Comparing Classical Portfolio Optimization and Robust Portfolio Optimization on Black Swan Events

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Black swan events, such as natural catastrophes and manmade market crashes, historically have a drastic negative influence on investments; and there is a discrepancy on losses caused by these two types of disasters. In general, there is a recovery and it is of interest to understand what type of investment strategies lead to better performance for investors.

In this thesis we study classical portfolio optimization, robust portfolio optimization and some historical black swan events. We compare two main strategies: mean variance optimization vs robust portfolio optimization on two types of black swan events: natural vs anthropogenic. The comparison illustrates that robust portfolio optimization is much more conservative, and has a shorter recovery time than classical portfolio optimization. Moreover, the losses in the stock investment resulted from a natural disaster are very minor compared to the losses resulted from an anthropogenic market crash.

Acknowledgements

I would like to express deep thanks to my supervisor, Professor Henry Wolkowicz, for his guidance and support during this special year when Covid-19 virus poses a drastic threat to people in all continents. Due to the virus, 2020 has been a hard time for everyone; and I haven't met Henry face to face for a long time. However, he has been very kind and encouraging when answering my questions. Hopefully the virus will go away soon, and everything can go back on track in the new year 2021.

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Part I

Introduction

In this part, we give an introduction to our thesis. We illustrate the motivation of the thesis, i.e., why we are interested in studying the classical portfolio optimization and robust portfolio optimization. Then we present some main results from our empirical experiments. We also outline the contents of our thesis.

Chapter 1

General

In this thesis we study classical models and robust models of portfolio optimization. We would like to compare these two approaches during and after a black swan event, i.e., an unexpected abnormal event. A black swan event can cause massive market losses. We study the mathematical tools that have been developed to reduce risks and find an optimal allocation of investments. The main conclusions that we see are that recovery when using robust portfolio optimization is faster than with classical portfolio optimization; and the effect of a black swan event arising from *nature* on the stock market is minor compared to an anthropogenic black swan event.

1.1 Motivation

In modern times, investment is common to individuals, families, and firms. Due to the globalization of financial markets, investments have become easily accessible, and the variety of investment opportunities has greatly increased. Major investment instruments include e.g., bonds, stocks, derivatives and mutual funds. The goal of investment is to maximize profits while minimizing risks, often by diversification of the investment instruments in the portfolio. However, the performance of the future market is highly unpredictable.

Portfolio optimization attempts to find the optimal portfolio strategy subject to minimizing risk. This is the classical Markowitz philosophy of maximizing profit while not exceeding an upper bound on the risk. Or conversely, one can minimize risk while maintaining a minimum level of profit. There are other modifications used. Sharpe ratio is one of the modifications discussed in this thesis.

A more modern approach uses *robust optimization* in order to ensure against catastrophic changes, *black swan events*. We incorporate uncertainties of the parameters into portfolio optimization. This is more of a min-max approach that looks at worst case scenarios.

We would like to compare these two strategies using the data from some real world black swan events. We make a comparison on the performance of these two different strategies during and after black swan events.

1.2 Main Results

We study classical portfolio optimization and robust portfolio optimization; and we apply data from historical black swan events to compare mean variance optimization (**MVO**) and robust mean variance optimization (**RMVO**).

We first obtain historical data of 50 stocks from the 2005 hurricane Katrina. We want to look at the performance of **MVO** and **RMVO** during and after this natural disaster. We compute the optimal portfolios utilizing **MVO** and **RMVO**. We observe that **MVO** strategy selects a narrow range of stocks and is heavily skewed to some assets. To the contrary, **RMVO** selects a diverse range of assets and has a shorter recovery period after the disaster. The impacts of this natural catastrophe are minor on the stock market; and the recovery is fast in general for both strategies.

Now we look at the performance of **MVO** and **RMVO** on the 2008 financial crisis. We also compute the optimal portfolios utilizing **MVO** and **RMVO**; and we conclude, as expected, that **RMVO** is much more conservative and has a much shorter recovery period than **MVO**. However, we observe that an anthropogenic disaster on the stock market has drastic impacts on the stock market, and the recovery is very slow.

The discrepancy between **MVO** and **RMVO** on black swan events is huge and profound. The losses resulted from a man made disaster takes a much longer time to recover than the losses resulted from a natural catastrophe.

1.3 Outline

In Part II, we study various classical portfolio optimization models. In Chapter 2, we introduce some financial concepts and present some fundamental portfolio optimization problems. In Chapter 3, we introduce different methods to measure risks.

In Part III, we add uncertainties of the parameters to the problems and study robust portfolio optimization. In Chapter 4, we introduce some background about robust optimization. In Chapter 5, we study robust portfolio optimization problems.

In Part IV, we look anthropogenic black swan events and those that arise from natural events. We compare the classical portfolio optimization with the robust portfolio optimization by testing data from real catastrophes. In Chapter 6, we study some historical black swan events and their effects on the financial markets. In Chapter 7, we compare the performance of the classical portfolio optimization with the robust portfolio optimization on black swan event.

In Part V, we conclude our thesis. In Chapter 8, we present our main results and discuss some future work.

Part II

Classical Portfolio Optimization

In this part we study classical portfolio optimization theory, including some widely used risk measures and the portfolio optimization problems associated with these risk measures. We first follow the book [13] to give an introduction on the background of portfolio optimization. Then we study some popular risk measures: Mean Absolute Deviation(**MAD**), Semi-Mean Absolute Deviation(Semi-**MAD**) Mean Variance Optimization(**MVO**), Value-at-Risk(**VaR**) and Conditional Value-at-Risk(**CVaR**). The main references are books [8, 13] and paper [17].

Chapter 2

Background on Portfolio Optimization

In this chapter, we introduce some terminologies and some fundamental portfolio optimization models to give a general idea of portfolio optimization.

2.1 Fundamental Models on Optimal Portfolios

In this section, we give some fundamental portfolio optimization models, and we analyze their objective functions and constraints. This section mainly follows from [8], [13] .

Following the concepts and notations from the book [13], we introduce some basic terminologies in finance:

- Capital: a certain amount of money that an investor wants to invest;
- Asset: any specific tradable financial instrument;
- Portfolio: the list of proportions of the total capital invested in the various assets.

We number the set of available assets using $N = \{1, 2, ..., n\}$. Let x_j denote the percentage of the available capital invested in asset j; and let $x = (x_j)_{j=1,...,n}$ be the vector of decision variables x_j, i.e., this defines the portfolio. We then also say that x is a portfolio or represents the portfolio. Following the book [13], we assume that we must use capital to buy assets, i.e., short sales are not allowed. Thus we have a non-negativity constraint:

$$x_j \geq 0 \quad j = 1, ..., n. \tag{2.1.1}$$

This is equivalent to

$$x \geq 0.$$

Moreover, the sum of the percentages invested in the assets is one, i.e., we have a basic *budget constraint*[1]:

$$\sum_{j=1}^{n} x_j = 1. \tag{2.1.2}$$

Let R_j be a random variable that represents the rate of return for asset j at the target time with given mean $\mu_j = \mathbb{E}\{R_j\}$. We denote the portfolio rate of return associated with portfolio x as $R_x = \sum_{j=1}^{n} R_j x_j$. This is also the weighted sum of the rates of the assets. The mean rate of return of portfolio x is defined as:

$$\mu(x) = \mathbb{E}\{R_x\} = \mu^T x = \sum_{j=1}^{n} \mu_j x_j,$$

where $\mu = (\mu_j)_{j=1,..,n}$ is a vector representing the mean rate of return of assets.

Denote the measure of risk associated with portfolio x as $\varrho(x)$. The risk measure expresses the uncertainty of the return of all the assets. A risk-free portfolio has $\varrho(x) = 0$, i.e., this means that the rate of return of portfolio x is known with certainty. For any other portfolio, we take positive values of $\varrho(x)$. The risk measure we consider here is also called the dispersion measure, which quantifies the level of variability of the portfolio rate of return around its expected value. We will introduce a few methods to measure risks in the following sections.

Now we can build up a mean-risk bi-criteria portfolio optimization problem:

$$\begin{aligned} \max \quad & [\mu(x), -\varrho(x)] \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} x_j = 1 \\ & x \geq 0. \end{aligned} \tag{2.1.3}$$

The objective function *maximizes* the mean rate of return of portfolio x and *minimizes* the risk measure. In reality, it is not possible to maximize the mean return and minimize

[1] This budget constraint is equivalent to $e^T x = 1$, where e is an n-dimensional vector of 1's.

the risk at the same time. A portfolio with high mean return is usually highly risky and vice-versa. Hence, we need to adjust the model to allow for the bi-criteria objective. We now present two views.

One common approach is to impose a lower bound μ_0 on the expected rate of return while minimizing the risk. This yields the following formulation:

$$\begin{array}{lll} \min & \varrho(x) & \text{(risk)} \\ \text{s.t.} & \mu(x) \geq \mu_0 & \text{(return)} \\ & \sum_{j=1}^{n} x_j = 1 & \text{(budget)} \\ & x \geq 0. & \end{array} \tag{2.1.4}$$

Another approach *flip*s the problem and bounds the risk while maximizing the return. This approach corresponds to paper [2] and yields the following problem:

$$\begin{array}{lll} \max & \mu(x) & \text{(return)} \\ \text{s.t.} & \varrho(x) \leq \varrho_0 & \text{(risk)} \\ & \sum_{j=1}^{n} x_j = 1 & \text{(budget)} \\ & x \geq 0. & \end{array} \tag{2.1.5}$$

In addition, we would like to introduce the mean-safety optimization problem. The concept of safety measure is introduced in order to overcome the weakness of risk measure. Each risk measure $\varrho(x)$ has a well-defined corresponding safety measure $\mu(x) - \varrho(x)$, and the mean-safety optimization problem is modeled as:

$$\begin{array}{ll} \max & [\mu(x), \mu(x) - \varrho(x)] \\ \text{s.t.} & \sum_{j=1}^{n} x_j = 1 \\ & x \geq 0. \end{array} \tag{2.1.6}$$

So far, we have given some fundamental portfolio optimization problems (2.1.3) to (2.1.6). In the rest of Part II, we will discuss methods to measure risk and use more sophisticated portfolio optimization models that capture the interests of investors.

Chapter 3

Measuring Risks

In Section 2.1, we introduce a few portfolio optimization models but do not specify risk measures. In this chapter, we discuss various methods to measure risks and build more sophisticated and useful portfolio optimization models.

3.1 LP : Mean Absolute Deviation

In the Mean Absolute Deviation (**MAD**) model, we measure the risk through the **MAD** of portfolio return. The **MAD** is a dispersion measure that measures the average of the absolute value of the difference between the random variable of portfolio return R_x and its expected value. The **MAD** is defined as:

$$\delta(x) := \mathbb{E}\{|R_x - \mathbb{E}\{R_x\}|\} = \mathbb{E}\{|\sum_{j=1}^{n} R_j x_j - \mathbb{E}\{\sum_{j=1}^{n} R_j x_j\}|\}, \tag{3.1.1}$$

where the random variable R_j is defined in Section 2.1. The references for the following material are [8, 13].

We introduce the concept of *scenario* to look at the uncertainty of the return rates of the assets at the target time. A scenario is defined as a possible situation that can happen at a target time t, $t = 1, ..., T$. Denote the probability of the scenario corresponding to target time t by p_t, then $\sum_{t=1}^{T} p_t = 1$. For each portfolio return R_j, we assume that its realization r_{jt} corresponding to scenario t is known. We can define the scenario t by a set of the returns of all the assets $\{r_{jt}, j = 1, ..., n, t = 1, ..., T\}$. The expected return of asset j is computed as:

$$\mu_j = \sum_{t=1}^{T} p_t r_{jt}. \tag{3.1.2}$$

The return y_t of a portfolio x in scenario t is computed as:

$$y_t = \sum_{j=1}^{n} r_{jt} x_j. \tag{3.1.3}$$

Moreover, the expected return of the portfolio $\mu(x)$ is computed as:

$$\mu(x) = \mathbb{E}\{R_x\} = \sum_{t=1}^{T} p_t y_t = \sum_{t=1}^{T} p_t \sum_{j=1}^{n} r_{jt} x_j = \sum_{j=1}^{n} x_j \sum_{t=1}^{T} p_t r_{jt} = \sum_{j=1}^{n} \mu_j x_j. \tag{3.1.4}$$

We can rewrite the **MAD** as:

$$\delta(x) = \sum_{t=1}^{T} p_t \left(\left| \sum_{j=1}^{n} r_{jt} x_j - \sum_{j=1}^{n} \mu_j x_j \right| \right) = \sum_{t=1}^{T} p_t(|y_t - \mu|). \tag{3.1.5}$$

The portfolio optimization problem is modelled as:

$$\begin{aligned}
\min \quad & \delta(x) = \textstyle\sum_{t=1}^{T} p_t(|y_t - \mu|) \\
\text{s.t.} \quad & \mu(x) = \mu^T x = \textstyle\sum_{j=1}^{n} \mu_j x_j \\
& \mu(x) \geq \mu_0 \\
& \textstyle\sum_{j=1}^{n} x_j = 1 \\
& x \geq 0.
\end{aligned} \tag{3.1.6}$$

Observe that this model is not a linear program. However, we can reformulate (3.1.6) into a linear model. Define $d_t = y_t - \mu$, and let $d_t = d_t^+ - d_t^-$ such that $d_t^+ \geq 0, d_t^- \geq 0$. Then

(3.1.6) can be formulated as:

$$
\begin{aligned}
\min \quad & \delta(x) = \textstyle\sum_{t=1}^{T} p_t |d_t| = \sum_{t=1}^{T} p_t (d_t^+ + d_t^-) \\
\text{s.t.} \quad & \mu(x) = \mu^T x = \textstyle\sum_{j=1}^{n} \mu_j x_j \geq \mu_0 \\
& d_t^+ - d_t^- + \mu = \textstyle\sum_{j=1}^{n} r_{jt} x_j \quad t = 1, ..., T \\
& \textstyle\sum_{j=1}^{n} x_j = 1 \\
& x \geq 0, d_t^+ \geq 0, d_t^- \geq 0.
\end{aligned} \tag{3.1.7}
$$

Now we have a linear program that minimizes the **MAD** .

3.1.1 Semi Mean Absolute Deviation

In the Semi Mean Absolute Deviation (Semi-**MAD**) model, we assume that the rate of return of the portfolio has a normal distribution [13]. Then, the proportionality relation between the **MAD** and the standard deviation is $\delta(x) = \sqrt{\frac{2}{\pi}}\sigma(x)$. Since investors are concerned with under-performance of a portfolio, we consider risks only that deviate below the expected return in the Semi-**MAD** model. The Semi-**MAD** is defined as:

$$\bar{\delta}(x) := \mathbb{E}\{\max\{0, \mathbb{E}\{\sum_{j=1}^{n} R_j x_j\} - \sum_{j=1}^{n} R_j x_j\}\}. \tag{3.1.8}$$

Adapted from the **MAD** optimization problem (3.1.7), the Semi-MAD optimization problem is formulated as follows:

$$
\begin{aligned}
\min \quad & \bar{\delta}(x) = \textstyle\sum_{t=1}^{T} p_t d_t \\
\text{s.t.} \quad & \mu(x) = \mu^T x = \textstyle\sum_{j=1}^{n} \mu_j x_j \\
& \mu(x) \geq \mu_0 \\
& d_t \geq \mu - y_t \qquad t = 1, ..., T \\
& y_t = \textstyle\sum_{j=1}^{n} r_{jt} x_j \quad t = 1, ..., T \\
& d_t \geq 0 \\
& \textstyle\sum_{j=1}^{n} x_j = 1 \\
& x \geq 0.
\end{aligned} \tag{3.1.9}
$$

Theorem 3.1.1 ([13]). *Minimizing the MAD is equivalent to minimizing the Semi-MAD as* $\delta(x) = 2\bar{\delta}(x)$. □

3.1.2 Accounting for Transaction Costs

When buying and selling an assent, transaction costs including commissions and other handling charges may incur. Here we present two types of transaction costs frequently used in practice: fixed and proportional transaction costs; and we build mixed integer models to incorporate transaction costs. Buying and selling an asset often incur transaction costs, and high transaction levels may result in expensive costs. Investors tend to invest capital in a relatively small number of assets since transaction costs may reduce the net portfolio return. The following material for this section is heavily based on the book [13].

We let variable X_j, $j = 1, ..., n$, be the amount of money invested in asset j. Let $\bar{C}$ be a constant representing the *available capital*, the total amount of money including the investment in assets and transaction costs. Let $K_j(Y_j)$ indicate the transaction costs paid for asset j. We assume that transaction costs for the assets are independent from each other, then the transaction cost function for a portfolio or the total amount of transaction costs is

$$K(X_1, ..., X_n) = \sum_{j=1}^{n} K_j(X_j).$$

First, we consider *fixed transaction cost* that is independent of the amount of money invested in an asset. Let u_j be a non-negative constant representing the transaction cost for asset j, and let u_j be 0 if selecting asset j does not incur any fixed transaction cost. We express the fixed transaction cost for asset j as

$$K_j(X_j) = \begin{cases} u_j & \text{if} \quad X_j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We call this cost structure *Pure Fixed Cost (PFC)*.

For each asset, we introduce a binary variable v_j, $j = 1, ..., n$. We assign 1 to variable v_j if asset j is selected in our portfolio, and 0 otherwise. Then we can express the above function in a linear form:

$$K_j(X_j) = u_j v_j.$$

Moreover, we have the following constraint:

$$L_j v_j \leq X_j \leq U_j v_j, \qquad j = 1, ..., n,$$

where L_j and U_j are positive lower and upper bounds. Observe that L_j cannot be 0, and U_j can possibly equal to $\bar{C}$.

Next, we introduce *proportional transaction costs* that are variables and depend on the amount of money invested in an asset. Let w_j denote the rate specified for asset j, and let w_j be 0 if proportional transaction cost does not incur for asset j. The transaction cost is a percentage of the quantity invested in asset j and can be expressed as:

$$K_j(X_j) = w_j X_j.$$

The above cost structure is called *Pure Proportional Cost (PPC)*.

When selecting an asset j for a portfolio, either fixed transaction cost or proportional transaction cost or both may incur. We express the total transaction cost in a portfolio as

$$\sum_{j=1}^{n} K_j(X_j) = \sum_{j=1}^{n} u_j v_j + \sum_{j=1}^{n} w_j X_j.$$

Now we consider the portfolio optimization problem accounting for transaction costs. We modify the constraint on expected return as follows:

$$\sum_{j=1}^{n} \mu_j X_j - \sum_{j=1}^{n} K_j(X_j) \geq \mu_0 \bar{C}.$$

The budget constraint is modified as

$$\sum_{j=1}^{n} X_j + \sum_{j=1}^{n} w_j X_j + \sum_{j=1}^{n} u_j v_j = \bar{C}.$$

The deviation constraint can be expressed as

$$\sum_{j=1}^{n} \mu_j X_j - \sum_{j=1}^{n} (r_{jt} - w_j) X_j + \sum_{j=1}^{n} u_j v_j \leq d_t \qquad t = 1, ..., T.$$

Suppose we use Semi-MAD for our risk measure, we can adapt (3.1.9) and express the portfolio optimization problem with transaction costs as

$$
\begin{aligned}
\min \quad & \textstyle\sum_{t=1}^{T} p_t d_t \\
\text{s.t.} \quad & L_j v_j \leq X_j \leq U_j v_j \qquad j = 1, ..., n \\
& \textstyle\sum_{j=1}^{n} (\mu_j - w_j) X_j - \sum_{j=1}^{n} u_j v_j \geq \mu_0 \bar{C} \\
& \textstyle\sum_{j=1}^{n} \mu_j X_j - \sum_{j=1}^{n} (r_{jt} - w_j) X_j + \sum_{j=1}^{n} u_j v_j \leq d_t \qquad t = 1, ..., T \\
& \textstyle\sum_{j=1}^{n} X_j + \sum_{j=1}^{n} w_j X_j + \sum_{j=1}^{n} u_j v_j = \bar{C} \\
& d_t \geq 0 \qquad t = 1, ..., T \\
& X_j \geq 0 \qquad j = 1, ..., n \\
& v_j \in \{0, 1\} \qquad j = 1, ..., n.
\end{aligned}
\tag{3.1.10}
$$

3.1.3 MAD Example

In this section, we present an example using **MAD**, and the *MATLAB Financial Toolbox* [14] to find optimal portfolios. We work with a list of 30 US stocks from diverse industries. We obtain the historical daily stock prices from the first trading day of year 2000 to the last trading day of year 2010, using Yahoo Finance. We assume that cash is risk free and has a zero interest rate. First, we calculate the daily rate of return for all the stocks. We use the following formula to compute *daily rate of return*(**DRoR**) for stocks based on the daily price values:

$$\mathbf{DRoR}(i) = \frac{\text{Price}(i+1) - \text{Price}(i)}{\text{Price}(i)}, \quad \text{(percentage return)}.$$

We create a matrix *AssetReturns* with the results for *all* the stocks. To obtain the mean and the covariance matrix, we use the MATLAB commands *mean(AssetReturns)* and *cov(AssetReturns)*. Now we have set up the preliminaries.

To implement the **MAD** model, we create a *PortfolioMAD* object p in MATLAB, using the MATLAB command *PortfolioMAD*. To set up *AssetScenarios*, we use the MATLAB command *stimulateNormalScenariosbyData* to generate $200,000$ number of scenarios based on our data. We also set up the budget constraint and the non-negativity constraint by MATLAB command *setDefaultConstraints*. After we construct the properties for p, we plot the efficient frontier in Figure 3.1.1. In this plot, the x-axis is the mean absolute deviation; and the y-axis is the daily mean rate of return. From the code we obtain: the optimal portfolio with daily target return 0.1% ; and the optimal portfolio with daily target risk 2% in Table 3.1.1.

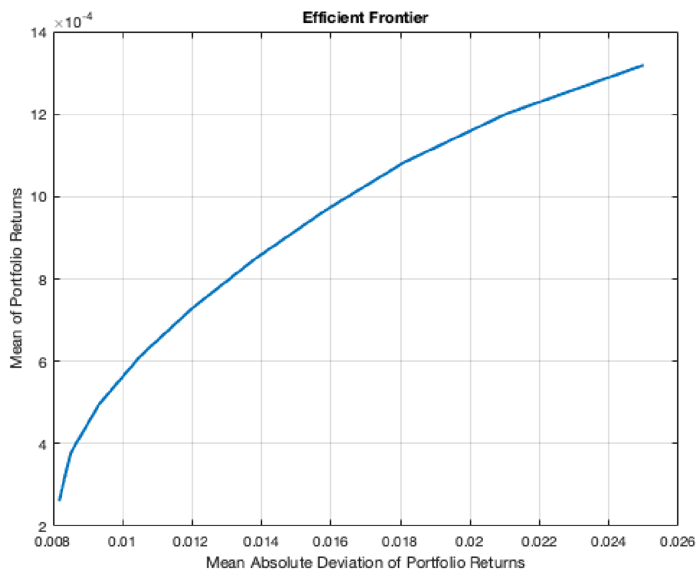

Figure 3.1.1: Example: **MAD** Efficient Frontier

Ticker	Weight(%)
BRK	11.8722
AAPL	44.2148
MCD	2.22366
CVX	13.9044
CAT	26.3887
AMZN	1.39628

Ticker	Weight(%)
AAPL	70.2808
CAT	28.5737
AMZN	1.14551

Table 3.1.1: Left: Optimal Portfolio with 0.1% Target Return; Right: Optimal Portfolio with 2% Target Risk

Now suppose that selling an asset incurs 0.05% transaction costs; and buying an asset also incurs 0.05% transaction costs. We add these properties to p by using MATLAB command *setCosts*. We obtain a new efficient frontier in Figure 3.1.2. We observe that the efficient frontier with transaction costs is below the original efficient frontier. With a given target return, the corresponding risk is higher in the efficient frontier with transaction costs. Conversely, with a given target risk, the corresponding return is lower.

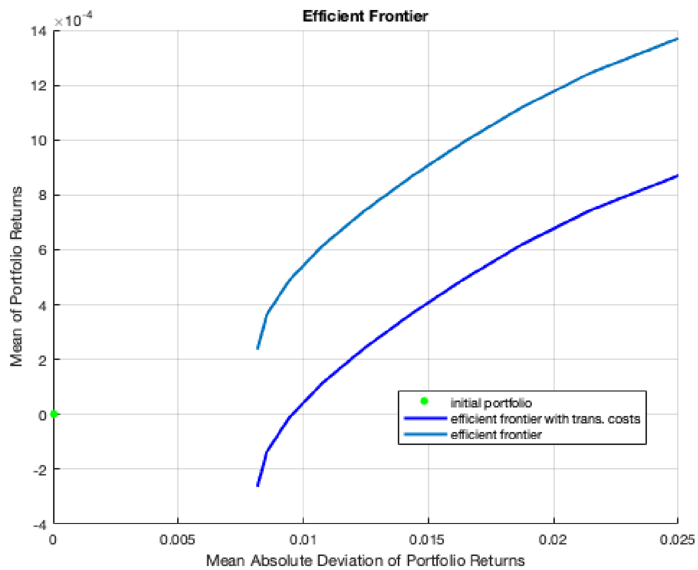

Figure 3.1.2: Example: **MAD** Efficient Frontier with Trans. Costs

3.2 QP: Mean Variance Optimization

In 1990, Harry Markowitz won the Nobel prize in Economics for his contributions in *Modern Portfolio Theory*. In his ground breaking work, he suggested to measure the risk ϱ based on the variance

$$\sigma^2 = \mathbb{E}\{(R - \mathbb{E}\{R\})^2\}. \tag{3.2.1}$$

The variance, or the risk, of a portfolio can be reduced through diversification. A rational investor spreads investments over different assets since investing the entire capital in a single asset is highly risky. This activity is called *diversification*. Portfolios from the same sector tend to move in the same direction, and diversification reduces volatility of portfolio performance. The following material in this section is heavily based on material from [8, 13] and the references therein.

Let ρ_{ij} represent the correlation coefficient between the returns of assets i and j, where we set $\rho_{ii} = 1$. The correlation coefficient ρ_{ij} is positive if pairs of assets belong to same sector; and it is negative if asset i and asset j move in opposite directions, i.e., are *negatively correlated*.

Let σ_j denote the standard deviation of the return of asset j, and let $\Sigma = (\sigma_{ij}) \in \mathbb{R}^{n\times n}$ be the symmetric covariance matrix such that $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \neq j$. Then, we can represent the variance of portfolio x as follows:

$$\sigma^2(x) = \sum_{i,j} \rho_{ij}\sigma_i\sigma_j x_i x_j = x^T\Sigma x. \tag{3.2.2}$$

Note that the variance is always non-negative, i.e., $x^T\Sigma x \geq 0, \forall x$, and it follows that Σ is positive semi-definite.

Recall that μ_j is the expected return of asset j, (3.1.2). The Markowitz mean-variance optimization (**MVO**) problem can now be formulated using (2.1.4) as:

$$\begin{array}{ll} \min & \frac{1}{2}x^T\Sigma x \\ \text{s.t.} & \mu(x) = \mu^T x = \sum_{j=1}^n \mu_j x_j \geq \mu_0 \quad \text{(lower bound on expected return)} \\ & \sum_{j=1}^n x_j = 1 \\ & x \geq 0. \end{array} \tag{3.2.3}$$

Now we have a quadratic optimization problem. The **MVO** problem (3.2.3) is equivalent to each of the following two problems:

$$\begin{array}{ll} \max & \mu^T x \\ \text{s.t.} & \frac{1}{2}x^T\Sigma x \leq \sigma_0^2 \\ & \sum_{j=1}^n x_j = 1 \\ & x \geq 0, \end{array} \tag{3.2.4}$$

where σ_0^2 is a given upper bound on the variance of the portfolio; and

$$\begin{array}{ll} \max & \mu^T x - \lambda x^T\Sigma x \\ \text{s.t.} & \sum_{j=1}^n x_j = 1 \\ & x \geq 0, \end{array} \tag{3.2.5}$$

where λ is a risk-aversion constant. The equivalence depends on the particular choices of the constants $\mu_0, \sigma_0^2, \lambda$.

Observe that (3.2.4) has a convex quadratic constraint and hence is a non-linear programming (NLP) problem. **QP**s and linear objectives with convex quadratic constraints (3.2.3) to (3.2.5) can be effectively solved by interior point methods.

3.2.1 Maximizing Sharpe Ratio

Before we discuss the Sharpe ratio, we first introduce the *efficient frontier* developed by Harry Markowitz. The efficient frontier graphically presents a set of optimal portfolios maximizing the expected return for a given level of risk or minimizing the risk for a defined level of expected return. An efficient frontier plots the risk on the x-axis and the mean rate of return on the y-axis. The risk is commonly depicted by the standard deviation.

Sharpe ratio introduced by William F. Sharpe is defined as the difference between the return of an investment and the risk-free return over the standard deviation of the investment. Mathematically, the formula of Sharpe ratio is expressed as:

$$h(x) = \frac{r_p - r_f}{\sigma_p}$$

where r_p is return of portfolio, r_f is risk-free rate, and σ_p is the standard deviation of the portfolio's excess return. In a graph, the Sharpe point is the tangency point of the efficient frontier and the line going through the point representing the risk-free asset. The Sharpe ratio is a measure of return characterizing how well the return compensates for the risk taken. More specifically, the ratio depicts the excess return when holding a riskier asset. A high Sharpe ratio is more attractive to investors as the return of a portfolio is better. A negative Sharpe ratio is possible when the risk-free rate (zero) is greater than the portfolio's rate of return. The portfolio optimization problem maximizing the Sharpe ratio is given below:

$$\begin{aligned} \max_x \quad & h(x) = \frac{\mu^T x - r_f}{(x^T \Sigma x)^{1/2}} \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^n x_j = 1 \\ & x \geq 0. \end{aligned} \tag{3.2.6}$$

3.2.2 MVO Example

In the following, we present an example using **MVO**. The example works with the same 30 US stocks in Section 3.1.3; and we set up the preliminaries using the same approach.

We create a *Portfolio* object p in MATLAB [14] by using command *Portfolio*. Note that this command *Portfolio* specifically implements the **MVO** model in MATLAB. We now would like to add properties to p. We implement the MATLAB commands " *estimateAssetMoments* to estimate mean and covariance of asset returns. We use the MATLAB function *setDefaultConstraints* to set up constraints such that the portfolio weights are non-negative and sum up to 1. Then, we apply the MATLAB function *estimateFrontier* and *estimatePortMoments* to draw the efficient frontier Figure 3.2.1. Recall that the efficient frontier plots the risks or the standard deviation on the x-axis and the mean rate of return on the y-axis. We use MATLAB functions *estimateFrontierByReturn*, *estimateFrontierByRisk*, *estimateMaxSharpeRatio* to emphasize the three dots in the plot. We find the optimal portfolios corresponding to the three dots in Table 3.2.1. The left table presents the optimal portfolio with 0.1% target return corresponding to the red dot; the middle table presents the optimal portfolio with 2.5% target risk corresponding to the yellow dot; the right table presents the optimal portfolio with maximum Sharpe ratio corresponding to the green dot. Figure 3.2.2 confirms that the green dot indeed maximizes Sharpe ratio.

Ticker	Weight(%)
BRK	12.3132
AAPL	45.7407
MCD	3.22771
CVX	6.29988
CAT	26.8188
AMZN	5.59967

Ticker	Weight(%)
AAPL	67.1053
CAT	25.1027
AMZN	7.79192

Ticker	Weight(%)
BRK	14.6793
AAPL	39.614
MCD	7.23032
CVX	10.5521
CAT	23.1875
AMZN	4.73677

Table 3.2.1: Left: Optimal Portfolio with 0.1% Target Return; Middle: Optimal Portfolio with 2.5% Target Risk; Right: Optimal Portfolio with Max Sharpe Ratio

3.3 SP: *VaR* and *CVaR*

In this section, we discuss *Value-at-Risk*, (**VaR**), and its relative *Conditional Value-at-Risk*, (**CVaR**), developed by financial engineers. **VaR** is used to reduce *risk of high losses*. **CVaR** is also known as expected shortfall, mean excess loss, or tail **VaR**. We present a

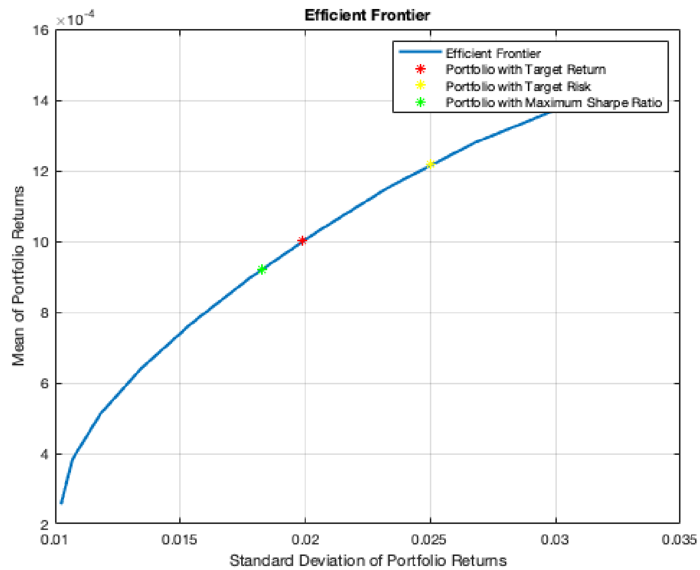

Figure 3.2.1: Example: **MVO** Efficient Frontier

stochastic programming, ***SP*** model with **CVaR** as risk measure. A **SP** is an optimization problem with data uncertainty, and we assume that the uncertain parameters are random variables with known probability distributions. The decision variables in an **SP** can be *anticipative* and/or *adaptive*. An *anticipative* decision variable cannot be made depending on future observations or partial realizations of the random parameters; an *adaptive* decision variable can be made after some or all of the random parameters are observed. An **SP** including both anticipative and adaptive variables is called a *recourse* model. A generic theoretical form of a *two-stage stochastic linear program with recourse* has the following form:

$$\begin{array}{ll} \max_x & a^T x + E[\max_{y(\omega)} c(\omega)^T y(\omega)] \\ \text{s.t.} & Ax = b \\ & B(\omega)x + C(\omega)y(\omega) = d(\omega) \\ & x \geq 0 \\ & y(\omega) \geq 0, \end{array} \tag{3.3.1}$$

where x is the first-stage decisions corresponding to the deterministic constraints, $Ax = b$, and $y(\omega)$ is the second-stage decisions that are made after a random event ω is observed corresponding to the stochastic constraints involving $B(\omega), c(\omega)$ and $d(\omega)$. The following material for this section mainly follows from the paper [17] and the book [8].

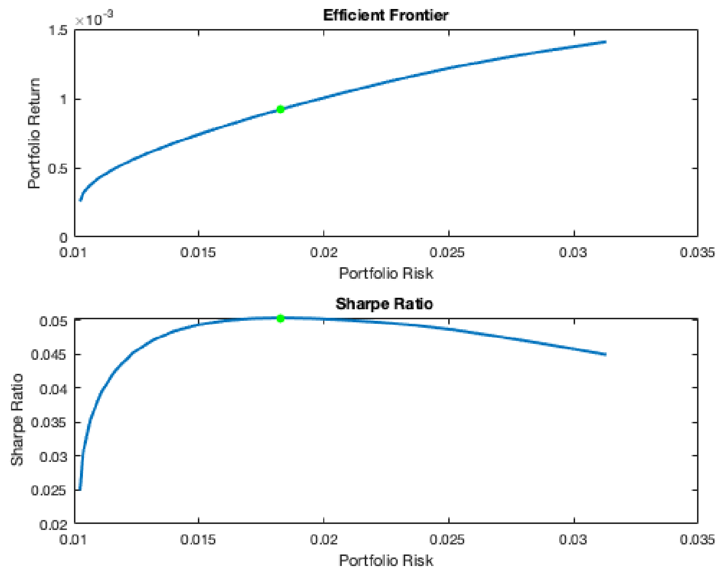

Figure 3.2.2: Example: Sharpe Ratio

VaR is a measure representing the risk of loss for investments; it estimates the maximum loss with a given probability level over a fixed period of time. Consider a random variable Y representing loss on a portfolio with a given probability level $\alpha \in (0, 1)$ over a certain time period (loss positive and gains negative). Mathematically, we use $\mathbf{VaR}_\alpha(Y)$, i.e., this is a function of the confidence level α and is defined as

$$\boldsymbol{VaR}_\alpha(Y) := \min\{\gamma : F_Y(\gamma) \geq \alpha\}, \tag{3.3.2}$$

where F_Y is the cumulative distribution function for Y. Informally, $\mathbf{VaR}_\alpha$, as expressed in (3.3.2), means that the probability of the maximum possible loss of a set of investments, in a given time period, is at most α. For example, if a portfolio has a one day 1% **VaR** of \$1000, that means that there is a 0.01 probability that the portfolio will lose a value of \$1000 or more over a one day period. Alternatively, a loss of \$1000 or more on this portfolio is expected to happen in one out of a hundred days.

The risk measure **VaR** is widely used in financial industries; however, it lacks the subadditive property defined for a function f as:

$$f(x + y) \leq f(x) + f(y), \forall x, y.$$

VaR of a combined portfolio can be larger than the sum of two individual **VaR**s. This violates the property that risks can be reduced through diversification. Moreover, **VaR** ignores the losses beyond the confidence level. In order to overcome the undesirable features of **VaR**, we introduce a coherent risk measure with superior mathematical properties — Conditional Value-at-Risk, (**CVaR**), also known as the *expected shortfall,* ***ES***.

Derived from **VaR**, **CVaR** is defined as the weighted average of **VaR** and losses exceeding **VaR**. **CVaR** is more sensitive to the loss distributions at the tails than **VaR**. Mathematically, we use $\mathbf{CVaR}_\alpha$ defined as

$$\mathbf{CVaR}_\alpha := \frac{1}{\alpha}\int_0^\alpha \mathbf{VaR}_\gamma(Y)d\gamma.$$

For example, if the **CVaR** for a portfolio is \$1000 for the 1% tail, that means that the average loss on the worst 1% of the possible outcomes for the portfolio is \$1000.

Now we want to build an optimization problem minimizing **CVaR**. Consider a portfolio $x \in \mathbb{R}^n$ and a random vector $y \in \mathbb{R}^m$ with a probability density function $q(y)$. The vector y represents the uncertainties that can affect the loss. Let $f(x, y)$ be a random variable representing the loss associated with portfolio x and induced by the random vector y. In this specific setting, the $\mathbf{VaR}_\alpha$ is defined as

$$\mathbf{VaR}_\alpha(x) := \min\{\gamma \in \mathbb{R} : \Psi(x, \gamma) \geq \alpha\},$$

where

$$\Psi(x, \gamma) := \int_{f(x,y)<\gamma} q(y)dy$$

is the *cumulative loss distribution function.* The $\mathbf{CVaR}_\alpha$ corresponding to portfolio x is defined as

$$\mathbf{CVaR}_\alpha(x) := \frac{1}{1-\alpha}\int_{f(x,y)\geq \mathbf{VaR}_\alpha(x)} f(x, y)q(y)dy.$$

Observe that

$$\begin{aligned}
\mathbf{CVaR}_\alpha(x) &= \frac{1}{1-\alpha}\int_{f(x,y)\geq \mathbf{VaR}_\alpha(x)} f(x, y)q(y)dy \\
&\geq \frac{1}{1-\alpha}\int_{f(x,y)\geq \mathbf{VaR}_\alpha(x)} \mathbf{VaR}_\alpha(x)q(y)dy \\
&= \frac{\mathbf{VaR}_\alpha(x)}{1-\alpha}\int_{f(x,y)\geq \mathbf{VaR}_\alpha(x)} q(y)dy \\
&\geq \mathbf{VaR}_\alpha(x),
\end{aligned}$$

indicating that **CVaR** of a portfolio is at least as large as its **VaR**.

In optimization, **CVaR** is a coherent risk measure [3] and thus superior to **VaR**. We will present an optimization problem minimizing **CVaR**. Since the definition of **CVaR** involves **VaR**, we consider an auxiliary function to simplify the problem:

$$\begin{aligned} F_\alpha(x,\gamma) := & \quad \gamma + \frac{1}{1-\alpha}\int_{f(x,y)\geq\gamma}(f(x,\gamma)-\gamma)q(y)dy \\ = & \quad \gamma + \frac{1}{1-\alpha}\int(f(x,y)-\gamma)^+q(y)dy \end{aligned}$$

where $(f(x,y)-\gamma)^+ = \max\{f(x,y)-\gamma, 0\}$.

Theorem 3.3.1. *([8, 17]) The function $F_\alpha(x,y)$ has the following properties for the computation of **VaR** and **CVaR**:*

1. *$F_\alpha(x,\gamma)$ is a convex function of γ.*

2. *$\boldsymbol{VaR}_\alpha(x)$ is a minimizer $F_\alpha(x,\gamma)$ with respect to γ, i.e., $\boldsymbol{VaR}_\alpha(x) = argmin_\gamma F_\alpha(x,\gamma)$.*

3. *$\boldsymbol{CVaR}_\alpha(x)$ equals the minimal value of the function $F_\alpha(x,\gamma)$ with respect to γ, i.e., $min_\gamma F_\alpha(x,\gamma) = \boldsymbol{CVaR}_\alpha(x)$.*

□

Consequently, we obtain

$$\min_{x\in Y}\mathbf{CVaR}_\alpha(x) = \min_{x\in Y,\gamma}F_\alpha(x,\gamma). \tag{3.3.3}$$

Since it is impossible to determine the function $p(y)$, we introduce scenarios $o = 1, ..., O$. We assume that all scenarios have the same probability, and each y_o represents some value from historical data or computer stimulation. Define

$$\tilde{F}_\alpha(x,\gamma) := \gamma + \frac{1}{(1-\alpha)O}\sum_{o=1}^{O}(f(x,y)-\gamma)^+.$$

as an approximation to the function $F_\alpha(x,\gamma)$. Now we approximate (3.3.3) with $\tilde{F}_\alpha(x,\gamma)$:

$$\min_{x\in Y,\gamma} \quad \tilde{F}_\alpha(x,\gamma) = \gamma + \frac{1}{(1-\alpha)O}\sum_{o=1}^{O}(f(x,y_o)-\gamma)^+.$$

We introduce an artificial variable z_s to simplify the problem:

$$\begin{aligned}
\min_{x,z,\gamma} \quad & \gamma + \tfrac{1}{(1-\alpha)O}\textstyle\sum_{o=1}^{O} z_o \\
\text{s.t.} \quad & z_o \geq f(x,y_o) - \gamma, \quad o = 1,...,O, \\
& z_o \geq 0, \quad o = 1,...,O, \\
& \textstyle\sum_{j=1}^{n} x_j = 1 \\
& x \geq 0.
\end{aligned} \tag{3.3.4}$$

Note that z_o can be larger than $\max\{f(x,y_o)-\gamma,0\}$ and still be feasible. However, the objective is a minimization involving a positive z_o. The optimal solution can never have z_o larger than $\max\{f(x,y_o)-\gamma,0\}$, and indeed the optimal solution will have $z_o = \max\{f(x,y_o)-\gamma,0\}$ precisely. Therefore, z_o is a valid substitution for $(f(x,y_o)-\gamma)^+$. If $f(x,y_o)$ is a linear function, the above problem (3.3.4) is simply an LP and can be solved by the simplex method.

We can also modify problem (3.3.4) to maximize the expected return as follows:

$$\begin{aligned}
\max_{x,z,\gamma} \quad & \mu^T x \\
\text{s.t.} \quad & \gamma + \frac{1}{(1-\alpha^j)O}\sum_{o=1}^{O} z_o \leq Q_{\alpha^j}, \quad j = 1,..,J, \\
& z_o \geq f(x,y_o) - \gamma, \quad o = 1,...,O, \\
& z_o \geq 0, \quad o = 1,...,O, \\
& \sum_{j=1}^{n} x_j = 1 \\
& x \geq 0,
\end{aligned}$$

where J is an index set for different confidence levels and Q_{α^j} is the maximum tolerable **CVaR** value at confidence level α^j.

In the paper [1], the authors apply model (3.3.4) to minimize *portfolio credit risk*. Credit risk is the risk of default that arises from a trading partner failing to fulfilling their obligations on the due date. The authors consider a portfolio of 197 bonds issued by 86 obligors in 29 countries. The portfolio is worth 8.8 billion, and the duration is approximately 5 years. As a result, the test portfolio has an expected portfolio return of 7.26% and an expected loss of 95 million dollars with standard deviation of 232 million of dollars for one year loss distribution.

3.3.1 CVaR Example

In this section, we present an example using **CVaR**, and the *MATLAB Financial Toolbox* [14] to find optimal portfolios. We work with the same data of 30 US stocks in Section 3.1.3, and we set up the preliminaries using the same approach.

To implement the **CVaR** model, we create a *PortfolioCVaR* object p in MATLAB, using the MATLAB command *PortfolioCVaR*. To set up *AssetScenarios*, we use the MATLAB command *stimulateNormalScenariosbyData* to generate $200,000$ number of scenarios based on our data. We set the probability level α to be 95% in the example by using the MATLAB command *setProbabilityLevel*. We also set up the budget constraint and the non-negativity constraint by MATLAB command *setDefaultConstraints*. After we construct the properties for p, we plot the efficient frontier in Figure 3.3.1. In this plot, the x-axis is the conditional value-at-risk; and the y-axis is the daily mean rate of return. For example, setting the x-axis (conditional value-at-risk) = 4% means that the average loss in 5% worst cases must not exceed 4% of the initial portfolio value; and the maximum rate of return corresponding to this point on the efficient frontier is approximately 0.092%. From the code we obtain: the optimal portfolio with daily target return 0.1% ; and the optimal portfolio with daily target risk 3% in Table 3.3.1.

Ticker	Weight(%)
BRK	11.162
AAPL	58.1207
CVX	8.69708
CAT	21.6539
AMZN	0.366326

Ticker	Weight(%)
BRK	19.1011
PG	9.92408
AAPL	29.6692
MCD	10.2427
CVX	20.174
CAT	10.8889

Table 3.3.1: Left: Optimal Portfolio with 0.1% Target Return; Right: Optimal Portfolio with 3% Target Risk

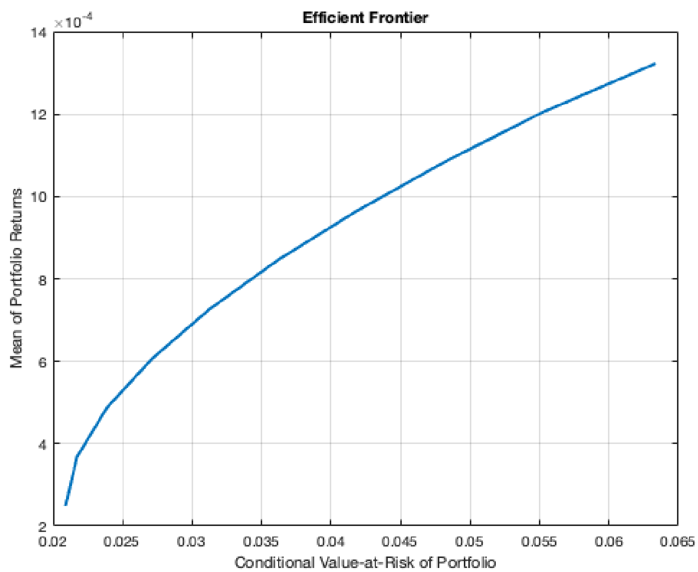

Figure 3.3.1: Example: **CVaR** Efficient Frontier

Part III

Robust Portfolio Optimization

In many optimization problems, the inputs (data) to the problem are unknown or uncertain. The data uncertainty has a great impact on the optimal solution we are looking for, as a small change in the data may result in a drastically different optimal solution. In this part of the thesis, we study the background of robust optimization and how to incorporate robustness into portfolio optimization.

Chapter 4

Background on Robust Optimization

This chapter gives some backgrounds on robust optimization, including how to include uncertainties in optimization problems and duality of robust optimization problems.

4.1 Optimizing with Uncertainties

This section follows closely from the book *Optimization Methods in Finance* by G. Cornuejols, J. Peña and R. Tütüncü [8].

Robust optimization is a field of optimization that deals with data uncertainty. The objective of a robust optimization problem is to find the best solution over all possible realizations with parameters restricted to the uncertainty sets. We use *uncertainty sets* to describe the uncertainty in the parameters. There are four common types of uncertainty sets in robust optimization problems for a specific parameter:

- Uncertainty sets representing a finite number of the possible values of the parameter: $\mathcal{U} = \{a_1, a_2, ..., a_k\}$.
- Uncertainty sets representing the convex hull for a finite number of the possible values of the parameter: $\mathcal{U} = \text{conv}(a_1, a_2, ..., a_k)$.
- Uncertainty sets representing an interval of the parameter: $\mathcal{U} = \{a : l \leq a \leq u\}$.
- Ellipsoidal uncertainty sets: $\mathcal{U} = \{a : a = a_0 + Mu, ||u|| \leq 1\}$.

The shape and the size of the uncertainty sets have a great impact on the robust solutions.

There are a few variations on the definitions and interpretations of robustness. Next, we will introduce *constraint robustness* and *objective robustness*. Data uncertainty affects the feasibility of potential solutions in constraint robustness and the proximity of the generated solutions to optimality in objective robustness.

Constraint robustness refers to the situation where the uncertainty of data is in the constraint. Consider the following optimization problem:

$$\begin{aligned} \min_x \quad & \xi(\omega) \\ \text{s.t.} \quad & G(\omega, a) \in H, \end{aligned} \tag{4.1.1}$$

where ω is the decision variable, ξ is the certain objective function, G and H are the certain structural elements of the constraints, and a is the uncertain parameter. Let $\mathcal{U}$ be the uncertainty set containing all the possible values of the uncertain parameter a. Then, a constraint-robust optimization problem of (4.1.1) is formed as:

$$\begin{aligned} \min_\omega \quad & \xi(\omega) \\ \text{s.t.} \quad & G(\omega, a) \in H, \qquad \forall a \in \mathcal{U}. \end{aligned} \tag{4.1.2}$$

We seek for a solution that is feasible for all possible values of the uncertain inputs in this problem.

Objective robustness refers to the situation where the uncertainty of data is in the objective function. Consider the following optimization problem:

$$\begin{aligned} \min_\omega \quad & \phi(\omega, a) \\ \text{s.t.} \quad & \omega \in I, \end{aligned} \tag{4.1.3}$$

where ϕ is the objective function depending on the uncertain parameter a, and I is the certain feasible set. As before, let $\mathcal{U}$ be the uncertainty set containing all the possible values of the uncertain parameter a. Then, an objective-robust optimization problem of (4.1.3) is formed as:

$$\min_{\omega \in I} \quad \max_{a \in \mathcal{U}} \quad \phi(\omega, a).$$

Here we seek for solutions that are close to the optimal solution for all possible realizations of the uncertain parameters. Such solutions are hard to find especially when the uncertainty set is large. Alternatively, we look for solutions whose worst-case behaviour is optimized.

The worst-case behaviour of a solution refers to the value of the objective function for the worst possible realization of the uncertain parameter.

Now consider the following optimization problem when we have uncertain parameters in both the objective function and the constraints:

$$\begin{aligned} \min_\omega \quad & \phi(\omega, a) \\ \text{s.t.} \quad & G(\omega, a) \in H. \end{aligned} \tag{4.1.4}$$

We can reformulate (4.1.4) to fit the form (4.1.2) as follows:

$$\begin{aligned} \min_\omega \quad & \iota \\ \text{s.t.} \quad & \iota - \phi(\omega, a) \geq 0 \\ & G(\omega, a) \in H. \end{aligned} \tag{4.1.5}$$

Note that (4.1.4) and (4.1.5) are equivalent, and (4.1.5) has its all uncertainties in the constraints.

4.2 Duality

In this section, we study the duality associated with robust counterparts of uncertain convex programs. We will show that the relation *primal worst equals dual best* is valid in robust optimization. The reference is the paper by Beck and Ben-Tal [5].

Consider a general uncertain optimization problem:

$$(P) \qquad \begin{aligned} \min_\omega \quad & \Omega(\omega, a) \\ \text{s.t.} \quad & g_i(\omega, c_i) \leq 0, \quad i = 1, ..., m, \\ & \omega \in \mathbb{R}^n, \end{aligned} \tag{4.2.1}$$

where ω is the decision variable, Ω and g_i are convex functions, $a \in \mathbb{R}^p$ and $c_i \in \mathbb{R}^{q_i}$ are the uncertain parameters restricted to convex compact uncertainty sets:

$$a \in \mathcal{A}, c_i \in \mathcal{C}_i, \quad i = 1, ..., m.$$

The primal uncertain problem (P) has an uncertain dual problem (D) with the same uncertain parameters:

$$(D) \qquad \max_{\theta \geq 0} \min_{\omega} \left\{ \Omega(\omega, a) + \sum_{i=1}^{m} \theta_i g_i(\omega, c_i) \right\}.$$

Define a vector ω to be a *robust feasible solution* of (P) if for every $i = 1, ..., m$:

$$g_i(\omega, c_i) \leq 0, \text{ for every } c_i \in \mathcal{C}_i.$$

We can rewrite the constraints in (4.2.1) as

$$G_i(\omega) \leq 0, \quad i = 1, .., m,$$

where $G_i(\omega) = \max_{c_i \in \mathcal{C}_i} g_i(\omega, c_i)$.

The *robust counterpart* (RC) of problem (4.2.1) is formulated as follows:

$$(RC) \qquad \begin{aligned} \min \quad & \chi(\omega) = \max_{a \in \mathcal{A}} \Omega(\omega, a) \\ \text{s.t.} \quad & G_i(\omega) \leq 0, \quad i = 1, ..., m, \\ & \omega \in \mathbb{R}^n. \end{aligned} \tag{4.2.2}$$

The functions χ and G_i are point-wise maxima of convex functions and thus convex. Hence, the robust counterpart (4.2.2) is a convex optimization problem and thus has a convex dual problem. The dual of RC (call it DRC) is formulated as:

$$(DRC) \qquad \max_{\theta \geq 0} \min_{\omega} \left\{ \chi(\omega) + \sum_{i=1}^{m} \theta_i G_i(\omega) \right\}.$$

The *Slater's condition* states that the feasible region must have an interior point. Assume that the Slater constraint qualification holds for (RC), and (RC) is bounded below. Then we have val(RC)=val(DRC) by the Strong Duality Theorem [5].

Define a vector ω to be an *optimistic feasible solution* of (P) if, and only if, for every $i = 1, ..., m$:

$$g(\omega, c_i) \leq 0 \text{ for some } c_i \in \mathcal{C}_i.$$

The *optimistic counterpart* (OC) of (P) consists of minimizing the best possible objective function over the set of optimistic feasible solutions. Then the optimistic counterpart of problem (4.2.1) is formulated as:

$$(OC) \qquad \begin{aligned} \min \quad & [\min_{a\in\mathcal{A}}\Omega(\omega,a)] \\ \text{s.t.} \quad & g_i(\omega,c_i) \leq 0 \text{ for some } c_i \in \mathcal{C}_i, \quad i=1,...,m, \\ & \omega \in \mathbb{R}^n. \end{aligned} \tag{4.2.3}$$

Let $\hat{\chi}(\omega) = \min_{a\in\mathcal{A}}\Omega(\omega,a)$ and $\hat{G}_i(\omega,c_i) = \min_{c_i\in\mathcal{C}_i} g(\omega,c_i)$. Then above problem (4.2.3) can be formulated as:

$$\begin{aligned} \min \quad & \hat{\chi}(\omega) \\ \text{s.t.} \quad & \hat{G}_i(\omega) \leq 0, \quad i=1,...,m, \\ & \omega \in \mathbb{R}^n. \end{aligned} \tag{4.2.4}$$

The above problem is not convex in general.

The optimistic counterpart of (D) (call it DOC) is

$$(DOC) \qquad \max_{\theta\geq 0}\max_{a\in\mathcal{A},c_i\in\mathcal{C}_i}\min_{\omega}\left\{\Omega(\omega,a)+\sum_{i=1}^{m}\theta_i g_i(\omega,c_i)\right\}.$$

Under standard assumptions [5], the optimal values of (DOC) and (DRC) are equal.

Theorem 4.2.1 ([5]). *Consider the general convex problem (P) (problem (4.2.1)), val(DOC)$\leq$ val(DRC). If in addition, the functions f and g_i are concave with respect to the unknown parameters u and v_i, then the following inequality holds:*

$$val(DOC) \leq val(DRC).$$

Chapter 5

Robust Portfolio Selection

The future values of security prices, interest rates, etc. are unknown in advance but can be estimated in many financial optimization problems, and robust optimization perfectly describes such characteristics. The references for this chapter are [6–9].

5.1 Robust Multi-Period Portfolio Selection

In this section, we follow [8] closely to come up with a robust multi-period portfolio selection model. Suppose an investor wants to adjust his portfolio selections in the next L investment periods and maximize his wealth at the end of period L. Let $x^0 = (x_1^0, x_2^0, ..., x_n^0)$ be the current portfolio that an investor holds where x_j^l represents the number of shares of asset i in the portfolio, for $j = 1, ..., n$, and let x_0^0 be his cash holdings. Let s_j^l denote the number of shares of asset j sold at the beginning of period l, and let b_j^l denote the number of shares of asset j bought at the beginning of period l, for $j = 1, ..., n$ and $l = 1, ..., L$. Then x_j^l represents the number of shares of asset j in the portfolio at the beginning of period l, and

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, \qquad j = 1, ..., n, \quad l = 1, ..., L.$$

Let P_j^l be the price of a share of asset j in period l, and assume that no interest is earned on cash account, i.e., $P_0^l = 1$ for all l. Since the objective is to maximize wealth at the end of period L, we can formulate the objective function as follows:

$$\max \sum_{j=1}^{n} P_j^L x_j^L.$$

We assume that selling and purchasing an asset occurs a proportional transaction cost, denoted η_j^l and τ_j^l respectively, that are known at the beginning of period 0, and all transaction costs are paid from the cash account. At the beginning of period l, the total available cash is the sum of cash balance from last period and the proceeds from sales minus the cost of new purchases. Therefore, we have the following balance equation:

$$x_0^l = x_0^{l-1} + \sum_{j=1}^{n}(1-\eta_j)P_j^l s_j^l - \sum_{j=1}^{n}(1+\tau_j)P_j^l b_j^l, \qquad l = 1, ..., L.$$

For technical reasons, we replace the above equation with an inequality:

$$x_0^l \leq x_0^{l-1} + \sum_{j=1}^{n}(1-\eta_j)P_j^l s_j^l - \sum_{j=1}^{n}(1+\tau_j)P_j^l b_j^l, \qquad l = 1, ..., L.$$

This inequality implies that the investor can burn some of his cash, but in reality this will never happen if the goal is to maximize wealth. Thus, this constraint will also be satisfied at equality.

If all the future prices P_j^L are known, then we can formulate a deterministic optimization problem:

$$\begin{array}{ll}
\max & \sum_{j=1}^{n} P_j^L x_j^L \\
\text{s.t.} & x_0^l \leq x_0^{l-1} + \sum_{j=1}^{n}(1-\eta_j)P_j^l s_j^l - \sum_{j=1}^{n}(1+\tau_j)P_j^l b_j^l, \qquad l = 1, ..., L, \\
& x_j^l = x_j^{l-1} - s_j^l + b_j^l, \qquad j = 1, ..., n, \quad l = 1, ..., L, \\
& x_j^l \geq 0, \qquad j = 0, ..., n, \quad l = 1, ..., L, \\
& s_j^l \geq 0, \qquad j = 1, ..., n, \quad l = 1, ..., L, \\
& b_j^l \geq 0, \qquad j = 1, ..., n, \quad l = 1, ..., L.
\end{array} \tag{5.1.1}$$

Observe that (5.1.1) is an LP that can be easily solved by the simplex method or an interior point method.

In reality, we do not know the future prices P_j^L. We can modify (5.1.1) into a robust optimization problem in order to incorporate the uncertain parameter P_j^L. Note that uncertainty is involved in both the objective function and the constraints. So we move all the uncertainty to the constraints and reformulate the problem as follows:

$$
\begin{aligned}
\max_{x,s,b,\zeta} \quad & \zeta \\
\text{s.t.} \quad & \zeta \leq \textstyle\sum_{j=1}^{n} P_j^L x_j^L \\
& x_0^l \leq x_0^{l-1} + \textstyle\sum_{j=1}^{n}(1-\eta_j)P_j^l s_j^l - \sum_{j=1}^{n}(1+\tau_j)P_j^l b_j^l, \qquad l = 1, ..., L, \\
& x_j^l = x_j^{l-1} - s_j^l + b_j^l, \qquad j = 1, ..., n, \quad l = 1, ..., L, \\
& x_j^l \geq 0, \qquad j = 0, ..., n, \quad l = 1, ..., L, \\
& s_j^l \geq 0, \qquad j = 1, ..., n, \quad l = 1, ..., L, \\
& b_j^l \geq 0, \qquad j = 1, ..., n, \quad l = 1, ..., L.
\end{aligned}
\tag{5.1.2}
$$

In order to find a solution for the above problem, we need to choose an appropriate uncertainty set for the uncertain parameter P_i^L. Assume that future prices can be random, and $P^l = \begin{bmatrix} P_1^l \\ \vdots \\ P_n^l \end{bmatrix}$. Denote the expected value of the vector P^l with $\mu^l = \begin{bmatrix} \mu_1^l \\ \vdots \\ \mu_n^l \end{bmatrix}$ and its covariance matrix with V^l. We follow a $3-\sigma$ approach, and the corresponding uncertainty set for P^l is

$$\mathcal{U}^l := \{P^l : \sqrt{(P^l - \mu^L)^T (V^l)^{-1} (P^l - \mu^l)} \leq 3\}, \quad l = 1, .., L.$$

The complete uncertainty set U is the Cartesian product of the sets defined as

$$\mathcal{U} = \mathcal{U}^1 \times ... \times \mathcal{U}^L.$$

The uncertainty is involved in the first two constraints. First, we consider the constraint:

$$\zeta \leq \sum_{i=1}^{n} P_j^L x_j^L.$$

Consider RHS, the expected value at the end of period L is

$$\mathbb{E}(\text{RHS}) = x_0^L + \sum_{j=1}^{n} \mu_j^L x_j^L = x_0^L + (\mu^L)^T x^L,$$

and the standard deviation is $\sigma = \sqrt{(x^L)^T V^T x^L}$. If P_i^L is normally distributed, the constraint becomes

$$\zeta \leq \mathbb{E}(\text{RHS}) - 3\sigma = x_0^L + (\mu^L)^T x^L - 3\sqrt{(x^L)^T V^T x^L}.$$

The inequality of the constraint is satisfied more than 99% of the time for a guarantee.

Now we consider the second constraint in (5.1.2). We rewrite the constraint to isolate all uncertain terms on RHS:

$$x_0^l - x_0^{l-1} \leq \sum_{j=1}^{n}(1-\eta_j)P_j^l s_j^l - \sum_{j=1}^{n}(1+\tau_j)P_j^l b_j^l, \qquad l = 1, ..., L.$$

The expected value of RHS is

$$\mathbb{E}(\text{RHS}) = (\mu^l)^T D_\eta^l s^l - (\mu^l)^T D_\tau^l b^l = (\mu^l)^T \begin{bmatrix} D_\eta^l & -D_\tau^l \end{bmatrix} \begin{bmatrix} s^l \\ b^l \end{bmatrix}$$

where $D_\eta^l = \begin{bmatrix} 1-\eta_1^l & & \\ & \ddots & \\ & & 1-\eta_n^l \end{bmatrix}$ and $D_\tau^l = \begin{bmatrix} 1+\tau_1^l & & \\ & \ddots & \\ & & 1+\tau_n^l \end{bmatrix}$ are diagonal matrices, and the standard deviation is

$$\sigma = \sqrt{\begin{bmatrix} s^l & b^l \end{bmatrix} \begin{bmatrix} D_\eta^l \\ D_\tau^l \end{bmatrix} V^l \begin{bmatrix} D_\eta^l & D_\tau^l \end{bmatrix} \begin{bmatrix} s^l \\ b^l \end{bmatrix}}.$$

Then the constraint becomes

$$x_0^l - x_0^{l-1} \leq (\mu^l)^T \begin{bmatrix} D_\eta^l & -D_\tau^l \end{bmatrix} \begin{bmatrix} s^l \\ b^l \end{bmatrix} - 3\sqrt{\begin{bmatrix} s^l & b^l \end{bmatrix} \begin{bmatrix} D_\eta^l \\ D_\tau^l \end{bmatrix} V^l \begin{bmatrix} D_\eta^l & D_\tau^l \end{bmatrix} \begin{bmatrix} s^l \\ b^l \end{bmatrix}}.$$

Again, if P_i^L is normally distributed, the inequality of the constraint is satisfied more than 99% of the time for a guarantee.

5.2 Robust MVO, RMVO

Recall that in Section 3.2, we have introduced the (equivalent) mean-variance optimization (**MVO**) problems (3.2.3) to (3.2.5). Since problem (3.2.4) is not quadratic, we will focus on problems (3.2.3) and (3.2.5). Now let

$$\mathcal{X} = \{x \in \mathbb{R}^n | \sum_{j=1}^{n} x_j = 1, x \geq 0\}. \tag{5.2.1}$$

We can rewrite (3.2.3) as below:

$$\begin{aligned} \min \quad & \tfrac{1}{2} x^T \Sigma x \\ \text{s.t.} \quad & \mu(x) = \mu^T x = \textstyle\sum_{j=1}^{n} \mu_j x_j \geq \mu_0 \\ & x \in \mathcal{X} \end{aligned} \tag{5.2.2}$$

where Σ is the covariance matrix, and μ_0 is lower bound on expected return. Problem (3.2.5) is equivalent to

$$\begin{aligned} \max \quad & \mu^T x - \lambda x^T \Sigma x \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{5.2.3}$$

where λ is a risk aversion constant. Now we would like to add robustness into this problem. We follow article [16] and book [8] to study robust mean-variance optimization (**RMVO**).

In general, the distribution of the population mean μ and the population covariance matrix Σ are often unknown. Thus, the sample mean $\bar{\mu}$ and the sample covariance matrix $\bar{\Sigma}$ may not be a good approximation. Yet the central limit theorem tells us that when size n is large, the distribution is normal. As an approach, we use intervals for uncertainty sets that contain possible values of these parameters ([4]). An uncertainty set for the expected return μ is given as

$$\mathcal{U}_\mu = \{\mu : \mu^L \leq \mu \leq \mu^U\}; \tag{5.2.4}$$

and uncertainty set for the covariance matrix Σ is taken as

$$\mathcal{U}_\Sigma = \{\Sigma : \Sigma^L \leq \Sigma \leq \Sigma^U, \Sigma \succeq 0\}, \tag{5.2.5}$$

where $\mu^L, \mu^U, \Sigma^L, \Sigma^U$ are the extreme values of the intervals. A compound uncertainty set describing uncertainty for both μ and Σ is given as

$$\mathcal{U} = \{(\mu, \Sigma) : \mu \in \mathcal{U}_\mu, \Sigma \in \mathcal{U}_\Sigma\}. \tag{5.2.6}$$

With the uncertainty sets $\mathcal{U}_\mu, \mathcal{U}_\Sigma, \mathcal{U}$, we can reformulate problem (3.2.3) into the **RMVO** below:

$$\begin{aligned} \min_x \quad & \{\max_{\Sigma \in \mathcal{U}_\Sigma} x^T \Sigma x\} \\ \text{s.t.} \quad & \min_{\mu \in \mathcal{U}_\mu} \mu^T x \geq \mu_0 \\ & x \in \mathcal{X}. \end{aligned} \tag{5.2.7}$$

This minimax problem (5.2.7) is discussed in article [9].

Moreover, we can reformulate (3.2.5) into **RMVO** as well:

$$\max_{x\in\mathcal{X}}\{\min_{\mu\in\mathcal{U}_\mu,\Sigma\in\mathcal{U}_\Sigma}\mu^T x-\lambda x^T\Sigma x\} \tag{5.2.8}$$

This **RMVO** problem (5.2.8) is introduced in article [11], and a solution algorithm is provided.

Proposition 5.2.1 ([16]). *Let $x^*(\lambda)$ denote an optimal solution of (5.2.8) for a given positive value of λ. Then, $x^*(\lambda)$ is also an optimal solution of (5.2.7) for*

$$\mu_0=\min_{\mu\in\mathcal{U}_\mu}\mu^T x^*(\lambda).$$

□

Proposition 5.2.2 ([16]). *Let $x\in\mathbb{R}^n$ be a non negative vector and let $\mathcal{U}$ be in (5.2.4) to (5.2.6) with a positive semidefinite matrix Σ^U. Then, an optimal solution of the problem*

$$\min_{(\mu,\Sigma)\in\mathcal{U}}\mu^T x-\lambda x^T\Sigma x$$

is $\mu^*=\mu^L$ and $\Sigma^*=\Sigma^U$ regardless of the values of the non negative scalar λ and the vector x. □

With the result of Proposition 5.2.2, we can reduce problem (5.2.8) to the following maximization problem:

$$\max_{x\in\mathcal{X}}(\mu^L)^T x-\lambda x^T\Sigma^U x. \tag{5.2.9}$$

This is a standard **QP** problem and can be solved by **QP** algorithms. Similarly, we can reduce the minimax problem (5.2.7) to the following minimization problem:

$$\begin{aligned}\min\quad & x^T\Sigma^U x\\ \text{s.t.}\quad & (\mu^L)^T x\geq\mu_0\\ & x\in\mathcal{X},\end{aligned} \tag{5.2.10}$$

where Σ^U is also assumed to be positive semidefine.

5.2.1 Robust Maximum Sharpe Ratio

We follow article [16] and book [8] to study robust maximum Sharpe ratio. Recall that in Section 3.2.1, we have introduced Sharpe ratio:

$$h(x) = \frac{\mu^T X - r_f}{(x^T \Sigma x)^{1/2}}.$$

The corresponding optimization problem for finding the highest Sharpe ratio is formulated as below:

$$\begin{aligned} \max_x \quad & \frac{\mu^T x - r_f}{(x^T \Sigma x)^{1/2}} \\ \text{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \tag{5.2.11}$$

where r_f is the known return for risk-free assets. Observe that this optimization problem has a nonlinear objective function. We follow an approach introduced by D. Goldfarb and G. Iyengar [9] to reduce (5.2.11) into a convex problem. First, we rewrite $h(x)$ as a homogeneous function:

$$h(x) = \frac{\mu^T x - r_f}{(x^T \Sigma x)^{1/2}} = \frac{(\mu - r_f e)^T x}{(x^T \Sigma x)^{1/2}}, \forall k > 0,$$

where e is an n-dimensional vector of 1's, and $e^T x = 1$. When $\mathcal{X}$ takes the form (5.2.1), the normalization constraint $e^T x = 1$ can be replaced by the alternative normalization constraint $(\mu^T x - r_f)^T x = 1$. Then the objective function is equivalent to minimizing $x^T \Sigma x$ that is convex and quadratic. When $\mathcal{X}$ is not in the form (5.2.1), we apply *lifting* technique to homogenize $\mathcal{X}$. Define

$$\mathcal{X}^+ := \{x \in \mathbb{R}^n, \kappa \in \mathbb{R} | \kappa > 0, \frac{x}{\kappa} \in \mathcal{X}\} \cup (0,0). \tag{5.2.12}$$

Note that $\mathcal{X}^+$ has a higher dimension than $\mathcal{X}$. We add $(0,0)$ to the set to get a closed set. Observe that $\mathcal{X}^+$ is a cone. Then, we formulate an equivalent problem to (5.2.11) below:

$$\begin{aligned} \max_x \quad & h(x) \\ \text{s.t.} \quad & (x, \kappa) \in \mathcal{X}^+. \end{aligned} \tag{5.2.13}$$

Adding the normalization constraint $(\mu^T x - r_f)^T x = 1$ does not affect the optimal solution of problem (5.2.13) since $h(x)$ is homogeneous in x. Thus, we formulate the following problem that is equivalent to problem (5.2.13):

$$
\begin{aligned}
\max_x \quad & \frac{1}{(x^T \Sigma x)^{1/2}} \\
\text{s.t.} \quad & (x, \kappa) \in \mathcal{X}^+ \\
& (\mu - r_f)^T x = 1.
\end{aligned}
\tag{5.2.14}
$$

Proposition 5.2.3 ([16])**.** *Given a set $\mathcal{X}$ of feasible portfolios with the property that $e^T X = 1, \forall x \in \mathcal{X}$, the portfolio x^* with the maximum Sharpe ratio in this set can be found by solving the following problem with a convex quadratic objective function*

$$
\begin{aligned}
min_x \quad & x^T \Sigma x \\
s.t. \quad & (x, \kappa) \in \mathcal{X}^+ \\
& (\mu - r_f)^T x = 1,
\end{aligned}
\tag{5.2.15}
$$

with $\mathcal{X}^+$ as in (5.2.1). If $(\hat{x}, \hat{\kappa})$ is the solution to (5.2.15), then $x^* = \frac{\hat{x}}{\hat{\kappa}}$. □

Following D. Goldfarb and G. Iyengar [9], we relax the normalization constraint $(\mu^T x - r_f)^T x = 1$ to $(\mu^T x - r_f)^T x \geq 1$ and formulate a robust optimization problem maximizing the Sharpe ratio:

$$
\begin{aligned}
\min \quad & \{\max_{\Sigma \in \mathcal{U}_\Sigma} x^T \Sigma x\} \\
\text{s.t.} \quad & (x, \kappa) \in \mathcal{X}^+ \\
& \min_{\mu \in \mathcal{U}_\mu} (\mu^T x - r_f)^T x \geq 1.
\end{aligned}
\tag{5.2.16}
$$

Other approaches including using an ellipsoidal uncertainty set is discussed in the article [18].

5.2.2 Example

We now present an example of portfolio optimization using the **RMVO** model. Here we use the same data of 30 US stocks as in Section 3.1.3. From the results in Section 3.2, we find that the sample covariance matrix $\bar{\Sigma}$ is positive semidefinite. We let $\Sigma^U = \bar{\Sigma} + \text{diag}(\epsilon)$ where $\epsilon \geq 0$. Thus, $\Sigma^U \succeq \bar{\Sigma}$, indicating that Σ^U is positive semidefinite. Now we can apply

Proposition 5.2.2 and reduce our problems as follows:

$$\begin{array}{ll} \max & (\mu^L)^T x - \lambda x^T \Sigma^U x \\ \text{s.t.} & \sum_{j=1}^n x_j = 1 \\ & x \geq 0, \end{array} \tag{5.2.17}$$

and

$$\begin{array}{ll} \min & x^T \Sigma^U x \\ \text{s.t.} & (\mu^L)^T x \geq \mu_0 \\ & \sum_{j=1}^n x_j = 1 \\ & x \geq 0. \end{array} \tag{5.2.18}$$

Then we create a *Portfolio* object p with such properties in MATLAB [14]. We use the MATLAB command *plotFrontier* to draw the efficient frontier in Figure 5.2.1. We use MATLAB functions *estimateFrontierByReturn*, *estimateFrontierByRisk*, *estimateMaxSharpeRatio* to emphasize the three dots in the plot. We find the optimal portfolios corresponding to the three dots in Table 5.2.1. The left table presents the optimal portfolio with 0.1% target return corresponding to the red dot; the middle table presents the optimal portfolio with 2.5% target risk corresponding to the yellow dot; the right table presents the optimal portfolio with maximum Sharpe ratio corresponding to the green dot.

Ticker	Weight(%)
BRK	5.74706
AAPL	48.499
MCD	1.73845
CVX	3.86328
CAT	20.992
AMZN	19.1602

Ticker	Weight(%)
BRK	8.59816
BA	1.18014
XOM	2.80894
AAPL	39.2792
MMM	0.827156
MCD	5.60612
CVX	6.98672
CAT	18.8363
AMZN	15.8772

Ticker	Weight(%)
BRK	8.8117
BA	1.54952
XOM	3.20649
AAPL	38.0925
MMM	1.3135
MCD	5.94015
CVX	7.21924
CAT	18.4459
AMZN	15.4209

Table 5.2.1: Robust **MVO**, **RMVO**; targets: return, risk and max Sharpe ratio

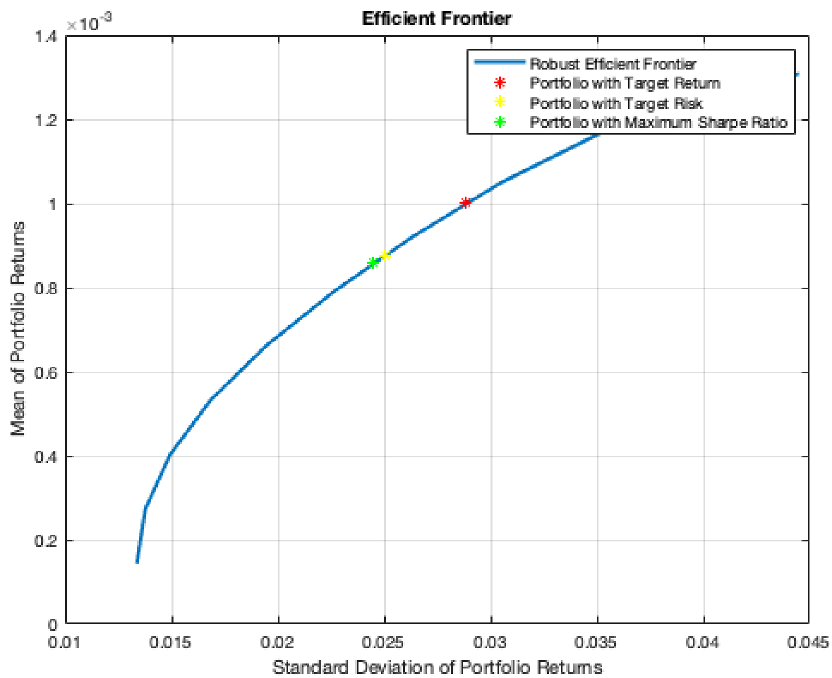

Figure 5.2.1: Robust **MVO**, **RMVO**, Efficient Frontier

Part IV

Black Swan Events

A black swan event is an extremely unpredictable and highly improbable event that has severe consequences. Taleb develop a black swan theory and discuss the impacts of black swan events on markets in his paper [15]. A central idea is to develop robustness to black swan events as economy is vulnerable when coping with hazardous events. In this part, we look at historical black swan events that cause major effects on world economy and then build robust optimization models to test data.

Chapter 6

History of Disasters: Effects on Markets

Historically, natural catastrophes and man-made disasters have great impacts on global markets. Natural disasters such as hurricanes and earthquakes can cause severe damage to properties, and can also lead to disruptions of economic activities. In fact, man-made risks often have greater impacts on market performances than natural catastrophes. In this chapter, we study several historical natural events and human disasters and look at market responses to these events.

6.1 Natural Catastrophes

Natural disasters, including hurricanes, tsunamis, droughts and earthquakes, kill 60,000 people per year on average globally, and can cause severe impacts on the world economy. Natural disasters damage physical properties such as buildings and equipment for firms, and can disrupt labour and production. The loss sometimes may be catastrophic to corporations and can result in bankruptcies. As modern business is interconnected worldwide, the economic downturn in one region may affect the global economy. The Cambridge Centre of Risk Studies explores six historical natural catastrophes that triggered market shocks and led to economic recessions in the report [12]. We follow this report to study two fatal natural catastrophes in recent history: 2005, Hurricane Katrina; 2011, The Great East Japan Earthquake.

6.1.1 2005 Hurricane Katrina

Hurricane Katrina is a tropical cyclone that occurred from August 23rd to August 31st in 2005 in the US and killed over $1,800$ civilians. Katrina caused massive damages in Louisiana and Mississippi; and the city of New Orleans was hit particularly hard by the storm. Many buildings were destroyed; and infrastructure was severely damaged. The total loss was estimated at $125 billion. The performance of the stock market was robust, with a slight decline in the Dow Jones in August as shown in Figure 6.1.1. As insurance companies were expected to pay claims, the stock prices of Berkshire Hathaway Inc. fell in August and September in Figure 6.1.2. Although the impacts of Hurricane Katrina seemed minimal on the stock market, it has serious political fallouts.

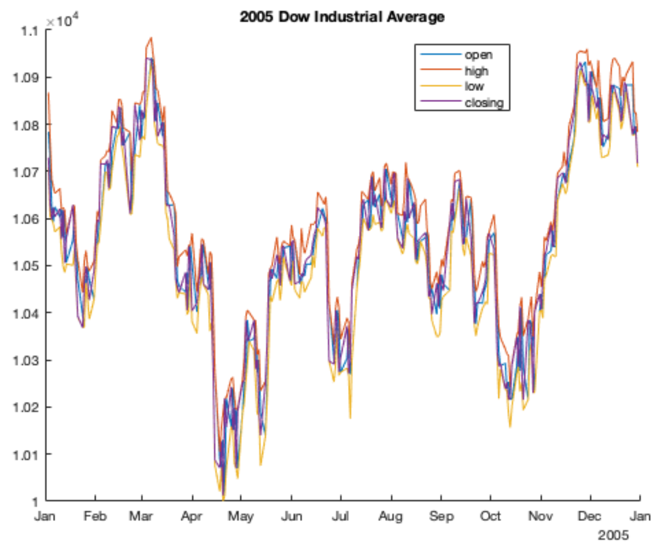

Figure 6.1.1: 2005 Dow Jones Industrial Average (Data Source: [27])

6.1.2 2011: The Great East Japan Earthquake

On 11 March 2011, a magnitude 9.0 undersea megathrust earthquake hit the Pacific coast of Japan. It still is the most powerful earthquake in the history of Japan, and is known as the Great East Japan Earthquake. The national crisis deepened as the earthquake triggered a powerful tsunami that caused enormous damage including a level 7 nuclear power

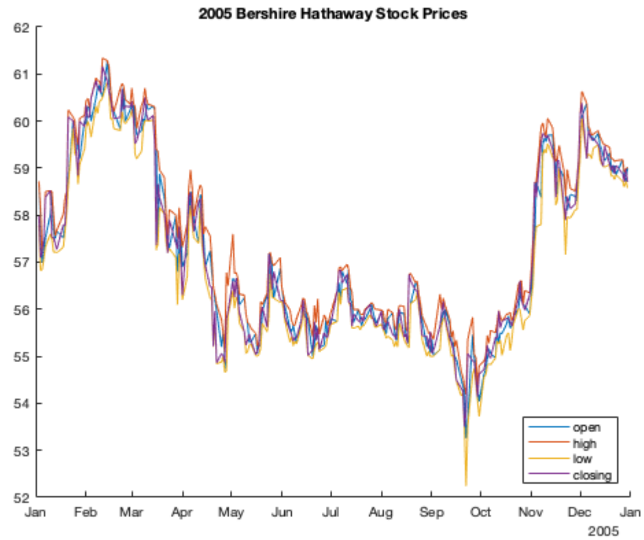

Figure 6.1.2: 2005 Berkshire Hathaway Inc. Stock Prices (Data Source: [25])

plant meltdown. The extent of damage affected millions of households and near 20,000 people were killed or disappeared. The losses from the earthquake and subsequent events were devastating to the domestic economy, though it had minimal effect on international markets. Hundreds of thousands of buildings were damaged, and infrastructure such as roads railways were destroyed. The World Bank estimated a $235 billion economic cost for this catastrophe, making it the costliest natural disaster recorded to date. Recovery from the devastating earthquake and follow-on disasters took several years.

As a result of the disaster, the Nikkei 225, the most prominent measure of the Japanese stock market, plunged more than 10%. It was the third worst one-day plunge in the history of the Nikkei. Figure 6.1.3 shows the sharp decline of the Nikkei 225 in March, and poor performance of Japanese stocks for the rest of year 2011. The stock market was closed for three full days. The huge devaluations in the stock market resulted in a panic among investors; and market sentiments also suffered from the catastrophe. According to the Cambridge report [12], for the year 2011, both personal consumption fell 79%, and national potential output declined up to 21%. Table 6.1.1 presents the GDP and annual GDP growth of Japan. As the government implemented stimulus packages to facilitate reconstruction and boost consumption and investment, the economy slowly recovered in the second year.

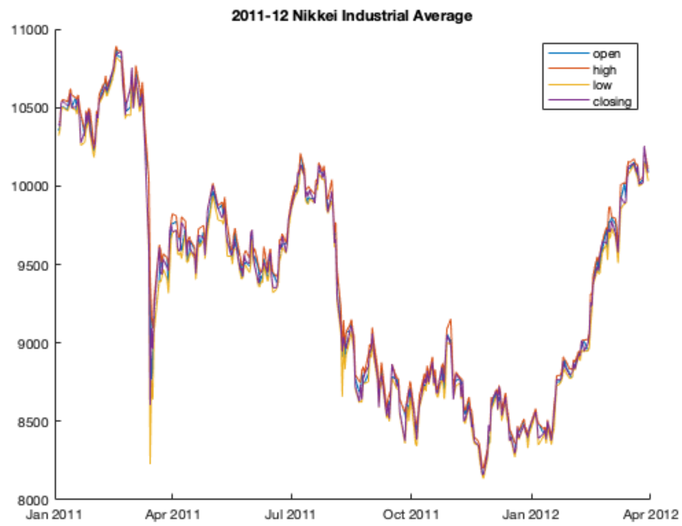

Figure 6.1.3: 2011 Nikkei 225 Index (Data Source: [31])

Year	Annual GDP (trillions of USD)	GDP Growth (%)
2010	5.7	4.192
2011	6.157	-0.115
2012	6.203	1.495
2013	5.156	2

Table 6.1.1: GDP of Japan (Data Source: [19, 21])

6.2 Anthropogenic Disasters

Anthropogenic disasters are hazards caused by human activities such as wars and terrorist attacks. Man-made disasters have a huge impact on our society including economy, ecosystems, etc.. The most defining event in the US history is probably the 9/11 attacks. Besides terrorist attacks, man-made financial crisis such as the dot-com bubble in early 2000s also brings devastating results. The 2008 global financial crisis is a bloody disaster causing huge economic losses in human history. We study these three significant human disasters in the following.

6.2.1 Dot-com Bubble

The dot-com bubble was a rapid rise in technology related stock market in 1990s, a period of rapid technological advancement in the US. In the late 1990s, the stock market of Internet-based companies grew massively as the Internet was widely adopted in the US. The Nasdaq Composite stock market index rose 400% between 1995 and 2000. Figure 6.2.1 shows that Nasdaq rose from under 1000 in 1995 to over 5000 in 2000; and reached its peak in March 2000. The year 1999 displays a massive growth. In 1999, shares of Qualcomm, a telecommunication corporation, increased $2,619\%$ in its value; and many other large-cap stocks grew more than 900% in value. The bubble burst in 2001 through 2002; Figure 6.2.1 shows a steep decline . During the crash, many online shopping companies and communication firms went bankrupt. Well known companies such as Cisco, Intel and Oracle lost more than 80% of their stock values. Figure 6.2.2 has a similar shape as Figure 6.2.1 with a tremendous growth from 1995 to 2000 and a steep decline in 2001. Figure 6.2.3 presents the stock prices of Microsoft Corporation from 1995 to 2002. It is obvious that Microsoft performs much more robust than Cisco during the crash. In 2001, equities entered a bear market; and US experienced a mild economic recession. The recovery was slow; Nasdaq did not return to its peak until 2015.

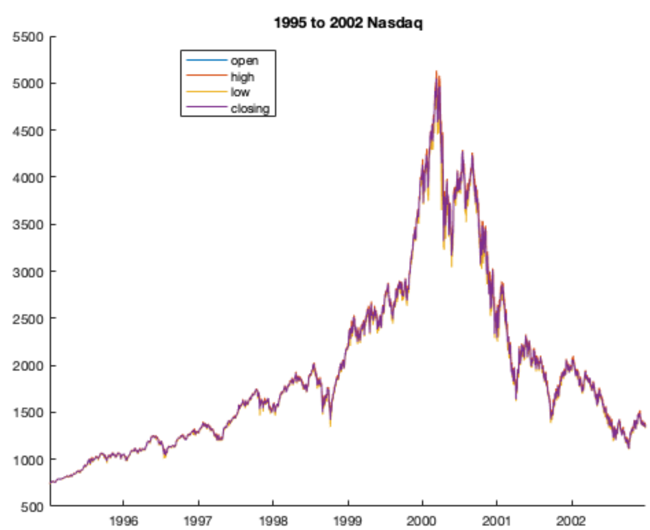

Figure 6.2.1: 1995 to 2002 Nasdaq Composite Index (Data Source: [30])

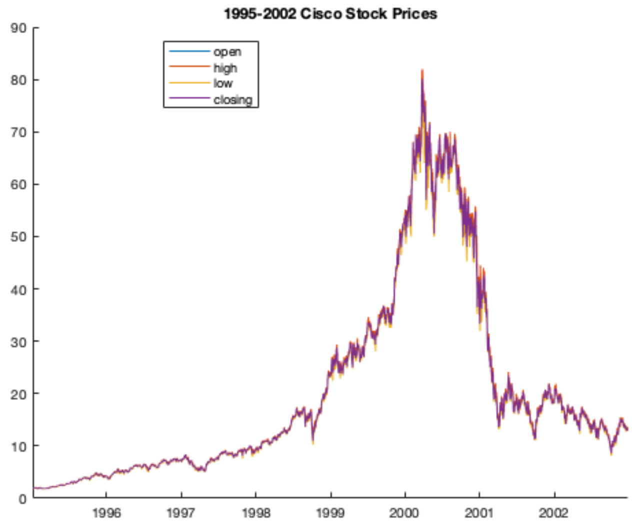

Figure 6.2.2: 1995 to 2002 Cisco Stock Prices (Data Source: [26])

6.2.2 2001 US 9/11 Attacks

On 11 September 2001, four passenger airliners were hijacked by Islamic terrorists, and two of the planes crashed into the World Trade Center in lower Manhattan. As a result, both 110-story towers collapsed, and thousands of people died and were injured. From an economic perspective, the attacks not only caused destruction to physical properties but also interrupted business. The 9/11 attack is the single deadliest terrorist attack in human history.

The 9/11 attack had a significant impact on US markets. Beginning in March 2001, the US suffered from a moderate economic recession; and the attacks worsened the recession. Stock markets were closed for the week following the attack to prevent a stock market meltdown. When the market reopened on 17 September 2001, the Dow Jones fell 14% and the S&P declined 11.6% in five trading days, with an estimated loss of 1.4 trillion. We can see the steep plunge of the stock markets in Figures 6.2.4 and 6.2.5. Airlines suffered the most from the attacks. American Airlines stock dropped 39%, and United Airlines fell 42%, as the demand drastically fell following the attack. The insurance industry was another area that suffered, as companies were expected to pay off claims. Figure 6.2.6 presents the stock prices of American International Group (AIG), an insurance corporation. The prices of AIG substantially fell after the attacks. According to Grossi [10], the destruction costs

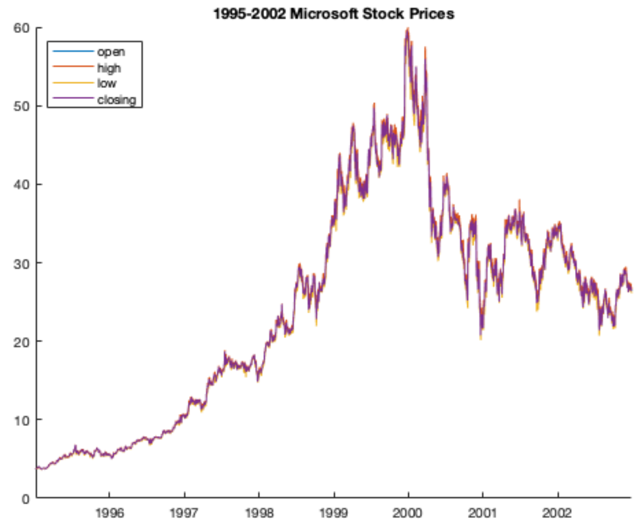

Figure 6.2.3: 1995 to 2002 Microsoft Stock Prices (Data Source: [29])

estimated over 90 billion, and insurance companies covered 32 billion. The recession ended in November 2001 as GDP grew 1.1% in the fourth quarter; however, the adverse influence lingered.

6.2.3 2008: Global Financial Crisis

The global financial crisis is a severe worldwide financial crisis following the Great Recession in the US. During the mid 2000s, as the housing prices fell, homeowners had less burden for their loans. Banks were willing to make large volumes of loans, and real estate developers excessively borrowed and built houses. As a result, American housing market boomed. Moreover, financial firms began marketing mortgage-backed securities and other financial products. As homeowners failed to pay off the loans, the housing bubble burst in 2007. The value of mortgage-backed securities held by the investment banks greatly declined. In September 2008, Lehman Brothers, one of the largest investment banks in the US, filed bankruptcy due to a downturn in the subprime lending market. Table 6.2.1 displays the GDP, GDP growth and unemployment rate of US during the recession. The growth rate of GDP was negative in 2008 and 2009; and unemployment once reached 10% at peak. The great recession in the US officially ended in June 2009; however, Figure 6.2.7 shows that Dow Jones did not regain its value pre-financial crisis until 2012.

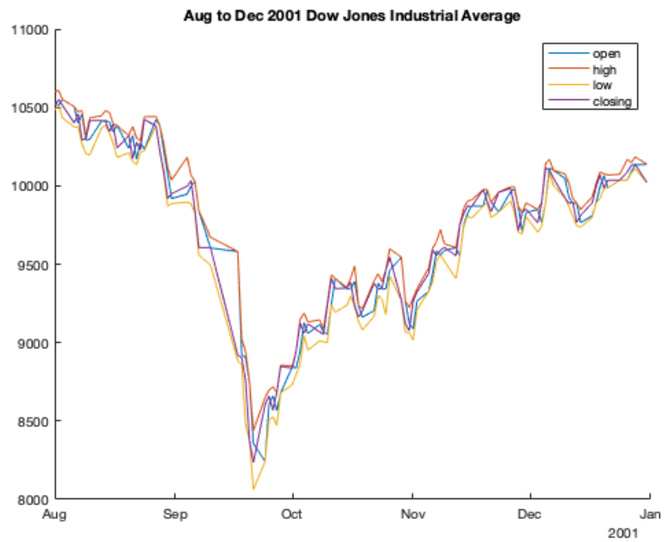

Figure 6.2.4: Aug to Dec 2001 **DJI** (Data Source: [27])

Year	Annual GDP (trillions of USD)	GDP Growth (%)	Unemployment Rate (%)
2007	14.452	1.876	4.622
2008	14.713	-0.137	5.784
2009	14.339	-2.537	9.254
2010	14.992	2.564	9.633
2011	15.542	1.551	8.949
2012	16.197	2.25	8.069

Table 6.2.1: GDP of United States (Data Source: [20, 22, 23])

In 2009, the European debt crisis followed the US Great Recession. The European debt crisis took place in most European Union member countries and lasted for several years; and this crisis was caused by devaluation in the currency of euros. In 2009, several eurozone member countries failed to repay their government debt or to bail out over-indebted banks. Greece suffered the most from the crisis; the Greek government called for external help due to high budget deficits in 2010. Figure 6.2.8 presents the stock market of EU during the recession. Unlike the US, the recovery was extremely slow in the EU due to inharmonization among the member countries.

Japan was another country hit hard by the financial crisis. As the trade structure

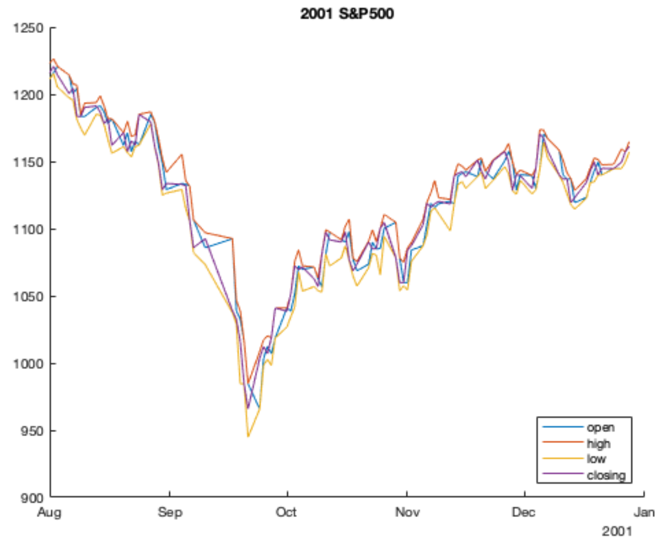

Figure 6.2.5: Aug to Dec 2001 S&P500 (Data Source: [32])

depended heavily on exports, Japaneses output was responsive and vulnerable to the outbreak of the crisis in the US and Western Europe. The demand for exports steeply fell as a result of the recession, leading to a shock in domestic industries. The stock market substantially fell in 2008 as shown in Figure 6.2.9. The result of the financial crisis was severe to Japaneses' economy; it took several years for Japan to recover.

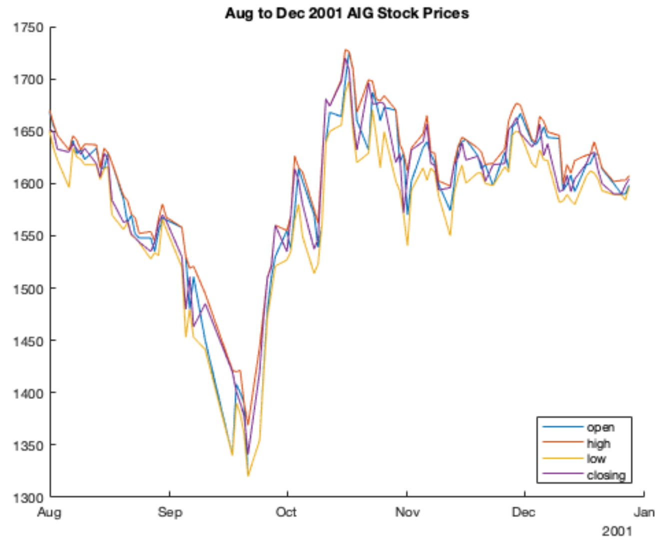

Figure 6.2.6: Aug to Dec 2001 AIG Stock Prices (Data Source: [24])

Figure 6.2.7: 2007 to 2012 Dow Jones (Data Source: [27])

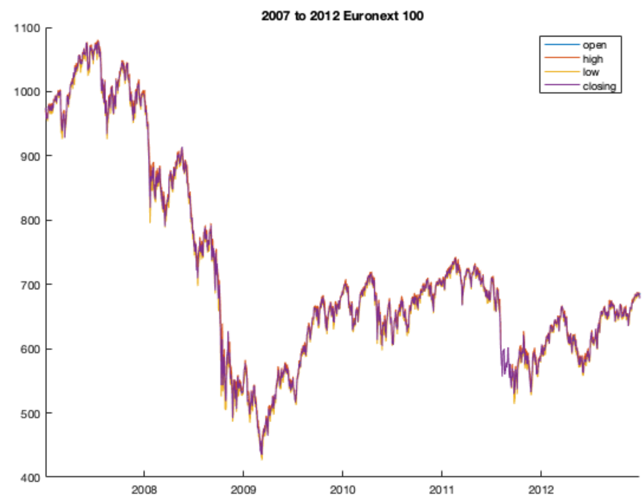

Figure 6.2.8: 2007 to 2012 Euronext 100 (Data Source: [28])

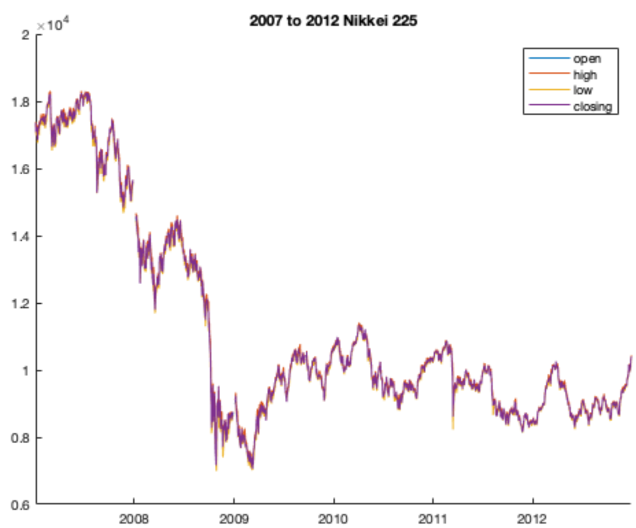

Figure 6.2.9: 2007 to 2012 Nikkei 225 (Data Source: [31])

Chapter 7

Numerics

In this chapter, we use **MVO** and **RMVO** models to test real data on a natural catastrophe and an anthropogenic black swan event. We first recall the **MVO** and **RMVO** models introduced in Section 3.2 and Section 5.2. We compute the optimal portfolios for the **MVO** and **RMVO** and then compare the differences between the two optimal portfolios. We also look at the performance of **MVO** and **RMVO** during and after the black swan event.

7.1 MVO vs RMVO on 2005 Hurricane Katrina

In this section, we compare the performance of **MVO** and **RMVO** on a natural disaster. Recall that we introduced the details about hurricane Katrina in Section 6.1.1. In August 2005, a severe hurricane hit New Orleans and caused massive damages. The response of the stock market to the catastrophe was robust as the Dow Jones Industrial Index had a slight decline in late August. We make a comparison between the **MVO** and the **RMVO** strategy.

We mainly study the blue chip stocks since they have a strong history of performance and thus are more attractive to investors. As we want to analyze the results and make comparisons, we control the number of stocks to be tractable. Suppose we have chosen 50 US stocks, and we invest into the US stock market from August 1st 2005 to September 15th 2005. We obtain the historical stock prices of the stocks from January 3rd 2000 to September 15th 2005, using Yahoo Finance. Assume that we have an initial balance $\$10,000$ in two accounts, one named **MVO** account and the other named **RMVO** account. We look at different targets: return 0.1%, risk 1.5% and max Sharpe ratio.

In the morning of August 1st 2005 , we have a balance of $\$10,000$ in both **MVO** account and **RMVO** account. We calculate the current optimal portfolios for the **MVO** and **RMVO** models, using known data (August 1st 2005), in Tables 7.1.1 and 7.1.2; and we invest all the available balance into the stock market. We first look at the approach of using a target return 0.1%. In Table 7.1.1, we observe that about the list of selected portfolio is narrow, and the last three stocks are heavily weighed with each having a over 20% weight. However, the **RMVO** strategy selects a wider and more diverse range of portfolios, and only 1 stock has a weight over 20% in Table 7.1.2. Looking at the approach of 1.5% risk, the difference in the range of **MVO** portfolio and the range of **RMVO** portfolio is even more obvious. In the second column of Table 7.1.1, the last stock EOG weighs over 50%. The maximum Sharpe ratio strategy selects a wider range of stocks in both cases; and the third column in Table 7.1.2 is more diverse than Table 7.1.1. The last three stocks in the third column of Table 7.1.1 are heavily weighed as well. We conclude that the **MVO** portfolios are heavily skewed on some stocks, and the **RMVO** strategy is much more conservative than the **MVO** strategy.

Ticker	Weight(%)
BA	0.0818046
PG	0.541192
AAPL	3.41713
CAT	10.9163
BAC	2.71015
OXY	8.45931
AVB	29.07
BXP	22.0523
EOG	22.7518

Ticker	Weight(%)
AAPL	6.79535
CAT	10.337
AVB	21.7466
BXP	1.59059
EOG	59.5304

Ticker	Weight(%)
BRK	1.77071
BA	0.374633
PG	2.32203
AAPL	3.00427
JNJ	1.13783
MMM	2.10247e-08
CAT	9.5898
BAC	2.89224
OXY	8.47711
WFC	0.689098
AVB	28.3225
BXP	21.9367
EOG	19.4831

Table 7.1.1: Three **MVO** portfolios: (i) return 0.1%; (ii) risk 1.5%; (iii) max Sharpe ratio

Ticker	Weight(%)
BRK	0.125824
BA	2.8032
AAPL	8.255
MMM	0.13774
CAT	8.89901
AMZN	1.61819
BAC	3.37015
OXY	13.376
WFC	0.68356
AVB	10.1973
EQR	3.1197
BXP	9.44852
EOG	26.4138
DVN	9.30468
MRO	2.2474

Ticker	Weight(%)
BRK	1.66937
BA	3.3615
XOM	0.142378
AAPL	7.31802
JNJ	0.836933
MMM	1.47542
CAT	8.2892
AMZN	1.50991
BAC	3.91604
OXY	11.9801
WFC	2.10184
AVB	10.0292
EQR	4.23584
BXP	9.45375
EOG	22.282
DVN	8.47743
MRO	2.92114

Ticker	Weight(%)
BRK	2.84356
BA	3.63264
XOM	1.40875
PG	0.849827
AAPL	6.30117
JNJ	2.10815
MMM	2.40602
CVX	0.618248
CAT	7.39889
AMZN	1.34997
BAC	4.16509
GS	0.441978
OXY	10.4506
WFC	3.15046
AVB	9.59293
EQR	5.01313
BXP	9.17394
EOG	18.2098
DVN	7.52668
MRO	3.35821

Table 7.1.2: Three **RMVO** portfolios: (i)return 0.1%; (ii)risk 1.5%; (iii)max Sharpe ratio

We now compare the performance of **MVO** and **RMVO** strategies during and after the black swan event. By the end of the trading day, we have the current prices of the 50 stocks. We compute the aggregate rate of return of the portfolios and calculate the current balance in both accounts. We do the same for the next trading day until Sep 15th 2005. We obtain the following three figures Figures 7.1.1 to 7.1.3 corresponding to three different approaches: target return 0.1%, target risk 1.5% and max Sharpe ratio. In all three figures, the x-axis is the number of days of investment, and the y-axis is the account balance. The blue curve corresponds to the **MVO** strategy and the red curve corresponds to the **RMVO** strategy. The hurricane originates around trading 16 and dissipates on trading day 22. In Figure 7.2.1, both **MVO** balance and **RMVO** balance drops slightly on trading day 16. The **RMVO** performs better, having a less loss. The red curve in the figure recovers to $\$10,000$ on trading day 20; and the blue curve recovers a day later. The disaster has very little influence on stock market since the recovery is fast. This observation applies to the other two approaches as well. We conclude that that the stock market is hardly influenced with hurricane Katrina; and the recovery period is very short.

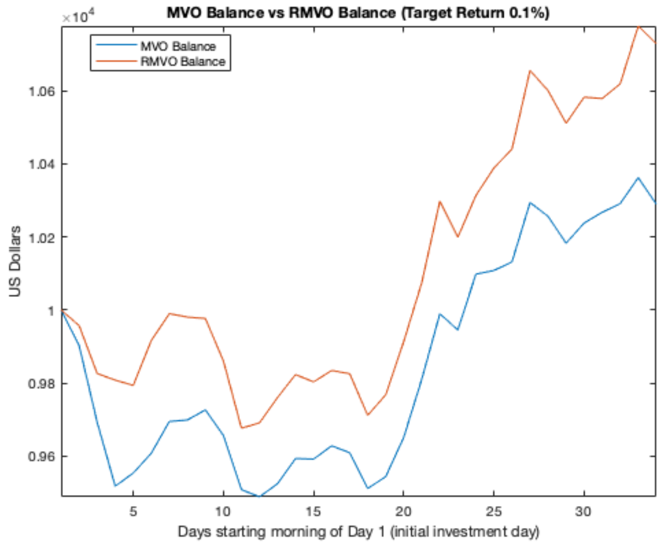

Figure 7.1.1: **RMVO** vs **MVO** with target return 0.1% on 2005 hurricane Katrina

7.2 MVO vs RMVO on 2008 Financial Crisis

In this section, we apply the **MVO** strategy and the robust **MVO** , **RMVO**, strategy to test real data to see the comparison between **MVO** and **RMVO** on an anthropogenic black swan event. First, we recall the 2008 Global Financial Crisis introduced in Section 6.2.3. In 2008, the US stock market crashed mainly due to a housing bubble; and there was a sharp decline for the Dow Jones Industrial Index. We would like to see how **RMVO** and **MVO** strategies perform when the crash happens.

Suppose we have chosen the same 50 stocks as in Section 7.1, and we want to make investments into the US stock market on Jan 2nd 2008, not knowing what will happen in the near future. We obtain the historical stock prices of 50 US stocks from Jan 3rd 2000 to Jun 30th 2010, using Yahoo Finance. For a comparison, assume that we have an initial balance $\$10,000$ in two accounts, one named **MVO** account and the other named **RMVO** account. We look at different targets: return 0.1%, risk 1.5% and max Sharpe ratio, and compare two different strategies: **MVO** vs **RMVO** .

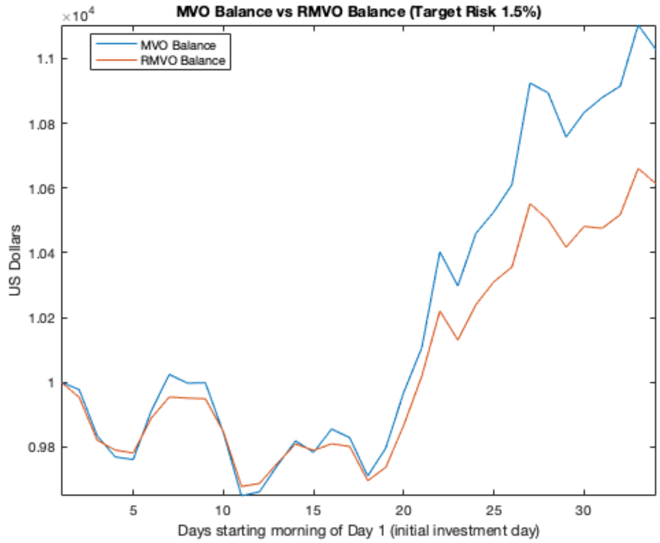

Figure 7.1.2: **RMVO** vs **MVO** with target risk 1.5% on 2005 hurricane Katrina

In the morning of Jan 2nd 2008, we have a balance of $10,000 in both **MVO** account and **RMVO** account. We compute the optimal portfolios for the **MVO** and **RMVO** models, using known data (Jan 3rd 2000 to Dec 31st 2007), in Tables 7.2.1 and 7.2.2; and we invest all the available balance into the stock market. We notice that Table 7.2.1 selects a narrow range of stocks while Table 7.2.2 includes a more diverse and wide range of stocks. Looking at the approach of using a minimum of 0.1% return, we observe in Table 7.2.1 that about half of the stocks have a weight over 10%, and the total weight of these stocks is over 80% of the portfolio. However, only 3 out of 14 stocks have a weight over 10% in Table 7.2.2. The same observation applies to the approach of 1.5% risk. The maximum Sharpe ratio strategy selects a wider range of stocks in both cases; and the third column in Table 7.2.2 is more diverse than Table 7.2.1. We conclude, as expected, that the **RMVO** strategy is much more conservative than the **MVO** strategy.

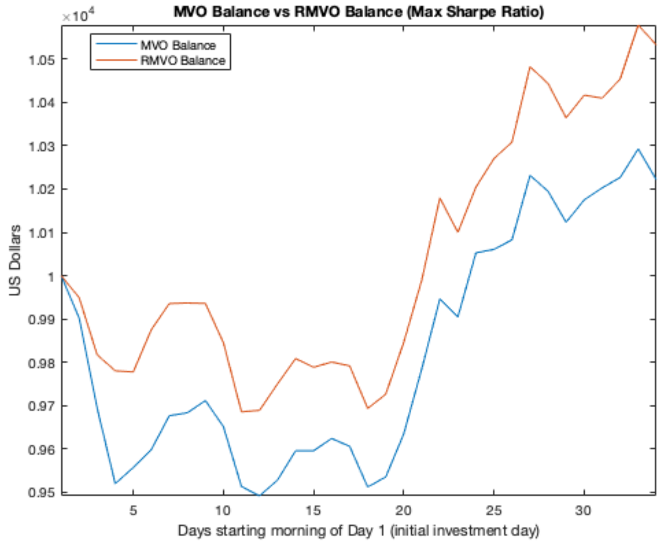

Figure 7.1.3: **RMVO** vs **MVO** with max Sharpe ratio on 2005 hurricane Katrina

Ticker	Weight(%)
BRK	14.8724
BA	1.39364
PG	1.9067
AAPL	13.2675
CVX	1.31665
CAT	8.8396
OXY	16.8255
AVB	4.33138
BXP	17.1386
EOG	20.108

Ticker	Weight(%)
BRK	1.14965
AAPL	24.9522
CAT	7.97393
OXY	22.701
BXP	5.30013
EOG	37.9231

Ticker	Weight(%)
BRK	16.446
BA	2.32222
XOM	0.136698
PG	6.00876
AAPL	9.76702
CVX	5.45075
CAT	6.70037
ORCL	0.0470805
GS	0.125283
OXY	14.0414
AVB	7.21334
BXP	16.8669
EOG	14.8742

Table 7.2.1: Three **MVO** portfolios: (i) return 0.1%; (ii) risk 1.5%; (iii) max Sharpe ratio

Ticker	Weight(%)
BRK	4.82866
BA	2.6511
XOM	1.94896
AAPL	17.8057
CVX	1.25289
CAT	6.54697
AMZN	3.46106
GS	3.36099
OXY	13.3651
AVB	4.40273
BXP	5.69126
EOG	18.8738
DVN	8.31705
MRO	7.49376

Ticker	Weight(%)
BRK	5.81997
BA	3.5615
XOM	3.06185
PG	0.857093
AAPL	15.3258
MMM	0.045177
CVX	2.60334
CAT	6.59742
AMZN	3.21542
GS	3.94609
OXY	12.0523
AVB	5.33262
BXP	6.42295
EOG	16.264
DVN	7.71313
MRO	7.18128

Ticker	Weight(%)
BRK	6.19226
BA	3.91126
XOM	3.63178
PG	2.2888
AAPL	12.5571
JNJ	0.919335
MMM	1.35756
MCD	0.902333
CVX	3.41889
CAT	6.09518
AMZN	2.73372
BAC	0.329861
ORCL	0.772999
GS	3.90162
OXY	10.3707
WFC	0.817085
AVB	5.63363
EQR	0.842926
BXP	6.52638
EOG	13.4126
DVN	6.84681
MRO	6.53717

Table 7.2.2: Three **RMVO** portfolios: (i)return 0.1%; (ii)risk 1.5%; (iii)max Sharpe ratio

We now illustrate the significance of **RMVO** when a black swan event *hits* the stock market. By the end of the trading day, we have the current prices of the 50 stocks. We compute the aggregate rate of return of the portfolios and calculate the current balance in both accounts. We do the same for the next trading day until Jun 30th 2010. We obtain the following three figures Figures 7.2.1 to 7.2.3 corresponding to three different approaches: target return 0.1%, target risk 1.5% and max Sharpe ratio. In all three figures, the x-axis is the number of days of investment, and the y-axis is the account balance. The blue curve corresponds to the **MVO** strategy and the red curve corresponds to the **RMVO** strategy. In Figure 7.2.1, when the market crashes, both **MVO** balance and **RMVO** balance drop steeply. Around investment day 300, our **MVO** balance falls to the lowest point about $4,700, incurring a 53% loss. The **RMVO** performs better, having a balance of $5,100, incurring a 49% loss. The red curve is above the blue curve after the black swan event, indicating that the **RMVO** balance increases more rapidly than the **MVO** balance. On Jun 30th 2010 (the last investment day in the graph), we have a **RMVO** balance of $11,000 and a **MVO** balance of $9,000. Clearly we have a better gain with the **RMVO** strategy and a greater lose with the **MVO** strategy. The loss recovers much faster with **RMVO** strategy. This observation applies to the other two approaches as well. In Figures 7.2.2 and 7.2.3, the red curve is above the blue curve after the crash. The **RMVO** strategy has a higher balance than the **MVO** strategy; and the recovery period is much shorter for the **RMVO** strategy compared to the **MVO** strategy.

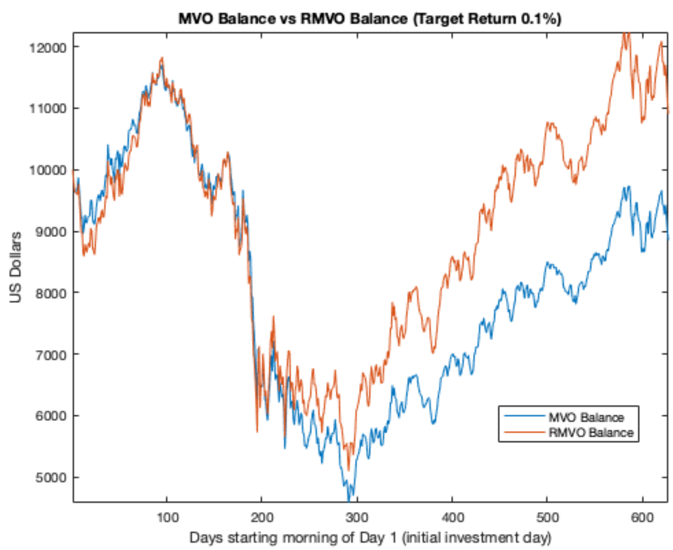

Figure 7.2.1: **RMVO** vs **MVO** with target return 0.1% on 2008 financial crisis

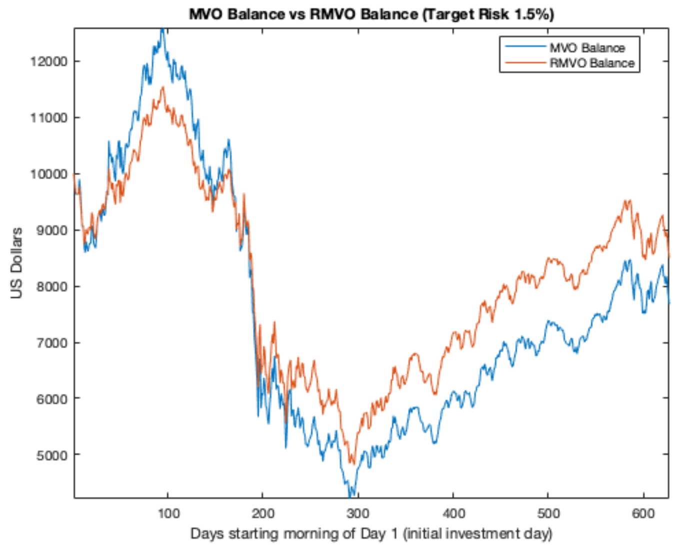

Figure 7.2.2: **RMVO** vs **MVO** with target risk 1.5% on 2008 financial crisis

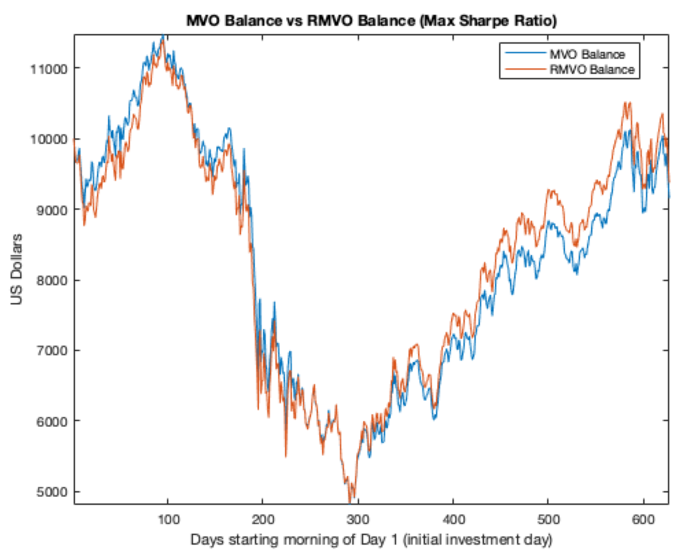

Figure 7.2.3: **RMVO** vs **MVO** with max Sharpe ratio on 2008 financial crisis

Part V

End

In this part, we conclude our thesis and discuss some future work. Covid-19 has caused massive economic losses, and interesting future works include studying the impacts of virus and making some predictions about the recovery.

Chapter 8

Conclusion

In this thesis, we have studied classical portfolio optimization, robust portfolio optimization and some historical black swan events. We have compared the **MVO** and **RMVO** strategy and how they influence/help investors during the period immediately after a black swan event in Chapter 7. We have seen that **RMVO** selects a much more conservative portfolio than **MVO** and recovers faster from a crash. Moreover, the recessions caused by anthropogenic black swan events are more significant. The recovery of the stock market is greatly slower on an anthropogenic disaster compared to a natural disaster.

8.1 Future Work

Covid-19 is the most recent and the most shocking black swan event in modern history. The virus has killed an enormous number of people around the world and has paralyzed the global economy. Many firms have failed and have gone into bankruptcies, and investors have experienced massive losses. The US stock market has gone into meltdowns four times in two weeks in March 2020. The meltdowns of the US stock market happened five times in history: once in 1997, four times in 2020. Not only the US but also many other countries suffer from the economic recessions resulted from the virus. One future work is to to study whether robust portfolio optimization helps investors to endure less loss from this black swan event and compare the results of classical portfolio optimization and robust portfolio optimization. Another direction is to study the impacts of Covid-19 and predict the recovery of the global economy.

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