# Investigations into quantum light-matter interactions, their approximations and applications 

by

Nicholas Funai

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## Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Miles Blencowe
Professor, Dept. of Physics and Astronomy,
Dartmouth College

Supervisor(s):
Eduardo Martín-Martínez
Associate Professor, Dept. of Applied Mathematics, University of Waterloo
$\left.\begin{array}{ll}\text { Internal Member: } & \begin{array}{l}\text { Achim Kempf } \\ \text { Professor, Dept. of Applied Matematics, } \\ \\ \\ \\ \\ \text { University of Waterloo } \\ \text { Robert Mann }\end{array} \\ & \text { Professor, Dept. of Physics and Astronomy, } \\ \text { University of Waterloo }\end{array}\right\} \begin{aligned} & \text { Internal-External Member: } \begin{array}{l}\text { Christopher Wilson } \\ \text { Associate Professor, Dept. of Electrical and Computer Engineering, } \\ \text { University of Waterloo }\end{array}\end{aligned}$

## Author's declaration

This thesis consists of material all of which I authored or co-authored. Details may be found in the Statement of Contributions included in this thesis. I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

All work presented in this thesis was supervised by Eduardo Martín-Martínez. The content of all chapters was developed in collaboration with Eduardo Martín-Martínez. Chapter 3 was developed in collaboration with Jorma Louko and Eduardo Martín-Martínez. I performed the analytical and numerical studies of the research presented in this thesis, as well as producing all the figures. I also developed the numerical tools \& tricks outlined in Chapter 5.

The results in chapter 2 can be found in my publication [1]. In chapters 3 and 4 there are substantial portions of text that paraphrase my past publications [2, 3], which were primarily written by me with textual and editing contributions from my fellow authors. Chapter 5 also closely follows a paper in preparation by Eduardo Martín-Martínez and me.


#### Abstract

Devoting some thought to the interactions between light and matter quickly conjure a myriad of different possibilities; different models for light and matter, different possible interaction Hamiltonians, different simplifying approximations and different initial conditions. This of course means that light-matter interactions has just as many uses and surprises yet to be explored. In this thesis we will approach light-matter interactions from three different perspectives: 1) How can light-matter interactions be used to manipulate quantum fields into exotic energy distributions, 2) How well does a classical light-matter approximation carry through to quantum light-matter interactions and 3) In a relativistic theory, what are the causal consequences of approximations commonly used to reduce the analytical and numerical complexity of quantum fields.

My first aim in this thesis is to develop an operationally feasible procedure for generating exotic energy distributions. Current proposals for generating negative energy densities include using the dynamical Casimir effects or by squeezed vacua, which require relativistically accelerating mirrors or non-linear crystals. Instead we seek an optimal protocol that does not require relativistically accelerating elements and exploits the coherent control of a detector. The quantum energy teleportation (QET) protocol is ideal for this task, and we present a QET protocol acting on a relativistic scalar field optimised for the generation of negative energy densities. We show that QET can be used to generate local negative energy densities using stationary qubits. In addition we discuss the consequences of detector smearings on the (QET generated) energy distribution, providing simple and intuitive guidelines for sculpting negative energy distributions. We also show that this protocol is capable of generating regions with an arbitrary amount of negative energy, with the total amount of negative energy $\Delta E$ increasing as the negative energy well's width $\Delta r$ is decreased by $\Delta E \sim \Delta r^{-3}$. However, this is accompanied by increasingly large ( $\sim \Delta r^{-3}$ ) positive energy peaks on either side of the well. We further find this energy-distance scaling saturates the quantum interest conjecture, suggesting the near optimality of QET for generating negative energy densities.

My second aim is to determine the validity of the dipole model of light-matter interaction with a quantum EM field. This model, classically derived from the more fundamental minimal model, is dependent on the existence of a dominant mode, whose wavelength is required to be much larger than the size of the atom it is interacting with (dipole approximation criterion). Quantisation of the EM field result in vacuum fluctuations without a dominant wavelength. Past works have attempted to overcome this by using point-like atoms [4], however this introduces UV divergences in the response of the atoms [5, 6] and is inadequate for describing light-matter interactions. The full influence of gauge consider-


ations and quantum behaviour in light-matter interaction is quite involved and complex, particularly when the atomic nucleus' mass is considered finite and the centre of mass degrees of freedom become relevant (See e.g., the recent work by my colleagues [7]). We consider effective models (infinite nucleus mass), which neglect the centre of mass degrees of freedom and their additional complexity, whilst allowing accurate investigations of the electronic orbital behaviour. Here we will attempt to determine under what circumstances the dipole and minimal models agree; and hope to clarify and extend the dipole approximation validity criteria for general use in quantum fields.

Using the dressed state formalism to remove gauge issues, we compare the transition probabilities of Hydrogen-like atoms under both models to determine the validity of the dipole model. We show that for atomic transitions with an initial EM vacuum state, both models noticeably disagree for short interaction times (i.e. shorter than the light-crossing time of the atom). We find the transition rates predicted by both models begin to converge for longer interaction times, particularly when considering vacuum excitations. Vacuum emissions have the additional requirement that the atomic energy gap must satisfy the dipole approximation criterion due to single mode approximation effects. In addition, we find that shrinking the atom (by increasing the atom's proton number) does not improve the accuracy of the dipole model, a result of the infinite number of UV modes in the vacuum fluctuations. We determine that the dipole model can be used with a quantum EM field, provided any intrinsically dominant modes, e.g. atomic energy gap (for vacuum emissions) or excited EM modes, satisfy the dipole approximation criterion and the interaction time is longer than the light crossing time of the atom.

Our final aim is to determine the causal consequences of the commonly used rotating wave approximation (RWA) and determine in what regimes it may be accurately used. The RWA removes terms from the (detector-field) interaction Hamiltonian that do not conserve excitation number, i.e. $\hat{\sigma}^{+} \hat{a}_{\boldsymbol{k}}^{\dagger}$ and $\hat{\sigma}^{-} \hat{a}_{\boldsymbol{k}}$. This is justified by noting that these terms oscillate quickly in time and therefore, for long interaction times ( T ), will integrate to zero, where long interaction time is defined by the RWA criterion $\Omega T \gg 1$, where $\Omega$ is the detector's energy gap. We wish to determine the extent of the non-locality and causality violation introduced into this originally local and causal theory; as well as determine in what regime the RWA model can be used as a faithful substitute for the unapproximated UDW model. We quantify these violations by studying the non-locality of the RWA interaction Hamiltonian, inspecting the acausal expectation values of field observables $\left\langle: \hat{\phi}^{2}(\boldsymbol{x}):\right\rangle$ and $\left\langle: \hat{T}_{00}(\boldsymbol{x}):\right\rangle$; and quantifying the superluminal communication in a 2 qubit communication protocol.

We verify and extend the previous results [8] by showing the RWA interaction Hamiltonian has a $1 / r^{2}$ polynomial non-locality, proportional to the vacuum Wightman function
of the field. We also show that this non-locality translates into causality violations of $1 / d^{4}$ and $1 / d^{6}$ for the expectation values $\left\langle: \hat{\phi}^{2}(\boldsymbol{x}):\right\rangle$ and $\left\langle: \hat{T}_{00}(\boldsymbol{x}):\right\rangle$ respectively; where $d$ is measured from the surface of the detector's lightcone and this polynomial causality violation is independent of time. We also find similar $1 / d^{2}$ non-localities in the channel capacity of the 2 qubit communication protocol, with non-local influence in the space-like and time-like regions of the first detector's interaction. We show that in setups that are not causally sensitive, e.g. measuring field observables within the bulk of a source detector's light-cone, then the RWA criterion $\Omega T \gg 1$ is sufficient for the RWA model with interaction strength $\lambda$ to converge to the unapproximated UDW model with interaction strength $\lambda / 2$. We establish that this factor of $1 / 2$ comes from a mathematically unsound commutation of limits and integrals.

In the case of cavity fields, we introduced a numerical trick to make infinite mode sums computationally possible without requiring single or few mode approximations. Using this trick we found the RWA non-local behaviour to be generally similar to the free space case. We find the RWA model introduces some severe polynomial non-localities and causality violations, but if used to model light-matter interactions well within the bulk of the detector's light-cone and $\Omega T \gg 1$ then it is a good approximation for the half strength UDW model.

## Acknowledgements

I would like to thank my former supervisor Nicolas Menicucci and the Perimeter Institute's PSI program for paving my way to studying relativistic quantum information at the University of Waterloo. I would also like to thank Mike and Ophelia Lazaridis, whose funding allowed me to follow my post-graduate studies. My thanks also go to the Insitute for Quantum Computing and the University of Waterloo for providing a wonderful learning and research environment.

I am deeply grateful to my supervisor Eduardo Martín-Martínez for guiding me through my degree and also to my friends and colleagues from Barrio-RQI, a genuine fountain of ideas, clarity and inspiration.

Finally, I would like to thank my family for their infinite patience and support; especially my father for sharing with me the joy of knowledge. Without their love, guidance and wisdom I would be all the poorer.

## Dedication

To Diego, my colleague, companion and brother. May your light always guide me.

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B. 1 A coarse sketch of the interaction lightcone under sudden switching (B.55). The dots indicate $t=-T, T$ respectively and the solid lines indicate the null-like propagation of these events. The RWA works best in the shaded regions, i.e. when $\left|(\mathrm{x}-\mathrm{y})^{\mu}(\mathrm{x}-\mathrm{y})_{\mu}\right| \Omega \gg 1$.
C. 1 The original integration contour of (C.8) is $-\infty<\xi<\infty$. Adding the contour via $+\mathrm{i} \infty$ (dashed line) closes the integration contour whilst Leaving the integral unchanged. This allows the direct use of the Residue theorem after which $\epsilon \rightarrow 0^{+}$can be taken.

## Glossary

$\hat{\sigma}^{ \pm}$The ladder operators of the $\mathfrak{z u}(2)$ algebra. In this thesis they are restricted to 2 dimensional qubits and act on states $\hat{\sigma}^{+}|g\rangle=|e\rangle, \hat{\sigma}^{+}|e\rangle=0, \hat{\sigma}^{-}|e\rangle=|g\rangle$ and $\hat{\sigma}^{-}|g\rangle=0$. In addition they relate to the Pauli matrices via $\hat{\sigma}^{+}+\hat{\sigma}^{-}=\hat{\sigma}_{x} .12$
annihilation and creation operators Operators associated with quantum harmonic oscillators. In quantum field theory each momentum mode is described by a quantum harmonic oscillator, resulting in: free space commutation relations (continuous $\boldsymbol{k}$ and Dirac normalised) $\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]=\delta^{n}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right)$ and cavity commutation relations (discrete $\boldsymbol{k}$ and Kronecker normalised) $\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]=\delta_{\boldsymbol{k} \boldsymbol{k}^{\prime}}$. With these commutation relations $\hat{a}_{\boldsymbol{k}}$ is said to annihilate an excitation of the $\boldsymbol{k}$ mode and $\hat{a}_{\boldsymbol{k}}^{\dagger}$ is said to create an excitation of the $\boldsymbol{k}$ mode. In EM fields, the commutation relations require an additional Kronecker delta for field polarisation: $\left[\hat{a}_{\lambda}(\boldsymbol{k}), \hat{a}_{\lambda^{\prime}}^{\dagger}\left(\boldsymbol{k}^{\prime}\right)\right]=\delta_{\lambda \lambda^{\prime}} \delta^{n}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) . \quad 7$
detector's light-cone Given an interacting detector, usually with compact smearing $\lambda(\boldsymbol{x})$ and switching $\chi(t)$, the detector's light-cone is the set of spacetime points that are null with respect to some point $(\boldsymbol{x}, t)$ in the support of $\lambda(\boldsymbol{x}) \chi(t)$, i.e. $\mathbf{y}$ is on the detector's light-cone if $\exists \mathrm{x}=(t, \boldsymbol{x})$, such that $(\mathrm{y}-\mathrm{x}) \cdot(\mathrm{y}-\mathrm{x})=0$ and $\lambda(\boldsymbol{x}) \chi(t) \neq 0$. The set of all such $y$ is the detector's light-cone, sometimes referred to as the bulk or inside of the detector's light-cone. The points outside this set are divided into time-like and space-like regions with respect to the detector's interaction. 137, 138
negative energy well A term often used in this thesis to describe the region of a wave packet that has a negative energy density. The usual shape of this region is that of a valley or well, from which the name derives. 24
normal ordering Given an operator $\hat{O}$ defined as a polynomial (or series) of creation and annihilation operators, normal ordering rearranges each term in the polynomial such that all creation operators are on the left (last to act on kets) and annihilation
operators on the right (first to act on kets). This process is denoted with 2 colons, i.e. : $\hat{O}$ :. E.g. given $\hat{O}=\hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{\boldsymbol{k}}^{\prime}$, then : $\hat{O}:=\hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}^{\prime}} 7$
position space vacuum By interpreting scalar field operators $\hat{\phi}(\boldsymbol{x})$ and $\hat{\pi}(\boldsymbol{x})$, respectively, as the position and momentum quadrature operators of a harmonic oscillator about point $\boldsymbol{x}$, one can define a spatial annihilation operator $\hat{\psi}(\boldsymbol{x})=(\hat{\phi}(\boldsymbol{x})+$ $\mathrm{i} \hat{\pi}(\boldsymbol{x})) / \sqrt{2}$ (and its adjoint, the creation operator) and then define a ground state for this spatial oscillator $|g(\boldsymbol{x})\rangle$. The position space vacuum is the tensor product over all these spatial ground states $|\Omega\rangle=\bigotimes_{\boldsymbol{x}}|g(\boldsymbol{x})\rangle$. This state can be constructed by squeezing the usual vacuum state $|0\rangle$. 10
quantum interest conjecture A conjectured relation describing the positive energy cost (interest) associated with generating a negative energy well. The conjecture describes the 'interest rate' of positive energy cost as a function of separation between the positive and negative wavepackets and the depth of the negative energy well. This conjecture was proposed and explored in [9]. 24
sonic-cone In massive quantum fields and spin lattice systems the propagation speed of energy and information is usually less than $c$. As such the propagation speed is sometimes called the speed of sound $c_{s}(\omega)$ (usually frequency dependent), and the resulting region of causal influence at this speed is the sonic-cone. If $c_{s}=c$ the sonic-cone is then called the light-cone. 50
vacuum state The ground state of the non-interacting, quantum scalar field; usually defined as the unique state $|0\rangle$ that satisfies $\hat{a}_{\boldsymbol{k}}|0\rangle=0 \forall \boldsymbol{k}$. It can be considered a tensor product over the ground states of the momentum mode harmonic oscillators. 7

## Abbreviations

EM Electromagnetism 61

QET Quantum Energy Teleportation 25
QIC quantum interest conjecture 24
RQI Relativistic Quantum Information 3
RWA Rotating Wave Approximation 88
SMA Single Mode Approximation 89

UDW Unruh-DeWitt 12

## Chapter 1

## Introduction

Light-matter interactions have played an important role in modern physics and continue to be central to many avenues of current research. With the development of quantum physics light-matter interactions took on a vast range of applications, from quantum electrodynamical (QED) models of high energy physics to the description of a laser. This range of applications was accompanied by similarly diverse mathematical models, from the relativistic quantum descriptions of QED to simpler, approximated models such as the Unruh-DeWitt coupling between a 2-level qubit and a scalar field. This thesis is generally aiming to use light-matter interactions as a tool for constructing quantum information protocols, especially exploiting the relativistic nature of the EM field, hence the field of study 'relativistic quantum information' (RQI).

Experimental developments [10] have begun to push the boundaries of previously established parameter regimes and in doing so raising questions as to the validity of previous commonly used approximations. RQI protocols find themselves at the forefront of these parameter boundaries, e.g. quantum energy teleportation [11], Fermi problem in circuit QED [12] and entanglement harvesting [13].

One such approximation is the dipole approximation, used when considering a 2nd quantised EM field interacting with a 1st quantised electron (usually in an atomic orbital). The use of this approximation is justified when the electron's orbital size is much smaller than the dominant field wavelength [14]. This approximation leads to the use of the dipole model, a very successful and unambiguous model. However the vacuum state of quantum fields lacks a clear definition of dominant wavelength, casting doubt over the validity of such an approximation. In classical fields the approximation was known to improve with smaller electron orbitals [15]; this raised the question if the same would hold in quantum
fields and if not then by how much.
Another prominent approximation is the rotating wave approximation (RWA), used generally for any interactions between a 2 nd quantised field and a 1 st quantised system (detector) provided the interaction time is 'long'. This approximation is implemented by Fermi [16] and forms the basis of commonly used models in condensed matter systems [17] and quantum optics [18] in the form of the Jaynes-Cummings model. The RWA produces a model that conserves excitation number (i.e. a field excitation can only be created by a detector de-excitation and vice versa), which is particularly appealing from a classical lightmatter interaction perspective. Given that many RQI protocols are realised in quantum optical systems [19], the use of an approximation capable of introducing non-local or acausal behaviour $[8,20]$ is of particular concern in RQI.

This thesis also considers a particular RQI protocol whose objective is to generate exotic EM field states which can then manipulate the spacetime geometry in unusual ways. The energy conditions introduced in the 1970s [21] ensured that only well behaved spacetime geometries would be allowed, however quantum fields posed a problem as they were known to violate these energy conditions [22]. Quantum energy teleportation (QET) [23] was a protocol introduced to transfer energy between two agents without requiring the energy carriers of the field, however it soon became evident that this protocol could be re-purposed as a RQI protocol for generating negative energy regions, capable of violating the classical energy conditions. In this thesis a quantitative analysis of the QET protocol is presented and compared to the restrictions imposed by the quantum energy conditions [24].

## Chapter 2

## Preliminary

Quantum fields continue to generate considerable research interest, yet some of their more interesting characteristics occur when they interact with matter. The pragmatic question of how fields are both observed and manipulated is answered by considering light-matter interaction, which is modelled with a first quantised detector interacting with a second quantised field, forming the basic blocks from which Relativistic Quantum Information (RQI) is made. This rough blueprint for interacting with a field is used extensively in this thesis and RQI in general; and is explained in greater detail in this chapter. This chapter also serves to familiarise the details of other concepts used in this thesis as well as defining any notations or conventions used in this thesis.

### 2.1 Review of quantum field theory

Quantum field theory, a union of quantum mechanics and special relativity, has been successfully used to describe electromagnetic fields as well as fermionic fields for nearly a century [16]. Whilst many physical scenarios employ electromagnetic fields, the shared bosonic properties of the simpler scalar field result, instead, in the confident use of the scalar field in most cases [25, 26, 27]. Here the quantisation of the scalar field will be reviewed (based on $[28,29]$ ) as well as further concepts relevant to this thesis.

### 2.1.1 Scalar field in flat spacetime

In $n+1$ spacetime dimensions, the Lagrangian density for the scalar field $\phi$ is

$$
\begin{equation*}
\mathcal{L}(x)=\frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x)-\frac{m^{2}}{2} \phi^{2}(x) \tag{2.1}
\end{equation*}
$$

where $\mathrm{x}=(t, \boldsymbol{x})$ is a generalised four-vector and $m$ is the 'field mass'. By minimising the action $S=\int \mathrm{d}^{n+1} \times \mathcal{L}(\mathrm{x})$ the field's Euler-Lagrange equations take the form

$$
\begin{equation*}
\left(\partial_{\mu} \partial^{\mu}+m^{2}\right) \phi(x)=0 \tag{2.2}
\end{equation*}
$$

i.e. the Klein-Gordon equation. Solutions to this equation are found by means of a mode decomposition

$$
\begin{equation*}
\phi(\mathrm{x})=\int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{k}\left(a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\mathrm{x})+a_{\boldsymbol{k}}^{*} u_{\boldsymbol{k}}^{*}(\mathrm{x})\right) \tag{2.3}
\end{equation*}
$$

where $a_{\boldsymbol{k}}$ are Fourier coefficients and the mode function $u_{\boldsymbol{k}}$ are usually chosen (in flat space) as plane waves $e^{-\mathrm{i} k_{\mu} x^{\mu}}$ and orthonormalised by the Klein-Gordon inner product

$$
\begin{equation*}
\left(\phi_{1}, \phi_{2}\right)_{\mathrm{KG}}=-\mathrm{i} \int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{x}\left(\phi_{1}(\mathrm{x}) \partial_{t} \phi_{2}^{*}(\mathrm{x})-\phi_{2}^{*}(\mathrm{x}) \partial_{t} \phi_{1}(\mathrm{x})\right) \tag{2.4}
\end{equation*}
$$

Under this inner product the (Dirac) normalised mode functions become

$$
\begin{equation*}
u_{\boldsymbol{k}}(\mathrm{x})=\frac{1}{(2 \pi)^{n / 2} \sqrt{2 \omega}} e^{-\mathrm{i} k_{\mu} x^{\mu}} \tag{2.5}
\end{equation*}
$$

where $\omega=k_{0}$. In this thesis the mode functions used for free space fields (i.e. no mirrors or boundaries) will always be plane wave modes. By convention $\omega \geq 0$ and positive frequency solutions are defined to satisfy

$$
\begin{equation*}
\partial_{t} u_{\boldsymbol{k}}(\mathrm{x})=-\mathrm{i} \omega u_{\boldsymbol{k}}(\mathrm{x}) \tag{2.6}
\end{equation*}
$$

With this definition the Klein-Gordon inner product can be said to be positive definite in the subset of positive frequency solutions.

Using the mode functions (2.5) to solve the Klein-Gordon equation (2.2), one can determine the dispersion relation for the scalar field

$$
\begin{equation*}
\omega=\sqrt{|\boldsymbol{k}|^{2}+m^{2}} \tag{2.7}
\end{equation*}
$$

In the massless limit $(m \rightarrow 0)$ the dispersion relation becomes that of light: $\omega=|\boldsymbol{k}|$.
From the Lagrangian density, the conjugate momentum can be found

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{t} \phi\right)}=\pi=\partial_{t} \phi \tag{2.8}
\end{equation*}
$$

and the corresponding Hamiltonian density is determined by a Legendre transformation

$$
\begin{align*}
\mathcal{H} & =\pi \partial_{t} \phi-\mathcal{L}  \tag{2.9}\\
& =\frac{1}{2}\left(\partial_{t} \phi\right)^{2}+\frac{1}{2}(\nabla \phi) \cdot(\nabla \phi)+\frac{m^{2}}{2} \phi^{2} . \tag{2.10}
\end{align*}
$$

The field's Hamiltonian is then obtained by integrating the density $H=\int \mathrm{d}^{n} \boldsymbol{x} \mathcal{H}$. The Hamiltonian density has physical significance as the time-time component of the stressenergy tensor $T_{00}$, generally interpreted as the local energy density of the field and a potential source of curvature when considering general relativity.

Summarising the field and its Fourier decomposition

$$
\begin{align*}
\phi(\mathrm{x}) & =\int_{\mathbb{R}^{n}} \frac{\mathrm{~d}^{n} \boldsymbol{k}}{(2 \pi)^{n / 2} \sqrt{2 \omega}}\left(a_{\boldsymbol{k}} e^{-\mathrm{i} k_{\mu} x^{\mu}}+a_{\boldsymbol{k}}^{*} e^{\mathrm{i} k_{\mu} x^{\mu}}\right)  \tag{2.11}\\
\pi(\mathrm{x}) & =-\mathrm{i} \int_{\mathbb{R}^{n}} \frac{\mathrm{~d}^{n} \boldsymbol{k}}{(2 \pi)^{n / 2}} \sqrt{\frac{\omega}{2}}\left(a_{\boldsymbol{k}} e^{-\mathrm{i} k_{\mu} x^{\mu}}-a_{\boldsymbol{k}}^{*} e^{\mathrm{i} k_{\mu} x^{\mu}}\right)  \tag{2.12}\\
a_{\boldsymbol{k}} & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{x}\left(\sqrt{\frac{\omega}{2}} \phi(\mathrm{x})+\frac{\mathrm{i}}{\sqrt{2 \omega}} \pi(\mathrm{x})\right) e^{\mathrm{i} k_{\mu} x^{\mu}}  \tag{2.13}\\
a_{\boldsymbol{k}}^{*} & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{x}\left(\sqrt{\frac{\omega}{2}} \phi(\mathrm{x})-\frac{\mathrm{i}}{\sqrt{2 \omega}} \pi(\mathrm{x})\right) e^{-\mathrm{i} k_{\mu} x^{\mu}} \tag{2.14}
\end{align*}
$$

### 2.1.2 Quantising the scalar field

Canonical quantisation involves promoting the (real) field $\phi$ and (real) momentum $\pi$ to (Hermitian) operators and imposing equal time canonical commutation relations

$$
\begin{align*}
{[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)] } & =\mathrm{i} \delta^{n}(\boldsymbol{x}-\boldsymbol{y})  \tag{2.15}\\
{[\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{y}, t)] } & =0  \tag{2.16}\\
{[\hat{\pi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)] } & =0 \tag{2.17}
\end{align*}
$$

The same Fourier decomposition can be made

$$
\begin{align*}
\hat{\phi}(\mathrm{x}) & =\int_{\mathbb{R}^{n}} \frac{\mathrm{~d}^{n} \boldsymbol{k}}{(2 \pi)^{n / 2} \sqrt{2 \omega}}\left(\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i} k_{\mu} x^{\mu}}+\hat{a}_{\boldsymbol{k}}^{\dagger} e^{\mathrm{i} k_{\mu} x^{\mu}}\right)  \tag{2.18}\\
\hat{\pi}(\mathrm{x}) & =-\mathrm{i} \int_{\mathbb{R}^{n}} \frac{\mathrm{~d}^{n} \boldsymbol{k}}{(2 \pi)^{n / 2}} \sqrt{\frac{\omega}{2}}\left(\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i} k_{\mu} x^{\mu}}-\hat{a}_{\boldsymbol{k}}^{\dagger} e^{\mathrm{i} \boldsymbol{k}_{\mu} x^{\mu}}\right)  \tag{2.19}\\
\hat{a}_{\boldsymbol{k}} & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{x}\left(\sqrt{\frac{\omega}{2}} \hat{\phi}(\mathrm{x})+\frac{\mathrm{i}}{\sqrt{2 \omega}} \hat{\pi}(\mathrm{x})\right) e^{\mathrm{i} k_{\mu} x^{\mu}}  \tag{2.20}\\
\hat{a}_{\boldsymbol{k}}^{\dagger} & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \mathrm{~d}^{n} \boldsymbol{x}\left(\sqrt{\frac{\omega}{2}} \hat{\phi}(\mathrm{x})-\frac{\mathrm{i}}{\sqrt{2 \omega}} \hat{\pi}(\mathrm{x})\right) e^{-\mathrm{i} k_{\mu} x^{\mu}} \tag{2.21}
\end{align*}
$$

which then implies another important set of commutation relations

$$
\begin{align*}
& {\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]=\delta^{n}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right),}  \tag{2.22}\\
& {\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}^{\prime}}\right]=0}  \tag{2.23}\\
& {\left[\hat{a}_{\boldsymbol{k}}^{\dagger}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]=0} \tag{2.24}
\end{align*}
$$

Using these new operators the Hamiltonian density can be written out as

$$
\begin{align*}
\hat{\mathcal{H}}(\mathrm{x}) & =\int_{\mathbb{R}^{2 n}} \frac{\mathrm{~d}^{n} \boldsymbol{k} \mathrm{~d}^{n} \boldsymbol{k}^{\prime}}{(2 \pi)^{n} \sqrt{4 \omega \omega^{\prime}}}\left(\omega \omega^{\prime}-\frac{k_{\mu} k^{\prime \mu}}{2}\right)\left[e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right. \\
& \left.-e^{-\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}-e^{\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}^{\dagger}\right] \\
& +\frac{m^{2}}{2} \int_{\mathbb{R}^{2 n}} \frac{\mathrm{~d}^{n} \boldsymbol{k} \mathrm{~d}^{n} \boldsymbol{k}^{\prime}}{(2 \pi)^{n} \sqrt{4 \omega \omega^{\prime}}}\left[e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right. \\
& \left.+e^{-\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}^{\dagger}\right] \tag{2.25}
\end{align*}
$$

and when integrated over all space the Hamiltonian becomes

$$
\begin{align*}
\hat{H} & =\int \mathrm{d}^{n} \boldsymbol{k} \frac{\omega}{2}\left(\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}}+\hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right)  \tag{2.26}\\
& =\int \mathrm{d}^{n} \boldsymbol{k} \frac{\omega}{2}\left(2 \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}}+\delta^{n}(0)\right) . \tag{2.27}
\end{align*}
$$

The $\delta^{n}(0)$ term is interpreted as a vacuum energy density of $\omega / 2$ for each mode and is then discarded as a renormalisation of the energy scale. This renormalisation process is called normal ordering and is denoted

$$
\begin{equation*}
: \hat{H}:=\int \mathrm{d}^{n} \boldsymbol{k} \omega \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}} \tag{2.28}
\end{equation*}
$$

Hence a quantised scalar field can be interpreted as a collection of independent harmonic oscillators in momentum space, each with an energy gap of $\omega$. This interpretation combined with the commutation relations of $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^{\dagger}$ lead to the names annihilation and creation operators respectively. Subsequently the scalar field is described in terms of excitations of these momentum oscillators.

### 2.1.3 States of the quantised field

Quantum states are chosen to model observed or desired conditions, with a large set from which to choose. In this section some of the important states used in this thesis are described.

## The vacuum state

The Hamiltonian (2.28) consists of a sum of commuting operators, which correspond to independent harmonic oscillators for each mode $\boldsymbol{k}$ with an energy gap $\omega$. Therefore the ground state of such a Hamiltonian is the tensor product of harmonic oscillator ground states

$$
\begin{equation*}
|0\rangle=\bigotimes_{k}|0\rangle_{k} \tag{2.29}
\end{equation*}
$$

where $|0\rangle_{k}$ is the ground state of the harmonic oscillator with energy gap $\omega$. The state $|0\rangle$ is the ground state of the Hamiltonian (2.28) and is called the vacuum state.

In general, experiments involve the ground state or systems near the ground state and hence it is a familiar and well studied state. One commonly recurring attribute is the two point correlator (also known as the Wightman function). It is given by

$$
\begin{equation*}
W\left(\mathrm{x}, \mathrm{x}^{\prime}\right)=\langle 0| \hat{\phi}(\mathrm{x}) \hat{\phi}\left(\mathrm{x}^{\prime}\right)|0\rangle=\int \frac{\mathrm{d}^{n} \boldsymbol{k}}{2 \omega(2 \pi)^{n}} e^{-\mathrm{i} k_{\mu}\left(x^{\mu}-x^{\prime \mu}\right)} \tag{2.30}
\end{equation*}
$$

which in the case of $3+1 \mathrm{D}$ spacetime and a massless field this reduces to

$$
\begin{equation*}
W\left(\mathrm{x}, \mathrm{x}^{\prime}\right)=\frac{1}{4 \pi^{2}} \frac{1}{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|^{2}-\left(t-t^{\prime}-\mathrm{i} \epsilon\right)^{2}} \tag{2.31}
\end{equation*}
$$

where $\epsilon>0$ is the usual pole prescription when $x-x^{\prime}$ becomes null. This quantity appears often, especially in perturbation theory of quantum fields.

The vacuum state is generally treated as the reference from which other states are constructed, e.g. Fock states

$$
\begin{equation*}
\left|n_{\boldsymbol{k}}, m_{\boldsymbol{k}^{\prime}}\right\rangle=\frac{\left(\hat{a}_{\boldsymbol{k}}^{\dagger}\right)^{n}\left(\hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right)^{m}}{\sqrt{n!m!}}|0\rangle \tag{2.32}
\end{equation*}
$$

i.e. excitations of the momentum space oscillators, which are usually used as a basis set.

## Coherent states

With field operators usually expressed in terms of momentum creation and annihilation operators it is useful to identify the eigenstates of the annihilation operator, i.e.

$$
\begin{equation*}
\hat{a}_{\boldsymbol{k}}|\boldsymbol{\alpha}\rangle=\alpha_{\boldsymbol{k}}|\boldsymbol{\alpha}\rangle, \quad \forall \boldsymbol{k} \in \mathbb{R}^{n} \tag{2.33}
\end{equation*}
$$

Such a state $|\boldsymbol{\alpha}\rangle$ is called a coherent state and faithfully represents the coherent emission of a laser source [30], as such the coherent state is often described as a coherent macroscopic quantum state [31]. In the case of a relativistic scalar field, the coherent state is a generalisation of the single harmonic oscillator coherent state, i.e. a tensor product of individual momentum space coherent states. Mathematically, these (normalised) states can be represented as

$$
\begin{align*}
|\boldsymbol{\alpha}\rangle & =\exp \left[\int \mathrm{d}^{n} \boldsymbol{k}\left(\alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}-\frac{1}{2}\left|\alpha_{\boldsymbol{k}}\right|^{2}\right)\right]|0\rangle  \tag{2.34}\\
& =\exp \left[\int \mathrm{d}^{n} \boldsymbol{k}\left(\alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}-\alpha_{\boldsymbol{k}}^{*} \hat{a}_{\boldsymbol{k}}\right)\right]|0\rangle \tag{2.35}
\end{align*}
$$

Both representations are equivalent only because $\hat{a}_{k}|0\rangle=0$. The unitary representation (2.35) is usually described as $\hat{D}(\boldsymbol{\alpha})|0\rangle$, as shall be the case in chapter 3. This unitary representation can be interpreted as the action of a linear Hamiltonian (i.e. linear in $\hat{\phi}$ and $\hat{\pi}$ ) acting on the vacuum state.

Coherent states are generally not orthogonal, i.e.

$$
\begin{align*}
\langle\boldsymbol{\beta} \mid \boldsymbol{\alpha}\rangle & =\exp \left[-\frac{1}{2} \int \mathrm{~d}^{n} \boldsymbol{k}\left(\left|\beta_{\boldsymbol{k}}\right|^{2}+\left|\alpha_{\boldsymbol{k}}\right|^{2}-2 \beta_{\boldsymbol{k}}^{*} \alpha_{\boldsymbol{k}}\right)\right]  \tag{2.36}\\
& \neq \delta^{n}(\boldsymbol{\beta}-\boldsymbol{\alpha}) \tag{2.37}
\end{align*}
$$

and $\hat{D}(\boldsymbol{\alpha}) \hat{D}(\boldsymbol{\beta})=e^{\mathrm{i} \phi} \hat{D}(\boldsymbol{\alpha}+\boldsymbol{\beta})$, where $\phi$ is some phase factor that is unused in this thesis.

## Squeezed states

Whilst coherent states are states generated by linear Hamiltonians acting on the vacuum, squeezed states are states generated by quadratic Hamiltonians [30], i.e.

$$
\begin{equation*}
\hat{S}(\hat{A})|0\rangle=\exp (-\mathrm{i} \hat{A})|0\rangle \tag{2.38}
\end{equation*}
$$

where $\hat{A}$ is a Hermitian operator, quadratic in creation and annihilation operators. An example of a squeezing operator is

$$
\begin{equation*}
\hat{S}(\boldsymbol{\beta})=\exp \left(\int \mathrm{d}^{n} \boldsymbol{k} \mathrm{~d}^{n} \boldsymbol{k}^{\prime} \frac{\beta_{\boldsymbol{k}} \beta_{\boldsymbol{k}^{\prime}}}{2}\left[\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}-\hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}^{\prime}}\right]\right) \tag{2.39}
\end{equation*}
$$

for some arbitrary real $\boldsymbol{\beta}$. The complexities introduced by quadratic terms means squeezed states do not have as many pleasant properties and equations as coherent states; however, tools from studying a single harmonic oscillator can also be used in fields (e.g. Wigner functions and Gaussian QM) [31].

If linear terms are included in $\hat{A}$ for (2.38) then this becomes a squeezed coherent state and can be separated into $\hat{S}(\hat{A})|0\rangle=\hat{S}\left(\hat{A}^{\prime}\right) \hat{D}(\boldsymbol{\alpha})|0\rangle$ for some $\boldsymbol{\alpha}$ and some purely quadratic $\hat{A}^{\prime}$. For the remainder of the thesis squeezed states will refer to purely quadratic $\hat{A}$ and will always be distinguished from squeezed coherent states.

Some simplifying properties can be established if, e.g. one considers the similarity transformation [32]

$$
\begin{align*}
e^{-\theta \hat{P}} \hat{B} e^{\theta \hat{P}} & =\hat{B}-\theta[\hat{P}, \hat{B}]+\frac{\theta^{2}}{2!}[\hat{P},[\hat{P}, \hat{B}]]+\ldots  \tag{2.40}\\
& =\sum_{n=0}^{\infty} \frac{(-\theta)^{n}}{n!} \hat{\hat{L}_{P}^{n}} \hat{B} \tag{2.41}
\end{align*}
$$

where

$$
\begin{align*}
\hat{\hat{L}}_{P}^{0} \hat{A} & =\hat{A}  \tag{2.42}\\
\hat{\hat{L}}_{P}^{1} \hat{A} & =[\hat{P}, \hat{A}]  \tag{2.43}\\
\hat{\hat{L}}_{P}^{n+1} \hat{A} & =\left[\hat{P}, \hat{\hat{L}}_{P}^{n} \hat{A}\right] . \tag{2.44}
\end{align*}
$$

By exploiting this similarity transformation with the squeezing operator in (2.39) the following useful property arises

$$
\begin{align*}
\hat{S}^{\dagger}(\boldsymbol{\beta}) \hat{a}_{\boldsymbol{k}} \hat{S}(\boldsymbol{\beta}) & =\hat{a}_{\boldsymbol{k}}-\beta_{\boldsymbol{k}} K^{-1} \int \mathrm{~d}^{n} \boldsymbol{k}^{\prime} \beta_{\boldsymbol{k}^{\prime}}{\hat{\boldsymbol{k}^{\prime}}}+\beta_{\boldsymbol{k}} K^{-1} \cosh (K) \int \mathrm{d}^{n} \boldsymbol{k}^{\prime} \beta_{\boldsymbol{k}^{\prime}}{\hat{\boldsymbol{k}^{\prime}}} \\
& +\beta_{\boldsymbol{k}} K^{-1} \sinh (K) \int \mathrm{d}^{n} \boldsymbol{k}^{\prime} \beta_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}  \tag{2.45}\\
K & =\int \mathrm{d}^{n} \boldsymbol{k} \beta_{\boldsymbol{k}}^{2} \tag{2.46}
\end{align*}
$$

i.e. the similarity transformation of an annihilation operator yields linear combinations of annihilation and creation operators. This property is central to simplifying expectation values of squeezed states and other algebraic manipulations. More generally

$$
\begin{align*}
& \hat{b}_{\boldsymbol{k}}=\hat{S}^{\dagger}(\hat{A}) \hat{a}_{\boldsymbol{k}} \hat{S}(\hat{A}),  \tag{2.47}\\
& \hat{b}_{\boldsymbol{k}}^{\dagger}=\hat{S}^{\dagger}(\hat{A}) \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{S}(\hat{A}), \tag{2.48}
\end{align*}
$$

are Bogoliubov transformations (i.e. $\left[\hat{b}_{\boldsymbol{k}}, \hat{b}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]=\delta^{n}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right)$ ). An example is to consider a massive scalar field as a lattice of spatial harmonic oscillators with annihilation operators $\hat{\psi}(\boldsymbol{x})=(\hat{\phi}(\boldsymbol{x})+\mathrm{i} \hat{\pi}(\boldsymbol{x})) / \sqrt{2}$ (Schrödinger picture), where $\left[\hat{\psi}(\boldsymbol{x}), \hat{\psi}^{\dagger}\left(\boldsymbol{x}^{\prime}\right)\right]=\delta^{n}\left(\boldsymbol{x}-\boldsymbol{x}^{\prime}\right)$. In the 'spatial Fock basis' defined by the 'position space vacuum' $\hat{\psi}(\boldsymbol{x})|\Omega\rangle=0\left(\forall \boldsymbol{x} \in \mathbb{R}^{n}\right)$, the scalar field vacuum is $|0\rangle=\hat{S}(\hat{A})|\Omega\rangle$, for some quadratic $\hat{A}$ [33]. In particular this means the vacuum state of a massive scalar field can be written as a squeezed state, where the squeezing operator $\hat{S}$ acts on the position space vacuum $|\Omega\rangle$. Therefore, by performing a specific squeezing operator on the global vacuum state, one can locally reduce the energy density of the scalar field below the vacuum expectations [22], which is the objective of chapter 3.

## Cat states

Within the set of superpositions of coherent states there is a particularly useful state, called the cat state, which is often described as a superposition of coherent macroscopic quantum
states. Using the 'macroscopic' coherent states $|\boldsymbol{\alpha}\rangle$, superpositions taking the form

$$
\begin{equation*}
|\operatorname{CAT}(\boldsymbol{\alpha})\rangle=\frac{1}{\mathcal{N}}\left(|\boldsymbol{\alpha}\rangle+e^{\mathrm{i} \varphi}|-\boldsymbol{\alpha}\rangle\right), \tag{2.49}
\end{equation*}
$$

are called cat states [31] with amplitude $\boldsymbol{\alpha}$ and where $\mathcal{N}$ is the normalisation factor. We will only concern ourselves with $\varphi=0$. A curious property of cat states becomes apparent when considering the Taylor expansion of $|\operatorname{CAT}(\boldsymbol{\alpha})\rangle$ for small $\boldsymbol{\alpha}$ :

$$
\begin{align*}
|\operatorname{CAT}(\boldsymbol{\alpha})\rangle & =\frac{e^{-\frac{1}{2} \int \mathrm{~d}^{n} \boldsymbol{k}\left|\alpha_{\boldsymbol{k}}\right|^{2}}}{\mathcal{N}}\left(1+\int \mathrm{d}^{n} \boldsymbol{k} \alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}+\frac{1}{2}\left[\int \mathrm{~d}^{n} \boldsymbol{k} \alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right]^{2}+o\left(\alpha^{3}\right)\right. \\
& \left.+1-\int \mathrm{d}^{n} \boldsymbol{k} \alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}+\frac{1}{2}\left[\int \mathrm{~d}^{n} \boldsymbol{k} \alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right]^{2}+o\left(\alpha^{3}\right)\right)|0\rangle  \tag{2.50}\\
& =\frac{1}{\mathcal{N}^{\prime}}\left(1+\frac{1}{2}\left[\int \mathrm{~d}^{n} \boldsymbol{k} \alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right]^{2}+o\left(\alpha^{4}\right)\right)|0\rangle . \tag{2.51}
\end{align*}
$$

This state contains only contains even powers of $\alpha$. Comparing the cat state Taylor series with that of the squeezed state (2.39)

$$
\begin{align*}
\hat{S}(\boldsymbol{\beta}) & =\left(1+\frac{1}{2} \int \mathrm{~d}^{n} \boldsymbol{k} \mathrm{~d}^{n} \boldsymbol{k}^{\prime} \beta_{\boldsymbol{k}} \beta_{\boldsymbol{k}^{\prime}}\left[\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}-\hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}^{\prime}}\right]+o\left(\beta^{4}\right)\right)|0\rangle  \tag{2.52}\\
& =\left(1+\frac{1}{2}\left[\int \mathrm{~d}^{n} \boldsymbol{k} \beta_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right]^{2}+o\left(\beta^{4}\right)\right)|0\rangle \tag{2.53}
\end{align*}
$$

In (2.51), when $\boldsymbol{\alpha}$ is small $\boldsymbol{N}^{\prime} \approx 1$ and then if $\boldsymbol{\beta}=\boldsymbol{\alpha}$ then (2.51) and (2.53) are equal (to second order), with some differences at 4th order and above. Hence a cat state is able to approximate a squeezed state to second order, with the advantage of only requiring a linear Hamiltonian for generation; and, depending on the situation, it may be experimentally easier to manipulate linear Hamiltonians over quadratic Hamiltonians. Their relative simplicity motivates their use in chapter 3 .

### 2.2 The UDW detector and the light-matter interaction

For a relativistic field, the question of measurement raises substantial concerns over causality and faster than light signalling [3, 34, 35, 36, 37]. A solution to this problem was to introduce a detector, which would interact with the field before being measured itself. This resolved the issue of local field measurement and any causality issues accompanying it.

### 2.2.1 Unruh-DeWitt model

The Unruh-DeWitt (UDW) model [38, 39] is the simplest approach to interacting a first quantised system (detector) with a second quantised field (scalar field). It involves using a qubit with energy gap $\Omega$ as a detector interacting linearly with the field. The Hamiltonian of the resulting detector-field system can be written as

$$
\begin{equation*}
\hat{H}=\int \mathrm{d}^{n} \boldsymbol{k} \omega \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}}+\frac{\Omega}{2} \hat{\sigma}_{z}+\lambda \chi(t) \hat{\sigma}_{x} \int \mathrm{~d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\phi}(t, \boldsymbol{x}) \tag{2.54}
\end{equation*}
$$

where $\lambda$ is the interaction strength, $\chi(t)$ dictates when the detector is in contact with the field (switching function) and $F(\boldsymbol{x})$ describes the region of the field that influences the detector (smearing function), e.g. an electron orbital of an atom [27].

When working in the interaction picture, the Hamiltonian is written out as

$$
\begin{equation*}
\hat{H}=\lambda \chi(t)\left(\hat{\sigma}^{+} e^{\mathrm{i} \Omega t}+\hat{\sigma}^{-} e^{-\mathrm{i} \Omega t}\right) \int \mathrm{d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\phi}(t, \boldsymbol{x}) \tag{2.55}
\end{equation*}
$$

where $\hat{\sigma}^{ \pm}$are ladder operators of the $\mathfrak{B u}(2)$ algebra. Generally the UDW model is used perturbatively (often to 2 nd order) to perform indirect measurements and other RQI protocols, i.e.

$$
\begin{align*}
\hat{U}(T) & =\mathcal{T} \exp \left[-\mathrm{i} \int_{-\infty}^{T} \mathrm{~d} t \hat{H}(t)\right]  \tag{2.56}\\
& =1-\mathrm{i} \int_{-\infty}^{T} \mathrm{~d} t_{1} \hat{H}\left(t_{1}\right)-\int_{-\infty}^{T} \mathrm{~d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \hat{H}\left(t_{1}\right) \hat{H}\left(t_{2}\right)+o\left(\lambda^{3}\right) \tag{2.57}
\end{align*}
$$

This perturbative approach is used in chapters 4,5 and 6 .

## Non-perturbative evolution

Under specific circumstances, a non-perturbative approach can be taken to compute the time evolution of a detector-field system. If the detector has no energy gap $(\Omega=0)$ or if $\chi(t)=\delta(t)$, i.e. if the detector very quickly interacts via a delta coupling then a nonperturbative calculation can be used [23]. In chapter 3 a delta coupling is used, allowing
for non-perturbative calculations. Assuming $T_{i}<0<T_{f}(\chi(t)=\delta(t))$ the unitary time evolution operator becomes

$$
\begin{align*}
\hat{U}\left(T_{i}, T_{f}\right) & =\mathcal{T} \exp \left[-\mathrm{i} \int_{T_{i}}^{T_{f}} \mathrm{~d} t \hat{H}(t)\right]  \tag{2.58}\\
& =\mathcal{T} \exp \left[-\mathrm{i} \lambda \int_{T_{i}}^{T_{f}} \mathrm{~d} t \delta(t)\left(\hat{\sigma}^{+} e^{\mathrm{i} \Omega t}+\hat{\sigma}^{-} e^{-\mathrm{i} \Omega t}\right) \int \mathrm{d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\phi}(t, \boldsymbol{x})\right]  \tag{2.59}\\
& =\exp \left[-\mathrm{i} \lambda\left(\hat{\sigma}^{+}+\hat{\sigma}^{-}\right) \int \mathrm{d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\phi}(0, \boldsymbol{x})\right] \tag{2.60}
\end{align*}
$$

where $\hat{\sigma}^{+}+\hat{\sigma}^{-}=\hat{\sigma}_{x}$. By expanding the $\hat{\phi}$ operator into creation and annihilation operators this becomes

$$
\begin{align*}
\hat{U}\left(T_{i}, T_{f}\right) & =\exp \left[-\mathrm{i} \lambda \hat{\sigma}_{x} \int \mathrm{~d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \int \frac{\mathrm{d}^{n} \boldsymbol{k}}{(2 \pi)^{n / 2} \sqrt{2 \omega}}\left(\hat{a}_{\boldsymbol{k}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}+\hat{a}_{\boldsymbol{k}}^{\dagger} e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}\right)\right]  \tag{2.61}\\
& =\exp \left[(|+x\rangle\langle+x|-|-x\rangle\langle-x|) \int \mathrm{d}^{n} \boldsymbol{k}\left(\alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}-\alpha_{\boldsymbol{k}}^{*} \hat{a}_{\boldsymbol{k}}\right)\right]  \tag{2.62}\\
& =|+x\rangle\langle+x| \exp \left[\int \mathrm{d}^{n} \boldsymbol{k}\left(\alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}-\alpha_{\boldsymbol{k}}^{*} \hat{a}_{\boldsymbol{k}}\right)\right] \\
& +|-x\rangle\langle-x| \exp \left[-\int \mathrm{d}^{n} \boldsymbol{k}\left(\alpha_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}-\alpha_{\boldsymbol{k}}^{*} \hat{a}_{\boldsymbol{k}}\right)\right]  \tag{2.63}\\
& =|+x\rangle\langle+x| \hat{D}(\boldsymbol{\alpha})+|-x\rangle\langle-x| \hat{D}(-\boldsymbol{\alpha}) \tag{2.64}
\end{align*}
$$

where

$$
\begin{equation*}
\alpha_{\boldsymbol{k}}=-\frac{\mathrm{i} \lambda}{(2 \pi)^{n / 2} \sqrt{2 \omega}} \int \mathrm{~d}^{n} \boldsymbol{x} F(\boldsymbol{x}) e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} . \tag{2.65}
\end{equation*}
$$

This non-perturbative approach results in a coherent state displacement operator controlled by the initial state of the qubit. One interesting case is if the qubit is initially in $|+z\rangle$ state and the field is initially vacuum then the unitary $\hat{U}\left(T_{i}, T_{f}\right)$, followed by a $\hat{\sigma}_{z}$ projective measurement would generate a cat state, as is discussed further in chapter 3.

### 2.2.2 Variations of UDW

Around the concept of UDW there are variations that can be made, with differing suitability in different scenarios.

## Derivative coupling

Whilst not as commonly used, another linear detector-field interaction is the derivative coupling,

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}=\lambda \chi(t) \hat{\sigma}_{x} \int \mathrm{~d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\pi}(t, \boldsymbol{x}) \tag{2.66}
\end{equation*}
$$

This coupling is used in chapter 3, resulting in an improvement in quantum energy teleportation efficiency.

## Harmonic oscillator

The UDW detector thus far has been described as a qubit, however it may be a qudit or even a harmonic oscillator [40]. Increasing the dimensionality of the detector allows more information to be passed to the detector, which can lead to improvements in RQI protocol capacities [41]. For the harmonic oscillator the interaction Hamiltonian becomes

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}=\lambda \chi(t) \hat{q} \int \mathrm{~d}^{n} \boldsymbol{x} F(\boldsymbol{x}) \hat{\phi}(t, \boldsymbol{x}), \tag{2.67}
\end{equation*}
$$

where $\hat{q}$ is a quadrature operator of the harmonic oscillator. Note $\hat{q}$ could have been replaced with any linear combination of $\hat{q}$ and $\hat{p}$.

### 2.3 Scalar field in a cavity

### 2.3.1 Field equations

When considering a scalar field in a cavity, the same Lagrangian density as a free space Klein-Gordon field is used

$$
\begin{equation*}
\mathcal{L}(\mathrm{x})=\frac{1}{2} \partial_{\mu} \phi(\mathrm{x}) \partial^{\mu} \phi(\mathrm{x})-\frac{m^{2}}{2} \phi^{2}(\mathrm{x}), \tag{2.68}
\end{equation*}
$$

however the action is now defined over a compact cavity

$$
\begin{equation*}
S=\int \mathrm{d} t \int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} \mathcal{L}(\mathrm{x}) \tag{2.69}
\end{equation*}
$$

where $\mathcal{C}$ denotes the cavity and the spacetime dimension is fixed to $3+1$. In this thesis $\mathcal{C}$ will always be a rectangular cavity of dimensions $\left(L_{1}, L_{2}, L_{3}\right)$ and the cavity fields will be treated as massless $(m=0)$.

As with free space fields, the minimisation of the action leads to the Euler-Lagrange equations

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi=0 \tag{2.70}
\end{equation*}
$$

In order to solve this Klein-Gordon equation the field is mode decomposed into

$$
\begin{equation*}
\phi(\mathrm{x})=\sum_{\boldsymbol{m} \in \mathcal{Z}}\left(a_{\boldsymbol{m}} u_{\boldsymbol{m}}(\mathrm{x})+a_{\boldsymbol{m}}^{*} u_{\boldsymbol{m}}^{*}(\mathrm{x})\right) \tag{2.71}
\end{equation*}
$$

where the finite size of the cavity ensures a discrete mode decomposition. $\mathcal{Z}$ is the index set of allowed mode functions, dependent on the boundary conditions and choice of mode decomposition. As with the free space case it is useful if the mode functions are (Kronecker) orthonormal under the Klein-Gordon product

$$
\begin{equation*}
\left(\phi_{1}, \phi_{2}\right)_{\mathrm{KG}}=-\mathrm{i} \int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x}\left(\phi_{1}(\mathrm{x}) \partial_{t} \phi_{2}^{*}(\mathrm{x})-\phi_{2}^{*}(\mathrm{x}) \partial_{t} \phi_{1}(\mathrm{x})\right) . \tag{2.72}
\end{equation*}
$$

and furthermore the positive frequency solutions satisfy ( $\omega_{m} \geq 0$ )

$$
\begin{equation*}
\partial_{t} u_{m}(\mathrm{x})=-\mathrm{i} \omega_{m} u_{m}(\mathrm{x}) \tag{2.73}
\end{equation*}
$$

## Field attributes

Most of the derivations in $\S 2.1 .1$ involve local equations, independent of the boundary conditions and given the cavity field is still continuous then most of the results can be directly translated. The conjugate momentum can then be written out as

$$
\begin{equation*}
\pi(\mathrm{x})=\partial_{t} \phi(\mathrm{x}) \tag{2.74}
\end{equation*}
$$

and the Hamiltonian density becomes

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2}\left(\partial_{t} \phi\right)^{2}+\frac{1}{2}(\nabla \phi) \cdot(\nabla \phi) \tag{2.75}
\end{equation*}
$$

Whilst the boundary conditions have not yet been fixed and therefore the mode functions remain ambiguous, the orthonormality of the mode functions can be exploited to determine the Fourier coefficients of the mode decomposition

$$
\begin{align*}
\phi(\mathrm{x}) & =\sum_{\boldsymbol{m} \in \mathcal{Z}}\left(a_{\boldsymbol{m}} u_{\boldsymbol{m}}(\mathrm{x})+a_{\boldsymbol{m}}^{*} u_{\boldsymbol{m}}^{*}(\mathrm{x})\right),  \tag{2.76}\\
\pi(\mathrm{x}) & =-\mathrm{i} \sum_{\boldsymbol{m} \in \mathcal{Z}} \omega_{\boldsymbol{m}}\left(a_{\boldsymbol{m}} u_{\boldsymbol{m}}(\mathrm{x})-a_{\boldsymbol{m}}^{*} u_{\boldsymbol{m}}^{*}(\mathrm{x})\right),  \tag{2.77}\\
a_{\boldsymbol{n}} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}^{*}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \phi(t, \boldsymbol{x})+\mathrm{i} \pi(t, \boldsymbol{x})\right),  \tag{2.78}\\
a_{\boldsymbol{n}}^{*} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \phi(t, \boldsymbol{x})-\mathrm{i} \pi(t, \boldsymbol{x})\right), \tag{2.79}
\end{align*}
$$

where the 4 -vector notation $\mathrm{x}=(t, \boldsymbol{x})$ is used interchangeably.

### 2.3.2 Boundary conditions and quantisation

Whilst boundary conditions can be entirely arbitrary, in this section only three boundary conditions are considered, Dirichlet, Neumann and Periodic boundary conditions. The cavity is defined as

$$
\begin{equation*}
\mathcal{C}=\left\{\boldsymbol{x} \in \mathbb{R}^{3}: 0 \leq x_{i} \leq L_{i}, i=1,2,3\right\}, \tag{2.80}
\end{equation*}
$$

and its boundary is denoted by $\partial \mathcal{C}$.

## Dirichlet Boundary conditions

Dirichlet boundary conditions require $\phi(t, \boldsymbol{x})=0$ for $\boldsymbol{x} \in \partial \mathcal{C}$. Implementing these boundary conditions into the Klein-Gordon equation yields mode functions

$$
\begin{equation*}
u_{\boldsymbol{m}}=\frac{2}{\sqrt{\omega_{m} V}} e^{-\mathrm{i} \omega_{\boldsymbol{m}} t} \prod_{i=1}^{3} \sin \left(k_{m}^{i} x_{i}\right) \tag{2.81}
\end{equation*}
$$

where $V=L_{1} L_{2} L_{3}$ and

$$
\begin{align*}
& k_{m}^{i}=\frac{m_{i} \pi}{L_{i}}  \tag{2.82}\\
& \omega_{m}=\left|\boldsymbol{k}_{\boldsymbol{m}}\right| \tag{2.83}
\end{align*}
$$

In addition the mode index set is

$$
\begin{equation*}
\mathcal{Z}=\left\{\boldsymbol{m} \in \mathbb{Z}^{3}: m_{i}>0, i=1,2,3\right\} . \tag{2.84}
\end{equation*}
$$

Therefore the field can be written out as

$$
\begin{equation*}
\phi(t, \boldsymbol{x})=\sum_{m \in \mathcal{Z}} \frac{2}{\sqrt{\omega_{\boldsymbol{m}} V}}\left(e^{-\mathrm{i} \omega_{\boldsymbol{m}} t} a_{\boldsymbol{m}}+e^{\mathrm{i} \omega_{\boldsymbol{m}} t} a_{\boldsymbol{m}}^{*}\right) \prod_{i=1}^{3} \sin \left(k_{\boldsymbol{m}}^{i} x_{i}\right) . \tag{2.85}
\end{equation*}
$$

## Dirichlet quantisation

The canonical quantisation process involves promoting the field functions into operators with the usual canonical commutation relations $[\hat{\phi}(t, \boldsymbol{x}), \hat{\phi}(t, \boldsymbol{y})]=0,[\hat{\pi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})]=0$ and $[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})]=\mathrm{i} \delta^{3}(\boldsymbol{x}-\boldsymbol{y})$. As in the free space case the Fourier coefficients also become operators with the harmonic oscillator commutation relations $\left[\hat{a}_{\boldsymbol{m}}, \hat{a}_{\boldsymbol{n}}^{\dagger}\right]=\delta_{\boldsymbol{m} \boldsymbol{n}}$ (Kronecker delta). Subsequently this leads to the renormalised Hamiltonian

$$
\begin{equation*}
: \hat{H}:=\sum_{m \in \mathbb{Z}} \omega_{m} \hat{a}_{\boldsymbol{m}}^{\dagger} \hat{a}_{\boldsymbol{m}} . \tag{2.86}
\end{equation*}
$$

For completeness

$$
\begin{align*}
\hat{\phi}(\mathrm{x}) & =\sum_{\boldsymbol{m} \in \mathcal{Z}}\left(\hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(\mathrm{x})+\hat{a}_{\boldsymbol{m}}^{\dagger} u_{\boldsymbol{m}}^{*}(\mathrm{x})\right)  \tag{2.87}\\
\hat{\pi}(\mathrm{x}) & =-\mathrm{i} \sum_{\boldsymbol{m} \in \mathcal{Z}} \omega_{\boldsymbol{m}}\left(\hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(\mathrm{x})-\hat{a}_{\boldsymbol{m}}^{\dagger} u_{\boldsymbol{m}}^{*}(\mathrm{x})\right),  \tag{2.88}\\
\hat{a}_{\boldsymbol{n}} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}^{*}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \hat{\phi}(t, \boldsymbol{x})+\mathrm{i} \hat{\pi}(t, \boldsymbol{x})\right),  \tag{2.89}\\
\hat{a}_{\boldsymbol{n}}^{\dagger} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \hat{\phi}(t, \boldsymbol{x})-\mathrm{i} \hat{\pi}(t, \boldsymbol{x})\right) . \tag{2.90}
\end{align*}
$$

## Neumann Boundary conditions

Neumann boundary conditions require $\vec{n} \cdot \nabla \phi(t, \boldsymbol{x})=0$ for $\boldsymbol{x} \in \partial \mathcal{C}$, where $\vec{n}$ is the normal vector to $\partial \mathcal{C}$ at $\boldsymbol{x}$. Implementing this boundary condition the mode functions become

$$
u_{\boldsymbol{m}}(t, \boldsymbol{x})=\frac{2}{\sqrt{\omega_{\boldsymbol{m}} V}} e^{-\mathrm{i} \omega_{\boldsymbol{m}} t} \prod_{i=1}^{3} \begin{cases}\frac{1}{\sqrt{2}} & \text { if } m_{i}=0  \tag{2.91}\\ \cos \left(k_{\boldsymbol{m}}^{i} x_{i}\right) & \text { otherwise }\end{cases}
$$

where the $m_{i}=0$ case ensures orthonormality. Also

$$
\begin{align*}
k_{m}^{i} & =\frac{m_{i} \pi}{L_{i}}  \tag{2.92}\\
\omega_{\boldsymbol{m}} & =\left|\boldsymbol{k}_{\boldsymbol{m}}\right|  \tag{2.93}\\
\mathcal{Z} & =\left\{\boldsymbol{m} \in \mathbb{Z}^{3}: m_{i} \geq 0, i=1,2,3 \& \boldsymbol{m} \neq \mathbf{0}\right\} \tag{2.94}
\end{align*}
$$

When $\boldsymbol{m}=\mathbf{0}$ then $\omega_{\boldsymbol{m}}=0$ but $u_{\boldsymbol{m}} \neq 0$. The $\boldsymbol{m}=\mathbf{0}$ mode is called the zero mode and involves displacing the field $\phi \rightarrow \phi+Q$, where $Q$ is a constant. In the massless limit presented here, all field equations rely on derivatives of $\phi$ and since the boundary conditions also rely on derivatives then $Q$ is arbitrary. In classical field theory it is not a major issue, however under quantisation it needs to be treated carefully.

## Neumann quantisation

If the standard quantisation process is used for Neumann boundary conditions then one would reach the incorrect conclusion that the zero mode $Q$ describes a harmonic oscillator with energy gap $\omega=0$. Inspection of the harmonic oscillator Hamiltonian shows that when $\omega=0$ the degree of freedom behaves as a free particle and the quantisation process must reflect this fact, therefore it is worthwhile to separate the zero mode contributions from the rest of the field:

$$
\begin{align*}
& \phi(t, \boldsymbol{x})=\phi_{\mathrm{OSC}}(t, \boldsymbol{x})+\phi_{\mathrm{ZM}}(t),  \tag{2.95}\\
& \pi(t, \boldsymbol{x})=\pi_{\mathrm{OSC}}(t, \boldsymbol{x})+\pi_{\mathrm{ZM}}, \tag{2.96}
\end{align*}
$$

where $\phi_{\text {Osc }}$ is defined by mode functions (2.91) and $\phi_{\mathrm{ZM}}$ describes a free particle. Under canonical quantisation the commutation relations become

$$
\begin{align*}
{[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})] } & =\mathrm{i} \delta^{3}(\boldsymbol{x}-\boldsymbol{y})  \tag{2.97}\\
{\left[\hat{\phi}_{\mathrm{ZM}}(t), \hat{\pi}_{\mathrm{ZM}}\right] } & =\mathrm{i} \tag{2.98}
\end{align*}
$$

$\hat{\pi}_{\mathrm{ZM}}$ does not depend on time and all other commutation relations are zero. From the Lagrangian

$$
\begin{align*}
\partial_{t} \hat{\phi}_{\mathrm{ZM}}(t) & =\frac{\hat{\pi}_{\mathrm{ZM}}}{V}  \tag{2.99}\\
\hat{\phi}_{\mathrm{ZM}}(t) & =\hat{\phi}_{\mathrm{ZM}}(0)+\frac{\hat{\pi}_{\mathrm{ZM}} t}{V} \tag{2.100}
\end{align*}
$$

Therefore the field operators become

$$
\begin{align*}
\hat{\phi}(\mathrm{x}) & =\hat{\phi}_{\mathrm{ZM}}(t)+\sum_{\boldsymbol{m} \in \mathcal{Z}} \frac{2}{\sqrt{\omega_{\boldsymbol{m}} V}}\left(e^{-\mathrm{i} \omega_{\boldsymbol{m}} t} \hat{a}_{\boldsymbol{m}}+e^{\mathrm{i} \omega_{\boldsymbol{m}} t} \hat{a}_{\boldsymbol{m}}^{\dagger}\right) \prod_{i=1}^{3} \begin{cases}\frac{1}{\sqrt{2}} & \text { if } m_{i}=0, \\
\cos \left(k_{\boldsymbol{m}}^{i} x_{i}\right) & \text { otherwise },\end{cases}  \tag{2.101}\\
\hat{\pi}(\mathrm{x}) & =\hat{\pi}_{\mathrm{ZM}}-\mathrm{i} \sum_{\boldsymbol{m} \in \mathcal{Z}} 2 \sqrt{\frac{\omega_{\boldsymbol{m}}}{V}}\left(e^{-\mathrm{i} \omega_{\boldsymbol{m}} t} \hat{a}_{\boldsymbol{m}}-e^{\mathrm{i} \omega_{\boldsymbol{m}} t} \hat{a}_{\boldsymbol{m}}^{\dagger}\right) \prod_{i=1}^{3} \begin{cases}\frac{1}{\sqrt{2}} & \text { if } m_{i}=0 \\
\cos \left(k_{\boldsymbol{m}}^{i} x_{i}\right) & \text { otherwise },\end{cases}  \tag{2.102}\\
\hat{a}_{\boldsymbol{n}} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}^{*}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \hat{\phi}(t, \boldsymbol{x})+\mathrm{i} \hat{\pi}(t, \boldsymbol{x})\right),  \tag{2.103}\\
\hat{a}_{\boldsymbol{n}}^{\dagger} & =\int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{x} u_{\boldsymbol{n}}(t, \boldsymbol{x})\left(\omega_{\boldsymbol{n}} \hat{\phi}(t, \boldsymbol{x})-\mathrm{i} \hat{\pi}(t, \boldsymbol{x})\right), \tag{2.104}
\end{align*}
$$

where $\mathrm{x}=(t, \boldsymbol{x})$. Finally the field's Hamiltonian becomes

$$
\begin{equation*}
: \hat{H}:=\frac{\hat{\pi}_{\mathrm{ZM}}^{2}}{2 V}+\sum_{\boldsymbol{m} \in \mathcal{Z}} \omega_{\boldsymbol{m}} \hat{a}_{\boldsymbol{m}}^{\dagger} \hat{a}_{\boldsymbol{m}} \tag{2.105}
\end{equation*}
$$

## Periodic Boundary conditions

Periodic boundary conditions require $\phi(t, \boldsymbol{x})=\phi(t, \boldsymbol{x}+\mathcal{K})$ where $\mathcal{K}=\left(n_{1} L_{1}, n_{2} L_{2}, n_{3} L_{3}\right)$ where $n_{i} \in \mathbb{Z}$. Under these circumstances

$$
\begin{equation*}
u_{\boldsymbol{m}}(t, \boldsymbol{x})=\frac{1}{\sqrt{2 \omega_{\boldsymbol{m}} V}} e^{-\mathrm{i} \omega_{\boldsymbol{m}} t+\mathrm{i} \boldsymbol{k}_{\boldsymbol{m}} \cdot \boldsymbol{x}} \tag{2.106}
\end{equation*}
$$

with

$$
\begin{align*}
k_{m}^{i} & =\frac{2 m_{i} \pi}{L_{i}}  \tag{2.107}\\
\omega_{m} & =\left|\boldsymbol{k}_{\boldsymbol{m}}\right|  \tag{2.108}\\
\mathcal{Z} & =\left\{\boldsymbol{m} \in \mathbb{Z}^{3}: \boldsymbol{m} \neq \mathbf{0}\right\} . \tag{2.109}
\end{align*}
$$

Similar to the Neumann boundary conditions case, the $\boldsymbol{m}=\mathbf{0}$ mode needs to be treated separately as a zero mode. Quantisation then follows the same steps as the Neumann quantisation.

## Zero mode

The zero mode appearing in Neumann and Periodic boundary conditions arises as all equations describing the field are indifferent to $\phi \rightarrow \phi+a$ where $a$ is a constant. The Dirichlet boundary conditions enforce $a=0$ on the boundaries, eliminating the appearance of the zero mode. Furthermore if using different boundary conditions on each of the cavity walls, as long as one of the walls has a Dirichlet condition the zero mode is eliminated.

Mathematically the zero mode needs to be treated carefully as it behaves as a free particle instead of a harmonic oscillator and in particular it should not be ignored. This is important when detector-field interactions are considered as the detector evolution experiences a non-vanishing influence from the zero mode [42]. Misuse of the zero mode can have additional consequences such as potential causality violations [43] or vacuum state ambiguities [44].

In Chapter 6 the zero mode will briefly be discussed, in particular with regards to its role in UDW field-detector interactions and its inconsistency with rotating wave approximations.

## Chapter 3

## Using Quantum Energy teleportation to shape spacetime

### 3.1 Exotic spacetimes and the energy conditions

### 3.1.1 Classical EC

The geometric construction of General Relativity has resulted in a tremendously successful model, satisfying all currently observed phenomena. However this geometric construction opened the way for disturbing spacetime solutions. The standard example of a problematic solution is the naked singularity [45]. In an effort to address these problems the singularity theorems [21] were derived, which necessitated the introduction of energy conditions. These energy conditions are restrictions on the stress-energy tensor, resulting in the exclusion of problematic geometric solutions whilst coinciding with the observed universal attraction of gravity. In particular, these energy conditions disallow the existence of repulsive gravity and bizarre spacetimes such as physically traversable wormholes [46] and warp drives [47]. All classical fields obey these energy conditions; however quantum fields are known, from first principles, to violate these energy conditions [22]. In principle, this opens up a door to exotic solutions of Einstein's equations that may be of fundamental interest.

In this chapter we will focus on an operational method of generating negative energy densities, which violate the weak energy condition (WEC). The weak energy condition states that given any time-like vector $\xi_{\mu}$ then $T_{\mu \nu} \xi^{\mu} \xi^{\nu} \geq 0$. The consequences of this violation are conveniently listed in [48] and include the possibility of closed time-like curves [49], warp-drives, and physically traversable wormholes.

### 3.1.2 An exotic spacetime

A popular and exciting exotic spacetime is the Alcubierre warp spacetime [47], a metric that permits a spaceship to travel superluminally according to a distant observer. This metric takes advantage of general relativity's edict that nothing can travel locally faster than light. Using a concept motivated by the inflationary phase of the early universe this metric expands the volume elements behind the spaceship whilst contracting the space before it, allowing a spaceship to travel a time-like trajectory whilst an observer distant enough to be unaffected by the 'warp bubble' perceives a superluminal spaceship whose internal clock ticks at the same rate as the distant observer.

Generally speaking the Alcubierre spacetime is a globally hyperbolic spacetime that at some early initial time is flat and as the time coordinate progresses a 'Warp bubble' appears and begins to move superluminally. Once the bubble has reached its destination it dissipates, leaving the space flat once more. The metric is given by

$$
\begin{equation*}
\mathrm{d} s^{2}=-\mathrm{d} t^{2}+\left[\mathrm{d} x-v_{s}(t) f\left(r_{s}(t)\right) \mathrm{d} t\right]^{2}+\mathrm{d} y^{2}+\mathrm{d} z^{2} \tag{3.1}
\end{equation*}
$$

where $x_{s}(t)$ is the trajectory of the centre of the bubble and $v_{s}(t)=\mathrm{d} x_{s}(t) / \mathrm{d} t$ is the bubble's velocity. The variable $r_{s}(t)$ measures the distance from the centre of the bubble

$$
\begin{equation*}
r_{s}(t)=\sqrt{\left(x-x_{s}(t)\right)^{2}+y^{2}+z^{2}} \tag{3.2}
\end{equation*}
$$

Finally $f\left(r_{s}\right)$ describes the shape of the bubble, which Alcubierre originally chose to be

$$
\begin{equation*}
f\left(r_{s}\right)=\frac{\tanh \left[\sigma\left(r_{s}-R\right)\right]-\tanh \left[\sigma\left(r_{s}+R\right)\right]}{2 \tanh (\sigma R)} \tag{3.3}
\end{equation*}
$$

$R$ is the radius of the warp bubble and $\sigma$ describes the 'thickness' of the bubble's walls, e.g. in the large $\sigma$ limit the walls become very thin and the bubble's shape approximates a top hat function, with $f\left(r_{s}\right)=1$ when $r_{s}<R$ and zero elsewhere.

This spacetime is globally hyperbolic and contains no closed time-like curves; but it does contain curious geodesics, namely

$$
\begin{align*}
\frac{\mathrm{d} x^{\mu}}{\mathrm{d} t}=u^{\mu} & =\left(1, v_{s}(t) f\left(r_{s}(t)\right), 0,0\right)  \tag{3.4}\\
u_{\mu} & =(-1,0,0,0) \tag{3.5}
\end{align*}
$$

are first integrals of the geodesic equation. Observers outside the influence of $f\left(r_{s}(t)\right)$ will experience conventional flat spacetime dynamics, whilst the spaceship at the centre of the
bubble moves along a time-like curve with proper time equal to coordinate time. This ensures the spaceship does not experience time dilation during the journey.

Given all these wonderful properties and promises of interstellar engines it is important to consider the energy density needed to generate this. Namely

$$
\begin{equation*}
T^{\mu \nu} u_{\mu} u_{\nu}=T^{00}=\frac{G^{00}}{8 \pi}=-\frac{1}{8 \pi} \frac{v_{s}^{2}(t) \rho^{2}}{4 r_{s}^{2}(t)}\left(\frac{\mathrm{d} f\left(r_{s}\right)}{\mathrm{d} r_{s}}\right)^{2} \tag{3.6}
\end{equation*}
$$

where $\rho=\sqrt{y^{2}+z^{2}}$. This energy density is negative everywhere and is most prominent about the leading and trailing regions of the bubble [24]. This energy distribution clearly violates the weak energy condition; however can this spacetime be approximately constructed using exotic quantum matter that is known to violate classical energy conditions [22]?

### 3.1.3 Quantum energy conditions

In the section above, an interesting spacetime was discussed, whose properties appeal strongly to potential interstellar travel; however it is classically forbidden as it violates the weak energy condition. Despite this setback there are still efforts to produce the Alcubierre Warp metric led by quantum theory. The work of Epstein et al. [22] demonstrated that quantum fields were not constrained by classical energy conditions and yet it was evident that certain limits must exist to satisfy the second law of thermodynamics [50]. Since then there have been concerted efforts to derive quantum energy conditions to better understand the physical limits constraining goals such as constructing an Alcubierre Warp drive.

Two main approaches have been used to derive energy conditions, the first employs energy-entropy relations and generally results in energy restrictions dependent on underlying entropic structures; and the second analyses the local stress-energy tensor operator itself and attempts to minimise it, leading to the so called 'quantum inequalities'. An early thermodynamic approach sought to prevent a macroscopic violation of the second law of thermodynamics as a black hole absorbed a negative energy flux [50]. This lead to the inequality ( $1+1 \mathrm{D}$ )

$$
\begin{equation*}
|F| \lesssim \tau^{-2} \tag{3.7}
\end{equation*}
$$

where $F$ is the flux of the negative energy wave packet and $\tau$ is the characteristic time width of the wave packet. More recent formulations have used the quantum focussing conjecture [51] to derive a quantum null energy condition [52]. This relates the stress-energy tensor
contracted twice with a null vector with the second derivative of the field's von Neumann entropy about a surface enclosing the point of interest (see [52] for details). This energy condition is fairly strict and is broadly valid over several types of fields, however this broad validity and somewhat unwieldy mathematical form diminishes its practicality; therefore we do not use it in this thesis.

The second approach constrains the stress-energy tensor in a specific field configurations, e.g. scalar fields in curved spacetimes, explicitly stating the correlative contributions to the energy conditions by means of a Euclidean Green's function [24]. This approach leads to a non-uniformly sampled weak energy inequality (3+1 D)

$$
\begin{equation*}
\frac{\tau_{0}}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \tau \frac{\left\langle: \hat{T}_{\mu \nu}:\right\rangle u^{\mu} u^{\nu}}{\tau^{2}+\tau_{0}^{2}} \geq-\frac{3}{32 \pi^{2} \tau_{0}^{4}} \tag{3.8}
\end{equation*}
$$

where $\left\langle: \hat{T}_{\mu \nu}:\right\rangle$ is the expectation value of the renormalised stress-energy tensor, $u^{\mu}$ is the tangent to a geodesic observer's world line, $\tau$ is the observer's proper time and $\tau_{0}$ is a sampling time parameter. When considering a pulse of negative energy passing an observer, this inequality tells us that if $\tau_{0}$ is large the integral must be larger than a small negative number, i.e. there cannot be a wide and deep negative energy wave packet alone, there must be accompanying positive energy peaks to satisfy the inequality. However if the wave packet is narrow, then taking $\tau 0$ small (as $\tau 0 \rightarrow 0$ the sampling becomes a delta) suggests that as a negative energy packet is made narrower the energy well can be made polynomially deeper; alternatively the narrower the energy well the smaller the positive energy flares on either side of the packet [9].

A consequence arising from quantum inequalities like (3.8) is the quantum interest conjecture (QIC) [9]. The vacuum state of a quantum field is known to be the unique ground state of the field's Hamiltonian, and the quantum interest conjecture (QIC) relates the positive energy cost associated with creating a negative energy wavepacket. The conjecture states relates the distance between the negative energy wavepacket and its accompanying positive energy wavepacket; and the depth of the negative energy well to the size of the positive energy wavepacket, which consists of the magnitude of the negative energy contained in the negative energy wavepacket (due to non-negative eigenvalues of Hamiltonian) plus the additional energy cost involved in creating the negative energy wavepacket, i.e. the interest. This conjecture, which arises from the quantum inequalities, provides intuition for the physically possible energy distributions, e.g. as the negative energy well becomes deeper or wider then the positive energy cost goes up. Also the further apart a positive and negative energy wavepacket are the larger the positive energy wave packet must be.

There has already been catalogued a number of states that violate the weak energy condition and other classical energy conditions [53], including the 2 excitation state $|0\rangle+\epsilon\left|2_{\boldsymbol{k}}\right\rangle$ and the squeezed vacuum. Generally these states are considered in a monochrome setting and are therefore impractically non-local. A more local state that also violates the weak energy condition is the dynamical Casimir effect [54], using relativistically accelerating mirrors to produce a pulse of negative energy, followed closely by a pulse of positive energy. Current experiments have significant technical limitations on potential dynamical Casimir effects. The method used in this chapter (QET) is a quantum information protocol that aims to produce violations of the weak energy condition using several detectors and linear interaction without resorting to currently unfeasible relativistic detector trajectories. In addition to violating the weak energy condition the QET protocol will be found to saturate the scaling relations of the quantum interest conjecture, an indicator of a near optimal protocol.

### 3.2 QET and creation of negative energy

Quantum Energy Teleportation (QET) is a quantum information protocol derived by Masahiro Hotta [23], inspired by quantum state teleportation [55], tasked with transferring energy from point A to point B more quickly than natural propagation of energy through the system. Traditional energy transportation involves injecting some energy into port A of a conduit (e.g. illuminating one end of an optic fibre), allowing time evolution to transport the energy to the output port B (e.g. photons travelling along the fibre) and finally energy extraction at port B (e.g. photodetector). In systems where energy propagation is much slower than information propagation the QET protocol allows for the extraction of energy at port $B$ before the energy has time to physically propagate from $A$ to $B$; instead entanglement and classical communication can be used, eliminating the need to wait for the energy to move from A to B. This can be particularly surprising if the system is initially in the ground state.

### 3.2.1 Minimal QET model

In order to unambiguously demonstrate QET, it is common to consider systems whose ground state is entangled, e.g. Klein-Gordon scalar fields or frustrated spin systems. The simplest example, presented here, consists of 2 interacting qubits.

## The model

Consider 2 qubits A and B with a Hamiltonian

$$
\begin{equation*}
\hat{H}=\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}}+V_{\mathrm{AB}}, \tag{3.9}
\end{equation*}
$$

where

$$
\begin{align*}
\hat{H}_{\mathrm{A}} & =h \hat{\sigma}_{z}^{\mathrm{A}}+f(h, k) \mathbb{I}  \tag{3.10}\\
\hat{H}_{\mathrm{B}} & =h \hat{\sigma}_{z}^{\mathrm{B}}+f(h, k) \mathbb{I}  \tag{3.11}\\
\hat{V}_{\mathrm{AB}} & =2 k \hat{\sigma}_{x}^{\mathrm{A}} \otimes \hat{\sigma}_{x}^{\mathrm{B}}+2 \frac{k^{2}}{h^{2}} f(h, k) \mathbb{I} \tag{3.12}
\end{align*}
$$

where $h$ and $k$ are positive constants and

$$
\begin{equation*}
f(h, k)=\frac{h^{2}}{\sqrt{h^{2}+k^{2}}} \tag{3.13}
\end{equation*}
$$

which has units of energy and has been chosen such that

$$
\begin{equation*}
\langle g| \hat{H}_{\mathrm{A}}|g\rangle=\langle g| \hat{H}_{\mathrm{B}}|g\rangle=\langle g| \hat{V}_{\mathrm{AB}}|g\rangle=\langle g| \hat{H}|g\rangle=0 . \tag{3.14}
\end{equation*}
$$

In particular note that $\hat{H} \geq 0$ and at the same time both $\hat{H}_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}}$ have a negative eigenvalue.

The ground state itself, given in the eigenbasis of $\hat{\sigma}_{z}^{\mathrm{A}} \otimes \hat{\sigma}_{z}^{\mathrm{B}}$ is

$$
\begin{equation*}
|g\rangle=\frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{f(h, k)}{h}}|+z\rangle_{\mathrm{A}}|+z\rangle_{\mathrm{B}}-\sqrt{1+\frac{f(h, k)}{h}}|-z\rangle_{\mathrm{A}}|-z\rangle_{\mathrm{B}}\right) . \tag{3.15}
\end{equation*}
$$

As required by QET, provided $k \neq 0,|g\rangle$ is an entangled state and therefore a prime candidate for a QET protocol.

## The protocol

The objective of the QET protocol is to extract energy from the system by a local operation acting on qubit B alone. Given that the system is initially in the ground state (3.15) this requires an injection of energy into the system with a local operation acting on qubit A alone. In this example the local operation will be a projective measurement (PVM) but it can be generalised to a positive operator value measure (POVM) [56], which becomes relevant for QET protocols in QFT (where field measurements are POVMs via UDW).

The procedure for the QET protocol involves

1. Alice performs a projective measurement of $\hat{\sigma}_{x}^{\mathrm{A}}$ with measurement result $\alpha= \pm 1$. This results in an injection of energy $E_{\mathrm{A}}$ into the system at qubit $A$.
2. The measurement result $\alpha$ is communicated to Bob, quickly enough so that the system does not significantly time evolve $(t \ll 1 / k$, although here the communication time is treated as instantaneous).
3. Using Alice's measurement outcome, Bob performs a local unitary operation $\left(\hat{U}_{\mathrm{B}}(\alpha)\right)$ on qubit $B$ that extracts energy from the system.

The unitary operation Bob performs is given by

$$
\begin{equation*}
\hat{U}_{\mathrm{B}}(\alpha)=\cos \theta \mathbb{I}-\mathrm{i} \alpha \sin \theta \hat{\sigma}_{y}^{\mathrm{B}}, \tag{3.16}
\end{equation*}
$$

where $\alpha$ is Alice's measurement result and

$$
\begin{align*}
& \cos (2 \theta)=\frac{h^{2}+2 k^{2}}{\sqrt{\left(h^{2}+2 k^{2}\right)^{2}+h^{2} k^{2}}}  \tag{3.17}\\
& \sin (2 \theta)=\frac{h k}{\sqrt{\left(h^{2}+2 k^{2}\right)^{2}+h^{2} k^{2}}} . \tag{3.18}
\end{align*}
$$

## Following the energy

Alice's projective measurement of qubit A can be described by the projector

$$
\begin{equation*}
\hat{P}_{\mathrm{A}}(\alpha)=\frac{1}{2}\left(1+\alpha \hat{\sigma}_{x}^{\mathrm{A}}\right), \tag{3.19}
\end{equation*}
$$

and the post-measurement state with measurement result $\alpha= \pm 1$ becomes

$$
\begin{equation*}
\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle=\frac{1}{\sqrt{p_{\mathrm{A}}(\alpha)}} \hat{P}_{\mathrm{A}}(\alpha)|g\rangle \tag{3.20}
\end{equation*}
$$

with $p_{\mathrm{A}}(\alpha)=\langle g| \hat{P}_{\mathrm{A}}(\alpha)|g\rangle$ being the probability of the measurement result $\alpha$.
As a result of Alice's PVM, the $\alpha$ post-selected energy expectation values become

$$
\begin{align*}
\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{H}_{\mathrm{A}}\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle & =f(h, k)>0,  \tag{3.21}\\
\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{H}_{\mathrm{B}}\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle & =0,  \tag{3.22}\\
\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{V}_{\mathrm{AB}}\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle & =0 . \tag{3.23}
\end{align*}
$$

The projective measurement injects $E_{\mathrm{A}}=f(h, k)>0$ energy into the system (independent of the measurement result $\alpha$ ) and given the energy expectation values above, it can be said the energy is entirely localised about qubit A, with no effect on the energy content about qubit B.

Following the PVM, the measurement result is communicated to Bob who performs a local unitary operation $\left(\hat{U}_{\mathrm{B}}(\alpha)\right)$ on qubit B. An evaluation of the energy expectation then demonstrates the amount of energy Bob is able to extract from the system. By exploiting $\left[\hat{U}_{\mathrm{B}}(\alpha), \hat{H}_{\mathrm{A}}\right]=0$ the expectation value becomes

$$
\begin{align*}
\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H} \hat{U}_{\mathrm{B}}(\alpha)\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle & =\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{U}_{\mathrm{B}}(\alpha)\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle \\
& +\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\left(\hat{H}_{\mathrm{B}}+\hat{V}_{\mathrm{AB}}\right) \hat{U}_{\mathrm{B}}(\alpha)\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle,  \tag{3.24}\\
& =E_{\mathrm{A}}+\left\langle\psi_{\mathrm{PM}}(\alpha)\right| \hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\left(\hat{H}_{\mathrm{B}}+\hat{V}_{\mathrm{AB}}\right) \hat{U}_{\mathrm{B}}(\alpha)\left|\psi_{\mathrm{PM}}(\alpha)\right\rangle  \tag{3.25}\\
& =E_{\mathrm{A}}-\frac{h k \sin (2 \theta)+\left(h^{2}+2 k^{2}\right)(1-\cos (2 \theta))}{\sqrt{h^{2}+k^{2}}}  \tag{3.26}\\
& =E_{\mathrm{A}}-E_{\mathrm{B}} .
\end{align*}
$$

By using the values of $\theta$ from (3.17) and (3.18) the energy expectation above is minimised. From this protocol Bob has extracted

$$
\begin{equation*}
E_{\mathrm{B}}=\frac{1}{\sqrt{h^{2}+k^{2}}}\left(h k \sin (2 \theta)+\left(h^{2}+2 k^{2}\right)(1-\cos (2 \theta))\right), \tag{3.27}
\end{equation*}
$$

where $E_{\mathrm{B}}>0$ by exploiting the information supplied by Alice's measurement and without the need for any system evolution or energy propagation.

It should be noted that the derivation above has implicitly assumed that expectation values are taken with a fixed post-selected $\alpha$ value. However, since the energy expectations $E_{\mathrm{A}}$ and $E_{\mathrm{B}}$ are independent of the value of $\alpha$ then an ensemble of identical setups can also be used to equal effect.

### 3.2.2 General theory of QET

In the previous section QET was introduced by way of the minimal QET model, which consisted of 2 qubits and involved an injection of energy at point A followed by an extraction
of energy at point B without the need to wait for physical energy propagation. QET has been used in several different contexts, including a quantum thermodynamic protocol to improve heat bath algorithmic cooling techniques in systems of $n$ qubits [57], all the way to relativistic scalar fields as a demonstration of stimulated Hawking radiation [58]. Broadly speaking, QET can be exploited in any system provided sufficient entanglement exists; usually this is taken as ground state entanglement (although not always [59]).

In [60] a general 1D discrete chain model with nearest neighbour interactions is used to describe the requirements of a general theory of QET. The local energy density operator is defined as

$$
\begin{equation*}
\hat{T}_{n}=\hat{X}_{n}+\sum_{j}\left(\frac{1}{2} g_{n-1 / 2, j} \hat{Y}_{n-1, j} \hat{Y}_{n, j}+\frac{1}{2} g_{n+1 / 2, j} \hat{Y}_{n, j} \hat{Y}_{n+1, j}\right), \tag{3.28}
\end{equation*}
$$

where $\hat{X}_{n}$ and $\hat{Y}_{n, j}$ are local Hermitian operators for subsystem $n$ and $g_{n \pm 1 / 2, j}$ are real coupling constants for the nearest neighbour interactions. $j$ is the index enumerating the different allowable nearest neighbour interactions. The total Hamiltonian is

$$
\begin{equation*}
\hat{H}=\sum_{n} \hat{T}_{n} \tag{3.29}
\end{equation*}
$$

In the minimal model (§3.2.1) above, the communication between Alice and Bob was instantaneous. In general the 'protocol time', i.e. time for Alice's measurement, communication of measurement result and Bob's local unitary, is assumed to satisfy:

$$
\begin{equation*}
t \ll \frac{1}{\Delta E} \tag{3.30}
\end{equation*}
$$

where $\Delta E$ is the energy difference between the highest and lowest eigenvalue of $\hat{H}$. Under this restriction the protocol time is fast enough to safely ignore any internal system dynamics. This restriction can be relaxed by replacing $\Delta E$ with $E_{i n}$, the average energy injected into the system as a result of Alice's measurement.

For simplicity, identity terms are added to the local energy density operators such that the ground state expectations can be renormalised to zero, i.e.

$$
\begin{equation*}
\langle g| \hat{T}_{n}|g\rangle=0 \tag{3.31}
\end{equation*}
$$

where $|g\rangle$ is the ground state of the system, $\hat{H}|g\rangle=0$.

## Non-separable state

Consider the local energy density $\hat{T}_{n}$ and an arbitrary observable $\hat{O}_{m}$ where $n \neq m$. One characteristic of an entangled state or non-separable state is the relation of broken factorisation

$$
\begin{equation*}
\langle g| \hat{T}_{n} \hat{O}_{m}|g\rangle \neq\langle g| \hat{T}_{n}|g\rangle\langle g| \hat{O}_{m}|g\rangle . \tag{3.32}
\end{equation*}
$$

If (3.32) is satisfied for all $n \neq m$ in the discrete chain, then the ground state is entangled, i.e. it cannot be written as a tensor product of local ground states.

One consequence of (3.32) is that $|g\rangle$ is not an eigenstate of $\hat{T}_{n}$. If $|g\rangle$ is assumed to be an eigenstate of $\hat{T}_{n}$ (with eigenvalue $\lambda_{g}=0$, given (3.31)) then

$$
\begin{align*}
\langle g| \hat{T}_{n} \hat{O}_{m}|g\rangle & =\lambda_{g}^{*}\langle g| \hat{O}_{m}|g\rangle  \tag{3.33}\\
& =0 \tag{3.34}
\end{align*}
$$

but the right hand side of (3.32) is also zero (a contradiction). Therefore if (3.32) is to be satisfied then $|g\rangle$ cannot be an eigenstate of $\hat{T}_{n}$.

## Negative energy density

If $|g\rangle$ is entangled (3.32), then $\hat{T}_{n}$ must have at least 1 positive eigenvalue and 1 negative eigenvalue.

As stated above, if $|g\rangle$ is entangled then it cannot be an eigenstate of $\hat{T}_{n}$. Using its Hermiticity $\hat{T}_{n}$ is eigendecomposed

$$
\begin{equation*}
\hat{T}_{n}=\sum_{\eta} \varepsilon_{\eta}|\eta\rangle\langle\eta|, \tag{3.35}
\end{equation*}
$$

where $\varepsilon_{\eta}$ are real and $\{|\eta\rangle\}$ form an orthonormal basis. Writing out (3.31) with this eigendecomposition

$$
\begin{align*}
\langle g| \hat{T}_{n}|g\rangle & =\sum_{\eta} \varepsilon_{\eta}|\langle\eta \mid g\rangle|^{2},  \tag{3.36}\\
& =0 \tag{3.37}
\end{align*}
$$

where $\langle\eta \mid g\rangle \neq 0$ for at least 2 different values of $\eta$ (otherwise $|g\rangle$ would be an eigenstate of $\hat{T}_{n}$ ).

Since $|\langle\eta \mid g\rangle|^{2} \geq 0$ and add up to 1 this implies that at least 1 eigenvalue $\varepsilon_{\eta}<0$ and $\varepsilon_{\eta^{\prime}}>0$ in order to obtain the expectation value (3.31). Therefore if the ground state is entangled then necessarily $\hat{T}_{n}$ must have at least 1 negative and 1 positive eigenvalue. Note however that for any state with a negative local energy density at site $n:\langle\xi| \hat{T}_{n}|\xi\rangle<0$ a corresponding positive energy density exists elsewhere on the chain such that $\langle\xi| \hat{H}|\xi\rangle>0$.

## General QET protocol

The system described above is ripe for QET, an entangled ground state with local energy density operators capable of negative energy densities.

The protocol begins with Alice performing a POVM [56] on the spin subsystem at site $n_{A}$. This POVM is $\left\{\hat{M}_{\mathrm{A}}(\alpha)\right\}$ where

$$
\begin{equation*}
\sum_{\mu} \hat{M}_{\mathrm{A}}^{\dagger}(\mu) \hat{M}_{\mathrm{A}}(\mu)=\mathbb{I} \tag{3.38}
\end{equation*}
$$

The probability of measurement is

$$
\begin{equation*}
p_{\mathrm{A}}(\alpha)=\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle, \tag{3.39}
\end{equation*}
$$

and the post measurement state becomes

$$
\begin{equation*}
|A(\alpha)\rangle=\frac{1}{\sqrt{p_{\mathrm{A}}(\alpha)}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle . \tag{3.40}
\end{equation*}
$$

In order to compute the average energy injected into the system by Alice, let $\hat{H}_{\mathrm{A}}=\hat{T}_{\mathrm{A}-1}+\hat{T}_{\mathrm{A}}+\hat{T}_{\mathrm{A}+1}$ such that $\hat{H}=\hat{H}_{\mathrm{A}}+\hat{H}_{\overline{\mathrm{A}}}$, then

$$
\begin{align*}
E_{\mathrm{A}} & =\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle,  \tag{3.41}\\
& =\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle+\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H}_{\overline{\mathrm{A}}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle,  \tag{3.42}\\
& =\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle+\langle g| \sum_{\alpha} \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha) \hat{H}_{\overline{\mathrm{A}}}|g\rangle,  \tag{3.43}\\
& =\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle+\langle g| \mathbb{I} \hat{H}_{\overline{\mathrm{A}}}|g\rangle,  \tag{3.44}\\
& =\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle, \tag{3.45}
\end{align*}
$$

where $\hat{H}_{\overline{\mathrm{A}}}$ commutes with $\hat{M}_{\mathrm{A}}(\alpha)$ as the operators act on different subsystems and the second term vanishes since $\langle g| \hat{T}_{n}|g\rangle=0$ for all $n$. Physically it is clear that $E_{\mathrm{A}} \geq 0$ and equal to zero only if the POVM is trivial.

Following Alice's measurement, the result $\alpha$ is quickly sent to Bob (3.30) (so that free dynamics can be ignored) and Bob uses this result to perform a unitary operation

$$
\begin{equation*}
\hat{U}_{\mathrm{B}}(\alpha)=e^{-\mathrm{i} \alpha \theta \hat{G}_{\mathrm{B}}} \tag{3.46}
\end{equation*}
$$

where $G_{\mathrm{B}}$ is a local Hermitian operator acting on the spin subsystem at site $n_{\mathrm{B}}$, where $\left|n_{\mathrm{A}}-n_{\mathrm{B}}\right| \geq 5$ to ensure Alice and Bob are 'space-like separated'. $\theta$ is a real constant whose value is chosen to optimise the QET's energy transport.

The post measurement state of the system will be

$$
\begin{equation*}
\hat{\rho}_{\mathrm{QET}}=\sum_{\alpha} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \tag{3.47}
\end{equation*}
$$

In order to compute the energy extracted by Bob let $\hat{H}_{\mathrm{B}}=\hat{T}_{\mathrm{B}-1}+\hat{T}_{\mathrm{B}}+\hat{T}_{\mathrm{B}+1}$ such that $\hat{H}=\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}}+\hat{H}_{\overline{\mathrm{AB}}}$. The difference in the system's energy prior to Bob's interaction $\left(E_{\mathrm{A}}\right)$ and after Bob's interaction $\operatorname{Tr}\left[\hat{H} \hat{\rho}_{\mathrm{QET}}\right]$ then becomes the energy extracted by Bob:

$$
\begin{align*}
E_{\mathrm{B}} & =E_{\mathrm{A}}-\operatorname{Tr}\left[\hat{H} \hat{\rho}_{\mathrm{QET}}\right]  \tag{3.48}\\
& =E_{\mathrm{A}}-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle,  \tag{3.49}\\
& =E_{\mathrm{A}}-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{A}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{B}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle \\
& -\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\overline{\mathrm{AB}}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle,  \tag{3.50}\\
& =E_{\mathrm{A}}-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}(\alpha) \hat{H}_{\mathrm{A}} \hat{M}_{\mathrm{A}}(\alpha)|g\rangle-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{B}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle \\
& -\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha) \hat{H}_{\overline{\mathrm{AB}}}|g\rangle,  \tag{3.51}\\
& =E_{\mathrm{A}}-E_{\mathrm{A}}-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{B}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle-0,  \tag{3.52}\\
& =-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{B}} \hat{U}_{\mathrm{B}}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)|g\rangle,  \tag{3.53}\\
& =-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha) \hat{U}_{\mathrm{B}}^{\dagger}(\alpha) \hat{H}_{\mathrm{B}} \hat{U}_{\mathrm{B}}(\alpha)|g\rangle, \tag{3.54}
\end{align*}
$$

where the commutation of operators acting on subsystems A, B and $\overline{\mathrm{AB}}$ is used. For the QET protocol to work, Bob needs to extract energy, i.e. $E_{\mathrm{B}}>0$.

Now consider $\theta$ small so that

$$
\begin{equation*}
\hat{U}_{\mathrm{B}}(\alpha) \approx 1-\mathrm{i} \alpha \theta \hat{G}_{\mathrm{B}}, \tag{3.55}
\end{equation*}
$$

then

$$
\begin{align*}
E_{\mathrm{B}} & \approx-\sum_{\alpha}\langle g| \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)\left(\hat{H}_{\mathrm{B}}+\mathrm{i} \alpha \theta\left[\hat{G}_{\mathrm{B}}, \hat{H}_{\mathrm{B}}\right]\right)|g\rangle,  \tag{3.56}\\
& =\langle g| \sum_{\alpha} \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha) \hat{H}_{\mathrm{B}}|g\rangle+\mathrm{i} \theta \sum_{\alpha}\langle g| \alpha \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha)\left[\hat{H}_{\mathrm{B}}, \hat{G}_{\mathrm{B}}\right]|g\rangle,  \tag{3.57}\\
& =\theta \sum_{\alpha}\langle g| \alpha \hat{M}_{\mathrm{A}}^{\dagger}(\alpha) \hat{M}_{\mathrm{A}}(\alpha) \hat{O}_{\mathrm{B}}|g\rangle, \tag{3.58}
\end{align*}
$$

where the first term is zero by $\langle g| \hat{T}_{n}|g\rangle$ and (3.38); and the second term has been simplified with the Hermitian operator $\hat{O}_{\mathrm{B}}=\mathrm{i}\left[\hat{H}_{\mathrm{B}}, \hat{G}_{\mathrm{B}}\right]$. In order to achieve $E_{\mathrm{B}}>0$ then the sign of $\theta$ must be chosen accordingly, therefore allowing QET to occur. When considering more specific cases the first order expansion of $\hat{U}_{\mathrm{B}}$ will not be required and a more accurate optimisation scheme can be used. Furthermore, the optimal value of $\theta$ depends on the choice of $\hat{G}_{\mathrm{B}}$, which is arbitrary.

In summary, QET involves an entanglement breaking measurement by Alice, communication of the measurement result to Bob and a local unitary operation based on the measurement result.

## Ising example

The general theory of QET above demonstrated the necessity of entanglement for energy teleportation. Inspection of (3.58) suggests the energy extracted should be roughly proportional to the 2 point correlation function of the chain. We consider a superficial view of the critical Ising model with transverse magnetic field (detail to be found in [61])

$$
\begin{equation*}
\hat{T}_{n}=-J \hat{\sigma}_{n}^{z}-\frac{J}{2} \hat{\sigma}_{n}^{x}\left(\hat{\sigma}_{n+1}^{x}+\hat{\sigma}_{n-1}^{x}\right)-\epsilon \tag{3.59}
\end{equation*}
$$

where $J$ is a positive constant and $\epsilon$ is a real constant ensuring $\langle g| \hat{T}_{n}|g\rangle=0$. In this system Alice's input energy is

$$
\begin{equation*}
E_{\mathrm{A}}=\frac{6}{\pi} J, \tag{3.60}
\end{equation*}
$$

while Bob's output energy is evaluated as

$$
\begin{equation*}
E_{\mathrm{B}}=\frac{2 J}{\pi}\left[\sqrt{1+\left(\frac{\pi}{2} \Delta\left(\left|n_{\mathrm{A}}-n_{\mathrm{B}}\right|\right)\right)^{2}}-1\right], \tag{3.61}
\end{equation*}
$$

with $\Delta(n)$ defined in [56]. When $\left|n_{\mathrm{B}}-n_{\mathrm{A}}\right| \gg 1$ the expression takes the asymptotic form

$$
\begin{equation*}
E_{\mathrm{B}} \sim J \frac{\pi}{64} \sqrt{\epsilon} 2^{1 / 6} c^{-6}\left|n_{\mathrm{B}}-n_{\mathrm{A}}\right|^{-9 / 2} \tag{3.62}
\end{equation*}
$$

where the constant $c \approx 1.28$.
The energy extracted by Bob decays polynomially as Alice and Bob are separated, which is the same behaviour seen by the 2 point correlation functions in critical Ising models. This highlights not only the importance of entanglement but also how QET quantitatively uses correlations as a resource in its protocols. In the section below we will note this characteristic persists in QFT systems.

### 3.3 Using QET to create optimal violations of the weak energy condition

In the section above the general QET protocol was discussed, including the necessity of ground state entanglement and negative eigenvalues of local energy operators. In this section QET is applied to a scalar field with the goal of creating states that violate the weak energy condition. However, the relativistic nature of the massless scalar field and the Alice-Bob communication speed limit (c) violates the assumption (3.30). This relativistic restriction forces us to consider the free field evolution and its consequences, particularly with regards to the quantum interest conjecture [9].

### 3.3.1 QET in QFT

When considering quantum energy teleportation in relativistic quantum field theories a major issue arises from the fact that the energy injected into the system by Alice now propagates at the same speed as the information communication between Alice and Bob, namely the speed of light $c$. This means (3.30) no longer holds and when Bob attempts to extract energy from the system, the local energy density of the system around Bob is now excited above the ground state due to the arrival of the energy wave packet introduced by

Alice. In this case 'quantum energy teleportation' seems an inaccurate description, however by using Alice's measurement result Bob is able to extract energy more efficiently than he would otherwise, enough to overcome the positive energy wave packet and introduce weak energy condition violations.

## The protocol

As suggested in $\S 3.2 .2$ the QET protocol will involve POVMs of the field and communication between Alice and Bob. In the case of scalar fields it is useful to describe Alice's POVM as an Unruh-DeWitt detector interacting with the field. Several variants of QET in QFT exist [11, 41, 58, 59], including the choice of projectively measuring Alice's qubit and classically communicating the result to Bob, or communicating the qubit to Bob. In the sections below the latter is arbitrarily chosen, although it is later shown to be equivalent in the scenarios considered.

Following Alice's interaction, the measurement result needs to be communicated to Bob and during this time the field's evolution propagates the energy wave packet introduced by Alice outward, including towards Bob. If the field is massless then this energy propagates at the speed of light, such that as Bob receives the information he is also engulfed in a positive energy wave packet.

## Timing

When Bob receives Alice's measurement result he has the choice to implement his unitary immediately, when the QET's entanglement enhancement is strongest although his attempts to create negative energy densities would need to compete with the positive energy wave packet already present around Bob (illustrated in Fig. 3.1); or he can wait for Alice's wave packet to pass and complete the QET protocol when Bob's local energy density has returned to ground state levels. This has the advantage of operating without positive wave packet interference; however, the QET protocol's efficiency decreases with distance from Alice's wave packet. This interplay between Alice's wave packet and the QET efficiency coincides with the quantum weak energy inequality (3.8) and the quantum interest conjecture [9].

With this concept of QET in QFT, the remainder of this section is dedicated to introducing QET as a tool to sculpt exotic energy densities in scalar field states, i.e. states that violate classical energy conditions, e.g. negative energy densities.

### 3.3.2 Negative energy in $1+1 \mathrm{D}$

Due to its mathematical simplicity, a study of the $1+1 \mathrm{D}$ massless field helps develop an intuition regarding how to estimate parameters for optimal performance. Unlike other QET protocols discussed above, this protocol will be a LOQC protocol, i.e. Alice will communicate a qubit state to Bob instead of a projective measurement result. Given the relativistic nature of the QFT, any QET protocol must account for the light-speed propagation of the initial energy injection (i.e. (3.30) does not hold) and the detectors will be positioned on the surface of the light-cone, in an attempt to maximise the content of negative energy.

## Protocol

a) Initially Alice prepares her qubit in the state $\left|A_{0}\right\rangle$. The choice of this state will serve as a control for the output of the QET. Then Alice interacts with the field via the Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{I}, \mathrm{~A}}=\delta(t) \hat{\sigma}_{x} \int_{-\infty}^{\infty} \mathrm{d} x \lambda(x) \hat{\pi}(x) \tag{3.63}
\end{equation*}
$$

Note, this is in the Schrödinger picture.
b) Alice's qubit is communicated to Bob, e.g. via quantum state teleportation [55]. During this time the field experiences the usual free field time evolution.
c) In this particular protocol Bob interacts with the field as soon as he has received Alice's qubit. The communication time is labelled $T$ and is restricted by $c$, i.e. the centres of $\lambda(x)$ and $\mu(x)$ are $c T$ apart. Bob's interaction Hamiltonian then becomes

$$
\begin{equation*}
\hat{H}_{\mathrm{I}, \mathrm{~B}}=\delta(t-T) \hat{\sigma}_{z} \int_{-\infty}^{\infty} \mathrm{d} x \mu(x) \hat{\phi}(x) . \tag{3.64}
\end{equation*}
$$

We use a standard UDW and a derivative coupling as this yields improved performance of the QET [11]. The choice of qubit operators $\hat{\sigma}_{x}$ and $\hat{\sigma}_{z}$ is also important, as the QET protocol's yield is proportional to the commutator of the two operators, i.e. eventually the QET will depend on $\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle$. Therefore, if $\left|A_{0}\right\rangle$ is chosen poorly, this will result
in an injection of energy instead of an extraction of energy if $\left|A_{0}\right\rangle$ is chosen carefully. In addition to $\left|A_{0}\right\rangle$, the shape and positioning of the detector smearings $\lambda(x)$ and $\mu(x)$ will play a role.

## States and energy

Given the derivative coupling and UDW are linear in the field operators, these interaction Hamiltonians will generate unitaries that create coherent state displacements (2.64), controlled by the state of the qubit. Broadly speaking this protocol exploits these controlled displacements, creating cat states that approximate a squeezed state, which we hope will violate the weak energy condition.

Initially the system is in a product state $|0\rangle \otimes\left|A_{0}\right\rangle$ and following Alice's interaction

$$
\begin{align*}
& |\psi(t)\rangle=e^{-\mathrm{i} \hat{\mathrm{o}}_{x} \int \mathrm{~d} x \lambda(x) \hat{\pi}(x)}|0\rangle\left|A_{0}\right\rangle \\
& \quad=\exp \left(-\hat{\sigma}_{x} \int \mathrm{~d} k \sqrt{\frac{|k|}{4 \pi}}\left[\hat{a}_{k} \tilde{\lambda}^{*}(k)-\hat{a}_{k}^{\dagger} \tilde{\lambda}(k)\right]\right)|0\rangle\left|A_{0}\right\rangle \\
& \quad=\exp \left(\hat{\sigma}_{x} \int \mathrm{~d} k\left[\alpha_{k} \hat{a}_{k}^{\dagger}-\alpha_{k}^{*} \hat{a}_{k}\right]\right)|0\rangle\left|A_{0}\right\rangle \\
& \quad=|\boldsymbol{\alpha}(t)\rangle|+x\rangle\left\langle+x \mid A_{0}\right\rangle+|-\boldsymbol{\alpha}(t)\rangle|-x\rangle\left\langle-x \mid A_{0}\right\rangle \tag{3.65}
\end{align*}
$$

where $\tilde{\lambda}(k)$ is the Fourier transformation of $\lambda(x)$. As can be seen, the unitary operator is a coherent displacement, controlled by $\hat{\sigma}_{x}$. Furthermore $|\boldsymbol{\alpha}(t)\rangle$ corresponds to a tensor product of momentum coherent states

$$
\begin{align*}
|\boldsymbol{\alpha}(t)\rangle & =\bigotimes_{k \in \mathbb{R}}\left|\alpha_{k}(t)\right\rangle  \tag{3.66}\\
\alpha_{k}(t) & =e^{-\mathrm{i}|k| t} \sqrt{\frac{|k|}{4 \pi}} \int \mathrm{~d} x \lambda(x) e^{-\mathrm{i} k x} \tag{3.67}
\end{align*}
$$

Notice that since this interaction has a delta switching the energy gap of the detector is irrelevant. This gap only influences the free qubit evolution whilst Bob is waiting for Alice's communication, and given the free Hamiltonian for the qubit would be $\Omega \hat{\sigma}^{+} \hat{\sigma}^{-}$then the choice for Bob's interaction $\hat{\sigma}_{z}$ is blind to the phase introduced by free evolution. Therefore the free evolution of the qubit can be entirely ignored.

The scalar field's stress-energy tensor is given by

$$
\begin{equation*}
\hat{T}_{\mu \nu}=\partial_{\mu} \hat{\phi} \partial_{\nu} \hat{\phi}-\eta_{\mu \nu}\left(\partial_{\rho} \hat{\phi} \partial^{\rho} \hat{\phi}\right) \tag{3.68}
\end{equation*}
$$

which we renormalise by normal ordering $\left[: \hat{T}_{\mu \nu}(x):\right]$. The $T_{00}$ component of the tensor is the "energy density", which shall be the main measure in this section. For the state above (3.65) the energy density is given by

$$
\begin{equation*}
\left\langle: \hat{T}_{00}(x, t):\right\rangle=\frac{1}{4}\left(\lambda^{\prime}(x-t)\right)^{2}+\frac{1}{4}\left(\lambda^{\prime}(x+t)\right)^{2} . \tag{3.69}
\end{equation*}
$$

As expected the first QET step injects energy into the system and also produces no negative energy densities of any sort. Indeed, when Alice interacts locally with the ground state of the field she elevates its energy density locally.

Following Alice's interaction she sends her qubit (e.g. via quantum state teleportation) to Bob, who now uses this qubit to locally interact with the system. Following Bob's interaction $(t>T)$ the system is in the state

$$
\begin{align*}
|\psi(t)\rangle & =e^{\mathrm{i} \xi} \frac{\left\langle+x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\beta}(t)+\boldsymbol{\alpha}(t)\rangle|+z\rangle+e^{-\mathrm{i} \xi} \frac{\left\langle-x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\beta}(t)-\boldsymbol{\alpha}(t)\rangle|+z\rangle \\
& +e^{-\mathrm{i} \xi} \frac{\left\langle+x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\alpha}(t)-\boldsymbol{\beta}(t)\rangle|-z\rangle-e^{\mathrm{i} \xi} \frac{\left\langle-x \mid A_{0}\right\rangle}{\sqrt{2}}|-\boldsymbol{\beta}(t)-\boldsymbol{\alpha}(t)\rangle|-z\rangle \tag{3.70}
\end{align*}
$$

where

$$
\begin{equation*}
\beta_{k}(t)=-\frac{\mathrm{i} e^{-\mathrm{i}|k|(t-T)}}{\sqrt{4 \pi|k|}} \int \mathrm{d} x \mu(x) e^{-\mathrm{i} k x} \tag{3.71}
\end{equation*}
$$

$\xi$ is a phase that results from combining two coherent displacement operators $\hat{D}(\boldsymbol{\beta}) \hat{D}(\boldsymbol{\alpha})=$ $e^{\mathrm{i} \xi} \hat{D}(\boldsymbol{\beta}+\boldsymbol{\alpha})$. Its specific value is not important as it is cancelled out when taking expectation values.

When studying the state (3.70) it has some similarities with (3.65). Both states consist of coherent states entangled with the qubit, however (3.70) has additional superpositions for each state of the qubit. The superposition $|\boldsymbol{\beta}+\boldsymbol{\alpha}\rangle+|\boldsymbol{\beta}-\boldsymbol{\alpha}\rangle$ has the ability to interfere with itself, with the cross terms of expectation values capable of producing exotic results. For the purposes of negative energy production, the state looks like a displaced cat state and therefore (to low order) is an approximation of a squeezed state. The resulting energy distribution can be written down as

$$
\begin{align*}
& \left\langle: \hat{T}_{00}(x, t):\right\rangle=\frac{\left(\lambda^{\prime}(x-t)\right)^{2}}{4}+\frac{\left(\lambda^{\prime}(x+t)\right)^{2}}{4}+\frac{(\mu(x-(t-T)))^{2}}{4}+\frac{(\mu(x+(t-T)))^{2}}{4} \\
+ & \underbrace{\frac{e^{-2\|\alpha\|}\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle}{2 \pi} \mu(x-(t-T)) \int \mathrm{d} y \lambda^{\prime}(y) \frac{\text { P.V. }}{y-x+t}}_{\text {Right moving QET term }} \\
+ & \underbrace{\frac{e^{-2\|\alpha\|}\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle}{2 \pi} \mu(x+(t-T)) \int \mathrm{d} y \lambda^{\prime}(y) \frac{\text { P.V. }}{y-x-t}}_{\text {Left moving QET term }}, \tag{3.72}
\end{align*}
$$

where $\|\alpha\|=\int \mathrm{d} k\left|\alpha_{k}\right|^{2}$. The energy density expression has been separated to highlight the different physical contributions; namely Alice's derivative coupling provides causal contributions, dependent on the derivative of the smearing $\left(\lambda^{\prime}\right)$; Bob's linear UDW coupling means the energy density contribution depends on the smearing function $\mu(x)$ and not its derivatives. The final two terms, labelled 'left/right moving QET term' depend on $\mu$ and $\lambda^{\prime}$, whilst also depending on the initial choice of $\left|A_{0}\right\rangle$. As these terms are a consequence of cross-terms from a cat state they are exponentially suppressed by $e^{-2\|\alpha\|}$. All these restrictions must be considered when attempting to optimise the QET protocol. Figure 3.1 provides an illustration of the contributions of each of the terms in (3.72).

## Principal value integrals

When studying the QET terms in (3.72) the principal value integral stands out. The integral is suggesting that QET will be polynomially suppressed with the distance from the edge of the light-cone. However, we can simplify the principal value integral by means of a series to help procure intuition into this protocol. This principal value integral can be written as

$$
\begin{equation*}
\int \mathrm{d} y \lambda^{\prime}(y) \frac{\mathrm{P.V.}}{y-x+t}=\lim _{\varepsilon \rightarrow 0}\left[\int_{-\infty}^{x-t-\varepsilon}+\int_{x-t+\varepsilon}^{\infty}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t}, \tag{3.73}
\end{equation*}
$$

where abbreviated notation has been introduced

$$
\begin{equation*}
\left[\int_{a}^{b}+\int_{c}^{d}\right] \mathrm{d} x f(x):=\int_{a}^{b} \mathrm{~d} x f(x)+\int_{c}^{d} \mathrm{~d} x f(x) . \tag{3.74}
\end{equation*}
$$

Direct numerical evaluation of this principal value integral poses problems around the singular value of the denominator. Instead, an efficient approach involves subdividing the integration domain as follows: $(-\infty, x-t-a],(x-t-a, x-t-\varepsilon],[x-t+\varepsilon, x-t+$ $a),[x-t+a, \infty)$ where $a$ is some positive real number

$$
\begin{equation*}
\int \mathrm{d} y \lambda^{\prime}(y) \frac{\mathrm{P.V} .}{y-x+t}=\left[\int_{-\infty}^{x-t-a}+\int_{x-t+a}^{\infty}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t}+\lim _{\varepsilon \rightarrow 0}\left[\int_{x-t-a}^{x-t-\varepsilon}+\int_{x-t+\varepsilon}^{x-t+a}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t} \tag{3.75}
\end{equation*}
$$

In the integration domains $(x-t-a, x-t-\varepsilon],[x-t+\varepsilon, x-t+a)$ it is further assumed that $\lambda^{\prime}(y)$ can be expanded using Taylor's theorem around $y=x-t$ in order to deal with the pole:

$$
\begin{align*}
& \int \mathrm{d} y \lambda^{\prime}(y) \frac{\mathrm{P.V} .}{y-x+t}=\left[\int_{-\infty}^{x-t-a}+\int_{x-t+a}^{\infty}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t}  \tag{3.76}\\
+ & \lim _{\varepsilon \rightarrow 0}\left[\int_{x-t-a}^{x-t-\varepsilon}+\int_{x-t+\varepsilon}^{x-t+a}\right] \frac{\mathrm{d} y}{y-x+t}\left[\sum_{n=0}^{2} \lambda^{(n+1)}(x-t) \frac{(y-x+t)^{n}}{n!}+\lambda^{(4)}(\xi) \frac{(y-x+t)^{3}}{3!}\right], \tag{3.77}
\end{align*}
$$

where $\xi$, between $x-t$ and $y$, bounds the error.
Following the Taylor expansion around $y=x-t$, the sum and integral are commuted; and principal value integral is analytically evaluated, resulting in

$$
\begin{equation*}
\int \mathrm{d} y \lambda^{\prime}(y) \frac{\mathrm{P} . \mathrm{V} .}{y-x+t}=\left[\int_{-\infty}^{x-t-a}+\int_{x-t+a}^{\infty}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t}+2 a \lambda^{\prime \prime}(x-t)+O\left(\lambda^{(4)}(\xi)\right) \frac{a^{3}}{9} \tag{3.78}
\end{equation*}
$$

Note that this can be done due to the fact that the principal value integrals are not divergent in the first place (yet difficult to evaluate numerically without using these tools). Since $\left|\lambda^{(4)}(\xi)\right|$ is bounded on the domain of interest, generally $a$ can be chosen small enough (the particular value will depend on the particular choice of smearing) such that only the first term of the Taylor polynomial is needed:

$$
\begin{equation*}
\int \mathrm{d} y \lambda^{\prime}(y) \frac{\mathrm{P} . \mathrm{V} .}{y-x+t}=\lim _{\varepsilon \rightarrow 0}\left[\int_{-\infty}^{x-t-a}+\int_{x-t+a}^{\infty}\right] \mathrm{d} y \frac{\lambda^{\prime}(y)}{y-x+t}+2 a \lambda^{\prime \prime}(x-t)+\mathcal{O}\left(a^{3} \lambda^{(4)}(\xi)\right) \tag{3.79}
\end{equation*}
$$

This helps eliminate overflow and precision issues that arise when programming this numerical integral.

Whilst we introduced a Taylor polynomial to simplify the numerical task of evaluating the principal value integral, it does provide broad intuition as to the shape of $\lambda(x)$ in order to improve QET extraction. Thus in $1+1 \mathrm{D}$ QET, we broadly aim for $\lambda(x)$ to have as large a second derivative as possible to enlarge the QET terms whilst keeping the first derivative as small as possible to minimise Alice's initial energy injections. Despite these goals, 'smooth enough' functions have restrictions that limit the output of QET as well as the allowed size of negative energy packets. These QET limits are connected with the inability to violate quantum energy conditions [9, 24, 62].

## Examples of energy distributions

We used 3 different functional forms for $\lambda(x)$ and $\mu(x)$, with the energy distributions (3.72) are shown in Fig. 3.2.

The 3 different parameterised smooth smearing functions $f, g$ and $h$ used were, a smooth compactly supported function, previously used in [63]

$$
f(z, \sigma, \delta)= \begin{cases}S\left(\frac{\frac{\sigma}{2}+\pi \delta+z}{\delta}\right) & \text { if }-\pi \delta<z+\frac{\sigma}{2}<0  \tag{3.80}\\ 1 & \text { if }-\frac{\sigma}{2} \leq z \leq \frac{\sigma}{2} \\ S\left(\frac{\frac{\sigma}{2}+\pi \delta-z}{\delta}\right) & \text { if } 0<z-\frac{\sigma}{2}<\pi \delta \\ 0 & \text { otherwise }\end{cases}
$$

where $S(x)=\frac{1}{2}(1-\tanh \cot (x))$; and the two analytic functions

$$
\begin{align*}
& g(z, \delta)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{z^{2}}{2 \delta^{2}}}  \tag{3.81}\\
& h(z, \delta)=\frac{1}{\pi} \frac{1}{1+\left(\frac{z}{\delta}\right)^{2}} \tag{3.82}
\end{align*}
$$

where $z=x-x_{0}$. That is we used a smooth class infinity bump function $(f)$, a Gaussian $(g)$ and a Lorentzian $(h)$ to model the interaction domain of the detectors. We considered the following three cases:

1. $\lambda(x)=\lambda_{0} f\left(x, \sigma_{\mathrm{A}}, \delta_{\mathrm{A}}\right), \quad \mu(x)=\mu_{0} f\left(x-x_{\mathrm{B}}, \sigma_{\mathrm{B}}, \delta_{\mathrm{B}}\right)$


Figure 3.1: Progression of the $1+1$ D energy wave packets are shown with each time slice offset in the y axis. As time progresses the wave can be seen propagating in both directions until $t=T=15.29 R_{\mathrm{A}}$ when Bob's interaction introduces the negative energy density, initially obscured by a left moving positive energy contribution. Later times $(t=$ $\left.25.49 R_{\mathrm{A}}\right)$ show Bob's contribution splitting into left and right propagators leaving the desired significant negative energy density. Here a Lorentzian smearing was used, with Alice's characteristic size $R_{\mathrm{A}}$.

$$
\begin{array}{ll}
\text { 2. } \lambda(x)=\lambda_{0} g\left(x, \delta_{\mathrm{A}}\right), & \mu(x)=\mu_{0} g\left(x-x_{\mathrm{B}}, \delta_{\mathrm{B}}\right) \\
\text { 3. } \lambda(x)=\lambda_{0} h\left(x, \delta_{\mathrm{A}}\right), & \mu(x)=\mu_{0} h\left(x-x_{\mathrm{B}}, \delta_{\mathrm{B}}\right) .
\end{array}
$$

In all cases the parameters $x_{\mathrm{B}}, \sigma_{\mathrm{A}, \mathrm{B}}$ and $\delta_{\mathrm{A}, \mathrm{B}}$ were optimised to attain the maximum negative energy creation. For plotting purposes $R_{\mathrm{A}}=\sigma_{\mathrm{A}}+\delta_{\mathrm{A}}$, i.e. the characteristic size of the smearing.

The plots in Fig. 3.2 themselves describe (3.72) sometime after Bob's interaction, in an effort to distance the negative energy well from any left moving positive contributions from Bob's interaction (see Fig. 3.1). From inspection of the y-axis we see that the Lorentzian ( $h$ )
smearing produces the deepest negative energy well, as expected since $h$ has a pronounced second derivative at its maximum, ideal conditions for effective QET. Inspection of the energy profiles of Fig. 3.2 shows clear violations of the weak energy condition and the accompanying positive energy peaks ensure compliance with (3.7). In the section below we consider the $3+1 \mathrm{D}$ case, which introduces additional mathematical complications with the physical character mostly unchanged.


Figure 3.2: Energy density at $x=T+\Delta T$, where Bob's interaction took place at $t=T$. The additional time $\Delta T$ is so that Bob's left moving energy packet moves away. These 3 plots (from the left $f, g$ and $h$ smearings) support the notion of large second derivatives relative to the first derivative of Alice's smearing $\lambda$. Also note the differing energy packet forms of each smearing function. Certain smearings seem inherently more effective at maximising negative energy yields. Here $R_{\mathrm{A}}$ is the characteristic size of Alice's smearing.

### 3.3.3 Negative energy in $3+1$ D

In principle QET in $3+1 \mathrm{D}$ is very similar to $1+1 \mathrm{D}$. The protocol is very much the same and most of the equations follow straightforward higher dimensional generalisations. However, the main issue is the presence of 3 dimensional integrals and the complexities they bring, both analytical and numerical. For this reason in this section we will analytically describe the $3+1 \mathrm{D}$ QET protocol as generally as possible and eventually adopt the restriction of considering spherically symmetric qubit smearings in an effort to reduce the complexity of integrals.

As in the previous section (§3.3.2), the $3+1$ D QET protocol is a LOQC protocol faced with the same challenge of communication speed vs energy propagation speed. For
numerical reasons we will be using a non-local, spherically symmetric smearing for Bob. Despite the hypocritical use of a non-local smearing in a causal theory, we will show later that the LOQC results are identical to LOCC results where a non-local smearing is modelled by a distribution of local detectors. The LOQC protocol is presented here as it is mathematically neater and easier to present, although the reader is free to follow the derivation below with a LOCC mindset.

## States and energy

Following the protocol described in §3.3.2 Alice will interact her qubit first, send the qubit to Bob, who will ultimately interact with the field to extract energy. This process, described in a single interaction Hamiltonian (Schrödinger picture) is given by

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}=\delta(t) \hat{\sigma}_{x} \int \mathrm{~d}^{3} \boldsymbol{x} \lambda(\boldsymbol{x}) \hat{\pi}(\boldsymbol{x})+\delta(t-T) \hat{\sigma}_{z} \int \mathrm{~d}^{3} \boldsymbol{x} \mu(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x}) \tag{3.83}
\end{equation*}
$$

Note that causality is implicitly satisfied by the choice of $T$ and the particular choices of $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$.

By generalising the derivations from the $1+1 \mathrm{D}$ case, we see that the state of the field after Bob's interaction has the same form as (3.70):

$$
\begin{align*}
|\psi(t)\rangle & =e^{\mathrm{i} \xi} \frac{\left\langle+x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\beta}(t)+\boldsymbol{\alpha}(t)\rangle|+z\rangle+e^{-\mathrm{i} \xi} \frac{\left\langle-x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\beta}(t)-\boldsymbol{\alpha}(t)\rangle|+z\rangle \\
& +e^{-\mathrm{i} \xi} \frac{\left\langle+x \mid A_{0}\right\rangle}{\sqrt{2}}|\boldsymbol{\alpha}(t)-\boldsymbol{\beta}(t)\rangle|-z\rangle-e^{\mathrm{i} \xi} \frac{\left\langle-x \mid A_{0}\right\rangle}{\sqrt{2}}|-\boldsymbol{\beta}(t)-\boldsymbol{\alpha}(t)\rangle|-z\rangle, \tag{3.84}
\end{align*}
$$

where

$$
\begin{align*}
& \alpha_{\boldsymbol{k}}(t)=e^{-\mathrm{i}|\boldsymbol{k}| t} \sqrt{\frac{|\boldsymbol{k}|}{4 \pi}} \frac{1}{2 \pi} \int \mathrm{~d}^{3} \boldsymbol{x} \lambda(\boldsymbol{x}) e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}  \tag{3.85}\\
& \beta_{\boldsymbol{k}}(t)=-\frac{\mathrm{i} e^{-\mathrm{i}|\boldsymbol{k}|(t-T)}}{2 \pi \sqrt{4 \pi|\boldsymbol{k}|}} \int \mathrm{d}^{3} \boldsymbol{x} \mu(\boldsymbol{x}) e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tag{3.86}
\end{align*}
$$

Thus far all the results are straightforward higher dimension generalisations of the $1+1 \mathrm{D}$ results. However when it comes time to calculate the energy distributions of the state above, the higher dimensional integration difficulties becomes evident:

$$
\begin{align*}
& \langle\psi(T+\Delta T)|: \hat{T}_{\mu \nu}:(\boldsymbol{x})|\psi(T+\Delta T)\rangle=\frac{1}{4^{4} \pi^{6}}\{\underbrace{\left(I_{\mu}^{1} I_{\nu}^{1}-\frac{\eta_{\mu \nu}}{2} I_{\lambda}^{1} I^{1 \lambda}\right)}_{\text {Bob's field contribution }} \\
- & \underbrace{e^{-2\|\alpha\|}\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle\left(I_{\mu}^{1} I_{\nu}^{3}-\frac{\eta_{\mu \nu}}{2} I_{\lambda}^{1} I^{3 \lambda}\right)-e^{-2\|\alpha\|}\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle\left(I_{\mu}^{3} I_{\nu}^{1}-\frac{\eta_{\mu \nu}}{2} I_{\lambda}^{3} I^{1 \lambda}\right)}_{\text {QET terms }} \\
- & \underbrace{\left(I_{\mu}^{2} I_{\nu}^{2}-\frac{\eta_{\mu \nu}}{2} I_{\lambda}^{2} I^{2 \lambda}\right)}_{\text {Alice's field contribution }}\}, \tag{3.87}
\end{align*}
$$

where $\|\alpha\|=\int \mathrm{d}^{3} \boldsymbol{k}\left|\alpha_{\boldsymbol{k}}\right|^{2}, t=T+\Delta T$ and

$$
\begin{align*}
& I_{\mu}^{1}(t, \boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} \frac{k_{\mu}}{|\boldsymbol{k}|}\left(e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i} \Delta T)+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}+e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i} \Delta T)-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \mu(\boldsymbol{r}),  \tag{3.88}\\
& I_{\mu}^{2}(t, \boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} k_{\mu}\left(e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}-e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \lambda(\boldsymbol{r}),  \tag{3.89}\\
& I_{\mu}^{3}(t, \boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} k_{\mu}\left(e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}+e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \lambda(\boldsymbol{r}) . \tag{3.90}
\end{align*}
$$

$\epsilon$ is a soft UV cutoff we introduced for mathematical simplicity only, which helps guide contours during integration, and at the end of the derivation the limit $\epsilon \rightarrow 0^{+}$is taken. When considering $\lambda$ and $\mu$ as spherically symmetric, from a deep inspection of the equations above we find that a reasonable but rough intuition is that Alice's energy injection will be proportional to $\partial_{r}(r \lambda(r))$ and the QET terms will ultimately be principal value integrals and therefore proportional to $\partial_{r}^{2}(r \lambda(r))$. This, admittedly rough intuition helps guide the functional form of the smearing functions.

## Spherical simplification

The derivation and equations shown above are generally valid for any choice of qubit smearings. Due to the complexity associated with the $I_{\mu}^{i}$ terms above we simplify the integrals by imposing spherical symmetry on the detector smearings:

$$
\begin{align*}
& \lambda(\boldsymbol{x})=\lambda\left(\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{A}}\right|\right),  \tag{3.91}\\
& \mu(\boldsymbol{x})=\mu\left(\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{B}}\right|\right), \tag{3.92}
\end{align*}
$$

that is, spherically symmetric functions with no restrictions on the central position $\boldsymbol{x}_{\mathrm{A}, \mathrm{B}}$. Using this assumption of spherical symmetry, we will present results below using functions $\lambda(r)$ and $\mu(r)$ whose Fourier transforms are known or easy to determine (e.g. Gaussian or Lorentzian functions). Under these restrictions the integrals describing the $I_{\mu}^{i}$ functions can be reduced to a single dimension semi-infinite integral. Let

$$
\begin{align*}
& \boldsymbol{y}_{\mathrm{A}}=\boldsymbol{x}-\boldsymbol{x}_{\mathrm{A}},  \tag{3.93}\\
& \boldsymbol{y}_{\mathrm{B}}=\boldsymbol{x}-\boldsymbol{x}_{\mathrm{B}}, \tag{3.94}
\end{align*}
$$

be vectors from the observer $\boldsymbol{x}$ to the centre of the smearings $\boldsymbol{x}_{\mathrm{A}, \mathrm{B}}$. Furthermore define the Fourier transformations

$$
\begin{align*}
& \lambda(|\boldsymbol{k}|)=\int \mathrm{d}^{3} \boldsymbol{r} \lambda\left(\boldsymbol{r}+\boldsymbol{x}_{\mathrm{A}}\right) e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}},  \tag{3.95}\\
& \mu(|\boldsymbol{k}|)=\int \mathrm{d}^{3} \boldsymbol{r} \mu\left(\boldsymbol{r}+\boldsymbol{x}_{\mathrm{B}}\right) e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}, \tag{3.96}
\end{align*}
$$

where the Fourier transformations depend only on $|\boldsymbol{k}|$ due to the spherical symmetry of $\lambda$ and $\mu$ about $\boldsymbol{x}_{\mathrm{A}}$ and $\boldsymbol{x}_{\mathrm{B}}$ respectively. Then the functions $I_{\mu}^{i}(t, \boldsymbol{x})$ can be simplified to

$$
\begin{align*}
I_{0}^{1}(t, \boldsymbol{x}) & =\frac{8 \pi}{\left|\boldsymbol{y}_{\mathrm{B}}\right|} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{B}}\right|\right) \cos (|\boldsymbol{k}| \Delta T)|\boldsymbol{k}| \mu(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|},  \tag{3.97}\\
I_{0}^{2}(t, \boldsymbol{x}) & =-\frac{8 \pi i}{\left|\boldsymbol{y}_{\mathrm{A}}\right|_{0}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \sin (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}|^{2} \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|},  \tag{3.98}\\
I_{0}^{3}(t, \boldsymbol{x}) & =\frac{8 \pi}{\left|\boldsymbol{y}_{\mathrm{A}}\right|} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \cos (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}|^{2} \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|},  \tag{3.99}\\
\vec{I}^{1}(t, \boldsymbol{x}) & =-\frac{8 \pi \boldsymbol{y}_{B}}{\left|\boldsymbol{y}_{B}\right|^{2}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \cos \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{B}}\right|\right) \sin (|\boldsymbol{k}| \Delta T)|\boldsymbol{k}| \mu(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|} \\
& +\frac{8 \pi \boldsymbol{y}_{B}}{\left|\boldsymbol{y}_{\mathrm{B}}\right|^{3}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{B}}\right|\right) \sin (|\boldsymbol{k}| \Delta T) \mu(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|},  \tag{3.100}\\
\vec{I}^{2}(t, \boldsymbol{x}) & =-\frac{8 \pi i \boldsymbol{y}_{A}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{2}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \cos \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \cos (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}|^{2} \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|} \\
& +\frac{8 \pi i \boldsymbol{y}_{A}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{3}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \cos (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}| \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|},  \tag{3.101}\\
\vec{I}^{3}(t, \boldsymbol{x}) & =-\frac{8 \pi \boldsymbol{y}_{A}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{2}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \cos \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \sin (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}|^{2} \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|} \\
& +\frac{8 \pi \boldsymbol{y}_{A}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{3}} \int_{0}^{\infty} \mathrm{d}|\boldsymbol{k}| \sin \left(|\boldsymbol{k}|\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \sin (|\boldsymbol{k}|(\Delta T+T))|\boldsymbol{k}| \lambda(|\boldsymbol{k}|) e^{-2 \varepsilon|\boldsymbol{k}|}, \tag{3.102}
\end{align*}
$$

where the orientation of the vector $\vec{I}^{i}$ is parallel to $\boldsymbol{y}_{\mathrm{A}, \mathrm{B}}$. Inspection of the above equations is not particularly enlightening, however with some algebraic manipulations we can obtain expressions (appendix A) that strongly suggest an intuitive general behaviour for the energy distributions, e.g. Alice's initial wave packet $\left(I_{\mu}^{2}\right)$ will roughly depend on $\partial_{r}(r \lambda)$ and the QET extraction $\left(I_{\mu}^{3}\right)$ will roughly depend on $\partial_{r}^{2}(r \lambda)$, as mentioned above.

The integrals above ( $I_{\mu}^{3}$ in particular) cannot be analytically solved in general, leading to reliance on numerical integration; however, the spherical symmetry assumption has helped reduce the dimensionality of the numerical integral, improving computation time and reducing errors.

It is worth noting that the functions $I_{\mu}^{i}$ in (3.87) are linear in the qubit smearing functions and therefore slightly more sophisticated 'effective smearings' could be engineered as linear combinations of spheres. In a mathematical sense this would be a crude Riemann sum over a general smearing, however physically we could interpret it as a coherent collection of multiple detectors, thereby equating a mathematical simplification with a viable experimental procedure (see LOCC vs LOQC discussion below). Whilst we do not explore this particular possibility here, it is one of our future aims in engineering spacetimes [1].

## Method

The simplest scenario possible under this spherical simplification was to consider both smearings $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ to be centred about the same point in space, i.e. $\boldsymbol{x}_{\mathrm{A}}=\boldsymbol{x}_{\mathrm{B}}$. Alice's smearing would be a conventional spherically symmetric smearing, e.g. $\lambda(\boldsymbol{x}) \propto$ $\exp \left(-|\boldsymbol{x}|^{2} / 2 \sigma^{2}\right)$; whilst, in order to emulate the $1+1 \mathrm{D}$ setup, Bob's smearing would be a shell, e.g. $\mu(\boldsymbol{x}) \propto \exp \left(-\left(|\boldsymbol{x}|-r_{0}\right)^{2} / 2 \sigma^{2}\right)$, providing Bob with a spherically symmetric presence on the leading edge of Alice's injected energy wave.

In figure 3.3 the radial slice of Alice's and Bob's smearings are shown, illustrating Alice's traditional smearing described by a spherically symmetric 'solid' function. Conversely, Bob's smearing is also spherically symmetric but is mostly concentrated on a spherical shell and is therefore unambiguously non-local. Later in this section we prove the equivalence between optimal LOCC and LOQC QET protocols; and we also prove a second equivalence between a non-local Bob under LOCC and a distribution of multiple local Bobs under LOCC. These two equivalences demonstrate how the results of this section can be physically justified and produced in terms of reasonable local and causal operations. In this section most of the derivations shown are LOQC as their derivations are simpler and less notationally tedious.

## $3+1$ D Gaussian smearing functions used (Normalised)



Figure 3.3: Radial slice of the normalised smearing functions (i.e. excluding relative interaction strength) used for $3+1$ D Gaussian smearing QET. With $r_{0}^{\mathrm{B}} \neq 0$ and $\boldsymbol{x}_{\mathrm{A}}=\boldsymbol{x}_{\mathrm{B}}, \mu(r)$ resembles a shell surrounding Alice.

For this protocol we use three different parameterised smooth functions $f, g$ and $h$, one of which $(f)$ is a smooth compactly supported function occasionally used for detectors [63],

$$
f(z, \sigma, \delta)= \begin{cases}S\left(\frac{\frac{\sigma}{2}+\pi \delta+z}{\delta}\right) & \text { if }-\pi \delta<z+\frac{\sigma}{2}<0  \tag{3.103}\\ 1 & \text { if }-\frac{\sigma}{2} \leq z \leq \frac{\sigma}{2} \\ S\left(\frac{\frac{\sigma}{2}+\pi \delta-z}{\delta}\right) & \text { if } 0<z-\frac{\sigma}{2}<\pi \delta \\ 0 & \text { otherwise }\end{cases}
$$

where $S(x)=\frac{1}{2}(1-\tanh \cot (x))$, and the two analytic functions

$$
\begin{align*}
& g(z, \delta)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{z^{2}}{2 \delta^{2}}}  \tag{3.104}\\
& h(z, \delta)=\frac{1}{\pi} \frac{1}{1+\left(\frac{z}{\delta}\right)^{2}} \tag{3.105}
\end{align*}
$$

We exploit these three different functions to generate the detector smearings, i.e. a smooth class infinity bump function $(f)$, a Gaussian $(g)$ and a Lorentzian ( $h$ ), in the hopes of gaining further intuition and possibly developing procedures for generating negative energy densities with QET.

Generally speaking, given $z_{\mathrm{A}}=\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{A}}\right|-r_{0}^{\mathrm{A}}, z_{\mathrm{B}}=\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{B}}\right|-r_{0}^{\mathrm{B}}$, the 3 scenarios for detector smearing are defined by

1. $\lambda(\boldsymbol{x})=\lambda_{0} f\left(z_{\mathrm{A}}, \sigma_{\mathrm{A}}, \delta_{\mathrm{A}}\right), \quad \mu(\boldsymbol{x})=\mu_{0} f\left(z_{\mathrm{B}}, \sigma_{\mathrm{B}}, \delta_{\mathrm{B}}\right)$
2. $\lambda(\boldsymbol{x})=\lambda_{0} g\left(z_{\mathrm{A}}, \delta_{\mathrm{A}}\right), \quad \mu(\boldsymbol{x})=\mu_{0} g\left(z_{\mathrm{B}}, \delta_{\mathrm{B}}\right)$
3. $\lambda(\boldsymbol{x})=\lambda_{0} h\left(z_{\mathrm{A}}, \delta_{\mathrm{A}}\right), \quad \mu(\boldsymbol{x})=\mu_{0} h\left(z_{\mathrm{B}}, \delta_{\mathrm{B}}\right)$.
$\sigma$ and $\delta$ are parameters describing the size and shape of the detector smearings; and $\lambda_{0}$ and $\mu_{0}$ describe the relative interaction strengths. In the results presented below we fix $\sigma$ and $\delta$; whereas $\lambda_{0}$ and $\mu_{0}$ are left undetermined so as to be optimised for improved QET efficiency.

In this section $\boldsymbol{x}_{\mathrm{A}}=\boldsymbol{x}_{\mathrm{B}}=\mathbf{0}$ and $r_{0}^{\mathrm{A}}=0$, i.e. $\lambda(\boldsymbol{x})$ is a solid smearing, whereas $r_{0}^{\mathrm{B}} \neq 0$ hence $\mu(\boldsymbol{x})$ is a shell. This is illustrated in Fig. 3.3. Computing the energy density of the field after QET for the 3 types of functions will help illustrate which of the function's characteristics are desirable for optimising QET efficiency.

## Results

By inspection of the spherical $I_{\mu}^{i}$ equations, the width of the smearing functions (parameterised by $\delta$ ) can be pulled out of the integrals (by adimensionalisation), allowing for additional free parameters to be isolated from numerical calculations, ultimately leading to a straightforward and analytic optimisation problem. The results presented here were determined by arbitrarily fixing $\delta_{\mathrm{A}}$ and $\sigma_{\mathrm{A}}$ (motivated by our intuition from $1+1 \mathrm{D}$ QET) then a combination of intuitive parameter estimation and optimisation for determining
parameters $\left(\delta_{\mathrm{B}}, \sigma_{\mathrm{B}}, r_{0}^{\mathrm{B}}, \lambda_{0}, \mu_{0}\right)$. This ensured reasonably optimal results without the need to run intensive and costly numerical optimisation algorithms.

Given the scalar field obeyed a $3+1 \mathrm{D}$ wave equation it is expected that after Alice perturbs the field with her measurement the out-going wave's energy density will decay according the inverse square law, whilst in-going waves will diverge as they reach the origin $(\boldsymbol{r}=\mathbf{0})$. This divergence is in the energy density only and physically such a divergence would require higher order interaction terms for a proper description; however the results presented here focus on the out-going waves where the QET protocol will be used.

The energy density for Gaussian smearings at different times is shown in Fig. 3.4, illustrating the $1 / r^{2}$ energy density decay as expected and the following Bob's interaction showing the presence of a negative energy well. Unfortunately this negative energy well is surrounded by significant amounts of positive energy, highlighting the main problem with using QET to create regions of negative energy in relativistic QETs, i.e. Alice's initial energy injection is very disruptive to Bob's negative energy ambitions. Whilst not shown in Fig. 3.4, if some time is allowed to pass then the positive energy contributions from Bob's interaction will divide into inward and outward moving, revealing slightly more negative energy.

In Fig. 3.5 the energy densities for the 3 types of smearings are plotted, in the radial axis and a contour plot to illustrate the shell-like nature of the propagating waves. In these plots some time has been allowed to pass since Bob's interaction, revealing the maximum negative energy density possible under these conditions. Despite our initial objective these results show that QET cannot be used to create an energy wave with a leading negative energy component, instead all negative energy wells are accompanied by positive energy bands on either side.

From our intuition learnt in $1+1 \mathrm{D}$ QET and by inspecting the general $I_{\mu}^{i}$ equations, placing Bob on the boundary of Alice's light-cone produces the optimal QET energy extraction by virtue of $\partial_{r}^{2}(r \lambda(r)) \neq 0$ at $r=x-t \approx 0$. Placing Bob behind the light-cone or superluminally in front would remove this second derivative contribution and the QET term in the energy density would then be roughly proportional to $\partial_{r}(r \lambda(r)) /(x-t)$, i.e. polynomially decaying away from the light-cone. In order to create an energy wave with leading negative energy (using QET) the only remaining possibility is to use a massive scalar field to slow the propagation of Alice's energy wave packet and place Bob in the light-cone but beyond Alice's 'sonic-cone'. However in this scenario the QET term then becomes exponentially decaying away from the sonic-cone, due to the exponential vs polynomial correlation lengths for massive vs massless fields, respectively.


Figure 3.4: $x$-axis slice of the progression of the $3+1 \mathrm{D}$ energy density wave at various times with Bob's interaction taking place at $t=T=18.85 R_{\mathrm{A}}$. The waves are 'in-going' and 'outgoing' with the spherical nature of these waves apparent by their decay with increasing $|x|$. Immediately following Bob's interaction there is a small amount of negative energy that is being partially obscured by Bob's in-going positive energy contribution. Here a Gaussian smearing was used. The energy distribution following the separation of in and out-going waves is shown in Fig. 3.5. Here $R_{\mathrm{A}}$ is the characteristic size of Alice's smearing.

## Scaling Law

Inspection of the various numerical plots generated under this protocol suggest a relation between the width of the negative energy well and its depth. The quantum interest conjecture [9] has a temporal scaling relation, $T^{3} \propto 1 / \Delta E$, i.e. a negative energy pulse with total negative energy $-\Delta E$ must be followed within time $T$ by a positive energy pulse with slightly more positive energy $(1+\epsilon) \Delta E$. While the quantum interest conjecture relates to the local energy observations over time, it can be also be interpreted to provide a relation between the width and depth of negative energy wells.

Consider (3.88),(3.89) and (3.90), the general 3+1 D $I_{\mu}^{i}$ definitions; and (3.87), the local energy density of the post QET field. Our objective is to scale the characteristic size of the detector smearings (e.g. $\delta$ and $\sigma$ ) and find scaling relations for the other free parameters in the model in order to obtain a polynomial scaling relation for the energy density. The first step is to ensure that $\|\alpha\|$ remains constant under scaling in order to avoid exponential suppression of the QET terms.

Let the scaling be described by the substitution $\lambda(\boldsymbol{x}) \rightarrow \Upsilon^{j} \lambda(\Upsilon \boldsymbol{x})$, where $\Upsilon$ is the scaling parameter and $j$ is unknown. Now consider

$$
\begin{equation*}
\|\alpha\|_{\Upsilon}=\frac{1}{(2 \pi)^{2}} \int \mathrm{~d}^{3} \boldsymbol{k} \int \mathrm{~d}^{3} \boldsymbol{x} \mathrm{~d}^{3} \boldsymbol{y} \frac{|\boldsymbol{k}|}{4 \pi} \Upsilon^{2 j} \lambda(\Upsilon \boldsymbol{x}) \lambda(\Upsilon \boldsymbol{y}) e^{-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{x}-\boldsymbol{y})}, \tag{3.106}
\end{equation*}
$$

and perform a change of variables $\tilde{\boldsymbol{x}}=\Upsilon \boldsymbol{x}, \tilde{\boldsymbol{y}}=\Upsilon \boldsymbol{y}, \Upsilon \tilde{\boldsymbol{k}}=\boldsymbol{k}$ :

$$
\begin{align*}
\|\alpha\|_{\Upsilon} & =\frac{1}{(2 \pi)^{2}} \int \mathrm{~d}^{3} \tilde{\boldsymbol{k}} \Upsilon^{3} \int \frac{\mathrm{~d}^{3} \tilde{\boldsymbol{x}}}{\Upsilon^{3}} \frac{\mathrm{~d}^{3} \tilde{\boldsymbol{y}}}{\Upsilon^{3}} \frac{\Upsilon \mid \tilde{\boldsymbol{k}}}{4 \pi} \Upsilon^{2 j} \lambda(\tilde{\boldsymbol{x}}) \lambda(\tilde{\boldsymbol{y}}) e^{-\mathrm{i} \tilde{\boldsymbol{k}} \cdot(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{y}})},  \tag{3.107}\\
& =\frac{\Upsilon^{2 j}}{\Upsilon^{2}}\|\alpha\|_{\Upsilon=1} . \tag{3.108}
\end{align*}
$$

Therefore by taking $j=1$ the exponential suppressor of the QET terms is left constant.
We also ask, can an arbitrarily large negative energy density be obtained by scaling alone? To proceed Bob's smearing must also be scaled $\mu(\boldsymbol{x}) \rightarrow \Upsilon^{\xi} \mu(\Upsilon \boldsymbol{x})$. This rescale
affects Bob's contributions (encoded in $I_{\mu}^{1}$ ) as follows

$$
\begin{align*}
\left.I_{\mu}^{1}\left(\Upsilon^{-1} t, \Upsilon^{-1} \boldsymbol{x}\right)\right|_{\Upsilon} & =\int \mathrm{d}^{3} \boldsymbol{r} \mathrm{~d}^{3} \boldsymbol{k} \frac{k_{\mu}}{|\boldsymbol{k}|}\left(e^{|\boldsymbol{k}|\left(-2 \epsilon+\mathrm{i} \Upsilon^{-1} \Delta T\right)+\mathrm{i} \boldsymbol{k} \cdot\left(\boldsymbol{r}-\Upsilon^{-1} \boldsymbol{x}\right)}\right. \\
& \left.+e^{|\boldsymbol{k}|\left(-2 \epsilon-\mathrm{i} \Upsilon^{-1} \Delta T\right)-\mathrm{i} \boldsymbol{k} \cdot\left(\boldsymbol{r}-\Upsilon^{-1} \boldsymbol{x}\right)}\right) \Upsilon^{\xi} \mu(\Upsilon \boldsymbol{r}),  \tag{3.109}\\
& =\int \frac{\mathrm{d}^{3} \tilde{\boldsymbol{r}}}{\Upsilon^{3}} \mathrm{~d}^{3} \tilde{\boldsymbol{k}} \Upsilon^{3} \frac{k_{\mu}}{|\boldsymbol{k}|}\left(e^{|\tilde{\boldsymbol{k}}|(-2 \Upsilon \epsilon+\mathrm{i} \Delta T)+\mathrm{i} \tilde{\boldsymbol{k}} \cdot(\tilde{\boldsymbol{r}}-\boldsymbol{x})}\right. \\
& \left.+e^{|\tilde{\boldsymbol{k}}|(-2 \Upsilon \epsilon-\mathrm{i} \Delta T)-\mathrm{i} \tilde{\boldsymbol{k}} \cdot(\tilde{\boldsymbol{r}}-\boldsymbol{x})}\right) \Upsilon^{\xi} \mu(\tilde{\boldsymbol{r}}),  \tag{3.110}\\
& =\left.\Upsilon^{\xi} I_{\mu}^{1}(t, \boldsymbol{x})\right|_{\Upsilon=1}, \tag{3.111}
\end{align*}
$$

where $\Delta T$ is the time that has transpired since Bob's interaction. Similarly
$\left.I_{\mu}^{2,3}\left(\Upsilon^{-1} t, \Upsilon^{-1} \boldsymbol{x}\right)\right|_{\Upsilon}=\left.\Upsilon^{2} I_{\mu}^{2,3}(t, \boldsymbol{x})\right|_{\Upsilon=1}$. Therefore after rescaling the smearings, $\boldsymbol{x}$ and $t$; the energy density becomes

$$
\begin{align*}
\left\langle\psi\left(\Upsilon^{-1} t\right)\right|: \hat{T}_{\mu \nu} & \left(\Upsilon^{-1} \boldsymbol{x}\right)\left|\psi\left(\Upsilon^{-1} t\right)\right\rangle=\frac{1}{4^{4} \pi^{6}}\left\{\Upsilon^{2 \xi}\left(I_{\mu}^{1} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{1, \lambda}}{2}\right)\right. \\
& -\Upsilon^{2+\xi}\left\langle A_{0}\right| \hat{\sigma}_{y}\left|A_{0}\right\rangle e^{-2\|\alpha\|}\left(\left(I_{\mu}^{1} I_{\nu}^{3}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{3, \lambda}}{2}\right)+\left(I_{\mu}^{3} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{3} I^{1, \lambda}}{2}\right)\right) \\
& \left.-\Upsilon^{4}\left(I_{\mu}^{2} I_{\nu}^{2}-\eta_{\mu \nu} \frac{I_{\lambda}^{2} I^{2, \lambda}}{2}\right)\right\} \tag{3.112}
\end{align*}
$$

where the $I_{\mu}^{i}$ terms on the RHS are evaluated at $(t, \boldsymbol{x})$. The goal of rescaling was to maximise the QET contributions (energy extraction) whilst minimising the energy injections of Bob and Alice. Consider the energy density when $\Upsilon \rightarrow \infty$, Bob's positive, Alice's positive and the QET's negative contributions will scale as $\Upsilon^{2 \xi}$, $\Upsilon^{4}$ and $\Upsilon^{2+\xi}$ respectively. In order to avoid the QET's contribution from being overwhelmed by Alice's energy injection $\xi \geq 2$; however if $\xi>2$ then the QET contributions are overwhelmed by Bob's contributions, therefore $\xi=2$ is the only choice available that preserves the negative contributions of QET under scaling. This implies that the energy density will scale as $\epsilon\left(\Upsilon^{-1} \boldsymbol{x}, \Upsilon^{-1} t\right)=\Upsilon^{4} \epsilon(\boldsymbol{x}, t)$.

Summarising the scalings we used

$$
\begin{align*}
& \lambda(\boldsymbol{r}) \rightarrow \Upsilon \lambda(\Upsilon \boldsymbol{r})  \tag{3.113}\\
& \mu(\boldsymbol{r}) \rightarrow \Upsilon^{2} \mu(\Upsilon \boldsymbol{r}) \tag{3.114}
\end{align*}
$$

When $\Upsilon>1$ these scalings compress the 'lab' where the QET protocol is executed ( $\mathrm{x} \rightarrow$ $\Upsilon^{-1} \mathrm{x}$ ) whilst scaling the positive peaks and negative troughs of the energy density by $\Upsilon^{4}$.

This consideration of the scaling law was motivated by our empirical observations of numerical results relating the width and depth of negative energy wells under the QET protocol. Careful inspection of the governing equations demonstrated that if narrower detector smearings are properly used then the width of a negative energy well decreases by a factor $\Upsilon$ whilst its depth increases by $\Upsilon^{4}$.

The results we have presented are for a QET protocol using spherically symmetric smearings for Alice and Bob in order to simplify numerical evaluations. However, in order to compare the scaling relations here with the quantum interest conjecture, we can consider instead a QET protocol consisting of a many Alices, forming a large plane and many Bobs, forming a parallel plane (like a UDW capacitor). Such a protocol would produce plane waves of negative and positive energy instead of spherical waves; and provided the planes are large enough then the $\Upsilon^{-1} x_{\mu}$ contraction under scaling would only affect the width of a negative energy well (dimension perpendicular to plane), leaving the dimensions parallel to the plane unchanged. Therefore a rough estimate of the negative energy (flux) contained locally in a well of depth $\Upsilon^{4} \epsilon$ and width $\Delta r=\Upsilon^{-1} d$ is $\Delta E \sim \Upsilon^{3} \epsilon d$. In particular this means $\Delta E \propto \frac{1}{\Delta r^{3}}$, which is entirely in agreement with the quantum interest conjecture, in particular the QET protocol's scaling relations saturates the limits imposed by quantum inequalities, demonstrating its efficacy as a generator of exotic stress-energy densities. It is also interesting to note that as $\Upsilon \rightarrow \infty$, the total allowed negative energy diverges, i.e. there is no limit to how much negative energy can be created, provided it is accompanied by an equally large positive energy contribution, like a gendarme and a prisoner.

Given the nature of the KG scalar field, it is well understood that creating regions of negative energy is best achieved by squeezing operations. In principle the QET protocol described above attempts to recreate this squeezing operation, using only linearly coupled detectors, which results in cat states, the best simple approximation of a squeezed state given linear couplings. Despite the linear limitations of the QET protocol these scaling laws demonstrate the protocol is already near the optimum of negative energy generation.

## LOCC vs LOQC

In this section we used a LOQC QET protocol, primarily due to the relative simplicity of the derivation, avoiding the need to resort to the typographic challenges of density matrices or mixed states. The LOQC protocol is particularly problematic given that the results presented require Bob's smearing function $\mu(\boldsymbol{x})$ to describe a highly non-local qubit. Whilst
the bulk of $\mu(\boldsymbol{x})$ lies within the causal cone of Alice's interaction, an 'effective' coherent, non-local qubit cannot be constructed from local detectors due to the restrictions imposed by the no-cloning theorem [64]. The only way an effective non-local coherent qubit can be constructed is to extract some classical information from Alice's qubit and send copies to each of these local detectors that make up Bob. In this subsection we prove a QET LOCC-LOQC equivalence and in the next subsection the non-local Bob vs multiple local Bob equivalence is proven.

The LOCC QET protocol is very similar to the LOQC (modelled by (3.83) with measurement and communication differences). After her interaction, Alice performs a projective measurement of her detector in the $\hat{\sigma}_{z}$ basis and sends her classical measurement result to the Bob. Bob prepares his qubit in the same $\hat{\sigma}_{z}$ eigenstate that Alice measured. Following Bob's interaction the resulting field state, dependent on Alice's measurement, becomes

$$
\begin{align*}
\left|\psi\left(T^{+}\right),+z\right\rangle & =\mathcal{N}_{+}\left\{e^{-\mathrm{i} T \Omega} e^{\mathrm{i} \xi}|\boldsymbol{\alpha}(T)+\boldsymbol{\beta}\rangle|+z\rangle\langle+z \mid+x\rangle\left\langle+x \mid A_{0}\right\rangle\right. \\
& \left.+e^{-\mathrm{i} T \Omega} e^{-\mathrm{i} \xi}|-\boldsymbol{\alpha}(T)+\boldsymbol{\beta}\rangle|+z\rangle\langle+z \mid-x\rangle\left\langle-x \mid A_{0}\right\rangle\right\},  \tag{3.115}\\
\left|\psi\left(T^{+}\right),-z\right\rangle & =\mathcal{N}_{-}\left\{e^{-\mathrm{i} \xi}|\boldsymbol{\alpha}(T)-\boldsymbol{\beta}\rangle|-z\rangle\langle-z \mid+x\rangle\left\langle+x \mid A_{0}\right\rangle\right. \\
& \left.+e^{\mathrm{i} \xi}|-\boldsymbol{\alpha}(T)-\boldsymbol{\beta}\rangle|-z\rangle\langle-z \mid-x\rangle\left\langle-x \mid A_{0}\right\rangle\right\}, \tag{3.116}
\end{align*}
$$

where $\mathcal{N}_{ \pm}$are the normalisation factors. Note the similarity between the equations above and (3.84). Using these states the expectation values of the stress-energy tensor can be computed $\left(\hat{T}_{\mu \nu}\right.$ and $I_{\mu}^{i}$ are evaluated at time $T+\Delta T$ and position $\left.\boldsymbol{x}\right)$

$$
\begin{align*}
& \left\langle\psi\left(T^{+}\right),+z\right|: \hat{T}_{\mu \nu}:\left|\psi\left(T^{+}\right),+z\right\rangle=\frac{\mathcal{N}_{+}^{2}}{4^{4} \pi^{6}}\left\{\left(I_{\mu}^{1} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{1, \lambda}}{2}\right)-\left(I_{\mu}^{2} I_{\nu}^{2}-\eta_{\mu \nu} \frac{I_{\lambda}^{2} I^{2, \lambda}}{2}\right)\right. \\
+ & \mathrm{i}\left(\left|\left\langle+x \mid A_{0}\right\rangle\right|^{2}-\left|\left\langle-x \mid A_{0}\right\rangle\right|^{2}\right)\left(\left(I_{\mu}^{1} I_{\nu}^{2}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{2, \lambda}}{2}\right)+\left(I_{\mu}^{2} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{2, \lambda}}{2}\right)\right) \\
+ & 2 \operatorname{Re}\left(\left\langle+x \mid A_{0}\right\rangle\left\langle A_{0} \mid-x\right\rangle\right) e^{-2\|\alpha\|}\left(-\left(I_{\mu}^{3} I_{\nu}^{3}-\eta_{\mu \nu} \frac{I_{\lambda}^{3} I^{3, \lambda}}{2}\right)+\left(I_{\mu}^{1} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{1, \lambda}}{2}\right)\right) \\
- & \left.2 \operatorname{Im}\left(\left\langle+x \mid A_{0}\right\rangle\left\langle A_{0} \mid-x\right\rangle\right) e^{-2\|\alpha\|}\left(\left(I_{\mu}^{1} I_{\nu}^{3}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{3, \lambda}}{2}\right)+\left(I_{\mu}^{3} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{3, \lambda}}{2}\right)\right)\right\}, \tag{3.117}
\end{align*}
$$

$$
\begin{align*}
& \left\langle\psi\left(T^{+}\right),-z\right|: \hat{T}_{\mu \nu}:|\psi(t),-z\rangle=\frac{\mathcal{N}_{-}^{2}}{4^{4} \pi^{6}}\left\{\left(I_{\mu}^{1} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{1, \lambda}}{2}\right)-\left(I_{\mu}^{2} I_{\nu}^{2}-\eta_{\mu \nu} \frac{I_{\lambda}^{2} I^{2, \lambda}}{2}\right)\right. \\
- & \mathrm{i}\left(\left|\left\langle+x \mid A_{0}\right\rangle\right|^{2}-\left|\left\langle-x \mid A_{0}\right\rangle\right|^{2}\right)\left(\left(I_{\mu}^{1} I_{\nu}^{2}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{2, \lambda}}{2}\right)+\left(I_{\mu}^{2} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{2, \lambda}}{2}\right)\right) \\
- & 2 \operatorname{Re}\left(\left\langle+x \mid A_{0}\right\rangle\left\langle A_{0} \mid-x\right\rangle\right) e^{-2\|\alpha\|}\left(-\left(I_{\mu}^{3} I_{\nu}^{3}-\eta_{\mu \nu} \frac{I_{\lambda}^{3} I^{3, \lambda}}{2}\right)+\left(I_{\mu}^{1} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{1, \lambda}}{2}\right)\right) \\
- & \left.2 \operatorname{Im}\left(\left\langle+x \mid A_{0}\right\rangle\left\langle A_{0} \mid-x\right\rangle\right) e^{-2\|\alpha\|}\left(\left(I_{\mu}^{1} I_{\nu}^{3}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{3, \lambda}}{2}\right)+\left(I_{\mu}^{3} I_{\nu}^{1}-\eta_{\mu \nu} \frac{I_{\lambda}^{1} I^{3, \lambda}}{2}\right)\right)\right\} . \tag{3.118}
\end{align*}
$$

Inspection of the equations above shows that if no post-selection is performed on Alice's measurement result and the post Bob interaction state is treated as mixed, then the LOQC stress-energy expectation value is recovered. However, if the results are post-selected and if $\left|A_{0}\right\rangle$ is an eigenstate of $\hat{\sigma}_{y}$ then the LOQC result will still be recovered.

This particular derivation shows that either an LOCC or LOQC protocol could have been used in this section with equal accuracy. Our final choice of LOQC was motivated primarily for aesthetic reasons and greater mathematical clarity.

## Non-local detector as distribution of local detectors

In the subsection above we demonstrated the equivalence between LOCC and LOQC for scalar field QET. In this subsection we justify the use of local detectors carefully distributed to model Bob's non-local detector. It is important to note that LOCC is important here to avoid no-cloning [64] complications.

Following Alice's interaction and subsequent projective measurement of the detector the field state becomes

$$
\begin{align*}
\left|\psi\left(T^{-}\right),+z\right\rangle & =\mathcal{N}_{+} e^{-\mathrm{i} T \Omega}\left\{|\boldsymbol{\alpha}(T)\rangle\langle+z \mid+x\rangle\left\langle+x \mid A_{0}\right\rangle+|-\boldsymbol{\alpha}(T)\rangle\langle+z \mid-x\rangle\left\langle-x \mid A_{0}\right\rangle\right\}  \tag{3.119}\\
\left|\psi\left(T^{-}\right),-z\right\rangle & =\mathcal{N}_{-}\left\{|\boldsymbol{\alpha}(T)\rangle\langle-z \mid+x\rangle\left\langle+x \mid A_{0}\right\rangle+|-\boldsymbol{\alpha}(T)\rangle\langle-z \mid-x\rangle\left\langle-x \mid A_{0}\right\rangle\right\} \tag{3.120}
\end{align*}
$$

Consider a modification of Bob's interaction term in (3.83), where Bob now consists of 2 detectors, of the form

$$
\begin{equation*}
\hat{H}_{\mathrm{I}, \mathrm{~B}}=\delta(t-T) \hat{\sigma}_{z}^{\mathrm{B}, 1} \int \mathrm{~d}^{3} \boldsymbol{x} \mu_{1}(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x})+\delta(t-T) \hat{\sigma}_{z}^{\mathrm{B}, 2} \int \mathrm{~d}^{3} \boldsymbol{x} \mu_{2}(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x}) \tag{3.121}
\end{equation*}
$$

where index $\{1,2\}$ labels Bob's 2 detectors and the original non-local smearing $\mu(\boldsymbol{x})=$ $\mu_{1}(\boldsymbol{x})+\mu_{2}(\boldsymbol{x})$. Following the LOCC QET protocol Bob prepares both qubits in the same $\hat{\sigma}_{z}$ state measured by Alice and interacts via the Hamiltonian described above. Given Bob's qubits are prepared in eigenstates of the interaction Hamiltonian, i.e.

$$
\begin{align*}
\hat{H}_{\mathrm{I}, \mathrm{~B}}\left| \pm z_{1}, \pm z_{2}\right\rangle & =\left(\delta(t-T)(-1)^{i} \int \mathrm{~d}^{3} \boldsymbol{x} \mu_{1}(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x})\right. \\
& \left.+\delta(t-T)(-1)^{i} \int \mathrm{~d}^{3} \boldsymbol{x} \mu_{2}(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x})\right)\left| \pm z_{1}, \pm z_{2}\right\rangle  \tag{3.122}\\
& =\delta(t-T)(-1)^{i} \int \mathrm{~d}^{3} \boldsymbol{x}\left(\mu_{1}(\boldsymbol{x})+\mu_{2}(\boldsymbol{x})\right) \hat{\phi}(\boldsymbol{x})\left| \pm z_{1}, \pm z_{2}\right\rangle  \tag{3.123}\\
& =\delta(t-T)(-1)^{i} \int \mathrm{~d}^{3} \boldsymbol{x} \mu(\boldsymbol{x}) \hat{\phi}(\boldsymbol{x})\left| \pm z_{1}, \pm z_{2}\right\rangle \tag{3.124}
\end{align*}
$$

where $i=0$ if Alice measured $+z$ and $i=1$ if Alice measured $-z$, i.e. Alice's measurement results acts as a controlling bit on Bob's interaction. Note that it is important Bob prepares his qubits in the eigenstate of the interaction term, i.e. $| \pm z\rangle$, that way the qubits are unchanged following the interaction and the interaction could have equivalently been described by a classically (bit) controlled operation. (3.124) is the Hamiltonian that generates a coherent state displacement of magnitude $\boldsymbol{\beta}$ (3.86), which when applied to the pre-Bob interaction field states above yield the same states as (3.115) and (3.116). By repeating this argument inductively, Bob's interaction Hamiltonian can be constructed from a distribution of detectors with local smearings $\mu_{\mathrm{I}}(\boldsymbol{x})$ and provided the smearings add up to the required non-local smearing $\mu(\boldsymbol{x})$ then the coherent displacement operator is the same, hence the final field state is the same; and subsequently the energy distributions for LOQC-QET, non-local LOCC-QET and local LOCC-QET are identical (provided $\left|A_{0}\right\rangle$ is an eigenstate of $\hat{\sigma}_{y}$, which holds true here).

### 3.4 Conclusions

In this chapter we described the quantum energy teleportation (QET) protocol, a protocol allowing energy to be transferred from Alice to Bob through a quantum system more quickly than the energy propagation time of the system. This protocol was demonstrated to rely on pre-existing entanglement, with the protocol's effectiveness dependent on the system's two point correlations.

When QET is implemented in relativistic QFTs it is no longer possible for Alice to communicate her measurement results to Bob more quickly than the energy propagation
time of the system, making 'teleportation' something of a misnomer. However the QET protocol allows Bob to enhance his energy extractions from the field, resulting in local violations of the weak energy condition. Therefore, despite a substantial energy wave packet arriving at Bob's location simultaneously with Alice's measurements, QET can be used effectively to generate negative energy densities.

In this chapter we explored QETs ability to generate negative energy density in $1+1 \mathrm{D}$ and $3+1 \mathrm{D}$, developing guidelines for generating energy density distributions, controlled by specific detector smearings. We also compared the resulting exotic energy distributions against the quantum interest conjecture [9] and found that, not only does the QET protocol generate violations of the weak energy condition, but also saturates the quantum interest conjecture's scaling relations, suggesting QET is a near optimum protocol for generating negative energy densities.

The nature of QET in a massless QFT produces negative energy wave packets surrounded by regions of positive energy density. Unlike the dynamical Casimir effect, massless QET cannot generate a wave packet with a leading negative energy pulse. One resolution to this problem would be to implement QET in a massive field, where communication is faster than energy propagation. This would allow Bob to create a leading negative energy wavepacket that would be followed by positive energy wavepackets from Alice. Despite this, QET has shown itself to be a relatively simple protocol to implement, requiring only linear qubits, unlike the dynamical Casimir effect that requires relativistically accelerating mirrors or squeezing operations that require non-linear crystals. This simplicity suggests that the QET protocol may serve as a fundamental quantum information operation, which if organised into networks of lattices may allow for the moulding of specific exotic energy distributions, ultimately with the hope of creating unusual spacetime geometries.


Figure 3.5: Energy density at $x=T+\Delta T$, where Bob's interaction took place at $t=T$. The additional time $\Delta T$ is to allow the in-going contribution $(r \approx 5)$ to move away from the out-going wave packet ( $r \approx 32 R_{\mathrm{A}}$ ), revealing the negative energy density that propagates radially outward. Unlike the $1+1 \mathrm{D}$ case the in-going and out-going wave packets will respectively increase and decay according to the inverse square law. As the in-going wave packet diverges this becomes a high energy scattering problem, which is beyond the scope of this thesis. The divergences at $r=0$ come as an artefact of the strict spherical symmetry of the problem and, furthermore, will not affect the out-going wave packet as this is a local field theory. However, if multiple sequential QET protocols are employed then the aftermath of this scattering may become important. In the out-going wave packet the negative energy well will then decay with time; however, the ratio between the depth of the negative energy well and its neighbouring positive energy peak will remain constant. Here a contour plot of the energy density for the $z=0$ surface is shown, accompanied by a radial plot. These three plots (from the left $f, g$ and $h$ smearings) have been truncated so that the divergent nature of the in-going wave at $r=0$ does not shadow the details of the out-going wave packet and its negative energy density. Here $R_{\mathrm{A}}$ is the characteristic size of Alice's smearing.

## Chapter 4

## Gauge issues in the light-matter interaction

### 4.1 Light-matter interaction with EM fields

### 4.1.1 Introduction to EM fields and gauge

In the introductory chapter to this thesis the Klein-Gordon scalar field model was introduced along with the Unruh-DeWitt detector model as a suitably descriptive and simple model for the light-matter interaction, especially when angular momentum exchange is not involved [27, 65]. Fundamentally, the prototypical example of light interactions with fermionic matter (e.g. electrons) should be modelled by QED with a quantum electromagnetic field and a Dirac field describing the electron. Outside of high-energy physics scenarios this theory is unwieldy. Instead the electron tends to be modelled by a first quantised, non-relativistic Schrödinger equation.

Classically, electromagnetism is described by the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, exerting a force on a classical electron $q \boldsymbol{E}$ and $q \boldsymbol{v} \times \boldsymbol{B}$ respectively. From Maxwell's equations the scalar and vector potentials for the fields can be defined

$$
\begin{align*}
& \boldsymbol{E}=-\nabla U-\frac{\partial \boldsymbol{A}}{\partial t},  \tag{4.1}\\
& \boldsymbol{B}=\nabla \times \boldsymbol{A} \tag{4.2}
\end{align*}
$$

As defined above the potentials $(U, \boldsymbol{A})$ are not unique and the class of potentials that
satisfy the definitions above are related via 'gauge transformations'

$$
\begin{align*}
& \tilde{\boldsymbol{A}}(\boldsymbol{x}, t)=\boldsymbol{A}(\boldsymbol{x}, t)-\nabla \chi(\boldsymbol{x}, t),  \tag{4.3}\\
& \tilde{U}(\boldsymbol{x}, t)=U(\boldsymbol{x}, t)+\dot{\chi}(\boldsymbol{x}, t), \tag{4.4}
\end{align*}
$$

where $\chi$ is an arbitrary, smooth enough function. Quantisation of the EM field involves 2nd quantising the potentials, instead of the fields [28]. Whilst $(U, \boldsymbol{A})$ and $(\tilde{U}, \tilde{\boldsymbol{A}})$ define the same physical $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, once the potential is quantised, changing gauge would require a quantum $\chi$ and constrained equations, which we do not cover here.

### 4.1.2 Light-matter interaction and gauge

The Hamiltonian operator that generates the time evolution of a quantised electron generally consists of $H=T+V$, kinetic energy and potential energy respectively. Classical Electromagnetism (EM) suggests $q U$ as a potential energy term. However there is ambiguity as to the gauge $U$ should take.

When considering a general Schrödinger equation

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \psi(\boldsymbol{x}, t)=\left\{\frac{\hat{\boldsymbol{p}}^{2}}{2 m_{e}}+V(\boldsymbol{x})\right\} \psi(\boldsymbol{x}, t) \tag{4.5}
\end{equation*}
$$

consider the local gauge transformation $\tilde{\psi}(t, \boldsymbol{x})=e^{-\mathrm{i} q \chi(\boldsymbol{x}, t)} \psi(\boldsymbol{x}, t)$, i.e. a spatially dependent phase factor. Then the Schrödinger equation becomes

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \tilde{\psi}(\boldsymbol{x}, t)=\left\{\frac{1}{2 m_{e}}(\hat{\boldsymbol{p}}+q \nabla \chi)+V(\boldsymbol{x})+q \dot{\chi}\right\} \tilde{\psi}(\boldsymbol{x}, t) . \tag{4.6}
\end{equation*}
$$

This suggests that given an EM-electron coupling of

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \psi(\boldsymbol{x}, t)=\left\{\frac{1}{2 m_{e}}(\hat{\boldsymbol{p}}-q \boldsymbol{A}(\boldsymbol{x}, t))^{2}+V(\boldsymbol{x})+q U(\boldsymbol{x}, t)\right\} \psi(\boldsymbol{x}, t) \tag{4.7}
\end{equation*}
$$

then the local gauge transformation on the wave function couples to the EM gauge transformation in a consistent fashion

$$
\begin{align*}
& \tilde{\psi}(t, \boldsymbol{x})=e^{-\mathrm{i} q \chi(\boldsymbol{x}, t)} \psi(\boldsymbol{x}, t),  \tag{4.8}\\
& \tilde{\boldsymbol{A}}(\boldsymbol{x}, t)=\boldsymbol{A}(\boldsymbol{x}, t)-\nabla \chi(\boldsymbol{x}, t),  \tag{4.9}\\
& \tilde{U}(\boldsymbol{x}, t)=U(\boldsymbol{x}, t)+\dot{\chi}(\boldsymbol{x}, t) . \tag{4.10}
\end{align*}
$$

Therefore by combining a classical notion of $q U$ for an electron's potential energy with the need for gauge invariance, an interacting equation of motion for light interacting with quantum matter is born. This model is known as the minimal coupling model (4.7) [15] and describes the interaction between a classical EM field and a first quantised electron in a potential $V$ with $m_{e}$ the mass of the electron.

Historically the minimal model has been misused by comparing results determined under different EM gauges whilst not performing the corresponding local gauge transformation [4]. This led to a misconception that a 'physical' gauge exists and can be determined by comparing light-matter experiments and theoretical predictions; e.g. when calculating a transition probability, e.g. $|1 s\rangle \rightarrow|2 p\rangle$, different EM gauges produce different results; in reality the failure to perform the wave function transformation means under different EM gauges the transition probability of different states are being compared. In practice these issues were sidestepped by means of a resonable approximation (dipole approximation), eliminating the need for EM potential in Schrödinger's equation by using a multipole expansion.

### 4.1.3 Multipole and dressing

When considering the interaction Hamiltonian between the EM field and the electron in (4.7) the ambiguities introduced by gauge freedom are problematic. A major step towards resolving this gauge issue was by Göppert-Mayer [66] in 1931 and its success throughout the 20th century has overshadowed the error causing the gauge dependent observables, i.e. the failure to couple the wave function transformation with the EM gauge. GöppertMayer's solution was to consider the Lagrangian from which (4.7) originates and to add a total time derivative. The change in form then allowed for multipole expansions to be taken, resulting in a Hamiltonian dependent only on physical observables.

Using a different derivation method, consider (4.7), where the EM field is in the Coulomb gauge $(U=0)$ and the EM field is still considered classical. By performing the gauge transformation [15]

$$
\begin{equation*}
\tilde{\psi}(\boldsymbol{x}, t)=e^{-\mathrm{i} q(\boldsymbol{A}(\boldsymbol{x}, t) \cdot \boldsymbol{x})} \psi(\boldsymbol{x}, t) \tag{4.11}
\end{equation*}
$$

then Schrödinger's equation takes the form

$$
\begin{equation*}
\mathrm{i} \frac{\partial \tilde{\psi}(\boldsymbol{x}, t)}{\partial t}=\left\{\frac{1}{2 m_{e}}\left(\hat{\boldsymbol{p}}+q\left[\left(x_{i} \nabla\right) A_{i}(\boldsymbol{x}, t)\right]\right)^{2}-q \boldsymbol{x} \cdot \boldsymbol{E}(\boldsymbol{x}, t)+V(r)\right\} \tilde{\psi}(\boldsymbol{x}, t) \tag{4.12}
\end{equation*}
$$

When the wavelength of light associated with $\boldsymbol{A}$ is much larger than the typical size of $\tilde{\psi}$ (e.g. if a Hydrogen atom with $R_{0} \approx 5.3 \times 10^{-2} \mathrm{~nm}$ is illuminated with UV light $\lambda \approx 91 \mathrm{~nm}$, i.e. ionising a 1 s electron) then $\boldsymbol{A}(\boldsymbol{x}) \approx \boldsymbol{A}(\mathbf{0})(\mathbf{0}=\langle\hat{\boldsymbol{x}}\rangle$ is the centre of the electron wave function) and therefore the $A_{i}$ terms above can be ignored, resulting in

$$
\begin{equation*}
\mathrm{i} \frac{\partial \tilde{\psi}(\boldsymbol{x}, t)}{\partial t}=\left\{\frac{\hat{\boldsymbol{p}}^{2}}{2 m_{e}}-q \boldsymbol{x} \cdot \boldsymbol{E}(\mathbf{0}, t)+V(r)\right\} \tilde{\psi}(\boldsymbol{x}, t) \tag{4.13}
\end{equation*}
$$

This Schrödinger equation has the distinction of EM gauge independence, with any predictions dependent on observable $\boldsymbol{E}$ fields. If the size $R_{0}$ of $\tilde{\psi}$ is not much smaller than the relevant EM wavelengths ( $R_{0} \nless \lambda_{0}$ ) then a power series of $\boldsymbol{A}$ may be required resulting in higher order multipole expansions. In this case a Power-Zienau-Woolley transformation [ $7,67,68,69]$ can be used, resulting in a (infinite) multipole expansion in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ only. This multipole approximation has many subtleties, especially when one considers a nucleus of finite mass, from which non-trivial dynamics arise for the centre of mass and relative motion degrees of freedom, including many gauge subtleties. This analysis can be found in [7]. In this thesis we take the nucleus' mass as infinite in order to avoid the complexities of the centre of mass degrees of freedom; focussing on the electronic orbital behaviour and presenting straightforward numerical results. The requirement $R_{0} \ll \lambda_{0}$ is sometimes called the dipole criterion, and when it is satisfied the multipole expansion can be truncated to consist only of $\boldsymbol{r} \cdot \boldsymbol{E}$. This truncation is known as the dipole approximation.

Despite the persistent ambiguity concerning the definition and exact local gauge of $\tilde{\psi}$, the dipole model has been of such success that experiments with Hydrogen-like atoms commonly use the electron orbital basis ( $|n, l, m\rangle$ : energy and angular momentum) to describe their electron wave functions. This success has also been the motivating force behind the Unruh-DeWitt detector as a good model for light-matter interaction [27, 65].

## When the dipole approximation is invalid?

As mentioned above the dipole model is only valid when the dipole approximation $R_{0} \ll \lambda_{0}$ is valid. Outside of this domain of validity a multipole power series is required, eliminating the main computational advantage of the dipole model. An example of this situation is when the EM field is quantised and therefore every wavelength undergoes vacuum fluctuations. This effectively eliminates the concept of dominant wavelength $\lambda_{0}$ and places major doubts as to whether the dipole model should be used in QFT scenarios. This chapter is focussed on addressing these doubts, but to do so requires a method of properly comparing predictions from different gauges.

The quest for gauge invariance begins with the tautological, gauge invariant 'physical observables'. The quantum theory of a single electron, the electron's momentum $\hat{\boldsymbol{p}}$ is an observable, whose expectation value has an irreproachably physical interpretation; however once electromagnetic fields and local gauge transformations are introduced the expectations of the usual momentum operator becomes gauge dependent, i.e.

$$
\begin{align*}
\langle\hat{\boldsymbol{p}}\rangle & =\int \mathrm{d}^{3} \boldsymbol{x} \tilde{\psi}^{*}(\boldsymbol{x})(-\mathrm{i} \nabla) \tilde{\psi}(\boldsymbol{x})  \tag{4.14}\\
& =\int \mathrm{d}^{3} \boldsymbol{x} \psi^{*}(\boldsymbol{x}) e^{\mathrm{i} q \chi(\boldsymbol{x}, t)}(-\mathrm{i} \nabla) \psi(\boldsymbol{x}) e^{-\mathrm{i} q \chi(\boldsymbol{x}, t)}  \tag{4.15}\\
& =\int \mathrm{d}^{3} \boldsymbol{x} \psi^{*}(\boldsymbol{x})(-\mathrm{i} \nabla-q \nabla \chi(\boldsymbol{x}, t)) \psi(\boldsymbol{x})  \tag{4.16}\\
& =\langle\hat{\boldsymbol{p}}-q \nabla \chi(\hat{\boldsymbol{x}}, t)\rangle \tag{4.17}
\end{align*}
$$

In order to resolve this issue Lamb et al. [4] introduced the notion of 'mechanical momentum', a physical observable whose operator is gauge dependent but with gauge independent expectations, given by $\hat{\boldsymbol{\pi}}=\hat{\boldsymbol{p}}-q \boldsymbol{A}(\boldsymbol{x}, t)$. Using this new definition Schrödinger's equation becomes

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \psi(\boldsymbol{x}, t)=\left\{\frac{1}{2 m_{e}} \hat{\boldsymbol{\pi}}^{2}(\boldsymbol{x}, t)+V(\boldsymbol{x})+q U(\boldsymbol{x}, t)\right\} \psi(\boldsymbol{x}, t) \tag{4.18}
\end{equation*}
$$

A consequence of gauge independence is the restriction of measurable attributes to those with a 'mechanical' or 'physical' observables. This usually means generalising existing observables by replacing any instances of $\hat{\boldsymbol{p}} \rightarrow \hat{\boldsymbol{\pi}}$. This is particularly important when considering the 'mechanical' Hamiltonian of the electron:

$$
\begin{equation*}
\hat{H}=\frac{1}{2 m_{e}} \hat{\boldsymbol{\pi}}^{2}(\boldsymbol{x}, t)+V(\boldsymbol{x}) \tag{4.19}
\end{equation*}
$$

The Hamiltonian above now has gauge independent expectation values and is the simplest generalisation of the usual standard, bounded electron Hamiltonian.

In order to compare physically equivalent states, whilst operating under different gauges, it becomes necessary to define the experiment in terms of 'physical' eigenstates of the 'mechanical' momentum. This would be the most logical way of defining a gauge invariant process. Due to the complexities associated with finding eigenstates of the 'mechanical' Hamiltonian a perturbation theory approach is used; resulting in so called 'dressed states':

$$
\begin{equation*}
\left|\tilde{\psi}_{t, l}\right\rangle=\sum_{k}\left(\delta_{l k}+\mathrm{i} q L_{l k}(t)\right)\left|\psi_{k}\right\rangle \tag{4.20}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{l k}(t)=-\frac{\left\langle\psi_{k}\right| \frac{\boldsymbol{A}(\hat{\boldsymbol{x}}, t) \cdot \hat{\boldsymbol{p}}+\hat{p} \cdot \boldsymbol{A}(\hat{\boldsymbol{x}}, t)}{2 m_{e}}\left|\psi_{l}\right\rangle}{\mathrm{i}\left(E_{l}-E_{k}\right)}, \tag{4.21}
\end{equation*}
$$

assuming the unperturbed $(q \rightarrow 0)$ eigenstates $\left|\psi_{k}\right\rangle$ are non-degenerate. This particular definition has the advantage of adhering closely to the notion of well defined energy and angular momentum states i.e. as $q \rightarrow 0,\left|\tilde{\psi}_{t, l}\right\rangle \rightarrow|n l m\rangle$.

Lamb et al. [4] demonstrated that by following this approach and by using pointlike atoms in classical EM fields (where the dipole approximation holds) the minimally coupled Hamiltonian yielded the same transition probabilities as the dipole Hamiltonian (4.13). However, to prove this, it was assumed in [4] that the atoms are point-like, again relying on an approximation that requires a dominant wavelength (or, rather, a range of dominant/relevant wavelengths) of the field to be much larger than the atomic size e.g., a coherent excitation of the field of peak wavelength much larger than the size of the atom, or a process of spontaneous emission where the gap of the atom has an associated wavelength again much larger than the atomic radius.

When considering a Hamiltonian (4.19) that has degenerate eigenstates when $q=0$, e.g. Hydrogen atom, the dressing operation needs some modifications. Strict adherence to perturbation theory would result in dressed states whose zeroth order terms are no longer eigenstates of $\hat{\boldsymbol{L}}^{2}$ or $\hat{L}_{z}$, i.e. not of the form $|n l m\rangle$, instead being linear combinations of $|n l m\rangle$ for a fixed $n$. In order to define dressed states close to $|n l m\rangle$ we choose a jury-rigged solution that adheres to the gauge invariant requirement whilst having an unperturbed component of the form $|n l m\rangle$, i.e.

$$
\begin{equation*}
\left|\tilde{\psi}_{t, l}\right\rangle=\sum_{k}\left(\delta_{l k}+\mathrm{i} q L_{l k}(t)\right)\left|\psi_{k}\right\rangle \tag{4.22}
\end{equation*}
$$

where

$$
L_{l k}(t)= \begin{cases}-\frac{\left\langle\psi_{k}\right| \frac{A(\hat{\boldsymbol{x}}, t) \cdot \hat{\boldsymbol{p}}+\hat{p} \cdot \boldsymbol{A}(\hat{\boldsymbol{x}}, t)}{2 m_{e}}\left|\psi_{l}\right\rangle}{\mathrm{i}\left(E_{l}-E_{k}\right)} & \text { if } E_{l} \neq E_{k},  \tag{4.23}\\ -\left\langle\psi_{k}\right| \int_{0}^{t} \mathrm{~d} s U(\hat{\boldsymbol{x}}, s)\left|\psi_{l}\right\rangle & \text { if } E_{l}=E_{k} .\end{cases}
$$

For completion this dressing becomes $L_{l k}(t)=0$ for the dipole model. Using this dressing provides a consistent way of comparing predictions from different gauges, without the need to directly locally gauge transform the electron wave function every time. This
dressing operation affects the preparation and measurement basis that can be used. The necessity of this complicated dressing is a reason why multipole models are favoured.

For the case of a quantised EM field the dressing coefficients $L_{l k}$ are slightly modified by replacing the field variables with field operators $A_{\mu} \rightarrow \hat{A}_{\mu}$.

## Validity of the dipole model

When the EM fields in (4.13) are quantised, the resulting Schrödinger equation describes an electron, whose energy levels correspond to a Hydrogen-like atom, which interacts with a single point in the EM field, i.e. $\hat{E}(\mathbf{0}, t)$. This interaction can be interpreted as a QFT interacting with a point-like detector. This situation produces serious divergences in the model (as seen in [5] for point-like UDW), which can be avoided by several different methods, most reasonably by replacing the point-like detector with a smeared detector. This equates to replacing $\hat{E}(\mathbf{0}, t)$ in (4.13) with $\hat{E}(\boldsymbol{x}, t)$ and then the smearings become e.g. functions of Hydrogen-like orbitals [27].

The evolution of the light-matter interaction model can be described as: originally modelled by an unapproximated minimal model (4.7), that was subject to the dipole approximation, resulting in the dipole model (4.13); and finally when the EM field is quantised the dipole model was broadened by introducing some spatial dependence $\hat{\boldsymbol{E}}(\boldsymbol{x}, t)$ in order to remove divergent results from the theory. The question raised in this chapter is whether this non-divergent dipole model predicts the same results as the original unapproximated minimal model and if so, for which particular physical scenarios. Using a mix of wave function and Dirac representations, consider the unapproximated minimal model

$$
\begin{equation*}
\mathrm{i} \frac{\partial \tilde{\psi}_{\mathrm{M}}(\boldsymbol{x}, t)}{\partial t}=\left\{\frac{1}{2 m_{e}}(\hat{\boldsymbol{p}}-q \hat{\boldsymbol{A}}(\boldsymbol{x}, t))^{2}+V(\boldsymbol{x})\right\} \tilde{\psi}_{\mathrm{M}}(\boldsymbol{x}, t) \tag{4.24}
\end{equation*}
$$

where $\hat{\boldsymbol{A}}$ is in the Coulomb gauge and $\tilde{\psi}_{\mathrm{m}}$ is the minimal model dressed state; and the generalised dipole model

$$
\begin{equation*}
\mathrm{i} \frac{\partial \tilde{\psi}_{\mathrm{D}}(\boldsymbol{x}, t)}{\partial t}=\left\{\frac{\hat{\boldsymbol{p}}^{2}}{2 m_{e}}-q \boldsymbol{x} \cdot \hat{\boldsymbol{E}}(\boldsymbol{x}, t)+V(\boldsymbol{x})\right\} \tilde{\psi}_{\mathrm{D}}(\boldsymbol{x}, t) \tag{4.25}
\end{equation*}
$$

where $\tilde{\psi}_{\mathrm{D}}$ is the dipole dressed state (note, the dipole model has trivial dressing). In the section, below we compare the various predictions such as transition probabilities and transition rates from both models in order to carefully determine the domain of validity for the dipole model, particularly as experiments continue to approach the limits of these domains.

### 4.2 Light-matter predictions, dipole vs minimal models

In the preceding section the gauge transformations of the electromagnetic field were expanded by coupling to an electron wave function's local gauge transformation. This general gauge freedom associated with light-matter interaction can be used to represent the lightmatter interaction as a multipole expansion, which can then be truncated via a dipole approximation for simplicity; however, as described above it is important that any gauge transformations be performed consistently, i.e. EM field and electron wave function. In this section we compare transition rates between 1s and 2p orbitals of a Hydrogen-like atom [7] as predicted by the dipole and minimal models, whilst carefully dressing the initial and final states to ensure a physically consistent comparison.

Our objective will be to explore the relative behaviour of the two predictions and determine when the dipole model may be accurately used. Our analysis will introduce new criteria for the validity of the dipole model, which compliments the current dipole criterion when dealing with quantised fields.

### 4.2.1 Time evolution and transition probabilities

By computing the transition probability $1 \mathrm{~s} \leftrightarrow 2 \mathrm{p}$ in various scenarios, additional criteria can be introduced for the proper use of the dipole model in RQI situations. This is particularly useful, given that all modes of a QFT experience fluctuations, thereby eliminating the concept of a dominant wavelength necessary for the usual dipole criterion.

## Setup

As mentioned in the previous section the dipole and minimal models can be compared fairly by using dressed states (4.22). Working in the Coulomb gauge the field operators are

$$
\begin{align*}
\hat{\boldsymbol{A}}(\boldsymbol{x}, t) & =\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_{\lambda}(\boldsymbol{k})\left(\hat{a}_{\lambda}(\boldsymbol{k}) e^{-\mathrm{i}(\omega t-\boldsymbol{k} \cdot \boldsymbol{x})}+\hat{a}_{\lambda}^{\dagger}(\boldsymbol{k}) e^{\mathrm{i}(\omega t-\boldsymbol{k} \cdot \boldsymbol{x})}\right),  \tag{4.26}\\
\boldsymbol{k} \cdot \boldsymbol{\epsilon}_{\lambda}(\boldsymbol{k}) & =0  \tag{4.27}\\
\hat{\boldsymbol{E}}(\boldsymbol{x}, t) & =-\frac{\partial}{\partial t} \hat{\boldsymbol{A}}(\boldsymbol{x}, t), \tag{4.28}
\end{align*}
$$

where $\hat{a}_{\lambda}$ are the usual EM creation and annihilation operators. The initial and final states of the detector, which ultimately will physically correspond to 1 s and 2 p respectively, are dressed (avoiding incorrect use of local gauge transformations of the electron wave function)

$$
\begin{equation*}
\left|\tilde{\psi}_{t, l}, \phi_{i}\right\rangle=\sum_{k}\left(\delta_{l k}+\mathrm{i} q \hat{L}_{l k}(t)\right)\left|\psi_{k}\right\rangle\left|\phi_{i}\right\rangle \tag{4.29}
\end{equation*}
$$

where $\left|\phi_{i}\right\rangle$ is the state of the EM field and $\hat{H}_{0}\left|\psi_{k}\right\rangle=E_{k}\left|\psi_{k}\right\rangle$ are the usual energy-angular momentum eigenstates of an electron in a Hydrogen-like atom (|nlm$\rangle)$. For the minimal case

$$
\hat{L}_{l k}(t)= \begin{cases}-\frac{\left\langle\psi_{k}\right| \frac{\hat{A} \hat{\boldsymbol{x}}, t) \cdot \hat{p}}{m_{e}}\left|\psi_{l}\right\rangle}{\mathrm{i}\left(E_{l}-E_{k}\right)} & \text { if } E_{l} \neq E_{k},  \tag{4.30}\\ 0 & \text { if } E_{l}=E_{k},\end{cases}
$$

since we are working in the Coulomb gauge $(U=0)$ and $\hat{L}_{l k}=0$ for the dipole case.
Given the states defined above the system is then subject to an interaction picture Hamiltonian

$$
\begin{align*}
\hat{H} & =\hat{H}_{0}+q \hat{H}_{\mathrm{I}}+O\left(q^{2}\right),  \tag{4.31}\\
\hat{H}_{0} & =\frac{1}{2 m_{e}} \hat{\boldsymbol{p}}^{2}+V(r),  \tag{4.32}\\
\hat{H}_{\mathrm{I}}^{\min } & =-\frac{1}{m_{e}} \hat{\boldsymbol{A}}(\boldsymbol{x}, t) \cdot \hat{\boldsymbol{p}},  \tag{4.33}\\
\hat{H}_{\mathrm{I}}^{\mathrm{dip}} & =-\hat{x} \cdot \hat{\boldsymbol{E}}(\boldsymbol{x}, t), \tag{4.34}
\end{align*}
$$

where the EM free field evolution is encoded into the field operators.

## The system state

The setup above presumes an ability to prepare the electron in a dressed state at time $t=0$ and to projectively measure the state at time $t=T$. Working to first order in perturbation theory the wave function of the system can be represented by

$$
\begin{equation*}
\left|\tilde{\psi}_{i}(t), \phi_{i}\right\rangle=\sum_{k}\left(\delta_{i k}+\mathrm{i} q \hat{K}_{i k}(t)\right) e^{-\mathrm{i} E_{k} t}\left|\psi_{k}\right\rangle\left|\phi_{i}\right\rangle \tag{4.35}
\end{equation*}
$$

i.e. $\left|\tilde{\psi}_{i}(t), \phi\right\rangle$ would be a time evolved representation of the dressed state corresponding to the undressed state $\left|\psi_{i}\right\rangle|\phi\rangle$. Note that $\hat{K}_{i k}(0)=\hat{L}_{i k}(0)$ and remember $\hat{K}_{i k}$ is an operator acting on the EM field's Hilbert space. Inserting (4.35) into Schrödinger's equation:

$$
\begin{align*}
\mathrm{i} \partial_{t}\left|\tilde{\psi}_{i}(t), \phi_{i}\right\rangle & =\sum_{k}\left(\delta_{i k}+\mathrm{i} q \hat{K}_{i k}(t)\right) E_{k} e^{-\mathrm{i} E_{k} t}\left|\psi_{k}\right\rangle\left|\phi_{i}\right\rangle+q \sum_{k}-\dot{\hat{K}}_{i k}(t) e^{-\mathrm{i} E_{k} t}\left|\psi_{k}\right\rangle\left|\phi_{i}\right\rangle \\
& =\left(\hat{H}_{0}+q \hat{H}_{1}\right)\left|\tilde{\psi}_{i}(t)\right\rangle\left|\phi_{i}\right\rangle \\
& =\sum_{k}\left(\delta_{i k}+\mathrm{i} q \hat{K}_{i k}(t)\right) E_{k} e^{-\mathrm{i} E_{k} t}\left|\psi_{k}\right\rangle\left|\phi_{i}\right\rangle+q \hat{H}_{1} e^{-\mathrm{i} E_{i} t}\left|\psi_{i}\right\rangle\left|\phi_{i}\right\rangle+O\left(q^{2}\right) \tag{4.36}
\end{align*}
$$

This leaves

$$
\begin{equation*}
\dot{\hat{K}}_{i k}(t)=-\left\langle\psi_{k}\right| \hat{H}_{1}\left|\psi_{i}\right\rangle e^{\mathrm{i}\left(E_{k}-E_{i}\right) t} \tag{4.37}
\end{equation*}
$$

Integration over time yields

$$
\begin{equation*}
\hat{K}_{i k}(T)=-\int_{0}^{T} \mathrm{~d} t\left\langle\psi_{k}\right| \hat{H}_{1}\left|\psi_{i}\right\rangle e^{\mathrm{i}\left(E_{k}-E_{i}\right) t}+\hat{L}_{i k}(0) \tag{4.38}
\end{equation*}
$$

By evaluating the coefficient $\hat{K}_{i k}(T)$ the state of a system, initially $(t=0)$ prepared in the dressed state $\left|\tilde{\psi}_{i}(0), \phi_{i}\right\rangle$, is well known at time $t=T$ (denoted as $\left.\left|\tilde{\psi}_{i}(T), \phi_{i}\right\rangle\right)$ and can now be compared to the measurement basis states $\left|\tilde{\psi}_{T, f}, \phi_{f}\right\rangle$.

## Transition probabilities

For future simplifications note $\hat{L}_{l k}^{\dagger}=\hat{L}_{k l}$. The inner product between the time evolved system state $\left|\tilde{\psi}_{i}(T), \phi_{i}\right\rangle$ and the dressed measurement basis $\left|\tilde{\psi}_{T, f}, \phi_{f}\right\rangle$ becomes

$$
\begin{equation*}
\left\langle\phi_{f}, \tilde{\psi}_{T, f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle=\mathrm{i} q\left\langle\phi_{f}\right|\left(\hat{K}_{i f}(T) e^{-\mathrm{i} E_{f} T}-\hat{L}_{i f}(T) e^{-\mathrm{i} E_{i} T}\right)\left|\phi_{i}\right\rangle \tag{4.39}
\end{equation*}
$$

where $\left|\phi_{i, f}\right\rangle$ correspond to the initial and final states of the EM field. Since there is no $O(1)$ term there is no need to keep track of the $O\left(q^{2}\right)$ terms. Expanding:

$$
\begin{align*}
\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle=\mathrm{i} q e^{-\mathrm{i} E_{f} T}\left\langle\phi_{f}\right| & \left(-\int_{0}^{T} \mathrm{~d} t\left\langle\psi_{f}\right| \hat{H}_{1}\left|\psi_{i}\right\rangle e^{\mathrm{i}\left(E_{f}-E_{i}\right) t}+\hat{L}_{i f}(0)\right. \\
& \left.-\hat{L}_{i f}(T) e^{\mathrm{i}\left(E_{f}-E_{i}\right) T}\right)\left|\phi_{i}\right\rangle \tag{4.40}
\end{align*}
$$

let $\Omega=E_{f}-E_{i}$,

$$
\begin{equation*}
\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle=\mathrm{i} q e^{-\mathrm{i} E_{f} T}\left\langle\phi_{f}\right|\left(-\int_{0}^{T} \mathrm{~d} t\left\langle\psi_{f}\right| \hat{H}_{1}\left|\psi_{i}\right\rangle e^{\mathrm{i} \Omega t}+\hat{L}_{i f}(0)-\hat{L}_{i f}(T) e^{\mathrm{i} \Omega T}\right)\left|\phi_{i}\right\rangle . \tag{4.41}
\end{equation*}
$$

For the cases of the minimal model and dipole model, the inner product above can be written out as

$$
\begin{align*}
\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle_{\min } & =T \frac{q}{m_{e}} e^{-\mathrm{i} E_{f} T} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2}} \sqrt{\frac{\omega}{2}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_{\lambda}(k) \cdot\left\langle\phi_{f}\right| \\
& \times\left(\hat{a}_{\lambda} e^{\mathrm{i}(\Omega-\omega) \frac{T}{2}} \operatorname{sinc}\left((\Omega-\omega) \frac{T}{2}\right)\left\langle\psi_{f}\right| \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \nabla}{\Omega}\left|\psi_{i}\right\rangle\right. \\
& \left.-\hat{a}_{\lambda}^{\dagger} e^{\mathrm{i}(\Omega+\omega) \frac{T}{2}} \operatorname{sinc}\left((\Omega+\omega) \frac{T}{2}\right)\left\langle\psi_{f}\right| \frac{e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \nabla}{\Omega}\left|\psi_{i}\right\rangle\right)\left|\phi_{i}\right\rangle,  \tag{4.42}\\
\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle_{\mathrm{dip}} & =-T q e^{-\mathrm{i} E_{f} T} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2}} \sqrt{\frac{\omega}{2}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_{\lambda}(k) \cdot\left\langle\phi_{f}\right| \\
& \times\left(\hat{a}_{\lambda} e^{\mathrm{i}(\Omega-\omega) \frac{T}{2}} \operatorname{sinc}\left((\Omega-\omega) \frac{T}{2}\right)\left\langle\psi_{f}\right| e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \hat{\boldsymbol{r}}\left|\psi_{i}\right\rangle\right. \\
& \left.-\hat{a}_{\lambda}^{\dagger} e^{\mathrm{i}(\Omega+\omega) \frac{T}{2}} \operatorname{sinc}\left((\Omega+\omega) \frac{T}{2}\right)\left\langle\psi_{f}\right| e^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \hat{\boldsymbol{r}}\left|\psi_{i}\right\rangle\right)\left|\phi_{i}\right\rangle, \tag{4.43}
\end{align*}
$$

where $\operatorname{sinc}(x)=\sin (x) / x, \Omega=E_{f}-E_{i}$ and $m_{e}$ is the electron mass. Of particular importance here is the sinc term. This introduces a weak polynomial type decay with increasing $\omega$. This decay is a consequence of the 'sudden switching' of the interaction between the atom and the EM field. The weakness of this decay is the source of the difference between the dipole and minimal models.

The forms of (4.42) and (4.43) encourage us to define new variables for derivational simplicity

$$
\begin{equation*}
\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle=\left\langle\phi_{f}\right| \int \mathrm{d}^{3} \boldsymbol{k} \sum_{\lambda=1}^{2}\left(h_{1, \lambda} \hat{a}_{\lambda}+h_{2, \lambda} \hat{a}_{\lambda}^{\dagger}\right)\left|\phi_{i}\right\rangle, \tag{4.44}
\end{equation*}
$$

where $h_{1, \lambda}$ and $h_{2, \lambda}$ are chosen to match up with (4.42) and (4.43) for each of the Hamiltonians under investigation. Using this compact expression and tracing out the final field
state we can determine the probability of transition

$$
\begin{align*}
\sum_{\phi_{f}}\left|\left\langle\tilde{\psi}_{T, f}, \phi_{f} \mid \tilde{\psi}_{i}(T), \phi_{i}\right\rangle\right|^{2} & =\sum_{\lambda, \lambda^{\prime}=1}^{2}\left\langle\phi_{i}\right| \int \mathrm{d}^{3} \boldsymbol{k}\left(h_{1, \lambda} \hat{a}_{\lambda}+h_{2, \lambda} \hat{a}_{\lambda}^{\dagger}\right)^{\dagger} \\
& \times \sum_{\phi_{f}}\left|\phi_{f}\right\rangle\left\langle\phi_{f}\right| \int \mathrm{d}^{3} \boldsymbol{k}^{\prime}\left(h_{1, \lambda^{\prime}} \hat{a}_{\lambda^{\prime}}+h_{2, \lambda^{\prime}} \hat{a}_{\lambda^{\prime}}^{\dagger}\right)\left|\phi_{i}\right\rangle \tag{4.45}
\end{align*}
$$

Now $\sum_{\phi_{f}}\left|\phi_{f}\right\rangle\left\langle\phi_{f}\right|=\mathbb{I}$ is the resolution of the identity for fields, therefore the probability of transition from initial to final states is

$$
\begin{align*}
& P(i \rightarrow f)=\sum_{\lambda, \lambda^{\prime}=1}^{2}\left\langle\int \mathrm{~d}^{3} \boldsymbol{k} \int \mathrm{~d}^{3} \boldsymbol{k}^{\prime}\left(h_{1, \lambda}^{*}(\boldsymbol{k}) \hat{a}_{\lambda}^{\dagger}(\boldsymbol{k})+h_{2, \lambda}^{*}(\boldsymbol{k}) \hat{a}_{\lambda}(\boldsymbol{k})\right)\right. \\
&\left.\times\left(h_{1, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) \hat{a}_{\lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right)+h_{2, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) \hat{a}_{\lambda^{\prime}}^{\dagger}\left(\boldsymbol{k}^{\prime}\right)\right)\right\rangle_{\phi_{i}} . \tag{4.46}
\end{align*}
$$

This expression can be further simplified by exploiting the commutation relations of the field operators

$$
\begin{align*}
P(i \rightarrow f) & =\overbrace{\left\langle: \sum_{\lambda, \lambda^{\prime}=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k} \int \mathrm{~d}^{3} \boldsymbol{k}^{\prime}\left(h_{1, \lambda}^{*}(\boldsymbol{k}) \hat{a}_{\lambda}^{\dagger}(\boldsymbol{k})+h_{2, \lambda}^{*}(\boldsymbol{k}) \hat{a}_{\lambda}(\boldsymbol{k})\right)\right.}^{P_{\phi}} \\
& \left.\times\left(h_{1, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) \hat{a}_{\lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right)+h_{2, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) \hat{a}_{\lambda^{\prime}}^{\dagger}\left(\boldsymbol{k}^{\prime}\right)\right):\right\rangle_{\phi_{i}} \\
& +\underbrace{\sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k} h_{2, \lambda}^{*}(\boldsymbol{k}) h_{2, \lambda}(\boldsymbol{k})}_{P_{0}}, \tag{4.47}
\end{align*}
$$

where the colons indicate normal ordering. The expression above is the probability of the dressed $\left|i_{0}\right\rangle$ state evolving over time $T$ into the dressed $\left|f_{T}\right\rangle$. The expression has been divided into two terms: $P_{0}$ is the contribution stemming from the vacuum fluctuations of the EM field and is independent of the field's state; and $P_{\phi}$ is the contribution dependent on the field state and is zero if the field is initially in the vacuum. By inspection note that the $P_{0}$ term consists of sums and integrals over non-negative numbers, where as $P_{\phi}$ is the product of sums and integrals over the complex plane. It is expected that $P_{0}$ will
generally be larger and more significant than $P_{\phi}$, particularly for the typical RQI scenarios where the initial EM field state is very close to the vacuum with few excitations. Furthermore, $P_{0}$ 's dependence on vacuum fluctuations suggest the dipole approximation criterion's shortcomings will be most strongly felt by $P_{0}$, in the form of noticeable differences in $P_{0}^{\text {dip }}$ vs $P_{0}^{\text {min }}$.

### 4.2.2 Analytical results and observations

By specifying particular field states and desired initial and final atomic configurations, the transition probabilities above (4.47) can be fleshed out into a descriptive form.

## Vacuum excitations and spontaneous emission

The natural first step is to compare predictions of the two models for $\left|\phi_{i}\right\rangle=|0\rangle$, i.e. the vacuum state. Under these circumstances we attempt to compute (4.47). For the transition $1 s \rightarrow 2 p$ we have

$$
\begin{align*}
P_{\mathrm{dip}} & =\frac{262144 \hbar^{3} \epsilon_{0}}{177147 c^{3} q^{2} Z^{2} m_{e}^{2}} \int_{0}^{\infty} \mathrm{d} \omega \frac{\omega^{3}}{\left(1+\frac{4 a_{0}^{2}}{9 c^{2} Z^{2}} \omega^{2}\right)^{6}} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}},  \tag{4.48}\\
P_{\min } & =\frac{262144 \hbar^{3} \epsilon_{0}}{177147 c^{3} q^{2} Z^{2} m_{e}^{2}} \int_{0}^{\infty} \mathrm{d} \omega \frac{\omega^{3}}{\left(1+\frac{4 a_{0}^{2}}{9 c^{2} Z^{2}} \omega^{2}\right)^{4}} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}}, \tag{4.49}
\end{align*}
$$

where $q$ is the electron charge, $Z$ is the proton number of this Hydrogen-like atom, $m_{e}$ is electron mass. Here we have reintroduced fundamental constants for completeness. We have also used $\Omega=E_{f}-E_{i}$, which for Hydrogen-like atoms is given by

$$
\begin{equation*}
E_{f}-E_{i}=\frac{1}{2} m_{e} Z^{2} \alpha^{2}\left(\frac{1}{n_{i}^{2}}-\frac{1}{n_{f}^{2}}\right), \tag{4.50}
\end{equation*}
$$

and $\alpha=q^{2} / 4 \pi$ (in natural units); where $\alpha$ is the fine structure constant, $n_{i}$ is the principal quantum number of the initial state and $n_{f}$ is the principal quantum number of the final state.
(4.48) and (4.49) have identical coefficients and one significant difference, i.e. a power in the $\omega$ integrand's denominator. This leads to a different decay rate of the integrand with respect to $\omega$, which generates the discrepancy between the two couplings. If high frequency contributions could be dampened, then these two integrands could be well approximated
by one another. This observation suggests more general conditions for the two models to predict the same probabilities.

Indeed, in this form, it is easy to see why when there's a dominant frequency, the dipole model approximates the minimal one for long times: Consider if $\Omega<0$ (atom initially in the excited state), then for $\omega=-\Omega$ we have resonance and consequently, for long times, one can apply Fermi's Golden rule (related to the single mode approximation) of the form

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\sin ^{2}(\eta t / 2)}{(\eta / 2)^{2} t}=\pi \delta(\eta / 2) \tag{4.51}
\end{equation*}
$$

In this case $\Omega$ becomes the field's 'dominant' frequency and the dipole approximation criterion becomes $\Omega a_{0} / Z \ll 1$. Such a condition, when coupled with the relevant zeroth order Taylor expansion of (4.48) and (4.49) yields equal predictions from both couplings. Notice, however, that if $\Omega>0$ then the transition is an excitation and the sinc contribution does not resonate (i.e. $\nexists \omega \geq 0$ such that $\omega+\Omega=0$ ). In other words, this single-mode like approximation would not be justified if considering the vacuum excitation probability of the field for finite times.

However, this golden rule/single mode approximation is not alone in isolating a single mode or ranges of modes that dominate EM field behaviour. One such phenomena is the introduction of a switching function that regulates the interaction strength over time, mathematically $q \rightarrow q(t)$. Implementing a switching function introduces its Fourier transform into the integral arguments of (4.48) and (4.49) where the Fourier transformation will dampen the contributions of higher frequencies. If the switching is smooth enough and turned on long enough then this high frequency damping becomes an effective UV cutoff, which could be used to satisfy the dipole approximation ( $\omega a_{0} / Z \ll 1$ ). When the UV cutoff satisfies the dipole approximation the integral arguments of (4.48) and (4.49) can be Taylor expanded to zeroth order to become

$$
\begin{align*}
& P_{\mathrm{dip}} \rightarrow \frac{262144 \hbar^{3} \epsilon_{0}}{177147 c^{3} q^{2} Z^{2} m_{e}^{2}} \int_{0}^{\omega \lll \frac{Z}{a_{0}}} \mathrm{~d} \omega \omega^{3} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}},  \tag{4.52}\\
& P_{\min } \rightarrow \frac{262144 \hbar^{3} \epsilon_{0}}{177147 c^{3} q^{2} Z^{2} m_{e}^{2}} \int_{0}^{\omega<\frac{Z}{a_{0}}} \mathrm{~d} \omega \omega^{3} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}}, \tag{4.53}
\end{align*}
$$

which, after substituting numerical values becomes

$$
\begin{equation*}
P_{\mathrm{dip}}, P_{\mathrm{min}} \rightarrow 2.68 \times 10^{-41} s^{2} \int_{0}^{\omega \ll \frac{1}{a_{0}}} \mathrm{~d} \omega \omega^{3} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}} . \tag{4.54}
\end{equation*}
$$

Under normal circumstances the dipole approximation criterion is not satisfied by higher frequency modes and we therefore ask how large are the contributions from these non-dipole approximation modes and how large is the subsequent difference between the models. This will identify scales for which the dipole approximation is accurate even if there is no dominant frequency. As we will discuss below, for example, this would include interactions where the atom and the field are in their ground states but the interaction lasts longer than the light-crossing time of the atom.

From (4.48) and (4.49) we cannot, a priori, know the exact effect of the high frequency modes on the transition probabilities. Our only expectation is that the introduction of higher order multipoles would reduce the discrepancy between the two models, however that is not the focus of this chapter. We re-emphasise that we want to assess the validity of this approximation in processes like vacuum excitations where there is no range of dominant frequencies and the duration of the interaction is what will dictate whether the approximation is good.

## Excited fields

When considering optical experiments, one of the most common excited fields considered is the coherent state. This is the state usually associated with a laser and for our purposes we will model it with a Gaussian frequency spectrum

$$
\begin{equation*}
\left|\phi_{i}\right\rangle=\mathcal{N} \exp \left(\int \mathrm{d} \boldsymbol{k} \sum_{\lambda=1}^{2} G_{\lambda}\left(\boldsymbol{k}, \boldsymbol{k}_{0}\right) \hat{a}_{\lambda}^{\dagger}(\boldsymbol{k})\right)|0\rangle, \tag{4.55}
\end{equation*}
$$

where $G_{\lambda}\left(\boldsymbol{k}, \boldsymbol{k}_{0}\right)$ is some Gaussian centred at $\boldsymbol{k}_{0}$ and $\mathcal{N}$ is the appropriate normalisation factor. Coherent states are eigenstates of the field annihilation operator, using this fact we
can simplify (4.47) as

$$
\begin{align*}
& P(i \rightarrow f)=\sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k}\left(h_{1, \lambda}^{*}(\boldsymbol{k}) G_{\lambda}^{*}\left(\boldsymbol{k}, \boldsymbol{k}_{0}\right)+h_{2, \lambda}^{*}(\boldsymbol{k}) G_{\lambda}\left(\boldsymbol{k}, \boldsymbol{k}_{0}\right)\right) \\
& \times \sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k}^{\prime}\left(h_{1, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) G_{\lambda^{\prime}}\left(\boldsymbol{k}^{\prime}, \boldsymbol{k}_{0}\right)+h_{2, \lambda^{\prime}}\left(\boldsymbol{k}^{\prime}\right) G_{\lambda^{\prime}}^{*}\left(\boldsymbol{k}^{\prime}, \boldsymbol{k}_{0}\right)\right)+\sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k} h_{2, \lambda}^{*}(\boldsymbol{k}) h_{2, \lambda}(\boldsymbol{k}), \\
&=\left|\sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k}\left(h_{1, \lambda}^{*}(\boldsymbol{k}) G_{\lambda}^{*}(\boldsymbol{k})+h_{2, \lambda}^{*}(\boldsymbol{k}) G_{\lambda}(\boldsymbol{k})\right)\right|^{2}+\sum_{\lambda=1}^{2} \int \mathrm{~d}^{3} \boldsymbol{k} h_{2, \lambda}^{*}(\boldsymbol{k}) h_{2, \lambda}(\boldsymbol{k}), \tag{4.56}
\end{align*}
$$

For the transition $1 s \rightarrow 2 p$ we have

$$
\begin{align*}
& h_{ \pm, \lambda}^{\operatorname{dip}}(\boldsymbol{k})= \pm \frac{128 \sqrt{2} \hbar^{3 / 2} \sqrt{\epsilon_{0}} t}{243 \sqrt{\pi} q Z m_{e}} \frac{e^{-\frac{i t}{2}\left(2 E_{f} / \hbar \pm \omega-\Omega\right)} \operatorname{sinc}\left(\frac{\omega \mp \Omega}{2} t\right) \sqrt{\omega} \sin \left(\theta_{k}\right)}{\left(1+\frac{4 a_{0}^{2}}{9 c^{2} Z^{2}} \omega^{2}\right)^{3}}  \tag{4.57}\\
& h_{ \pm, \lambda}^{\min }(\boldsymbol{k})= \pm \frac{128 \sqrt{2} \hbar^{3 / 2} \sqrt{\epsilon_{0}} t}{243 \sqrt{\pi} q Z m_{e}} \frac{e^{-\frac{i t}{2}\left(2 E_{f} / \hbar \pm \omega-\Omega\right)} \operatorname{sinc}\left(\frac{\omega \mp \Omega}{2} t\right) \sqrt{\omega} \sin \left(\theta_{k}\right)}{\left(1+\frac{4 a_{0}^{2}}{9 c^{2} Z^{2}} \omega^{2}\right)^{2}} \tag{4.58}
\end{align*}
$$

where the $\pm$ subscript refers to $+\rightarrow 1,-\rightarrow 2$; and $\theta_{k}$ is the spherical coordinate polar angle of $\boldsymbol{k}$ (relative to 2 p's orientation). Here we have reintroduced the physical constants for completeness.

As with the vacuum contributions the minimal coupling and the dipole approximation differ only in the decay rate of $h_{ \pm, \lambda}$ with respect to $\omega$. Unlike the vacuum contributions (4.48) and (4.49), the asymptotic behaviour of $G_{\lambda}\left(\boldsymbol{k}, \boldsymbol{k}_{0}\right)$ make it possible to enforce the dipole criterion $\omega a_{0} / Z \ll 1$ for all significantly contributing modes. In this case we can implement a zeroth order Taylor expansion to obtain

$$
\begin{align*}
& h_{ \pm, \lambda}^{\mathrm{dip}} \rightarrow \pm \frac{128 \sqrt{2} \hbar^{3 / 2} \sqrt{\epsilon_{0}} t}{243 \sqrt{\pi} q Z m_{e}} e^{-\frac{\mathrm{it}}{2}\left(2 E_{f} / \hbar \pm \omega-\Omega\right)} \operatorname{sinc}\left(\frac{\omega \mp \Omega}{2} t\right) \sqrt{\omega} \sin \left(\theta_{k}\right),  \tag{4.59}\\
& h_{ \pm, \lambda}^{\min } \rightarrow \pm \frac{128 \sqrt{2} \hbar^{3 / 2} \sqrt{\epsilon_{0}} t}{243 \sqrt{\pi} q Z m_{e}} e^{-\frac{\mathrm{it}}{2}\left(2 E_{f} \pm \omega-\Omega\right)} \operatorname{sinc}\left(\frac{\omega \mp \Omega}{2} t\right) \sqrt{\omega} \sin \left(\theta_{k}\right), \tag{4.60}
\end{align*}
$$

which are equal, as expected when the dipole approximation criterion is satisfied.
Therefore, if the field dependent term of (4.47) is dominant over the vacuum contribution, and the field is excited in dipole criterion satisfying modes, then we expect that the dipole model can be successfully and reliably used.

### 4.2.3 Numerical results

In order to fully understand the discrepancies introduced by the dipole approximation it is necessary to resort to numerical integration. Particular attention is given to the relation between discrepancies in transition probabilities vs size of the electron orbital. Scully and Zubairy [15] seem to suggest that in the limit of an infinitely small atom the two models should converge; however that derivation was based around classical fields whilst assuming the energy gap $\Omega$ remains constant as the atom is shrunk.

## Vacuum field excitation

Consider first the transition $1 s \rightarrow 2 p$ with the initial EM field in the vacuum.
In Fig. 4.1 the transition probabilities have been plotted as a function of time. At a glance these two figures appear similar; however, the two graphs are offset by $2.6 \times 10^{-4}$. Since the graphs do not detail extremely small times we must presume that this offset arises in the very early evolution of the electron, an artefact of using different models combined with a 'sudden switching'. This already suggests that a significant difference is present for short time scales, i.e. $t<\Omega^{-1}$.

In Fig. 4.2, as expected, the transition probability decays with $Z$ given that the energy gap increases without an increase in the interaction strength. However in spite of the electron orbital size decreasing as $1 / Z$ the predictions of the dipole and minimal model remain distant, in particular the relative error is also seen to increase.

This behaviour goes against our expectations given the dipole criterion $R|\boldsymbol{k}| \ll 1$. Mathematically (see (4.48) and (4.49)) this behaviour is a consequence of the $\left(1+\frac{4 a_{0}^{2}}{9 c^{2} Z^{2}} \omega^{2}\right)$ term increasing the number of dipole approximation satisfying modes, whilst increasing the sensitivity to previously 'dormant' UV modes via the $\omega^{3} \operatorname{sinc}^{2}\left((\omega+\Omega) \frac{t}{2}\right) \sim \omega$ growth. These competing effects ensure that the two predictions never coincide.

## Vacuum field emission

Now consider the transition $2 p \rightarrow 1 s$ in the vacuum. In this case we would expect that if $\Omega a_{0} / Z \ll 1$ then the single mode approximation would limit the integration domain of (4.48) and (4.49) to a dipole criterion satisfying domain and therefore we expect the dipole model to be good.

In Fig. 4.3 the emission probabilities as a function of time are plotted. Their long time behaviour coincides with Fermi's golden rule, where both models yield the same


Figure 4.1: Vacuum transition probability $1 s \rightarrow 2 p$ for $Z=1$ atom. The short time behaviour, i.e. $t \in\left(0,40 \Omega^{-1}\right)$, corresponds to all vacuum modes constructively contributing with very minor phase differences. In the long time limit all modes are dephased with one another, leading to a constant transition probability. The dashed lines correspond to analytic results obtained from (4.48) and (4.49) by replacing the quickly oscillating function (large $t) \sin ^{2}((\omega+\Omega) t / 2)$ with its average value $1 / 2$. From these analytic expressions we know that the probability offset is $2.58 \times 10^{-4}$.


Figure 4.2: Long time transition probability as a function of $Z$ (proton number) and the relative error between the two models. As $Z$ increases the atom becomes smaller; however, contrary to Scully and Zubairy, the models diverge. Minimal model is dashed.
gradient, implying coincidence of the models. This can be justified using the single mode approximation, which is valid for long times, i.e.

$$
\begin{equation*}
\lim _{t \rightarrow \infty} t^{2} \operatorname{sinc}^{2}\left((\omega+\Omega) \frac{t}{2}\right) \sim \pi t \delta\left(\frac{\omega+\Omega}{2}\right) \tag{4.61}
\end{equation*}
$$



Figure 4.3: Vacuum transition probability $2 p \rightarrow 1 s$ for $Z=1$ atom. The short time behaviour corresponds to the region where the single mode approximation is invalid. During this time the dipole criterion is violated and the observed offset is generated. The dashed line corresponds to the single mode approximation. Note that for longer times the curve becomes linear as dictated by the single mode approximation.
and the fact that $\Omega(Z=1)$ satisfies the dipole criterion. For short times the single mode approximation is no longer applicable and this generates the offset seen in the graphs.


Figure 4.4: Vacuum emission transition rates as a function of $Z$. Note that as $Z$ increases and the atom becomes smaller the two models diverge. Minimal model is dashed. By implementing the single mode approximation in (4.48) and (4.49) one can show that the transition rates are given by $R=6.26 \times 10^{8} Z^{4}\left(1+3.33 \times 10^{-6}\right)^{-n}$, where $n=4,6$ for the minimal and dipole coupling respectively.

Fig. 4.4 shows the progression of the asymptotic emission rate with $Z$. As $Z$ increases the atom size decreases; however when implementing the single mode approximation the dipole approximation criterion becomes $\Omega a_{0} / Z \ll 1$, so the atomic size decreases as $1 / Z$
but the energy gap $\Omega$ increases as $Z^{2}$, therefore as $Z \rightarrow \infty$ the dipole criterion is increasingly violated. Of interest is the relative error graph in Fig. 4.4. The linear vs quadratic behaviour results in a minimum in the relative error occurring at $Z \approx 3$, the optimal proton number for coincidence of minimal and dipole models.

Note that if a temporal switching is introduced this may help the dipole model converge on the minimal model, at the potential cost of invalidating any use or interpretations of the single mode approximation.

## Excited fields

Finally consider the transitions concerning excited fields, i.e. field states where $P_{\phi}$ from (4.47) is not zero. In particular we focus on 'spatial pulses' of coherent 'light'. In order to explore the effects of model choice on $P_{\phi}$ alone the following section will involve plots and discussions of $P_{\phi}$ alone, note that $P_{0}$ is independent of the field state so the discussion in previous sections generally holds.

The initial field state used was

$$
\begin{equation*}
|\phi\rangle=\mathcal{N} \exp \left(\sum_{\lambda} \int \mathrm{d}^{3} \boldsymbol{k} G_{\lambda}(\boldsymbol{k}) \hat{a}_{\lambda}^{\dagger}(\boldsymbol{k})\right)|0\rangle, \tag{4.62}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{\lambda}(\boldsymbol{k})=\frac{\delta_{\lambda, \lambda_{0}} e^{\mathrm{i} \omega T^{*}}}{(2 \pi)^{3 / 4}} \frac{e^{-\frac{\left(k_{x}-k_{x}^{0}\right)^{2}}{4 \sigma_{x}^{2}}} e^{-\frac{\left(k_{y}-k_{y}^{0}\right)^{2}}{4 \sigma_{y}^{2}}} e^{-\frac{\left(k_{z}-k_{z}^{0}\right)^{2}}{4 \sigma_{z}^{2}}}}{\sqrt{\sigma_{x} \sigma_{y} \sigma_{z}}}, \tag{4.63}
\end{equation*}
$$

where $\boldsymbol{k}^{0}$ is the central wavevector of the wave packet, $\lambda_{0}$ is the polarisation of the field excitation, $T^{*}$ dictates the wave packets initial position and $\mathcal{N}$ is the appropriate normalisation factor. Note that $G_{\lambda}(\boldsymbol{k})$ is $L_{2}$ normalised.

For the numerical work presented here $\sigma_{x}=\sigma_{y}=\sigma_{z}=\Omega / 100$ and $\boldsymbol{k}^{0}=\Omega \boldsymbol{e}_{x}$, i.e. a resonant wave packet. Fig. 4.5 shows the $P_{\phi}$ contribution to the transition probability as a function of time. There is a rapid increase in the transition probability as the wave packet passes through the atom, finally the probability becomes almost constant as the field locally returns to the vacuum. The relative error between the two models is very small, in fact it is very similar to the relative error shown in Fig. 4.4 for small $Z$. Note that as we change $Z$ then $\Omega \sim Z^{2}$; this includes changing the EM field.


Figure 4.5: Excitation probability ( $P_{\phi}$ only) for a Gaussian coherent pulse with central frequency $\Omega$ (resonant). As the pulse arrives the transition probability increases to $3 \times 10^{-10}$. Once the pulse is far from the atom the transition probability remains roughly constant. Note that the relative error remains small throughout $2 \times 10^{-3}$. Also note, the bump at early times is believed to be a numerical imprecision.

In Fig. 4.6 the asymptotic transition probability is shown along with the relative error between the models as a function of $Z$. As in previous cases as $Z$ increases the dominant mode $\Omega$ no longer satisfies the dipole criterion and therefore there is no expectation that


Figure 4.6: Long time excitation probability ( $P_{\phi}$ only) as a function of $Z$. Unlike the vacuum cases the dominant frequency is dictated by the field excitation and not the single mode approximation; however, since we chose the field excitation to remain resonant with the atomic transition the relative error increases with $Z$ as $\Omega$ ceases to satisfy the dipole criterion.
the two models should give the same predictions.

### 4.2.4 Analysing the dipole approximation in vacuum fields

As can be seen from the plots above there seem to be cases when the dipole model is valid and others when it is not. These can be explained by the existence or not of a dominant mode and whether this mode satisfies the dipole criterion.

In the case of vacuum excitations there is no notion of a dominant mode in the EM field. The vacuum fluctuations cause all modes of all wavelengths to interact with the electron, with short wavelength modes suppressed by the Fourier properties of the atom itself. In particular the equation describing the contributions of each mode is given by (4.48) and (4.49). It can be rewritten to highlight key aspects as

$$
\begin{equation*}
P=K \int \mathrm{~d} \omega \underbrace{\frac{1}{\left(1+\frac{4 a_{0}^{2} \omega^{2}}{9 Z^{2}}\right)^{n}}}_{\text {Geometry \& coupling }} \underbrace{\omega^{3} \frac{\sin ^{2}\left((\omega+\Omega) \frac{t}{2}\right)}{\left((\omega+\Omega) \frac{1}{2}\right)^{2}}}_{\text {Intrinsic \& Switching }}, \tag{4.64}
\end{equation*}
$$

where $K$ is some constant and $n=4,6$ depending on the model. The intrinsic \& switching term dictates the 'dominant' or 'range of dominant' modes. When considering the dipole criterion $\omega \ll Z / a_{0}$ this can be interpreted as saying "we want the intrinsic \& switching factors to decay long before the geometry \& coupling term begins to decay." As was shown in (4.54), treating the geometry \& coupling term as constant reduces the minimal model to the dipole model. Inspection of (4.64) shows that the intrinsic \& switching term actually grows with increasing $\omega$, contrary to our needs, therefore creating this discrepancy between the two models. If a smooth switching function $f(t)$ (with characteristic width $T$ ) could be introduced, then the intrinsic \& switching term would be modified to include its Fourier transformation $F(\omega)$, which would suppress UV modes with $\omega \gtrsim T^{-1}$, thereby reducing the contributions to the transition probability from high frequency modes and diminishing the difference between the dipole and minimal models:

$$
\begin{equation*}
P=K \int \mathrm{~d} \omega \underbrace{\frac{1}{\left(1+\frac{4 a_{0}^{2} \omega^{2}}{9 Z^{2}}\right)^{n}}}_{\text {Geometry \& coupling }} \underbrace{\omega^{3} F(\omega)}_{\text {Intrinsic \& Switching }} . \tag{4.65}
\end{equation*}
$$

We illustrate this in figure 4.7, where we introduce a cutoff $\Lambda$ (that would be proportional to $1 / T$ ) and we see that the two models yield identical predictions for small enough $\Lambda$. This characteristic appears in figure 4.1, where the high frequency modes cause an offset in the transition probabilities on a very short timescale. However for longer times the two


Figure 4.7: Difference in asymptotic behaviour between dipole and minimal models for vacuum excitations, as a function of a hard UV cutoff. The x-axis is normalised to dipole criterion frequency cutoff. $\Lambda \ll 1$ corresponds to dominant modes that satisfy the dipole criterion. This reflects the conclusion of (4.54).
models predict similar trends (i.e. for $T$ such that IR unsuppressed modes $\omega \lesssim T^{-1}$ satisfy the dipole criterion).

In the case of spontaneous emission $\Omega<0$ and so the intrinsic \& switching terms of (4.64) have a dominant frequency $(\omega=-\Omega)$. It is the identity $\lim _{t \rightarrow \infty} \frac{\sin ^{2}(\eta t / 2)}{(\eta / 2)^{2} t}=\pi \delta(\eta / 2)$ that gives us the dominant mode, not by suppressing the higher frequency modes but by elevating a single mode, i.e. the single mode approximation. If this dominant mode satisfies the dipole criterion then the geometry \& coupling term will have the approximate value of 1 and both models will be equivalent. However, if $\Omega$ no longer satisfies the dipole criterion then the models will begin to differ and this is what is shown in Fig. 4.3. As $Z$ becomes larger, $\Omega$ grows quadratically in $Z$ and therefore at some point $\Omega>Z / a_{0}$.

When excited field contributions are significant (as discussed above), the predictions of the two models become very similar when there is a dominant mode that satisfies the dipole criterion. In particular Fig. 4.6 shows how the models begin to diverge as $\Omega$ ceases to satisfy the dipole criterion. These are the effective predictions given under the rotating wave approximation and they hold true for stronger coherent amplitudes. This is the regime most commonly found in experiments and therefore justifies the widespread use of
the dipole model.
In the case of spontaneous emission or stimulated excitation we say that the dipole model is good because the dominant frequency $\Omega$ satisfies the dipole criterion; however, there is still a relative error of $2 \times 10^{-3}$. If we consider the basis of the dipole approximation, i.e. approximating $e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \approx 1+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x} \approx 1$ then we note that the first order error will be of $O(\boldsymbol{k} \cdot \boldsymbol{x})$, which can be rewritten as $O\left(\Omega a_{0} / Z\right)$. Therefore the amount by which the dipole criterion is satisfied provides an estimate for the error between the two models. In particular note that for the Hydrogen atom $\Omega a_{0}=2.7 \times 10^{-3}$. In the cases where a dominant frequency exists, or a finite range of effective modes exists, the dipole criterion can be used as a first order estimate for the relative error. Hence if, experimentally, $a_{0}$ can be made smaller without changing $\Omega$ then the dipole model would become exact for all cases except for the vacuum excitation case, where the probability of transition would become divergent.

### 4.3 Conclusions

Whilst studying light-matter interactions there are many different models used, usually depending on which particular physical phenomena is of interest and generally the simplest of these is then used. In this chapter we reviewed the minimal model for light-matter interaction as well its simpler approximation, the dipole model. The idea of dressed states [4] as a means to compare the predictions of both models in a physically consistent manner was also reviewed, in particular demonstrating the advantages of working with the dipole model, which is ignorant of gauge.

In this chapter we have evaluated the differences in the predictions of the minimal coupling Hamiltonian ( $\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{A}}$ ) and the dipole coupling ( $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{E}}$ ) of light-matter interaction for the Hydrogen-like atom. This comparison is non-trivial for extended atoms due to the explicit gauge non-invariant nature of the minimal coupling.

We have confirmed the validity of the dipole model for spontaneous emission and stimulated excitation in the long time regimes. These setups admit dominant wavelengths (and their corresponding frequency $\omega_{0}$ ), by means of the single mode approximation for spontaneous emission or from the excited modes of the field itself. In these situations the relative difference between the models is approximately given (to first order) by $\omega_{0} a_{0}$, i.e. the magnitude evaluated to assess the validity of the dipole approximation.

Interestingly, we have found the situation to be much different in the case of vacuum excitation, where the atom starts in the ground state and the field in the vacuum. For the
cases of vacuum excitation, a dominant mode is absent, and even considering a very small (point-like) atom does not guarantee that the dipole approximation is accurate, i.e. there can still be a discrepancy between dipole and minimal model predictions. One cannot justify the dipole model by claiming that the atom is 'small', since there is no characteristic field wavelength dominating the interaction with which to compare. Entanglement harvesting and the Fermi problem are two such scenarios where one considers finite-time evolution of the ground state of atoms and the field vacuum, and one may then need to be further justified to use the dipole approximation.

For this case, we have characterised the regimes where the dipole model does not suffice to predict the physics of the light-matter interaction. In particular we found that when considering vacuum excitations for short time interactions the dipole coupling does not yield the same results as the minimal coupling. This characteristic also holds for spontaneous emission, where short time interactions forbid the single mode approximation and hence there is no characteristic field wavelength dominating the interaction.

As (4.64) shows, this difference cannot be removed by shrinking the atom, and we have shown that it is the contribution of arbitrarily high frequency modes that makes the two predictions diverge. However, in practice, most light-matter interactions are finite in time (preparation to measurement). We have shown that the introduction of a smooth switching (which in turn suppresses the influence of the higher frequency modes on the atomic dynamics) ensures satisfaction of the dipole criterion as long as the interaction time between atom and field is longer than the light-crossing time of the radius of the atom. This would justify the widespread use of the dipole approximation in modelling light-matter interactions, even for vacuum fluctuations (as in the case of entanglement harvesting $[13,70,71]$ and the Fermi problem $[12,16])$. However, it is not because of an argument of a 'small atom', but instead for 'sufficiently long interaction time'.

## Chapter 5

## Locality, causality and the approximations of quantum optics in free space

When considering quantum light-matter interaction it is often found that some simplifying approximations are needed due to the high numerical cost of computing, especially when a simpler model 'captures the physics' of the interaction. The phrase 'captures the physics' is intentionally vague to help justify the use of approximated models via some analytic derivation and general intuition. The purpose of this chapter is to review one such approximation, namely the Rotating Wave Approximation (RWA); more precisely determine when it can be used and how much physics does this approximation capture.

### 5.1 RWA and interaction Hamiltonian non-locality

### 5.1.1 Rotating wave approximation

The UDW model is a scalar theory commonly used to model interactions between light and electrons; however despite the relative simplicity of UDW it is common to apply additional approximations for simplicity. Consider the general UDW coupling

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\lambda \chi(t)\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \hat{\phi}(t, \boldsymbol{y}) \tag{5.1}
\end{equation*}
$$

with an interaction strength parameter $\lambda$, detector energy gap $\Omega>0$, a switching function $\chi(t)$ and an effective detector smearing $F(\boldsymbol{y})$. Expanding $\hat{\phi}(t, \boldsymbol{y})$ in terms of orthonormal modes results in

$$
\begin{align*}
\hat{H}_{\mathrm{I}}(t) & =\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}(\underbrace{\hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+} e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-} e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}}_{\mathrm{RW}} \\
& +\underbrace{\hat{a}_{\boldsymbol{k}} \hat{\sigma}^{-} e^{-\mathrm{i}(\omega+\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{+} e^{\mathrm{i}(\omega+\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}}_{\mathrm{CRW}}) . \tag{5.2}
\end{align*}
$$

When expanded and separated in this particular way it is evident that the terms labelled RW oscillate slowly in time $\left(e^{ \pm \mathrm{i}(\omega-\Omega) t}\right)$ due to the difference between two positive frequencies and are called 'co-rotating terms'. More accurately, for 'close-to-resonance' modes, i.e. $\omega \approx \Omega$, this exponential oscillates very slowly and given that Hamiltonians are ultimately integrated over time $(-\infty<t<\infty)$ these co-rotating terms are expected to integrate to large numbers. Conversely the CRW terms oscillate with a frequency of at least $\Omega$ and are called 'counter-rotating terms'. These counter-rotating terms are generally treated as rapidly oscillating terms and are expected to integrate to insignificant small numbers. Strictly, when the characteristic time $T$ of the switching $\chi(t)$ becomes large, i.e. $\Omega T \gg 1$ then by the Riemann-Lebesgue lemma the CRW terms can be discarded with a negligible error. The resulting Hamiltonian is known as the rotating wave approximation and is given by

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(\hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+} e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-} e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}\right) . \tag{5.3}
\end{equation*}
$$

A related approximation, exploiting the same underlying maths is to restrict the integration domain above to frequencies where $|\omega-\Omega|<\delta$ where $\delta$ is an arbitrary parameter. This restriction of frequency modes interacting with the detector is known as the single (few) mode approximation (Single Mode Approximation (SMA)) and is known to cause significant causality violations [36]; however it is not the focus of this chapter.

Inspection of RWA interaction term (5.3) shows some features that agree with classical notions of light-matter interactions. The field and detector operators are coupled so as to preserve excitation number, i.e. EM excitation results in detector de-excitation and vice versa. The full UDW contains terms $\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{+}$and $\hat{a}_{\boldsymbol{k}} \hat{\sigma}^{-}$, that allow for spontaneous vacuum excitation of a detector, a quantum phenomena no longer seen in RWA. Also if the SMA is considered, the naïve concept of energy conservation is maintained with field excitations of energy $\approx \Omega$ being interchanged with detector excitations of the same energy.

In this chapter the RWA criterion $\Omega T \gg 1$ will be tested, in particular for causal issues that become important in RQI experiments

### 5.1.2 Non-locality of RWA interactions

One of the earliest uses of the RWA was in Fermi's 'Quantum theory of radiation' [16] in order to calculate emission rates from an excited atom. In spite of the issues associated with the rotating wave approximation and the single mode approximation, they have been commonly used in condensed matter [17] as well as in quantum optics [18] in the form of Jaynes-Cummings models or as rotating-frame approximations. Due to the frequent use of finite size cavities within these communities, the RWA and SMA have generally been considered good approximations when causality and relativistic considerations are not of paramount importance, i.e. extremely long times (chapter 6 considers RWA in a cavity).

## Non-local behaviour

Whilst the RWA is a convenient and tempting approximation, the RWA introduces noticeable non-local behaviour, which is especially problematic in relativistic settings that are sensitive to non-locality. Compagno et al. [20, 72] considered such a setting, by placing an RWA approximated detector in an EM field. They found that the counter-rotating contributions interfered with the co-rotating terms and enforced causality, both when considering the excitation of a detector (detection of a field excitation) or the emission from a detector and the propagation of the support of the resulting renormalised stress-energy tensor.

More experimentally analogous scenarios have been considered to study RWA nonlocality, i.e. by considering non-local influences of 1 detector on another. In [73] Buscemi \& Compagno consider 2 point correlation functions in a scalar field whilst under the RWA and show that the detectors respond non-locally to the other's presence, a behaviour absent from the unapproximated model. In [74] Dolce et al. considered 3 atoms with the first in an excited state as a source and the other 2 as detectors for the excitation propagated from atom 1. This paper demonstrates well the concept of RWA non-locality, but does so in a cavity, which seems to contradict the notion that RWA works well in a cavity. This chapter will consider non-cavity situations, whilst chapter 6 will consider cavity fields.

The works described above qualitatively found RWA to be acausal and non-local, with no quantitative analysis. One of the few works to perform any quantitative analysis was by Clerk \& Sipe [8], as described below.

## Explicit non-locality

Whilst looking at the interaction terms themselves (5.3), Clerk \& Sipe [8] demonstrated that the RWA's non-locality stems from the RWA interaction Hamiltonian's non-locality, a result that is independent of the field measurement device used. Following their proof, consider the RWA interaction Hamiltonian (5.3). This equation is defined in terms of creation and annihilation operators over momentum space; and by writing out these momentum operators in terms of local field operators $\hat{\phi}$ and $\hat{\pi}$ the spatial support of the RWA interaction can be determined. Namely,

$$
\begin{align*}
& \hat{a}_{\boldsymbol{k}}=\int \frac{\mathrm{d}^{3} \boldsymbol{y}}{\sqrt{2}(2 \pi)^{3 / 2}} e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}\left(\sqrt{\omega} \hat{\phi}(\boldsymbol{y}, t)+\frac{\mathrm{i}}{\sqrt{\omega}} \hat{\pi}(\boldsymbol{y}, t)\right),  \tag{5.4}\\
& \hat{a}_{\boldsymbol{k}}^{\dagger}=\int \frac{\mathrm{d}^{3} \boldsymbol{y}}{\sqrt{2}(2 \pi)^{3 / 2}} e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}\left(\sqrt{\omega} \hat{\phi}(\boldsymbol{y}, t)-\frac{\mathrm{i}}{\sqrt{\omega}} \hat{\pi}(\boldsymbol{y}, t)\right), \tag{5.5}
\end{align*}
$$

which when substituted into (5.3) yields

$$
\begin{align*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}} & =\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int \frac{\mathrm{d}^{3} \boldsymbol{z}}{2} \hat{\phi}(\boldsymbol{z}, t) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}}\left(e^{\mathrm{i} \Omega t} e^{\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} e^{-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})} \hat{\sigma}^{-}\right) \\
& +\mathrm{i} \lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int \frac{\mathrm{d}^{3} \boldsymbol{z}}{2} \hat{\pi}(\boldsymbol{z}, t) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} \omega}\left(e^{\mathrm{i} \Omega t} e^{\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})} \hat{\sigma}^{+}-e^{-\mathrm{i} \Omega t} e^{-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})} \hat{\sigma}^{-}\right) . \tag{5.6}
\end{align*}
$$

At this point the interaction Hamiltonian consists of the usual interaction strength $\lambda$ and time switching function. Also present is the detector's spatial smearing $F(\boldsymbol{y})$, however this is additionally smeared by the $\mathrm{d}^{3} \boldsymbol{k}$ integral on the right of (5.6). These integrals can be evaluated directly (using a soft UV cutoff and complex analysis)

$$
\begin{align*}
\int \mathrm{d}^{3} \boldsymbol{k} e^{\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})} & =(2 \pi)^{3} \delta(\boldsymbol{y}-\boldsymbol{z})  \tag{5.7}\\
\int \mathrm{d}^{3} \boldsymbol{k} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})}}{\omega} & =\frac{4 \pi}{|\boldsymbol{y}-\boldsymbol{z}|^{2}} \tag{5.8}
\end{align*}
$$

which when substituted back into (5.6) yields [8]

$$
\begin{align*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}} & =\frac{\lambda \chi(t)}{2} \int \mathrm{~d}^{3} \boldsymbol{y} F(\boldsymbol{y})\left[\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \hat{\phi}(\boldsymbol{y}, t)\right. \\
& \left.-\frac{2 \mathrm{i}}{(2 \pi)^{2}}\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}-e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \int \mathrm{d}^{3} \boldsymbol{z} \frac{\hat{\pi}(\boldsymbol{z}, t)}{|\boldsymbol{y}-\boldsymbol{z}|^{2}}\right] . \tag{5.9}
\end{align*}
$$

What started out as a simple UDW interaction acting on the support of the detector's smearing $F(\boldsymbol{y})$ has become delocalised by the RWA to become $1 / 2$ of the original UDW interaction with an additional non-local term, whose non-locality is proportional to the vacuum Wightman function of the field (in the case of massless field this means $1 / r^{2}$ ). In terms of the Wightman function $W(t, \boldsymbol{y}, \boldsymbol{z})=\langle 0| \hat{\phi}(t, \boldsymbol{y}) \hat{\phi}(t, \boldsymbol{z})|0\rangle$ the interaction Hamiltonian can be written as

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y})\left[\frac{1}{2} \hat{\sigma}_{x}(t) \hat{\phi}(t, \boldsymbol{y})-\hat{\sigma}_{y}(t) \int \mathrm{d}^{3} \boldsymbol{z} W(t, \boldsymbol{y}, \boldsymbol{z}) \hat{\pi}(t, \boldsymbol{z})\right] \tag{5.10}
\end{equation*}
$$

Importantly this shows no sign of improving over long times. If the RWA is to be validated then predictions using RWA should become more accurate for long interaction times, however inspection of the interaction Hamiltonian above suggests that if RWA does converge to the unapproximated UDW then this convergence will be pointwise, and not uniform.

## Hamiltonian adimensionalisation

From (5.1) the detector smearing $F(\boldsymbol{y})$ can be seen to have dimensions of $[L]^{-3}$. By assuming the detector smearing is only non-negligible over a length scale $R$ (e.g. the size of the atom) the smearing will be rewritten in terms of a dimensionless smearing as follows

$$
\begin{equation*}
F(\boldsymbol{y})=\frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \tag{5.11}
\end{equation*}
$$

where the dimensionless function $G(\boldsymbol{\zeta})$ is localised around $|\boldsymbol{\zeta}| \lesssim 1$. With this in mind the interaction Hamiltonian can be written as

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+}+e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-}\right) . \tag{5.12}
\end{equation*}
$$

### 5.2 Acausal behaviour of RWA

As previously stated, evidence of RWA's acausal behaviour was qualitatively noted by Compagno et al. [20, 72] In this section the scale of the RWA's causality violations are determined. Additionally the RWA criterion $\Omega T \gg 1$ is challenged as insufficient for the successful use of the RWA.

In order to see the extent of RWA acausal behaviour a study of the observables $\hat{T}_{00}$ and $\hat{\phi}^{2}$, i.e. the local energy and field amplitude squared respectively, is undertaken.

### 5.2.1 Nonperturbative very short time regimes

The use of a $\delta$ switching in UDW interactions allows for a non-perturbative description of the system. It also means a very short interaction time for an approximation whose validity demands long interaction times; however by inspecting the field observables of a $\delta$ switching one can gain some intuition of the structure of long interaction time scenarios where the $\delta$ switching would resemble the leading edge of the initial perturbation of the field. The $\delta$ switching also provides an analytic opportunity to see how the interaction Hamiltonian's non-locality translates into field observables.

## Time evolution

The RWA interaction Hamiltonian, for the $\delta$-switching case takes the form

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}=\tilde{\lambda} \delta(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+}+e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-}\right), \tag{5.13}
\end{equation*}
$$

where $R$ is the characteristic size of the smearing function and $\tilde{\lambda}:=\lambda \eta$ is the overall interaction strength. In order to compress notation define

$$
\begin{align*}
\tilde{F}(\boldsymbol{k}) & :=\int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}},  \tag{5.14}\\
\hat{\alpha}(t) & :=\tilde{\lambda} \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}} . \tag{5.15}
\end{align*}
$$

This allows the interaction Hamiltonian to be written in a very compact form

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}=\delta(t)\left(\hat{\alpha}(t) \hat{\sigma}^{+}(t)+\hat{\alpha}^{\dagger}(t) \hat{\sigma}^{-}(t)\right), \tag{5.16}
\end{equation*}
$$

where $\hat{\sigma}^{ \pm}(t)=e^{ \pm i \Omega t} \hat{\sigma}^{ \pm}$. Observe that $\hat{\alpha}(t)$ consists of the sum of annihilation operators (5.15), i.e. $\hat{\alpha}(t)$ acts similarly to an annihilation operator (annihilating the same vacuum as all of the $\hat{a}_{\boldsymbol{k}}$ ). In this form (5.16) resembles a Jaynes-Cummings model.

Taking advantage of the $\delta$-switching we can evaluate the time evolution operator,

$$
\begin{equation*}
\hat{U}=\mathcal{T} \exp \left(-\mathrm{i} \int \mathrm{~d} t \hat{H}_{\mathrm{I}}^{\mathrm{RWA}}(t)\right)=\exp \left[-\mathrm{i}\left(\hat{\alpha} \hat{\sigma}^{+}+\hat{\alpha}^{\dagger} \hat{\sigma}^{-}\right)\right] \tag{5.17}
\end{equation*}
$$

where $\hat{\alpha}$ and $\hat{\sigma}^{ \pm}$are evaluated at $t=0$. As shown in appendix B. 2 the exponential above can be expanded and simplified when acting on the vacuum state:

$$
\begin{equation*}
\hat{U}|\varphi\rangle|0\rangle=\left[\hat{\Pi}_{g} \otimes \hat{\mathbb{I}}+\hat{\Pi}_{e} \otimes \hat{\mathbb{I}} \cos K-\mathrm{i} \frac{\hat{\sigma}^{-} \otimes \hat{\alpha}^{\dagger}(0)}{K} \sin K\right]|\varphi\rangle|0\rangle, \tag{5.18}
\end{equation*}
$$

where $|\varphi\rangle$ is the initial detector state and

$$
\begin{equation*}
K^{2} \hat{\mathbb{I}}:=\left[\hat{\alpha}(0), \hat{\alpha}^{\dagger}(0)\right]=\tilde{\lambda}^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}|\tilde{F}(\boldsymbol{k})|^{2} \hat{\mathbb{I}} \tag{5.19}
\end{equation*}
$$

i.e. $K \geq 0$. $\hat{\Pi}_{g, e}$ are the projection operators onto the detector ground and excited state respectively.

Note that in the RWA, (5.18) yields 0 and 1 field excitations, conditional on the initial state of the detector. This is particularly interesting when comparing with the nonapproximated full interaction Hamiltonian, which has the form

$$
\begin{align*}
\hat{H}_{\mathrm{I}}^{\mathrm{FULL}} & =\tilde{\lambda} \delta(t) \hat{\sigma}_{x}(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right)  \tag{5.20}\\
& =\delta(t) \hat{\sigma}_{x}(t)\left(\hat{\alpha}(t)+\hat{\alpha}^{\dagger}(t)\right) \tag{5.21}
\end{align*}
$$

where $\hat{\sigma}_{x}(t)=e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}$. This particular Hamiltonian allows for terms of the form $\hat{\alpha}^{\dagger} \hat{\sigma}^{+}$, i.e. emission of a field excitation via a detector excitation. The corresponding time evolution operator then becomes

$$
\begin{align*}
\hat{U} & =\mathcal{T} \exp \left(-\mathrm{i} \int \hat{H}_{\mathrm{I}} \mathrm{~d} t\right) \\
& =|+x\rangle\langle+x| \otimes \exp \left(-\mathrm{i}\left(\hat{\alpha}+\hat{\alpha}^{\dagger}\right)\right)+|-x\rangle\langle-x| \otimes \exp \left(\mathrm{i}\left(\hat{\alpha}+\hat{\alpha}^{\dagger}\right)\right) \tag{5.22}
\end{align*}
$$

where $|+x\rangle$ and $|-x\rangle$ are the $\pm$ eigenstates of $\hat{\sigma}_{x}$ and $\hat{\alpha}, \hat{\alpha}^{\dagger}$ are evaluated at $t=0$. In contrast to the (5.18), the full interaction time evolution operator generates phase-space displacements conditioned to the state of the detector, which applied to the vacuum state generates superpositions of coherent states and therefore states with multiple field excitations. This is in stark contrast with the RWA where only single-photon excitations are produced.

However, note that if $\tilde{\lambda}$ is very small then the coherent state displacements in (5.22) approximate zero and one excitation states, but as the coupling increases the approximation becomes exponentially worse. In fact the final states produced by these two Hamiltonians, $\left|\psi_{\text {Rwa }}\right\rangle$ and $\left|\psi_{\text {Full }}\right\rangle$ have the following overlap:

$$
\left\langle\psi_{\mathrm{RWA}} \mid \psi_{\mathrm{Full}}\right\rangle=\langle\varphi| \hat{\Pi}_{g}|\varphi\rangle e^{-K^{2} / 2}+\langle\varphi| \hat{\Pi}_{e}|\varphi\rangle e^{-K^{2} / 2}(\cos K+K \sin K)
$$

which, regardless of the detector's initial state, goes to zero exponentially fast with $\tilde{\lambda}$.

## Faster-than-light effects in physical observables

This subsection will focus on the energy deposited in the field and the amplitude of the field. In particular the expectation of the stress-energy density and the square of the field amplitude are evaluated. The expectation values of interest correspond to the operators

$$
\begin{align*}
&: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):=\int \frac{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} \sqrt{4 \omega \omega^{\prime}}}\left(k_{\mu} k_{\nu}^{\prime}-\frac{\eta_{\mu \nu}}{2} k_{\gamma} k^{\prime \gamma}\right)\left[e^{-\mathrm{i}\left(\omega-\omega^{\prime}\right) t+\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}\right. \\
&\left.\quad+e^{\mathrm{i}\left(\omega-\omega^{\prime}\right) t-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}^{\prime}}-e^{-\mathrm{i}\left(\omega+\omega^{\prime}\right) t+\mathrm{i}\left(\boldsymbol{k}+\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}-e^{\mathrm{i}\left(\omega-\omega^{\prime}\right) t-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}^{\dagger}\right],  \tag{5.23}\\
&: \phi^{2}(\boldsymbol{x}, t):=\int \frac{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} \sqrt{4 \omega \omega^{\prime}}}\left[e^{-\mathrm{i}\left(\omega-\omega^{\prime}\right) t+\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i}\left(\omega-\omega^{\prime}\right) t-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}^{\prime}}\right. \\
&\left.+e^{-\mathrm{i}\left(\omega+\omega^{\prime}\right) t \mathrm{i}\left(\boldsymbol{k}+\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i}\left(\omega-\omega^{\prime}\right) t-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}, \hat{a}_{\boldsymbol{k}}^{\dagger}\right], \tag{5.24}
\end{align*}
$$

where $k_{\gamma}=(\omega,-\boldsymbol{k})$ is a 4 -vector. When considering a normalised initial detector state $|\varphi\rangle=a_{g}|g\rangle+a_{e}|e\rangle$ tensored with an initial vacuum field state, i.e. $|\varphi\rangle \otimes|0\rangle$, the final (RWA evolved) state is described by (5.18). This in turn leads to

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {RWA }} & =\tilde{\lambda}^{2} \frac{\left|a_{e}\right|^{2} \sin ^{2} K}{K^{2}} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{6} 4 \omega \omega^{\prime}}\left(k_{\mu} k_{\nu}^{\prime}-\frac{\eta_{\mu \nu}}{2} k_{\gamma} k^{\prime \gamma}\right) \\
\times & {\left[e^{-\mathrm{i}\left(\omega-\omega^{\prime}\right) t+\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \tilde{F}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}^{*}(\boldsymbol{k})+e^{\mathrm{i}\left(\omega-\omega^{\prime}\right) t-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{x}} \tilde{F}(\boldsymbol{k}) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right)\right], }  \tag{5.25}\\
\left\langle: \phi^{2}(\boldsymbol{x}, t):\right\rangle_{\text {RWA }} & =\tilde{\lambda}^{2}\left|a_{e}\right|^{2} \frac{\sin ^{2} K}{K^{2}} \frac{1}{2(2 \pi)^{6}}\left|\int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{\omega} e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tilde{F}(\boldsymbol{k})\right|^{2} . \tag{5.26}
\end{align*}
$$

These expressions are non-zero only if $a_{e} \neq 0$, that is the detector must be excited in order to deposit energy in the field. In contrast the final state following the non-approximated interaction Hamiltonian leads to

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\tilde{\lambda}^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{6} 4 \omega \omega^{\prime}}\left(k_{\mu} k_{\nu}^{\prime}-\frac{\eta_{\mu \nu}}{2} k_{\gamma} k^{\prime \gamma}\right)\left(e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tilde{F}(\boldsymbol{k})+e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tilde{F}^{*}(\boldsymbol{k})\right) \\
& \times\left(e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \tilde{F}\left(\boldsymbol{k}^{\prime}\right)+e^{-\mathrm{i} \omega^{\prime} t+\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right)\right),  \tag{5.27}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =-\frac{\tilde{\lambda}^{2}}{4(2 \pi)^{6}}\left[\int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{\omega}\left(e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tilde{F}(\boldsymbol{k})-e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tilde{F}^{*}(\boldsymbol{k})\right)\right]^{2} \tag{5.28}
\end{align*}
$$

Note that these results are independent of the detector energy gap $\Omega$ (as expected from a delta switching). In this delta switching scenario an important qualitative difference between the RWA case and the full model is the full Hamiltonian's results are independent of the initial detector state unlike the RWA's results. Indeed, inspection of (5.25) and (5.26) vs (5.27) and (5.28) respectively demonstrates two important points. Firstly, the RWA expectation values are dependent on the detector being excited. The RWA does not permit spontaneous excitation of a detector via a field excitation emission. Secondly, the RWA expectation values take the form $\varphi_{i} \varphi_{j}^{\dagger}+\varphi_{i}^{\dagger} \varphi_{j}$, whilst the full model expectations take the form $\left(\varphi_{i}+\varphi_{j}^{\dagger}\right)\left(\varphi_{i}^{\dagger}+\varphi_{j}\right)$, which is a direct reflection of the differences in the Hamiltonians (5.16) and (5.21).

## Numerical Results

Using the equations above consider the situation of a spherically symmetric detector smearing

$$
G(\boldsymbol{\zeta})= \begin{cases}1 & \text { if }|\boldsymbol{\zeta}|<1  \tag{5.29}\\ 0 & \text { otherwise }\end{cases}
$$

I.e. a "hard sphere" extending to radius $R$. The length scale $R$ is used as a reference scale to adimensionalise all the dimensionful parameters in our setup. The expectation values of the renormalised energy density, and : $\hat{\phi}^{2}$ : at $t=0^{+}$, i.e., immediately following the $\delta$-coupling interaction are numerically studied below. Note that (in order that the RWA yields a non-trivial result) the detector is assumed to be initially excited.

In figure 5.1 the renormalised energy density, for the full interaction model, is plotted as a function of distance from the detector's distribution centre at time $t=0^{+}$, i.e. just after the $\delta$-coupling interaction has taken place. As expected from a local relativistic theory there is no acausal propagation or perturbations of energy beyond the support of the detector distribution (and their support remains always strictly inside the light-cone of the detector).

In contrast in figures 5.2 and 5.3 the renormalised energy density and normal ordered $\phi^{2}$ distributions are plotted respectively, for the RWA model, at time $t=0^{+}$. Particularly noteworthy are the non-zero values for $|\boldsymbol{x}|>R$, demonstrating acausal behaviour in physically measurable quantities. Moreover, these acausal tails decay only polynomially, severely limiting the situations when the RWA can be treated as local in this regime.


Figure 5.1: Energy density distribution immediately following $\delta$-coupling interaction with no approximations. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. a hard sphere. Note that the interaction has no non-local field consequences.

### 5.2.2 Perturbative evolution and long time regime

The previous section demonstrated that for extremely short interaction times the RWA's Hamiltonian non-locality is reflected in the non-locality of physically measurable field quantities. However, one may perhaps expect the RWA to work well when considering long interaction times. In this section it will be shown that this is not quite the case whilst the long time regime effects of the RWA on causality shall be explored.

In this section consider an extended switching function with the RWA Hamiltonian

$$
\begin{align*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}(t) & =\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+}+e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-}\right) \\
& =\chi(t)\left(\hat{\alpha}(t) \hat{\sigma}^{+}(t)+\hat{\alpha}^{\dagger}(t) \hat{\sigma}^{-}(t)\right) \tag{5.30}
\end{align*}
$$

where we take $\chi(t)=\Theta(t+T / 2)-\Theta(t-T / 2)$, i.e. an interaction of duration $T$ switched on at $t=-T / 2$. Here $\hat{\alpha}$ and its conjugate are defined by:

$$
\begin{align*}
\tilde{F}(\boldsymbol{k}) & :=\int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}},  \tag{5.31}\\
\hat{\alpha}(t) & :=\lambda \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}} . \tag{5.32}
\end{align*}
$$



Figure 5.2: Energy density distribution immediately following $\delta$-coupling interaction under the RWA. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. The spike at $|\boldsymbol{x}|=R$ is a consequence of the $F(\boldsymbol{x})$ discontinuity. Note the polynomially decaying tail for $|\boldsymbol{x}|>R$.

Under these conditions the corresponding time evolution operator becomes, up to second order in the Dyson expansion,

$$
\begin{align*}
& \hat{U}^{\mathrm{RWA}}(t)=\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\sigma}^{+}(t)+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\sigma}^{-}(t)\right) \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\alpha}^{\dagger}\left(t_{2}\right) \hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\alpha}\left(t_{2}\right) \hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{5.33}
\end{align*}
$$

Here $\hat{\Pi}_{g, e}$ are the detector projection operators onto the ground and excited state respectively.

In contrast, the full model has a Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\chi(t) \hat{\sigma}_{x}(t)\left(\hat{\alpha}(t)+\hat{\alpha}^{\dagger}(t)\right), \tag{5.34}
\end{equation*}
$$



Figure 5.3: : $\hat{\phi}^{2}$ : distribution immediately following $\delta$-coupling interaction under the RWA. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. Note the polynomially decaying tail for $|\boldsymbol{x}|>R$.
and the second order Dyson expansion of the time evolution operator yields

$$
\begin{align*}
& \hat{U}^{\mathrm{FULL}}=\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) \hat{\sigma}_{x}\left(t_{1}\right)\left(\hat{\alpha}\left(t_{1}\right)+\hat{\alpha}^{\dagger}\left(t_{1}\right)\right) \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) \hat{\sigma}_{x}\left(t_{1}\right) \hat{\sigma}_{x}\left(t_{2}\right)\left(\hat{\alpha}\left(t_{1}\right)+\hat{\alpha}^{\dagger}\left(t_{1}\right)\right)\left(\hat{\alpha}\left(t_{2}\right)+\hat{\alpha}^{\dagger}\left(t_{2}\right)\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{5.35}
\end{align*}
$$

If in the long time regime it were found that $\left\|\hat{U}^{\mathrm{RWA}}-\hat{U}^{\mathrm{FLLL}}\right\| \rightarrow 0$, then the rotating wave approximation would be guaranteed to work. Consider this difference perturbatively
(2nd order) and particularise for the initial state $|\psi\rangle=|\varphi\rangle|0\rangle$ :

$$
\begin{align*}
& \quad\left(\hat{U}^{\mathrm{RWA}}-\hat{U}^{\mathrm{FULL}}\right)|\varphi\rangle|0\rangle=\left(\mathrm{i} \lambda \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}^{*}(\boldsymbol{k}) \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{+} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) e^{\mathrm{i}(\omega+\Omega) t_{1}}\right. \\
& + \\
& \lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} \sqrt{4 \omega \omega^{\prime}}} \tilde{F}^{*}(\boldsymbol{k}) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right) \hat{a}_{\boldsymbol{k}}^{\dagger}{\hat{\boldsymbol{k}^{\prime}}}^{\dagger} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i}(\omega+\Omega) t_{1}+\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}\right.  \tag{5.36}\\
& + \\
& \left.\left.+\hat{\Pi}_{g} e^{\mathrm{i}(\omega-\Omega) t_{1}} e^{\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{2}}\right)+\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2}\left[\hat{\alpha}\left(t_{1}\right), \hat{\alpha}^{\dagger}\left(t_{2}\right)\right] \hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right)|\varphi\rangle|0\rangle
\end{align*}
$$

which (if RWA is valid) should converge to zero for long times. $\hat{\Pi}_{g, e}$ are the usual ground and excited state detector projection operators. This does not turn out to be the case.

For the first term in (5.36) the time integral in the $T \rightarrow \infty$ limit will become a Dirac delta of a positive argument $\delta(\omega+\Omega)$, which after integration over $\boldsymbol{k}$ yields zero. Therefore, the difference in predictions (5.36) vanishes to order $\mathcal{O}(\lambda)$ as $T \rightarrow \infty$; as dictated by the RWA.

However, the second-order terms feature a double integral over a semi-infinite domain, which will not yield a delta-like contribution of an always positive argument. Instead its contribution is governed by the expression (B.53) in appendix B.4. This ensures a persistent non-zero difference between the RWA and the exact prediction, of order $\mathcal{O}\left(\lambda^{2}\right)$, even when $T \rightarrow \infty$. Notice that this implies that as the coupling strength approaches non-perturbative regimes, the RWA validity becomes more and more questionable.

Whilst the unitaries may remain substantially different, consider how this difference between the two models manifests in the field's observables' expectation values, with a focus on causality.

As in section 5.2.1, consider an initial state given by $\left(a_{g}|g\rangle+a_{e}|e\rangle\right) \otimes|0\rangle$ then, as shown in appendix B.3, the second order expectation values are

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{4(2 \pi)^{6}}\left|a_{e}\right|^{2}\left[J_{\mu, e}^{1}\left(J_{\nu, e}^{1}\right)^{*}+\left(J_{\mu, e}^{1}\right)^{*} J_{\nu, e}^{1}-\eta_{\mu \nu}\left(J_{\gamma, e}^{1}\right)^{*} J_{e}^{1 \gamma}\right]+\mathcal{O}\left(\lambda^{3}\right),  \tag{5.37}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{2(2 \pi)^{6}}\left|a_{e}\right|^{2}\left|M_{e}^{1}\right|^{2}+\mathcal{O}\left(\lambda^{3}\right), \tag{5.38}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}}\left|a_{i}\right|^{2}\left[J_{\mu, i}^{1}\left(J_{\nu, i}^{1}\right)^{*}+\left(J_{\mu, i}^{1}\right)^{*} J_{\nu, i}^{1}-\frac{\eta_{\mu \nu}}{2}\left[J_{\gamma, i}^{1}\left(J_{i}^{1 \gamma}\right)^{*}\right.\right. \\
& \left.\left.+\left(J_{\gamma, i}^{1}\right)^{*} J_{i}^{1 \gamma}\right]+J_{\mu \nu, i}^{2}+\left(J_{\mu \nu, i}^{2}\right)^{*}-\frac{\eta_{\mu \nu}}{2}\left(J_{\gamma, i}^{2 \gamma}+\left(J_{\gamma, i}^{2 \gamma}\right)^{*}\right)\right]+\mathcal{O}\left(\lambda^{3}\right),  \tag{5.39}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}}\left|a_{i}\right|^{2}\left(2\left|M_{i}^{1}\right|^{2}-M_{i}^{2}-\left(M_{i}^{2}\right)^{*}\right)+\mathcal{O}\left(\lambda^{3}\right), \tag{5.40}
\end{align*}
$$

where

$$
\begin{align*}
J_{\mu, e}^{1}(\boldsymbol{x}, t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{\omega} k_{\mu} \tilde{F}(\boldsymbol{k}) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) e^{-\mathrm{i}(\omega-\Omega) t_{1}},  \tag{5.41}\\
J_{\mu \nu, e}^{2}(\boldsymbol{x}, t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{\omega \omega^{\prime}} k_{\mu} k_{\nu}^{\prime} \tilde{F}(\boldsymbol{k}) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \\
& \times \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}+e^{-\mathrm{i}(\omega-\Omega) t_{2}-\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{1}}\right),  \tag{5.42}\\
M_{e}^{1}(\boldsymbol{x}, t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{\omega} \tilde{F}(\boldsymbol{k}) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) e^{-\mathrm{i}(\omega-\Omega) t_{1}},  \tag{5.43}\\
M_{e}^{2}(\boldsymbol{x}, t) & =\int_{-\infty}^{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}} \tilde{\omega \omega^{\prime}} \tilde{F}(\boldsymbol{k}) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}}  \tag{5.44}\\
& \times \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}+e^{-\mathrm{i}(\omega-\Omega) t_{2}-\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{1}}\right), \tag{5.45}
\end{align*}
$$

with $J_{\mu, g}^{1}, J_{\mu \nu, g}^{2}, M_{g}^{1}$ and $M_{g}^{2}$ differing from those above by a swap $\Omega \rightarrow-\Omega$ and $\tilde{F}(\boldsymbol{k})$ defined in equation (5.14). In the equations above the repeated Greek subindex and superindex pairs follow Einstein's summation convention.

## Numerical evaluations

As with the $\delta$-coupling case, consider the situation of a spherically symmetric detector spatial distribution (5.29), i.e. a hard sphere with radius $R$; and a sudden switching
function, i.e.

$$
\begin{equation*}
\chi(t)=\Theta(t) \Theta(T-t), \tag{5.46}
\end{equation*}
$$

which means that the interaction starts at $t=0$ and we evaluate as if the interaction stops at time $t=T$, hence $T$ represents the duration of the interaction. Here, instead of the energy density and $\hat{\phi}^{2}$ distributions at $t=0^{+}$, consider the distributions at $t=T=150 R$. This lies within the RWA criterion $T \Omega \gg 1$ as $\Omega=4 R^{-1}$, such that $T \Omega=600$. Note that the detector is initially assumed to be excited.

In figures 5.4 and 5.5 the normal ordered energy density and $\phi^{2}$ distributions are plotted respectively for the full model at $t=T=150 R$. The field observable expectations should be zero outside of the light-cone of the detector. Hence, from the support of the switching and smearing functions chosen, the field expectations should vanish for $|\boldsymbol{x}|>151 R$. This is the case for the full model prediction, as can be seen in the aforementioned figures.

In figures 5.6 and 5.7 the normal ordered energy density and $\phi^{2}$ distributions are plotted respectively for the RWA model at $t=T=150 R$. In this case the violations of causality in physically measurable quantities is apparent, especially so in figure 5.7, with an obvious polynomial tail extending well beyond $|\boldsymbol{x}|=151 R$.

The results presented above satisfied the RWA criterion $T \Omega=600 \gg 1$, and yet causality violations have not been lessened. This could be perhaps more surprising than the $\delta$-coupling case, as it is usually stated that "the RWA corrects itself over long times". However, due to second order effects coming from the nested integration in time appearing in terms such as (5.42), as discussed in the previous section and in appendix B.4, this is not the case. It is worth noting that figures 5.4 and 5.6 appear very similar over large scales, especially when far from the leading edge of the causal sphere. It is also equally important to note that figures 5.5 and 5.7 are wildly different. This can be attributed to the longer range effects of the qubit and the leading edge of the causal sphere on $\phi^{2}$. However, theoretically for sufficiently long times the two figures should begin to converge when far from the qubit and light-cone surface. Nevertheless, they will always be different near the light-cone no matter how long the interaction time. Particularly, the faster-than-light tails that the RWA wrongfully predicts will not disappear for large $T$ (See more details in appendix B.4).

### 5.2.3 Communication under RWA

From the perspective of a relativistic quantum information theorist the non-localities in the field state make little impression if they do not translate into causality violations during


Figure 5.4: Energy density distribution from a second order perturbative interaction where $\chi(t)=\Theta(t) \Theta(T-t)$ with no approximations and $T=150 R$. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. Note that the interaction has no non-local field consequences, i.e. no effect beyond $|\boldsymbol{x}|>151 R$. The vertical line at $|\boldsymbol{x}|=151 R$ indicates the locality limit.
exchanges of information. For example, does the RWA allow for superluminal signalling between 2 detectors that communicate via 'exchanging field quanta'? This section answers this question by considering two detectors coupling to the field at different times, communicating with each other through that interaction. We will see the emergence and behaviour of superluminal signalling when the RWA is assumed.

The leading order communication between two particle detectors has been formalised in [75]. We will follow a similar scheme here comparing the RWA with the full model in a much more detailed way.


Figure 5.5: : $\hat{\phi}^{2}$ : distribution from a second order perturbative interaction where $\chi(t)=\Theta(t) \Theta(T-t)$ with no approximations and $T=150 R$. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. Note that the interaction has no non-local field consequences, i.e. no effect beyond $|\boldsymbol{x}|>151 R$ (to numerical precision). The vertical line at $|\boldsymbol{x}|=151 R$ indicates the locality limit.

In the case of two detectors the RWA Hamiltonian is naturally extended to

$$
\begin{align*}
& \hat{H}_{\mathrm{I}}^{\mathrm{RWA}}(t)=\lambda_{\mathrm{A}} \chi_{\mathrm{A}}(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R_{\mathrm{A}}^{3}} G_{\mathrm{A}}\left(\frac{\boldsymbol{y}}{R_{\mathrm{A}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \\
& \times\left(e^{-\mathrm{i}\left(\omega-\Omega_{\mathrm{A}}\right) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}_{\mathrm{A}}^{+}+e^{\mathrm{i}\left(\omega-\Omega_{\mathrm{A}}\right) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}_{\mathrm{A}}^{-}\right) \\
& +\lambda_{\mathrm{B}} \chi_{\mathrm{B}}(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R_{\mathrm{B}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}}{R_{\mathrm{B}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \\
& \times\left(e^{-\mathrm{i}\left(\omega-\Omega_{\mathrm{B}}\right) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}_{\mathrm{B}}^{+}+e^{\mathrm{i}\left(\omega-\Omega_{\mathrm{B}}\right) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}_{\mathrm{B}}^{-}\right), \tag{5.47}
\end{align*}
$$

where $\lambda_{\mathrm{A}, \mathrm{B}}, \chi_{\mathrm{A}, \mathrm{B}}, G_{\mathrm{A}, \mathrm{B}}$ are the interaction strength, switching function and spatial smearing functions of the two detectors respectively; and $\hat{\sigma}_{\mathrm{A}, \mathrm{B}}^{ \pm}$are the usual ladder operators acting on detectors A and B respectively with their associated energy gaps $\Omega_{\mathrm{A}, \mathrm{B}}$ respectively.


Figure 5.6: Energy density distribution from a second order perturbative interaction where $\chi(t)=\Theta(t) \Theta(T-t)$ under the RWA and $T=150 R$. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. Note the polynomial decaying tail for $|\boldsymbol{x}|>151 R$. The vertical line at $|\boldsymbol{x}|=151 R$ indicates the locality limit.

Similarly for the full Hamiltonian:

$$
\begin{align*}
\hat{H}_{\mathrm{I}}^{\mathrm{FULL}} & =\frac{\lambda_{\mathrm{A}} \chi_{\mathrm{A}}(t)}{R_{\mathrm{A}}^{3}} \hat{\sigma}_{\mathrm{A}, x}(t) \int \mathrm{d}^{3} \boldsymbol{y} G_{\mathrm{A}}\left(\frac{\boldsymbol{y}}{R_{\mathrm{A}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right) \\
& +\frac{\lambda_{\mathrm{B}} \chi_{\mathrm{B}}(t)}{R_{\mathrm{B}}^{3}} \hat{\sigma}_{\mathrm{B}, x}(t) \int \mathrm{d}^{3} \boldsymbol{y} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}}{R_{\mathrm{B}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right), \tag{5.48}
\end{align*}
$$

where we recall that

$$
\begin{equation*}
\hat{\sigma}_{\kappa, x}(t):=e^{\mathrm{i} \Omega_{\kappa} t} \hat{\sigma}_{\kappa}^{+}+e^{-\mathrm{i} \Omega_{\kappa} t} \hat{\sigma}_{\kappa}^{-} \tag{5.49}
\end{equation*}
$$

acts on the subspace of states of detector $\kappa$. In order to compress notation we can encompass the Hamiltonians of both cases with the following expression:

$$
\begin{equation*}
\hat{H}=\chi_{\mathrm{A}}(t)\left(\hat{\sigma}_{\mathrm{A}}^{+}(t) \hat{\psi}_{\mathrm{A}}+\hat{\sigma}_{\mathrm{A}}^{-}(t) \hat{\psi}_{\mathrm{A}}^{\dagger}\right)+\chi_{\mathrm{B}}(t)\left(\hat{\sigma}_{\mathrm{B}}^{+}(t) \hat{\psi}_{\mathrm{B}}+\hat{\sigma}_{\mathrm{B}}^{-}(t) \hat{\psi}_{\mathrm{B}}^{\dagger}\right), \tag{5.50}
\end{equation*}
$$

$$
\begin{aligned}
& \lambda^{-2} R^{2}\left\langle: \hat{\phi}^{2}(\boldsymbol{x}):\right\rangle \quad \hat{\phi}^{2} \text { distribution - RWA, } t=150 R \\
& \Omega t=600
\end{aligned}
$$

Figure 5.7: : $\hat{\phi}^{2}$ : distribution from a second order perturbative interaction where $\chi(t)=$ $\Theta(t) \Theta(T-t)$ under the RWA and $T=150 R$. Here $G(\boldsymbol{\zeta})=\Theta(1-\boldsymbol{\zeta})$, i.e. spherically symmetric with a sudden cutoff. Note the polynomial decaying tail for $|\boldsymbol{x}|>151$. The vertical line at $|\boldsymbol{x}|=151 R$ indicates the locality limit.
where

$$
\begin{align*}
\tilde{F}_{\kappa}(\boldsymbol{k}) & :=\lambda_{\kappa} \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R_{\kappa}^{3}} G_{\kappa}\left(\frac{\boldsymbol{y}}{R_{\kappa}}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}},  \tag{5.51}\\
\hat{\alpha}_{\kappa}(t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}_{\kappa}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}},  \tag{5.52}\\
\hat{\psi}_{\kappa}(t) & := \begin{cases}\hat{\alpha}_{\kappa} & \text { if RWA, } \\
\hat{\alpha}_{\kappa}+\hat{\alpha}_{\kappa}^{\dagger} & \text { otherwise. }\end{cases} \tag{5.53}
\end{align*}
$$

With this notation we only need to perform one formal second order Dyson expansion of the time evolution operator in order to investigate the possibilities of superluminal signalling.

The corresponding second order Dyson expansion of the time evolution operator takes
the form

$$
\begin{align*}
\hat{U}(t) & =\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1}\left(\chi_{\mathrm{A}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right)\right. \\
& \left.+\chi_{\mathrm{B}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right)\right) \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2}\left(\chi_{\mathrm{A}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right)\right. \\
& \left.+\chi_{\mathrm{B}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right)\right)\left(\chi_{\mathrm{A}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right)\right. \\
& \left.+\chi_{\mathrm{B}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{2}\right) \hat{\psi}_{\mathbf{B}}\left(t_{2}\right)+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right)\right)\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{5.54}
\end{align*}
$$

As usual we assume the initial field state is the vacuum and we consider the initial state to be a completely uncorrelated state i.e. $\hat{\rho}=\hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}} \otimes|0\rangle\langle 0|$. For brevity we will also define $\hat{\rho}^{0}=\hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}}$. Additionally, since we are investigating the causality of the signalling, we set up the detectors' switching and smearing functions to be compactly supported and their domains to be space-like separated. With the extra assumption that the supports of the switching functions are non-overlapping in the frame $(t, \boldsymbol{x})$, we can de-nest the time integrals in the same fashion as in [75], and we can assume without loss of generality that $\chi_{\mathrm{A}}$ switches on and off before $\chi_{\mathrm{B}}$ switches on in that frame. This plays a large role in simplifying the time ordered integral above.

Following the application of the time evolution operator we trace out the field and detector 2 and focus our attention on the reduced density matrix terms that involve communication, i.e. the $\lambda_{A} \lambda_{B}$ dependent terms. This leads us to

$$
\begin{align*}
& \hat{\rho}_{\mathrm{B}}^{1}(t)=\operatorname{Tr}_{\mathrm{A}}\left(\hat{\rho}^{0}\right)+\mathcal{O}\left(\lambda_{\mathrm{A}}^{2}\right)+\mathcal{O}\left(\lambda_{\mathrm{B}}^{2}\right)+\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2}\left\{\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\right. \\
& \quad \operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle\right. \\
& \left.\quad+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle\right) \\
& \quad+\operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle\right. \\
& \left.\left.\quad+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle\right)\right\}+\mathcal{O}\left(\lambda_{i}^{3}\right) \tag{5.55}
\end{align*}
$$

where the expectation values are taken over the field vacuum.
At this point we can examine the differences between the RWA and the full model by referring to our definitions in (5.53). In the RWA, only expectation values of the form $\left\langle\psi \psi^{\dagger}\right\rangle$ will be non-zero, meaning that only 1 of the 2 terms in the commutators above would actually contribute. In these cases we have

$$
\begin{align*}
\left\langle\left[\hat{\psi}_{\kappa}\left(t_{1}\right), \hat{\psi}_{\xi}^{\dagger}\left(t_{2}\right)\right]\right\rangle \stackrel{\mathrm{RWA}}{=}\left\langle\hat{\psi}_{\kappa}\left(t_{1}\right) \hat{\psi}_{\xi}^{\dagger}\left(t_{2}\right)\right\rangle & =\lambda_{\kappa} \lambda_{\xi} \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\kappa}^{3} R_{\xi}^{3}} G_{\kappa}\left(\frac{\boldsymbol{y}_{1}}{R_{\kappa}}\right) G_{\xi}\left(\frac{\boldsymbol{y}_{2}}{R_{\xi}}\right) \\
& \times \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega} e^{-\mathrm{i} \omega\left(t_{1}-t_{2}\right)} e^{\mathrm{i} \boldsymbol{k} \cdot\left(\boldsymbol{y}_{1}-\boldsymbol{y}_{2}\right)}, \tag{5.56}
\end{align*}
$$

where the indices $\kappa$ and $\xi$ take values in $\{\mathrm{A}, \mathrm{B}\}$.
In contrast, for the full model none of the expectations of the commutators are zero. Since the $\hat{\psi}_{\kappa}$ are self-adjoint in the full model (see (5.53)), then all the commutators are of the form:

$$
\begin{align*}
\left\langle\left[\hat{\psi}_{\kappa}\left(t_{1}\right), \hat{\psi}_{\xi}\left(t_{2}\right)\right]\right\rangle & \stackrel{\text { Full }}{=} \lambda_{\kappa} \lambda_{\xi} \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\kappa}^{3} R_{\xi}^{3}} G_{\kappa}\left(\frac{\boldsymbol{y}_{1}}{R_{\kappa}}\right) G_{\xi}\left(\frac{\boldsymbol{y}_{2}}{R_{\xi}}\right) \\
& \times \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}\left(e^{-\mathrm{i} \omega\left(t_{1}-t_{2}\right)}-e^{\mathrm{i} \omega\left(t_{1}-t_{2}\right)}\right) e^{\mathrm{i} \boldsymbol{k} \cdot\left(\boldsymbol{y}_{1}-\boldsymbol{y}_{2}\right)} . \tag{5.57}
\end{align*}
$$

The difference between (5.56) and (5.57) is the fact that the sole exponential $e^{-\mathrm{i} \omega \Delta t}$ in the RWA case is replaced by the difference $e^{-\mathrm{i} \omega \Delta t}-e^{\mathrm{i} \omega \Delta t}$. To understand the implications
of this difference, let us evaluate the following integral:

$$
\begin{align*}
& \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega} e^{\mp \mathrm{i} \omega \Delta t} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}=\int \frac{\mathrm{d} \omega \mathrm{~d} z \omega}{2(2 \pi)^{2}} e^{\mp \mathrm{i} \omega \Delta t} e^{\mathrm{i} \omega|\boldsymbol{x}| z}  \tag{5.58}\\
& =\int \frac{\mathrm{d} \omega}{(2 \pi)^{2}} e^{\mp \mathrm{i} \omega \Delta t} \frac{\sin (\omega|\boldsymbol{x}|)}{d}  \tag{5.59}\\
& =\int \frac{\mathrm{d} \omega}{(2 \pi)^{2} 2 \mathrm{i}|\boldsymbol{x}|}\left(e^{\mp \mathrm{i} \omega(\Delta t \mp|\boldsymbol{x}|)}-e^{\mp \mathrm{i} \omega(\Delta t \pm|\boldsymbol{x}|)}\right)  \tag{5.60}\\
& =\frac{1}{8 \pi^{2}|\boldsymbol{x}|}\left(\frac{\text { P.V. }}{|\boldsymbol{x}|+\Delta t}+\frac{\text { P.V. }}{|\boldsymbol{x}|-\Delta t}\right) \pm \frac{\mathrm{i}}{8 \pi|\boldsymbol{x}|}(\delta(|\boldsymbol{x}|+\Delta t)-\delta(|\boldsymbol{x}|-\Delta t)) \tag{5.61}
\end{align*}
$$

where P.V. indicates principal value integral when read under an integral sign.
The coefficient of $\operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\right)$from (B.65) takes the form

$$
\begin{equation*}
\mathcal{C}_{\mathrm{AB}}=\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)} . \tag{5.62}
\end{equation*}
$$

This is a good estimator for a lower bound on the channel capacity between detectors A and B. As discussed in [75], when this quantity is non-zero there is communication between the operator of detector $A$ and the operator of detector $B$ (i.e. a local measurement on detector B can reveal information about the state of detector A through a simple protocol).

By combining (5.56) and (5.61) then the capacity becomes (with non-overlapping switching functions)

$$
\begin{align*}
& \mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}}=\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{A}}^{3} R_{\mathrm{B}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right) e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)} \\
& \quad \times\left(\frac{1}{8 \pi^{2}|\boldsymbol{x}|}\left(\frac{\text { P.V. }}{|\boldsymbol{x}|+\Delta t}+\frac{\text { P.V. }}{|\boldsymbol{x}|-\Delta t}\right)+\frac{\mathrm{i}}{8 \pi|\boldsymbol{x}|}(\delta(|\boldsymbol{x}|+\Delta t)-\delta(|\boldsymbol{x}|-\Delta t))\right) \tag{5.63}
\end{align*}
$$

where $\boldsymbol{x}=\boldsymbol{y}_{1}-\boldsymbol{y}_{2}$ and $\Delta t=t_{1}-t_{2}$. This can be compared to the unapproximated channel capacity

$$
\begin{align*}
& \mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}}=2 \lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{A}}^{3} R_{\mathrm{B}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right) e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)} \\
& \quad \times\left(\frac{\mathrm{i}}{8 \pi|\boldsymbol{x}|}(\delta(|\boldsymbol{x}|+\Delta t)-\delta(|\boldsymbol{x}|-\Delta t))\right) . \tag{5.64}
\end{align*}
$$

As can be seen the principal value terms in (5.63) yield non-local polynomially decaying terms that are present in the RWA case and enable superluminal communication. However, in the non-approximated model (5.57), the terms $e^{-\mathrm{i} \omega \Delta t}-e^{\mathrm{i} \omega \Delta t}$ work together to eliminate these polynomial tails, leaving only delta functions and subsequently communication on the light-cone, as is expected from a non-approximated (and therefore causal) interaction theory [75].

Close inspection suggest that upon $t_{1}$ and $t_{2}$ integration, the principal value terms' contribution to the channel capacity will consist of a constant and a rapidly decaying function of interaction time. In contrast the Dirac delta terms will grow linearly with interaction time, eventually overwhelming the principal value contributions. In this way the RWA converges harmonically with time (§ 5.1.1). In addition for long times, once these polynomial tails are considered small, $\mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}}$ is twice as large as $\mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}}$. This is due to RWA forbidding a vacuum excitation path of communication, a path capable of contributing a similar amount to the channel capacity. This factor of $1 / 2$ is more thoroughly addressed in appendix B.6.

## Quantifying signalling through channel capacity

In this subsection we illustrate with plots for particular cases the effect of the causalityviolating tails in signalling for the RWA model. For simplicity we use spherically symmetric detector distributions, the same as in (5.29), with detector A centred around $\boldsymbol{x}=0$ and detector B centred around $\boldsymbol{x}=\boldsymbol{d}$. The switching functions where chosen to have compact, non-overlapping supports:

$$
\begin{align*}
& \chi_{\mathrm{A}}\left(t_{2}\right)= \begin{cases}1 & \text { if } 0<t_{2} \Omega_{2}<10 \\
0 & \text { otherwise },\end{cases}  \tag{5.65}\\
& \chi_{\mathrm{B}}\left(t_{1}\right)= \begin{cases}1 & \text { if } 13<t_{1} \Omega_{1}<23 \\
0 & \text { otherwise }\end{cases} \tag{5.66}
\end{align*}
$$

where $\Omega_{\mathrm{A}}=\Omega_{\mathrm{B}}=R^{-1}$. Also $R_{\mathrm{A}}=R_{\mathrm{B}}=R$ and the precise numerical values for the support are chosen to maximise visibility in the plots.

In figure 5.8 we plot, for the case of the RWA, the magnitude $\mathcal{C}_{\mathrm{AB}}$ (5.62). As we expect from [75] the communication between 2 detectors arises from the commutators of the $\hat{\psi}_{\kappa}$ operators, as seen in (B.65). We see in figure 5.8 how the non-locality in these commutators induce a non-vanishing signalling estimator $\mathcal{C}_{\mathrm{AB}}$ outside the causal contact between A and B $(|\boldsymbol{d}|>25 R)$, demonstrating communication beyond the light-cone in the approximated
model. The results from section 5.1.2 and [8] coincide exactly with figure 5.8 in describing superluminal communication at $|\boldsymbol{d}|>25 R$ with polynomial decay.

Conversely in figure 5.9 we plot (5.62) for a non-approximated model. These results are indeed consistent with causality. In fact a close look at $|\boldsymbol{d}|<R$ verifies the strong Huygens principle [76, 77, 78] at work.


Figure 5.8: Integrated RWA detector response function. Here we used $F_{\mathrm{A}}(\boldsymbol{x})=\Theta\left(R_{\mathrm{A}}-|\boldsymbol{x}|\right)$ and $F_{\mathrm{B}}(\boldsymbol{x})=\Theta\left(R_{\mathrm{B}}-|\boldsymbol{x}-\boldsymbol{d}|\right)$. In addition the detector interaction times where $\chi_{\mathrm{A}}\left(t_{2}\right)=1$ for $t_{2} \Omega_{\mathrm{A}} \in(0,10)$ and zero otherwise; and $\chi_{\mathrm{B}}\left(t_{1}\right)=1$ for $t_{1} \Omega_{\mathrm{B}} \in(13,23)$ and zero otherwise. Given that both detectors have a radius of $R$ the light-cone should only reach $|\boldsymbol{d}|=25 R$. The polynomial decay beyond this is a consequence of the RWA. The vertical line at $|\boldsymbol{d}|=R$ indicates the superior limit of the strong Huygens principle and the vertical line at $|\boldsymbol{d}|=25 R$ indicates the causal limit. Here $R_{\mathrm{A}}=R_{\mathrm{B}}=R$ and $\Omega_{\mathrm{A}}=\Omega_{\mathrm{B}}=R^{-1}$. It is worth noting that $\Omega T \ngtr 1$.

### 5.2.4 The persistent violation of causality in the RWA

From the results and plots above we have seen that the Hamiltonian non-locality introduced by the RWA translates into physically measurable non-causal effects such as non-causal field expectation values and superluminal communication between two particle detectors. Remarkably this is true regardless of how long the interaction lasts.


Figure 5.9: Integrated no approx detector response function. Here we used $F_{\mathrm{A}}(\boldsymbol{x})=$ $\Theta\left(R_{\mathrm{A}}-|\boldsymbol{x}|\right)$ and $F_{\mathrm{B}}(\boldsymbol{x})=\Theta\left(R_{\mathrm{B}}-|\boldsymbol{x}-\boldsymbol{d}|\right)$. In addition the detector interaction times where $\chi_{\mathrm{A}}\left(t_{2}\right)=1$ for $t_{2} \Omega_{\mathrm{A}} \in(0,10)$ and zero otherwise; and $\chi_{\mathrm{B}}\left(t_{1}\right)=1$ for $t_{1} \Omega_{\mathrm{B}} \in(13,23)$ and zero otherwise. Given that both detectors have a radius of $R$ the light-cone should only reach $|\boldsymbol{d}|=25 R$. Note how when considering the full model then causality is maintained. The vertical line at $|\boldsymbol{d}|=R$ indicates the superior limit of the strong Huygens principle and the vertical line at $|\boldsymbol{d}|=25 R$ indicates the causal limit. Here $R_{\mathrm{A}}=R_{\mathrm{B}}=R$ and $\Omega_{\mathrm{A}}=\Omega_{\mathrm{B}}=R^{-1}$.

For very short interactions, we have seen that for the $\delta$-coupling, when considering spherically symmetric smearings of compact support of the form (5.29) and under the assumption $|\boldsymbol{x}| \gg R$ the expressions (5.25) and (5.26) yield

$$
\begin{align*}
\left\langle: \hat{T}_{00}(\boldsymbol{x}, 0):\right\rangle_{\mathrm{RWA}} & \sim \frac{4 \lambda^{2} \sin ^{2}\left(R^{2}\right)}{9 \pi^{2} R^{4}|\boldsymbol{x}|^{6}}  \tag{5.67}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, 0):\right\rangle_{\mathrm{RWA}} & \sim \frac{2 \lambda^{2} \sin ^{2}\left(R^{2}\right)}{9 \pi^{2} R^{4}|\boldsymbol{x}|^{4}} \tag{5.68}
\end{align*}
$$

This behaviour is perhaps unsurprising given $\hat{H}_{\mathrm{I}}^{\mathrm{RwA}}$ has a $1 / r^{2}$ non-locality, combined with the quadratic nature of $\hat{\phi}^{2}$ should result in a $1 / r^{4}$ non-locality. As for the stress-energy tensor, it is composed of $\partial_{\mu} \hat{\phi} \partial_{\nu} \hat{\phi}$, i.e. the two derivative operators act of the canonical commutation relations to produce a $1 / r^{6}$ non-locality.

Remarkably, even for long timescales, when considering perturbative evolution under
the assumption $|\boldsymbol{x}| \gg t, R$ (events far ahead of the light-cone) and when considering spherically symmetric smearings of compact support of the form (5.29), the expectation values (5.37) and (5.38) asymptote to

$$
\begin{align*}
& \left\langle: \hat{T}_{00}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} \sim \frac{16 \lambda^{2} \sin ^{2}\left(\frac{t \Omega}{2}\right)}{9 \pi^{2}|\boldsymbol{x}|^{6} \Omega^{2}}  \tag{5.69}\\
& \left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} \sim \frac{8 \lambda^{2} \sin ^{2}\left(\frac{t \Omega}{2}\right)}{9 \pi^{2} \Omega^{2}|\boldsymbol{x}|^{4}} . \tag{5.70}
\end{align*}
$$

Hence, perhaps not expected under the usual 'RWA works for long times' belief, the asymptotic behaviour of the expectation values is the same for long interaction times as it is for very short interaction times. The satisfaction of the RWA's criterion does not improve the causality violation in any way.

By applying the same asymptotic analysis and assumptions $\left(|\boldsymbol{d}| \gg t, R_{\mathrm{A}}, R_{\mathrm{B}}\right.$, and smearings of the form (5.29)) to the case of 2 detector communication (5.56), we find that

$$
\begin{equation*}
\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right)\right\rangle \sim \frac{1}{|\boldsymbol{d}|^{2}}, \tag{5.71}
\end{equation*}
$$

where $|\boldsymbol{d}|$ is the inter-detector spatial distance in the detector's comoving frame. This should not be a surprise, given that the communication capacity is given by the commutator of the respective field operators and (5.9) tells us that this commutation relations will decay as $1 / r^{2}$.

Whilst the presence of a polynomially decaying non-locality should be a deal breaker for the RWA models we expect the behaviour of 'resonant-rotating' terms to be more significant when considering situations well within the bulk of the interaction light-cone (see appendix B.4.2). In these regions the RWA model's field observable predictions will converge to the full model for long interaction times; however when considering 2 qubit communication the RWA model's channel capacity will converge to $1 / 2$ the full model, since the full model allows an additional communication method to contribute. It is for this reason that cavity setups are considered ideal for RWA, provided the interaction timescales are larger than the light crossing time of the cavity itself. However, as the realm of relativistic quantum information and ultra fast optical experiments expands [10], the usefulness of the RWA diminishes and will become unsuitable for modelling experimental situations.

### 5.2.5 Conclusions

In this chapter we studied the non-local behaviour of the rotating wave approximation (RWA), a commonly used approximation in quantum optics. The work in this chapter combined the qualitative approach and results of Compagno et al. [20, 72] with the quantitative approach of Clerk and Sipe [8] so as to quantify the non-local behaviour of a RWA detector, both in the field observables and in acausal communications. Our work extended the results of Compagno et al. quantitatively illustrating the non-local physical effects of a RWA detector acting on a vacuum field, including an asymptotic study of the non-locality of field observables and detector responses. The non-locality of the RWA interaction was found to be of the form $1 / r^{2}$, as stated in [8] and this non-locality carried through into the unitary time evolution by means of $1 / r^{4}$ and $1 / r^{6}$ non-localities for $\hat{\phi}^{2}(\boldsymbol{x})$ and $\hat{T}_{00}(\boldsymbol{x})$ field expectation values. This polynomial decay is independent of time, demonstrating that waiting for long times does not fix the causality violations of the RWA when looking at field observables.

Additionally, we have also studied the fundamental relativistic quantum information scenario consisting of 2 detectors communicating through their coupling with a quantum field. In this situation the RWA predicts superluminal signalling, introducing a potentially severe $1 / r^{2}$ non-locality, which becomes particularly important in vacuum field experiments, such as entanglement harvesting [13, 27, 70, 71]. Again, no matter how long we wait, there are always polynomial tails that allow for faster-than-light signalling in the RWA.

Finally, as shown in the appendix B.4.2, the RWA model's field observables converges to those of the unapproximated UDW model within the null-like region of the interaction light-cone, i.e. very deep inside the light-cone of the detector. However we find that for 2 qubit communication, the RWA's channel capacity converges to half that predicted by the UDW model: a warning that unrestricted use of the RWA will lead to incorrect predictions, especially as there is no region where RWA and UDW models unconditionally agree.

The RWA may provide a certain simplification to the mathematical description of the physics as discussed in appendix B.3.3; however, the non-localities introduced by RWA make it incompatible with any setup with relativistic considerations are relevant (such is the case in relativistic quantum information). Furthermore, these considerations are becoming more relevant with the improvement of fast switching light-matter interaction experimental technologies [10].

## Chapter 6

## Locality, causality and the approximations of quantum optics in optical cavities

When modelling fields in cavities a major computational issue arises from the discrete nature of the momentum modes. The inability to use integration to simplify calculations can cause time and precision issues when numerically evaluating results, especially when considering 3 dimensional cavities. This realisation encourages the use of certain approximations in an effort to minimise the work load whilst 'retaining as much physics' as possible. This chapter emulates the previous chapter in reviewing the rotating wave approximation, however it does so in the context of cavity fields where the approximation is considered to fair better. As a consequence there will substantial overlap, cross-referencing and comparison with chapter 5.

### 6.1 RWA and interaction Hamiltonian non-locality

### 6.1.1 Light-matter interaction, RWA and SMA

The Unruh-DeWitt (UDW) model, a common scalar approach to the light-matter interaction was introduced in $\S 2.2 .1$. This model captures most of the features of light-matter interaction, especially when angular momentum exchange is not relevant [25, 27, 65]. The
cavity UDW Hamiltonian is given by

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\lambda \chi(t)\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \int_{\mathcal{C}} \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \hat{\phi}(t, \boldsymbol{y}), \tag{6.1}
\end{equation*}
$$

with an interaction strength parameter $\lambda$, detector energy gap $\Omega>0$, a switching function $\chi(t)$ and an effective detector smearing $F(\boldsymbol{y})$. Expanding $\hat{\phi}(t, \boldsymbol{y})$ in terms of orthonormal modes (§2.1.1) we get

$$
\begin{align*}
\hat{H}_{\mathrm{I}}(t) & =\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \sum_{\boldsymbol{m} \in \mathcal{Z}}(\underbrace{\hat{a}_{\boldsymbol{m}} \hat{\sigma}^{+} e^{\mathrm{i} \Omega t} u_{\boldsymbol{m}}(t, \boldsymbol{y})+\hat{a}_{\boldsymbol{m}}^{\dagger} \hat{\sigma}^{-} e^{-\mathrm{i} \Omega t} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{y})}_{\mathrm{RW}} \\
& +\underbrace{\hat{a}_{\boldsymbol{m}} \hat{\sigma}^{-} e^{-\mathrm{i} \Omega t} u_{\boldsymbol{m}}(t, \boldsymbol{y})+\hat{a}_{\boldsymbol{m}}^{\dagger} \hat{\sigma}^{+} e^{\mathrm{i} \Omega t} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{y})}_{\text {CRW }}) . \tag{6.2}
\end{align*}
$$

Inspection of the time behaviour of the mode functions reveals that the terms above labelled RW, known as the co-rotating terms, oscillate slowly in time as $e^{ \pm i\left(\omega_{m}-\Omega\right) t}$ for close-toresonance modes $\omega_{m} \approx \Omega$. Conversely the terms labelled CRW, known as counter-rotating, oscillate more quickly as $e^{ \pm \mathrm{i}\left(\omega_{m}+\Omega\right) t}$. A rough argument to justify the rotating-wave approximation (RWA) is to claim that as time-evolution will be given by (time-ordered exponentials of) integrals of the interaction Hamiltonian, the slowly oscillating terms (RW) will be much larger than the rapidly oscillating terms (CRW) and therefore one can discard the CRW terms with small errors (motivated by the Riemann-Lebesgue lemma). This approximation can only be taken when $\chi(t)$ describes a long smooth switching with characteristic time $T$ and $T \Omega \gg 1$. Following the application of the RWA, the interaction Hamiltonian becomes

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}^{\mathrm{RWA}}(t)=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \sum_{\boldsymbol{m} \in \mathcal{Z}}\left(\hat{a}_{\boldsymbol{m}} \hat{\sigma}^{+} e^{\mathrm{i} \Omega t} u_{\boldsymbol{m}}(t, \boldsymbol{y})+\hat{a}_{\boldsymbol{m}}^{\dagger} \hat{\sigma}^{-} e^{-\mathrm{i} \Omega t} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{y})\right) . \tag{6.3}
\end{equation*}
$$

A further common approximation is the so-called single mode approximation (SMA), which exploits the fact that for some modes that are close to resonance ( $\omega_{m} \approx \Omega$ ), the oscillations are much slower than for non-resonant modes. This leads to truncating the sum with a subset of $\mathcal{Z}$, usually by the criterion $\left|\omega_{m}-\Omega\right|<\delta$.

It is well known that SMA (or even a few-mode approximation around resonance) causes significant causality violations even in cavity settings (See [79]), therefore the focus of this chapter is to assess the violations of causality and locality introduced by the rotating-wave approximation. Despite these cautions, the RWA and SMA are commonly used in quantum optics, generally in experiments where causality or locality are not at issue. We will not directly discuss the SMA.

### 6.1.2 Hamiltonian non-locality

The use of the RWA introduces non-local behaviour in detector-field interactions. Outside of cavity settings these non-localities have been studied in, see e.g., [20, 72] (also chapter 5), and concretely it has been recently discussed how these non-localities are polynomially decaying facilitating superluminal signalling [3].

One may wonder whether the already local nature of cavity settings and the fact that we have a discrete set of field modes rather than a continuum spectrum of them may ameliorate the non-local and acausal behaviour of the light-matter interaction under the RWA in terms of its non-locality and their repercussions on causality in quantum optics. A pioneering analysis of these non-localities in a $1+1 \mathrm{D}$ cavity settings can be found in $[8,74]$. [8] also analysed non-localities in a 3+1D cavity with a final closed expression in the limit of very large cavities. Here we will revisit these studies for the $3+1$ D scalar case, explicitly evaluating the strength of the non-locality beyond the formal results in [8] which focus on large cavity regimes. In this section the non-locality of the interaction Hamiltonian will be studied with a particular focus on the effects from the cavity boundaries.

## The non-locality of the RWA Hamiltonian

In (6.3) the RWA interaction Hamiltonian is written out in terms of momentum creation and annihilation operators. If we the use (2.89) and (2.90) to express the interaction Hamiltonian in terms of local field operators we obtain

$$
\begin{align*}
& \hat{H}_{I}^{\mathrm{RWA}}=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int_{0}^{\boldsymbol{L}} \mathrm{d}^{3} \boldsymbol{z}\left[e^{\mathrm{i} \Omega t} \hat{\sigma}^{+} \hat{\phi}(t, \boldsymbol{z}) \sum_{\boldsymbol{m} \in \mathcal{Z}} \omega_{\boldsymbol{m}} u_{\boldsymbol{m}}(t, \boldsymbol{y}) u_{\boldsymbol{m}}^{*}(t, \boldsymbol{z})\right. \\
& \quad+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-} \hat{\phi}(t, \boldsymbol{z}) \sum_{\boldsymbol{m} \in \mathcal{Z}} \omega_{\boldsymbol{m}} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{y}) u_{\boldsymbol{m}}(t, \boldsymbol{z})+\mathrm{i} e^{\mathrm{i} \Omega t} \hat{\sigma}^{+} \hat{\pi}(t, \boldsymbol{z}) \sum_{\boldsymbol{m} \in \mathcal{Z}} u_{\boldsymbol{m}}(t, \boldsymbol{y}) u_{\boldsymbol{m}}^{*}(t, \boldsymbol{z}) \\
& \left.\quad-\mathrm{i} e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-} \hat{\pi}(t, \boldsymbol{z}) \sum_{\boldsymbol{m} \in \mathcal{Z}} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{y}) u_{\boldsymbol{m}}(t, \boldsymbol{z})\right] . \tag{6.4}
\end{align*}
$$

Recalling $\partial_{t} u_{\boldsymbol{m}}(t, \boldsymbol{x})=-\mathrm{i} \omega_{m} u_{\boldsymbol{m}}(t, \boldsymbol{x})$, careful inspection of the sums in the first two terms (i.e. the $\hat{\phi}(t, \boldsymbol{z})$ terms) shows that these are equal to half the magnitude of the expectation of the equal-time commutator $|\langle[\hat{\phi}(t, \boldsymbol{y}), \hat{\pi}(t, \boldsymbol{z})]\rangle|$ and therefore equal to $\delta^{(3)}(\boldsymbol{y}-\boldsymbol{z}) / 2$.

Further careful inspection of the sums in the final two terms (i.e. the $\hat{\pi}(t, \boldsymbol{z})$ terms)
allows for some simplification. The interaction Hamiltonian then becomes

$$
\begin{align*}
\hat{H}_{I}^{\mathrm{RWA}} & =\frac{\lambda}{2} \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y})\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \hat{\phi}(t, \boldsymbol{y}) \\
& +\lambda \chi(t)\left(\mathrm{i} e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}-\mathrm{i} e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y}) \int \mathrm{d}^{3} \boldsymbol{z} \underbrace{\sum_{m \in \mathcal{Z}} u_{\boldsymbol{m}}(t, \boldsymbol{y}) u_{\boldsymbol{m}}^{*}(t, \boldsymbol{z})}_{W(t, \boldsymbol{y}, \boldsymbol{z})} \hat{\pi}(t, \boldsymbol{z}),  \tag{6.5}\\
& =\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} F(\boldsymbol{y})\left[\frac{1}{2} \hat{\sigma}_{x}(t) \hat{\phi}(t, \boldsymbol{y})-\hat{\sigma}_{y}(t) \int \mathrm{d}^{3} \boldsymbol{z} W(t, \boldsymbol{y}, \boldsymbol{z}) \hat{\pi}(t, \boldsymbol{z})\right] \tag{6.6}
\end{align*}
$$

where $W(t, \boldsymbol{y}, \boldsymbol{z}):=\langle 0| \hat{\phi}(t, \boldsymbol{y}) \hat{\phi}(t, \boldsymbol{z})|0\rangle$ is the vacuum equal-time Wightman function of the cavity. The work above shows how the RWA transforms the local interaction Hamiltonian into half the original interaction with an additional non-local component governed by the Wightman function of the field in the cavity. This is a more general analogue to the non-locality result obtained in [8] for Dirichlet boundary conditions.

## Wightman function in a cavity

In computing the non-locality of the RWA interaction Hamiltonian our attentions are directed to the Wightman function, namely

$$
\begin{equation*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\sum_{\boldsymbol{m} \in \mathcal{Z}} u_{\boldsymbol{m}}(t, \boldsymbol{y}) u_{\boldsymbol{m}}^{*}(t, \boldsymbol{z}) \tag{6.7}
\end{equation*}
$$

If we now substitute the Dirichlet mode functions (2.81) into this sum

$$
\begin{equation*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\sum_{\boldsymbol{m} \in \mathcal{Z}} \frac{4}{V \omega_{\boldsymbol{m}}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right) \tag{6.8}
\end{equation*}
$$

where $\mathcal{Z}=\left\{\boldsymbol{m} \in \mathbb{Z}: m_{i} \geq 1\right\}$ and therefore we have a 3 D semi-infinite sum that cannot be expressed in closed form. The summands are slowly decaying and oscillatory, which makes it difficult to evaluate numerically by brute force in a computer without significant truncation errors or immense computation times (this direct approach was done in [74] for $1+1 \mathrm{D}$ cavity where convergence is faster and truncation errors are less serious).

A possible approach is to treat the sum as a Riemann sum and integrate (as done in [8]), which is valid if $\left|y_{i}-z_{i}\right| \ll L_{i}$ for all $i$. This leads to the $1 / r^{2}$ behaviour seen in free-space (non-cavity) cases [3]. However we are interested in more general domains and precisely
what effects the cavity walls have on the non locality of the interaction. To evaluate (6.8) we will therefore introduce a couple of formal tricks to gain analytical intuition about the Hamiltonian non-locality and to numerically evaluate the Wightman function in practical light-matter communication scenarios.

First trick Courtesy of complex analysis and the residue theorem one can show that

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right)=\frac{e^{\mathrm{i} \omega a}}{\omega}, \tag{6.9}
\end{equation*}
$$

where $\epsilon$ is introduced to move the poles off the real axis (see (C.8) and appendix C.1.1 for clarification) and $\omega=|\boldsymbol{k}|$. Proof of this expression can be found in the Appendix C.1.1.

When this is applied to the Wightman function

$$
\begin{align*}
& W(t, \boldsymbol{y}, \boldsymbol{z})=\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{V \pi^{2}} \int_{\mathbb{R}^{3}} \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\frac{1}{\xi-\mathrm{i} \epsilon}+\frac{1}{\xi+\mathrm{i} \epsilon}\right) \\
& \times \sum_{m \in\left(\mathbb{Z}^{+}\right)^{3}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right) . \tag{6.10}
\end{align*}
$$

The sum is now in an analytically manageable form (appendix C.2)

$$
\begin{align*}
W(t, \boldsymbol{y}, \boldsymbol{z}) & =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{V \pi^{2}} \int_{\mathbb{R}^{3}} \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\frac{1}{\xi-\mathrm{i} \epsilon}+\frac{1}{\xi+\mathrm{i} \epsilon}\right) \\
& \times \prod_{i=1}^{3} \frac{L_{i}}{2} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+y_{i}-z_{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+y_{i}+z_{i}+2 n_{i} L_{i}\right)\right) . \tag{6.11}
\end{align*}
$$

Inspection of the behaviour under $\epsilon \rightarrow 0^{+}$shows that once the $\delta$ functions have been eliminated under the $\boldsymbol{\xi}$ integral then the residue contributions from $\xi=0$ are cancelled out by the two fractions leaving only their principal values:

$$
\begin{equation*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\frac{1}{4 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{\mathrm{P} . \mathrm{V} .}{\xi^{2}} \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+y_{i}-z_{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+y_{i}+z_{i}+2 n_{i} L_{i}\right)\right) \tag{6.12}
\end{equation*}
$$

c.f. in free space cases: $W(t, \boldsymbol{y}, \boldsymbol{z})=1 /\left(4 \pi^{2}|\boldsymbol{y}-\boldsymbol{z}|^{2}\right)$. Equation (6.12) tells us that the contributions to the Wightman function decay under an inverse square law. The negative
sign for the second $\delta$ function tells us that contribution to the Wightman that are reflected have a $\pi$ phase shift to introduce the negative sign. The sum over $n_{i}$ tells us that the boundaries reflect the $1 / r^{2}$ contributions to the Wightman resulting in an infinite decaying contribution to the Wightman function. Thus, perhaps as expected, cavities differ from free space fields by the presence of mirrors that predictably and endlessly add reflected contributions to non-local functions.

2nd numerical trick Despite the success of the first trick above in illustrating the properties of the Wightman function in a cavity, (6.12) is still a difficult function to evaluate. Instead of a 3 D semi-infinite sum of a $1 / \omega$ oscillating sequence we now have a 3 D infinite sum of a $1 / r^{2}$ oscillating sequence, which is not good enough. We therefore introduce the 2nd numerical trick, which does not provide great intuition, but is however convenient for the numerical analysis of the setup. By means of complex analysis and the residue theorem

$$
\begin{align*}
\frac{e^{\mathrm{i} \omega a}}{\omega} & =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{\rho e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\sinh [\rho(\xi-a-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+a+\mathrm{i} \epsilon)]}\right) \\
& +\frac{2 \cos (\omega a)}{\omega} \frac{1}{1+e^{\pi \omega / \rho}}, \tag{6.13}
\end{align*}
$$

where $\omega=|\boldsymbol{k}|$ and $\rho$ is a positive real number. Notice that despite the explicit appearance of a parameter $\rho$ in the expression above, the right-hand side of (6.13) does not really depend on $\rho^{1}$. Rather, $\rho$ is explicitly introduced as a helpful parameter to aid the numerical evaluation of this expression. Proof of this trick can be found in the Appendix C.1.2.

When applied to the Wightman function (in almost identical steps to the first trick; also implicit $\epsilon \rightarrow 0^{+}$)

$$
\begin{align*}
W(t, \boldsymbol{y}, \boldsymbol{z}) & =\frac{\rho}{8 \pi^{2}} \int_{\mathbb{R}^{3}} \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\frac{1}{\sinh [\rho(\xi-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+\mathrm{i} \epsilon)]}\right) \\
\times & \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+y_{i}-z_{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+y_{i}+z_{i}+2 n_{i} L_{i}\right)\right) \\
& +\frac{8}{V} \sum_{m \in\left(\mathbb{Z}^{+}\right)^{3}} \frac{1}{\omega} \frac{1}{1+e^{\pi \omega / \rho}} \prod_{i=1}^{3} \sin \left(k_{i} y_{i}\right) \sin \left(k_{i} z_{i}\right) . \tag{6.14}
\end{align*}
$$

[^0]Now we can see that $\rho$ is a useful numerical parameter that can be used to hasten the exponential decay rate of the summands of the $\boldsymbol{n}$ sum, at the cost of slowing the exponential decay of the summands in the $\boldsymbol{m}$ sum. However, with this 2 nd trick we now have (after eliminating the $\delta$ functions with $\boldsymbol{\xi}$ integrals) two 3D sums to evaluate, both of which now converge exponentially fast. This makes the calculations amenable without the need of substantial computing power.

Figures 6.1 and 6.2 both exploit the 2nd numerical trick to illustrate the Wightman's behaviour and the resulting non-locality caused by the RWA on the interaction Hamiltonian. Note that when $\boldsymbol{z} \approx \boldsymbol{y}$ the cavity Wightman asymptotes to the free space Wightman, i.e. over very short distances the field is not affected by the presences of cavities. Also note that the Wightman vanishes at the cavity's boundaries, unsurprising given Dirichlet boundary conditions.


Figure 6.1: The Wightman function $\boldsymbol{y}=(0.3,0.5,0.5)$ and $\boldsymbol{z}=(s, 0.5,0.5)$ with $s$ on the xaxis of the diagram and $L_{1}=L_{2}=L_{3}=1$. As $\boldsymbol{z} \rightarrow \boldsymbol{y}$ the function diverges as $1 /|\boldsymbol{y}-\boldsymbol{z}|^{2}$. Close view.

Wightman function for cavity vacuum


Figure 6.2: The Wightman function $\boldsymbol{y}=(0.3,0.5,0.5)$ and $\boldsymbol{z}=(s, 0.5,0.5)$ with $s$ on the xaxis of the diagram and $L_{1}=L_{2}=L_{3}=1$. As $\boldsymbol{z} \rightarrow \boldsymbol{y}$ the function diverges as $1 /|\boldsymbol{y}-\boldsymbol{z}|^{2}$. Far view.

### 6.2 Qubit communication

We will now analyse the impact of the non-locality of the model on superluminal signalling between two particle detectors placed at two different positions in the cavity.

### 6.2.1 Two-detector interaction Hamiltonian

We will consider a typical communication scenario where two particle detectors communicate via their interaction with the quantum field in the cavity. In its simplest form a communication protocol can be established by coupling a first detector to the field, which perturbs it. That perturbation propagates and modifies the state of a second detector (see, e.g., $[75,78]$ ). We know that if there are no approximations such as single (or few) mode approximation or the RWA the first detector cannot signal the second one if they are space-like separated [75, 79]. However, the RWA non-locality will enable the model to (unphysically) transmit a faster-than-light signal to the second detector. We quantify by how much in this section.

The interaction Hamiltonian of 2 two-level particle detectors coupling to the field is

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}=\sum_{j \in\{\mathrm{~A}, \mathrm{~B}\}} \lambda_{j} \chi_{j}(t)\left(e^{\mathrm{i} \Omega_{j} t} \hat{\sigma}_{j}^{+}+e^{-\mathrm{i} \Omega_{j} t} \hat{\sigma}_{j}^{-}\right) \int \mathrm{d}^{3} \boldsymbol{x} F_{j}(\boldsymbol{x}) \hat{\phi}(t, \boldsymbol{x}), \tag{6.15}
\end{equation*}
$$

with $j$ indexing the detectors. In order to compress notation we define the operators

$$
\begin{align*}
& \hat{\psi}_{j}^{\mathrm{UDW}}(t)=\int_{V} \mathrm{~d}^{3} \boldsymbol{x} F_{j}(\boldsymbol{x}) \sum_{\boldsymbol{m} \in \mathcal{Z}}\left(\hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(t, \boldsymbol{x})+\hat{a}_{\boldsymbol{m}}^{\dagger} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{x})\right)  \tag{6.16}\\
& \hat{\psi}_{j}^{\mathrm{RWA}}(t)=\int_{V} \mathrm{~d}^{3} \boldsymbol{x} F_{j}(\boldsymbol{x}) \sum_{\boldsymbol{m} \in \mathcal{Z}} \hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(t, \boldsymbol{x}) \tag{6.17}
\end{align*}
$$

With these operator definitions we can rewrite the interaction Hamiltonian as

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}=\sum_{j \in\{\mathrm{~A}, \mathrm{~B}\}} \lambda_{j} \chi_{j}(t)\left(e^{\mathrm{i} \Omega_{j} t} \hat{\sigma}_{j}^{+} \hat{\psi}_{j}(t)+e^{-\mathrm{i} \Omega_{j} t} \hat{\sigma}_{j}^{-} \hat{\psi}_{j}^{\dagger}\right), \tag{6.18}
\end{equation*}
$$

where substituting $\hat{\psi}_{j} \rightarrow \hat{\psi}_{j}^{\text {UDW }}$ yields the unapproximated interaction and $\hat{\psi}_{j} \rightarrow \hat{\psi}_{j}^{\mathrm{RWA}}$ yields the RWA interaction Hamiltonian.

### 6.2.2 Communication

We will follow the procedure introduced in [75] and used in the continuum case above (chapter 5). To compute the effect that the presence of detector A has on the state of detector $B$ we start from an arbitrary uncorrelated state of the two detectors and the field $\hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}} \otimes \hat{\rho}_{\phi}$. We then evolve the system to leading order in perturbation theory and trace out the field and detector A; and inspect the reduced density matrix of detector B following the interaction. This partial density matrix for the second is given by

$$
\begin{align*}
& \hat{\rho}_{\mathrm{B}}^{1}(t)=\operatorname{Tr}_{\mathrm{A}}\left(\hat{\rho}^{0}\right)+\mathcal{O}\left(\lambda_{\mathrm{A}}^{2}\right)+\mathcal{O}\left(\lambda_{\mathrm{B}}^{2}\right)+\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2}\left\{\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\right. \\
& \quad \operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle\right. \\
& \left.\quad+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle\right) \\
& \quad+\operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle\right. \\
& \left.\left.\quad+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle\right)\right\}+\mathcal{O}\left(\lambda_{i}^{3}\right), \tag{6.19}
\end{align*}
$$

This is identical to the continuum expression (B.65). To obtain this expression we have made the simplifying assumption that the supports of the switching functions $\chi_{j}$ do not overlap, i.e. the first detector (A) turns off before the second detector (B) turns on. The magnitude

$$
\mathcal{C}_{\mathrm{AB}}=\int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle
$$

which is the coefficient of $\hat{\sigma}_{A}^{-} \rho^{0} \hat{\sigma}_{\mathrm{B}}^{+}$, is a good estimator for a lower bound on the channel capacity between detectors A and B . As discussed in [3, 75] and chapter 5, when this quantity is non-zero there is communication between the detectors through the field.

When we consider communication under the RWA and Dirichlet boundary conditions we have

$$
\begin{align*}
\mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}} & =\int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int_{V} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{B}} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{A}} F_{\mathrm{B}}\left(\boldsymbol{x}_{\mathrm{B}}\right) F_{\mathrm{A}}\left(\boldsymbol{x}_{\mathrm{A}}\right) \\
& \times \sum_{\boldsymbol{m} \in \mathcal{Z}} u_{\boldsymbol{m}}\left(t_{1}, \boldsymbol{x}_{\mathrm{B}}\right) u_{\boldsymbol{m}}^{*}\left(t_{2}, \boldsymbol{x}_{\mathrm{A}}\right),  \tag{6.20}\\
& =\int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int_{V} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{B}} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{A}} F_{\mathrm{B}}\left(\boldsymbol{x}_{\mathrm{B}}\right) F_{\mathrm{A}}\left(\boldsymbol{x}_{\mathrm{A}}\right) \\
& \times \frac{4}{V} \sum_{\boldsymbol{m} \in\left(\mathbb{Z}^{+}\right)^{3}} \frac{e^{-\mathrm{i} \omega\left(t_{1}-t_{2}\right)}}{\omega} \prod_{i=1}^{3} \sin \left(k_{i} x_{\mathrm{B}}^{i}\right) \sin \left(k_{i} x_{\mathrm{A}}^{i}\right) . \tag{6.21}
\end{align*}
$$

Noting the similarities between the sum above and that of the Wightman function we implement the first trick developed in Section 6.1.2:

$$
\begin{align*}
\mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}} & =\lim _{\epsilon \rightarrow 0^{+}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int_{V} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{B}} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{A}} F_{\mathrm{B}}\left(\boldsymbol{x}_{\mathrm{B}}\right) F_{\mathrm{A}}\left(\boldsymbol{x}_{\mathrm{A}}\right) \\
& \times \frac{1}{8 \pi^{2}} \int \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\frac{1}{\xi+\Delta t-\mathrm{i} \epsilon}+\frac{1}{\xi-\Delta t+\mathrm{i} \epsilon}\right) \\
& \times \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}-x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}+x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)\right), \tag{6.22}
\end{align*}
$$

where $\Delta t=t_{1}-t_{2}$. We can also eliminate the $\epsilon$ by separating the principal value contribution from the delta function (Sokhotski-Plemelj theorem [80], §5.7)

$$
\begin{align*}
\mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}} & =\int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int_{V} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{B}} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{A}} F_{\mathrm{B}}\left(\boldsymbol{x}_{\mathrm{B}}\right) F_{\mathrm{A}}\left(\boldsymbol{x}_{\mathrm{A}}\right) \\
& \times \frac{1}{8 \pi^{2}} \int \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\mathrm{i} \pi \delta(\xi+\Delta t)-\mathrm{i} \pi \delta(\xi-\Delta t)+\frac{\mathrm{P} . \mathrm{V} .}{\xi+\Delta t}+\frac{\mathrm{P} . \mathrm{V} .}{\xi-\Delta t}\right) \\
& \times \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}-x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}+x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)\right), \tag{6.23}
\end{align*}
$$

which when compared to (5.63) and (5.64) shows that the signalling estimator for the cavity case is the same as in free space but with the added contribution from the reflections off the boundaries.

We can compare this result with the signalling estimator for the full model

$$
\begin{align*}
\mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}} & =\int_{-\infty}^{\infty} \mathrm{d} t_{1} \mathrm{~d} t_{2} e^{\mathrm{i} \Omega_{\mathrm{B}} t_{1}-\mathrm{i} \Omega_{\mathrm{A}} t_{2}} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \int_{V} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{B}} \mathrm{~d}^{3} \boldsymbol{x}_{\mathrm{A}} F_{\mathrm{B}}\left(\boldsymbol{x}_{\mathrm{B}}\right) F_{\mathrm{A}}\left(\boldsymbol{x}_{\mathrm{A}}\right) \\
& \times \frac{1}{8 \pi^{2}} \int \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}(2 \mathrm{i} \pi \delta(\xi+\Delta t)-2 \mathrm{i} \pi \delta(\xi-\Delta t)) \\
& \times \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}-x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+x_{\mathrm{A}}^{i}+x_{\mathrm{B}}^{i}+2 n_{i} L_{i}\right)\right) . \tag{6.24}
\end{align*}
$$

As we show in Figs. 6.3 and 6.4, we see how the full model displays no faster-than-light signalling, whereas the RWA approximated version has faster-than-light contributions to communication originating in the tails of the principal value contributions that appear suppressing the counter-rotating terms. Recalling that the duration timescale (the support
of the switching functions) is considered to be $T$, we note that when $\Omega T \gg 1$ the RWA signalling estimator converges to $1 / 2$ of the unapproximated model, similar to the noncavity case and explained in detail in appendix B.6. As in [3] and chapter 5 the RWA is plausible when in the null-like interior of the first detector's interaction light-cone (see Fig. B.1), far from the light-like boundaries. We illustrate this in Figs. 6.5 and 6.6, where a larger value of $\Omega T$ is considered and we see how the error in the approximation is reduced.

### 6.2.3 Numerical results

Here we clarify the particular choices taken to plot Figs. 6.3 to 6.6. Our setup will involve placing the first detector (A) in the cavity at $\boldsymbol{x}_{\mathrm{A}}=\left(0.4 L_{1}, 0.5 L_{2}, 0.5 L_{3}\right)$ and placing the second detector $(\mathrm{B})$ at $\boldsymbol{x}_{\mathrm{B}}=\left(s, 0.5 L_{2}, 0.5 L_{3}\right)$ where $s$ is a free parameter indicating the position of Bob's detector in the x-axis. We choose the cavity to be cubic $L_{1}=L_{2}=L_{3} \equiv$ $L$. The switching functions are chosen to have compact and non-overlapping supports (as in chapter 5), to avoid complications with cavity wall reflections we choose the switching functions to be short enough so that there is not enough time for light to go from Alice to Bob through a reflection off the walls:

$$
\begin{align*}
& \chi_{\mathrm{A}}\left(t_{2}\right)= \begin{cases}1 & \text { if } 0<t_{2} L^{-1}<0.1 \\
0 & \text { otherwise },\end{cases}  \tag{6.25}\\
& \chi_{\mathrm{B}}\left(t_{1}\right)= \begin{cases}1 & \text { if } 0.3<t_{1} L^{-1}<0.4 \\
0 & \text { otherwise }\end{cases} \tag{6.26}
\end{align*}
$$

The detectors' energy gap $\Omega$ was chosen as

$$
\begin{equation*}
\Omega=\frac{\pi}{L} \sqrt{m_{1}^{2}+m_{2}^{2}+m_{3}^{2}}, \tag{6.27}
\end{equation*}
$$

with $m_{i}$ not integer to avoid resonant $\boldsymbol{k}_{\boldsymbol{m}}$ modes. Hence $\Omega=9.02 L^{-1}, 871.19 L^{-1}$ are used, although preliminary numerical tests showed resonance was not relevant.

By using the second numerical trick from Section 6.1.2 we obtain Figures 6.3,6.4,6.5 and 6.6. When $\Omega T$ is small as in Fig. 6.4 we note significant causality violations as expected from the failure to meet the RWA's requirements. However in Fig. 6.6 we have $\Omega T \gg 1$ and whilst the resulting $\mathcal{C}_{\mathrm{AB}}$ behaves in a similar way for both RWA and UDW we see that small non-localities persist. At the same time within the light-like causal regions the large (RWA) contributions from the $\delta$ function in (6.23) are half that of the UDW, highlighting the contributions of the factor of 2 in (6.24).


Figure 6.3: $\mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}}$, with the first detector at $\boldsymbol{x}_{\mathrm{A}}=(0.4,0.5,0.5) L$ and the second detector at $\boldsymbol{x}_{\mathrm{B}}=(s, 0.5,0.5) L$ with $s$ on the x-axis of the diagram and $L_{1}=L_{2}=L_{3} \equiv L$. The vertical lines $s=0,0.8$ indicate causal limit $(c t), s=0.2,0.6$ indicate time-like null-like signalling boundary and $s=0.4$ indicates the location of the first detector. The interaction times are $0<t_{\mathrm{A}} L^{-1}<0.1,0.3<t_{\mathrm{B}} L^{-1}<0.4$ and $\Omega=9.02 L^{-1}$. Here $\Omega T=0.9 \ngtr 1$.

### 6.3 Discussion

The results presented here help to clarify when the RWA can be used in cavity scenarios, whilst quantifying the penalties associated with its usage.

### 6.3.1 RWA non-locality

In the results presented above the non-local behaviour of the RWA interaction Hamiltonian was demonstrated, highlighting the significant polynomial non-localities introduced by the approximation (Fig. 6.1). Unlike the free space case, cavities by definition have bounded dimensions, therefore it is unreasonable to presume that one can consider sufficiently distant points for which the interaction non-locality can be discarded (as is possible in free space fields).

The consequences of the substantial non-localities of the RWA interaction Hamiltonian


Figure 6.4: $\mathcal{C}_{\mathrm{AB}}^{\mathrm{RwA}}$, with the first detector at $\boldsymbol{x}_{\mathrm{A}}=(0.4,0.5,0.5) L$ and the second detector at $\boldsymbol{x}_{\mathrm{B}}=(s, 0.5,0.5) L$ with $s$ on the x-axis of the diagram and $L_{1}=L_{2}=L_{3} \equiv L$. The vertical lines $s=0,0.8$ indicate causal limit $(c t), s=0.2,0.6$ indicate time-like null-like signalling boundary and $s=0.4$ indicates the location of the first detector. The interaction times are $0<t_{\mathrm{A}} L^{-1}<0.1,0.3<t_{\mathrm{B}} L^{-1}<0.4$ and $\Omega=9.02 L^{-1}$. Note significant causality violations and time-like communication not present in $\mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}}$. Also note the sharp peak coincides with the UDW peak. Here $\Omega T=0.9 \ngtr 1$.
appear in Fig. 6.3-6.6. As expected from the RWA when $\Omega T$ is small the approximation is invalid, Fig. 6.4 in particular shows how the non-local influences dominate the communication measure to an unrecognisable degree. Significant acausal tails are present as well as large time-like communication (violation of the strong Huygens principle). A coarse analysis shows that the causality violations in $\mathcal{C}_{\mathrm{AB}}$ decay polynomially, i.e. $1 / r^{2}$ from the edge of the light-cone, whilst constrained by $\mathcal{C}_{\mathrm{AB}}=0$ on the cavity's boundaries.

When $\Omega T \gg 1$ RWA is expected to work well, however in [3] it was shown that for free space fields this is not the case when considering $|\boldsymbol{x}| \approx T$, i.e. when the second detector was close to the edge of the light-cone of the first detector. In Fig. 6.6 we can see that this phenomena still holds, i.e. for $s>0.8$ a polynomial tail of acausal communications persists, although it is dwarfed by the causal contributions. Importantly, most physical scenarios involve detector-field interaction times that are larger than the light-crossing time of the cavity, in which case these small acausal tails will be secondary to the large


Figure 6.5: $\mathcal{C}_{\mathrm{AB}}^{\mathrm{UDW}}$, with the first detector at $\boldsymbol{x}_{\mathrm{A}}=(0.4,0.5,0.5) L$ and the second detector at $\boldsymbol{x}_{\mathrm{B}}=(s, 0.5,0.5) L$ with $s$ on the x-axis of the diagram and $L_{1}=L_{2}=L_{3} \equiv L$. The vertical lines $s=0,0.8$ indicate causal limit $(c t), s=0.2,0.6$ indicate time-like null-like signalling boundary and $s=0.4$ indicates the location of the first detector. The interaction times are $0<t_{\mathrm{A}} L^{-1}<0.1,0.3<t_{\mathrm{B}} L^{-1}<0.4$ and $\Omega=871.19 L^{-1}$. Here $\Omega T=87.1 \gg 1$.
causal contributions (in these cases causality is not a primary concern). Furthermore, the RWA enjoys the obvious fact that cavities have bounded size and therefore we cannot have $|\boldsymbol{x}| \approx T$ for large $T$.

Another observation from the results is that proper use of the RWA yields a factor of 2 discrepancy with the UDW model in channel capacity predictions. When properly considered the counter-rotating terms approach zero under long-time integration (RiemannLebesgue lemma) but in a pointwise manner. Given that $3+1 \mathrm{D}$ quantum field communications can only be performed on null-like trajectories it is important to consider the uniform convergence properties of the channel capacity. As shown in appendix B. 6 the RWA discards a significant avenue of communication on the assumption that counter-rotating terms will converge to zero. Instead, given the abundance of null-like communication the counter-rotating terms eventually contribute an equal amount to the overall channel capacity, resulting in a factor 2 difference. Generally this factor of 2 is not a issue as it can be absorbed into the interaction strength parameter $\lambda$; however the RWA predictions for qubit de-excitation rates coincides with UDW without the factor of 2 discrepancy. Therefore the


Figure 6.6: $\mathcal{C}_{\mathrm{AB}}^{\mathrm{RWA}}$, with the first detector at $\boldsymbol{x}_{\mathrm{A}}=(0.4,0.5,0.5) L$ and the second detector at $\boldsymbol{x}_{\mathrm{B}}=(s, 0.5,0.5) L$ with $s$ on the x-axis of the diagram and $L_{1}=L_{2}=L_{3} \equiv L$. The vertical lines $s=0,0.8$ indicate causal limit $(c t), s=0.2,0.6$ indicate time-like null-like signalling boundary and $s=0.4$ indicates the location of the first detector. The interaction times are $0<t_{\mathrm{A}} L^{-1}<0.1,0.3<t_{\mathrm{B}} L^{-1}<0.4$ and $\Omega=871.19$. Here we have $\Omega T \gg 1$. Note the y-axis. The causal regions are $1 / 2$ the size of the UDW. Since $\Omega T<\infty$ acausal tails still persist. Here $\Omega T=87.1 \gg 1$.

RWA model can only be used in very specific, limited cases and then only with great care. This demonstrates the necessity of carefully considering when sums and integrals can be commuted, in particular, here the time integral and mode sum cannot be commuted.

### 6.3.2 Zero-mode and RWA inconsistency

The bulk of the above work concerned the acausal behaviour of the rotating wave approximation when applied to a rectangular Dirichlet cavity. When considering other boundary conditions (§2.3.2), e.g. Neumann or Periodic, one has to include the zero mode [43] to avoid acausal behaviour. However the presence of the zero mode creates several ambiguities; e.g. what is the state of the vacuum mode [44], or how do we divide the zero mode into co-rotating and counter-rotating terms?

One approach is to focus on physical observables, e.g. the 2 qubit communication measure $\mathcal{C}_{\mathrm{AB}}$. Assuming that $\mathcal{C}_{\mathrm{AB}}$ should decay polynomially to zero (as with Dirichlet BC ) this leads to (using Neumann BC for illustration) a RWA decomposition of

$$
\begin{align*}
\hat{H}_{\mathrm{I}}(t) & =\lambda \chi(t)\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \hat{\phi}(t, \boldsymbol{x}),  \tag{6.28}\\
& =\lambda \chi(t)\left(e^{\mathrm{i} \Omega t} \hat{\sigma}^{+}+e^{-\mathrm{i} \Omega t} \hat{\sigma}^{-}\right) \\
& \times\left(\sum_{\boldsymbol{m} \in \mathcal{Z}}\left(\hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(t, \boldsymbol{x})+\hat{a}_{\boldsymbol{m}}^{\dagger} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{x})\right)+\hat{\phi}_{\mathrm{ZM}}(t)\right),  \tag{6.29}\\
& \xrightarrow{\mathrm{RWA}} \lambda \chi(t)\left[\sum_{\boldsymbol{m} \in \mathcal{Z}}\left(\hat{\sigma}^{+} e^{\mathrm{i} \Omega t} \hat{a}_{\boldsymbol{m}} u_{\boldsymbol{m}}(t, \boldsymbol{x})+\hat{\sigma}^{-} e^{-\mathrm{i} \Omega t} \hat{a}_{\boldsymbol{m}}^{\dagger} u_{\boldsymbol{m}}^{*}(t, \boldsymbol{x})\right)\right. \\
& \left.+\hat{\sigma}^{+} e^{\mathrm{i} \Omega t}\left(\frac{1+\mathrm{i}}{2} \hat{\phi}_{\mathrm{ZM}}(t)-\mathrm{i} c_{2} \hat{\pi}_{\mathrm{ZM}}\right)+\hat{\sigma}^{-} e^{-\mathrm{i} \Omega t}\left(\frac{1-\mathrm{i}}{2} \hat{\phi}_{\mathrm{ZM}}(t)+\mathrm{i} c_{2} \hat{\pi}_{\mathrm{ZM}}\right)\right], \tag{6.30}
\end{align*}
$$

where $c_{2}$ is divergent and would need to be treated under a renormalisation theory approach

$$
\begin{equation*}
c_{2}=-\frac{1}{\pi V} \int_{0}^{\infty} \mathrm{d} \xi \tag{6.31}
\end{equation*}
$$

Under this choice of RWA decomposition $\mathcal{C}_{\mathrm{AB}}$ assumes the form of (6.24), without the $\pi$ phase shift upon reflection.

Renormalisation issues aside, this RWA decomposition seems a good choice considering $\mathcal{C}_{\mathrm{AB}}$ is well behaved. However using this very same decomposition (6.30) to study the nonlocality of the interaction Hamiltonian results in an expression very similar to (6.6), with an additional, stray $-\hat{\sigma}_{y}(t) \hat{\phi}_{\mathrm{ZM}}(t) / 2$ term, which introduces a DC non-locality that cannot be removed, even in the long time regime. Therefore, we find that fields that require a zero mode description are inherently incompatible with the RWA; however, our introduction of numerical tricks means we shouldn't need to use RWA for these fields.

### 6.4 Conclusion

In this chapter we introduced two numerical tricks that provided physical intuition and numerical simplicity respectively. In particular this new method can be used to compute the cavity Wightman function efficiently and accurately, a useful skill given the omnipresence of the Wightman function in RQI.

By exploiting these numerical tricks we have been able to analyse the non-locality and causality violations introduced by the RWA, expanding on past free space or approximated results $[3,8,20,72]$. We have found that for a rectangular Dirichlet cavity the non-localities can be thought of as identical to the free space case, i.e. where the Hamiltonian non-locality decays as $1 / r^{2}$ but is reflected by the boundaries, even introducing $\pi$ phase shifts for these 'reflected non-localities', interfering with itself and resulting in an infinite series. This polynomial decay was present in both the RWA interaction Hamiltonian non-locality and the causality violations observed in the 2 qubit communication protocol.

This polynomially decaying non-locality can be quite severe, especially with locality sensitive protocols e.g. entanglement harvesting [13, 70]; however our results in this chapter have shown examples of when the RWA can be used safely within a cavity, in particular when causality is not a major issue (as in the free space case [3]). As with the free space case, we found the channel capacity predictions of the RWA with coupling strength $\lambda$ pointwise converges to the UDW with coupling strength $\lambda / 2$, with this factor of 2 a consequence of commuting limits and integrals without uniform and absolute convergence. However, since cavities are spatially compact, this strengthens the convergence, hence providing a strong justification for the use of RWA in cavities (with care).

We also briefly discussed boundary conditions that allow the field to have a non-trivial zero-mode. In these cases the RWA is not well defined and efforts to implement a RWAtype approximation that, at least weakly, converges to the UDW model in the long time limit are found to be inconsistent. Hence, any boundary conditions that admit a zero mode cannot reasonably implement the RWA, without ignoring the zero mode and introducing substantial causality issues [43].

In cavities, the RWA and importantly the SMA provide mathematical and numerical simplifications to all theoretical predictions. The presence of discrete sums without closed forms make these approximations particularly inviting; however the non-localities they introduce, especially in regimes that are becoming experimentally accessible [10, 81] are slowly demanding their retirement. Furthermore, as we have shown above, with the right numerical trick and current computational power these approximations can be considered obsolete.

## Chapter 7

## Conclusion

This thesis has sought to explore three aspects of the light-matter interaction: how lightmatter interactions can be used to manipulate quantum fields into exotic energy distributions (chapter 3); how well do classical light-matter approximations carry through to quantum light-matter interactions (chapter 4); and what are the relativistic consequences of approximations commonly used to reduce analytic and numerical complexity in relativistic quantum fields (chapters 5 and 6 ). In a broad sense our goal was to determine the limitations of light-matter interactions, not only by exploring the potential states accessible by linear Unruh-DeWitt interactions; but also by determining where, when and how the complexities of quantum light-matter interactions can be faithfully reduced by certain approximations.

## Quantum energy teleportation and exotic spacetime

This first aspect of light-matter interaction involves the manipulation of a quantum field using control detectors (qubits). The objective of this manipulation was to generate specific states of a quantum field capable of violating classical energy conditions, providing an operational description of such states as found in [22]. Previous proposals to generate these states used relativistically accelerating mirrors (dynamical Casimir effect [54]) or nonlinear crystals (squeezed states [53]). Instead we wanted to see if a quantum state could be generated with stationary, linear Unruh-DeWitt (UDW) detectors, which could violate a classical energy condition. For this we used an existing quantum information protocol, namely quantum energy teleportation (QET), for a massless relativistic scalar field and optimised for negative energy generation. Quantum energy teleportation involves

Alice performing a local measurement (POVM) on an entangled ground state, transmitting the measurement result to Bob who then uses this information to perform a local unitary of the system, resulting in an extraction of energy. In order to explore the efficacy of QET, both analytically and numerically, we applied the protocol to $1+1 \mathrm{D}$ and $3+1 \mathrm{D}$ scalar fields.

We found that for the specific QET protocol we used, the expectation of the $1+1 \mathrm{D}$ energy density operator has positive contributions proportional to $\left[\lambda^{\prime}(x \pm t)\right]^{2}$ (from Alice's $\lambda(x)$ smearing and derivative coupling), $[\mu(x \pm t)]^{2}$ (from Bob's $\mu(x)$ smearing and UDW coupling); and negative (QET) contributions proportional to $\lambda^{\prime \prime}(x \pm t) \mu(x)$. In a more elaborate $3+1 \mathrm{D}$ field, we found that when restricted to spherically symmetric smearing functions, similar functional dependence of the energy density occurred. The energy density injected into the system by Alice's (derivative coupled) POVM was approximately dependent on $\left[\partial_{r}(r \lambda(r))\right]^{2}$ and the subsequent QET contributions were approximately proportional to $\partial_{r}^{2}(r \lambda(r))$, where $r=|\boldsymbol{x}|-t$. Thus with QET we can generate negative energy densities and have simple intuitive guidelines of how to distribute and smear the various UDW detectors to sculpt a desired energy density distribution. This general functional intuition can also be numerically observed in the various energy distributions we plotted in chapter 3.

We also explored the size and depth of the negative energy wells, especially with its relation to the quantum energy conditions and quantum interest conjecture. We established that QET (in $3+1 \mathrm{D}$ ) is capable of generating regions containing arbitrary amounts of negative energy, with the total amount of negative energy $\Delta E$ increasing as the negative energy well's width $\Delta r$ is decreased by $\Delta E \sim \Delta r^{-3}$. However, this is accompanied by increasingly large $\left(\sim \Delta r^{-3}\right)$ positive energy peaks on either side of the well. This scaling relation saturates the quantum interest conjecture, suggesting the near optimality of QET for generating negative energy densities.

Thus QET is an operationally feasible quantum information protocol for generating negative energy densities, for which we introduced guidelines for energy distribution control by means of the detector smearings. We hope this protocol can be used as a fundamental operation for sculpting specific exotic energy density distributions with the goal of generating unusual spacetime geometries. A potential variation of this protocol would use a massive scalar field, so that the protocol may generate wave packets whose leading edge has a negative energy density (like the dynamical Casimir effect). Future work would need to consider 1) the gravitational back-reaction to the energy distribution generated by the protocol and 2) what additional guidelines are needed if 2 QET protocols are implemented in series?

## Dipole model with quantum EM fields

This second aspect of light-matter interaction considers if the dipole approximation, a well defined and valid approximation in light-matter interactions with classical EM fields, is still valid when considering a quantum EM field. The dipole approximation requires that the dominant wavelength $(\lambda)$ of the EM field be much larger than the typical size of the atom it is interacting with (e.g. Bohr radius $a_{0}$ ), mathematically $\lambda \gg a_{0}$. Quantisation of EM theory introduces a field with vacuum fluctuation, with no good definition for a dominant wavelength. Past works have attempted to overcome this by using point-like atoms [4], however this introduces UV divergences in the response of the atoms [5, 6] and is inadequate for describing light-matter interactions. We consider effective models (infinite nucleus mass), which neglect the centre of mass degrees of freedom and their additional complexity [7], whilst allowing accurate investigations of the electronic orbital behaviour. By using the dressed state formalism proposed by Lamb et al. [4] to remove gauge issues, we computed atomic transition probabilities for Hydrogen-like atoms using both dipole and minimal models; and used these results to introduce additional criteria for the validity of the dipole model in quantum EM.

Our analysis of atomic transitions has found that if a dominant wavelength exists (with frequency $\omega_{0}$ ), then the difference between the minimal and dipole models will be of order $\omega_{0} a_{0}$, therefore if the dipole criterion is met the two models will converge (as $\omega_{0} a_{0} \ll 1$ ). Therefore our attention is focussed on determining when a quantum EM field interacting with an atom with energy gap $\Omega$ (modelled by a qubit) has a dominant wavelength. A straightforward case is an excited field, whose excitations define the dominant wavelength. However, unlike classical EM, the excitations need to be strong enough to overwhelm the contributions from the vacuum fluctuations.

An important RQI situation involves placing an atom in the vacuum EM field, the prototypical example where the field has no dominant wavelength. When considering fast interactions, i.e. faster than the light crossing time of the atom, the minimal and dipole models differ significantly. We find the Fourier transformation of the switching function (in this case sudden switching on and off coinciding with state preparation and measurement respectively) serves to cutoff UV modes, and for fast interactions this cutoff is virtually non-existent. When considering longer interaction times, the Fourier transformation of the switching function begins to suppress UV modes, and if the atom is initially excited this leads to a dominant wavelength (with frequency $\Omega$ ) emerging from the single mode approximation. When the atom is initially unexcited, we found that as the UV cutoff $\omega_{c}$ moves to the IR, the difference between the minimal and dipole models became of order $\omega_{c} a_{0}$, i.e. $\omega_{c}$ could be used to test the dipole criterion.

The discrepancy between the dipole and minimal models in the vacuum excitation scenario revolves around the lack of a dominant wavelength in the EM vacuum and the failure of the switching function's Fourier transform to filter out UV modes that violate the dipole criterion. Our work shows that shrinking the atom (by increasing the proton number) does not improve the difference as there remain an arbitrary number of UV modes that contribute to this difference. Instead we found that a switching function capable of suppressing UV modes (where $\omega_{c} a_{0} \ll 1$ ) is needed, indicating a need for a long interaction time and preferably a smooth switching. In this way the Fourier transformation of the switching function would isolate a band of dominant modes that, given a sufficient long interaction time, will satisfy the dipole criterion.

Generally, the validity of the dipole model in quantum EM fields relies on ancillary methods for defining a dominant mode, followed by satisfaction of the dipole criterion by that mode. We have shown that the introduction of a (preferably smooth) switching in vacuum excitation scenarios ensures satisfaction of the dipole criterion as long as the interaction time between atom and field is longer than the light-crossing time of the diameter of the atom. Similarly, spontaneous emission scenarios require long interaction times to isolate a dominant mode $\Omega$ (the atomic energy gap), which would in turn have to satisfy the dipole criterion to ensure the validity of the dipole model. This would justify the widespread use of the dipole approximation in modelling light-matter interactions, even for vacuum fluctuations (as in the case of entanglement harvesting [13, 70, 71] and the Fermi problem $[12,16]$ ), but crucially not because of an argument of a 'small atom', but instead for 'sufficiently long interaction time'. Therefore, we extend the criteria for the validity of the dipole model in quantum EM field, requiring any intrinsically dominant modes, e.g. atomic energy gap (for vacuum emissions) or excited EM modes, satisfy the dipole approximation criterion and that the interaction time is longer than the light crossing time of the atom.

## RWA's causality violations

This third aspect of light-matter interaction considers the causality violations introduced into a relativistic theory by a common, simplifying approximation, namely the rotating wave approximation and in what regimes it may be accurately used. Working in the scalar field, the rotating wave approximation eliminates terms from the interaction Hamiltonian that do not conserve excitation number, i.e. $\hat{\sigma}^{+} \hat{a}_{\boldsymbol{k}}^{\dagger}$ and $\hat{\sigma}^{-} \hat{a}_{\boldsymbol{k}}$, arguing that for long interaction times these terms will vanish. The criterion for evaluating long interaction time $T$ is given as $\Omega T \gg 1$, where $\Omega$ is the detector's energy gap.

Our study of the RWA interaction Hamiltonian showed a non-locality proportional
to the vacuum Wightman function of the field, verifying and extending the results of Clerk and Sipe [8]. We also found that this corresponding $1 / r^{2}$ Hamiltonian non-locality translates into $1 / d^{4}$ and $1 / d^{6}$ causality violations in the expectations values $\left\langle: \hat{\phi}^{2}(\boldsymbol{x}):\right\rangle$ and $\left\langle: \hat{T}_{00}(\boldsymbol{x}):\right\rangle$ respectively, quantitatively extending the results of Compagno et al. [20, 72]; where $d$ is the spatial distance from $\boldsymbol{x}$ to the surface of the detector's light-cone. This means the RWA causality violations persist even if $\Omega T \gg 1$. Instead, if we examine a spacetime point that is in the bulk of the detector's light-cone, i.e. a point that is null to the detector when it was still on, and long interaction times, we find that the counterrotating terms in the UDW expectations vanish, as argued by RWA. However, in this long time limit, in the bulk of the light-cone, the RWA does not converge to the UDW; instead the RWA with interaction strength $\lambda$ converges to the UDW with interaction strength $\lambda / 2$.

We also considered a 2 qubit communication protocol, and evaluated its channel capacity for UDW and RWA models. We found that under the RWA, the channel capacity experienced superluminal communications, with decay $1 / d^{2}$. The resulting causality violations were polynomially decaying in the first detector's space-like and time-like regions, violating the strong Huygens principle with time-like communication in a massless $3+1 \mathrm{D}$ field. As with the field observables the RWA model coincided with the half-strength UDW model when both detectors satisfied $\Omega T \gg 1$ and the second detector was placed in the bulk of the first detector's light-cone.

Our study of RWA continued into the cavity (Dirichlet boundaries), where discrete modes suggest the RWA should be even more reliable. As with the free space case the RWA interaction Hamiltonian non-locality was proportional to the equal time, vacuum Wightman function and the 2 qubit communication's channel capacity was proportional to the integral of the unequal time, vacuum Wightman function. In order to evaluate these Wightman functions we introduced two exact numerical tricks, outlined in §6.1.2. The first of these tricks translates the mode sum defining the Wightman function into a sum of free space Wightman functions, reflected infinitely off the mirrored boundaries (including a $\pi$ phase shift). Intuitively this meant the non-localities introduced by the RWA in free space can be directly translated to non-localities of cavity fields by considering all the superposition from reflections.

Our second numerical trick transformed the Wightman function's mode sum into an exponentially decaying series, an ideal sum for numerical evaluation. Using this 2nd trick we were able to plot the non-locality of the RWA interaction Hamiltonian and the superluminal communications of the 2 qubit communication protocol. One of our results is a plot of the 2 qubit channel capacity when $\Omega T \gg 1$, i.e. the usual RWA criterion, where $T$ is less than the light crossing time of the cavity. It numerically illustrates how the RWA
converges to the half-strength UDW model when in the bulk of the first detector's lightcone, and how the non-localities persist, leaking into the space-like and time-like regions of the cavity (relative the the first detector's interaction).

In addition to the causality issues introduced by the RWA, there results suggest that when $\Omega T \gg 1$ and within the bulk of the detector's light-cone the RWA converges to the half-strength UDW model. This is a consequence of the RWA argument, which commutes a limit with an integral without absolute and uniform convergence. Despite this mathematical faux pas, the RWA is a valid model, provided the coupling strength is properly scaled; however this should be considered a warning to take greater care in derivations.

Hence, we find that the criteria for the validity of the RWA requires additions, i.e. RWA is valid for long times $\Omega T \gg 1$, within the bulk of the detector' light-cone and requires a coupling strength rescaling for convergence with UDW predictions. We also find that in cavity fields, where discrete sums without closed forms encourage the use of RWA and SMA, there are numerical tricks available that remove the need for such approximations. Our 2nd numerical trick transforms an oscillatory and slowly decaying mode sum into an exponentially decaying sum, improving computation time whilst reducing intrinsic computational errors. With our 2nd numerical trick and current computational power we believe the RWA is no longer necessary in cavities, especially important in regimes that are becoming experimentally accessible [10, 81].

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## APPENDICES

## Appendix A

## QET in 3+1 D: a numerical perspective

## A. 1 Simplifications under spherical symmetry

In the main text the expectation of the stress-energy tensor was dependent on 3 functions (3.88),(3.89) and (3.90):

$$
\begin{align*}
& I_{\mu}^{1}(\boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} \frac{k_{\mu}}{|\boldsymbol{k}|}\left(e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i} \Delta T)+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}+e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i} \Delta T)-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \mu(\boldsymbol{r}),  \tag{A.1}\\
& I_{\mu}^{2}(\boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} k_{\mu}\left(e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}-e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \lambda(\boldsymbol{r}),  \tag{A.2}\\
& I_{\mu}^{3}(\boldsymbol{x})=\int \mathrm{d}^{3} \boldsymbol{r} d^{3} \boldsymbol{k} k_{\mu}\left(e^{|\boldsymbol{k}|(-2 \varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}+e^{|\boldsymbol{k}|(-2 \varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{r}-\boldsymbol{x})}\right) \lambda(\boldsymbol{r}) . \tag{A.3}
\end{align*}
$$

In the main text these equations were simplified under the assumption of spherical symmetry to become equations (3.97) to (3.102). In this form the equations are pleasant 1D integrals; and if $\mu(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ are smooth and large enough, their Fourier transforms will suppress UV modes in the integrals, allowing for ease of computation.

A different approach to simplification is to eliminate the momentum integrals and numerically deal with spatial integrals instead. This has a numerical advantage is $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ are small and sharp, i.e. their Fourier transforms do not decay quickly. In this
case, recalling that

$$
\begin{align*}
\lambda(\boldsymbol{x}) & =\lambda\left(\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{A}}\right|\right),  \tag{A.4}\\
\mu(\boldsymbol{x}) & =\mu\left(\left|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{B}}\right|\right),  \tag{A.5}\\
\boldsymbol{y}_{\mathrm{A}} & =\boldsymbol{x}-\boldsymbol{x}_{\mathrm{A}},  \tag{A.6}\\
\boldsymbol{y}_{\mathrm{B}} & =\boldsymbol{x}-\boldsymbol{x}_{\mathrm{B}}, \tag{A.7}
\end{align*}
$$

then the $I_{\mu}^{i}$ functions evaluated at time $T+\Delta T$ become

$$
\begin{align*}
I_{0}^{1}(\boldsymbol{x}) & =\frac{8 \pi^{3}}{\left|\boldsymbol{y}_{\mathrm{B}}\right|}\left[\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|-\Delta T\right) \Theta\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|-\Delta T\right) \mu\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|-\Delta T\right)\right. \\
& +\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|-\Delta T\right) \Theta\left(-\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T\right) \mu\left(-\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T\right) \\
& \left.+\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T\right) \Theta\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T\right) \mu\left(\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T\right)\right],  \tag{A.8}\\
\vec{I}_{i}^{1} & =-\frac{8 \pi^{3} \boldsymbol{y}_{\mathrm{B}}}{\left|\boldsymbol{y}_{\mathrm{B}}\right|^{3} \int_{\boldsymbol{y}_{\mathrm{B}}|-\Delta T|}^{\left|\boldsymbol{y}_{\mathrm{B}}\right|+\Delta T} \mathrm{~d} r r \mu(r),}  \tag{A.9}\\
I_{0}^{2} & =\frac{8 \pi^{3} \mathrm{i}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|}\left[-\left|T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|\right| \lambda^{\prime}\left(| | \boldsymbol{y}_{\mathrm{A}}|-T-\Delta T|\right)-\lambda\left(| | \boldsymbol{y}_{\mathrm{A}}|-T-\Delta T|\right)\right. \\
& \left.+\left(T+\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \lambda^{\prime}\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right)+\lambda\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right)\right],  \tag{A.10}\\
\vec{I}_{i}^{2} & =-\frac{8 \pi^{3} \mathrm{i} \boldsymbol{y}_{\mathrm{A}}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{3}}\left[-\left|\boldsymbol{y}_{\mathrm{A}}\right|\left|T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|\right| \lambda^{\prime}\left(| | \boldsymbol{y}_{\mathrm{A}}|-T-\Delta T|\right)\right. \\
& -(T+\Delta T) \lambda\left(| | \boldsymbol{y}_{\mathrm{A}}|-T-\Delta T|\right)-\left|\boldsymbol{y}_{\mathrm{A}}\right|\left(T+\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \lambda^{\prime}\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right) \\
& \left.+(T+\Delta T) \lambda\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right)\right],  \tag{A.11}\\
I_{0}^{3} & =\frac{8 \pi^{2}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|} \mathrm{P} . \mathrm{V} . \int_{0}^{\infty} \mathrm{d} r\left[\frac{-\lambda(r)+\left(T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \lambda^{\prime}(r)}{r+T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|}+\frac{\lambda(r)+\left(T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \lambda^{\prime}(r)}{r-T-\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|}\right. \\
& -\frac{-\lambda(r)+\left(T+\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|\right) \lambda^{\prime}(r)}{r+T+\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|} \tag{A.12}
\end{align*}
$$

$$
\begin{align*}
\vec{I}_{i}^{3} & =\frac{8 \pi^{2} \boldsymbol{y}_{\mathrm{A}}}{\left|\boldsymbol{y}_{\mathrm{A}}\right|^{3}} \mathrm{P} \cdot \mathrm{~V} \cdot \int_{0}^{\infty} \mathrm{d} r\left[\frac{-(T+\Delta T) \lambda(r)+\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|-T-\Delta T\right)\left|\boldsymbol{y}_{\mathrm{A}}\right| \lambda^{\prime}(r)}{r-T-\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|}\right. \\
& +\frac{(T+\Delta T) \lambda(r)+\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|-T-\Delta T\right)\left|\boldsymbol{y}_{\mathrm{A}}\right| \lambda^{\prime}(r)}{r+T+\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|} \\
& -\frac{-(T+\Delta T) \lambda(r)+\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right)\left|\boldsymbol{y}_{\mathrm{A}}\right| \lambda^{\prime}(r)}{r-T-\Delta T-\left|\boldsymbol{y}_{\mathrm{A}}\right|} \\
& \left.-\frac{(T+\Delta T) \lambda(r)+\left(\left|\boldsymbol{y}_{\mathrm{A}}\right|+T+\Delta T\right)\left|\boldsymbol{y}_{\mathrm{A}}\right| \lambda^{\prime}(r)}{r+T+\Delta T+\left|\boldsymbol{y}_{\mathrm{A}}\right|}\right] \tag{A.13}
\end{align*}
$$

where the orientation of $\vec{I}^{i}$ is parallel to $\boldsymbol{y}_{\mathrm{A}, \mathrm{B}}$. In this form the functions $I_{\mu}^{i}$ become significantly cheaper to compute, especially if $\lambda$ and $\mu$ have small supports (e.g. with respect to $T$ ). In this form it is easier to see how Alice's energy contributions $\left(I^{2}\right)$ have a broad $\partial_{r}(r \lambda(r))$ dependence and how the QET contributions $\left(I^{3}\right)$ have a broad $\partial_{r}^{2}(r \lambda(r))$ dependence.

## Appendix B

## Locality, causality and the approximations of quantum optics in free space

## B. 1 RWA Hamiltonian non-locality integrals

In section 5.1.2 we made use of equations (5.7) and (5.8) to demonstrate that the RWA Hamiltonian has non-local interaction terms. Equation (5.7) is the well known result, whilst equation (5.8) requires a couple of careful considerations. For brevity let $\boldsymbol{r}:=\boldsymbol{y}-\boldsymbol{z}$,

$$
\begin{align*}
\int \mathrm{d}^{3} \boldsymbol{k} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}}{\omega} & =\int_{0}^{\infty} \mathrm{d} \omega \int_{0}^{2 \pi} \mathrm{~d} \phi \int_{-1}^{1} \mathrm{~d} z \omega^{2} \frac{e^{\mathrm{i} \omega r} z}{\omega}  \tag{B.1}\\
& =2 \pi \int_{0}^{\infty} \mathrm{d} \omega \omega \frac{e^{\mathrm{i} \omega r}-e^{-\mathrm{i} \omega r}}{\mathrm{i} \omega r}  \tag{B.2}\\
& =\frac{2 \pi}{\mathrm{i} r} \int_{0}^{\infty} \mathrm{d} \omega\left(e^{\mathrm{i} \omega r}-e^{-\mathrm{i} \omega r}\right) \tag{B.3}
\end{align*}
$$

At this point we introduce a soft UV cutoff as a regulariser to facilitate the $\omega$ integral. This cutoff takes the form of $e^{-\varepsilon \omega}(\varepsilon>0)$, where following the $\omega$ integration we will take $\varepsilon \rightarrow 0^{+}$.

$$
\begin{align*}
\frac{2 \pi}{\mathrm{i} r} \int \mathrm{~d} \omega\left(e^{\mathrm{i} \omega r}-e^{-\mathrm{i} \omega r}\right) & =\lim _{\varepsilon \rightarrow 0} \frac{2 \pi}{\mathrm{i} r} \int \mathrm{~d} \omega\left(e^{\omega(\mathrm{i} r-\varepsilon)}-e^{\omega(-\mathrm{i} r-\varepsilon)}\right)  \tag{B.4}\\
& =\lim _{\varepsilon \rightarrow 0} \frac{2 \pi}{\mathrm{i} r}\left(-\frac{1}{\mathrm{i} r-\varepsilon}+\frac{1}{-\mathrm{i} r-\varepsilon}\right)  \tag{B.5}\\
& =\lim _{\varepsilon \rightarrow 0} \frac{2 \pi}{\mathrm{i} r}\left(-\frac{2 \mathrm{i} r}{-r^{2}-\varepsilon^{2}}\right)=\frac{4 \pi}{r^{2}} \tag{B.6}
\end{align*}
$$

This leaves us with equation (5.8),

$$
\begin{equation*}
\int \mathrm{d}^{3} \boldsymbol{k} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot(\boldsymbol{y}-\boldsymbol{z})}}{\omega}=\frac{4 \pi}{|\boldsymbol{y}-\boldsymbol{z}|^{2}} \tag{B.7}
\end{equation*}
$$

## B. 2 RWA $\delta$-switching unitary time evolution operator

In section 5.2 .1 we stated that the time evolution operator generated by the RWA Hamiltonian under a $\delta$-switching, after considering that it will be acting on the vacuum (i.e. the time evolution operator restricted to that particular state of the field), is given by equation (5.18). Its derivation follows:

$$
\begin{align*}
\hat{U} & =\mathcal{T} \exp \left(-\mathrm{i} \int \hat{H}_{\mathrm{I}} \mathrm{~d} t\right)  \tag{B.8}\\
& =\exp \left(-\mathrm{i}\left(\hat{\alpha} \hat{\sigma}^{+}+\hat{\alpha}^{\dagger} \hat{\sigma}^{-}\right)\right)  \tag{B.9}\\
& =\sum_{n=0}^{\infty} \frac{(-\mathrm{i})^{2 n}}{(2 n)!}\left(\hat{\sigma}^{+} \hat{\alpha}+\hat{\sigma}^{-} \hat{\alpha}^{\dagger}\right)^{2 n}+\sum_{n=0}^{\infty} \frac{(-\mathrm{i})^{2 n+1}}{(2 n+1)!}\left(\hat{\sigma}^{+} \hat{\alpha}+\hat{\sigma}^{-} \hat{\alpha}^{\dagger}\right)^{2 n+1}  \tag{B.10}\\
& =\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n)!}\left(\hat{\Pi}_{e}\left(\hat{\alpha} \hat{\alpha}^{\dagger}\right)^{n}+\hat{\Pi}_{g}\left(\hat{\alpha}^{\dagger} \hat{\alpha}\right)^{n}\right)-\mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!}\left(\hat{\sigma}^{+}\left(\hat{\alpha} \hat{\alpha}^{\dagger}\right)^{n} \hat{\alpha}+\hat{\sigma}^{-} \hat{\alpha}^{\dagger}\left(\hat{\alpha} \hat{\alpha}^{\dagger}\right)^{n}\right)  \tag{B.11}\\
& =\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n)!}\left(\hat{\Pi}_{e}\left(\hat{\alpha}^{\dagger} \hat{\alpha}+K^{2} \hat{\mathbb{I}}\right)^{n}+\hat{\Pi}_{g}\left(\hat{\alpha}^{\dagger} \hat{\alpha}\right)^{n}\right) \\
& -\mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!}\left(\hat{\sigma}^{+}\left(\hat{\alpha}^{\dagger} \hat{\alpha}+K^{2} \hat{\mathbb{I}}\right)^{n} \hat{\alpha}+\hat{\sigma}^{-} \hat{\alpha}^{\dagger}\left(\hat{\alpha}^{\dagger} \hat{\alpha}+K^{2} \hat{\mathbb{I}}\right)^{n}\right) . \tag{B.12}
\end{align*}
$$

Here $\hat{\Pi}_{g}:=|g\rangle\langle g|, \hat{\Pi}_{e}:=|e\rangle\langle e|$ refer to projection operators on the detector Hilbert space. Note that all the field operators $\hat{\alpha}$ are evaluated at $t=0$. Further note that

$$
\begin{equation*}
K^{2} \hat{\mathbb{I}}:=\left[\hat{\alpha}(0), \hat{\alpha}^{\dagger}(0)\right]=\tilde{\lambda}^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}|\tilde{F}(\boldsymbol{k})|^{2} \hat{\mathbb{I}} \tag{B.13}
\end{equation*}
$$

Acting with (B.12) on the vacuum we can cancel all terms that annihilate it and therefore

$$
\begin{align*}
\hat{U}|0\rangle & =\left[\hat{\Pi}_{g}+\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n)!} \hat{\Pi}_{e} K^{2 n}-\mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!} \hat{\sigma}^{-} \hat{\alpha}^{\dagger} K^{2 n}\right]|0\rangle  \tag{B.14}\\
& =\left[\hat{\Pi}_{g}+\hat{\Pi}_{e} \cos K-\mathrm{i} \frac{\hat{\sigma}^{-} \hat{\alpha}^{\dagger}(0)}{K} \sin K\right]|0\rangle, \tag{B.15}
\end{align*}
$$

where in the final step the time dependence of $\hat{\alpha}$ is explicitly shown for clarity.

## B. 3 Field expectations under perturbative expansions

Here we present a derivation of the expectation values $\left\langle\hat{T}_{\mu \nu}\right\rangle$ and $\left\langle\hat{\phi}^{2}\right\rangle$ when using second order perturbation theory both for the full model and under the RWA.

## B.3.1 Full model expectations

Without the RWA approximation the interaction Hamiltonian is

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\lambda \chi(t) \hat{\sigma}_{x}(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right) \tag{B.16}
\end{equation*}
$$

where, in order to simplify, we can define

$$
\begin{align*}
\tilde{F}(\boldsymbol{k}) & :=\int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}},  \tag{B.17}\\
\hat{\alpha}(t) & :=\lambda \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}} \tag{B.18}
\end{align*}
$$

then

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\chi(t) \hat{\sigma}_{x}(t)\left(\hat{\alpha}+\hat{\alpha}^{\dagger}\right) \tag{B.19}
\end{equation*}
$$

The corresponding second order time evolution operator becomes

$$
\begin{align*}
\hat{U} & =\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) \hat{\sigma}_{x}\left(t_{1}\right)\left(\hat{\alpha}\left(t_{1}\right)+\hat{\alpha}^{\dagger}\left(t_{1}\right)\right) \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) \hat{\sigma}_{x}\left(t_{1}\right) \hat{\sigma}_{x}\left(t_{2}\right)\left(\hat{\alpha}\left(t_{1}\right)+\hat{\alpha}^{\dagger}\left(t_{1}\right)\right)\left(\hat{\alpha}\left(t_{2}\right)+\hat{\alpha}^{\dagger}\left(t_{2}\right)\right)+\mathcal{O}\left(\lambda^{3}\right), \tag{B.20}
\end{align*}
$$

where the interaction time is encoded in the shape and support of $\chi(t)$.
Taking into account that

$$
\begin{equation*}
\left[\hat{\alpha}\left(t_{1}\right), \hat{\alpha}^{\dagger}\left(t_{2}\right)\right]=\lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}|\tilde{F}(\boldsymbol{k})|^{2} e^{-\mathrm{i} \omega\left(t_{1}-t_{2}\right)} \tag{B.21}
\end{equation*}
$$

$\hat{U}$ acting on the vacuum can be simplified to

$$
\begin{align*}
\hat{U}|0\rangle & =\left[\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) \hat{\sigma}_{x}\left(t_{1}\right) \hat{\alpha}^{\dagger}\left(t_{1}\right)-\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right.\right. \\
& \left.\left.+\Pi_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right)\left(\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\alpha}^{\dagger}\left(t_{2}\right)+\lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega} e^{-\mathrm{i} \omega\left(t_{1}-t_{2}\right)}|\tilde{F}(\boldsymbol{k})|^{2}\right)\right]|0\rangle . \tag{B.22}
\end{align*}
$$

This yields components with 0,1 and 2 excitations. By taking the expectation values and using the relation

$$
\begin{equation*}
\left[\hat{a}_{\boldsymbol{k}}, \hat{\alpha}^{\dagger}\left(t_{1}\right)\right]=\lambda \frac{e^{\mathrm{i} \omega t_{1}} \tilde{F}^{*}(\boldsymbol{k})}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tag{B.23}
\end{equation*}
$$

we can write

$$
\begin{align*}
& \left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {Full }}=\lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{6} 4 \omega \omega^{\prime}}\left(k_{\mu} k_{\nu}^{\prime}-\frac{\eta_{\mu \nu}}{2} k_{\gamma} k^{\prime \gamma}\right) \\
& {\left[e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{1}^{\prime} \chi\left(t_{1}\right) \chi\left(t_{1}^{\prime}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}\right) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}^{*}(\boldsymbol{k}) e^{-\mathrm{i} \omega^{\prime} t_{1}+\mathrm{i} \omega t_{1}^{\prime}}\right.} \\
& +e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{1}^{\prime} \chi\left(t_{1}\right) \chi\left(t_{1}^{\prime}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}\right) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i} \omega t_{1}+\mathrm{i} \omega^{\prime} t_{1}^{\prime}} \\
& +e^{-\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}^{*}(\boldsymbol{k}) \\
& \times\left(e^{\mathrm{i} \omega t_{1}+\mathrm{i} \omega^{\prime} t_{2}}+e^{\mathrm{i} \omega^{\prime} t_{1}+\mathrm{i} \omega t_{2}}\right)+e^{\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) \\
& \left.\times\left(\hat{\Pi}_{e} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}+\hat{\Pi}_{g} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}(\boldsymbol{k})\left(e^{-\mathrm{i} \omega t_{1}-\mathrm{i} \omega^{\prime} t_{2}}+e^{-\mathrm{i} \omega^{\prime} t_{1}-\mathrm{i} \omega t_{2}}\right)\right],  \tag{B.24}\\
& \left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\text {Full }}=\lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{6} 4 \omega \omega^{\prime}} \\
& {\left[e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{1}^{\prime} \chi\left(t_{1}\right) \chi\left(t_{1}^{\prime}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}\right) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}^{*}(\boldsymbol{k}) e^{-\mathrm{i} \omega^{\prime} t_{1}+\mathrm{i} \omega t_{1}^{\prime}}\right.} \\
& +e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{1}^{\prime} \chi\left(t_{1}\right) \chi\left(t_{1}^{\prime}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{1}^{\prime}\right)}\right) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i} \omega t_{1}+\mathrm{i} \omega^{\prime} t_{1}^{\prime}} \\
& -e^{-\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\Pi}_{e} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}+\hat{\Pi}_{g} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right) \tilde{F}^{*}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}^{*}(\boldsymbol{k}) \\
& \times\left(e^{\mathrm{i} \omega t_{1}+\mathrm{i} \omega^{\prime} t_{2}}+e^{\mathrm{i} \omega^{\prime} t_{1}+\mathrm{i} \omega t_{2}}\right)-e^{\mathrm{i}\left(k_{\mu}+k_{\mu}^{\prime}\right) x^{\mu}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) \\
& \left.\times\left(\hat{\Pi}_{e} e^{-\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}+\hat{\Pi}_{g} e^{\mathrm{i} \Omega\left(t_{1}-t_{2}\right)}\right) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) \tilde{F}(\boldsymbol{k}) \times\left(e^{-\mathrm{i} \omega t_{1}-\mathrm{i} \omega^{\prime} t_{2}}+e^{-\mathrm{i} \omega^{\prime} t_{1}-\mathrm{i} \omega t_{2}}\right)\right] . \tag{B.25}
\end{align*}
$$

In the equations above the contributions to the expectations from 1 excitation states are those of the form $\tilde{F} \tilde{F}^{*}$, where as the remainder, i.e. $\tilde{F} \tilde{F}$ and $\tilde{F}^{*} \tilde{F}^{*}$, are contributions from the superposition of 0 and 2 excitation states.
Here we assumed that $t$ (the observable measurement time) is larger than the maximum $t$ in the support of $\chi(t)$, i.e. post interaction measurement.

In order to simplify this rather long expression and further compare with the RWA, we define the following:

$$
\begin{align*}
J_{\mu, e}^{1}(\boldsymbol{x}, t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{\omega} k_{\mu} \tilde{F}(\boldsymbol{k}) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) e^{-\mathrm{i}(\omega-\Omega) t_{1}},  \tag{B.26}\\
J_{\mu \nu, e}^{2}(\boldsymbol{x}, t) & :=\int_{-\infty} \frac{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{\omega \omega^{\prime}} k_{\mu} k_{\nu}^{\prime} \tilde{F}(\boldsymbol{k}) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \\
& \times \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}+e^{-\mathrm{i}(\omega-\Omega) t_{2}-\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{1}}\right),  \tag{B.27}\\
M_{e}^{1}(\boldsymbol{x}, t) & :=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{\omega} \tilde{F}(\boldsymbol{k}) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) e^{-\mathrm{i}(\omega-\Omega) t_{1}},  \tag{B.28}\\
M_{e}^{2}(\boldsymbol{x}, t) & =\int^{\frac{\mathrm{d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{\omega \omega^{\prime}} \tilde{F}(\boldsymbol{k}) \tilde{F}\left(\boldsymbol{k}^{\prime}\right) e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}}} \\
& \times \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}+e^{-\mathrm{i}(\omega-\Omega) t_{2}-\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{1}}\right), \tag{B.29}
\end{align*}
$$

with $J_{\mu, g}^{1}, J_{\mu \nu, g}^{2}, M_{g}^{1}$ and $M_{g}^{2}$ differing from those above by a swap $\Omega \rightarrow-\Omega$. This way

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}} \hat{\Pi}_{i}\left(J_{\mu, i}^{1} J_{\nu, i}^{1 *}+J_{\mu, i}^{1 *} J_{\nu, i}^{1}-\frac{\eta_{\mu \nu}}{2}\left(J_{\gamma, i}^{1} J_{i}^{1 \gamma *}+J_{\gamma, i}^{1 *} J_{i}^{1 \gamma}\right)\right. \\
& \left.+J_{\mu \nu, i}^{2}+J_{\mu \nu, i}^{2 *}-\frac{\eta_{\mu \nu}}{2}\left(J_{\gamma, i}^{2 \gamma}+J_{\gamma, i}^{2 \gamma *}\right)\right)+\mathcal{O}\left(\lambda^{3}\right)  \tag{B.30}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}} \hat{\Pi}_{i}\left(2\left|M_{i}^{1}\right|^{2}-M_{i}^{2}-M_{i}^{2 *}\right)+\mathcal{O}\left(\lambda^{3}\right) \tag{B.31}
\end{align*}
$$

The projection operators meant that if we consider an initial state given by $|0\rangle \otimes\left(a_{g}|g\rangle+\right.$ $a_{e}|e\rangle$ ) then the equations above simplify to

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}}\left|a_{i}\right|^{2}\left(J_{\mu, i}^{1} J_{\nu, i}^{1 *}+J_{\mu, i}^{1 *} J_{\nu, i}^{1}-\frac{\eta_{\mu \nu}}{2}\left(J_{\gamma, i}^{1} J_{i}^{1 \gamma *}+J_{\gamma, i}^{1 *} J_{i}^{1 \gamma}\right)\right. \\
& \left.+J_{\mu \nu, i}^{2}+J_{\mu \nu, i}^{2 *}-\frac{\eta_{\mu \nu}}{2}\left(J_{\gamma, i}^{2 \gamma}+J_{\gamma, i}^{2 \gamma *}\right)\right)+\mathcal{O}\left(\lambda^{3}\right),  \tag{B.32}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\text {Full }} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \sum_{i \in\{e, g\}}\left|a_{i}\right|^{2}\left(2\left|M_{i}^{1}\right|^{2}-M_{i}^{2}-M_{i}^{2 *}\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{B.33}
\end{align*}
$$

## B.3.2 RWA expectations

The RWA interaction Hamiltonian is (see (5.12))

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i}(\omega-\Omega) t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}} \hat{\sigma}^{+}+e^{\mathrm{i}(\omega-\Omega) t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{\sigma}^{-}\right) \tag{B.34}
\end{equation*}
$$

where, in order to simplify, we can define

$$
\begin{align*}
\tilde{F}(\boldsymbol{k}) & :=\int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}},  \tag{B.35}\\
\hat{\alpha}(t) & :=\lambda \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}(\boldsymbol{k}) e^{-\mathrm{i}(\omega-\Omega) t} \hat{a}_{\boldsymbol{k}} \tag{B.36}
\end{align*}
$$

then

$$
\begin{equation*}
\hat{H}_{\mathrm{I}}(t)=\chi(t)\left(\hat{\alpha}(t) \hat{\sigma}^{+}+\hat{\alpha}^{\dagger}(t) \hat{\sigma}^{-}\right) \tag{B.37}
\end{equation*}
$$

The corresponding second order time evolution operator becomes

$$
\begin{align*}
\hat{U} & =\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\sigma}^{+}+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\sigma}^{-}\right) \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\sigma}^{+}+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\sigma}^{-}\right)\left(\hat{\alpha}\left(t_{2}\right) \hat{\sigma}^{+}+\hat{\alpha}^{\dagger}\left(t_{2}\right) \hat{\sigma}^{-}\right)+\mathcal{O}\left(\lambda^{3}\right)  \tag{B.38}\\
& =\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\sigma}^{+}+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\sigma}^{-}\right)
\end{align*}
$$

$$
\begin{equation*}
-\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(\hat{\alpha}\left(t_{1}\right) \hat{\alpha}^{\dagger}\left(t_{2}\right) \hat{\Pi}_{e}+\hat{\alpha}^{\dagger}\left(t_{1}\right) \hat{\alpha}\left(t_{2}\right) \hat{\Pi}_{g}\right)+\mathcal{O}\left(\lambda^{3}\right) \tag{B.39}
\end{equation*}
$$

where $\hat{\Pi}_{g, e}$ are the projection operators onto the ground and excited states of the detector respectively. Note the interaction duration is encoded in the shape and support of $\chi(t)$.

For compactness we define

$$
\begin{equation*}
\hat{\xi}:=-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi\left(t_{1}\right) \hat{\alpha}^{\dagger}\left(t_{1}\right) \tag{B.40}
\end{equation*}
$$

Using the relation

$$
\begin{equation*}
\left[\hat{\alpha}\left(t_{1}\right), \hat{\alpha}^{\dagger}\left(t_{2}\right)\right]=\lambda^{2} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}|\tilde{F}(\boldsymbol{k})|^{2} e^{-\mathrm{i}(\omega-\Omega) t_{1}} e^{\mathrm{i}(\omega-\Omega) t_{2}} \tag{B.41}
\end{equation*}
$$

the time evolution operator acting on the vacuum state simplifies to

$$
\begin{align*}
\hat{U}|0\rangle & =\left(\hat{\mathbb{I}}+\hat{\xi} \hat{\sigma}^{-}\right. \\
& \left.-\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) \hat{\Pi}_{e} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega}|F(\boldsymbol{k})|^{2} e^{-\mathrm{i}(\omega-\Omega) t_{1}} e^{\mathrm{i}(\omega-\Omega) t_{2}}\right)|0\rangle \tag{B.42}
\end{align*}
$$

For computational purposes we only need to focus on the $\hat{\xi}$ term, given it is the only one with a field excitation. Since we only have 0 and 1 field excitations, and using the relation

$$
\begin{equation*}
\left[\hat{a}_{\boldsymbol{k}}, \hat{\xi}\right]=-\mathrm{i} \lambda \int_{-\infty}^{\infty} \mathrm{d} t_{1} e^{\mathrm{i}(\omega-\Omega) t_{1}} \chi\left(t_{1}\right) \frac{\tilde{F}^{*}(\boldsymbol{k})}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tag{B.43}
\end{equation*}
$$

the stress-energy tensor and $\hat{\phi}^{2}$ expectations reduce to

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\hat{\Pi}_{e} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} \sqrt{4 \omega \omega^{\prime}}}\left(k_{\mu} k_{\nu}^{\prime}-\frac{\eta_{\mu \nu}}{2} k_{\gamma} k^{\prime \gamma}\right)\left(e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}}\left[\hat{\xi}^{\dagger}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]\left[\hat{a}_{\boldsymbol{k}}, \hat{\xi}\right]\right. \\
& \left.+e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}}\left[\hat{\xi}^{\dagger}, \hat{a}_{\boldsymbol{k}}^{\dagger}\right]\left[\hat{a}_{\boldsymbol{k}^{\prime}}, \hat{\xi}\right]\right)+\mathcal{O}\left(\lambda^{3}\right),  \tag{B.44}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\hat{\Pi}_{e} \int \frac{\mathrm{~d}^{3} \boldsymbol{k} \mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} \sqrt{4 \omega \omega^{\prime}}}\left(e^{-\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}}\left[\hat{\xi}^{\dagger}, \hat{a}_{\boldsymbol{k}^{\prime}}^{\dagger}\right]\left[\hat{a}_{\boldsymbol{k}}, \hat{\xi}\right]\right. \\
& \left.+e^{\mathrm{i}\left(k_{\mu}-k_{\mu}^{\prime}\right) x^{\mu}}\left[\hat{\xi}^{\dagger}, \hat{a}_{\boldsymbol{k}}^{\dagger}\right]\left[\hat{a}_{\boldsymbol{k}^{\prime}}, \hat{\xi}\right]\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{B.45}
\end{align*}
$$

Here we assumed that $t$ (the observable measurement time) is larger than the maximum $t$ in the support of $\chi(t)$, i.e. post interaction measurement. Also note that these results require the initial state of the detector to have some excited state component. If the initial state is the ground state then the expectation of the stress-energy density and the field amplitude squared (normal ordered) are exactly zero. In order to simplify (B.44) and (B.45) we use (B.26) and (B.28), where the expectation values then become

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{4(2 \pi)^{6}} \hat{\Pi}_{e}\left[J_{\mu, e}^{1} J_{\nu, e}^{1 *}+J_{\mu, e}^{1 *} J_{\nu, e}^{1}-\eta_{\mu \nu} J_{\gamma, e}^{1 *} J_{e}^{1 \gamma}\right]+\mathcal{O}\left(\lambda^{3}\right)  \tag{B.46}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{2(2 \pi)^{6}} \hat{\Pi}_{e}\left|M_{e}^{1}\right|^{2}+\mathcal{O}\left(\lambda^{3}\right) \tag{B.47}
\end{align*}
$$

The projection operators meant that if we consider an initial state given by $|0\rangle \otimes\left(a_{g}|g\rangle+\right.$ $\left.a_{e}|e\rangle\right)$ then the equations above simplify to

$$
\begin{align*}
\left\langle: \hat{T}_{\mu \nu}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{4(2 \pi)^{6}}\left|a_{e}\right|^{2}\left[J_{\mu, e}^{1} J_{\nu, e}^{1 *}+J_{\mu, e}^{1 *} J_{\nu, e}^{1}-\eta_{\mu \nu} J_{\gamma, e}^{1 *} J_{e}^{1 \gamma}\right]+\mathcal{O}\left(\lambda^{3}\right)  \tag{B.48}\\
\left\langle: \hat{\phi}^{2}(\boldsymbol{x}, t):\right\rangle_{\mathrm{RWA}} & =\frac{\lambda^{2}}{2(2 \pi)^{6}}\left|a_{e}\right|^{2}\left|M_{e}^{1}\right|^{2}+\mathcal{O}\left(\lambda^{3}\right) \tag{B.49}
\end{align*}
$$

## B.3.3 Why RWA?

As shown above the implementation of the RWA avoids the need to calculate $J_{\mu \nu, e}^{2}$ when determining the stress-energy expectations. The great advantage to this is that a 2 D semiinfinite integral can be avoided, i.e. $0<\omega<\infty$ and $0<\omega^{\prime}<\infty$. Unlike the $J_{\mu, e}^{1}$ terms that can be separated, the terms $J_{\mu \nu, e}^{2}$ (as shown in (B.53)) contain a denominator that cannot be separated.

## B. 4 RWA in large time limit

In the discussion of $\S 5.2 .4$ the field expectations are Laurent expanded in the limit $|\boldsymbol{x}| \gg$ $t, R$, showing that the RWA continues to violate causality in the long time limit. Here we demonstrate why this occurs.

## B.4.1 Persistence of RWA causality violations

Consider equation (B.29), with

$$
\begin{equation*}
\chi(t)=\Theta(t+T)-\Theta(t-T), \tag{B.50}
\end{equation*}
$$

i.e. a simple switching function of duration $2 T$.

Again, a hand-wavy argument can be put together along the lines of that as $T \rightarrow \infty$ then the $t_{1}$ integral will resemble $\delta(\omega+\Omega)$, and the $t_{2}$ integral will resemble $\delta(\omega-\Omega)$ given the Fourier transform definition of the Dirac $\delta$. This in turn would mean that in the long time limit the contribution from that integral would be zero once one integrates over $\boldsymbol{k}$ since the argument of the delta is always strictly positive, and hence one can just throw away the contribution from those counter-rotating terms. In the same fashion the emergent $\delta(\omega-\Omega)$ would allow one to keep only one frequency in the field (the so-called single mode approximation) for the integrals involving de-excitation probabilities (e.g. (B.26) and (B.28)) in the same long time limit.

This may be true if we keep the position at which we evaluate the observables fixed and we take the limit of large $T$. However this will not be true if we take the limit of long times and long spatial separation simultaneously as to evaluate field observables near the light-cone of the detector. In this particular situation it is important to consider the terms outside the integrals, i.e. $e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} t-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}}$, which are evaluated at $t=T$ and since as $T \rightarrow \infty$ these terms will begin to oscillate wildly such as to unravel the integral definition of the Dirac delta introducing polynomial decays in $\omega$, eliminating the foundations on which the RWA (and the single mode approximation) and SMA are based. That is to say, the long time limit of the integrals of $J_{\mu \nu, e}^{2}$ do not converge uniformly to zero when considering the external exponentials.

Mathematically, consider the following expression (central to $J_{\mu \nu, e}^{2}$ and $M_{e}^{2}$ )

$$
\begin{equation*}
e^{\mathrm{i} \omega T-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} T-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right)\left(e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}}+e^{-\mathrm{i}(\omega-\Omega) t_{2}-\mathrm{i}\left(\omega^{\prime}+\Omega\right) t_{1}}\right) . \tag{B.51}
\end{equation*}
$$

First note that when evaluating the expectations of $\hat{T}_{00}$ or $\hat{\phi}^{2}$ that we can swap $\omega \leftrightarrow \omega^{\prime}$ in the second term of (B.51) without affecting the result of (B.32) and (B.33) (although not
for the off diagonal stress-energy terms). We perform this swap to simplify the equations in this derivation, i.e. the expression becomes

$$
\begin{equation*}
I:=2 e^{\mathrm{i} \omega T-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} T-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi\left(t_{1}\right) \chi\left(t_{2}\right) e^{-\mathrm{i}(\omega+\Omega) t_{1}-\mathrm{i}\left(\omega^{\prime}-\Omega\right) t_{2}} . \tag{B.52}
\end{equation*}
$$

When we perform the integrals in question we obtain

$$
\begin{equation*}
I=e^{\mathrm{i} \omega T-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} e^{\mathrm{i} \omega^{\prime} T-\mathrm{i} \boldsymbol{k}^{\prime} \cdot \boldsymbol{x}} \frac{4 \mathrm{i}}{\Omega-\omega^{\prime}}\left(\frac{e^{\mathrm{i}\left(\omega^{\prime}-\Omega\right) T} \sin [(\omega+\Omega) T]}{\omega+\Omega}-\frac{\sin \left[\left(\omega+\omega^{\prime}\right) T\right]}{\omega+\omega^{\prime}}\right) . \tag{B.53}
\end{equation*}
$$

Here we note that the sinc functions in the brackets usually pointwise converge to delta functions (as $T \rightarrow \infty$ ) and since $\omega, \omega^{\prime}, \Omega>0$ then these will naturally be zero, making $J_{\mu \nu, e}^{2} \rightarrow 0$ and therefore seemingly demonstrating that the RWA predictions tend to the full model predictions in the infinite time limit. However, we must consider the exponentials outside the brackets. Since we are looking at the violations of causality near the surface of the light-cone and the interaction lasts from $-T$ to $T$, we must set $|\boldsymbol{x}| \approx 2 T$, which is the leading edge of the detectors perturbation on the field.

Therefore, when we consider the integrals in momentum space in equations (B.26) and (B.28), the oscillatory terms outside of the $t_{1}, t_{2}$ integrals, near the light-cone, go as $e^{2 \mathrm{i} \omega T}-e^{-2 \mathrm{i} \omega T}$ (this difference between two exponentials emerges from $e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$ after integrating the angular variables in momentum space). Therefore the terms of (B.53) near the lightcone, approximately oscillate as:

A quick inspection reveals that within $I$ there will be terms that oscillate slowly (or not at all), i.e. if $|\boldsymbol{x}|=2 T+\epsilon$, terms of the form $e^{\mathrm{i} \omega \epsilon}$ will appear that oscillate slowly with respect to the significant sections of the smearing Fourier transforms (i.e. $\tilde{F}(\boldsymbol{k})$ ) and other terms within the $\boldsymbol{k}, \boldsymbol{k}^{\prime}$ integrals. This of course means that even if $T \rightarrow \infty$ the polynomial decay remains. With more rigorous working it can be shown that the non-locality introduced by the RWA Hamiltonian persists, polynomially decaying from the surface of the lightcone/sphere, similar to the plots shown in chapter 5. This of course should not be surprising considering the explicit non-locality of the interaction Hamiltonian.

Note, however, that if $|\boldsymbol{x}| \ll 2 T$ then the arguments above no longer hold and we can take the pointwise limit of Dirac $\delta$, i.e. for $|\boldsymbol{x}| \ll 2 T$ : RWA $\rightarrow$ UDW, as described in the next section.

Mathematically, this failure of RWA to converge to UDW is due to the mathematically unsound step of commuting the infinite time integral and momentum integral in $J_{\mu \nu, e}^{2}$ (B.27) and $M_{e}^{2}$ (B.29). These integrals fail the absolute and uniform convergence tests and therefore can only be commuted if the time integral is taken as finite (i.e. $T<\infty$ ), which results in the pointwise convergence of (B.54).

## B.4.2 RWA convergence to the full model

The derivation above showed that the second order counter-rotating terms do not vanish for long times near the light-cone. One can ask under what conditions there are points where the second order counter-rotating terms do vanish.

Consider the counter-rotating contributions to the expectation of $\hat{\phi}^{2}$ (B.33), evaluated at spacetime $(\boldsymbol{x}, \tilde{T})$ (where $\tilde{T} \geq T$ ), which are given by the real part of (B.29). Consider a simple switching of duration $2 T$ :

$$
\begin{equation*}
\chi(t)=\Theta(t+T)-\Theta(t-T) \tag{B.55}
\end{equation*}
$$

and a spherically symmetric detector smearing. Then

$$
\begin{align*}
M_{e}^{2} & =\frac{2(2 \pi)^{2}}{|\boldsymbol{x}|^{2}} \int \mathrm{~d} \omega \mathrm{~d} \omega^{\prime} F(\omega) F\left(\omega^{\prime}\right) \\
& \times\left\{\frac{e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)|\boldsymbol{x}|}}{\left(\omega^{\prime}-\Omega\right)}\left[\frac{e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}-T)}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{\left(\omega+\omega^{\prime}\right)}-\frac{e^{-\mathrm{i}(\omega+\Omega) T+\mathrm{i}\left(\omega^{\prime}-\Omega\right) T+\mathrm{i}\left(\omega+\omega^{\prime}\right) \tilde{T}}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{(\omega+\Omega)}\right]\right. \\
& -\frac{e^{\mathrm{i}\left(\omega-\omega^{\prime}\right)|\boldsymbol{x}|}}{\left(\omega^{\prime}-\Omega\right)}\left[\frac{e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}-T)}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{\left(\omega+\omega^{\prime}\right)}-\frac{e^{-\mathrm{i}(\omega+\Omega) T+\mathrm{i}\left(\omega^{\prime}-\Omega\right) T+\mathrm{i}\left(\omega+\omega^{\prime}\right) \tilde{T}}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{(\omega+\Omega)}\right] \\
& -\frac{e^{-\mathrm{i}\left(\omega-\omega^{\prime}\right)|\boldsymbol{x}|}}{\left(\omega^{\prime}-\Omega\right)}\left[\frac{e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}-T)}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{\left(\omega+\omega^{\prime}\right)}-\frac{e^{-\mathrm{i}(\omega+\Omega) T+\mathrm{i}\left(\omega^{\prime}-\Omega\right) T+\mathrm{i}\left(\omega+\omega^{\prime}\right) \tilde{T}}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{(\omega+\Omega)}\right] \\
& \left.+\frac{e^{-\mathrm{i}\left(\omega+\omega^{\prime}\right)|\boldsymbol{x}|}}{\left(\omega^{\prime}-\Omega\right)}\left[\frac{e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}-T)}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{\left(\omega+\omega^{\prime}\right)}-\frac{e^{-\mathrm{i}(\omega+\Omega) T+\mathrm{i}\left(\omega^{\prime}-\Omega\right) T+\mathrm{i}\left(\omega+\omega^{\prime}\right) \tilde{T}}-e^{\mathrm{i}\left(\omega+\omega^{\prime}\right)(\tilde{T}+T)}}{(\omega+\Omega)}\right]\right\} . \tag{B.56}
\end{align*}
$$

Written in this explicitly expanded way the general behaviour of $M_{e}^{2}$ can be observed. Standard RWA requires $\Omega T \gg 1$ in order to ensure convergence of the RWA model to the exact model, which can be interpreted in the equation above as $T$ large, causing the $\omega$ and $\omega^{\prime}$ integrals to oscillate very quickly with respect to $F(\omega)$, therefore causing $M_{e}^{2} \rightarrow 0$. However, as seen in §B.4.1 above, this convergence is dependent on the value of $|\boldsymbol{x}|$.

Case 1: If $|x|=\tilde{T}+T+\varepsilon$ (where $\varepsilon$ is small), i.e. if evaluating the expectation of $\phi^{2}$ near the leading edge of the interaction light-cone, then terms of the form

$$
\begin{equation*}
\frac{2(2 \pi)^{2}}{|\boldsymbol{x}|^{2}} \int \mathrm{~d} \omega \mathrm{~d} \omega^{\prime} F(\omega) F\left(\omega^{\prime}\right) \frac{e^{-\mathrm{i} \varepsilon\left(\omega+\omega^{\prime}\right)}}{\omega+\Omega} \tag{B.57}
\end{equation*}
$$

contribute to $M_{e}^{2}$. These terms do not oscillate quickly in $\omega$ or $\omega^{\prime}$ and therefore will not integrate to zero. Terms such as these contribute to the polynomial causality violations seen in the main text's diagrams, and as can be seen they are time independent. In this region the RWA does not work well and leads to superluminal observations.

Case 2: If $|x|=\tilde{T}-T+2 k T$ where $0<k<1$, i.e. evaluating the expectation of $\phi^{2}$ deep within the bulk of the interaction light-cone, all terms oscillate quickly, e.g. $e^{\mathrm{i} k \omega T}$. In this region the RWA works well, yielding field observables equal to the exact model.

Case 3: If $|x|=\tilde{T}-T+\varepsilon$ (where $\varepsilon$ is small), i.e. if evaluating the expectation of $\phi^{2}$ near the trailing edge of the interaction light-cone, then as in case 1 there will be slowly oscillating terms, e.g. $e^{-\mathrm{i} \varepsilon\left(\omega+\omega^{\prime}\right)}$, which ensure $M_{e}^{2}$ is non-zero. This ensures the RWA is unreliable near the trailing edge of the interaction light-cone.

Case 4: If $|x|=\varepsilon$ (where $\varepsilon$ is small), i.e. evaluating the expectation of $\phi^{2}$ well within the time-like region of the detector interaction. In this case $M_{e}^{2}$ converges to zero, similar to case 2 .

Analysis of the $M_{e}^{2}$ term leads to the conclusion that the RWA is best used deep within the bulk of the interaction lightcone. The polynomial non-locality of the RWA interaction Hamiltonian is branded onto the field with the initial sudden switching, initiating the interaction and a second polynomial non-locality is branded onto the field by the final sudden switching, terminating the interaction. We conclude that provided the spacetime location of observations is 'spacetime' distant (i.e. $\left.\left|(x-y)^{\mu}(x-y)_{\mu}\right| \Omega \gg 1\right)$ from insufficiently
smooth regions of the switching function ( t ) (with $\boldsymbol{x}$ given roughly by the support of the detector smearing) then the RWA can be used with some confidence.


Figure B.1: A coarse sketch of the interaction lightcone under sudden switching (B.55). The dots indicate $t=-T, T$ respectively and the solid lines indicate the null-like propagation of these events. The RWA works best in the shaded regions, i.e. when $\left|(\mathrm{x}-\mathrm{y})^{\mu}(\mathrm{x}-\mathrm{y})_{\mu}\right| \Omega \gg 1$.

## B.5 RWA signalling - 2 detector perturbative expansion

Here we go step by step over the 2 detector perturbative expansion, resulting in the reduced density matrix for 1 of the 2 detectors with the field completely traced out.

$$
\begin{equation*}
\tilde{F}(\boldsymbol{k}):=\int \mathrm{d}^{3} \boldsymbol{y} \frac{1}{R^{3}} G\left(\frac{\boldsymbol{y}}{R}\right) e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}} . \tag{B.58}
\end{equation*}
$$

The Unruh-DeWitt interaction Hamiltonian has the form

$$
\begin{equation*}
\hat{H}=\lambda \chi(t) \int \mathrm{d}^{3} \boldsymbol{x} G(\boldsymbol{x}) \hat{\sigma}_{x}(t) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(e^{-\mathrm{i} \omega t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}}+e^{\mathrm{i} \omega t-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \hat{a}_{\boldsymbol{k}}^{\dagger}\right) . \tag{B.59}
\end{equation*}
$$

In order to proceed we define the following, if Unruh-DeWitt coupling:

$$
\begin{equation*}
\hat{\psi}_{i}=\lambda_{i} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}}\left(\tilde{F}_{i}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}}+\tilde{F}_{i}^{*}(\boldsymbol{k}) e^{\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}}^{\dagger}\right) \tag{B.60}
\end{equation*}
$$

if RWA coupling:

$$
\begin{equation*}
\hat{\psi}_{i}=\lambda_{i} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3 / 2} \sqrt{2 \omega}} \tilde{F}_{i}(\boldsymbol{k}) e^{-\mathrm{i} \omega t} \hat{a}_{\boldsymbol{k}} \tag{B.61}
\end{equation*}
$$

c.f. (5.53).

This way the interaction Hamiltonian becomes

$$
\begin{equation*}
\hat{H}=\chi_{\mathrm{A}}(t)\left(\hat{\sigma}_{\mathrm{A}}^{+}(t) \hat{\psi}_{\mathrm{A}}+\hat{\sigma}_{\mathrm{A}}^{-}(t) \hat{\psi}_{\mathrm{A}}^{\dagger}\right)+\chi_{\mathrm{B}}(t)\left(\hat{\sigma}_{\mathrm{B}}^{+}(t) \hat{\psi}_{\mathrm{B}}+\hat{\sigma}_{\mathrm{B}}^{-}(t) \hat{\psi}_{\mathrm{B}}^{\dagger}\right) \tag{B.62}
\end{equation*}
$$

The time evolution operator then looks like

$$
\begin{align*}
\hat{U}(t) & =\hat{\mathbb{I}}-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} t_{1}\left(\chi_{\mathrm{A}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right)+\chi_{\mathrm{B}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right.\right. \\
& \left.\left.+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right)\right)-\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2}\left(\chi_{\mathrm{A}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right)\right. \\
& \left.+\chi_{\mathrm{B}}\left(t_{1}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right)\right)\left(\chi_{\mathrm{A}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right)\right. \\
& \left.+\chi_{\mathrm{B}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{2}\right)+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right)\right)\right)+\mathcal{O}\left(\lambda^{3}\right) . \tag{B.63}
\end{align*}
$$

By assuming the initial field state is the vacuum and the initial detector states is $\hat{\rho}_{0}$, the reduced detector density matrix becomes

$$
\begin{aligned}
& \hat{\rho}_{q}(t)=\hat{\rho}_{0}+\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2}\{ \\
& \quad \chi_{\mathrm{A}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} e^{\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}+t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} \mathrm{i}^{\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle\right. \\
& + \\
& \left.+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} e^{-\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} e^{-\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}+t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle\right) \\
& + \\
& +\chi_{\mathrm{A}}\left(t_{1}\right) \chi_{\mathrm{B}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{B}} t_{2}+\Omega_{\mathrm{A}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{B}} t_{2}-\Omega_{\mathrm{A}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle\right. \\
& \left.+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{B}} t_{2}-\Omega_{\mathrm{A}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{B}} t_{2}+\Omega_{\mathrm{A}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle\right)
\end{aligned}
$$

$$
\begin{align*}
& +\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle\right. \\
& \left.+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle\right) \\
& +\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{B}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}+t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} e^{\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle\right. \\
& \left.\left.+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{-\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} e^{-\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}+t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle\right)\right\} \\
& -\int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2}\{ \\
& \chi_{\mathbf{A}}\left(t_{1}\right) \chi_{\mathbf{A}}\left(t_{2}\right)\left(\hat{\Pi}_{e}^{1} \hat{\rho}_{0} e^{\mathrm{i} \Omega_{\mathbf{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathbf{A}}\left(t_{1}\right) \hat{\psi}_{\mathbf{A}}^{\dagger}\left(t_{2}\right)\right\rangle+\hat{\Pi}_{g}^{1} \hat{\rho}_{0} e^{-\mathrm{i} \Omega_{\mathbf{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathbf{A}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathbf{A}}\left(t_{2}\right)\right\rangle\right. \\
& \left.+\hat{\rho}_{0} \hat{\Pi}_{e}^{1} e^{-\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\Pi}_{g}^{1} e^{\mathrm{i} \Omega_{\mathrm{A}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \psi_{\mathrm{A}}\left(t_{1}\right)\right\rangle\right) \\
& +\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{B}}^{+} \hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\sigma}_{\mathrm{A}}^{+} \hat{\rho}_{0} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right\rangle\right. \\
& +\hat{\sigma}_{\mathrm{B}}^{+} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right\rangle \\
& +\hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{+} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle \\
& \left.+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} \hat{\sigma}_{\mathrm{B}}^{-} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}-\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{+} \hat{\sigma}_{\mathrm{B}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{2}+\Omega_{\mathrm{B}} t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle\right) \\
& +\chi_{\mathrm{A}}\left(t_{1}\right) \chi_{\mathrm{B}}\left(t_{2}\right)\left(\hat{\sigma}_{\mathrm{A}}^{+} \hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}+\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{2}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{+} \hat{\rho}_{0} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}-\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}\left(t_{2}\right)\right\rangle\right. \\
& +\hat{\sigma}_{\mathrm{A}}^{+} \hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}-\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right)\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{-} \hat{\rho}_{0} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}+\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right)\right\rangle \\
& +\hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} \hat{\sigma}_{\mathrm{A}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}+\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{-} \hat{\sigma}_{\mathrm{A}}^{+} e^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}-\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle \\
& \left.+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} \hat{\sigma}_{\mathrm{A}}^{-} e^{-\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}-\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} \hat{\sigma}_{\mathrm{A}}^{+} \mathrm{e}^{\mathrm{i}\left(\Omega_{\mathrm{A}} t_{1}+\Omega_{\mathrm{B}} t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}\left(t_{1}\right)\right\rangle\right) \\
& +\chi_{\mathbf{B}}\left(t_{1}\right) \chi_{\mathbf{B}}\left(t_{2}\right)\left(\hat{\Pi}_{e}^{2} \hat{\rho}_{0} e^{\mathrm{i} \Omega_{\mathbf{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathbf{B}}\left(t_{1}\right) \hat{\psi}_{\mathbf{B}}^{\dagger}\left(t_{2}\right)\right\rangle+\hat{\Pi}_{g}^{2} \hat{\rho}_{0} e^{-\mathrm{i} \Omega_{\mathbf{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathbf{B}}^{\dagger}\left(t_{1}\right) \hat{\psi}_{\mathbf{B}}\left(t_{2}\right)\right\rangle\right. \\
& \left.\left.+\hat{\rho}_{0} \hat{\Pi}_{e}^{2} e^{-\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right\rangle+\hat{\rho}_{0} \hat{\Pi}_{g}^{2} e^{\mathrm{i} \Omega_{\mathrm{B}}\left(t_{1}-t_{2}\right)}\left\langle\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{2}\right) \psi_{\mathrm{B}}\left(t_{1}\right)\right\rangle\right)\right\}+\mathcal{O}\left(\lambda_{i}^{3}\right) \tag{B.64}
\end{align*}
$$

Now in order to gauge the non-local effects we consider the following scenario, $\chi_{\mathrm{A}}$ occurs before $\chi_{\mathrm{B}}$ and in our frame of reference their supports do not overlap, as shown in [75]. This allows us to eliminate the terms $\chi_{\mathbf{A}}\left(t_{1}\right) \chi_{\mathbf{B}}\left(t_{2}\right)$ from the ordered integral and allows us to compare terms from the integrals that proportional to $\lambda_{\mathrm{A}} \lambda_{\mathrm{B}}$, note that our choice of switching means that the time-ordering becomes trivial for $\lambda_{A} \lambda_{B}$ terms. Furthermore we trace out the second detector and inspect the first detector's density matrix,

$$
\begin{align*}
& \hat{\rho}_{\mathrm{B}}^{1}(t)=\operatorname{Tr}_{\mathrm{A}}\left(\hat{\rho}^{0}\right)+\mathcal{O}\left(\lambda_{\mathrm{A}}^{2}\right)+\mathcal{O}\left(\lambda_{\mathrm{B}}^{2}\right)+\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2}\left\{\chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right)\right. \\
& \quad \operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right]\right\rangle\right. \\
& \left.\quad+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right)\left\langle\left[\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right), \hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right)\right]\right\rangle\right) \\
& \quad+\operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{+}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}\left(t_{2}\right)\right]\right\rangle\right. \\
& \left.\left.\quad+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle+\hat{\sigma}_{\mathrm{A}}^{-}\left(t_{2}\right) \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{-}\left(t_{1}\right)\left\langle\left[\hat{\psi}_{\mathrm{B}}^{\dagger}\left(t_{1}\right), \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right]\right\rangle\right)\right\}+\mathcal{O}\left(\lambda_{i}^{3}\right), \tag{B.65}
\end{align*}
$$

In this last step we have used the cyclic property of the partial trace to arrange terms nicely. All that remains is to evaluate the commutators, all of which can be accomplished easily, as shown in (5.56) and (5.57).

## B. 6 RWA channel capacity convergence

In the main text the channel capacity of a 2 qubit communication protocol was presented for RWA (5.63) and UDW (5.64). A suprising result is that the RWA channel capacity converges to half the UDW channel capacity for long times. This appendix will show where this factor of $1 / 2$ comes from.

In the main text a measure of channel capacity was chosen as the magnitude of the coefficient of the $\operatorname{Tr}_{\mathrm{A}}\left(\hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}^{0} \hat{\sigma}_{\mathrm{B}}^{+}\right)$. From appendix B. 5 we find that two terms contribute to this coefficient (for simplicity let $\Omega_{\mathrm{A}}=\Omega_{\mathrm{B}}=\Omega$ ):

$$
\begin{align*}
& \Xi_{1}=\operatorname{Tr}_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \chi_{\mathrm{A}}\left(t_{1}\right) \chi_{\mathrm{B}}\left(t_{2}\right) \hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{\mathrm{i} \Omega\left(t_{2}-t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{2}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{1}\right)\right\rangle,  \tag{B.66}\\
& \Xi_{2}=-\operatorname{Tr}_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \int_{-\infty}^{t_{1}} \mathrm{~d} t_{2} \chi_{\mathrm{B}}\left(t_{1}\right) \chi_{\mathrm{A}}\left(t_{2}\right) \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{+} e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle, \tag{B.67}
\end{align*}
$$

where we have left the detectors operators and density matrix for illustration.
If we rename the integration variables and observe that the non-overlapping switchings allow for extension of the integration domain:

$$
\begin{align*}
& \Xi_{1}=\operatorname{Tr}_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi_{\mathrm{A}}\left(t_{2}\right) \chi_{\mathrm{B}}\left(t_{1}\right) \hat{\sigma}_{\mathrm{A}}^{-} \hat{\rho}_{0} \hat{\sigma}_{\mathrm{B}}^{+} e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{B}}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right\rangle,  \tag{B.68}\\
& \Xi_{2}=-\operatorname{Tr}_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi_{\mathrm{A}}\left(t_{2}\right) \chi_{\mathrm{B}}\left(t_{1}\right) \hat{\rho}_{0} \hat{\sigma}_{\mathrm{A}}^{-} \hat{\sigma}_{\mathrm{B}}^{+} e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)}\left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle . \tag{B.69}
\end{align*}
$$

From (5.56) we know that $\left(\hat{\psi}_{\kappa}(t)=\hat{\alpha}_{\kappa}+\hat{\alpha}_{\kappa}^{\dagger}\right)$ :

$$
\begin{align*}
& \left\langle\hat{\psi}_{\mathrm{B}}\left(t_{1}\right) \hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right)\right\rangle=\lambda_{\mathrm{B}} \lambda_{\mathrm{A}} \int \frac{\mathrm{~d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{B}}^{3} R_{\mathrm{A}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega} e^{\mathrm{i} \omega\left(t_{2}-t_{1}\right)} e^{\mathrm{i} \boldsymbol{k} \cdot\left(\boldsymbol{y}_{1}-\boldsymbol{y}_{2}\right)}, \\
& \left\langle\hat{\psi}_{\mathrm{A}}^{\dagger}\left(t_{2}\right) \hat{\psi}_{\mathrm{B}}\left(t_{1}\right)\right\rangle=\lambda_{\mathrm{B}} \lambda_{\mathrm{A}} \int \frac{\mathrm{~d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{B}}^{3} R_{\mathrm{A}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3} 2 \omega} e^{-\mathrm{i} \omega\left(t_{2}-t_{1}\right)} e^{\mathrm{i} \cdot \cdot\left(\boldsymbol{y}_{1}-\boldsymbol{y}_{2}\right)} . \tag{B.70}
\end{align*}
$$

In the $\Xi_{2}$ term, the counter-rotating term $e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)} e^{-\mathrm{i} \omega\left(t_{2}-t_{1}\right)}$ appears, a term discarded by RWA. If we inspect the $\operatorname{SU}(2)$ operators in $\Xi_{2}$ they reveal $\Xi_{2}$ is produced by a field density matrix term $\sim|0\rangle\langle 0| \hat{\psi}_{\mathrm{A}}^{\dagger} \hat{\psi}_{\mathrm{B}}$, i.e. $\Xi_{2}$ is the result of tracing out a matrix element that involves a (counter-rotating) vacuum emission from detector A and a (counter-rotating) field absorbtion with detector de-excitation from B. The apparent violation of energy conservation from detector $A$ is quickly remedied by detector $B$, resulting in a significant avenue for Alice to communicate to Bob.

In contrast, the $\Xi_{1}$ term arises from a field matrix element $\sim \hat{\psi}_{\mathrm{A}}^{\dagger}|0\rangle\langle 0| \hat{\psi}_{\mathrm{B}}$, an energy conserving term (in classical notions).

As shown in $\S B .4 .2$, the convergence of RWA is reliant on the choice of position, i.e. $\boldsymbol{y}_{1}$ and $\boldsymbol{y}_{1}$. This casts doubts on the common perception that $\Xi_{1}$ should be dominant over the counter-rotating $\Xi_{2}$ term. If we evaluate the momentum integrals in $\Xi_{1,2}$, then

$$
\begin{align*}
\Xi_{1} & =\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi_{\mathrm{A}}\left(t_{2}\right) \chi_{\mathrm{B}}\left(t_{1}\right) e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)} \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{B}}^{3} R_{\mathrm{A}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right)  \tag{B.72}\\
& \times \frac{1}{8 \pi^{2}|\boldsymbol{x}|}\left(\frac{\text { P.V. }}{|\boldsymbol{x}|+\Delta t}+\frac{\text { P.V. }}{|\boldsymbol{x}|-\Delta t}+\mathrm{i} \pi(\delta(|\boldsymbol{x}|+\Delta t)-\delta(|\boldsymbol{x}|-\Delta t))\right), \\
\Xi_{2} & =-\lambda_{\mathrm{A}} \lambda_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d} t_{2} \int_{-\infty}^{\infty} \mathrm{d} t_{1} \chi_{\mathrm{A}}\left(t_{2}\right) \chi_{\mathrm{B}}\left(t_{1}\right) e^{-\mathrm{i} \Omega\left(t_{2}-t_{1}\right)} \int \frac{\mathrm{d}^{3} \boldsymbol{y}_{1} \mathrm{~d}^{3} \boldsymbol{y}_{2}}{R_{\mathrm{B}}^{3} R_{\mathrm{A}}^{3}} G_{\mathrm{B}}\left(\frac{\boldsymbol{y}_{1}}{R_{\mathrm{B}}}\right) G_{\mathrm{A}}\left(\frac{\boldsymbol{y}_{2}}{R_{\mathrm{A}}}\right)  \tag{B.73}\\
& \times \frac{1}{8 \pi^{2}|\boldsymbol{x}|}\left(\frac{\text { P.V. }}{|\boldsymbol{x}|+\Delta t}+\frac{\text { P.V. }}{|\boldsymbol{x}|-\Delta t}-\mathrm{i} \pi(\delta(|\boldsymbol{x}|+\Delta t)-\delta(|\boldsymbol{x}|-\Delta t))\right),
\end{align*}
$$

where $\boldsymbol{x}=\boldsymbol{y}_{1}-\boldsymbol{y}_{2}$ and $\Delta t=t_{1}-t_{2}$. From these expressions we can see that if $|x|$ lies within the time-like or space-like (but not null-like) regions described by Fig. B.1, the Dirac delta functions would yield zero and the principal value terms become dominant. Under these circumstances the UDW model would predict a channel capacity of $\Xi_{1}+\Xi_{2}=0$, where as RWA would yield some non-zero superluminal signalling, with a $1 /|\boldsymbol{x}|^{2}$ decay.

However, when considering the null-like regions described by Fig. B.1, regions where RWA field observables converged to UDW field observables, the Dirac delta functions play a significant role. Assuming sudden switchings, as in (5.65) and (5.66); a change of coordinates

$$
\begin{align*}
\sigma_{t} & =t_{1}+t_{2},  \tag{B.74}\\
\Delta_{t} & =t_{1}-t_{2}, \tag{B.75}
\end{align*}
$$

can then be used to analytically evaluate one of the time integrals. This results in terms of the rough form:

$$
\begin{equation*}
\Xi_{1,2} \sim \int \mathrm{~d} \Delta_{t} e^{\mathrm{i} \Omega \Delta_{t}}\left(\frac{\mathrm{P.V.}}{|\boldsymbol{x}|+\Delta_{t}}+\frac{\mathrm{P} . \mathrm{V} .}{|\boldsymbol{x}|-\Delta_{t}} \pm \mathrm{i} \pi\left(\Delta\left(|\boldsymbol{x}|+\Delta_{t}\right)-\Delta\left(|x|-\Delta_{t}\right)\right)\right)\left(a_{1} \Delta_{t}+a_{2}\right) \tag{B.76}
\end{equation*}
$$

where $a_{1,2}$ are constants dependent on the detector interaction timings. From here we can see that the Dirac delta terms will have a linear leading contribution. On the other hand
the principal value terms will integrate into an oscillatory term with constant amplitude and a quickly decaying term. This results in a dominant Dirac delta contribution for both $\Xi_{1}$ and $\Xi_{2}$ in the long time limits, demonstrating that $\Xi_{2}$ contributes equally to the final channel capacity predicted by the UDW model, despite consisting of counter-rotating terms. Since the RWA eliminates this avenue of communication the RWA converges to $1 / 2$ the channel capacity of UDW.

Note: this result is also valid in the cavity case, and can be see with the aid of the first numerical trick §C.1.1.

## Appendix C

## Locality, causality and the approximations of quantum optics in optical cavities

## C. 1 Mathematical tricks from contour integrals

The results presented above revolve around evaluating sums over mode functions, e.g. the Wightman function

$$
\begin{equation*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\sum_{\boldsymbol{m} \in \mathcal{Z}} \frac{4}{V \omega_{\boldsymbol{m}}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right) \tag{C.1}
\end{equation*}
$$

Functions of this form have no closed form solution and are generally only evaluated in regimes where the sum could be said to resemble a Riemann sum and evaluated by integration. In this appendix two simplifying tricks are derived the first of which helps interpret cavity results with notions native to free field configurations; and the second of which serves a significant role in improving numerical computation time and precision by introducing exponential decay to the sequence being summed.

## C.1.1 First trick

Consider the integral in (6.9) and introduce spherical coordinates

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{0}^{\infty} \mathrm{d} \xi \xi^{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{~d} \theta \sin (\theta) \int_{0}^{2 \pi} \mathrm{~d} \phi \frac{e^{\mathrm{i} \omega \xi \cos (\theta)}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right)  \tag{C.2}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{0}^{\infty} \mathrm{d} \xi \xi^{2} \frac{e^{\mathrm{i} \omega \xi}-e^{-\mathrm{i} \omega \xi}}{\mathrm{i} \omega \xi} \frac{2 \pi}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right)  \tag{C.3}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi \mathrm{i}} \int_{0}^{\infty} \mathrm{d} \xi \frac{e^{\mathrm{i} \omega \xi}-e^{-\mathrm{i} \omega \xi}}{\omega}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \tag{C.4}
\end{align*}
$$

Now from the expression above, consider one of the terms

$$
\begin{align*}
& \frac{1}{2 \pi \mathrm{i}} \int_{0}^{\infty} \mathrm{d} \xi \frac{e^{-\mathrm{i} \omega \xi}}{\omega}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \\
& =\frac{1}{2 \pi \mathrm{i}} \int_{0}^{-\infty} \mathrm{d} \eta(-1) \frac{e^{\mathrm{i} \omega \eta}}{\omega}\left(\frac{1}{-\eta-a-\mathrm{i} \epsilon}+\frac{1}{-\eta+a+\mathrm{i} \epsilon}\right)  \tag{C.5}\\
& =-\frac{1}{2 \pi \mathrm{i}} \int_{-\infty}^{0} \mathrm{~d} \eta \frac{e^{\mathrm{i} \omega \eta}}{\omega}\left(\frac{1}{\eta+a+\mathrm{i} \epsilon}+\frac{1}{\eta-a-\mathrm{i} \epsilon}\right) \tag{C.6}
\end{align*}
$$

where the change of variables $\eta=-\xi$ was made. Now inserting (C.6) into (C.4) gives

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}\right. & \left.+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d} \xi \frac{e^{\mathrm{i} \omega \xi}}{\omega}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \tag{C.7}
\end{align*}
$$

The integral above is known to converge as it is an oscillatory, decaying integrand. Since $a$ will represent time it is a real parameter and also worth noting is that $\omega>0$. The fact that $\omega>0$ means that the integrand decays exponentially as $\xi \rightarrow+\mathrm{i} \infty$, therefore the integration contour can be extended from the real line $(-\infty, \infty)$ to include $+\mathrm{i} \infty$ without changing the value of the integral, i.e. the real line contour can be closed via $+\mathrm{i} \infty$, i.e.

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}\right. & \left.+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right) \\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi \mathrm{i} \omega} \int_{\mathcal{C}^{+}} \mathrm{d} \xi e^{\mathrm{i} \omega \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right), \tag{C.8}
\end{align*}
$$

where $\mathcal{C}^{+}$denotes a positively oriented contour that runs along the real axis and is then closed via $+\mathrm{i} \infty$ (Fig. C.1).


Figure C.1: The original integration contour of (C.8) is $-\infty<\xi<\infty$. Adding the contour via $+\mathrm{i} \infty$ (dashed line) closes the integration contour whilst Leaving the integral unchanged. This allows the direct use of the Residue theorem after which $\epsilon \rightarrow 0^{+}$can be taken.

Now with a closed contour this integral can be evaluated by the residue theorem. By taking the limit $\epsilon \rightarrow 0^{+}$the integral becomes

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \boldsymbol{\xi} \frac{e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\xi-a-\mathrm{i} \epsilon}+\frac{1}{\xi+a+\mathrm{i} \epsilon}\right)=\frac{e^{\mathrm{i} \omega a}}{\omega} \tag{C.9}
\end{equation*}
$$

as required. With this trick, sums like the Wightman function take the form

$$
\begin{align*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{V \pi^{2}} & \int_{\mathbb{R}^{3}} \frac{\mathrm{~d}^{3} \boldsymbol{\xi}}{\xi}\left(\frac{1}{\xi-\mathrm{i} \epsilon}+\frac{1}{\xi+\mathrm{i} \epsilon}\right) \\
& \times \sum_{\boldsymbol{m} \in\left(\mathbb{Z}^{+}\right)^{3}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right), \tag{C.10}
\end{align*}
$$

where the sum above does have a closed form solution, namely the geometric series solutions. As mentioned in the main text this leads to

$$
\begin{equation*}
W(t, \boldsymbol{y}, \boldsymbol{z})=\frac{1}{4 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{\mathrm{P} . \mathrm{V} .}{\xi^{2}} \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+y_{i}-z_{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+y_{i}+z_{i}+2 n_{i} L_{i}\right)\right) \tag{C.11}
\end{equation*}
$$

which can be interpreted as a free space Wightman function reflected infinitely many times off the cavity mirrors.

## C.1.2 2nd trick

This second trick is similar to the 1st trick in its derivation however the fractions $1 /(\xi \pm$ $a \pm \mathrm{i} \epsilon$ ) is replaced by a function that has $\xi=a$ as its only real pole, does not exponentially grow as $\xi \rightarrow+\mathrm{i} \infty$ (i.e. does not grow faster than $e^{\mathrm{i} \omega \xi}$ decays) and decays quickly for large real $\xi$. Instinctively one might immediately consider a Gaussian $e^{-\xi^{2}} /(\xi \pm a \pm \mathrm{i} \epsilon)$, however this diverges exponentially as $\xi \rightarrow+\mathrm{i} \infty$. One compromise is found in the hyperbolic trigonometric functions:

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{\rho e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\sinh [\rho(\xi-a-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+a+\mathrm{i} \epsilon)]}\right) \\
= & \lim _{\epsilon \rightarrow 0^{+}} \frac{\rho}{2 \pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d} \xi \frac{e^{\mathrm{i} \omega \xi}}{\omega}\left(\frac{1}{\sinh [\rho(\xi-a-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+a+\mathrm{i} \epsilon)]}\right), \tag{C.12}
\end{align*}
$$

where spherical coordinates are used and the coordinate change $\xi \rightarrow-\xi$ is used on half the expression as done in $\S$ C.1.1. The integral expression of this second trick replaces the polynomial decay of (C.9) with an exponential decay along the real axis. This exponential decay is dictated by the real parameter $\rho>0$, which at first glance suggests we choose a large value for stronger exponential decay.

As in the previous section the contour of the integral can be extended and closed via $+\mathrm{i} \infty$ allowing the use of the residue theorem. This is where this trick becomes more complicated; unlike the simple polynomial of $\S$ C.1.1, $\sinh (z)$ has zeros whenever $z=\mathrm{i} n \pi$. This means the closure of the contour will not only include the pole at $\xi=a+\mathrm{i} \epsilon$, but also
at $\xi= \pm(a+\mathrm{i} \epsilon)+\mathrm{i} n \pi / \rho$ (provided $\operatorname{Im}(\xi)>0)$. When all the poles are accounted for

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0^{+}} \frac{\rho}{2 \pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d} \xi \frac{e^{\mathrm{i} \omega \xi}}{\omega}\left(\frac{1}{\sinh [\rho(\xi-a-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+a+\mathrm{i} \epsilon)]}\right) \\
& =\frac{1}{\omega}\left[\sum_{n=0}^{\infty}(-1)^{n} e^{\mathrm{i} \omega(a+\mathrm{i} n \pi / \rho)}+\sum_{n=1}^{\infty}(-1)^{n} e^{\mathrm{i} \omega(-a+\mathrm{i} n \pi / \rho)}\right]  \tag{C.13}\\
& =\frac{1}{\omega}\left[\frac{e^{\mathrm{i} \omega a}}{1+e^{-\frac{\pi \omega}{\rho}}}-\frac{e^{-\mathrm{i} \omega a} e^{-\frac{\pi \omega}{\rho}}}{1+e^{-\frac{\pi \omega}{\rho}}}\right]  \tag{C.14}\\
& =\frac{e^{\mathrm{i} \omega a}}{\omega}-\frac{1}{\omega} \frac{2 \cos (\omega a)}{1+e^{\frac{\pi \omega}{\rho}}} \tag{C.15}
\end{align*}
$$

where the geometric sum closed forms have been used.
Rearranging this leads to

$$
\begin{align*}
\frac{e^{\mathrm{i} \omega a}}{\omega} & =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{2 \pi^{2}} \int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} \frac{\rho e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}}}{2 \xi}\left(\frac{1}{\sinh [\rho(\xi-a-\mathrm{i} \epsilon)]}+\frac{1}{\sinh [\rho(\xi+a+\mathrm{i} \epsilon)]}\right) \\
& +\frac{2 \cos (\omega a)}{\omega} \frac{1}{1+e^{\pi \omega / \rho}} . \tag{C.16}
\end{align*}
$$

Now there is an exponentially decaying integrand, which is always a desirable feature when numerically integrating. Also the role of $\rho$ is more obvious, if $\rho$ is made larger the integrand decays faster, however it is balanced by the a large remainder term (the $\cos (\omega a)$ term); and given that all expressions containing $\omega$ will be summed over all the cavity modes this means having a weaker exponentially decaying mode sum. These competing requirements dictate how to chose the parameter $\rho$.

Hence by using the second trick, slow polynomially decaying series can be replaced with exponentially decaying series so as to improve numerical computation time and minimise truncation errors, allowing for quick and reliable results.

## C. 2 Geometric sums and Dirac comb

Following implementation of the numerical tricks above $\S$ C. 1 the mode sums under consideration assume the form of geometric series, which can be explicitly evaluated. In this
chapter 6 the expressions of interest follow the form of (6.10)

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \mathrm{~d}^{3} \boldsymbol{\xi} f(|\boldsymbol{\xi}|) \sum_{\boldsymbol{m} \in\left(\mathbb{Z}^{+}\right)^{3}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right), \tag{C.17}
\end{equation*}
$$

i.e. the replacement $\xi_{i} \rightarrow-\xi_{i}$ has no influence on the integral or on $f(|\boldsymbol{\xi}|)$. In the derivation below this property will be exploited in order to simplify the geometric sum of the integrand. Consider,

$$
\begin{equation*}
\sum_{m \in\left(\mathbb{Z}^{+}\right)^{3}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}} \prod_{i=1}^{3} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right)=\prod_{i=1}^{3} \sum_{m_{i}=0}^{\infty} e^{\mathrm{i} \frac{m_{i} \pi}{L_{i}} \xi_{i}} \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right) \tag{C.18}
\end{equation*}
$$

note the sum can be extended to include $m_{i}=0$ without changing the value of the sum. Let

$$
\begin{align*}
& \frac{\xi \pi}{L}=\tilde{\xi}  \tag{C.19}\\
& \frac{y \pi}{L}=\tilde{y}  \tag{C.20}\\
& \frac{z \pi}{L}=\tilde{z} \tag{C.21}
\end{align*}
$$

and consider the $i=1$ term of the expression above,

$$
\begin{align*}
& \sum_{m=0}^{\infty} e^{\mathrm{i} m \tilde{\xi}} \sin (m \tilde{y}) \sin (m \tilde{z}) \\
= & -\sum_{m=1}^{\infty} e^{\mathrm{i} m \tilde{\xi}} \frac{1}{4}\left(e^{\mathrm{i} m(\tilde{y}+\tilde{z})}-e^{\mathrm{i} m(\tilde{y}-\tilde{z})}-e^{-\mathrm{i} m(\tilde{y}-\tilde{z})}+e^{-\mathrm{i} m(\tilde{y}+\tilde{z})}\right),  \tag{C.22}\\
= & -\frac{1}{4} \sum_{m=0}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}-\tilde{y}+\tilde{z})}\right)^{m}+\left(e^{\mathrm{i}(\tilde{\xi}-\tilde{y}-\tilde{z})}\right)^{m} . \tag{C.23}
\end{align*}
$$

Recalling the bigger picture of (C.17), perform the replacement $\tilde{\xi} \rightarrow-\tilde{\xi}$ for the last two terms

$$
\begin{align*}
& \sum_{m=0}^{\infty} e^{\mathrm{i} m \tilde{\xi}} \sin (m \tilde{y}) \sin (m \tilde{z}) \\
= & -\frac{1}{4} \sum_{m=0}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(-\tilde{\xi}-\tilde{y}+\tilde{z})}\right)^{m}+\left(e^{\mathrm{i}(-\tilde{\xi}-\tilde{y}-\tilde{z})}\right)^{m}  \tag{C.24}\\
= & -\frac{1}{4} \sum_{m=0}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}+\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{-m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{-m},  \tag{C.25}\\
= & -\frac{1}{4}\left[\sum_{m=0}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}+\sum_{m=1}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{-m}+1\right. \\
- & \left.\sum_{m=0}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m}-\sum_{m=1}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{-m}-1\right]  \tag{C.26}\\
= & -\frac{1}{4} \sum_{m=-\infty}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m} . \tag{C.27}
\end{align*}
$$

This geometric sum is a known Fourier series for the Dirac Comb

$$
\begin{align*}
& \sum_{m=-\infty}^{\infty}\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}+\tilde{z})}\right)^{m}-\left(e^{\mathrm{i}(\tilde{\xi}+\tilde{y}-\tilde{z})}\right)^{m} \\
= & 2 \pi \sum_{n=-\infty}^{\infty} \delta(\tilde{\xi}+\tilde{y}+\tilde{z}+2 \pi n)-\delta(\tilde{\xi}+\tilde{y}-\tilde{z}+2 \pi n)  \tag{C.28}\\
= & 2 \pi \sum_{n=-\infty}^{\infty} \delta\left(\frac{\pi}{L}(\xi+y+z+2 n L)\right)-\delta\left(\frac{\pi}{L}(\xi+y-z+2 n L)\right),  \tag{C.29}\\
= & 2 L \sum_{n=-\infty}^{\infty} \delta(\xi+y+z+2 n L)-\delta(\xi+y-z+2 n L) . \tag{C.30}
\end{align*}
$$

Therefore

$$
\begin{align*}
\sum_{m=0}^{\infty} e^{\mathrm{i} m \tilde{\xi}} \sin (m \tilde{y}) \sin (m \tilde{z}) & =-\frac{2 L}{4} \sum_{n=-\infty}^{\infty} \delta(\xi+y+z+2 n L)-\delta(\xi+y-z+2 n L)  \tag{C.31}\\
& =-\frac{L}{2} \sum_{n=-\infty}^{\infty} \delta(\xi+y+z+2 n L)-\delta(\xi+y-z+2 n L) \tag{C.32}
\end{align*}
$$

Combining this result into the 3 component product in (C.18) gives

$$
\begin{align*}
\sum_{m \in\left(\mathbb{Z}^{+}\right)^{3}} e^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\xi}} \prod_{i=1}^{3} & \sin \left(\frac{m_{i} \pi}{L_{i}} y_{i}\right) \sin \left(\frac{m_{i} \pi}{L_{i}} z_{i}\right)=\frac{V}{8} \\
& \times \prod_{i=1}^{3} \sum_{n_{i}=-\infty}^{\infty}\left(\delta\left(\xi_{i}+y_{i}-z_{i}+2 n_{i} L_{i}\right)-\delta\left(\xi_{i}+y_{i}+z_{i}+2 n_{i} L_{i}\right)\right) . \tag{C.33}
\end{align*}
$$

Note this result in only valid for expressions of the form (C.17), i.e. with a $\xi_{i} \rightarrow-\xi_{i}$ symmetry. Fortunately these are the exact expressions found in this chapter 6.

The main purpose of the mathematical tricks was to circumvent the 3 D semi-infinite mode sum required to evaluate the Wightman function. In this appendix we showed how the consequence of these mathematical tricks was to replace the 3D semi-infinite sum with a 3D infinite sum; however as the second trick shows §C.1.2 this 3D infinite sum will converge exponentially quickly, providing a significant computational advantage.


[^0]:    ${ }^{1}$ Not unlike the way in which $\frac{1+x}{x}-\frac{1}{x}$ does not depend on $x$.

