Selected Topics in Variable Annuities

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Variable annuities are long-term insurance products. They have become one of the most popular savings/investment vehicles over the past two decades due to their flexible investment options, stable long-term guarantees, and favorable tax-deferral treatment. The popularity of variable annuities has catalyzed an extensive amount of research papers looking into pricing and hedging of embedded guarantees and attempting to better understand policyholder behavior. This thesis aims to contribute to these two strands of research.

While an ample amount of literature has covered various topics in variable annuities, two important market trends/phenomena need further investigation: 1). the variable annuity market has seen decreasing sales ever since the year 2013; 2). there is evident discordance between the theoretical and empirical insurance fees and policyholder behavior. The theme of this thesis is to address these two market phenomena by offering potential remedy or reasonable explanation.

Chapter 3 aims to offer an appropriate potential remedy to the declining demand of the variable annuity market. In this chapter, we propose a novel high-water mark fee structure and examine its impact on variable annuity marketability. We apply a mean-variance preference model to evaluate policyholder welfare from holding a variable annuity. By also evaluating policyholder welfare from holding two alternative investments, we introduce a quantitative measure, namely a compatible set of risk aversions, to assess the marketability of the variable annuity under a certain fee structure. We find that the high-water mark fee structure improves the variable annuity's marketability compared to a constant and a state-dependent fee structure.

Chapters 4 and 5 aim to address the aforementioned discordance by investigating the impact of policyholders' uncertainty in discount rates on their welfare and behavior from holding variable annuities. In Chapter 4, we consider policyholders of a variable annuity with guaranteed minimum death and maturity benefits whose subjective discount rates follow a Gamma distribution. We use a constant relative risk aversion utility model to evaluate policyholder welfare and surrender behavior from holding the variable annuity. We also compute an insurer's profit under given insurance fees and policyholder surrender behavior. We find that when sharing the same expected discount rate, Gamma discounting policyholders delay surrender behavior and value the variable annuity more than than exponential discounting policyholders. Moreover, the insurer makes higher profit when policyholders are Gamma discounting than when they are exponential discounting.

Chapter 5 considers policyholders of a variable annuity with guaranteed minimum withdrawal benefit whose subjective discount rates follow a Gamma distribution. Policyholder

welfare and withdrawal behavior are quantified by the expected present value of variable annuity payouts. Different from the last two chapters, we deal with stochastic optimal control problems in this chapter due to the withdrawal type guarantees. Consistent with Chapter 4, we find that when having the same expected discount rate, Gamma discounting policyholders withdraw less and value the variable annuity more than exponential discounting policyholders.

To keep a smooth flow of the thesis, Chapter 1 presents the existing strands of literature and introduces the main motivation of this thesis. Chapter 2 presents the core mathematical preliminaries for the latter chapters. Chapter 6 concludes this thesis and proposes potential future research avenues.

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Dedication

To my parents, Xiaomei Wang and Sheng Yu.

Table of Contents

List of Figures **xi**

List of Tables **xiii**

1 Introduction **1**

1.1 Background and Motivation 1

1.2 Structure of the Thesis 5

2 Mathematical Preliminaries **8**

2.1 Itô's Formula and Stochastic Differential Equations 8

2.2 Feynman-Kac Formula 10

2.3 Stochastic Control Problems 10

2.4 Optimal Stopping Problems 12

3 High-Water Mark Fee Structure in Variable Annuities **14**

3.1 Introduction 14

3.2 Valuation of Fair Fees 18

3.2.1 Guaranteed Benefits 18

3.2.2 HWM Fee Structure and Investment Account Dynamics 19

3.2.3 Mortality Model 20

3.2.4 Fair Fees 22

3.3 Policyholder's Welfare . . . 26
3.3.1 Model Setup . . . 27
3.3.2 Compatible Set of Risk Aversions . . . 30
3.3.3 Sensitivity Analysis . . . 35
3.3.4 Surrender Behavior . . . 41
3.4 Conclusion . . . 42

4 Gamma Discounting in Variable Annuities with Guaranteed Minimum Death and Maturity Benefits 43
4.1 Introduction . . . 43
4.2 Valuation of Fair Fees . . . 45
4.3 Policyholder with Gamma Discounting . . . 47
4.3.1 Model Setup . . . 47
4.3.2 Policyholders' Time Preferences and Welfare . . . 51
4.4 Insurer's Profit . . . 57
4.5 Conclusion . . . 62

5 Gamma Discounting in Variable Annuities with Guaranteed Minimum Withdrawal Benefits 63
5.1 Introduction . . . 63
5.2 Valuation of Fair Fees . . . 65
5.2.1 Pricing Model . . . 65
5.3 Policyholder with Gamma Discounting . . . 68
5.3.1 Model Setup . . . 69
5.3.2 Policyholders' Time Preferences and Welfare . . . 73
5.4 Conclusion . . . 79

6 Conclusion and Future Work 80

References 82

APPENDICES **89**

A Appendix for Chapter 3 **90**

A.1 Verification Theorem for Optimization Problem (3.16) 90

A.2 Algorithm for the System of EHJB Equations (A.1)-(A.7) 98

A.3 Robustness of Initial Investment F_0 105

B Appendix for Chapter 4 **108**

B.1 Proof of Verification Theorem 6 108

B.2 Derivation of Eq. (4.20) 110

C Appendix for Chapter 5 **111**

C.1 Derivation of the System of EHJB Equations (5.15)-(5.22) 111

C.2 Algorithm for the System of EHJB Equations (5.15)-(5.22) 114

List of Figures

3.1 A graphical illustration of fee mechanisms for constant, state-dependent and HWM fee structures. 21

3.2 PH time-0 welfare of holding a risk-free bond, pure fund, or VA under constant or state-dependent fee structures in the base case parameter setting. The left panel is the constant fee structure case, and the right panel is the state-dependent fee structure case. Observe that the VA with the constant fee structure is preferable to the alternative investments for PHs with levels of risk aversion within the set $[1.33, 1.7]$. With the state-dependent fee structure, the set is $[1.28, 1.59]$. 31

3.3 PH time-0 welfare of holding a risk-free bond, pure fund, or VA under different fee structures in the base case parameter setting. "VA (HWM Fee $\alpha = 0.5$)" corresponds to PH time-0 welfare under the HWM fee structure with $\alpha = 0.5$. We can observe that the HWM fee structure records a wider compatible set than the other two fee structures. 33

3.4 Optimal surrender regions under different fee structures at $t = 3$ and $t = 9$. The symbol "**c**" corresponds to the continuation region and "**s**" to the surrender region. The horizontal axis "m" stands for the HWM of the investment account and the vertical axis "z" stands for the state variable introduced in Eq. (3.8). The surrender risk is insignificant, and surrender only occurs when the investment account value is very large. 40

4.1 Decreasing impatience for exponential and Gamma discounting PHs with the same discount rate expectation. Observe that Gamma discounting PHs display a decreasing impatience level while exponential discounting PHs exhibit a constant impatience level. Also, Gamma discounting PHs have a higher level of decreasing impatience than that of exponential discounting PHs. 53

4.2 Optimal surrender regions for PHs with four levels of discount rate uncertainty as above. "**S**" corresponds to the surrender region and "**C**" to the continuation region. For each curve in the figure, the upper right part is the surrender region and the lower left part the continuation region. Observe that, when having the same expected discount rate, Gamma discounting PHs are less likely to surrender the VA policy than exponential discounting PHs, and the more uncertain PHs are regarding the discount rate, the less likely they will surrender the policy. 54

4.3 Optimal surrender regions for three kinds of PHs as in Table 4.4. Note that, PHs whose discount rate follows $\Gamma(1.13, 2.67\%)$ do not surrender in this domain. Observe that, when having the same level discount rate uncertainty, PHs with a higher discount rate expectation are more likely to surrender the VA policy. 56

4.4 Insurer's profit over a range of insurance fees below the fair fee c^{f} when PHs have the same expected discount rate but different discount rate variance. c^* denotes the insurance fee that generates the highest profit for the insurer. Observe that the insurer has a higher profit and a higher c^* when PHs have higher discount rate uncertainty. 60

4.5 Insurer's profit over a range of insurance fees below the fair fee c^{f} when PHs have the same different discount rate variance but different discount rate expectations. Observe that the insurer in general has a lower profit when PHs have a higher discount rate expectation for a range of insurance fees. . 62

5.1 Contour graph of withdrawal regions for exponential discounting PHs (blue), Gamma discounting PHs of $\Gamma(2, 2\%)$ (red), and Gamma discounting PHs of $\Gamma(1, 4\%)$ (pink) with the same expected discount rate at time $t = 1$ (left two panels) and time $t = 13$ (right two panels). Observe that Gamma discounting PHs withdraw less from the VA policy than exponential discounting PHs. 75

5.2 Contour graph of withdrawal regions at time $t = 1$ for PHs with the same discount rate uncertainty but different discount rate expectation. The left panel compares the base case (red) to the case with a lower expected discount rate (blue), and the right panel compares the base case to the case with a higher expected discount rate (black). Observe that the higher discount rate expectation PHs have, the more they will withdraw from the VA policy. . . 78

List of Tables

3.1 Parameter inputs 25

3.2 Fair constant fees c under different fee structures. "Constant" corresponds to the constant fee structure and "SD" to the state-dependent fee structure (the case $\alpha = 0$). We also consider four other levels of HWM fee α, namely 0.05, 0.2, 0.35 and 0.5. Observe that the constant fee c decreases as the HWM fee α increases for all cases. Also, a higher risk-free rate corresponds to a lower constant fee and higher volatility corresponds to a higher constant fee. 26

3.3 Compatible sets for different fee structures. This table presents the location and width of compatible sets for different fee structures in the base case parameter setting. Observe that a HWM fee of $\alpha = 0.5$ produces the widest compatible set. 34

3.4 Optimal HWM fee and welfare comparison. This table summarizes sets of the HWM fees α (i.e., "Compatible α") that make given levels of risk aversion γ belong to the corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare in the base case parameter setting. We observe that the HWM fee structure generates the highest welfare for the PH when the VA is preferred to the alternative investments under all three fee structures. 35

3.5 Compatible sets for different risk-free rates. This table summarizes the location and width of compatible sets under different fee structures for different risk-free rates r. Observe that only the HWM fee structure generates a non-empty compatible set when the risk-free rate is low (i.e., $r = 1\%$), and the HWM fee structure still has the widest compatible set. 36

3.6 Compatible sets for different performance of the policy fund. This table summarizes the location and width of compatible sets of the VA with different fee structures for different performance of the underlying policy fund. For a more volatile policy fund, the compatible set shifts to the left and a HWM fee of 0.05 generates the widest set. For a policy fund with a higher return, compatible sets become wider. 37

3.7 Optimal HWM fee and welfare comparison under various market conditions. This table summarizes sets of HWM fees α that make given levels of risk aversion γ belong to corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare under various market conditions. Observe that the HWM fee structure generates the highest PH welfare under various levels of risk aversion (in the corresponding compatible sets). . . . 39

4.1 Parameter inputs 47

4.2 Parameter inputs under Gamma discounting 51

4.3 PHs' time-0 welfare with increasing uncertainty of $\boldsymbol{\beta}$. Note that $\mathbb{E}[\boldsymbol{\beta}] = 4\%$ is kept the same for all four cases and $\text{Var}[\boldsymbol{\beta}]$ is increasing. "E.D." is short for "exponential discounting PHs". Observe that Gamma discounting PHs value the VA policy more than exponential discounting ones, and the larger variance a PH's subjective discount rate has, the more the PH will value the VA policy. 52

4.4 Time-0 welfare of PHs with increasing discount rate expectation. We compare the base case with those that PHs have the same discount rate variance but different discount rate expectation. Observe that the higher discount rate expectation PHs have, the lower their time-0 welfare becomes. 55

4.5 Insurer's profit valuated at $(t, \omega, F) = (0, 0, F_0)$ when PHs have the same expected discount rate but different discount rate variance. The insurer charges the fair fee $c^{\text{f}} = 0.0186$ obtained in Eq. (4.5). Parameter inputs are given by Table 4.1 and the base case in Table 4.2. Observe that the insurer generates a higher profit when PHs are Gamma discounting than exponential discounting, and the profit is higher when PHs have a higher level of discount rate uncertainty. 59

4.6 Insurer's profit when PHs have the same discount rate variance but different discount rate expectations. The insurer charges the fair fee $c^{\text{f}} = 0.0186$. Observe that the insurer's profit is reduced if PHs have higher discount rate expectations. 61

5.1 Parameter inputs 69

5.2 Parameter inputs under Gamma discounting 73

5.3 PHs' time-0 welfare with increasing uncertainty of $\boldsymbol{\beta}$. Note that $\mathbb{E}[\boldsymbol{\beta}] = 4\%$ is kept the same for all four cases and $\mathrm{Var}[\boldsymbol{\beta}]$ is increasing. "E.D." is short for "exponential discounting PH". Observe that Gamma discounting PHs value the VA policy more than exponential discounting ones, and the larger variance a PH's subjective discount rate has, the more the PH will value the VA policy. 74

5.4 Time-0 welfare of PHs with increasing discount rate expectation. We compare the base case with those that have the same discount rate variance but different discount rate expectation. Observe that the higher discount rate expectation PHs have, the lower their time-0 welfare becomes. 77

Chapter 1

Introduction

1.1 Background and Motivation

Variable annuities (VAs) are long-term equity-linked insurance products. In a typical VA policy, policyholders (PHs) get access to various investment options through different subaccounts, and the performance of these accounts are protected from below by embedded guarantees which normally contain some insurance features (such as death benefits). Apart from flexible investment opportunities and stable long-term guarantees, VAs also enjoy favorable tax-deferral treatment. Specifically, in a VA policy, taxes are only applicable when PHs (or their beneficiaries) receive withdrawals, income payments, or death benefits. These advantages of VAs have made them one of the most prevalent savings and investment vehicles for investors with different motives. By the second quarter of the year 2017, the net U.S. assets have amounted to approximately 1.98 trillion dollars (see IRI (2017) for more details). Even in year 2020, with the detrimental impact of the COVID-19 pandemic and low interest rate environment, VA providers still achieved nearly 100 billions in sales.

Not surprisingly, the popularity of VAs has catalyzed an extensive amount of research papers looking into various aspects of this product. One aspect of great importance is pricing and hedging of different kinds of guarantees embedded in VAs. To name a few, Milevsky & Posner (2001) price guaranteed minimum death benefits (GMDB) in VAs, Milevsky & Salisbury (2006) and Dai et al. (2008) investigate risk-neutral pricing for guaranteed minimum withdrawal benefits (GMWB) in VAs, Marshall et al. (2010) explore valuation of guaranteed minimum income benefits (GMIB) in VAs, and Huang & Kwok (2016) determine the no-arbitrage price of guaranteed lifetime withdrawal benefits (GLWB) in VAs. In particular, Bauer et al. (2008) establish a universal framework to price various guarantees in VAs. The most important common theme of the aforementioned research is that they make use of the resemblance of VAs to financial derivatives and price the embedded guarantees using risk-neutral pricing techniques, which is an improvement to deal with the financial risk undertaken by VA providers over the traditional reserving approach in insurance (see Boyle & Hardy (2003) for more details).

Since numerical implementation of pricing models can be very challenging, another strand of research focuses on developing numerical tools to facilitate the pricing of guarantees in VAs. For instance, Huang & Forsyth (2012) proposes a penalty method to solve a singular control problem corresponding to the pricing of GMWB, while a Willow tree algorithm is proposed in Dong et al. (2019) for the same guarantees. Huang & Kwok (2016) and Shen & Weng (2020) develop regression-based Monte Carlo methods to price GLWB and similar products. Hedging of guarantees in VAs are analyzed far less often than pricing. We list some research papers here for interested readers. Coleman et al. (2006), Coleman et al. (2007), and Trottier et al. (2018) apply the local risk minimization method to hedge guarantees in VAs, and a semi-static hedging method is used in Kolkiewicz & Liu (2012) and Bernard & Kwak (2016).

Another very important aspect is to understand PH behavior in VAs. Since guarantees in VAs offer PHs the flexibility to access sub-accounts (through withdrawing or lapsing), it is of significant importance for insurers to properly understand PH behavior so that they can improve VA contract design to better manage PH behavior risk and increase VA marketability. It is common in actuarial science literature to investigate PH behavior assuming a complete market. In this setting, Dai et al. (2008) and Huang & Kwok (2014) analyze PH withdrawal behavior for VAs with GMWB, Huang et al. (2014) and Huang et al. (2017) determine optimal initiation of income phase for VA PHs with GLWB, Azimzadeh & Forsyth (2015) prove the existence of a bang-bang solution for optimal PH withdrawal behavior of VAs with GLWB, and numerical examples of a similar problem are given in Huang & Kwok (2016). As for PH surrender behavior, MacKay et al. (2017) propose a state-dependent fee structure that can make surrender behavior sub-optimal when applying a marketable surrender penalty, Moenig & Zhu (2018) demonstrate that a state-dependent fee structure or a Rachet guarantee works the best to contain lapse-and-reentry behavior, Bernard & Moenig (2019) introduce a time-dependent fee structure that reduces insurance fees without reducing insurers' profit through containing PH surrender behavior, and Knoller et al. (2016) provide empirical evidence on PH surrender behavior for the Japanese VA market. While Bauer et al. (2017) point out the main drive for PH behavior in VAs is market friction (e.g., taxes), Moenig & Bauer (2015) examine PH withdrawal behavior by incorporating the effect of taxes into the analysis.

In addition to the aforementioned literature, there is also a strand of research analyzing

PH behavior using a life cycle consumption approach in a potentially incomplete market. For example, Gao & Ulm (2012) and Gao & Ulm (2015) determine optimal allocation between fixed and variable sub-accounts for VA PHs with GMDB, Horneff et al. (2015) consider optimal life cycle consumption and portfolio allocations when VAs with GMWB are available, and Steinorth & Mitchell (2015) explore how PHs should optimally withdraw from VAs with GLWB to maximize expected utility. Although a fair amount of research has been done to better understand PH behavior, it is safe to point out that PH behaviour remains a complex problem given that VAs are very complicated products for PHs to properly understand, which impedes their ability to evaluate VAs and behave accordingly (see Brown et al. (2017)).

While most of the aforementioned literature focuses on valuation of various guarantees for a single VA, some research papers propose novel methods to evaluate a large portfolio of VAs. To name some representatives, Gan (2013) introduces an approach based on data clustering and machine learning to price guarantees for a large portfolio of VAs, Gan & Lin (2015) combine a clustering technique with a functional data analysis technique to speed up nested simulation of a large VA portfolio, Gan & Lin (2017) propose a two-level meta-modeling approach to calculate partial dollar Deltas of a portfolio of VAs, and Feng et al. (2020) offer efficient simulation designs for valuation of large VA portfolios.

With existing literature covering various topics in multiple aspects of VAs, there are two important market trends/phenomena this thesis aims to address:

(i) The VA market has been experiencing decreasing demand since year 2013;

(ii) There is evident discordance between the theoretical and empirical insurance fees and PH behavior in the VA market.

Specifically, for the first trend of the VA market, "investors are surrendering policies faster than new money is coming into the market" (see Bernard & Moenig (2019) for the source). In year 2020, with the impact of the COVID-19 pandemic and years of low interest rate environment, VAs with traditional living and death benefits are very expensive and providers are shifting away from these business lines. In fact, the major driver for VA sales in year 2020 is registered index-linked annuities (RILAs), a hybrid of fixed-indexed and variable annuities. Although it remains relevant to investigate RILAs, we focus on VAs with traditional living and death benefits in this thesis.

To address the declining demand, Bernard & Moenig (2019) introduce a time-dependent fee structure to improve the prevalent constant fee structure. The new fee structure is shown to reduce the high insurance fees while keeping VAs profitable for insurers. As alluded to above, MacKay et al. (2017) directly tackle surrender behavior by making it sub-optimal via a state-dependent fee structure. Although each aforementioned paper focuses on improving one aspect of the constant fee structure's deficiencies, little is known about the impact that the corresponding newly proposed fee structures may have on PH welfare. By merely focusing on the risk management implications of the proposed fee structures from an insurer's perspective, the analysis fails to fully address the issue of declining demand. Therefore, **the first objective of this thesis is to contribute to this line of research by proposing a novel high-water mark (HWM) fee structure and investigating its impact on VA marketability.** By focusing on PH welfare from holding VAs under the HWM fee structure, we hope to better address the decreasing demand in the VA market.

As for the second market phenomenon, it is well-documented that there is evident disconnect between theoretical and empirical insurance fees and PH behavior. More specifically, PHs exercise embedded options (e.g. withdrawal option) in VAs less often than risk-neutral pricing approaches suggest and market prices for VAs are lower than these approaches indicate. In fact, Milevsky & Salisbury (2006) points out that market fees for GMWB are significantly less than the ones obtained from a risk-neutral pricing approach. The seminal work of Moenig & Bauer (2015) demonstrates that this discordance/disconnect can be explained by "the very friction that arguably explains the rapid expansion of the VA market: taxes". **The second objective of this thesis is to offer an alternative explanation to the discordance: time inconsistency in PHs' time preferences.** By considering PHs whose subjective discount rates follow a Gamma distribution, we hope to also explain the discordance.

Another motivation for the second objective is that there is substantial evidence that individuals' time preferences are time-inconsistent (i.e., non-constant discount rates), especially in the long run. Time inconsistency in time preference has been widely observed for a group of decision makers when valuing long-term investment. For instance, the seminal work of Weitzman (2001) attempts to address the "perennial dilemma of what discount rate to use". The author conducts a survey from 2000 economists, including 52 Nobel Laureates, and the result shows an evidently dispersed discount rate distribution. Similar results can also be found in Frederick et al. (2002) and Falk et al. (2015). Apart from group heterogeneity in time preferences, individuals' uncertainty about what discount rate to use has also been observed. For example, Grenadier & Wang (2007) believe that individuals are "more prone to time-inconsistent behavior than firms". Consistent with this, Laibson (1997), Brocas & Carrillo (2004), and DellaVigna & Malmendier (2004) have all considered time-inconsistent individuals. In particular, DellaVigna & Malmendier (2004) consider time-inconsistent individuals but time-consistent firms.

Since VAs are typical long-term insurance products with investment features, we incorporate PHs' time preferences into the analysis, and aim to investigate the impact of PHs' time-inconsistent time preferences on their welfare from holding VAs and the impact on insurers' profit.

1.2 Structure of the Thesis

With the aforementioned motivation, this thesis is a collection of three research projects. The remainder of this thesis is organized as follows. Chapter 2 presents the core mathematical preliminaries for the latter chapters. Chapter 3 proposes a novel HWM fee structure and investigates its impact on the marketability of VAs. Chapters 4 and 5 analyze the impact of Gamma discounting on PH welfare and insurer profit for VAs with GMDB and guaranteed minimum maturity benefit (GMMB) and on PH welfare for VAs with GMWB, respectively. Potential future research avenues are discussed in Chapter 6. Specific methodology and main results of each research project are summarized in the following.

Chapter 3 proposes a novel HWM fee structure and examines its impact on VA marketability. We first obtain the fair insurance fees under the HWM fee structure using risk-neutral pricing approach. To evaluate a PH's welfare from holding a VA with GMDB and GMMB, we set up a mean-variance (MV) preference model with state-dependent risk aversion and solve a time-inconsistent optimal stopping problem. We overcome the intricacy stemming from the random time horizon and high dimension, and derive a system of extended Hamilton-Jacobi-Bellman (EHJB) equations corresponding to the problem. Furthermore, numerically solving the system of EHJB equations using standard finite difference methods is extremely challenging due to the complex form of the system and a time-consuming iterative procedure. We circumvent the complications by proposing a numerical algorithm based on Wang & Forsyth (2011) and further prove that a solution of the algorithm is a solution of the system. By also determining the PH's welfare from holding two alternative investments, a risk-free bond and a pure fund, we introduce a quantitative measure, namely a compatible set of risk aversions, to assess VA marketability under a certain fee structure.

Comparing the PH's welfare from holding the VA under the HWM fee structure with those under a constant and a state-dependent fee structure, we find that the HWM fee structure improves VA marketability in the following two aspects. First, the HWM fee structure makes the VA preferable to the alternative investment options for a broader range of policyholders. Second, when the VA is preferred over the alternative investment options, the HWM fee structure produces the highest welfare for the PH.

Chapter 4 considers PHs whose subjective discount rates follow a Gamma distribution. The analysis is based on a similar VA with the constant fee structure as in Chapter 3. We determine the impact of Gamma discounting on PHs' surrender behavior and welfare from holding the VA as well as on an insurer's profit. To investigate PHs' welfare and surrender behavior, we use a constant relative risk aversion utility function. Since PHs' subjective discount rates follow a Gamma distribution, the corresponding discount function is no longer exponential and consequently incurs a time-inconsistent optimal stopping problem with a random horizon. Similar to Chapter 3, we adopt the game theoretic framework proposed by Björk & Murgoci (2010) to derive a system of EHJB equations. Numerically solving this system of equations, we obtain PHs' welfare and the corresponding surrender behavior. Under PHs' surrender behavior, we determine the insurer's profit quantified by total fees collected deducting the VA payout.

Our results show that, when PHs share the same expected discount rate, Gamma discounting PHs delay surrender and value the VA policy more than exponential discounting PHs. And, the more uncertain they are with respect to the discount rate, the more they will value the VA policy and the less likely they will surrender the policy. Moreover, the insurer makes higher profit when PHs are Gamma discounting than when they are exponential discounting, and the insurer's profit increases as PH discount rate uncertainty increases.

In Chapter 5, we go a step further from Chapter 4 and consider risk-neutral PHs of a VA with GMWB whose subject discount rates follow a Gamma distribution. Different from the last two chapters, we formulate the risk-neutral pricing of the VA within a singular stochastic optimal control framework as in Dai et al. (2008). The fair fee of the VA policy is numerically determined via the penalty method in Huang & Forsyth (2012). We set up a time-inconsistent stochastic optimal control model to determine risk-neutral PHs' withdrawal behavior and welfare from holding the VA. Again, we derive a system of EHJB equations and propose an efficient algorithm to numerically solve the system so that the time-consuming iterative procedure to attain an equilibrium withdrawal strategy in standard finite difference methods can be avoided.

Consistent with those in Chapter 4, our results show that, when having the same expected discount rate, Gamma discounting PHs withdraw less and value the VA policy more than exponential discounting PHs. And, the higher discount rate variance PHs have, the less they withdraw and the more they will value the VA policy. Our findings in Chapters 4 and 5 indicate that, in addition to the effect of taxes, Gamma discounting can offer an alternative explanation to the phenomenon that PHs exercise embedded options in VAs less frequently and market prices for VAs are lower than risk-neutral pricing approaches suggest.

Finally, it is important to mention that Chapters 3-5 are written independently of one another, as each chapter contains the main material of a research paper. While efforts have been made to keep the notation consistent throughout this thesis, some inconsistency may remain. Hence, we invite readers to treat each chapter independently in terms of its notation.

Chapter 2

Mathematical Preliminaries

2.1 Itô's Formula and Stochastic Differential Equations

Consider a stochastic process $X = \{X_t\}_{t\geq 0}$ which is defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ satisfying the usual conditions. The state space of the process X is assumed to be $\mathbb{R}$ throughout this chapter.

We first state the famous Itô's formula which is defined on semimartingales, the most general class of stochastic processes.

Definition 1. (Semimartingale) *A càdlàg adapted process X is a semimartingale if it admits the following decomposition:*

$$X_t = X_0 + M_t + A_t,$$

where X_0 is finite and $\mathcal{F}_0$-measurable, M is a càdlàg local martingale, A is an adapted process with finite variation, and $M_0 = A_0 = 0$.

Theorem 1. (Itô's Formula) *Let $X = \{X_t\}_{t\geq 0}$ be a semimartingale with state space $\mathbb{R}$ and f be a $\mathcal{C}^2$ function on $\mathbb{R}$. Then $f(X)$ is also a semimartingale, and*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_{0^+}^{t} f'(X_{s^-}) dX_s + \frac{1}{2}\int_{0^+}^{t} f''(X_{s^-}) d[X,X]_s^c \\ &+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s \right\}, \end{aligned} \tag{2.1}$$

where f' (f'' resp.) is the first (second resp.) order derivative of f with respect to X, $[X,X]^c$ is the continuous part of the quadratic variation process of X, and $\Delta X_t = X_t - X_{t^-}$.

Note that Itô's formula (2.1) is applied in the following chapters to prove verification theorems and derive partial differential equations.

We present the solution of a stochastic differential equation (SDE) and formulation of some stochastic optimization problems for the rest of this chapter. Note that these optimization problems are based on a one-dimensional diffusion process driven by a one-dimensional Brownian motion. Interested readers are referred to Fleming & Soner (2006) and Øksendal & Sulem (2007) for stochastic optimization problems on more general processes.

Given deterministic functions b and σ that are progressively measurable and are from $[0,T]\times\Omega\times\mathbb{R}$ to $\mathbb{R}$, we consider the following SDE:

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t\in[0,T], \tag{2.2}$$

where $\{W_t\}_{t\geq 0}$ is a one-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. We present the definition of a strong solution of Eq. (2.2) (see Touzi (2012) Definition 2.1 on page 5) in the following.

Definition 2. (Strong Solution of an SDE) *A strong solution of SDE (2.2) is a progressively measurable process X such that $\int_0^T \left(|b_t(X_t)| + |\sigma_t(X_t)|^2\right)dt < \infty$, a.s., and*

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t\in[0,T].$$

A sufficient condition for the existence and uniqueness of a strong solution of SDE (2.2) is given in the following (see Touzi (2012) Theorem 2.2 on page 6).

Theorem 2. *Let X_0 be a random variable that is independent of W such that $\mathbb{E}\left[|X_0|^2\right] < \infty$. Assume that the processes $b.(0)$ and $\sigma.(0)$ are in $\mathbb{H}^2$ and that for some $K>0$,*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leq K|x-y|, \quad \text{for all } \; t\in[0,T], \quad x,y\in\mathbb{R},$$

where $\mathbb{H}^2$ denotes the collection of all progressively measurable processes ψ with finite dimension such that $\mathbb{E}\left[\int_0^T |\psi_t|^2\right] < \infty$. Then, for all $T>0$, there exists a unique strong solution of SDE (2.2) in $\mathbb{H}^2$. Moreover,

$$\mathbb{E}\left[\sup_{t\leq T}|X_t|^2\right] \leq C(1+\mathbb{E}|X_0|^2)e^{CT},$$

for some constant C=C(T,K) depending on T and K.

2.2 Feynman-Kac Formula

For SDE (2.2), we introduce the the infinitesimal generator of the diffusion process (2.2) in the following:

$$(\mathcal{L}_t\psi)(x) = b_t(x)\psi'(x) + \frac{1}{2}\sigma_t^2(x)\psi''(x), \quad \text{for all} \;\; t \in [0,T], \quad \psi \in \mathcal{C}^2(\mathbb{R}).$$

We consider the Cauchy linear parabolic partial differential equation (PDE):

$$\begin{cases} \nu_t(t,x) + (\mathcal{L}_t\nu)(t,x) + f(t,x) - r(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ \nu(T,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $\nu \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$, g is a given function from $\mathbb{R}$ to $\mathbb{R}$, and r and f are functions from $\mathbb{R}$ to $\mathbb{R}$ where r is further assumed to be non-negative. We present a simple version of the Feynman-Kac representation theorem in the following.

Theorem 3. (Feynman-Kac Representation Theorem) *Let ν be a $\mathcal{C}^{1,2}([0,T) \times \mathbb{R})$ solution of Cauchy PDE (2.3). Then, ν admits the following representation*

$$\nu(t,x) = \mathbb{E}\left[\int_t^T e^{-\int_t^s r(u,X_u^{t,x})du} f(s,X_s^{t,x})ds + e^{-\int_t^T r(s,X_s^{t,x})ds} g(X_T^{t,x})\right], \;\; \textit{for all} \;\; (t,x) \in [0,T] \times \mathbb{R},$$

where $X_s^{t,x}$ for $t \leq s \leq T$ indicates process X is with initial state $X_t = x$.

Note that the Feynman-Kac Representation Theorem is applied in Chapter 4 to derive the PDE corresponding to the profit of an insurer.

2.3 Stochastic Control Problems

A controlled diffusion process $X = \{X_t\}_{t\geq 0}$ can be described by the following SDE:

$$dX_t = b_t(X_t,\alpha_t)dt + \sigma_t(X_t,\alpha_t)dW_t, \quad 0 \leq t \leq T, \tag{2.4}$$

where $T > 0$ is a fixed finite horizon, $\alpha = \{\alpha_t\}_{t\geq 0}$ is the progressively measurable control process valued in $A \subset \mathbb{R}$, and b and σ are progressively measurable functions from $[0,T] \times \mathbb{R} \times A$ to $\mathbb{R}$. We assume that the functions b and σ satisfy the following two conditions

(i) $|b_t(x,a) - b_t(y,a)| + |\sigma_t(x,a) - \sigma_t(y,a)| \le C|x-y|$ for $\forall x, y \in \mathbb{R}$ and $\forall a \in A$, where C is a uniform constant that is independent of x, y, and a;

(ii) the set $\mathcal{A}$ of controls α is restricted to

$$\mathcal{A} = \left\{ \alpha : \mathbb{E}\left[\int_0^T |b_t(0,\alpha_t)|^2 + |\sigma_t(0,\alpha_t)|^2 dt \right] < \infty \right\};$$

to ensure that the SDE (2.4) has a unique strong solution. For the rest of this section, we denote this solution by $\{X_s^{t,x}\}_{s\in[t,T]}$ to indicate process X is at state $x \in \mathbb{R}$ at time $t \in [0,T]$.

The objective function is defined as

$$J(t,x,a) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s) ds + g(X_T^{t,x}) \right], \tag{2.5}$$

where f is a function from $[0,T] \times \mathbb{R} \times A$ to $\mathbb{R}$ and g is a function from $\mathbb{R}$ to $\mathbb{R}$. To ensure that $J(t,x,a) < \infty$, we further assume

(i) a quadratic growth condition, $|g(x)| \le C(1 + |x|^2)$ for $\forall x \in \mathbb{R}$ and a constant C independent of x;

(ii) the set of controls $\{\alpha_t\}_{t\ge0}$ is restricted to

$$\mathcal{A}(t,x) = \left\{ \alpha \in \mathcal{A} : E\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| ds \right] < \infty \right\},$$

for $(t,x) \in [0,T] \times \mathbb{R}$.

The value function associated with Eq. (2.5) is

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha), \tag{2.6}$$

and

$$\hat{\alpha} = \arg \max_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha) \tag{2.7}$$

is called an optimal control. Note that a control process α in the form of $\alpha_s = a(s, X_s^{t,x})$ for some measurable function a from $[0,T] \times \mathbb{R}$ to A is called a Markovian (feedback) control. We are only interested in Markovian controls in this thesis.

Theorem 4. (HJB Equation) *Suppose the functions b, σ, f, and g are uniformly continuous, and the value function $V \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$. Then, the value function V solves the Hamilton-Jacobi-Bellman (HJB) equation:*

$$\begin{cases} -V_t - H(t,x,V_x,V_{xx}) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ V(T,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where, for $(t,x,p.M) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$H(t,x,p,M) = \sup_{a \in A} \left\{ b(t,x,a)p + \frac{1}{2}\sigma^2(t,x,a)M + f(t,x,a) \right\}.$$

H is called the Hamiltonian of the control problem.

Interested readers are referred to Touzi (2012) (Propositions 3.4 and 3.5, and Theorem 3.6) for formal statements and derivation. Note that a singular control framework where the Hamiltonian $H = \infty$ is applied to establish the pricing value function for variable annuities with guaranteed minimum withdrawal benefits in Chapter 5.

2.4 Optimal Stopping Problems

For a finite horizon $T > 0$, we denote by $\mathcal{T}_{[t,T]}$ the collection of all stopping times τ valued in $[t,T]$. We assume that a stochastic process $X = \{X_t\}_{t \geq 0}$ is defined by SDE (2.2) and the functions b and σ satisfy the conditions in Theorem 2 such that the SDE has a unique strong solution.

Let g be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and assume that

$$\mathbb{E}\left[\sup_{t \in [0,T]} |g(X_t)| \right] < \infty. \tag{2.8}$$

Under condition (2.8), the objective function

$$J(t,x,\tau) = \mathbb{E}[g(X_\tau^{t,x})],$$

is well defined for all $(t,x) \in [0,T] \times \mathbb{R}$. Again, $X_s^{t,x}$ indicates that the process X is at state $x \in \mathbb{R}$ at time $t \in [0,T]$. The optimal stopping problem is defined by

$$V(t,x) = \sup_{\tau \in \mathcal{T}_{[t,T]}} J(t,x,\tau), \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{2.9}$$

A stopping time $\hat{\tau}$ is called an optimal stopping rule if $V(t,x) = J(t,x,\hat{\tau})$. The set

$$S = \{(t,x) : V(t,x) = g(x)\} \tag{2.10}$$

is called the optimal stopping region, and its complement S^c is called the continuation region.

Theorem 5. (Variational Inequality) *Assume that the value function* $V \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ *and let* $g : \mathbb{R} \to \mathbb{R}$ *be continuous. Then value function* V *solves the following variational inequality:*

$$\min\{-V_t - \mathcal{L}V, V - g\} = 0. \tag{2.11}$$

A formal proof can be found in Touzi (2012) (see Theorem 4.5). Note that an optimal stopping framework is applied to solve for fair insurance fees of variable annuities in Chapter 3 and Chapter 4.

Chapter 3

High-Water Mark Fee Structure in Variable Annuities

3.1 Introduction

As alluded to in Chapter 1, flexible investment options, favorable tax-deferral treatment, and stable long-term guarantees have made variable annuities (VAs) one of the most prevalent investment vehicles over the last two decades. In a typical VA contract, a policyholder (PH) pays a lump sum initial premium to an insurer who invests the sum into a basket of preassigned mutual funds (often referred to as the policy fund) by setting up an investment account to track the performance of the policy fund. Payouts under the VA contract are often subject to some minimum guarantees which kick in when the performance of the policy fund is poor. To fund these guarantees, the insurer periodically depletes the investment account by charging insurance fees. The PH is also given the option to surrender the VA contract before maturity, subject to some predetermined surrender penalty. In light of the above, a standard VA contract with minimum guaranteed payouts offers the PH protection against bearish market conditions while allowing the PH to gain (financially) from bullish market movements.

Nevertheless, even with the seemingly great advantages to PHs, the VA market has experienced dwindling sales over the past half decade (see Bernard & Moenig (2019)). Many reasons have been evoked to explain this trend, most notably unfavorable fee structures associated with VA products. Therefore, the sluggish market highlights the importance of fee structure design for VAs.

The most prevalent fee structure for VAs is the so-called constant fee structure which periodically levies a fixed percentage of the investment account as insurance fees. This time-invariant and state-invariant fee structure suffers from a number of drawbacks, which served as the main catalyst for a number of researchers to look into alternative (and possibly more favorable) fee structures in VAs. For instance, to deal with the high insurance fees, Bernard & Moenig (2019) proposes a time-dependent fee structure where the insurance fee decreases after a certain time threshold. This fee structure is shown to reduce insurance fees while keeping the VA contract profitable to insurers. To manage the volatility risk, Cui et al. (2017) considers a VIX-linked fee structure in a Heston-type stochastic volatility setting. It is demonstrated that, compared to the constant fee structure, the VIX-linked fee structure lowers the sensitivity of insurers' liabilities to market volatility. To contain the surrender risk, Bernard et al. (2014) introduces a state-dependent fee structure that charges a constant fee only when the value of the investment account is below a certain threshold. MacKay et al. (2017) later shows that, in a complete market, the state-dependent fee structure can render surrender behavior fully sub-optimal by imposing a certain marketable surrender penalty. Moreover, Moenig & Zhu (2018) finds that the state-dependent fee structure offers the best potential remedy for lapse-and-reentry in VAs.

Although each aforementioned paper focuses on improving one aspect of the constant fee structure's deficiencies (e.g., reducing high insurance fees, lowering the volatility risk, or discouraging PH surrender behavior, etc.), little is known about the impact that the corresponding newly proposed fee structures may have on PH welfare. Examining the impact of fee design on the welfare effects of holding a VA helps determine whether fee design can improve VA marketability in comparison to alternative investments, and possibly address the issue of dwindling sales in the VA market. However, by merely focusing on risk management implications of the proposed fee structures from the insurer's perspective, the analysis fails to fully address the issue of declining demand.

In light of the above, the goal of this chapter is twofold. First, we propose a VA with a novel *high-water mark fee structure* and evaluate the welfare effects of holding the VA. This high-water mark (HWM) fee structure[1] is assumed to have a blend of features, from a state-dependent constant fee to a pure HWM fee. More specifically, in addition to a constant fee charged when the investment account is below a certain threshold, the HWM fee structure is designed to also apply an HWM fee on the increase in the record highs of the investment account above the threshold. This fee structure can be considered as a

[1]The HWM fee structure is frequently applied in the hedge fund industry where a Two-and-Twenty fee scheme (2% constant fee for assets under management and 20% HWM fee for newly created HWM) is widely accepted. The impact of the HWM fee on investors and fund managers in the hedge fund has been well documented; see e.g. Guasoni & Obłój (2016) and references therein.

generalization of both the constant and the state-dependent fee structures, and it will be shown to reduce the uncertainty of the VA payout.

Second, we introduce a quantitative measure, namely a *compatible set of risk aversions*, to assess the marketability of a VA under a certain fee structure. For the fee structure, the compatible set is defined as a collection of risk aversions where, for any level of risk aversion within this set, the PH's welfare of holding the VA is higher than that of holding either of two alternative investments, namely a risk-free bond and a pure fund. The compatible set offers a benchmark to compare the impact of different fee structures on VA marketability. By definition, a wider compatible set indicates better marketability of the corresponding fee structure as it makes the VA preferable to the alternative investment options for a broader range of PHs. We also conduct sensitivity analysis on compatible sets under various parameter settings to investigate VA marketability under different fee structures in various market conditions.

By comparing the compatible sets and the welfare effects of holding the VA under the HWM fee structure with those under the constant and the state-dependent fee structures, we find that the HWM fee structure improves VA marketability in the following two aspects. First, the HWM fee structure generates the widest compatible set under various parameter settings. Second, when the VA is preferred over the alternative investment options, the HWM fee structure produces the highest welfare for the PH.

As for the analysis, we first adopt the risk-neutral pricing approach to determine the fair insurance fees[2] for the VA with the HWM fee structure. This is the conventional approach to price VAs in the literature; see for example Milevsky & Salisbury (2001), Milevsky & Salisbury (2006), Bauer et al. (2008), Dai et al. (2008), Huang & Kwok (2016), and references therein. Within the risk-neutral pricing framework, the insurer assumes that the PH aims to maximize the expected present value (EPV) of the VA's future cash payouts. Corresponding to insurers' worst-case liabilities, this assumption has important risk management implications as it renders that insurers can, at least in theory, hedge every possible cash outflow with no risk.

Next, from the PH's perspective, we carry out the welfare effects of holding the VA contract under the HWM fee structure. We carry a comparative analysis by also considering PH welfare under the constant and the state-dependent fee structures. In a complete and frictionless market, it is opportune for PHs to maximize the EPV of the VA's future payouts (see Bauer et al. (2017) for more details). However, for PHs, the life insurance market is neither complete nor frictionless, and they might deviate from maximizing future

[2]The fair fee is also called the break-even fee. It is the fee that equates the expected present value of PHs' future cash payouts to their initial premiums.

payouts. Therefore, we evaluate PH welfare by a mean-variance (MV) preference model in a potentially incomplete market. More specifically, the PH is assumed to maximize their welfare quantified by a MV objective function.

The MV preference model has been widely used in tackling investment optimization problems (e.g. Zhou & Li (2000), Basak & Chabakauri (2010), and Björk et al. (2014)), ever since the seminal work by Markowitz (1952). In this chapter, we adopt this model for the following three reasons. First, as alluded to above, one merit of the HWM fee structure is that its inherent design reduces the uncertainty of the VA payout, while the MV preference model is explicit in assessing PH aversion against the variance of the VA payout. Second, the MV preference model has been often applied as a benchmark to assess long-term intertemporal investments (see, e.g. Miller & Whitman (1970), Cochrane (2014), and Munk (2020)). The present work is therefore consistent with this line of thought as VAs are typically long-term investments. Last, compared to a utility maximization model, the risk aversion coefficient in the MV preference model is simpler to gauge (see Dai et al. (2020) for more details).

To evaluate the welfare effects of holding a VA under the HWM fee structure, we resort to solving an MV optimal stopping problem. Adopting the game theoretic framework proposed in Björk & Murgoci (2010), we overcome the intricacy stemming from the random time horizon and high dimension, and derive a system of extended Hamilton-Jacobi-Bellman (EHJB) equations corresponding to the problem. Furthermore, numerically solving the system of EHJB equations using standard finite difference methods is extremely challenging due to the complex form of the system and a time-consuming iteration procedure. We circumvent the complications by proposing a numerical algorithm based on Wang & Forsyth (2011) and further prove that a solution of the algorithm is a solution of the system.

The main contributions of this chapter are summarized in the following four points. First, we propose a VA with a novel HWM fee structure and evaluate PH welfare with the VA. Second, we introduce an innovative quantitative measure to assess the impact of fee structure on VA marketability. Third, we derive a system of EHJB equations corresponding to an MV optimal stopping problem with two state variables over a random time horizon. Last, we propose a numerical algorithm to solve the system of EHJB equations and establish the connection between the algorithm and the system.

The remainder of this chapter is organized as follows. Section 3.2 formally introduces the VA with an HWM fee structure along with its risk-neutral pricing framework. Numerical examples of fair insurance fees under the HWM fee structure are also presented. For comparative purposes, we present the corresponding results for the constant and the state-dependent fee structures. Section 3.3 sets up an MV preference model to quantify the welfare effects of holding the VA with the HWM fee structure. Furthermore, the concept of compatible sets is introduced to measure the impact of a fee structure on the VA's marketability. A comparative analysis of the compatible sets of VAs with HWM, constant, and state-dependent fee structures is then conducted under various parameter settings. For completeness, we compare the PH's optimal surrender behavior for VAs with the aforementioned three fee structures. Section 3.4 concludes. A verification theorem of the system of EHJB equations corresponding to the MV optimal stopping problem, the algorithm, and proofs are relegated to Appendix A.

3.2 Valuation of Fair Fees

In this section, we give a detailed description of the VA policy under consideration and formulate its risk-neutral pricing model within a continuous-time stochastic optimal stopping framework. As a result, the fair fees of the VA policy under the HWM fee structure are obtained.

3.2.1 Guaranteed Benefits

We consider a VA policy with maturity $T > 0$. At inception of the contract, the PH pays an initial premium P which is invested in a predetermined basket of funds, namely the policy fund. The insurer sets up an investment account F to track the performance of the underlying policy fund with $F_0 = P$. The VA policy is assumed to have the following two embedded guarantees:

- **Guaranteed minimum death benefit (GMDB)**: in the event of PH death at time $t \in (0, T)$, the policy stipulates that the death benefit is the greater of the investment account F_t and guaranteed amount G_t;
- **Guaranteed minimum maturity benefit (GMMB)**: at policy maturity T, a payment equal to the greater of the investment account F_T and guaranteed amount G_T is made to the PH.

In what follows, the guaranteed amount is assumed to roll up continuously at a rate $g \geq 0$ before the contract matures, i.e.,

$$G_t = F_0 e^{gt}, \quad \text{for} \quad t \in [0, T]. \tag{3.1}$$

Note that the roll-up rate g should be no greater than the risk-free rate r to exclude arbitrage opportunities.

Meanwhile, the PH is allowed to surrender the VA at any time before maturity. If this surrender right is exercised, no guarantees are applicable and a penalty κ is levied on the balance of the investment account. Thus, upon surrender, the PH is entitled to receive $(1-\kappa_t)F_t$ at $t \in [0, T)$. To disincentivize early surrender behavior, the surrender penalty κ_t is usually assumed to decrease with time. Note that the VA under consideration does not allow for partial withdrawals from the investment account as guarantees. In other words, the only ways for the PH (or their beneficiary) to access funds in the VA are to hold the contract to maturity, pass away holding the contract, or fully surrender the contract.

3.2.2 HWM Fee Structure and Investment Account Dynamics

Suppose the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfies the usual conditions, where $\mathbb{Q}$ is an equivalent martingale measure. Under the $\mathbb{Q}$ measure, we assume that the policy fund value process S follows a geometric Brownian motion with dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}},$$

where $r > 0$, $\sigma > 0$ and $\{W^{\mathbb{Q}}\}_{t\ge 0}$ is a Brownian motion under $\mathbb{Q}$.

Under the HWM fee structure, the insurance fee has two components, namely a constant fee of rate $c \ge 0$ and an HWM fee of rate $\alpha \ge 0$. More precisely, the constant fee c is continuously charged as a fixed percentage of the investment account F whenever the investment account is lower than a certain predetermined threshold θ. Alternatively, the HWM fee α is levied as a proportion of the increase of the investment account's HWM only when the investment account is above the threshold θ [3]. In other words, the HWM fee α is not charged continuously, and for it to become effective, the investment account F has to be above the threshold θ and reach new record highs. In light of the above, under $\mathbb{Q}$, the dynamics of the investment account under the HWM fee structure is assumed to be

$$dF_t = (r - c\mathbb{1}_{\{F_t\le\theta\}})F_t dt + \sigma F_t dW_t^{\mathbb{Q}} - \alpha\mathbb{1}_{\{F_t\ge\theta\}}dM_t, \tag{3.2}$$

where

$$M_t = \sup_{0\le s\le t} F_s$$

[3]The HWM fee structure can be generalized to the case with two thresholds, for example θ_1 and θ_2. Only the constant fee c is effective when the investment account is lower than θ_1 and only the HWM fee is effective when the investment account reaches new record highs above θ_2.

denotes the HWM of the investment account at time t.

Note that the above HWM fee structure is to a certain extent related to the fee structure of some VAs in the market. Indeed, apart from insurance fees, it is also customary for insurers to charge performance-based fees for the underlying policy fund. These performance fees are normally higher for high returns of the policy fund and smaller for low returns (see, for instance, page 11 of Prudential (2020) for details), which resembles the mechanism of the HWM fee structure. Therefore, the HWM fee structure can be regarded as the union of "insurance fees" (c) and "performance fees" (α).

For comparative purposes, we also introduce the dynamics of the investment account under two other fee structures for VAs:

- A constant fee structure:

$$dF_t = (r - c)F_t dt + \sigma F_t dW_t^{\mathbb{Q}};$$

- A state-dependent fee structure (Bernard et al. (2014)):

$$dF_t = (r - c\mathbb{1}_{\{F_t \le \theta\}})F_t dt + \sigma F_t dW_t^{\mathbb{Q}}.$$

Note that, for notational convenience, we use the same parameter c to represent the constant fee in all three fee structures. However, under the risk-neutral pricing model, the parameter c in the HWM fee structure will be different from the ones in the constant or state-dependent fee structures.

A graphical illustration of the mechanism of the three fee structures is presented in Figure 3.1. Note that the HWM fee structure is a generalization of both the constant and state-dependent fee structures. Indeed, the HWM fee structure reduces to the state-dependent fee structure if $\alpha = 0$. Also, the constant fee structure can be viewed as a limiting case of the HWM fee structure when $\theta \to \infty$.

3.2.3 Mortality Model

From the inclusion of the GMDB, the insurer is exposed to mortality risk. Trading only stocks and bonds cannot fully replicate mortality-related insurance claims since mortality-based assets are scarce in the financial market (see Møller (1998) for more details). This gives rise to market incompleteness. To deal with this, it is common in the actuarial literature to make the following two assumptions: independence between financial market

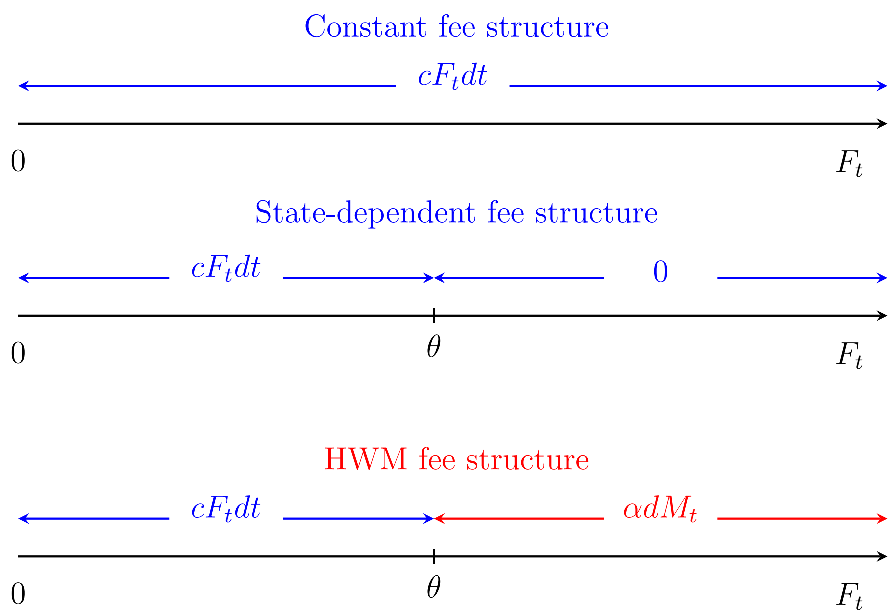

Figure 3.1: A graphical illustration of fee mechanisms for constant, state-dependent and HWM fee structures.

risk and mortality risk, and risk-neutrality of the insurer to mortality risk; see, for instance, Aase & Persson (1994) and Bauer et al. (2008). With these two assumptions, the mortality risk of each PH can be diversified by measuring the expected value of mortality-based claims for a group of insured lives. In this chapter, we abide by this convention for VA pricing. Hence, the risk-neutral measure of the financial and insurance markets combined is the product measure of $\mathbb{Q}$ and the physical measure of mortality risk. With slight abuse of notation, we henceforth denote this product measure by $\mathbb{Q}$.

For convenience, let ρ_x denote the future lifetime of an x-year-old PH with survival function

$${}_sp_x = \mathbb{P}(\rho_x > s) = e^{-\int_0^s \lambda_{x+u} du}, \quad \text{for} \quad s \geq 0,$$

where λ_{x+u} is the PH's force of mortality at age $x + u$. In what follows, we omit the subscript x of ρ_x for brevity. For illustrative purposes, we consider the same Makeham mortality model as in MacKay et al. (2017) i.e.,

$$\lambda_x = A + B \cdot C^x, \quad \text{for} \quad x > 0,$$

with $A = 0.0001$, $B = 0.00035$, and $C = 1.075$.

3.2.4 Fair Fees

To obtain the fair fees for the VA policy, we make use of its resemblance to American-type financial derivatives and apply the risk-neutral pricing approach. More specifically, within the risk-neutral pricing framework, the insurer is able to replicate every possible VA cash payout, and they assume a PH that maximizes the EPV of the future cash payouts for risk management purposes as this corresponds to the insurer's worst-case scenario from a liability standpoint (see Bauer et al. (2017) for more details). In light of the above, we formulate the pricing of the VA policy within a continuous-time stochastic optimal stopping framework where the PH is assumed to surrender the contract optimally to maximize the EPV of the future cash payouts.

For the HWM fee structure, the VA's cash payouts depend on the HWM of the investment account. Therefore, a state variable m, defined as

$$m := M_t = \sup_{0 \le s \le t} F_s,$$

is further added to preserve the Markov property. Denote the set of all stopping times τ valued in $[t, T]$ by $\mathcal{T}_{[t,T]}$, and let $\bar{\mathcal{O}}$ be the closure of the set $\mathcal{O} = \{(F, m) \in \mathbb{R}_+^2 : 0 < F < m\}$. The pricing value function V of the VA policy at time t over the domain $(t, F, m) \in [0, T] \times \bar{\mathcal{O}}$ is:

$$V(t, F, m) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}^{\mathbb{Q}}_{t,F,m} \left[\underbrace{{}_{\tau-t}p_{x+t}\psi(\tau, F_\tau) e^{-r(\tau-t)}}_{\text{(a)}} + \int_t^\tau \underbrace{{}_{s-t}p_{x+t}\lambda_{x+s} \max(F_s, G_s) e^{-r(s-t)}}_{\text{(b)}} ds \right], \tag{3.3}$$

where

$$\psi(t, F_t) = \begin{cases} (1 - \kappa_t) F_t, & 0 \le t < T, \\ \max(G_T, F_T), & t = T, \end{cases} \tag{3.4}$$

corresponds to the surrender value or the maturity payout of the VA policy, and G_s is the guaranteed amount defined in Eq. (3.1). Note that, in Eq. (3.3), (a) contains the surrender and maturity payouts while (b) contains the payout triggered by the PH's death. Define

$$\mathcal{L}V = (r - c\mathbb{1}_{\{F<\theta\}}) F V_F + \frac{1}{2}\sigma^2 F^2 V_{FF} - (\lambda_{x+t} + r) V,$$

where V_t, V_F, and V_{FF} are the first-order derivatives of V with respect to t and F, and the second-order derivative with respect to F, respectively. By the dynamic programming principle (as in Fleming & Soner (2006)), the pricing value function V over the domain $[0,T] \times \bar{\mathcal{O}}$ satisfies the following variational inequality

$$\min\left\{-V_t - \mathcal{L}V - \lambda_{x+t}\max\left(F,G\right), V-\psi\right\} = 0, \tag{3.5}$$

with boundary conditions

$$\begin{cases} V(T,F,m) = \max(F,G_T), & \text{for} \quad (F,m) \in [0,m]\times[0,\infty), \\ V_t\big|_{F=0} = (\lambda_{x+t}+r)V\big|_{F=0} - \lambda_{x+t}G, & \text{for} \quad (t,m)\in[0,T)\times[0,\infty), \\ \frac{\partial V}{\partial m}\big|_{F=m} = \alpha \mathbb{1}_{\{F\geq\theta\}}\frac{\partial V}{\partial F}\big|_{F=m}, & \text{for} \quad (t,m)\in[0,T)\times[0,\infty), \\ \lim_{m\to\infty} V(t,m,m) = \psi(t,m), & \text{for} \quad t\in[0,T). \end{cases} \tag{3.6}$$

The first boundary condition in Eq. (3.6) corresponds to the maturity payout of the VA policy (GMMB). The second boundary condition corresponds to the case in which the investment account is ruined, i.e. the degeneration of Eq. (3.5) when $F=0$. The third boundary condition reflects the continuity of the value function V before and after the investment account F adjusts for a new HWM. Heuristically, when the investment account is above the threshold θ and reaches a new HWM from m to $m+\Delta m$, a portion $\alpha\cdot\Delta m$ of the increase is deducted as the HWM fee, namely

$$V(t,m+\Delta m,m) = V(t, m+\Delta m \underbrace{-\alpha\mathbb{1}_{\{F\geq\theta\}}\cdot\Delta m}_{\text{HWM fee deduction}}, \underbrace{m+\Delta m}_{\text{HWM update}}). \tag{3.7}$$

Taking the limit as Δm approaches zero and applying Taylor's expansion on Eq. (3.7) yields the third boundary condition. A similar condition can also be found in Goetzmann et al. (2003), Panageas & Westerfield (2009), and Lan et al. (2013). The last boundary condition holds as it is optimal for the PH to surrender the contract for large investment account values when the HWM fee α is positive ($\alpha>0$)[4].

Note that state variables F and m in the partial differential equation (PDE) (3.5) are such that $F\leq m$, which makes PDE (3.5) undesirable to solve numerically. To *disentangle the dependence* between these two variables, we consider the following change of variable:

$$z = \frac{F}{m}, \tag{3.8}$$

[4] For the state-dependent fee structure ($\alpha=0$), it is optimal for the PH to hold the VA for large investment account values as suggested in MacKay et al. (2017). Therefore, the last boundary condition in Eq. (3.6) becomes $\lim_{m\to\infty} V(t,m,m) = m$.

where $z \in [0,1]$. Evidently, the term $1 - z$ represents a drawdown[5] measure of the investment account with respect to its HWM. Define a function

$$J(t, z, m) = V(t, F, m),$$

and it is not difficult to see that Eq. (3.5) becomes

$$\min\left\{-J_t - \mathcal{L}J - \lambda_{x+t}\max(mz, G), J - \psi(t, mz)\right\} = 0, \tag{3.9}$$

with boundary conditions

$$\begin{cases} J(T, z, m) = \max(zm, G), & \text{for } (z, m) \in [0,1] \times [0, \infty), \\ J_t\big|_{z=0} = (\lambda_{x+t} + r)J\big|_{z=0} - \lambda_{x+t}G, & \text{for } (t, m) \in [0, T) \times [0, \infty), \\ \frac{\partial J}{\partial m}\big|_{z=1} = \frac{\alpha \mathbb{1}_{\{m \geq \theta\}} + 1}{m}\frac{\partial J}{\partial z}\big|_{z=1}, & \text{for } (t, m) \in [0, T) \times [0, \infty), \\ \lim_{m \to \infty} J(t, 1, m) = \psi(t, m), & \text{for } t \in [0, T), \end{cases} \tag{3.10}$$

where $\mathcal{L}J = (r - c\mathbb{1}_{\{mz<\theta\}})zJ_z + \frac{1}{2}\sigma^2 z^2 J_{zz} - (\lambda_{x+t} + r)J$. We propose numerically solving PDE (3.9) with boundary conditions (3.10) to obtain the fair fees for the VA under the HWM fee structure.

For a VA policy, a fair fee is one that makes the EPV of the cash inflows and outflows perfectly balanced. Specifically, we present the following definition of fair insurance fees for a VA with an HWM fee structure.

Definition 3. *For a VA with the HWM fee structure, $(c, \alpha) \in [0,1]^2$ is called a pair of fair fees if it satisfies*

$$V(0, F_0, F_0; c, \alpha) = F_0. \tag{3.11}$$

To determine a pair of fair insurance fees (c, α) under the HWM fee structure, we fix the HWM fee $\alpha \in [0, 1]$ and derive the state-dependent constant fee $c \in [0, 1]$ (if it exists) such that Eq. (3.11) is satisfied. The parameter inputs are as specified in Table 3.1. Note that the roll-up rate g is zero, which means the guarantees are effectively return-of-initial-premium ($G_t = F_0$ for $t \in [0, T]$). The threshold is set to be $\theta = 125$ in the following numerical analysis but other values could have as easily been considered. The surrender penalty is chosen to be a decreasing smooth function to generate smooth

[5]Drawdown is a risk metric measuring the magnitude of the decline in portfolio value relative to its historic HWM.

Description	Parameter	Base Case	Sensitivity Analysis
Contract length	T	10	
PH's age	x	60	
Initial premium	F_0	100	
Roll-up rate	g	0	
Threshold for HWM fee	θ	125	
Surrender penalty	κ_t	$0.05 \times (1 - \frac{t}{T})^3$	
Interest rate	r	2%	1%, 3%
Expected return	μ	8%	10%
Volatility	σ	15%	20%

Table 3.1: Parameter inputs

surrender regions. The same surrender penalty function is also applied in MacKay et al. (2017) as it simulates market surrender penalties which start high and dip rapidly in the later years of the contract. For parameters r, μ, and σ, we conduct sensitivity tests in addition to the base case to comprehensively investigate their impact on the fair fees and the PH's valuation of the VA contract.

A list of fair fees under different fee structures with various risk-free rates is summarized in Table 3.2: where "Constant" corresponds to the constant fee structure and "SD" to the state-dependent fee structure (the case $\alpha = 0$). We also consider four other levels of HWM fee α, namely 0.05, 0.2, 0.35 and 0.5, and derive the corresponding constant fee c. As expected, we first observe that the constant fee c decreases as the HWM fee α increases for all three risk-free rates r and two different volatilities σ. This is immediate from Definition 3. Now focusing on the risk-free rate effect, we note that, for a given HWM fee α, the fair constant fee c decreases from the case $r = 1\%$ to $r = 3\%$. This is intuitive as the value of these guarantees tends to decrease with the interest rate. Lastly, for the same HWM fee α, higher volatility corresponds to a higher constant fee c (all else being equal). This confirms the intuition that more fees are needed to finance the guarantees in more volatile

	$r = 1\%$		$r = 2\%$		$r = 3\%$	
Fee Structure	$\sigma = 15\%$	$\sigma = 20\%$	$\sigma = 15\%$	$\sigma = 20\%$	$\sigma = 15\%$	$\sigma = 20\%$
Constant	0.0584	0.1008	0.0290	0.0541	0.0163	0.0332
SD ($\alpha = 0$)	0.0593	0.1028	0.0320	0.0587	0.0203	0.0400
$\alpha = 0.05$	0.0587	0.1015	0.0298	0.0549	0.0174	0.0352
$\alpha = 0.2$	0.0582	0.1003	0.0281	0.0525	0.0142	0.0304
$\alpha = 0.35$	0.0579	0.0994	0.0275	0.0513	0.0133	0.0287
$\alpha = 0.5$	0.0576	0.0988	0.0271	0.0504	0.0126	0.0275

Table 3.2: Fair constant fees c under different fee structures. "Constant" corresponds to the constant fee structure and "SD" to the state-dependent fee structure (the case $\alpha = 0$). We also consider four other levels of HWM fee α, namely 0.05, 0.2, 0.35 and 0.5. Observe that the constant fee c decreases as the HWM fee α increases for all cases. Also, a higher risk-free rate corresponds to a lower constant fee and higher volatility corresponds to a higher constant fee.

market conditions.

3.3 Policyholder's Welfare

In this section, we examine PH welfare from holding the VA with an HWM fee structure under an MV preference model in a potentially incomplete market. For comparative purposes, we also consider PH welfare under VAs with constant and state-dependent fee structures. Note that we assume the insurer still prices the VA using risk-neutral pricing and the fair fees are as given in Section 3.2 when measuring PH welfare. In contrast to the insurer, it is appropriate to assume the PH faces an incomplete market as they might not be able replicate every VA payout (e.g., they cannot replicate mortality-based claims through diversification).

3.3.1 Model Setup

In what follows, we formulate the MV preference of the PH to quantify the welfare from holding the VA with an HWM fee structure. Specifically, the PH is assumed to adopt an MV objective function with a state-dependent risk aversion function.

Suppose that the dynamics of the underlying policy fund under a physical probability measure $\mathbb{P}$ follows a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3.12}$$

where $\{W_t\}_{\{t\geq 0\}}$ is a Brownian motion under measure $\mathbb{P}$. As mentioned above, the PH is assumed to optimally surrender the VA contract to maximize the following MV objective function with a state-dependent risk aversion function:

$$V(t,F,m) = \sup_{\tau\in\mathcal{T}_{[t,T]}} \left\{ \mathbb{E}_{t,F,m}\left[H(F_{\tau\wedge\rho})\middle|\rho > t\right] - \frac{\gamma(F)}{2}\mathrm{Var}_{t,F,m}\left[H(F_{\tau\wedge\rho})\middle|\rho > t\right] \right\} \tag{3.13}$$

for $(t,F,m) \in [0,T]\times\bar{\mathcal{O}}$, where H corresponds to the VA payout defined as

$$H(F_{\tau\wedge\rho}) = \begin{cases} (1-\kappa_\tau)F_\tau, & \tau < \rho\wedge T, \\ \max(G_\rho, F_\rho), & \rho < \tau\wedge T, \\ \max(G_T, F_T), & T < \tau\wedge\rho, \end{cases} \tag{3.14}$$

$\mathcal{T}_{[t,T]}$ denotes the set of all stopping times τ valued in $[t,T]$, ρ stands for the future lifetime of an x-year-old PH, and the risk aversion function $\gamma(F)$ [6] is given as

$$\gamma(F) = \frac{\gamma}{\max(F,G)}. \tag{3.15}$$

Note that the risk aversion function $\gamma(F)$ can be regarded as a measure of PH aversion to the variance of the VA payout. The choice of state-dependent risk aversion ensures that good performances of the investment account are penalized less than poor performances,

[6] The state term $\max(F,G)$ in Eq. (3.15) can be replaced by F or $(1-\kappa)F$ with little change to the results. This is because it is not optimal for the PH to surrender if the fund value is lower than the guaranteed amount, and the probability of early surrender is accordingly very small (see Section 3.3.4). Therefore, the state term $\max(F,G)$ can be interpreted as a simple approximation of the real-time value of the VA.

which is an improvement on the standard constant risk aversion MV preference model with symmetric risk penalty. As can be seen from Eq. (3.15), higher investment account values reduce penalties associated with the variance term in the MV objective function (3.13). Moreover, the functional form of $\gamma(F)$ tailors to the VA with embedded guarantees. State-dependent risk aversion also guarantees the value function is state-homogeneous, namely $V(0, F_0, F_0) = F_0 \cdot V(0, 1, 1)$, a property that further ensures the comparison between the VA and aforementioned alternative investments is robust with respect to the initial investment F_0 (for a detailed discussion, see Appendix A.3). Interested readers are referred to see Björk et al. (2014) for a more comprehensive discussion on the MV preference model with state-dependent risk aversion.

Note that a stochastic optimal stopping problem can be viewed as an optimal control problem with binary controls (see, e.g., Tan et al. (2018) and Ebert et al. (2020)). Therefore, we can regard our optimal stopping problem as an optimal control problem by introducing the following stopping rule.

Definition 4. *A stopping rule is a measurable function $u : [0, T] \times \bar{\mathcal{O}} \to \{0, 1\}$ where 0 corresponds to continuation and 1 corresponds to stopping.*

More specifically, our setup's continuation and stopping regions at time $t \in [0, T)$ are defined as

$$\begin{cases} \mathcal{C}_t = \{(F, m) \in \bar{\mathcal{O}} : u(t, F, m) = 0\}, \\ \mathcal{S}_t = \{(F, m) \in \bar{\mathcal{O}} : u(t, F, m) = 1\}, \end{cases}$$

with corresponding stopping time τ^u defined as

$$\tau^u = \inf\{s \geq t : u(s, F_s, M_s) = 1\}.$$

In light of the above, the optimal value function in Eq. (3.13) can be rewritten as

$$\begin{aligned} V(t, F, m) &= \sup_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \left\{ \mathbb{E}_{t,F,m} \left[H\left(F_{\tau^{\boldsymbol{u}} \wedge \rho}\right) \middle| \rho > t \right] - \frac{\gamma(F)}{2} \mathrm{Var}_{t,F,m} \left[H\left(F_{\tau^{\boldsymbol{u}} \wedge \rho}\right) \middle| \rho > t \right] \right\} \\ &:= \sup_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \mathcal{J}\left(t, F, m; \tau^{\boldsymbol{u}}\right), \end{aligned} \tag{3.16}$$

where $\boldsymbol{\mathcal{A}}$ is the admissible set of stopping rules.

One of the major intricacies of the stochastic MV optimization problem (3.16) is the well-known issue of *time inconsistency* (see, e.g., Basak & Chabakauri (2010), and Björk & Murgoci (2014)). To deal with this, we adopt the game theoretic approach proposed in

Björk & Murgoci (2010) and derive an equilibrium stopping strategy by solving a system of extended Hamilton-Jacobi-Bellman (EHJB) equations. Nevertheless, deriving the system of EHJB equations is much more complicated in this case than a standard MV optimization problem since the MV optimization problem (3.16) has two state variables (F and m) with a random time horizon (due to the PH's future lifetime ρ).

To derive the system of EHJB equations, we next define an equilibrium stopping rule over a random horizon by extending the definition introduced in Landriault et al. (2018).

Definition 5. *For a fixed point $(t, F, m) \in [0, T) \times \bar{\mathcal{O}}$, a small $\varepsilon > 0$, and an admissible stopping rule $\hat{\boldsymbol{u}} = \{\hat{u}_s\}_{s\in[t,T)}$ conditional on $\rho > t$, define a stopping rule $\boldsymbol{u}^\varepsilon$ by*

$$\boldsymbol{u}^\varepsilon(s, y, z) = \begin{cases} u \in \{0, 1\}, & \text{for } \; t \leq s < (t + \varepsilon) \wedge \rho, \\ \hat{\boldsymbol{u}}(s, y, z), & \text{for } \; (t + \varepsilon) \wedge \rho \leq s < T \wedge \rho, \end{cases}$$

in which $(y, z) \in [0, m] \times [0, \infty)$. If

$$\liminf_{\varepsilon \downarrow 0} \frac{\mathcal{J}(t, F, m; \tau^{\hat{\boldsymbol{u}}}) - \mathcal{J}(t, F, m; \tau^{\boldsymbol{u}^\varepsilon})}{\varepsilon} \geq 0$$

for all $(t, F, m) \in [0, T) \times \bar{\mathcal{O}}$, then $\hat{\boldsymbol{u}}$ is called an equilibrium feedback stopping rule and the corresponding equilibrium value function V is given by $V(t, F, m) = \mathcal{J}(t, F, m; \tau^{\hat{\boldsymbol{u}}})$.

Via a formal verification theorem, which can be found in Appendix A.1, we show that the PH value function (a.k.a. optimal welfare) $V(t, F, m)$ can be determined by solving the system of EHJB equations (A.1)-(A.7). We resort to numerically solving this system of EHJB equations which corresponds to an optimal stopping problem with a finite horizon. We point out that computing numerical solutions for the system of EHJB equations is a challenging task. Indeed, the system of equations displays a highly complex form and involves three state variables (the additional state variable stems from the state-dependent risk aversion function). In this case, standard finite difference methods would be extremely time-consuming to determine an equilibrium stopping strategy.

To circumvent these difficulties in solving the system of EHJB equations, we develop an efficient algorithm based on Wang & Forsyth (2011) tailored to MV optimal control problems. We additionally go one step further by formally establishing a connection between the system of EHJB equations (A.1)-(A.7) and the algorithm. More specifically, we show that a solution derived by this algorithm is indeed a solution to the system of EHJB equations (A.1)-(A.7). The details of the algorithm and the proof of this connection are relegated to Appendix A.2.

3.3.2 Compatible Set of Risk Aversions

In this section, we evaluate the welfare effects of holding a VA with different fee structures in comparison to holding two alternative investment options, namely a *risk-free bond* and a *pure fund*. In line with the MV preference model with state-dependent risk aversion given by Eq. (3.13), PH welfare from holding the risk-free bond at time $t \in [0, T]$ is given by

$$V_{\mathrm{RFB}}(t, R) = \mathbb{E}\left[R \cdot (1+r)^{(T\wedge\rho)-t} \middle| \rho > t\right] - \frac{\gamma}{2R}\mathrm{Var}\left[R \cdot (1+r)^{(T\wedge\rho)-t} \middle| \rho > t\right], \quad R \in [0, \infty), \tag{3.17}$$

where r is the risk-free rate, and R denotes the amount of money invested in the risk-free bond at time t. Similarly, the welfare effect of holding the pure fund[7] at time $t \in [0, T]$ is given by

$$V_{\mathrm{PF}}(t, S) = \mathbb{E}_{t,S}\left[S_{T\wedge\rho} \middle| \rho > t\right] - \frac{\gamma}{2S}\mathrm{Var}_{t,S}\left[S_{T\wedge\rho} \middle| \rho > t\right], \quad S \in [0, \infty), \tag{3.18}$$

where S is the pure fund value at time t, and its value process is assumed to follow the same dynamics as the policy fund given by Eq. (3.12). Note that, for proper comparison, we consider the *same investment horizon* $T \wedge \rho$ for the alternative investments as with the VA. Similarly, we impose the same amount of initial input for each of the VA and the alternative investments, i.e., $F_0 = R_0 = S_0$.

Figure 3.2 presents PH welfare at time $t = 0$ from holding the VA under the constant and the state-dependent fee structures and compares them with that of holding the risk-free bond and the pure fund. The left panel is the constant fee structure case where "VA (Constant Fee)" corresponds to PH welfare under the constant fee structure. The right panel is the state-dependent fee structure case where "VA (SD $\alpha = 0$)" represents PH welfare under the state-dependent fee structure. The parameter inputs are those of the base case scenario in Table 3.1.

From the left panel, it is evident that, under the constant fee structure, PHs whose levels of risk aversion γ are in a region of approximately $[1.33, 1.7]$ would find the VA generating higher welfare than both of the alternative investments. In other words, a rational PH in this region should prefer the VA to the alternative investments. This observation offers one a rough idea of the VA's target clientele, namely PHs who are sufficiently risk-averse ($\gamma \geq 1.33$) to prefer the VA over the pure fund but are not too risk-averse ($\gamma \leq 1.7$) to choose the risk-free bond over the VA.

[7] The pure fund can be regarded as a mutual fund that invests in a portfolio of both risky and risk-free assets.

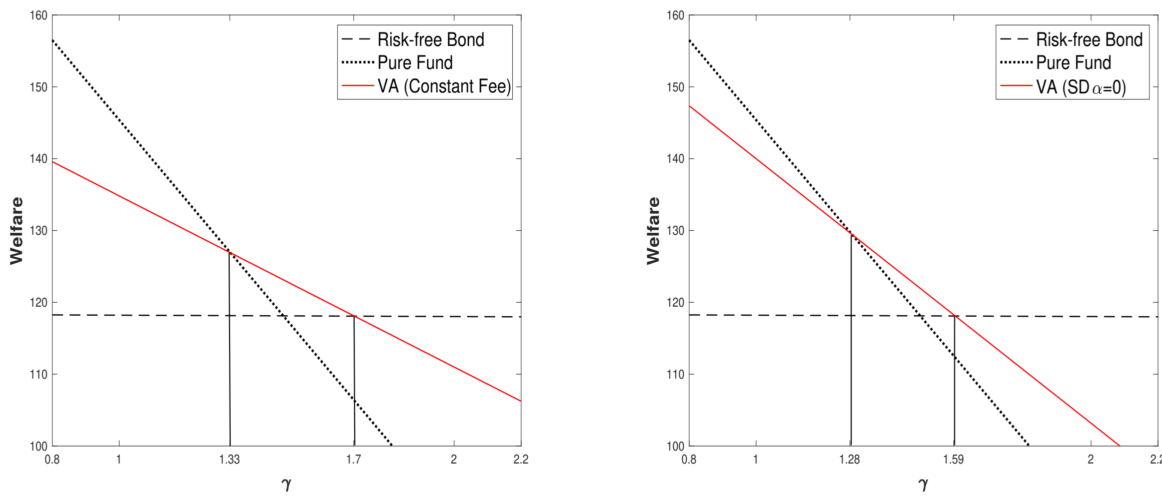

Figure 3.2: PH time-0 welfare of holding a risk-free bond, pure fund, or VA under constant or state-dependent fee structures in the base case parameter setting. The left panel is the constant fee structure case, and the right panel is the state-dependent fee structure case. Observe that the VA with the constant fee structure is preferable to the alternative investments for PHs with levels of risk aversion within the set $[1.33, 1.7]$. With the state-dependent fee structure, the set is $[1.28, 1.59]$.

From the right panel, we see that PHs with a level of risk aversion in the set $[1.28, 1.59]$ would prefer the VA with the state-dependent fee structure, over the two alternative investments. The set of risk aversions for the state-dependent fee structure is narrower than that of using the constant fee structure (of size $1.59 - 1.28 = 0.31$ vs. $1.7 - 1.33 = 0.37$, respectively), meaning that the state-dependent fee structure may even impair the VA's marketability. Further, we observe that the set for the state-dependent fee structure is located to the left of that of the constant fee structure (in the vicinity of $[1.28, 1.59]$ vs. $[1.33, 1.7]$, respectively). This indicates that the VA's target clientele becomes less risk-averse under the state-dependent fee structure. One can attribute these two observations to the inherent design of the state-dependent fee structure. Specifically, the fee structure charges more insurance fees for low values of the investment account and none for high values. This mechanism largely increases the uncertainty of the VA payout as the investment account is more likely to achieve extreme highs or extreme lows, consequently making the VA a less desirable investment for more risk-averse PHs and impeding the VA's marketability.

In light of the above discussion, we formally propose a novel concept called *compatible set of risk aversions* (compatible set, for short), to quantify the marketability of various VA fee structures.

Definition 6. *The compatible set of a given fee structure is a collection of risk aversions such that, for any γ within this set, the PH time-0 welfare of holding the VA is higher than that of holding either the risk-free bond or the pure fund, i.e.*

$$\Big\{\gamma \geq 0 : V(0, F_0, F_0; \gamma) \geq \max\big(V_{RFB}(0, R_0; \gamma), V_{PF}(0, S_0; \gamma)\big)\Big\}.$$

First, a compatible set, if not empty, should be of an *interval form.* That is, holding a pure fund offers a lower bound for the compatible set, while holding a risk-free bond offers an upper bound. On the one hand, a PH with a sufficiently high level of risk aversion (i.e., a large γ) would prefer the risk-free bond to the VA since the VA is essentially a risky investment. On the other hand, a PH with a low level of risk aversion (i.e., a small γ) would choose the pure fund over the VA because the latter charges fees which lower its expected return rate.

Second, the compatible set offers a benchmark to measure the marketability of the VA under different fee structures. A wider compatible set implies that the corresponding fee structure makes the VA more appealing to a broader range of PHs with various risk appetites.

Third, the compatible set also helps one locate the VA's target clientele, which in turn, should help to provide a better understanding of the VA market. Indeed, as the compatible set provides an interval of risk aversions γ where the VA is preferred over the alternative investments, it clearly reveals information about the risk appetite of VA-holding PHs. As will be shown later, we can also discover some interesting implications on VA marketability based on changes in the compatible set's location and width when market conditions change.

Lastly, results on the compatible set are *independent* of the scale of the initial investment F_0 due to the choice of the state-dependent MV model. We formally establish this statement via Theorem 9, which we relegate to Appendix A.3.

Through the lens of the compatible set of risk aversions, we now investigate how the inclusion of the HWM fee α affects the marketability of the VA. In Figure 3.3, we present PH welfare from holding the VA under three different fee structures as well as from holding the two alternative investments. The parameters are those of the base case scenario. We

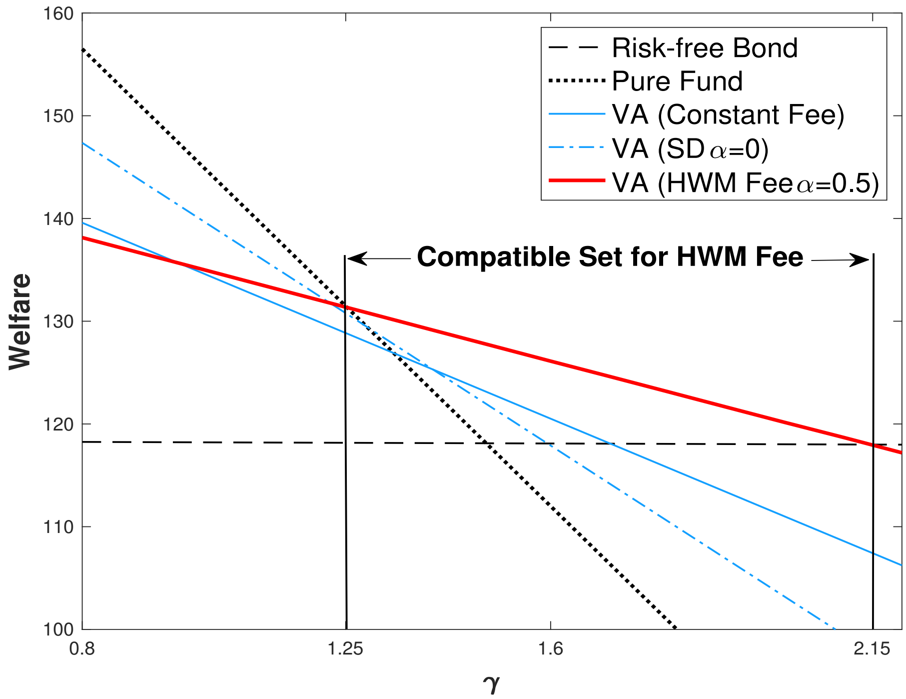

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Figure 3.3: PH time-0 welfare of holding a risk-free bond, pure fund, or VA under different fee structures in the base case parameter setting. "VA (HWM Fee $\alpha = 0.5$)" corresponds to PH time-0 welfare under the HWM fee structure with $\alpha = 0.5$. We can observe that the HWM fee structure records a wider compatible set than the other two fee structures.

observe that the HWM fee structure with $\alpha = 0.5$ records a noticeably wider compatible set (roughly $\gamma \in [1.25, 2.15]$) than either the constant or state-dependent fee structures. And this set fully covers the ones generated by the other two fee structures. The main reason behind this observation rests on the mechanism of the fee structure. Specifically, the HWM fee structure purposefully contains the growth rate of the investment account to aid in its recovery from poor performance through a lowered state-dependent constant fee when the value of the investment account is low. Hence, the HWM fee structure contains the volatility of the investment account, and as such, reduces the uncertainty of the VA payout, a virtue for risk-averse PHs. In line with the previous discussion, a wider compatible set implies that the introduction of an HWM fee does improve VA marketability.

Fee Structure	Location	Width
Constant	$[1.3322, 1.7022]$	0.3700
SD ($\alpha = 0$)	$[1.2862, 1.5946]$	0.3084
$\alpha = 0.05$	$[1.2302, 1.6590]$	0.4288
$\alpha = 0.2$	$[1.2185, 1.8096]$	0.5911
$\alpha = 0.35$	$[1.2334, 1.9718]$	0.7384
$\alpha = 0.5$	$[\mathbf{1.2525}, \mathbf{2.1500}]$	**0.8975**

Table 3.3: Compatible sets for different fee structures. This table presents the location and width of compatible sets for different fee structures in the base case parameter setting. Observe that a HWM fee of $\alpha = 0.5$ produces the widest compatible set.

Moreover, from Figure 3.3, we notice that the state-dependent fee structure displays a shortcoming called *risk-aversion mismatch*. We observe that the state-dependent fee structure generates higher welfare than the constant and HWM fee structures for PHs with a level of risk aversion γ approximately below 1.2. This means that the state-dependent fee structure's target clientele is less risk-averse. Nevertheless, the pure fund produces higher welfare than the VA in this region. In other words, PHs would choose the pure fund over the VA with the state-dependent fee structure.

Table 3.3 displays the specific location and width of compatible sets for different levels of HWM fees and compares them with those of the constant and state-dependent fee structures. Again, the parameters are as given by the base case. Comparing the width of compatible sets with those of the constant and state-dependent fee structures, we find that the HWM fee structure makes the VA the preferred option over the alternative investments for a broader range of PHs. Furthermore, we observe that increasing the HWM fee α increases the width of the compatible set and all four levels of HWM fee α generate a compatible set that fully covers the ones corresponding to the other two fee structures. However, we stress that this relationship does not hold in general (as will be shown in the sensitivity analysis).

From the numerical examples above, we observe that all fee structures produce a non-

			Welfare		
Risk Aversion	Compatible α	α^*	α^*	Constant	SD ($\alpha = 0$)
$\gamma = 1.3$	$[0, 0.5]$	0.2	**131.13**	127.66	128.95
$\gamma = 1.4$	$[0, 0.5]$	0.35	**129.14**	125.28	125.26
$\gamma = 1.5$	$[0, 0.5]$	0.5	**127.62**	122.90	121.58
$\gamma = 1.6$	$[0.05, 0.5]$	0.5	**126.11**	120.51	117.90

Table 3.4: Optimal HWM fee and welfare comparison. This table summarizes sets of the HWM fees α (i.e., "Compatible α") that make given levels of risk aversion γ belong to the corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare in the base case parameter setting. We observe that the HWM fee structure generates the highest welfare for the PH when the VA is preferred to the alternative investments under all three fee structures.

empty compatible set. Therefore, it is also interesting to search for the optimal fee structure in terms of PH welfare. Table 3.4 presents sets of HWM fees α that make given levels of risk aversion γ belong to the corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare under the base case parameter inputs. It is apparent that, when the VA is preferable to the alternative investments (i.e., PH level of risk aversion γ belongs to the compatible set) under more than one fee structure (including the HWM fee structure), the HWM fee structure generates the highest welfare for the PH. Also, we see that PHs with a higher level of risk aversion prefer a higher HWM fee α, which is not surprising as a higher HWM fee further reduces the uncertainty of VA payout.

3.3.3 Sensitivity Analysis

In this section, we conduct a sensitivity analysis of the results of Section 3.3.2 under different parameter settings. Specifically, we first examine the impact of the risk-free rate r on the compatible sets for the VA under different fee structures (Table 3.5). We then investigate how the performance of the underlying policy fund (μ and σ) affects the

	$r = 1\%$		$r = 2\%$		$r = 3\%$	
Fee Structure	Location	Width	Location	Width	Location	Width
Constant	$\varnothing$	0	$[1.3322, 1.7022]$	0.3700	$[1.0511, 1.5295]$	0.4784
SD ($\alpha = 0$)	$\varnothing$	0	$[1.2862, 1.5946]$	0.3084	$[0.9151, 1.4990]$	0.5839
$\alpha = 0.05$	$\varnothing$	0	$[1.2302, 1.6590]$	0.4288	$[0.9077, 1.5565]$	0.6488
$\alpha = 0.2$	$[1.6365, 1.6904]$	0.0539	$[1.2185, 1.8096]$	0.5911	$[0.9887, 1.6701]$	0.6814
$\alpha = 0.35$	$[1.5656, 1.8691]$	0.3035	$[1.2334, 1.9718]$	0.7384	$[1.0568, 1.7667]$	0.7099
$\alpha = 0.5$	$[\mathbf{1.5350}, \mathbf{2.0655}]$	**0.5305**	$[\mathbf{1.2525}, \mathbf{2.1500}]$	**0.8975**	$[\mathbf{1.1055}, \mathbf{1.8572}]$	**0.7517**

Table 3.5: Compatible sets for different risk-free rates. This table summarizes the location and width of compatible sets under different fee structures for different risk-free rates r. Observe that only the HWM fee structure generates a non-empty compatible set when the risk-free rate is low (i.e., $r = 1\%$), and the HWM fee structure still has the widest compatible set.

corresponding results (Table 3.6). Lastly, we determine the optimal fee structure in terms of PH welfare from holding the VA under different parameter settings (Table 3.7). The aim of this section is to explore how introducing the HWM fee affects VA marketability in various market conditions.

Table 3.5 presents the compatible sets for the VA with different fee structures under different risk-free rates. All other parameters are as given in the base case.

First, consistent with Section 3.3.2, we observe that the HWM fee structure generates the widest compatible set under all three risk-free rates (in comparison to the constant and state-dependent fee structures). Second, when the risk-free rate decreases from the base case (i.e., when r decreases from 2% to, say, 1%), all compatible sets shrink, and only the HWM fee structure generates a noticeably non-empty compatible set. Not surprisingly, in a low interest rate market, "savings instruments" such as annuities and VAs become less appealing to investors. The mechanism of the HWM fee structure is closer to the fee scheme adopted in risky fund investment (e.g. mutual funds or hedge funds), and hence can be helpful for improving VA marketability in the current low risk-free rate environment.

	$(\mu,\sigma)=(8\%,20\%)$		$(\mu,\sigma)=(8\%,15\%)$		$(\mu,\sigma)=(10\%,15\%)$	
Fee Structure	Location	Width	Location	Width	Location	Width
Constant	$[0.7726, 0.9787]$	0.2061	$[1.3322, 1.7022]$	0.3700	$[1.0900, 1.9459]$	0.8559
SD ($\alpha = 0$)	$[0.6445, 1.0638]$	0.4193	$[1.2862, 1.5946]$	0.3084	$[0.8780, 1.8715]$	0.9935
$\alpha = 0.05$	$[\mathbf{0.6427}, \mathbf{1.0950}]$	**0.4523**	$[1.2302, 1.6590]$	0.4288	$[0.8981, 1.9274]$	1.0293
$\alpha = 0.2$	$[0.6843, 1.1201]$	0.4358	$[1.2185, 1.8096]$	0.5911	$[0.9896, 2.0658]$	1.0762
$\alpha = 0.35$	$[0.7151, 1.1453]$	0.4302	$[1.2334, 1.9718]$	0.7384	$[1.0492, 2.2422]$	1.1930
$\alpha = 0.5$	$[0.7371, 1.1731]$	0.4360	$[\mathbf{1.2525}, \mathbf{2.1500}]$	**0.8975**	$[\mathbf{1.0898}, \mathbf{2.4613}]$	**1.3715**

Table 3.6: Compatible sets for different performance of the policy fund. This table summarizes the location and width of compatible sets of the VA with different fee structures for different performance of the underlying policy fund. For a more volatile policy fund, the compatible set shifts to the left and a HWM fee of 0.05 **generates the widest set. For a policy fund with a higher return, compatible sets become wider.**

Third, as the risk-free rate increases, the compatible set shifts to the left (compare, e.g., columns $r = 2\%$ and $r = 3\%$). When the risk-free rate is larger, the VA, as a risky asset, is only preferable for PHs who are less risk-averse, since its risk-free alternative could generate higher welfare. Moreover, the compatible set also shrinks when the risk-free rate r increases from the base case. This can be attributed to the fact that only a smaller group of PHs would still prefer holding the VA, given that the opportunity cost of holding the VA increases as holding the risk-free bond generates higher returns.

To investigate how the performance of the underlying policy fund affects VA marketability, Table 3.6 provides the compatible sets (of the VA with different fee structures) under different expected returns and volatilities of the underlying policy fund. We compare the base case with a case of higher volatility (i.e., $\sigma = 20\%$, implying a lower Sharpe ratio) and a case of higher expected return (i.e., $\mu = 10\%$, implying a higher Sharpe ratio).

We first compare the base case ($\mu = 8\%$, $\sigma = 15\%$) with the case of a more volatile policy fund ($\mu = 8\%$, $\sigma = 20\%$). First, for a given fee structure, the compatible set shifts to the left when the policy fund is more volatile. With higher volatility σ, VA payout becomes more uncertain. Hence, for a PH with a given risk appetite, the VA becomes less favorable against the risk-free bond, and compared to the base case, only PHs with lower risk aversion would prefer the VA. Second, we observe that the widest compatible set appears at a low HWM fee of $\alpha = 0.05$ when the policy fund is more volatile. Notice that this is different from the base case (see Table 3.3), where a higher HWM fee α corresponds to a wider compatible set. This can be explained as follows: For a highly volatile policy fund, only PHs with low risk aversion would prefer the VA over the risk-free bond, and these PHs would prefer the state-dependent fee structure ($\alpha = 0$) or a low HWM fee α as they generate higher rates of return.

Next, we compare the base case ($\mu = 8\%$, $\sigma = 15\%$) with the case where the policy fund has a higher expected return ($\mu = 10\%$, $\sigma = 15\%$). We see that every fee structure has a compatible set that fully covers that of the base case, where the policy fund's expected return μ is higher. This is not surprising as the VA becomes more favorable when the policy fund performs better.

Another observation from Table 3.6 worth mentioning is that, for a policy fund with either higher volatility σ or higher expected return μ, the state-dependent fee structure generates a compatible set with larger width than the constant fee structure, which is different from the base case where the opposite conclusion was observed (see Table 3.3). The reason is that, with either a more volatile or a more performant policy fund, the investment account is more likely to grow to exceed the threshold θ beyond which no insurance fee is charged under the state-dependent fee structure. Therefore, in these two cases, the state-dependent fee structure can be seen as improving VA marketability from the constant fee structure. However, the improvement is relatively marginal and the compatible set of the state-dependent fee structure is still narrower than that of the HWM fee structure.

From the previous numerical analysis, we see that all fee structures generate a non-empty compatible set except for $r = 1\%$. Therefore, it is interesting to search for the optimal fee structure in terms of PH welfare in the other cases (higher interest $r = 3\%$, higher volatility $\sigma = 20\%$, and higher return $\mu = 10\%$). Table 3.7 presents sets of HWM fees α that make given levels of risk aversion γ belong to corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare. Consistent with the base case in Table 3.4, we observe that the HWM fee structure generates the highest PH welfare under various levels of risk aversion.

				Welfare		
	Risk Aversion	Compatible α	α^*	α^*	Constant	SD ($\alpha = 0$)
$r = 3\%$	$\gamma = 1.2$	$[0, 0.5]$	0.05	**140.40**	138.03	139.36
	$\gamma = 1.4$	$[0, 0.5]$	0.35	**135.32**	131.99	131.82
	$\gamma = 1.6$	$[0.2, 0.5]$	0.5	**131.86**	125.95	124.29
$\sigma = 20\%$	$\gamma = 0.7$	$[0, 0.2]$	0.05	**133.45**	123.96	133.19
	$\gamma = 0.8$	$[0, 0.5]$	0.05	**129.58**	121.89	129.07
	$\gamma = 0.9$	$[0, 0.5]$	0.05	**125.72**	119.83	124.95
$\mu = 10\%$	$\gamma = 1.2$	$[0, 0.5]$	0.05	**150.20**	143.33	149.96
	$\gamma = 1.5$	$[0, 0.5]$	0.5	**138.33**	133.15	135.70
	$\gamma = 1.8$	$[0, 0.5]$	0.5	**131.94**	122.98	121.45

Table 3.7: Optimal HWM fee and welfare comparison under various market conditions. This table summarizes sets of HWM fees α that make given levels of risk aversion γ belong to corresponding compatible sets as well as the optimal HWM fee α^* that maximizes PH welfare under various market conditions. Observe that the HWM fee structure generates the highest PH welfare under various levels of risk aversion (in the corresponding compatible sets).

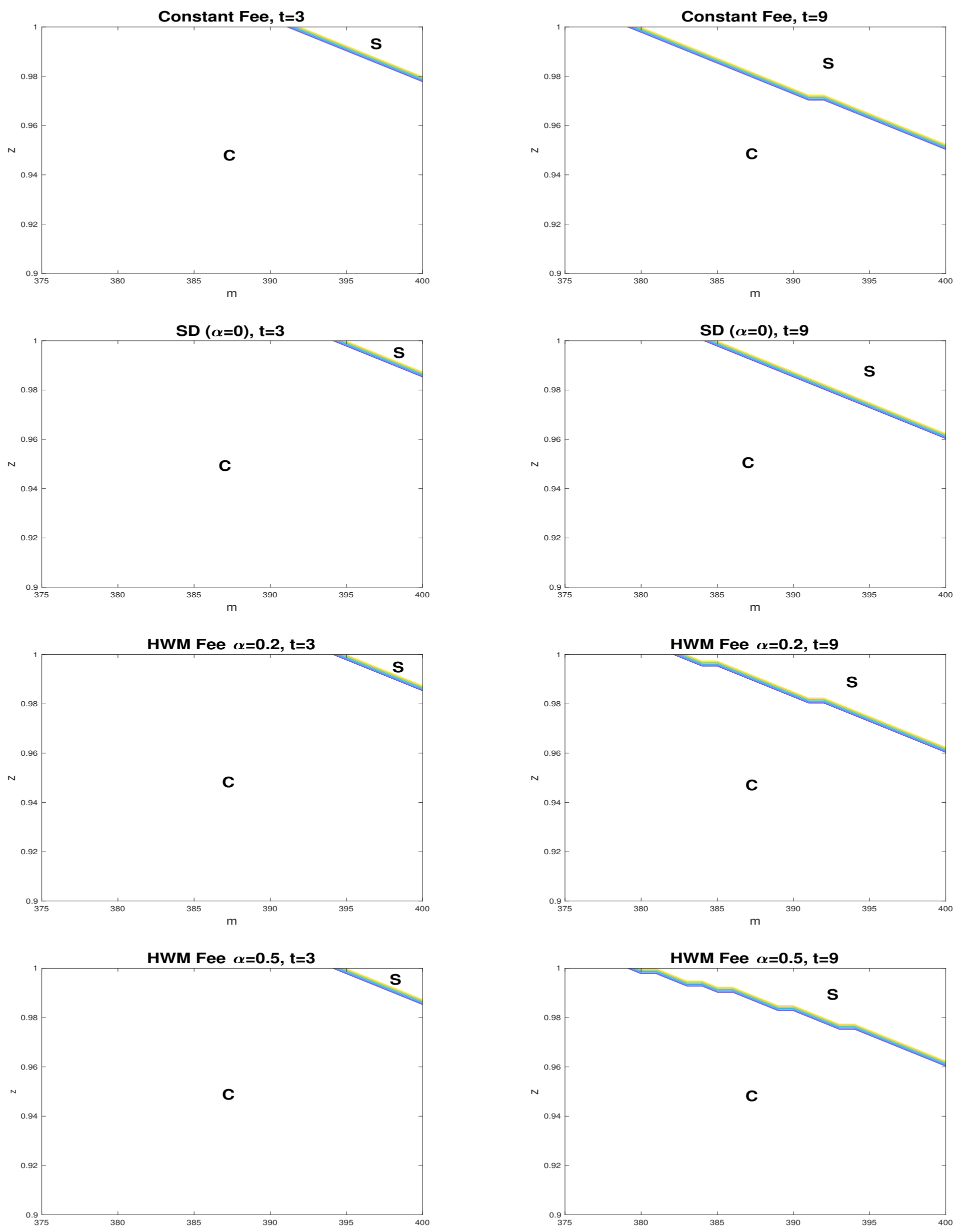

Figure 3.4: Optimal surrender regions under different fee structures at $t = 3$ and $t = 9$. The symbol "c" corresponds to the continuation region and "s" to the surrender region. The horizontal axis "m" stands for the HWM of the investment account and the vertical axis "z" stands for the state variable introduced in Eq. (3.8). The surrender risk is insignificant, and surrender only occurs when the investment account value is very large.

3.3.4 Surrender Behavior

Figure 3.4 presents the optimal surrender behavior of the PH holding the VA with MV preferences under different fee structures. In this figure, "**c**" corresponds to the continuation region and "**s**" to the surrender region. The x-axis "m" stands for the HWM of the investment account, and the y-axis "z" stands for the state variable introduced in Eq. (3.8).

The left and right panels of this figure are for times $t = 3$ and $t = 9$, respectively. The risk aversion parameter γ is taken as 1.7, the expected return rate μ is 10%, and the other parameters are as in the base case. We shall mention that the parameters in Figure 3.4 are chosen so that all fee structures under consideration are *compatible to the market* in the sense that holding the VA with these fee structures offers greater welfare for the PH than holding the risk-free bond or the pure fund. In this way, the comparison for the surrender behavior is meaningful.

There are a few noteworthy observations. First and foremost, the surrender risk is insignificant when all three fee structures are compatible to the market, and surrender only occurs when the investment account value is very large. As can be seen, all optimal surrender regions are located in the upper right corner of each panel, namely, when the investment account value $F = m \cdot z$ is very large. Note that the initial account value is $F_0 = 100$, and the expected return on the investment account is $\mu = 10\%$ without insurance fees. Hence, the odds of entering the surrender regions (roughly $F_3 > 390$ or $F_9 > 375$) are slim.

This behavior is not surprising. On the one hand, when the VAs are preferred over the alternative investments, rational PHs have little incentive to surrender such good assets, let alone the presence of surrender charges that further disincentivize surrender. On the other hand, it is still opportune for PHs to surrender when the investment account reaches higher levels because the variance of the VA payout increases simultaneously with the account level. This surrender behavior is also in line with the empirical study in Knoller et al. (2016), which indicates that surrender is more likely when the investment account value becomes greater than the guaranteed amount and, in turn, helps justify the selection of the state-dependent MV preference model.

Second, the HWM fee structure slightly increases the surrender risk only when the VA is approaching maturity. From the left panels of Figure 3.4, we can see that, when $t = 3$, the difference of the surrender regions under different fee structures is negligible. From the last three panels on the right, it is shown that, when $t = 9$, the surrender region slightly increases with respect to the HWM fee α. This is also very intuitive because charging more

HWM fees in “rich” states will encourage surrender. However, as shown in this figure, the incentive to surrender is very little. HWM fees are charged only when the investment account reaches a new historical peak. Even for an HWM fee of $\alpha = 0.5$, PHs can still receive 50% in an HWM increase, while surrender means PHs will have to forego these profits.

Third, although not presented, we should mention that all surrender regions for earlier times ($t < 3$) are of a similar shape but even smaller. In other words, the early surrender risk is rather small across the board. This is favorable for the insurer because there might exist some initial policy acquisition expenses in practice which may be recovered through collecting long-term fee streams.

Fourth, the constant (resp. state-dependent) fee structure generates a larger (resp. smaller) surrender region than the HWM fee structure, though the difference is very small. Note that the state-dependent fee structure also generates a non-empty surrender region for MV PHs, which differs from MacKay et al. (2017) showing that the state-dependent fee structure can completely eliminate the surrender risk for risk-neutral PHs in a complete market.

Finally, we should mention that similar surrender regions (a.k.a. maximizers) do not imply similar PH welfare (a.k.a. value functions), because different fee structures generate different dynamics of the investment account values. This does not contradict welfare improvement from the HWM fee structure.

3.4 Conclusion

This chapter introduces a novel HWM fee structure and show its merits with respect to improving VA marketability with the aim of possibly addressing issues related to declining demand.

For the analysis, we first determine various pairs of fair fees for the VA under the HWM fee structure by adopting the conventional risk-neutral pricing framework. We later conduct a comprehensive analysis of PH welfare from holding the VA, which is quantified by an MV preference model. More importantly, we introduce the concept of a compatible set as an innovative measure to quantitatively assess VA marketability under a certain fee structure. By comparing the corresponding compatible sets in various market conditions, we find that the HWM fee structure improves VA marketability relative to the constant and state-dependent fee structures.

Chapter 4

Gamma Discounting in Variable Annuities with Guaranteed Minimum Death and Maturity Benefits

4.1 Introduction

As alluded to in Chapter 1, the popularity of VAs has catalyzed an extensive amount of research papers looking into their valuation. To name a few, Milevsky & Posner (2001) investigate the valuation of VAs with guaranteed minimum death benefits; Milevsky & Salisbury (2006) and Dai et al. (2008) focus on the risk-neutral pricing of VAs with guaranteed minimum withdrawal benefits; Steinorth & Mitchell (2015) discuss how risk-averse policyholders (PHs) value VAs with guaranteed minimum lifetime withdrawal benefits in a life-cycle consumption model. While these papers offer useful techniques and insightful implications on valuation of VAs with various embedded guarantees, they fail to incorporate PHs' time preferences into the analysis. However, there is substantial evidence that individuals' time preferences are of significant importance when making investment decisions, and it has been well documented that individuals' *time preferences are time-inconsistent* (i.e., non-constant discount rates), especially in the long run.

Time inconsistency in time preference has been widely observed for decision makers in a group when valuing long-term investment. For example, Weitzman (2001) attempts to address the "perennial dilemma of what discount rate to use" and conducts a survey from over 2000 economists (including 52 Nobel Laureates). The result shows a significantly dispersed discount rate distribution (see Figure 1 in Weitzman (2001)). Similar evidence is also found in Frederick et al. (2002) and Falk et al. (2015). In addition to group heterogeneity in discounting preference, inconsistency in time preference has also been spotted for individuals in the sense that they are uncertain about what discount rate to use (i.e., intra-personal disagreement). In fact, Grenadier & Wang (2007) believe that individuals are "more prone to time-inconsistent behavior than firms". In line with this, Laibson (1997) investigates the consumption decisions made by a hyperbolic discounting consumer, Brocas & Carrillo (2004) assume hyperbolic discounting entrepreneurs, and DellaVigna & Malmendier (2004) consider time-inconsistent individuals but time-consistent firms. With the strong evidence in the existing literature, it is of great importance to study VAs, typical long-term investment/savings vehicles, considering that PHs are uncertain about what discount rate to use.

In light of the above, we consider risk-averse PHs whose subjective discount rates follow a *Gamma distribution* in this chapter. More specifically, we determine the impact of *Gamma discounting* on PHs' welfare and surrender behavior of holding a VA policy. Moreover, we investigate the impact on the profit of an insurer issuing the policy. Our analysis shows that, when PHs share the same expected discount rate, Gamma discounting PHs delay surrender behavior and value the VA policy more than exponential discounting PHs. And, the more uncertain they are with respect to the discount rate, the less likely they will surrender the policy and the more they will value the VA policy. Moreover, the insurer makes higher profit when PHs are Gamma discounting than when they are exponential discounting, and the insurer's profit increases as PH discount rate uncertainty increases.

Our results are consistent with findings obtained in the real option settings which indicate that a higher level of discount rate uncertainty (for the same expected discount rate) generates a larger investment threshold (i.e., later investment time); see, for example, Ebert et al. (2020). More importantly, our results are in line with the empirical phenomena that PHs exercise their options embedded in VAs less frequently than risk-neutral pricing approaches suggested and market fee rates are lower than the risk-neural insurance fees. We deem that, in addition to market friction (i.e., taxes) suggested by Moenig & Bauer (2015), PHs' inconsistency in time preferences might offer an alternative explanation to the above market phenomena.

As for the analysis, we consider a VA policy with two embedded guarantees, namely guaranteed minimum death benefits and guaranteed minimum maturity benefit. We first adopt the risk-neutral pricing approach to determine the fair insurance fee for this policy (a conventional approach adopted in the literature, e.g. Bauer et al. (2008)). We then consider Gamma discounting PHs whose welfare from holding the VA policy is quantified by a constant relative risk aversion utility function. Since PHs' subjective discount rates follow a Gamma distribution, the corresponding discount function is no longer exponential and consequently incurs a time-inconsistent optimal stopping problem with a random horizon. We adopt the game theoretic framework proposed by Björk & Murgoci (2010) to derive a system of extended Hamilton-Jacobi-Bellman (EHJB) equations. Numerically solving this system of equations, we obtain PHs' welfare and surrender behavior from holding the VA policy under Gamma discounting. Lastly, under PHs' surrender behavior, we compute the insurer's profit quantified by total fees collected deducting the VA payout.

The main contribution of this chapter is in the following. First, we incorporate uncertainty in discount rates into the analysis of VAs. Second, overcoming the challenge of high dimensionality of the state space, we investigate the effect of Gamma discounting on the insurer's profit. Last, in addition to market friction (i.e., taxes), our results show that Gamma discounting can offer an alternative explanation for the discrepancy between theoretical PH behavior and market fee rates and empirically observed ones.

The remainder of this chapter is organized as follows. Section 4.2 formally introduces the VA policy and determines the corresponding fair insurance fee. Section 4.3 sets up a utility-based framework to quantify Gamma discounting PHs' welfare from holding the VA and characterize their corresponding surrender behavior. Section 4.4 presents the insurer's profit under Gamma discounting PHs' surrender behavior. Section 4.5 concludes. The proof of a verification theorem for the system of EHJB equations corresponding to the time-inconsistent optimal stopping problem and the derivation of the partial differential equation (PDE) satisfied by the insurer's profit are relegated to Appendix B.

4.2 Valuation of Fair Fees

In this chapter, we consider the same VA policy as in Chapter 3 with the constant fee structure. Therefore, we again formulate its risk-neutral pricing model within a continuous-time stochastic optimal stopping framework and obtain the fair fee of this policy.

Applying the same mortality model and assuming the same dynamics for the underlying policy fund under the risk-neutral measure $\mathbb{Q}$, the pricing value function V of the VA policy is similar to the one in Chapter 3:

$$V(t,F) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}^{\mathbb{Q}}_{t,F}\left[{}_{\tau-t}p_{x+t}\psi(\tau, F_\tau)e^{-r(\tau-t)} + \int_t^\tau {}_{s-t}p_{x+t}\lambda_{x+s}\max(F_s, G_s)e^{-r(s-t)}ds \right] \tag{4.1}$$

over the domain $\mathcal{O} = \{F \in \mathbb{R}_+ : 0 < F < \infty\}$, where

$$\psi(t, F_t) = \begin{cases} (1-\kappa_t)F_t, & 0 \le t < T, \\ \max(G_T, F_T), & t = T, \end{cases}$$

corresponds to the surrender value or the maturity payout of the VA policy. Define

$$\mathcal{L}V := (r-c)FV_F + \frac{1}{2}\sigma^2 F^2 V_{FF} - (\lambda_{x+t} + r)V,$$

where V_t, V_F, and V_{FF} are the first-order derivative of V with respect to t, F, and second-order derivative with respect to F, respectively. By dynamic programming principle (see, e.g. Fleming & Soner (2006)), pricing value function V over the domain $[0, T] \times \bar{\mathcal{O}}$ satisfies the following variational inequality

$$\min\{-V_t - \mathcal{L}V - \lambda_{x+t} \max(F, G), V - \psi\} = 0, \tag{4.2}$$

with boundary conditions

$$\begin{cases} V(T, F) = \max(F, G_T), & \text{for} \quad F \in [0, \infty), \\ V_t\big|_{F=0} = (\lambda_{x+t} + r)V\big|_{F=0} - \lambda_{x+t}G, & \text{for} \quad t \in [0, T), \\ \lim_{F\to\infty} V(t, F) = \psi(t, F), & \text{for} \quad t \in [0, T). \end{cases} \tag{4.3}$$

As in Chapter 3, we propose to numerically solve Eq. (4.2) with boundary conditions (4.3) to obtain the fair fees for the VA policy. Again, we solve for constant fee $c^{\mathrm{f}} \in [0, 1]$ such that

$$V(0, F_0; c^{\mathrm{f}}) = F_0. \tag{4.4}$$

Parameter inputs are as specified in Table 4.1. Different from Chapter 3, we consider a VA policy that matures in 15 years and suppose a PH who enters the VA policy at age 50. We also take higher values for both the volatility σ and the risk-free rate r than their counterparts in Chapter 3. Solving Eq. (4.4), the fair fee under the parameter settings specified in Table 4.1 is

$$c^{\mathrm{f}} = 0.0186. \tag{4.5}$$

Description	Parameter	Base Case
Contract length	T	15
PH's age	y	50
Initial premium	F_0	100
Roll-up rate	g	0
Surrender penalty	κ_t	$0.05 \times (1 - \frac{t}{T})^3$
Interest rate	r	3%
Volatility	σ	20%

Table 4.1: Parameter inputs

4.3 Policyholder with Gamma Discounting

In this section, we investigate PHs' valuation of the VA policy when they are uncertain about what discount rates to use in a potentially incomplete market. More specifically, inspired by the seminal work of Weitzman (2001), we consider that PHs' subjective discount rates follow a Gamma distribution and we evaluate PHs' welfare from holding the VA policy in a utility-based framework. As in the spirit of last chapter, we assume that the insurer still adopts the risk-neutral pricing approach to price the VA policy and charges the fair fee c^{f} as given in Eq. (4.5). Different from the insurer, it is opportune to assume an incomplete market for PHs as they might not be able replicate every VA payout (e.g., mortality-based payouts or claims related to time-inconsistent time preferences).

4.3.1 Model Setup

In what follows, we set up the utility model to quantify PHs' welfare from holding the VA policy when their discount rates follow a Gamma distribution.

Suppose that, under the physical measure, the dynamics of the investment account F follows

$$dF_t = (\mu - c)F_t dt + \sigma F_t dW_t,$$

where $\mu > 0$ and $\{W_t\}_{\{t\geq 0\}}$ is a Brownian motion under measure $\mathbb{P}$. As alluded to above, a PH's subjective discount rate $\boldsymbol{\beta} \geq 0$, when evaluating the VA policy, follows a Gamma distribution $\Gamma(k, \theta)$ where $k > 0$ corresponds to the shape parameter and $\theta > 0$ to the scale parameter. Different choices of the parameters k and θ reflect different time preferences of PHs, which will be discussed in detail in the next section. Thereafter, we refer to the PH as a *Gamma discounting PH*. Note that, for a standard exponential discounting PH, the subjective discount rate β is just a fixed constant. Let h denote the discount function of the Gamma discounting PH with subjective discount rate $\boldsymbol{\beta}$ such that

$$h(t) = \mathbb{E}\left[e^{-\boldsymbol{\beta} t}\right] = \int_0^\infty e^{-xt} f(x; k, \theta) dx = (1 + \theta t)^{-k}, \quad \text{for} \quad t \in [0, T], \tag{4.6}$$

where $f(x; k, \theta) = \frac{x^{k-1}}{\Gamma(k)\theta^k} e^{-\frac{x}{\theta}}$ for $x > 0$ represents the probability density function (pdf) of the subjective discount rate $\boldsymbol{\beta}$, and $\Gamma(k) = \int_0^\infty z^{k-1} e^{-z} dz$ represents the Gamma function valued at k. Comparatively, for an exponential discounting PH, $h(t) = e^{-\beta t}$ with β being a fixed constant for $t \in [0, T]$. Effectively, $h(t)$ stands for the expected present value of a monetary unit at time t. As for PHs' risk preferences, we consider a constant relative risk aversion (CRRA) utility function

$$U(X) = \begin{cases} \frac{X^{1-\gamma} - 1}{1-\gamma}, & \text{for} \quad \gamma \geq 0 \quad \text{and} \quad \gamma \neq 1, \\ \ln(X), & \text{for} \quad \gamma = 1, \end{cases} \qquad \text{for} \quad X > 0,$$

where γ is the coefficient of relative risk aversion. Note that, in this chapter, we assume that PHs' time preferences and risk preferences are independent of each other.

We further assume that PHs optimally surrender the VA policy to maximize the EPV of the welfare generated by the VA payout. More specifically, PHs are assumed to maximize the following objective function

$$J(t, F; \tau) = \mathbb{E}_{t,F}\left[h(\tau \wedge \rho - t) \cdot U\Big(H(F_{\tau\wedge\rho})\Big) \,\Big|\, \rho > t\right], \tag{4.7}$$

where $(t, F) \in [0, T] \times \bar{\mathcal{O}}$, $\tau \in \mathcal{T}_{t,T}$ is a stopping time valued in $[t, T]$, ρ is the future lifetime of a y-year-old PH, and

$$H(F_{\tau\wedge\rho}) = \begin{cases} \psi(\tau, F_\tau), & \text{for} \quad \tau < \rho, \\ \max(F_\rho, G_\rho), & \text{for} \quad \tau \geq \rho, \end{cases} \tag{4.8}$$

represents the VA payout. Note that a stochastic optimal stopping problem can be viewed as an optimal control problem with binary controls. Therefore, we can regard our optimal stopping problem as an optimal control problem by introducing the following stopping rule (as in Ebert et al. (2020)).

Definition 7. *A stopping rule is a measurable function* $u : [0,T] \times \bar{\mathcal{O}} \to \{0,1\}$ *where* 0 *corresponds to continuation and* 1 *corresponds to stopping.*

More specifically, in our setup, the continuation and stopping regions at time $t \in [0,T)$ are defined as

$$\begin{cases} \mathcal{C}_t = \{F \in \bar{\mathcal{O}} : u(t,F) = 0\}, \\ \mathcal{S}_t = \{F \in \bar{\mathcal{O}} : u(t,F) = 1\}, \end{cases} \tag{4.9}$$

with the corresponding stopping time τ^u defined as

$$\tau^u = \inf\{s \geq t : u(s, F_s) = 1\}.$$

In light of the above, the value function of the objective (4.7) (a.k.a. PHs' optimal welfare from holding the VA policy) can be written as

$$V(t,F) = \max_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \mathbb{E}_{t,F} \left[h(\tau_{\boldsymbol{u}} \wedge \rho - t) \cdot U\Big(H(F_{\tau_{\boldsymbol{u}} \wedge \rho})\Big) \Big| \rho > t \right] = \max_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} J(t,F;\tau_{\boldsymbol{u}}), \tag{4.10}$$

where $\boldsymbol{\mathcal{A}}$ is the admissible set of stopping rules.

It is well-documented that stochastic optimal control problems with non-exponential discounting functions display the well-known issue of *time inconsistency*. We tackle this issue by adopting the game theoretic approach proposed in Björk & Murgoci (2010) and deriving an equilibrium stopping strategy via solving a system of extended Hamilton-Jacobi-Bellman (EHJB) equations.

Nevertheless, deriving the system of EHJB equations is much more complicated in this case due to a random time horizon (through a PH's future lifetime ρ). To deal with this, we next define an equilibrium stopping rule over a random horizon by extending the definition introduced in Landriault et al. (2018) and Chapter 3.

Definition 8. *For a fixed point* $(t,F) \in [0,T) \times \bar{\mathcal{O}})$*, a small* $\varepsilon > 0$*, and an admissible stopping rule* $\hat{\boldsymbol{u}} = \{\hat{u}_s\}_{s \in [t,T)}$ *conditional on* $\rho > t$*, define a stopping rule* $\boldsymbol{u}^\varepsilon$ *by*

$$\boldsymbol{u}^{\varepsilon,a}(s,z) = \begin{cases} a \in \{0,1\}, & \text{for } \; t \leq s < (t+\varepsilon) \wedge \rho, \\ \hat{\boldsymbol{u}}(s,z), & \text{for } \; (t+\varepsilon) \wedge \rho \leq s < T \wedge \rho, \end{cases}$$

in which $z \in \bar{\mathcal{O}}$*. Suppose that*

$$\liminf_{\varepsilon \downarrow 0} \frac{J(t,F;\tau_{\hat{\boldsymbol{u}}}) - J(t,F;\tau_{\boldsymbol{u}^{\varepsilon,a}})}{\varepsilon} \geq 0 \tag{4.11}$$

for all $(t,x) \in [0,T) \times [0,\infty)$*, then* $\hat{\boldsymbol{u}}$ *is called an equilibrium feedback stopping rule and the corresponding equilibrium value function* V *is given by* $V(t,F) = J(t,F;\tau_{\hat{\boldsymbol{u}}})$*.*

Via a formal verification theorem, we show that the value function (a.k.a. PHs' optimal welfare) $V(t,F)$ can be determined by solving the system of EHJB equations (4.12)-(4.16). The proof of the theorem is relegated to Appendix B.1.

Theorem 6 (Verification Theorem)**.** *Consider the discount function* (4.6)*, a stopping rule* $\hat{u}$*, and functions*

$$\begin{cases} g(t,F;x) = \mathbb{E}_{t,F}\left[e^{-x(\tau_{\hat{u}}\wedge\rho - t)} \cdot U\Big(H(F_{\tau_{\hat{u}}\wedge\rho})\Big)\Big|\rho > t\right], \\ V(t,F) = \int_0^\infty g(t,F;x)\cdot f(x;k,\theta)dx. \end{cases}$$

Let $\mathcal{L}f(t,F) = f_t + (\mu - c)Ff_F + \frac{1}{2}F^2\sigma^2 f_{FF}$ *and suppose that* $(V,g,\hat{u})$ *solves*

$$\max\{\mathcal{L}V(t,F) - \int_0^\infty (x+\lambda_{y+t})\cdot g(t,F;x)\cdot f(x;k,\theta)dx + \lambda_{y+t}\cdot U\Big(\max(F,G)\Big), U\Big(\psi(t,F)\Big) - V(t,F)\} = 0 \tag{4.12}$$

with boundary conditions

$$\begin{cases} V(T,F) = U\Big(\max(F,G_T)\Big), \\ V_t\big|_{F=0} - \int_0^\infty (x+\lambda_{y+t})\cdot g(t,0;x)\cdot f(x;k,\theta)dx + \lambda_{y+t}\cdot U(G) = 0, \\ \lim_{F\to\infty} V(t,F) = U\left[\psi(t,F)\right], \end{cases} \tag{4.13}$$

$$\hat{u}(t,F) = \begin{cases} 1, & \text{if } V(t,F) = U\Big(\psi(t,F)\Big), \\ 0, & \text{otherwise}, \end{cases} \tag{4.14}$$

and for each $x \in [0,\infty)$*,*

$$\mathcal{L}g(t,F;x) - (\lambda_{y+t} + x)\cdot g(t,F;x) + \lambda_{y+t}\cdot U\Big(\max(F,G)\Big) = 0 \tag{4.15}$$

with boundary conditions

$$\begin{cases} g(T,F) = U\Big(\max(F,G_T)\Big), \\ g_t\big|_{F=0} - (\lambda_{y+t}+x)g(t,0;x) + \lambda_{y+t}U(G) = 0, \\ \lim_{F\to\infty} g(t,F;x) = U\left[\psi(t,F)\right]. \end{cases} \tag{4.16}$$

Description	Parameter	Base Case	Sensitivity Analysis
Expected Return	μ	8%	
Risk Aversion	γ	0.35	
Expected Discount Rate	$\mathbb{E}[\boldsymbol{\beta}]$	4%	3%, 5%
Gamma Distribution	(k, θ)	$(2, 2\%)$	$(4, 1\%)$, $(1, 4\%)$, $(1.13, 2.67\%)$, $(3.13, 1.6\%)$

Table 4.2: Parameter inputs under Gamma discounting

Then $\hat{u}$ is an equilibrium stopping rule and the value function of (4.10) *is given by*

$$V(t, F) = \mathbb{E}_{t,F}\left[h(\tau_{\hat{u}} \wedge \rho - t) \cdot U\Big(H(F_{\tau_{\hat{u}} \wedge \rho})\Big)\Big|\rho > t\right].$$

4.3.2 Policyholders' Time Preferences and Welfare

As alluded to above, with the subjective discount rate $\boldsymbol{\beta}$ following a Gamma distribution, PHs' time preferences can be reflected by different choices of the parameters k and θ of the distribution. In this section, we discuss PHs' welfare and surrender behavior under different time preferences.

We present the relevant parameter inputs in a Gamma discounting context in Table 4.2. Note that the expected subjective discount rate is taken to be 4% for a Gamma discounting PH. Similarly, the subjective discount rate of an exponential discounting PH is also taken to be $\beta = 4\%$. Moreover, the base case Gamma distribution is taken to be $\Gamma(2, 2\%)$. This is calibrated from "2160 replies from economists in 48 countries" summarized Table 1 in Weitzman (2001). To conduct more comprehensive analysis, we also test different subjective discount rates and different parameters of a Gamma distribution. Lastly, the risk aversion parameter γ is taken as 0.35 to be consistent with findings in Holt & Laury (2002) indicating that the risk aversion level γ is "centered around the $0.3 - 0.5$ range".

For the rest of this section, we analyze effects of Gamma discounting on PHs' welfare and surrender behavior. We first compare cases in which they have the same discount rate expectation but different discount rate variance from the base case (see Table 4.3 and Figure 4.2). We then compare the base case with cases in which PHs have the same

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\text{Var}[\boldsymbol{\beta}]$	$V(0, F_0)$
E.D.	4%	0	29.2402
$\Gamma(4, 1\%)$	4%	0.04%	30.1665
$\Gamma(2, 2\%)$	4%	0.08%	30.8932
$\Gamma(1, 4\%)$	4%	0.16%	32.2254

Table 4.3: PHs' time-0 welfare with increasing uncertainty of $\boldsymbol{\beta}$. Note that $\mathbb{E}[\boldsymbol{\beta}] = 4\%$ is kept the same for all four cases and Var$[\boldsymbol{\beta}]$ is increasing. "E.D." is short for "exponential discounting PHs". Observe that Gamma discounting PHs value the VA policy more than exponential discounting ones, and the larger variance a PH's subjective discount rate has, the more the PH will value the VA policy.

discount rate variance but different discount rate expectation (see Table 4.4 and Figure 4.3).

Table 4.3 summarizes PHs' time-0 welfare (a.k.a. $V(0, F_0)$) when they are more and more uncertain about the subjective discount rate $\boldsymbol{\beta}$. In this table, we adjust the parameters k and θ simultaneously to make sure $\mathbb{E}[\boldsymbol{\beta}] = 4\%$ is kept the same for all four cases, and we use Var$[\boldsymbol{\beta}]$ to quantitatively measure the uncertainty level of PHs with respect to the discount rate $\boldsymbol{\beta}$. Note that "E.D." is short for "exponential discounting PHs" in this table and for the rest of this chapter.

It is evident from Table 4.3 that, when having the same belief towards the expectation of the discount rate $\boldsymbol{\beta}$, Gamma discounting PHs value the VA more than exponential discounting ones, and the more uncertain (i.e. larger Var$[\boldsymbol{\beta}]$) a PH is with respect to the discount rate $\boldsymbol{\beta}$, the more they will value the VA policy. This can be attributed to the fact that, when $\mathbb{E}[\boldsymbol{\beta}]$ remains the same, a higher Var$[\boldsymbol{\beta}]$ corresponds to a higher discount function $h(t)$ for $t \in [0, T]$ (not hard to deduce from Eq. (4.6)). For a typical long-term investment like VA, it is opportune to assume Gamma distributed time preferences other than standard exponential ones since PHs are likely to be uncertain about their discount rates in the long run. Therefore, the observation might help one justify the popularity of VAs as Gamma discounting PHs value the VA policy more than exponential discounting PHs do when having the same expected discount rate. The observation also helps explain

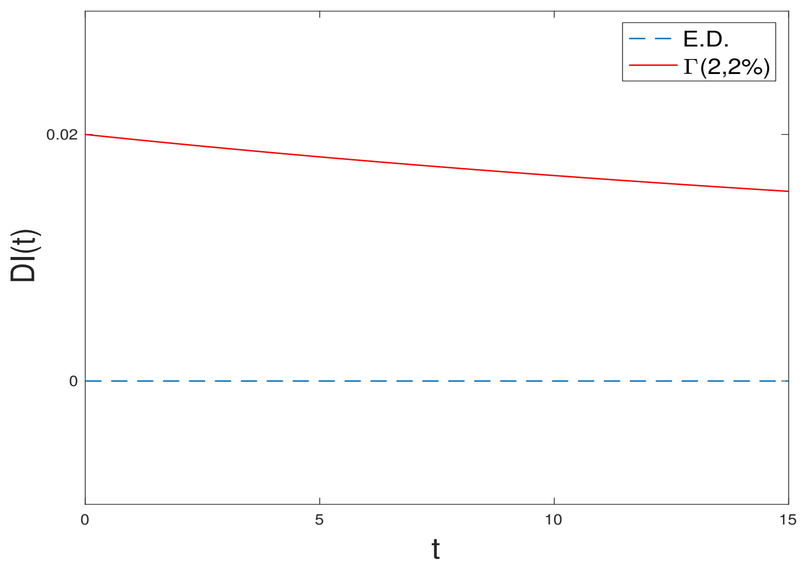

Figure 4.1: Decreasing impatience for exponential and Gamma discounting PHs with the same discount rate expectation. Observe that Gamma discounting PHs display a decreasing impatience level while exponential discounting PHs exhibit a constant impatience level. Also, Gamma discounting PHs have a higher level of decreasing impatience than that of exponential discounting PHs.

why the VA market has seen significant net assets and sales even with relatively high fees.

To compare PHs' surrender behavior under different time preferences, we first present the concept of *decreasing impatience* in the Gamma discounting context. This concept was initially introduced in Prelec (2004), and it is defined as

$$\mathrm{DI}(t) = -\frac{(\log h(t))''}{(\log h(t))'} = \frac{\theta}{1+\theta t}, \quad \text{for} \quad t \in [0, T], \tag{4.17}$$

when the subjective discount rate $\boldsymbol{\beta}$ follows a Gamma distribution. Decreasing impatience measures an investor's impatience level of making an investment decision over time t. A decreasing impatience level indicates that the investor focuses more on the near future than on the distant future. More specifically, the investor is more impatient on making investment decisions in the short run than in the long run. And a higher decreasing impatience level indicates a more patient investor.

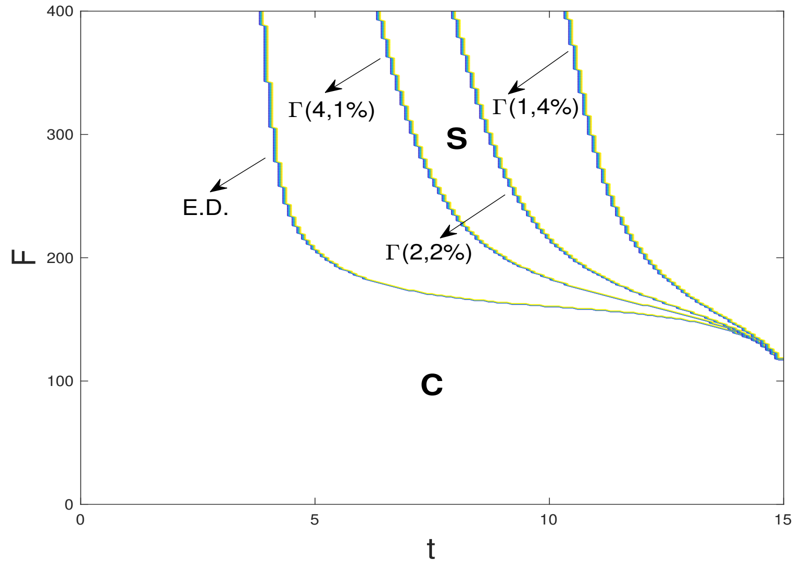

Figure 4.2: Optimal surrender regions for PHs with four levels of discount rate uncertainty as above. "S" corresponds to the surrender region and "C" to the continuation region. For each curve in the figure, the upper right part is the surrender region and the lower left part the continuation region. Observe that, when having the same expected discount rate, Gamma discounting PHs are less likely to surrender the VA policy than exponential discounting PHs, and the more uncertain PHs are regarding the discount rate, the less likely they will surrender the policy.

Figure 4.1 presents levels of decreasing impatience for exponential and Gamma discounting (i.e., $\boldsymbol{\beta}$ follows $\Gamma(2, 2\%)$) PHs with the same discount rate expectation. We can observe that Gamma discounting PHs display a decreasing impatience level while exponential discounting PHs exhibit a constant impatience level (meaning that exponential PHs have the same impatience level over time). Furthermore, Gamma discounting PHs have a higher level of decreasing impatience than that of exponential discounting PHs. This means that exponential discounting PHs are more impatient, especially in the long run, than Gamma discounting PHs. Gamma discounting PHs might be more likely to procrastinate on making surrender decisions; similar procrastination behavior of hyperbolic discounting agents is also observed in O'Donoghue & Rabin (1999*a*) and O'Donoghue & Rabin (1999*b*).

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\mathrm{Var}[\boldsymbol{\beta}]$	$V(0, F_0)$
$\Gamma(1.13, 2.67\%)$	3%	0.08%	34.7954
$\Gamma(2, 2\%)$	4%	0.08%	30.8932
$\Gamma(3.13, 1.6\%)$	5%	0.08%	29.0169

Table 4.4: Time-0 welfare of PHs with increasing discount rate expectation. We compare the base case with those that PHs have the same discount rate variance but different discount rate expectation. Observe that the higher discount rate expectation PHs have, the lower their time-0 welfare becomes.

Figure 4.2 shows PHs' optimal surrender regions under four levels of discount rate uncertainty as above. In this figure, "**S**" corresponds to the surrender region and "**C**" to the continuation region. For each curve, the upper right part is the surrender region and the lower left part the continuation region.

As can be seen from Figure 4.2, when PHs share the same discount rate expectation, Gamma discounting PHs are less likely to surrender the VA policy than exponential discounting PHs. In other words, Gamma discounting PHs are more likely to procrastinate on making surrender decisions than exponential discounting PHs, which is consistent with similar findings in O'Donoghue & Rabin (1999*a*) and O'Donoghue & Rabin (1999*b*), and consistent with the decreasing impatience observation in Figure 4.1. This observation also echoes the empirical phenomenon that PHs surrender the VA policy less often than option pricing approaches suggest (see Moenig & Bauer (2015) for more details) as Gamma distributed time preferences are more reasonable than exponential ones, especially for long-term investments such as VAs.

Also, we can observe that the more uncertain they are regarding the discount rate $\boldsymbol{\beta}$, the less likely they will surrender the VA policy. This reason is that, when sharing the same expected discount rate, a higher level of discount rate uncertainty corresponds to higher welfare from the VA policy (see Table 4.3), and makes PHs more reluctant to surrender the policy.

Table 4.4 summarizes cases in which PHs have the same level of discount rate uncertainty (variance) but different discount rate expectations. We observe that, when PHs share the same level of uncertainty with respect to the discount rate, the higher the discount rate expectation they have, the lower they value the VA policy. This is because

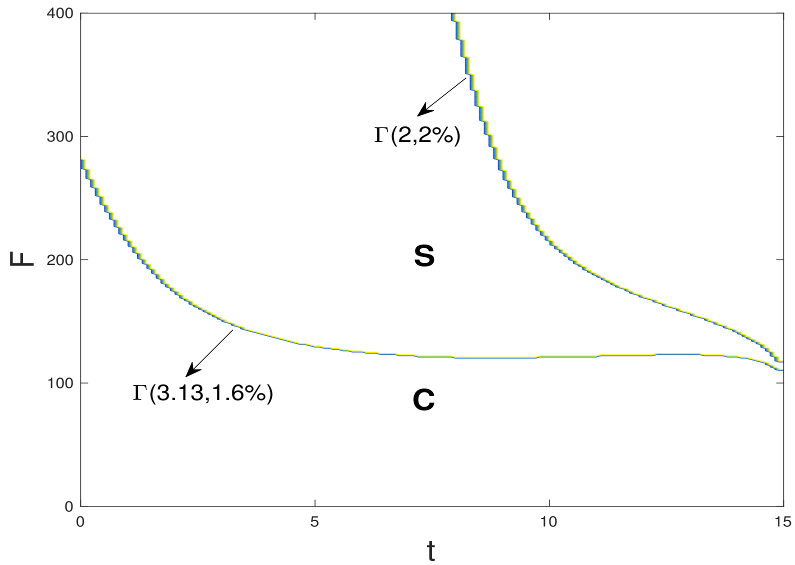

Figure 4.3: Optimal surrender regions for three kinds of PHs as in Table 4.4. Note that, PHs whose discount rate follows $\Gamma(1.13, 2.67\%)$ do not surrender in this domain. Observe that, when having the same level discount rate uncertainty, PHs with a higher discount rate expectation are more likely to surrender the VA policy.

a higher discount rate expectation corresponds to a lower discount function (not hard to check from Eq. (4.6)), which decreases PHs' welfare.

Figure 4.3 presents optimal surrender regions for PHs with the same level of discount rate uncertainty but different discount rate expectations (the same three cases as in Table 4.4). Note that PHs whose discount rates follow $\Gamma(1.13, 2.67\%)$ do not surrender in this domain. We can observe that PHs with a higher discount rate expectation are more likely to surrender the VA policy. When the level of discount rate uncertainty is the same, a higher expectation corresponds to a lower value for the parameter θ, which indicates a lower discount function, reduces PHs' welfare from the VA policy (see Table 4.4), and consequently stimulates earlier surrender.

4.4 Insurer's Profit

In the previous section we demonstrated how Gamma discounting affects PHs' welfare and surrender behavior from holding the VA policy. It is natural to also determine how the insurer is impacted by PHs' time preferences. Therefore, in this section, we investigate the insurer's profit from issuing the VA policy when PHs' subjective discount rates follow Gamma distributions. The aim is to determine the effect of PHs' time preferences on the insurer's profit and on the fees the insurer deems sufficient to charge. However, obtaining the insurer's profit is complicated as it involves numerically solving a PDE with two state variables.

We assume that the insurer is risk-neutral and their profit is measured by the total fees collected deducting the payout to PHs (specified by Eq. (4.8)). For a certain fixed insurance fee c, PHs' optimal surrender times $\hat{\tau}$, which affect the VA payout, are determined by Eq. (4.10). Moreover, since the insurer's profit depends on the total fees collected, we add a state variable to preserve the Markov property, i.e.,

$$\omega_t = \int_0^t cF_s ds = \omega,$$

where ω_t denotes the total fees collected by the insurer at time t under the insurance fee c. In light of the above, the insurer's profit at time t can be written as

$$I(t,\omega,F;c) = \mathbb{E}_{t,\omega,F}\left[\int_t^{\hat{\tau}} {}_{s-t}p_{x+t}\lambda_{x+s}\left(\omega_s - (G_s - F_s)_+\right)ds + {}_{\hat{\tau}-t}p_{x+t}\phi\left(\hat{\tau},\omega_{\hat{\tau}},F_{\hat{\tau}}\right)\right], \quad (4.18)$$

where

$$\phi(t,\omega_t,F_t) = \begin{cases} \omega_t + \kappa_t F_t, & t \in [0,T), \\ \omega_T - (G_T - F_T)_+, & t = T, \end{cases}$$

denotes the insurer's profit upon PHs' surrender (i.e., $t \in [0,T)$) or the policy maturity (i.e., $t = T$). Indeed, when a PH surrenders the VA policy, the insurer's profit is the total fees collected plus the surrender penalty income (since no guarantees are applicable), while the insurer's profit is the total fees collected deducting the GMMB payment[1] at maturity T. Moreover, if the PH dies prematurely, the insurer's profit is the total fees collected deducting the GMDB payment, which corresponds to the integral part in Eq. (4.18). Define

$$\mathcal{L}I := F(\mu - c)I_F + \frac{1}{2}F^2\sigma^2 I_{FF} + cFI_\omega - \lambda_{x+t}I. \quad (4.19)$$

[1]Note that the insurer only needs to pay for the difference of the guaranteed amount G and the balance of the investment account F when G is larger than F (i.e., when the guarantee is "in-the-money").

The PDE satisfied by I in Eq. (4.18) is

$$I_t + \mathcal{L}I + \lambda_{x+t}\left(\omega - \max(0, (G - F))\right) = 0, \tag{4.20}$$

with boundary conditions

$$\begin{cases} I(T, \omega, F) = \omega - (G_T - F)_+, & \text{for } (\omega, F) \in [0, \infty) \times [0, \infty), \\ I_t\Big|_{F=0} = \lambda_{x+t} I\Big|_{F=0} - \lambda_{x+t}\left(\omega - G\right), & \text{for } (t, \omega) \in [0, T) \times [0, \infty), \\ I(t, \omega, F) = \omega + \kappa_t F, & \text{for } (t, \omega, F) \in [0, T) \times [0, \infty) \times [\hat{F}, \infty), \\ \lim_{\omega \to \infty} I(t, \omega, F) = \omega, & \text{for } (t, F) \in [0, T) \times [0, \infty), \end{cases} \tag{4.21}$$

where

$$\hat{F} = \min\left\{F \in [0, \infty) : F \in \mathcal{S}_t\right\}$$

denotes the PHs' optimal surrender boundary at time t and $\mathcal{S}_t$ is the PHs' optimal surrender region defined in Eq. (4.9). Derivation of Eq. (4.20) is relegated to Appendix B.2. The first boundary condition in Eq. (4.21) corresponds to the insurer's terminal profit. The second condition is the degeneration of Eq. (4.20) when $F = 0$. The third condition corresponds to the insurer's profit when a PH surrenders the VA policy. The last condition holds since the insurer's profit converges to the total fees collected as they approach to infinity. In what follows, we numerically solve PDE (4.20) with boundary conditions (4.21) to analyze the impact of PHs' time preferences on the insurer's profit.

Table 4.5 presents the insurer's profit at point $(t, \omega, F) = (0, 0, F_0)$ [2] when PHs have the same discount rate expectation (i.e., $\mathbb{E}[\boldsymbol{\beta}] = 4\%$) but different levels of discount rate uncertainty (the same four cases as in Table 4.3). The insurer's profit is calculated under the fair insurance fee $c^{\mathrm{f}} = 0.0186$ obtained in Eq. (4.5) and parameter inputs are given by Table 4.1 and the base case in Table 4.2.

It is evident that the insurer enjoys significantly positive profit when charging the fair insurance fee for PHs with four different discount rate uncertainty levels. And from Figure 4.2, PHs with these discount rate uncertainty levels will participate in the VA policy (in other words, not surrender the policy) at the point $(t, \omega, F) = (0, 0, F_0)$, which means the insurer's profit will be realized. This observation indicates that the insurer might even make a profit from the VA policy after paying some policy-related initial expenses (e.g. administration costs and agent commission) out of their own pocket (which is common in VA practice). This means that it might be sufficient to charge a lower fee than the fair insurance fee for the insurer to make a profit.

[2] Note that at time $t = 0$, the total fees collected by the insurer is 0, i.e., $\omega_0 = 0$.

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\text{Var}[\boldsymbol{\beta}]$	$I(0, \omega_0, F_0)$
E.D.	4%	0	13.9478
$\Gamma(4, 1\%)$	4%	0.04%	18.0184
$\Gamma(2, 2\%)$	4%	0.08%	20.0935
$\Gamma(1, 4\%)$	4%	0.16%	23.2538

Table 4.5: Insurer's profit valuated at $(t, \omega, F) = (0, 0, F_0)$ when PHs have the same expected discount rate but different discount rate variance. The insurer charges the fair fee $c^{\text{f}} = 0.0186$ obtained in Eq. (4.5). Parameter inputs are given by Table 4.1 and the base case in Table 4.2. Observe that the insurer generates a higher profit when PHs are Gamma discounting than exponential discounting, and the profit is higher when PHs have a higher level of discount rate uncertainty.

More importantly, we observe that the insurer makes a higher profit when PHs are Gamma discounting than exponential discounting, and the profit becomes higher when PHs are more uncertain with respect to the discount rate. This is because, when sharing the same discount rate expectation, PHs are less likely to surrender the VA policy (as shown in Figure 4.2) if they are less certain regarding the discount rate, which means the insurer can collect more insurance fees. This observation indicates that the insurer might have more room to adjust the insurance fee and still make a profit when PHs are Gamma discounting.

Figure 4.4 presents the insurer's profit over a range of insurance fees *below* the fair insurance fee c^{f} when PHs have the same expected discount rate but different discount rate variance. In this figure, c^* denotes the insurance fee that generates the highest profit for the insurer.

We first observe from Figure 4.4 that the insurer can make a positive profit for a wide range of insurance fees below the fair fee c^{f} (roughly from 0.0016 to 0.0186). Note that this might not hold in general if initial costs are incorporated into the consideration, which might likely lift the insurance fee at which the insurer makes zero profit (i.e., $c = 0.0016$ in this figure). However, this observation still indicates that the insurer has the room to reduce the insurance fee below the fair fee and still make a profit.

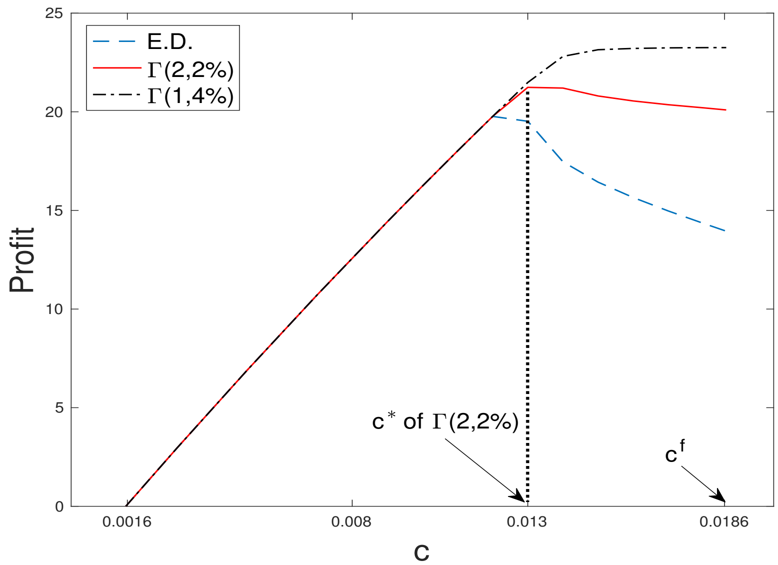

Figure 4.4: Insurer's profit over a range of insurance fees below the fair fee c^{f} when PHs have the same expected discount rate but different discount rate variance. c^* denotes the insurance fee that generates the highest profit for the insurer. Observe that the insurer has a higher profit and a higher c^* when PHs have higher discount rate uncertainty.

Second, for small insurance fees ($c \leq 0.012$), the insurer makes the same profit regardless of PHs' discount rate uncertainty levels, while, for larger insurance fees ($c \in [0.012, 0.0186]$), the insurer has a higher profit when PHs have higher discount rate uncertainty. For small insurance fees, PHs do not surrender the VA policy for all three cases, and therefore the insurer makes the same profit. For larger insurance fees, PHs' surrender behavior starts to differ (surrendering less likely for PHs with higher discount rate uncertainty) and the observation regarding the insurer's profit is consistent with that in Table 4.5.

Third, under the base case (i.e., $\Gamma(2, 2\%)$), the insurer's highest profit is achieved roughly at $c^* = 0.013$, which is lower than the fair fee $c^{\mathrm{f}} = 0.0186$. In other words, it is beneficial for the insurer to charge a lower insurance fee than the fair fee. This observation is in line with the empirical finding that the market fees of VAs are lower than the fair fees, which consequently justifies the adoption of Gamma discounting when evaluating the VA policy.

Last, we observe that, when PHs have the same discount rate expectation, the insurer's

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\text{Var}[\boldsymbol{\beta}]$	$I(0, \omega_0, F_0)$
$\Gamma(1.13, 2.67\%)$	3%	0.08%	30.6466
$\Gamma(2, 2\%)$	4%	0.08%	20.0935
$\Gamma(3.13, 1.6\%)$	5%	0.08%	9.1507

Table 4.6: Insurer's profit when PHs have the same discount rate variance but different discount rate expectations. The insurer charges the fair fee $c^{\text{f}} = 0.0186$. Observe that the insurer's profit is reduced if PHs have higher discount rate expectations.

c^* is higher when PHs have higher discount rate uncertainty (i.e., approximately $c^* = 0.012$ when PHs are exponential discounting and $c^* = c^{\text{f}}$ when discount rate $\boldsymbol{\beta}$ follows $\Gamma(1, 4\%)$). PHs with higher discount rate uncertainty are less likely to surrender the VA policy, and the insurer can charge a higher insurance fee and still make a higher profit.

Table 4.6 summarizes the insurer's profit when PHs have the same discount rate variance but different discount rate expectations (the same three cases as in Table 4.4). Note that the insurer charges the fair fee $c^{\text{f}} = 0.0186$. We can observe that the insurer's profit is reduced when PHs have a higher expected discount rate. This is because PHs who have a higher expected discount rate are more likely to surrender the VA policy (see Figure 4.3), which decreases the insurer's total fees collected.

Figure 4.5 displays the insurer's profit over a range of insurance fees below the fair fee c^{f} when PHs have the same discount rate variance but different discount rate expectation. It is evident that, in general, the insurer has a higher profit when PHs have a lower discount rate expectation. The reason is that PHs are more likely to surrender the VA policy when they have a higher expected discount rate (see Figure 4.3) and this reduces the total fees collected. Note that exceptions occur for very low insurance fees (when $c \leq 0.003$) where a higher discount rate expectation incurs a higher insurer's profit. This might be because the case where the discount rate $\boldsymbol{\beta}$ follows $\Gamma(3.13, 1.6\%)$ generates a moderate surrender region for extremely low insurance fees while the other two cases do not, and for extremely low insurance fees, the income from surrender penalties surpasses that from the insurance fees.

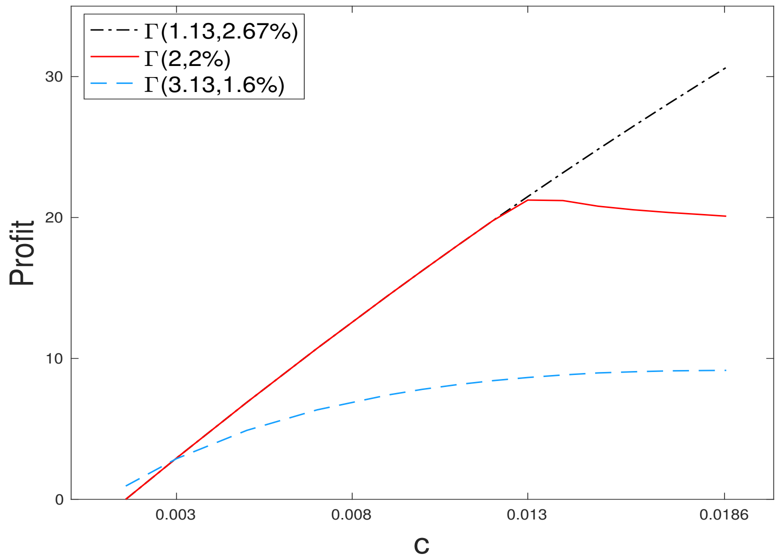

Figure 4.5: Insurer's profit over a range of insurance fees below the fair fee c^{f} when PHs have the same different discount rate variance but different discount rate expectations. Observe that the insurer in general has a lower profit when PHs have a higher discount rate expectation for a range of insurance fees.

4.5 Conclusion

In this chapter, we study the impact of Gamma discounting on PHs' welfare and surrender behavior from holding the VA policy as well as on the insurer's profit from the policy.

For the analysis, we set up a utility-based framework to quantify PHs' welfare and the corresponding surrender behavior from holding the VA policy when their subjective discount rates follow a Gamma distribution. By numerically solving a PDE with two state variables, we also compute the insurer's profit from the VA policy when PHs are Gamma discounting. We find that discount rate uncertainty makes PHs value the VA contract more, delays their surrender behavior, and increases the insurer's profit. Our results indicate that Gamma discounting can offer an alternative explanation for the discrepancy between theoretical PH behavior and market fee rates and empirically observed ones.

Chapter 5

Gamma Discounting in Variable Annuities with Guaranteed Minimum Withdrawal Benefits

5.1 Introduction

A guaranteed minimum withdrawal benefit (GMWB) offers a policyholder (PH) the right to withdraw the initial premium from a variable annuity (VA) policy over the life of the policy regardless of the performance of the investment. This guarantee is extremely prevalent. Specifically, a GMWB is included in roughly 70% of all VAs issued in 2004 (see Brothers (2005) for more details). In fact, withdrawal guarantees (also including guaranteed lifetime withdrawal benefit) in VAs have been the most popular guarantees over the past two decades among various guaranteed living benefits which have significantly expanded ever since late 1990s (see Bauer et al. (2008)).

The prevalence of GMWBs has sparked great interest from academics. For example, Milevsky & Salisbury (2006) and Dai et al. (2008) investigate the pricing of VAs with GMWB. While the latter formulates the pricing within a singular stochastic optimal control framework, the former points out that the market insurance fee is significantly below the one obtained from a risk-neutral pricing approach. The seminal work of Moenig & Bauer (2015) demonstrates that this under-priced phenomenon can be explained by a market friction: taxes. The reason is that the tax-deferral treatment can stimulate PHs to withdraw less frequently from VA policies. Among many others, Horneff et al. (2015) and Steinorth & Mitchell (2015) incorporate the analysis of GMWB into a life-cycle consumption model.

As in the spirit of Chapter 4, we study the time-inconsistent time preferences for VA PHs with GMWB in this chapter. More specifically, we consider risk-neutral PHs whose subjective discount rates follow a *Gamma distribution*, and determine how this will impact their welfare and withdrawal behavior from holding the VA policy.

Consistent with their counterparts in Chapter 4, our results show that, when having the same expected discount rate, *Gamma discounting* PHs value the VA policy more and withdraw less often than exponential discounting PHs. Also, we observe that when the variability of the discount rate increases (for a given mean), PHs value the policy more and they tend to make fewer withdrawals from the VA policy. Our findings indicate that Gamma discounting can offer an alternative explanation, in addition to the effect of taxes, to the phenomenon that PHs withdraw less frequently than the risk-neutral pricing approach suggests.

As for the analysis, we first adopt the conventional risk-neutral pricing approach and formulate the pricing of a VA with GMWB within a singular stochastic optimal control framework as in Dai et al. (2008). The fair fee of the VA policy is numerically determined via the penalty method in Huang & Forsyth (2012). We later consider risk-neutral PHs whose subjective discount rates follow a Gamma distribution. The reasons to consider risk-neutral PHs are two-fold. First, the impact of PHs' risk preferences is not significant for products with downside protection such as VAs. Second, incorporating a utility-based framework into the analysis would causes us some technical difficulties, e.g., we might lose bang-bangness of PHs' optimal withdrawal strategies, which might massively increase computational cost. Therefore, we consider risk-neutral PHs which are a simplification of risk-averse PHs, and we deem that the results on the effect of Gamma discounting are unlikely to be different.

To analyze PHs' welfare and withdrawal behavior, we set up a time-inconsistent stochastic optimal control model. To deal with the time-inconsistency from the non-exponential discount function, we adopt the game theoretic framework proposed by Björk & Murgoci (2010) to derive a system of extended Hamilton-Jacobi-Bellman (EHJB) equations. Due to the absence of explicit solutions, we resort to numerically solving the system of equations. However, numerically solving this system using standard finite difference methods is extremely challenging due to its complex form and time-consuming iterative procedure to attain an equilibrium withdrawal strategy. We circumvent these difficulties by proposing an efficient algorithm, and show that an optimal withdrawal strategy obtained from this algorithm is indeed an equilibrium withdrawal strategy from the system of equations.

The main contribution of this chapter is as follows. First, we incorporate time inconsistency of PHs' time preferences into the analysis of VAs with GMWB. Second, our results indicate that Gamma discounting can offer an alternative explanation (to the effect of taxes) to the phenomenon that PHs withdraw less often from VAs. Last, we propose an efficient numerical algorithm to solve the system of EHJB equations and establish the connection between the algorithm and the system.

The remainder of this chapter is organized as follows. Section 5.2 formally introduces the VA policy and determines the corresponding fair insurance fee. Section 5.3 sets up a time-inconsistent stochastic optimal control model and presents numerical examples to analyze the welfare and withdrawal behavior of PHs whose subjective discount rates follow a Gamma distribution. Section 5.4 concludes. Derivation of the system of EHJB equations, the algorithm, and a proof that establishes the connection between the algorithm and the system are relegated to Appendix C.

5.2 Valuation of Fair Fees

In this section, we describe the VA policy under consideration in detail and formulate its risk-neutral pricing model within a continuous-time singular stochastic optimal control framework (as in Dai et al. (2008)). Consequently, we obtain the fair fee of the VA policy.

5.2.1 Pricing Model

We consider a T-year VA policy with *guaranteed minimum withdrawal benefit* (GMWB). Suppose the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfies the usual conditions, where $\mathbb{Q}$ is a risk-neutral measure. Under the measure $\mathbb{Q}$, we assume that the value process of the underlying policy fund of the VA policy S follows a geometric Brownian motion with dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}},$$

where $r > 0$ is the risk-free rate, $\sigma > 0$ represents the policy fund volatility, and $\{W^{\mathbb{Q}}\}_{t\ge 0}$ stands for the standard Brownian motion under $\mathbb{Q}$.

The GMWB stipulates that the VA policy returns initial premium P to a PH via withdrawals. Specifically, the PH can continuously withdraw from the policy up to the amount of the upfront initial premium before the policy matures.[1] In light of the above,

[1] Note that the time value of each withdrawal is not considered.

we introduce A_t to denote the balance of the guarantee account at time $t \in [0, T]$, which records the cash value left in the guarantee account that the PH is still eligible to withdraw. At the inception of the policy, A_0 equals P, and A_t decreases over the life of the policy as withdrawal continues. Once A_t hits 0, the PH can no longer withdraw from the policy. In view of this, A_t is a non-negative and non-increasing process which is right-continuous with left limit. Similar to Dai et al. (2008), we first consider a restricted class of withdrawal policies where A_t is constrained to be absolutely continuous with bounded derivatives, i.e.,

$$A_t = A_0 - \int_0^t \gamma_s ds, \quad 0 \leq \gamma_s \leq \lambda, \tag{5.1}$$

where γ_s represents the withdrawal rate at time s. Note that we take the penalty approximation approach by imposing an upper bound for the continuous withdrawal rate γ. We later take the limit $\lambda \to \infty$ to establish the singular stochastic control formulation for the pricing of the VA policy (as in Dai et al. (2008)).

To fund the guarantees, the insurer is assumed to continuously charge a constant insurance fee with rate c as a fixed percentage against investment account F. Therefore, the dynamics of the investment account follows

$$\begin{cases} dF_t = (r - c)F_t dt + \sigma F_t dW_t^{\mathbb{Q}} + dA_t, \\ F_0 = P. \end{cases} \tag{5.2}$$

Effectively, the value of the investment account is affected by the performance of the underlying policy fund, the insurance fee, and PHs' withdrawal behavior. The initial value of the investment account F_0 is equal to the initial premium P.

Moreover, the policy imposes a contractual withdrawal rate G, and a withdrawal penalty κ is applied on the part of a withdrawal γ above G. More specifically, when a withdrawal γ is higher than the contractual rate G, the net amount received by a PH is $G+(1-\kappa)(\gamma-G)$. In light of the above, denote the rate of cash flow received by the PH by $f(\gamma)$, and we then have

$$f(\gamma) = \begin{cases} \gamma, & \text{for} \quad 0 \leq \gamma \leq G, \\ G + (1 - \kappa)(\gamma - G), & \text{for} \quad G \leq \gamma \leq \lambda. \end{cases} \tag{5.3}$$

Before the policy matures, the PH receives this withdrawal cash flow $f(\gamma)$, and at maturity T, the maximum between the balance of the investment account and the balance of the guarantee account is paid to the PH, subject to the withdrawal penalty, i.e., the maturity payout of the VA policy is $\max(F_T, A_T) - \kappa A_T$.[2]

[2]We adopt the terminal payout introduced by Huang & Kwok (2014) instead of Dai et al. (2008). Note that the one in the latter is simply a special case of the one in the former.

We adopt the risk-neutral pricing framework to obtain the fair fee of the VA policy. Specifically, within this framework, the insurer is able to replicate every possible VA cash payout, and they assume a PH that chooses the optimal withdrawal strategy to maximize the expected present value (EPV) of cash flows generated by holding the VA policy. Therefore, the risk-neutral pricing value function V is given by

$$V(t,F,A) = \max_{\gamma} \mathbb{E}^{\mathbb{Q}}_{t,F,A} \left[\int_t^T e^{-r(s-t)} f(\gamma_s) ds + e^{-r(T-t)} M(F_T, A_T) \right], \tag{5.4}$$

where $F_t = F$, $A_t = A$, γ is the control variable that maximizes Eq. (5.4), and

$$M(F,A) = \max(F,A) - \kappa A \tag{5.5}$$

denotes the terminal payout function of the VA policy. By the dynamic programming principle (see, e.g. Fleming & Soner (2006) for more details), the Hamilton-Jacobi-Bellman (HJB) equation satisfied by Eq. (5.4) is

$$V_t + \mathcal{L}V + \max_{\gamma} h(\gamma) = 0, \tag{5.6}$$

where

$$\mathcal{L}V = (r-c)FV_F + \frac{1}{2}F^2\sigma^2 V_{FF} - rV,$$

and

$$h(\gamma) = f(\gamma) - \gamma V_F - \gamma V_A = \begin{cases} (1 - V_F - V_A)\gamma, & \text{for } \; 0 \le \gamma < G, \\ \kappa G + (1 - \kappa - V_F - V_A)\gamma, & \text{for } \; G \le \gamma \le \lambda. \end{cases} \tag{5.7}$$

Note that V_t, V_F, and V_{FF} correspond to the partial derivatives of V with respect to t and F. As can be observed from Eq. (5.7), the function $h(\gamma)$ is piece-wise linear with respect to γ. Hence, $\max_\gamma h(\gamma)$ will only be achieved at $\gamma = 0$, $\gamma = G$, or $\gamma = \lambda$, and

$$\max_{\gamma} h(\gamma) = \begin{cases} \kappa G + (1 - \kappa - V_F - V_A)\lambda, & \text{for } \; 1 - V_F - V_A \ge \kappa, \\ (1 - V_F - V_A)G, & \text{for } \; 0 < 1 - V_F - V_A < \kappa, \\ 0, & \text{for } \; 1 - V_F - V_A \le 0. \end{cases}$$

By taking the limit $\lambda \to \infty$ in Eq. (5.6), we obtain the linear complementarity formulation of the pricing value function V (as in Dai et al. (2008)) in the following,

$$\min\left\{-V_t - \mathcal{L}V - \max(1 - V_F - V_A, 0)G, V_F + V_A - (1-\kappa)\right\} = 0, \tag{5.8}$$

where $0 < t < T$, $F > 0$, and $0 < A < P$. Note that Eq. (5.6) is a penalty approximation of Eq. (5.8). The terminal and boundary conditions for Eq. (5.8) are summarized in the following:

$$\begin{cases} V(T,F,A) = \max(F,A) - \kappa A, & \text{for} \quad (F,A) \in [0,\infty) \times [0,P], \\ V(t,F,0) = e^{-c(T-t)}F, & \text{for} \quad (t,F) \in [0,T) \times (0,\infty), \\ \lim_{F\to\infty} V(t,F,A) = e^{-c(T-t)}F, & \text{for} \quad (t,A) \in [0,T) \times (0,P], \end{cases} \tag{5.9}$$

and for $(t,A) \in [0,T) \times (0,P]$ when $F = 0$, Eq. (5.8) degenerates to

$$\begin{cases} \min\left\{ V_t\big|_{F=0} + rV\big|_{F=0} - \max(1 - V_A\big|_{F=0}, 0)G, V_A\big|_{F=0} - (1-\kappa) \right\} = 0, \\ V(T,A)\big|_{F=0} = (1-\kappa)A, \quad \text{and} \quad V(t,0)\big|_{F=0} = 0. \end{cases} \tag{5.10}$$

The first equation in Eq. (5.9) is the terminal condition which can be easily obtained from Eq. (5.4) by letting $t = T$. The second equation in Eq. (5.9) holds since the PH can no longer withdraw from the VA policy and the risk-neutral value function V will only be discounted with the insurance fee rate c before maturity. The third equation holds because the withdrawal guarantee is negligible when F becomes very large and therefore it is similar to the second equation. Eq. (5.10) corresponds to the degeneration of Eq. (5.8) when $F = 0$.

Since it is difficult to solve Eq. (5.8) with boundary conditions (5.9) and (5.10), we solve for its penalty approximation Eq. (5.6), which is the penalty method proposed in Dai et al. (2008) and Huang & Forsyth (2012). Similar to the last two chapters, we solve for the fair insurance fee c^{f} which satisfies

$$V(0, F_0, A_0; c^{\mathrm{f}}) = F_0. \tag{5.11}$$

Parameters required to obtain the fair fee of the VA policy are summarized in Table 5.1. We consider a VA policy that matures in 15 years. The contractual withdrawal rate, withdrawal penalty, and the upper bound for the withdrawal rate are taken the same as in Huang & Forsyth (2012). Adopting the penalty method to numerically solve Eq. (5.6) with boundary conditions (5.9) and (5.10) such that Eq. (5.11) is satisfied, we obtain the fair insurance fee

$$c^{\mathrm{f}} = 0.0184. \tag{5.12}$$

5.3 Policyholder with Gamma Discounting

In this section, we consider risk-neutral PHs whose subjective discount rates follow a Gamma distribution, and determine the impact of their time preferences on the welfare and

Description	Parameter	Base Case
Contract length	T	15
Initial premium	P	100
Contractual withdrawal rate	G	10
Withdrawal penalty	κ	0.1
Interest rate	r	3%
Volatility	σ	20%
Upper bound of withdrawal rate	λ	10^{-5}

Table 5.1: Parameter inputs

withdrawal behaviour from holding the VA policy with GMWB. We consider risk-neutral PHs for the following two reasons. First, we can avoid the expensive computational cost that might stem from a utility-based framework. Second, we deem that the results on the effect of Gamma discounting are unlikely to be different under a utility-based framework. As in the spirit of the last two chapters, we assume that the insurer still adopts the risk-neutral pricing approach to price the policy and charges the fair fee c^{f} as given in Eq. (5.12).

5.3.1 Model Setup

In what follows, we setup the model to quantify risk-neutral PHs' welfare from holding the VA policy when their subjective discount rates follow a Gamma distribution.

Suppose that, under the physical measure $\mathbb{P}$, the dynamics of the investment account F follows

$$dF_t = (\mu - c)F_t dt + \sigma F_t dW_t + dA_t,$$

where $\mu > 0$ denotes the expected return rate of the underlying policy fund and $\{W_t\}_{t\geq 0}$ is a Brownian motion under $\mathbb{P}$. Similar to Chapter 4, PHs' subjective discount rates $\boldsymbol{\beta}$ follow a Gamma distribution $\Gamma(k, \theta)$ with $k > 0$ and $\theta > 0$ being the shape and scale parameters, respectively. We refer to these PHs as *Gamma discounting* PHs as in Chapter 4. Comparatively, PHs who discount with a fixed rate β are referred to as exponential discounting PHs. The discount function h of the Gamma discounting PH with subjective discount rate $\boldsymbol{\beta}$ is

$$h(t) = \mathbb{E}\left[e^{-\boldsymbol{\beta}t}\right] = (1+\theta t)^{-k}, \quad \text{for} \quad t \in [0,T]. \tag{5.13}$$

For exponential discounting PHs, $h(t) = e^{-\beta t}$ for $t \in [0,T]$.

We further assume that PHs are risk-neutral and optimally withdraw from the VA policy to maximize the following objective function representing their welfare

$$J(t,F,A;\gamma) = \mathbb{E}_t\left[\int_t^T h(s-t)\cdot f(\gamma_s)ds + h(T-t)\cdot M(F_T,A_T)\right], \tag{5.14}$$

where the terminal payout function M is defined in Eq. (5.5). Since we now face a stochastic optimal control problem with a non-exponential discount function which possesses the well-known issue of *time inconsistency*, we adopt the game theoretic approach proposed in Björk & Murgoci (2010) and derive an equilibrium withdrawal strategy via solving a system of extended Hamilton-Jacobi-Bellman (EHJB) equations. In light of the above, we first define an equilibrium withdrawal law in the following.

Definition 9. *Consider a withdrawal law $\hat{\boldsymbol{\gamma}}$. Choose a fixed $\gamma \in [0,\lambda]$, and a fixed real number $\varepsilon > 0$. Also, fix an arbitrary chosen initial point $(t,F,A) \in [0,T) \times [0,\infty) \times [0,P]$. Define a withdrawal law $\boldsymbol{\gamma}$ by*

$$\boldsymbol{\gamma}(s,y,z) = \begin{cases} \gamma, & \textit{for} \;\; t \le s < t+\varepsilon, \\ \hat{\boldsymbol{\gamma}}(s,y,z), & \textit{for} \;\; t+\varepsilon \le s < T, \end{cases}$$

in which $(y,z) \in [0,\infty) \times [0,P]$. If

$$\liminf_{\varepsilon \to 0} \frac{J(t,F,A;\hat{\boldsymbol{\gamma}}) - J(t,F,A;\boldsymbol{\gamma})}{\varepsilon} \ge 0,$$

for all $\gamma \in [0,\lambda]$, we say that $\hat{\boldsymbol{\gamma}}$ is an equilibrium withdrawal law. The equilibrium value function is defined by

$$V(t,F,A) = J(t,F,A;\hat{\boldsymbol{\gamma}}).$$

Before presenting the system of EHJB equations, we first define auxiliary functions in the following. Suppose that an equilibrium withdrawal law $\hat{\boldsymbol{\gamma}}$ exists. For each $s \in [t,T]$, we define

$$k^s(t,F,A) := \mathbb{E}_{t,F,A}\left[f(\hat{\boldsymbol{\gamma}}(s,F_s^{\hat{\boldsymbol{\gamma}}},A_s^{\hat{\boldsymbol{\gamma}}}))\right], \quad \text{for} \quad 0 \le t \le s.$$

As for the terminal payout of the VA policy, we define

$$H(t,F,A) = \mathbb{E}_{t,F,A}\left[M(F_T^{\hat{\gamma}}, A_T^{\hat{\gamma}})\right], \quad \text{for} \quad 0 \le t \le T.$$

Note that, under this equilibrium withdrawal law, each k^s function corresponds to the instantaneous expected cash flow rate at time $s \in [t,T]$, which gives us infinitely many k^s functions, and the function H corresponds to the expected terminal payout of the VA policy. Therefore, under an equilibrium withdrawal law $\hat{\boldsymbol{\gamma}}$, the system of EHJB equations satisfied by (V, k^s, H) is:

- V satisfies

$$V_t + \mathcal{A}V + \int_t^T h'(s-t)\cdot k^s(t,F,A)du + h'(T-t)\cdot H(t,F,A) + \sup_{\gamma}\{f(\gamma) - \gamma V_F - \gamma V_A\} = 0, \tag{5.15}$$

  with boundary conditions

$$\begin{cases} V(T,F,A) = \max(F,A) - \kappa A, & \text{for} \quad (F,A) \in [0,\infty) \times [0,P], \\ V(t,F,0) = F \cdot h(T-t) \cdot e^{(\mu-c)(T-t)}, & \text{for} \quad (t,F) \in [0,T) \times (0,\infty), \\ \lim_{F\to\infty} V(t,F,A) = F \cdot h(T-t) \cdot e^{(\mu-c)(T-t)}, & \text{for} \quad (t,A) \in [0,T) \times (0,P], \end{cases} \tag{5.16}$$

  and with the boundary condition when $F = 0$ for $(t,A) \in [0,T) \times (0,P]$ such that

$$V_t\Big|_{F=0} + \int_t^T h'(s-t)\cdot k^s(t,0,A)ds + h'(T-t)\cdot H(t,0,A) + \sup_{\gamma}\left\{f(\gamma) - \gamma V_A\Big|_{F=0}\right\} = 0, \tag{5.17}$$

  with conditions

$$\begin{cases} V(T,A)\Big|_{F=0} = (1-\kappa)A, & \text{for} \quad A \in [0,P], \\ V(t,0)\Big|_{F=0} = 0, & \text{for} \quad t \in [0,T), \end{cases} \tag{5.18}$$

  where $\mathcal{A}V = (\mu - c)FV_F + \frac{1}{2}\sigma^2 F^2 V_{FF}$ and $h'(t) = \frac{dh(t)}{dt}$;

- For each $s \in [0,T)$, k^s satisfies

$$k_t^s + \mathcal{A}k^s - (k_F^s + k_A^s)\hat{\gamma} = 0, \tag{5.19}$$

with boundary conditions

$$\begin{cases} k^s(s,F,A) = f(\hat{\gamma}_s), & \text{for } \ (F,A) \in [0,\infty) \times [0,P], \\ k^s(t,F,0) = 0, & \text{for } \ (t,F) \in [0,s) \times (0,\infty), \\ \lim_{F\to\infty} k^s(t,F,A) = 0, & \text{for } \ (t,A) \times [0,s) \times (0,P], \\ k^s_t\Big|_{F=0} - \hat{\gamma} \cdot k^s_A\Big|_{F=0} = 0, & \text{for } \ (t,A) \times [0,s) \times (0,P], \\ k^s(s,A)\Big|_{F=0} = f(\hat{\gamma}_s), & \text{for } \ A \in (0,P], \\ k^s(t,0)\Big|_{F=0} = 0, & \text{for } \ s \in [0,s), \end{cases} \tag{5.20}$$

where $\mathcal{A}k^s = (\mu - c)Fk^s_F + \frac{1}{2}\sigma^2 F^2 k^s_{FF}$;

- H satisfies

$$H_t + \mathcal{A}H - (H_F + H_A)\hat{\gamma} = 0, \tag{5.21}$$

with boundary conditions

$$\begin{cases} H(T,F,A) = \max(F,A) - \kappa A, & \text{for } \ (F,A) \in [0,\infty) \times [0,P], \\ H(t,F,0) = Fe^{(\mu-c)(T-t)}, & \text{for } \ (t,F) \in [0,T) \times (0,\infty), \\ \lim_{F\to\infty} H(t,F,A) = Fe^{(\mu-c)(T-t)}, & \text{for } \ (t,A) \in [0,T) \times (0,P], \\ H_t\Big|_{F=0} - \hat{\gamma} H_A\Big|_{F=0} = 0, & \text{for } \ (t,A) \times [0,T) \times (0,P], \\ H(T,A)\Big|_{F=0} = (1-\kappa)A, & \text{for } \ A \in (0,P], \\ H(t,0)\Big|_{F=0} = 0, & \text{for } \ t \in [0,T), \end{cases} \tag{5.22}$$

where $\mathcal{A}H = (\mu - c)FH_F + \frac{1}{2}\sigma^2 F^2 H_{FF}$.

This system of EHJB equations contains one partial differential equation (PDE) for V, infinitely many PDEs for k^s (one for each $s \in [t,T]$), and one PDE for H. The derivation of this system is relegated to Appendix C.1. Note that, in this system, PDEs (5.19) and (5.21) corresponding to k^s and H are under an equilibrium withdrawal strategy $\hat{\gamma}$, and $\hat{\gamma}$ is determined by Eq. (5.15). Similar to Eq. (5.6), we can observe from Eq. (5.15) that the expression $f(\gamma) - \gamma V_F - \gamma V_A$ is piece-wise linear in γ. Therefore, an equilibrium withdrawal strategy can only be taken from $\{0, G, \lambda\}$. Again, by taking the limit $\lambda \to \infty$, we can obtain a linear complementary formulation for the value function V (a.k.a. PHs' optimal welfare) similar to Eq. (5.8). However, it is difficult to solve for this formulation. Hence, we solve for its penalty approximation Eq. (5.15) instead.

Description	Parameter	Base Case	Sensitivity Analysis
Expected Return	μ	8%	
Expected Discount Rate	$\mathbb{E}[\boldsymbol{\beta}]$	4%	3%, 5%
Gamma Distribution	(k, θ)	$(2, 2\%)$	$(4, 1\%)$, $(1, 4\%)$, $(1.13, 2.67\%)$, $(3.13, 1.6\%)$

Table 5.2: Parameter inputs under Gamma discounting

Using a standard finite difference approach to numerically solve the system of EHJB equations (5.15)-(5.22) is extremely challenging due to its complex form, high dimensionality (i.e., two state variables F and A), and a time-consuming iterative procedure to achieve an equilibrium withdrawal strategy $\hat{\gamma}$ in Eq. (5.15). To circumvent these difficulties, we propose an efficient algorithm to solve this system. Moreover, we show that a solution derived by this algorithm is indeed a solution to the system of EHJB equations (5.15)-(5.22). The details of the algorithm are relegated to Appendix C.2.

5.3.2 Policyholders' Time Preferences and Welfare

In what follows, we give out numerical examples to illustrate how PHs' time preferences (a.k.a. Gamma discounting) affect their welfare and withdrawal behavior from holding the VA policy. We adopt similar parameter inputs for Gamma discounting PHs as in Chapter 4. The only difference is that PHs are now assumed to be risk-neutral. We reiterate that the base case Gamma distribution $\Gamma(2, 2\%)$ is calibrated from Table 1 in Weitzman (2001). For the rest of this section, we first present the welfare and withdrawal behavior of PHs with the same discount rate expectation but different discount rate uncertainty (i.e., variance); see Table 5.3 and Figure 5.1. We then display the welfare and withdrawal behavior of PHs with the same discount rate uncertainty but different expected discount rate; see Table 5.4 and Figure 5.2.

Table 5.3 summarizes time-0 welfare of PHs with the same discount rate expectation but different discount rate uncertainty. "E.D." represents exponential discounting PHs in this table. It is evident that, when sharing the same expected discount rate, Gamma discounting PHs value the VA policy higher than exponential discounting PHs, and the higher level of discount rate uncertainty PHs have, the more they will value the VA policy. The reason is that, when having the same expected discount rate, a higher discount rate

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\text{Var}[\boldsymbol{\beta}]$	$V(0, F_0, A_0)$
E.D.	4%	0	139.0549
$\Gamma(4, 1\%)$	4%	0.04%	144.3018
$\Gamma(2, 2\%)$	4%	0.08%	148.9121
$\Gamma(1, 4\%)$	4%	0.16%	156.6813

Table 5.3: PHs' time-0 welfare with increasing uncertainty of $\boldsymbol{\beta}$. Note that $\mathbb{E}[\boldsymbol{\beta}] = 4\%$ is kept the same for all four cases and Var[$\boldsymbol{\beta}$] is increasing. "E.D." is short for "exponential discounting PH". Observe that Gamma discounting PHs value the VA policy more than exponential discounting ones, and the larger variance a PH's subjective discount rate has, the more the PH will value the VA policy.

uncertainty level corresponds to a higher discount function h, which consequently increases PHs' welfare. Note that these findings are consistent with those in Chapter 4. And these results also imply that Gamma discounting might help justify the popularity of VAs with withdrawal-type guarantees.

Figure 5.1 displays withdrawal regions for exponential discounting PHs (blue), Gamma discounting PHs of $\Gamma(2, 2\%)$ (red), and Gamma discounting PHs of $\Gamma(1, 4\%)$ (pink) at time $t = 1$ (the left two panels) and at time $t = 13$ (the right two panels) when they have the same discount rate expectation. We summarize some important observations in the following.

First, as alluded to above, the optimal regions of PHs are consist of three withdrawal strategies, namely no withdrawal ($\gamma = 0$), withdraw at the contractual withdrawal rate ($\gamma = G$), or instantaneously withdraw with a discrete amount ($\gamma = \infty$). Note that these results are consistent with literature on withdrawal rates of GMWBs (e.g. Huang & Forsyth (2012)).

Second, our results show that PHs do not withdraw from the VA policy very often, in particular when the policy is at the early stage (the left two panels). As can be seen at time $t = 1$ from this figure, PHs only withdraw from the policy when the investment account is very small (which is similar to the withdrawal behavior found in Moenig & Bauer (2015)) and when the balance of the guarantee account A is approximately two times higher than

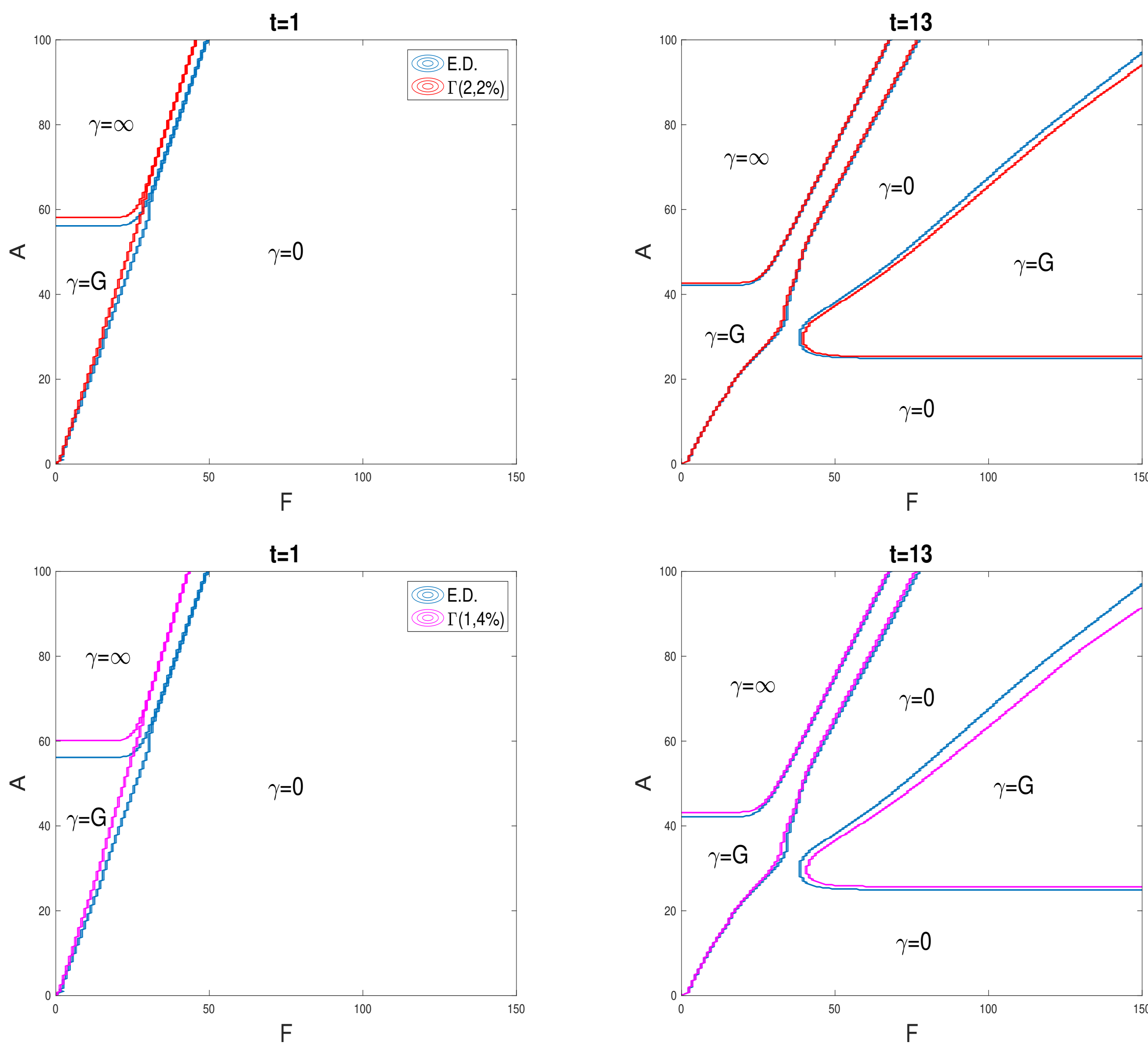

Figure 5.1: Contour graph of withdrawal regions for exponential discounting PHs (blue), Gamma discounting PHs of $\Gamma(2, 2\%)$ (red), and Gamma discounting PHs of $\Gamma(1, 4\%)$ (pink) with the same expected discount rate at time $t = 1$ (left two panels) and time $t = 13$ (right two panels). Observe that Gamma discounting PHs withdraw less from the VA policy than exponential discounting PHs.

the balance of the investment account F (i.e., $A > 2F$). In the case that PHs do withdraw from the policy, they withdraw at contractual rate G when the balance of the guarantee account is approximately below 60, i.e., when both guarantee and investment accounts are small. This is because the guarantee is not too out-of-the-money when both accounts are small and PHs still expect the investment account to grow higher than the guarantee account. Therefore, they refrain from withdrawing too much. Nevertheless, PHs withdraw at a discrete amount instantaneously ($\gamma = \infty$), when the guarantee account is roughly higher than 60, i.e., when the guarantee account is much higher than the investment account. The reason is that, when the guarantee is too out-of-the-money, PHs no longer expect the investment account to grow higher than the guarantee account in the future. Hence, to avoid future insurance fee charges and further value depreciation of the policy due to discounting, PHs opt to withdraw as much as possible.

Note that when the investment account grows higher (roughly when $F > \frac{1}{2}A$), PHs do not withdraw from the policy. This is different from the findings in Huang & Forsyth (2012) and Huang & Kwok (2014), where non-zero withdrawal regions could be observed for large investment account value. The reason is that we consider an incomplete market where PHs might not be able to replicate the VA payouts corresponding to their time preferences. With a high expected return rate (i.e., $\mu = 8\%$) of the underlying policy fund, the VA policy is more attractive than that in a complete market.

Third, we observe that PHs withdraw more from the policy as it approaches maturity. From the right panel of this figure, the contractual withdrawal rate region and region of withdrawing with a discrete amount are enlarged, while no withdrawal region is reduced. In particular, when the investment account becomes larger and the balance of the guarantee account is approximately between 25 and 90, PHs withdraw at the contractual withdrawal rate G. This is because, at the late stage of the policy, PHs do not anticipate significant growth of the investment account. Therefore, it is opportune for them to withdraw from the policy to avoid the penalty at maturity. Withdrawal penalty is also the reason for them to only withdraw at rate G but not higher.

Last, as for the difference of withdrawal behavior between exponential discounting PHs and Gamma discounting PHs, we see that Gamma discounting PHs withdraw less from the VA policy. Indeed, at time $t = 1$, Gamma discounting PHs have a larger no withdrawal region and a smaller non-zero withdrawal region (a combination region of $\gamma = G$ and $\gamma = \infty$) than exponential discounting PHs. Note that this difference is more evident for Gamma discounting PHs with a higher discount rate uncertainty (i.e., $\Gamma(1, 4\%)$). This can be attributed to the fact that Gamma discounting PHs enjoy higher welfare than exponential PHs (see Table 5.3), and are therefore more reluctant to withdraw from the policy. Moreover, similar to Chapter 4, this can also be explained by the procrastination

Distribution of $\boldsymbol{\beta}$	$\mathbb{E}[\boldsymbol{\beta}]$	$\text{Var}[\boldsymbol{\beta}]$	$V(0, F_0)$
$\Gamma(1.13, 2.67\%)$	3%	0.08%	171.4610
$\Gamma(2, 2\%)$	4%	0.08%	148.9121
$\Gamma(3.13, 1.6\%)$	5%	0.08%	129.6498

Table 5.4: Time-0 welfare of PHs with increasing discount rate expectation. We compare the base case with those that have the same discount rate variance but different discount rate expectation. Observe that the higher discount rate expectation PHs have, the lower their time-0 welfare becomes.

effect for PHs with time-inconsistent time preferences (a.k.a. Gamma discounting) who are more likely to delay investment decisions. Our results also resonate with the empirical finding that PHs withdraw less often than option pricing approaches suggest, indicating that Gamma discounting can be an alternative explanation to this mismatch (in addition to the effect of taxes suggested by Moenig & Bauer (2015)).

However, we observe that the difference in withdrawal behavior between exponential discounting PHs and Gamma discounting PHs is reduced when the policy approaches to maturity. Indeed, from the right two panels of Figure 5.1, regions of $\gamma = G$ and $\gamma = \infty$ are almost the same when the balance of the investment account is small for Gamma discounting and exponential discounting PHs. The difference is only evident when the investment account is large, where a slightly larger no withdrawal region (resp. smaller contractual withdrawal rate region) can be observed for Gamma discounting PHs. The reason is that, in the short term, the impact of PHs' time preferences on the welfare and withdrawal behavior from holding the VA policy also becomes smaller than in the long term (e.g. the difference in the discount functions is smaller).

For the reminder of this section, we present PHs' welfare and withdrawal behavior when they have the same discount uncertainty but different expected discount rates. Table 5.4 summarizes PHs' time-0 welfare when they share the same discount rate variance but different discount rate expectation. Similar to Chapter 4, we observe that the higher expected discount rates PHs have, the lower their welfare from holding the VA policy will be. This is not surprising since, when having the same discount rate uncertainty, a higher discount rate expectation corresponds to a lower discount function h, which decreases PHs' welfare.

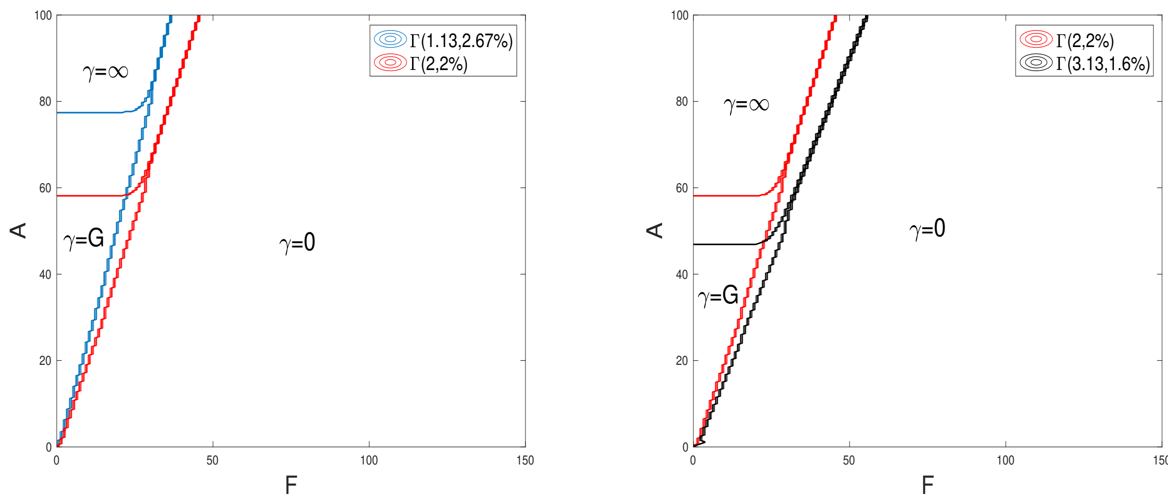

Figure 5.2: Contour graph of withdrawal regions at time $t = 1$ for PHs with the same discount rate uncertainty but different discount rate expectation. The left panel compares the base case (red) to the case with a lower expected discount rate (blue), and the right panel compares the base case to the case with a higher expected discount rate (black). Observe that the higher discount rate expectation PHs have, the more they will withdraw from the VA policy.

Figure 5.2 presents PHs' withdrawal regions at time $t = 1$ when they have the same discount rate uncertainty but different discount rate expectation. The left panel compares the base case (i.e., $\Gamma(2, 2\%)$) with the case of a lower expected discount rate (i.e., $\Gamma(1.13, 2.67\%)$), while the right panel compares the base case with the case of a higher expected discount rate (i.e., $\Gamma(3.13, 1.6\%)$).

It is apparent that a higher expected discount rate corresponds to a smaller no withdrawal region and a larger region of non-zero withdrawal. Not surprisingly, as can be seen from Table 5.4, a higher expected discount rate produces lower welfare for PHs given the corresponding lower discount function h, which means the VA policy becomes less attractive and stimulates PHs to withdraw more from the policy.

5.4 Conclusion

In this chapter, we study the impact of Gamma discounting on the welfare and withdrawal behavior for risk-neutral PHs of VAs with GMWB.

For the analysis, we set up a time-inconsistent stochastic optimal control model and present numerical examples to analyze the welfare and withdrawal behavior of PHs whose subjective discount rates follow a Gamma distribution. We propose an efficient numerical algorithm to circumvent the difficulties of solving the system of EHJB equations using standard finite difference methods.

Our results show that Gamma discounting PHs value the VA policy more and withdraw less often than their exponential discounting counterparts. And, when having the same expected discount rate, the higher discount rate variance PHs have, the more they will value the policy and the less often they withdraw from the policy. Our findings indicate that Gamma discounting can offer an alternative explanation to the phenomenon that PHs withdraw less often than the risk-neutral pricing approach suggests, in addition to the effect of taxes.

Chapter 6

Conclusion and Future Work

The main objective of this thesis is centered around proposing potential remedies and alternative explanations to two current VA phenomena: declining demand for VA products and existence of discordance between some theoretical and empirical findings in the VA market, most notably those pertaining to insurance fees and PH behavior. We hope the work of this thesis can contribute to better VA contract design and better understanding of PH behavior.

More specifically, Chapter 3 proposes a VA with a novel HWM fee structure and evaluates PH welfare with the VA. We introduce an innovative quantitative measure to assess the impact of fee structure on VA marketability. A system of EHJB equations corresponding to an MV optimal stopping problem with two state variables over a random time horizon is derived. Moreover, we propose a numerical algorithm to solve the system of EHJB equations and establish the connection between the algorithm and the system. Our results show that the HWM fee structure can generally improve VA marketability.

Chapters 4 and 5 incorporate Gamma discounting into the analysis of VAs with various guarantees. In these two chapters, we solve both a time-inconsistent optimal stopping and a time-inconsistent stochastic control problem. The impact of Gamma discounting on the insurer's profit is investigated in Chapter 4, and an efficient algorithm is proposed to numerically obtain an equilibrium withdrawal strategy for risk-neutral PHs in Chapter 5. Consistent results are obtained in these chapters regarding PH welfare and PH surrender and withdrawal behavior. Collectively, these results indicate that Gamma discounting can offer an alternative explanation, in addition to the effect of taxes, to the aforementioned discordance between theoretical and empirical insurance fees and PH behavior.

While there are many potential directions to extend the work of this thesis, we list some future research avenues that are most relevant in the following:

- For Chapter 3, it would be interesting to incorporate more practical features such as initial expenses (see, e.g., Bernard & Moenig (2019)) and VA tax benefits (see, e.g., Moenig & Bauer (2015)) into our model. The presence of initial expenses would increase total fees of VAs which could impair the resulting VA marketability. However, this is likely offset by VA tax benefits. It could also be interesting to look into the question of lapse-and-reentry (see, e.g., Moenig & Zhu (2018)) and examine its impact on the performance of the HWM fee structure. Moreover, exploring the influence of the HWM fee structure on withdrawal type guarantees (see, e.g., Dai et al. (2008)) remains an open and very relevant research problem.

- For Chapter 4, it would be interesting to investigate the impact of Gamma discounting within a principal-agent framework. Specifically, we can determine the participant constraints for both PHs and insurers by figuring out the set of insurance fees that both parties would be willing to participate in the contract and examining how Gamma discounting would affect this set. A reasonable conjecture would be that Gamma discounting will enlarge this set.

- For Chapter 5, it would be interesting to relax the assumption of risk-neutral PHs and establish an expected utility model to quantify PH welfare. Although this relaxation is unlikely to provide significantly different implications on the impact of Gamma discounting, the results would be more general (with the interplay of PHs' time and risk preferences). Another relevant future direction is to also consider insurers' profit for VAs with GMWB. However, this is very challenging as it involves numerically solving a PDE with three state variables.

Essentially, when addressing the decreasing demand of the VA market by improving VA contract design, it would be meaningful to also incorporate practical features and market friction into the analysis. Moreover, when trying to better understand PH behavior, it would also be important to consider the problem within an equilibrium framework (i.e., consider interactions of interest of both insurers and PHs in a VA policy).

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APPENDICES

Appendix A

Appendix for Chapter 3

A.1 Verification Theorem for Optimization Problem (3.16)

Theorem 7. *Suppose that there exist functions $V, g \in C^{1,2,1}([0,T]\times\bar{\mathcal{O}})$, $f \in C^{1,2,1,2}([0,T]\times\bar{\mathcal{O}}\times[0,m])$ and a stopping rule $\hat{u}$ satisfying the following conditions:*

- *For any $(t,F,m) \in [0,T]\times\bar{\mathcal{O}}$, V solves*

$$\begin{aligned}\min\Big\{&-V_t-\big(\mu-c\mathbb{1}_{\{F\le\theta\}}\big)F\Big(V_F-f_y-\frac{\gamma_F}{2}g^2\Big)-\frac{1}{2}\sigma^2F^2\Big(V_{FF}-f_{yy}-2f_{Fy}-\frac{\gamma_{FF}}{2}g^2\\&-\gamma(F)g_F^2-2\gamma_F g g_F\Big)+\frac{\lambda_{x+t}\gamma(F)}{2}\left(g-\max(F,G)\right)^2+\lambda_{x+t}V\\&-\lambda_{x+t}\max(F,G),V-\psi\Big\}=0,\end{aligned} \tag{A.1}$$

  with boundary conditions

$$\begin{cases}V(T,F,m)=\max(F,G), & \textit{for}\ \ (F,m)\in\bar{\mathcal{O}},\\ V_t\big|_{F=0}=\frac{\lambda_{x+t}\gamma(0)}{2}(g\big|_{F=0}-G)^2+\lambda_{x+t}V\big|_{F=0}-\lambda_{x+t}G, & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ V_m\big|_{F=m}=\alpha\mathbb{1}_{\{F\ge\theta\}}\left(V_F\big|_{F=m}-f_y\big|_{F=m}-\frac{\gamma_F}{2}g^2\big|_{F=m}\right), & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ \lim_{m\to\infty}V(t,m,m)=\psi(t,m), & \textit{for}\ \ t\in[0,T).\end{cases} \tag{A.2}$$

- *For any* $(t,F,m)\in[0,T)\times\bar{\mathcal{O}}$,

$$\hat{u}(t,F,m)=\begin{cases}1, & \text{if } V(t,F,m)=\psi(t,F),\\ 0, & \text{otherwise.}\end{cases} \tag{A.3}$$

- *For fixed* $y\in[0,m]$ *and for any* $(t,F,m)\in[0,T]\times\bar{\mathcal{O}}$ *s.t.* $\hat{u}=0$, f *solves*

$$f_t+\left(\mu-c\mathbb{1}_{\{F\le\theta\}}\right)Ff_F+\frac{1}{2}\sigma^2F^2f_{FF}-\lambda_{x+t}f+\lambda_{x+t}\left[\max(F,G)-\frac{\gamma(y)}{2}\max(F,G)^2\right]=0, \tag{A.4}$$

with boundary conditions

$$\begin{cases} f(T,F,m,y)=\max(F,G)-\frac{\gamma(y)}{2}\max(F,G)^2, & \text{for } (F,m,y)\in\bar{\mathcal{O}}\times[0,m],\\ f_t\big|_{F=0}=\lambda_{x+t}f\big|_{F=0}-\lambda_{x+t}\left(G-\frac{\gamma(y)}{2}G^2\right), & \text{for } (t,m,y)\in[0,T)\times[0,\infty)\times[0,m],\\ f_m\big|_{F=m}=\alpha\mathbb{1}_{\{F\ge\theta\}}f_F\big|_{F=m}, & \text{for } (t,m,y)\in[0,T)\times[0,\infty)\times[0,m],\\ \lim_{m\to\infty}f(t,m,m,y)=\psi(t,m)-\frac{\gamma(y)}{2}\psi^2(t,m), & \text{for } t\in[0,T), \end{cases} \tag{A.5}$$

and for $\hat{u}=1$, $f(t,F,m,y)=\psi(t,F)-\frac{\gamma(y)}{2}\psi^2(t,F)$.

- *For any* $(t,F,m)\in[0,T]\times\bar{\mathcal{O}}$ *s.t.* $\hat{u}=0$, g *solves*

$$g_t+\left(\mu-c\mathbb{1}_{\{F\le\theta\}}\right)Fg_F+\frac{1}{2}\sigma^2F^2g_{FF}-\lambda_{x+t}g+\lambda_{x+t}\max(F,G)=0, \tag{A.6}$$

with boundary conditions

$$\begin{cases} g(T,F,m)=\max(F,G), & \text{for } (F,m)\in\bar{\mathcal{O}},\\ g_t\big|_{F=0}=\lambda_{x+t}g\big|_{F=0}-\lambda_{x+t}G, & \text{for } (t,m)\in[0,T)\times[0,\infty),\\ g_m\big|_{F=m}=\alpha\mathbb{1}_{\{F\ge\theta\}}g_F\big|_{F=m}, & \text{for } (t,m)\in[0,T)\times[0,\infty),\\ \lim_{m\to\infty}g(t,m,m)=\psi(t,m), & \text{for } t\in[0,T), \end{cases} \tag{A.7}$$

and for $\hat{u}=1$, $g(t,F,m)=\psi(t,F)$.

Then $\hat{u}$ *is an equilibrium stopping rule such that*

$$\begin{cases} V(t,F,m)=\mathcal{J}(t,F,m;\tau^{\hat{u}})\\ g(t,F,m)=\mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)\big|\rho>t\right]\\ f(t,F,m,y)=\mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)\big|\rho>t\right]-\frac{\gamma(y)}{2}\mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)^2\big|\rho>t\right]. \end{cases} \tag{A.8}$$

This system of EHJB equations corresponds to a high dimensional free boundary problem with three unknown functions, V, f, and g, where $\hat{u} = 0$ corresponds to the continuation region and $\hat{u} = 1$ to the stopping region.

To prove Theorem 7, we first introduce some notation and results for brevity of illustration.

- $\mathbb{E}_{t,F,m}[\cdot] = \mathbb{E}_t[\cdot]$ and $\mathrm{Var}_{t,F,m}[\cdot] = \mathrm{Var}_t[\cdot]$
- $\mathcal{L}h(t,F,m) := h_t + (\mu - c\mathbb{1}_{\{F<\theta\}})Fh_F + \frac{1}{2}\sigma^2F^2h_{FF}$ for a function h s.t. $h \in C^{1,2,1}([0,T]\times\bar{\mathcal{O}})$
- $\mathcal{L}f(t,F,m,y) := f_t + (\mu - c\mathbb{1}_{\{F<\theta\}})(Ff_F + yf_y) + \frac{1}{2}\sigma^2(F^2f_{FF} + y^2f_{yy}) + f_{Fy}Fy\sigma^2$ for a function f s.t. $f \in C^{1,2,1,2}([0,T]\times\bar{\mathcal{O}}\times[0,m])$
- $\mathbb{E}_t\left[h(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon})\right] = h(t,F,m) + \varepsilon\mathcal{L}h(t,F,m) + \mathbb{E}_t\left[\int_t^{t+\varepsilon}(h_m - \alpha\mathbb{1}_{\{F\geq\theta\}}h_F)dM_s\right] + o(\varepsilon)$
- $\mathbb{E}_t\left[f(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}, y_{t+\varepsilon})\right] = f(t,F,m,y) + \varepsilon\mathcal{L}f(t,F,m,y)$ $+\, \mathbb{E}_t\left[\int_t^{t+\varepsilon} f_m - (f_F + f_y)\alpha\mathbb{1}_{\{F\geq\theta\}}dM_s\right] + o(\varepsilon)$

Proof. The proof of Theorem 7 can be divided into two parts. In part one, we prove that V, g, and f are indeed the solution of their probabilistic interpretation (A.8) under stopping rule $\hat{u}$. In part two, we prove that $\hat{u}$ is an equilibrium stopping rule. First, we prove that

$$
\begin{aligned}
g(t,F,m) &= \mathbb{E}_t\left[H(F_{\tau^{\hat{u}}\wedge\rho})\middle|\rho > t\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right],
\end{aligned}
$$

where

$$
H(F_{\tau^{\hat{u}}\wedge\rho}) = \begin{cases} \max(F_\rho, G_\rho), & \text{if } \rho < \tau^{\hat{u}}, \\ \psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}), & \text{if } \rho \geq \tau^{\hat{u}}. \end{cases}
$$

Suppose that $g \in C^{1,2,1}([0,T]\times\hat{\mathcal{O}})$ satisfies Eq. (A.6) with boundary conditions (A.7), and define a sequence of stopping times

$$
\tau_t^n := \tau^{\hat{u}} \wedge \inf\left\{s > t : |F_s^{t,F} - F| > n\right\}.
$$

Applying Itô's formula to $e^{-\int_0^s \lambda_u du} g(s, F_s, M_s)$, we get

$$
\begin{aligned}
& e^{-\int_t^{\tau_t^n} \lambda_{x+u} du} g(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) - g(t, F, m) \\
&= \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} (\mathcal{L}g - \lambda_{x+s} g) ds + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} (g_m - \alpha \mathbb{1}_{\{F_s > \theta\}} g_F) dM_s \\
&+ \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} \sigma F_s g_F dW_s.
\end{aligned}
$$

Taking $\mathbb{E}_t[\cdot]$ on both sides of the equation and invoking Eq. (A.6) with boundary conditions (A.7), we obtain

$$
g(t, F, m) = \mathbb{E}_t \left[e^{-\int_t^{\tau_t^n} \lambda_{x+u} du} g(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} \lambda_{x+s} \max(F_s, G_s) ds \right].
$$

Letting $n \to \infty$ gives $\tau_t^n \to \tau^{\hat{u}}$ which further leads to

$$
\begin{aligned}
g(t, F, m) &= \mathbb{E}_t \left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u} du} g(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u} du} \lambda_{x+s} \max(F_s, G_s) ds \right] \\
&= \mathbb{E}_t \left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u} du} \psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u} du} \lambda_{x+s} \max(F_s, G_s) ds \right] \\
&= \mathbb{E}_t \left[H(F_{\tau^{\hat{u}} \wedge \rho}) \middle| \rho > t \right],
\end{aligned}
$$

since

$$
\tau^{\hat{u}} = \inf\{s \geq t : \hat{u}(t, F, m) = 1\}.
$$

Define $f^y(t, F, m) := f(t, F, m, y)$ for a fixed y and similarly, we apply Itô's formula to $e^{-\int_0^s \lambda_{x+u} du} f^y(s, F_s, M_s)$ and obtain

$$
\begin{aligned}
& e^{-\int_t^{\tau_t^n} \lambda_{x+u} du} f^y(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) - f^y(t, F, m) \\
&= \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} (\mathcal{L}f^y - \lambda_{x+s} f^y) ds + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} (f_m^y - \alpha \mathbb{1}_{\{F_s > \theta\}} f_F^y) dM_s \\
&+ \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u} du} \sigma F_s f_F^y dW_s.
\end{aligned}
$$

Taking $\mathbb{E}_t[\cdot]$ on both sides of the equation and invoking Eq. (A.4) with boundary conditions

(A.5) results in

$$f^y(t,F,m) = \mathbb{E}_t\Bigg[e^{-\int_t^{\tau_t^n}\lambda_{x+u}du} f^y(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\left(\max(F_s,G_s) - \frac{\gamma(y)}{2}\max(F_s,G_s)^2\right)\Bigg].$$

By letting $n\to\infty$, we have $\tau_t^n \to \tau^{\hat{u}}$ which in turn leads to

$$\begin{aligned}
f^y(t,F,m) &= \mathbb{E}_t\Bigg[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du} f^y(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}) \\
&\quad + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\left(\max(F_s,G_s)ds - \frac{\gamma(y)}{2}\max(F_s,G_s)^2\right)\Bigg] \\
&= \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right] \\
&\quad - \frac{\gamma(y)}{2}\mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\psi^2(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)^2ds\right] \\
&= \mathbb{E}_t\left[H(F_{\tau^{\hat{u}}\wedge\rho})\middle|\rho>t\right] - \frac{\gamma(y)}{2}\mathbb{E}_t\left[H^2(F_{\tau^{\hat{u}}\wedge\rho})\middle|\rho>t\right].
\end{aligned}$$

To verify the probabilistic interpretation of V, we need to show that

$$V(t,F,m) = \frac{\gamma(x)}{2}g^2(t,F,m) + f(t,F,m,F).$$

Since $V(T,F,m) = \psi(T,F)$, we have $\tau^{\hat{u}} \in \mathcal{T}_{[t,T]}$. Applying Itô's formula to $e^{-\int_0^s \lambda_u du}V(s,F_s,M_s)$, we get

$$\begin{aligned}
&e^{-\int_t^{\tau_t^n}\lambda_{x+u}du}V(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) - V(t,F,m) \\
&= \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}(\mathcal{L}V - \lambda_{x+s}V)ds + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}(V_m - \alpha\mathbb{1}_{\{F_s>\theta\}}V_F)dM_s \\
&+ \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}\sigma F_s V_F dW_s.
\end{aligned}$$

Since $[t, \tau_t^n) \subset [t, \tau^{\hat{u}})$, we deduce that $V > \psi$ on $[t, \tau_t^n)$. Using Eq. (A.1) for $V > \psi$, taking $\mathbb{E}_t[\cdot]$ on both sides of the equation, and letting $n \to \infty$, we obtain

$$
\begin{aligned}
V(t,F,m) = {} & \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}V(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}})\right] - \mathbb{E}_t\Bigg\{\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\Big[(\mu - c\mathbb{1}_{\{F_s<\theta\}})F_s\left(f_y + \frac{\gamma_F}{2}g^2\right) \\
& + \frac{1}{2}\sigma^2 F_s^2\left(f_{yy} + 2f_{Fy} + \frac{\gamma_{FF}}{2}g^2 + \gamma(F_s)g_F^2 + 2\gamma_F g g_F\right) + \frac{\gamma(F_s)}{2}\lambda_{x+s}\left(g - \max(F_s,G_s)\right)^2 \\
& - \lambda_{x+s}\max(F_s,G_s)\Big]ds\Bigg\} - \mathbb{E}_t\left[\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\left(V_m - \alpha\mathbb{1}_{\{F_s\geq\theta V_F\}}\right)dM_s\right].
\end{aligned}
$$

Now, applying Itô's formula and conducting operations as above on $e^{-\int_0^s \lambda_{x+u}du}f(s,F_s,M_s,F_s)$, one deduces that

$$
\begin{aligned}
f(t,F,m,F) = {} & \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}f(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}, F_{\tau^{\hat{u}}})\right] - \mathbb{E}_t\Bigg\{\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\Big[-\lambda_{x+s}\max(F_s,G_s) \\
& + \lambda_{x+s}\frac{\gamma(F_s)}{2}\max(F_s,G_s)^2 + f_yF_s(\mu - \mathbb{1}_{\{F_s<\theta\}}) + \frac{1}{2}F_s^2\sigma^2 f_{yy} + F_s^2\sigma^2 f_{Fy}\Big]ds\Bigg\} \\
& + \mathbb{E}_t\left(\int_t^{\tau^{\hat{u}}} f_y\alpha\mathbb{1}_{\{F_s\geq\theta\}}dM_s\right).
\end{aligned}
$$

Again, applying Itô's formula and proceeding as above on $e^{-\int_0^s \lambda_{x+u}du}\frac{\gamma(F_s)}{2}g(s,F_s,M_s)$, we have

$$
\begin{aligned}
\frac{\gamma(F)}{2}g^2(t,F,m) = {} & \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\frac{\gamma(F_{\tau^{\hat{u}}})}{2}g\left(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}\right)\right] - \mathbb{E}_t\Bigg\{\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du} \\
& \Big[-\gamma(F_s)g\lambda_{x+s}\max(F_s,G_s) + \frac{\gamma(F_s)}{2}\lambda_{x+s}g^2 + \frac{\gamma(F_s)}{2}g_F^2\sigma^2F_s^2 + \frac{g_F^2}{2}\gamma_F(\mu - c\mathbb{1}_{\{F_s<\theta\}})F_s \\
& + \frac{1}{4}g^2\gamma_{FF}\sigma^2F_s^2 + \gamma_F\sigma^2F_s^2 g g_F)\Big]ds\Bigg\} + \mathbb{E}_t\left[\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\frac{\gamma_F}{2}g^2\alpha\mathbb{1}_{\{F_s\geq\theta\}}dM_s\right].
\end{aligned}
$$

Combining the expressions for $f(t,F,m,F)$ and $\frac{\gamma(F)}{2}g^2(t,F,m)$, it is easy to establish that

$$
V(t,F,m) = \frac{\gamma(x)}{2}g^2(t,F,m) + f(t,F,m,F).
$$

Now we prove that the stopping rule $\hat{\boldsymbol{u}}$ defined in Eq. (A.3) is an equilibrium stopping rule. Recall that we define a stopping rule $\boldsymbol{u}^{\varepsilon}$ such that

$$\boldsymbol{u}^{\varepsilon}(s,y,z) = \begin{cases} u \in \{0,1\}, & \text{for } \ t \le s < (t+\varepsilon)\wedge\rho \\ \hat{\boldsymbol{u}}(s,y,z), & \text{for } \ (t+\varepsilon)\wedge\rho \le s < T\wedge\rho. \end{cases}$$

If $u = 1$, then $\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\boldsymbol{\varepsilon}}}) = \psi(t,F) \le \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}}) = V(t,F,m)$. Therefore, $\hat{\boldsymbol{u}}$ automatically satisfies the definition of an equilibrium stopping rule. If $u = 0$, we have

$$\begin{aligned}
\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}}) &= \mathbb{E}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t\right] - \frac{\gamma(F)}{2}\mathrm{Var}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t\right] \\
&= \mathbb{E}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho}) - \frac{\gamma(F)}{2}H^2(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t\right] + \frac{\gamma(F)}{2}\left\{\mathbb{E}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t\right]\right\}^2 \\
&= e^{-\int_t^{t+\varepsilon}\lambda_{x+u}du}\mathbb{E}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho}) - \frac{\gamma(F)}{2}H^2(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t+\varepsilon\right] \\
&+ \int_t^{t+\varepsilon} e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\mathbb{E}_t\left[H(F_s) - \frac{\gamma(F)}{2}H^2(F_s))\right]ds \\
&+ \frac{\gamma(F)}{2}\left\{e^{-\int_t^{t+\varepsilon}\lambda_{x+u}du}\mathbb{E}_t\left[H(F_{\tau^{\boldsymbol{u}^{\varepsilon}}\wedge\rho})\middle|\rho>t+\varepsilon\right] + \int_t^{t+\varepsilon} e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\mathbb{E}_t\left[H(F_s)\right]\right\}^2 \\
&= (1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F)}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right]\right\} \\
&+ \varepsilon\lambda_{x+t}\left[H(F) - \frac{\gamma(F)}{2}H^2(F)\right] \\
&+ \frac{\gamma(F)}{2}\left\{(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[\mathbb{E}_{t+\varepsilon}\left(H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right)\right] + \lambda_{x+t}\varepsilon H(F)\right\}^2 + o(\varepsilon).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathcal{J}(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon};\tau^{\hat{\boldsymbol{u}}}) &= \mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F_{t+\varepsilon})}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right] \\
&+ \frac{\gamma(F_{t+\varepsilon})}{2}\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right]\right\}^2.
\end{aligned}$$

Taking $\mathbb{E}_t[\cdot]$ on both sides of the equation, we obtain

$$\begin{aligned}
\mathbb{E}_t\left[\mathcal{J}(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon};\tau^{\hat{\boldsymbol{u}}})\right] &= \mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F_{t+\varepsilon})}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right]\right\} \\
&+ \mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\middle|\rho>t+\varepsilon\right]\right\}^2\right].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}}) &= (1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})-\frac{\gamma(F)}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho>t+\varepsilon\right]\right\}\\
&+\varepsilon\lambda_{x+t}\left[H(F)-\frac{\gamma(F)}{2}H^2(F)\right]\\
&+\frac{\gamma(F)}{2}\left\{(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[\mathbb{E}_{t+\varepsilon}\left(H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho>t+\varepsilon\right)\right]+\lambda_{x+t}\varepsilon H(F)\right\}^2\\
&+\mathbb{E}_t\left[\mathcal{J}(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon};\tau^{\hat{\boldsymbol{u}}})\right]-\mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})-\frac{\gamma(F_{t+\varepsilon})}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho>t+\varepsilon\right]\right\}\\
&-\mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho>t+\varepsilon\right]\right\}^2\right]+o(\varepsilon)\\
&=(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[f(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon},F)\right]+\varepsilon\lambda_{x+t}\left[H(F)-\frac{\gamma(F)}{2}H^2(F)\right]\\
&+\frac{\gamma(F)}{2}\left\{(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[g(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon})\right]+\lambda_{x+t}\varepsilon H(F)\right\}^2+\mathbb{E}_t\left[V(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon})\right]\\
&-\mathbb{E}_t\left[f(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon},F_{t+\varepsilon})\right]-\mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}g^2(t+\varepsilon,F_{t+\varepsilon},M_{t+\varepsilon})\right]+o(\varepsilon).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}})-\mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}})\\
&=\varepsilon\mathcal{L}V(t,F,m)-\varepsilon\lambda_{x+t}f^F(t,F,m)+\varepsilon\mathcal{L}f^F(t,F,m)+\varepsilon\lambda_{x+t}\left[H(F)-\frac{\gamma(F)}{2}H^2(F)\right]\\
&-\varepsilon\gamma(F)\lambda_{x+t}g^2(t,F,m)+\varepsilon\gamma(F)\lambda_{x+t}H(F)g(t,F,m)-\varepsilon\mathcal{L}f(t,F,m,y)-\varepsilon\mathcal{L}\frac{\gamma(F)}{2}g^2(t,F,m)\\
&+\mathbb{E}_t\left[\int_t^{t+\varepsilon}V_m-\alpha\mathbb{1}_{\{F\geq\theta\}}\left(V_F-f_y-\frac{\gamma_F}{2}g^2(t,F,m)\right)dM_s\right]+o(\varepsilon),
\end{aligned}
$$

where $f^y(t,F,m) = f(t,F,m,y)$ with the fourth variable y fixed. By rearranging the equation above, we get

$$
\begin{aligned}
&\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}}) - \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}}) \\
&= \varepsilon\Bigg\{ V_t + (\mu - c\mathbb{1}_{\{F\geq\theta\}})FV_F\left(V_F - f_y - \frac{\gamma_F}{2}g^2\right) \\
&+ \frac{1}{2}\left(V_{FF} - f_{yy} - 2f_{Fy} - \frac{\gamma_{FF}}{2}g^2 - \gamma(F)g_F^2 - 2\gamma_F g g_F\right) \\
&- \frac{\lambda_{x+t}\gamma(F)}{2}(g - \max(F,G))^2 - \lambda_{x+t}V + \lambda_{x+t}\max(F,G)\Bigg\} \\
&+ \mathbb{E}_t\Big[\int_t^{t+\varepsilon} V_m - \alpha\mathbb{1}_{\{F\geq\theta\}}\big(V_F - f_y - \frac{\gamma_F}{2}g^2(t,F,m)\big)dM_s\Big] + o(\varepsilon).
\end{aligned}
$$

Invoking Eq. (A.1) and (A.2), dividing both sides of the equation by ε, and letting ε go to 0, we get

$$
\lim_{\varepsilon\downarrow 0}\frac{\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}}) - \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}})}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{o(\varepsilon)}{\varepsilon} = 0.
$$

This completes the proof. □

A.2 Algorithm for the System of EHJB Equations (A.1)-(A.7)

Given the complex nature of the system of EHJB equations (A.1)-(A.7), we propose to numerically solve it via the development of an algorithm. Inspired by the work of Wang & Forsyth (2011) on the development of an algorithm for MV optimal control problems, the proposed algorithm will circumvent the complexity arising from directly solving the system of EHJB equations (A.1)-(A.7) corresponding to our MV optimal stopping problem. We go a step further than Wang & Forsyth (2011) by establishing a connection between the system of EHJB equations (A.1)-(A.7) and the algorithm.

Directly solving the system of EHJB equations (A.1)-(A.7) numerically using a finite difference method roughly entails the following steps:

1. Partition the interval $[0,T]$ into N equal sub-intervals each with length $\Delta t = \frac{T}{N}$, and endpoints $t_n = T - n\Delta t$ for $n = 0,1,...,N$.

2. At time t_n, solve g numerically using Eq. (A.6) with boundary conditions (A.7) and solve f numerically for each fixed $y \in [0, m]$ using Eq. (A.4) with boundary conditions (A.5), by assuming that $\hat{u} = 0$ (no surrender) over the domains $[0, m) \times [0, \infty)$.

3. Substitute the resulting f and g in Eq. (A.1) and solve V numerically using Eq. (A.1) with boundary conditions (A.2) to obtain an equilibrium stopping rule $\hat{u}$ (or, equivalently, an equilibrium stopping region).

4. Update f and g so that their values correspond to the equilibrium stopping rule $\hat{u}$. Specifically, if $\hat{u}(t_n, F, m) = 0$, keep $f(t_n, F, m, y)$ and $g(t_n, F, m)$ unchanged; if $\hat{u}(t_n, F, m) = 1$, let $f(t_n, F, m, y) = \psi(t_n, F) - \frac{\gamma(F)}{2}\psi^2(t, F)$ and $g(t_n, F, m) = \psi(t_n, F)$.

5. Substitute the updated f and g into Eq. (A.1) to update V.

6. Repeat steps 4 and 5 until V converges.

7. Move onto time t_{n+1} and repeat steps 2-6.

The challenges of solving the system of EHJB equations (A.1)-(A.7) numerically using the above steps are three-fold. First, solving f is time-consuming as it entails solving an unknown function with two state variables (F and m) for each fixed $y \in [0, m]$. Second, the complex form of Eq. (A.1) makes the computational process highly non-trivial. Third, at each t_n, a time-consuming iterative procedure is required to solve for V and identify an equilibrium stopping rule $\hat{u}$.

In light of the above, we develop an algorithm based on Wang & Forsyth (2011) to circumvent the complexity of numerically solving the system of EHJB equations (A.1)-(A.7) directly. The basic idea is to first represent the mean and variance of VA payouts H by their first and second moments. Therefore, the value function V in Eq. (3.16) can be rewritten as

$$V(t, F, m) = \sup_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \left\{ \nu^1(t, F, m; \tau^{\boldsymbol{u}}) - \frac{\gamma(F)}{2} \left[\nu^2(t, F, m; \tau^{\boldsymbol{u}}) - \left(\nu^1(t, F, m; \tau^{\boldsymbol{u}}) \right)^2 \right] \right\}, \quad \text{(A.9)}$$

where

$$\nu^p(t, F, m; \tau^{\boldsymbol{u}}) = \mathbb{E}_{t,F,m} \left[H^p \left(F_{\tau^{\boldsymbol{u}} \wedge \rho} \right) \middle| \rho > t \right], \quad \text{for} \quad p = 1, 2 \quad \text{(A.10)}$$

denote the first and second moments of the VA payout function H under a certain stopping rule $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, respectively. Furthermore, we observe from Eq. (A.8) that the value functions

V, f, and g are all combinations of ν^1 and ν^2 under an equilibrium stopping rule. We also observe that, over the domain $[0,T] \times \bar{\mathcal{O}}$ s.t. $\hat{u} = 0$, ν^p $(p = 1, 2)$ satisfy

$$\frac{\partial \nu^p}{\partial t} + \mathcal{L}\nu^p + \lambda_{x+t} \max{(F, G)^p} = 0, \tag{A.11}$$

for $\mathcal{L}\nu^p = (\mu - c\mathbb{1}_{\{F<\theta\}})F(\nu^p)_F + \frac{1}{2}\sigma^2 F^2 (\nu^p)_{FF} - \lambda_{x+t}\nu^p$, with boundary conditions

$$\begin{cases} \nu^p(T, F, m) = \max(F, G_T)^p, & \text{for} \quad (F, m) \in [0, m] \times [0, \infty), \\ \frac{\partial \nu^p}{\partial t}|_{F=0} = (\lambda_{x+t} + \gamma)\nu^p|_{F=0} - \lambda_{x+t} G^p, & \text{for} \quad (t, m) \in [0, T) \times [0, \infty), \\ \frac{\partial \nu^p}{\partial m}\Big|_{F=m} = \alpha \mathbb{1}_{\{F \geq \theta\}} \frac{\partial \nu^p}{\partial F}\Big|_{F=m}, & \text{for} \quad (t, m) \in [0, T) \times [0, \infty), \\ \lim_{m \to \infty} \nu^p(t, m, m) = \psi^p(t, m), & \text{for} \quad (t, F) \in [0, T). \end{cases} \tag{A.12}$$

In view of this, instead of numerically solving the system of EHJB equations (A.1)-(A.7) in a direct manner, one can get ν^1 and ν^2 by solving their corresponding PDEs to obtain an optimal stopping rule. We break this procedure into the following steps:

Step 1 At time t_n, solve ν^p $(p = 1, 2)$ numerically using Eq. (A.11) with boundary conditions (A.12), by assuming that $u = 0$ (no surrender) over the domain $\bar{\mathcal{O}}$.

Step 2 Obtain an optimal stopping rule $\hat{u}$ by substituting the corresponding ν^1 and ν^2 in Eq. (A.9).

Step 3 Update ν^1 and ν^2 s.t. if $\hat{u}(t_n, F, m) = 0$, keep $\nu^1(t_n, F, m)$ and $\nu^2(t_n, F, m)$ unchanged; if $\hat{u}(t_n, F, m) = 1$, let $\nu^1(t_n, F, m) = \psi(t_n, F)$ and $\nu^2(t_n, F, m) = \psi^2(t_n, F)$.

Step 4 Move onto time t_{n+1} and repeat **Step 1** to **Step 3**.

This iterative procedure circumvents many of the challenges resulting from solving the system of EHJB equations (A.1)-(A.7) directly. More specifically, we no longer deal with the high dimensionality of the function f, and the iterative procedure at each t_n has been eliminated. Also, the determination of an equilibrium stopping rule $\hat{u}$ from Eq. (A.9) is much simpler than solving the complex Eq. (A.1). However, since we bypass solving the system of EHJB equations (A.1)-(A.7) directly by solving for ν^1 and ν^2 (through their PDEs), we need to establish that an optimal stopping rule generated by going through **Step 1** to **Step 4** is the same as an equilibrium stopping rule resulting from the solution of the system of EHJB equations (A.1)-(A.7). The following theorem establishes this equivalence.

Theorem 8. *Let $\hat{u}$ be an optimal stopping rule resulting from **Step 1** to **Step 4**. For any $(t, F, m) \in [0, T] \times \bar{\mathcal{O}}$ s.t. $\hat{u} = 0$, let ν^p ($p = 1, 2$) satisfy Eq. (A.11) with boundary conditions (A.12). For any $(t, F, m) \in [0, T] \times \bar{\mathcal{O}}$ s.t. $\hat{u} = 1$, let $\nu^1(t, F, m) = \psi(t, F)$ and $\nu^2(t, F, m) = \psi^2(t, F)$. Then,*

$$\begin{cases} V(t,F,m) := \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\hat{u}}) - \left(\nu^1(t,F,m;\tau^{\hat{u}})\right)^2\right] \\ f(t,F,m,y) := \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(t,F,m;\tau^{\hat{u}}) \\ g(t,F,m) := \nu^1(t,F,m;\tau^{\hat{u}}) \end{cases} \tag{A.13}$$

satisfy the system of EHJB equations (A.1)-(A.7).

Proof. Recall that, for $\hat{u} = 0$, the first two moments ν^p ($p = 1, 2$) satisfy Eq. (A.11) with boundary conditions (A.12). Therefore, the derivatives of V with respect to t, F, and m can then be expressed in terms of those of ν^1 and ν^2 as follows:

$$\begin{cases} V_t = \nu^1_t - \frac{\gamma(F)}{2}\left(\nu^2_t - 2\nu^1\nu^1_t\right), \\ V_F = \nu^1_F - \frac{\gamma_F}{2}\left(\nu^2 - (\nu^1)^2\right) - \frac{\gamma(F)}{2}\left(\nu^2_F - 2\nu^1\nu^1_F\right), \\ V_{FF} = \nu^1_{FF} - \frac{\gamma_{FF}}{2}\left(\nu^2 - (\nu^1)^2\right) - \gamma_F\left(\nu^2_F - 2\nu^1\nu^1_F\right) - \frac{\gamma(F)}{2}\left(\nu^2_{FF} - 2(\nu^1_F)^2 - 2\nu^1\nu^1_{FF}\right), \\ V_m = \nu^1_m - \frac{\gamma(F)}{2}\left(\nu^2_m - 2\nu^1\nu^1_m\right). \end{cases}$$

Similarly, for f, its derivatives can be expressed as

$$\begin{cases} f_t = \nu^1_t - \frac{\gamma(y)}{2}\nu^2_t, \\ f_F = \nu^1_F - \frac{\gamma(y)}{2}\nu^2_F, \\ f_y\big|_{y=F} = -\frac{\gamma'(F)}{2}\nu^2, \\ f_{FF} = \nu^1_{FF} - \frac{\gamma(y)}{2}\nu^2_{FF}, \\ f_{yy}\big|_{y=F} = -\frac{\gamma'(FF)}{2}\nu^2, \\ f_{Fy}\big|_{y=F} = -\frac{\gamma'(F)}{2}\nu^2_F, \\ f_m = \nu^1_m - \frac{\gamma(y)}{2}\nu^2_m. \end{cases}$$

First, we check if linear combination (A.13) solves Eq. (A.1) with boundary conditions (A.2). For $\hat{u} = 1$,

$$\begin{aligned} V(t,F,m) &= \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\hat{u}}) - \left(\nu^1(t,F,m;\tau^{\hat{u}})\right)^2\right] \\ &= \max(F,G) - \frac{\gamma(F)}{2}\left(\max(F,G)^2 - \max(F,G)^2\right) \\ &= \max(F,G) = \psi(t,F). \end{aligned}$$

For $\hat{u} = 0$, we know that

$$V_t + \left(\mu - c\mathbb{1}_{\{F\leq\theta\}}\right) F \left(V_F - f_y - \frac{\gamma_F}{2} g^2\right) + \frac{1}{2}\sigma^2 F^2 \Big(V_{FF} - f_{yy} - 2 f_{Fy} - \frac{\gamma_{FF}}{2} g^2 - \gamma(F) g_F^2 - 2\gamma_F g g_F\Big) - \frac{\lambda_{x+t}\gamma(F)}{2} \left(g - \max(F,G)\right)^2 - \lambda_{x+t} V + \lambda_{x+t} \max(F,G) = 0.$$

We plug in the derivatives of V, f, and g in terms of ν^1 and ν^2 and verify that the equality still holds:

$$\begin{aligned}
\nu_t^1 - &\frac{\gamma(F)}{2}\left(\nu_t^2 - 2\nu^1\nu_t^1\right) + \left(\mu - c\mathbb{1}_{\{F\leq\theta\}}\right) F \left(\nu_F^1 - \frac{\gamma(F)}{2}\nu_F^2 + \gamma(F)\nu^1\nu_F^1\right) \\
&+ \frac{1}{2}\sigma^2 F^2 \left(\nu_{FF}^1 - \frac{\gamma(F)}{2}\nu_{FF}^2 + \gamma(F)\nu^1\nu_{FF}^1\right) - \frac{\lambda_{x+t}\gamma(F)}{2}\left(\nu^1 - \max(F,G)\right)^2 \\
&- \lambda_{x+t}\left[\nu^1 - \frac{\gamma(F)}{2}\left(\nu^2 - (\nu^1)^2\right)\right] + \lambda_{x+t}\max(F,G) \\
&= \left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^1 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F,G)\right] \\
&- \frac{\gamma(F)}{2}\left[\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^2 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}\max(F,G)^2\right] \\
&+ \gamma(F)\nu^1\left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^1 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F,G)\right] \\
&= 0.
\end{aligned}$$

We must now check boundary conditions (A.2):

- for the terminal condition,

$$V(T,F,m) = \nu^1(T,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(T,F,m;\tau^{\hat{u}}) - (\nu^1(T,F,m;\tau^{\hat{u}})^2\right] = \max(F,G),$$

- for the condition where $F = 0$,

$$
\begin{aligned}
V_t - &\frac{\lambda_{x+t}\gamma(0)}{2}\left(\nu^1 - G\right)^2 - \lambda_{x+t}V + \lambda_{x+t}G \\
&= \nu_t^1 - \frac{\gamma(0)}{2}\nu_t^2 + \gamma(0)\nu^1\nu_t^1 - \frac{\lambda_{x+t}\gamma(0)}{2}(\nu^1)^2 + \lambda_{x+t}\gamma(0)\nu^1 G - \frac{\lambda_{x+t}\gamma(0)}{2}G^2 \\
&- \lambda_{x+t}\left(\nu^1 - \frac{\gamma(0)}{2}(\nu^2 - (\nu^1)^2)\right) + \lambda_{x+t}G \\
&= \left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}G\right) - \frac{\gamma(0)}{2}\left(\nu^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}G^2\right) \\
&+ \gamma(0)\nu^1\left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}G\right) \\
&= 0,
\end{aligned}
$$

- for the condition where $F = m$,

$$
\begin{aligned}
V_m - &\alpha\mathbb{1}_{\{F\geq\theta\}}\left(V_F - f_y - \frac{\gamma'(F)}{2}(\nu^1)^2\right) \\
&= \nu_m^1 - \frac{\gamma(F)}{2}\left(\nu_m^2 - 2\nu^1\nu_m^1\right) \\
&- \alpha\mathbb{1}_{\{F\geq\theta\}}\left[\nu_F^1 - \frac{\gamma_F}{2}\left(\nu^2 - (\nu^1)^2\right) - \frac{\gamma(F)}{2}\left(\nu_F^2 - 2\nu^1\nu_F^1\right) + \frac{\gamma_F}{2}\nu^2 - \frac{\gamma_F}{2}(\nu^1)^2\right] \\
&= \nu_m^1 - \frac{\gamma(F)}{2}\left(\nu_m^2 - 2\nu^1\nu_m^1\right) - \alpha\mathbb{1}_{\{F\geq\theta\}}\left[\nu_F^1 - \frac{\gamma(F)}{2}\left(\nu_F^2 - 2\nu^1\nu_F^1\right)\right] \\
&= \left(\nu_m^1 - \alpha\mathbb{1}_{\{F\geq\theta\}}\nu_F^1\right) - \frac{\gamma(F)}{2}\left(\nu_m^2 - \alpha\mathbb{1}_{\{F\geq\theta\}}\nu_F^2\right) + \gamma(F)\nu^1\left(\nu_m^1 - \alpha\mathbb{1}_{\{F\geq\theta\}}\nu_F^1\right) \\
&= 0,
\end{aligned}
$$

- and for the last condition of Eq. (A.2), we have

$$
\begin{aligned}
\lim_{m\to\infty} V(t,m,m) &= \lim_{m\to\infty}\left\{\nu^1(T,m,m;\tau^{\hat{u}}) - \frac{\gamma(m)}{2}\left[\nu^2(T,m,m;\tau^{\hat{u}}) - (\nu^1)^2(T,m,m;\tau^{\hat{u}})\right]\right\} \\
&= \psi(t,m).
\end{aligned}
$$

Finally, we check if the function f defined in linear combination (A.13) solves Eq. (A.4) with boundary conditions (A.5). For $\hat{u} = 1$,

$$\begin{aligned} f(t, F, m, y) &= \nu^1(t, F, m; \tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(t, F, m; \tau^{\hat{u}}) \\ &= \max(F, G) - \frac{\gamma(y)}{2}\max(F, G)^2 \\ &= \psi(t, F) - \frac{\gamma(y)}{2}\psi(t, F). \end{aligned}$$

For $\hat{u} = 0$, one obtains

$$f_t + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right) F f_F + \frac{1}{2}\sigma^2 F^2 f_{FF} - \lambda_{x+t} f + \lambda_{x+t}\left[\max(F, G) - \frac{\gamma(y)}{2}\max(F, G)^2\right] = 0.$$

As before, we plug in the derivatives of f in terms of ν^1 and ν^2 and verify that the equality still holds:

$$\begin{aligned} \nu_t^1 - &\frac{\gamma(y)}{2}\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right) F \left(\nu_F^1 - \frac{\gamma(y)}{2}\nu_F^2\right) + \frac{1}{2}\sigma^2 F^2 \left(\nu_{FF}^1 - \frac{\gamma(y)}{2}\nu_{FF}^2\right) \\ &- \lambda_{x+t}\left(\nu^1(t, F, m) - \frac{\gamma(y)}{2}\nu^2(t, F, m)\right) + \lambda_{x+t}\left[\max(F, G) - \frac{\gamma(y)}{2}\max(F, G)^2\right] \\ &= \left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^1 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F, G)\right] \\ &- \frac{\gamma(y)}{2}\left[\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^2 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}\max(F, G)^2\right] \\ &= 0. \end{aligned}$$

Now, we check boundary conditions (A.5):

- for the terminal condition,

$$f(T, F, m, y) = \nu^1(T, F, m; \tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(T, F, m; \tau^{\hat{u}}) = \max(F, G) - \frac{\gamma(y)}{2}\max(F, G)^2,$$

- for the condition where $F = 0$,

$$\begin{aligned} f_t - &\lambda_{x+t} f + \lambda_{x+t} G - \frac{\gamma(y)}{2}\lambda_{x+t} G^2 \\ &= \nu_t^1 - \frac{\gamma(y)}{2}\nu_t^2 - \lambda_{x+t}\left(\nu^1 - \frac{\gamma(y)}{2}\nu^2\right) + \lambda_{x+t} G - \frac{\gamma(y)}{2}\lambda_{x+t} G^2 \\ &= \left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t} G\right) - \frac{\gamma(y)}{2}\left(\nu^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t} G\right) \\ &= 0, \end{aligned}$$

- for the condition where $F = m$,

$$
\begin{aligned}
f_m - \alpha \mathbb{1}_{\{F \geq \theta\}} f_F &= \nu_m^1 - \frac{\gamma(y)}{2}\nu_m^2 - \alpha \mathbb{1}_{\{F \geq \theta\}} \left(\nu_F^1 - \frac{\gamma(y)}{2}\nu_F^2 \right) \\
&= \left(\nu_m^1 - \alpha \mathbb{1}_{\{F \geq \theta\}} \nu_F^1 \right) - \frac{\gamma(y)}{2} \left(\nu_m^2 - \alpha \mathbb{1}_{\{F \geq \theta\}} \nu_F^2 \right) \\
&= 0,
\end{aligned}
$$

- and for the last condition of Eq. (A.5), we have

$$
\begin{aligned}
\lim_{m \to \infty} f(t, m, m, y) &= \lim_{m \to \infty} \left\{ \nu^1(T, m, m; \tau^{\hat{u}}) - \frac{\gamma(y)}{2} \nu^2(T, m, m; \tau^{\hat{u}}) \right\} \\
&= \psi(t, m) - \frac{\gamma(y)}{2} \psi^2(t, m).
\end{aligned}
$$

This completes the proof. □

Upon the establishment of Theorem 8, we can determine an equilibrium stopping rule numerically by going through **Step 1** to **Step 4**. A detailed algorithmic procedure is as follows.

Partition $[0, 1]$ into K equal intervals each with length $\Delta z = \frac{1}{K}$ and endpoints $z_i = i\Delta z$ for $i = 0, 1, ..., K$. Truncate the domain of m at $m_{\max}$ and partition $[0, m_{\max}]$ into M equal intervals each with length $\Delta m = \frac{m_{\max}}{M}$ and endpoints $m_j = j\Delta m$ for $j = 0, 1, ..., M$. Define $V_{i,j}^n := V(t_n, z_i, m_j)$, $(\nu_u^1)_{i,j}^n := \nu^1(t_n, z_i, m_j; \tau^u)$, $(\nu_u^2)_{i,j}^n := \nu^2(t_n, z_i, m_j; \tau^u)$, and $\psi_{i,j}^n := \psi(t_n, z_i, m_j)$ for $n = 1, ..., N$, $i = 0, ..., K$, $j = 0, ..., M$, and $u \in \{0, 1\}$. Moreover, if $u = 1$, we have $(\nu_u^1)_{i,j}^n = (\nu_1^1)_{i,j}^n = \psi_{i,j}^n$ and $(\nu_u^2)_{i,j}^n = (\nu_1^2)_{i,j}^n = (\psi_{i,j}^n)^2$.

A.3 Robustness of Initial Investment F_0

In this section, we show that the results of fair fees and compatible sets are independent of the initial investment F_0.

Theorem 9. *Define $\theta_0 = \frac{\theta}{F_0}$ as the threshold ratio for the HWM fee. The pair of fair fees (c, α), in Definition 3, and the compatible set, in Definition 6, are independent of initial investment F_0.*

Algorithm 1 Equilibrium Stopping Rule

Set $V_{i,j}^0 = z_i m_j \vee G$, $(\nu^1)_{i,j}^0 = z_i m_j \vee G$, and $(\nu^2)_{i,j}^0 = (z_i m_j \vee G)^2$ for $i = 0, ..., K$ and $j = 0, ..., M$.
for $n = 0, ..., N-1$ **do**
  Obtain $(\nu_0^1)_{i,j}^{n+1}$ and $(\nu_0^2)_{i,j}^{n+1}$ for $i = 0, ..., K$ and $j = 0, ..., M$ using a finite difference method to solve Eq. (A.11) and (A.12) with $p = 1, 2$ for ν^1 and ν^2, respectively.
  Determine $\hat{u}$ by $V_{i,j}^{n+1} = \max_{u \in \{0,1\}} \left[(\nu_u^1)_{i,j}^{n+1} - \frac{\gamma(z_i m_j \vee G)}{2} \left((\nu_u^2)_{i,j}^{n+1} - ((\nu_u^1)_{i,j}^{n+1})^2 \right) \right]$ $(*)$.
  if $\hat{u} = 0$ **then**
    $(\nu_{\hat{u}}^1)_{i,j}^{n+1} = (\nu_0^1)_{i,j}^{n+1}$ and $(\nu_{\hat{u}}^2)_{i,j}^{n+1} = (\nu_0^2)_{i,j}^{n+1}$
  else
    $(\nu_{\hat{u}}^1)_{i,j}^{n+1} = \psi_{i,j}^{n+1}$ and $(\nu_{\hat{u}}^2)_{i,j}^{n+1} = (\psi_{i,j}^{n+1})^2$.
  end if
end for

Proof. We first prove that the pair of fair fees (c, α) is independent of the initial investment F_0. Applying Itô's formula, we can obtain

$$F_t = F_0 \cdot e^{Y_t},$$

where Y_t solves the stochastic integral

$$Y_t = \int_0^t \left(\mu - c \mathbb{1}_{\{Y_s \le \log \theta_0\}} - \frac{1}{2}\sigma^2 \right) ds + \sigma W_t - \alpha \int_0^t \mathbb{1}_{\{Y_s \ge \log \theta_0\}} dY_s^*,$$

and $Y_t^* = \max_{0 \le s \le t} Y_s$. In light of the above, the pricing value function (3.3) at time $t = 0$ becomes

$$\begin{aligned} V(0, F_0, F_0; c, \alpha) &= \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}^{\mathbb{Q}} \left[{}_\tau p_x \psi(\tau, F_0 \cdot e^{Y_\tau}) e^{-r\tau} + \int_0^\tau {}_s p_x \lambda_{x+s} \max(F_0 \cdot e^{Y_s}, F_0 \cdot e^{gs}) e^{-rs} ds \right] \\ &= \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}^{\mathbb{Q}} \left[{}_\tau p_x \cdot F_0 \cdot \psi(\tau, e^{Y_\tau}) e^{-r\tau} + F_0 \cdot \int_0^\tau {}_s p_x \lambda_{x+s} \max(e^{Y_s}, e^{gs}) e^{-rs} ds \right] \\ &= F_0 \cdot V(0, 1, 1; c, \alpha). \end{aligned}$$

Therefore, by Eq. (3.11), the pair of fair fees (c, α) is determined by

$$V(0, 1, 1; c, \alpha) = 1,$$

which is independent of F_0.

Next we show the compatible set is also independent of F_0. By Eq. (3.14), PH welfare from holding the VA is

$$\begin{aligned} V(0,F_0,F_0;\gamma) &= \sup_{\tau\in\mathcal{T}_{[0,T]}} \left\{ \mathbb{E}\left[H(F_0\cdot e^{Y_{\tau\wedge\rho}})\right] - \frac{\gamma}{2\cdot F_0}\mathrm{Var}\left[H(F_0\cdot e^{Y_{\tau\wedge\rho}})\right] \right\} \\ &= F_0 \cdot \sup_{\tau\in\mathcal{T}_{[0,T]}} \left\{ \mathbb{E}\left[H(e^{Y_{\tau\wedge\rho}})\right] - \frac{\gamma}{2}\mathrm{Var}\left[H(e^{Y_{\tau\wedge\rho}})\right] \right\} \\ &= F_0 \cdot V(0,1,1;\gamma). \end{aligned}$$

Similarly, PH welfare from holding the two alternative investments are given by

$$\begin{cases} V_{\mathrm{RFB}}(0,R_0;\gamma) = R_0 \cdot V_{\mathrm{RFB}}(0,1;\gamma), \\ V_{\mathrm{PF}}(0,S_0;\gamma) = S_0 \cdot V_{\mathrm{PF}}(0,1;\gamma). \end{cases}$$

Since $F_0 = R_0 = S_0$, the compatible set in Definition 6 can be rewritten as

$$\begin{aligned} & \{\gamma \geq 0 : V(0,F_0,F_0;\gamma) \geq \max\left(V_{\mathrm{RFB}}(0,R_0;\gamma), V_{\mathrm{PF}}(0,S_0;\gamma)\right)\} \\ = \; & \{\gamma \geq 0 : F_0 V(0,1,1;\gamma) \geq F_0 \max\left(V_{\mathrm{RFB}}(0,1;\gamma), V_{\mathrm{PF}}(0,1;\gamma)\right)\} \\ = \; & \{\gamma \geq 0 : V(0,1,1;\gamma) \geq \max\left(V_{\mathrm{RFB}}(0,1;\gamma), V_{\mathrm{PF}}(0,1;\gamma)\right)\}, \end{aligned}$$

which is also independent of F_0. □

Appendix B

Appendix for Chapter 4

B.1 Proof of Verification Theorem 6

Proof. It is easy to establish that

$$\begin{aligned}
V(t,F) &= \int_0^\infty g(t,F;x)\cdot f(x;k,\theta)dx \\
&= \int_0^\infty \mathbb{E}_{t,F}\left[e^{-x(\tau_{\hat{u}}\wedge\rho-t)}U\left[H(F_{\tau_{\hat{u}}\wedge\rho})\right]\middle|\rho>t\right]\cdot f(x;k,\theta)dx \\
&= \mathbb{E}_{t,F}\left[\int_0^\infty e^{-x(\tau_{\hat{u}}\wedge\rho-t)}f(x;k,\theta)dx\cdot U\left[H(F_{\tau_{\hat{u}}\wedge\rho})\right]\middle|\rho>t\right] \\
&= \mathbb{E}_{t,F}\left[h(\tau_{\hat{u}}\wedge\rho-t)\cdot U\left[H(F_{\tau_{\hat{u}}\wedge\rho})\right]\middle|\rho>t\right].
\end{aligned}$$

Now we only need to show that $\hat{u}$ is indeed an equilibrium stopping rule. Note that if $a=1$, then $J(t,F,\boldsymbol{u}^{\varepsilon,1}) = U\left[\psi(t,F)\right] \le V(t,F)$, based on the variational inequality (4.12). It follows that Eq. (4.11) is automatically satisfied. We turn to the case that $a=0$. It follows that

$$
\begin{aligned}
J(t,F,\boldsymbol{u}^{\varepsilon,0}) &= \mathbb{E}_{t,F}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t\right]\\
&= \mathbb{E}_{t,F}\Big[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\cdot\mathbb{1}_{\{\rho>t+\varepsilon\}}\\
&+ h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\cdot\mathbb{1}_{\{t\le\rho<t+\varepsilon\}}\Big|\rho>t\Big]\\
&= e^{-\int_t^{t+\varepsilon}\lambda_{y+u}du}\mathbb{E}_{t,F}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\\
&+ \int_t^{t+\varepsilon}\lambda_{y+s}e^{-\int_t^s\lambda_{y+u}du}\mathbb{E}_{t,F}\left\{h(s-t)\cdot U\left[H(F_s)\right]\right\}ds\\
&= (1-\lambda_{y+t}\varepsilon)\cdot\mathbb{E}_{t,F}\left\{\mathbb{E}_{t+\varepsilon}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right\}\\
&+ \varepsilon\lambda_{y+t}\cdot U\left[H(F)\right]+o(\varepsilon).
\end{aligned}
$$

We also have

$$
J(t,F_{t+\varepsilon},\boldsymbol{u}^{\varepsilon,0}) = \mathbb{E}_{t+\varepsilon}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t-\varepsilon)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right].
$$

Grouping them together, we obtain

$$
\begin{aligned}
J(t,F,\boldsymbol{u}^{\varepsilon,0}) &= \mathbb{E}_{t,F}\left[J(t,F_{t+\varepsilon},\hat{\boldsymbol{u}})\right]\\
&- \Bigg\{\mathbb{E}_{t,F}\left[\mathbb{E}_{t+\varepsilon}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t-\varepsilon)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right]\\
&- \mathbb{E}_{t,F}\left[\mathbb{E}_{t+\varepsilon}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right]\Bigg\}\\
&- \lambda_{y+t}\varepsilon\cdot\mathbb{E}_{t,F}\left\{\mathbb{E}_{t+\varepsilon}\left[h(\tau_{\boldsymbol{u}^{\varepsilon,0}}\wedge\rho-t)\cdot U\left[H(F_{\boldsymbol{u}^{\varepsilon,0}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right\}\\
&+ \varepsilon\lambda_{y+t}\cdot U\left[H(F)\right]+o(\varepsilon)\\
&= \mathbb{E}_{t,F}\left[V(t+\varepsilon,F_{t+\varepsilon})\right]\\
&+ \int_0^\infty(e^{-x\varepsilon}-1)\mathbb{E}_{t,F}\left[\mathbb{E}_{t+\varepsilon}\left[e^{-x(\tau_{\hat{\boldsymbol{u}}}\wedge\rho-t-\varepsilon)}\cdot U\left[H(F_{\hat{\boldsymbol{u}}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right]\cdot f(x;k,\theta)dx\\
&- \lambda_{y+t}\varepsilon\cdot\int_0^\infty e^{-x\varepsilon}\mathbb{E}_{t,F}\left[\mathbb{E}_{t+\varepsilon}\left[e^{-x(\tau_{\hat{\boldsymbol{u}}}\wedge\rho-t-\varepsilon)}U\left[H(F_{\hat{\boldsymbol{u}}\wedge\rho})\right]\Big|\rho>t+\varepsilon\right]\right]\cdot f(x;k,\theta)dx\\
&+ \varepsilon\lambda_{y+t}\cdot U\left[H(F)\right]+o(\varepsilon).
\end{aligned}
$$

Since

$$
g(t,F;x) = \mathbb{E}_{t,F}\left[e^{-x(\tau_{\hat{\boldsymbol{u}}}\wedge\rho-t)}\cdot U\left[H(F_{\hat{\boldsymbol{u}}\wedge\rho})\right]\Big|\rho>t\right],
$$

it follows that

$$\begin{aligned}
J(t,F,\boldsymbol{u}^{\varepsilon,0}) =& \mathbb{E}_{t,F}\left[V(t+\varepsilon, F_{t+\varepsilon})\right] + \int_0^\infty (e^{-x\varepsilon}-1)\mathbb{E}_{t,F}\left[g(t+\varepsilon, F_{t+\varepsilon};x)\right]\cdot f(x;k,\theta)dx \\
& - \lambda_{y+t}\varepsilon \cdot \int_0^\infty e^{-x\varepsilon}\mathbb{E}_{t,F}\left[g(t+\varepsilon, F_{t+\varepsilon};x)\right]\cdot f(x;k,\theta)dx + \varepsilon\lambda_{y+t}\cdot U\left[H(F)\right] + o(\varepsilon) \\
=& V(t,F) + \varepsilon\cdot\mathcal{L}V(t,F) + \int_0^\infty (e^{-x\varepsilon}-1)\mathbb{E}_{t,F}\left[g(t+\varepsilon, F_{t+\varepsilon};x)\right]\cdot f(x;k,\theta)dx \\
& - \lambda_{y+t}\varepsilon \cdot \int_0^\infty e^{-x\varepsilon}\mathbb{E}_{t,F}\left[g(t+\varepsilon, F_{t+\varepsilon};x)\right]\cdot f(x;k,\theta)dx + \varepsilon\lambda_{y+t}\cdot U\left[H(F)\right] + o(\varepsilon).
\end{aligned}$$

We can then derive that

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\frac{V(t,F)-J(t,F,\boldsymbol{u}^{\varepsilon,0})}{\varepsilon} =& -\mathcal{L}V(t,F) + \int_0^\infty (x+\lambda_{y+t})g(t,F;x)\cdot f(x;k,\theta)dx \\
& - \lambda_{y+t}\cdot U\left[H(F)\right] \geq 0.
\end{aligned}$$

□

B.2 Derivation of Eq. (4.20)

In what follows, we heuristically derive Eq. (4.20) from the insurer's profit Eq. (4.18). The statement that a solution of Eq. (4.20) has the form of Eq. (4.18) can be easily established by Feynman-Kac formula.

For $s \in [t, \hat{\tau}]$, by Itô's formula, we have

$$d\left(e^{-\int_t^s \lambda_{x+u}ds} I\right) = e^{-\int_t^s \lambda_{x+u}ds}\left(I_s + \mathcal{L}I\right)ds + F\sigma I_F e^{-\int_t^s \lambda_{x+u}ds} dW_s,$$

where $\mathcal{L}I$ is defined by Eq. (4.19). Taking integral from t to $\hat{\tau}$ and conditional expectation $\mathbb{E}_{t,\omega,F}[\cdot]$ on both sides of the equation, we get

$$-\mathbb{E}_{t,\omega,F}\left[\int_t^{\hat{\tau}} {}_{s-t}p_{x+t}\lambda_{x+s}\left(\omega_s - (G_s - F_s)_+\right)ds\right] = \mathbb{E}_{t,\omega,F}\left[\int_t^{\hat{\tau}} {}_{s-t}p_{x+t}\left(I_s + \mathcal{L}I\right)ds\right].$$

The integrand on both sides of the above equation must be equivalent for the equation to hold. Therefore, we have

$$I_t + \mathcal{L}I + \lambda_{x+t}\left(\omega - (G-F)_+\right) = 0,$$

which is the PDE Eq. (4.20) satisfied by the insurer's profit Eq. (4.18).

Appendix C

Appendix for Chapter 5

C.1 Derivation of the System of EHJB Equations (5.15)-(5.22)

In what follows, we derive the system of EHJB equations (5.15)-(5.22). We first assume that an equilibrium withdrawal law $\hat{\boldsymbol{\gamma}}$ exists and we start from the withdrawal law $\boldsymbol{\gamma}$ defined in Definition 9. Under the withdrawal law $\boldsymbol{\gamma}$, the objective function (5.14) becomes

$$\begin{aligned} J(t, F, A; \boldsymbol{\gamma}) &= \mathbb{E}_t \left[\int_t^T h(u-t) \cdot f\left[\boldsymbol{\gamma}(F_u^{\boldsymbol{\gamma}}, A_u^{\boldsymbol{\gamma}})\right] du + h(T-t) \cdot M(F_T^{\boldsymbol{\gamma}}, A_T^{\boldsymbol{\gamma}}) \right] \\ &= \int_t^T h(u-t) \mathbb{E}_t \left[\mathbb{E}_{t+\varepsilon} \left(f\left[\boldsymbol{\gamma}(F_u^{\boldsymbol{\gamma}}, A_u^{\boldsymbol{\gamma}})\right] \right) \right] du + h(T-t) \cdot \mathbb{E}_t \left[\mathbb{E}_{t+\varepsilon} \left(M(F_T^{\boldsymbol{\gamma}}, A_T^{\boldsymbol{\gamma}}) \right) \right] \\ &= \int_t^T h(u-t) \mathbb{E}_t \left[k^{\boldsymbol{\gamma}, u} \left(t+\varepsilon, F_{t+\varepsilon}^{\boldsymbol{\gamma}}, A_{t+\varepsilon}^{\boldsymbol{\gamma}}\right) \right] du + h(T-t) \cdot \mathbb{E}_t \left[H^{\boldsymbol{\gamma}}(t+\varepsilon, F_{t+\varepsilon}^{\boldsymbol{\gamma}}, A_{t+\varepsilon}^{\boldsymbol{\gamma}}) \right], \end{aligned} \tag{C.1}$$

where $\mathbb{E}_t[\cdot]$ is short for $\mathbb{E}_{t,F,A}[\cdot]$. Also, at time $t+\varepsilon$, the objective function becomes

$$
\begin{aligned}
J(t+\varepsilon, F^{\boldsymbol{\gamma}}_{t+\varepsilon}, A^{\boldsymbol{\gamma}}_{t+\varepsilon}; \boldsymbol{\gamma}) &= \mathbb{E}_{t+\varepsilon}\left[\int_{t+\varepsilon}^{T} h(u-t-\varepsilon)\cdot f\left[\boldsymbol{\gamma}(F^{\boldsymbol{\gamma}}_u, A^{\boldsymbol{\gamma}}_u)\right]du + h(T-t-\varepsilon)\cdot M(F^{\boldsymbol{\gamma}}_T, A^{\boldsymbol{\gamma}}_T)\right] \\
&= \int_{t+\varepsilon}^{T} h(u-t-\varepsilon)\cdot \mathbb{E}_{t+\varepsilon}\left[f\left[\boldsymbol{\gamma}(F^{\boldsymbol{\gamma}}_u, A^{\boldsymbol{\gamma}}_u)\right]\right]du \\
&\quad + h(T-t-\varepsilon)\cdot \mathbb{E}_{t+\varepsilon}\left[M(F^{\boldsymbol{\gamma}}_T, A^{\boldsymbol{\gamma}}_T)\right] \\
&= \int_{t+\varepsilon}^{T} h(u-t-\varepsilon)\cdot k^{\boldsymbol{\gamma},u}\left(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}\right)du \\
&\quad + h(T-t-\varepsilon)\cdot H^{\boldsymbol{\gamma}}(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}).
\end{aligned}
$$

Since, from time $t+\varepsilon$, the withdrawal law is equivalent to the equilibrium withdrawal law (from the definition of $\boldsymbol{\gamma}$ in Definition 9), the above equation can be written as

$$
V(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}) = \int_{t+\varepsilon}^{T} h(u-t-\varepsilon)\cdot k^{u}\left(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}\right)du + h(T-t-\varepsilon)\cdot H(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}).
$$

Taking expectation $\mathbb{E}_t$ on both sides of the above equation, we get

$$
\begin{aligned}
\mathbb{E}_t\left[V(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon})\right] &= \int_{t+\varepsilon}^{T} h(u-t-\varepsilon)\cdot \mathbb{E}_t\left[k^{u}\left(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}\right)\right]du \\
&\quad + h(T-t-\varepsilon)\cdot \mathbb{E}_t\left[H(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon})\right]. \qquad \text{(C.2)}
\end{aligned}
$$

Grouping Eq. (C.1) and Eq. (C.2) together, we can obtain

$$
\begin{aligned}
\mathbb{E}_t\left[V(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon})\right] &= J(t, F, A; \boldsymbol{\gamma}) \\
&\quad - \int_{t+\varepsilon}^{T} \left(h(u-t) - h(u-t-\varepsilon)\right)\cdot \mathbb{E}_t\left[k^{u}\left(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}\right)\right]du \\
&\quad - \int_{t}^{t+\varepsilon} h(u-t)\cdot \mathbb{E}_t\left[k^{u}\left(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon}\right)\right]du \\
&\quad - \left(h(T-t) - h(T-t-\varepsilon)\right)\cdot \mathbb{E}_t\left[H(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon})\right].
\end{aligned}
$$

By Dynkin's formula, the left hand side of the above equation becomes

$$
\mathbb{E}_t\left[V(t+\varepsilon, F^{\gamma}_{t+\varepsilon}, A^{\gamma}_{t+\varepsilon})\right] = V(t,F,A) + \varepsilon\cdot V_t(t,F,A) + \varepsilon\cdot \mathcal{A}^{\gamma}V(t,F,A) + o(\varepsilon),
$$

where

$$
\mathcal{A}^{\gamma}V(t,F,A) = (\mu - c)FV_F + \frac{1}{2}\sigma^2F^2V_{FF} - \gamma V_F - \gamma V_A.
$$

Plugging this in the above equation, we have

$$\begin{aligned}V(t,F,A)+\varepsilon\cdot V_t(t,F,A)+\varepsilon\cdot\mathcal{A}^{\gamma}V(t,F,A)+o(\varepsilon)&=J(t,F,A;\boldsymbol{\gamma})\\&-\int_{t+\varepsilon}^{T}\left(h(u-t)-h(u-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du\\&-\int_{t}^{t+\varepsilon}h(u-t)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du\\&-\left(h(T-t)-h(T-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[H(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon})\right],\end{aligned}$$

which, after some rearrangement, becomes

$$\begin{aligned}V(t,F,A)-J(t,F,A,\boldsymbol{\gamma})&=-\varepsilon\cdot V_t(t,F,A)-\varepsilon\cdot\mathcal{A}^{\gamma}V(t,F,A)\\&-\int_{t+\varepsilon}^{T}\left(h(u-t)-h(u-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du\\&-\int_{t}^{t+\varepsilon}h(u-t)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du\\&-\left(h(T-t)-h(T-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[H(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon})\right]+o(\varepsilon).\end{aligned}$$

Dividing both sides of the above equation by ε and taking the limit $\varepsilon\to 0$, we get

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{V(t,F,A)-J(t,F,A,\boldsymbol{\gamma})}{\varepsilon}&=-V_t(t,F,A)-\mathcal{A}^{\gamma}V(t,F,A)\\&-\lim_{\varepsilon\to0}\frac{\int_{t+\varepsilon}^{T}\left(h(u-t)-h(u-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du}{\varepsilon}\\&-\lim_{\varepsilon\to0}\frac{\int_{t}^{t+\varepsilon}h(u-t)\cdot\mathbb{E}_t\left[k^u\left(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon}\right)\right]du}{\varepsilon}\\&-\lim_{\varepsilon\to0}\frac{\left(h(T-t)-h(T-t-\varepsilon)\right)\cdot\mathbb{E}_t\left[H(t+\varepsilon,F^{\gamma}_{t+\varepsilon},A^{\gamma}_{t+\varepsilon})\right]}{\varepsilon}\\&=-V_t(t,F,A)-A^{\gamma}V(t,F,A)-\int_t^T h'(u-t)\cdot k^u(t,F,A)du\\&-f(\gamma)-h'(T-t)\cdot H(t,F,A)\geq 0.\end{aligned}$$

Note that the inequality holds due to the definition of an equilibrium withdrawal law in Definition 9. Therefore, PHs' welfare (a.k.a. the value function) V satisfies

$$V_t+\mathcal{A}V+\int_t^T h'(s-t)\cdot k^s(t,F,A)du+h'(T-t)\cdot H(t,F,A)+\sup_{\gamma}\left\{f(\gamma)-\gamma V_F-\gamma V_A\right\}=0.$$

And, under the equilibrium withdrawal law $\hat{\boldsymbol{\gamma}}$, PDEs (5.19) and (5.21) satisfied by k^s and H respectively are simply application of Feynman-Kac formula.

C.2 Algorithm for the System of EHJB Equations (5.15)-(5.22)

Due to the absence of an explicit solution, we resort to numerically solving the system of EHJB equations (5.15)-(5.22). Nevertheless, due to its complex form (a PIDE with many terms in Eq. (5.15)), high dimensionality (i.e., two state variables), and a time-consuming iterative procedure to attain an equilibrium withdrawal strategy in Eq. (5.15), numerically solving the system is extremely challenging using standard finite difference methods. In light of the above, we propose an efficient numerical algorithm that circumvents the aforementioned iterative procedure to obtain an equilibrium withdrawal strategy. We also show that the solution obtained by the algorithm is indeed an equilibrium withdrawal strategy.

It involves the following steps to numerically solve the system of EHJB equations (5.15)-(5.22) using a standard finite difference method.

1. Partition the interval $[0, T]$ into N equal sub-intervals each with length $\Delta t = \frac{T}{N}$, and endpoints $t_n = T - n\Delta t$ for $n = 0, 1, ..., N$.

2. At time t_n, fix a withdrawal strategy $\gamma \in \{0, G, \lambda\}$, obtain $H(t_n, F, A; \gamma)$ via Eq. (5.21) and Eq. (5.22) using a finite difference method, and obtain $k^s(t_n, F, A; \gamma)$ for each $s \in \{t_1, ..., t_n\}$ via Eq. (5.19) and Eq. (5.20) using finite difference method.

3. Plug the results of $H(t_n, F, A; \gamma)$ and $k^s(t_n, F, A; \gamma)$ for each $s \in \{t_1, ..., t_n\}$ into Eq. (5.15), and obtain $V(t_n, F, A)$ and the corresponding equilibrium withdrawal strategy $\hat{\gamma}_n$.

4. Update H and k^s so that they become $H(t_n, F, A; \hat{\gamma}_n)$ and $k^s(t_n, F, A; \hat{\gamma}_n)$ for each $s \in \{t_1, ..., t_n\}$.

5. Plug the updated $H(t_n, F, A; \hat{\gamma}_n)$ and $k^s(t_n, F, A; \hat{\gamma}_n)$ for each $s \in \{t_1, ..., t_n\}$ into Eq. (5.15) and obtain new values of $V(t_n, F, A)$ and a new withdrawal strategy $\hat{\gamma}_n$.

6. Repeat steps 4 and 5 until the values of $V(t_n, F, A)$ converge (i.e., an equilibrium withdrawal strategy $\hat{\gamma}_n$ is obtained).

7. Move to time t_{n+1} and repeat steps 2-6.

Note that, in addition to the high dimensionality and complicated form of the PIDE (i.e., it is necessary to solve one H function and n functions k^s at each time t_n), the iterative procedure steps 4-6 to obtain an equilibrium withdrawal strategy is very time-consuming.

In light of the above, we propose an efficient algorithm that circumvents this iterative procedure. The basic idea is that, at time t_n, we obtain $H(t_n, F, A; \gamma)$ and $k^s(t_n, F, A; \gamma)$ with each $s \in \{t_1, ..., t_n\}$ for each $\gamma \in \{0, G, \lambda\}$ using a standard finite difference method, and then determine values of $V(t_n, F, A)$ and an optimal withdrawal strategy $\hat{\gamma}_n$ by

$$V(t_n, F, A) = \max_{\gamma \in \{0,G,\lambda\}} \left[\sum_{s=t_n}^{t_1} h(s - t_n) \cdot k^s(t_n, F, A; \gamma) \cdot \Delta t + h(T - t_n) \cdot H(t_n, F, A; \gamma) \right]. \tag{C.3}$$

We break this procedure into the following steps:

Step 1 At time t_n, obtain $H(t_n, F, A; \gamma)$ and $k^s(t_n, F, A; \gamma)$ with each $s \in \{t_1, ..., t_n\}$ for each $\gamma \in \{0, G, \lambda\}$ using a standard finite difference method.

Step 2 Obtain $V(t_n, F, A)$ and an optimal withdrawal strategy $\hat{\gamma}_n$ via Eq. (C.3).

Step 3 Update H and k^s such that they become $H(t_n, F, A; \hat{\boldsymbol{\gamma}})$ and $k^s(t_n, F, A; \hat{\boldsymbol{\gamma}})$ for each $s \in \{t_1, ..., t_n\}$.

Step 4 Move to time t_{n+1} and repeat **Step 1** to **Step 3**.

The above procedure circumvents the time-consuming iterative procedure to determine an equilibrium withdrawal strategy using a standard finite difference method. Also, we no longer handle the complicated PDE (5.15) directly. In what follows, we establish that an optimal withdrawal strategy generated by going through **Step 1** to **Step 4** is the same as an equilibrium withdrawal strategy resulting from the solution of the system of EHJB equations (5.15)-(5.22). The following theorem establishes this equivalence.

Theorem 10. *Let $\hat{\gamma}$ be an optimal withdrawal strategy obtained from* ***Step 1****-****Step 4****. Then*

$$\begin{cases} V(t, F, A) := \int_t^T h(s - t) \cdot k^s(t, F, A; \hat{\gamma}) ds + h(T - t) \cdot H(t, F, A; \hat{\gamma}) \\ k^s(t, F, A; \hat{\gamma}) \\ H(t, F, A; \hat{\gamma}) \end{cases}$$

satisfy the system of EHJB equations (5.15)-(5.22)*.*

Proof. Under an optimal withdrawal strategy $\hat{\gamma}$, it is evident from the Feynman-Kac formula that k^s satisfies Eq. (5.19) with boundary conditions (5.20) for each $s \in [t, T)$, and H satisfies Eq. (5.21) with boundary conditions (5.22). Therefore, we only need to prove that

Under an optimal withdrawal strategy $\hat{\gamma}$, V satisfies Eq. (5.15) with boundary conditions (5.16)-(5.18).

We first summarize the derivatives of V with respect to t, F and A in terms of those of k^s and H in the following:

$$\begin{cases} V_F = \int_t^T h(s-t) \cdot k_F^s ds + h(T-t) \cdot H_F, \\ V_{FF} = \int_t^T h(s-t) \cdot k_{FF}^s ds + h(T-t) \cdot H_{FF}, \\ V_A = \int_t^T h(s-t) \cdot k_A^s ds + h(T-t) \cdot H_A, \\ V_t = -f(\hat{\gamma}) - \int_t^T h'(s-t) k^s ds + \int_t^T h(s-t) k_t^s ds - h'(T-t)H + h(T-t)H_t, \end{cases}$$

where k^s and H are short for $k^s(t, F, A; \hat{\gamma})$ and $H(t, F, A; \hat{\gamma})$, respectively. In light of the above, Eq. (5.15) becomes

$$\begin{aligned}
& V_t + (\mu - c)FV_F + \frac{1}{2}\sigma^2 F^2 V_{FF} + \int_t^T h'(s-t) \cdot k^s ds + h'(T-t) \cdot H + f(\hat{\gamma}) - \hat{\gamma} V_F - \hat{\gamma} V_A \\
&= -f(\hat{\gamma}) - \int_t^T h'(s-t) k^s ds + \int_t^T h(s-t) k_t^s ds - h'(T-t)H + h(T-t)H_t \\
&+ (\mu - c)F \cdot \int_t^T h(s-t) \cdot k_F^s ds + (\mu - c)F \cdot h(T-t) \cdot H_F \\
&+ \frac{1}{2}\sigma^2 F^2 \cdot \int_t^T h(s-t) \cdot k_{FF}^s ds + \frac{1}{2}\sigma^2 F^2 \cdot h(T-t) \cdot H_{FF} + \int_t^T h'(u-t) \cdot k^s ds + h'(T-t) \cdot H \\
&+ f(\hat{\gamma}) - \hat{\gamma} \cdot \left[\int_t^T h(s-t) \cdot k_F^s ds + h(T-t) \cdot H_F \right] - \hat{\gamma} \cdot \left[\int_t^T h(s-t) \cdot k_A^s ds + h(T-t) \cdot H_A \right] \\
&= \int_t^T h(s-t) \left[k_t^s + (\mu - c)F \cdot k_F^s + \frac{1}{2}\sigma^2 F^2 \cdot k_{FF}^s - \hat{\gamma} \cdot (k_F^s + k_A^s) \right] \\
&+ h(T-t) \cdot \left[H_t + (\mu - c)F \cdot H_F + \frac{1}{2}\sigma^2 F^2 \cdot H_{FF} - \hat{\gamma} \cdot (H_F + H_A) \right] = 0.
\end{aligned}$$

Therefore, we can conclude that $V(t, F, A) := \int_t^T h(s-t) \cdot k^s(t, F, A; \hat{\gamma}) ds + h(T-t) \cdot H(t, F, A; \hat{\gamma})$ satisfies Eq. (5.15). Now we check the boundary conditions. For $t = T$, we have

$$V(T, F, A) = \int_T^T h(s-t) \cdot 0 ds + h(T-T) \cdot [\max(F, A) - \kappa A] = \max(F, A) - \kappa A.$$

For $A = 0$, we have

$$V(t, F, 0) = 0 + h(T-t) \cdot Fe^{(\mu-c)(T-t)} = h(T-t) \cdot Fe^{(\mu-c)(T-t)}.$$

For $W \to \infty$, we have

$$V(t, W, A) \to 0 + h(T-t) \cdot Fe^{(\mu-c)(T-t)} = h(T-t) \cdot Fe^{(\mu-c)(T-t)}.$$

For $W = 0$, we have

$$\begin{aligned}
&V_t\Big|_{F=0} - \hat{\gamma} \cdot V_A\Big|_{F=0} + f(\hat{\gamma}) + \int_t^T h'(s-t) \cdot k^s\Big|_{F=0} ds + h'(T-t) \cdot H\Big|_{F=0} \\
&= -f(\hat{\gamma}) - \int_t^T h'(s-t)k^s\Big|_{F=0} ds + \int_t^T h(s-t)k_t^s\Big|_{F=0}) ds - h'(T-t)H\Big|_{F=0} + h(T-t)H_t\Big|_{F=0} \\
&- \hat{\gamma} \cdot \left[\int_t^T h(s-t) \cdot k_A^s\Big|_{F=0} ds + h(T-t) \cdot H_A\Big|_{F=0}\right] + f(\hat{\gamma}) + \int_t^T h'(u-t) \cdot k^s\Big|_{F=0} ds \\
&+ h'(T-t) \cdot H\Big|_{F=0} = 0.
\end{aligned}$$

This completes the proof. □

Upon the establishment of Theorem 10, we can determine an equilibrium withdrawal strategy numerically by going through **Step 1** to **Step 4**. A detailed algorithmic procedure is as follows.

Partition $[0, P]$ into K equal intervals each with length $\Delta A = \frac{1}{K}$ and endpoints $A_j = j\Delta A$ for $j = 0, 1, ..., K$. Truncate the domain of F at $F_{\max}$ and partition $[0, F_{\max}]$ into M equal intervals each with length $\Delta F = \frac{F_{\max}}{M}$ and endpoints $F_i = i\Delta F$ for $j = 0, 1, ..., M$. Define $V_{i,j}^n := V(t_n, F_i, A_j)$, $(k^s)_{i,j}^{n,\gamma} := k^s(t_n, F_i, A_j; \gamma)$, and $H_{i,j}^{n,\gamma} := H(t_n, F_i, A_j; \gamma)$.

Algorithm 2 **Equilibrium Withdrawal Strategy**

1: Set $V_{i,j}^0 = F_i \vee A_j - \kappa A_j$, and $H_{i,j}^0 = F_i \vee A_j - \kappa A_j$ for $i = 0, ..., M$ and $j = 0, ..., K$.

2: **for** $n = 0, ..., N-1$ **do**

3: For each $\gamma \in \{0, G, \lambda\}$, obtain $(k^s)_{i,j}^{n+1,\gamma}$ for each $s \in \{t_1, ..., t_{n+1}\}$ and $H_{i,j}^{n+1,\gamma}$ for $i = 0, ..., M$ and $j = 0, ..., K$ via Eqs. (5.19) and (5.21) with boundary conditions (5.20) and (5.22) using a standard finite difference method.

4: Determine $V_{i,j}^{n+1}$ and $\hat{\gamma}_{n+1}$ by

$$V_{i,j}^{n+1} = \max_{\gamma_{n+1} \in \{0, G, \lambda\}} \left[\sum_{s=t_{n+1}}^{t_1} h(s - t_{n+1}) \cdot (k^s)_{i,j}^{n+1,\gamma} \cdot \Delta t + h(T - t_{n+1}) \cdot H_{i,j}^{n+1,\gamma} \right].$$

5: Update k^s and H so that they become $(k^s)_{i,j}^{n+1,\hat{\gamma}_{n+1}}$ for $s \in \{t_1, ..., t_{n+1}\}$ and $H_{i,j}^{n+1,\hat{\gamma}_{n+1}}$ for $i = 0, ..., M$ and $j = 0, ..., K$.

6: **end for**