Monolith: A Monolithic Pressure-Viscosity-Contact Solver for Strong Two-Way Rigid-Rigid Rigid-Fluid Coupling – Supplementary Material

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1 RELATED WORK

Fluid and rigid body dynamics form the foundation of our method. For completeness, we therefore review prior techniques for viscous fluid simulation and rigid body simulation.

1.1 Viscous Fluid Simulation

The unique behaviors of viscous fluids, which can differ significantly from (nearly) inviscid fluids, have long been a focus in animation. Stam [1999] first proposed a stable Eulerian grid-based method using implicit integration of the basic Laplacian (i.e., componentwise diffusion) form of viscosity for fluids without free surfaces, which Carlson et al. [2002] and Fält and Roble [2003] extended to free surface flows. Rasmussen et al. [2004] proposed an implicit/explicit (IMEX) scheme that includes cross-derivative terms for variable viscosity. Batty and Bridson [2008] presented a fully implicit variational method to improve support for rotational surface effects. It was later accelerated with adaptive schemes [Batty and Houston 2011; Goldade et al. 2019] and multigrid [Aanjaneya et al. 2019]. Larionov et al. [2017] proposed a unified pressure-viscosity (Stokes) formulation to further improve free surface motion, including capturing coiling behaviors and avoiding artificial melting.

Approaches to simulate fluids with more exotic properties, such as viscoelasticity and shear-dependent viscosity, have also been explored. Early on, Goktekin et al. [2004] added elastic forces to an Eulerian fluid simulator. More recently, numerous variations of the material point method (MPM) have been proposed to support a wide range of materials, including snow [Stomakhin et al. 2013], foams and shear-dependent fluids [Nagasawa et al. 2019; Ram et al. 2015; Yue et al. 2015], elastoplastic solids [Fang et al. 2019; Gao et al. 2017], temperature-dependent solids [Ding et al. 2019; Stomakhin et al. 2014], granular materials [Daviet and Bertails-Descoubes 2016; Gao et al. 2018; Klár et al. 2016; Tampubolon et al. 2017; Yue et al. 2018], and frictional contact [Ding and Schroeder 2020; Guo et al. 2018; Han et al. 2019; Jiang et al. 2017].

While we focus on Eulerian and hybrid schemes, purely Lagrangian methods are a possible alternative. Smoothed Particle Hydrodynamics (SPH) is a popular approach with several methods having been proposed for efficient, robust, and accurate viscous effects, e.g., [Peer et al. 2015; Peer and Teschner 2017; Takahashi et al. 2015; Weiler et al. 2018]. Other particle-based viscosity methods have relied on spring forces, moving-least-squares approximations, or position-based schemes, e.g., [Clavet et al. 2005; Gerszewski et al. 2009; Takahashi et al. 2014]. With the help of dynamic remeshing, Lagrangian Finite Element Method (FEM) can also handle viscous materials [Bargteil et al. 2007; Clausen et al. 2013; Wicke et al. 2010; Wojtan and Turk 2008]. Similarly, Lagrangian techniques based on simplicial complexes have been proposed to capture viscous threads and sheets [Batty et al. 2012; Bergou et al. 2010; Zhu et al. 2015].

We employ a grid-based, unified pressure-viscosity formulation [Larionov et al. 2017] as the basis for our monolithic solver since it allows for reasonably efficient and realistic animation of incompressible viscous liquids via the solution of a single SPD linear system per step. Nevertheless, our concept of full monolithic coupling could also be adapted to other implicit methods, such as MPM, SPH, or Lagrangian FEM simulators.

1.2 Rigid Body Simulation

Rigid body models are vital in countless applications, and research towards their accurate, robust, and efficient simulation has a long history. Bender et al. [2014] offers a useful overview of rigid body dynamics in computer graphics; we focus on contact handling below.

A popular approach to prevent solid interpenetrations is the penalty method [Bell et al. 2005; Terzopoulos et al. 1987]. While it has advantages (e.g., simplicity, efficiency, ease of implementation, and tolerance of redundant contacts) the need to tune contact-stiffness parameters can make it difficult to ensure plausible simulations: low stiffness may leave penetrations unresolved whereas high stiffness can cause overshooting instabilities. Continuous penalty forces [Tang et al. 2012] and implicit integration [Otaduy and Lin 2005; Xu et al. 2014] yield improved stability, yet it remains challenging to choose appropriate parameters for a wide range of scenarios.

Another common collision-resolution strategy is to model non-penetrations as (hard) constraints. While such constraints can be

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modeled at the position level, yielding a differential algebraic equation (DAE) [Ascher and Petzold 1998], the non-smooth nature of the non-penetration constraints generally necessitates specialized, expensive solvers [Anitescu and Hart 2004]. Alternatively, non-penetration constraints have been modeled as an acceleration-level LCP [Baraff 1991, 1994] or velocity-level LCP [Stewart 2000]. It is not always guaranteed that the (NP-hard) acceleration-level LCP will have a solution [Baraff 1993], so recent work builds on velocity-level formulations [Smith et al. 2012; Vouga et al. 2017; Zhang et al. 2015]. An overview of LCP methods is provided by Erleben [2013].

While LCPs can be solved with direct methods with some extensions (e.g, Lemke's method [Lloyd 2005], Cholesky factorization with block pivoting [Enzenhöfer et al. 2019], and Schur complement-based substructuring [Peiret et al. 2019]), as the number of contacts increases, iterative approaches tend to be preferred due to scaling and memory usage. Arguably, one of the most commonly used iterative approaches is Projected Gauss-Seidel (PGS) due to its robustness, simplicity, and efficiency in each iteration. However, a known issue for PGS is slow convergence, which has been actively investigated. Erleben [2007] presented a shock propagation technique for the velocity-level LCP, by extending the position-level shock propagation method of Guendelman et al. [2003]. Silcowitz et al. [2010] proposed to reduce the dimension of the problem in each GS iteration, Tonge et al. [2012] proposed a mass-splitting technique, Erleben [2017] presented PROX schemes based on proximal operators, and Müller et al. [2017] proposed long range constraints.

Different types of iterative solvers have also been investigated for efficient solution of rigid body LCPs. Renouf and Alart [2005] proposed reformulating the LCP as a QP and solved it with a conjugate projected gradient algorithm. Silcowitz-Hansen et al. [2009; 2010] presented an LCP solver based on non-smooth nonlinear conjugate gradient. Mazhar et al. [2015] proposed using Nesterov's method to accelerate the projected gradient descent method.

Realistic contact simulation also requires friction modeling. To address both non-penetration constraints and friction effects, Kaufman et al. [2008] proposed the *staggered projections* approach, formulating each as a convex QP and solving them iteratively and alternately. To improve the efficiency and accuracy of frictional contacts, Verschoor and Jalba [2019] presented a collision handling method using conjugate residual with dynamic update of constraints while Macklin et al. [2019] proposed non-smooth Newton methods by modeling contacts as nonlinear complementarity problems.

The velocity-level contact formulation using staggered projections [Kaufman et al. 2008] forms the basis for our monolithic formulation, because of its simplicity and effectiveness. Nonetheless, our monolithic strategy could be combined with other contact solvers.

2 VOLUME FRACTION COMPUTATION

Given the solid geometry, represented as a signed distance function, we approximate the integral by summing fractions of cell-sized control volumes. We divide the simulation domain Ω into non-overlapping solid domains $\Omega_{S1}, \Omega_{S2}, \ldots, \Omega_{Sn}$ (n: number of rigid bodies) and a fluid domain Ω_F, (i.e., $\cup_i^n \Omega_{Si} \cup \Omega_F = \Omega$, $\Omega_{Si} \cap \Omega_{Sj} = \emptyset$, and $\Omega_{Si} \cap \Omega_F = \emptyset$, where i, j $(i \neq j)$ denote solid domain indices). Given the usual staggered arrangement of velocity variables on the Cartesian grid and their surrounding cubic control volumes, we assemble their partial fluid and solid volume fraction weights into diagonal matrices $\mathbf{W}_F^u$ and $\mathbf{W}_S^u$ (whose range is $[0, 1]$), respectively. Here, the superscript and subscript indicate weight matrices for velocity variables u and fluid domain F or solid domain S, respectively (for more detail, see e.g., [Larionov et al. 2017; Takahashi and Lin 2019]). Using fractions rather than true volumes avoids numerical issues from very small values caused by Δx^3 factors (where Δx denotes grid cell width). Later, we will scale rigid body terms for consistency. We note that $\mathbf{W}_S^u$ is defined per rigid body, i.e., $\mathbf{W}_{S1}^u, \ldots, \mathbf{W}_{Sn}^u$ and $\sum_i \mathbf{W}_{Si}^u + \mathbf{W}_F^u = \mathbf{I}$ (where $\mathbf{I}$ denotes the identity matrix). Similarly, we compute weight matrices for viscous stress control volume fractions defined in a staggered manner [Goktekin et al. 2004], for fluid domains $\mathbf{W}_F^s$ and solid domains $\mathbf{W}_S^s$. These volume fractions can be consistently evaluated using the method of Takahashi and Lin [2019]. Given the free surface geometry, also defined by a signed distance function, we can likewise decompose the simulation domain into the liquid domain Ω_L and air domain Ω_A (now ignoring solids), and compute corresponding volume fraction weight matrices for pressure $\mathbf{W}_L^p$ and $\mathbf{W}_A^p$ (satisfying $\mathbf{W}_L^p + \mathbf{W}_A^p = \mathbf{I}$), velocity $\mathbf{W}_L^u$ and $\mathbf{W}_A^u$, and viscous stress $\mathbf{W}_L^s$ and $\mathbf{W}_A^s$.

3 DETAILS OF $\mathbf{Q}$ AND $\mathbf{J}_p$ AGGREGATION

To ensure consistent force exchange between fluids and rigid bodies, we define pressure forces applied from fluids to rigid bodies as $-\mathbf{Q}\mathbf{W}_S^u\mathbf{G}\mathbf{W}_L^p\mathbf{p} = \mathbf{J}_p\mathbf{W}_L^p\mathbf{p}$, i.e., $\mathbf{J}_p = -\mathbf{Q}\mathbf{W}_S^u\mathbf{G}$. In practice, assembling $\mathbf{J}_p$ via an explicitly constructed $\mathbf{Q}$ is not ideal since we can significantly reduce the number of entries in $\mathbf{J}_p$ by exploiting cancellation of forces inside of the rigid bodies. We illustrate this below.

For simplicity, we consider three consecutive grid cells in one dimension. Considering the pressure force applied from fluids to a rigid body, F_x, we can compute F_x by aggregating fluid pressure forces at each fluid velocity sample $\mathbf{F}_p = (F_{p,-1/2}, F_{p,1/2}, F_{p,3/2}, F_{p,5/2})^T$, and we can write this relation with an aggregation matrix $\mathbf{Q} = (1, 1, 1, 1)$ as $F_x = \mathbf{Q}\mathbf{F}_p$. Since $\mathbf{F}_p$ can be computed by $\mathbf{F}_p = -\mathbf{W}_S^u\mathbf{G}\mathbf{W}_L^p\mathbf{p}$, i.e.,

$$
\begin{aligned}
\begin{bmatrix} F_{p,-1/2} \\ F_{p,1/2} \\ F_{p,3/2} \\ F_{p,5/2} \end{bmatrix} &= -\begin{bmatrix} W^u_{S,-1/2} & 0 & 0 & 0 \\ 0 & W^u_{S,1/2} & 0 & 0 \\ 0 & 0 & W^u_{S,3/2} & 0 \\ 0 & 0 & 0 & W^u_{S,5/2} \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} W^p_{L,0} & 0 & 0 \\ 0 & W^p_{L,1} & 0 \\ 0 & 0 & W^p_{L,2} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \\
&= \begin{bmatrix} -W^u_{S,-1/2}(W^p_{L,0}p_0) \\ -W^u_{S,1/2}(W^p_{L,1}p_1 - W^p_{L,0}p_0) \\ -W^u_{S,3/2}(W^p_{L,2}p_2 - W^p_{L,1}p_1) \\ -W^u_{S,5/2}(-W^p_{L,2}p_2) \end{bmatrix}, \qquad (1)
\end{aligned}
$$

we have

$$\begin{aligned} F_x &= -W^u_{S,-1/2}(W^p_{L,0}p_0) - W^u_{S,1/2}(W^p_{L,1}p_1 - W^p_{L,0}p_0) \\ &\quad - W^u_{S,3/2}(W^p_{L,2}p_2 - W^p_{L,1}p_1) - W^u_{S,5/2}(-W^p_{L,2}p_2) \\ &= (W^u_{S,1/2} - W^u_{S,-1/2})W^p_{L,0}p_0 \\ &\quad + (W^u_{S,3/2} - W^u_{S,1/2})W^p_{L,1}p_1 \\ &\quad + (W^u_{S,5/2} - W^u_{S,3/2})W^p_{L,2}p_2. \end{aligned} \tag{2}$$

Typically, deep inside of the rigid bodies, the volume fraction is 1, and if neighboring volume fractions are the same, some of the terms in F_x can be canceled, e.g., if $W^u_{S,3/2} = W^u_{S,1/2}$, the term $(W^u_{S,3/2} - W^u_{S,1/2})W^p_{L,1}p_1$ can be completely ignored. Thus, we directly assemble $\mathbf{J}_p$ canceling some of the terms, without explicitly forming $\mathbf{Q}$, since that would have led to a larger number of entries in $\mathbf{J}_p$.

This approach can be naturally extended into 2D. Considering one grid cell in 2D, $\mathbf{F}_p = (F_{p,-1/2,0}, F_{p,1/2,0}, F_{p,0,-1/2}, F_{p,0,1/2})^T$, pressure forces applied from fluids to a rigid body $\mathbf{F}_r = (F_x, F_y, F_{rz})^T$ (where F_y denotes y directional force and F_{rz} denotes rotational force), and $\mathbf{Q}$ is given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -(y_j - Y) & -(y_j - Y) & (x_i - X) & (x_i - X) \end{bmatrix}, \tag{3}$$

where x_i and y_j denote the coordinates of the center of grid cell (i, j) and X and Y denote the center of mass of the rigid body. Similarly, this idea can be extended into 3D and adapted to viscous stress forces.

4 SOLUTION CONSISTENCY

PC-iterative solve is guaranteed to generate the same results as the PC-unified solve *at convergence*. This is because both the pressure and contact problems are SPD, and since pressure is unconstrained, we can eliminate it and define the normal contact force directly as $(\mathbf{A}_{33} - \mathbf{A}^T_{23}\mathbf{A}^{-1}_{22}\mathbf{A}_{23})\boldsymbol{\lambda} = \mathbf{b}_3 - \mathbf{A}^T_{23}\mathbf{A}^{-1}_{22}\mathbf{b}_2$, s.t. $0 \le \boldsymbol{\lambda}$. This manipulation is not possible for staggered projections for frictional contact [Kaufman et al. 2008]: both normal contact and friction forces involve bound constraints so neither can be eliminated.

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