively promote ξ and $\tilde{\xi}$ into integration variables to be fixed by the bosonic equations. The relationship between the variables α, $\tilde{\alpha}$ and the components ξ, $\tilde{\xi}$ can be read off from (J.40):

$$\alpha = \langle \Xi\,\xi\rangle\,, \quad \tilde{\alpha} = [\tilde{\Xi}\,\tilde{\xi}]. \tag{J.56}$$

Including the scaling of the soft factor $S^{++\hat{+}} \to \alpha\tilde{\alpha}S^{++\hat{+}}$ and putting everything together, we find the following formula for the $\mathcal{N}=(1,1)$ integrand:

$$\frac{1}{2\pi i}\oint_{|\hat{E}_{n+1}|=\varepsilon}\frac{d\sigma_{n+1}}{E_{n+1}}\frac{\mathcal{I}^{+\hat{+}}_{2m+1}}{S^{+\hat{+}}}=\mathcal{J}_{2m+1}\times\int d\widehat{\Omega}^{(1,1)}_{\mathrm{F}}. \tag{J.57}$$

The Vandermonde factor V_n has been absorbed into the fermionic measure $d\widehat{\Omega}^{(1,1)}_{\mathrm{F}}$, which is defined as:

$$d\widehat{\Omega}^{(1,1)}_{F}=V_n\,dg\,d\tilde{g}\prod_{k=0}^{m-1}d^2\chi_k\,d^2\tilde{\chi}_k\prod_{i=1}^{n}\delta^4\!\left(q_i^A-\frac{\langle\rho^A(\sigma_i)\,\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}}\right)\delta^4\!\left(\tilde{q}_{i,A}-\frac{[\tilde{\rho}_A(\sigma_i)\,\tilde{\chi}(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}}\right).$$

The bosonic part of the integrand $\mathcal{J}_{2m+1}$ is given by

$$\mathcal{J}_{2m+1}=\mathrm{PT}(\mathbb{I}_n)\frac{\sigma_{1n}}{2\pi i}\oint_{|\hat{E}_{n+1}|=\varepsilon}\frac{d\sigma_{n+1}}{E_{n+1}}\frac{1}{S^{+\hat{+}}}\frac{x\,\tilde{x}}{\langle\Xi\xi\rangle[\tilde{\Xi}\tilde{\xi}]}\frac{1}{\prod_{i=1}^{n}\sigma_{n+1,i}}\left(\frac{\mathrm{Pf}'\hat{A}}{\sigma_{1,n+1}\sigma_{n+1,n}}\right), \tag{J.58}$$

which encodes the complete σ_{n+1} and $\hat{p}_{n+1}$ dependence. It is now straightforward to repeat these steps for other polarizations $(a,\hat{a})$. In fact, from (J.47) we see that for the choice $a=-$, the τ^{-1} contribution will dominate, yielding no factor of x in the numerator. At the same time, the different τ dependence of this integrand will be compensated by the different τ behaviour of the soft factor $S^{a\hat{a}}$. For a general polarization we have:

$$\frac{x\tilde{x}}{S^{+\hat{+}}}\;\to\;\frac{x^a\tilde{x}^{\hat{a}}}{S^{a\hat{a}}}, \tag{J.59}$$

where we have defined $x^a=(x,-1)$ and $\tilde{x}^{\hat{a}}=(\tilde{x},-1)$. Setting $\sigma_{n+1}=z$ and removing the fermionic delta functions, the integrand becomes

$$\mathcal{J}_{2m+1}=\frac{1}{S^{a\hat{a}}}\mathrm{PT}(\mathbb{I}_n)\,\frac{\sigma_{1n}}{2\pi i}\oint_{|\hat{\mathcal{E}}_{n+1}|=\varepsilon}\frac{dz}{\mathcal{E}_{n+1}}\frac{\mathrm{Pf}'A_{n+1}}{(z-\sigma_1)(z-\sigma_n)}\,\frac{x^a}{\langle\xi\,\Xi\rangle}\,\frac{\tilde{x}^{\hat{a}}}{[\tilde{\xi}\,\tilde{\Xi}]}, \tag{J.60}$$

where $\mathcal{E}_{n+1}=\tau\hat{\mathcal{E}}_{n+1}=p(z)\cdot p_{n+1}$ is the scattering equation for the $(n+1)$th particle, valid on the support of the equations associated to hard particles. In this form the τ dependence cancels between the soft factor and the scattering equation. This form is taken as the starting point in Section 7.2.

From Odd to Even Multiplicity and the Number of Solutions

Here we consider taking a soft limit of the odd-point measure. The goal is to prove that the relation

$$\int d\mu_{n+1}^{\text{6D}} = \delta(p_{n+1}^2) \int d\mu_n^{\text{6D}} \frac{1}{2\pi i} \oint_{|\hat{E}_{n+1}|=\varepsilon} \frac{d\sigma_{n+1}}{E_{n+1}} + \mathcal{O}(\tau^0), \tag{J.61}$$

holds for any n, whether it is even or odd. (The corresponding measures were defined in Sections 5 and 7.1). This result can be used to prove that the equations for the maps and the punctures of n particles have $(n-3)!$ solutions,[1] as claimed in Section 5. Since we have already shown that integrating out the coefficients of the maps $\rho_k^{A,a}$ leaves delta functions localizing the σ_i's, this implies that this measure should correspond to the CHY measure (5.7) up to a trivial Jacobian. Such a Jacobian must not carry a nontrivial $\mathrm{SL}(2,\mathbb{C})$ weight or mass dimension. This has been checked numerically.

The reasoning used to find the number of solutions follows closely the inductive proof in [64]. For $n=3$ one can analytically check that there is one solution for the moduli $\{\rho, \sigma\}$. We then assume that the lower-point measure $d\mu_n$ in (J.61) has support on exactly $(n-3)!$ solutions. Then, we use the fact that in the soft limit $d\mu_{n+1}$ decouples into the lower-point measure and $\delta(E_{n+1})$. In the previous section we recognized E_{n+1} as the soft limit of the scattering equation for σ_{n+1}, which has been shown to yield $n-2$ solutions for given hard data [64]. This can also be seen directly from (J.37). Since the number of solutions cannot change in the soft limit, we conclude that $d\mu_{n+1}$ has support on $(n-2)!$ solutions, which completes the argument.

In order to show the validity of (J.61) for odd n we begin with the same identity used in the previous section for n odd:

$$\Delta_B^{(n+1)} = \Delta_B^{(n)} \delta(p_{n+1}^2) \int d^4 M_{n+1} |M_{n+1}|^3 \\ \times \delta^8 \left(\rho^{A,a}(\sigma_{n+1}) - (M_{n+1})^a_b \lambda_{n+1}^{A,b} \right) \delta \left(|M_{n+1}| - \prod_{i=1}^{n} \sigma_{n+1\, i} \right), \tag{J.62}$$

where we have used the odd-point parametrization of the rational maps,

$$\rho^{A,a}(z) = \sum_{k=0}^{m-1} \rho_k^{A,a} z^k + \omega'^A \xi'^a z^m \,, \tag{J.63}$$

[1] This assumes generic kinematics in the sense of the discussion we give in Section 10.

and $m = (n-1)/2$. To avoid confusion we have labeled the highest-degree coefficient using primed variables. As before, we parametrize the $(n+1)$th soft particle for $\tau \to 0$ using a 6D spinor of the form $\lambda_{n+1}^{A,a} = \xi^a v^A + \tau q^{A,a}$, which gives $p_{n+1}^{AB} \sim O(\tau)$. We also define $q^{A,a}\xi_a = q^A$. For the odd-point parametrization of the maps, the symmetry group G includes the T-shift redundancy parametrized by the $\mathrm{GL}(1,\mathbb{C})$ parameter α. $\rho(z)$ and M_i both transform under the T shift, as shown in (8.20) for $W_i = M_i^{-1}$.

Much of the soft-limit analysis for n odd is similar to the case of n even; the coefficients of the rational maps are fixed by the data of the hard particles while M_{n+1} is allowed to have a singular piece in the soft limit. We may repeat the steps of Section J, inserting an ansatz for M_{n+1} and decomposing it in a basis of spinors Ξ^a and a modulus x. The dependence of the measure on the $(n+1)$th particle can we written in the soft limit as

$$\frac{1}{\tau}\delta(p_{n+1}^2)\int \frac{\prod_{k=0}^{m-1} d^8\rho_k \, d^4\omega' \, \langle \xi' d\xi' \rangle \, d\sigma_{n+1}}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T})} \frac{\prod_{i=1}^n \sigma_{n+1,i}}{V_n^2} \Delta_B^{(n)}$$
$$\times \int dx \, d^2\Xi \, \delta^8\left(\rho^{A,a}(\sigma_{n+1}) - \Xi^a(q^A + x v^A)\right) . \tag{J.64}$$

After decomposing M_{n+1} in the soft limit as done here, the transformation rule for M_{n+1} becomes one for Ξ^a:

$$\delta\Xi^a = \alpha \, \sigma_{n+1} \, \xi'^a \langle \xi' \Xi \rangle \, . \tag{J.65}$$

Having isolated the singular τ dependence in the soft limit, let us now examine the behavior of the even-point rational maps arising from the soft limit of odd-point amplitudes. At each point in the $d^2\Xi$ integration, we expand the odd-point map in a special basis, the one determined by the two preferred spinors Ξ^a and ξ'^a. This basis is not orthonormal, and $\langle \Xi \xi' \rangle \neq 1$. Changing variables to (π^A, ω^A) spinor coordinates, the odd-point map ω'^A becomes the last component of the latter:

$$\rho^{A,a}(z) = \Xi^a \pi^A(z) + \xi'^a \omega^A(z), \tag{J.66}$$

or more explicitly

$$\rho^{A,a}(z) = \Xi^a \sum_{k=0}^{m-1} \pi_k^A z^k + \xi'^a \left(\sum_{k=0}^{m-1} \omega_k^A z^k + \omega'^A z^m \right) . \tag{J.67}$$

By taking linear combinations of the eight-dimensional constraint equations for $\rho^{A,a}$, we arrive at a split form involving the basis:

$$\delta^8\left(\rho^{A,a}(\sigma_{n+1})-\Xi^a(q^A+xv^A)\right)=\frac{1}{\langle\Xi\,\xi'\rangle^4}\delta^4\left(\omega^A(\sigma_{n+1})\right)\,\delta^4\left(\pi^A(\sigma_{n+1})-(q^A+xv^A)\right). \tag{J.68}$$

Additionally, the remaining bosonic delta functions also change under this basis transformation:

$$\Delta_B^{(n)}=\prod_{i=1}^{n}\delta^6\left(p_i^{AB}-\langle\Xi\xi'\rangle\frac{\pi^{[A}(\sigma_i)\omega^{B]}(\sigma_i)}{\prod_{j=1}^{n}\sigma_{n+1,\,j}}\right)\,, \tag{J.69}$$

along with the integration measure, which acquires a Jacobian

$$\left(\prod_{k=0}^{m-1}d^8\rho_k\right)\,d^4\omega'\rightarrow\langle\Xi\,\xi'\rangle^{4m}\left(\prod_{k=0}^{m-1}d^4\pi_k\,d^4\omega_k\right)\,d^4\omega'. \tag{J.70}$$

As in the case of taking a soft limit from even n to odd n, we may now use the delta functions to reduce the degree of the map. To see this, we parametrize the map evaluated at the $(n+1)$th puncture as:

$$\omega^A(z)=(z-\sigma_{n+1})\hat{\omega}^A(z)+r^A, \tag{J.71}$$

$$\prod_{k=0}^{m-1}d^4\omega_k\,d^4\omega'\,\delta^4\left(\omega^A(\sigma_{n+1})\right)\rightarrow d^4r^A\prod_{k=0}^{m-1}d^4\hat{\omega}_k\,\delta^4(r^A)\,. \tag{J.72}$$

The r^A integrations are trivial, and now the ω' component has dropped out of the problem in favor of the $\hat{\omega}$ variables. This means we may now use the hatted variables in the remaining bosonic delta functions.

Having reduced the degree of the map, we may now switch back to the ρ variables through another change of basis:

$$\hat{\rho}^{A,a}(z)=\Xi^a\pi^A(z)+\xi'^a\hat{\omega}^A(z), \tag{J.73}$$

$$\hat{\rho}^{A,a}(z)\Xi_a=\langle\Xi\xi'\rangle\,\hat{\omega}^A(z), \tag{J.74}$$

$$\hat{\rho}^{A,a}(z)\xi'_a=\langle\xi'\Xi\rangle\,\pi^A(z). \tag{J.75}$$

This has the effect of undoing several of the Jacobians acquired earlier, and the relevant piece of the measure and integrand becomes

$$\frac{\prod_{i=1}^{n}\sigma_{n+1\,i}}{V_n^2}\int d\sigma_{n+1}\,\frac{\prod_{k=0}^{m-1}d^8\hat{\rho}_k\langle\xi' d\xi'\rangle\,d^2\Xi\,dx}{\text{vol}(\text{SL}(2,\mathbb{C})_\sigma,\text{SL}(2,\mathbb{C})_\rho,\text{T})}\prod_{i=1}^{n}\delta^6\left(p_i^{AB}-\frac{\langle\hat{\rho}^A(\sigma_i)\hat{\rho}^B(\sigma_i)\rangle}{\prod_{j=1}^{n}\sigma_{n+1\,j}}\right)$$
$$\times\,\frac{1}{\langle\Xi\xi'\rangle^4}\delta^4\left(\frac{\langle\hat{\rho}^A(\sigma_{n+1})\,\xi'\rangle}{\langle\Xi\,\xi'\rangle}-q^A-xv^A\right). \tag{J.76}$$

The freedom to projectively scale ξ' allows us to set the first component to 1 and define the second as ξ' so that $\langle\xi' d\xi'\rangle = d\xi'$. Now we may focus on the last piece, which can be written as

$$\prod_{i=1}^{n}\sigma_{n+1\,i}\int d\sigma_{n+1}\,d\xi'\,d^2\Xi\,dx\,\delta^4\big(\hat{\rho}^{A,a}(\sigma_{n+1})\xi'_a-\langle\Xi\xi'\rangle(q^A-xv^A)\big)\,. \tag{J.77}$$

There are now five integrations, four delta functions, and the T redundancy to cancel. The strategy is to isolate the scattering equation for the last particle, integrate out the other delta functions, and cancel the T-shift symmetry. The scattering equation for the soft particle is supported on the solution of $E_{n+1}=\epsilon_{ABCD}\hat{\rho}^{A,+}(\sigma_{n+1})\hat{\rho}^{B,-}(\sigma_{n+1})v^Cq^D=0$. To get this, we first make the change of variables

$$\begin{aligned}\langle\xi'\Xi\rangle=\Xi^- -\xi'\Xi^+ &\to u,\\ x&\to\frac{x'}{u},\\ d\xi'\,d\Xi^+\,d\Xi^-\,dx&\to\frac{d\Xi^+}{u}\,d\xi'\,du\,dx',\\ \delta^4\big(\hat{\rho}^{A,a}(\sigma_{n+1})\xi'_a-\langle\Xi\xi'\rangle(q^A-xv^A)\big)&\to\delta^4\big(\hat{\rho}^{A,+}(\sigma_{n+1})\xi'-\hat{\rho}^{A,-}(\sigma_{n+1})-uq^A-x'v^A\big)\,.\end{aligned} \tag{J.78}$$

Now we would like to evaluate the integrals over u, x', and ξ'. As in the even-point case, we observe that these integrations give the scattering equation for the last particle after taking the appropriate linear combinations:

$$\frac{\prod_{i=1}^{n}\sigma_{n+1\,i}}{\tau}\int d\sigma_{n+1}\,d\xi'\,du\,dx'\,\delta^4\big(\hat{\rho}^{A,+}(\sigma_{n+1})\xi'-\hat{\rho}^{A,-}(\sigma_{n+1})-uq^A-x'v^A\big)$$
$$=\frac{\prod_{i=1}^{n}\sigma_{n+1\,i}}{\tau}\int d\sigma_{n+1}\,\delta\big(\epsilon_{ABCD}\hat{\rho}^{A,+}(\sigma_{n+1})\hat{\rho}^{B,-}(\sigma_{n+1})v^Cq^D\big)=\int d\sigma_{n+1}\,\delta(E_{n+1}). \tag{J.79}$$

So we are left with

$$\delta(p_{n+1}^2) V_n^{-2} \int \frac{\prod_{k=0}^{m-1} d^8 \hat{\rho}_k}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})_\sigma, \mathrm{SL}(2,\mathbb{C})_\rho, \mathrm{T})} \frac{d\Xi^+}{u} \times \prod_{i=1}^{n} \delta^6 \left(p_i^{AB} - \frac{\langle \hat{\rho}^A(\sigma_i) \hat{\rho}^B(\sigma_i) \rangle}{\prod_{j=1}^{n} \sigma_{n+1\,j}} \right) \int d\sigma_{n+1}\, \delta(E_{n+1}). \tag{J.80}$$

In this expression $u = \langle \xi' \Xi \rangle$ has a value determined by the constraints after doing the integral. Since T acts as a $\mathrm{GL}(1,\mathbb{C})$ shift on the components of Ξ, we can absorb u and cancel the symmetry. The result is the expected measure for n even:

$$\int \frac{\prod_{i=1}^{n} d\sigma_i \prod_{k=0}^{m-1} d^8 \hat{\rho}_k}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\rho)} \prod_{i=1}^{n} \delta^6 \left(p_i^{AB} - \frac{\langle \hat{\rho}^A(\sigma_i) \hat{\rho}^B(\sigma_i) \rangle}{\prod_{j=1}^{n} \sigma_{n+1\,j}} \right) \delta(p_{n+1}^2) V_n^{-2} \int d\sigma_{n+1}\, \delta(E_{n+1}). \tag{J.81}$$