Budget-Constrained Optimal Insurance with an Upper Limit on the Insurer's Exposure

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis studies the problem of budget-constrained optimal insurance indemnification when the insurer imposes an upper limit on disbursement. In balancing the trade-off between the cost of paying the insurance premium and the benefit of receiving the indemnity, the risk-averse insured aims to maximize his/her subjective expected utility of terminal wealth, subject to a budget constraint, and to the constraint that the insurer has an upper bound on the indemnification disbursement. We assume that the insured's subjective probability measure is obtained from the insurer's probability measure by a transformation such as in Furman and Zitikis ([8]), and that the insurer uses a distortion-type premium principle. We also assume that the insurer can observe the realized loss by incurring a state-verification cost. We show that in the presence of an upper limit on disbursement, the optimal indemnification function is a limited variable deductible. We then examine three numerical applications, and we illustrate the optimal indemnity and retention function in each case.

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I would like to personally thank my supervisor, Professor Mario Ghossoub: thank you for taking me as a student, helping and supporting me throughout my thesis. I am elated to say that during your supervision I learned a lot. I could not have wished for a better supervisor. I would also like to thank all the little people who made this thesis possible.

Dedication

This is dedicated to my family who supported me through my program.

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Chapter 1

Introduction

It is generally understood that the problem of optimal insurance design is about how a risk-averse decision maker (DM, i.e. the insured) balances his/her trade-off between the cost of paying the insurance premium and the benefit of receiving the indemnity, for a realized loss, so as to make his/her well off in the future. The literature on optimal insurance design can be broadly split into two main approaches to the problem: the equilibrium model and the optimization model. Under the equilibrium model, the optimal insurance contract is determined by the optimal risk sharing between the insured and the insurer. For the latter, the optimal insurance design problem is formulated as a problem of maximizing the insured's expected utility or minimizing his/her residual risk, subject to various constraints.

The problem of optimal insurance design for the risk-averse DM, based on the optimization model, can be traced back to the classical approach of Arrow ([1]). In his seminal paper, he formulated a problem in which the risk-averse insured seeks to maximize his/her expected utility of terminal wealth with the constraint of a non-negative indemnity that is bounded above by the realized loss variable. With the assumption that the insurer's premium is proportional to the net premium (i.e. the expected value principle) with a loading factor, Arrow concluded that if the DM and the insurer have similar probabilistic beliefs about the random loss variable, then full coverage above a fixed deductible is optimal.

Since then, researchers have extended the classical problem, with the aim of replicating practices in the real-world insurance market. Case in point, it is important for the insurer to control his/her risk exposure when he designs an insurance contract. Therefore, limited coverage on insurance and reinsurance contracts, among other things, are fairly standard features of contracts in the real-world insurance market. The limited coverage feature of insurance contracts may be attributed to the limited financial capacity of the insurer

(Cummins and Mahul [5]), as well as the existence of regulatory constraints. Along this line, studies have extended the Arrow model to account for the insurer's risk constraint. The literature on optimal insurance design with an explicit upper limited coverage amount dates back to Cummins and Mahul ([5]). The authors used a two-step approach to solve the problem of maximizing the insured's expected utility of terminal wealth with the constraint of a predetermined upper limit on coverage. First, they solved the problem using a fixed premium. The result obtained is then used to determine the optimal premium level, thus completing the determination of the solution to the problem. The effect on the optimal deductible of a change in the insured's initial wealth and risk aversion were examined in the study. Moreover, the authors examined the impact of changes in the coverage cap on the optimal deductible. They showed that the optimal insurance contract for an expected utility maximizing DM with an upper-bound on coverage amount can be characterized as full insurance above a deductible up to the predetermined cap. Changes in risk aversion and initial wealth were shown to have ambiguous effects on the optimal deductible. Finally, the authors showed that an increase in the upper limit on coverage would increase the demand for insurance against small losses through a decrease in the deductible, for an insured that possesses a constant absolute risk aversion utility function.

Zhou, Wu and Wu ([20]) showed that in the presence of an insurer's loss limit, the optimal insurance indemnity function displays full coverage above a deductible, up to a cap. While the finding of Zhou et al ([20]) is similar to Cummins and Mahul ([5]), they differ in that the coverage limit was endogenously determined through the imposition of a prespecified limit on the insurer's net loss amount, unlike Cummins and Mahul ([5]), in which the coverage limit was prespecified. Similar to Cummins and Mahul ([5]), the authors embarked on the two-step approach in solving the problem. First, the problem is solved for the optimal insurance while keeping the insurance premium fixed. Then, the result is used to determine the optimal deductible and cap. Moreover, the authors examined the connections between the Arrow model and their proposed model, and they assessed the impact on insurance consumption of a change in the insured's initial wealth, for cases where the utility function is of the Decreasing Absolute Risk Aversion (DARA) or Increasing Absolute Risk Aversion (IARA) type. The authors showed that under expected value premium principle, when an insured's preference displays increasing (decreasing) absolute risk aversion, the optimal insurance policy provides more (less) coverage for small losses, and less coverage for large losses when compared to Arrow model. Moreover, for an insured with a DARA (IARA) utility function, the authors concluded that the optimal cap will decrease (increase) while the optimal deductible will increase (decrease) as the insured's initial wealth increases, suggesting that the insured will decrease (increase) the insurance coverage for both the small losses and large losses.

Zhou and Wu ([18]) explored the classical problem of optimal insurance design, subject to the constraint that insurer's expected loss after the insurance payment is maintained below some predetermined level. As in Cummins and Mahul ([5]), a two-step approach was used to solve the proposed problem. Moreover, they provided numerical illustrations using the exponential utility function for the cases of a two-valued loss distribution and an exponential loss distribution. The authors showed that if the insurer's risk constraint is binding, then the optimal solution to the problem is characterized as a piecewise linear deductible. The authors further concluded that it can be shown that the insured's optimal expected utility will increase if the insurer's risk tolerance increases.

In addressing the problem of optimal insurance design from a probabilistic point of view, Zhou and Wu ([19]) extended the Arrow model by imposing an additional the constraint that the insurer's Value-at-risk (VaR) of his/her terminal wealth falls below a certain predetermined threshold. As in Zhou et al ([18]), a two-step approach was used to solve the problem. As an example, Zhou and Wu ([19]) illustrated the calculation process of the optimal insurance for an absolute risk aversion utility maximizer and an exponential loss distribution. The authors showed that, in the presence of the VaR constraint, the optimal solution to the problem is a piecewise linear deductible, and the insured's optimal expected utility will increase as the insurer becomes more risk tolerant. They also showed that when the insured has an exponential utility function, the optimal insurance based on the VaR constraint results in larger losses for the insurer when compared to a similar problem without this risk constraint.

Ghossoub ([14]) examined the problem of budget-constrained demand for insurance when the insured and insurer disagree about the likelihoods associated with the realizations of the insurable loss. Moreover, Ghossoub ([14]) assumed that the insurer distorts his/her probability measure and uses a distortion premium principle, while the insured has a fixed insurance budget and imposes an upper limit constraint on the retained loss. Ghossoub ([14]) imposed a state-verification cost that the insurer can incur to verify the loss severity, in order to rule out any ex-post moral hazard issues. As a numerical example, Ghossoub ([14]) considers two special cases of the proposed problem based on variations in the transformation of the insurer's probability measure from the DM's probability measure. First, he examines a setting in which the insurer's probability measure is an Esscher-type transformation of the DM's probability measure for a truncated exponential distributed random loss variable. Also, the assumption that the DM has a high aversion to losses was made; thus, reflecting a stringent risk management constraint. He then re-examined the problem using a transformation as in Furman and Zitikis ([8]), using a similar setting as in the first numerical case. He concluded that the optimal retention function has a simple two-part structure: zero retention (full insurance) on events for which the insurer assigns zero probability, and a retention function that could be described as a limited variable deductible on the complement of that event. Moreover, in the case of the Esscher-type transformation, Ghossoub ([14]) showed that the optimal indemnity function for the problem being examined follows a three-part structure.

In this thesis, we examine the problem of budget-constrained optimal insurance indemnification when the insurer imposes an upper limit on disbursement. The problem is similar in spirit to that of Cummins and Mahul ([5]), but extends it in two directions: first, we use a distortion premium principle; and second, we allow for heterogeneity in beliefs between the insurer and the insured. The formulation of the problem follows closely with that of Ghossoub ([14]); however, we assume that the DM's subjective probability measure is obtained from the insurer's probability measure by the transformation such as in Furman and Zitikis ([8]). Similar to Ghossoub ([14]), we assume that the insurer uses a distortion-type premium principle and the DM has a fixed insurance budget. We also assume that the insurer can observe the realized loss by incurring a state-verification cost. The approach taken in obtaining the closed-form characterization of the optimal indemnification was guided by Ghossoub ([14]) and Xu ([17]). First, we convert the problem to its associated quantile formulation. The literature on quantile formulation generally relies on the use of the calculus of variation or the making of monotonicity assumptions. Like Ghossoub ([14]) and Xu ([17]), we perform a change of variable to the quantile formulation and then apply a relaxation method, via the use of a concave envelope function, to solve the problem without the use of the calculus of variation or making any monotonicity assumptions. We show that in the presence of an upper limit on disbursement, the optimal indemnification function is a limited variable deductible.

Moreover, we consider some numerical applications based on three different types of distortion function: convex power distortion, linear distortion, and concave power distortion. For each numerical example, we assume that the DM's probability measure is obtained from the insurer's probability measure through the Esscher-type model and that the loss variable is distributed according to the truncated exponential distribution. We illustrate the optimal indemnity in each case.

The rest of this thesis is organized as follows. In Chapter 2, we present the model, formulate the optimal insurance problem with an upper limit on the insurable loss, and characterize the optimal indemnification in closed form. Chapter 3 provides numerical illustrations, and Chapter 4 concludes. Some background material is presented in the Appendices.

Chapter 2

Model Formulation and Solution

2.1 Model Setup

A risk-averse EU-maximizing DM has initial wealth W_0 and is exposed to an insurable random loss $X : (S, \Sigma) \mapsto [0, M]$, a non-negative random variable on a measurable space (S, Σ) that is bounded by some $M \in \mathbb{R}^+$. Denote by $B(\Sigma)$ the set of all bounded Σ-measurable functions $f : (S, \Sigma) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $B(\Sigma)$ is a vector space and $B^+(\Sigma)$ denotes the cone of all positive elements of $B(\Sigma)$. The DM transfers part of her exposure to an insurer by purchasing insurance. The insurer provides insurance indemnification $Y(s) = I(X(s))$ against the realized loss $X(s)$, for each state of the world $s \in S$, based on the agreed insurance policy $Y \in B^+(\Sigma)$. The DM is assumed to have a fixed insurance budget of $0 < \Pi < W_0$, (see [4, 16]). After purchasing insurance, the DM's wealth is given by the random variable

$$W(s) := W_0 - \Pi - X(s) + Y(s), \ \forall s \in S.$$

Let $u : \mathbb{R} \to \mathbb{R}$ denote the DM's utility function, which satisfies the following assumption.

Assumption 2.1.1. *The utility function $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and satisfies:*

i) $u'(x) > 0$ *and* $u''(x) < 0$ *for all* $x \in \mathbb{R}^+$,*and*

ii) $u'(0) = +\infty$ *and* $\lim_{x \to +\infty} u'(x) = 0$.

We assume that the DM has a subjective probability measure Q on (S, Σ) and the insurer premium is based on a distortion-type premium principle, with probability measure P on (S, Σ) and a distortion function[1] T. In particular, the distortion-type premium principle $\aleph : B^+(\Sigma) \to \mathbb{R}^+$ is given by

$$\aleph(Y) := \int Y dT \circ P, \text{ for all } Y \in B^+(\Sigma),$$

where the integration is in the sense of Choquet, as defined below.

Definition 2.1.2. *The Choquet integral of $f \in B^+(\Sigma)$ with respect to the distorted probability measure $T \circ P$ is defined as*

$$\int f \, dT \circ P := \int_0^{+\infty} T(P(\{s \in S : f(s) > t\}))dt.$$

Assumption 2.1.3. *T is strictly increasing and continuously differentiable.*

We make the assumption that the DM's probability measure Q is obtained from the insurer's probability measure by the transformation such as the ones in Furman and Zitikis [7, 8].

Assumption 2.1.4. *Q is obtained from P by a transformation of the type*

$$\frac{dQ}{dP} = \frac{w(X)}{\int w(X)dP} := \phi(X) > 0,$$

where $w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a non-decreasing and strictly positive function.

Then P and Q are mutually absolutely continuous measures on Σ, $\phi(X) > 0$, and $\frac{dQ}{dP}$ is comonotonic with X. Moreover, for all $Z \in B(\Sigma)$,

$$\int Z dQ = \int Z\phi(X)dP.$$

[1] A probability distortion function is an increasing function $T : [0, 1] \to [0, 1]$ that satisfies $T(0) = 0$ and $T(1) = 1$.

Remark 2.1.5. *A special case of the above transformation is the Esscher transformation*[2], *defined as*

$$\frac{dQ}{dP} := \phi_E(X) = \frac{e^{bX}}{\int e^{bX} dP} \in B^+(\Sigma).$$

This transformation will be examined in section 3.

$$\frac{dQ}{dP} = \frac{w(X)}{\displaystyle\int w(X) dP},$$

for a non-decreasing and strictly positive weighting function $w : \mathbb{R}^+ \to \mathbb{R}^+$.

We make the standard assumption that DM is well-diversified so that the particular loss exposure X against which she is seeking an insurance coverage is sufficiently small. Such an assumption can also be interpreted as a limited liability constraint since it guarantees a nonnegative terminal wealth for indemnity functions that pay no more than the value of the loss. Assumptions of limited liability are commonly used in the literature (e.g., [4]). Specifically, in our setting, we assume the following.

Assumption 2.1.6. $X \leq W_0 - \Pi$.

Similar to Ghossoub ([14]), we assume that the insurer can observe the realized loss $X(s)$, by incurring a State-Verification Cost (SVC) $C(X(s))$, for each state of the world $s \in S$. We also assume that $C(0) = 0$ and that the administrative cost of processing an indemnity Y is given by ρY, where the factor loading $\rho \in (0, 1)$ is exogenously determined. In using the distortion-type premium principle, we assume that the expected SVC will therefore reduce the premium charged by the insurer, while the total expected indemnity payment will increase by the factor loading ρ. Consequently, the DM's budget constraint is given by

$$\int Y \, dT \circ P \leq \tilde{\Pi} := \frac{\Pi - \displaystyle\int C \, dT \circ P}{1 + \rho}.$$

[2]See, e.g.,m Bühlmann [2].

Definition 2.1.7. *Two functions $Y_1, Y_2 \in B(\Sigma)$ are said to be comonotonic (resp., anti-comonotonic) if*

$$\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \geq 0 \text{ (resp., } \leq 0\text{), for all } s, s' \in S.$$

For any $V \in B(\Sigma)$, we denote by F_V the cumulative distribution function of V with respect to the probability measure P, defined by

$$F_V(t) := P\big(\{s \in S : V(s) \leq t\}\big), \ \forall t \geq 0,$$

and we denote by $F_V^{-1}(t)$ the left-continuous inverse of the distribution function F_V (that is, the quantile function of V w.r.t. Q), defined by

$$F_V^{-1}(t) = \inf\Big\{z \in \mathbb{R}^+ : F_V(z) \geq t\Big\}, \ \forall t \in [0,1]\,.$$

2.2 Optimal Indemnification

2.2.1 The DM's Demand Problem

The insurer imposes an upper limit $L \in (0, M)$ on possible indemnification disbursement, and the DM's problem is that of finding an indemnity function that maximizes her subjective expected utility of terminal wealth, subject to the budget constraint, and to the constraint that the indemnity exceeds neither the total loss (indemnity constraint) or the insurer's upper limit on disbursement (insurer's risk limit):

Problem 2.2.1.

$$\sup_{Y \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - X + Y\Big)\ dQ : 0 \leq Y \leq \min(L, X); \int Y\ dT \circ P \leq \tilde{\Pi} \right\}.$$

By Assumption 2.1.4, Problem 2.2.1 can be re-written as follows:

Problem 2.2.2.

$$\sup_{Y \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - X + Y\Big)\phi(X)\ dP : 0 \leq Y \leq \min(L, X); \int Y\ dT \circ P \leq \tilde{\Pi} \right\}.$$

We proceed with the following lemma:

Lemma 2.2.3.

i) If $\tilde{\Pi} < 0$, then the set of feasible solutions for Problem 2.2.2 is empty.

ii) If $\tilde{\Pi} \geq \int \min(L, X) dT \circ P$, then $Y^* := \min(L, X)$ is optimal for Problem 2.2.2.

Proof.

i) Suppose that $\tilde{\Pi} < 0$. If Y feasible, then in particular

$$0 \leq \int Y dT \circ P \leq \tilde{\Pi}.$$

Since $\tilde{\Pi} < 0$, we have $\int Y dT \circ P < 0$, which is a contradiction. Therefore, the set of feasible solutions for Problem 2.2.2 is empty.

ii) Suppose that $Y^* := \min(L, X)$ and $\int \min(L, X) dT \circ P \leq \tilde{\Pi}$. Since $Y^* \in [0, \min(L, X)]$, it is feasible for Problem 2.2.2. To show optimality, let Y' be any feasible solution for Problem 2.2.2. Then we have $0 \leq Y' \leq \min(L, X)$ and $\int Y' dT \circ P \leq \tilde{\Pi}$. Moreover, since u is concave and increasing,

$$u(W_0 - \Pi - X + Y') \leq u(W_0 - \Pi - X + \min(L, X)) = u(W_0 - \Pi - X + Y^*).$$

Therefore, since $\phi \geq 0$,

$$\int u(W_0 - \Pi - X + Y')\phi(X) dP \leq \int u(W_0 - \Pi - X + Y^*)\phi(X) dP.$$

Hence, Y^* is optimal for Problem 2.2.2.

□

Consequently, in light of Lemma 2.2.3, we examine the case $0 \leq \tilde{\Pi} < \int \min(L, X) dT \circ P$:

Assumption 2.2.4. $0 \leq \tilde{\Pi} < \int \min(L, X) dT \circ P$

2.2.2 Optimal Indemnities

Our main result, Theorem 2.2.6, provides a closed-form characterization of the optimal indemnification, in the presence of an upper limit on the possible indemnification disbursement. First, we provide the following definition for a concave envelope.

Definition 2.2.5. *For a real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,\alpha]$, for some $\alpha > 0$, the concave envelope of f on the interval $[0,\alpha]$ is the smallest concave function g on $[0,\alpha]$ such that $g(x) \geq f(x)$, for each $x \in [0,\alpha]$.*

Theorem 2.2.6. *Let*

- *$\mathcal{L}$ denote the Lebesgue measure on $[0,1]$;*
- *$\psi : [0,1] \mapsto [0,1]$ be defined by $\psi(t) := \int_0^t \phi(F_{X,P}^{-1}(x))dx$;*
- *$m : [0,1] \mapsto [0,1]$ be defined by $m(t) := 1 - T(1 - \psi^{-1}(t))$, for all $t \in [0,1]$;*
- *δ be the concave envelope on $[0,1]$ of the function m;*
- *For each $\lambda \geq 0$ and for all $t \in [0,1]$,*

  $$q_\lambda^*(t) = \min\left[\min(L, F_{X,P}^{-1}(\psi^{-1}(t))), \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F_{X,P}^{-1}(\psi^{-1}(t)) + \Pi - W_0\right\}\right];$$

- *$\lambda^* > 0$ be chosen such that $\int_0^1 q_{\lambda^*}^*(t)m'(t)dt = \tilde{\Pi}$; and*
- *For each $\lambda \geq 0$, $\mathcal{E}_\lambda$ be the subset on $[0,1]$ defined by*

$$\begin{aligned}\mathcal{E}_\lambda = &\{t \in [0,1] : \delta(t) > m(t), L \leq F_{X,P}^{-1}(\psi^{-1}(t)), 0 < K_\lambda(t) < L\} \\ &\cup \{t \in [0,1] : \delta(t) > m(t), L > F_{X,P}^{-1}(\psi^{-1}(t)), K_\lambda(t) > 0\}.\end{aligned}$$

If assumptions 2.1.3, 2.1.5, 2.1.1, 2.1.6, and 2.2.4 hold, and if

$$\mathcal{L}(\mathcal{E}_{\lambda^*}) = 0, \tag{2.1}$$

then the indemnity function

$$Y^* := q_{\lambda*}^*(\psi(U)) \tag{2.2}$$

is optimal for Problem 2.2.2 and comonotonic with X, where $U := F_{X,P}(X)$.

2.2.3 Proof of Theorem 2.2.6

The goal of this section is to reformulate Problem 2.2.2 as a quantile optimization problem, solve for the optimal quantile, and then revert back to the corresponding optimal indemnity function. Denote the feasibility set for Problem 2.2.2 by

$$\mathcal{F}_Y = \left\{ Y \in B(\Sigma) : 0 \leq Y \leq \min(L, X); \int Y \, dT \circ P \leq \tilde{\Pi} \right\}$$

and define

$$\mathcal{F}_Y^{=} = \left\{ Y \in B(\Sigma) : 0 \leq Y \leq \min(L, X); \int Y \, dT \circ P = \tilde{\Pi} \right\}.$$

Lemma 2.2.7. *For a given $Y \in \mathcal{F}_Y$, denote $\tilde{Y} := F_{Y,P}^{-1}(F_{X,P}(X))$. Then $\tilde{Y} \in \mathcal{F}_Y$ and*

$$\int u(W_0 - \Pi - X + \tilde{Y})\phi(X)dP \geq \int u(W_0 - \Pi - X + Y)\phi(X)dP.$$

Proof. The proof follows closely that of Ghossoub ([14]). Since X is continuous, $U := F_{X,P}(X)$ has a uniform distribution over $(0,1)$, that is, $P(\{s \in S : F_{X,P}X(s) \leq t\}) = t$ for each $t \in (0,1)$, and $X = F_{X,P}^{-1}(U)$, P-a.s, see Föllmer and Schied ([6]). Let $\tilde{Y} := F_{Y,P}^{-1}(U)$ be a non-decreasing P- rearrangement of Y with respect to X. Then $\tilde{Y} \in \mathcal{F}_Y$ is comonotonic with X, by Proposition (1.1). Since u and $\phi(\cdot)$ are increasing, the map $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $L(x,y) := u(W_0 - \Pi - x + y)\phi(x)$ is supermodular by concavity of u (see Example 1.3). Then by Lemma (1.4),

$$\begin{aligned} \int u(W_0 - \Pi - X + Y)\phi(X)dP &= \int L(X,Y)dP \\ &\leq \int L(X,\tilde{Y})dP = \int u(W_0 - \Pi - X + \tilde{Y})\phi(X)dP, \end{aligned}$$

as required. □

Let $\mathcal{Q}$ denote the collection of all quantile functions f satisfying $0 \leq f(t) \leq \min(L, F_{X,P}^{-1}(t))$, for all $t \in [0,1]$. That is,

$$\mathcal{Q} := \Big\{ f : [0,1] \to \mathbb{R} \mid f \text{ is non-decreasing and left continuous;}$$

$$0 \le f(t) \le \min(L, F^{-1}_{X,P}(t)), \text{for each}\, t \in [0,1]\Big\}.$$

For each $f \in \mathcal{Q}$, we have $Y := f(U) \in B(\Sigma)$, where $U = F_{X,P}(X)$. Therefore, since $X = F^{-1}_{X,P}(U)$, we have

$$\begin{aligned}\int u(W_0 - \Pi - X + Y)\phi(X)dP &= \int u(W_0 - \Pi - F^{-1}_{X,P}(U) + f(U))\phi(F^{-1}_{X,P}(U))dP \\ &= \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))\phi(F^{-1}_{X,P}(t))dt.\end{aligned}$$

Likewise,

$$\int Y \; dT \circ P = \int f(U)T'(1-U)dP = \int_0^1 f(t)T'(1-t)dt.$$

In light of Lemma 2.2.7, solving Problem 2.2.2 reduces to solving the following quantile optimization problem

Problem 2.2.8.

$$\sup_{f \in \mathcal{Q}} \left\{ \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t)\right) \phi(F^{-1}_{X,P}(t))dt : \int_0^1 f(t)T'(1-t)dt \le \tilde{\Pi} \right\}.$$

The following Lemma shows the relationship between Problems 2.2.2 and 2.2.8.

Lemma 2.2.9. *If f^* is optimal for Problem 2.2.8, then $Y^* := f^*(F_{X,P}(X))$ is optimal for Problem 2.2.2.*

Proof. Let f^* be optimal for Problem 2.2.8 and $Y^* := f^*(U)$, where $U = F_{X,P}(X)$. Since $f^* \in \mathcal{Q}$, $0 \le Y^* \le \min(L, X)$. Moreover,

$$\begin{aligned}\int Y^* dT \circ P = \int F^{-1}_{Y^*,P}(U)T'(1-U)dP &= \int f^*(U)T'(1-U)dP \\ &= \int_0^1 f^*(t)T'(1-t)dt \le \tilde{\Pi}.\end{aligned}$$

Hence, Y^* is feasible for Problem 2.2.2. Take $Y \in B(\Sigma)$ with quantile function $F^{-1}_{Y,P}$ to be a feasible solution for Problem 2.2.2. Then $F^{-1}_{Y,P}$ is feasible for Problem 2.2.8. Observe that

$$\begin{aligned}
\int u(W_0 - \Pi - X + Y)\phi(X)dP &\leq \int u(W_0 - \Pi - F^{-1}_{X,P}(U) + F^{-1}_{Y,P}(U))\phi(F^{-1}_{X,P}(U))dP \\
&= \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + F^{-1}_{Y,P}(t))\phi(F^{-1}_{X,P}(t))dt \\
&\leq \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f^*(t))\phi(F^{-1}_{X,P}(t))dt \\
&= \int u(W_0 - \Pi - X + Y^*)\phi(X)dP,
\end{aligned}$$

where we used Lemma 2.2.7 and the optimality of f^* for Problem 2.2.8. Therefore Y^* is optimal for Problem 2.2.2. □

Next, we make a change of variable to remove the quantile function of ϕ from the objective of Problem 2.2.8. Let $\psi : [0,1] \to [0,1]$ be defined by

$$\psi(t) := \int_0^t \phi(F^{-1}_{X,P}(x))dx, \text{ with } \psi(0) = 0, \ \psi(1) = \int \phi(X)dP = Q(S) = 1.$$

Then,

$$\int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))\phi(F^{-1}_{X,P}(t))dt = \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))d\psi(t).$$

Taking $v(t) = \psi^{-1}(t)$,

$$\int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))d\psi(t) = \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))dv^{-1}(t),$$

and $z = v^{-1}(t) \Rightarrow t = v(z)$, we have

$$\begin{aligned}
\int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t))dv^{-1}(t) &= \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(v(z)) + f(v(z)))dz \\
&= \int_0^1 u(W_0 - \Pi - F^{-1}_{X,P}(v(t)) + q(t))dt,
\end{aligned}$$

where $q(t) = f(v(z))$. Furthermore, let $m : [0,1] \to [0,1]$ be defined by

$$m(t) := 1 - T(1 - v(t)) = 1 - T(1 - \psi^{-1}(t)).$$

Then

$$m'(t) = T'(1 - v(t))v'(t),$$

and

$$\begin{aligned}
\int_0^1 T'(1-t)f(t)dt &= \int_0^1 T'(1-v(z))f(v(z))dv(z),\\
&= \int_0^1 f(v(z))T'(1-v(z))v'(z)dz,\\
&= \int_0^1 q(t)T'(1-v(t))v'(t)dt,\\
&= \int_0^1 q(t)m'(t)dt.
\end{aligned}$$

Let $\mathcal{Q}^*$ be the feasibility set for the transformed quantile function, defined as

$$\mathcal{Q}^* := \Big\{ q : [0,1] \to \mathbb{R} \mid q \text{ is non-decreasing and left-continuous,}$$

$$0 \le q(t) \le \min\left(L, F^{-1}_{X,P}(v(t))\right), \text{for each } t \in [0,1] \Big\},$$

where $v = \psi^{-1}$. By making the change of variable for Problem 2.2.8 we now have the following problem.

Problem 2.2.10.

$$\sup_{q\in\mathcal{Q}^*} \left\{ \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(v(t)) + q(t)\right) dt : \int_0^1 q(t)m'(t)dt \le \tilde{\Pi} \right\}.$$

Next we show the relation between Problems 2.2.8 and 2.2.10 through the following lemma.

Lemma 2.2.11. *If q^* is optimal for Problem 2.2.10, then $f^* := q^* \circ \psi$ is optimal for Problem 2.2.8.*

Proof. Let $q^* \in \mathcal{Q}^*$ is optimal for Problem 2.2.10, and $f^* := q^* \circ \psi$. Then

$$\begin{aligned}
\int_0^1 T'(1-t)f^*(t)dt &= \int_0^1 T'(1-v(z))f^*(v(z))dv(z) = \int_0^1 T'(1-v(t))f^*(v(t))v'(t)dt\\
&= \int_0^1 q^*(\psi(v(t)))m'(t)dt = \int_0^1 q^*(t)m'(t)dt \le \tilde{\Pi},
\end{aligned}$$

where the first equality is based on the change of variable $z = v^{-1}(t)$ and $\psi^{-1} = v$.

Moreover, q^* feasible for Problem 2.2.10 implies that it is non-decreasing, left-continuous, and satisfies $0 \leq q^*(t) \leq \min(L, F^{-1}_{X,P}(v(t)))$, for each $t \in [0,1]$. Since ψ is increasing and continuous, by the Inverse Function Theorem, v is also increasing and continuous. Therefore, $f^* := q^* \circ v^{-1}$ is non-decreasing, left-continuous, and satisfies $0 \leq f^* = q^*(v^{-1}(t)) \leq \min(L, F^{-1}_{X,P}(t))$, for each $t \in [0,1]$. Therefore, f^* is feasible for Problem 2.2.8. Hence, $f^* \in \mathcal{Q}$. Next we show that f^* is optimal for Problem 2.2.8. Let f be feasible for Problem 2.2.8 and $q := f \circ v$. Then

$$\tilde{\Pi} \geq \int_0^1 T'(1-t)f(t)dt = \int_0^1 T'(1-v(z))f(v(z))dv(z) = \int_0^1 q(t)m'(t)dt,$$

which shows that q is feasible for Problem 2.2.10. Consequently,

$$\begin{aligned}
\int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t)\right)\phi(F^{-1}_{X,P}(t))dt &= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t)\right)d\psi(t) \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f(t)\right)dv^{-1}(t) \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(v(z)) + f(v(z))\right)dz \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(v(t)) + q(t)\right)dt \\
&\leq \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(v(t)) + q^*(t)\right)dt \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f^*(t)\right)dv^{-1}(t) \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f^*(t)\right)d\psi(t) \\
&= \int_0^1 u\left(W_0 - \Pi - F^{-1}_{X,P}(t) + f^*(t)\right)\phi(F^{-1}_{X,P}(t))dt,
\end{aligned}$$

where $q := f \circ v$. Hence, f^* is optimal for Problem 2.2.8. □

So far Lemma 2.2.11 shows that by solving Problem 2.2.10, we also indirectly solve Problem 2.2.8. In what follows, a similar approach to the work of Xu ([17]) and Ghossoub ([14]) is adapted to the current setting. First, we need the following result.

Lemma 2.2.12. *Let m be a real-valued function on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0, \alpha]$, for some $\alpha > 0$, and let δ be its concave envelope on the interval $[0, \alpha]$. Then:*

i) *$\delta(\cdot)$ dominates $m(\cdot)$ on $[0, \alpha]$, with $\delta(0) = m(0)$ and $\delta(\alpha) = m(\alpha)$;*

ii) *$\delta(\cdot)$ is concave on $[0, \alpha]$;*

iii) *$\delta(\cdot)$ is affine on $\{x \in [0, \alpha] : \delta(x) > m(x)\}$; and*

iv) *for all $x \in [0, \alpha]$, $\delta(x) \geq m(x)$.*

Moreover,

v) *If m is increasing, then so is δ;*

vi) *If m is continuously differentiable on $(0, \alpha)$, then δ is continuously differentiable on $(0, \alpha)$.*

Resulting from Lemma 2.2.12, we have the following lemma.

Lemma 2.2.13. *If $\delta(\cdot)$ is the concave envelope of $m(\cdot)$ on $[0, 1]$, then for any $q \in \mathcal{Q}^*$,*

$$\int_0^1 q(t)m'(t)dt \geq \int_0^1 q(t)\delta'(t)dt.$$

Proof. Let $\delta(\cdot)$ be the concave envelope of $m(\cdot)$ on $[0, 1]$. Then, $\delta(t) \geq m(t)$ for all $t \in [0, 1]$ with $\delta(1) = m(1)$. Hence, for every $q \in \mathcal{Q}^*$ we have

$$\begin{aligned} 0 &\leq \int_0^1 [\delta(t) - m(t)]dq(t) = \int_0^1 \int_y^1 [m'(x) - \delta'(x)]dxdq(y) \\ &= \int_0^1 \left[\int_0^x dq(y)\right] [m'(x) - \delta'(x)]dx = \int_0^1 q(t)[m'(t) - \delta'(t)]dt, \end{aligned}$$

where the second equality follows from applying Fubini's Theorem. □

In light of Lemma 2.2.13 we consider the following relaxed problem.

Problem 2.2.14.

$$\sup_{q\in\mathcal{Q}^*}\left\{\int_0^1 u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q(t)\right)dt : \int_0^1 q(t)\delta'(t)dt \le \tilde{\Pi}\right\}.$$

In what follows, we solve Problem 2.2.14 and then show that the solution is also optimal for Problem 2.2.10.

Lemma 2.2.15. *Let $q^*\in\mathcal{Q}^*$ satisfies:*

i) $\int_0^1 \delta'(t)q^*dt=\tilde{\Pi}$; and,

ii) there exists some $\lambda\ge 0$ such that for all $t\in[0,1]$,

$$q^*(t)=\underset{0\le y\le \min\left(L,F^{-1}_{X,P}(v(t))\right)}{\arg\max}\left\{u(W_0-\Pi-F^{-1}_{X,P}(v(t))+y)-\lambda y\delta'(t)\right\},$$

then q^* is optimal for Problem 2.2.14.

Proof. Suppose conditions i) and ii) holds. Then q^* is feasible for Problem 2.2.14. Let $q\in\mathcal{Q}^*$ be feasible for Problem 2.2.14. Then for each $t\in[0,1]$,

$$u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q^*(t)\right)-\lambda\delta'(t)q^*(t)\ge u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q(t)\right)-\lambda\delta'(t)q(t).$$

Therefore

$$u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q^*(t)\right)-u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q(t)\right)\ge\lambda[\delta'(t)q^*(t)-\delta'(t)q(t)].$$

Hence, by integrating over $[0,1]$ we have

$$\begin{aligned}\int_0^1\Big[u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q^*(t)\right)&-u\left(W_0-\Pi-F^{-1}_{X,P}(v(t))+q(t)\right)\Big]dt\\ &\ge\lambda\left[\int_0^1\delta'(t)q^*(t)dt-\int_0^1\delta'(t)q(t)dt\right]\\ &\ge\lambda\left[\tilde{\Pi}-\int_0^1\delta'(t)q(t)dt\right]\ge 0.\end{aligned}$$

Hence, q^* is optimal for Problem 2.2.14. □

By pointwise maximization of the Lagrangian for Problem 2.2.14 we obtain, for each $t \in [0,1]$

$$q_\lambda^*(t) = \underset{0 \le y \le \min\left(L, F_{X,P}^{-1}(v(t))\right)}{\arg\max} \left\{ u(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + y) - \lambda y \delta'(t) \right\}$$
$$= \min\left[\min\left(L, F_{X,P}^{-1}(v(t))\right), \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F_{X,P}^{-1}(v(t)) + \Pi - W_0\right\}\right].$$

Note that u is strictly increasing and strictly concave implies that u' is decreasing. Hence, by the Inverse Function Theorem $(u')^{-1}$ is decreasing. Therefore, for $q_\lambda^*(t)$ to be non-decreasing, δ' has to be a decreasing function over $[0,1]$, which follows from the concavity of δ.

Lemma 2.2.16. *For each $\lambda \ge 0$, define the function q_λ^* on $[0,1]$ by*

$$q_\lambda^*(t) := \min\left[\min\left(L, F_{X,P}^{-1}(v(t))\right), \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F_{X,P}^{-1}(v(t)) + \Pi - W_0\right\}\right]. \quad (2.3)$$

Then:

i) For each $\lambda \ge 0$, $q_\lambda^* \in \mathcal{Q}^*$;

ii) There exist $\lambda^* > 0$ such that $\int_0^1 \delta'(t) q_{\lambda^*}^*(t)dt = \tilde{\Pi}$; and

iii) For all $t \in [0,1]$,

$$q_\lambda^*(t) = \underset{0 \le y \le \min\left(L, F_{X,P}^{-1}(v(t))\right)}{\arg\max} \left\{ u(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + y) - \lambda y \delta'(t) \right\}.$$

For the proof of Lemma 2.2.16, we will need the following lemma.

Lemma 2.2.17. *For q_λ^*, as defined by eq. (2.3), there exists a $\lambda^* > 0$ such that $\int_0^1 q_{\lambda^*}^* m'(t)dt = \tilde{\Pi}$.*

Proof. Define the function $d : \mathbb{R}^+ \mapsto \mathbb{R}$ by

$$d(\lambda) := \int_0^1 q_\lambda^* m'(t)dt$$
$$= \int_0^1 m'(t) \times \min\left[\min\left(L, F_{X,P}^{-1}(v(t))\right), \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F_{X,P}^{-1}(v(t)) + \Pi - W_0\right\}\right] dt.$$

Since $(u')^{-1}$ is decreasing, $d(\lambda)$ is a decreasing function of λ. Furthermore, Assumption 2.1.1 implies that $\lim_{\lambda\mapsto 0} u'(\lambda\delta'(t)) = +\infty$ and $\lim_{\lambda\mapsto\infty} u'(\lambda\delta'(t)) = 0$, for a given $t \in [0,1]$. Therefore,

$$\lim_{\lambda\mapsto 0}((u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0) = +\infty,$$

and

$$\lim_{\lambda\mapsto\infty}((u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0) = F^{-1}_{X,P}(v(t)) + \Pi - W_0.$$

Furthermore, Assumption 2.1.6 implies that $F_{X,P}(W_0 - \Pi) = 1$, and so $W_0 - \Pi = F^{-1}_{X,P}(1) = F^{-1}_{X,P}(v(1)) \geq F^{-1}_{X,P}(v(t))$, for all $t \in [0,1]$. Hence, $F^{-1}_{X,P}(v(t)) + \Pi - W_0 \leq 0$. Therefore, Assumption 2.2.4 implies that

$$\begin{aligned}
\lim_{\lambda\mapsto 0} d(\lambda) &= \lim_{\lambda\mapsto 0}\int_0^1 q^*_\lambda m'(t)dt = \int_0^1 \lim_{\lambda\mapsto 0} q^*_\lambda m'(t)dt \\
&= \int_0^1 m'(t) \times \min\left[\min\left(L, F^{-1}_{X,P}(v(t))\right), \max\{0,\infty\}\right] dt \\
&= \int_0^1 m'(t) \times \min(L, F^{-1}_{X,P}(v(t)))dt \\
&= \int_0^1 \min(L, F^{-1}_{X,P}(z))T'(1-z)dz \\
&= \int \min(L,X)dT\circ P > \tilde{\Pi},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda\mapsto\infty} d(\lambda) &= \int_0^1 \lim_{\lambda\mapsto\infty} q^*_\lambda m'(t)dt \\
&= \int_0^1 m'(t) \times \min\left[\min\left(L, F^{-1}_{X,P}(v(t))\right), \max\left\{0, F^{-1}_{X,P}(v(t)) + \Pi - W_0\right\}\right] dt \\
&= \int_0^1 m'(t) \times \min\left[\min(L, F^{-1}_{X,P}v(t)), 0\right] dt = 0.
\end{aligned}$$

Therefore,

$$0 = \lim_{\lambda\mapsto+\infty} d(\lambda) \leq \tilde{\Pi} < \int \min(L,X)dT\circ P = \int_0^1 \min(L, F^{-1}_{X,P}(z))T'(1-z)dz = \lim_{\lambda\mapsto 0} d(\lambda).$$

Hence, the existence of a $\lambda^* > 0$ such that $\int Y^* dT\circ P = \tilde{\Pi}$, follows from the Intermediate Value Theorem. □

Proof of Lemma 2.2.16.

We begin by proving parts i) and iii), respectively. Denote $M(t) = \min(L, F^{-1}_{X,P}(v(t)))$. By construction, for each $\lambda \geq 0$, $q^*_\lambda(\mathrm{t})$ is bounded between 0 and $M(t)$, for all $t \in [0,1]$. Since $(u')^{-1}$ is decreasing and δ' non-increasing for $t \in [0,1]$, the composition $(u')^{-1}(\lambda\delta')$ is non-decreasing for a given $\lambda \geq 0$. Moreover, $F^{-1}_{X,P}(v(t)))$ and $v(t)$ are also increasing functions of t. Therefore, $\max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0\right\}$ and $\min\left(L, F^{-1}_{X,P}(v(t))\right)$ are both non-decreasing for $t \in [0,1]$. Moreover, since $F^{-1}_{X,P}(v(t)))$ and $v(t)$ are continuous for all $t \in [0,1]$, then they are also left-continuous. Likewise,$(u')^{-1}(\lambda\delta')$ is continuous and is also left-continuous. Therefore, the functions

$$t \mapsto \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0\right\}$$

and

$$t \mapsto \min\left(L, F^{-1}_{X,P}(v(t))\right)$$

are both left-continuous. Hence, q^*_λ is non-decreasing and left-continuous. Thus, $q^*_\lambda \in \mathcal{Q}^*$.

Now, for each $\lambda > 0$ and fixed $t \in [0,1]$, consider the problem

$$\underset{0 \leq y \leq M(t)}{\arg\max}\, f(y) := u(W_0 - \Pi - F^{-1}_{X,P}(v(t)) + y) - \lambda y \delta'(t).$$

Since u is strictly concave, f is also strictly concave function of y, that is f' is (strictly) decreasing in y. Hence, the first-order condition on f yields a global maximum for f at

$$y^* := (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0.$$

If $y^* < 0$, then since f' is decreasing, it is negative on $[0, M(t)]$. Therefore, f is decreasing on $[0, M(t)]$, and hence, attains a local maximum of $f(0)$ at $y = 0$. If $y^* > M(t)$, then since f' is decreasing, it is positive on the interval $[0, M(t)]$. Therefore, f is increasing on the interval $[0, M(t)]$, and hence, attains a local maximum of $f(M(t))$ at $y = M(t)$. If $0 < y^* < M(t)$, then f attains a global maximum of $f(y^*)$ on $[0, M(t)]$. Consequently, the function $y^{**} := \min[M(t), \max(0, y^*)]$ solves the problem, for a given choice of λ, t and L. Hence, part iii) is proved.

We now turn to the proof of part ii). To this end, note that eq. (2.3) implies that for a given $t \in [0,1]$, if $L \leq F^{-1}_{X,P}(v(t))$,

$$q^*_\lambda(t) = \begin{cases} 0, & \text{if } K_\lambda(t) \leq 0, \\ (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0 & \text{if } 0 < K_\lambda(t) < L, \\ L & \text{if } K_\lambda(t) \geq L; \end{cases}$$

where

$$K_\lambda(t) := (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0, \text{ for a given } t \in [0,1].$$

Likewise, if $F^{-1}_{X,P}(v(t)) < L$, then

$$q^*_\lambda(t) = \begin{cases} 0, & \text{if } K_\lambda(t) \le 0, \\ (u')^{-1}(\lambda\delta'(t)) + F^{-1}_{X,P}(v(t)) + \Pi - W_0 & \text{if } 0 < K_\lambda(t) < F^{-1}_{X,P}(v(t)), \\ F^{-1}_{X,P}(v(t)) & \text{if } K_\lambda(t) \ge F^{-1}_{X,P}(v(t)). \end{cases}$$

Define the sets

$$\begin{aligned} \mathcal{A} :=& \{t \in [0,1] : \delta(t) > m(t)\}; \\ \mathcal{B} :=& \{t \in [0,1] : L \le F^{-1}_{X,P}(v(t))\} = \{t \in [0,1] : t \ge \psi(F_{X,P}(L))\}. \end{aligned} \tag{2.4}$$

For a given $\lambda \ge 0$, define the sets

$$\begin{aligned} \mathcal{C}_\lambda :=& \{t \in [0,1] : 0 < K_\lambda(t) < L\}; \\ \mathcal{D}_\lambda :=& \{t \in [0,1] : K_\lambda(t) > 0\}; \end{aligned} \tag{2.5}$$

and

$$\mathcal{E}_\lambda := (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_\lambda) \cup ((\mathcal{A} \setminus \mathcal{B}) \cap \mathcal{D}_\lambda). \tag{2.6}$$

Observe that

$$\begin{aligned} (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_\lambda) &= \{x : x \in \mathcal{A}, x \in \mathcal{B} \text{ and } x \in \mathcal{C}_\lambda\} \\ &= \left\{t \in [0,1] : \delta(t) > m(t), 0 < K_\lambda(t) < L, L \le F^{-1}_{X,P}(v(t))\right\} \\ &= \{x : x \in \mathcal{A} \text{ and } x \in \mathbf{V}\} = \mathcal{A} \cap \mathbf{V}, \end{aligned}$$

where $\mathbf{V} := \left\{t \in [0,1] : 0 < K_\lambda(t) < L, L \le F^{-1}_{X,P}(v(t))\right\} = \mathcal{B} \cap \mathcal{C}_\lambda$. Similarly,

$$\begin{aligned} (\mathcal{A} \setminus \mathcal{B}) \cap \mathcal{D}_\lambda &= \{x : x \in \mathcal{A}, x \notin \mathcal{B} \text{ and} x \in \mathcal{D}_\lambda\} \\ &= \{x : x \in \mathcal{A}, x \in \mathcal{B}^c \text{ and } x \in \mathcal{D}_\lambda\} \\ &= \mathcal{A} \cap \mathcal{B}^c \cap \mathcal{D}_\lambda \\ &= \left\{t \in [0,1] : \delta(t) > m(t), K_\lambda(t) > 0, L > F^{-1}_{X,P}(v(t))\right\} \\ &= \{x : x \in \mathcal{A} \text{ and } x \in \mathbf{Q}\} = \mathcal{A} \cap \mathbf{Q}, \end{aligned}$$

where $\mathbf{Q} := \left\{t \in [0,1] : K_\lambda(t) > 0, L > F^{-1}_{X,P}(v(t))\right\} = \mathcal{D}_\lambda \cap \mathcal{B}^c$. Therefore

$$\begin{aligned}
\mathcal{E}_\lambda =& (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_\lambda) \cup ((\mathcal{A} \setminus \mathcal{B}) \cap \mathcal{D}_\lambda) \\
=& (\mathcal{A} \cap \mathbf{V}) \cup (\mathcal{A} \cap \mathbf{Q}) \\
=& (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_\lambda) \cup (\mathcal{A} \cap \mathcal{B}^c \cap \mathcal{D}_\lambda) \\
=& \mathcal{A} \cap [(B \cap \mathcal{C}_\lambda) \cup (B^c \cap \mathcal{D}_\lambda)] \\
=& \{t \in [0,1] : \delta(t) > m(t), L \le F^{-1}_{X,P}(v(t)), 0 < K_\lambda(t) < L\} \\
& \cup \{t \in [0,1] : \delta(t) > m(t), L > F^{-1}_{X,P}(v(t)), K_\lambda(t) > 0\}.
\end{aligned}$$

With the help of Lemma 2.2.17 and eq. (2.1), we are now able to provide the proof of Lemma 2.2.16 part ii). Lemma 2.2.17 shows that there exists a $\lambda^* > 0$ such that

$$\int_0^1 q^*_{\lambda^*} m'(t) dt = \tilde{\Pi}.$$

Since for each $\lambda \ge 0$ q^*_λ is monotone, then it is differentiable *a.e.* Therefore, for a given $t \in [0,1]$, if $L \le F^{-1}_{X,P}(v(t))$, then

$$dq^*_\lambda(t) = \begin{cases} 0, & \text{if } K_\lambda(t) \le 0, \\ \lambda((u')^{-1})'(\lambda\delta'(t)) d\delta'(t) + (F^{-1}_{X,P})'(v(t)) v'(t) dt & \text{if } 0 < K_\lambda(t) < L, \\ 0 & \text{if } K_\lambda(t) \ge L; \end{cases} \tag{2.7}$$

and, if $F^{-1}_{X,P}(v(t)) < L$, then

$$dq^*_\lambda(t) = \begin{cases} 0, & \text{if } K_\lambda(t) \le 0, \\ \lambda((u')^{-1})'(\lambda\delta'(t)) d\delta'(t) + (F^{-1}_{X,P})'(v(t)) v'(t) dt & \text{if } 0 < K_\lambda(t) < F^{-1}_{X,P}(v(t)), \\ (F^{-1}_{X,P})'(v(t)) v'(t) dt & \text{if } K_\lambda(t) \ge F^{-1}_{X,P}(v(t)). \end{cases} \tag{2.8}$$

Then for $\lambda > 0$,

$$\begin{aligned}
\int_0^1 [\delta(t) - m(t)] dq^*_\lambda(t) &= \int_{\mathcal{A}} [\delta(t) - m(t)] dq^*_\lambda(t) + \int_{[0,1]\setminus\mathcal{A}} [\delta(t) - m(t)] dq^*_\lambda(t) \\
&= \int_{\mathcal{A}} [\delta(t) - m(t)] dq^*_\lambda(t) \\
&= \int_{\mathcal{A}\cap\mathcal{B}} [\delta(t) - m(t)] dq^*_\lambda(t) + \int_{\mathcal{A}\setminus\mathcal{B}} [\delta(t) - m(t)] dq^*_\lambda(t),
\end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in eq. (2.4) above and

$$[0,1] \setminus \mathcal{A} := \{t \in [0,1] : \delta(t) = m(t)\}.$$

Since δ is affine on the set $\mathcal{A}$, we have $d\delta' = 0$ on $\mathcal{A}$. Moreover, by eqs. (2.7) and (2.8),

$$\begin{aligned}
\int_0^1 [\delta(t) - m(t)] dq_\lambda^*(t) &= \int_{\mathcal{A}\cap\mathcal{B}} [\delta(t) - m(t)] dq_\lambda^*(t) + \int_{\mathcal{A}\setminus\mathcal{B}} [\delta(t) - m(t)] dq_\lambda^*(t) \\
&= \int_{\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}_\lambda} [\delta(t) - m(t)] dF_{X,P}^{-1}(v(t)) + \int_{(\mathcal{A}\setminus\mathcal{B})\cap\mathcal{D}_\lambda} [\delta(t) - m(t)] dF_{X,P}^{-1}(v(t)) \\
&= \int_{(\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}_\lambda)\cup((\mathcal{A}\setminus\mathcal{B})\cap\mathcal{D}_\lambda)} [\delta(t) - m(t)] dF_{X,P}^{-1}(v(t)) \\
&= \int_{\mathcal{E}_\lambda} [\delta(t) - m(t)] dF_{X,P}^{-1}(v(t)),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}_\lambda = &\{t \in [0,1] : \delta(t) > m(t), L \leq F_{X,P}^{-1}(v(t)), 0 < K_\lambda(t) < L\} \\
&\cup \{t \in [0,1] : \delta(t) > m(t), L > F_{X,P}^{-1}(v(t)), K_\lambda(t) > 0\},
\end{aligned}$$

and $\mathcal{C}_\lambda$ and $\mathcal{D}_\lambda$ are defined in eq. (2.5), previously.

Therefore, by eq. (2.1), we have:

$$\int_0^1 [\delta(t) - m(t)] dq_{\lambda^*}^*(t) = \int_{\mathcal{E}_{\lambda^*}} [\delta(t) - m(t)] dF_{X,P}^{-1}(v(t)) = 0,$$

where λ^* is defined in Lemma 2.2.16. Applying Fubini's Theorem, as in the proof of Lemma 2.2.13, gives

$$\begin{aligned}
0 = \int_0^1 [\delta(t) - m(t)] dq_{\lambda^*}^*(t) &= \int_0^1 \int_y^1 [m'(x) - \delta'(x)] dx dq_{\lambda^*}^*(y) \\
= \int_0^1 \left[\int_0^x dq_{\lambda^*}^*(y) \right] [m'(x) - \delta'(x)] dx &= \int_0^1 q_{\lambda^*}^* [m'(x) - \delta'(x)] dx.
\end{aligned}$$

Consequently, $\int_0^1 q_{\lambda^*}^* m'(t) dt = \int_0^1 q_{\lambda^*}^* \delta'(t) dt$. Therefore,

$$\int_0^1 q_{\lambda^*}^* \delta'(t) dt = \int_0^1 q_{\lambda^*}^* m'(t) dt = \tilde{\Pi},$$

which completes the proof. $\square$

Lemma 2.2.16 and Lemma 2.2.13 imply that for all $q \in \mathcal{Q}^*$, and for all $\lambda \geq 0$,

$$\begin{aligned}\int_0^1 \Big[u\left(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + q(t)\right) - \lambda q(t) m'(t)\Big] dt &\leq \int_0^1 \Big[u\left(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + q(t)\right) \\ &\qquad - \lambda q(t)\delta'(t)\Big] dt \\ &\leq \int_0^1 \Big[u\left(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + q_\lambda^*(t)\right) \\ &\qquad - \lambda q_\lambda^*(t)\delta'(t)\Big] dt.\end{aligned}$$

Therefore, if $\lambda^* > 0$ is such that $\int_0^1 q_{\lambda^*}^* m'(t)dt = \tilde{\Pi}$, then by eq. (2.1), $\int_0^1 q_{\lambda^*}^* m'(t)dt = \int_0^1 q_{\lambda^*}^* \delta'(t)dt$, and so for all $q \in \mathcal{Q}^*$,

$$\begin{aligned}\int_0^1 \Big[u\left(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + q(t)\right) - \lambda^* q(t) m'(t)\Big] dt &\leq \int_0^1 \Big[u\left(W_0 - \Pi - F_{X,P}^{-1}(v(t)) + q_{\lambda^*}^*(t)\right) \\ &\qquad - \lambda^* q_{\lambda^*}^*(t)\ m'(t)\Big] dt.\end{aligned}$$

Then, the optimal solution to Problem 2.2.10 is given by $q_{\lambda^*}^*$. Thus, by Lemma 2.2.9, 2.2.11, 2.2.15, and 2.2.16, the function

$$Y^* := q_{\lambda^*}^*(\psi(U))$$

is optimal for Problem 2.2.2 and comonotonic with X, where:

i) $U := F_{X,P}(X)$ is uniformly distributed on $(0,1)$;

ii) For all $t \in [0,1]$,

$$q_\lambda^*(t) = \min\left[\min(L, F_{X,P}^{-1}(\psi^{-1}(t))), \max\left\{0, (u')^{-1}(\lambda\delta'(t)) + F_{X,P}^{-1}(\psi^{-1}(t)) + \Pi - W_0\right\}\right];$$

iii) δ is a concave envelope on $[0,1]$ of the function m defined by $m(t) := 1 - T(1-\psi^{-1}(t))$, where $\psi(t) := \int_0^t \phi(F_{X,P}^{-1}(x))dx$, for all $t \in [0,1]$; and,

iv) $\lambda^* > 0$ is chosen such that $\int_0^1 q_{\lambda^*}^*(t) m'(t)dt = \tilde{\Pi}$, and so $\int Y^* dT \circ P = \tilde{\Pi}$.

2.2.4 Optimality of Full Insurance with an Upper Limit

Lemma 2.2.18. *Let Y^* denote the optimal indemnity for Problem 2.2.2, given in eq. (2.2), where $\lambda^* > 0$ is chosen such that $\int_0^1 q^*_{\lambda^*}(t)m'(t)dt = \int Y^* \, dT \circ P = \tilde{\Pi}$. If*

$$\lambda^* \leq \frac{u'(W_0 - \Pi)}{\delta'(0)}, \tag{2.9}$$

then $Y^* = \min(L, X)$.

Proof. First note that the indemnity the optimal Y^* can be re-written as $Y^* = I^* \circ X$, where the Borel-measurable function $I^* : [0, M] \to [0, M]$ is given by:

$$I^*(x) := \min\left[\min(L, x), \max\left\{0, x + \Pi - W_0 + (u')^{-1}(\lambda^* \delta'(\psi(F_{X,P}(x))))\right\}\right], \ \forall x \in [0, M].$$

Let $f : [0, M] \mapsto \mathbb{R}$ be the function defined by

$$f(x) := \frac{u'(\min(L, x) - x - \Pi + W_0)}{\delta'(\psi(F_{X,P}(x)))} = \begin{cases} \frac{u'(W_0 - \Pi)}{\delta'(\psi(F_{X,P}(x)))} & \text{for } x \leq L; \\ \frac{u'(L - x - \Pi + W_0)}{\delta'(\psi(F_{X,P}(x)))} & \text{for } x > L, \end{cases}$$

By (strict) monotonicity and concavity of the functions u and δ, it is easily verified that the function f is increasing on its domain. Therefore,

$$f(0) = \frac{u'(W_0 - \Pi)}{\delta'(0)} \leq f(x) \leq f(M), \ \forall x \in [0, M].$$

Consequently, eq. (2.9) implies that

$$\lambda^* \leq f(x) = \frac{u'(\min(L, x) - x - \Pi + W_0)}{\delta'(\psi(F_{X,P}(x)))}, \ \forall x \in [0, M].$$

By monotonicity and concavity of u, this then implies that

$$x + \Pi - W_0 + (u')^{-1}(\lambda^* \delta'(\psi(F_{X,P}(x)))) \geq \min(L, x) \geq 0, \ \forall x \in [0, M],$$

and so

$$Y^* = I^* \circ X = \min(L, X).$$

□

2.2.5 When is Equation (2.1) satisfied?

Here we examine some situations in which eq. (2.1) is satisfied. Eq. (2.1) is satisfied for example whenever the set $\mathcal{E}_{\lambda^*}$ is empty over $[0, 1]$. The cases examined in this section span across three subgroups: $\delta = m$ (when m is concave), $\delta > m$, and $\delta = t$ for $t \in [0, 1]$.

In general, if the distortion function T is the identity function or a convex function over $[0, 1]$, then we have $\delta = m$ over $[0, 1]$. Further details are captured in propositions 2.2.19 and 2.2.20 below.

Proposition 2.2.19. *If $T(t) = t$, for all $t \in [0, 1]$, then eq. (2.1) is satisfied.*

Proof. Suppose $T(t) = t$, then $m(t) = 1 - T(1 - v(t)) = v(t)$. Since $\psi'(t) = \phi(F^{-1}_{X,P}(t))$ is increasing and positive, then ψ is increasing and convex. By the Inverse Function Theorem, $v = \psi^{-1}$ is concave and increasing. Therefore, $\delta = m$ on $[0, 1]$ and so $\mathcal{E}_\lambda = \emptyset$ for all λ. □

Proposition 2.2.20. *If the function T is convex, then eq. (2.1) is satisfied.*

Proof. Suppose that T is convex. Let $c(t) := T(1-v(t))$ and $m(t) = 1-T(1-v(t))$. Then $m(t) = 1 - c(t)$. Since T is convex, then $c(t)$ is also convex and $c''(t) \geq 0$. Therefore,

$$c''(t) = T''(1 - v(t))[v'(t)]^2 - T'(1 - v(t))v''(t) \geq 0.$$

This implies that,

$$\frac{T''(1 - v(t))}{T'(1 - v(t))} \geq \frac{v''(t)}{[v'(t)]^2}.$$

Moreover, $m''(t) = -c''(t)$ implies that

$$m''(t) = -T''(1 - v(t))[v'(t)]^2 + T'(1 - v(t))v''(t) \leq 0.$$

Hence, m is concave and we have $\delta = m$ on [0,1]. Therefore, $\mathcal{E}_\lambda = \emptyset$ for all λ. □

The third case in which eq. (2.1) is satisfied follows closely some of the results of Ghossoub ([14]) and Carlier & Dana ([3]).

Proposition 2.2.21. *If the likelihood ratio $t \mapsto \frac{\phi(F^{-1}_{X,P}(t))}{T'(1-t)}$ is non-decreasing on $[0, 1]$, then eq. (2.1) is satisfied.*

Proof. Recall that $\psi(t) = \int_0^t \phi(F_{X,P}^{-1}(x))dx$ and $m(t) = 1 - T(1-\psi^{-1}(t)) = 1 - T(1-v(t))$, for all $t \in [0,1]$. If the function $t \mapsto \frac{\phi(F_{X,P}^{-1}(t))}{T'(1-t)}$ is non-decreasing on $[0,1]$, then for all $t \in [0,1]$, we have

$$-\frac{T''(1-t)}{T'(1-t)} \leq \frac{(\phi(F_{X,P}^{-1}))'(t)}{\phi(F_{X,P}^{-1}(t))} = \frac{\psi''(t)}{\psi'(t)}.$$

By the Inverse Function Theorem we have

$$\psi'(t) = \frac{1}{v'(\psi(t))}.$$

Therefore, by taking the second derivative with respect to t, this gives

$$\frac{T''(1-v(t))}{T'(1-v(t))} \geq \frac{v''(t)}{[v'(t)]^2}, \ \forall t \in [0,1].$$

Therefore, for all $t \in [0,1]$,

$$m''(t) = T'(1-v(t))v''(t) - T''(1-v(t))[v'(t)]^2 \leq 0,$$

and so m is concave, implying that $\delta = m$ on $[0,1]$. Therefore, $\mathcal{E}_\lambda = \emptyset$ for all λ. □

The fourth case in which eq. (2.1) is satisfied is when L is small enough relative to the loss.

Proposition 2.2.22. *Let $\lambda^* > 0$ be chosen such that $\int_0^1 q_{\lambda^*}^* m'(t)dt = \tilde{\Pi}$. If $L \leq F_{X,P}^{-1}(\psi^{-1}(0))$ and $K_{\lambda^*}(t) \geq L$ for all $t \in [0,1]$, then eq. (2.1) is satisfied.*

Proof. Let $v = \psi^{-1}$, and suppose that $L \leq F_{X,P}^{-1}(v(0))$. This implies that $L \leq F_{X,P}^{-1}(v(t))$ for all $t \in [0,1]$. Hence, $\mathcal{B} = [0,1]$ and so $\mathcal{B}^c = \emptyset$. Therefore, for $\lambda \geq 0$,

$$\mathcal{E}_\lambda = (\mathcal{A} \cap [0,1] \cap \mathcal{C}_\lambda) \cup (\emptyset \cap \mathcal{D}_\lambda) = \mathcal{A} \cap \mathcal{C}_\lambda.$$

Moreover, if $K_{\lambda^*}(t) \geq L$, for all $t \in [0,1]$, then $\mathcal{C}_{\lambda^*} = \emptyset$. Hence, $\mathcal{E}_{\lambda^*} = \emptyset$. □

The fifth case in which eq. (2.1) is satisfied is when λ^* is bounded from below.

Proposition 2.2.23. *Let $\lambda^* > 0$ be chosen such that $\int_0^1 q^*_{\lambda^*} m'(t)dt = \tilde{\Pi}$. If $\lambda^* > \frac{u'(W_0 - \Pi - F^{-1}_{X,P}(v(1)))}{\delta'(1)}$, then eq. (2.1) is satisfied.*

Proof. Let $\lambda^* > 0$ be defined such that $\int_0^1 q^*_{\lambda^*} m'(t)dt = \tilde{\Pi}$ and suppose that $\lambda^* > \frac{u'(W_0 - \Pi - F^{-1}_{X,P}(v(1)))}{\delta'(1)}$. Since δ' is decreasing, then it follows that $\delta'(1) \le \delta'(t)$, for all $t \in [0,1]$. Hence,

$$\lambda^* \delta'(t) \ge \lambda^* \delta'(1) > u'(W_0 - \Pi - F^{-1}_{X,P}(v(1))),$$

for all $t \in [0,1]$. Therefore,

$$(u')^{-1}(\lambda^* \delta'(t)) < W_0 - F^{-1}_{X,P}(v(1)) - \Pi \le W_0 - F^{-1}_{X,P}(v(t)) - \Pi,$$

for all $t \in [0,1]$. This implies that $K_{\lambda^*}(t) \le 0$, for all $t \in [0,1]$. Hence, $\mathcal{E}_{\lambda^*} = \emptyset$. □

Remark 2.2.24. *If T concave, then m is convex and increasing for $t \in [0,1]$. In this case $\delta = t$ for $t \in [0,1]$, which results in $\delta' = 1$. Moreover, $\delta > m$ on $(0,1)$. Therefore, provided that proposition (2.2.22) or (2.2.23) hold, then eq. (2.1) is satisfied. Alternatively, if T is concave and $\delta = t$ for $t \in [0,1]$, then eq. (2.1) holds for $\{t \in (0,1) : L \le F^{-1}_{X,P}(v(t)), 0 \le K_{\lambda^*}(t) \le L\} \cup \{t \in (0,1) : L > F^{-1}_{X,P}(v(t)), K_{\lambda^*}(t) > 0\}$.*

Chapter 3

Numerical Examples

The examples explored in this section are based on the distorted Esscher-type premium principle. The variation in the examples provided is based on three different types of distortion function T examined. We assume that the transformation between the insurer's probability measure and the DM's probability measure follows the Esscher type model. That is, for a given $b \in (0, \infty)$,

$$\phi_E(X) := \frac{dQ}{dP} = \frac{e^{bX}}{\int e^{bX} dP},$$

and so ϕ_E is comonotonic with X.

Here we let X be distributed according to the truncated exponential distribution on the interval on $[0, M]$, with density function $f_{X,P}$ given by

$$f_{X,P}(x) := \frac{\nu e^{-\nu x}}{1 - e^{-\nu M}},$$

for a fixed $\nu > 0$ under the probability measure P. The cumulative distribution function of X is given by

$$F_{X,P}(x) := \int_0^x \frac{\nu e^{-\nu t}}{1 - e^{-\nu M}} dt = \frac{1 - e^{-\nu x}}{1 - e^{-\nu M}},$$

for $x \in [0, M]$, and the associated quantile function is given by

$$F_{X,P}^{-1}(z) = -\frac{1}{\nu} \ln(1 - z(1 - e^{-\nu M})).$$

For a given $b \in (0, \nu)$ we have

$$\mathbb{E}^P[e^{bX}] = \int e^{bX} dP = \int_0^M e^{bx} \times \frac{\nu e^{-\nu x}}{1 - e^{-\nu M}} dx = \left(\frac{\nu}{\nu - b}\right)\left(\frac{1 - e^{(b-\nu)M}}{1 - e^{-\nu M}}\right).$$

Consequently,

$$\phi_E(F^{-1}_{X,P}(z)) = \frac{[1 - z(1 - e^{-\nu M})]^{-b/\nu}}{\mathbb{E}^P[e^{bX}]}.$$

Therefore, for a given $b \in (0, \nu)$,

$$\psi(t) = \int_0^t \phi_E(F^{-1}_{X,P}(z))dz = \frac{1}{\mathbb{E}^P[e^{bX}]}\int_0^t [1 - z(1 - e^{-\nu M})]^{-b/\nu} dz = \frac{1 - [1 - t(1 - e^{-\nu M})]^{\frac{\nu - b}{\nu}}}{1 - e^{(b-\nu)M}},$$

which is increasing and convex on $[0, 1]$, with $\psi(0) = 0$ and $\psi(1) = 1$. The associated inverse function, which is concave and increasing on $[0, 1]$, is given by

$$\psi^{-1}(t) = \frac{1 - [1 - t(1 - e^{(b-\nu)M})]^{\frac{\nu}{\nu - b}}}{1 - e^{-\nu M}}.$$

3.0.1 Convex Distortion Function

The first case considered in the numerical examples is when $T(t) = t^2$, for all $t \in [0, 1]$. Then T satisfies eq. (2.1) and by Proposition (2.2.20) we have $m(t) = \delta(t)$ is concave for all $t \in [0, 1]$. Therefore, for all $t \in [0, 1]$, we have

$$m(t) = 1 - T(1 - \psi^{-1}(t)) = 1 - (1 - \psi^{-1}(t))^2 = 1 - \left[1 - \frac{[1 - t(1 - e^{(b-\nu)M})]^{\frac{\nu}{\nu - b}} - e^{-\nu M}}{1 - e^{-\nu M}}\right]^2,$$

and

$$m'(t) = \delta'(t) = \left(\frac{2\nu}{\nu - b}\right)\left(\frac{1 - e^{(b-\nu)M}}{(1 - e^{-\nu M})^2}\right)\left[1 - t(1 - e^{(b-\nu)M})\right]^{\frac{b}{\nu - b}}\left[(1 - t(1 - e^{(b-\nu)M}))^{\frac{\nu}{\nu - b}} - e^{-\nu M}\right].$$

Moreover, we assume that the DM's utility function is given by $u(x) = x^\alpha$, where $\alpha = 0.5$. Since u satisfies Assumption 2.1.1, then for each $\lambda > 0$,

$$\begin{aligned}(u')^{-1}(\lambda\delta'(t)) =& \frac{1}{4(\lambda\delta'(t))^2} \\ =& \left(\frac{\nu - b}{4\lambda\nu}\right)^2 \times \frac{(1-e^{-\nu M})^4}{\left(1-e^{(b-\nu)M}\right)^2} \times \left[1 - t\left(1 - e^{(b-\nu)M}\right)\right]^{\frac{-2b}{\nu-b}} \\ & \times \left[\left(1 - t(1-e^{(b-\nu)M})\right)^{\frac{\nu}{\nu-b}} - e^{-\nu M}\right]^{-2} \\ =&: d(t,\lambda).\end{aligned}$$

As expected, $d(t, \lambda)$ is increasing in t for all $t \in [0, 1]$ and decreasing in λ, for a given $0 < b < \nu$. Therefore, for all $t \in [0, 1]$,

$$q^*_\lambda(t) := \min\left[\min\left(L, F^{-1}_{X,P}(\psi^{-1}(t))\right), \max\left(0, d(t,\lambda) + F^{-1}_{X,P}(\psi^{-1}(t)) + \Pi - W_0\right)\right],$$

where

$$F^{-1}_{X,P}(\psi^{-1}(t)) := -\frac{1}{\nu}\ln\left[1 - \left(1 - t(1-e^{(b-\nu)M})\right)^{\frac{\nu}{\nu-b}}\right].$$

For each $\lambda > 0$, it follows that

$$\begin{aligned}h(\lambda) :=& \int_0^1 q^*_\lambda(t)m'(t)dt \\ =& \int_0^1 \min\left[\min\left(L, F^{-1}_{X,P}(\psi^{-1}(t))\right), \max\left(0, d(t,\lambda) + F^{-1}_{X,P}(\psi^{-1}(t)) + \Pi - W_0\right)\right] \\ & \times \left(\frac{2\nu}{\nu-b}\right)\left(\frac{1-e^{(b-\nu)M}}{(1-e^{-\nu M})^2}\right)\left[1 - t(1-e^{(b-\nu)M})\right]^{\frac{b}{\nu-b}}\left[(1-t(1-e^{(b-\nu)M}))^{\frac{\nu}{\nu-b}} - e^{-\nu M}\right]dt.\end{aligned}$$

We take $M = 10$, $L = 8$, $\nu = 0.15$, $b = 0.8 * \nu$, $W_0 = 17$, $\Pi = 5$, and $\tilde{\Pi} = 2$. Solving for z^* from $L = F^{-1}_{X,P}(z)$ results in

$$z^* := \frac{1-e^{-\nu L}}{1-e^{-\nu M}} \approx 0.8995.$$

Therefore, by numerically integrating over $[0, 1]$ results in,

$$\begin{aligned}\int \min(L, X) dT \circ P &= \int_0^1 \min(L, F_{X,P}^{-1}(z)) T'(1-z) dz \\ &= 2\int_0^1 \min(L, -\frac{1}{\nu}\ln[1 - z(1 - e^{-\nu M})])(1-z)dz \\ &= 2\int_0^{z^*} -\frac{1}{\nu}\ln[1 - z(1-e^{-\nu M})] * (1-z)dz + 2\int_{z^*}^1 L * (1-z)dz \\ &\approx 2.2373,\end{aligned}$$

and

$$\begin{aligned}\int X dT \circ P = \int_0^1 F_{X,P}^{-1} T'(1-t)dt &= -\frac{2}{\nu}\int_0^1 \ln[1 - t(1 - e^{-\nu M})](1-t)dt \\ \approx 2.2435.\end{aligned}$$

By similar principle as above, solving for t^* such that $L = F_{X,P}^{-1}(\psi^{-1}(t))$, yields

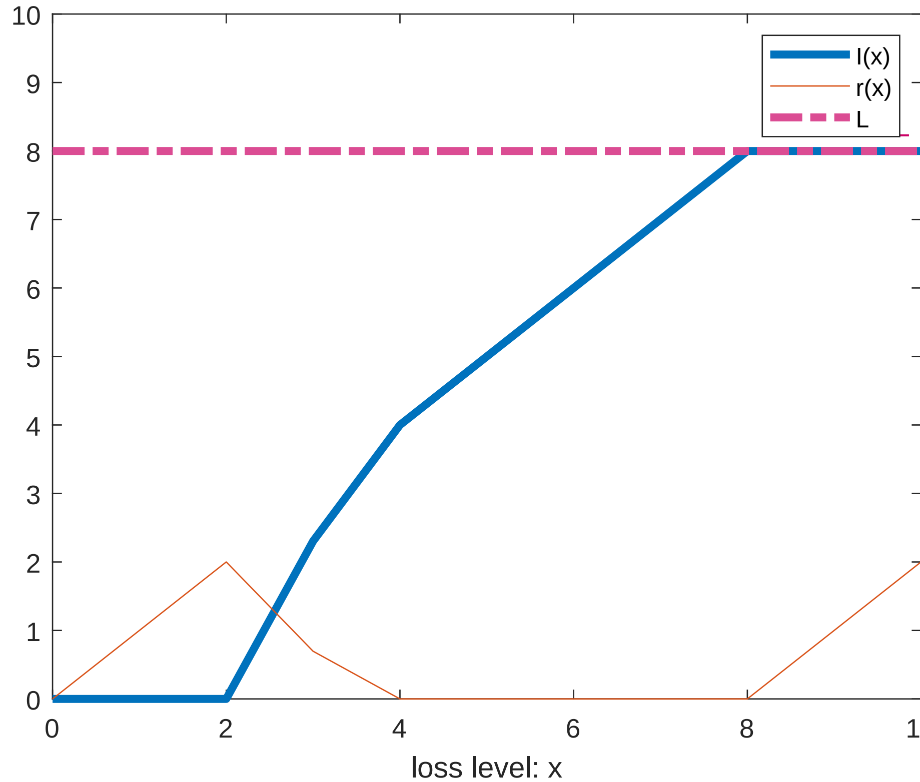
$$t^* = \frac{1 - (1 - e^{-\nu L})^{\frac{\nu - b}{\nu}}}{1 - e^{(b-\nu)M}} \approx 0.2669.$$

Moreover,

$$\begin{aligned}h(\lambda) = &\int_0^{t^*} \min\left[F_{X,P}^{-1}(\psi^{-1}(t)), \max\left(0, d(t,\lambda) + F_{X,P}^{-1}(\psi^{-1}(t)) + \Pi - W_0\right)\right] \\ &\times \left(\frac{2\nu}{\nu - b}\right)\left(\frac{1 - e^{(b-\nu)M}}{(1 - e^{-\nu M})^2}\right)\left[1 - t(1 - e^{(b-\nu)M})\right]^{\frac{b}{\nu - b}}\left[(1 - t(1 - e^{(b-\nu)M}))^{\frac{\nu}{\nu - b}} - e^{-\nu M}\right] dt \\ &+ \int_{t^*}^1 \min\left[L, \max\left(0, d(t,\lambda) + F_{X,P}^{-1}(\psi^{-1}(t)) + \Pi - W_0\right)\right] \\ &\times \left(\frac{2\nu}{\nu - b}\right)\left(\frac{1 - e^{(b-\nu)M}}{(1 - e^{-\nu M})^2}\right)\left[1 - t(1 - e^{(b-\nu)M})\right]^{\frac{b}{\nu - b}}\left[(1 - t(1 - e^{(b-\nu)M}))^{\frac{\nu}{\nu - b}} - e^{-\nu M}\right] dt.\end{aligned}$$

Note that if $\lambda^* > 0$ is chosen such that $h(\lambda^*) = \tilde{\Pi}$, then $q_{\lambda^*}^*(t)$ is optimal for Problem (2.2.2). Therefore, numerically integrating $h(\lambda^*)$ and solving for λ^* from $h(\lambda^*) = \tilde{\Pi}$ yields $\lambda^* \approx 0.1197$. Consequently, the optimal solution $q_{\lambda^*}^*$ for Problem (2.2.2) is depicted in Figure (3.1) below.

Figure 3.1: Plot of Optimal Indemnity function for $\lambda^* = 0.1197$



$I(x)$: indemnity function, $r(x)$: retention function, L : Loss Limit on indemnity function.

The retention, which is defined as $r(x) = x - q^*_{\lambda^*}(\psi(F_{X,P}(x)))$, shows an indemnity function of full coverage for losses between 4 and the upper limit of L. Thereafter, the graph depicts an increasing retention function. Losses below 2 are fully retained (not insured). Moreover, the indemnity function may be considered as a special case of deductible insurance with an upper limit on the loss amount.

3.0.2 Linear Distortion Function

We consider the case where the DM's distortion function is given as $T(t) = t$ for all $t \in [0, 1]$. Then T satisfies eq. (2.1) and likewise, by Proposition (2.2.20), $m(t) = \delta(t)$ for all $t \in [0,1]$. Consequently,

$$\begin{aligned} m(t) &= 1 - T(1-\psi^{-1}(t)) = \psi^{-1}(t) \\ &= \frac{1-\left(1-t\left(1-e^{(b-\nu)M}\right)\right)^{\frac{\nu}{\nu-b}}}{1-e^{-\nu M}}, \end{aligned}$$

and

$$m'(t) = \left(\frac{\nu}{\nu-b}\right)\left(\frac{1-e^{(b-\nu)M}}{1-e^{-\nu M}}\right)\left[1-t\left(1-e^{(b-\nu)M}\right)\right]^{\frac{b}{\nu-b}}.$$

Similar to subsection (2.2.20), we assume that the DM's utility function is given by $u(x) = x^{\alpha}$, where $\alpha = 0.5$. Therefore, by Assumption 2.1.1 we have,

$$\begin{aligned} (u')^{-1}(\lambda\delta'(t)) &= \frac{1}{4(\lambda\delta'(t))} \\ &= \frac{1}{(2\lambda)^2}\left(\frac{\nu-b}{\nu}\right)^2\left(\frac{1-e^{-\nu M}}{1-e^{(b-\nu)M}}\right)^2\left[\left(1-t(1-e^{(b-\nu)M})\right)^{\frac{b}{\nu-b}}\right]^{-2} =: k(t,\lambda). \end{aligned}$$

Thus,

$$q^*_{\lambda}(t) := \min\left[\min\left(L, F^{-1}_{X,P}(\psi^{-1}(t))\right), \max\left(0, k(t,\lambda) + F^{-1}_{X,P}(\psi^{-1}(t)) + \Pi - W_0\right)\right],$$

where

$$F^{-1}_{X,P}(\psi^{-1}(t)) := -\frac{1}{\nu}\ln\left[1-\left(1-t(1-e^{(b-\nu)M})\right)^{\frac{\nu}{\nu-b}}\right],$$

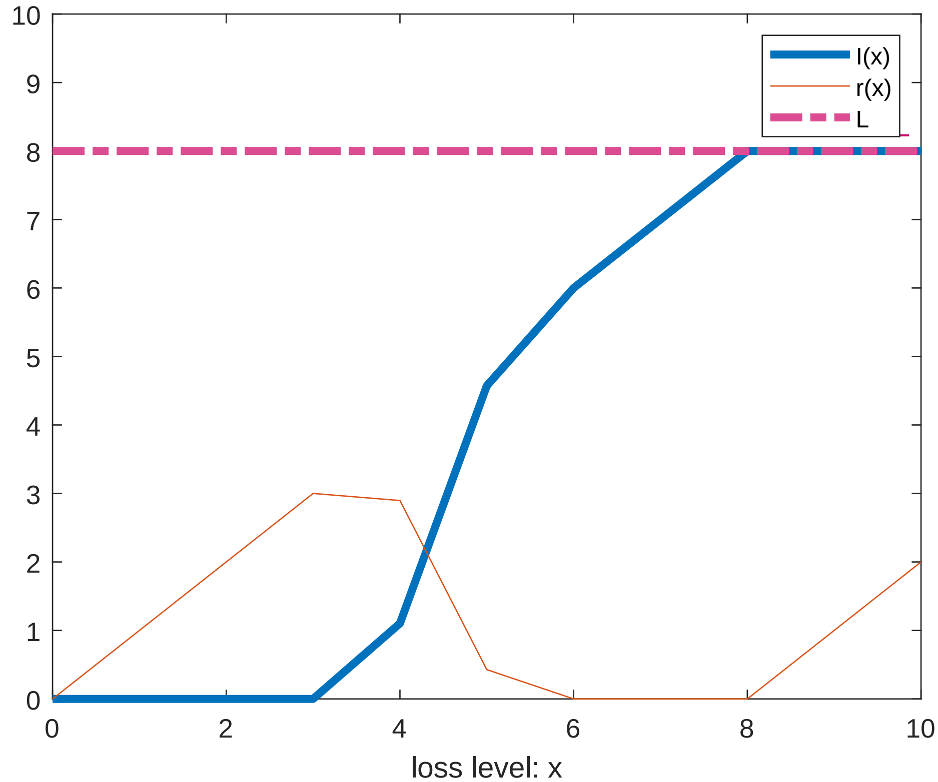
and

$$\begin{aligned} h(\lambda) &:= \int_0^1 q^*_{\lambda}(t)m'(t)dt \\ &= \int_0^1 \min\left[\min\left(L, F^{-1}_{X,P}(\psi^{-1}(t))\right), \max\left(0, k(t,\lambda) + F^{-1}_{X,P}(\psi^{-1}(t)) + \Pi - W_0\right)\right] \\ &\times \left(\frac{\nu}{\nu-b}\right)\left(\frac{1-e^{(b-\nu)M}}{1-e^{-\nu M}}\right)\left[1-t\left(1-e^{(b-\nu)M}\right)\right]^{\frac{b}{\nu-b}} dt. \end{aligned}$$

Using similar parameter values as in subsection(2.2.20), results in $\lambda^* \approx 0.1606$ from solving the numerical integration $h(\lambda) = \tilde{\Pi} = 2$. Therefore, $q^*_{\lambda^*}$ is optimal for Problem (2.2.2), as illustrated in Figure (3.2).

As depicted in Figure (3.2), the indemnity function for the random loss variable is non-linear in the loss levels leading up to the upper limit of L. Moreover, losses below a

Figure 3.2: Plot of Optimal Indemnity function for $\lambda^* = 0.1606$



$I(x)$: indemnity function, $r(x)$: retention function, L : Loss Limit on indemnity function.

threshold of level 3 are fully retained (not insured), while losses between 6 and L are fully insured. Similar to Figure (3.1), the optimal indemnity function exhibits similar features of deductible insurance with an upper limit on the loss amount.

3.0.3 Condition on T for which eq. (2.1) is staisfied

Proposition (2.2.21) states that eq. (2.1) is satisfied provided that $\frac{\phi(F^{-1}_{X,P}(t))}{T'(1-t)}$ is non-decreasing on $[0, 1]$. In keeping with the Esscher transformation for the DM's probability measure P and the truncated exponential distribution for the random loss variable, we note that the likelihood ratio is non-decreasing if

$$\frac{d}{dt}\left(\frac{\phi_E(F_{X,P}^{-1}(t))}{T'(1-t)}\right) \geq 0.$$

Note that

$$\frac{\phi_E(F_{X,P}^{-1}(t))}{T'(1-t)} = \frac{[1-t(1-e^{-\nu M})]^{-\frac{b}{\nu}}}{\mathbb{E}^P[e^{bX}]T'(1-t)},$$

implying that

$$\frac{d}{dt}\left(\frac{\phi_E(F_{X,P}^{-1}(t))}{T'(1-t)}\right) = \frac{T'(1-t)\phi_E'(F_{X,P}^{-1}(t))(F_{X,P}^{-1})'(t) + \phi_E(F_{X,P}^{-1}(t))T''(1-t)}{(T'(1-t))^2},$$

where

$$\phi_E'(F_{X,P}^{-1}(t)) = \frac{(b/\nu)(1-e^{-\nu M})[1-t(1-e^{-\nu M})]^{-\frac{\nu+b}{\nu}}}{\mathbb{E}^P[e^{bX}]},$$

and

$$(F_{X,P}^{-1})'(t) = \frac{(1/\nu)(1-e^{-\nu M})}{1-t(1-e^{-\nu M})}.$$

Therefore,

$$\frac{T'(1-t)\phi_E'(F_{X,P}^{-1}(t))(F_{X,P}^{-1})'(t) + \phi_E(F_{X,P}^{-1}(t))T''(1-t)}{(T'(1-t))^2} \geq 0$$

implies that

$$\frac{T'(1-t)(b/\nu^2)(1-e^{-\nu M})[1-t(1-e^{-\nu M})]^{-\frac{\nu+b}{\nu}}(1-e^{-\nu M})}{\mathbb{E}^P[e^{bX}](1-t(1-e^{-\nu M}))} + \frac{T''(1-t)[1-t(1-e^{-\nu M})]^{-\frac{b}{\nu}}}{\mathbb{E}^P[e^{bX}]} \geq 0.$$

Therefore,

$$-\frac{T''(1-t)}{T'(1-t)} \leq \frac{\phi_E'(F_{X,P}^{-1}(t))(F_{X,P}^{-1})'(t)}{\phi_E(F_{X,P}^{-1}(t))} = \frac{(b/\nu^2)(1-e^{-\nu M})^2}{[1-t(1-e^{-\nu M})]^2} := l(t). \quad (*)$$

Next we examine the case of a concave power distortion function for which eq. (*) holds. Define the distortion function as $T = t^{\gamma}$, which is concave with $\gamma \in (0,1)$. By taking the first and second derivatives with respect to t results in

$$\frac{-T''(t)}{T'(t)} = \frac{-(\gamma-1)}{t}.$$

Therefore,

$$\frac{-T''(1-t)}{T'(1-t)} = \frac{-(\gamma-1)}{1-t},$$

which has a limit of ∞ as $t \mapsto 1$. Taking $M = 10$, $\nu = 4.5$ and $b = 0.5 * \nu$, then we have $(1 - e^{\nu M})$ being very close to 1. Therefore,

$$\frac{-T''(1-t)}{T'(1-t)} - l(t) = \frac{(1-\gamma)}{1-t} - \frac{0.1111(1-e^{-45.0})^2}{[1-t(1-e^{-45.0})]^2}.$$

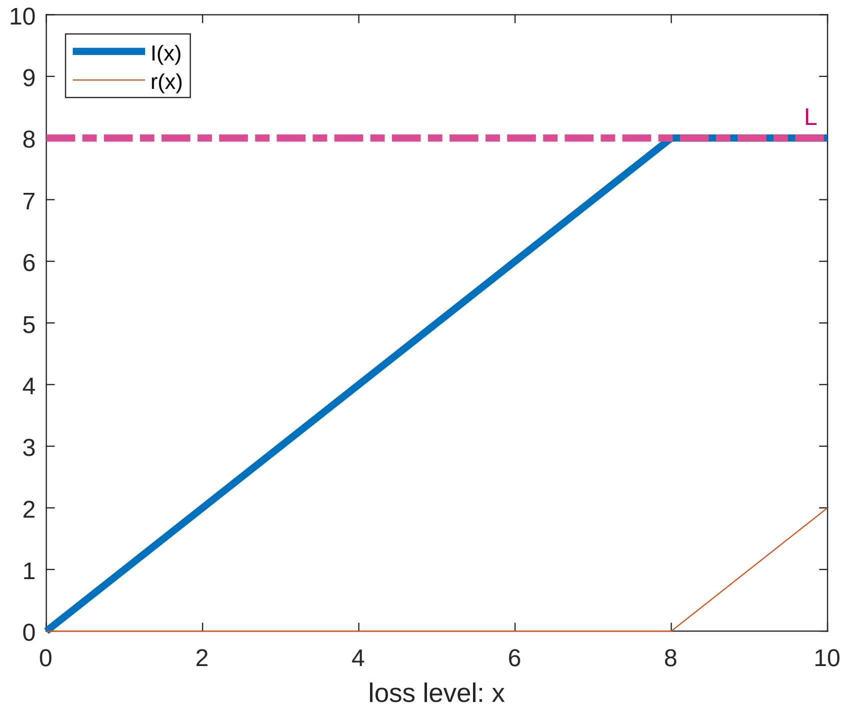
The results of our numerical assessment showed that eq. ($*$) holds for $\gamma \in [0.89, 1)$, for all $t \in [0, 1]$. Therefore, by Proposition (2.2.21), eq. (2.1) is satisfied for a concave distortion function for $\gamma \in [0.89, 1)$.

Against the backdrop of the Esscher-type premium principle, we proceed to obtain the optimal indemnification for the DM. Similar to section 3.0.1, we assumed that the DM's utility function is given by $u(x) = x^\alpha$, where $\alpha = 0.5$. Moreover, we take $M = 10$, $L = 8$, $\nu = 4.5$, $b = \nu/2$, $W_0 = 17$, $\Pi = 5$, and $\tilde{\Pi} = 2$. We proceed by selecting $\gamma = 0.9$, to obtain the optimal solution $q^*_{\lambda^*}$ for Problem (2.2.2). Since T is concave, then $m(t) \leq \delta(t) = t$ for all $t \in [0, 1]$. Applying similar reasoning as in section 3.0.1, results in

$$q^*_\lambda(t) := \min\left[\min\left(L, F^{-1}_{X,P}(\psi^{-1}(t))\right), \max\left(0, \frac{1}{(2\lambda)^2} + F^{-1}_{X,P}(\psi^{-1}(t)) + \Pi - W_0\right)\right],$$

where $\lambda^* \approx 0.1342$ is obtained from solving the numerical integration $h(\lambda) = \tilde{\Pi} = 2$. Moreover, $\lambda^* < 0.1443 \approx \frac{u'(W_0-\Pi)}{\delta'(0)} \leq \frac{u'(\min(L,x)+W_0-x-\Pi)}{\delta'(\psi(F_{X,P}(x)))}$, for all $x \in [0, M]$. Therefore, by Lemma (2.2.18), $Y^* = \min(L, X)$ is optimal for Problem (2.2.2). This result is illustrated in Figure (3.3) below.

Figure 3.3: Plot of Optimal Indemnity function for $\lambda^* \approx 0.1342$



$I(x)$: indemnity function, $r(x)$: retention function, L : Loss Limit on indemnity function.

Figure (3.3) shows that optimal indemnity function provides full coverage for losses below the insurer's risk limit of L. Moreover, losses in excess of L result in a piecewise increasing linear retention function.

Chapter 4

Conclusion

This thesis studies the problem of optimal insurance indemnification for a DM with subjective expected utility preferences, in the presence of a budget constraint and an additional upper limit on possible indemnification disbursement by the insurer. We assumed that the DM's probability measure is obtained from the insurer's probability measure via a transformation as in Furman and Zitikis ([8]). In solving the problem, we employed the methodology proposed by Xu ([17]) within the context of the portfolio choice problem and extended by Ghossoub [14] to an insurance framework. First, we transformed the proposed problem into its associated quantile formulation. Next, we conducted a change of variable and a relaxation approach, without the reliance on the calculus of variation or making any monotonicity assumptions. We showed that in the presence of an upper limit on disbursement, the optimal indemnification function is a limited variable deductible contract.

We then examined three numerical applications, in which we assumed that the insurable loss random variable follows a truncated exponential distribution under the probability measure of the insurer, and that the DM's subjective probability measure is obtained by an Esscher-type transformation from the insurer's probability measure. In the first example examined, the insurer's distortion function is a convex power function, in the second a linear distortion function, and in the third a concave power function. We illustrated the optimal indemnity and retention function in each case.

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APPENDICES

Appendix A

APPENDICES

1 Equimeasurable Rearrangements and Supermodularity

All of the results in this Appendix are taken from Ghossoub [9] and references therein, to which we refer for proofs, additional results, and additional references.

1.1 The Rearrangement

Let $(S, \mathcal{G}, \mu)$ be a probability space and let $V \in L^\infty(S, \mathcal{G}, \mu)$ be a continuous random variable (i.e., $\mu \circ V^{-1}$ is nonatomic) with range $V(S) \subset \mathbb{R}^+$.

For each $Z \in L^\infty(S, \mathcal{G}, \mu)$, let $F_{Z,\mu}(t) = \mu\big(\{s \in S : Z(s) \le t\}\big)$ denote the cumulative distribution function of Z with respect to the probability measure μ, and let $F^{-1}_{Z,\mu}(t)$ be the left-continuous inverse of the distribution function $F_{Z,\mu}$ (that is, the quantile function of Z w.r.t. μ), defined by

$$F^{-1}_{Z,\mu}(t) = \inf\left\{z \in \mathbb{R}^+ : F_{Z,\mu}(z) \ge t\right\}, \ \forall t \in [0,1]. \tag{A.1}$$

Proposition 1.1. *For any $Y \in L^\infty(S, \mathcal{G}, \mu)$, define $\tilde{Y}_\mu$ and $\overline{Y}_\mu$ as follows:*

$$\overline{Y}_\mu = F^{-1}_{Y,\mu}(F_{V,\mu}(V)) \quad \text{and} \quad \tilde{Y}_\mu = F^{-1}_{Y,\mu}(1 - F_{V,\mu}(V)).$$

Then,

(i) Y*,* $\tilde{Y}_\mu$*, and* $\overline{Y}_\mu$ *have the same distribution under* μ*.*

(ii) $\overline{Y}_\mu$ *is comonotonic with* V*.*

(iii) $\tilde{Y}_\mu$ *is anti-comonotonic with* V*.*

(iv) For each $L \in \mathbb{R}$*, if* $0 \leq Y \leq L$*, then* $0 \leq \tilde{Y}_\mu \leq L$*, and* $0 \leq \overline{Y}_\mu \leq L$*.*

(v) For each $Z \in L^\infty(S, \mathcal{G}, \mu)$*, If* $0 \leq Y \leq Z$*, then* $0 \leq \tilde{Y}_\mu \leq \tilde{Z}_\mu$*, and* $0 \leq \overline{Y}_\mu \leq \overline{Z}_\mu$*.*

(vi) If Z^* *is any other element of* $L^\infty(S, \mathcal{G}, \mu)$ *that has the same distribution as* Y *under* μ *and that is comonotonic with* V*, then* $Z^* = \overline{Y}_\mu, \mu$*-a.s.*

(vii) If Z^{**} *is any other element of* $L^\infty(S, \mathcal{G}, \mu)$ *that has the same distribution as* Y *under* μ *and that is anti-comonotonic with* V*, then* $Z^{**} = \tilde{Y}_\mu, \mu$*-a.s.*

$\tilde{Y}_\mu$ is called the nonincreasing μ-rearrangement of Y with respect to V, and $\overline{Y}_\mu$ is called the nondecreasing μ-rearrangement of Y with respect to V.

Since $\mu \circ V^{-1}$ is nonatomic, it follows that $F_{V,\mu}(V)$ has a uniform distribution over $(0, 1)$ [6, Lemma A.25]. Letting $U := F_{V,\mu}(V)$, it follows that U is a random variable on the probability space (S, Σ, μ) with a uniform distribution on $(0, 1)$ and that $V = F^{-1}_{V,\mu}(U)$, μ-a.s., that is, $\overline{V}_\mu = V$, μ-a.s.

1.2 Supermodularity and Hardy-Littlewood-Pólya Inequalities

A *partially ordered set* (*poset*) is a pair $(T, \succcurlyeq)$ where $\succcurlyeq$ is a reflexive, transitive and antisymmetric binary relation on T. For any $x, y \in S$ denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound, or supremum (resp. greatest lower bound, or infimum) of the set $\{x, y\}$. A poset $(T, \succcurlyeq)$ is called a *lattice* when $x \vee y, x \wedge y \in T$, for each $x, y \in T$. For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\succcurlyeq$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, write $x \succcurlyeq y$ when $x_i \geq y_i$, for each $i = 1, \ldots, n$. It is then easy to see that $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

Definition 1.2. *Let* $(T, \succcurlyeq)$ *be a lattice. A function* $L : T \to \mathbb{R}$ *is said to be supermodular if for each* $x, y \in T$,

$$L(x \vee y) + L(x \wedge y) \geq L(x) + L(y). \tag{A.2}$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ *is supermodular if for any* $x_1, x_2, y_1, y_2 \in \mathbb{R}$ *with* $x_1 \leq x_2$ *and* $y_1 \leq y_2$*, one has*

$$L(x_2, y_2) + L(x_1, y_1) \geq L(x_1, y_2) + L(x_2, y_1). \tag{A.3}$$

Equation (A.3) then implies that a function $L : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if and only if the function $\eta(y) := L(x + h, y) - L(x, y)$ is nondecreasing on $\mathbb{R}$, for any $x \in \mathbb{R}$ and $h \geq 0$.

Example 1.3. *The following are supermodular functions:*

1. *If* $g : \mathbb{R} \to \mathbb{R}$ *is concave and* $a \in \mathbb{R}$*, then the function* $L_1 : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $L_1(x, y) = g(a - x + y)$ *is supermodular. Moreover, if* g *is strictly concave, then* L_1 *is strictly supermodular.*

2. *If* $g : \mathbb{R} \to \mathbb{R}$ *is concave and increasing,* $h : \mathbb{R} \to \mathbb{R}$ *is increasing and nonnegative, and* $a \in \mathbb{R}$*, then the function* $L_2 : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $L_2(x, y) = g(a - x + y)h(x)$ *is supermodular. Moreover, if* g *is strictly concave and strictly increasing, and if* h *is strictly increasing and positive, then* L_2 *is strictly supermodular.*

3. *If* $\psi, \phi : \mathbb{R} \to \mathbb{R}$ *are both nonincreasing or both nondecreasing functions, then the function* $L_3 : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $L_3(x, y) = \phi(x)\psi(y)$ *is supermodular.*

Lemma 1.4 (Hardy-Littlewood-Pólya Inequality)**.** *Let* $Y \in L^{\infty}(S, \mathcal{G}, \mu)$*, let* $\tilde{Y}_\mu$ *be the nonincreasing* μ*-rearrangement of* Y *with respect to* V*. If* L *is supermodular then*

$$\int L(V, \tilde{Y}_\mu)\, d\mu \leq \int L\Big(V, Y\Big)\, d\mu \leq \int L(V, \overline{Y}_\mu)\, d\mu,$$

provided the integrals exist.