Differential Operators on Manifolds with G_2-Structure

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2020

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study differential operators on manifolds with torsion-free G_2-structure. In particular, we use an identification of the spinor bundle $\mathbb{S}$ of such a manifold M with the bundle $\underline{\mathbb{R}} \oplus T^*M$ to reframe statements regarding the Dirac operator in terms of three other first order differential operators: the divergence, the gradient, and the curl operators. We extend these three operators to act on tensors of one degree higher and study the properties of the extended operators. We use the extended operators to describe a Dirac bundle structure on the bundle $T^*M \oplus (T^*M \otimes T^*M) = T^*M \otimes (\underline{\mathbb{R}} \oplus T^*M)$ as well as its Dirac operator. We show that this Dirac operator is equivalent to the twisted Dirac operator D_T defined using the original identification of $\mathbb{S}$ with $\mathbb{R} \oplus T^*M$.

As the two Dirac operators are equivalent, we use the $T^*M \oplus (T^*M \otimes T^*M) = T^*M \otimes (\underline{\mathbb{R}} \oplus T^*M)$ description of the bundle of spinor-valued 1-forms to examine the properties of the twisted Dirac operator D_T. Using the extended divergence, gradient, and curl operators, we study the kernel of the twisted Dirac operator when M is compact and provide a proof that $\dim \ker D_T = b^2 + b^3$.

Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Spiro Karigiannis for his knowledge and guidance during my time as a masters student. In addition, I would like to thank Prof. Benoit Charbonneau and Prof. Ruxandra Moraru for introducing me to the wonders of differential geometry and fostering my interest in the subject. I would also like to thank Prof. Laurent Marcoux as well for his help in developing my mathematical research ability. The conversations I have had with each of them regarding mathematics, academics, and general advice have been enlightening and I would not be here today without them.

I would like to thank the friends I have made during my studies at Waterloo for the fun times we have had hanging out and playing board games over the years. Even though lately we have not been able to meet up in person, I hope that these game nights can continue into the future.

I want to thank Amy Tan for her support. She has kept me focused on my work when I needed to be and also given me a distraction from work when I needed one.

I am especially grateful for my parents for their support and encouragement in my studies. Their love and care for me have made me the person I am now.

Last but not least, I would like to thank God for His blessings and favour upon me.

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Chapter 1

Introduction

The aim of this thesis is to study and collect results regarding the Dirac and twisted Dirac operators on manifolds with torsion-free G_2-structure. In doing so, we will define and come across other differential operators defined on such manifolds such as the divergence, gradient, and curl operators.

The following two chapters consist of the preliminary material related to this thesis. Chapter 2 is dedicated to material on spin geometry and Chapter 3 provides an introduction to the G_2-geometry that will be used throughout the rest of the thesis. Most of the information from each of these chapters have been taken from their sources, with their proofs slightly adapted in the attempt to improve readability and clarity.

Chapter 4 provides the groundwork and justification for identifying the spinor bundle $\mathbb{S}$ with the bundle $\mathbb{R} \oplus T^*M$. We define the first order operators div, grad, and curl here and extend them to act on spinor-valued 1-forms. Moreover, we prove identities involving these operators and the Dirac operators on their respective bundles when the underlying G_2-structure is torsion-free.

In Chapter 5, we take a deeper look at the twisted Dirac operator defined on the bundle of spinor-valued 1-forms. In particular, we compute its kernel in the compact and torsion-free case. Moreover, we note some characteristics of harmonic forms on manifolds with torsion-free G_2-structure.

1.1 Notation and Conventions

When left unspecified, M will denote a smooth 7-dimensional manifold admitting a G_2-structure with metric g, associative 3-form φ, and coassociative 4-from ψ. (See Chapter 3 for an introduction to G_2-structures.) All structures involved will be smooth unless stated otherwise. Using the metric, we identify vector fields and 1-forms. Tensor calculations will be done pointwise and tensors on M will be expressed with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ with respect to the metric g such that $\nabla_i e_j = 0$ at the center. As such, all indices will be subscripts. We employ the Einstein summation convention throughout, so repeated indices will be summed over the values 1 to $\dim M$. Additionally, at times we will, through an abuse of notation, identify a global object with its local coordinate representation.

The trivial rank 1 real bundle over a manifold M will be denoted by $\underline{\mathbb{R}}$. Given a vector bundle E over M, we use $\Gamma(E)$ to denote the space of smooth sections of E. In some cases, we denote these spaces in different ways. For example:

- $\Omega^k = \Gamma(\Lambda^k(T^*M))$ is the space of smooth k-forms on M;
- $\mathfrak{X} = \Gamma(TM)$ is the space of smooth vector fields on M;
- $\mathcal{T}^2 = \Gamma(T^*M \otimes T^*M)$ is the space of smooth 2-tensors on M;
- $\mathcal{S}^2 = \Gamma(S^2(T^*M))$ is the subspace of $\mathcal{T}^2$ of smooth symmetric 2-tensors on M.

With respect to the metric g on M, we let $\mathcal{S}^2_0$ denote the subspace of traceless symmetric 2-tensors. That is, $\mathcal{S}^2_0$ consists of symmetric 2-tensors h with $\operatorname{tr} h = h_{ii} = 0$. Using the induced metric on $T^*M \otimes T^*M$ induced by g, we have $\mathcal{T}^2 = \Omega^2 \oplus \mathcal{S}^2$ where the splitting is pointwise orthogonal.

The Levi-Civita connection induced by g will be denoted by ∇. We will encounter several Laplacian operators throughout this thesis. The symbol Δ will always denote the rough Laplacian $\Delta = -\nabla_i \nabla_i$. The Hodge Laplacian on differential forms will be denoted by $\Delta_d = dd^* + d^*d$ and the Lichnerowicz Laplacian (see Chapter 3 of [CK04]) will be denoted by Δ_L.

The labelling convention for the Riemann curvature tensor R_{ijkl} used is such that the Ricci tensor is $\mathrm{Ric}_{jk} = R_{ijki}$.

Chapter 2

Preliminaries on Spin Geometry

Much of the focus of this thesis is on differential operators on G_2-manifolds such as Laplacian and Dirac operators. Moreover we will utilize certain notions from spin geometry. In order to describe these, we need to understand the underlying theory of Clifford algebras. Since our applications concern real bundles, we will only consider the real cases here. Further, we will focus on results pertaining to odd dimensional spaces, in particular 7-dimensional spaces, as we aim to import these results onto manifolds with G_2-structure. This chapter closely follows [LM89], though results not relevant to odd dimensions have generally been omitted. Other sources for this section include [ABS64], [Har90], [HS19], [Nic13], and [Roe98].

2.1 Clifford Algebras

Definition 2.1.1. Let V be a finite-dimensinal real inner product space with a positive-definite inner product denoted by $\langle \cdot, \cdot \rangle$. A ***Clifford algebra*** for V is a unital algebra A with a map $\phi \colon V \to A$ such that $\phi(v)^2 = -\langle v, v\rangle 1$ that satisfies the universal property. In other words, if there were another unital algebra A' and map $\phi' \colon V \to A'$ satisfying $\phi'(v)^2 = -\langle v, v\rangle 1$, then there is a unique algebra homomorphism $f \colon A \to A'$ such that the diagram below commutes.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

For $(V, \langle\cdot,\cdot\rangle)$, a Clifford algebra exists and is unique up to algebra isomorphism by universality. As such, we may refer to the Clifford algebra of an inner product space. More concretely, we can consider the Clifford algebra of $(V, \langle\cdot,\cdot\rangle)$ to be to quotient of the tensor algebra $\oplus_{k\geq 0} V^{\otimes k}$ by the ideal generated by all elements of the form

$$v \otimes v + \langle v, v\rangle 1.$$

The map ϕ in this case is given by the composition of the embedding $V \hookrightarrow \oplus_{k\geq 0} V^{\otimes k}$ with the quotient map described above. We suppress the tensor notation here, as is usually done, and represent it via concatenation. Additionally, by polarizing the defining identity of the Clifford algebra, we get that

$$vw + wv = -2\langle v, w\rangle 1. \tag{2.1}$$

We denote the Clifford algebra of V by $\mathrm{Cl}(V)$ and note that if V has dimension n, then $\mathrm{Cl}(V)$ has dimension 2^n. The map ϕ is injective (see [LM89]), and so we identify V with its image in $\mathrm{Cl}(V)$ and consider it as a subspace of $\mathrm{Cl}(V)$.

The map

$$V \to \mathrm{Cl}(V) \colon v \mapsto -v$$

extends to an algebra automorphism $\alpha\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ by the universal property. Since negation on the vector space V is an involution, so is α. This gives a natural $\mathbb{Z}_2$-grading of the Clifford algebra

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V) \tag{2.2}$$

where $\mathrm{Cl}^0(V) = \ker(\alpha - 1)$ and $\mathrm{Cl}^1(V) = \ker(\alpha + 1)$. Indeed, since α is an algebra homomorphism, we see that the relations

$$\begin{aligned} \mathrm{Cl}^0(V)\cdot \mathrm{Cl}^0(V) \subseteq \mathrm{Cl}^0(V), \quad \mathrm{Cl}^1(V)\cdot \mathrm{Cl}^1(V) \subseteq \mathrm{Cl}^0(V), \\ \mathrm{Cl}^0(V)\cdot \mathrm{Cl}^1(V) \subseteq \mathrm{Cl}^1(V), \quad \mathrm{Cl}^1(V)\cdot \mathrm{Cl}^0(V) \subseteq \mathrm{Cl}^1(V), \end{aligned}$$

are satisfied. We call $\mathrm{Cl}^0(V)$ the ***even*** part of the Clifford algebra and $\mathrm{Cl}^1(V)$ the ***odd*** part. We note that $\mathrm{Cl}^0(V)$ is a subalgebra of $\mathrm{Cl}(V)$.

Using our concrete notion of the Clifford algebra as a quotient of the tensor algebra, we have that $\mathrm{Cl}(V)$ has a basis consisting of elements of the form $v_1 v_2 \cdots v_k$ where $\{v_1, \ldots, v_n\}$ is a basis of V. We define a ***transposition*** operator on such elements by

$$(v_1 v_2 \cdots v_k)^t = v_k \cdots v_2 v_1 \tag{2.3}$$

and extend it to all of $\mathrm{Cl}(V)$ linearly. This map is well-defined on the Clifford algebra since it preserves the ideal by which we quotient out. It is clear that transposition is an involution on $\mathrm{Cl}(V)$. We note that it is also an antiautomorphism, that is, $(ab)^t = b^t a^t$.

Simple calculations allow us to compute the Clifford algebras of low dimensional inner product spaces. The following table lists $\mathrm{Cl}(V)$ up to isomorphism when V has dimension between 1 and 8 where $\mathbb{K}(m)$ denotes the algebra of $m \times m$ matrices over a field $\mathbb{K}$ (or skew-field in the case of the quaternions $\mathbb{H}$).

$\dim V$	1	2	3	4	5	6	7	8
$\mathrm{Cl}(V)$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

Table 2.1: Clifford Algebras of Low Dimensional Spaces

2.2 Pin and Spin Groups

We now turn our attention to the multiplicative groups of units $\mathrm{Cl}^\times(V)$ in the Clifford algebra $\mathrm{Cl}(V)$ which is the subset

$$\mathrm{Cl}^\times(V) = \{x \in \mathrm{Cl}(V) \mid \text{there exists } x^{-1} \in \mathrm{Cl}(V) \text{ such that } xx^{-1} = x^{-1}x = 1\}. \tag{2.4}$$

Since $\mathrm{Cl}(V)$ is subject to the relation

$$v^2 = -2\langle v, v\rangle 1, \qquad v \in V$$

we see that $\mathrm{Cl}^\times(V)$ contains all elements $v \in V$ with $\langle v, v\rangle \neq 0$. This group acts on the algebra $\mathrm{Cl}(V)$ as automorphisms via the ***adjoint representation***

$$\begin{aligned} \mathrm{Ad}\colon \mathrm{Cl}^\times(V) &\to \mathrm{Aut}(\mathrm{Cl}(V)) \\ x &\mapsto \mathrm{Ad}_x \end{aligned} \tag{2.5}$$

where

$$\mathrm{Ad}_x(y) = xyx^{-1}. \tag{2.6}$$

Additionally, $\mathrm{Cl}^\times(V)$ can act on $\mathrm{Cl}(V)$ via the ***twisted adjoint representation*** given by

$$\begin{aligned} \mathrm{Ad}\colon \mathrm{Cl}^\times(V) &\to \mathrm{Aut}(\mathrm{Cl}(V)) \\ x &\mapsto \widetilde{\mathrm{Ad}}_x \end{aligned} \tag{2.7}$$

where

$$\widetilde{\mathrm{Ad}}_x(y) = \alpha(x) y x^{-1}. \tag{2.8}$$

The difference between the two representations is that for odd elements of $\mathrm{Cl}(V)$, in particular for $v \in V$, the involution α acts as negation. Though the adjoint representation seems more natural, we will focus largely on the twisted adjoint representation as the notation and calculations end up being cleaner.

Using the twisted adjoint representation, we can define the Clifford group $\Gamma(V)$.

Definition 2.2.1. The ***Clifford group*** $\Gamma(V)$ is the subgroup of $\mathrm{Cl}^\times(V)$ whose elements leave V invariant under the twisted adjoint representation. In other words,

$$\Gamma(V) = \{u \in \mathrm{Cl}^\times(V) \mid \widetilde{\mathrm{Ad}}_u(v) = \alpha(u) v u^{-1} \in V \text{ for all } v \in V\}. \tag{2.9}$$

Proposition 2.2.2. *Let $0 \neq v \in V$. Then $\widetilde{\mathrm{Ad}}_v(V) \subseteq V$. In particular, we have for $w \in V$ that*

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{\langle v, w\rangle}{\|v\|^2} v. \tag{2.10}$$

Proof. The Clifford identity $v^2 = -\langle v, v\rangle$ tells us that $v^{-1} = -\frac{v}{\|v\|^2}$. Direct computation using (2.1) then yields

$$\widetilde{\mathrm{Ad}}_v(w) = -vwv^{-1} = \frac{vwv}{\|v\|^2} = -\frac{v^2 w}{\|v\|^2} - 2\frac{\langle v, w\rangle}{\|v\|^2} v = w - 2\frac{\langle v, w\rangle}{\|v\|^2} v.$$

□

Proposition 2.2.3. *Let $0 \neq v \in V$. Then $\widetilde{\mathrm{Ad}}_v$ preserves the bilinear form $\langle \cdot, \cdot \rangle$.*

Proof. Let $x, y \in V$. We compute using (2.10) that

$$\begin{aligned}
\langle \widetilde{\mathrm{Ad}}_v(x), \widetilde{\mathrm{Ad}}_v(y)\rangle &= \langle x - 2\frac{\langle v, x\rangle}{\|v\|^2} v, y - 2\frac{\langle v, y\rangle}{\|v\|^2} v\rangle \\
&= \langle x, y\rangle - 2\frac{\langle v, y\rangle}{\|v\|^2}\langle v, x\rangle - 2\frac{\langle v, x\rangle}{\|v\|^2}\langle v, y\rangle + 4\frac{\langle v, x\rangle\langle v, y\rangle}{\|v\|^2\|v\|^2}\|v\|^2 \\
&= \langle x, y\rangle.
\end{aligned}$$

□

Geometrically, we can see that the right hand side of (2.10) is just the reflection of w across the hyperplane perpendicular to v which matches up with the fact that the twisted adjoint representation preserves the bilinear form. Since it is just a reflection, we also have that for each non-zero vector $v \in V$ that $\widetilde{\mathrm{Ad}}_v(V) = V$. As a corollary, we observe that $\widetilde{\mathrm{Ad}}$ is actually a map

$$\Gamma(V) \to O(V)$$

where $O(V)$ denotes the orthogonal group of V with respect to the bilinear form $\langle \cdot, \cdot \rangle$. A classical result of Cartan and Dieudonné (see [Har90]) tells us that the image of $\widetilde{\mathrm{Ad}}$ is the entire orthogonal group.

Theorem 2.2.4 (Cartan–Dieudonné)**.** *Every element of $O(V)$ can be written as a composition of at most n reflections.*

We now define the Pin and Spin groups.

Definition 2.2.5. The ***Pin group*** of V is the subgroup $\mathrm{Pin}(V)$ of $\Gamma(V)$ generated by the elements $v \in V$ with $\langle v, v \rangle = 1$. The ***Spin group*** of V is the subgroup $\mathrm{Spin}(V)$ of $\mathrm{Pin}(V)$ containing only even elements of $\mathrm{Cl}(V)$, that is $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V)$. In other words,

$$\mathrm{Pin}(V) = \{v_1 \cdots v_k \in \Gamma(V) \mid \langle v_i, v_i \rangle = 1 \text{ for each } i\} \tag{2.11}$$

and

$$\mathrm{Spin}(V) = \{v_1 \cdots v_{2k} \in \Gamma(V) \mid \langle v_i, v_i \rangle = 1 \text{ for each } i\}. \tag{2.12}$$

Because $\widetilde{\mathrm{Ad}}_{\lambda v} = \widetilde{\mathrm{Ad}}_v$ for any scalar λ, we still see that the image of $\mathrm{Pin}(V)$ under the twisted adjoint representation results in the entire orthogonal group $O(V)$. We work to get a similar result for $\mathrm{Spin}(V)$. Recall that the special orthogonal group $SO(V)$ is the subgroup of $O(V)$ consisting of elements with unit determinant. Thus to show that we have a similar relationship between the Spin group $\mathrm{Spin}(V)$, the twisted adjoint representation $\widetilde{\mathrm{Ad}}$, and the special orthogonal group $SO(V)$, it suffices to show that $\det \widetilde{\mathrm{Ad}}_v = -1$ for each $v \in V$. To prove this, extend v to an orthogonal basis $\{v = v_1, \cdots, v_n\}$. We compute that $\widetilde{\mathrm{Ad}}_v(v_1) = -v v_1 v^{-1} = -v_1$ and for each $i > 1$ we have $\widetilde{\mathrm{Ad}}_v(v_i) = -v v_i v^{-1} = v_i v v^{-1} = v_i$. It follows that $\det \widetilde{\mathrm{Ad}}_v = -1$. Thus $\widetilde{\mathrm{Ad}}$ maps $\mathrm{Spin}(V)$ onto $SO(V)$.

More can be said about the twisted adjoint representation $\widetilde{\mathrm{Ad}}$. In order to discuss this, we need some other results.

Proposition 2.2.6. *The kernel of the twisted adjoint representation $\widetilde{\mathrm{Ad}}$ is the group $\mathbb{R}^\times$ of non-zero multiples of the unit 1.*

Proof. Let $\{v_1, \dots, v_n\}$ be an orthogonal basis for V. Suppose that $u \in \mathrm{Cl}^\times(V)$ is in the kernel of $\widetilde{\mathrm{Ad}}$. Then we have that $\alpha(u)vu^{-1} = \widetilde{\mathrm{Ad}}_u(v) = v$ for all $v \in V$, that is, $\alpha(u)v = vu$ for all $v \in V$. We may write $u = u_0 + u_1$ where u_0 is even and u_1 is odd. By equating odd and even parts, we see that

$$\begin{aligned} u_0 v &= v u_0 \\ -u_1 v &= v u_1 \end{aligned}$$

for all $v \in V$. We can write u_0 and u_1 as polynomial expressions of the basis elements $v_1, \dots, v_n$. By repeatedly using orthogonality of the basis and the identity $v_i v_j = -v_j v_i - 2\langle v_i, v_j \rangle$, we may assume that these polynomial expressions are of the form $u_0 = a_0 + v_1 a_1$ and $u_1 = b_1 + v_1 b_0$ where a_0, a_1, b_0, and b_1 are polynomial expressions of $v_2, \dots, v_n$. We also see that a_0 and b_0 are even while a_1 and b_1 are odd.

If we let $v = v_1$ in the relations above, we get

$$v_1 a_0 + v_1^2 a_1 = a_0 v_1 + v_1 a_1 v_1 = v_1 a_0 - v_1^2 a_1.$$

where the second equality follows from repeatedly using orthogonality of the basis and the identity $v_i v_j = -v_j v_i - 2\langle v_i, v_j \rangle$. This shows that $v_1^2 a_1 = -\langle v_1, v_1 \rangle a_1 = 0$ and so $a_1 = 0$. This shows that u_0 does not involve v_1. Since we arbitrarily chose 1 as the index here, we can repeat this argument for each index to show that u_0 must be a scalar.

We apply a similar argument for u_1 and get

$$v_1 b_1 + v_1^2 b_0 = -b_1 v_1 - v_1 b_0 v_1 = v_1 b_1 - v_1^2 b_0.$$

So $v_1^2 b_0 = 0$ which implies that $b_0 = 0$. Hence u_1 does not involve v_1. Repeating this for each index shows that u_1 must also be a scalar. Since u_1 is odd, it follows that $u_1 = 0$.

Putting the above together, we get that $u = u_0 + u_1 = u_0$ is a scalar. Since by assumption u is non-zero, the result follows. □

Next, we introduce the ***norm*** map N on $\mathrm{Cl}(V)$ by setting

$$N(u) = u\alpha(u^t). \tag{2.13}$$

We note that in the case that $v \in V$, then $v^t = v$ and $\alpha(v) = -v$ so $N(v) = -v^2 = \langle v, v \rangle$.

Though defined on all of the Clifford algebra, we are mostly interested to its restriction to the Clifford group and its subgroups.

Proposition 2.2.7. *The restriction of the norm map N to the Clifford subgroup $\Gamma(V)$ is a group homomorphism onto the group $\mathbb{R}^\times$ of non-zero multiples of the unit 1. Moreover $N(\alpha(u)) = N(u)$.*

Proof. First we show that N maps $\Gamma(V)$ to $\mathbb{R}^\times$. If $u \in \Gamma(V)$, then for $v \in V$ we have $\alpha(u)vu^{-1} = v'$ for some $v' \in V$. We apply transposition to both sides of this equation, taking note that transposition acts as the identity on V to get

$$\alpha(u)vu^{-1} = (\alpha(u)vu^{-1})^t = (u^{-1})^t v^t \alpha(u)^t = (u^t)^{-1} v \alpha(u^t).$$

Since α is an involutive homomorphism, rearranging the above gives

$$\alpha([\alpha(u^t)u])v = u^t\alpha(u)v = v\alpha(u^t)u,$$

which implies that $\alpha(u^t)u$ is in the kernel of the twisted adjoint representation. By Proposition 2.2.6, $\alpha(u^t)u$ is a non-zero scalar and so $u^t\alpha(u)$ is a non-zero scalar. Since $N(u^t) = u^t\alpha((u^t)^t) = u^t\alpha(u)$ and $\Gamma(V)$ is closed under transposition, it follows that $N(\Gamma(V)) \subseteq \mathbb{R}^\times$.

To show that N is a homomorphism, let $x, y, u \in \Gamma(V)$. Then we have

$$\begin{aligned} N(xy) = xy\alpha((xy)^t) &= xy\alpha(y^t x^t) = xy\alpha(y^t)\alpha(x^t) \\ &= xN(y)\alpha(y^t) = N(y)x\alpha(x^t) = N(x)N(y). \end{aligned}$$

We also have

$$N(\alpha(u)) = \alpha(u)\alpha(\alpha(u)^t) = \alpha(u)u^t = \alpha([u\alpha(u^t)]) = \alpha(N(u)) = N(u).$$

$\square$

Putting some of the previous results together we get the following proposition.

Proposition 2.2.8. *There exist short exact sequences*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \operatorname{Pin}(V) \xrightarrow{\widetilde{\operatorname{Ad}}} O(V) \longrightarrow 1 \tag{2.14}$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \operatorname{Spin}(V) \xrightarrow{\widetilde{\operatorname{Ad}}} SO(V) \longrightarrow 1. \tag{2.15}$$

Additionally, depending on the dimension of the vector space V, we can say more about the relationship between $\operatorname{Spin}(V)$ and $SO(V)$. We state the following proposition without proof (see [Nic13]).

Proposition 2.2.9. *The twisted adjoint representation $\widetilde{\operatorname{Ad}}$ defines a covering map. Further, $\operatorname{Spin}(V)$ is connected if V has dimension at least 2 and it is simply connected if V has dimension at least 3. In particular, if $\dim V \geq 3$, $\operatorname{Spin}(V)$ is the universal cover of $SO(V)$.*

2.3 Spinor Representations and Spin Structures

In working towards defining the bundle of spinors on a manifold, we first need to consider the representations of Clifford algebras.

Definition 2.3.1. A (real) ***representation*** of the Clifford algebra $\mathrm{Cl}(V)$ is an algebra homomorphism

$$\rho\colon \mathrm{Cl}(V) \to \mathrm{Hom}(W, W)$$

into the algebra of linear transformations of a finite dimensional real vector space W. We call W a (real) left $\mathrm{Cl}(V)$-***module***. We often suppress the map ρ by writing

$$\rho(u)(w) = u \cdot w$$

for $u \in \mathrm{Cl}(V)$ and $w \in W$. The product above is called ***Clifford multiplication***.

The representation (ρ, W) is said to be ***reducible*** if there exists a proper invariant subspace, that is, there exists a subspace $\{0\} \subsetneq Z \subsetneq W$ such that $\rho(u)(Z) \subseteq Z$ for each $u \in \mathrm{Cl}(V)$. If no such invariant subspace exists, we say that the representation is ***irreducible***.

Lastly, two representations $\rho_1\colon \mathrm{Cl}(V) \to \mathrm{Hom}(W_1, W_1)$ and $\rho_2\colon \mathrm{Cl}(V) \to \mathrm{Hom}(W_2, W_2)$ are considered ***equivalent*** if there exists a linear isomorphism $L\colon W_1 \to W_2$ such that $L \circ \rho_1(u) \circ L^{-1} = \rho_2(u)$ for each $u \in \mathrm{Cl}(V)$.

We note that the Clifford algebra $\mathrm{Cl}(V)$ is almost the group algebra of a finite group. Consider the elements $e_1, \cdots, e_n$ and -1 where the e_i form an orthonormal basis of V. We can form a group of these elements by declaring a presentation given by these elements subject to the relations

$$(-1)^2 = 1, (e_i)^2 = -1, e_i e_j = (-1) e_j e_i \text{ for } i \neq j, \tag{2.16}$$

as well as stipulating that -1 be central. By taking the group algebra of this group and quotienting out by the subspace spanned by $1 + (-1)$, we get the Clifford algebra $\mathrm{Cl}(V)$. Using this, we can apply some of the representation theory of finite groups. In particular, we can decompose a representation of a Clifford algebra into a direct sum of irreducible ones.

Proposition 2.3.2. *Every representation ρ of a Clifford algebra $\mathrm{Cl}(V)$ can be decomposed into a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_k$ of irreducible representations.*

From Table 2.1, we notice that the Clifford algebras $\mathrm{Cl}(V)$ are all of the form $\mathbb{K}(m)$ or $\mathbb{K}(m) \oplus \mathbb{K}(m)$ for some field or skew-field $\mathbb{K}$. Arguments in [LM89] show that Clifford algebras generally have this form and can be computed from the entries in Table 2.1 using "periodicity" isomorphisms with period 8. The following theorem shows us that the representations of these algebras are quite simple. A proof of this result can be found in [Lan02].

Theorem 2.3.3. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and consider the $\mathbb{R}$-algebras $\mathbb{K}(m)$ and $\mathbb{K}(m) \oplus \mathbb{K}(m)$. The natural representation ρ of $\mathbb{K}(m)$ on the vector space $\mathbb{K}^m$ is, up to equivalence, the only irreducible representation of $\mathbb{K}(m)$. The algebra $\mathbb{K}(m) \oplus \mathbb{K}(m)$ has two equivalence classes of irreducible representations given by*

$$\rho_1(u_1, u_2) = \rho(u_1) \text{ and } \rho_2(u_1, u_2) = \rho(u_2)$$

acting on $\mathbb{K}^m$.

To further this discussion on spin representations, we need to consider an important element of the Clifford algebra $\mathrm{Cl}(V)$ called the ***volume element***. To define the volume element, choose an orientation for the vector space V and let $e_1, \ldots, e_n$ be an oriented orthonormal basis for V. The volume element is defined to be

$$\omega = e_1 \cdots e_n. \tag{2.17}$$

We check that this definition is independent of the basis chosen. If $e'_1, \ldots, e'_n$ is another oriented orthonormal basis, then we can transform one basis into the other via a linear transformation L. In particular, since it preserves orientation and orthonormality of the basis, it follows that $L \in SO(V)$. Writing this out, we have $e'_i = \sum_j L_{ij} e_j$. Using the identities $e_i e_j + e_j e_i = -2\delta_{ij}$, we can compute that

$$e'_1 \cdots e'_n = (\det L) e_1 \cdots e_n = e_1 \cdots e_n.$$

An important property of the volume element is that it squares to ± 1 and either commutes or anticommutes with vectors based on the dimension of V. This can be seen by repeated use of the identities $e_i e_j + e_j e_i = -2\delta_{ij}$.

Proposition 2.3.4. *The volume element ω satisfies the following properties:*

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}} \tag{2.18}$$

$$v\omega = (-1)^{n-1} \omega v. \tag{2.19}$$

We see that if $n \equiv 3, 4 \pmod 4$ then $\omega^2 = 1$ and if $n \equiv 1, 2, \pmod 4$ then $\omega^2 = -1$. In the former case, we can define a pair of idempotent elements which will allow us to decompose $\mathrm{Cl}(V)$ in another way.

Lemma 2.3.5. *Suppose the volume element ω satisfies $\omega^2 = 1$. Set*

$$\pi^+ = \frac{1}{2}(1 + \omega), \quad \pi^- = \frac{1}{2}(1 - \omega). \tag{2.20}$$

Then π^+ and π^- satisfy

$$\pi^+ + \pi^- = 1, \tag{2.21}$$

$$(\pi^+)^2 = \pi^+, \quad (\pi^-)^2 = \pi^-, \tag{2.22}$$

$$\pi^+\pi^- = \pi^-\pi^+ = 0. \tag{2.23}$$

Using these two elements π^+ and π^- we get the following decomposition

Proposition 2.3.6. *Suppose the volume element ω satisfies $\omega^2 = 1$ and that n is odd. Then $\mathrm{Cl}(V)$ can be decomposed as a direct sum*

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V) \tag{2.24}$$

of isomorphic subalgebras, where $\mathrm{Cl}^\pm(V) = \pi^\pm \cdot \mathrm{Cl}(V) = \mathrm{Cl}(V) \cdot \pi^\pm$. Further $\alpha(\mathrm{Cl}^\pm(V)) = \mathrm{Cl}^\mp(V)$.

Proof. From Proposition 2.3.4, ω is central, so π^+ and π^- are central. The decomposition follows from the properties of π^+ and π^- seen in (2.21), (2.22), and (2.23). Since n is odd, ω is an odd element so $\alpha(\omega) = -\omega$. This gives us that $\alpha(\pi^\pm) = \pi^\mp$ and hence $\alpha(\mathrm{Cl}^\pm(V)) = \mathrm{Cl}^\mp(V)$. The fact that $\mathrm{Cl}^+(V)$ and $\mathrm{Cl}^-(V)$ are isomorphic comes from α being an automorphism. $\square$

Using the above, we get a result on irreducible representations of the Clifford algebra when $n \equiv 3 \pmod 4$.

Proposition 2.3.7. *Let $\rho\colon \mathrm{Cl}(V) \to \mathrm{Hom}(W, W)$ be an irreducible representation where $\dim V \equiv 3 \pmod 4$. Then either $\rho(\omega) = 1$ or $\rho(\omega) = -1$. Both possibilities can occur and they result in inequivalent representations.*

Proof. Since $\dim V \equiv 3 \pmod 4$, $\omega^2 = 1$, so we get $\rho(\omega)^2 = \rho(\omega^2) = 1$. We can then decompose W into $W = W^+ \oplus W^-$ where W^+ and W^- are the $+1$- and -1-eigenspaces for $\rho(\omega)$ respectively. On $W^\pm$, the map $\rho(\omega)$ acts as ± 1 so for $u \in \mathrm{Cl}(V)$ and $w^\pm \in W^\pm$ we get

$$\rho(u)(w^\pm) = \rho(u)((\pm 1)^2 w^\pm) = \rho(u)\rho(\omega)(\pm w^\pm) = \pm\rho(\omega)\rho(u)(w^\pm).$$

where we used the fact that ω is central. This shows that both W^+ and W^- are invariant. Since we assumed that ρ was irreducible, we must either have $W = W^+$ or $W = W^-$.

To show that these representations are inequivalent, we note that if ρ_+ and ρ_- are representations with $\rho_\pm(\omega) = \pm 1$, then given any linear isomorphism L between the spaces, we get $L \circ \rho_+(\omega) \circ L^{-1} = 1 \neq -1 = \rho_-(\omega)$.

To see that both possibilities exist, we can precompose a representation ρ with the automorphism α as $\alpha(\omega) = -\omega$. □

In the $n \equiv 3 \pmod 4$ case, it turns out that these are the only two inequivalent irreducible representations of $\mathrm{Cl}(V)$ (see Theorem 2.3.3). We recall the following containment $\mathrm{Spin}(V) \subseteq \mathrm{Cl}^0(V) \subseteq \mathrm{Cl}(V)$.

Definition 2.3.8. The ***spinor representation*** of $\mathrm{Spin}(V)$ is the homomorphism

$$\Delta_V \colon \mathrm{Spin}(V) \to GL(W) \tag{2.25}$$

given by restricting an irreducible representation $\mathrm{Cl}(V) \to \mathrm{Hom}(W, W)$ to $\mathrm{Spin}(V) \subseteq \mathrm{Cl}^0(V) \subseteq \mathrm{Cl}(V)$.

Previous arguments have shown that when $n \equiv 3 \pmod 4$ we have two different irreducible representations of $\mathrm{Cl}(V)$. The next proposition shows that it does not matter which one we start with as they result in the same spinor representation.

Proposition 2.3.9. *When $n \equiv 3 \pmod 4$ the definition of Δ_V is independent of the irreducible Clifford representation used.*

Proof. From Proposition 2.3.6, we have that the involution α interchanges $\mathrm{Cl}^+(V)$ and $\mathrm{Cl}^-(V)$. Since α acts as the identity on $\mathrm{Cl}^0(V)$, it follows that $\mathrm{Cl}^0(V)$ is diagonal in the decomposition $\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$, that is

$$\mathrm{Cl}^0(V) = \{(u, \alpha(u)) \colon u \in \mathrm{Cl}^+(V)\}. \tag{2.26}$$

The irreducible representations of $\mathrm{Cl}(V)$ differ by α so they match when each is restricted to $\mathrm{Cl}^0(V)$. Since $\mathrm{Spin}(V) \subseteq \mathrm{Cl}^0(V)$, the result holds. □

An important property of Clifford representations is that we are able to endow it with an inner product that makes Clifford multiplication skew-adjoint.

Proposition 2.3.10. *Let* $\mathrm{Cl}(V) \to \mathrm{Hom}(W, W)$ *be a representation of* $\mathrm{Cl}(V)$*. Then there exists an inner product* $\langle \cdot, \cdot \rangle$ *on* W *such that Clifford multiplication by unit vectors is orthogonal. Additionally, with respect to this inner product, Clifford multiplication by any vector* $v \in V$ *is skew-adjoint.*

Proof. Choose an orthonormal basis $e_1, \ldots, e_n$ of V and recall that we can define a group with these elements and -1 via the presentation in (2.16). Call this finite group G. Choose an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on W and average it over this finite group. This results in another inner product on W. That is, we can define a new inner product $\langle \cdot, \cdot \rangle$ by

$$\langle w, w' \rangle = \frac{1}{|G|} \sum_{g \in G} \langle\langle g \cdot w, g \cdot w' \rangle\rangle$$

for $w, w' \in W$.

We note that the new inner product has a couple of important properties. First, we have

$$\begin{aligned}
\langle e_i \cdot w, e_i \cdot w' \rangle &= \frac{1}{|G|} \sum_{g \in G} \langle\langle g e_i \cdot w, g e_i \cdot w' \rangle\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \langle\langle (g e_i^{-1}) e_i \cdot w, (g e_i^{-1}) e_i \cdot w' \rangle\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \langle\langle g \cdot w, g \cdot w' \rangle\rangle \\
&= \langle w, w' \rangle.
\end{aligned}$$

Secondly, we also have

$$\begin{aligned}
\langle e_i \cdot w, w' \rangle &= \frac{1}{|G|} \sum_{g \in G} \langle\langle g e_i \cdot w, g \cdot w' \rangle\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \langle\langle (g e_i) e_i \cdot w, g e_i \cdot w' \rangle\rangle \\
&= -\frac{1}{|G|} \sum_{g \in G} \langle\langle g \cdot w, g e_i \cdot w' \rangle\rangle \\
&= -\langle w, e_i \cdot w' \rangle.
\end{aligned}$$

To check that Clifford multiplication by unit vectors is orthogonal with respect to this inner product, write a unit vector e as $e = \sum_i a_i e_i$ with $\sum_i a_i^2 = 1$. For $w \in W$, we compute

$$\langle e \cdot w, e \cdot w \rangle = \sum_i a_i^2 \langle e_i \cdot w, e_i \cdot w \rangle + \sum_{i \neq j} a_i a_j \langle e_i \cdot w, e_j \cdot w \rangle = \sum_i a_i^2 \langle w, w \rangle = \langle w, w \rangle \quad (2.27)$$

where terms in the second summation vanish as $\langle e_i \cdot w, e_j \cdot w \rangle = -\langle e_j e_i \cdot w, w \rangle = \langle e_i e_j \cdot w, w \rangle = -\langle e_j \cdot w, e_i \cdot w \rangle$.

To show the skew-adjointness of Clifford multiplication, let $0 \neq v \in V$ and $w, w' \in W$. Using the above we can check that

$$\langle v \cdot w, w' \rangle = \langle \frac{v}{\|v\|} v \cdot w, \frac{v}{\|v\|} \cdot w' \rangle = \frac{1}{\|v\|^2} \langle v^2 \cdot w, v \cdot w' \rangle = -\langle w, v \cdot w' \rangle. \quad (2.28)$$

□

Since the spin group $\mathrm{Spin}(V)$ is generated by unit vectors it follows from the previous proposition that the spinor representation Δ_V is orthogonal.

Remark 2.3.11. By choosing an appropriate basis for the inner product space $V \cong \mathbb{R}^n$ we may assume that the inner product space is just $\mathbb{R}^n$ equipped with the standard inner product. It is conventional to suppress the notation by specifying the dimension of the space V instead of the space V itself. We write

$$\begin{aligned} &\mathrm{Cl}(n) \equiv \mathrm{Cl}(V), \quad O(n) \equiv O(V), \quad SO(n) \equiv SO(V), \\ &\mathrm{Pin}(n) \equiv \mathrm{Pin}(V), \quad \mathrm{Spin}(n) \equiv \mathrm{Spin}(V), \quad \Delta_n \equiv \Delta_V. \end{aligned} \quad (2.29)$$

We recall that Proposition 2.2.9 tells us that when the dimension of an inner product space is at least 3 then its spin group $\mathrm{Spin}(V)$ is the universal cover of its special orthogonal group $SO(V)$. Let M be an oriented Riemannian manifold of dimension $n \geq 3$. We can consider the principal $SO(n)$-bundle $P_{SO}(M)$ of oriented orthonormal frames of its tangent bundle. We define a spin structure on M by extending the universal covering map $\xi_0 \colon \mathrm{Spin}(n) \to SO(n)$ to a manifold setting.

Definition 2.3.12. A ***spin structure*** on an oriented Riemannian manifold M of dimension n is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(M)$ with a 2-sheeted covering

$$\xi \colon P_{\mathrm{Spin}}(M) \to P_{SO}(M)$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{Spin}}(M)$ and $g \in \mathrm{Spin}(n)$.

If such a structure exists on a manifold M, we say that M is ***spinnable***. A spinnable manifold with a choice of spin structure is called a ***spin manifold***.

An important result regarding spinnable manifolds is the following theorem. We state it below though the result is not of particular importance in this thesis.

Theorem 2.3.13. *Let M be an oriented Riemannian manifold of dimension n. Then M is spinnable if and only if its second Stiefel–Whitney class $w_2(M)$ vanishes.*

Using the spinor representations discussed above, we can define the spinor bundle of a spin manifold.

Definition 2.3.14. Let M be a spin manifold of dimension n. The ***spinor bundle*** $\mathbb{S}(M)$ of M is the vector bundle associated to the principal $\mathrm{Spin}(n)$-bundle via the spinor representation Δ_n. That is

$$\mathbb{S}(M) = P_{\mathrm{Spin}}(M) \times_{\Delta_n} W. \tag{2.30}$$

A ***spinor field*** is a section of the spinor bundle.

In the sequel, we omit the M in notation for the spinor bundle when it is clear which manifold it lies over. We also omit the word field and refer to sections of the spinor bundle simply as spinors.

From Table 2.1, we have that the Clifford algebras $\mathrm{Cl}(n)$ are of the form $\mathbb{K}(m)$ or $\mathbb{K}(m) \oplus \mathbb{K}(m)$. Using Theorem 2.3.3 and the fact that there is a linear isomorphism between the fibre of an associated bundle and its representation, we can compute the rank of the spinor bundle $\mathbb{S}$. We list these in the table below.

n	1	2	3	4	5	6	7	8
$\mathrm{Cl}(n)$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\operatorname{rank} \mathbb{S}$	2	4	4	8	8	8	8	16

Table 2.2: Rank of the Spinor Bundle for Low Dimensional Manifolds

We recall that a connection on a Riemannian manifold M induces a unique Ehresmann connection on the principal $SO(n)$-bundle $P_{SO}(M)$ of oriented orthonormal frames of TM. If M is also spin, we can use the map ξ to lift this Ehresmann connection onto one on $P_{\mathrm{Spin}}(M)$. This procedure allows us to define the spin connection on $P_{\mathrm{Spin}}(M)$ and on the spinor bundle.

Definition 2.3.15. Let M be a spin manifold of dimension $n \geq 3$. The ***spin connection*** ω^S on $P_{\mathrm{Spin}}(M)$ is the Ehresmann connection obtained by lifting the Ehresmann connection on $P_{SO}(M)$ induced by the Levi-Civita connection via the map ξ. The ***spin connection*** ∇^S on the spinor bundle $\mathbb{S}$ is the connection associated to the spin connection on $P_{\mathrm{Spin}}(M)$.

We list two important properties of the spin connection in the following proposition (proofs of these results can be found in Chapter 2 of [LM89]).

Proposition 2.3.16. *The spin connection ∇^S on the spinor bundle $\mathbb{S}$ is compatible with the metric on $\mathbb{S}$ induced by the spin representation as well as with the Levi-Civita connection on M, that is,*

$$\nabla_X\langle s, s'\rangle = \langle \nabla^S_X s, s'\rangle + \langle s, \nabla^S_X s'\rangle \tag{2.31}$$

and

$$\nabla^S_X(Y \cdot s) = (\nabla_X Y)\cdot s + Y \cdot (\nabla^S_X s) \tag{2.32}$$

for vector fields $X, Y \in \mathfrak{X}$ and spinors $s, s' \in \Gamma(\mathbb{S})$.

2.4 Dirac Bundles and Dirac Operators

We can transport the algebraic structures considered above onto Riemannian manifolds. Since the fibres of the tangent bundle TM of a Riemannian manifold M are inner product spaces, it makes sense to construct the ***bundle of Clifford algebras*** $\mathrm{Cl}(TM)$ whose fibre over a point $x \in M$ is the Clifford algebra $\mathrm{Cl}(T_xM)$ associated to its tangent space T_xM. Additionally we can consider bundles of Clifford modules where the fibre over a point $x \in M$ is a left $\mathrm{Cl}(T_xM)$-module. We would like to differentiate sections of these bundles, as such we require a connection on such a bundle. We choose a connection which satisfies certain compatibility requirements.

Definition 2.4.1. A ***Dirac bundle*** S over a Riemannian manifold M is a bundle of left modules over $\mathrm{Cl}(TM)$ equipped with a Riemannian metric and metric compatible connection satisfying the following conditions:

- The Clifford multiplication of a vector $v \in T_xM$ on S_x is skew-adjoint, that is

$$\langle v \cdot s, s'\rangle = -\langle s, v \cdot s'\rangle; \tag{2.33}$$

- The connection on S is compatible with the Levi-Civita connection on M, that is

$$\nabla_X(Y \cdot s) = (\nabla_X Y)\cdot s + Y \cdot (\nabla_X s) \tag{2.34}$$

  for vector fields X, Y and sections $s \in \Gamma(S)$.

Example 2.4.2. By Proposition 2.3.16, the spinor bundle $\mathbb{S}$ of a spin manifold M equipped with the spin connection ∇^S defined in the previous section is a Dirac bundle.

When M is compact, the inner product on each fibre of a Dirac bundle induces an inner product on $\Gamma(S)$ by integrating over the manifold M given by

$$(s, s') = \int_M \langle s, s' \rangle. \tag{2.35}$$

Definition 2.4.3. The ***Dirac operator*** D of a Dirac bundle S is the first order differential operator on $\Gamma(S)$ defined by the composition

$$\Gamma(S) \longrightarrow \Gamma(T^*M \otimes S) \longrightarrow \Gamma(TM \otimes S) \longrightarrow \Gamma(S) \tag{2.36}$$

where the first arrow is given by the connection, the second arrow is given by using the metric to identify T^*M and TM and the last arrow is given by Clifford multiplication.

With respect to a local orthonormal frame $e_1, \ldots, e_n$ for the tangent bundle, we can write the Dirac operator as

$$Ds = \sum_i e_i \cdot \nabla_i s. \tag{2.37}$$

We have several important properties of the Dirac operator which we summarize in the following results and definitions.

First, we recall the definition of the principal symbol of a differential operator and the definition of an elliptic operator.

Definition 2.4.4. Let E and F be vector bundles over M. The ***principal symbol*** of a differential operator $L \colon \Gamma(E) \to \Gamma(F)$ associates each point $x \in M$ and each cotangent vector $\xi \in T_x^*M$ to a linear map $\sigma_\xi(L) \colon S_x \to S_x$ defined as follows. If L is of order m, we may write in local coordinates

$$L = \sum_\alpha A_\alpha(x) \partial_{|\alpha|}, \quad \xi = \xi_i dx_i$$

where α is taken over all k-tuples of indices with $k \leq m$. The principal symbol is defined to be

$$\sigma_\xi(L) = \sum_{|\alpha|=m} A_\alpha(x) \xi_\alpha. \tag{2.38}$$

Though the principal symbol is defined locally, it is well-defined independent of the choice of coordinate chart.

The operator L is said to be ***elliptic*** if $\sigma_\xi(L)$ is an isomorphism for each $\xi \neq 0$.

The principal symbol satisfies the following identity under compositions of differential operators (see [Nic13]). If E, E', and E'' are vector bundles over M and $L_1 \colon \Gamma(E) \to \Gamma(E')$, $L_2 \colon \Gamma(E') \to \Gamma(E'')$ are differential operators, then for any cotangent vector $\xi \in T^*_x M$, we have

$$\sigma_\xi(L_2 L_1) = \sigma_\xi(L_2)\sigma_\xi(L_1). \tag{2.39}$$

We use this identity in the proof of the following lemma.

Lemma 2.4.5. *The Dirac operator D of a Dirac bundle S is elliptic, as is its square D^2. More specifically, for any $\xi \in T^*_x M$, we have*

$$\sigma_\xi(D) = \xi \cdot \tag{2.40}$$

$$\sigma_\xi(D^2) = -\|\xi\|^2 \tag{2.41}$$

Proof. Pick local coordinates centered at x. For any local trivialization of S with these coordinates, we have $\nabla_i = \partial_i +$ zeroth order terms. Then using the local description of D given by (2.37), we get that at x we have

$$D = \sum_i e_i \cdot \partial_i.$$

It then follows that $\sigma_\xi(D) = \sum_i \xi_i e_i \cdot = \xi \cdot$. By direct calculation, we also see that

$$\sigma_\xi(D^2) = \sigma_\xi(D)\sigma_\xi(D) = \xi \cdot \xi \cdot = -||\xi||^2.$$

□

Proposition 2.4.6. *The Dirac operator D of a Dirac bundle S is formally self-adjoint. That is, for compactly supported sections $s, s' \in \Gamma(S)$,*

$$(Ds, s') = (s, Ds'). \tag{2.42}$$

Proof. We use a local orthonormal frame $\{e_1, \ldots, e_n\}$ such that $\nabla_i e_j = 0$ at the point p to compute that, at the point p, we have

$$\begin{aligned}
\langle Ds, s' \rangle - \langle s, Ds' \rangle &= \sum_i \langle e_i \cdot \nabla_i s, s' \rangle - \langle s, e_i \cdot \nabla_i s' \rangle \\
&= \sum_i \langle \nabla_i (e_i \cdot s), s' \rangle - \langle (\nabla_i e_i) \cdot s, s' \rangle + \langle e_i \cdot s, \nabla_i s' \rangle \\
&= \sum_i \langle \nabla_i (e_i \cdot s), s' \rangle + \langle e_i \cdot s, \nabla_i s' \rangle \\
&= \sum_i \nabla_i \langle e_i \cdot s, s' \rangle \\
&= d^* \omega
\end{aligned}$$

where ω is the 1-form given by $\omega(X) = -(X \cdot s, s')$. By the divergence theorem, the integral over all of M will vanish, proving the result. $\square$

2.5 The Twisted Dirac Operator

As noted in Example 2.4.2 of the previous section, the spinor bundle $\mathbb{S}$ of an n-dimensional spin manifold M is a Dirac bundle. We may embed $\mathbb{S}$ into the bundle $T^*M \otimes \mathbb{S}$ of spinor-valued 1-forms via the map $\iota\colon \mathbb{S} \to T^*M \otimes \mathbb{S}$ defined by

$$\iota(s) = -\frac{1}{n} \sum_i e_i \otimes [e_i \cdot s] \tag{2.43}$$

for a spinor s where the e_i form a local orthonormal frame. The factor of $\frac{1}{n}$ is chosen such that the map μ given by Clifford multiplication defines a left inverse for ι. That is, for a vector field X and a spinor s,

$$\mu(X \otimes s) = X \cdot s, \tag{2.44}$$

so, we have that

$$\mu \circ \iota(s) = \mu(-\sum_i \frac{1}{n} e_i \otimes [e_i \cdot s]) = -\frac{1}{n} \sum_i e_i \cdot (e_i \cdot s) = -\frac{1}{n}(-ns) = s.$$

We get a decomposition

$$T^*M \otimes \mathbb{S} = \mathbb{S}_{\frac{1}{2}} \oplus \mathbb{S}_{\frac{3}{2}} \tag{2.45}$$

of this bundle where we identify $\mathbb{S}_{\frac{1}{2}} = \mathbb{S}$ with its image under ι and set $\mathbb{S}_{\frac{3}{2}} = \ker \mu$. We call these spaces $\mathbb{S}_{\frac{1}{2}}$ and $\mathbb{S}_{\frac{3}{2}}$ the spaces of ***$\frac{1}{2}$-spinors*** and ***$\frac{3}{2}$-spinors*** and denote the projections onto them by $\mathrm{pr}_{\frac{1}{2}}$ and $\mathrm{pr}_{\frac{3}{2}}$ respectively. One can see that $\mathrm{pr}_{\frac{1}{2}} = \iota \circ \mu$ and $\mathrm{pr}_{\frac{3}{2}} = \mathrm{id} - \iota \circ \mu$.

We can use the Dirac operator D on $\Gamma(\mathbb{S}_{\frac{1}{2}})$ to define the ***twisted Dirac operator*** $D_T\colon T^*M \otimes \mathbb{S} \to T^*M \otimes \mathbb{S}$ which is given by $D_T = (\mathrm{id} \otimes \mu) \circ \nabla$. Locally on decomposable elements, we have

$$D_T(X \otimes s) = X \otimes Ds + \sum_i \nabla_i X \otimes [e_i \cdot s]. \tag{2.46}$$

The twisted Dirac operator can be split up with respect to the decomposition $T^*M \otimes \mathbb{S} = \mathbb{S}_{\frac{1}{2}} \oplus \mathbb{S}_{\frac{3}{2}}$ (see [HS19] and [Wan91]). Doing so results in the block matrix form

$$D_T = \begin{bmatrix} \frac{2-n}{n} \iota \circ D \circ \mu & 2\iota \circ P^* \\ \frac{2}{n} P \circ \mu & Q \end{bmatrix}, \tag{2.47}$$

where $P\colon \Gamma(\mathbb{S}_{\frac{1}{2}}) \to \Gamma(\mathbb{S}_{\frac{3}{2}})$ is a first order differential operator called the ***Penrose*** or ***twistor operator*** defined by $P = \mathrm{pr}_{\frac{3}{2}} \circ \nabla$. Its adjoint $P^*\colon \Gamma(\mathbb{S}_{\frac{3}{2}}) \to \Gamma(\mathbb{S}_{\frac{1}{2}})$ can be written as

$$P^*(\tilde{s}) = -\sum_i (\nabla_i \tilde{s})(e_i).$$

The operator $Q\colon \Gamma(\mathbb{S}_{\frac{3}{2}}) \to \Gamma(\mathbb{S}_{\frac{3}{2}})$ is called the ***Rarita–Schwinger operator*** and has importance in physics (see [AGW84] and [Wit85] for example). In addition to D being self-adjoint, the operators D_T and Q are also self-adjoint.

Chapter 3

Preliminaries on G_2-Structures

In this chapter, we introduce the group G_2 by first considering algebraic structures on the imaginary octonions. We then generalize these notions to the tangent space of 7-dimensional Riemannian manifolds to form G_2-structures. Our conventions in this section are consistent with that of [Kar09], [Kar20], and [KLL], which were the main sources for this chapter. Other sources for G_2-geometry include [Bry87],[Bry06], and [Joy00], although their sign conventions run opposite to the ones used here.

3.1 Structures from the Octonions

Let $\mathbb{O}$ denote the normed division algebra of the octonions. As a vector space, we may identify $\mathbb{O}$ with $\mathbb{R}^8$. We recall that the real octonions $\operatorname{Re}\mathbb{O}$ are the real span of the multiplicative identity 1 and that the imaginary octonions $\operatorname{Im}\mathbb{O}$ are the orthogonal complement of $\operatorname{Re}\mathbb{O}$ with respect to the inner product $\langle\cdot,\cdot\rangle$. This gives us the orthogonal splitting

$$\mathbb{O} = \operatorname{Re}\mathbb{O} \oplus \operatorname{Im}\mathbb{O}$$

and we have the ***conjugation*** operator on $\mathbb{O}$ defined by

$$\overline{a} = \operatorname{Re} a - \operatorname{Im} a \tag{3.1}$$

where Re and Im denote the orthogonal projections onto $\operatorname{Re}\mathbb{O}$ and $\operatorname{Im}\mathbb{O}$ respectively. We define two other operators, the ***commutator*** $[\cdot,\cdot]\colon \mathbb{O}\times\mathbb{O}\to\mathbb{O}$ and ***associator*** $[\cdot,\cdot,\cdot]\colon \mathbb{O}\times\mathbb{O}\times\mathbb{O}\to\mathbb{O}$, by

$$[a,b] = ab - ba, \qquad a,b\in\mathbb{O}, \tag{3.2}$$

$$[a, b, c] = (ab)c - a(bc), \qquad a, b, c \in \mathbb{O}. \tag{3.3}$$

Since the octonions are neither commutative nor associative, these operators are non-zero.

In order to define the cross-product and the associative and coassociative forms, we require several identities and lemmas and state them without proof. (For the proofs of these results see Section 3 of [Kar20].) The following results hold more generally for normed division algebras, however, since we are particularly interested in the group G_2 we will continue to use notation specific to the octonions $\mathbb{O}$.

Lemma 3.1.1. *Let $a, b, c \in \mathbb{O}$. Then the following identities hold*

$$\langle ac, bc \rangle = \langle ca, cb \rangle = \|c\|^2 \langle a, b \rangle, \tag{3.4}$$

$$\begin{aligned} \langle a, bc \rangle &= \langle a\bar{c}, b \rangle, \\ \langle a, cb \rangle &= \langle \bar{c}a, b \rangle, \end{aligned} \tag{3.5}$$

$$\overline{ab} = \bar{b}\bar{a}. \tag{3.6}$$

Lemma 3.1.2. *Let $a, b \in \mathbb{O}$. Then we have*

$$\langle a, b \rangle = \mathrm{Re}\,(a\bar{b}) = \mathrm{Re}\,(b\bar{a}) = \mathrm{Re}\,(\bar{b}a) = \mathrm{Re}\,(\bar{a}b) \tag{3.7}$$

and

$$\|a\|^2 = a\bar{a} = \bar{a}a. \tag{3.8}$$

Further $a^2 = aa$ is real if and only if a is either real or imaginary.

Lemma 3.1.3. *Let $a, b \in \mathbb{O}$. Then we have*

$$\begin{aligned} (ab)\bar{b} = a(b\bar{b}) = \|b\|^2 a = a(\bar{b}b) = (a\bar{b})b \\ a(\bar{a}b) = (a\bar{a})b = \|a\|^2 b = (\bar{a}a)b = \bar{a}(ab). \end{aligned} \tag{3.9}$$

Using these identities, we are able to prove certain important properties involving the commutator and associator and their restrictions to the imaginary octonions.

Proposition 3.1.4. *The commutator and associator are both alternating.*

Proof. It is clear from the definition of the commutator that it is alternating. Since $\mathbb{O}$ is an algebra over $\mathbb{R}$, one can see that the associator will vanish if one of the arguments is completely real. Hence we only need to consider the case where all its arguments are imaginary.

Let $a, b \in \operatorname{Im} \mathbb{O}$. Then we have $\overline{a} = -a$ and $\overline{b} = -b$. Using (3.9), we see

$$-[a, a, b] = [a, \overline{a}, b] = (a\overline{a})b - a(\overline{a}b) = 0.$$

Similarly, we have

$$-[a, b, b] = [a, \overline{b}, b] = (a\overline{b})b - a(\overline{b}b) = 0.$$

This shows that the associator is alternating in its first two and last two arguments. To check the first and third arguments, we see $[a, b, a] = -[a, a, b] = 0$, so the associator is alternating. $\square$

Lemma 3.1.5. *If $a, b, c \in \operatorname{Im} \mathbb{O}$, then $[a, b] \in \operatorname{Im} \mathbb{O}$ and $[a, b, c] \in \operatorname{Im} \mathbb{O}$.*

Proof. Using (3.5), we can compute that

$$\langle [a, b], 1 \rangle = \langle ab - ba, 1 \rangle = \langle b, \overline{a} \rangle - \langle b, \overline{a} \rangle = 0.$$

This shows that $[a, b] \in \operatorname{Im} \mathbb{O}$. Similarly, since b, c are imaginary, we have $\overline{b} = -b$ and $\overline{c} = -c$ so (3.6) tells us that $\overline{bc} = \overline{c}\overline{b} = (-c)(-b) = cb$. Hence we see that

$$\begin{aligned} \langle [a, b, c], 1 \rangle &= \langle (ab)c - a(bc), 1 \rangle = \langle ab, \overline{c} \rangle - \langle bc, \overline{a} \rangle \\ &= -\langle ab, c \rangle + \langle bc, a \rangle = -\langle a, c\overline{b} \rangle + \langle a, bc \rangle = \langle a, cb \rangle + \langle a, bc \rangle \\ &= \langle a, cb + bc \rangle = \langle a, \overline{bc} + bc \rangle = 2\langle a, \operatorname{Re}(bc) \rangle = 0. \end{aligned}$$

This finishes the proof. $\square$

The next proposition shows that we can use the commutator and associator in conjunction with the inner product to define multilinear forms on the space of octonions.

Proposition 3.1.6. *Let $a, b, c, d \in \mathbb{O}$. The expressions $\langle a, [b, c] \rangle$ and $\langle a, [b, c, d] \rangle$ are alternating.*

Proof. Proposition 3.1.4 tells us that the commutator and associator are alternating so it suffices to show that $\langle a, [a, b] \rangle = \langle a, [a, b, c] \rangle = 0$. The identities in Lemma 3.1.1 give us that

$$\langle a, [a, b] \rangle = \langle a, ab - ba \rangle = \|a\|^2 \langle 1, b \rangle - \|a\|^2 \langle 1, b \rangle = 0$$

and that

$$\begin{aligned} \langle a, [a, b, c] \rangle &= \langle a, (ab)c - a(bc) \rangle = \langle a\overline{c}, ab \rangle - \|a\|^2 \langle 1, bc \rangle \\ &= \|a\|^2 \langle \overline{c}, b \rangle - \|a\|^2 \langle \overline{c}, b \rangle = 0 \end{aligned}$$

as desired. $\square$

The above allows us to define the associative 3-form and coassociative 4-form on the imaginary octonions.

Definition 3.1.7. Define a 3-form φ and a 4-form ψ on $\operatorname{Im}\mathbb{O}$ by

$$\varphi(a,b,c) = \frac{1}{2}\langle a, [b,c]\rangle = \frac{1}{2}\langle [a,b], c\rangle, \qquad a,b,c \in \operatorname{Im}\mathbb{O} \tag{3.10}$$

$$\psi(a,b,c,d) = \frac{1}{2}\langle a, [b,c,d]\rangle = -\frac{1}{2}\langle [a,b,c], d\rangle, \qquad a,b,c,d \in \operatorname{Im}\mathbb{O}. \tag{3.11}$$

The form φ is called the ***associative* 3*-form*** and the form ψ is called the ***coassociative* 4*-form***.

Identifying $\mathbb{O}$ with $\mathbb{R}^8$ and $\operatorname{Im}\mathbb{O}$ with $\mathbb{R}^7$, we are able to define a cross-product akin to the one on $\mathbb{R}^3$.

Definition 3.1.8. We define the ***octonionic cross-product*** $\times \colon \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \to \operatorname{Im}\mathbb{O}$ to be the bilinear map given by

$$a \times b = \operatorname{Im}(ab). \tag{3.12}$$

This cross-product shares several properties with the familiar cross-product in 3 dimensions.

Lemma 3.1.9. *Let $a,b \in \operatorname{Im}\mathbb{O}$. Then we have*

$$a \times b = -b \times a, \tag{3.13}$$

$$\langle a \times b, a\rangle = 0, \tag{3.14}$$

$$\operatorname{Re}(ab) = -\langle a,b\rangle 1. \tag{3.15}$$

Lastly, we have a couple of nice relations between φ, ψ, and $\times$ (see [Kar20] for more details.)

Proposition 3.1.10. *Let $a,b,c \in \operatorname{Im}\mathbb{O}$. Then*

$$\varphi(a,b,c) = \langle a \times b, c\rangle = \langle ab, c\rangle, \tag{3.16}$$

$$a \times (b \times c) = -\langle a,b\rangle c + \langle a,c\rangle b - \frac{1}{2}[a,b,c] = -\langle a,b\rangle c + \langle a,c\rangle b + (\psi(a,b,c,\cdot))^\sharp \tag{3.17}$$

where $\alpha^\sharp$ denotes the vector dual to α with respect to the inner product.

3.2 The Group G_2

The previous subsection defined several structures on the imaginary octonions $\operatorname{Im}\mathbb{O}$. By making the identification $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$, we can describe the standard G_2-structure on $\mathbb{R}^7$. The ingredients needed to do so are:

- the standard Euclidean metric g_0,
- the standard volume form $\mu_0 = e^1 \wedge \cdots \wedge e^7$ associated to g_0 and the standard orientation, where $e^1, \cdots, e^7$ is the standard orthonormal basis,
- the associative 3-form φ_0,
- the coassociative 4-form ψ_0,
- the octonionic cross-product $\times_0$.

By using the standard dual basis on $(\mathbb{R}^7)^*$ and the octonionic multiplication table we can write φ_0 and ψ_0 as the sum of decomposable forms. In particular, we get

$$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437}, \tag{3.18}$$

$$\psi_0 = e^{4567} - e^{4523} - e^{4163} - e^{4127} - e^{2637} - e^{1537} - e^{1526}, \tag{3.19}$$

where we have written $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$. From the above equations, we see that $\psi_0 = \star_0 \varphi_0$ where $\star_0$ denotes the Hodge star operator induced from the metric g_0 and the volume form μ_0.

We use this standard structure on $\mathbb{R}^7$ to define the group G_2.

Definition 3.2.1. The group G_2 is the subgroup of $GL(7,\mathbb{R})$ that preserves the standard G_2-structure on $\mathbb{R}^7$. Symbolically, we have

$$G_2 = \{A \in GL(7,\mathbb{R}) \mid A^* g_0 = g_0,\ A^*\mu_0 = \mu_0,\ A^*\varphi_0 = \varphi_0,\ A^*\psi_0 = \psi_0,\ A^*\times_0 = \times_0\}. \tag{3.20}$$

Firstly, we note that we can simplify the above definition since certain components of the standard G_2-structure are defined in terms of others. In particular, the metric g_0 and the volume form μ_0 determine the Hodge star operator $\star_0$ which then, along with the associated φ_0 determines the 4-form ψ_0 from the relation $\psi_0 = \star_0\varphi_0$. Additionally, (3.16) shows that the cross-product $\times_0$ is also determined from the metric g_0 and the 3-form φ_0. Hence we can write

$$G_2 = \{A \in GL(7,\mathbb{R}) \mid A^* g_0 = g_0,\ A^*\mu_0 = \mu_0,\ A^*\varphi_0 = \varphi_0\}. \tag{3.21}$$

We also note that since G_2 preserves both the metric and orientation, it is a subgroup of $SO(7,\mathbb{R})$. A theorem found in [Bry87] tells us that we can simplify this definition even further.

Theorem 3.2.2. *Let $A \in GL(7,\mathbb{R})$. If A preserves φ_0, then A also preserves g_0 and μ_0. In particular, $G_2 = \{A \in GL(7,\mathbb{R}) \mid A^*\varphi_0 = \varphi_0\}$.*

Proof. Using the expression for φ_0 given by (3.18), direct computation shows that for $a, b \in \mathbb{R}^7$

$$(a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(a,b)\mu_0. \tag{3.22}$$

If A preserves φ_0, then by applying A^* to both sides of the above equation and scaling we see that

$$\begin{aligned} g_0(a,b)(\det A)\mu_0 &= g_0(a,b)A^*\mu_0 \\ &= -\frac{1}{6}A^*(a \lrcorner \varphi_0) \wedge A^*(b \lrcorner \varphi_0) \wedge A^*\varphi_0 \\ &= -\frac{1}{6}(A^{-1}a \lrcorner A^*\varphi_0) \wedge (A^{-1}b \lrcorner A^*\varphi_0) \wedge A^*\varphi_0 \\ &= -\frac{1}{6}(A^{-1}a \lrcorner \varphi_0) \wedge (A^{-1}b \lrcorner \varphi_0) \wedge \varphi_0 \\ &= g_0(A^{-1}a, A^{-1}b)\mu_0 \\ &= (A^{-1})^* g(a,b)\mu_0. \end{aligned} \tag{3.23}$$

Hence $(\det A) g_0(Aa, Ab) = g_0(a,b)$. Looking at the matrices associated with these operations we have $g_0 = (\det A)A^T g_0 A$. We take the determinants of both sides to get that $\det g_0 = (\det A)^9 \det g_0$. It follows that $\det A = 1$ and so $A^* g_0 = g_0$. Equation (3.23) then says that $A^*\mu_0 = \mu_0$. $\square$

3.3 G_2-Structures on Manifolds

In this section, we move the G_2-structure defined above on $\mathbb{R}^7$ onto a smooth 7-dimensional manifold.

Definition 3.3.1. A G_2***-structure*** on a smooth 7-dimensional manifold M is a smooth 3-form φ on M such that at every $x \in M$, there exists a linear isomorphism $T_xM \cong \mathbb{R}^7$ with respect to which $\varphi_x \in \Lambda^3(T_x^*M)$ corresponds to the associative 3-form $\varphi_0 \in \Lambda^3((\mathbb{R}^7)^*)$.

Since φ_0 induces both g_0 and μ_0, the 3-form φ on M induces a Riemannian metric g_φ and a Riemannian volume form μ_φ on M. In turn, these determine a Hodge star $\star_\varphi$ and the coassociative 4-form $\psi_\varphi = \star_\varphi \varphi$. We often omit the φ-subscript when discussing φ and its associated structures.

As such, if φ is a G_2-structure then at every point $x \in M$, there exists a basis $\{e_1, \ldots, e_7\}$ of T_xM with respect to which $\varphi_x = \varphi_0$. In general, this can only be done at a single point. We cannot choose a local frame on an open set with this property.

Not every smooth 7-dimensional manifold admits a G_2-structure. A G_2-structure is equivalent to a reduction of the structure group of the frame bundle of M from $GL(7, \mathbb{R})$ to G_2. This means that the existence of a G_2-structure is dependent on the topology of the manifold. The following result characterizes which manifolds admit such a structure (see [LM89]).

Proposition 3.3.2. *A smooth* 7*-dimensional manifold* M *admits a* G_2*-structure if and only if* M *is orientable and spinnable or equivalently if its first two Stiefel–Whitney classes* $w_1(M)$ *and* $w_2(M)$ *vanish.*

There are several important identities which will be used heavily throughout this thesis involving contractions of the tensors φ and ψ with each other. We list them below without proof. Details discussing their derivations can be found in [Kar09].

Theorem 3.3.3. *In local coordinates on* M*, the tensors* $g, \varphi,$ *and* ψ *sastisfy the following relations:*

Contractions of φ *with* φ*:*

$$\varphi_{ijk}\varphi_{abk} = g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}, \tag{3.24}$$

$$\varphi_{ijk}\varphi_{ajk} = 6g_{ia}, \tag{3.25}$$

$$\varphi_{ijk}\varphi_{ijk} = 42. \tag{3.26}$$

Contractions of φ *with* ψ*:*

$$\varphi_{ijk}\psi_{abck} = g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj} - g_{aj}\varphi_{ibc} - g_{bj}\varphi_{aic} - g_{cj}\varphi_{abi}, \tag{3.27}$$

$$\varphi_{ijk}\psi_{abjk} = -4\varphi_{iab}, \tag{3.28}$$

$$\varphi_{ijk}\psi_{aijk} = 0. \tag{3.29}$$

Contractions of ψ with ψ:

$$\psi_{ijkl}\psi_{abkl} = 4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}, \tag{3.30}$$

$$\psi_{ijkl}\psi_{ajkl} = 24g_{ia}, \tag{3.31}$$

$$\psi_{ijkl}\psi_{ijkl} = 168. \tag{3.32}$$

3.4 Decomposition of Forms on Manifolds with G_2-Structure

On a manifold M with G_2-structure φ, the bundle $\Lambda^{\cdot}(T^*M) = \oplus_{k=1}^{7}\Lambda^k(T^*M)$ decomposes fibrewise into irreducible representations of the group G_2. This gives a decomposition of the space Ω^k of smooth k-forms on M. These decompositions of the spaces of 2, 3, and 4-forms on a manifold M with G_2-structure will be important as they are closely related to the space of 2-tensors on M. The identities proven in this section will be especially useful when we study the bundle of spinor-valued 1-forms on M due to the identification $T^*M \otimes \mathbb{S} \cong T^*M \oplus (T^*M \otimes T^*M)$.

Theorem 3.2.2 has shown that each tensor that can be determined by the 3-form φ will be invariant under the group G_2. As such, any subspaces of Ω^k defined using φ and its associated structures will be G_2 representations. In particular, the space Ω^k decompose non-trivially into G_2 subrepresentations as follows:

$$\begin{aligned} \Omega^2 &= \Omega^2_7 \oplus \Omega^2_{14}, \qquad & \Omega^3 &= \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}, \\ \Omega^4 &= \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}, \qquad & \Omega^5 &= \Omega^5_7 \oplus \Omega^5_{14}. \end{aligned} \tag{3.33}$$

The spaces Ω^k_l of k-l-forms have pointwise dimension l and the decomposition is orthogonal with respect to the metric g. Further, the decompositions of Ω^4 and Ω^5 are obtained by taking the Hodge star of those of Ω^3 and Ω^2 respectively.

Invariantly, we can describe the decompositions of Ω^2 and Ω^3 by

$$\Omega^2_7 = \{X \lrcorner \varphi \mid X \in \mathfrak{X}\}, \tag{3.34}$$

$$\Omega^2_{14} = \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\}, \tag{3.35}$$

$$\Omega^3_1 = \{f\varphi \mid f \in C^\infty(M)\}, \tag{3.36}$$

$$\Omega^3_7 = \{X \lrcorner \psi \mid X \in \mathfrak{X}\}, \tag{3.37}$$

$$\Omega^3_{27} = \{\gamma \in \Omega^3 \mid \gamma \wedge \varphi = \gamma \wedge \psi = 0\}. \tag{3.38}$$

3.4.1 Decomposition of 2-Forms

There exist alternate characterizations of the subspaces shown above. We begin with the subspaces of Ω^2. Consider the map $\mathcal{P}\colon \Omega^2 \to \Omega^2$ given by $\mathcal{P}\beta = 2 \star (\varphi \wedge \beta)$. Write β in local coordinates as $\beta = \frac{1}{2}\beta_{ij} dx^i \wedge dx^j$. Then we have

$$\begin{aligned}
\mathcal{P}\beta &= \beta_{ij} \star (dx^i \wedge dx^j \wedge \varphi) = \beta_{ij}\partial_i \lrcorner \star(dx^j \wedge \varphi) \\
&= -\beta_{ij}\partial_i \lrcorner \partial_j \lrcorner \star\varphi = -\beta_{ij}\left(\frac{1}{2}\psi_{jikl} dx^k \wedge dx^l\right) \\
&= \frac{1}{2}\beta_{ij}\psi_{ijkl} dx^k \wedge dx^l.
\end{aligned}$$

So if $\mathcal{P}\beta = \frac{1}{2}(\mathcal{P}\beta)_{ij} dx^i \wedge dx^j$, we have

$$(\mathcal{P}\beta)_{kl} = \beta_{ij}\psi_{ijkl}. \tag{3.39}$$

Using the inner product on 2-forms induced by g, we see that

$$\langle \mathcal{P}\beta, \widetilde{\beta} \rangle = \frac{1}{2}\beta_{ij}\psi_{ijkl}\widetilde{\beta}_{kl} = \langle \beta, \mathcal{P}\widetilde{\beta} \rangle. \tag{3.40}$$

Hence $\mathcal{P}$ is self-adjoint and thus orthogonally diagonalizable with real eigenvalues. Computing $\mathcal{P}^2$ in coordinates gives

$$\begin{aligned}
(\mathcal{P}^2\beta)_{ab} &= (P\beta)_{kl}\psi_{klab} = \beta_{ij}\psi_{ijkl}\psi_{klab} \\
&= \beta_{ij}(4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}) = 4\beta_{ab} - 4\beta_{ba} - 2\beta_{ij}\psi_{ijab} \\
&= 8\beta_{ab} - 2(\mathcal{P}\beta)_{ab}.
\end{aligned}$$

We see that the operator $\mathcal{P}$ satisfies $\mathcal{P}^2 = 8I - 2\mathcal{P}$. Factoring this gives $(\mathcal{P}+4I)(\mathcal{P}-2I) = 0$ so $\mathcal{P}$ has two eigenvalues, -4 and $+2$. We now verify that these eigenspaces correspond to the spaces Ω^2_7 and Ω^2_{14} defined earlier.

Proposition 3.4.1. *The following descriptions of Ω^2_7 and Ω^2_{14} hold:*

$$\Omega^2_7 = \{\beta \in \Omega^2 \mid \mathcal{P}\beta = -4\beta\}, \tag{3.41}$$

$$\Omega^2_{14} = \{\beta \in \Omega^2 \mid \mathcal{P}\beta = 2\beta\}. \tag{3.42}$$

Proof. Let $X \in \mathfrak{X}$ and consider the 2-form $\beta = X \lrcorner \varphi$. Direct computation using (3.28) yields

$$(\mathcal{P}\beta)_{kl} = \beta_{ij}\psi_{ijkl} = X_m\varphi_{mij}\psi_{ijkl} = -4X_m\varphi_{mkl} = -4\beta_{kl}.$$

Conversely, suppose β satisfies $\beta_{ij}\psi_{ijkl} = -4\beta_{kl}$. Define a vector field X by $X_m = \frac{1}{6}\beta_{kl}\varphi_{mkl}$. We compute using (3.24) that

$$\begin{aligned}(X \lrcorner \varphi)_{ij} = X_m\varphi_{mij} &= \frac{1}{6}\beta_{kl}\varphi_{mkl}\varphi_{mij} = \frac{1}{6}\beta_{kl}(g_{ik}g_{jl} - g_{il}g_{jk} - \psi_{ijkl}) \\ &= \frac{1}{6}\beta_{ij} - \frac{1}{6}\beta_{ji} - \frac{1}{6}\beta_{kl}\psi_{klij} = \frac{1}{6}\beta_{ij} + \frac{1}{6}\beta_{ij} + \frac{4}{6}\beta_{ij} = \beta_{ij}.\end{aligned}$$

This proves the first relation.

For the second relation, suppose that $\beta \wedge \psi = 0$. In local coordinates, this becomes

$$\begin{aligned}0 = \beta \wedge \psi &= \frac{1}{2}\beta_{ij}dx^i \wedge dx^j \wedge \psi = -\frac{1}{2}\beta_{ij} \star (\partial_i \lrcorner \star(dx^j \wedge \psi)) \\ &= -\frac{1}{2}\beta_{ij} \star (\partial_i \lrcorner \partial_j \lrcorner \star\psi) = -\frac{1}{2} \star (\beta_{ij}\partial_i \lrcorner \partial_j \lrcorner \varphi).\end{aligned}$$

Since the Hodge star is an isomorphism, it follows that this is equivalent to $\beta_{ij}\varphi_{ijm} = 0$. Next, since the eigenspace decomposition is orthogonal, it follows that if $\mathcal{P}\beta = 2\beta$, then $\beta_{ij}X_m\varphi_{mij} = 0$ for each vector field X. Hence $\beta_{ij}\varphi_{ijm} = 0$, giving one side of the equivalence. Conversely if $\beta_{ij}\varphi_{ijm} = 0$, (3.24) gives that

$$\beta_{ij}\psi_{ijab} = \beta_{ij}(g_{ia}g_{jb} - g_{ib}g_{ja} - \varphi_{ijk}\varphi_{abk}) = \beta_{ab} - \beta_{ba} = 2\beta_{ab}$$

as desired. □

As corollaries of the above, we have the following equivalences in local coordinates, as well as expressions for the projection operators pr_7 and pr_{14} from Ω^2 onto Ω^2_7 and Ω^2_{14} respectively. We write $\beta_7 = \mathrm{pr}_7\beta$ and $\beta_{14} = \mathrm{pr}_{14}\beta$ for $\beta \in \Omega^2$.

Corollary 3.4.2. *Let $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j \in \Omega^2$. The following equivalences hold:*

$$\beta \in \Omega^2_7 \iff \beta_{ij}\psi_{ijkl} = -4\beta_{kl} \iff \beta_{ij} = X_k\varphi_{ijk}, \tag{3.43}$$

$$\beta \in \Omega^2_{14} \iff \beta_{ij}\psi_{ijkl} = 2\beta_{kl} \iff \beta_{ij}\varphi_{ijm} = 0. \tag{3.44}$$

Furthermore, we have that

$$\beta = X \lrcorner \varphi \iff X_k = \frac{1}{6}\beta_{ij}\varphi_{ijk}. \tag{3.45}$$

Corollary 3.4.3. *Let* $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j \in \Omega^2$. *Then*

$$\beta_7 = \frac{1}{3}\beta - \frac{1}{6}\mathcal{P}\beta = \frac{1}{2}\left(\frac{1}{3}\beta_{kl} - \frac{1}{6}\beta_{ij}\psi_{ijkl}\right)dx^k \wedge dx^l, \tag{3.46}$$

$$\beta_{14} = \frac{2}{3}\beta + \frac{1}{6}\mathcal{P}\beta = \frac{1}{2}\left(\frac{2}{3}\beta_{kl} + \frac{1}{6}\beta_{ij}\psi_{ijkl}\right)dx^k \wedge dx^l. \tag{3.47}$$

We have one other important relation pertaining to 2-14-forms.

Lemma 3.4.4. *Let* $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j \in \Omega^2_{14}$. *Then*

$$\beta_{am}\varphi_{mij} = \beta_{im}\varphi_{maj} - \beta_{jm}\varphi_{mai}. \tag{3.48}$$

Proof. Since $\beta \in \Omega^2_{14}$, we have that $\beta_{am} = \frac{1}{2}\beta_{kl}\psi_{klam}$. Using Equations (3.27) and (3.44), we compute that

$$\begin{aligned}
\beta_{am}\varphi_{mij} &= \frac{1}{2}\beta_{kl}\psi_{klam}\varphi_{mij} \\
&= \frac{1}{2}\beta_{kl}(g_{ik}\varphi_{jla} + g_{il}\varphi_{kja} + g_{ia}\varphi_{klj} - g_{kj}\varphi_{ila} - g_{lj}\varphi_{kia} - g_{aj}\varphi_{kli}) \\
&= \frac{1}{2}\beta_{il}\varphi_{jla} + \frac{1}{2}\beta_{ki}\varphi_{kja} - \frac{1}{2}\beta_{jl}\varphi_{ila} - \frac{1}{2}\beta_{kj}\varphi_{kia} \\
&= \beta_{im}\varphi_{maj} - \beta_{jm}\varphi_{mai}.
\end{aligned}$$

□

3.4.2 Decomposition of 3- and 4-Forms

We now turn our attention to the decomposition of the spaces of 3- and 4-forms on manifolds with G_2-structures which we analyze using maps from the space of 2-tensors to the spaces of 3- and 4-forms. The methods and notations presented here are largely taken from [KLL].

Using the orthogonal decomposition of Ω^2 from the previous section, we can decompose the space $\mathcal{T}^2$ of 2-tensors further as

$$\mathcal{T}^2 = \Omega^0 \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}. \tag{3.49}$$

With respect to this splitting, we can write a 2-tensor A as

$$A = A_1 + A_{27} + A_7 + A_{14} = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7 + A_{14}. \tag{3.50}$$

By using the coordinate expression of the operator $\mathcal{P}$ in (3.39), we can extend it to all of $\mathcal{T}^2$ by setting

$$(\mathcal{P}A)_{kl} = A_{ij}\psi_{ijkl}. \tag{3.51}$$

We can see that $\ker \mathcal{P} = \mathcal{S}^2$ and from the discussion above, we get

$$\mathcal{P}\left(\frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7 + A_{14}\right) = -4A_7 + 2A_{14}. \tag{3.52}$$

We look at ways to obtain 3- and 4-forms from a 2-tensor. Given a 3-form $\gamma \in \Omega^3$, a 4-form $\eta \in \Omega^4$ and 2-tensor $A \in \mathcal{T}^2$, we define a new 3- and 4-tensor by

$$(A \diamond \gamma)_{ijk} = A_{im}\gamma_{mjk} + A_{jm}\gamma_{imk} + A_{km}\gamma_{ijm}. \tag{3.53}$$

and

$$(A \diamond \eta)_{ijkl} = A_{im}\eta_{mjkl} + A_{jm}\eta_{imkl} + A_{km}\eta_{ijml} + A_{lm}\eta_{ijkm}. \tag{3.54}$$

Routine calculations show that $A \diamond \gamma$ and $A \diamond \eta$ are skew in each of their indices. Using the 3-form φ and the 4-form ψ, this gives us linear maps $\mathcal{T}^2 \to \Omega^3$ given by $A \mapsto A \diamond \varphi$ and $\mathcal{T}^2 \to \Omega^4$ given by $A \mapsto A \diamond \psi$. We also note that the definitions above are such that

$$A \diamond \gamma = A_{ij}dx^i \wedge (\partial_j \lrcorner \gamma) \tag{3.55}$$

and

$$A \diamond \eta = A_{ij}dx^i \wedge (\partial_j \lrcorner \eta). \tag{3.56}$$

Proposition 3.4.5. *Let $A, B \in \mathcal{T}^2$. Then we have*

$$\langle A \diamond \varphi, B \diamond \varphi \rangle = \frac{9}{7}(\operatorname{tr} A)(\operatorname{tr} B) + 2\langle A_{27}, B_{27}\rangle + 6\langle A_7, B_7\rangle \tag{3.57}$$

and

$$\langle A \diamond \psi, B \diamond \psi \rangle = \frac{16}{7}(\operatorname{tr} A)(\operatorname{tr} B) + 2\langle A_{27}, B_{27}\rangle + 6\langle A_7, B_7\rangle \tag{3.58}$$

Proof. We note that each of φ, $A \diamond \varphi$, and $B \diamond \varphi$ are skew-symmetric in their indices. Using this, (3.24) and (3.25), we compute that

$$\begin{aligned}
\langle A \diamond \varphi, B \diamond \varphi \rangle &= \frac{1}{6}(A \diamond \varphi)_{ijk}(B \diamond \varphi)_{ijk} \\
&= \frac{1}{6}(A_{im}\varphi_{mjk} + A_{jm}\varphi_{imk} + A_{km}\varphi_{ijm})(B \diamond \varphi)_{ijk} \\
&= \frac{3}{6}A_{im}\varphi_{mjk}(B \diamond \varphi)_{ijk} \\
&= \frac{1}{2}A_{im}\varphi_{mjk}(B_{il}\varphi_{ljk} + B_{jl}\varphi_{ilk} + B_{kl}\varphi_{ijl}) \\
&= \frac{1}{2}A_{im}B_{il}\varphi_{mjk}\varphi_{ljk} + \frac{2}{2}A_{im}B_{jl}\varphi_{mjk}\varphi_{ilk} \\
&= \frac{1}{2}A_{im}B_{il}(6g_{ml}) + A_{im}B_{jl}(g_{mi}g_{jl} - g_{ml}g_{ji} - \psi_{mjil}) \\
&= 3\langle A, B\rangle + (\operatorname{tr} A)(\operatorname{tr} B) - \langle A, B\rangle - \langle \mathcal{P}A, B\rangle \\
&= 2\langle A, B\rangle + (\operatorname{tr} A)(\operatorname{tr} B) + 4\langle A_7, B\rangle - 2\langle A_{14}, B\rangle.
\end{aligned}$$

Using the orthogonality of the decomposition (3.49), we can expand the above to get

$$\begin{aligned}
\langle A \diamond \varphi, B \diamond \varphi \rangle = \frac{2}{49}&(\operatorname{tr} A)(\operatorname{tr} B)\langle g, g\rangle + 2\langle A_{27}, B_{27}\rangle + 2\langle A_7, B_7\rangle + 2\langle A_{14}, B_{14}\rangle \\
&+ (\operatorname{tr} A)(\operatorname{tr} B) + 4\langle A_7, B_7\rangle - 2\langle A_{14}, B_{14}\rangle \\
= \frac{9}{7}&(\operatorname{tr} A)(\operatorname{tr} B) + 2\langle A_{27}, B_{27}\rangle + 6\langle A_7, B_7\rangle.
\end{aligned}$$

as desired. The second identity (3.58) is proved in a similar manner. □

We get the following as a corollary to the above proposition.

Corollary 3.4.6. *A 2-tensor A is in Ω^2_{14} if and only if $A \diamond \varphi = 0$ if and only if $A \diamond \psi = 0$. Furthermore, when restricted to the orthogonal complement of Ω^2_{14}, the map $A \mapsto A \diamond \varphi$ defines a linear isomorphism onto Ω^3 and the map $A \mapsto A \diamond \psi$ defines a linear isomorphism onto Ω^4.*

Proof. If $A \in \Omega^2_{14}$, then $A = A_{14}$ and so Proposition 3.4.5 tells us that $|A \diamond \varphi|^2 = 0$. Conversely, if $A \diamond \varphi = 0$, then $|A \diamond \varphi|^2 = \frac{9}{7}(\operatorname{tr} A)^2 + 2|A_{27}|^2 + 6|A_7|^2 = 0$. It follows that $A \in \Omega^2_{14}$. From this, we see that the map $A \mapsto A \diamond \varphi$ is injective on $\mathcal{S}^2 \oplus \Omega^2_7$. Similarly, the map $A \mapsto A \diamond \psi$ is injective. By counting dimensions, we see that each of these spaces are (pointwise) 35-dimensional, hence the maps are isomorphisms. □

The above results establish orthogonal decompositions of the spaces Ω^3 and Ω^4. We verify that this decomposition matches that of the one given in (3.36), (3.37), and (3.38).

Proposition 3.4.7. *The following descriptions of Ω^3_1, Ω^3_7 and Ω^3_{27} hold:*

$$\Omega^3_1 = \{A \diamond \varphi \mid A \in \Omega^0\}, \tag{3.59}$$

$$\Omega^3_7 = \{A \diamond \varphi \mid A \in \Omega^2_7\}, \tag{3.60}$$

$$\Omega^3_{27} = \{A \diamond \varphi \mid A \in \mathcal{S}^2_0\}. \tag{3.61}$$

Proof. The first equivalence holds since if $A \in \Omega^0$, then $A = fg$ for some $f \in C^\infty(M)$, and so $A \diamond \varphi = f(g \diamond \varphi) = 3f\varphi$. To see the second equivalence, suppose $A \in \Omega^2_7$, so $A_{ij} = X_m \varphi_{mij}$ for some vector field X. Using (3.24) we can compute that

$$\begin{aligned}
(A \diamond \varphi)_{ijk} &= A_{im}\varphi_{mjk} + A_{jm}\varphi_{imk} + A_{km}\varphi_{ijm} \\
&= X_l \varphi_{lim}\varphi_{mjk} + X_l \varphi_{ljm}\varphi_{imk} + X_l \varphi_{lkm}\varphi_{ijm} \\
&= X_l(g_{lj}g_{ik} - g_{lk}g_{ij} - \psi_{lijk}) + X_l(g_{lk}g_{ji} - g_{li}g_{jk} - \psi_{ljki}) \\
&\quad + X_l(g_{li}g_{kj} - g_{lj}g_{ki} - \psi_{lkij}) \\
&= X_j g_{ik} - X_k g_{ij} - X_l \psi_{lijk} + X_k g_{ij} - X_i g_{jk} - X_l\psi_{lijk} + X_i g_{jk} - X_j g_{ik} - X_l \psi_{lijk} \\
&= -3X_l \psi_{lijk} = -3(X \lrcorner \psi)_{ijk}
\end{aligned}$$

The calculations above also show that if $\gamma = X \lrcorner \psi$ for some vector field X, then $\gamma = -\frac{1}{3}(X \lrcorner \varphi) \diamond \varphi$. This gives the second equivalence.

Next, let $A \in \mathcal{S}^2 \oplus \Omega^2_7$ be a 2-tensor. We can calculate that

$$\begin{aligned}
\star[(A \diamond \varphi) \wedge \varphi] &= (A \diamond \varphi)_{ijk} \star (dx^i \wedge dx^j \wedge dx^k \wedge \varphi) \\
&= -(A \diamond \varphi)_{ijk} \partial_i \lrcorner \star (dx^j \wedge dx^k \wedge \varphi) \\
&= -(A \diamond \varphi)_{ijk} \partial_i \lrcorner \partial_j \lrcorner \star (dx^k \wedge \varphi) \\
&= (A \diamond \varphi)_{ijk} \partial_i \lrcorner \partial_j \lrcorner \partial_k \lrcorner \psi \\
&= (A \diamond \varphi)_{ijk} \psi_{kjil} dx^l \\
&= -(A \diamond \varphi)_{ijk} \psi_{ijkl} dx^l.
\end{aligned}$$

Plugging in our expression for $(A \diamond \varphi)_{ijk}$ we get

$$\begin{aligned}
-(A \diamond \varphi)_{ijk}\psi_{ijkl} &= -A_{im}\varphi_{mjk}\psi_{ijkl} - A_{jm}\varphi_{imk}\psi_{ijkl} - A_{km}\varphi_{ijm}\psi_{ijkl} \\
&= 4A_{im}\varphi_{mil} + 4A_{jm}\varphi_{mjl} + 4A_{km}\varphi_{mkl} \\
&= -12 A_{ij}\varphi_{ijl},
\end{aligned}$$

and so $(A \diamond \varphi) \wedge \varphi = 0$ if and only if A is symmetric.

Similarly, we compute that

$$\begin{aligned}
\star[(A \diamond \varphi) \wedge \psi] &= (A \diamond \varphi)_{ijk} \star (dx^i \wedge dx^j \wedge dx^k \wedge \psi) \\
&= (A \diamond \varphi)_{ijk} \partial_i \lrcorner \star (dx^j \wedge dx^k \wedge \psi) \\
&= -(A \diamond \varphi)_{ijk} \partial_i \lrcorner \partial_j \lrcorner \star (dx^k \wedge \psi) \\
&= -(A \diamond \varphi)_{ijk} \partial_i \lrcorner \partial_j \lrcorner \partial_k \lrcorner \varphi \\
&= -(A \diamond \varphi)_{ijk} \varphi_{kji} \\
&= (A \diamond \varphi)_{ijk} \varphi_{ijk}.
\end{aligned}$$

Substituting in our expression for $(A \diamond \varphi)_{ijk}$ again, we get

$$\begin{aligned}
(A \diamond \varphi)_{ijk} \varphi_{ijk} &= A_{im} \varphi_{mjk} \varphi_{ijk} + A_{jm} \varphi_{imk} \varphi_{ijk} + A_{km} \varphi_{ijm} \varphi_{ijk} \\
&= 6A_{ii} + 6A_{jj} + 6A_{kk} \\
&= 18 \operatorname{tr} A,
\end{aligned}$$

hence $(A \diamond \varphi) \wedge \psi = 0$ if and only if A is traceless. $\square$

Since we can obtain a decomposition of Ω^4 from that of the one on Ω^3 via the Hodge star, we have the following as well:

Corollary 3.4.8. *We have an orthogonal decomposition of* $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$ *where*

$$\Omega^4_1 = \star\Omega^3_1 = \{f\psi \mid f \in C^\infty(M)\} = \{A \diamond \psi \mid A \in \Omega^0\}, \tag{3.62}$$

$$\Omega^4_7 = \star\Omega^3_7 = \{X \wedge \varphi \mid X \in \mathfrak{X}\} = \{A \diamond \psi \mid A \in \Omega^2_7\}, \tag{3.63}$$

$$\Omega^4_{27} = \star\Omega^4_{27} = \{\eta \in \Omega^4 \mid \star\eta \wedge \varphi = \star\eta \wedge \psi = 0\}. \tag{3.64}$$

Proof. As in the proof of Proposition 3.4.7, we notice that if $A = fg$, then $A \diamond \psi = 4f\psi$ and that if $A = X \lrcorner \varphi$ then

$$\begin{aligned}
(A \diamond \psi)_{ijkl} &= A_{in} \psi_{njkl} + A_{jn} \psi_{inkl} + A_{kn} \psi_{ijnl} + A_{ln} \psi_{ijkn} \\
&= X_m \varphi_{min} \psi_{njkl} + X_m \varphi_{mjn} \psi_{inkl} + X_m \varphi_{mkn} \psi_{ijnl} + X_m \varphi_{mln} \varphi_{ijkn} \\
&= -X_m (g_{mj} \varphi_{ikl} + g_{mk} \varphi_{jil} + g_{ml} \varphi_{jki} - g_{ij} \varphi_{mkl} - g_{ik} \varphi_{jml} - g_{il} \varphi_{jkm}) \\
&\quad + X_m (g_{mi} \varphi_{jkl} + g_{mk} \varphi_{ijl} + g_{ml} \varphi_{ikj} - g_{ji} \varphi_{mkl} - g_{jk} \varphi_{iml} - g_{jl} \varphi_{ikm}) \\
&\quad - X_m (g_{mi} \varphi_{kjl} + g_{mj} \varphi_{ikl} + g_{ml} \varphi_{ijk} - g_{ki} \varphi_{mjl} - g_{kj} \varphi_{iml} - g_{kl} \varphi_{ijm}) \\
&\quad + X_m (g_{mi} \varphi_{ljk} + g_{mj} \varphi_{ilk} + g_{mk} \varphi_{ijl} - g_{li} \varphi_{mjk} - g_{lj} \varphi_{imk} - g_{lk} \varphi_{ijm}) \\
&= 3(X_i \varphi_{jkl} - X_j \varphi_{ikl} + X_k \varphi_{ijl} - X_l \varphi_{ijk})
\end{aligned}$$

and so $A \diamond \psi$ is some non-zero scalar multiple of $X \wedge \varphi$.

Now, let $\eta \in \Omega^4$. We have that $\eta = A \diamond \psi$ for some 2-tensor $A \in \mathcal{S}^2 \oplus \Omega^2_7$. We compute that

$$\begin{aligned}
\star(\star\eta \wedge \psi) &= \langle \eta, \psi \rangle \\
&= \frac{1}{24}\eta_{ijkl}\psi_{ijkl} \\
&= \frac{1}{24}(A_{im}\psi_{mjkl}\psi_{ijkl} + A_{jm}\psi_{imkl}\psi_{ijkl} + A_{km}\psi_{ijml}\psi_{ijkl} + A_{lm}\psi_{ijkm}\psi_{ijkl}) \\
&= \frac{1}{24}(24A_{im}g_{mi} + 24A_{jm}g_{mj} + 24A_{km}g_{mk} + 24A_{lm}g_{ml}) \\
&= 4\operatorname{tr} A.
\end{aligned}$$

Using (3.55) and (3.56), we have

$$\begin{aligned}
\star(A \diamond \psi) &= \star(A_{ij}dx^i \wedge (\partial_j \lrcorner \psi)) \\
&= -A_{ij}\partial_i \lrcorner \star(\partial_j \lrcorner \psi) \\
&= A_{ij}\partial_i \lrcorner (dx^j \wedge \varphi) \\
&= A_{ij}(\partial_i \lrcorner dx^j) \wedge \varphi - A_{ij}dx^j \wedge (\partial_i \lrcorner \varphi) \\
&= (\operatorname{tr} A)\varphi - A^T \diamond \varphi,
\end{aligned}$$

where $A^T_{ij} = A_{ji}$. Taking the wedge product with φ then gives

$$\star(A \diamond \psi) \wedge \varphi = (\operatorname{tr} A)\varphi \wedge \varphi - (A^T \diamond \varphi) \wedge \varphi = -(A^T \diamond \varphi) \wedge \varphi.$$

From the above and by the proof of Proposition 3.4.7, we see that $\star\eta \wedge \psi = 0$ if and only if A is traceless and that $\star\eta \wedge \varphi = 0$ if and only if A is symmetric. $\square$

We end this section with explicit descriptions of the inverses of the maps $A \mapsto A \diamond \varphi$ and $A \mapsto A \diamond \psi$.

Lemma 3.4.9. *Let $A \in \mathcal{S}^2 \oplus \Omega^2_7$ and write $A = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7$. Then*

$$(A \diamond \varphi)_{ijk}\varphi_{ajk} = \frac{18}{7}(\operatorname{tr} A)g_{ia} + 4(A_{27})_{ia} + 12(A_7)_{ia} \tag{3.65}$$

and

$$(A \diamond \psi)_{ijkl}\psi_{ajkl} = \frac{96}{7}(\operatorname{tr} A)g_{ia} + 12(A_{27})_{ia} + 36(A_7)_{ia}. \tag{3.66}$$

Proof. Direct computation using the contraction identities (3.24) and (3.25) and the expression (3.52) for the operator $\mathcal{P}$ yields

$$\begin{aligned}
(A \diamond \varphi)_{ijk}\varphi_{ajk} &= (A_{im}\varphi_{mjk} + A_{jm}\varphi_{imk} + A_{km}\varphi_{ijm})\varphi_{ajk} \\
&= A_{im}\varphi_{mjk}\varphi_{ajk} + 2A_{jm}\varphi_{imk}\varphi_{ajk} \\
&= A_{im}(6g_{ma}) + 2A_{jm}(g_{ia}g_{mj} - g_{ij}g_{ma} - \psi_{imaj}) \\
&= 6A_{ia} + 2(\operatorname{tr} A)g_{ia} - 2A_{ia} - 2A_{jm}\psi_{jmia} \\
&= 4A_{ia} + 2(\operatorname{tr} A)g_{ia} - 2(\mathcal{P}A)_{ia} \\
&= 4(\frac{1}{7}(\operatorname{tr} A)g_{ia} + (A_{27})_{ia} + (A_7)_{ia}) + 2(\operatorname{tr} A)g_{ia} - 2(-4(A_7)_{ia}) \\
&= \frac{18}{7}(\operatorname{tr} A)g_{ia} + 4(A_{27})_{ia} + 12(A_7)_{ia}.
\end{aligned}$$

Similar computations yield the second identity. □

Corollary 3.4.10. *Let $\gamma \in \Omega^3$ and $\eta \in \Omega^4$. Then $\gamma = A \diamond \varphi$ and $\eta = B \diamond \psi$ for some unique $A = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7$ and $B = \frac{1}{7}(\operatorname{tr} B)g + B_{27} + B_7$. Define the 2-tensors $\gamma \cdot \varphi$ and $\eta \cdot \psi$ by*

$$(\gamma \cdot \varphi)_{ia} = \gamma_{ijk}\varphi_{ajk}$$

and

$$(\eta \cdot \psi)_{ia} = \eta_{ijkl}\psi_{ajkl}$$

then

$$\begin{aligned}
\operatorname{tr} A &= \frac{1}{18}\operatorname{tr}(\gamma \cdot \varphi), \\
(A_{27})_{ia} &= \frac{1}{8}((\gamma \cdot \varphi)_{ia} + (\gamma \cdot \varphi)_{ai}) - \frac{1}{28}\operatorname{tr}(\gamma \cdot \varphi)g_{ia}, \\
(A_7)_{ia} &= \frac{1}{24}((\gamma \cdot \varphi)_{ia} - (\gamma \cdot \varphi)_{ai}).
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr} B &= \frac{1}{96}\operatorname{tr}(\eta \cdot \psi), \\
(B_{27})_{ia} &= \frac{1}{24}((\eta \cdot \psi)_{ia} + (\eta \cdot \psi)_{ai}) - \frac{1}{84}\operatorname{tr}(\eta \cdot \psi)g_{ia}, \\
(B_7)_{ia} &= \frac{1}{72}((\eta \cdot \psi)_{ia} - (\eta \cdot \psi)_{ai}).
\end{aligned}$$

Proof. We compute using the expression in (3.65)

$$\frac{1}{18}\mathrm{tr}\,(\gamma\cdot\varphi) = \frac{1}{18}(\frac{18}{7}(\mathrm{tr}\,A)7) = \mathrm{tr}\,A$$

to show the first equality. Using the symmetry of g and A_{27} and the skew-symmetry of A_7 we get

$$\begin{aligned}\frac{1}{8}((\gamma\cdot\varphi)_{ia} + (\gamma\cdot\varphi)_{ai}) - \frac{1}{28}\mathrm{tr}\,(\gamma\cdot\varphi)g_{ia} &= \frac{1}{8}(\frac{36}{7}(\mathrm{tr}\,A)g_{ia} + 8(A_{27})_{ia}) - \frac{1}{28}(18(\mathrm{tr}\,A)g_{ia})\\ &= \frac{9}{14}(\mathrm{tr}\,A)g_{ia} + (A_{27})_{ia} - \frac{9}{14}(\mathrm{tr}\,A)g_{ia} = (A_{27})_{ia},\end{aligned}$$

as well as

$$\frac{1}{24}((\gamma\cdot\varphi)_{ia} - (\gamma\cdot\varphi)_{ai}) = \frac{1}{24}(24(A_7)_{ia}) = (A_7)_{ia}.$$

Similar computations yield the analogous results involving B. □

3.5 Torsion of a G_2-structure

We recall that a G_2-structure on a 7-dimensional smooth manifold is a 3-form φ which determines a Riemannian metric g. Using the metric, we can look at its Levi-Civita covariant derivative ∇. In particular, the tensor $\nabla\varphi \in \Gamma(T^*M \otimes \Lambda^3 T^*M)$ is of heavy importance in G_2-geometry.

Definition 3.5.1. A G_2-structure is said to be ***torsion-free*** if $\nabla\varphi = 0$. We note that the equation $\nabla\varphi = 0$ is a fully non-linear first order partial differential equation since the metric g depends non-linearly on the 3-form φ.

We have the following important observation about the Levi-Civita covariant derivative of a G_2-structure.

Theorem 3.5.2. *Let $X \in \mathfrak{X}$ be a vector field on M. Then the 3-form $\nabla_X\varphi$ lies in Ω^3_7.*

Proof. Let $A = \frac{1}{7}(\mathrm{tr}\,A)g + A_0 + A_7$ be a 2-tensor on M. We compute the inner product of $A\diamond\varphi$ and $\nabla_X\varphi$.

$$\begin{aligned}\langle A\diamond\varphi, \nabla_X\varphi\rangle &= \frac{1}{6}(A\diamond\varphi)_{ijk}(\nabla_X\varphi)_{ijk}\\ &= \frac{1}{6}(A_{im}\varphi_{mjk} + A_{jm}\varphi_{imk} + A_{km}\varphi_{ijm})X_l(\nabla_l\varphi_{ijk})\\ &= \frac{1}{2}A_{im}\varphi_{mjk}X_l(\nabla_l\varphi_{ijk}).\end{aligned}$$

Since g is parallel with respect to ∇, taking the covariant derivative of (3.25) gives that

$$(\nabla_l \varphi_{ijk})\varphi_{mjk} = -\varphi_{ijk}(\nabla_l \varphi_{mjk})$$

and so the expression $(\nabla_l \varphi_{ijk})\varphi_{mjk}$ is skew in m and i. It follows that the symmetric part of A does not contribute to the inner product and so $\nabla_X \varphi$ is orthogonal to each element of $\Omega^3_1 \oplus \Omega^3_{27}$. $\square$

From (3.37), we have that each Ω^3_7-form is of the form $X \lrcorner \psi$ for some vector field X. This allows us to define the torsion tensor.

Definition 3.5.3. Let X be a vector field on M. We can write

$$\nabla_X \varphi = T(X) \lrcorner \psi$$

for some vector field $T(X)$ on M. Hence there exists a 2-tensor T such that

$$\nabla_l \varphi_{ijk} = T_{lm}\psi_{mijk}. \tag{3.67}$$

The tensor T is called the ***full torsion tensor*** of φ.

Contracting (3.67) with ψ on three indices allows us to obtain an expression for T.

$$T_{lm} = \frac{1}{24}(\nabla_l \varphi_{ijk})\psi_{mijk}. \tag{3.68}$$

Equations (3.67) and (3.68) show that $\nabla\varphi = 0$ if and only if $T = 0$. Hence φ is torsion-free if and only if $T = 0$. Since T is a 2-tensor, we can decompose it using (3.50) as

$$T = \frac{1}{7}(\operatorname{tr} T)g + T_0 + T_7 + T_{14}. \tag{3.69}$$

We have the following theorem of Fernández and Gray [FG82] which gives another characterization of torsion-free G_2-structures.

Corollary 3.5.4. *A G_2-structure φ on M is torsion-free if and only if φ is closed and co-closed or equivalently, both φ and ψ are closed.*

Proof. We first recall that $d\psi = d \star \varphi = - \star d^*\varphi$. Since the Hodge star is an isomorphism, we get that $d\psi = 0$ if and only if $d^*\varphi = 0$. Additionally, we note that every parallel form is both closed and co-closed, so it only remains to prove the converse. Since $d\varphi$ and $d^*\varphi$ are both linear in $\nabla\varphi$, both are linear in T as well. We can decompose the spaces $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$ of 4-forms and $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$ of 2-forms and so it follows that the independent components of $d\varphi$ and $d^*\varphi$ correspond to the components of T up to scaling. Hence if φ is both closed and co-closed, we must have $T = 0$. $\square$

Remark 3.5.5. Though it will not come into play in this thesis as we will largely be considering the torsion-free case, we note that the decomposition (3.69) gives rise to 16 classes of G_2-structures, determined by whether each component of T is either zero or non-zero.

We would like to determine the relation between the torsion tensor T and the Riemannian curvature tensor R of a G_2-manifold. As such, we collect several identities involving both for later use. We start with the "G_2 Bianchi identity" from [Kar09].

Theorem 3.5.6. *The following identity holds:*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ia} T_{jb} \varphi_{abk} + \frac{1}{2} R_{ijab} \varphi_{abk}. \tag{3.70}$$

Proof. We take the covariant derivative of the identity (3.24) and use (3.67) and (3.27) to get that

$$\begin{aligned}
\nabla_m \psi_{ijab} &= -\nabla_m(\varphi_{ijk}\varphi_{abk}) \\
&= -(\nabla_m \varphi_{ijk})\varphi_{abk} - \varphi_{ijk}(\nabla_m \varphi_{abk}) \\
&= -T_{mp}\psi_{pijk}\varphi_{abk} - T_{mp}\psi_{pabk}\varphi_{ijk} \\
&= -T_{mp}(g_{ap}\varphi_{bij} + g_{ai}\varphi_{pbj} + g_{aj}\varphi_{pib} - g_{pb}\varphi_{aij} - g_{ib}\varphi_{paj} - g_{jb}\varphi_{pia}) \\
&\qquad - T_{mp}(g_{ip}\varphi_{jab} + g_{ia}\varphi_{pjb} + g_{ib}\varphi_{paj} - g_{pj}\varphi_{iab} - g_{aj}\varphi_{pib} - g_{bj}\varphi_{pai}) \\
&= -T_{ma}\varphi_{ijb} + T_{mb}\varphi_{ija} - T_{mi}\varphi_{abj} + T_{mj}\varphi_{abi}.
\end{aligned} \tag{3.71}$$

Next, we take the covariant derivative of (3.67) and substitute in the above to get

$$\begin{aligned}
\nabla_l \nabla_m \varphi_{ijk} &= \nabla_l (T_{mp}\psi_{pijk}) \\
&= (\nabla_l T_{mp})\psi_{pijk} + T_{mp}(\nabla_l \psi_{pijk}) \\
&= (\nabla_l T_{mp})\psi_{pijk} + T_{mp}(-T_{lp}\varphi_{ijk} + T_{li}\varphi_{pjk} - T_{lj}\varphi_{pik} + T_{lk}\varphi_{pij}).
\end{aligned}$$

We can then apply the Ricci identity to $(\nabla_l \nabla_m - \nabla_m \nabla_l)\varphi_{ijk}$ to get

$$\begin{aligned}
&- R_{lmia}\varphi_{ajk} - R_{lmja}\varphi_{iak} - R_{lmka}\varphi_{ija} \\
&= (\nabla_l T_{mp} - \nabla_m T_{lp})\psi_{pijk} + (T_{mp}T_{li} - T_{lp}T_{mi})\varphi_{pjk} \\
&\qquad (T_{lp}T_{mj} - T_{mp}T_{lj})\varphi_{pik} + (T_{mp}T_{lk} - T_{lp}T_{mk})\varphi_{pij}.
\end{aligned} \tag{3.72}$$

By contracting both sides with ψ on the indices i, j, and k, we get

$$\begin{aligned}
&- R_{lmia}\varphi_{ajk}\psi_{qijk} - R_{lmja}\varphi_{iak}\psi_{qijk} - R_{lmja}\varphi_{ija}\psi_{qijk} \\
&= -R_{lmia}(-4\varphi_{aqi}) - R_{lmja}(-4\varphi_{aqj}) - R_{lmka}(-4\varphi_{aqk}) \\
&= 12 R_{lmia}\varphi_{qia}
\end{aligned}$$

and

$$\begin{aligned}
&(\nabla_l T_{mp} - \nabla_m T_{lp})\psi_{pijk}\psi_{qijk} + (T_{mp}T_{li} - T_{lp}T_{mi})\varphi_{pjk}\psi_{qijk} \\
&\quad + (T_{lp}T_{mj} - T_{mp}T_{lj})\varphi_{pik}\psi_{qijk} + (T_{mp}T_{lk} - T_{lp}T_{mk})\varphi_{pij}\psi_{qijk} \\
&= (\nabla_l T_{mp} - \nabla_m T_{lp})(24 g_{pq}) + (T_{mp}T_{li} - T_{lp}T_{mi})(-4\varphi_{pqi}) \\
&\quad + (T_{lp}T_{mj} - T_{mp}T_{lj})(4\varphi_{pqj}) + (T_{mp}T_{lk} - T_{lp}T_{mk})(-4\varphi_{pqk}) \\
&= 24(\nabla_l T_{mq} - \nabla_m T_{lq}) - 12 T_{mp}T_{li}\varphi_{ipq} + 12 T_{lp}T_{mi}\varphi_{ipq} \\
&= 24(\nabla_l T_{mq} - \nabla_m T_{lq}) - 24 T_{mp}T_{li}\varphi_{ipq}.
\end{aligned}$$

Reindexing and rearranging gives the desired identity. □

More generally, we can use the G_2 Bianchi identity to relate the Ricci tensor Ric and the full torsion tensor T. We first prove a lemma involving a contraction of the Riemann tensor and ψ.

Lemma 3.5.7. *The following identity holds:*

$$R_{ijkl}\psi_{ajkl} = 0. \tag{3.73}$$

Proof. By the first Bianchi identity, we compute that

$$\begin{aligned}
R_{ijkl}\psi_{ajkl} &= -R_{jkil}\psi_{ajkl} - R_{kijl}\psi_{ajkl} \\
&= -R_{ilkj}\psi_{alkj} - R_{ikjl}\psi_{akjl} \\
&= -R_{ijkl}\psi_{ajkl} - R_{ijkl}\psi_{ajkl} \\
&= -2R_{ijkl}\psi_{ajkl}
\end{aligned}$$

So $3R_{ijkl}\psi_{ajkl} = 0$. □

Corollary 3.5.8. *The Ricci tensor of a manifold with G_2-structure is given by*

$$\mathrm{Ric}_{jm} = (\nabla_i T_{jk} - \nabla_j T_{ik})\varphi_{ikm} - T_{ji}T_{im} + (\operatorname{tr} T)T_{jm} + T_{ia}T_{jb}\psi_{miab}. \tag{3.74}$$

In particular, if the G_2-structure on M is torsion-free, then M is Ricci-flat.

Proof. We rearrange and contract (3.70) with φ on the indices i and k to get

$$\begin{aligned}
(\nabla_i T_{jk} - \nabla_j T_{ik})\varphi_{mik} &= T_{ia}T_{jb}\varphi_{abk}\varphi_{mik} + \frac{1}{2}R_{ijab}\varphi_{abk}\varphi_{mik} \\
&= T_{ia}T_{jb}(g_{am}g_{bi} - g_{ai}g_{bm} - \psi_{abmi}) + \frac{1}{2}R_{ijab}(g_{am}g_{bi} - g_{ai}g_{bm} - \psi_{abmi}) \\
&= T_{im}T_{ji} - (\operatorname{tr} T)T_{jm} - T_{ia}T_{jb}\psi_{miab} + \frac{1}{2}R_{ijmi} - \frac{1}{2}R_{ijim} - \frac{1}{2}R_{ijab}\psi_{abmi} \\
&= T_{im}T_{ji} - (\operatorname{tr} T)T_{jm} - T_{ia}T_{jb}\psi_{miab} + \mathrm{Ric}_{jm},
\end{aligned}$$

where the final term vanishes by the previous lemma. □

Chapter 4

Differential Operators on Manifolds with Torsion-Free G_2-Structure

In this chapter, we discuss several differential operators which can be defined on certain vector bundles over a manifold M with torsion-free G_2-structure. In particular, we look at analogues of the divergence, gradient, and curl operators on the spinor bundle of M as well as the Dirac operator of this bundle. We then extend these operators and ideas onto the bundle of spinor-valued 1-forms.

4.1 Differential Operators on the Spinor Bundle

We follow the discussion of the Dirac operator on the spinor bundle of a manifold with torsion-free G_2-structure in [Kar10] and use this as the basis to extend some definitions onto the bundle of spinor-valued 1-forms.

4.1.1 The Divergence, Gradient, and Curl Operators

On any Riemannian manifold, we are able to define the divergence of a vector field and the gradient of a function. In particular, let M be a manifold with G_2-structure. The ***divergence*** of a vector field X is the function given by

$$\operatorname{div} X = \nabla_i X_i. \tag{4.1}$$

Invariantly, we have the formula

$$\operatorname{div} X = -d^* X^\flat = \star d \star X^\flat. \tag{4.2}$$

The ***gradient*** of a function f is the vector field given in local coordinates by

$$(\operatorname{grad} f)_a = \nabla_a f. \tag{4.3}$$

Invariantly, we have

$$\operatorname{grad} f = (df)^\sharp. \tag{4.4}$$

There is a cross-product $\times$ on vector fields of M given by the analogue of (3.16) in the manifold setting, that is, for vector fields X and Y, we have a cross-product defined by

$$(X \times Y)_a = X_i Y_j \varphi_{ija}. \tag{4.5}$$

Invariantly, this is given by

$$(X \times Y)^\flat = Y \lrcorner X \lrcorner \varphi. \tag{4.6}$$

Using this cross-product, analogues of (3.16) and (3.17) hold and so for another vector field Z, we have

$$\langle X \times Y, Z \rangle = \varphi(X, Y, Z) \tag{4.7}$$

and

$$X \times (Y \times Z) = -\langle X, Y \rangle Z + \langle X, Z \rangle Y - (X \lrcorner Y \lrcorner Z \lrcorner \psi)^\sharp. \tag{4.8}$$

We can use the cross-product to define the curl operator, which is another first order differential operator acting on vector fields.

Definition 4.1.1. The ***curl*** of a vector field X is the vector field $\operatorname{curl} X$ given by

$$(\operatorname{curl} X)_a = (\nabla_i X_j)\varphi_{ija}, \tag{4.9}$$

or invariantly by

$$\operatorname{curl} X = [\star(dX^\flat \wedge \psi)]^\sharp. \tag{4.10}$$

Remark 4.1.2. We note that in dimension 3 that the curl of a vector field X can be defined by

$$\operatorname{curl} X = (\star dX^\flat)^\sharp.$$

In dimension 7, the analagous expression $\star dX^\flat$ results in a 5-form. Since we have isomorphisms $\Omega^5 = \Omega^5_7 \oplus \Omega^5_{14} \cong \Omega^2_7 \oplus \Omega^2_{14} \cong \mathfrak{X} \oplus \Omega^2_{14}$, one could consider the projection of $\star dX^\flat$ onto its $\mathfrak{X}$-component. Using Corollary 3.4.2, we compute in local coordinates that

$$\begin{aligned}\frac{1}{6}(\star\star dX^\flat)_{ij}\varphi_{ijk} &= \frac{1}{6}(dX^\flat)_{ij}\varphi_{ijk} \\ &= \frac{1}{6}(\nabla_i X_j - \nabla_j X_i)\varphi_{ijk} \\ &= \frac{1}{3}(\nabla_i X_j)\varphi_{ijk}.\end{aligned}$$

Hence these two expressions for the curl of a vector field on a manifold with G_2-structure are equivalent up to a scalar multiple.

If we further assume that the G_2-structure on M is torsion-free, we get several relations between the operators div, grad, and curl (see Section 4 of [Kar10]). We recall that in the torsion-free case, both φ and ψ are parallel and that M is Ricci-flat.

Proposition 4.1.3. *Let $f \in \Omega^0$ be a function and $X \in \mathfrak{X}$ be a vector field on a manifold with torsion-free G_2-structure. The following relations hold:*

$$\operatorname{curl}\operatorname{grad} f = 0, \tag{4.11}$$

$$\operatorname{div}\operatorname{curl} X = 0, \tag{4.12}$$

$$\operatorname{curl}\operatorname{curl} X = \operatorname{grad}\operatorname{div} X + \Delta X. \tag{4.13}$$

Proof. We can use the invariant definitions of div, grad, and curl to establish the first two equations. We notice that

$$\operatorname{curl}\operatorname{grad} f = [\star(d(\operatorname{grad} f)^\flat \wedge \psi)]^\sharp = [\star(d^2 f \wedge \psi)]^\sharp = 0$$

since $d^2 = 0$ and

$$\operatorname{div}\operatorname{curl} X = \star d \star (\operatorname{curl} X)^\flat = \star d \star [\star(dX^b \wedge \psi)] = \star(d^2 X^\flat \wedge \psi) - \star(dX^\flat \wedge d\psi) = 0$$

since by torsion-freeness and Corollary 3.5.4 we have that ψ is closed (or that φ is co-closed).

To show the last of the identities, we compute in local coordinates that

$$\begin{aligned}
(\operatorname{curl}\operatorname{curl} X)_a &= (\nabla_i(\operatorname{curl} X)_j)\varphi_{ija} \\
&= (\nabla_i[(\nabla_k X_l)\varphi_{klj}])\varphi_{ija} \\
&= \nabla_i\nabla_k X_l(g_{ka}g_{li} - g_{la}g_{ki} - \psi_{klai}) \\
&= \nabla_i\nabla_a X_i - \nabla_i\nabla_i X_a + \nabla_i\nabla_k X_l\psi_{ikla} \\
&= \nabla_i\nabla_a X_i + (\Delta X)_a + \nabla_i\nabla_k X_l\psi_{ikla}
\end{aligned}$$

Since ψ is skew in all of its indices, we may rewrite the last term and use the Ricci identity and Lemma 3.5.7 to compute

$$\begin{aligned}
\nabla_i\nabla_k X_l\psi_{ikla} &= \frac{1}{2}(\nabla_i\nabla_k - \nabla_k\nabla_i)X_l\psi_{ikla} \\
&= -\frac{1}{2}R_{iklm}X_m\psi_{ikla} = 0.
\end{aligned}$$

Finally, we note that $(\operatorname{grad}\operatorname{div} X)_a = \nabla_a\nabla_i X_i$ and the Ricci identity gives that

$$\nabla_i\nabla_a X_i - \nabla_a\nabla_i X_i = -R_{iaim}X_m = \operatorname{Ric}_{am}X_m = 0$$

by Ricci-flatness so $\nabla_i\nabla_a X_i = (\operatorname{grad}\operatorname{div} X)_a$. Putting everything together gives the result. □

Remark 4.1.4. As in [Kar10], we note that (4.13) required the full torsion-freeness condition, (4.12) only required that φ be co-closed, and (4.11) did not require any of the torsion-freeness condition. Further, these identites mirror those of vector calculus.

4.1.2 The Dirac Bundle Structure and Dirac Operator on Spinors

From Proposition 3.3.2, we have that any manifold M admitting a G_2-structure is spinnable. Additionally, the spinor bundle $\mathbb{S}$ of a 7-dimensional manifold is a rank 8 real vector bundle. When M is a G_2-manifold, using a unit norm spinor we have an identification of $\mathbb{S}$ with the bundle $\underline{\mathbb{R}} \oplus TM$ whose sections consist of a function and a vector field. This is done in [Kar10] and [Gri17] by identifying $\mathbb{S}$ with the octonions $\mathbb{O}$, as such we get a Dirac bundle structure on the bundle $\underline{\mathbb{R}} \oplus TM$. A concrete description of the bundle isomorphism is given in Section 8 of [Gri17]. We review this structure below. As an abuse of notation, we use $\langle \cdot, \cdot \rangle$ to represent all inner products, the arguments will determine whether the inner product is acting on global sections of a bundle or on a fibre over a point.

Let Y be a vector field on M and let (f, Z) be a spinor. We define the Clifford product on this bundle using octonion multiplication, that is,

$$Y \cdot (f, Z) = (-\langle Y, Z \rangle, fY + Y \times Z) \tag{4.14}$$

in coordinates, we have

$$Y \cdot (f, Z) = (-Y_i Z_i, fY_a + Y_i Z_j \varphi_{ija}). \tag{4.15}$$

Additionally, if X is another vector field on M, we can compute that

$$\begin{aligned}
&X \cdot (Y \cdot (f, Z)) + Y \cdot (X \cdot (f, Z)) \\
&= X \cdot (-\langle Y, Z \rangle, fY + Y \times Z) + Y \cdot (-\langle X, Z \rangle, fX + X \times Z) \\
&= (-\langle X, fY \rangle - \langle X, Y \times Z \rangle, -\langle Y, Z \rangle X + X \times fY + X \times (Y \times Z)) \\
&\qquad + (-\langle Y, fX \rangle - \langle Y, X \times Z \rangle, -\langle X, Z \rangle Y + Y \times fX + Y \times (X \times Z)) \\
&= (-2f\langle X, Y \rangle - \varphi(X, Y, Z) - \varphi(Y, X, Z), \\
&\qquad - \langle Y, Z \rangle X - \langle X, Z \rangle Y + fX \times Y + fY \times X \\
&\qquad - \langle X, Y \rangle Z + \langle X, Z \rangle Y - (X \lrcorner Y \lrcorner Z \lrcorner \psi)^\sharp \\
&\qquad - \langle Y, X \rangle Z + \langle Y, Z \rangle X - (Y \lrcorner X \lrcorner Z \lrcorner \psi)^\sharp) \\
&= -2\langle X, Y \rangle (f, Z)
\end{aligned}$$

and so the Clifford identity holds.

Next, we check that Clifford multiplication is skew-adjoint. Let $v \in T_x M$ be a vector and let $(a_1, z_1), (a_2, z_2) \in \mathbb{R} \oplus T_x M$. We check

$$\begin{aligned}
&\langle v \cdot (a_1, z_1), (a_2, z_2) \rangle + \langle (a_1, z_1), v \cdot (a_2, z_2) \rangle \\
&= \langle (-\langle v, z_1 \rangle, a_1 v + v \times z_1), (a_2, z_2) \rangle + \langle (a_1, z_1), (-\langle v, z_2 \rangle, a_2 v + v \times z_2) \rangle \\
&= -a_2 \langle v, z_1 \rangle + a_1 \langle v, z_2 \rangle + \langle v \times z_1, z_2 \rangle - a_1 \langle v, z_2 \rangle + a_2 \langle v, z_1 \rangle + \langle v \times z_2, z_1 \rangle = 0.
\end{aligned}$$

Finally, we check compatibility with the Levi-Civita connection. Let X and Y be vector fields and let (f, Z) be a spinor. An argument in Section 8 of [Gri17] shows that the spin connection ∇^S under the bundle isomorphism $\mathbb{S} \cong \underline{\mathbb{R}} \oplus TM$ is just given by the Levi-Civita connection on each component when the G_2-structure is torsion-free. That is, for a spinor (f, Z) we have

$$\nabla^S (f, Z) = \nabla (f, Z) = (\nabla f, \nabla Z). \tag{4.16}$$

Using local coordinates, we compute that

$$
\begin{aligned}
\nabla^S_X[Y \cdot (f, Z)] &= \nabla_X(-\langle Y, Z\rangle, fY + Y \times Z) \\
&= (-X_k \nabla_k (Y_i Z_i), X_k \nabla_k (f Y_a) + X_k \nabla_k (Y_i Z_j)\varphi_{ija}) \\
&= (-X_k (\nabla_k Y_i) Z_i, X_k f(\nabla_k Y_a) + X_k (\nabla_k Y_i) Z_j \varphi_{ija}) \\
&\qquad + (-X_k Y_i (\nabla_k Z_i), X_k (\nabla_k f) Y_a + X_k Y_i (\nabla_k Z_j)\varphi_{ija}) \\
&= (\nabla_X Y) \cdot (f, Z) + Y \cdot [\nabla^S_X (f, Z)].
\end{aligned}
$$

Since we have established that this bundle has a Dirac bundle structure, we can apply results from Section 2.4. In particular, we can define its Dirac operator D_0 and compute its action on a spinor (f, Z). In local coordinates, we have

$$
\begin{aligned}
D_0(f, Z) &= \sum_i e_i \cdot \nabla^S_i (f, Z) \\
&= \sum_i e_i \cdot (\nabla_i f, \nabla_i Z_a) \\
&= (-\nabla_i Z_i, \nabla_i f \delta_{ia} + \delta_{ik} \nabla_i Z_l \varphi_{kla}) \\
&= (-\nabla_i Z_i, \nabla_a f + \nabla_k Z_l \varphi_{kla}) \\
&= (-\operatorname{div} Z, \operatorname{grad} f + \operatorname{curl} Z),
\end{aligned}
\tag{4.17}
$$

which allows us to express D_0 in terms of the operators div, grad, and curl.

We have that D_0 is a self-adjoint operator from Proposition 2.4.6. Further, we have a result regarding the Dirac Laplacian D_0^2.

Proposition 4.1.5. *On a manifold with torsion-free G_2-structure, we have that*

$$
D_0^2(f, Z) = \Delta(f, Z) = (\Delta f, \Delta Z). \tag{4.18}
$$

Proof. We can use the identities (4.11), (4.12), and (4.13) and Equation (4.17) to compute directly that

$$
\begin{aligned}
D_0^2(f, Z) &= D_0(-\operatorname{div} Z, \operatorname{grad} f + \operatorname{curl} Z) \\
&= (-\operatorname{div}\operatorname{grad} f - \operatorname{div}\operatorname{curl} Z, -\operatorname{grad}\operatorname{div} Z + \operatorname{curl}\operatorname{grad} f + \operatorname{curl}\operatorname{curl} Z) \\
&= (-\operatorname{div}\operatorname{grad} f, \Delta Z).
\end{aligned}
$$

Since $-\operatorname{div}\operatorname{grad} f = -\nabla_i (\operatorname{grad} f)_i = -\nabla_i \nabla_i f = \Delta f$, the result follows. $\square$

Since M is Ricci-flat, we have that $\Delta = \Delta_d$ when acting on 1-forms. Indeed, if Z is a 1-form, we calculate using coordinate representations of d and d^* that

$$\begin{aligned}(\Delta_d Z)_a &= (dd^*Z)_a + (d^*dZ)_a \\ &= \nabla_a(d^*Z) - \nabla_i(dZ)_{ia} \\ &= \nabla_a(-\nabla_i Z_i) - \nabla_i(\nabla_i Z_a - \nabla_a Z_i) \\ &= -\nabla_i\nabla_i Z_a + (\nabla_i\nabla_a - \nabla_a\nabla_i)Z_i \\ &= (\Delta Z)_a - R_{iail}Z_l \\ &= (\Delta Z)_a.\end{aligned}$$

Hence $\Delta Z = 0$ if and only if Z is harmonic. Similarly, $\Delta f = 0$ if and only if f is harmonic.

Self-adjointness of the Dirac operator D_0 tells us that D_0 and $D_0^2 = \Delta$ have the same kernel. As such, using the above, we can see that $\ker D_0$ has dimension $b^0 + b^1$.

4.2 Extensions to Spinor-Valued 1-Forms

The previous section introduced the operators div, grad, and curl which act on smooth functions and vector fields. Using these operators, and identifying the spinor bundle with $\underline{\mathbb{R}} \oplus TM$, we were able to describe its Dirac operator D_0. In this section, we extend the operators div, grad, and curl by one degree, so they act on 1-forms and 2-tensors and use these to define a Dirac bundle structure on the space of spinor-valued 1-forms.

4.2.1 The Extended Divergence, Gradient, and Curl Operators

Here, we define new first order operators which are analogues of the div, grad, and curl operators from Section 4.1.1. Let Y be a 1-form and C be a 2-tensor on a manifold M with torsion-free G_2-structure. We define the operators as follows:

$$(\operatorname{div} C)_a = \nabla_i C_{ai}, \tag{4.19}$$

$$(\operatorname{grad} Y)_{ab} = \nabla_b Y_a, \tag{4.20}$$

$$(\operatorname{curl} C)_{ab} = (\nabla_i C_{aj})\varphi_{ijb}. \tag{4.21}$$

The definitions above are the transposes of what one might guess these operators to be, however, we define the div, grad, and curl operators in this manner so that the identities that follow are cleaner and mirror those of the previous section.

Remark 4.2.1. We note that similar extensions of the divergence and curl operators can be found in the literature, for example in [Gri13] and [Gri20].

These new versions of div, grad, and curl satisfy similar identities to those in Proposition 4.1.3. Before stating and proving these identities, we first recall that the torsion-free condition ensures that M is Ricci-flat, and so the Lichnerowicz Laplacian Δ_L acting on symmetric 2-tensors (see Chapter 3 of [CK04]) can be simplified. Let C_+ be a symmetric 2-tensor, then we have

$$\begin{aligned}(\Delta_L C_+)_{ab} &= (\Delta C_+)_{ab} + \mathrm{Ric}_{ai}(C_+)_{ib} + \mathrm{Ric}_{bi}(C_+)_{ai} - 2R_{iabj}(C_+)_{ij} \\ &= (\Delta C_+)_{ab} - 2R_{iabj}(C_+)_{ij}.\end{aligned} \tag{4.22}$$

Moreover, if C_- is a 2-form on M, then the Hodge Laplacian Δ_d acts on C_- in a similar manner (see [Pet16]) as

$$\begin{aligned}(\Delta_d C_-)_{ab} &= (\Delta C_-)_{ab} + \mathrm{Ric}_{ai}(C_-)_{ib} + \mathrm{Ric}_{bi}(C_-)_{ai} - 2R_{iabj}(C_-)_{ij} \\ &= (\Delta C_-)_{ab} - 2R_{iabj}(C_-)_{ij}.\end{aligned} \tag{4.23}$$

The torsion-free condition also simplifies other identities from Section 3.5. We note that the G_2 Bianchi identity (3.70) becomes

$$R_{ijab}\varphi_{abk} = 0. \tag{4.24}$$

We can use the above and the first Bianchi identity to show that the contraction of the Riemann tensor and φ on any two indices vanishes:

$$\begin{aligned}R_{iabj}\varphi_{abk} &= -R_{abij}\varphi_{abk} - R_{biaj}\varphi_{abk} \\ &= -R_{biaj}\varphi_{abk} \\ &= -R_{ibaj}\varphi_{bak} \\ &= -R_{iabj}\varphi_{abk},\end{aligned}$$

and so

$$R_{iabj}\varphi_{abk} = 0. \tag{4.25}$$

We also get the following lemma

Lemma 4.2.2. *On a manifold with torsion-free G_2-structure, we have*

$$R_{ijab}\psi_{abkl} = 2R_{ijkl}. \tag{4.26}$$

Proof. In the torsion-free case, the G_2 Bianchi identity (3.70) becomes

$$R_{ijab}\varphi_{abk} = 0.$$

Contracting both sides with φ on the index k gives

$$\begin{aligned} 0 &= R_{ijab}\varphi_{abc}\varphi_{klc} \\ &= R_{ijab}(g_{ak}g_{bl} - g_{al}g_{bk} - \psi_{abkl}) \\ &= R_{ijkl} - R_{ijlk} - R_{ijab}\psi_{abkl} \\ &= 2R_{ijkl} - R_{ijab}\psi_{abkl}. \end{aligned}$$

□

Using the above and first Bianchi identity again, we can compute the contraction of the Riemann tensor and ψ on the middle two indices. We compute

$$\begin{aligned} R_{iabj}\psi_{abkl} &= -R_{abij}\psi_{abkl} - R_{biaj}\psi_{abkl} \\ &= -2R_{ijkl} - R_{ibaj}\psi_{bakl} \\ &= -2R_{ijkl} - R_{iabj}\psi_{abkl}. \end{aligned}$$

Rearranging gives

$$R_{iabj}\psi_{abkl} = -R_{ijkl}. \tag{4.27}$$

We now state the identities between the extended operators div, grad, and curl.

Proposition 4.2.3. *Let $Y \in \Omega^1$ be a 1-form and $C = C_+ + C_- \in \mathcal{T}^2$ be a 2-tensor with symmetric part $C_+ \in \mathcal{S}^2$ and skew part $C_- \in \Omega^2$ on a manifold with torsion-free G_2-structure. The following relations hold:*

$$\operatorname{curl}\operatorname{grad} Y = 0, \tag{4.28}$$

$$\operatorname{div}\operatorname{curl} C = 0, \tag{4.29}$$

$$\operatorname{curl}\operatorname{curl} C = \operatorname{grad}\operatorname{div} C + \Delta_L C_+ + \Delta_d C_-. \tag{4.30}$$

Proof. We compute each of these in local coordinates. First, we get

$$\begin{aligned} (\operatorname{curl}\operatorname{grad} Y)_{ab} &= [\nabla_i(\operatorname{grad} Y)_{aj}]\varphi_{ijb} \\ &= \nabla_i\nabla_j Y_a\varphi_{ijb} \\ &= \frac{1}{2}(\nabla_i\nabla_j - \nabla_j\nabla_i)Y_a\varphi_{ijb} \\ &= -\frac{1}{2}R_{ijam}Y_m\varphi_{ijb} = 0, \end{aligned}$$

where we have used the fact that φ is skew in all of its indices and (4.24). Using the same method, we have

$$\begin{aligned}
(\operatorname{div}\operatorname{curl} C)_a &= \nabla_i(\operatorname{curl} C)_{ai} \\
&= \nabla_i\nabla_k C_{al}\varphi_{kli} \\
&= \frac{1}{2}(\nabla_i\nabla_k - \nabla_k\nabla_i)C_{al}\varphi_{kli} \\
&= -\frac{1}{2}(R_{ikam}C_{ml}\varphi_{kli} + R_{iklm}C_{am}\varphi_{kli}) = 0.
\end{aligned}$$

Lastly, we check that

$$\begin{aligned}
(\operatorname{curl}\operatorname{curl} C)_{ab} &= [\nabla_i(\operatorname{curl} C)_{aj}]\varphi_{ijb} \\
&= \nabla_i\nabla_k C_{al}\varphi_{klj}\varphi_{ijb} \\
&= \nabla_i\nabla_k C_{al}(g_{kb}g_{li} - g_{ki}g_{lb} - \psi_{klbi}) \\
&= \nabla_i\nabla_b C_{ai} - \nabla_i\nabla_i C_{ab} - \nabla_i\nabla_k C_{al}\psi_{klbi} \\
&= \nabla_i\nabla_b C_{ai} + (\Delta C)_{ab} - \frac{1}{2}(\nabla_i\nabla_k - \nabla_k\nabla_i)C_{al}\psi_{klbi} \\
&= \nabla_i\nabla_b C_{ai} + (\Delta C)_{ab} + \frac{1}{2}(R_{ikam}C_{ml}\psi_{klbi} + R_{iklm}C_{al}\psi_{klbi}) \\
&= \nabla_i\nabla_b C_{ai} + (\Delta C)_{ab} + R_{ambl}C_{ml} \\
&= \nabla_i\nabla_b C_{ai} + (\Delta C)_{ab} - R_{iabj}C_{ij}.
\end{aligned}$$

We note that $(\operatorname{grad}\operatorname{div} C)_{ab} = \nabla_b\nabla_i C_{ai}$ and the Ricci identity gives that

$$\begin{aligned}
\nabla_i\nabla_b C_{ai} - \nabla_b\nabla_i C_{ai} &= -R_{ibam}C_{mi} - R_{ibim}C_{am} \\
&= -R_{mabi}C_{mi} + \mathrm{Ric}_{bm}C_{am} \\
&= -R_{iabj}C_{ij}.
\end{aligned}$$

Rearranging and using (4.22) and (4.23) yields the result. □

4.2.2 The Dirac Bundle Structure and Dirac Operator on Spinor-Valued 1-Forms

As on the spinor bundle, there exists a Dirac bundle structure on the bundle $T^*M \oplus (T^*M \otimes T^*M) = T^*M \otimes (\underline{\mathbb{R}} \oplus T^*M)$. The Clifford multiplication on this bundle is induced by that on the bundle $\underline{\mathbb{R}} \oplus TM$. Let W and X be 1-forms and (f, Z) be a spinor. The induced Clifford multiplication is given by

$$W \cdot [X \otimes (f, Z)] = X \otimes (W \cdot (f, Z)) \tag{4.31}$$

and extending linearly. This is indeed a Clifford product since if W_1 and W_2 are 1-forms, then

$$\begin{aligned}
&W_1 \cdot (W_2 \cdot [X \otimes (f, Z)]) + W_2 \cdot (W_1 \cdot [X \otimes (f, Z)]) \\
&= W_1 \cdot (X \otimes [W_2 \cdot (f, Z)]) + W_2 \cdot (X \otimes [W_1 \cdot (f, Z)]) \\
&= [X \otimes (W_1 \cdot (W_2 \cdot (f, Z)))] + [X \otimes (W_2 \cdot (W_1 \cdot (f, Z)))] \\
&= X \otimes [(W_1 \cdot (W_2 \cdot (f, Z))) + (W_2 \cdot (W_1 \cdot (f, Z)))] \\
&= X \otimes (-\langle W_1, W_2 \rangle (f, Z)) \\
&= -\langle W_1, W_2 \rangle [X \otimes (f, Z)].
\end{aligned}$$

In local coordinates, the Clifford product is given by

$$W \cdot [X \otimes (f, Z)] = (-W_i X_a Z_i, f X_a W_b + W_i X_a Z_j \varphi_{ija}).$$

By linearity, we can write this in terms of 1-forms and 2-tensors. Let X and Y be 1-forms and C be a 2-tensor, then in local coordinates,

$$X \cdot (Y, C) = (-X_i C_{ai}, X_b Y_a + X_i C_{aj} \varphi_{ijb}). \tag{4.32}$$

We now check skew-adjointness of Clifford multiplication. Let v be a cotangent vector and let $(w_1, A_1), (w_2, A_2) \in T_x^*M \oplus (T_x^*M \otimes T_x^*M)$. We check that

$$\begin{aligned}
&\langle v \cdot (w_1, A_1), (w_2, A_2) \rangle + \langle (w_1, A_1), v \cdot (w_2, A_2) \rangle \\
&= -v_i (A_1)_{ai} (w_2)_a + v_b (w_1)_a (A_2)_{ab} + v_i (A_1)_{aj} \varphi_{ijb} (A_2)_{ab} \\
&\qquad - v_i (A_2)_{ai} (w_1)_a + v_b (w_2)_a (A_1)_{ab} + v_i (A_2)_{aj} \varphi_{ijb} (A_1)_{ab} \\
&= v_i [(A_1)_{aj} (A_2)_{ab} + (A_1)_{ab} (A_2)_{aj}] \varphi_{ijb} = 0.
\end{aligned}$$

Lastly, for compatibility with the Levi-Civita connection we let W, X, and Y be 1-forms and C be a 2-tensor. Simple computations in the torsion-free case show that the induced connection on the tensor product $T^*M \oplus (T^*M \otimes T^*M) = T^*M \otimes (\underline{\mathbb{R}} \oplus T^*M)$ using the spin connection from (4.16) is again just the Levi-Civita connection on each component. Indeed, if we denote the induced connection on this bundle by ∇^S we see that

$$\begin{aligned}
\nabla^S (X \otimes (f, Z)) &= \nabla X \otimes (f, Z) + X \otimes \nabla^S (f, Z) \\
&= (\nabla X \otimes f, (\nabla X) \otimes Z) + X \otimes (\nabla f, \nabla Z) \\
&= ((\nabla f) X + f (\nabla X), (\nabla X) \otimes Z + X \otimes (\nabla Z)) \\
&= (\nabla (f X), \nabla (X \otimes Z)).
\end{aligned}$$

By linearity of connections and since $X \otimes (f, Z) = (fX, X \otimes Z)$, we have

$$\nabla^S(Y, C) = \nabla(Y, C) = (\nabla Y, \nabla C). \tag{4.33}$$

Using local coordinates, we compute that

$$\begin{aligned}
&\nabla^S_W[X \cdot (Y, C)] \\
&= \nabla^S_W(-X_i C_{ai}, X_b Y_a + X_i C_{aj} \varphi_{ijb}) \\
&= (-W_k \nabla_k (X_i C_{ai}), W_k \nabla_k (X_b Y_a) + W_k \nabla_k (X_i C_{aj}) \varphi_{ijb}) \\
&= (-W_k (\nabla_k X_i) C_{ai}, W_k (\nabla_k X_b) Y_a + W_k (\nabla_k X_i) C_{aj} \varphi_{ijb}) \\
&\qquad + (-W_k X_i (\nabla_k C_{ai}), W_k X_b (\nabla_k Y_a) + W_k X_i (\nabla_k C_{aj}) \varphi_{ijb}) \\
&= (\nabla_W X) \cdot (Y, C) + X \cdot [\nabla^S_W (Y, C)].
\end{aligned}$$

Thus $T^*M \oplus (T^*M \otimes T^*M)$ is a Dirac bundle.

As before, we can define the Dirac operator D_1 for this Dirac bundle and determine how it acts on (Y, C). In local coordinates, we get

$$\begin{aligned}
D_1(Y, C) &= \sum_i e_i \cdot \nabla^S_i (Y, C) \\
&= \sum_i e_i \cdot (\nabla_i Y_a, \nabla_i C_{ab}) \\
&= (-\nabla_i C_{ai}, \nabla_i Y_a \delta_{ib} + \delta_{ik} \nabla_i C_{al} \varphi_{klb}) \\
&= (-\nabla_i C_{ai}, \nabla_b Y_a + \nabla_k C_{al} \varphi_{klb}) \\
&= (-\operatorname{div} C, \operatorname{grad} Y + \operatorname{curl} C).
\end{aligned} \tag{4.34}$$

The extended definitions of div, grad, and curl show that the Dirac operator D_1 acts in a similar manner as to the Dirac operator D_0 on spinors. Using these definitions, we also get a similar relation to Proposition 4.1.5.

Proposition 4.2.4. *On a manifold with torsion-free G_2-structure, we have*

$$D_1^2(Y, C) = (\Delta Y, \Delta_L C_+ + \Delta_d C_-). \tag{4.35}$$

Proof. This follows from the identities (4.28), (4.29), and (4.30) in a similar manner to the proof of Proposition 4.1.5. We compute directly that

$$\begin{aligned}
D_1^2(Y, C) &= D_1(-\operatorname{div} C, \operatorname{grad} Y + \operatorname{curl} C) \\
&= (-\operatorname{div} \operatorname{grad} Y - \operatorname{div} \operatorname{curl} C, -\operatorname{grad} \operatorname{div} C + \operatorname{curl} \operatorname{grad} Y + \operatorname{curl} \operatorname{curl} C) \\
&= (-\operatorname{div} \operatorname{grad} Y, \Delta_L C_+ + \Delta_d C_-).
\end{aligned}$$

Since $-(\operatorname{div} \operatorname{grad} Y)_a = -\nabla_i (\operatorname{grad} Y)_{ai} = -\nabla_i \nabla_i Y_a = (\Delta Y)_a$, we get $-\operatorname{div} \operatorname{grad} Y = \Delta Y$. The result follows. $\square$

Chapter 5

The Kernel of the Twisted Dirac Operator on Manifolds with Torsion-Free G_2-Structures

In this chapter, we look at the curl operator more closely and decompose it with respect to the decomposition (3.49) of 2-tensors as well as gather several equivalences showing when each component vanishes. The results using the identification $T^*M \otimes \mathbb{S} \cong T^*M \oplus (T^*M \otimes T^*M)$ from Section 5.2 to Section 5.4 are, to the author's knowledge, original.

In addition, we use the structures defined in Chapter 4 to analyze the twisted Dirac operator on spinor-valued 1-forms as defined in Section 2.5 when M is a manifold with torsion-free G_2-structure. In particular, we compute the kernel of the twisted Dirac operator and its dimension, a result of Theorem 3.7 of [Wan91]. The computations in Section 5.6 are similar to those found in Section 3 of [Wan91], however we focus more on the structure of the $\diamond$ operator from Section 3.4.2 and write them in the notation of this thesis.

5.1 The Twisted Dirac Operator on Spinor-Valued 1-Forms

Since the spinor bundle $\mathbb{S}$ of a G_2-manifold M can be identified with the bundle $\underline{\mathbb{R}} \oplus TM$, we are also able to identify the bundle $T^*M \otimes \mathbb{S}$ of spinor-valued 1-forms with the bundle $T^*M \oplus (T^*M \otimes T^*M)$. As such, we can compute the action of D_T and compare it to that of the operator D_1 above due to the identifications of these bundles.

Let X be a 1-form and let (f, Z) be a spinor. Using the local description of D_T in (2.46), we see that

$$\begin{aligned}
D_T(X \otimes (f, Z)) &= X \otimes D_0(f, Z) + \sum_i \nabla_i X \otimes [e_i \cdot (f, Z)] \\
&= X \otimes (-\operatorname{div} Z, \operatorname{grad} f + \operatorname{curl} Z) + \sum_i \nabla_i X \otimes (-Z_i, f e_i + e_i \times Z) \\
&= (-X_a(\nabla_k Z_k), X_a(\nabla_b f) + X_a(\nabla_k Z_l)\varphi_{klb}) \\
&\qquad + \sum_i \nabla_i X \otimes (-Z_i, f\delta_{ib} + \delta_{ik} Z_l \varphi_{klb}) \\
&= (-X_a(\nabla_k Z_k), X_a(\nabla_b f) + X_a(\nabla_k Z_l)\varphi_{klb}) \\
&\qquad + (-(\nabla_i X_a) Z_i, (\nabla_i X_a) f \delta_{ib} + (\nabla_i X_a)\delta_{ik} Z_l \varphi_{klb}) \\
&= (-X_a(\nabla_k Z_k), X_a(\nabla_b f) + X_a(\nabla_k Z_l)\varphi_{klb}) \\
&\qquad + (-(\nabla_k X_a) Z_k, (\nabla_b X_a) f + (\nabla_k X_a) Z_l \varphi_{klb}) \\
&= (-\nabla_k(X_a Z_k), \nabla_b(f X_a) + \nabla_k(X_a Z_l)\varphi_{klb}) \\
&= (-\operatorname{div}(X \otimes Z), \operatorname{grad}(fX) + \operatorname{curl}(X \otimes Z)) \\
&= D_1(fX, X \otimes Z).
\end{aligned}$$

Since $X \otimes (f, Z) = (fX, X \otimes Z)$ and the operators D_T and D_1 are linear in their arguments, it follows that $D_T = D_1$. We get another block matrix form of the twisted Dirac operator, but this time with respect to the decomposition $T^*M \otimes \mathbb{S} = T^*M \oplus (T^*M \otimes T^*M)$ as follows:

$$D_T = \begin{bmatrix} 0 & -\operatorname{div} \\ \operatorname{grad} & \operatorname{curl} \end{bmatrix}. \tag{5.1}$$

Since $D_T = D_1$, we can use Proposition 4.2.4 to describe the action of D_T^2. It follows that if Y is a 1-form and C is a 2-tensor with symmetric part C_+ and skew part C_- then

$$D_T^2(Y, C) = (\Delta Y, \Delta_L C_+ + \Delta_d C_-). \tag{5.2}$$

5.2 The Spaces of $\frac{1}{2}$- and $\frac{3}{2}$-Spinors

The decompositions $\mathcal{T}^2 = \Omega^0 \oplus \mathcal{S}_0^2 \oplus \Omega_7^2 \oplus \Omega_{14}^2$ and $T^*M \otimes \mathbb{S} = \mathbb{S}_{\frac{1}{2}} \oplus \mathbb{S}_{\frac{3}{2}}$ from (3.49) and (2.45) give rise to two different decompositions of spinor-valued 1-forms. That is, we have

$$\Gamma(T^*M \otimes \mathbb{S}) = \Gamma(\mathbb{S}_{\frac{1}{2}}) \oplus \Gamma(\mathbb{S}_{\frac{3}{2}}) = \Omega^1 \oplus \Omega^0 \oplus \mathcal{S}_0^2 \oplus \Omega_7^2 \oplus \Omega_{14}^2. \tag{5.3}$$

We see how these decompositions relate to each other.

Let Y be a 1-form and C be a 2-tensor. We compute $\mathrm{pr}_{\frac{1}{2}}(Y, C)$.

$$\begin{aligned}
\mathrm{pr}_{\frac{1}{2}}(Y, C) &= \iota \circ \mu(Y, C) \\
&= \iota\Big[\sum_i e_i \cdot (Y_i, C_{ib})\Big] \\
&= \iota(-C_{ii}, Y_i\delta_{ib} + \delta_{ik}C_{il}\varphi_{klb}) \\
&= \iota(-C_{ii}, Y_b + C_{kl}\varphi_{klb}) \\
&= -\frac{1}{7}\sum_j e_j \otimes [e_j \cdot (-C_{ii}, Y_b + C_{kl}\varphi_{klb})] \\
&= -\frac{1}{7}\sum_j e_j \otimes (-Y_j - C_{kl}\varphi_{klj}, -C_{ii}\delta_{jb} + \delta_{jk}Y_l\varphi_{klb} + \delta_{jp}C_{kl}\varphi_{klq}\varphi_{pqb}) \\
&= -\frac{1}{7}\sum_j e_j \otimes (-Y_j - C_{kl}\varphi_{klj}, -C_{ii}\delta_{jb} + \delta_{jk}Y_l\varphi_{klb} + \delta_{jp}C_{kl}(g_{kb}g_{lp} - g_{kp}g_{lb} - \psi_{klbp})) \\
&= -\frac{1}{7}\sum_j e_j \otimes (-Y_j - C_{kl}\varphi_{klj}, -C_{ii}\delta_{jb} + Y_l\varphi_{jlb} + C_{bj} - C_{jb} - C_{kl}\psi_{klbj}) \\
&= -\frac{1}{7}(-Y_a - C_{kl}\varphi_{kla}, -C_{kk}g_{ab} - Y_l\varphi_{lab} + C_{ba} - C_{ab} + C_{kl}\psi_{klab}).
\end{aligned}$$

Replacing C with $fg \in \Omega^0$, $C_{27} \in \mathcal{S}^2_0$, $Z \lrcorner \varphi \in \Omega^2_7$, and $C_{14} \in \Omega^2_{14}$ and using their properties, we get that

$$\begin{aligned}
\mathrm{pr}_{\frac{1}{2}}(Y, 0) &= \left(\frac{1}{7}Y, \frac{1}{7}Y \lrcorner \varphi\right), \\
\mathrm{pr}_{\frac{1}{2}}(0, fg) &= (0, fg), \\
\mathrm{pr}_{\frac{1}{2}}(0, C_{27}) &= (0, 0), \\
\mathrm{pr}_{\frac{1}{2}}(0, Z \lrcorner \varphi) &= \left(\frac{6}{7}Z, \frac{6}{7}Z \lrcorner \varphi\right), \\
\mathrm{pr}_{\frac{1}{2}}(0, C_{14}) &= (0, 0).
\end{aligned} \tag{5.4}$$

That is, we have that

$$\mathrm{pr}_{\frac{1}{2}}(Y, fg + C_{27} + Z \lrcorner \varphi + C_{14}) = \left(\frac{1}{7}Y + \frac{6}{7}Z, fg + \left(\frac{1}{7}Y + \frac{6}{7}Z\right) \lrcorner \varphi\right) \tag{5.5}$$

and

$$\mathrm{pr}_{\frac{3}{2}}(Y, fg + C_{27} + Z \lrcorner \varphi + C_{14}) = \left(\frac{6}{7}Y - \frac{6}{7}Z, C_{27} - \left(\frac{1}{7}Y - \frac{1}{7}Z\right) \lrcorner \varphi + C_{14}\right). \tag{5.6}$$

It follows that $\Gamma(\mathbb{S}_{\frac{1}{2}})$ consists of elements of the form $(Y, fg+Y \lrcorner \varphi)$ and and $\Gamma(\mathbb{S}_{\frac{3}{2}})$ consists of elements of the form $(Y, C_{27} - \frac{1}{6}Y \lrcorner \varphi + C_{14})$. We see that this is consistent with the pointwise dimensions of these spaces and that $\mathbb{S} = \mathbb{S}_{\frac{1}{2}}$ is a rank 8 real vector bundle over M.

5.3 Decomposition of the Curl Operator

Since the curl operator maps 2-tensors to 2-tensors, we can use the decomposition (3.49) to write curl in a block matrix form. Further, since D_T can be written in terms of the operators div, grad, and curl, understanding this block matrix form will give further insight into the properties of $\ker D_T$.

We begin by computing how curl acts on the multiples of the metric g. If f is a function, we can see that

$$\begin{aligned}
(\mathrm{curl}\,(fg))_{ab} &= (\nabla_i (fg)_{aj})\varphi_{ijb} \\
&= \nabla_i f g_{aj} \varphi_{ijb} \\
&= (\mathrm{grad}\, f)_i \varphi_{iab}.
\end{aligned}$$

That is,

$$\mathrm{curl}\,(fg) = (\mathrm{grad}\, f) \lrcorner \varphi, \tag{5.7}$$

and so curl maps Ω^0 to Ω^2_7.

Next, we look at the action of curl on Ω^2_7. Let Z be a 1-form. Computing in local coordinates yields

$$\begin{aligned}
(\mathrm{curl}\,(Z \lrcorner \varphi))_{ab} &= (\nabla_i (Z \lrcorner \varphi)_{aj})\varphi_{ijb} \\
&= \nabla_i Z_m \varphi_{maj} \varphi_{ijb} \\
&= \nabla_i Z_m (g_{mb} g_{ai} - g_{mi} g_{ab} - \psi_{mabi}) \\
&= \nabla_a Z_b - \nabla_i Z_i g_{ab} + \nabla_i Z_j \psi_{ijab}.
\end{aligned} \tag{5.8}$$

To decompose $\mathrm{curl}\,(Z \lrcorner \varphi)$ further, we take the trace of the above expression to find its Ω^0-component and we contract it with φ on two indices to find its Ω^2_7-component. Taking the trace gives

$$\begin{aligned}\operatorname{tr}(\operatorname{curl}(Z \lrcorner \varphi)) &= (\operatorname{curl}(Z \lrcorner \varphi))_{aa} \\ &= \nabla_a Z_a - \nabla_i Z_i g_{aa} + \nabla_i Z_j \psi_{ijaa} \\ &= -6\nabla_i Z_i \\ &= -6 \operatorname{div} Z,\end{aligned}$$

so

$$[\operatorname{curl}(Z \lrcorner \varphi)]_1 = -\frac{6}{7}(\operatorname{div} Z) g. \tag{5.9}$$

Contracting with φ gives

$$\begin{aligned}(\operatorname{curl}(Z \lrcorner \varphi))_{ab}\varphi_{abk} &= \nabla_a Z_b \varphi_{abk} - \nabla_i Z_i g_{ab}\varphi_{abk} + \nabla_i Z_j \psi_{ijab}\varphi_{abk} \\ &= \nabla_a Z_b \varphi_{abk} - 4\nabla_i Z_j \varphi_{ijk} \\ &= -3\nabla_a Z_b \varphi_{abk} \\ &= -3(\operatorname{curl} Z)_k.\end{aligned}$$

By (3.45), we then see that

$$[\operatorname{curl}(Z \lrcorner \varphi)]_7 = -\frac{1}{2}(\operatorname{curl} Z) \lrcorner \varphi \tag{5.10}$$

We can compute the $\mathcal{S}_0$- and Ω^2_{14}-components of $\operatorname{curl}(Z \lrcorner \varphi)$ by subtracting the above expressions from the symmetric and skew parts of $\operatorname{curl}(Z \lrcorner \varphi)$ respectively. Since g is symmetric and ψ is skew, this gives

$$([\operatorname{curl}(Z \lrcorner \varphi)]_{27})_{ab} = \frac{1}{2}(\nabla_a Z_b + \nabla_b Z_a) - \frac{1}{7}(\operatorname{div} Z) g_{ab} \tag{5.11}$$

and

$$\begin{aligned}([\operatorname{curl}(Z \lrcorner \varphi)]_{14})_{ab} &= \frac{1}{2}(\nabla_a Z_b - \nabla_b Z_a) + \nabla_i Z_j \psi_{ijab} + \frac{1}{2}(\operatorname{curl} Z)_i \varphi_{iab} \\ &= \frac{1}{2}(\nabla_a Z_b - \nabla_b Z_a) + \nabla_i Z_j \psi_{ijab} + \frac{1}{2}(\nabla_k Z_l)\varphi_{kli}\varphi_{iab} \\ &= \frac{1}{2}(\nabla_a Z_b - \nabla_b Z_a) + \nabla_i Z_j \psi_{ijab} + \frac{1}{2}\nabla_k Z_l (g_{ka}g_{lb} - g_{kb}g_{la} - \psi_{klab}) \\ &= \nabla_a Z_b - \nabla_b Z_a + \frac{1}{2}\nabla_i Z_j \psi_{ijab}.\end{aligned} \tag{5.12}$$

We can get other expressions for $[\operatorname{curl}(Z \lrcorner \varphi)]_7$ and $[\operatorname{curl}(Z \lrcorner \varphi)]_{14}$ by using the expression $(dZ)_{ab} = \nabla_a Z_b - \nabla_b Z_a$ and (3.52). Indeed, we notice that

$$\begin{aligned}
\nabla_i Z_j \psi_{ijab} &= \frac{1}{2}(\nabla_i Z_j - \nabla_j Z_i)\psi_{ijab} \\
&= \frac{1}{2}(dZ)_{ij}\psi_{ijab} \\
&= -2[(dZ)_7]_{ab} + [(dZ)_{14}]_{ab}.
\end{aligned}$$

Using (5.8) and (5.12), this gives that

$$[\operatorname{curl}(Z \lrcorner \varphi)]_7 = -\frac{3}{2}(dZ)_7 \tag{5.13}$$

and

$$[\operatorname{curl}(Z \lrcorner \varphi)]_{14} = \frac{3}{2}(dZ)_{14}. \tag{5.14}$$

We now look at how curl acts on Ω^2_{14}. Let $C_{14} \in \Omega^2_{14}$. Using (3.48), we can write

$$\begin{aligned}
(\operatorname{curl} C_{14})_{ab} &= (\nabla_i (C_{14})_{aj})\varphi_{ijb} \\
&= \nabla_i[(C_{14})_{aj}\varphi_{ijb}] \\
&= \nabla_i[(C_{14})_{bj}\varphi_{jai} - (C_{14})_{ij}\varphi_{jab}] \\
&= (\nabla_i (C_{14})_{bj})\varphi_{ija} + (\nabla_i (C_{14})_{ji})\varphi_{jab} \\
&= (\operatorname{curl} C_{14})_{ba} + ((\operatorname{div} C_{14}) \lrcorner \varphi)_{ab}.
\end{aligned}$$

Hence the skew part of $\operatorname{curl} C_{14}$ is given by

$$\frac{1}{2}[(\operatorname{curl} C_{14})_{ab} - (\operatorname{curl} C_{14})_{ba}] = \frac{1}{2}((\operatorname{div} C_{14}) \lrcorner \varphi)_{ab}.$$

It then follows that

$$[\operatorname{curl} C_{14}]_7 = \frac{1}{2}(\operatorname{div} C_{14}) \lrcorner \varphi \tag{5.15}$$

and that $\operatorname{curl} C_{14}$ has no Ω^2_{14}-component. Computing the trace of $\operatorname{curl} C_{14}$ gives

$$\begin{aligned}
\operatorname{tr}(\operatorname{curl} C_{14}) &= (\operatorname{curl} C_{14})_{aa} \\
&= (\nabla_i (C_{14})_{aj})\varphi_{ija} \\
&= \nabla_i[(C_{14})_{aj}\varphi_{ija}] \\
&= 0.
\end{aligned}$$

Since this expression is traceless, the $\mathcal{S}^2_0$-part is just the symmetric part of the expression. Hence

$$([\operatorname{curl} C_{14}]_{27})_{ab} = \frac{1}{2}[(\nabla_i (C_{14})_{aj})\varphi_{ijb} + (\nabla_i (C_{14})_{bj})\varphi_{ija}]. \tag{5.16}$$

Lastly, we look at the action of curl on $\mathcal{S}^2_0$. Let $C_{27} \in \mathcal{S}_0$. We compute

$$\begin{aligned}
\operatorname{tr}(\operatorname{curl} C_{27}) &= (\operatorname{curl} C_{27})_{aa} \\
&= (\nabla_i (C_{27})_{aj})\varphi_{ija} \\
&= \nabla_i [(C_{27})_{aj}\varphi_{ija}] \\
&= 0.
\end{aligned}$$

Thus $\operatorname{curl} C_{27}$ also has no Ω^0-component and we get a similar expression to the Ω^2_{14}-case

$$([\operatorname{curl} C_{27}]_{27})_{ab} = \frac{1}{2}[(\nabla_i (C_{27})_{aj})\varphi_{ijb} + (\nabla_i (C_{27})_{bj})\varphi_{ija}]. \tag{5.17}$$

To find the Ω^2_7-component, we contract $\operatorname{curl} C_{27}$ with φ on two indices.

$$\begin{aligned}
(\operatorname{curl} C_{27})_{ab}\varphi_{abk} &= (\nabla_i (C_{27})_{aj})\varphi_{ijb}\varphi_{abk} \\
&= (\nabla_i (C_{27})_{aj})(g_{ik}g_{ja} - g_{ia}g_{jk} - \psi_{ijka}) \\
&= \nabla_k (C_{27})_{aa} - \nabla_i (C_{27})_{ik} - \nabla_i [(C_{27})_{aj}\psi_{ijka}] \\
&= -\nabla_i (C_{27})_{ki} \\
&= -(\operatorname{div} C_{27})_k.
\end{aligned}$$

Equation (3.45) gives

$$[\operatorname{curl} C_{27}]_7 = -\frac{1}{6}(\operatorname{div} C_{27}) \lrcorner \varphi. \tag{5.18}$$

Subtracting the above from the skew part of $\operatorname{curl} C_{27}$ gives the Ω^2_{14}-component,

$$([\operatorname{curl} C_{27}]_{14})_{ab} = \frac{1}{2}[(\nabla_i (C_{27})_{aj})\varphi_{ijb} - (\nabla_i (C_{27})_{bj})\varphi_{ija}] + \frac{1}{6}(\nabla_i (C_{27})_{ji})\varphi_{jab}. \tag{5.19}$$

We summarize the findings of this section in Table 5.1 below.

5.4 Some Equivalences

Since the twisted Dirac operator D_T is defined in terms of the operators div, grad, and curl, it is natural to consider when these operators vanish. We look at their restrictions to the

	pr_1	pr_{27}	pr_7	pr_{14}
$(\mathrm{curl}\, fg)_{ab}$	0	0	$((\mathrm{grad}\, f) \lrcorner \varphi)_{ab}$	0
$(\mathrm{curl}\, C_{27})_{ab}$	0	$\frac{1}{2}[\nabla_i (C_{27})_{aj}\varphi_{ijb} + \nabla_i (C_{27})_{bj}\varphi_{ija}]$	$-\frac{1}{6}((\mathrm{div}\, C_{27}) \lrcorner \varphi)_{ab}$	$\frac{1}{2}[\nabla_i (C_{27})_{aj}\varphi_{ijb} - \nabla_i (C_{27})_{bj}\varphi_{ija}]$ $+\frac{1}{6}((\mathrm{div}\, C_{27}) \lrcorner \varphi)_{ab}$
$(\mathrm{curl}\,(Z \lrcorner \varphi))_{ab}$	$-\frac{6}{7}(\mathrm{div}\, Z) g_{ab}$	$\frac{1}{2}(\nabla_a Z_b + \nabla_b Z_a) - \frac{1}{7}(\mathrm{div}\, Z) g_{ab}$	$-\frac{1}{2}((\mathrm{curl}\, Z) \lrcorner \varphi)_{ab}$ $= -\frac{3}{2}((dZ)_7)_{ab}$	$\nabla_a Z_b - \nabla_b Z_a + \frac{1}{2}\nabla_i Z_j \psi_{ijab}$ $= \frac{3}{2}((dZ)_{14})_{ab}$
$(\mathrm{curl}\, C_{14})_{ab}$	0	$\frac{1}{2}[\nabla_i (C_{14})_{aj}\varphi_{ijb} + \nabla_i (C_{14})_{bj}\varphi_{ija}]$	$\frac{1}{2}((\mathrm{div}\, C_{14}) \lrcorner \varphi)_{ab}$	0

Table 5.1: Decomposition of the Curl Operator

various components of a 2-tensor with respect to our decomposition (3.49). Starting with the divergence operator, we compute its kernel when restricted to the space of traceless symmetric 2-tensors.

Proposition 5.4.1. *Let M be a manifold with torsion-free G_2-structure. If $C = C_{27} \in \mathcal{S}^2_0$ is a traceless symmetric 2-tensor on M with associated 3-form $\gamma = C \diamond \varphi \in \Omega^3_{27}$, then*

$$\operatorname{div} C = 0 \iff (d\gamma)_7 = 0 \iff (d^*\gamma)_7 = 0. \tag{5.20}$$

Proof. Since $(d\gamma)_{ijkl} = \nabla_i \gamma_{jkl} - \nabla_j \gamma_{ikl} + \nabla_k \gamma_{ijl} - \nabla_l \gamma_{ijk}$, using (3.53) we can compute

$$\begin{aligned}
[(d\gamma) \cdot \psi]_{ia} &= (d\gamma)_{ijkl} \psi_{ajkl} \\
&= (\nabla_i \gamma_{jkl}) \psi_{ajkl} - 3(\nabla_j \gamma_{ikl}) \psi_{ajkl} \\
&= [\nabla_i (C_{jp} \varphi_{pkl})] \psi_{ajkl} + [\nabla_i (C_{kp} \varphi_{jpl})] \psi_{ajkl} + [\nabla_i (C_{lp} \varphi_{jkp})] \psi_{ajkl} \\
&\quad - 3[\nabla_j (C_{ip} \varphi_{pkl})] \psi_{ajkl} - 3[\nabla_j (C_{kp} \varphi_{ipl})] \psi_{ajkl} - 3[\nabla_j (C_{lp} \varphi_{ikp})] \psi_{ajkl}.
\end{aligned}$$

Using the contractions (3.27) and (3.28), the fact that both φ and ψ are parallel with respect to ∇, and the identities $C_{ij} \varphi_{ija} = 0$ and $\operatorname{tr} C = C_{aa} = 0$ we get

$$\begin{aligned}
[(d\gamma) \cdot \psi]_{ia} &= \nabla_i C_{jp} (-4\varphi_{paj}) + \nabla_i C_{kp} (-4\varphi_{pak}) + \nabla_i C_{lp} (-4\varphi_{pal}) \\
&\quad - 3\nabla_j C_{ip} (-4\varphi_{paj}) \\
&\quad - 3\nabla_j C_{kp} (g_{ia} \varphi_{pjk} + g_{ij} \varphi_{apk} + g_{ik} \varphi_{ajp} - g_{ap} \varphi_{ijk} - g_{jp} \varphi_{aik} - g_{kp} \varphi_{aji}) \\
&\quad - 3\nabla_j C_{lp} (g_{ia} \varphi_{pjl} + g_{ij} \varphi_{apl} + g_{il} \varphi_{ajp} - g_{ap} \varphi_{ijl} - g_{jp} \varphi_{ail} - g_{lp} \varphi_{aji}) \\
&= 12 \nabla_j C_{ip} \varphi_{jpa} \\
&\quad - 3\nabla_j C_{ip} \varphi_{jpa} + 3\nabla_j C_{ak} \varphi_{jki} - 3\nabla_j C_{kj} \varphi_{kia} \\
&\quad - 3\nabla_j C_{ip} \varphi_{jpa} + 3\nabla_j C_{al} \varphi_{jli} - 3\nabla_j C_{lj} \varphi_{lia} \\
&= 6\nabla_j C_{ik} \varphi_{jka} + 6\nabla_j C_{ak} \varphi_{jki} - 6\nabla_j C_{kj} \varphi_{kia}.
\end{aligned}$$

Suppose that $\operatorname{div} C = 0$. Then in coordinates, $\nabla_i C_{ai} = 0$. By our hypothesis, the final term in $[(d\gamma) \cdot \psi]_{ia}$ vanishes which leaves us with

$$[(d\gamma) \cdot \psi]_{ia} = 6\nabla_j C_{ik} \varphi_{jka} + 6\nabla_j C_{ak} \varphi_{jki},$$

which is symmetric in i and a. Using the inverse of $\diamond$ in Corollary 3.4.10, we see that $(d\gamma)_7 = 0$. Conversely, suppose $(d\gamma)_7 = 0$. It follows that $(d\gamma) \cdot \psi$ is a symmetric 2-tensor. Our earlier calculation shows that $((\operatorname{div} C) \lrcorner \varphi)_{ia} = \nabla_j C_{kj} \varphi_{kia} = 0$. Since the map $X \mapsto X \lrcorner \varphi$ defines an isomorphism between $\mathfrak{X}$ and Ω^2_7, we must have that $\operatorname{div} C = 0$. This proves the first equivalence.

We have $(d^*\gamma)_{jk} = -\nabla_i \gamma_{ijk}$. We can again use (3.53) to compute the contraction of $d^*\gamma$ with φ on two indices.

$$\begin{aligned}
&(d^*\gamma)_{jk}\varphi_{jka}\\
&= -(\nabla_i \gamma_{ijk})\varphi_{jka}\\
&= -[\nabla_i(C_{ip}\varphi_{pjk}) - \nabla_i(C_{jp}\varphi_{ipk}) - \nabla_i(C_{kp}\varphi_{ijp})]\varphi_{jka}\\
&= -[\nabla_i(C_{ip}\varphi_{pjk})]\varphi_{jka} - 2[\nabla_i(C_{jp}\varphi_{ipk})]\varphi_{jka}\\
&= -\nabla_i C_{ip}(6g_{pa}) - 2\nabla_i C_{jp}(g_{ia}g_{pj} - g_{ij}g_{pa} - \psi_{ipaj})\\
&= -6\nabla_i C_{ai} + 2\nabla_i C_{ai}\\
&= -4\nabla_i C_{ai}.
\end{aligned}$$

If $\operatorname{div} C = 0$, we see that the above expression vanishes, so $(d^*\gamma) \in \Omega^2_{14}$. Conversely, if $(d^*\gamma)_7 = 0$, then $(d^*\gamma) \in \Omega^2_{14}$, so its contraction with φ on two indices vanishes. Hence $\nabla_i C_{ai} = (\operatorname{div} C)_a = 0$, and so $\operatorname{div} C = 0$. This proves the second equivalence. $\square$

Using a similar approach, we can also characterize when the various components of $\operatorname{curl} C$ vanish for a traceless symmetric 2-tensor.

Proposition 5.4.2. *Let M be a manifold with torsion-free G_2-structure. If $C = C_{27} \in \mathcal{S}^2_0$ is a traceless symmetric 2-tensor on M with associated 3-form $\gamma = C \diamond \varphi \in \Omega^3_{27}$, then the following equivalences hold:*

$$\operatorname{div} C = 0 \iff [\operatorname{curl} C]_7 = 0 \iff (d\gamma)_7 = 0 \iff (d^*\gamma)_7 = 0; \tag{5.21}$$

$$[\operatorname{curl} C]_{14} = 0 \iff (d^*\gamma)_{14} = 0; \tag{5.22}$$

$$[\operatorname{curl} C]_{27} = 0 \iff (d\gamma)_{27} = 0. \tag{5.23}$$

Proof. The equivalences (5.21) follow from (5.18) and the fact that $X \mapsto X \lrcorner \varphi$ defines an isomorphism between $\mathfrak{X}$ and Ω^2_7.

We have that

$$(d^*\gamma)_{jk} = -\nabla_i \gamma_{ijk} = -\nabla_i C_{ip}\varphi_{pjk} - \nabla_i C_{jp}\varphi_{ipk} - \nabla_i C_{kp}\varphi_{ijp}.$$

From the proof of the previous proposition, we also have that

$$[(d^*\gamma)_7]_{jk} = -\frac{2}{3}\nabla_i C_{ai}\varphi_{ajk}.$$

Taking the difference of the above two expressions, we get

$$[(d^*\gamma)_{14}]_{jk} = -\nabla_i C_{jp}\varphi_{ipk} + \nabla_i C_{kp}\varphi_{ipj} - \frac{1}{3}\nabla_i C_{ai}\varphi_{ajk}$$

which by comparing with (5.19) is $-2([\operatorname{curl} C]_{14})_{jk}$. Hence $(d^*\gamma)_{14}$ vanishes if and only if $[\operatorname{curl} C]_{14}$ vanishes.

Again, from the proof of the previous proposition, we have

$$[(d\gamma)\cdot\psi]_{ia} = 6\nabla_j C_{ik}\varphi_{jka} + 6\nabla_j C_{ak}\varphi_{jki} - 6\nabla_j C_{kj}\varphi_{kia}.$$

We then see that the 2-tensor $(d\gamma)\cdot\psi$ is traceless. Using the inverse of $\diamond$ in Corollary 3.4.10, it follows that the Ω^4_{27} component of $d\gamma$ corresponds to the traceless symmetric 2-tensor

$$\begin{aligned}
B_{ia} &= \frac{1}{24}([d\gamma)\cdot\psi]_{ia} + [(d\gamma)\cdot\psi]_{ai}) \\
&= \frac{1}{24}(6\nabla_j C_{ik}\varphi_{jka} + 6\nabla_j C_{ak}\varphi_{jki} - 6\nabla_j C_{kj}\varphi_{kia} \\
&\qquad + 6\nabla_j C_{ak}\varphi_{jki} + 6\nabla_j C_{ik}\varphi_{jka} - 6\nabla_j C_{kj}\varphi_{kai}) \\
&= \frac{1}{2}(\nabla_j C_{ik}\varphi_{jka} + \nabla_j C_{ak}\varphi_{jki}),
\end{aligned}$$

which by comparing with (5.17) is $([\operatorname{curl} C]_{27})_{ia}$. Thus $(d\gamma)_{27}$ vanishes if and only if $[\operatorname{curl} C]_{27}$ vanishes. □

The previous proposition allows us to fully characterize which traceless symmetric 2-tensors are in the kernel of curl.

Corollary 5.4.3. *Let M be a compact manifold with torsion-free G_2-structure. If $C = C_{27} \in \mathcal{S}^2_0$ is a traceless symmetric 2-tensor on M with associated 3-form $\gamma = C \diamond \varphi \in \Omega^3_{27}$, then*

$$\operatorname{curl} C = 0 \iff d\gamma = d^*\gamma = 0 \iff \Delta_d\gamma = 0. \tag{5.24}$$

Proof. The second equivalence follows from results from Hodge theory and so we only need to prove the first equivalence.

If $d\gamma = d^*\gamma = 0$, then each of their components with respect to the decompositions $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$ and $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$ are also all 0. From Table 5.1, $\operatorname{curl} C$ has no Ω^0-part, and so we get that $\operatorname{curl} C = 0$ from Proposition 5.4.2.

To show the other direction of the first equivalence, it suffices to show that $d\gamma$ has no Ω^4_1-part. We show this by taking the inner product of $d\gamma$ and ψ. We notice that $\langle d\gamma, \psi\rangle = \frac{1}{24}(d\gamma)_{ijkl}\psi_{ijkl} = \frac{1}{24}\operatorname{tr}((d\gamma)\cdot\psi)$. The 2-tensor $(d\gamma)\cdot\psi$ was shown to be traceless in the proof of Proposition 5.4.2 and so the result follows. □

Similar results about the kernel of div and curl restricted to Ω^2_{14} can be shown.

Proposition 5.4.4. *Let M be a manifold with torsion-free G_2-structure. If $C = C_{14} \in \Omega^2_{14}$ is a 2-14-form on M, then*

$$\operatorname{div} C = 0 \iff (dC)_7 = 0 \iff (d^*C) = 0. \tag{5.25}$$

Proof. We have the formula $(dC)_{ijk} = \nabla_i C_{jk} - \nabla_j C_{ik} + \nabla_k C_{ij}$. We use the inverse of $\diamond$ from Corollary 3.4.10 to find the Ω^2_7-form associated with dC. Contracting dC with φ on two indices yields

$$\begin{aligned}
[(dC) \cdot \varphi]_{ia} &= (dC)_{ijk}\varphi_{ajk} \\
&= [(\nabla_i C_{jk}) - (\nabla_j C_{ik}) + (\nabla_k C_{ij})]\varphi_{ajk} \\
&= \nabla_i C_{jk}\varphi_{jka} - 2\nabla_j C_{ik}\varphi_{jka} \\
&= -2\nabla_j C_{ik}\varphi_{jka}.
\end{aligned}$$

Using (3.48), we then see that dC corresponds to the Ω^2_7-form A given by

$$\begin{aligned}
A_{ia} &= \frac{1}{24}([(dC) \cdot \varphi]_{ia} - [(dC) \cdot \varphi]_{ai}) \\
&= \frac{1}{24}(-2\nabla_j C_{ik}\varphi_{jka} + 2\nabla_j C_{ak}\varphi_{jki}) \\
&= -\frac{1}{12}(\nabla_j C_{ik}\varphi_{jka} - \nabla_j C_{ak}\varphi_{kij}) \\
&= -\frac{1}{12}(\nabla_j C_{ik}\varphi_{jka} - (\nabla_j C_{ik}\varphi_{kaj} - \nabla_j C_{jk}\varphi_{kai})) \\
&= -\frac{1}{12}\nabla_j C_{kj}\varphi_{kia} \\
&= -\frac{1}{12}((\operatorname{div} C) \lrcorner \varphi)_{ia}.
\end{aligned}$$

This shows that $(dC)_7 = 0$ if and only if $\operatorname{div} C = 0$.

To prove the other equivalence, we notice that

$$(\operatorname{div} C)_a = \nabla_i C_{ai} = -\nabla_i C_{ia} = (d^*C)_a.$$

So $\operatorname{div} C = d^*C$, and the result follows. □

Proposition 5.4.5. *Let M be a manifold with torsion-free G_2-structure. If $C = C_{14} \in \Omega^2_{14}$ is a 2-14-form on M, then the following equivalences hold:*

$$\operatorname{div} C = 0 \iff [\operatorname{curl} C]_7 = 0 \iff (dC)_7 = 0 \iff d^*C = 0; \tag{5.26}$$

$$[\operatorname{curl} C]_{27} = 0 \iff (dC)_{27} = 0. \tag{5.27}$$

Proof. The equivalences (5.26) follow from (5.15) and the fact that $X \mapsto X \lrcorner \varphi$ defines an isomorphism between $\mathfrak{X}$ and Ω^2_7.

As computed in the proof of the previous proposition, we have

$$[(dC) \cdot \varphi]_{ia} = -2\nabla_j C_{ik}\varphi_{jka}.$$

Since $C \in \Omega^2_{14}$, $(dC) \cdot \varphi$ is traceless. It follows by Corollary 3.4.10 that $(dC)_{27}$ corresponds via $\diamond$ to the 2-tensor $A \in \mathcal{S}_0$ given by

$$\begin{aligned} A_{ia} &= \frac{1}{8}([(dC) \cdot \varphi]_{ia} + [(dC) \cdot \varphi]_{ai}) \\ &= \frac{1}{8}(-2\nabla_j C_{ik}\varphi_{jka} - 2\nabla_j C_{ak}\varphi_{jki}) \\ &= -\frac{1}{4}(\nabla_j C_{ik}\varphi_{jka} + \nabla_j C_{ak}\varphi_{jki}). \end{aligned}$$

Comparing this with (5.16), we see that A is a non-zero scalar multiple of $[\operatorname{curl} C]_{27}$. The result follows. $\square$

We can, in a similar fashion to Corollary 5.4.3, describe exactly which Ω^2_{14}-forms are in the kernel of curl.

Corollary 5.4.6. *Let M be a compact manifold with torsion-free G_2-structure. If $C = C_{14} \in \Omega^2_{14}$ is a 2-14-form on M, then*

$$\operatorname{curl} C = 0 \iff dC = d^*C = 0 \iff \Delta_d C = 0. \tag{5.28}$$

Proof. The second equivalence follows from results from Hodge theory, so we prove the first equivalence.

If $dC = d^*C = 0$, then each of $(dC)_1, (dC)_7$, and $(dC)_{27}$ are 0. Since $\operatorname{curl} C$ has no Ω^0- or Ω^2_{14}-part (see Table 5.1), it follows from Proposition 5.4.5 that $\operatorname{curl} C = 0$.

In order to show the other direction, we see that it suffices to show that $(dC)_1 = 0$. We take the inner product of dC and φ. Notice that $\langle dC, \varphi \rangle = \frac{1}{6}(dC)_{ijk}\varphi_{ijk} = \frac{1}{6}\operatorname{tr}((dC) \cdot \varphi) = 0$, hence $(dC)_1 = 0$ as required. $\square$

5.5 Splitting of Laplacians

In Sections 4.2 and 5.1, we showed (see (5.2)) that the square of the twisted Dirac operator D_T^2 acts as the Lichnerowicz Laplacian Δ_L on symmetric 2-tensors and as the Hodge

Laplacian Δ_d on 2-forms. Moreover, in the previous section, we noted that the kernels of the divergence and curl operators are closely tied with harmonic 2- and 3-forms. In this, we shall show that when the G_2-structure φ is torsion-free, the Lichnerowicz and Hodge Laplacians commute with the projections onto the spaces $\Omega^0, \mathcal{S}^2_0, \Omega^2_7$, and Ω^2_{14} as well as analogous results for our decomposition of 3-forms.

We begin with the Lichnerowicz Laplacian. Let f be a function. Direct computations using (4.22) yields

$$(\Delta_L(fg))_{ab} = (\Delta(fg))_{ab} - 2R_{iabj}(fg)_{ij} = (\Delta f)g_{ab}, \tag{5.29}$$

where the second term vanishes by Ricci-flatness. We see that Δ_L maps Ω^0 to Ω^0. Next, let $C_{27} \in \mathcal{S}_0$. We can check that

$$\begin{aligned}(\Delta_L C_{27})_{ab} &= (\Delta C_{27})_{ab} - 2R_{iabj}C_{ij}\\ &= (\Delta C_{27})_{ba} - 2R_{jabi}C_{ji}\\ &= (\Delta_L C_{27})_{ba},\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(\Delta C_{27}) &= (\Delta_L C_{27})_{aa}\\ &= (\Delta C_{27})_{aa} - 2R_{iaaj}C_{ij}\\ &= 0\end{aligned}$$

by Ricci-flatness. Hence $\Delta_L C_{27}$ is traceless and symmetric. This shows that Δ_L splits with respect to the decomposition $\mathcal{S}^2 = \Omega^0 \oplus \mathcal{S}^2_0$.

Remark 5.5.1. This splitting of the Lichnerowicz Laplacian still holds without the torsion-free assumption. Indeed, using the more general form of (4.22), we see that if f is a function then

$$\begin{aligned}(\Delta_L(fg))_{ab} &= (\Delta(fg))_{ab} + \mathrm{Ric}_{ai}(fg)_{ib} + \mathrm{Ric}_{bi}(fg)_{ai} - 2R_{iabj}(fg)_{ij}\\ &= (\Delta f)g_{ab} + f\mathrm{Ric}_{ab} + f\mathrm{Ric}_{ba} - 2f\mathrm{Ric}_{ab}\\ &= (\Delta f)g_{ab}.\end{aligned}$$

Additionally, if $C_{27} \in \mathcal{S}^2_0$, we get

$$\begin{aligned}\operatorname{tr}(\Delta_L C_{27}) &= (\Delta_L C_{27})_{aa}\\ &= (\Delta C_{27})_{aa} + \mathrm{Ric}_{ai}(C_{27})_{ia} + \mathrm{Ric}_{ai}(C_{27})_{ai} - 2R_{iaaj}(C_{27})_{ij}\\ &= 2\mathrm{Ric}_{ai}(C_{27})_{ai} - 2\mathrm{Ric}_{ij}(C_{27})_{ij}\\ &= 0,\end{aligned}$$

so Δ_L is again traceless. Symmetry follows from an argument similar to the torsion-free case where we also use the symmetry of the Ricci tensor and of C_{27}.

Next, we check that the Hodge Laplacian splits with respect to our decompositions of k-forms. This allows us to consider harmonic k-l-forms and refined Betti numbers b^k_l. A more general proof of this fact in the torsion-free case can be found in Chapter 3 of [Joy00]. We provide a direct proof using local coordinates here for the cases $k = 2$ and $k = 3$.

We start with the $k = 2$ case. Let X be a vector field. Using (4.23) and (4.24), we can see that

$$\begin{aligned}(\Delta_d(X \lrcorner \varphi))_{ab} &= (\Delta(X \lrcorner \varphi))_{ab} - 2R_{iabj}(X \lrcorner \varphi)_{ij} \\ &= -\nabla_i\nabla_i(X_l\varphi_{lab}) - 2R_{iabj}X_l\varphi_{lab} \\ &= -(\nabla_i\nabla_i X_l)\varphi_{lab} \\ &= ((\Delta X) \lrcorner \varphi)_{ab},\end{aligned} \tag{5.30}$$

so Δ_d maps Ω^2_7 to Ω^2_7. Lastly, if we let $C_{14} \in \Omega^2_{14}$, then

$$\begin{aligned}(\Delta_d C_{14})_{ab}\varphi_{abk} &= (\Delta C_{14})_{ab}\varphi_{abk} - 2R_{iabj}(C_{14})_{ij}\varphi_{abk} \\ &= -(\nabla_i\nabla_i(C_{14})_{ab})\varphi_{abk} \\ &= -\nabla_i\nabla_i[(C_{14})_{ab}\varphi_{abk}] \\ &= 0.\end{aligned}$$

Hence the Hodge Laplacian splits with respect to the decomposition $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$.

Now we consider the $k = 3$ case. We recall (see [Pet16]) that for a 3-form η that the Hodge Laplacian Δ_d acts by

$$\begin{aligned}(\Delta_d\eta)_{abc} = &-\nabla_i\nabla_i\eta_{abc} + \mathrm{Ric}_{ai}\eta_{ibc} + \mathrm{Ric}_{bi}\eta_{aic} + \mathrm{Ric}_{ci}\eta_{abi} \\ &- 2R_{iabj}\eta_{ijc} - 2R_{ibcj}\eta_{ija} - 2R_{icaj}\eta_{ijb}.\end{aligned}$$

By Ricci-flatness, the above reduces to

$$(\Delta_d\eta)_{abc} = -\nabla_i\nabla_i\eta_{abc} - 2R_{iabj}\eta_{ijc} - 2R_{ibcj}\eta_{ija} - 2R_{icaj}\eta_{ijb}. \tag{5.31}$$

Since contractions of the Riemann tensor and φ on two indices vanish, we see that if f is a function then

$$\begin{aligned}(\Delta_d(f\varphi))_{abc} &= -\nabla_i\nabla_i(f\varphi)_{abc} - 2R_{iabj}(f\varphi)_{ijc} - 2R_{ibcj}(f\varphi)_{ija} - 2R_{icaj}(f\varphi)_{ijb} \\ &= -(\nabla_i\nabla_i f)\varphi_{abc} = (\Delta f)\varphi_{abc}.\end{aligned} \tag{5.32}$$

Hence Δ_d maps Ω^3_1 to Ω^3_1.

Now, if X is a vector field, we can compute using (4.27) and the first Bianchi identity that

$$\begin{aligned}
(\Delta_d(X \lrcorner \psi))_{abc} &= -\nabla_i\nabla_i(X \lrcorner \psi)_{abc} - 2R_{iabj}(X \lrcorner \psi)_{ijc} - 2R_{ibcj}(X \lrcorner \psi)_{ija} \\
&\qquad - 2R_{icaj}(X \lrcorner \psi)_{ijb} \\
&= -\nabla_i\nabla_i(X_l\psi_{labc}) - 2R_{iabj}(X_l\psi_{lijc}) - 2R_{ibcj}(X_l\psi_{lija}) - 2R_{icaj}(X_l\psi_{lijb}) \\
&= -(\nabla_i\nabla_i X_l)\psi_{labc} + 2X_lR_{ablc} + 2X_lR_{bcla} + 2X_lR_{calb} \\
&= (\Delta X)_l\psi_{labc} + 2X_lR_{labc} + 2X_lR_{lbca} + 2X_lR_{lcab} \\
&= ((\Delta X) \lrcorner \psi)_{abc}.
\end{aligned} \tag{5.33}$$

It follows that Δ_d also maps Ω^3_7 to Ω^3_7.

Lastly, if $C = C_{27} \in \mathcal{S}^2_0$ is a traceless symmetric 2-tensor, then

$$\begin{aligned}
&[(\Delta_d(C \diamond \varphi)) \cdot \varphi]_{ia} \\
&= (\Delta_d(C \diamond \varphi))_{ijk}\varphi_{ajk} \\
&= -\nabla_p\nabla_p(C \diamond \varphi)_{ijk}\varphi_{ajk} - 2R_{pijq}(C \diamond \varphi)_{pqk}\varphi_{ajk} - 2R_{pjkq}(C \diamond \varphi)_{pqi}\varphi_{ajk} \\
&\qquad - 2R_{pkiq}(C \diamond \varphi)_{pqj}\varphi_{ajk} \\
&= -\nabla_p\nabla_p C_{im}\varphi_{mjk}\varphi_{ajk} - \nabla_p\nabla_p C_{jm}\varphi_{imk}\varphi_{ajk} - \nabla_p\nabla_p C_{km}\varphi_{ijm}\varphi_{ajk} \\
&\qquad - 2R_{pijq}C_{pm}\varphi_{mqk}\varphi_{ajk} - 2R_{pijq}C_{qm}\varphi_{pmk}\varphi_{ajk} - 2R_{pijq}C_{km}\varphi_{pqm}\varphi_{ajk} \\
&\qquad - 2R_{pjkq}C_{pm}\varphi_{mqi}\varphi_{ajk} - 2R_{pjkq}C_{qm}\varphi_{pmi}\varphi_{ajk} - 2R_{pjkq}C_{im}\varphi_{pqm}\varphi_{ajk} \\
&\qquad - 2R_{pkiq}C_{pm}\varphi_{mqj}\varphi_{ajk} - 2R_{pkiq}C_{qm}\varphi_{pmj}\varphi_{ajk} - 2R_{pkiq}C_{jm}\varphi_{pqm}\varphi_{ajk} \\
&= -\nabla_p\nabla_p C_{im}(6g_{ma}) \\
&\qquad - \nabla_p\nabla_p C_{jm}(g_{ia}g_{mj} - g_{ij}g_{ma} - \psi_{imaj}) - \nabla_p\nabla_p C_{km}(g_{mk}g_{ia} - g_{ma}g_{ik} - \psi_{mika}) \\
&\qquad - 2R_{pijq}C_{pm}(g_{ma}g_{qj} - g_{mj}g_{qa} - \psi_{mqaj}) - 2R_{pijq}C_{qm}(g_{pa}g_{mj} - g_{pj}g_{ma} - \psi_{pmaj}) \\
&\qquad - 2R_{pkiq}C_{pm}(g_{mk}g_{qa} - g_{ma}g_{qk} - \psi_{mqka}) - 2R_{pkiq}C_{qm}(g_{pk}g_{ma} - g_{pa}g_{mk} - \psi_{pmka}) \\
&= -6\nabla_p\nabla_p C_{ia} + \nabla_p\nabla_p C_{ia} + \nabla_p\nabla_p C_{ia} \\
&\qquad + 2R_{pima}C_{pm} - 4R_{piam}C_{pm} - 2R_{aimq}C_{qm} + 2R_{iqma}C_{qm} \\
&\qquad - 2R_{pmia}C_{pm} - 2R_{pima}C_{pm} + 2R_{amiq}C_{qm} + 4R_{iqam}C_{qm} \\
&= -4\nabla_p\nabla_p C_{ia} - 8R_{piam}C_{pm}.
\end{aligned}$$

The tracelessness of C and Ricci-flatness show that this expression is traceless. The symmetry of C along with the symmetries of the Riemann tensor together show that this expression is also symmetric. Using the identities from Corollary 3.4.10 we see that $\Delta_d(C \diamond \varphi) \in \Omega^3_{27}$. This proves that the Hodge Laplacian splits with respect to the decomposition $\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$.

5.6 The Kernel and its Dimension

In order to compute the kernel of the twisted Dirac operator D_T on a compact manifold M with torsion-free G_2-structure, we make use of the fact that it is self-adjoint which we showed in Section 4.2.2. Indeed if $(Y, C) \in \ker D_T^2$, then we have

$$\begin{aligned} 0 &= \langle D_T^2(Y, C), (Y, C)\rangle \\ &= \langle D_T(Y, C), D_T^*(Y, C)\rangle \\ &= \langle D_T(Y, C), D_T(Y, C)\rangle \\ &= \|D_T(Y, C)\|^2 \end{aligned}$$

Hence $D_T(Y, C) = 0$ and $\ker D_T^2 \subseteq \ker D_T$. The reverse containment is evident as D_T is a linear operator thus $\ker D_T = \ker D_T^2$.

Equation (5.2) provides us with a nice form of D_T^2 to work with due to the Lichnerowicz and Hodge Laplacians splitting with respect to the decompositions $\mathcal{S}^2 = \Omega^0 \oplus \mathcal{S}^2_0$ and $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$ respectively (see Section 5.5). In particular, we see that

$$D_T(Y, C) = 0 \iff D_T^2(Y, C) = 0 \iff \Delta Y = 0, \Delta_L C_1 = \Delta_L C_{27} = 0, \Delta_d C_7 = \Delta_d C_{14} = 0.$$

Using the equations on the right side of the above, we would like to characterize when (Y, C) is in the kernel of D_T.

We recall that since M is Ricci-flat, Δ and Δ_d agree on 1-forms. This shows that $\Delta Y = 0$ if and only if Y is harmonic. Since (5.30) tells us that $\Delta_d(X \lrcorner \varphi) = (\Delta X) \lrcorner \varphi$, it also follows that $\Delta_d(X \lrcorner \varphi) = 0$ if and only if $\Delta X = 0$ if and only if X is harmonic. Next, using (5.29), we can see that if $C_1 = fg$ for some function f, then $\Delta_L(fg) = 0$ if and only if f is harmonic as well.

To compute the dimension of $\ker D_T$, we need to be able to characterize when the Lichnerowicz Laplacian of a symmetric 2-tensor vanishes. The next proposition provides such a characterization.

Proposition 5.6.1. *Let $C = C_{27} \in \mathcal{S}_0$ be a traceless symmetric 2-tensor and let $\gamma = C \diamond \varphi \in \Omega^3_{27}$ be its associated 3-form. If M is a manifold with torsion-free G_2-structure, the following equivalence holds:*

$$\Delta_L C = 0 \iff \Delta_d \gamma = 0. \tag{5.34}$$

Proof. We show the forward direction first. Computing $\Delta_d\gamma = dd^*\gamma + d^*d\gamma$ in coordinates, we have that the first term is

$$\begin{aligned}(dd^*\gamma)_{abc} &= \nabla_a(d^*\gamma)_{bc} - \nabla_b(d^*\gamma)_{ac} + \nabla_c(d^*\gamma)_{ab} \\ &= \nabla_a(-\nabla_i\gamma_{ibc}) - \nabla_b(-\nabla_i\gamma_{iac}) + \nabla_c(-\nabla_i\gamma_{iab}) \\ &= -\nabla_a\nabla_i(C_{ij}\varphi_{jbc} + C_{bj}\varphi_{ijc} + C_{cj}\varphi_{ibj}) \\ &\quad + \nabla_b\nabla_i(C_{ij}\varphi_{jac} + C_{aj}\varphi_{ijc} + C_{cj}\varphi_{iaj}) \\ &\quad - \nabla_c\nabla_i(C_{ij}\varphi_{jab} + C_{aj}\varphi_{ijb} + C_{bj}\varphi_{iaj}).\end{aligned}$$

The second term gives

$$\begin{aligned}(d^*d\gamma)_{abc} &= -\nabla_i(d\gamma)_{iabc} \\ &= -\nabla_i(\nabla_i\gamma_{abc} - \nabla_a\gamma_{ibc} + \nabla_b\gamma_{iac} - \nabla_c\gamma_{iab}) \\ &= -\nabla_i\nabla_i(C_{aj}\varphi_{jbc} + C_{bj}\varphi_{ajc} + C_{cj}\varphi_{abj}) \\ &\quad + \nabla_i\nabla_a(C_{ij}\varphi_{jbc} + C_{bj}\varphi_{ijc} + C_{cj}\varphi_{ibj}) \\ &\quad - \nabla_i\nabla_b(C_{ij}\varphi_{jac} + C_{aj}\varphi_{ijc} + C_{cj}\varphi_{iaj}) \\ &\quad + \nabla_i\nabla_c(C_{ij}\varphi_{jab} + C_{aj}\varphi_{ijb} + C_{bj}\varphi_{iaj}).\end{aligned}$$

By combining the above, we get

$$\begin{aligned}(\Delta_d\gamma)_{abc} &= -\nabla_i\nabla_iC_{aj}\varphi_{jbc} - \nabla_i\nabla_iC_{bj}\varphi_{ajc} - \nabla_i\nabla_iC_{cj}\varphi_{abj} \\ &\quad + (\nabla_i\nabla_a - \nabla_a\nabla_i)[C_{ij}\varphi_{jbc} + C_{bj}\varphi_{ijc} + C_{cj}\varphi_{ibj}] \\ &\quad - (\nabla_i\nabla_b - \nabla_b\nabla_i)[C_{ij}\varphi_{jac} + C_{aj}\varphi_{ijc} + C_{cj}\varphi_{iaj}] \\ &\quad + (\nabla_i\nabla_c - \nabla_c\nabla_i)[C_{ij}\varphi_{jab} + C_{aj}\varphi_{ijb} + C_{bj}\varphi_{iaj}] \\ &= -\nabla_i\nabla_iC_{aj}\varphi_{jbc} - \nabla_i\nabla_iC_{bj}\varphi_{ajc} - \nabla_i\nabla_iC_{cj}\varphi_{abj} \\ &\quad - (R_{iaik}C_{kj} + R_{iajk}C_{ik})\varphi_{jbc} - (R_{iabk}C_{kj} + R_{iajk}C_{bk})\varphi_{ijc} \\ &\quad - (R_{iack}C_{kj} + R_{iajk}C_{ck})\varphi_{ibj} \\ &\quad + (R_{ibik}C_{kj} + R_{ibjk}C_{ik})\varphi_{jac} + (R_{ibak}C_{kj} + R_{ibjk}C_{ak})\varphi_{ijc} \\ &\quad + (R_{ibck}C_{kj} + R_{ibjk}C_{ck})\varphi_{iaj} \\ &\quad - (R_{icik}C_{kj} + R_{icjk}C_{ik})\varphi_{jab} - (R_{icak}C_{kj} + R_{icjk}C_{ak})\varphi_{ijb} \\ &\quad - (R_{icbk}C_{kj} + R_{icjk}C_{bk})\varphi_{iaj}\end{aligned}$$

We recall that contractions of the Riemann curvature tensor and φ on any two indices vanish. Additionally, Equation (3.72) in the torsion-free setting gives us the following Bianchi-type identity:

$$R_{abil}\varphi_{ljk} + R_{abjl}\varphi_{lki} + R_{abkl}\varphi_{lij} = 0. \tag{5.35}$$

Using the above identity and Ricci-flatness, we can simplify our expression for $(\Delta_d\gamma)_{abc}$. We have

$$\begin{aligned}
(\Delta_d\gamma)_{abc} &= -\nabla_i\nabla_i C_{aj}\varphi_{jbc} - \nabla_i\nabla_i C_{bj}\varphi_{jca} - \nabla_i\nabla_i C_{cj}\varphi_{jab} \\
&\quad - R_{iajk}C_{ik}\varphi_{jbc} - R_{ibjk}C_{ik}\varphi_{jca} - R_{icjk}C_{ik}\varphi_{jab} \\
&\quad + (-R_{iabk}\varphi_{ijc} - R_{icbk}\varphi_{iaj})C_{kj} + (R_{ibck}\varphi_{iaj} - R_{iack}\varphi_{ibj})C_{kj} \\
&\quad + (-R_{icak}\varphi_{ijb} + R_{ibak}\varphi_{ijc})C_{kj} \\
&= -\nabla_i\nabla_i C_{aj}\varphi_{jbc} - \nabla_i\nabla_i C_{bj}\varphi_{jca} - \nabla_i\nabla_i C_{cj}\varphi_{jab} \\
&\quad - R_{iajk}C_{ik}\varphi_{jbc} - R_{ibjk}C_{ik}\varphi_{jca} - R_{icjk}C_{ik}\varphi_{jab} \\
&\quad + (-R_{kbai}\varphi_{ijc} - R_{kbci}\varphi_{iaj})C_{kj} + (-R_{kcbi}\varphi_{ija} - R_{kcai}\varphi_{ibj})C_{kj} \\
&\quad + (-R_{kaci}\varphi_{ijb} - R_{kabi}\varphi_{icj})C_{kj} \\
&= -\nabla_i\nabla_i C_{aj}\varphi_{jbc} - \nabla_i\nabla_i C_{bj}\varphi_{jca} - \nabla_i\nabla_i C_{cj}\varphi_{jab} \\
&\quad - R_{iajk}C_{ik}\varphi_{jbc} - R_{ibjk}C_{ik}\varphi_{jca} - R_{icjk}C_{ik}\varphi_{jab} \\
&\quad + R_{kbji}C_{kj}\varphi_{ica} + R_{kcji}C_{kj}\varphi_{iab} + R_{kaji}C_{kj}\varphi_{ibc} \\
&= -\nabla_i\nabla_i C_{aj}\varphi_{jbc} - \nabla_i\nabla_i C_{bj}\varphi_{jca} - \nabla_i\nabla_i C_{cj}\varphi_{jab} \\
&\quad - 2R_{iajk}C_{ik}\varphi_{jbc} - 2R_{ibjk}C_{ik}\varphi_{jca} - 2R_{icjk}C_{ik}\varphi_{jab} \\
&= (-\nabla_i\nabla_i C_{aj} - 2R_{iajk}C_{ik})\varphi_{jbc} + (-\nabla_i\nabla_i C_{bj} - 2R_{ibjk}C_{ik})\varphi_{jca} \\
&\quad + (-\nabla_i\nabla_i C_{cj} - 2R_{icjk}C_{ik})\varphi_{jab}.
\end{aligned}$$

Since $\Delta_L C = 0$, it follows that $-\nabla_i\nabla_i C_{ab} - 2R_{iabj}C_{ij} = 0$. Substituting this into the above tells us that $(\Delta_d\gamma)_{abc} = 0$, and so $\Delta_d\gamma = 0$.

To show the other direction, suppose that $\Delta_d\gamma = 0$. Using the computation from above, we have

$$\begin{aligned}
&(\Delta_d\gamma)_{ijk}\varphi_{ajk} \\
&= (-\nabla_p\nabla_p C_{iq} - 2R_{piqr}C_{pr})\varphi_{qjk}\varphi_{ajk} + (-\nabla_p\nabla_p C_{jq} - 2R_{ijqr}C_{pr})\varphi_{qki}\varphi_{ajk} \\
&\qquad + (-\nabla_p\nabla_p C_{kq} - 2R_{pkqr}C_{pr})\varphi_{qij}\varphi_{ajk} \\
&= (-\nabla_p\nabla_p C_{iq} - 2R_{piqr}C_{pr})(6g_{qa}) \\
&\qquad + (-\nabla_p\nabla_p C_{jq} - 2R_{pjqr}C_{pr})(g_{ia}g_{qj} - g_{ij}g_{qa} - \psi_{iqaj}) \\
&\qquad + (-\nabla_p\nabla_p C_{kq} - 2R_{pkqr}C_{pr})(g_{qk}g_{ia} - g_{qa}g_{ik} - \psi_{qika}).
\end{aligned}$$

We can use (4.27), the symmetry of C and the anti-symmetry of the Riemann tensor in its first two indices to simplify this and so

$$\begin{aligned}
&(\Delta_d\gamma)_{ijk}\varphi_{ajk}\\
&= (-6\nabla_p\nabla_p C_{ia} - 12R_{piar}C_{pr})\\
&\qquad + (\nabla_p\nabla_p C_{ia} + 2R_{piar}C_{pr} + 2R_{pjqr}\psi_{jqia}C_{pr})\\
&\qquad + (\nabla_p\nabla_p C_{ia} + 2R_{piar}C_{pr} + 2R_{pkqr}\psi_{kqia}C_{pr})\\
&= -4\nabla_p\nabla_p C_{ia} - 8R_{piar}C_{pr} - 4R_{pria}C_{pr}\\
&= 4(-\nabla_p\nabla_p C_{ia} - 2R_{piar}C_{pr}).
\end{aligned}$$

Since $\Delta_d\gamma = 0$, it follows that $(\Delta_L C)_{ia} = -\nabla_p\nabla_p C_{ia} - 2R_{piar}C_{pr} = 0$. Hence $\Delta_L C = 0$. $\square$

The result of this proposition in addition to the the other results of this section allow us to compute the kernel of the twisted Dirac operator. Our earlier computations shows that if $D_T(Y, C) = 0$ on a compact manifold with torsion-free G_2-structure, then by writing $C = C_1 + C_{27} + C_7 + C_{14} = fg + C_{27} + X \lrcorner \varphi + C_{14}$, we must have that f is a harmonic function, X and Y are harmonic 1-forms, C_{14} is a harmonic Ω^2_{14}-form, and C_{27} corresponds to a harmonic Ω^3_{27}-form. We have proven the following:

Theorem 5.6.2. *Let M be a compact manifold with torsion-free G_2-structure, then*

$$\dim\ker D_T = b^0 + b^1 + b^1 + b^2_{14} + b^3_{27} = b^2 + b^3. \tag{5.36}$$

Remark 5.6.3. The above provides an alternate proof for Theorem 3.7 of [Wan91]. Indeed, if M is connected, we have $b^0 = 1$. Furthermore, we have from Proposition 10.1.3 of [Joy00] that when M is a compact manifold with torsion-free G_2-structure with $\mathrm{Hol}(M) = G_2$, then $b^1 = 0$. We also have from Section 5.2 that $\mathbb{S}_{\frac{3}{2}}$ is consists of elements of the form $(Y, C) = (Y, C_{27} - \frac{1}{6}Y \lrcorner \varphi + C_{14})$. Our previous analysis tells us that if $D_T(Y, C) = 0$, then Y is a harmonic 1-form, C_{14} is a harmonic Ω^2_{14}-form, and C_{27} corresponds to a harmonic Ω^3_{27}-form, hence $\dim\ker D_T|_{\mathbb{S}_{\frac{3}{2}}} = b^2_{14} + b^3_{27} = b^2 + b^3 - 1$.

Remark 5.6.4. The quantity $b^2 + b^3$ is of importance in G_2 mirror symmetry. As seen in [BDZ18], if M and M' are G_2-mirror manifolds then they satisfy the Shatashvili–Vafa relation:

$$b^2(M) + b^3(M) = b^2(M') + b^3(M'). \tag{5.37}$$

Remark 5.6.5. The arguments presented in this section provide alternate proofs of Corollaries 5.4.3 and 5.4.6 from Section 5.4. In particular, if $C_{27} \in \mathcal{S}^2_0$ with $\gamma = C_{27} \diamond \varphi$, then we have

$$\begin{aligned}
\operatorname{curl} C_{27} = 0 &\implies \operatorname{div} C_{27} = \operatorname{curl} C_{27} = 0 \implies D_T(0, C_{27}) = 0\\
&\implies D_T^2(0, C_{27}) = 0 \implies \Delta_L C_{27} = 0 \implies \Delta_d\gamma = 0.
\end{aligned}$$

Conversely

$$\begin{aligned}\Delta_d \gamma = 0 \implies \Delta_L C_{27} = 0 &\implies D_T^2(0, C_{27}) = 0 \\ &\implies D_T(0, C_{27}) = 0 \implies \operatorname{curl} C_{27} = 0.\end{aligned}$$

A similar chain of implications shows the result for $C_{14} \in \Omega^2_{14}$.

5.7 Possible Applications and Extensions

We have shown that identifying the spinor bundle of a manifold with G_2-structure with the bundle $\underline{\mathbb{R}} \oplus T^*M$ is a useful framework for studying spin-theoretic objects. Since the analysis done in this thesis assumed torsion-freeness of the G_2-structure, a natural extension would be to explore analogous results in the general torsion case or in any of the 15 other torsion cases described in Remark 3.5.5. Further, since the Dirac and twisted Dirac operators are purely spin-theoretic and require no underlying G_2-structure, one could potentially use the dimension of the kernel of the Dirac and twisted Dirac operators as an obstruction to the existence of certain types of compatible G_2-structure on 7-dimensional spin manifolds.

As the twisted Dirac operator and the connection induced from the spin connection are first order differential operators on the bundle of spinor-valued 1-forms, it may be fruitful to study their respective Laplacians and Weitzenböck formulae. By comparing them, one could possibly obtain further results in G_2-geometry.

Finally, one could explore if analogous results could be obtained on manifolds with different holonomy groups such as Spin(7).

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