

# Fractional refinements of integral theorems

by

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## **Author's Declaration**

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

## **Statement of Contributions**

This thesis contains joint work with Logan Grout, Evelyne Smith-Roberge, Richard Brewster, and Douglas B. West. In particular, Chapter 2 is based on work with Logan Grout, Chapter 3 and 4 is based on work with Evelyne Smith Roberge, Chapter 6 is based on work with Richard Brewster, and Chapter 7 is based on work with Douglas B. West.

## Abstract

The focus of this thesis is to take theorems which deal with “integral” objects in graph theory and consider fractional refinements of them to gain additional structure.

A classic theorem of Hakimi says that for an integer  $k$ , a graph has maximum average degree at most  $2k$  if and only if the graph decomposes into  $k$  pseudoforests. To find a fractional refinement of this theorem, one simply needs to consider the instances where the maximum average degree is fractional.

We prove that for any positive integers  $k$  and  $d$ , if  $G$  has maximum average degree at most  $2k + \frac{2d}{k+d+1}$ , then  $G$  decomposes into  $k + 1$  pseudoforests, where one of pseudoforests has every connected component containing at most  $d$  edges, and further this pseudoforest is acyclic. The maximum average degree bound is best possible for every choice of  $k$  and  $d$ .

Similar to Hakimi’s Theorem, a classical theorem of Nash-Williams says that a graph has fractional arboricity at most  $k$  if and only if  $G$  decomposes into  $k$  forests. The Nine Dragon Tree Theorem, proven by Jiang and Yang, provides a fractional refinement of Nash-Williams Theorem. It says, for any positive integers  $k$  and  $d$ , if a graph  $G$  has fractional arboricity at most  $k + \frac{d}{k+d+1}$ , then  $G$  decomposes into  $k + 1$  forests, where one of the forests has maximum degree  $d$ .

We prove a strengthening of the Nine Dragon Tree Theorem in certain cases. Let  $k = 1$  and  $d \in \{3, 4\}$ . Every graph with fractional arboricity at most  $1 + \frac{d}{d+2}$  decomposes into two forests  $T$  and  $F$  where  $F$  has maximum degree  $d$ , every component of  $F$  contains at most one vertex of degree  $d$ , and if  $d = 4$ , then every component of  $F$  contains at most 8 edges  $e = xy$  such that both  $\deg(x) \geq 3$  and  $\deg(y) \geq 3$ .

In fact, when  $k = 1$  and  $d = 3$ , we prove that every graph with fractional arboricity  $1 + \frac{3}{5}$  decomposes into two forests  $T, F$  such that  $F$  has maximum degree 3, every component of  $F$  has at most one vertex of degree 3, further if a component of  $F$  has a vertex of degree 3 then it has at most 14 edges, and otherwise a component of  $F$  has at most 13 edges.

Shifting focus to problems which partition the vertex set, circular colouring provides a way to fractionally refine colouring problems. A classic theorem of Tuza says that every graph with no cycles of length  $1 \pmod k$  is  $k$ -colourable. Generalizing this to circular colouring, we get the following:

Let  $k$  and  $d$  be relatively prime, with  $k > 2d$ , and let  $s$  be the element of  $\mathbb{Z}_k$  such that  $sd \equiv 1 \pmod k$ . Let  $xy$  be an edge in a graph  $G$ . If  $G - xy$  is  $(k, d)$ -circular-colorable and  $G$  is not, then  $xy$  lies in at least one cycle in  $G$  of length congruent to  $is \pmod k$  for some  $i$  in  $\{1, \dots, d\}$ . If this does not occur with  $i \in \{1, \dots, d - 1\}$ , then  $xy$  lies in at least two cycles of length  $1 \pmod k$  and  $G - xy$  contains a cycle of length  $0 \pmod k$ .

This theorem is best possible with regards to the number of congruence classes when  $k = 2d + 1$ .

A classic theorem of Grötzsch says that triangle free planar graphs are 3-colourable. There are many generalizations of this result, however fitting the theme of fractional refinements, Jaeger conjectured that every planar graph of girth  $4k$  admits a homomorphism to  $C_{2k+1}$ . While we make no progress on this conjecture directly, one way to approach the conjecture is to prove critical graphs have large average degree. On this front, we prove:

Every 4-critical graph which does not have a  $(7, 2)$ -colouring and is not  $K_4$  or  $W_5$  satisfies  $e(G) \geq \frac{17v(G)}{10}$ , and every triangle free 4-critical graph satisfies  $e(G) \geq \frac{5v(G)+2}{3}$ .

In the case of the second theorem, a result of Davies shows there exists infinitely many triangle free 4-critical graphs satisfying  $e(G) = \frac{5v(G)+4}{3}$ , and hence the second theorem is close to being tight. It also generalizes results of Thomas and Walls, and also Thomassen, that girth 5 graphs embeddable on the torus, projective plane, or Klein bottle are 3-colourable.

Lastly, a theorem of Cereceda, Johnson, and van den Heuvel, says that given a 2-connected bipartite planar graph  $G$  with no separating four-cycles and a 3-colouring  $f$ , then one can obtain all 3-colourings from  $f$  by changing one vertices' colour at a time if and only if  $G$  has at most one face of size 6.

We give the natural generalization of this to circular colourings when  $\frac{p}{q} < 4$ .

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Others are also very deserving of individual shout outs, but to do them justice I would have to make the acknowledgements longer than my thesis, and so simply say thanks Alan Wong, Florian Hoersch, Lukas Nabergall, Iain Crump, Sabrina Lato, Ronen Wdowinski, Tim Miller, Martin Pei, Tina Chen, Rose McCarty, Nick Olsen Harris, Matthew Kroeker, Sina Baghal, Amena Assem, Brett Nassaraden, Arnott Kidner, Soffiá Árnadóttir, Jason Qu, Tao Jiang, James Davies and Kazuhiro Nomoto for being great (and anyone I missed by accident as well!)

For this thesis, I would also like to thank two (ex)-graduate students of C and O in particular, Evelyne Smith Roberge and Logan Grout, who contributed greatly to content of this thesis, and without whom it is not clear the thesis would have been completed (at least without many errors).

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## Dedication

This thesis is dedicated to all my family members in Kamloops, especially my Mom and Dad.



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# Chapter 1

## Introduction

In this thesis we are interested in partitioning graphs into elementary pieces. We assume throughout all graphs are simple and finite. As graphs contain both edges and vertices, there are two natural ways to partition a graph, one either partitions the edge set or the vertex set. We will deal with both throughout this thesis. As such we introduce the following standard definitions which describe partitioning into edges or into vertices.

**Definition 1.1.** For any graph  $G$ , a *decomposition* of  $G$  is a set of edge disjoint subgraphs of  $G$  such that the union of their edge sets is the edge set of the graph.

**Definition 1.2.** For two graphs  $G$  and  $H$ , a *graph homomorphism* from  $G$  to  $H$  is a map  $f : V(G) \rightarrow V(H)$  such that for all  $e = xy \in E(G)$  we have  $f(x)f(y) \in E(H)$ . If  $G$  admits a homomorphism to  $H$ , then we say  $G$  admits an  $H$ -*colouring*, and we write this as  $G \rightarrow H$ . In the particular case where  $H$  is the complete graph on  $k$  vertices, which we denote as  $K_k$ , we say a  $K_k$ -*colouring* is a  $k$ -*colouring*.

Here we note the name  $H$ -colouring is due to the fact that homomorphisms to  $K_k$  are referred to as  $k$ -colourings.

In general, it is hard to partition graphs, regardless of whether we are partitioning the vertices or edges. For partitioning vertices, the celebrated Hell-Nešetřil dichotomy [20] says that if  $H$  is not bipartite, then it is NP-complete to decide if  $G$  admits a homomorphism to  $H$ .

For decompositions, Dor and Tarsi showed that it is NP-complete to determine if a graph  $G$  admits a decomposition into copies of a graph  $H$  when  $H$  is connected and has at least three edges, and polynomial when  $H$  has at most two edges [8].

In the case of graph decompositions, one very natural way to get past the NP-completeness barrier is to ask if  $G$  admits a decomposition  $H_1, \dots, H_k$  such that all  $H_i$  belong to some special family  $\mathcal{F}$ , where  $|\mathcal{F}| \geq 2$ .

In the case of homomorphisms, a natural strategy is to restrict the input graphs to classes of graphs closed under taking subgraphs, and also having few edges, as intuitively it is easier to have a colouring if there are fewer constraints to satisfy.

Both strategies result in very pretty theorems. We split up the rest of the introduction depending on whether the primary focus is decomposition theorems relevant to this thesis or homomorphism results related to this thesis.

Before continuing, we note any undefined graph theory terms can be found in Bondy and Murty's graduate graph theory text [1]. We will occasionally direct our graphs, and we will refer to a directed graph as a digraph. Note as all our graphs are simple, a directed graph is the same as taking a graph and equipping it with an orientation. As notation, given a graph  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ , we will always use  $v(G)$  to denote  $|V(G)|$  and  $e(G)$  to denote  $|E(G)|$ .

## 1.1 On the Nine Dragon Tree Theorem

We start this section by highlighting two pretty theorems from the world of graph decompositions. We need some standard definitions.

**Definition 1.3.** The *fractional arboricity* of a graph  $G$ , denoted  $\Gamma_f(G)$ , is

$$\Gamma_f(G) := \max_{H \subseteq G, v(H) \geq 2} \frac{e(H)}{v(H) - 1}.$$

**Definition 1.4.** The *maximum average degree* of a graph  $G$ , denoted  $\text{mad}(G)$ , is

$$\text{mad}(G) := \max_{H \subseteq G} \frac{2e(H)}{v(H)}.$$

**Definition 1.5.** A *pseudoforest* is a graph where each connected component contains at most one cycle.

With this we can state Nash-Williams' Theorem and Hakimi's Theorem, which give necessary and sufficient conditions for when a graph decomposes into  $k$  forests, and  $k$  pseudoforests, respectively.

**Theorem 1.6** ([40], Nash-Williams Theorem). *A graph  $G$  decomposes into  $k$  forests if and only if  $\Gamma_f(G) \leq k$ .*

**Theorem 1.7** ([19] Hakimi's Theorem). *A graph  $G$  decomposes into  $k$  pseudoforests if and only if  $\text{mad}(G) \leq 2k$ .*

These theorems are classical examples of results where obvious necessary conditions become sufficient. At a glance, it is not clear how to generalize these theorems. One

approach is to make the easy observation that both the fractional arboricity and maximum average degree need not be integral. Thus, intuitively, if, say the fractional arboricity of a graph is  $k + \varepsilon$  for a small value of  $\varepsilon$ , one barely needs  $k + 1$  forests, and hence you might anticipate that  $G$  actually decomposes into  $k + 1$  forests, where one of the forests is a matching. This intuition is correct, and there is a series of papers which prove results of this nature. We present (a subset) of them in roughly chronological order. In 2012, Montassier, Ossona de Mendez, Raspaud, and Zhu opened the area.

**Theorem 1.8** ([35]). *Every graph with fractional arboricity at most  $\frac{4}{3}$  decomposes into a forest and a matching. Additionally, every graph with fractional arboricity at most  $\frac{3}{2}$  decomposes into a forest and a path.*

One year later, Kim, Kostochka, West, Wu and Zhu extended this result to allow forests of larger degree.

**Theorem 1.9** ([26]). *Fix positive integers  $k$  and  $d$  where either  $d = k + 1$  or  $k = 1$  and  $d \leq 6$ . Every graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests where one of the forests has maximum degree  $d$ .*

In 2018, Yang extended Theorem 1.8 in a different way (the date is due to publishing lag times).

**Theorem 1.10** ([50]). *Let  $k$  be a positive integer. Every graph with fractional arboricity at most  $k + \frac{1}{k+2}$  decomposes into  $k + 1$  forests where one of the forests is a matching.*

In 2015, Chen, Kim, Kostochka, West and Zhu generalized Theorem 1.9.

**Theorem 1.11** ([7]). *Let  $k$  and  $d$  be integers where  $k \leq 2$ , and if  $k = 2$ ,  $d \neq 1$ . Then every graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests where one of the forests has maximum degree  $d$ .*

At this point you may see a pattern emerging. In 2017, the Nine Dragon Tree Theorem (conjectured in [35]) was proved by Jiang and Yang which unifies all above results (the theorem is named after a tree in Taiwan which is far from acyclic):

**Theorem 1.12** ([24], Nine Dragon Tree Theorem). *Let  $k$  and  $d$  be positive integers. Every graph  $G$  with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests where one of the forests has maximum degree  $d$ .*

In [35], it was observed that the fractional arboricity bound in the Nine Dragon Tree Theorem cannot be strengthened for any choice of  $k$  and  $d$ . In particular, they showed:

**Theorem 1.13** ([35]). *For any positive integers  $k$  and  $d$ , there are arbitrarily large graphs  $G$  and a set of edges  $S = \{e_1, \dots, e_d\}$  such that  $\Gamma_f(G - S) = k + \frac{d}{k+d+1}$  and  $G$  does not decompose into  $k + 1$  forests where one of the forests has maximum degree  $d$ .*

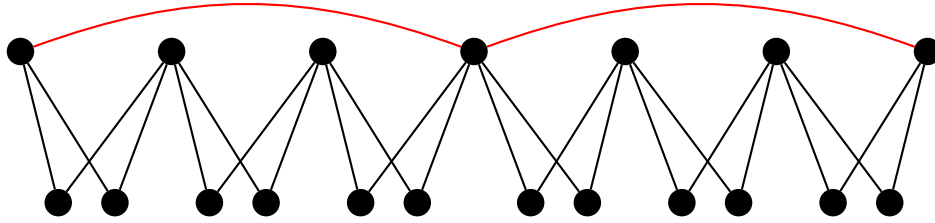


Figure 1.1: An example of a graph which does not decompose into a forest and a matching, and where the deletion of both red edges gives a graph with fractional arboricity  $1 + \frac{1}{3}$ .

See Figure 1.1 for one of the graphs [35] constructed for Theorem 1.13 in the  $k = 1$ ,  $d = 1$  case.

Despite this, an extremely strong conjecture was proposed in [35], aptly named the Strong Nine Dragon Tree Conjecture.

**Conjecture 1.14** ([35], Strong Nine Dragon Tree Conjecture). *Let  $k$  and  $d$  be positive integers. Every graph  $G$  with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests where one of the forests has every connected component containing at most  $d$  edges.*

This conjecture is wide open. It is known to be true when  $d = 1$ , as that follows immediately from the Nine Dragon Tree Theorem. It has also been proved when  $k = 1$  and  $d = 2$  [26]. All other cases are open. Interestingly, in [35], they showed that the Strong Nine Dragon Tree Conjecture implies (and is in fact equivalent to) the naively stronger conjecture:

**Conjecture 1.15** ([35]). *Let  $k$  and  $d$  be positive integers. For every graph  $G$  with fractional arboricity at most  $k + \frac{d}{k+d+1}$ , and every vertex  $v \in V(G)$ ,  $G$  decomposes into  $k + 1$  forests where one of the forests has every connected component containing at most  $d$  edges, and  $v$  is an isolated vertex in this forest.*

The Strong Nine Dragon Tree Conjecture is open even if you replace  $d$  edges with a constant depending on  $d$  and  $k$ . It is open even when  $d \leq k + 1$ , which should be a fairly easy case to handle. As some semblance of progress towards the conjecture, with Evelyne Smith-Roberge, I proved the following strengthening of the Nine Dragon Tree Theorem in the  $k = 1$  and  $d \in \{3, 4\}$  cases:

**Theorem 1.16.** *Let  $k = 1$ ,  $d \in \{3, 4\}$ , and  $w = d + 4$ . Every graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into two forests  $T, F$  such that  $F$  has maximum degree  $d$ , every component of  $F$  contains at most one vertex of degree  $d$ , and further if  $d \neq 3$ , every component of  $F$  has at most  $w$  edges  $e = xy$  such that the degree of both  $x$  and  $y$  is at least  $d - 1$ .*

It seems believable that Theorem 1.16 will generalize to the following using similar techniques:

**Conjecture 1.17.** *Let  $k$  and  $d$  be integers such that  $d \geq \frac{3(k+1)}{2}$  and  $d \leq 2(k+1)$ . Let  $w = d + 4$ . Then every graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests  $T_1, \dots, T_k, F$  such that  $F$  has maximum degree  $d$ , every component of  $F$  contains at most one vertex of degree  $d$ , and further if  $d \neq 3$ , every component of  $F$  has at most  $w$  edges  $e = xy$  such that the degree of both  $x$  and  $y$  is at least  $d - 1$ .*

In the particular case when  $k = 1$  and  $d = 3$ , we managed to get close to the Strong Nine Dragon Tree Conjecture:

**Theorem 1.18.** *Every graph with fractional arboricity at most  $1 + \frac{3}{5}$  decomposes into two forests,  $T, F$  such that  $F$  has maximum degree three, every component of  $F$  contains at most one vertex of degree three, if a component contains a vertex of degree three, then it has at most 14 edges, and otherwise a component of  $F$  has at most 13 edges.*

It would be nice to extend this Theorem beyond  $k = 1$  and  $d = 3$ , however, this so far has been elusive.

Theorem 1.16 gives rise to a slightly different question than the Strong Nine Dragon Tree Conjecture. We attribute this question to Xuding Zhu, who gave a weaker version of this question at the Watercolor conference in 2019.

**Question 1.19** (Xuding Zhu). *Let  $\mathcal{F}_d$  be the set of trees with maximum degree  $d$ . What is the smallest  $S \subseteq \mathcal{F}_d$  such that every graph with fractional arboricity  $k + \frac{d}{k+d+1}$  decomposes into  $k + 1$  forests, where one of the forests has every connected component from  $S$ ? Can  $S$  be taken to be the stars on at most  $d$  edges?*

It seems unlikely that  $S$  can be taken to just be stars on at most  $d$  edges, or that  $S$  can be taken to be the star on  $d$  edges and all trees on at most  $d - 1$  edges, however no example of this is known.

With this, let us turn our attention to pseudoforests and Hakimi's Theorem. Of course, one can ask Nine Dragon Tree type questions for pseudoforests. In fact, on the journey to proving the Nine Dragon Tree Theorem, it turned out to be useful to prove a pseudoforest analogue for the Nine Dragon Tree Theorem.

**Theorem 1.20** ([13]). *Let  $k$  and  $d$  be integers. Every graph with  $\text{mad}(G) \leq 2k + \frac{2d}{k+d+1}$  decomposes into  $k + 1$  pseudoforests, where one of the pseudoforests has maximum degree  $d$ . Further, for every pair  $k$  and  $d$ , there exists arbitrarily large graphs  $G$  containing an edge  $e$  such that  $\text{mad}(G - e) = 2k + \frac{2d}{k+d+1}$  and  $G$  does not decompose into  $k + 1$  pseudoforests where one of the pseudoforests has maximum degree  $d$ .*

See Figure 1.2 for one of the graphs constructed in [13] as a tightness example.

The ideas in [13] are instrumental to the proof of the Nine Dragon Tree Theorem. Building on this, Logan Grout and I proved the pseudoforest analogue of the Strong Nine Dragon Tree Conjecture.



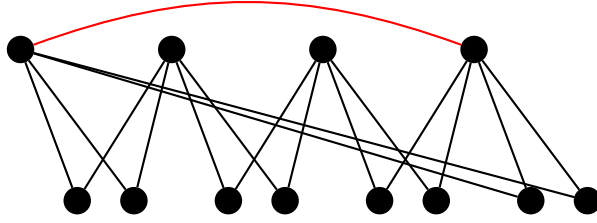


Figure 1.2: A graph which does not decompose into a pseudoforest and a matching, but the deletion of the red edge results in a graph with maximum average degree exactly  $2 + \frac{2}{3}$ .

**Theorem 1.21** ([18]). *Let  $k$  and  $d$  be positive integers. Every graph with  $\text{mad}(G) \leq 2k + \frac{2d}{k+d+1}$  decomposes into  $k + 1$  pseudoforests, where one of the pseudoforests has every connected component containing at most  $d$  edges.*

All instances of this theorem were open aside from the  $d = 1$  case (which follows from the pseudoforest analogue of the Nine Dragon Tree Theorem), and rather remarkably the  $k = 1$  and  $d = 2$  case, which follows from the proof of the Strong Nine Dragon Tree Theorem [26] in the  $k = 1$  and  $d = 2$  case.

Later on, it became apparent that the proof of Theorem 1.21 actually proves a slightly stronger result.

**Theorem 1.22.** *Let  $k$  and  $d$  be positive integers. Every graph with  $\text{mad}(G) \leq 2k + \frac{2d}{k+d+1}$  decomposes into  $k + 1$  pseudoforests, where one of the pseudoforests has every connected component containing at most  $d$  edges and is a forest.*

It would be particularly pleasing if the following strengthening of Theorem 1.21 was true.

**Conjecture 1.23.** *Let  $k$  and  $d$  be positive integers. Every graph with  $\text{mad}(G) \leq 2k + \frac{2d}{k+d+1}$  decomposes into  $k + 1$  pseudoforests where one of the pseudoforests is a forest, has every connected component containing at most  $d$  edges, and if a component contains  $d$  edges, then it is isomorphic to a star on  $d + 1$  vertices.*

However a much easier problem is the following:

**Conjecture 1.24.** *Let  $k$  and  $d$  be integers. Every graph with  $\text{mad}(G) \leq 2k + \frac{2d}{k+d+1}$  decomposes into  $k + 1$  pseudoforests, where one of the pseudoforests is acyclic, has every connected component containing at most  $d$  edges, and if a component contains  $d$  edges, then it does not have a vertex of degree  $d - 1$ .*

This conjecture seems to follow from a modification of the proof of Theorem 1.21.

By this point, the reader may have some questions. First, they may wonder why just pseudoforests and forests. Are there not more general families for which we can ask Nine

Dragon Tree type questions? For this, an answer is to consider matroids (we refer the reader to [42] for an introduction to matroids).

**Definition 1.25.** A matroid  $\mathcal{M}$  is an ordered pair  $(E, \mathcal{I})$  consisting of a finite set  $E$  (called the groundset) and a collection  $\mathcal{I}$  of subsets of  $E$  having the following three properties.

- $\emptyset \in \mathcal{I}$
- If  $I \in \mathcal{I}$  and  $I' \subseteq I$ , then  $I' \in \mathcal{I}$
- If  $I_1$  and  $I_2$  are in  $\mathcal{I}$  and  $|I_1| < |I_2|$ , then there is an element  $e$  of  $I_2 - I_1$  such that  $I_1 \cup e \in \mathcal{I}$ .

**Definition 1.26.** Given a matroid  $\mathcal{M} = (E, \mathcal{I})$ , the *rank function* of matroid denoted  $r_{\mathcal{M}}$ , is defined so that for a subset  $S \subseteq E$ ,  $r_{\mathcal{M}}(S)$  is  $|I|$ , where  $I \subseteq S$ , and  $I$  is maximal subject to  $I \in \mathcal{I}$ .

Jack Edmonds generalized both Hakimi's and Nash-Williams Theorem to matroids.

**Definition 1.27.** Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid. The *covering number* of a matroid is

$$\beta(\mathcal{M}) := \max_{X \subseteq E, r_{\mathcal{M}}(X) \neq 0} \frac{|X|}{r_{\mathcal{M}}(X)}.$$

**Theorem 1.28** ([10]). *A matroid  $\mathcal{M} = (E, \mathcal{I})$  has  $k$  sets in  $\mathcal{I}$  whose union is  $E(\mathcal{M})$  if and only if the covering number of  $\mathcal{M}$  is at most  $k$ .*

To see the connection, we note the following easy facts:

**Observation 1.29** ([42]). Let  $G$  be a graph. If  $\mathcal{I}$  is the set of forests of  $G$ , then  $(E(G), \mathcal{I})$  is the *graphic matroid* of  $G$ . If  $\mathcal{I}$  is the set of pseudoforests of  $G$ , then  $(E(G), \mathcal{I})$  forms the *bicircular matroid* of  $G$ .

It now follows that Edmond's theorem generalizes both Nash-Williams' and Hakimi's Theorem. Despite this, it is not clear how to formulate an appropriate analogue of the (Strong) Nine Dragon Tree Conjecture for matroids. This is due to a lack of a good notion of a connected component for matroids or the degree of vertex (since we no longer have vertices).

However, there is still a (single) result on the general problem.

**Theorem 1.30** ([12]). *Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid. If  $\beta(\mathcal{M}) = k + \varepsilon$ , where  $k$  is a non-negative integer and  $0 \leq \varepsilon < 1$ , then  $E$  can be partitioned into  $k + 1$  sets in  $\mathcal{I}$  with one of size at most  $\varepsilon r_{\mathcal{M}}(E)$ .*

We note this theorem does not quite capture the strength of the Strong Nine Dragon Tree Conjecture, or even the Nine Dragon Tree Theorem, but it does show that possibly there is a general theorem lurking which would recover the pseudoforest and forest theorems.

Aiming slightly below the generality of all matroids, one could aim for an interpolation between pseudoforests and forests. For this we need the idea of a biased graph.

**Definition 1.31.** A *biased graph* is a pair  $(G, \beta)$  where  $\beta$  is a collection of cycles of  $G$  satisfying the *theta property*, which is if  $C_1, C_2 \in \beta$  and  $C_1$  and  $C_2$  intersect in a path with at least one edge, then the unique third cycle is in  $\beta$ .

Let  $(G, \beta)$  be a biased graph. If a cycle  $C \in \beta$ , we say  $C$  is *balanced*. From this, we can define the *frame matroid* of a biased graph  $(G, \beta)$  as being the matroid  $\mathcal{M}(G, \beta)$  which has ground set  $E(G)$ , and a subset of edges  $I \subseteq E(G)$  is in  $\mathcal{I}$  if each connected component of  $I$  contains at most one cycle, and if a connected component of  $I$  contains a cycle, then that cycle is unbalanced.

An easy observation (see [42]) is that given a graph  $G$ , if we let  $\beta$  be empty, then the associated frame matroid is the bicircular matroid, and if  $\beta$  is all cycles, then the associated frame matroid is the graphic matroid. Hence the following conjecture is quite natural.

**Conjecture 1.32.** Let  $(G, \beta)$  be a biased graph, and let  $\mathcal{M}$  be the associated frame matroid. Let  $k$  and  $d$  be positive integers. Suppose the covering number of  $\mathcal{M}$  is at most  $k + \frac{d}{k+d+1}$ . Then  $G$  decomposes into  $k + 1$  sets  $I_1, \dots, I_k, F$  such that each set is independent in the frame matroid, and further  $F$  has maximum degree  $d$ .

Of course, one could pose a strong version of this conjecture as well. We mention one final way of possibly generalizing the Nine Dragon Tree Theorem which was pointed out to me recently by Ronen Wdowinski. One could ask for a list version of the (Strong) Nine Dragon Tree Conjecture. We need a couple definitions to state the proposed generalization. Let  $G$  be a graph, let  $\mathcal{C}$  be a set and  $L$  a function which assigns to each edge of  $G$  a subset of  $\mathcal{C}$ . If for every edge  $e \in E(G)$ , and an integer  $k$ , we have  $|L(e)| \geq k$ , then we say  $L$  is a  $k$ -list-edge-assignment for  $G$ . If there is a function  $f : E(G) \rightarrow \mathcal{C}$  such that for all  $e \in E(G)$ ,  $f(e) \in L(e)$ , and further for every  $c \in \mathcal{C}$ , the inverse image of  $f$  is a forest, we say that  $f$  is a  $k$ -forest-list-colouring. The list version of Nash-Williams Theorem is known to be true by a short argument of Paul Seymour.

**Theorem 1.33** ([44]). Suppose  $G$  is a graph that decomposes into  $k$  forests. Then for any  $k$ -list-edge-assignment of  $G$ ,  $L$ , there is a  $k$ -forest-list-colouring of  $G$ .

Ronen Wdowinski and Penny Haxell asked if a list version of the Nine Dragon tree theorem is true:

**Conjecture 1.34.** *Let  $k$  and  $d$  be positive integers. Let  $G$  be a graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$ , and  $L$  any  $(k+1)$ -list-edge-assignment of  $G$ . Then there is a  $(k+1)$ -forest-list-colouring  $f$  which partitions  $E(G)$  into forests where one of these forests has maximum degree  $d$ .*

Of course, one could pose a strong version of the above conjecture and one could pose similar list versions for pseudoforests and signed graphs, which would also be interesting.

It would also be natural to ask if the Nine Dragon Tree Theorem (and related results) have any applications, aside from just being nice themselves. It turns out, the Nine Dragon Tree Theorem was motivated by the game chromatic number, which we define now.

**Definition 1.35.** Let  $G$  be a graph and  $C$  a set of colours. Consider the two player game, where we have players Alice and Bob, Alice starts the game, and they take turns picking a colour  $c$  from  $C$  and a vertex  $v$  of  $G$  and colouring  $v$  with  $c$  in such a way that no vertex adjacent to  $v$  is coloured  $c$ . We say Alice wins the game if all vertices of  $G$  end up coloured. Otherwise, if there are no valid moves but not all vertices are coloured Bob wins. The *game chromatic number* of a graph  $G$  is the least number of colours needed so that Alice has a strategy which always wins the game.

A rather remarkable result of Xuding Zhu relates the game chromatic number to forest decompositions of bounded degree.

**Theorem 1.36** ([55]). *If  $G$  decomposes into two forests  $T_1$  and  $T_2$ , where  $T_2$  has maximum degree  $d$ , then the game chromatic number of  $G$  is at most  $d + 4$ .*

This bound is best possible, and in fact gives best possible bound in planar graphs of girth at least eight [25]. Here, recall that the *girth* of a graph is the length of the shortest cycle in  $G$ . Note that it follows from the Nine Dragon Tree Theorem and Euler's formula that planar graphs of girth 8 decompose into a forest and a matching, and hence have game chromatic number at most 5. It would be interesting to see if Theorem 1.36 can be generalized to a wider class of forest decompositions, or pseudoforest decompositions.

To see additional applications of the Strong Nine Dragon Tree Conjecture, we need more definitions.

**Definition 1.37.** Let  $G$  be a connected graph. Let  $0 < \varepsilon < 1$  and let  $T$  be a spanning tree of  $G$ . We say that  $T$  is  $\varepsilon$ -thin if for every vertex set  $A \subseteq V(G)$ , if  $t$  is the number of edges with one endpoint in  $A$  and the other in  $G - A$  which are in  $T$ , and  $t'$  the number of edges with one endpoint in  $A$  and the other in  $G - A$ , then  $\frac{t}{t'} \leq \varepsilon$ .

There is a famous conjecture of Luis Goddyn asserting the existence of thin trees in highly edge-connected graphs (see [15])

**Conjecture 1.38** (Goddyn's Thin Tree Conjecture). *For every  $\varepsilon > 0$ , there exists a constant  $c$  depending only on  $\varepsilon$  such that all graphs which are  $c$ -edge connected have a  $\varepsilon$ -thin tree.*

Despite the conjecture's fame, there is not much progress towards proving the conjecture. This is likely due to the fact that the conjecture, if true, would have strong implications towards constant-factor approximations of the Asymmetric Travelling Salesman problem, as well as the existence of nowhere zero flows (see [15], for a description of these applications). In the first case, we note very recently a constant-factor approximation algorithm was given for Asymmetric Travelling Salesman, see [46]. Therefore it is natural to restrict the graph class to see if some partial progress can be made.

Merker and Postle proved that every planar 6-edge connected graph contains two edge-disjoint  $\frac{18}{19}$ -thin trees [33]. The  $\frac{18}{19}$  bound was later strengthened by Ramin Mousavi to  $\frac{12}{13}$  [38]. Carsten Thomassen proved that there is no  $\varepsilon$  such that every planar 4-edge connected graph contains an  $\varepsilon$ -thin tree (this is unpublished, see the comments in [33] for the construction). Merker and Postle made the following natural conjecture:

**Conjecture 1.39** ([33]). *There exists an  $\varepsilon$  such that all planar 5-edge connected graphs have two edge disjoint  $\varepsilon$ -thin trees.*

However even the following weaker statement is open:

**Conjecture 1.40.** *There exists a  $0 < \varepsilon < 1$  such that all planar 5-edge connected graphs have an  $\varepsilon$ -thin tree.*

From the arguments in [33], it is not hard to see that Conjecture 1.40 could be obtained by showing that every girth 5 planar graph decomposes into a forest and a bounded diameter forest. Here, a forest has *bounded diameter* if every connected component of the forest has bounded diameter.

A straightforward calculation from Euler's formula says that every planar graph of girth at least five has fractional arboricity at most  $1 + \frac{2}{3}$ . Hence a positive answer to the Strong Nine Dragon Tree Conjecture in the  $k = 1$  and  $d = 4$  case would give an affirmative answer to Conjecture 1.40 (since bounded size implies bounded diameter).

An interesting related theorem talks about partitioning the vertex set of a graph into an independent set and an induced forest.

**Theorem 1.41** ([22]). *Every planar graph of girth 5 admits a partitioning of its vertex set into two sets  $I$  and  $T$  such that  $I$  induces an independent set, and  $T$  is an induced forest.*

Perhaps the following is true:

**Question 1.42.** *Is it true that for every planar graph of girth 5, there exists a decomposition into two forests  $F_1$  and  $F_2$ , such that every component of  $F_1$  is isomorphic to a star, the centres of the star (making a choice if necessary) induce independent set  $I$ , and the vertices not in  $I$  induce a forest?*

This would simultaneously imply Conjecture 1.40 and generalize Theorem 1.41 if true. We note that in [33], the following was conjectured:

**Conjecture 1.43.** *Every planar graph of girth 5 decomposes into two forests, such that both forests have bounded diameter.*

This conjecture seems to be difficult if true.

These applications have focused on forests, which to date have had all of the applications. It would be nice to extend some of the applications to pseudoforests, but at the moment these are lacking.

A third question would be if on certain graph classes we can do better than the Nine Dragon Tree Theorem. There are not many results of this nature, and those that are known are negative. For example, we have:

**Theorem 1.44** ([35]). *There exist planar graphs of girth at least 7 which do not decompose into a forest and a matching.*

**Theorem 1.45** ([35]). *There exist planar graphs of girth at least 5 which do not decompose into a forest and a forest of maximum degree two.*

It is possible these theorems can be strengthened to pseudoforests.

**Question 1.46.** *Is it true that every planar graph of girth 7 decomposes into a pseudoforest and a matching?*

**Question 1.47.** *Is it true that every planar graph of girth 5 decomposes into a pseudoforest and a pseudoforest of maximum degree two?*

Another interesting class to consider is graphs with large girth. Perhaps the following is true:

**Conjecture 1.48.** *Let  $k, d, g$  be positive integers. There exists a function  $f(k, d, g)$  such that every graph  $G$  with maximum average degree at most  $2k + \frac{2d}{d+k+1} + f(k, d, g)$  and girth  $g$  decomposes into  $k + 1$  pseudoforests where one of the pseudoforests has maximum degree  $d$ .*

Of course, one can ask a strong version of the above conjecture.

## 1.2 Colouring sparse graphs

In this section we look at colouring graphs which have bounded maximum average degree.

We start this section off with an amazing theorem of Grötzsch. Recall, a graph is *triangle-free* if it contains no subgraph isomorphic to  $K_3$ .

**Theorem 1.49** ([17]). *Every triangle-free planar graph is 3-colourable.*

The original proof of Grötzsch’s theorem is relatively difficult. A particularly nice and short proof of this theorem was given by Kostochka and Yancey in [27]. Kostochka and Yancey give an easy reduction to the case where  $G$  is planar and has no faces of length 4, and then prove a stronger theorem about the average degree of vertex and edge minimal graphs with no 3-colouring. To state the theorem, we first define the notion of a  $k$ -critical graph. A graph  $G$  is  $k$ -critical if  $G$  is  $k$ -colourable,  $G$  is not  $(k - 1)$ -colourable, but all proper subgraphs are  $(k - 1)$ -colourable. The Kostochka-Yancey Theorem states:

**Theorem 1.50** ([28]). *If  $G$  is  $k$ -critical, then*

$$e(G) \geq \frac{(k + 1)(k - 2)v(G) - k(k - 3)}{2(k - 1)}.$$

In the particular case of 4-critical graphs, the theorem says that if  $G$  is 4-critical, then  $e(G) \geq \frac{5v(G)-2}{3}$ . Using Euler’s formula, it is easy to see that planar graphs with no face of length at smaller than  $g$  have maximum average degree at most  $\frac{2g}{g-2}$ , and hence if  $G$  has no triangles or faces of length four then  $G$  has maximum average degree strictly less than  $\frac{10}{3}$ , and using some easy observations about minimal counterexamples to Grötzsch’s Theorem, we get the result.

One might wonder if other known results about colouring graphs on surfaces can be refined to a result about maximum average degree in a similar way. While there are many such results, of particular note for this thesis are two results, one by Carsten Thomassen, and the other by Robin Thomas and Barrett Walls.

**Theorem 1.51** ([48]). *Every graph of girth at least five embeddable on the torus or the projective plane is 3-colourable.*

**Theorem 1.52** ([47]). *Every graph of girth at least five embeddable on the Klein Bottle is 3-colourable.*

We recall a well known fact that follows from Euler’s formula for surfaces and the fact that the Euler characteristic of the Klein Bottle and Torus is 0, and the Euler characteristic of the projective plane is 1 (see Theorem 10.37 in [1] and the surrounding discussion).

**Observation 1.53.** *If  $G$  is a graph embeddable on the Klein Bottle, torus, or projective plane, and  $G$  has girth at least 5, then*

$$2e(G) \leq \frac{10v(G)}{3}.$$

Note the Kostochka-Yancey bound for 4-critical graphs is just slightly too low to deduce these results from the above bound. Further, it is not hard to see that the Kostochka-Yancey bound is best possible, for instance,  $K_4$ . Hence it might look like a dead end, but Liu and Postle showed that 4-critical graphs with girth at least 5 have large density.

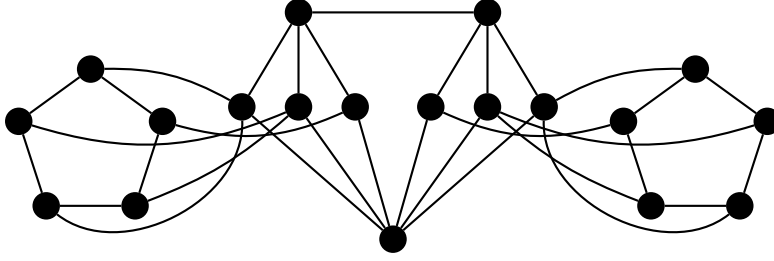


Figure 1.3: A example of Davies of a triangle free 4-critical graph satisfying  $e(G) = \frac{5v(G)+4}{3}$ .

**Theorem 1.54** ([31]). *Every 4-critical graph with girth at least 5 has  $e(G) \geq \frac{5v(G)+2}{3}$ .*

Theorem 1.51 and Theorem 1.52 follow immediately from Theorem 1.54 and Observation 1.53.

An unpublished result of Postle shows that asymptotically, the average degree of 4-critical graphs with girth at least 5 is strictly higher than  $\frac{10}{3}$ , and hence the girth at least five hypothesis in Theorem 1.54 is not the most natural.

It is natural then to ask if triangle-free 4-critical graphs achieve the same density bound as in Theorem 1.54. Even stronger than this, Liu and Postle conjectured the following:

**Conjecture 1.55** ([31]). *Every triangle-free 4-critical graph satisfies  $e(G) \geq \frac{5v(G)+5}{3}$ .*

Unfortunately, this conjecture is false, and James Davies found an infinite family of counterexamples (personal communication).

**Theorem 1.56** (Davies). *There exist infinitely many triangle-free 4-critical graphs satisfying  $e(G) = \frac{5v(G)+4}{3}$ .*

Figure 1.3 gives the smallest example he constructed. Nevertheless, we obtained the following theorem:

**Theorem 1.57** ([36]). *Every triangle-free 4-critical graph satisfies  $e(G) \geq \frac{5v(G)+2}{3}$ .*

We in fact proved a stronger theorem mostly for the purposes of induction. We need some definitions and theorems first to state the theorem.

**Definition 1.58.** An *Ore Composition* of two graphs  $H_1$  and  $H_2$  is the graph  $H$  obtained by deleting an edge  $xy \in E(H_1)$ , splitting a vertex  $z \in V(H_2)$  into two vertices  $z_1$  and  $z_2$  of positive degree such that  $N(z) = N(z_1) \cup N(z_2)$  and  $N(z_1) \cap N(z_2) = \emptyset$ , and then identifying  $x$  with  $z_1$  and  $y$  with  $z_2$ . We say that  $H_1$  is the *edge side of the composition* and  $H_2$  is the *split side of the composition*, and we denote the graph obtained from  $H_2$  by splitting  $z$  as  $H_2^z$ .



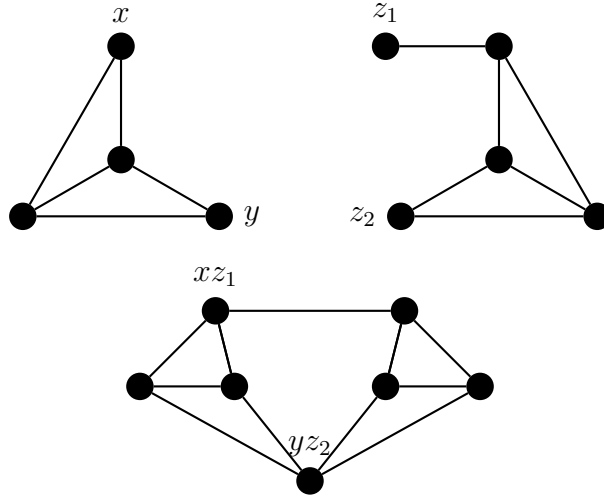


Figure 1.4: The Ore Composition operation applied to two  $K_4$ 's, resulting in the 4-Ore graph called the Moser Spindle. Here we deleted the edge  $xy$  of  $K_4$ , and split a vertex  $z$  into two vertices  $z_1$  and  $z_2$ .

**Definition 1.59.** A graph  $G$  is  $k$ -Ore if it is obtained from some fixed number of copies of  $K_k$  via Ore compositions.

We note in [29] that Ore Compositions are called DGHO-compositions, owing to the fact that Dirac, Gallai, Hajos and Ore all used a similar construction. However, we will simply use the term Ore Composition.

For context, it is important to note the rather remarkable result of Kostochka and Yancey who proved that the instances where equality holds in Theorem 1.50 are precisely the  $k$ -Ore graphs.

**Theorem 1.60** ([29]). *If  $G$  is  $k$ -critical, and*

$$e(G) = \frac{(k+1)(k-2)v(G) - k(k-3)}{2(k-1)},$$

*then  $G$  is  $k$ -Ore.*

Here and throughout, we let  $W_k$  denote the *wheel on  $k+1$  spokes*, that is, the graph obtained from a cycle on  $k$  vertices by adding a vertex adjacent to all other vertices in the cycle.

**Definition 1.61.** Given a graph  $G$ , and an integer  $k$ , the function  $T^k(G)$  is the size of the largest collection of vertex-disjoint  $K_k$ -subgraphs in  $G$ .

**Definition 1.62.** Let  $T_8$  be the graph with vertex set  $V(T_8) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$  and  $E(T_8) = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_3, u_2u_4, u_2u_5, u_3u_8, u_4u_7, u_5u_6, u_6u_7, u_6u_8, u_7u_8\}$ . Let

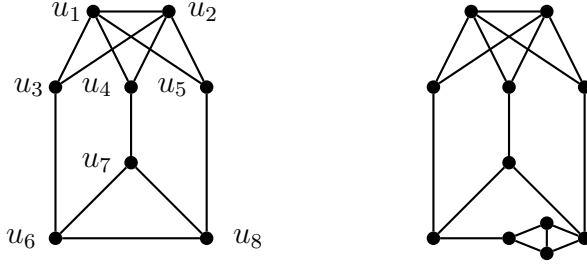


Figure 1.5: The graph on the left is  $T_8$ , and the graph on the right is an example of a graph in  $\mathcal{B}$ , which we call  $T_{11}$ .

$\mathcal{B}$  be defined as follows: the graph  $T_8$  is in  $\mathcal{B}$ , and given a graph  $G \in \mathcal{B}$  and a 4-Ore graph  $H$ , the Ore composition  $G'$  of  $G$  and  $H$  is in  $\mathcal{B}$  if  $T^3(G') = 2$ . See Figure 1.5.

**Definition 1.63.** Let  $a, b, c$  and be non-negative integers. The  $(a, b, c)$ -potential of a graph  $G$ , denoted  $p_{a,b,c}(G)$ , is  $av(G) - be(G) - cT^3(G)$ .

Now we can state the stronger theorem.

**Theorem 1.64** ([36]). *Let  $p(G)$  denote the  $(5, 3, 1)$ -potential. Let  $G$  be a 4-critical graph. Then*

- $p(K_4) = 1$ ,
- $p(G) = 0$  if  $T^3(G) = 2$  and  $G$  is 4-Ore,
- $p(G) = -1$  if  $G = W_5$ , or  $G \in \mathcal{B}$ , or  $G$  is 4-Ore with  $T^3(G) = 3$ , and
- $p(G) \leq -2$  otherwise.

In the case that  $G$  is 4-critical and triangle-free, it follows that  $p(G) \leq -2$ , and thus we recover Theorem 1.57 from Theorem 1.64. We note that Liu and Postle proved a similar theorem in [31].

**Theorem 1.65** ([31]). *Let  $p'(G) := 5v(G) - 3e(G) - T'(G)$ , where  $T'(G)$  is the maximum number of vertex disjoint 4-cycles or triangles. Let  $G$  be a 4-critical graph. Then*

- $p'(K_4) = 1$
- $p'(G) = 0$  if  $G$  is the Ore-composition of two  $K_4$ 's
- $p'(G) = -1$  if  $G$  is 4-Ore with  $T'(G) = 3$ , or  $G = W_5$ , or  $G = T_8$ , or  $G = T_{11}$ , and
- $p'(G) \leq -2$  otherwise.

Here we let  $T_{11}$  be defined as in Figure 1.5. It is easy to see that Theorem 1.64 implies Theorem 1.65 (and the proof of Theorem 1.64 does not rely on Theorem 1.65).

It is of interest to determine the exact density of triangle free 4-critical graphs. If a characterization of when equality holds could be found, this would be particularly pleasing.

**Question 1.66.** *What is the best  $c \in \{2, 3, 4\}$  such that every triangle free 4-critical graph satisfies  $e(G) \geq \frac{5v(G)+c}{3}$ ? Is there a nice characterization of the 4-critical graphs for which equality holds.*

A second obvious question is to extend the result from 4-critical graphs with no  $K_3$  to  $k$ -critical graphs with no  $K_{k-1}$ -subgraph. We conjecture a bound:

**Conjecture 1.67.** *Every  $k$ -critical graph  $G$  with no  $K_{k-1}$  satisfies*

$$e(G) \geq \frac{(k+1)(k-2)v(G) + k(k-3)}{2(k-1)}.$$

The conjecture is only known to hold when  $k = 4$ . James Davies (personal communication) found that for  $k \geq 5$ , the conjecture is best possible infinitely often.

This conjecture does not (seemingly) have as nice applications as Theorem 1.64, however it still seems to be quite interesting.

Returning to Grötzsch's Theorem, there is a different way to make a fractional refinement of the theorem, and this is the dual form of Jaeger's modular orientation conjecture.

**Conjecture 1.68** ([23]). *Every planar graph of girth at least  $4k$  admits a homomorphism to the odd cycle on  $2k+1$  vertices.*

This conjecture has attracted a large amount of attention (see [32] for a good survey). The following is the best progress so far (stated in a weaker form than what was proved):

**Theorem 1.69** ([32]). *Every planar of girth at least  $6k$  admits a homomorphism to the odd cycle on  $2k+1$  vertices.*

Observing that the  $k = 1$  case of Conjecture 1.68 is Grötzsch's Theorem, one might think there is a proof of Jaeger's conjecture similar to the Kostochka-Yancey proof of Grötzsch's Theorem. This motivates a more general notion of a critical graph.

**Definition 1.70.** For a fixed graph  $H$ , we say that a graph  $G$  is  $H$ -critical if  $G$  does not admit a homomorphism to  $H$ , but all proper subgraphs of  $G$  do.

In [9], Zdeněk Dvořák and Postle investigated  $C_5$ -critical graphs, and proved:

**Theorem 1.71** ([9]). *If  $G$  is  $C_5$ -critical and not  $K_3$ , then*

$$e(G) \geq \frac{5v(G) - 2}{4}.$$

Via Euler’s formula this implies:

**Corollary 1.72** ([9]). *If  $G$  is planar and has girth at least 10, then  $G$  admits a homomorphism to  $C_5$ .*

They conjectured the following bound:

**Conjecture 1.73.** *If  $G$  is  $C_5$ -critical, then*

$$e(G) \geq \frac{14v(G) - 9}{11}.$$

Further, they observed using a Hell-Nešetřil indicator construction (see [20]) that if this conjecture were true, it would generalize the  $k = 6$  case of Theorem 1.50. Later, Postle and Smith-Roberge asked the natural extension of Conjecture 1.73 to arbitrary odd cycles.

**Question 1.74** ([43]). *Is it true that if  $G$  is  $C_{2t+1}$ -critical, for  $t \geq 2$ , then*

$$e(G) \geq \frac{t(2t + 3)v(G) - (t + 1)(2t - 1)}{2t^2 + 2t - 1}?$$

This naively is not strong enough to prove Jaeger’s conjecture. Nevertheless it is still of interest as it generalizes the odd  $k$  cases of Theorem 1.50. Using signed graphs [39] suggest that Question 1.74 is not true, however no counterexamples have been constructed yet. The best result towards Question 1.74 is:

**Theorem 1.75** ([43]). *Let  $G$  be a  $C_7$ -critical graph. If  $G$  is not  $C_3$  or  $C_5$ , then*

$$e(G) \geq \frac{17v(G) - 2}{15}.$$

Homomorphisms to odd cycles is a special instance of a certain circular colouring, which is a natural refinement of colouring.

**Definition 1.76.** Let  $p$  and  $q$  be positive integers such that  $\frac{p}{q} \geq 2$ . We say the  $(p, q)$ -circular-clique, denoted  $G_{p,q}$ , has vertices  $\{0, 1, 2, \dots, p - 1\}$  and an edge  $ij$  if  $q \leq |i - j| \leq p - q$ . We say  $G$  admits a  $(p, q)$ -circular-colouring if  $G$  admits a homomorphism to  $G_{p,q}$ . As there will be no confusion, we will refer to  $(p, q)$ -circular-colourings as  $(p, q)$ -colourings.

We refer the reader to [53] for a comprehensive overview of circular colouring.

It is easy to see that  $K_k$  is isomorphic to  $G_{k,1}$ , and that the odd cycle on  $2k + 1$  vertices is isomorphic to  $G_{2k+1,k}$ . Thus circular colourings allow for a refinement of regular colouring, as well as homomorphisms to odd cycles. In particular, a useful and easy fact is that if  $\frac{p}{q} \leq \frac{p'}{q'}$ , then  $G_{p,q} \rightarrow G_{p',q'}$  (see [21]). Combining this with the fact that homomorphisms compose, we see that circular colouring is a “fractional” refinement of colouring.

If Question 1.74 is true, then it implies a statement about  $G_{7,2}$ -critical graphs.

**Observation 1.77.** If the  $t = 3$  case of Question 1.74 is true, then every  $G_{7,2}$ -critical graph  $G$  satisfies

$$e(G) \geq \frac{27v(G) - 20}{15}.$$

There is a special case which I will focus on.

**Observation 1.78.** Let  $p$  and  $q$  be integers such that  $3 \leq \frac{p}{q} < 4$ . Any 4-critical graph with no  $(p, q)$ -colouring is  $G_{p,q}$ -critical.

*Proof.* Let  $G$  be such a graph. By the assumption  $G$  has no  $(p, q)$ -colouring. By 4-criticality, for any edge  $e \in E(G)$ ,  $G - e \rightarrow K_3$ , and  $K_3 \rightarrow G_{p,q}$  as  $3 \leq \frac{p}{q}$ . As homomorphisms compose,  $G - e \rightarrow G_{p,q}$ , and hence  $G - e$  has a  $(p, q)$ -circular colouring. Therefore  $G$  is  $G_{p,q}$ -critical.  $\square$

We note that Observation 1.77 is tight with regards to the choice of  $(7, 2)$ , as we find for any integers  $p$  and  $q$  satisfying  $3 \leq \frac{p}{q} < \frac{7}{2}$  there exists a 4-critical graph with no  $(p, q)$ -colouring which satisfies

$$e(G) < \frac{27v(G) - 20}{15}.$$

One can ask if every 4-critical graph with no  $(7, 2)$ -colouring satisfies  $e(G) \geq \frac{27v(G) - 20}{15}$ . This would be an asymptotic improvement in the Kostochka-Yancey bound, and show that sparse 4-critical graphs have low circular chromatic number. Here, the *circular chromatic number* is

$$\chi_c(G) := \inf\left\{\frac{p}{q} \mid G \rightarrow G_{p,q}\right\}.$$

One can show that the infimum is always attained, and hence can be taken to be a minimum (see for instance Corollary 6.8 of [21] for a proof.) Most readers will be familiar with the *chromatic number*, denoted  $\chi(G)$ , which is the minimum  $k$  such that  $G$  admits a  $k$ -colouring. We note that  $\chi(G)$  is the ceiling of  $\chi_c(G)$ , showing that circular colouring is a fractional refinement of colouring.

We do not get particularly close to this, but we do manage to show an asymptotic improvement over the Kostochka-Yancey bound.

**Theorem 1.79.** *Let  $G$  be a 4-critical graph that does not have a  $(7, 2)$ -colouring. Then either  $G = K_4$ , or  $G = W_5$ , or*

$$e(G) \geq \frac{17v(G)}{10}.$$

As a cute application, this result does show the complements of sparse 4-critical graphs contain hamiltonian cycles. Recall a *Hamiltonian cycle* is a cycle whose vertex set is the entire graph.

Let  $\bar{G}$  denote the *complement* of  $G$ , that is the graph with the same vertex set of  $G$ , but if  $uv \in E(G)$ , then  $uv \notin E(\bar{G})$ , and if  $uv \notin E(G)$ , then  $uv \in E(\bar{G})$ .

**Theorem 1.80** ([11]). *If a graph  $G$  has circular chromatic number  $\frac{p}{q}$  which is not integral, then  $\bar{G}$  contains a hamiltonian cycle.*

**Corollary 1.81.** *If  $G$  is 4-critical,  $G$  is not isomorphic to  $K_4$  or  $W_5$ , and  $e(G) < \frac{17v(G)}{10}$ , then  $\bar{G}$  contains a Hamiltonian cycle.*

Aside from Theorem 1.79, we prove some structural results about the types of graphs which can appear in the subgraph induced by the degree three vertices of a graph  $G$ , which we denote  $D_3(G)$ . Recall that the *claw* is the unique tree on four vertices containing a vertex of degree three.

**Theorem 1.82.** *If  $G$  is a 4-critical graph that does not have a  $(7, 2)$ -colouring, then either  $G$  is isomorphic to an odd wheel, or every component of  $D_3(G)$  is isomorphic to a path or a claw. Further, there are 4-critical graphs with no  $(7, 2)$ -colouring that have components of  $D_3(G)$  isomorphic to either a claw, or arbitrarily long paths.*

We can generalize the ideas of part of Theorem 1.82 to  $k$ -critical graphs.

**Theorem 1.83.** *If  $G$  is a  $k$ -critical graph that does not have a  $(2k - 1, 2)$ -colouring, then  $D_{k-1}(G)$  does not contain a clique of size  $k - 1$ .*

In light of the above, we propose an extremely strong conjecture.

**Conjecture 1.84.** *Let  $p$  and  $q$  be integers where  $3 < \frac{p}{q} < 4$ . Let  $G$  be a 4-critical graph with no  $(p, q)$ -colouring. There exists positive rational numbers  $\varepsilon_{p,q}$  and  $c_{p,q}$  depending on  $p$  and  $q$  such that*

$$e(G) \geq \frac{(5 + \varepsilon_{p,q}) - c_{p,q}}{3}.$$

### 1.3 Cycles and paths between colourings

The first part of this section is devoted to discussing cycles in graphs with large circular chromatic number.

The first result we note is one that almost everyone learns in an introductory graph theory course.

**Observation 1.85.** A graph  $G$  is 2-colourable if and only if  $G$  has no cycle of length  $1 \pmod 2$ .

A natural generalization of this was given by Tuza [49] who proved the following:

**Theorem 1.86** ([49]). *Every graph with no cycle of length  $1 \pmod k$  is  $k$ -colourable.*

Tuza in fact proved a stronger statement. For context we include Minty's Theorem [34].

**Theorem 1.87** ([34]). *A graph  $G$  is  $k$ -colourable if and only if it has an orientation in which no cycle of  $G$  has more than  $k - 1$  times as many forward edges as backwards edges.*

Tuza showed that you only need to consider cycles of length  $1 \pmod k$ .

**Theorem 1.88** ([49]). *A graph  $G$  is  $k$ -colourable if and only if it has an orientation in which no cycle of length  $1 \pmod k$  has more than  $k - 1$  times as many forward edges as backwards edges.*

Focusing back on Tuza's basic result, there are a few ways we could aim to refine it. You could ask how many cycles of length  $1 \pmod k$  can you have before you cannot ensure that your graph is  $k$ -colourable. We manage to answer this and do a bit better. We show:

**Theorem 1.89** ([37]). *Let  $e$  be an edge such that  $G - e$  is  $k$ -colourable and  $G$  is not. Then for  $2 \leq r \leq k$ ,  $e$  lies in at least  $\prod_{i=1}^{r-1} (k - i)$  cycles of length congruent to 1 modulo  $r$ .*

We point out that in a  $(k + 1)$ -critical graph, every edge satisfies the condition of Theorem 1.89. As a special case, this implies that every graph with fewer than  $(k - 1)!$  cycles of length congruent to 1 modulo  $k$  is  $k$ -colourable, and hence we generalize Tuza's result.

We note that the fact that these cycles can be assumed to be going through a particular edge is important. A referee suggested a probabilistic argument showing that every graph with fewer than  $k!/2$  cycles of length  $1 \pmod k$  is  $k$ -colourable. Although  $k!/2$  is greater than  $(k - 1)!$ , the probabilistic argument does not yield  $k!/2$  such cycles *through every edge* in a  $(k + 1)$ -critical graph, so it does not imply Theorem 1.89. Nevertheless we sketch the probabilistic argument here as it is nice.

For the probabilistic argument, randomly order the vertices and orient each edge toward its later endpoint in the order. Now, bound the probability that some cycle of length  $1 \pmod k$  (or its reverse) has more than  $k - 1$  times as many forward edges as backwards edges. If this probability is at most  $2/k!$  and there are fewer than  $k!/2$  such cycles, then some orientation has no cycle of length  $1 \pmod k$  with more than  $k - 1$  times as many forward edges as backward edges. By Tuza's strengthening of Minty's Theorem,  $G$  is then  $k$ -colourable.

An ordering of a cycle  $C$  of length  $qk + 1$  is *bad* if following the vertices along  $C$  involves at most  $q$  backward steps and more than  $(k - 1)q$  forward steps in the ordering. Since the vertices outside the cycle are irrelevant, it suffices to show that at most  $(qk + 1)!/k!$  of the  $(qk + 1)!$  orderings of  $v_1, \dots, v_{qk+1}$  have at most  $q$  instances of  $v_i$  preceding  $v_{i-1}$ .

From one backward step to the next is an increasing run. Hence to form an ordering with at most  $q$  backward steps we assign the vertices to bins 1 through  $q$  and place the

vertices within a bin in increasing order. Such assignments include the orderings where there are fewer backwards steps. Since we are following a cycle, we can start at any of the  $qk+1$  vertices in the resulting linear order. The probability is thus bounded by  $q^{qk+1}/(qk)!$ . Multiplying by 2 accounts for following the cycle in the opposite order.

The bound is at most  $2/k!$ , with equality only when  $q = 1$ . This yields the additional observation that if a non- $k$ -colourable graph has only  $k!/2$  cycles of length congruent to 1 modulo  $k$ , then those cycles all must have length exactly  $k + 1$ . This occurs, for example, in  $K_{k+1}$ , so the result is sharp.

More in line with the idea of fractionally refining integral theorems, one can ask if there is an extension of Tuza's result to circular colouring.

Zhu [54] extended Tuza's result to circular colouring. Given an orientation of a graph  $G$ , and a cycle  $C$  in  $G$  viewed in a consistent direction, let  $C^-$  denote the set of edges in  $C$  that  $C$  follows opposite to the orientation of  $G$ .

**Theorem 1.90** ([54]). *If  $G$  has an orientation such that  $|e(C)|/|C^-| \leq k/d$  for every cycle  $C$  (in each direction) such that  $d|E(C)|$  is congruent modulo  $k$  to some value in  $\{1, \dots, 2d - 1\}$ , then  $G$  is  $(k, d)$ -colourable.*

When  $d$  and  $k$  are relatively prime, let  $s$  be the congruence class such that  $sd \equiv 1 \pmod k$ . It follows from Zhu's result that if  $G$  has no cycle whose length is congruent modulo  $k$  to  $is$  for any  $i$  with  $1 \leq i \leq 2d - 1$ , then  $G$  is  $(k, d)$ -colourable. That is,  $(k, d)$ -colourability holds when  $2d - 1$  congruence classes of cycle lengths modulo  $k$  are forbidden. We manage to strengthen this result:

**Theorem 1.91** ([37]). *Let  $k$  and  $d$  be relatively prime, and  $s$  be the congruence class such that  $sd \equiv 1 \pmod k$ . If  $G - e$  is  $(k, d)$ -colourable, and  $G$  is not, then  $e$  lies in a cycle with length congruent to  $is \pmod k$  for some  $i \in \{1, \dots, d\}$ . Further, if  $G$  has no cycle through  $e$  with length  $is \pmod k$ , when  $i \in \{1, \dots, d - 1\}$ , then  $e$  lies in at least two cycles with length  $1 \pmod k$  and  $G - e$  contains a cycle of length  $0 \pmod k$ .*

The special case of  $(2d + 1, d)$ -colouring shows that the result is sharp. Here  $-2d \equiv 1 \pmod{2d + 1}$ , so  $s = -2$ , and we seek a cycle length congruent to  $-2i$  for some  $i \in \{1, \dots, d\}$ . These lengths are the odd values from  $2d - 1$  to 1. The graph  $G_{2d+1, d}$  is isomorphic to the odd cycle  $C_{2d+1}$ . Thus  $\chi_c(G) \leq 2 + 1/d$  for a  $C_{2d+1}$ -colourable graph  $G$ . All shorter odd cycles are critical non- $C_{2d+1}$ -colourable graphs. This means that although possibly only one cycle length among the listed classes of lengths occurs, we cannot omit any of those classes from the list.

A different generalization would be to extend our result to either Tuza's or Zhu's general result. We just state the possible generalization of Tuza's result.

**Question 1.92.** *Is it true that a graph  $G$  is  $k$ -colourable if it has an orientation where at most  $(k - 1)! - 1$  cycles of length  $1 \pmod k$  have more than  $k - 1$  times as many forward edges as backwards edges?*



If this question is too hard, one can change  $(k-1)!-1$  to a smaller value, and it would still be interesting.

In the remaining part of the section, we discuss finding paths between colourings.

**Definition 1.93.** Let  $G$  be a graph which is  $H$ -colourable. Let  $f, g$  be two  $H$ -colourings of  $G$ . A reconfiguration sequence from  $f$  to  $g$  is a sequence of  $H$ -colourings  $f = f_1, \dots, f_t = g$  where  $f_i$  differs from  $f_{i+1}$  on at most one vertex. If for a fixed  $H$ -colouring  $f$ , and for every  $H$ -colouring  $g$ , there is a reconfiguration sequence from  $f$  to  $g$ , then we say  $G$  is  $H$ -mixing. If  $H = K_k$ , we say  $G$  is  $k$ -mixing. If  $H = G_{p,q}$ , we say  $G$  is  $(p, q)$ -mixing.

We note that if  $G$  is not  $H$ -colourable,  $G$  is not  $H$ -mixing.

A very nice result of Cereceda, Johnson, and van den Heuvel gives a structural characterization of 3-mixing in planar graphs, as well as showing all 3-mixing graphs are bipartite (not necessarily requiring planarity)

**Theorem 1.94** ([5]). *Every graph which is not bipartite is not 3-mixing.*

Recall that given a planar graph  $G$ , with a given planar embedding, a cycle  $C$  in  $G$  is separating if both the interior and exterior of the cycle contain a vertex. Here we note that the interior and exterior do not contain the cycle  $C$ .

**Theorem 1.95** ([5]). *Let  $G$  be a 2-connected planar bipartite graph with no separating 4-cycle. Otherwise,  $G$  is 3-mixing if and only if  $G$  has a plane embedding where there is at most one face of length at most 6.*

It may not look like a full characterization, but one can show that there is no loss in generality by restricting to the 2-connected case and having no separating 4-cycle. Hence this does give a complete characterization of 3-mixing in planar graphs. Cereceda, Johnson and van den Heuvel observed that 3-mixing in general is co-NP-complete, so the characterization for planar graphs is surprising [5]. We extend this to the setting of circular colourings.

**Theorem 1.96** ([3]). *Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the minimal non- $(p, q)$ -mixing even cycle. Let  $G$  be a 2-connected bipartite planar graph with no separating  $C_{2i}$ -cycles for  $i \in \{2, \dots, k-1\}$ . The graph  $G$  is  $(p, q)$ -mixing if and only if for every planar embedding of  $G$ ,  $G$  has at most one facial cycle with length greater than  $2k$ .*

This theorem, as with the 3-mixing result, gives a full characterization of  $(p, q)$ -mixing in planar graphs when  $3 \leq \frac{p}{q} < 4$ , which we present when we prove this result. The restriction to bipartite graphs is justified as a consequence of a result in [4], which implies the following:

**Theorem 1.97** ([4]). *Let  $p$  and  $q$  be positive integers such that  $\frac{p}{q} < 4$ . If  $G$  is  $(p, q)$ -mixing, then  $G$  is bipartite.*

When  $\frac{p}{q} \geq 4$ , the proof technique completely breaks down. In general, Cereceda, Johnson and van den Heuvel conjectured that the 4-mixing problem is PSPACE-complete, and it seems believable that it is PSPACE-complete even when restricted to planar graph inputs.

There is a wide selection of literature on reconfiguration results, and instead of surveying them, we refer the reader to the brilliant survey of [\[41\]](#).

## 1.4 Structure of the Thesis

In Chapter [2](#) we prove Theorem [1.21](#). In Chapter [3](#) we prove Theorem [1.16](#). In Chapter [4](#) we prove Theorem [1.64](#). In Chapter [5](#) we prove Theorem [1.79](#). In Chapter [6](#) we prove Theorem [1.96](#). In Chapter [7](#) we prove Theorem [1.89](#) and Theorem [1.91](#).

# Chapter 2

## The Pseudoforest Strong Nine Dragon Tree Theorem

The work in this chapter is joint work with Logan Grout.

### 2.1 Introduction

The main result of this chapter is a proof of the pseudoforest analogue of the Strong Nine Dragon Tree Conjecture (Theorem [1.22](#))

As a template for how the proof of Theorem [1.22](#) will proceed, we will give a proof of the non-trivial direction of Hakimi's Theorem which acts as a framework. The proof we give appears in [\[13\]](#).

Before proceeding, we need some definitions. Given a graph  $G$ , an *orientation* of  $G$  is obtained from  $E(G)$  by taking each edge  $xy$ , and replacing  $xy$  with exactly one of the arcs  $(x, y)$  or  $(y, x)$ . To *reverse* the direction of an arc  $(x, y)$  is to replace  $(x, y)$  with the arc  $(y, x)$ . For any vertex  $v$ , let  $d(v)$  denote the degree of  $v$  in  $G$ , and if  $G$  is oriented, we let  $d^+(v)$  ( $d^-(v)$ ) denote the outdegree (indegree) of  $v$ . A directed path  $P$  from  $u$  to  $v$  is a path  $P$  with endpoints  $u$  and  $v$  oriented so that  $v$  is the only vertex with no outgoing edge. The next observation is easy and well known.

**Observation 2.1.** A graph  $G$  is a pseudoforest if and only if  $G$  admits an orientation where every vertex has outdegree at most one.

From this observation, we get an important corollary.

**Corollary 2.2.** *A graph admits a decomposition into  $k$  pseudoforests if and only if it admits an orientation such that every vertex has outdegree at most  $k$ .*

For a proof of Corollary 2.2 we refer the reader to Corollary 1.2 and Theorem 1.1 of [13]. Alternatively, here is a short proof due to a referee. Given an orientation where each vertex has outdegree at most  $k$ , colour the arcs  $(u, v)$  incident to each vertex  $u$  with distinct colours from  $1, \dots, k$ . This is possible as the orientation has outdegree at most  $k$ . Now each colour class of edges induces a subgraph with maximum outdegree at most one, hence it is a pseudoforest. For the converse, given  $k$  pseudoforests, we orient each vertex to have maximum outdegree at most 1. The union of these oriented pseudoforests is an orientation of  $G$  with maximum outdegree at most  $k$ . We will use Corollary 2.2 repeatedly and implicitly throughout our proofs. With that, we can give a proof of Hakimi's Theorem.

*Proof of Theorem 1.7.* We only prove that a graph with maximum average degree  $2k$  decomposes into  $k$  pseudoforests, as the other direction follows immediately from the fact that an edge maximal pseudoforest  $P$  has  $e(P) = v(P)$ .

Suppose towards a contradiction that  $G$  has maximum average degree at most  $2k$ , but  $G$  does not decompose into  $k$  pseudoforests. Then  $G$  does not admit an orientation such that each vertex has outdegree at most  $k$ .

Consider an orientation  $\vec{G}$  of  $G$  that minimizes the sum

$$\rho := \sum_{v \in V(G)} \max\{0, d^+(v) - k\}.$$

If this sum is zero, then we have a desired decomposition, a contradiction. Thus there is a vertex  $v \in V(G)$  such that  $v$  has outdegree at least  $k + 1$ . If there is a directed path  $P$  from  $v$  to  $x$  such that  $x$  has outdegree at most  $k - 1$ , then we can reverse the directions on all of the arcs on  $P$  and obtain a decomposition with smaller  $\rho$  value, a contradiction. Consider the subgraph  $H$  induced by vertices which are reachable from directed paths of  $v$ . That is,  $x \in V(H)$  if there is a directed path  $P$  which starts at  $v$  and ends at  $x$ . Then all vertices in  $H$  have outdegree at least  $k$ , and  $v$  has outdegree at least  $k + 1$ . But this implies that the average degree of  $H$  is strictly larger than  $2k$ , a contradiction.  $\square$

Now we will give a high level overview of how our proof will proceed. We will take a pseudoforest decomposition  $C_1, \dots, C_t, F$  where we will try and bound the size of each connected component in  $F$ . In the above proof of Hakimi's Theorem, the bad situation was a vertex which had too large outdegree. Now, the bad situation is that there is a component which is too large. In the proof of Hakimi's Theorem, we searched for special paths to augment on from a vertex which had too large outdegree, and in our proof, we will search for paths to augment on from a component which is too large. In the proof of Hakimi's theorem, we identified a situation where we could augment our decomposition and obtain a better decomposition, namely, directed paths from a vertex with too large outdegree to a vertex with small outdegree. In our proof, we will identify similar situations, namely, finding two components which are small enough to augment our decomposition, or finding a large component which has at least two small components nearby to perform augmentations. Then we will show that when these configurations are removed, either

we have a decomposition satisfying Theorem 1.22 or our graph actually had too large maximum average degree to begin with.

The chapter is structured as follows. In Section 2.2 we describe how we pick our hypothetical counterexample and prove basic properties said counterexample. In Section 2.3, we describe how we will augment our decomposition in certain situations. In Section 2.4, we show how to use this augmentation strategy to either find an optimal decomposition or show that our graph has too large maximum average degree.

## 2.2 Picking the counterexample

In this section we describe how we will pick our counterexample. Fix positive integers  $k$  and  $d$ , and suppose that  $G$  is a vertex minimal counterexample to Theorem 1.22 for the values of  $k$  and  $d$ .

Our first step will be to obtain desirable orientations of  $G$ . In particular, the orientations we will demand will imply that  $G$  decomposes into  $k$  pseudoforests each with  $v(G)$  edges, and one left over pseudoforest. For this, we use a lemma proved in [13] (Lemma 2.1). Technically, we need a stronger lemma, however the same proof as Lemma 2.1 will suffice. We give a proof for completeness sake only, there is no new idea needed.

**Lemma 2.3** ([13]). *If  $G$  is a vertex minimal counterexample to Theorem 1.22, then there exists an orientation of  $G$  such that for all  $v \in V(G)$ , we have  $k \leq d^+(v) \leq k + 1$ .*

*Proof.* Suppose no such orientation exists. As  $G$  has maximum average degree at most  $2k + 2$ , by Hakimi's Theorem,  $G$  admits an orientation so that every vertex has outdegree at most  $k + 1$ . Orient  $G$  so that every vertex has outdegree at most  $k + 1$ , and that the sum

$$\rho := \sum_{v \in V(G)} \max\{0, k - d^+(v)\}$$

is minimized. Observe that if  $\rho$  is zero, then we have a desirable orientation.

First we claim there is a vertex  $v$  with outdegree  $k + 1$ . If not, then all vertices have outdegree at most  $k$ , and by Hakimi's Theorem  $G$  decomposes into  $k$  pseudoforests, contradicting that  $G$  is a counterexample to Theorem 1.22.

Now we claim there is no directed path  $P$  from a vertex  $v$  with outdegree  $k + 1$  to a vertex  $u$  with outdegree at most  $k - 1$ . Suppose towards a contradiction that  $P$  is such a path. Then reversing the orientation on all of the arcs in  $P$  gives a new orientation, where  $v$  has outdegree  $k$ , all internal vertices have the same outdegree, and the outdegree of  $u$  increases by one. But this contradicts that we picked our orientation to minimize  $\rho$ .

Let  $S$  be the set of vertices in  $G$  with outdegree at most  $k - 1$ , and let  $S'$  be the set of vertices which have a directed path to a vertex in  $S$ . Observe that every vertex in  $S'$  has

outdegree at most  $k$ . Let  $\bar{S}' = V(G) - S'$ . Then every edge with one endpoint lying in  $S'$  and one endpoint in  $\bar{S}'$  is directed from  $S'$  to  $\bar{S}'$ . Observe that  $|\bar{S}'| < v(G)$ .

As  $G$  is a vertex minimal counterexample we can decompose  $G[\bar{S}']$  into  $k + 1$  pseudoforests such that one of the pseudoforests has each connected component containing at most  $d$  edges and is acyclic. Additionally, as every vertex in  $S'$  has outdegree at most  $k$ , by Hakimi's Theorem we can decompose  $G[S']$  into  $k$  pseudoforests  $C_1, \dots, C_k$ . Thus we only need to deal with the edges between  $\bar{S}'$  and  $S'$ . Observe that if  $v$  has  $t$  arcs  $(v, u_1), \dots, (v, u_t)$  where  $u_i \in \bar{S}'$  for all  $i \in \{1, \dots, t\}$ , then  $v$  has outdegree at most  $k - t$  in  $G[S']$ . Thus  $v$  has outdegree zero in at least  $t$  of the pseudoforests  $C_1, \dots, C_k$ . Therefore we can add the arcs  $(v, u_1), \dots, (v, u_t)$  to  $t$  of the pseudoforests so that the result is a pseudoforest. As all arcs between  $S'$  and  $\bar{S}'$  are oriented from  $S'$  to  $\bar{S}'$ , we now have a decomposition of  $G$  which satisfies Theorem 1.22. But this contradicts that  $G$  is a counterexample to Theorem 1.22.  $\square$

Let  $\mathcal{F}$  be the set of orientations of  $E(G)$  with  $k \leq d^+(v) \leq k + 1$  for each vertex  $v \in V(G)$ . A useful way of keeping track of our pseudoforest decomposition will be to colour the edges blue and red, where the edges coloured red will induce a pseudoforest. This will be the pseudoforest where we will want to bound the size of each connected component.

**Definition 2.4.** Suppose  $G$  is oriented such that  $k \leq d^+(v) \leq k + 1$  for each vertex  $v \in V(G)$ . Then a *red-blue colouring* of  $G$  is a (non-proper) colouring of the edges where for any vertex  $v \in V(G)$ , we colour  $k$  outgoing arcs of  $v$  blue; if after this there is an uncoloured outgoing arc, colour this arc red.

Note that given an orientation in  $\mathcal{F}$ , one can generate many different red-blue colourings. As a graph decomposes into  $k$  pseudoforests if and only if it admits an orientation where each vertex has outdegree at most  $k$ , we obtain the following observation.

**Observation 2.5.** Given a red-blue colouring of  $G$ , we can decompose our graph  $G$  into  $k + 1$  pseudoforests such that  $k$  of the pseudoforests have all of their edges coloured blue, and the other pseudoforest has all of its edges coloured red.

Observe that one red-blue colouring can give rise to many different pseudoforest decompositions. Given a pseudoforest decomposition obtained from Observation 2.5 we will say a pseudoforest which has all arcs coloured blue is a *blue pseudoforest*, and the pseudoforest with all arcs coloured red is the *red pseudoforest*.

**Definition 2.6.** Let  $f$  be a red-blue colouring of  $G$ , and let  $C_1, \dots, C_k, F$  be a pseudoforest decomposition obtained from  $f$  by Observation 2.5. Then we say that  $C_1, \dots, C_k, F$  is a *pseudoforest decomposition generated from  $f$* . We will always use the convention that  $F$  is the red pseudoforest, and each  $C_i$  is a blue pseudoforest.

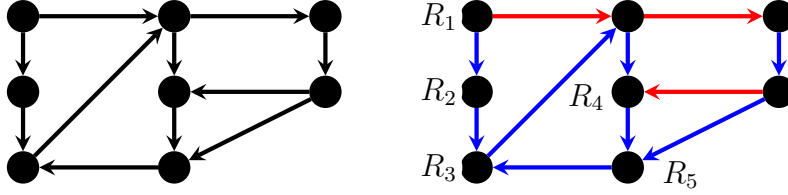


Figure 2.1: In this example we assume  $k = 1$  and  $d = 1$ . On the left, the orientation is in  $\mathcal{F}$ . On the right we have one possible red-blue colouring generated by this orientation. Here, the entire graph would be the exploration subgraph, and assuming  $R_1$  is the root,  $(R_1, R_2, R_3, R_4, R_5)$  is the smallest legal order. Lastly, the isolated vertices are the small components (and are in fact the only possible small components when  $k = 1$  and  $d = 1$ )

As  $G$  is a counterexample, in every pseudoforest decomposition generated from a red-blue colouring, there is a component of the red pseudoforest which has more than  $d$  edges, or a component that contains a cycle, or a component with  $d$  edges that has a vertex of degree  $d-1$ . We define a *residue function* which simply measures how close a decomposition is to satisfying Theorem 1.22.

**Definition 2.7.** Let  $f$  be a red-blue colouring and  $C_1, \dots, C_k, F$  be a pseudoforest decomposition generated by  $f$ . Let  $\mathcal{T}$  be the set of components of  $F$ . Then the *residue function*, denoted  $\rho$ , is

$$\rho(F) = \sum_{K \in \mathcal{T}} \max\{e(K) - d, 0\}.$$

Using a red-blue colouring, and the resulting pseudoforest decomposition, we define an induced subgraph of  $G$  on which we will focus our attention. Intuitively, this subgraph should be thought of as an “exploration” subgraph similar to how in the proof of Hakimi’s theorem we “explored” from a vertex which had too large outdegree. Here we will “explore” from a component which is too large.

**Definition 2.8.** Suppose that  $f$  is a red-blue colouring of  $G$ , and suppose  $D = (C_1, \dots, C_k, F)$  is a pseudoforest decomposition generated from  $f$ . If it exists, let  $R$  be a component of  $F$  such that  $e(R) > d$ . Otherwise, take  $R$  to be a component which contains a cycle. We define the *exploration subgraph*  $H_{f,D,R}$  in the following manner. Let  $S \subseteq V(G)$  where  $v \in S$  if and only if there exists a path  $P = v_1, \dots, v_m$  such that  $v_m = v$ ,  $v_1 \in V(R)$ , and either  $v_i v_{i+1}$  is an arc  $(v_i, v_{i+1})$  coloured blue, or  $v_i v_{i+1}$  is an arbitrarily directed arc coloured red. Then we let  $H_{f,D,R}$  be the graph induced by  $S$ .

Given a particular exploration subgraph  $H_{f,D,R}$ , we say  $R$  is the *root component*. We say the *red components* of  $H_{f,D,R}$  are the components of  $F$  contained in  $H_{f,D,R}$ .

It might not be clear why we made this particular definition for  $H_{f,D,R}$ , however the next observation shows that for any exploration subgraph  $H_{f,D,R}$ , the red edge density

must be low. Before stating the observation, we fix some notation. Given a subgraph  $K$  of  $G$ , we will let  $E_b(K)$  and  $E_r(K)$  denote the sets of edges of  $K$  coloured blue and red, respectively. We let  $e_b(K) = |E_b(K)|$  and  $e_r(K) = |E_r(K)|$ .

**Observation 2.9.** For any red-blue colouring  $f$ , any pseudoforest decomposition  $D$  generated from  $f$ , and any choice of root component  $R$ , the exploration subgraph  $H_{f,D,R}$  satisfies

$$\frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} \leq \frac{d}{d+k+1}.$$

*Proof.* Suppose towards a contradiction that

$$\frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} > \frac{d}{d+k+1}.$$

As  $H_{f,D,R}$  is an induced subgraph defined by directed paths, and every vertex  $v \in V(G)$  has  $k$  outgoing blue arcs, each vertex in  $H_{f,D,R}$  has  $k$  outgoing blue arcs. Thus,

$$\frac{e_b(H_{f,D,R})}{v(H_{f,D,R})} = k.$$

Then we have

$$\frac{\text{mad}(G)}{2} \geq \frac{e(H_{f,D,R})}{v(H_{f,D,R})} = \frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} + \frac{e_b(H_{f,D,R})}{v(H_{f,D,R})} > k + \frac{d}{d+k+1}.$$

But this contradicts that  $G$  has  $\text{mad}(G) \leq 2k + \frac{2d}{d+k+1}$ .

□

For the entire proof, we will be attempting to show that we can augment a given decomposition in such a way that either we obtain a decomposition satisfying Theorem 1.22 or we can find a exploration subgraph  $H_{f,D,R}$  which contradicts Observation 2.9.

As Observation 2.9 allows us to focus only on red edges, it is natural to focus on red components which have small average degree. With this in mind, we define the notion of a *small red component*.

**Definition 2.10.** Let  $C_1, \dots, C_t, F$  be a pseudoforest decomposition generated by a red-blue colouring. Let  $K$  be a subgraph of  $F$ . Then  $K$  is a *small red subgraph* if

$$\frac{e_r(K)}{v(K)} < \frac{d}{d+k+1}.$$

If  $K$  is connected, we say  $K$  is a *small red component*.



In particular, we will be interested in the case when  $K$  is connected and a small red component. When  $K$  is connected and small, the red subgraph is actually isomorphic to a tree, and we can rewrite the density bound in the definition in a more convenient manner.

**Observation 2.11.** Let  $K$  be a connected small red subgraph. Then  $K$  is a tree, and further

$$e_r(K) < \frac{d}{k+1}.$$

*Proof.* First, suppose that  $K$  is not a tree. Then  $K$  contains exactly one cycle. As  $K$  is connected, it follows that

$$\frac{e_r(K)}{v(K)} = 1.$$

But  $\frac{d}{k+d+1} < 1$  by the assumption, and as  $d$  and  $k$  are positive integers, it follows that  $K$  is not a small red subgraph. Thus we can assume that  $K$  is a tree, and hence  $e(K) = v(K) - 1$ . Thus

$$\frac{e(K)}{v(K)} = \frac{e(K)}{e(K) + 1} < \frac{d}{d + k + 1}.$$

Therefore

$$e(K)(d + k + 1) < d(e(K) + 1).$$

Simplifying, we see that this is equivalent to

$$e(K) < \frac{d}{k+1}.$$

□

We will want to augment our decomposition, and we will want a measure of progress that our decomposition is improving. Of course, if we reduce the residue function that clearly improves the decomposition. However, this might not always be possible, so we will introduce a notion of a “legal order” of the red components. This order keeps track of the number of edges in components which are “close” to the root component, with the idea being that if we can continually perform augmentations to make components “closer” to the root component have fewer edges without creating any large components, then we eventually reduce the number of edges in the root component, which improves the residue function. We formalize this in the following manner.

**Definition 2.12.** We call an ordering  $(R_1, \dots, R_t)$  of the red components of  $H_{f,D,R}$  *legal* if all red components are in the ordering,  $R_1$  is the root component, and for all  $j \in \{2, \dots, t\}$  there exists an integer  $i$  with  $1 \leq i < j$  such that there is a blue arc  $(u, v)$  such that  $u \in V(R_i)$  and  $v \in V(R_j)$ .

Let  $(R_1, \dots, R_t)$  be a legal ordering. We will say that  $R_i$  is a *possible parent* of  $R_j$  if  $i < j$  and there is a blue arc  $(v_i, v_j)$  where  $v_i \in R_i$  and  $v_j \in R_j$ . From this definition a red component may have possible parents. To remedy this, if a red component has many possible parents, we arbitrarily pick one such red component and designate it as the *parent*. If  $R_i$  is the parent of  $R_j$ , then we say that  $R_j$  is a *child* of  $R_i$ . We say a red component  $R_i$  is an *ancestor* of  $R_j$  if we can find a sequence of red components  $R_{i_1}, \dots, R_{i_m}$  such that  $R_{i_1} = R_i$ ,  $R_{i_m} = R_j$ , and  $R_{i_q}$  is the parent of  $R_{i_{q+1}}$  for all  $q \in \{1, \dots, m-1\}$ . An important definition is that of vertices witnessing a legal order.

**Definition 2.13.** Given a legal order  $(R_1, \dots, R_t)$ , we say a vertex  $v$  *witnesses the legal order for  $R_j$*  if there is a blue arc  $(u, v)$  such that  $u \in R_i$  and  $v \in R_j$  and  $i < j$ .

Observe that there may be many vertices which witness the legal order for a given red component. More importantly, for every component which is not the root, there exists a vertex which witnesses the legal order. We also want to compare two different legal orders.

**Definition 2.14.** Let  $(R_1, \dots, R_t)$  and  $(R'_1, \dots, R'_t)$  be two legal orders. We will say  $(R_1, \dots, R_t)$  is *smaller* than  $(R'_1, \dots, R'_t)$  if the sequence  $(e(R_1), \dots, e(R_t))$  is smaller lexicographically than  $(e(R'_1), \dots, e(R'_t))$ .

With Definition 2.14, we will pick our minimal counterexample  $G$  in the following manner. First,  $v(G)$  is minimized. After this we pick an orientation in  $\mathcal{F}$ , a red-blue colouring  $f$  of this orientation, a pseudoforest decomposition  $D = (C_1, \dots, C_k, F)$  generated from  $f$ , such that the number of cycles in  $F$  is minimized. Subject to this, we minimize the residue function  $\rho$ . Finally, we pick a smallest possible legal order  $(R_1, \dots, R_t)$ .

From here on out, we will assume we are working with a counterexample picked in the manner described. The point of minimizing the number of cycles in  $F$  is slightly unintuitive compared to minimizing the residue function and minimizing the legal order. However, we minimize the number of cycles because when we augment we will need to ensure our decomposition is in fact a pseudoforest decomposition, and our augmentations will never create more cycles in  $F$ . Hence by minimizing the number of cycles in  $F$  first, we can easily take care of the cases where cycles occur, which allows us to focus on the more important cases where the components are acyclic.

## 2.3 Augmenting the decomposition

In this section we describe a very simple operation which will mostly be how we augment our decomposition. Let  $f$  be the red-blue colouring of our counterexample, and let  $C_1, \dots, C_k$  be the blue pseudoforests, and  $F$  the red pseudoforest. Let  $(R_1, \dots, R_t)$  be the legal ordering picked for our counterexample. As some notation, given a vertex  $x \in V(H_{f,D,R})$ , we let  $R^x$  denote the red component of  $H_{f,D,R}$  containing  $x$ .

**Definition 2.15.** Let  $(x, y)$  be a blue arc. Further suppose that  $R^y$  is a tree, and suppose that  $e = xv$  is an arbitrarily oriented red arc incident to  $x$ . To *exchange  $e$  and  $(x, y)$*  is to perform the following procedure. First, take the maximal directed red path in  $R^y$  starting at  $y$ , say  $Q = y, v_1, \dots, v_l$  where  $(v_i, v_{i+1})$  is a red arc for  $i \in \{1, \dots, l\}$  and  $(y, v_1)$  is a red arc, and reverse the direction of all arcs of  $Q$ . Second, change the colour of  $(x, y)$  to red and reorient  $(x, y)$  to  $(y, x)$ . Finally, change the colour of  $e$  to blue, and if  $e$  is oriented  $(v, x)$ , reorient to  $(x, v)$ .

See Figure 2.2 for an illustration. We note that exchanging on an edge  $e$  and  $(x, y)$  is well-defined. This is because  $R^y$  is acyclic (and hence a tree), and thus there is a unique maximal directed path in  $R^y$  which starts at  $y$ .

**Observation 2.16.** Suppose we exchange the edge  $e = xv$  and  $(x, y)$ . Then the resulting orientation is in  $\mathcal{F}$ , and the resulting colouring is a red-blue colouring of this orientation.

*Proof.* Let us first check the outdegrees of vertices after the exchange. Let  $Q = y, v_1, \dots, v_l$  be the maximal directed red path in  $R^y$  before exchanging the edges. First suppose that  $Q$  is not just  $y$ . Then all of the internal vertices on this path have the same outdegree after reversing as before. On the other hand, the outdegree of  $y$  decreases by 1, and the outdegree of  $v_l$  increases by one. As  $Q$  is a maximal red path, this implies that the outdegree of  $v_l$  before reversing the arcs on  $Q$  was  $k$ , and hence after reversing the arcs this outdegree is  $k + 1$ . The outdegree of  $y$  drops by one after reversing the arcs on  $Q$ , but we reverse the arc  $(x, y)$  to  $(y, x)$ , and hence the outdegree of  $y$  remains the same as before.

If  $Q$  is just  $y$ , then the outdegree of  $y$  before exchanging was  $k$ , and then as we reorient  $(x, y)$  to  $(y, x)$ , the outdegree of  $y$  is now  $k + 1$ .

Focusing on  $x$  now, if  $e$  is oriented from  $x$  to  $v$ , then the outdegree of  $x$  is initially  $k + 1$ , and after the exchange it ends up being  $k$ ; otherwise, we reorient  $(v, x)$  to  $(x, v)$  and so the outdegree of  $x$  remains the same as before the exchange. Lastly, the outdegree of  $v$  remains the same if  $e$  was oriented  $(x, v)$ , and otherwise the outdegree of  $v$  initially was  $k + 1$ , and after reorienting becomes  $k$ . Thus the resulting orientation is in  $\mathcal{F}$ .

Now we will see that this new colouring is a red-blue colouring of the orientation. Note after exchanging  $e$  and  $(x, y)$ ,  $y$  has exactly one outgoing edge coloured red. If  $x$  had no outgoing red edge before, it still has no outgoing red edge, and if it did have an outgoing red edge, then the outdegree of  $x$  dropped by one, and now  $x$  has no outgoing red edge. Finally, if  $Q$  was not just  $y$ , then  $v_l$  now has no outgoing red edge. It follows that the resulting colouring is in fact a red-blue colouring.  $\square$

To avoid repetitively mentioning it, we will implicitly make use of Observation 2.16. Now we begin to impose some structure on our decomposition. First we make an observation which allows us to effectively ignore parent components with cycles (such components will still exist, but for the purposes of our argument we will not need to worry about them).

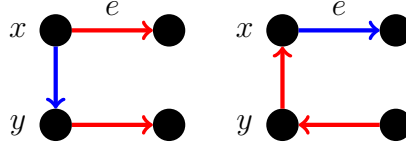


Figure 2.2: An example of an exchange on an edge  $e$  and  $(x, y)$

**Observation 2.17.** Let  $(x, y)$  be a blue arc such that  $R^x$  is distinct from  $R^y$  and further  $R^y$  is a tree. Then  $x$  does not lie in a cycle of  $F$ .

*Proof.* Suppose towards a contradiction that  $x$  lies in a cycle of  $F$ . Let  $e$  be an edge incident to  $x$  which lies in the cycle coloured red. Now exchange  $(x, y)$  and  $e$ . As  $(x, y)$  was an arc between two distinct red components, and  $e$  was in the cycle coloured red, after performing the exchange, we reduce the number of cycles in  $F$  by one. However, this contradicts that we picked our counterexample to have the fewest number of red cycles.  $\square$

With this we can show that given two components  $K$  and  $C$ , where  $K$  is the parent of  $C$ , and  $C$  is acyclic, that  $e_r(K) + e_r(C) \geq d$ .

**Lemma 2.18.** Let  $R^x$  and  $R^y$  be red components such that  $R^y$  is the child of  $R^x$ ,  $R^y$  does not contain a cycle, and  $(x, y)$  is a blue arc from  $x$  to  $y$ . Then  $e_r(R^x) + e_r(R^y) \geq d$ .

*Proof.* Suppose towards a contradiction that  $e_r(R^x) + e_r(R^y) < d$ . Hence  $e_r(R^x) < d$ . Thus  $R^x$  is not the root component. Let  $w$  be a vertex which witnesses the legal order for  $R^x$  ( $w$  exists as  $R^x$  is not the root component). By Observation 2.17 we know that  $x$  does not belong to a cycle of  $R^x$ .

**Case 1:**  $w \neq x$ .

Let  $e$  be the edge incident to  $x$  in  $R^x$  such that  $e$  lies on the path from  $x$  to  $w$  in  $R^x$ . Then exchange  $(x, y)$  and  $e$ . As  $e_r(R^x) + e_r(R^y) < d$ , all resulting red components have fewer than  $d$  edges, and hence we do not increase the residue function. Furthermore, we claim we can find a smaller legal order. Let  $R_i$  be the component in the legal order corresponding to  $R^x$ . Then consider the new legal order where the components  $R_1, \dots, R_{i-1}$  remain in the same position, we replace  $R_i$  with the new red component containing  $w$ , and then complete the order arbitrarily. By how we picked  $e$ ,  $e(R^w)$  is strictly smaller than  $e(R_i)$ , and hence we have found a smaller legal order, a contradiction.

**Case 2:**  $w = x$ .

We refer the reader to Figure 2.3 for an illustration. As  $R^x$  is not the root component, let  $R^{x_1}$  be the closest ancestor of  $R^x$  such that  $e(R^{x_1}) \geq 1$  (there is an ancestor with this property, as the root component has at least one edge). Let  $R^{x_n}, R^{x_{n-1}}, \dots, R^{x_1}$  be a sequence of red components such that for  $i \in \{2, \dots, n\}$ ,  $R^{x_i}$  is the child of  $R^{x_{i-1}}$  and  $R^x$  is the child of  $R^{x_n}$ . Up to relabelling the vertices, there is a path  $P = x_1, \dots, x_n, x, y$  such

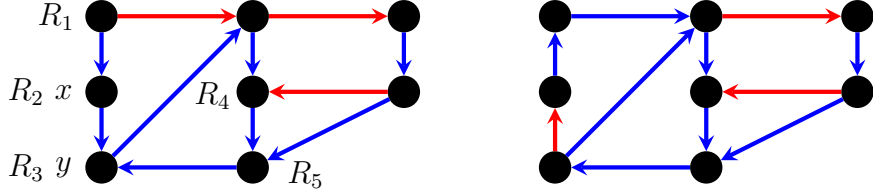


Figure 2.3: An illustration of Case 2 in Lemma 2.18, where  $k = 1$  and  $d = 1$  and  $x = w$ . A vertex in the root component was the ancestor, so in this case we reduce the residue function.

that  $(x_i, x_{i+1})$  is an arc coloured blue, and  $(x_n, x)$  is an arc coloured blue. Let  $e$  be a red edge incident to  $x_n$ . Now do the following. Colour  $(x, y)$  red, and reverse the direction of all arcs in  $P$ . Colour  $e$  blue, and orient  $e$  away from  $x_1$ . By the argument in our proof of Observation 2.16, the resulting orientation is in  $\mathcal{F}$  and the colouring described is a red-blue colouring. Furthermore as  $e_r(R^x) + e_r(R^y) < d$ , all resulting red components have at most  $d$  edges, and hence the residue function did not increase (in the event that  $R^{x_1}$  is the root, the residue function strictly decreases, so we assume that  $R^{x_1}$  is not the root). Finally, we can find a smaller legal order in this orientation, as we simply take the same legal order up to the component containing  $x_1$ , and then complete the remaining order arbitrarily. As the component containing  $x_1$  has at least one fewer edge now, this order is a smaller legal order, a contradiction.  $\square$

We note the following important special case of Lemma 2.18, that small red components do not have small red children.

**Corollary 2.19.** *If  $K$  is a small red component, then  $K$  does not have any small red children.*

*Proof.* Suppose towards a contradiction that  $K$  has a small red child  $C$ . As  $K$  is small, then  $e_r(K) < \frac{d}{k+1}$ . Similarly,  $e_r(C) < \frac{d}{k+1}$ . But then  $e_r(K) + e_r(C) < \frac{2d}{k+1} \leq d$ , contradicting Lemma 2.18.  $\square$

Now we will show that every red component has at most  $k$  small red children.

**Lemma 2.20.** *If  $K$  is a red component, then  $K$  has at most  $k$  small children.*

*Proof.* Suppose towards a contradiction that  $K$  has at least  $k + 1$  distinct small children. Then by the pigeon-hole principle, there are two distinct small children  $C_1$  and  $C_2$  such that there are blue arcs  $(x, x')$ ,  $(y, y')$  so that  $x \neq y$ ,  $x, y \in V(K)$ ,  $x' \in V(C_1)$  and  $y' \in V(C_2)$ . By Observation 2.17 we can assume that neither  $x$  nor  $y$  lies in a red cycle in  $K$ . Consider a path  $P_{x,y}$  in  $K$  from  $x$  to  $y$  in the underlying graph (that is, ignoring the directions of the arcs). Let  $e_x$  be the edge incident to  $x$  in  $P_{x,y}$  and  $e_y$  be the edge incident to  $y$  in  $P_{x,y}$ . Let  $K_x$  denote the component of  $K - e_y$  which contains  $x$ , and let  $K_y$  denote the component of  $K - e_x$  which contains  $y$ .

**Claim 2.21.**  $e_r(K_x) \leq e_r(T_2)$  and  $e_r(K_y) \leq e_r(T_1)$ .

*Proof.* By symmetry, we will only show that  $e_r(K_x) \leq e_r(T_2)$ . So suppose towards a contradiction that  $e_r(K_x) > e_r(T_2)$ . Then exchange on  $(y, y')$  and  $e_y$ . As  $e_r(K_x) > e_r(T_2)$ , the residue function does not increase. Observe that if  $K$  is the root, then the residue function will decrease, and that will give a contradiction. Thus we may assume that  $K$  is not the root. We claim we can find a smaller legal order. If there is a vertex which witnesses the legal order for  $K$  in  $K_x$ , then taking the same legal order up to  $K$  and then replacing  $K$  with  $K_x$  gives a smaller legal order. Similarly, if there is no vertex which witnesses the legal order in  $K_x$ , then because  $e_r(K_x) > e_r(T_2)$ , taking the same legal order up to  $K$  and replacing  $K$  with the component containing  $y$  of  $K - e_y$  after the exchange, and filling in the rest of the order arbitrarily gives a smaller legal order. In both cases, this is a contradiction.  $\square$

Note that either  $e_r(K) \leq e_r(K_x) + e_r(K_y)$  or  $e_r(K) \leq e_r(K_x) + e_r(K_y) - 1$  (the first case occurs if  $e_y \neq e_x$ , and the second occurs if  $e_y = e_x$ ). Since each  $T_i$  is a small child, Claim 2.21 (together with Observation 2.11) implies that

$$e_r(K_x) + e_r(K_y) < \frac{d}{k+1} + \frac{d}{k+1} \leq d.$$

Hence,  $e_r(K) \leq d$ . Thus we can assume that  $K$  is not the root. Let  $w$  be a vertex which witnesses the legal order. Without loss of generality, we can assume that  $w \in V(K_x)$ . Then exchange on  $(y, y')$  and  $e_y$ . We do not increase the residue function as  $e_r(K_y) \leq e_r(T_1) < \frac{d}{k+1}$ . However, we can find a smaller legal order by taking the same legal order up to  $K$ , and replacing  $K$  with  $K_x$ , and completing this order arbitrarily. But this contradicts our choice of legal order, a contradiction.  $\square$

We are now in position to prove the theorem.

## 2.4 Bounding the maximum average degree

In this section, we give a counting argument to show that our chosen exploration subgraph has too large average degree. We make the following definition for ease of notation.

**Definition 2.22.** Let  $K$  be a red component, and let  $K_1, \dots, K_q$  be the small red children of  $K$ . We will let  $K_C$  denote the subgraph with vertex set  $V(K_C) = V(K) \cup V(K_1) \cup \dots \cup V(K_q)$ , that contains all red edges from  $K, K_1, \dots, K_q$ .

**Lemma 2.23.** *Let  $K$  be a red component which is not small. Then  $K_C$  is not small. Further, if  $e_r(K) > d$  or  $K$  contains a cycle, then*

$$\frac{e_r(K_C)}{v(K_C)} > \frac{d}{d+k+1}.$$

*Proof.* First, observe that if  $K$  has no small children then  $K_C = K$  and hence is not small. If  $e_r(K) > d$ , then as  $K$  is connected,  $v(K) \leq e_r(K) + 1$  and hence  $e_r(K)/v(K) \geq (d+1)/(d+2) > d/(d+k+1)$ . If  $K$  contains a cycle, then  $e(K) = v(K)$  and we have  $1 > d/(d+k+1)$ . Thus we can suppose that  $K$  has small children  $K_1, \dots, K_q$ . By Lemma 2.20,  $q \leq k$ . By Lemma 2.18, for every  $i \in \{1, \dots, q\}$  the inequality  $e_r(K) + e_r(K_i) \geq d$  holds. As  $e_r(K_i) \geq 0$  for all  $i \in \{1, \dots, q\}$ , it follows that  $e_r(K_i) \geq \max\{0, d - e_r(K)\}$  for all  $i \in \{1, \dots, q\}$ .

Then a quick calculation shows

$$\begin{aligned} \frac{e_r(K_C)}{v(K_C)} &= \frac{e_r(K) + \sum_{i=1}^q e_r(K_i)}{v(K) + \sum_{i=1}^q v(K_i)} \\ &\geq \frac{e_r(K) + \sum_{i=1}^q \max\{0, d - e_r(K)\}}{e_r(K) + 1 + q + \sum_{i=1}^q \max\{0, d - e_r(K)\}} \\ &\geq \frac{e_r(K) + \sum_{i=1}^q \max\{0, d - e_r(K)\}}{e_r(K) + 1 + k + \sum_{i=1}^q \max\{0, d - e_r(K)\}}. \end{aligned}$$

The first equality is simply applying the definition of  $K_C$ . The first inequality uses that  $e_r(K_i) \geq \max\{0, d - e_r(K)\}$ , and that as  $K_i$  is small,  $K_i$  is a tree, so  $v(K_i) = e_r(K_i) + 1$ . Finally, the second inequality is using that  $q \leq k$ . Note that in this calculation, we also used that  $v(K) \leq e(K) + 1$ . In the case that  $K$  contains a cycle, this inequality is strict, and hence if  $K$  contains a cycle all calculations beyond this point are strict. Now we split this into two cases based on whether or not  $\max\{0, d - e_r(K)\}$  is 0 or  $d - e_r(K)$ .

**Case 1:**  $\max\{0, d - e_r(K)\} = 0$ .

If  $\max\{0, d - e_r(K)\} = 0$ , then  $e_r(K) \geq d$ . Thus it follows that,

$$\begin{aligned} \frac{e_r(K) + \sum_{i=1}^q \max\{0, d - e_r(K)\}}{e_r(K) + 1 + k + \sum_{i=1}^q \max\{0, d - e_r(K)\}} &= \frac{e_r(K)}{e_r(K) + k + 1} \\ &\geq \frac{d}{d + k + 1}. \end{aligned}$$

Further, if  $e_r(K) > d$ , the above inequality is strict.

**Case 2:**  $\max\{0, d - e_r(K)\} = d - e_r(K)$ . As  $e_r(K) \leq d$ , we only need to show that  $K_C$  is not small. Calculating we obtain,

$$\begin{aligned} \frac{e_r(K) + \sum_{i=1}^q \max\{0, d - e_r(K)\}}{e_r(K) + 1 + k + \sum_{i=1}^q \max\{0, d - e_r(K)\}} &= \frac{e_r(K) + q(d - e_r(K))}{e_r(K) + q(d - e_r(K)) + k + 1} \\ &\geq \frac{d - e_r(K) + e_r(K)}{e_r(K) + d - e_r(K) + k + 1} \\ &= \frac{d}{d + k + 1}. \end{aligned}$$

□

Now we finish the proof. Let  $\mathcal{R}$  denote the set of red components of  $H_{f,D,R}$  which are not small. By Corollary 3.19 it follows that,

$$V(H_{f,D,R}) = \bigcup_{K \in \mathcal{R}} V(K_C).$$

This follows since a small component cannot have a small child. Therefore it follows that:

$$E_r(H_{f,D,R}) = \bigcup_{K \in \mathcal{R}} E(K_C).$$

Now we bound the maximum average degree of  $H_{f,D,R}$ . By Lemma 3.47, we have

$$\begin{aligned} \frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} &= \frac{\sum_{K \in \mathcal{R}} e_r(K_C)}{\sum_{K \in \mathcal{R}} v(K_C)} \\ &> \frac{d}{d+k+1}. \end{aligned}$$

Here, equality holds in the first line because parents are unique. The strict inequality follows as  $K_C$  is not small for any  $K \in \mathcal{R}$  by Lemma 3.47, and further the root component satisfies  $e_r(R) > d$ , or the root component is a cycle,. However, this contradicts Observation 2.9. Theorem 1.22 follows. □



# Chapter 3

## Strong Dragons are tough to slay

All work in this chapter is joint work with Evelyne Smith-Roberge.

### 3.1 Introduction

In this Chapter we prove Theorem 1.16 and Theorem 1.18.

The outline of the proof is the same as the pseudoforest Strong Nine Dragon Tree Theorem (hence to understand this section, it is easier to first understand the proof of the pseudoforest Strong Nine Dragon Tree Theorem). The only reason we are not able to prove the Strong Nine Dragon Tree Conjecture in full is due to a lack of control when trying to reconfigure the decomposition, unlike in pseudoforests. In other words, we cannot find a reasonable analogue of Lemma 2.20, and so we have to make do with weaker statements. Due to no suitable structural lemma like Lemma 2.20, we have to dig much deeper into the counting argument to obtain the result.

We write the proof as if we are proving Conjecture 1.17, and we only specialize to  $d \in \{3, 4\}$  when we do the counting argument. Thus all of the structural reductions in this section can be used to prove Conjecture 1.17 without any modification. We note even most of the counting argument holds in general, and only very few spots do not. Despite this, it is much easier to read if one only considers the  $k = 1$  and  $d \in \{3, 4\}$  cases, as a lot of technical details become simpler in these cases.

We overload the terminology from Chapter 2, however the definitions subtly change, (although they are analogous).

### 3.2 Picking the minimal counterexample

Suppose Conjecture 1.17 is false. Fix integers  $k$  and  $d$  such that there exists a counterexample to Conjecture 1.17. Let  $w$  be defined as in Conjecture 1.17. Let  $G$  be a vertex

minimal counterexample to Conjecture 1.17.

Let  $\mathcal{F}$  be the set of decompositions of  $G$  of the form  $(T_1, \dots, T_k, F)$  such that  $T_i$  is a spanning tree for all  $i \in \{1, \dots, k\}$  and  $F$  is forest with maximum degree  $d$ . Below, we will show  $\mathcal{F}$  is non-empty. For convenience, we use (non-proper) edge colourings to keep track of the forest decomposition. In particular, for any decomposition  $(T_1, \dots, T_k, F) \in \mathcal{F}$ , for  $i \in \{1, \dots, k\}$ , we refer to  $T_i$  as a *blue forest*, and  $F$  as the red forest. For  $i \in \{1, \dots, k\}$ , we refer to edges of  $T_i$  as *blue edges* and edges of  $F$  as *red edges*. We call such a colouring a *red-blue colouring*. As before, we will use  $E_r(G)$  and  $E_b(G)$  to denote the set of red and blue edges of  $G$ , and we use  $e_r(G)$  and  $e_b(G)$  to denote the number of red and blue edges respectively of  $G$ . Here we note that unlike in the pseudoforest strong nine dragon tree theorem, the red-blue colouring serves only the purpose of distinguish the red forest. The next observation is essentially proven in [24]. We provide a proof for completeness.

**Observation 3.1.** The set  $\mathcal{F}$  is non-empty.

*Proof.* Suppose not. By the Nine Dragon Tree Theorem,  $G$  has a decomposition into  $k + 1$  forests where one of the forests has maximum degree  $d$ . Let  $D = (T_1, \dots, T_k, F)$  be such a decomposition, where  $\Delta(F) \leq d$ , and  $D$  is chosen so that  $e(F)$  is minimum. Let  $\mathcal{T}$  be an auxiliary rooted tree whose vertices are subsets of  $V(G)$ , defined algorithmically as follows.

1. Set  $u_{V(G)} = V(G)$  to be the root of  $\mathcal{T}$ .
2. Let  $i \in \{1, 2, \dots, k\}$  be an index such that  $T_i$  is disconnected. By assumption,  $i$  exists. Add a vertex  $u_{V(C)}$  to  $V(\mathcal{T})$  for each component  $C$  of  $T_i$ , and add an edge from  $u_{V(G)}$  to  $u_{V(C)}$ . Associate to  $u_{V(G)}$  the index  $t(u_{V(G)}) = i$ .
3. While there exists a leaf  $u_X$  of  $\mathcal{T}$  and an index  $j \in \{1, 2, \dots, k\}$  such that  $T_j[X]$  is disconnected, add a vertex  $u_{V(C)}$  to  $V(\mathcal{T})$  for each component  $C$  of  $T_j[X]$ , and associate to  $u_{V(X)}$  the index  $t(u_{V(X)}) = j$ .

**Claim 3.2.** Let  $X \subseteq V(G)$  and  $Y, Z \subset X$ . If  $u_X$  is a non-leaf in  $\mathcal{T}$  and  $u_Y$  and  $u_Z$  are children of  $u_X$ , then there does not exist a red edge from  $X$  to  $Y$  in  $G$ .

*Proof.* Suppose not. Let  $u_X$  be a non-leaf of  $\mathcal{T}$ , and let  $u_Y$  and  $u_Z$  be children of  $u_X$  with  $t(u_Y) = t(u_Z) = i$  for some  $i \in \{1, 2, \dots, k\}$ . Let  $c$  be a counter, initially set to  $c = \text{dist}(u_Y, u_{V(G)})$  (that is, the distance in  $\mathcal{T}$  between  $u_Y$  and  $u_{V(G)}$ ). Suppose there is a red edge  $e$  between  $X$  and  $Y$  in  $G$ . If  $T_i + e$  is a forest, then the decomposition obtained from  $D$  by replacing  $T_i$  with  $T_i + e$  and  $F$  by  $F - e$  contradicts our choice of  $D$ . Thus  $T_i + e$  contains a cycle,  $H$ . Since  $Y$  and  $Z$  are the vertex-sets of distinct components of  $T_i[X]$ , it follows that  $H$  is not contained in  $T_i[X]$ . Thus  $\mathcal{T}$  contains an ancestor  $u_{X'}$  of  $u_X$  such that there is an edge  $e'$  of  $H$  between two children of  $u_{X'}$ . Now redefine  $c$  to be so that  $c = \text{dist}(u_{X'}, u_{V(G)})$ . Note that since  $u_{X'}$  is an ancestor of  $u_X$ , the value of  $c$  has decreased. Starting from  $D$ , we replace  $F$  by  $F - e$ , replace  $T_i$  by  $T_i + e - e'$ , and replace  $T_{t(u_{X'})}$  by  $T_{t(u_{X'})} + e'$  to obtain a decomposition  $D'$ . Note that  $T_{t(u_{X'})} + e'$  is not a forest, as otherwise

$D'$  contradicts our choice of  $D$ . Thus  $T_{t(u_{X'})} + e'$  contains a cycle  $H'$ , which as above implies that there is an ancestor  $u_{X''}$  of  $u_{X'}$  such that there is an edge  $e''$  of  $H'$  between two children of  $u_{X''}$ . Again, redefine  $c$  such that  $c = \text{dist}(u_{X''}, u_{V(G)})$ . As before, since  $u_{X''}$  is an ancestor of  $u_{X'}$ , the value of  $c$  has decreased. Again, we remove  $e''$  from  $T_{t(u_{X'})+e'}$  and add it instead to  $T_{t(u_{X''})}$ , necessarily creating a cycle  $H''$ . We then find that there is an ancestor of  $u_{X''}$  that has two children with an edge of  $H''$  between them. The process does not terminate: otherwise, as the decomposition obtained at the end of the procedure includes  $F - e$ , it contradicts our choice of  $D$ . But this is a contradiction, since the value of the counter  $c$  is strictly decreasing throughout the process and  $c$  is lower-bounded by 0.  $\square$

**Claim 3.3.** *Let  $u_X$  and  $u_Y$  be leaves of  $\mathcal{T}$ . Then there is no edge  $e = xy \in E(G)$  such that  $x \in X$  and  $y \in Y$ .*

*Proof.* Suppose not. Let  $u$  be the closest common ancestor of  $u_X$  and  $u_Y$ . Note that by Claim 3.2,  $u$  is not the parent of both  $u_X$  and  $u_Y$ ; without loss of generality, suppose that  $u$  is not a parent of  $u_X$ . Let  $u'_X$  be a child of  $u$  that is also an ancestor of  $x$ . If  $u$  is the parent of  $y$ , set  $u'_Y = y$ . Otherwise, let  $u'_Y$  be the child of  $u$  that is also an ancestor of  $u_Y$ . Since  $xy$  is an edge, by definition of  $\mathcal{T}$  there is an edge between the associated components of  $u'_Y$  and  $u'_X$ . As  $u'_Y$  and  $u'_X$  are children of  $u$ , this contradicts Claim 3.2.  $\square$

Let  $L$  be the set of leaves of  $\mathcal{T}$ . By definition of  $\mathcal{T}$ ,  $L$  corresponds to a partition of  $V(G)$ . Let  $G' = \cup_{v \in L} G[v]$ . Since  $G$  is a minimum counterexample to Theorem 1.17 and  $G'$  is disconnected, it follows that  $G'$  has a decomposition  $(H_1, \dots, H_k, H)$  such that for each  $i = 1, 2, \dots, k$ ,  $H_i$  is a forest and  $H$  is a forest of maximum degree at most  $d$  such that each component of  $H$  has at most one vertex of degree  $d$ , at most  $w$  edges which are incident to two vertices of degree at least  $d - 1$ .

In each forest  $T_i$  with  $i = 1, 2, \dots, k$ , we replace  $E(T_i) \cap E(G')$  with  $E(H_i)$  to obtain a new forest  $T'_i$ . Similarly, we replace  $E(F) \cap E(G')$  with  $E(H)$  to obtain a new forest  $F'$ . Note that by definition the leaves of  $\mathcal{T}$  are connected subgraphs of  $T_i$  for all  $i \in \{1, 2, \dots, k\}$ : this implies that  $T'_i$  is indeed a forest for all  $i \in \{1, 2, \dots, k\}$ . Since there are no red edges between leaves of  $\mathcal{T}$ , it follows that  $F'$  is a forest maximum degree at most  $d$  where each component of  $F'$  has at most one vertex of degree  $d$ , at most  $w$  edges which are incident to two vertices of degree at least  $d - 1$ . This contradicts the fact that  $G$  is a counterexample to Theorem 1.17.  $\square$

We will need to make a few definitions. It is convenient to identify the components which cause problems for us.

**Definition 3.4.** A component  $C \in F$  is *bad* if  $C$  contains at least two vertices of degree  $d$  or if  $d \geq 4$ , and  $C$  contains more than  $w$  edges whose endpoints both have degree at least  $d - 1$ .

We start by noting that bad components have to have a significant number of edges.

**Observation 3.5.** Every bad component contains at least  $2d - 1$  edges.

*Proof.* Let  $K$  be a bad component. If  $K$  contains more than  $w$  edges whose endpoints both have degree at least  $d - 1$ , then  $K$  has at least  $w + 1$  vertices of degree  $d - 1$ . Thus  $e(K) \geq \frac{(w+1)(d-1)}{2}$  by the handshaking lemma. As  $w = d + 4$ , we have

$$\frac{(w+1)(d-1)}{2} \geq 2d - 1$$

if and only if  $d^2 \geq 3$ , which holds when  $d \geq 4$ . In the other case,  $K$  contains two vertices of degree  $d$ , and as  $K$  is a simple graph,  $K$  contains at least  $2d - 1$  edges.  $\square$

In the case when  $d = 4$  and  $k = 1$ , we will need a more general statement that applies to more than just bad components.

**Observation 3.6.** If  $Q$  is a forest, and there are at least 6 edges between vertices of degree at least 3, then  $e(Q) \geq 15$ .

*Proof.* Let  $Q$  be a vertex minimal counterexample. If a leaf  $\ell$  of  $Q$  is adjacent to a vertex of degree at most 2, then consider  $Q - \ell$ . By minimality,  $e(Q - \ell) \geq 15$ , and hence  $e(Q) \geq 16$  as desired. If  $Q$  contains a vertex of degree two, say  $x$  then let  $xy \in E(Q)$ , and let  $Q'$  be the forest obtained by contracting  $xy$ . Then by minimality,  $e(Q') \geq 15$ , and thus  $e(Q) \geq 16$  as desired. Therefore  $Q$  only has vertices of degree at least 3 and degree one. As there are 6 edges between vertices of degree at least 3, it follows that there are at least 7 vertices of degree three as  $Q$  is a forest. Let  $Q_3$  be the subtree in  $Q$  induced by the vertices of degree at least 3, and let  $Q_1$  be the graph induced by the leaves in  $Q$ . Then by the handshaking lemma, we have

$$2e(Q) \geq 3v(Q_3) + v(Q_1).$$

As  $e(Q) = v(Q) - 1 = v(Q_3) + v(Q_1) - 1$  we have

$$v(Q_1) \geq v(Q_3) + 2.$$

Therefore as  $v(Q_3) \geq 7$  we have at least 9 leaves, and thus  $e(Q) \geq 15$ , as desired.  $\square$

We define a *residue function* to measure how close our decomposition is to satisfying Theorem 1.17.

**Definition 3.7.** Let  $(T_1, T_2, \dots, T_k, F)$  be a decomposition of  $G$ . Let  $\mathcal{L}$  be the set of bad components of  $F$ . The *residue function*  $\rho$  is defined as

$$\rho(F) = \sum_{C \in \mathcal{L}} (e_r(C) - d).$$

As in the pseudoforest section, given a red-blue colouring of  $G$  and a subgraph  $K$  of  $G$ , we let  $e_b(K)$  denote the number of blue edges in  $K$ , and  $e_r(K)$  denote the number of red edges in  $K$ . Additionally, for a vertex  $v \in V(G)$ , we let  $d_r(v)$  be the *red degree of  $v$*  (that is, the number of red edges incident to  $v$ ), and similarly we denote by  $d_b(v)$  the *blue degree of  $v$*  (the number of blue edges incident to  $v$ ). We will orient the graph, and in the oriented graph, the degree refers to the degree in the underlying graph, and similarly the number of blue (red) edges in the directed graph refers to the number of such edges in the underlying graph.

Now we define an exploration graph for forest decompositions.

**Definition 3.8.** Let  $(T_1, \dots, T_k, F) \in \mathcal{F}$ . Let  $R$  be a bad component of  $F$ . Let  $s$  be any vertex of largest degree in  $R$ . For all  $i \in \{1, \dots, k\}$  orient  $T_i$  such that  $s$  is the only vertex with outdegree zero. We recursively define a set of components  $\mathcal{Q}$ . Initially  $\mathcal{Q}$  is  $\{R\}$ . While there is a component  $C$  of  $F$  such that there is a blue directed edge  $(x, y)$  where  $x$  is in a component in  $\mathcal{Q}$  and  $y \in V(C)$ , we add  $C$  to  $\mathcal{Q}$ . Once this procedure has terminated, let  $H_{R,F,s}$  be the digraph induced by the components of  $\mathcal{Q}$ . We call  $H_{R,F,s}$  an *exploration subgraph*, and  $R$  a *root component*.

The next observation allows us to forget about the blue edges and simply try to give a bound on the red edges.

**Observation 3.9.** Given a decomposition  $(T_1, T_2, \dots, T_k, F) \in \mathcal{F}$ , a root component  $R$ , and a vertex  $s \in V(R)$  of largest degree in  $V(R)$ , the graph  $H_{R,F,s}$  satisfies

$$\frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} \leq \frac{d}{d + k + 1}.$$

*Proof.* Suppose not, then

$$\begin{aligned} \Gamma_f(G) &\geq \frac{e(H_{R,F,s})}{v(H_{R,F,s}) - 1} \\ &= \frac{e_b(H_{R,F,s})}{v(H_{R,F,s}) - 1} + \frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} \\ &\geq k + \frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} \\ &> k + \frac{d}{d + k + 1}. \end{aligned}$$

Here

$$\frac{e_b(H_{R,F,s})}{v(H_{R,F,s}) - 1} \geq k,$$

since  $H_{R,F,s}$  is an induced subgraph, and by the definition of  $H_{R,F,s}$ , every vertex other than  $s$  has an outgoing blue edge for each of the  $k$  spanning trees  $T_1, \dots, T_k$ , and  $s$  has zero

outgoing blue edges. The strict inequality follows as we assumed that

$$\frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} > \frac{d}{d + k + 1}.$$

However, by assumption  $G$  satisfies

$$\Gamma_f(G) \leq k + \frac{d}{d + k + 1},$$

a contradiction. □

In general we will want to modify our decomposition to reduce the residue function. However, we might not always be able to reduce the residue function immediately. We will put a natural ordering on the red components of  $H_{R,F,s}$ , and will use this ordering to give a measure of progress towards improving the residue function.

**Definition 3.10.** Given the graph  $H_{R,F,s}$ , an ordering of the red components  $(R_1, \dots, R_t)$  is a *legal order* if all red components in  $H_{R,F,s}$  are in the ordering,  $R_1 = R$ , and for any component  $R_i$  with  $i > 1$ , there exists an  $R_j$  such that  $j < i$  with a directed blue arc  $(x, y)$  such that  $x \in V(R_j)$  and  $y \in V(R_i)$ .

The next definition is for convenience.

**Definition 3.11.** Let  $(R_1, \dots, R_t)$  be a legal order. For a component  $R_i$ , a component  $R_j$  is a *parent* of  $R_i$  if  $j < i$  and there is a blue directed arc  $(x, y)$  where  $x \in V(R_j)$  and  $y \in V(R_i)$ . If  $R_j$  is a parent of  $R_i$  then we say  $R_i$  is a *child* of  $R_j$ .

Note that in the above definition, a component may have numerous parents. As with pseudoforests, we will want to compare two different legal orders, and to do this we will use a lexicographic ordering.

**Definition 3.12.** Given two legal orders  $(R_1, \dots, R_t)$  and  $(R'_1, \dots, R'_{t'})$ , we say  $(R_1, \dots, R_t)$  is *smaller than*  $(R'_1, \dots, R'_{t'})$  if  $(v(R_1), \dots, v(R_t))$  is lexicographically smaller than  $(v(R'_1), \dots, v(R'_{t'}))$ .

Overall, our goal is to use Observation 3.9 to derive a contradiction. As such, we want to bound the number of red edges in exploration graphs. To this end, the following definition is useful:

**Definition 3.13.** A subgraph  $K$  is *small* if  $e_r(K) < \frac{d}{k+1}$ .

As we assumed  $d \leq 2(k+1)$ , we get a simple characterization of small components.

**Observation 3.14.** If  $d \leq 2(k+1)$ , and  $K$  is a small red component, then  $e_r(K) \leq 1$ .

*Proof.* As  $K$  is small, we have  $e_r(K) < \frac{d}{k+1}$ . As  $d \leq 2(k+1)$ , we have  $e_r(K) < 2$ , which implies  $e_r(K) \leq 1$ .  $\square$

We now describe how we pick our counterexample. A counterexample is a tuple  $(G, D, H_{R,F,s}, L)$  where  $G$  is a graph with fractional arboricity at most  $k + \frac{d}{k+d+1}$ ;  $D = (T_1, \dots, T_k, F)$  is a decomposition of  $G$  into  $k+1$  forests with  $D \in \mathcal{F}$ ;  $H_{R,F,s}$  is an exploration subgraph; and where  $L$  is a legal order of the exploration subgraph. A minimal counterexample is a tuple  $(G, D, H_{R,F,s}, L)$ , where  $v(G)$  is minimized; subject to this,  $D = (T_1, \dots, T_k, F)$  is chosen to minimize the residue function; subject to that, the number of vertices of degree  $d$  in bad components is minimized; subject to that, if  $d \geq 4$ , the number of vertices of degree  $d-1$  in bad components is minimized; subject to this a bad component  $R$  of  $F$  and a vertex  $s \in V(R)$  of highest degree in  $R$  are chosen such that there exists a legal order  $L$  of  $H_{R,F,s}$  which is smaller than all other legal orders possible.

For the rest of the chapter, we will assume we are working with a minimal counterexample as described above.

### 3.3 Reducible Configurations

In this section we build up series of configurations which do not occur in a minimal counterexample  $(G, D, H_{R,F,s}, L)$ . Throughout we assume that  $L = (R_1, \dots, R_t)$ .

First we argue that we have at least two components in our legal order.

**Observation 3.15.** If  $(G, H_{R,F,s}, D, L)$  is a minimal counterexample and  $L = (R_1, \dots, R_t)$ , then  $t \geq 2$ .

*Proof.* Suppose not. Then by definition of the exploration subgraph  $H_{R,F,s}$ , there is no blue arc  $(x, y) \in E(G)$  with  $x \in V(R_1)$  and  $y \notin V(R_1)$ . Since  $T_1, \dots, T_k$  are spanning subgraphs of  $G$ , it follows that  $T_i[R_1]$  is a tree for each  $i = 1, \dots, k$ . Thus  $G[V(R_1)]$  contains  $k(v(R_1) - 1)$  blue edges. Since  $R_1$  is a connected component of  $F$ , it follows further that  $e(R_1) = k(v(R_1) - 1) + (v(R_1) - 1)$ . Therefore

$$\begin{aligned} \Gamma_f(G) &\geq \frac{e(R_1)}{v(R_1) - 1} \\ &= k + 1. \end{aligned}$$

This is a contradiction, since  $\Gamma_f(G) \leq k + \frac{d}{d+k+1}$  by assumption.  $\square$

We say a blue edge  $xy$  is *saturated* if  $\rho(F + xy) > \rho(F)$  or  $F + xy$  has a vertex of degree  $d+1$ . Otherwise, we say  $xy$  is *unsaturated*. Our first point of order is to show that if  $K$  is a parent of a component  $C$ , then any blue directed edge from  $K$  to  $C$  is saturated. This turns out to be rather technical, and we need quite a few definitions.

Let  $R_i$  be a component such that  $i > 1$ . For each parent  $R_j$  of  $R_i$ , let  $S_j^i$  be the set of vertices  $x \in V(R_i)$  such that there exists a blue arc  $(y, x)$  where  $y \in V(R_j)$ . Let  $\mathcal{P}$  denote the set of parent components of  $R_i$ . Then we define:

$$S^i = \bigcup_{R_j \in \mathcal{P}} S_j^i.$$

For each  $i \in \{2, \dots, t\}$ , pick an arbitrary vertex  $x_i \in S^i$ . For this choice of vertices, we will say the *auxiliary digraph for  $L$* , denoted  $\text{Aux}(L)$ , has vertex set  $V(H_{R,F,s})$  and the edge set is obtained by including all red edges in  $H_{R,F,s}$ , and for each  $i \in \{2, \dots, t\}$ , we include exactly one arc  $(y, x_i)$  where  $y$  lies in a parent component of  $R_i$ . Then we direct all of the edges in  $\text{Aux}(L)$  towards  $s$ . This digraph is only needed as a tiebreaking mechanism when doing reconfiguration arguments.

Observe that  $\text{Aux}(L)$  is a tree, and there are many possible trees that could be generated from a single legal order. For every legal order, we will associate the legal order to some auxiliary digraph, chosen arbitrarily.

We let the *index* of a vertex  $x \in V(R_i)$ , denoted  $\text{ind}(x)$ , be  $i$ . Let  $K$  be a red component of  $H_{R,F,s}$ , and  $C$  be a child of  $K$ .

A directed walk  $P = v_0, \dots, v_t$  in  $H_{R,F,s}$  where all edges in  $P$  are blue is *special with respect to  $L$  and  $(x, y)$*  if the last directed edge  $(v_{t-1}, v_t)$  is  $(x, y)$ , where  $(x, y)$  is unsaturated;  $\text{ind}(v_t) > \text{ind}(v_0)$ ; and all vertices except  $y$  appear at most once in  $P$ .

Let  $P = v_0, \dots, v_t$  and  $P' = u_0, \dots, u_{t'}$  be two special walks with respect to  $L$  and  $(x, y)$ . Note that by definition,  $v_0 \neq v_t$ . We say  $P$  is smaller than  $P'$ , denoted  $P < P'$ , if  $\text{ind}(v_0) < \text{ind}(u_0)$  or  $\text{ind}(v_0) = \text{ind}(u_0)$  and  $v_0$  is an ancestor of  $u_0$  in  $\text{Aux}(L)$ . A special path with respect to  $L$  and  $(x, y)$  is minimal if there are no smaller special walks with respect to  $L$  and  $(x, y)$ .

The next lemma is extremely important, and we note that it follows ideas taken from [24].

**Lemma 3.16.** *Let  $K$  be a parent of a component  $C$  in  $H_{R,F,s}$ , where  $(x, y)$  is a blue directed arc such that  $x \in V(K)$  and  $y \in V(C)$ . Then  $(x, y)$  is saturated.*

*Proof.* Suppose not. Let  $P = v_0, \dots, v_t$  be a minimal special walk with respect to  $L$  and  $(x, y)$ . Observe that this special walk exists, because the path  $P' = x, y$  is itself a special walk with respect to  $L$  and  $(x, y)$  (note that  $(x, y)$  is unsaturated by assumption).

Let  $v_{-1}$  be the ancestor of  $v_0$  in  $\text{Aux}(L)$ . Observe that  $v_{-1}$  exists. If not,  $v_0 = s$ , but  $P$  is a directed blue walk, and  $s$  has no outgoing blue edges by construction. Thus  $v_{-1}$  exists. Now we claim that  $v_{-1}v_0$  is a red edge. If not, the path  $P' = v_{-1}, P$  is a smaller special walk, a contradiction.

Now write  $P = P_1, \dots, P_m$  where for  $i \in \{1, \dots, m\}$  we have  $P_i = v_{a_{i-1}}, \dots, v_{a_i}$ ; where all edges of  $P_i$  are in some  $T_{b_i}$ ; where  $a_0 = 0$ ; and where  $a_m = t$ . We call each  $P_i$  a



segment. We may assume that we picked our minimal special walk such that the number of segments is minimized.

Let  $\mu(P) = (a_0, \dots, a_m)$ . Subject to picking a minimal special walk with minimal number of segments, we may assume that  $\mu(P)$  is lexicographically minimal. Note that this choice of  $P$  implies  $d_P(y) \leq 3$ .

**Claim 3.17.** *For a segment  $P_i = v_{a_{i-1}}, \dots, v_{a_{i-1}}, v_{a_i}$  with  $a_i \neq t$ , we have that  $v_{a_{i-1}}$  is the first vertex in  $P$  that has a directed path in  $T_{b_i}$  to  $v_{a_i}$ .*

*Proof.* If not, let  $v_j$  be a vertex such that  $j < a_{i-1}$  and there is a directed path in  $T_{b_i}$  to  $v_{a_i}$ . Then replace the segment  $P_i$  with the path from  $v_j$  to  $v_{a_i}$  (removing all segments in between). Then either we reduce the number of segments, or we reduce  $\mu(P)$ , either way contradicting our choice of  $P$ .  $\square$

Note that this implies the following.

**Observation 3.18.** The cycle in  $T_{b_i} + v_{a_{i-1}-1}v_{a_{i-1}}$  contains the edge  $v_{a_i}v_{a_{i-1}}$ .

Otherwise, there is a directed path in  $T_{b_i}$  from  $v_{a_{i-1}-1}$  to  $v_{a_i}$ , a contradiction.

Now make the edge  $v_{a_m-1}v_{a_m}$  red, and for all  $i \in \{1, \dots, t-1\}$  add the edge  $v_{a_{i-1}}v_{a_i}$  to the tree  $T_{b_{i+1}}$ .

Let  $T'_1, \dots, T'_k, F'$  be the resulting decomposition. We claim this decomposition is in  $\mathcal{F}$ . The only edge that becomes red is  $xy$  and since  $(x, y)$  is unsaturated by assumption, it follows that  $F'$  is a forest with maximum degree  $d$ . It follows from Claim 3.17 and Observation 3.18 that for each  $i \in \{1, 2, \dots, k\}$ ,  $T'_i$  is a tree. Since the total number of edges in each tree remains unchanged, we have that each  $T'_i$  is spanning. Hence  $T'_1, \dots, T'_k, F' \in \mathcal{F}$ .

Moreover, since  $(x, y)$  is unsaturated by assumption, the residue function of  $(T'_1, \dots, T'_k, F')$  does not increase, as the only new red component is  $K + C + xy$ . If the residue function decreased, then we get a contradiction. Thus we can assume that the red components created by deleting  $v_{-1}v_0$  did not decrease the number of edges in a bad component, or split a bad component into two bad components. Observe that this implies that  $v_{-1}$  is not in  $R$ . Then in this new decomposition, taking the same root  $R$  and vertex  $s$ , we create an exploration subgraph from  $s$ . Then there is a legal order  $L'$  where  $L$  and  $L'$  are identical for all components before the  $\text{ind}(v_{-1})$  component, and then we can take  $R'_{\text{ind}(v_{-1})}$  to be one of the two components of  $R_{\text{ind}(v_{-1})} - v_{-1}v_0$ . This component is of course strictly smaller than  $R_{\text{ind}(v_{-1})}$ . (Note if  $K = R_{\text{ind}(v_{-1})}$  we can still take the component of  $K + C - v_0v_{-1} + xy$  that does not contain any of  $C$ , as otherwise there is a smaller special walk, a contradiction.) Hence there is a decomposition with a smaller legal order, a contradiction.  $\square$

**Corollary 3.19.** *Let  $K$  be a red component and  $C$  a child of  $K$ . Then  $e_r(K) + e_r(C) \geq d$ .*

*Proof.* By Lemma 3.16,  $(x, y)$  is saturated, and so by definition  $K + C + xy$  is a bad component or has a vertex of degree  $d + 1$ . If  $K + C + xy$  is bad, then by Observation 3.5,  $e_r(K + C + xy) > d$ . Otherwise  $K + C + xy$  has a vertex of degree  $d + 1$ , and hence trivially  $e_r(K + C + xy) > d$ . The result follows.  $\square$

**Definition 3.20.** Let  $T \in \{T_1, \dots, T_k\}$ . Let  $xy \in E(T)$  and  $uv \in E(F)$ . To *exchange  $xy$  and  $uv$*  is to construct two new forests  $T'$  and  $F'$  such that  $T' = T + uv - xy$  and  $F' = T' + xy - uv$ . Given an exchange of edges, the *resulting decomposition* is the decomposition obtained from  $(T_1, \dots, T_k, F)$  by replacing  $T$  with  $T'$  and  $F$  with  $F'$ .

**Definition 3.21.** Let  $K$  be a red component, and  $C$  a child of  $K$ . We will say a directed arc  $(x, y)$  *generates*  $C$  if  $x \in V(K)$  and  $y \in V(C)$ . We will say that  $(x, y)$  *generates  $C$  by  $T$*  if  $(x, y)$  is in  $T$ .

Note for a particular child  $C$ , many arcs may generate  $C$ . The next two lemmas are essential. They allow for the exchange of edges in certain situations.

The following notation will be frequently used. Given two vertices  $x, y$  in a red component  $K$ , let  $P_{x,y}$  denote the path from  $x$  to  $y$ .

**Lemma 3.22.** *Let  $R$  be the root component of  $H_{R,F,s}$ , and let  $x \in V(R)$ . If  $(x, x')$  generates a small child  $C$  by  $T$ , then there exists an edge in  $P_{x,s}$  that can be exchanged with  $xx'$ .*

*Proof.* Suppose not. Let  $P_{x,s} = x_1x_2 \dots x_n$ , where  $x_1 = x$  and  $x_n = s$ . If  $n = 2$ , then since all blue trees are oriented towards  $s$ , we can exchange  $xx'$  with  $xs$  and reduce the residue function, a contradiction. Thus we assume  $n \geq 3$ . We claim that for all  $i \in \{2, \dots, n-1\}$ , the vertex  $x_i$  does not lie in the same component of  $T - x$  as  $s$ . If not, choose the smallest  $i$  for which the statement fails: then  $xx'$  lies in the fundamental cycle of  $T + x_ix_{i-1}$ , from which it follows that we can exchange  $xx'$  with  $x_{i-1}x_i$ . Thus it follows that  $x_{n-1}s$  lies in the fundamental cycle of  $T + xx'$  (again this follows as all blue trees are oriented towards  $s$ ), and hence we can exchange  $x_{n-1}s$  with  $xx'$ , a contradiction.  $\square$

**Corollary 3.23.** *Let  $R$  be the root component of  $H_{R,F,s}$ , and let  $x \in V(R)$ . Suppose  $(x, x')$  generates a small child  $C$  from tree  $T$ . If  $e_r(C) = 0$ , then  $d_r(x) = d$ .*

*Proof.* Suppose not. By Lemma 3.22, there is an edge  $e \in E(P_{x,s})$  that can be exchanged with  $(x, x')$ . Since  $d_r(x) < d$ , the resulting decomposition is in  $\mathcal{F}$ . Since  $e_r(C) = 0$  and  $d_r(x) \leq d_r(s)$ , all resulting components have fewer edges than the original root component, and hence the residue function decreases—a contradiction.  $\square$

In a similar vein, we have the following.

**Lemma 3.24.** *Let  $K$  be a component,  $C_1$  and  $C_2$  be children of  $K$  where  $(x, x')$  is a blue directed arc from  $K$  to  $C_1$ ,  $(y, y')$  is a blue directed arc from  $K$  to  $C_2$ , and both arcs are generated by  $T \in \{T_1, \dots, T_k\}$ . At least one of the following holds.*

1. There is an edge  $e \in E(P_{x,y})$  such that we can exchange  $xx'$  with  $e$ .
2. There is an edge  $e \in E(P_{x,y})$  such that we can exchange  $yy'$  with  $e$ .

*Proof.* Let  $P_{x,y} = x_1, x_2, x_3, \dots, x_n$  where  $x_1 = x$  and  $x_n = y$ . We may assume that  $s \notin V(P_{x,y})$ , as otherwise the lemma holds by Lemma 3.22. Suppose towards a contradiction that neither of the two outcomes occur.

First, we claim that for all  $i \in \{2, \dots, n\}$ , that  $x_i$  does not lie in the same component of  $T - x$  as  $s$ . To see this, suppose not: let  $i \in \{2, \dots, n\}$  be the smallest  $i$  such that  $x_i$  and  $s$  lie in the same component of  $T - x$ . Then as  $i$  is picked to be minimal,  $xx'$  lies in the fundamental cycle of  $T + x_ix_{i-1}$ . But this is a contradiction, as then we can exchange  $xx'$  with  $x_{i-1}x_i$ .

Since  $x_n = y$ , this implies that  $s$  and  $y$  lie in distinct components of  $T - x$ . By symmetry,  $s$  and  $x$  lie in distinct components of  $T - y$ . However, both of these cannot happen simultaneously —a contradiction.  $\square$

**Observation 3.25.** If  $C$  is a small component, then  $C$  contains no vertex of degree  $d - 1$ . Further, unless  $d = 3$  and  $k = 1$ ,  $C$  contains no vertex of degree  $d - 2$ .

*Proof.* Observe that  $d \geq 3$ , and  $e_r(C) \leq 1$ . If  $d = 3$ , then as  $d \geq \frac{3(k+1)}{2}$ , we have  $k = 1$ . The observation now follows.  $\square$

**Observation 3.26.** If  $K$  is not a bad component, and  $(x, y)$  generates a small child  $C$ , then  $d_r(x) \geq d - 2$ .

*Proof.* Suppose  $(x, y)$  generates a small child and  $d(x) \leq d - 3$ . If  $d = 3$ , then  $k = 1$  and we have  $d(x) = 0$ , which implies that  $V(K) = \{x\}$ , and  $K$  itself is a small child. But then as  $C$  is small, we have  $e_r(K) + e_r(C) \leq 2$ , contradicting Corollary 3.19. Therefore we can assume that  $d \geq 4$ . Now we claim that  $(x, y)$  is unsaturated. If  $(x, y)$  is saturated, then  $K + C + xy$  is a bad component (as  $d_r(x) \leq d - 3$  and  $d_r(y) \leq d - 2$ ). As  $C$  is small,  $K$  is not a bad component, and  $x$  has red degree at most  $d - 3$ , we do not create any new vertices with red degree  $d - 1$  or higher. Thus the only way for  $K + C + xy$  to be a bad component is if  $K + C + xy$  has more than  $w$  edges between vertices of red degree at least  $d - 1$ . By Observation 3.25,  $C$  contains no vertex of red degree  $d - 2$ , and hence after adding the edge  $xy$ , has no vertex of red degree larger than  $d - 1$ , and thus contributes no edges whose endpoints both have red degree at least  $d - 1$ . As  $d_r(x) \leq d - 3$ , adding  $xy$  leaves the red degree of  $x$  being at most  $d - 2$ , and hence we do not create any new edges whose endpoints both have red degree at least  $d - 1$ . As  $K$  is not bad, with the above discussion we have  $K + C + xy$  has at most  $w$  edges whose endpoints have degree at least  $d - 1$ , and thus  $K + C + xy$  is not a bad component. This implies that  $(x, y)$  is unsaturated, contradicting Lemma 3.16.  $\square$

**Lemma 3.27.** *If  $K$  is a non-bad red component with two arcs  $(x, x')$  and  $(y, y')$  which generate small children  $C_1$  and  $C_2$ , respectively, by  $T$ , and both  $x$  and  $y$  have red degree at least  $d - 2$ , then  $xy \notin E(K)$ .*

*Proof.* Suppose not. Thus  $xy \in E(K)$ . By Lemma 3.16, both of  $(x, x')$  and  $(y, y')$  are saturated. Note in the case where  $d = 3$  and  $k = 1$ , this implies that neither  $x$  nor  $y$  has degree one, as otherwise as  $K$  is not bad, either  $(x, x')$  or  $(y, y')$  is not saturated. By Lemma 3.24 either we can exchange  $xx'$  with  $xy$  or  $yy'$  with  $xy$ . Without loss of generality, suppose we can exchange  $xx'$  with  $xy$ . Exchange  $xx'$  with  $xy$ , and without loss of generality let  $(T_1, \dots, T'_k, F')$  be the resulting decomposition. Note that the red degree of  $x$  remains the same, and the red degree of  $y$  decreases. As  $C_1$  is small, so long as  $(d, k) \neq (3, 1)$ , the red degree of  $x'$  is strictly less than  $d - 1$ , and hence we did not increase the residue function. If we decreased the residue function, we get a contradiction. As  $d_r(y) \geq d - 2$ , and  $C_1$  is small, both of the resulting new components in  $F'$  have strictly fewer edges than  $K$  (unless  $(d, k) \in \{(3, 1), (4, 1)\}$ ). Then we can find a legal order that is the same up until the new components, which are strictly smaller, and take the smaller component at this point, contradicting our choice of legal order.

The argument breaks down when  $(d, k) \in \{(3, 1), (4, 1)\}$ . However in this case we can take the same legal order up until the new component, and then take the component containing  $x$  first, and then the component containing  $y$  next in the order, and complete the rest of the order arbitrarily (if we can take the component containing  $y$  first, then we get a smaller legal order, a contradiction). Let  $K^x$  denote the red component containing  $x$ . Note that  $v(K^x) \leq v(K)$ , and as we do not get a smaller legal order,  $v(K^x) = v(K)$ . In this case, note that  $(x, y, y')$  is a special path. Hence we can apply the argument from Lemma 3.16, and obtain a smaller legal order, a contradiction.

□

**Lemma 3.28.** *Let  $K$  be a bad component. Let  $(x, x')$  and  $(y, y')$  generate small children  $C_1$  and  $C_2$ , respectively, by  $T$ . Suppose that we can exchange  $e \in E(P_{x,y})$  with  $xx'$ . Let  $K'$  be the component of  $K - e$  containing  $y$ . Then either  $d_r(x) = d$  or  $e_r(K') \leq e_r(C_1)$ .*

*Proof.* Suppose not, so  $e_r(K') > e_r(C_1)$  and  $d_r(x) < d$ . Exchange  $xx'$  with  $e$ . Note we do not create a vertex of degree  $d + 1$ , as  $d_r(x) < d$ . If  $e_r(K') > e_r(C_1)$ , then since  $e_r(C_1) < d$ , the resulting decomposition has a lower residue function—a contradiction. □

**Corollary 3.29.** *Let  $K$  be a bad component and suppose that  $(d, k) \neq (3, 1)$ . If  $(x, x')$  and  $(y, y')$  generate small children  $C_1$  and  $C_2$  by a tree  $T$ , then the red degree of at least one of  $x$  or  $y$  is not  $d - 1$ .*

*Proof.* If so, Lemma 3.24 and Lemma 3.28 give rise to a contradiction as small children have at most one edge. □

**Lemma 3.30.** *Let  $K$  be a bad component with two arcs  $(x, x')$  and  $(y, y')$  which generate small children  $C_1$  and  $C_2$  respectively by  $T$ . Suppose  $xy \in E(K)$  and we can exchange  $xx'$  with  $xy$ . Let  $K^y$  be the component of  $K - xy$  containing  $y$ . Then  $e_r(K^y) \leq e_r(C_1) - 1$ .*

*Proof.* Suppose not. Exchange  $xx'$  with  $xy$ . If  $e_r(K^y) > e_r(C_1)$ , or if  $K^y$  is itself a bad component, we immediately reduce the residue function, a contradiction. Therefore  $e_r(K^y) = e_r(C_1)$ . In this case observe that we do not increase the residue function (but also do not decrease it). Further observe in this new decomposition, if we can take the same legal order up until  $K$  but then take  $K^y$  instead of  $K$ , we have a smaller legal order—a contradiction. Thus we may assume that we can take the same legal order up until  $K$ , but then take the new component containing  $x$ , and after that  $K^y$ . This legal order is possibly worse than before. But now note that since  $e_r(K^y) + e_r(C_2) < d$ , it follows from Observation 3.5 that  $(y, y')$  is unsaturated. Thus  $x, y, y'$  is a special path, and apply the special path argument from Lemma 3.16 to get a new decomposition with a strictly smaller legal order than  $L$ , a contradiction.  $\square$

**Lemma 3.31.** *Let  $K$  be a bad component. Suppose that  $v_1, \dots, v_t$  are vertices of  $K$  such that for each  $i \in \{1, \dots, t\}$ ,  $(v_i, v'_i)$  generates a small child  $C_i$ . Further, suppose that  $\max_i e(C_i) \leq j$ , all  $v_i$  have red degree at least  $j + 1$  and at most  $d - 1$ . Then  $t \leq 2$ . Moreover, if  $t = 2$ , then at least one of  $v_1$  and  $v_2$  has red degree  $j + 1$ .*

*Proof.* We first assume that, without loss of generality, both  $v_1$  and  $v_2$  have red degree at least  $j + 2$ . Up to relabelling  $v_1$  as  $v_2$ , by Lemma 3.24 there is an edge  $e \in E(P_{v_1, v_2})$  that can be exchanged with  $v_1 v'_1$ . By Lemma 3.28, since  $d_r(v_1) < d$  by assumption it follows that  $e(K^{v_2}) \leq e(C_1)$ , where  $K^{v_2}$  is the component of  $K - e$  containing  $v_2$ . But this is a contradiction, since  $d(v_2) \geq j + 2$  implies that  $e(K^{v_2}) \geq j + 1$ , and  $e(C_1) \leq j$ .

Now suppose that  $t \geq 3$ . By Lemma 3.24, we may assume without loss of generality that we can exchange an edge  $e_1$  on  $P_{v_1, v_2}$  with the edge  $v_1 v'_1$  and we can assume we can exchange an edge  $e_2$  on  $P_{v_2, v_3}$  with the edge  $v_2 v'_2$  (up to relabelling the vertex names).

**Claim 3.32.** *For both  $i \in \{1, 2\}$ ,  $e_i$  is the edge incident to  $v_{i+1}$  in  $P_{v_i, v_{i+1}}$ , and  $e_i$  is the only edge in  $E(P_{v_i, v_{i+1}})$  which can be exchanged with  $v_i v'_i$ . Moreover,  $e(P_{v_i, v_{i+1}}) \geq 2$ .*

*Proof.* If we can exchange any edge other than the edge incident to  $v_{i+1}$  in  $P_{v_i, v_{i+1}}$ , then do so with  $v_i v'_i$  and the residue function decreases, a contradiction. If  $e(P_{v_i, v_{i+1}}) = 1$ , then we contradict Lemma 3.30.  $\square$

Now exchange  $v_1 v'_1$  with  $e_1$  and then exchange  $e_2$  with  $v_2 v'_2$ . Let  $D'$  be the resulting decomposition, and let  $K'$  be the new component containing  $v_1$ . Note that  $e(K') < e(K)$ . Moreover, the component of  $K - e_2$  containing  $v_3$  has at most  $d - 2$  edges, since  $v_3$  has degree at most  $d - 1$ . The component of  $K - e_1 + v_2 v'_2$  containing  $v_2$  similarly has at most  $d - 1$  edges. It follows from Lemma 3.5 that neither of these components are bad: and since  $e(K') < e(K)$ , it follows further that  $F' = F - e_1 - e_2 + v_1 v'_1 + v_2 v'_2$  has strictly

smaller residue function than  $F$ . This is a contradiction unless  $D'$  is *not* a decomposition where  $F'$  is a forest of maximum degree  $d$  and  $T' = T + e_1 + e_2 - v_1v'_1 - v_2v'_2$  is a spanning tree. But that  $F'$  has maximum degree  $d$  follows from the fact that  $d_r(v_1) \leq d - 1$ . Below, we show that  $T'$  is a tree, thus completing the proof.

**Claim 3.33.**  $T'$  is a tree.

*Proof.* Since  $e(T') = e(T)$  and  $T$  is a spanning tree, it suffices to show that  $T'$  is connected (since we assume that  $V(G) = V(T')$ ). We begin by showing that there are no isolated vertices in  $T'$ . Suppose not. Since only  $v_1v'_1$  and  $v_2v'_2$  were the only edges deleted from  $T$  when constructing  $T'$ , we have only to show that there is an edge of  $T'$  incident to each vertex in  $\{v_1, v_2, v'_1, v'_2\}$ . Note that since  $T$  is directed towards  $s$  and  $(v_1, v'_1)$  and  $(v_2, v'_2)$  are arcs in  $T$ , and neither  $v'_1$  nor  $v'_2$  are  $s$ , it follows that  $T'$  spans both  $v'_1$  and  $v'_2$ . Therefore to prove that there are no isolated vertices in  $T'$ , it suffices to show that neither  $v_1$  nor  $v_2$  is a leaf of  $T$ .

Let  $P_{v_1, v_2} = x_1, \dots, x_n$ , where  $v_1 = x_1$ ,  $v_2 = x_n$ , and  $x_{n-1}x_n = e_1$ . Let  $P_{v_2, v_3} = y_1, \dots, y_m$ , where  $y_1 = v_2$  and  $y_m = v_3$ . Note that by Claim 3.32, both  $m$  and  $n$  are at least three. Moreover, no edge in  $P_{v_1, v_2}$  except  $e_1$  can be exchanged with  $(v_1, v'_1)$ . In particular,  $x_1x_2$  cannot be exchanged with  $(v_1, v'_1)$ : that is,  $(v_1, v'_1)$  is not in the fundamental cycle of  $T + x_1x_2$ . It follows that  $v_1$  is not a leaf in  $T$ . Similarly,  $v_2$  is not a leaf in  $T$ . Thus  $T' = T - v_1v'_1 - v_2v'_2 + e_1 + e_2$  spans  $V(G)$ .

To show  $T'$  is a tree, it remains only to show that  $T'$  is connected. Suppose not. Since  $v(T) = v(T')$  and  $e(T) = e(T')$ , it follows that  $T$  contains a cycle  $C$ . Moreover, since both  $T + e_1 - v_1v'_1$  and  $T + e_2 - v_2v'_2$  are trees by definition of exchange, it follows that  $C$  contains both  $e_1$  and  $e_2$ .

Since  $T + e_2 - v_2v'_2$  is a tree,  $T + e_1 + e_2 - v_2v'_2$  is connected. As  $T' = T + e_2 + e_1 - v_2v'_2 - v_1v'_1$  is disconnected, it follows that  $T'$  has exactly two components  $H_1, H_2$ , and that  $v_1v'_1$  is the only edge in  $T + e_1 + e_2 - v_2v'_2$  with exactly one endpoint in  $V(H_1)$  and one endpoint in  $V(H_2)$ . Thus  $v_1v'_1$  is not contained in a cycle in  $T + e_1 + e_2 - v_2v'_2$ . Since  $v_1v'_1$  is contained in the fundamental cycle  $C'$  of  $T + e_1$ , it follows that  $v_2v'_2$  and  $v_1v'_1$  are *both* contained in the fundamental cycle of  $T + e_1$ . But then  $v_2v'_2$  can be exchanged with  $e_1$  in  $T$ , contradicting Claim 3.32. □

This completes the proof. □

**Lemma 3.34.** *Suppose  $K$  is a bad component with arcs  $(x, x')$  and  $(y, y')$  that generate small children  $C_1$  and  $C_2$  respectively by  $T$ . Further suppose that  $e(C_i) = 0$ , and both  $x$  and  $y$  have degree at most  $d - 2$ , then there exists an edge in  $P_{x, y}$  which is not incident to a vertex of degree at least  $d$ .*

*Proof.* Suppose not. Then without loss of generality there is an edge  $e \in E(P_{x,y})$  such that we can exchange  $xx'$  with  $e$  by Lemma 3.24. Do so. By assumption,  $e$  is incident to a vertex of degree  $d$ , and hence we either reduce the residue function, or the residue function stays the same and we reduce the number of degree  $d$  vertices in bad components. In either case, a contradiction occurs.  $\square$

**Definition 3.35.** Let  $K$  be a bad red component where  $(x, x')$ ,  $(y, y')$  generate small children  $C_1$  and  $C_2$ . Let  $E \subseteq E(P_{x,y})$  be the set of edges that can be exchanged either with  $xx'$  or  $yy'$ . For each  $e \in E$ , let  $D_e$  be the resulting decomposition after exchanging  $e$  with the corresponding edge  $f \in \{xx', yy'\}$ , and  $F_e = F - e + f$ . We say  $P_{x,y}$  is an *exchangeable path* if for each  $e \in E$ , we have that  $D_e \in \mathcal{F}$  and one of the following hold:

- $\rho(F_e) < \rho(F)$ , or
- $\rho(F_e) = \rho(F)$  and the number of vertices of degree  $d$  in  $F_e$  is less than that in  $F$ , or
- $\rho(F_e) = \rho(F)$ ,  $d \geq 4$ , the number of vertices of degree  $d$  in  $F_e$  is the same as that in  $F$ , and the number of vertices of degree  $d - 1$  in  $F_e$  is less than that in  $F$ .

**Observation 3.36.** Let  $K$  be a red component where  $(x, x')$ ,  $(y, y')$  generate small children and both  $d_r(x) < d$  and  $d_r(y) < d$ . Then  $P_{x,y}$  is not an exchangeable path.

*Proof.* If not, Lemma 3.24 ensures that there is an edge  $e$  in  $P_{x,y}$  that we can exchange with either  $(x, x')$  or  $(y, y')$ . In either case we contradict our choice of decomposition.  $\square$

We highlight some important cases of Observation 3.36

**Definition 3.37.** We will say a bad component  $K$  is *d-fragile* if  $K$  has at most  $w$  edges between vertices of degree at least  $d - 1$ , and exactly two vertices of degree  $d$ .

**Lemma 3.38.** *Let  $K$  be a bad component which is d-fragile. Let  $u$  and  $v$  be the two vertices of degree  $d$ , and suppose  $uv \in E(K)$ . Then there does not exist an  $x, y \in V(K) \setminus \{u, v\}$  such that  $x$  and  $y$  are neighbours of either  $u$  or  $v$  the degree of both  $x$  and  $y$  is at most  $d - 2$ , and  $(x, x')$  and  $(y, y')$  generate small children  $C_1, C_2$ , respectively from a tree  $T$ .*

*Proof.* Suppose so. Observe that  $P_{x,y}$  is an exchangeable path, contradicting Observation 3.36.  $\square$

**Lemma 3.39.** *The following cannot occur. Let  $K$  be a bad component. Let  $x, y, z$  be three vertices in  $K$  such that  $x, y, z$  is a path and  $y$  has degree  $d$ , or  $d - 1$  if  $d \geq 4$ , and  $x$  and  $z$  both are leaves of  $K$ . Suppose that  $(x, x')$ ,  $(y, y')$  and  $(z, z')$  generate small children  $C_1, C_2$ , and  $C_3$  from a tree  $T$ .*



*Proof.* Suppose not. Since we cannot reduce the residue function, and by Lemma 3.24, we have that  $yy'$  can be exchanged with  $xy$  and  $yy'$  can be exchanged with  $yz$ . Further, without loss of generality by Lemma 3.24 we can assume that  $xx'$  can be exchanged with  $yz$  (otherwise, we reduce the residue function). Now exchange  $xx'$  with  $yz$ , and  $yy'$  with  $xy$ . If we reduce the residue function, then we get a contradiction. Thus the residue function does not decrease. Note we do not increase the residue function. Further, we do reduce the number of vertices of degree  $d$  (or  $d - 1$  if  $d \geq 4$ ) in bad components, a contradiction.  $\square$

**Definition 3.40.** A bad component  $K$  is *w-fragile* if  $K$  has at most one vertex of degree  $d$ .

**Lemma 3.41.** *Let  $K$  be w-fragile. Let  $u, v$  be two vertices of degree at least  $d - 1$ . Let  $u', v'$  be neighbours of  $u, v$  respectively such that  $u', v'$  are leaves of  $K$ . Then at least one of  $u'$  or  $v'$  does not generate a small child from a tree  $T$ .*

*Proof.* Suppose not. By Lemma 3.24, there is an edge on  $P_{u',v'}$  which is exchangeable and from the assumptions, the path is an exchangeable path, a contradiction.  $\square$

**Lemma 3.42.** *Let  $K$  be a bad component. Let  $u, v$  be vertices of degree at least three. Let  $\ell$  be a neighbour of  $u$  which is a leaf of  $K$ . Let  $y$  be a neighbour of  $v$  and  $y'$  a neighbour of  $y$  such that the degree of  $y$  is two and the  $y'$  is a leaf. Then at least one of  $y, y'$  and  $\ell$  does not generate a small child from tree  $T$ .*

*Proof.* Suppose not. Let  $(\ell, \ell'), (y, x), (y', x')$  be blue arcs which generate small children. By Lemma 3.24, there is an edge in  $P_{y,\ell}$  which we can exchange with either  $(y, x)$  or  $(\ell, \ell')$ . Note if we can exchange  $(\ell, \ell')$  with any edge in  $P_{y,\ell}$ , then we reduce either the number of vertices of degree at least 3, or we reduce the residue function. In either case, we get a contradiction. Thus we can exchange  $(y, x)$  with an edge in  $P_{y,\ell}$ , and there are only two edges in  $P_{y,\ell}$  which does not reduce the residue function or reduce the number of vertices of degree three. So either we can exchange  $(y, x)$  with  $u\ell$  or the other edge incident to  $u$  in  $P_{y,\ell}$ .

Note by Lemma 3.24, we can exchange  $(y, x)$  with  $(y, y')$  (otherwise we can reduce the residue function, or find a smaller legal order).

Now suppose we can exchange  $(y', x')$  with an edge in  $P_{\ell,y'}$ . The only edge we can exchange  $(y', x')$  with without improving the decomposition is  $u\ell$ . Then exchange  $(y', x')$  with  $u\ell$ , and  $(y, x)$  with  $(y, y')$ . Then the number of edges in the new component is the same as originally, however we have reduced the number of vertices of degree at least three, a contradiction.

Therefore we can exchange  $(\ell, \ell')$  with an edge in  $P_{\ell,y'}$ , which implies that we can exchange  $(\ell, \ell')$  with  $yy'$ . Then exchange  $(\ell, \ell')$  with  $yy'$  and  $(y, x)$  with the edge in  $P_{y,\ell}$ . Then we either reduce the residue function, or reduce the number of vertices of degree at least three, a contradiction in either case.  $\square$



The following corollary summarizes the main results of this section.

**Corollary 3.43.** *Let  $K$  be a red component. Suppose  $(x, y)$  generates a small child  $C$  by the tree  $T$ . If  $K$  is not bad, then  $d_r(x) \geq d - 2$ . If  $x$  has a neighbour of red degree at least  $d - 2$ , then this neighbour does not generate a small child by the tree  $T$ . If  $x$  has red degree  $d - 2$ , then either  $K$  is a bad component, or  $K$  contains at least  $w - d + 2$  edges whose endpoints have degree at least  $d - 1$ . If  $x$  has red degree  $d - 1$  then either  $K$  is a bad component, or there is a vertex in  $K$  with red degree  $d$ .*

*Proof.* Let  $K$ ,  $(x, y)$ , and  $C$  be as in the statement. If  $K$  is not bad, then  $d_r(x) \geq d - 2$ . If  $x$  has a neighbour of degree at least  $d - 2$ , then by Lemma 3.27 this neighbour does not generate a small child. If  $x$  has red degree  $d - 2$ , then since  $(x, y)$  is saturated by Lemma 3.16 either  $K$  is a bad component or contains at least  $w - d + 2$  edges whose endpoints have degree at least  $d - 1$ . If  $d_r(x) = d - 1$  then again since  $(x, y)$  is saturated either  $K$  is a bad component or there is a vertex in  $K$  with red degree  $d$ .  $\square$

With this, we have all of the structural components we need. We will finish the argument with a lengthy counting argument to show that given all of the structure we have derived, the exploration graph has too large density.

### 3.4 The counting argument for non-bad components

We start with some notation.

**Definition 3.44.** Let  $K$  be a red component, and  $C_1, \dots, C_q$  be the small children of  $K$ . Then we let  $K_C$  be the subgraph with  $V(K_C) = V(K) \cup V(C_1) \cup \dots \cup V(C_q)$  and  $E(K_C)$  is the set of all red edges on  $V(K_C)$ .

We will first bound the average degree of  $K_C$  when  $K$  is not a bad component. For the purposes of induction, we make the following definition. In the definition, we use  $\mathcal{P}(X)$  to refer to the powerset of  $X$ . Note that this definition is simulating the properties of  $K_C$  when  $K$  is not bad.

**Definition 3.45.** Let  $P$  be a forest with components  $K, C_1, \dots, C_t$ . Suppose that  $K$  contains at least  $\frac{d}{k+1}$  edges, and that no other component in  $P$  contains at least  $\frac{d}{k+1}$  edges. Let  $\mathcal{C} = \{C_1, \dots, C_t\}$ . Let  $A : V(K) \rightarrow \mathcal{P}(\mathcal{C} \times \{1, \dots, k\})$  be a function satisfying the following properties.

1. For each component  $C' \in \mathcal{C}$ , there exists exactly one vertex  $v \in V(K)$  such that there is an  $i \in \{1, \dots, k\}$  so that  $(C', i) \in A(v)$ .
2. For every vertex  $v \in V(K)$ ,  $|A(v)| \leq k$ .

3. If  $uv \in E(K)$ , then there does not exist an  $i \in \{1, \dots, k\}$  and two distinct components  $C, C' \in \mathcal{C}$  such that  $(C, i) \in A(u)$  and  $(C', i) \in A(v)$ .
4. For  $v \in V(K)$ , if  $d(v) \leq d - 3$ , then  $A(v) = \emptyset$ .
5. For  $v \in V(K)$ , if  $d(v) = d - 2$  and  $A(v) \neq \emptyset$ , then  $d \neq 3$  and  $K$  contains at least  $w - d + 2$  edges where both endpoints have degree at least  $d - 1$ .
6. For  $v \in V(K)$ , if  $d(v) = d - 1$ , and  $A(v) \neq \emptyset$ , then there exists a vertex in  $K$  with  $d(v) = d$ .
7. For every vertex  $v \in V(K)$ , if  $(C, i) \in A(v)$ , then there does not exist a  $C' \neq C$  such that  $(C', i) \in A(v)$ .

We say such a pair  $(P, A)$  is an *admissible forest*.

As notation, if we have an admissible forest  $(P, A)$ , we will always let  $K$  be the component of  $P$  with more than one edge.

**Observation 3.46.** Let  $K$  be a red component that is not small and not bad. Let  $K_C$  have components  $K, C_1, \dots, C_r$  and let  $\mathcal{C} = \{C_1, \dots, C_r\}$ . For each component  $C_i$ , pick exactly one arc  $(u, v)$  such that  $u \in V(K)$  and  $v \in V(C_i)$ . Let  $A : V(K) \rightarrow \mathcal{P}(\mathcal{C} \times \{1, \dots, k\})$  be the function such that  $(C_i, j) \in A(v)$  for  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, k\}$  if  $uv \in E(T_j)$ . Then  $(K_C, A)$  is an admissible forest.

*Proof.* The conditions follow from Corollary 3.43 and the fact that each vertex has at most one outgoing arc from each  $T \in \{T_1, \dots, T_k\}$ .  $\square$

While Observation 3.46 only applies when  $K$  is not bad, if  $K$  is bad and  $(K_C, A)$  is an admissible forest, then any lemma dealing with  $(K_C, A)$  applies. We restrict to  $d \in \{3, 4\}$  and  $k = 1$  to finish this section, however we note that the majority of the upcoming lemma works in general (with more work).

**Lemma 3.47.** *Suppose that  $d \in \{3, 4\}$  and  $k = 1$ . Then for any admissible forest  $(P, A)$ , we have*

$$\frac{e(P)}{v(P)} \geq \frac{d}{d+2}.$$

*Proof.* Suppose not. Let  $(P, A)$  be an admissible forest where  $K$  is the only component with at least 2 edges. Of all of the admissible forests that are counterexamples to Lemma 3.47, pick  $(P, A)$  such that  $v(P)$  is minimized.

Observe there exists a component of  $P$  other than  $K$ . If not, then

$$\frac{e(P)}{v(P)} \geq \frac{\frac{d}{2}}{\frac{d}{2} + 1} \geq \frac{d}{d+2},$$

a contradiction. Now let  $\mathcal{C} = \{C_1, \dots, C_q\}$  be the set of components of  $P$  that are not  $K$ . We make some structural claims.

**Claim 3.48.** For all  $i \in \{1, \dots, q\}$ ,  $e(C_i) = 0$ .

*Proof.* Suppose there is a component  $C_i$  such that  $e(C_i) \geq 1$ . Let  $v$  be a leaf of  $C_i$ . Consider  $P - v$ . Let  $A'$  be the function where  $A'$  agrees with  $A$  aside from on  $x \in V(K)$  where  $(C_i, 1) \in A(x)$ , and in this case we let  $A'(x) = A(x) - (C_i, 1) \cup (C_i - v, 1)$ . Then  $(P - v, A')$ , is an admissible forest, and hence by minimality we have

$$\frac{e(P)}{v(P)} = \frac{e(P - v) + 1}{v(P - v) + 1} \geq \frac{e(P - v)}{v(P - v)} \geq \frac{d}{d + 2}.$$

a contradiction. □

We now build towards proving  $q \geq 3$ . Note that  $q \geq 1$  since there exists a component of  $P$  other than  $K$ .

**Claim 3.49.** We have  $q \geq 2$ .

*Proof.* Suppose  $q = 1$ . Let  $v \in V(K)$  be the vertex such that  $A(v) \neq \emptyset$ . By Properties (4) and (5) of admissible forests, there are three possibilities: either  $d(v) = d$ ,  $d(v) = d - 1$ , or  $d(v) = d - 2$ ,  $d = 4$ , and  $K$  contains at least 6 edges whose endpoints both have degree at least  $d - 1$ . If  $d(v) = d$ , then trivially  $e(K) \geq d$ . Similarly, if  $d(v) = d - 1$ , then by Property (6) of admissible forests there exists a vertex  $u$  of degree  $d$  in  $K$  and so again  $e(K) \geq d$ . Recall that by Claim 3.48,  $e(C_1) = 0$ . Thus since  $q = 1$  by assumption, in either case we have

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{e(K)}{e(K) + 2} \\ &\geq \frac{d}{d + 2} \end{aligned}$$

as desired. Therefore we may assume that  $v$  has degree  $d - 2$ , that  $d = 4$ , and that there are at least 6 edges with both endpoints having degree at least 3. Thus by Observation 3.6  $e(K) \geq 15$ . Then we have

$$\frac{e(P)}{v(P)} \geq \frac{15}{17} > \frac{2}{3},$$

as desired. □

**Claim 3.50.** We have  $q \geq 3$ .

*Proof.* Suppose not. We may assume that  $q = 2$  by Claim 3.49. Let  $v_1, v_2$  be the two vertices in  $K$  where  $A(v_i) \neq \emptyset$  for  $i = 1, 2$ . We split into cases depending on the degrees of  $v_1$  and  $v_2$ . Note that by Property (4) of admissible forests,  $d(v_i) \geq d - 2$  for  $i = 1, 2$ . If

both vertices have degree  $d - 1$ , then by Property (6) of admissible forests there exists a vertex in  $K$  with degree  $d$ . It follows that  $e(K) \geq 3d - 4$  and we have

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{3d - 4}{3d - 1} \\ &\geq \frac{d}{d + 2}. \end{aligned}$$

Now suppose we have (without loss of generality)  $d(v_1) = d$  and  $d(v_2) \in \{d - 1, d\}$ . Note by Property (3),  $v_1$  and  $v_2$  are not adjacent. It follows that we have  $e(K) \geq 2d - 1$  and thus

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{2d - 1}{2d + 2} \\ &\geq \frac{d}{d + 2}. \end{aligned}$$

Now suppose (without loss of generality) that  $d(v_1) = d - 2$ . Then by Property (5),  $d = 4$ , and  $K$  contains at least 6 edges between vertices of degree at least 3. Hence by Observation 3.6,  $e(P) \geq 15$ . We have

$$\frac{e(P)}{v(P)} \geq \frac{15}{18} > \frac{2}{3}.$$

□

Now assume that  $q \geq 3$ . Let  $q'$  be the number of vertices  $v$  of degree  $d - 2$  such that  $A(v) \neq \emptyset$  and let  $q'' = q - q'$ .

**Claim 3.51.** *We have  $q' \geq 1$ .*

*Proof.* Suppose not. Then there are no vertices  $v$  where  $A(v) \neq \emptyset$  and  $d(v) = d - 2$ , so we have

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{e(K) + \sum_{i=1}^q e(C_i)}{e(K) + 1 + \sum_{i=1}^q (e(C_i) + 1)} \\ &\geq \frac{e(K)}{e(K) + 1 + q} \quad \text{by Claim 3.48} \\ &\geq \frac{(d - 1)q}{dq + 1} \\ &\geq \frac{3(d - 1)}{3d + 1} \\ &\geq \frac{d}{d + 2}, \end{aligned}$$

as desired. □

Since  $q' \geq 1$ , by Property (5) we have that  $d = 4$  and that  $K$  has at least 6 edges where both endpoints have degree at least  $d - 1$ . Additionally,  $K$  contains at least  $q'$  vertices of degree at least  $(d - 2)$  and at least  $q'' = q - q'$  vertices of degree at least  $d - 1$ , and these  $q$  vertices form an independent set by Property (3). Thus  $e(K) \geq \min\{q'(d - 2) + q''(d - 1), q'(d - 2) + 6\}$ . Moreover, by Claim 3.48 we have that  $e(C_i) = 0$  for each  $i \in \{1, \dots, q\}$ .

It follows that

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{q'(d - 2) + q''(d - 1)}{q'(d - 2) + q''(d - 1) + 1 + q} \\ &= \frac{q'(d - 2) + q''(d - 1)}{q'(d - 1) + q''d + 1} \\ &= \frac{2q' + 3q''}{3q' + 4q'' + 1}. \end{aligned}$$

**Claim 3.52.** *We have  $q'' \leq 1$ .*

*Proof.* Suppose not, so  $q'' \geq 2$ . We claim that

$$\frac{2q' + 3q''}{3q' + 4q'' + 1} \geq \frac{2q' + 6}{3q' + 9}.$$

Note that it suffices to show that

$$\frac{3q''}{4q'' + 1} \geq \frac{2}{3},$$

and this holds when  $q'' \geq 2$ , with strict inequality when  $q'' \geq 3$ .

By Claim 3.51 we have that  $\frac{2q'}{3q'} = \frac{2}{3}$ , and hence

$$\frac{e(P)}{v(P)} \geq \frac{2q' + 3q''}{3q' + 4q'' + 1} \geq \frac{2}{3},$$

as desired. □

Now we finish the proof. By Claim 3.51, we have that  $q' \geq 2$ . Moreover, by Property (5)  $K$  contains at least 6 edges between vertices of degree at least  $d - 1$ . Thus  $e(K) \geq 2q' + 6$ , so

$$\frac{e(P)}{v(P)} \geq \frac{2q' + 6}{3q' + 8} > \frac{2}{3},$$

as desired. □

### 3.5 The counting argument for bad components

As with the above section, we create a notion of bad admissible forests for the purposes of induction. Again the definition given is simply designed to simulate the properties of a bad component. We overload the terminology.

**Definition 3.53.** Let  $P$  be a forest with components  $K, C_1, \dots, C_t$ . Suppose that  $K$  is a bad component, and that no other component in  $P$  contains at least  $\frac{d}{k+1}$  edges. Let  $\mathcal{C} = \{C_1, \dots, C_t\}$ . Let  $A : V(K) \rightarrow \mathcal{P}(\mathcal{C} \times \{1, \dots, k\})$  be a function satisfying the following properties.

1. For each component  $C \in \mathcal{C}$ , there exists a vertex  $v \in V(K)$  such that there is an  $i \in \{1, \dots, k\}$  so that  $(C, i) \in A(v)$ .
2. For every vertex  $v \in V(K)$ ,  $|A(v)| \leq k$ .
3. Suppose  $uv \in E(K)$ , and there exists an  $i \in \{1, \dots, k\}$  with two distinct components  $C, C' \in \mathcal{C}$  such that  $(C, i) \in A(u)$  and  $(C', i) \in A(v)$ . Let  $K^u$  denote the component of  $K - uv$  containing  $u$ , and let  $K^v$  be defined analogously. Then at least one of the following holds:  $e(K^u) \leq e(C') - 1$ , or  $e(K^v) \leq e(C) - 1$ .
4. There are at most two distinct vertices  $u, v \in V(K)$  such that  $(C, i) \in A(u)$  and  $(C', i) \in A(v)$ ,  $d(u) \leq d - 1$ ,  $d(v) \leq d - 1$ , and  $e(C) = e(C') = 0$ .
5. There are at most two distinct vertices  $u, v \in V(K)$  such that  $(C, i) \in A(u)$  and  $(C', i) \in A(v)$ ,  $d(u) \leq d - 1$ ,  $d(v) \leq d - 1$ ,  $d(u) \geq 2$ , and  $d(v) \geq 2$ .
6. If  $u, v \in V(K)$ ,  $(C, i) \in A(u)$ ,  $(C', i) \in A(v)$ , where both  $d(u) \leq d - 2$  and  $d(v) \leq d - 2$  and  $e(C) = 0$ ,  $e(C') = 0$ , then either  $K$  has at least  $w - d + 2$  edges incident to vertices of degree at least  $d - 1$  and at least one of  $u$  or  $v$  has degree  $d - 2$ , or there is an edge in  $P_{u,v}$  that is not incident to a vertex of degree  $d$  (or  $d - 1$  if  $d = 4$ ).
7. For every vertex  $v \in V(K)$ , if  $(C, i) \in A(v)$ , then there does not exist a  $C' \neq C$  such that  $(C', i) \in A(v)$ .
8. If  $K$  is  $d$ -fragile, and  $u, v$  are the two vertices of degree  $d$ , and  $uv \in E(K)$ , then there does not exist two neighbours of  $u, v$  which are not  $u$  or  $v$ , say  $x, y$ , where  $d(x) \leq d - 2$  and  $d(y) \leq d - 2$  such that  $(C, i) \in A(x)$  and  $(C', i) \in A(y)$  for some  $C, C' \in \mathcal{C}$ .
9. There do not exist three vertices  $x, y, z$  such that  $x, y, z$  is a path in  $K$ ,  $y$  has degree  $d$  (or  $d - 1$  if  $d \geq 4$ ), both  $x$  and  $z$  are leaves in  $K$  and  $(C, i) \in A(x)$ ,  $(C', i) \in A(y)$ , and  $(C'', i) \in A(z)$ .
10. Let  $K$  be  $w$ -fragile. Let  $u, v$  be two vertices of degree at least  $d - 1$ . Let  $u', v'$  be neighbours of  $u, v$  respectively such that  $u', v'$  are leaves of  $K$ . Then if  $(C, i) \in A(u')$ , there is no component  $(C', i) \in A(v')$ .

11. Let  $u, v$  be two vertices of degree at least three. Let  $x$  be a neighbour of either  $u$  or  $v$ ,  $y$  a neighbour of  $v$  distinct from  $x$ , and  $\ell$  a neighbour of  $y$  where  $\ell$  is a leaf of  $K$ . Suppose that  $y$  has degree two. Then either  $A(x) = \emptyset$ ,  $A(y) = \emptyset$  or  $A(\ell) = \emptyset$ .
12. Let  $K$  be  $d$ -fragile. Let  $u, v$  be two vertices of degree at least three in  $K$ . Suppose  $u', v'$  are leaves adjacent to  $u$  and  $v$  respectively. Then either  $A(u') = \emptyset$  or  $A(v') = \emptyset$ .

We say such a pair  $(P, A)$  is an *bad admissible forest*. If additionally  $(P, A)$  satisfies the property that for any leaf  $\ell$  of  $K$ , if  $(C, i) \in A(\ell)$ , then  $e(C) = 1$ , we say  $K$  is a *root component*.

The following observation is immediate.

**Observation 3.54.** Let  $K$  be a bad component. Let  $K_C$  have components  $K, C_1, \dots, C_r$ , and let  $\mathcal{C} = \{C_1, \dots, C_r\}$ . For each component  $C_i$ , pick exactly one blue arc  $(u_i, v_i)$  such that  $u_i \in V(K)$  and  $v_i \in V(C_i)$ . Let  $A : V(K) \rightarrow \mathcal{P}(\mathcal{C} \times \{1, \dots, k\})$  be the function such that  $(C_i, j) \in A(v)$  for  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, k\}$  if  $(u_i, v_i) \in E(T_j)$ . Then  $(K_C, A)$  is a bad admissible forest.

We restrict to  $k = 1$  and  $d \in \{3, 4\}$  to finish the counting argument. Here we note that it is not obvious how to extend the next lemma to larger values of  $k$  and  $d$ .

**Lemma 3.55.** *Let  $(P, A)$  be a bad admissible forest, where  $k = 1$ , and  $d \in \{3, 4\}$ . Then*

$$\frac{e(P)}{v(P)} \geq \frac{d}{d+2}.$$

*Strict inequality holds if the bad component of  $P$  is a root component.*

*Proof.* Suppose not. Let  $(P, A)$  be a bad admissible forest where  $K$  is the bad component. Of all of the bad admissible forests that are counterexamples to Lemma 3.55, pick  $(P, A)$  such that the number of vertices in  $P$  is minimized. Subject to this, minimize the number of vertices  $v$  where  $A(v) \neq \emptyset$ . Let  $q$  be the number of vertices in  $K$  such that  $A(v) \neq \emptyset$ . Note that  $q \geq 1$ . We begin with some structural claims.

**Claim 3.56.** *There is no leaf  $\ell$  of  $K$ , such that  $(C, i) \in A(\ell)$ ,  $e(C) = 1$ , and  $(P - \ell, A|_{V(K-\ell)})$  is a bad admissible forest.*

*Proof.* Suppose so. Then  $(P - \ell, A|_{V(K-\ell)})$  is a bad admissible forest and by our choice we have

$$\frac{e(P - \ell)}{v(P - \ell)} \geq \frac{d}{d+2}.$$

Hence,

$$\frac{e(P)}{v(P)} = \frac{e(P - \ell) + 2}{v(P - \ell) + 3} \geq \frac{d}{d+2}.$$

To see the last inequality, it suffices to note that  $\frac{2}{3} \geq \frac{d}{d+2}$  when  $d \in \{3, 4\}$ . Note we need strict inequality if  $K$  was a root component. In this case, observe that there is no leaf  $\ell$  of  $K$  such that  $(C, 1) \in A(\ell)$  and  $e(C) = 0$ . Hence by Property (3), it follows that if  $u$  is a neighbour of  $\ell$  in  $K$  and  $A(u) = \{(C, 1)\}$  for some  $C \in \mathcal{C}$ , then  $e(C) = 1$ . Thus  $K - \ell$  is a root component and by minimality, we now have

$$\frac{e(P - \ell) + 2}{v(P - \ell) + 3} > \frac{d}{d + 2},$$

as desired.  $\square$

**Corollary 3.57.** *Every leaf  $\ell$  of  $K$  such that  $(C, 1) \in A(\ell)$  and  $e(C) = 1$  has a neighbour of degree at least 3. Further, if  $d = 4$  and  $k = 1$ , and  $\ell$  has a neighbour of degree three, then  $K$  is  $w$ -fragile.*

*Proof.* Suppose not. We claim we can apply Claim 3.56. Let  $\ell'$  be the neighbour of  $\ell$ , and suppose that the degree of  $\ell'$  is 2 (note that the degree of  $\ell'$  is not 1, as otherwise  $K$  is isomorphic to  $K_2$ .) Observe that if  $(C', 1) \in A(\ell')$ , then  $e(C') = 1$  as otherwise as otherwise Property (3) is violated. Observe in  $K - \ell$ ,  $\ell'$  is a leaf. Thus  $K - \ell$  either still contains at least two vertices of degree  $d$ , or more than  $w$  edges between vertices of degree  $d - 1$ . Thus  $K - \ell$  is a bad component. Therefore we simply have to check that all of the properties still hold for a bad admissible forest. Observe that if any of the properties were violated, then  $(P, A)$  also violated one of the properties (not necessarily the same one), and hence we get a contradiction.

So now suppose  $d = 4$  and  $k = 1$ , and  $\ell'$  has degree three. We want to show that  $K$  is  $w$ -fragile. If not, then  $K - \ell$  is still a bad component, and the same arguments as above imply we have a bad admissible forest, a contradiction.  $\square$

**Claim 3.58.** *If  $v \in V(K)$  has degree  $d$ , and  $uv \in E(K)$  such that  $d(u) = 1$ , and  $A(u) = \{(C', 1)\}$  where  $e(C') = 1$ . Then  $A(v) = \emptyset$ .*

*Proof.* Suppose not. By Property (3), we have that  $(C'', 1) \in A(v)$  and  $e(C'') = 1$ . Now define  $A'$  where  $A' = A$  on all vertices  $K - u - v$ , and  $A'(v) = (C''', 1)$  where  $C'''$  is a new component with  $e(C''') = 0$ , and  $A'(u) = \emptyset$ . Let  $(P', A')$  denote this new pair. If  $(P', A')$  is a bad admissible forest, then by our choice we have

$$\frac{e(P')}{v(P')} = \frac{e(P') + 2}{v(P') + 3} \geq \frac{d}{d + 2}$$

as desired (observe that strict inequality holds if  $K$  is a root component, since  $K$  remains a root component after the modification).

Hence  $(P', A')$  is not a bad admissible forest. If this happens, then there is a neighbour  $x \neq u$  such that  $A(x) \neq \emptyset$ . By Property (3),  $x$  is a leaf. But then we violate Property (9), a contradiction.  $\square$



**Corollary 3.59.** *When  $d = 3$  and  $k = 1$ , and  $q \geq 7$ , then  $K$  contains at most four distinct vertices  $v_i$  such that  $d(v_i) \neq 3$  and  $A(v_i) \neq \emptyset$ .*

*Proof.* Suppose not. By Property (4) and (5) there are at most four vertices  $v_1, \dots, v_t$  such that  $d(v_i) = 2$  and  $A(v_i) \neq \emptyset$  or  $d(v_i) = 1$   $A(v_i) \neq \emptyset$  and  $(C', 1) \in A(v_i)$  where  $e(C') = 0$ . As we assumed that the claim does not hold, then there exists a vertex  $v$  such that  $d(v) = 1$ , and  $(C', 1) \in A(v)$ , where  $e(C') = 1$ . By Corollary 3.57, we have that the existence of such a  $v$  implies that  $v$  has a neighbour  $u$  of degree 3. If we cannot simply apply Claim 3.56, then  $K$  has exactly two vertices of degree  $d$ , and hence  $K$  is  $d$ -fragile. Now without loss of generality we may suppose that  $v = v_5$ . Then by the same analysis, either  $v_6$  and  $v_7$  are adjacent to vertices of degree  $d$ , or have degree  $d$  themselves. If both of these vertices have degree  $d$ , then there are three vertices of degree  $d$ , by Claim 3.58, a contradiction. Therefore say  $v_6$  has degree one and  $(C'', 1) \in A(v_6)$  with  $e(C'') = 1$ . Then  $v_6$  is adjacent to a vertex of degree  $d$  say  $v'_6$ . We claim that  $v'_6 \neq u$ . If  $v'_6 = u$ , then as  $K$  is  $d$ -fragile, we contradict Property (8). By applying the same argument to  $v_7$  we see that  $K$  has three vertices of degree  $d$ , a contradiction.  $\square$

**Claim 3.60.** *If  $K$  is  $w$ -fragile, then  $q \leq 6$ .*

*Proof.* Suppose not. Let  $v_1, \dots, v_7$  be vertices such that  $A(v_i) \neq \emptyset$ . Then by Properties (4) and (5) there are two vertices say  $v_1$  and  $v_2$  such that  $d(v_1) = d(v_2) = 1$  and  $(C_i, 1) \in A(v_i)$  where  $e(C_i) = 1$ . By Corollary 3.57 both  $v_1$  and  $v_2$  are adjacent to vertices of at least degree three. But this contradicts Property (10).  $\square$

**Claim 3.61.** *When  $d = 4$  and  $k = 1$ , and  $q \geq 7$ , then  $K$  contains at most four vertices  $v_i$  such that  $d(v_i) \neq 4$  and  $A(v_i) \neq \emptyset$ .*

*Proof.* Suppose not. Let  $v_1, \dots, v_t$  be vertices such that  $A(v_i) \neq \emptyset$  and  $d(v_i) \leq d - 1$ . By Property (4) and (5) at most two of these vertices do not have degree one, and at most two of these vertices have degree one and have a component  $(C', 1)$  such that  $e(C') = 0$ . Thus as  $K$  is not  $w$ -fragile (as  $q \geq 7$ ), it follows that  $K$  contains at least three vertices of degree  $d$  by Corollary 3.57. But then as  $t \geq 5$ , there exists a leaf which satisfies Claim 3.56, a contradiction.  $\square$

Now we build towards showing that we can assume  $q \geq 7$ .

**Claim 3.62.**  $q \geq 3$ .

*Proof.* Suppose not. Then  $q \leq 2$ , and we have  $e(K) \geq 2d - 1$ . Then

$$\frac{e(P)}{v(P)} \geq \frac{2d - 1}{2d + 2} > \frac{d}{d + 2}$$

as desired.  $\square$

**Claim 3.63.**  $q \geq 4$ .

*Proof.* Suppose not. Then we can assume that  $q = 3$ . We claim that  $e(P) \geq 2d$ , and  $e(P) \geq 2d + 1$  unless  $K$  is a root component. If  $e(P) \geq 2d + 1$ , then we have

$$\frac{e(P)}{v(P)} \geq \frac{2d+1}{2d+5} > \frac{d}{d+2},$$

as desired. If  $e(P) \geq 2d$ , and  $K$  is not a root component, we have

$$\frac{e(P)}{v(P)} \geq \frac{2d}{2d+4} = \frac{d}{d+2},$$

as desired.

If  $K$  has more than 6 edges between endpoints of degree at least 3, then  $e(K) \geq 15 > 2d + 1$ , as desired. Hence  $K$  contains two vertices of degree  $d$  and is  $d$ -fragile. Thus  $e(K) \geq 2d - 1$ . First suppose that  $e(K) = 2d - 1$ . This implies that the two vertices of degree  $d$  are adjacent, say  $u, v$ . Then by Property (3), without loss of generality we may assume that  $A(u) = \emptyset$ . Then as  $q = 3$ , there are two leaves  $\ell_1$  and  $\ell_2$  such that  $A(\ell_i) \neq \emptyset$  for  $i \in \{1, 2\}$ . Then either  $e(P) \geq 2d + 1$ , in which case we are done, or  $K$  is not a root component, and we have  $e(P) \geq 2d$ , or we violate Property (8).

Therefore we may assume that  $e(K) = 2d$ . Thus we are done unless  $K$  is a root component. This implies that for any leaf  $\ell$  of  $K$ , where  $(C, 1) \in A(\ell)$ , we have  $e(C) = 1$ . Additionally, if  $e(P) = 2d$ , this implies that for any vertex  $x \in V(K)$ , if  $(C', 1) \in A(x)$ , then  $e(C') = 0$ . Then it follows that by Property (3), no two adjacent vertices  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ . As there are at most 3 vertices of degree at least 2 in  $K$ , and these vertices induce a path, this implies there is a leaf  $\ell$  such that  $A(\ell) \neq \emptyset$ , which implies either  $K$  is not a root component, or  $e(P) \geq 2d + 1$ , and so we are done.  $\square$

**Claim 3.64.**  $q \geq 5$

*Proof.* Suppose not. Then we can assume that  $q = 4$ . We claim that  $e(P) \geq 2d + 2$ , if  $K$  is not a root component, and otherwise we have  $e(P) \geq 2d + 3$ . Note that  $15 > 2d + 3$ , so  $K$  has less than 6 edges between vertices of degree three, and hence we can assume that  $K$  has two vertices, say  $u, v$ , of degree  $d$ . We split into cases depending on if  $uv \in E(K)$ .

**Case 1:**  $uv \in E(K)$

Then  $e(K) \geq 2d - 1$ . Note as  $q = 4$ , we cannot have  $e(K) = 2d - 1$  as otherwise we violate Property (8). Suppose  $e(K) = 2d$ . Note that  $K$  is  $d$ -fragile. As  $q = 4$ , two neighbours of  $u$  and  $v$ , say  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ . But this violates Property (8).

Thus  $e(K) \geq 2d + 1$ . Suppose  $e(K) = 2d + 1$ . There are four possibilities. First suppose that  $u$  is adjacent to a vertex  $u'$  with degree two, and similarly  $v$  is adjacent to a vertex  $v'$  with degree two. Note by Property (8), at most one vertex  $x$  adjacent to  $u$  or  $v$  has  $A(x) \neq \emptyset$ . Further, without loss of generality we can assume that  $A(u) = \emptyset$  by Property (3).

Then at least two leaves  $\ell_1$  and  $\ell_2$  have  $A(\ell_i) \neq \emptyset$ . If  $(C_i, 1) \in A(\ell)$ , and  $e(C_i) = 0$ , then  $K$  is not a root component. If neither  $\ell_i$  have  $e(C_i) = 0$ , then  $e(P) \geq 2d + 3$  as desired. Otherwise, if exactly one  $\ell_i$  has  $e(C_i) = 0$ , then,  $e(P) \geq 2d + 2$ , and  $K$  is not a root component, as desired. Therefore, we can assume that both  $\ell_i$  have  $e(C_i) = 0$ . Further, for every vertex  $x$  with  $(C, 1) \in A(x)$ , we have  $e(C) = 0$  as otherwise  $e(P) \geq 2d + 2$ . By Property (3), this implies if  $xy \in E(K)$ , either  $A(x) = \emptyset$  or  $A(y) = \emptyset$ . By Property (5), there are at most two leaves  $x, y$  such that  $(C, 1) \in A(x)$  and  $e(C) = 0$ , and  $(C', 1) \in A(y)$  where  $e(C') = 0$ . As  $q = 4$ , and  $e(K) = 2d + 1$ , it is not possible to satisfy all these constraints.

The same argument applies if  $v$  (or  $u$ ) is adjacent to two vertices of degree two.

Now suppose that  $v$  is adjacent to a vertex  $v'$  of degree two,  $v'$  is adjacent to a vertex  $v''$  of degree two. Let  $v'''$  be the leaf adjacent to  $v''$ . Then at most one neighbour of  $u$  or  $v$ , say  $x$  has  $A(x) \neq \emptyset$ . Then either we can apply the previous argument, or  $A(v''') \neq \emptyset$ , and  $A(v'') \neq \emptyset$ . In this case, note that if  $(C, 1) \in A(v'')$ , we have  $e(C) = 1$  by Property (3). Then  $e(P) \geq 2d + 2$ , and either  $K$  is not a root component, or  $e(P) \geq 2d + 3$ , as desired.

Now for the last case, which is that without loss of generality,  $u$  is adjacent to a vertex  $u'$  with degree three and  $d = 4$ . Let  $u''$  and  $u'''$  be the two leaves adjacent to  $u'$ . First suppose  $A(u'') = \emptyset$ . Then there are two neighbours of  $u$  and  $v$ , say  $x, y$  where  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$  which contradicts Property (8) unless  $x = u'$ . If we do not contradict Property (8) in some manner, then  $A(u''') \neq \emptyset$ , and in this case, the path from  $u'''$  to  $y$  contradicts Property (6), or we have  $e(P) \geq 2d + 3$ , or  $e(P) \geq 2d + 2$  and  $K$  is not a root component.

Thus  $A(u'') \neq \emptyset$  and by symmetry  $A(u''') \neq \emptyset$ . Still if a neighbour of  $u$  or  $v$  which is not  $u'$ , say  $x$  has  $A(x) \neq \emptyset$ , then the path from  $P_{u'',x}$  contradicts Property (6), or we have  $e(P) \geq 2d + 3$ , or  $e(P) \geq 2d + 2$  and  $K$  is not a root component. But then  $A(u') \neq \emptyset$ , and that contradicts Property (8).

Therefore  $e(K) = 2d + 2$ . In this case, if  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ , then  $xy \notin E(K)$  as otherwise  $e(P) \geq 2d + 3$  by Property (3). Further if  $e(P) = 2d + 2$ , this means for all  $(C', 1) \in A(x)$ ,  $e(C') = 0$ . By Property (4), this implies that both  $A(u) \neq \emptyset$  and  $A(v) \neq \emptyset$ . But this contradicts Property (3). Thus  $e(P) \geq 2d + 3$ , as desired.

**Case 2:**  $uv \notin E(K)$

In this case  $e(K) \geq 2d$ . First assume  $e(K) = 2d$ . Let  $u'$  be the vertex such that  $u, u', v$  is a path in  $K$ . Suppose that  $A(u') \neq \emptyset$ . Then by Property (3), both  $A(u) = \emptyset$  and  $A(v) = \emptyset$ . But then two leaf neighbours of  $u$  and  $v$ , say  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ , and we contradict Property (12). Thus  $A(u') = \emptyset$ . However in this case again two leaves that are neighbours of  $u$  and  $v$ , say  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ , and again we contradict Property (12).

Therefore  $e(K) \geq 2d + 1$ . We consider cases depending on how many vertices there are on the path from  $u$  to  $v$ .

If there are four internal vertices on the path  $P_{u,v}$  then  $e(K) \geq 2d + 3$ , as desired. Suppose  $u, u', u'', u''', v$  is the path from  $u$  to  $v$  in  $K$ . Then  $e(K) \geq 2d + 2$ . If equality

holds, then Property (4) implies that  $e(P) \geq 2d + 3$ , as desired.

Suppose  $u, u', u'', v$  is the path from  $u$  to  $v$  in  $K$ . Then  $e(K) \geq 2d + 1$ . If  $e(K) = 2d + 1$ , then two leaves adjacent to  $u$  or  $v$ , say  $x, y$  have  $A(x) \neq \emptyset$  or  $A(y) \neq \emptyset$ . Then we contradict Property (12). Therefore  $e(K) \geq 2d + 2$  in this case, and Property (4) and Property (5) imply that  $e(P) \geq 2d + 3$ .

Now suppose  $u, u', v$  is the path from  $u$  to  $v$  in  $K$ . If  $u'$  has degree four (implying  $d = 4$ ), then  $e(K) \geq 2d + 2$  and if equality holds, then Property (4) ensures that  $e(P) \geq 2d + 3$ . So suppose that  $u'$  has degree three. If  $e(K) = 2d + 1$ , then in this case, two leaves  $x, y$  have  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ , and thus we contradict Property (12). Then  $e(K) = 2d + 2$ . We are done if  $K$  is not a root component. So  $K$  is a root component, and thus for any leaf  $\ell$ , either  $A(\ell) = \emptyset$  or  $(C, 1) \in A(\ell)$  and  $e(C) = 1$ . So suppose  $e(P) = 2d + 2$  and  $K$  is a root component. Then all vertices  $x \in V(K)$  such that  $(C, 1) \in A(x)$  satisfies  $e(C) = 0$ . Then if  $xy \in E(K)$ , either  $A(x) \neq \emptyset$  or  $A(y) \neq \emptyset$ . As  $e(K) = 2d + 2$ , this implies that both  $A(u) \neq \emptyset$  and  $A(v) \neq \emptyset$ . As  $e(K) = 2d + 2$ , there is a leaf  $\ell$  such that  $A(\ell) \neq \emptyset$ , but then either  $K$  is not a root component, or  $e(P) \geq 2d + 3$ , as desired.

Thus  $u'$  has degree two. First suppose that  $e(K) = 2d + 1$ . Suppose that  $u$  is adjacent to a vertex  $u'' \neq u'$  where  $u''$  has degree two. Note there is at most one pair  $x, y$  such that  $xy \in E(K)$  and  $A(x) \neq \emptyset$  and  $A(y) \neq \emptyset$ , as otherwise  $e(P) \geq 2d + 3$ . Let us consider the case where  $A(u'') \neq \emptyset$ . Let  $(C, 1) \in A(u'')$ . If  $e(C) = 0$ , then the leaf, adjacent to  $u''$ , say  $u'''$  has  $A(u''') = \emptyset$  by Property (3), and as  $q = 4$ , this implies we have two leaves  $\ell$  and  $\ell'$  where both  $A(\ell)$  and  $A(\ell')$  are not empty, and this contradicts Property (12). Now suppose that  $e(C) = 1$ . Then if  $K$  is not a root component then we are done as  $e(P) \geq 2d + 2$ . If the argument from the above case does not apply, then  $A(u''') \neq \emptyset$ , and if  $K$  is not a root component then  $(C_{u'''}, 1) \in A(u''')$  and  $e(C_{u'''}) = 1$ . But then we contradict Corollary 3.57.

Thus we can assume that  $A(u'') = \emptyset$ . Therefore the case which has not been covered is if  $A(v) \neq \emptyset$ ,  $A(u) \neq \emptyset$ , a leaf of  $v$ , say  $v'$  has  $A(v') \neq \emptyset$ , and the leaf neighbour of  $u''$ , say  $u'''$  has  $A(u''') \neq \emptyset$ . Note then that  $(C', 1) \in A(v)$  has  $e(C') = 1$ , and thus we have  $e(P) \geq 2d + 2$ , and if equality holds, we have a leaf say  $u'''$  where  $(C'', 1) \in A(u''')$  and  $e(C'') = 0$ , hence  $K$  cannot be a root component.

Thus  $e(K) = 2d + 2$ , and either there is a leaf  $\ell$  with  $(C, 1) \in A(\ell)$  and  $e(C) = 0$ , and hence  $K$  is not a root component, or  $e(P) \geq 2d + 3$ , as desired.

Finally observe that if  $e(P) \geq 2d + 2$  we have

$$\frac{e(P)}{v(P)} \geq \frac{2d + 2}{2d + 7} \geq \frac{d}{d + 2}.$$

Note strict inequality holds if  $e(P) \geq 2d + 3$ . □

**Claim 3.65.**  $q \geq 6$

*Proof.* Suppose not. We may assume that  $q = 5$  in this case. Let  $v_1, \dots, v_5$  be the five vertices such that  $A(v_i) = \{(C_i, 1)\}$ . We claim that  $e(P) \geq 3d$ , and that strict inequality holds if  $K$  is a root component. Note if  $K$  has more than 6 edges between vertices of degree three, then by Observation 3.6,  $e(P) \geq 15 > 3d$ . Hence  $K$  contains at least two vertices of degree  $d$ , say  $u, v$ .

**Case 1:  $K$  contains 3 vertices of degree  $d$**

Let  $z$  be the other vertex of degree three. Note as we have three vertices of degree  $d$ , there is no leaf  $\ell \in \{v_1, \dots, v_5\}$  such that  $(C, 1) \in A(\ell)$  and  $e(C) = 1$ , as otherwise we can apply Claim 3.56.

**Subcase 1:  $u, v, z$  is a path in  $K$**

Then  $e(K) \geq 3d - 2$ . First suppose that equality holds. Then  $K$  is the graph with just  $u, v, z$  and leaves. Then by Property (3), there are three leaves  $\ell_1, \ell_2, \ell_3 \in \{v_1, \dots, v_5\}$ . Let  $(C_i, 1) \in A(\ell_i)$  for  $i \in \{1, 2, 3\}$ . By Property (6), at least two of  $e(C_i) = 1$ , and hence we obtain our desired bound.

So  $e(K) \geq 3d - 1$ . If equality holds, then one of  $u, v, z$  is adjacent to a vertex of degree two. First suppose that  $A(v) \neq \emptyset$ . Then  $A(u) = A(z) = \emptyset$ , by Property (3). Then by Property (4) and Property (5), there are two vertices in  $\{v_1, \dots, v_5\} - v$  such that  $e(C_i) = 1$ . Then  $e(P) \geq 3d + 1$ , as desired. Therefore we can suppose that  $A(v) = \emptyset$ . Note if either  $A(u) = \emptyset$  or  $A(z) = \emptyset$ , the same argument as above gives the desired bound. Thus we assume that  $u, z \in \{v_1, \dots, v_5\}$ . By the pigeon hole principle, Property (3), Property (9), and as  $q = 5$ , we can assume that  $v_1v_2 \in E(K)$  and  $v_3v_4 \in E(K)$ . Then there are two  $i \in \{1, \dots, 5\}$  such that  $e(C_i) = 1$ , and hence  $e(P) \geq 3d + 1$ , as desired.

Therefore  $e(K) \geq 3d$ . Suppose equality holds. Then  $K$  has at most four vertices of degree  $d$ . If  $K$  has four vertices of degree  $d$ , then these vertices induce a path, and by Property (3), this implies that three of the vertices in  $\{v_1, \dots, v_5\}$  have degree less than  $d$ , and Property (4) and Property (5) now imply that  $e(P) \geq 3d + 1$ . Similarly, if  $K$  has only three vertices of degree  $d$ , then Property (4) and Property (5) imply that  $e(P) \geq 3d + 1$ , as desired.

**Subcase 2:  $uv \in E(K)$  and  $z$  is not adjacent to either  $u$  or  $v$ , or none of  $u, v, z$  are adjacent**

Then we have  $e(K) \geq 3d - 1$ . Suppose equality holds. Then without loss of generality there is a vertex  $v'$  of degree two such that  $v, v', z$  is a path in  $K$ . If  $e(P) \leq 3d$ , at most one  $v_i$  has  $e(C_i) = 1$ . Further, we can assume there are exactly three vertices of degree  $d$ , as otherwise  $e(P) \geq 3d + 1$ . By Property (3), at most two of  $u, v, v', z$  are in  $\{v_1, \dots, v_5\}$ . It follows then by Property (4), there exists an  $i \in \{1, \dots, 5\}$  such that  $e(C_i) = 1$ . Thus  $e(P) \geq 3d$ . Further, as  $v'$  is the only vertex of degree two, it follows that either  $e(P) \geq 3d + 1$ , or there is a leaf  $\ell$  of  $K$  in  $\{v_1, \dots, v_5\}$  such that  $e(C_i) = 0$ , and hence  $K$  is not a root component.

Therefore  $e(K) \geq 3d$ . But now, if equality holds by Property (3), Property (4), and Property (5), at least one of the  $e(C_i) = 1$ , and we have  $e(P) \geq 3d + 1$ , as desired.

**Case 2:  $K$  contains at most 2 vertices of degree  $d$**

In this case  $K$  is  $d$ -fragile. First suppose that  $uv \in E(K)$ . Then by Property (8), at

most one neighbour of  $u$  or  $v$  is in  $v_1, \dots, v_5$ . Further without loss of generality  $A(u) = \emptyset$  by Property (3). It follows that  $e(K) \geq 2d + 2$ , since otherwise by Property (3), Property (8) and Property (9) we have that  $q \leq 4$ . Suppose equality holds. Then by Property (3), Property (4), it follows that at least two  $i \in \{1, 2, 3, 4, 5\}$  we have  $e(C_i) = 1$ . Then  $e(P) \geq 2d + 4 \geq 3d$ . Further if equality holds, then there is a vertex  $v_i$  with  $d(v_i) = 1$  and  $e(C_i) = 0$ , hence  $K$  is not a root component.

Now suppose that  $e(K) \geq 2d + 3$ . Again at least two  $i \in \{1, 2, 3, 4, 5\}$  have  $e(C_i) = 1$ , and  $e(P) \geq 2d + 5 \geq 3d + 1$ , as desired.

Therefore we can assume that  $uv \notin E(K)$ . We split into cases depending on how many vertices  $P_{u,v}$  has. First suppose that  $P_{u,v}$  has at most one internal vertex, say  $v'$ . By Property (3), at most two of  $u, v', v$  are in  $\{v_1, \dots, v_5\}$  and at most one neighbour of both  $u$  and  $v$  is in  $\{v_1, \dots, v_5\}$ . Then by Property (4) and Property (5), it follows that at least two  $i \in \{1, 2, 3, 4, 5\}$  have  $e(C_i) = 1$ . Now we claim that  $e(K) \geq 2d + 2$ . First observe that if  $e(K) = 2d$ , then as  $q = 5$ , we contradict Property (12). Thus  $e(K) = 2d + 1$ , and then by Property (3) and Property (11), we have three  $i \in \{1, 2, 3, 4, 5\}$  with  $e(C_i) = 1$ , and thus  $e(P) \geq 2d + 4 \geq 3d$ . If equality holds, observe that there is a leaf  $\ell$  in  $\{v_1, \dots, v_5\}$  such that  $e(C) = 0$ , and hence  $K$  is not a root component as desired. Therefore  $e(K) \geq 2d + 2$ . In this case, observe that there are at least two  $i \in \{1, 2, 3, 4, 5\}$  such that  $e(C_i) = 1$ , and thus  $e(P) \geq 2d + 4$ , and if equality holds there is a leaf  $\ell \in \{v_1, \dots, v_5\}$  which has  $(C, 1) \in A(\ell)$  and  $e(C) = 0$ , implying  $K$  is not a root component. Thus  $e(P) \geq 3d$  with strict inequality when  $K$  is root component.

Therefore  $P_{u,v}$  contains at least two internal vertices. Suppose there are exactly two internal vertices. Let  $u'$  and  $u''$  be the two internal vertices of  $P_{u,v}$ . Then  $e(K) \geq 2d + 1$ . Note if equality holds, then  $u'$  and  $u''$  both have degree two, and by Property (3), it follows that (without loss of generality)  $u$  is adjacent to two leaves, and both leaves are in  $v_1, \dots, v_5$ . But this contradicts Property (12).

Thus  $e(K) \geq 2d + 2$ . Now suppose  $e(K) = 2d + 2$ . If (without loss of generality)  $u$  is adjacent to a vertex  $u_1$  not in  $V(P_{u,v})$ , with degree two then as  $q = 5$ , Property (9), Property (12) and Property (3) force that there are at least two  $i \in \{1, 2, 3, 4, 5\}$  such that  $e(C_i) = 1$ , and thus  $e(P) \geq 2d + 4 \geq 3d$ . In the case equality holds, there is a  $v_i$  such that  $d(v_i) = 1$  and  $e(C_i) = 0$ , so  $K$  cannot be a root component. Thus we can assume that  $u'$  has degree three. However now if  $e(K) = 2d + 2$ , as  $q = 5$ , two leaf neighbours of  $u$  and  $v$  are in  $v_1, \dots, v_5$ , and thus we violate Property (12).

Therefore  $e(K) \geq 2d + 3$ . Property (4) implies that  $e(P) \geq 2d + 4$ , and if equality holds we have a leaf  $v_i$  such that  $e(C_i) = 0$ , implying that  $K$  cannot be a root component.

Therefore we can assume that  $P_{u,v}$  has three internal vertices. Thus  $e(K) \geq 2d + 2$ . If equality holds then two leaves adjacent to  $u$  or  $v$  are in  $v_1, \dots, v_5$ , which implies we violate Property (11). Thus  $e(K) \geq 2d + 3$ . Suppose equality holds. Then by Property (4) and Property (5), we have  $e(P) \geq 2d + 4$ . Further if equality holds, we have a leaf  $\ell \in \{v_1, \dots, v_5\}$  where  $e(C_i) = 0$ , and thus  $K$  is not a root component. Thus  $e(P) \geq 2d + 4$ , and Property (4) and Property (5) imply that  $e(P) \geq 2d + 5$ .

Assume  $P_{u,v}$  has four internal vertices. Then  $e(K) \geq 2d + 3$ . If equality holds, we have two leaves adjacent to  $u$  and  $v$  in  $v_1, \dots, v_5$ , and hence we have an exchangeable path. Thus  $e(P) \geq 2d + 4$ , and Property (4) implies that  $e(P) \geq 2d + 5$  as desired.

In the last case,  $P_{u,v}$  has more than five internal vertices, and thus  $e(K) \geq 2d + 4$ . Then Property (4) implies  $e(P) \geq 2d + 5$  as desired.

To finish, simply observe that

$$\frac{e(P)}{v(P)} \geq \frac{3d}{3d + 1 + 5} \geq \frac{d}{d + 2},$$

and strict inequality holds if  $e(P) \geq 3d + 1$ . □

**Claim 3.66.**  $q \geq 7$

*Proof.* Suppose not. Then we may assume that  $q = 6$ . We claim that  $e(P) \geq 3d + 2$  and that strict inequality holds when  $d = 4$  and  $K$  is not a root component. If  $K$  has more than 6 edges between vertices of degree at least three, then by Observation 3.6, we have  $e(K) \geq 15 > 3d + 2$ . So  $K$  has at least two vertices of degree  $d$  say  $u, v$ . Let  $v_1, \dots, v_6$  be the six vertices such that  $A(v_i) \neq \emptyset$ . We split into cases depending on how many vertices in  $v_1, \dots, v_6$  have degree  $d$ .

**Case 1: At least three vertices in  $v_1, \dots, v_6$  have degree  $d$**

In this case, as  $K$  is connected, and by Property (3), we have  $e(K) \geq 3d + 3$  as desired.

**Case 2: Exactly two vertices in  $v_1, \dots, v_6$  have degree  $d$**

Without loss of generality, let these vertices be  $v_1$  and  $v_2$ . Then  $v_1v_2 \notin E(K)$  by Property (3). If for every  $i, j$  we have  $v_iv_j \notin E(K)$ , then  $\sum_{i=3}^6 d(v_i) \geq 4$ , and if equality holds by Property (4) there are two  $i \in \{3, 4, 5, 6\}$  such that  $e(C_i) = 1$ . But then we have a leaf  $\ell$  with  $e(C_i) = 1$  and we can apply Claim 3.56. Therefore we can assume that  $\sum_{i=3}^6 d(v_i) \geq 5$ . By the same argument as above, there are two  $i \in \{1, \dots, 6\}$  such that  $e(C_i) = 1$ . Thus we have  $e(P) \geq 2d + 7 \geq 3d + 3$  as desired.

Now assume that  $v_1v_3 \in E(K)$  and no other pairs  $v_i, v_j$  are adjacent. Then  $e(C_1) = 1$ . Then  $\sum_{i=4}^6 d(v_i) \geq 3$ , and if equality holds, at least two  $i \in \{3, 4, 5, 6\}$  have  $e(C_i) = 1$ . Then again there is a leaf where Claim 3.56 applies. Therefore we can assume that  $\sum_{i=4}^6 d(v_i) \geq 4$ , and there are two  $i \in \{3, 4, 5, 6\}$  such that  $e(C_i) = 1$ . Thus  $e(P) \geq 3d + 3$  in this case.

Now suppose that  $v_3v_4 \in E(K)$ . Then without loss of generality,  $d(v_3) = 1$ . By Claim 3.56, this implies that  $d(v_4) = d$ , as  $K$  is not  $w$ -fragile, a contradiction as  $d(v_4) < d$ . A similar argument works for all pairs in  $\{v_3, v_4, v_5, v_6\}$ .

Now for the last case, assume without loss of generality that that  $v_1v_3 \in E(K)$  and  $v_2v_4 \in E(K)$ . Then  $e(C_1) = 1$  and  $e(C_2) = 1$ . Then  $d(v_5) + d(v_6) \geq 2$ . Note that by Property (4) and Property (5) we have at least two  $i \in \{3, 4, 5, 6\}$  such that  $e(C_i) = 1$ . Then  $e(P) \geq 2d + 6 = 3d + 2$ , and if equality holds there is a leaf  $\ell$  in  $\{v_1, \dots, v_6\}$  such that  $e(C_i) = 0$ , as desired.

**Case 3:  $v_1$  is the only vertex in  $v_1, \dots, v_6$  with degree  $d$**

Then without loss of generality we have  $d(v_2) = 1$  and  $e(C_2) = 1$ . If  $v_2$  is not adjacent to a vertex of degree  $d$ , then by Claim 3.56, we get a contradiction. Thus  $v_2$  is adjacent to a vertex of degree  $d$  that is not  $v_1$ , say  $v'_2$  and  $K$  is  $d$ -fragile. By the same analysis as above, we have that  $\sum_{i=3}^6 d(v_i) \geq 4$ . Further, there are at least two  $i \in \{3, 4, 5, 6\}$  with  $e(C_i) = 1$ . Then  $e(P) \geq 3d + 3$  as desired.

**Case 4: No vertex in  $v_1, \dots, v_6$  has degree  $d$**

Then up to relabelling we have for  $i \in \{1, 2\}$ ,  $d(v_i) = 1$  and  $e(C_i) = 1$ . Then  $K$  is  $d$ -fragile, and both  $v_1$  and  $v_2$  are adjacent to (distinct) vertices of degree  $d$ , say  $v'_1$  and  $v'_2$ . Further these two vertices are not adjacent as otherwise we have an exchangeable path. Further no other neighbour of  $v'_1$  and  $v'_2$  is in  $v_3, v_4, v_5, v_6$ . Hence since  $\sum_{i=3}^6 d(v_i) \geq 4$  and we have two  $i \in \{3, 4, 5, 6\}$  where  $e(C_i) = 1$ . It follows that  $e(P) \geq 3d + 3$ .

Finally, observe that if  $e(P) \geq 3d + 2$ , we have

$$\frac{e(P)}{v(P)} \geq \frac{3d + 2}{3d + 9} \geq \frac{d}{d + 2}$$

and strict inequality holds if  $e(P) \geq 3d + 3$ . □

Therefore we have  $q \geq 7$ . Then there are at most four vertices  $v_1, v_2, v_3, v_4$  such that  $A(v_i) \neq \emptyset$  and  $d(v_i) < d$ . We claim that  $e(P) \geq (q - 4)d + 6$ . By Property (3), the vertices  $v$  of degree  $d$  with  $A(v) \neq \emptyset$  form an independent set, and are not adjacent to any of  $v_1, \dots, v_4$  by Claim 3.58. Then at least two of  $v_1, \dots, v_4$  have  $e(C_i) = 1$  by Properties (4) and (5), and thus it follows that  $e(P) \geq (q - 4)d + 6$ . Thus we have

$$\begin{aligned} \frac{e(P)}{v(P)} &\geq \frac{(q - 4)d + 6}{(q - 4)d + 7 + (q - 4)} \\ &\geq \frac{d}{d + 2}. \end{aligned}$$

Here the second inequality holds when  $q \geq 7$ . □

Restrict to the case where  $k = 1$  and  $d \in \{3, 4\}$ . Let  $\mathcal{R}$  denote the set of red components of  $H_{R,F,s}$  which are not small. By Corollary 3.19 it follows that,

$$V(H_{R,F,s}) = \bigcup_{K \in \mathcal{R}} V(K_C).$$

Therefore it follows that:

$$E_r(H_{R,F,s}) = \bigcup_{K \in \mathcal{R}} E_r(K_C).$$



Now we bound the fractional arboricity of  $H_{R,F,s}$ . By Lemma 3.47 and Lemma 3.55 it follows that have

$$\begin{aligned} \frac{e_r(H_{R,F,s})}{v(H_{R,F,s})} &\geq \frac{\sum_{K \in \mathcal{R}} e_r(K_C)}{\sum_{K \in \mathcal{R}} v(K_C)} \\ &> \frac{d}{d+2}. \end{aligned}$$

The strict inequality follows as  $\mathcal{R}$  contains the root component. However, this contradicts Observation 3.9. Now Theorem 1.16 follows.

### 3.6 Strengthening the $d = 3, k = 1$ case

In this section, we prove Theorem 1.18. We overload the terminology from before to prove Theorem 1.18.

Let  $G$  be a vertex minimum counterexample to Theorem 1.18. Note  $G$  is connected as otherwise we apply minimality to each connected component and obtain a desired decomposition. Let  $\mathcal{F}$  be the set of decompositions  $(T, F)$  of  $G$  satisfying Theorem 1.16 when  $k = 1$  and  $d = 3$ , where  $T$  is a spanning tree. Observe that  $\mathcal{F}$  is not empty, since given any decomposition  $(T', F')$  satisfying Theorem 1.16, if  $T'$  is not a spanning tree, we can simply add edges from  $F'$  to  $T'$  until we obtain a spanning tree.

As before we define a notion of a bad component.

**Definition 3.67.** Given a decomposition  $(T, F) \in \mathcal{F}$ , a component  $C$  of  $F$  is *bad* if either  $C$  contains a vertex of degree three and has more than 14 edges, or  $C$  has no vertex of degree three and more than 13 edges.

We define a residue function to describe how close a decomposition in  $\mathcal{F}$  is to satisfying Theorem 1.18.

**Definition 3.68.** Let  $(T, F) \in \mathcal{F}$ . Let  $\mathcal{L}$  be the set of bad components of  $F$ . The *residue function*, denoted  $\rho(F)$ , is defined as

$$\rho(F) = \sum_{K \in \mathcal{L}} (e_r(K) - 3).$$

In the same manner as before, we define the notion of an exploration subgraph.

**Definition 3.69.** Let  $(T, F) \in \mathcal{F}$ . Let  $R$  be a bad component. Let  $s$  be any vertex of maximum degree in  $R$ . Orient  $T$  such that  $s$  is the only vertex with outdegree zero. We recursively define a set of components  $\mathcal{Q}$  of  $F$ . Initially  $\mathcal{Q}$  is  $\{R\}$ . While there is a component  $C$  of  $F$  such that there is a blue directed edge  $(x, y)$  where  $x$  is in a component in  $\mathcal{Q}$  and  $y \in V(C)$ , we add  $C$  to  $\mathcal{Q}$ . Once this procedure has terminated, let  $H_{R,F,s}$  be the digraph induced by the components of  $\mathcal{Q}$ . We will call  $H_{R,F,s}$  an *exploration subgraph*, and  $R$  a *root component*.

**Observation 3.70.** Given a decomposition  $(T, F) \in \mathcal{F}$ , a root component  $R$  and a vertex  $s \in V(R)$ , the graph  $H_{R,F,s}$  satisfies:

$$\frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} \leq \frac{3}{5}.$$

*Proof.* The argument is identical to that of Observation 3.9.  $\square$

A counterexample is a tuple  $(G, D, H_{R,F,s}, L)$  where  $G$  is a graph with fractional arboricity at most  $1 + \frac{3}{5}$ , where  $D = (T, F)$  is a decomposition in  $\mathcal{F}$ , where  $H_{R,F,s}$  is an exploration subgraph, and where  $L$  is a legal order of  $H_{R,F,s}$ . A minimum counterexample is one where  $v(G)$  is minimized, subject to that  $D$  is taken to minimize the residue function, subject to that our choice of  $D$  minimizes the number of bad components in  $F$  with no vertex of degree three, and finally subject to all of this,  $H_{R,F,s}$  and  $L$  are taken such that  $L$  is smaller than all other possible legal orders. From here on out, assume we are working with a minimum counterexample.

As before, a *small component* is a red component with at most one edge. A blue edge  $(x, y)$  is *saturated* if  $\rho(F + xy) > \rho(F)$ , or  $F + xy$  either has maximum degree four, or has a component containing two vertices of degree three. Otherwise  $(x, y)$  is *unsaturated*.

**Lemma 3.71.** *If  $K$  is a red component and  $C$  is a child of  $K$ , and  $(x, y)$  is a blue arc where  $x \in V(K)$  and  $y \in V(C)$ , then  $(x, y)$  is saturated.*

*Proof.* The argument is identical to that of Lemma 3.16, with the one exception that we note that when we delete a red edge  $e$  from a component  $K$ , if  $K$  is not a bad component, then  $K - e$  cannot be bad.  $\square$

**Corollary 3.72.** *If  $K$  is a component, and  $C$  is a child of  $K$ , then  $e_r(K) + e_r(C) \geq 3$ .*

*Proof.* Suppose not. Then  $e_r(K) + e_r(C) \leq 2$ , and this implies any blue directed arc from  $K$  to  $C$  is unsaturated, a contradiction.  $\square$

**Lemma 3.73.** *Let  $K$  be a red component with no vertex of degree three. If  $e_r(K) \leq 11$ , then there does not exist an arc  $(x, y)$  with  $x \in V(K)$  that generates a small child. If  $e_r(K) = 12$  and there exists an arc  $(x, y)$  with  $x \in V(K)$  that generates a small child  $C$ , then  $d_r(x) = 1$  and  $e_r(C) = 1$ .*

*Proof.* Otherwise, since  $e_r(C) \leq 1$  it follows that  $(x, y)$  is unsaturated, contradicting Lemma 3.71.  $\square$

**Lemma 3.74.** *Let  $K$  be a red component with two arcs  $(x, x')$  and  $(y, y')$  which generate small children  $C_1$  and  $C_2$ , respectively. Then one of the following situations occurs:*

- $xy \notin E(K)$

- Up to relabelling  $x$  as  $y$  (and  $C_1$  with  $C_2$ ),  $xy$  can only be exchanged with  $(x, x')$ ,  $e_r(C_1) = 1$ ,  $K$  contains a vertex of degree three,  $e_r(K) \geq 14$ , and  $d(y) = 1$
- Up to relabelling  $x$  as  $y$  (and  $C_1$  with  $C_2$ ),  $xy$  can only be exchanged with  $(x, x')$ ,  $e_r(C_1) = 1$ ,  $K$  is isomorphic to a path,  $e_r(K) \geq 13$ , and  $d_r(y) = 1$ .

*Proof.* Suppose not. Then without loss of generality we may assume that  $xy \in E(K)$ . By Lemma 3.24, either we can exchange  $(x, x')$  with  $xy$  or  $(y, y')$  with  $xy$ . If we can exchange both, then let  $K^x$  be the component of  $K - xy$  containing  $x$ , and  $K^y$  be the component containing  $y$ . If both  $e_r(K^y) \leq 1$  and  $e_r(K^x) \leq 1$ ,  $K$  is a path of length two and so  $(x, x')$  is unsaturated, contradicting Lemma 3.71. Otherwise we may assume without loss of generality that  $e_r(K^y) \geq 2$ , and we apply the argument from Lemma 3.27, to  $(x, x')$  and  $xy$  to get a contradiction. Therefore without loss of generality, we may assume that  $xy$  can only be exchanged with  $(x, x')$ . Then if the argument from Lemma 3.27 does not improve the legal order or reduce the residue function,  $e_r(K^x + xx') > e_r(K)$  and  $K^x + xx'$  is a bad component. As  $e_r(C_1) \leq 1$ , this implies that  $d(y) = 1$ , and as  $K^x + xx'$  is bad,  $e_r(K) \geq 14$  if  $K$  has a vertex of degree three. If  $K$  does not have a vertex of degree three, then the same argument gives  $e_r(K) \geq 13$ .  $\square$

**Lemma 3.75.** *Let  $K$  be a red component containing a vertex of degree three, where  $(x, x')$  generates a small child  $C$ , and  $d_r(x) = 1$ . Then  $e_r(K) \geq 13$ , and if  $e_r(K) = 13$ , then  $e_r(C) = 1$ .*

*Proof.* Note that if  $e_r(K) \leq 12$ ,  $(x, x')$  is unsaturated—a contradiction. Finally, if  $e_r(K) = 12$  and  $e_r(C) = 0$ , then  $(x, x')$  is unsaturated, a contradiction.  $\square$

**Corollary 3.76.** *Let  $K$  be a red component. Suppose  $(x, y)$  generates a small child  $C$ . Then at most two neighbours of  $x$  generate a small child, and such neighbours are leaves in  $K$ . Further  $(x, y)$  is a saturated edge, which implies that  $e_r(K) + e_r(C) \geq 3$ .*

*Proof.* By Lemma 3.74 and Corollary 3.72 the only part of this corollary that does not immediately follow is that  $x$  has at most two neighbours that generate a small child. However, if three neighbours of  $x$  generate small children, then  $x$  has degree three, and all neighbours of  $x$  are leaves. But then  $K$  is isomorphic to the star on 3 edges, and so  $(x, y)$  is unsaturated, contradicting Lemma 3.71.  $\square$

We require one final lemma before we finish with a counting argument.

**Lemma 3.77.** *Let  $R$  be the root component of an exploration subgraph  $H_{R,F,s}$ , and suppose  $s$  has degree three. There does not exist an arc  $(x, x')$  that generates a small child such that  $x \in V(R)$ .*

*Proof.* Suppose not. Note that since  $T$  is oriented so that  $s$  has outdegree zero, it follows that  $x \neq s$ . By Lemma 3.22, there exists an edge  $e$  on  $P_{xs}$  that can be exchanged with  $(x, x')$ , resulting in the decomposition  $(T', F')$ . Let  $K^x$  and  $K^s$  be the components of  $F'$  containing  $x$  and  $s$ , respectively. Since  $d_r(s) = 3$ , we have that  $e_r(K^s) \geq 3$ . Since  $s \neq x$ , the exchange of  $e$  and  $(x, x')$  does not create a vertex of degree at least four. Since each component in  $F$  has at most one vertex of degree three, we have in fact that  $(T', F') \in \mathcal{F}$ . However, since  $e_r(K^s) \geq 3$ , it follows that  $\rho(F') < \rho(F)$ . This contradicts our choice of minimum counterexample.  $\square$

Now we finish with a counting argument.

**Lemma 3.78.** *For any component  $K$  that is not small, we have*

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{3}{5},$$

*and the inequality is strict if  $K$  is the root component of an exploration subgraph.*

*Proof.* Let  $q$  be the number of small children of  $K$ . We consider cases.

**Case 1:**  $q = 0$ .

In this case  $e_r(K) \geq 2$  as  $K$  is not small, and then we have  $\frac{2}{3} \geq \frac{3}{5}$  as desired.

**Case 2:**  $q = 1$ .

Let  $(x, x')$  generate a small child  $C_1$ . Then  $(x, x')$  is saturated by Lemma 3.71, and so either  $K$  contains a vertex of degree three or it does not and  $K + C_1 + xx'$  is bad. In either case,  $e_r(K) \geq 3$ , and so we have  $\frac{e_r(K_C)}{v(K_C)} \geq \frac{3}{5}$  as desired. Note if  $K$  is a root component, then  $e_r(K) \geq 4$ , and so we have strict inequality.

**Case 3:**  $q = 2$ .

When  $q = 2$ , we have a few possibilities. In all cases, we aim to show  $e_r(K_C) \geq 5$ . Let  $(v_1, v'_1)$  and  $(v_2, v'_2)$  generate small children. If  $v_1 v_2 \in E(K)$ , then  $e_r(K) \geq 12$  by Lemma 3.74 and the result follows. Otherwise,  $v_1$  and  $v_2$  are not adjacent. If  $K$  does not contain a vertex of degree three, then by Lemma 3.73,  $e_r(K_C) \geq 12$ , and again the result follows. Thus we may assume that  $K$  contains a vertex of degree three, and that  $v_1$  and  $v_2$  are not adjacent. If either of  $v_1$  or  $v_2$  has degree one, then by Lemma 3.75 we have that  $e_r(K_C) \geq 13$ , and so  $\frac{e_r(K_C)}{v(K_C)} > \frac{3}{5}$ , as desired. Thus we may assume neither  $v_1$  nor  $v_2$  has degree one, and so since  $v_1$  and  $v_2$  are not adjacent,  $e_r(K) \geq 4$ . Since moreover  $K$  contains a vertex of degree three, again we have that  $e_r(K) \geq 5$ . Hence  $\frac{e_r(K_C)}{v(K_C)} \geq \frac{5}{8} > \frac{3}{5}$ , as desired.

**Case 4:**  $q \geq 3$ .

We split into subcases. First suppose that  $K$  has no vertex of degree three. Thus  $K$  is isomorphic to a path, and since  $q \neq 0$ , by Lemma 3.73 we have that  $e_r(K) \geq 12$ .

Let  $b$  be the number of edges where both endpoints generate a small child, and let  $c$  be the number of leaves that generate a small child whose neighbours in  $K$  do not generate a small child. Let  $\ell = b + c$ . By Lemma 3.74, for any edge whose endpoints both produce small children, one of the endpoints is a leaf of  $K$ . Hence  $\ell \leq 2$ . Further, if  $xy \in E_r(K)$

is an edge where  $(x, x')$  and  $(y, y')$  generate small children  $C_1$  and  $C_2$  respectively, where  $d_r(x) \geq 2$ , then again by Lemma 3.74, we have  $e_r(C_1) = 1$ . Note that since  $e_r(K) \geq 12$ , no vertex in  $K$  is adjacent to both leaves. Additionally, by Lemma 3.73 and since  $q \geq 3$ , we have  $e_r(K) \geq 12$  and  $e_r(K_C) \geq 13$ . Thus it follows that  $e_r(K_C) \geq \max\{13, 2(q - \ell) + \ell\}$ , and  $v(K_C) \leq e_r(K_C) + 1 + q$ . Therefore, if  $q \geq 8$  we have:

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{e_r(K_C)}{e_r(K_C) + 1 + q} \geq \frac{2(q - \ell) + \ell}{2(q - \ell) + \ell + 1 + q} = \frac{2q - \ell}{3q - \ell + 1} > \frac{3}{5},$$

where the last line holds because  $q \geq 8$  and  $\ell \leq 2$ . So we may assume instead  $q \leq 7$ . Then since  $e_r(K_C) \geq 13$ , we have: we have

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{13}{21} > \frac{3}{5}.$$

Thus we can assume  $K$  has a vertex of degree three, and by our choice of decomposition,  $K$  has exactly one vertex of degree three. Note that by Lemma 3.77, since  $q \geq 3$  it follows that  $K$  is not a root component. Let  $b$  be the number of edges where both endpoints generate a small child, and let  $c$  be the number of leaves that generate a small child whose neighbours in  $K$  do not generate a small child. Let  $\ell = b + c$ . As  $K$  has at most one vertex of degree three, it follows that  $K$  has at most 3 leaves, and hence by Lemma 3.74, it follows that  $\ell \leq 3$ . Further by Lemma 3.74, if  $xy \in E_r(K)$ ,  $d_r(x) \geq 2$  and there exist arcs  $(x, x')$ ,  $(y, y')$  that generate small children  $C_1$  and  $C_2$ , then  $e_r(C_1) = 1$ , and  $d_r(y) = 1$ . If the vertex of degree three does not generate a small child, we have  $e_r(K_C) \geq 2(q - \ell) + \ell$  and  $v(K_C) \leq e_r(K_C) + 1 + q$ . Similarly, if the vertex of degree three generates a child, then this vertex is adjacent to at most two leaves which generate small children. Thus, we get  $e_r(K_C) \geq 2(q - \ell - 1) + 3 + \ell - 1 = 2(q - \ell) + \ell$ , and  $v(K_C) \leq e_r(K_C) + 1 + q$ . Therefore if  $q \geq 9$  and no vertex of degree three generates a small child, we have:

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{2(q - \ell) + \ell}{2(q - \ell) + \ell + 1 + q} = \frac{2q - \ell}{3q - \ell + 1} \geq \frac{3}{5}.$$

Again, this last inequality holds because  $\ell \leq 3$  and we assumed  $q \geq 9$ . Note this inclusive inequality is sufficient, since  $K$  is not a root component by assumption. Now suppose  $q \leq 8$ .

If no leaves generate small children, then  $\ell = 0$ ,  $e_r(K_C) \geq 2q$ , and thus

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{2q}{3q + 1} \geq \frac{3}{5}.$$

Therefore we can assume that a leaf generates a small child. In this case, by Lemma 3.75, we have that  $e_r(K) \geq 13$  and  $e_r(K_C) \geq 14$ . Since  $q \leq 8$ , we have

$$\frac{e_r(K_C)}{v(K_C)} \geq \frac{14}{14 + 1 + 8} > \frac{3}{5}.$$

□

Now we finish the proof. Let  $\mathcal{R}$  denote the set of red components of  $H_{R,F,s}$  which are not small. By Corollary 3.72 it follows that,

$$V(H_{R,F,s}) = \bigcup_{K \in \mathcal{R}} V(K_C).$$

Therefore it follows that:

$$E_r(H_{R,F,s}) = \bigcup_{K \in \mathcal{R}} E_r(K_C).$$

Now we bound the fractional arboricity of  $H_{R,F,s}$ . By Lemma 3.78, it follows that have

$$\begin{aligned} \frac{e_r(H_{R,F,s})}{v(H_{R,F,s})} &\geq \frac{\sum_{K \in \mathcal{R}} e_r(K_C)}{\sum_{K \in \mathcal{R}} v(K_C)} \\ &> \frac{3}{5}. \end{aligned}$$

The strict inequality follows as  $\mathcal{R}$  contains the root component. However, this contradicts Observation 3.70. Now the main theorem follows.

# Chapter 4

## A density bound for 4-critical triangle free graphs

This chapter is joint work with Evelyne Smith-Roberge.

### 4.1 Introduction

In this chapter we prove Theorem 1.64. We start with a brief outline of the proof of Theorem 1.64.

Our main tool is the potential method. This is a counting argument combined with a quotient argument. This argument shows that all subgraphs with fewer vertices are sparse. From this, we can show that a vertex minimal counterexample to Theorem 1.64 does not contain a  $K_4 - e$  subgraph, or any cycle of degree three vertices. We will eventually be able to show that subgraph induced by the vertices of degree three has no component with more than six vertices. Further, if the components have more than two vertices, then the local structure around the components is heavily constrained (there must be a subgraph “close” that contains an  $M$ -gadget, which we define later). Once enough structure is established, a discharging argument rules out the existence of a minimum counterexample, thereby completing the proof.

The chapter is organized as follows. In Section 4.2, we present structural lemmas regarding  $(k - 1)$ -cliques in  $k$ -Ore graphs, focusing on the case where  $k = 4$ . In Section 4.3, we present results regarding the triangles of graphs in  $\mathcal{B}$ . In Section 4.4, we give a brief overview of the potential method and results specific to its use towards Theorem 1.64. Section 4.5 contains the critical arguments which impose structure on a vertex minimal counterexample. Finally, the discharging portion of the proof is found in Section 4.6.

## 4.2 $(k - 1)$ -cliques in $k$ -Ore graphs

In this section we prove many structural results about  $(k - 1)$ -cliques in  $k$ -Ore graphs. An important graph is the unique 4-Ore graph on seven vertices, called the *Moser spindle*. We denote the Moser spindle as  $M$ . The first two observations are easy and follow from effectively the same proofs as similar statements about 4-Ore graphs in [31].

**Observation 4.1.** Let  $G$  be  $k$ -Ore and  $v$  be a vertex in  $V(G)$ . Then  $G - v$  contains a  $K_{k-1}$  subgraph.

*Proof.* We proceed by induction on  $v(G)$ . If  $G = K_k$ , then this is immediate. Otherwise,  $G$  is the Ore composition of two graphs  $H_1$  and  $H_2$  where  $H_1$  is the edge side obtained by deleting edge  $e = xy$  and  $H_2$  is the split side obtained by splitting a vertex  $z$  into vertices  $z_1$  and  $z_2$ . By induction both  $H_1 - e$  and  $H_2 - z$  contain a  $K_{k-1}$  subgraph.

Now let  $v \in V(G)$ . If  $v \in V(H_1 \setminus \{x, y\})$ , then  $G - v$  contains a  $K_{k-1}$  subgraph as there is a  $K_{k-1}$  subgraph in  $H_2 - z$ . A similar argument applies if  $v \in V(H_2 - z)$ . Therefore  $v \in \{z_1, z_2\}$ . Since  $H_2 - z$  contains a  $K_{k-1}$  subgraph, it then follows that  $G - v$  contains a  $K_{k-1}$ -subgraph.  $\square$

**Observation 4.2.** If  $G$  is  $k$ -Ore and not isomorphic to  $K_k$ , then for any subgraph  $K$  in  $G$  isomorphic to  $K_{k-1}$ ,  $G - K$  contains a  $K_{k-1}$  subgraph.

*Proof.* We proceed by induction on  $v(G)$ . As  $G$  is not isomorphic to  $K_k$ ,  $G$  is the Ore composition of two graphs  $H_1$  and  $H_2$ . Let  $H_1$  be the edge side of  $G$  where we delete the edge  $xy$ , and let  $H_2$  be the split side where we split the vertex  $z$  into two vertices  $z_1$  and  $z_2$ . Let  $K$  be any  $K_{k-1}$  subgraph in  $G$ .

**Case 1:**  $H_1 = K_k$ .

First suppose that  $H_2$  is also isomorphic to  $K_k$ . Then each of  $H_2 - z$  and  $H_1 - xy$  contains a  $K_{k-1}$  subgraph. Note that either  $x \notin V(K)$  or  $y \notin V(K)$  since  $xy \notin E(G)$ . Additionally, either  $V(K) \subseteq V(H_1 - xy)$  or  $V(K) \subseteq V(H_2^z)$ . If  $V(K) \subseteq V(H_1 - xy)$ , then in  $G - K$ , there is a  $K_{k-1}$  subgraph in  $H_2 - z$ . Therefore we may assume that  $V(K) \subseteq V(H_2^z)$ , and without loss of generality that  $x \notin V(K)$ . But then  $H_1 - y$  contains a  $K_{k-1}$  subgraph that is contained in  $G - K$ , as desired.

Hence we can assume that  $H_2 \neq K_k$ . By induction, deleting any  $K_{k-1}$  subgraph in  $H_2$  leaves a  $K_{k-1}$  subgraph in  $H_2$ . Let  $K$  be any  $K_{k-1}$  subgraph in  $G$ . If  $V(K) \subseteq V(H_1 - xy)$ , then  $G - K$  contains a  $K_{k-1}$  subgraph in  $H_2 - z$  by Observation 4.1. If  $K$  lies in  $H_2^z$ , then there is a  $K_{k-1}$  subgraph in at least one of  $H_1 - x$  or  $H_1 - y$ , depending on if  $x \in V(K)$  or  $y \in V(K)$  or neither is.

**Case 2:**  $H_2 = K_k$ .

From the previous case we may assume that  $H_1 \neq K_k$ . Hence by induction, deleting any  $K_{k-1}$  subgraph in  $H_1$  leaves a  $K_{k-1}$  subgraph in  $H_1$ . If  $V(K) \subseteq V(H_1 - xy)$ , then  $G - K$  has a  $K_{k-1}$  subgraph in  $H_2^z - \{z_1, z_2\}$ . Therefore  $K \subseteq V(H_2^z)$ . But since  $xy \notin E(G)$ ,  $K$



uses at most one of  $z_1$  and  $z_2$ . Thus by Observation 4.1 there is a  $K_{k-1}$  subgraph in  $G - K$  that lies in  $H_1 - xy$ .

**Case 3: Neither  $H_1$  nor  $H_2$  is  $K_k$ .**

In this case, since the induction hypothesis holds for both  $H_1$  and  $H_2$  and any  $K_{k-1}$  subgraph uses at most one of  $x$  or  $y$ , it is easy to see the same arguments as above give the result.  $\square$

The following definition is helpful to avoid repetition.

**Definition 4.3.** Let  $G$  be a graph. A  $(k-1)$ -clique packing of  $G$  is a maximum collection of vertex-disjoint  $K_{k-1}$  subgraphs in  $G$ .

Thus  $T^{k-1}(G)$  is the size of a  $(k-1)$ -clique packing of a graph  $G$ . It will be useful to bound the size of a  $(k-1)$ -clique packing of an Ore composition, which we do now.

**Proposition 4.4.** *If  $G$  is an Ore composition of  $H_1$  and  $H_2$  where  $H_1$  is the edge side and uses edge  $xy$  and  $H_2$  is the split side using vertex  $z$ , then*

$$T^{k-1}(G) \geq T^{k-1}(H_1) + T^{k-1}(H_2) - f(H_1, H_2),$$

where  $f(H_1, H_2)$  is defined as follows.

- If every  $(k-1)$ -clique packing of  $H_1$  contains a clique using the edge  $xy$ , and every  $(k-1)$ -clique packing of  $H_2$  contains a clique using the vertex  $z$ , then  $f(H_1, H_2) = -2$ .
- If there exists a  $(k-1)$ -clique packing of  $H_1$  where no clique uses the edge  $xy$ , but every  $(k-1)$ -clique packing of  $H_2$  contains a clique using the vertex  $z$ , then  $f(H_1, H_2) = -1$ .
- If every  $(k-1)$ -clique packing of  $H_1$  contains a clique using the edge  $xy$  but there exists a  $(k-1)$ -clique packing of  $H_2$  where no clique uses the vertex  $z$ , then  $f(H_1, H_2) = -1$ .
- If none of these occur, then  $f(H_1, H_2) = 0$ .

*Proof.* Let  $H_1$  be the edge side of the composition with edge  $xy$  and  $H_2$  the split side, where we split  $z$  into vertices  $z_1$  and  $z_2$ .

For  $i \in \{1, 2\}$ , let  $\mathcal{T}_i$  be a  $(k-1)$ -clique packing of  $H_i$ , and let  $\mathcal{T}'_i$  be the  $(k-1)$ -clique packing obtained from  $\mathcal{T}_i$  by removing a clique if it uses either  $z$  or the edge  $xy$ . Note that  $|\mathcal{T}'_i| \geq |\mathcal{T}_i| - 1$ , and equality holds if and only if  $\mathcal{T}_i$  contains a clique using either the edge  $xy$  or  $z$ .

Now  $\mathcal{T}'_1 \cup \mathcal{T}'_2$  is a collection of vertex-disjoint  $(k-1)$ -cliques by construction. It follows that

$$T^{k-1}(G) \geq T^{k-1}(H_1) + T^{k-1}(H_2) - f(H_1, H_2).$$

$\square$

**Corollary 4.5.** *If  $G$  is the Ore composition of a graph  $H$  and  $K_k$ , then  $T^{k-1}(G) \geq T^{k-1}(H)$ .*

*Proof.* Note that  $T^{k-1}(K_k) = 1$ , and that for any vertex  $v \in V(K_k)$ , there is a  $(k-1)$ -clique in  $K_k - v$ . Hence regardless of whether  $K_k$  is the split side or edge side of the composition, by Proposition 4.4

$$T^{k-1}(G) \geq T^{k-1}(H) + T^{k-1}(K_k) - 1 = T^{k-1}(H),$$

as desired. □

**Corollary 4.6.** *The only  $k$ -Ore graph  $G$  with  $T^{k-1}(G) = 1$  is  $K_k$ .*

*Proof.* Let  $G$  be a vertex-minimum counterexample. Since  $G \neq K_k$ ,  $G$  is the Ore composition of two graphs  $H_1$  and  $H_2$ . If neither  $H_1$  nor  $H_2$  are  $K_k$ , then by induction  $T^{k-1}(H_i) \geq 2$  for  $i \in \{1, 2\}$  and hence by Proposition 4.4 we have  $T^{k-1}(G) \geq T^{k-1}(H_1) + T^{k-1}(H_2) - 2 \geq 2$ .

Now consider the case where  $H_1 = K_k$  but  $H_2 \neq K_k$ . Then  $T^{k-1}(G) \geq T^{k-1}(H_2) \geq 2$  by Corollary 4.5. Therefore both  $H_1$  and  $H_2$  are isomorphic  $K_k$ .

In this case, without loss of generality we may assume  $H_2$  is the split side where we split vertex  $z$ . Observe that  $H_2 - z$  contains a  $K_{k-1}$  subgraph, and there is a  $K_{k-1}$  subgraph in  $H_1$  after deleting an edge. Hence every  $(k-1)$ -clique packing of  $G$  has size at least two, a contradiction. □

For the rest of this section we restrict our attention to 4-Ore graphs, since there is no straightforward generalization of our lemmas to  $k$ -Ore graphs for  $k > 4$ .

**Definition 4.7.** A *kite* in  $G$  is a  $K_4 - e$  subgraph  $K$  such that the vertices of degree three in  $K$  have degree three in  $G$ . The *spar* of a kite  $K$  is the unique edge in  $E(K)$  contained in both triangles of  $K$ .

The following two lemmas partially describe the structure of 4-Ore graphs with 3-clique packings of size two.

**Lemma 4.8.** *If  $G$  is a 4-Ore graph with  $T^3(G) = 2$ , then  $G$  contains two edge-disjoint kites that share at most one vertex. Furthermore, if  $G \neq M$ , then  $G$  contains two vertex-disjoint kites.*

*Proof.* We proceed by induction on  $v(G)$ . As  $T^3(G) = 2$ , by Corollary 4.6 we have that  $G \neq K_4$ . Hence  $G$  is the Ore composition of two graphs  $H_1$  and  $H_2$ . Up to relabelling, we may assume that  $H_1$  is the edge side of the composition where we delete the edge  $xy$ , that  $H_2$  is the split side where the vertex  $z$  is split into two vertices  $z_1$  and  $z_2$ , and that  $x$  is identified with  $z_1$  and  $y$  with  $z_2$  in  $G$ . We break into cases depending on which (if any) of  $H_1$  and  $H_2$  is isomorphic to  $K_4$ .

**Case 1:**  $H_1 = H_2 = K_4$ .

In this case,  $G$  is isomorphic to the Moser spindle, which contains two edge-disjoint kites that share exactly one vertex.

**Case 2:**  $H_1 = K_4$ , and  $H_2 \neq K_4$ .

First suppose that  $H_2 \neq M$ . Then by the induction hypothesis,  $H_2$  contains two vertex-disjoint kites. Note that  $z$  belongs to at most one of these kites, and hence  $H_2^z$  contains a kite not containing  $z_1$  or  $z_2$ . Observe that  $H_1 - xy$  is a kite, and thus in this case  $G$  contains two vertex-disjoint kites. Therefore we may assume that  $H_2 = M$ . If  $z$  is the unique vertex of degree four in  $M$ , then  $T^3(G) = 3$ , a contradiction. But if  $z$  is not the unique vertex of degree four, then there is a kite in  $M^z$  not containing  $z_1$  or  $z_2$ , and thus this kite and  $H_1 - xy$  give two vertex-disjoint kites.

**Case 3:**  $H_1 \neq K_4$ , and  $H_2 = K_4$ .

First suppose that  $H_1 \neq M$ . Then by the induction hypothesis there are two vertex-disjoint kites in  $H_1$ , say  $D_1$  and  $D_2$ . Thus either there is a kite in  $H_1 - xy$  that does not contain  $x$  or  $y$ , or up to relabelling  $x \in V(D_1)$  and  $y \in V(D_2)$ . In either case, since  $H_2^z$  contains a kite that contains at most one of  $z_1$  and  $z_2$ , it follows that  $G$  contains two vertex-disjoint kites.

Therefore  $H_1 = M$ . If  $T^3(H_1 - xy) = 2$ , then  $T^3(G) = 3$ , a contradiction. Hence in  $H_1$ , both  $x$  and  $y$  have degree three. Further,  $x$  and  $y$  are not the two vertices that have degree three and are not incident to a spar of a kite. Thus it follows that there is a kite in  $H_1 - xy$  that does not contain  $x$  or  $y$ . Since there is a kite in  $H_2^z$ , there are two vertex-disjoint kites in  $G$ .

**Case 4:**  $H_1 \neq K_4$ , and  $H_2 \neq K_4$ .

First suppose  $H_2 \neq M$ . Then by induction,  $H_2$  contains two vertex-disjoint kites, and hence  $H_2^z$  contains a kite that does not contain  $z_1$  or  $z_2$ . Similarly,  $H_1$  contains two edge-disjoint kites by induction. Thus  $H_1 - xy$  contains a kite, and so  $G$  contains two vertex-disjoint kites.

Therefore  $H_2 = M$ . If  $z$  is the unique vertex of degree four in  $M$ , then  $T^3(H_2^z - z_1 - z_2) = 2$ , and it follows that  $T^3(G) \geq 3$ , a contradiction. Thus  $z$  is not the degree four vertex in  $M$ , and thus there is a kite in  $H_2^z$  that does not contain  $z_1$  or  $z_2$ . By induction, there are two edge-disjoint kites in  $H_1$ , and hence there is a kite in  $H_1 - xy$ . Thus it follows that there are two vertex-disjoint kites in  $G$ , as desired.  $\square$

**Lemma 4.9.** *Let  $G$  be 4-Ore with  $T^3(G) = 2$ . Let  $v \in V(G)$  and let  $G^v$  be the graph obtained by splitting  $v$  into two vertices of positive degree  $v_1$  and  $v_2$ , with  $N(v_1) \cup N(v_2) = N(v)$  and  $N(v_1) \cap N(v_2) = \emptyset$ . Then either*

(i)  $T^3(G^v) \geq 2$ , or

(ii)  $d(v) = 3$ , there is an  $i \in \{1, 2\}$  such that  $d(v_i) = 1$ , and the edge  $e$  incident to  $v_i$  is the spar of a kite in  $G$ .

*Proof.* We proceed by induction on the number of vertices. First suppose that  $G = M$ . If  $v$  is the unique vertex of degree four, then it is easy to verify that any split leaves two vertex-disjoint triangles, and hence  $T^3(G^v) \geq 2$ . Now suppose that  $v$  is incident to one of the two spars of kites. Again, it is easy to check that if the vertex of degree one is incident to the spar of the kite, then  $T^3(G^v) = 1$ , and otherwise  $T^3(G^v) \geq 2$ . Lastly, if  $v$  is either of the other two vertices in  $M$ , then one simply checks that  $T^3(G^v) \geq 2$  for any split.

Therefore we can assume that  $G \neq M$ . Let  $G$  be the Ore composition of  $H_1$  and  $H_2$  where  $H_1$  is the edge side, we delete the edge  $xy$ , and  $H_2$  is the split side where we split the vertex  $z$  into two vertices  $z_1$  and  $z_2$ .

**Case 1:**  $H_1 = K_4$ .

We can assume that  $H_2 \neq K_4$ , as otherwise  $G = M$ . This implies that  $T^3(H_2) = 2$ : if  $T^3(H_2) \geq 3$ , then Proposition 4.4 implies that  $T^3(G) \geq 3$ , a contradiction. First suppose that  $v \in \{x, y\}$ . Without loss of generality, let  $v = x$ . By Observation 4.1 there is a triangle in  $H_2 - z$ . Since there is also a triangle in  $H_1 - x$ , we have that  $T^3(G^v) \geq 2$  as desired.

Now suppose that  $v \in V(H_1) \setminus \{x, y\}$ . Note  $H_1 - xy$  is a kite. Assume  $d(v_1) = 1$ . If  $v_1$  is not incident to the spar of a kite, then there is a triangle on  $v_2, x, y$ . By Observation 4.1 there is also a triangle in  $H_2 - z$ . This implies that  $T^3(G^v) \geq 2$ , as desired.

Thus we can assume that  $v \in V(H_2 - z)$ . We apply induction to  $H_2$ . If  $T^3(H_2^v) \geq 2$ , this implies  $T^3(H_2^v - z) \geq 1$ , and hence  $T^3(G^v) \geq 2$ . Otherwise we split  $v$  in such a way that in  $H_2^v$ ,  $d(v_1) = 1$  and  $v_1$  is incident to a spar of a kite in  $H_2$ . Let  $K$  be this kite. Note if  $z \notin V(K)$  then the same split occurs in  $G^v$  and (ii) occurs. Hence  $z \in V(K)$ . If  $T^3(H_2 - z) \geq 2$ , then  $T^3(H_2^v - z) \geq 1$ , and it follows that  $T^3(G^v) \geq 2$ . Hence every 3-clique packing of  $H_2$  uses the vertex  $z$ . If  $vz$  is the spar in  $K$ , then without loss of generality we may assume  $z_1v \in E(G)$  and  $z_2$  is incident to the other two vertices of  $K$  (we may make this assumption as otherwise  $T^3(G) \geq 3$ ). But then since  $d(v) = 3$  in both  $H_2$  and in  $G$ ,  $v$  is not in any triangle in  $G$ . Thus  $T^3(G^v) = T^3(G) = 2$ , as desired. Therefore  $z$  is not incident to the spar of  $K$ . Since every 3-clique packing of  $G$  contains a triangle using  $z$  and  $z$  is not incident to the spar of  $K$ , we have that  $H_2 \neq M$ . To see this, note that if  $H_2 = M$ , then  $H_2 - z$  contains two vertex disjoint triangles and then  $T^3(G) = 3$ . Thus by Lemma 4.8, since  $H_2 \neq M$  it follows that  $H_2$  contains two vertex-disjoint kites,  $D_1$  and  $D_2$ . If neither  $D_1$  nor  $D_2$  is  $K$ , then  $T^3(H_2) \geq 3$ , as there is a set of three vertex-disjoint triangles in  $D_1 \cup D_2 \cup K$ , a contradiction. Thus without loss of generality we may assume  $D_1 = K$ . But again,  $H_2 - z$  contains two vertex-disjoint triangles, namely the triangle in  $K - z$ , and one triangle in  $D_2$ . Hence  $T^3(G) \geq 3$ , a contradiction.

**Case 2:**  $H_2 = K_4$ .

In this case,  $T^3(H_1) = 2$  and every 3-clique packing of  $H_1$  uses the edge  $xy$  since otherwise  $G = M$  or  $T^3(G) \geq 3$  by Proposition 4.4.

First suppose that  $v \in \{z_1, z_2\}$ . Without loss of generality, let  $v = z_1$ . Then since  $T^3(H_1 - z_1) \geq 1$  by Observation 4.1 and  $T^3(H_2^z - \{z_1, z_2\}) = 1$ , it follows that  $T^3(G^v) \geq 2$ , as desired.

Now suppose that  $v \in V(H_1) \setminus \{x, y\}$ . Consider  $H_1^v$ , where we perform the same split as in  $G^v$ . First suppose that  $T^3(H_1^v) \geq 2$ . Hence  $H_1^v - xy$  has at least one triangle. Since there is a triangle in  $H_2^z - \{z_1, z_2\}$ , it follows that  $T^3(G^v) \geq 2$  as desired. Therefore we may assume that  $T^3(H_1^v) < 2$ , and so by induction  $v$  is incident to the spar in a kite  $K$  in  $H_1$ , and that after splitting  $v_1$  has degree one and is incident to the spar of the kite. If  $K$  is in  $G$ , then we are done. Therefore we may assume that  $xy \in E(K)$ , and so that  $\{x, y, v\}$  induces a triangle in  $G$ . Since  $H_1 \neq K_4$  by assumption, by Observation 4.2 we have that  $H_1 - \{x, y, v\}$  contains a triangle. As there is also a triangle in  $H_2 - z$  by Observation 4.1, it follows that  $T^3(G^v) \geq 2$ , as desired.

The final case to consider is if  $v \in V(H_2) - \{z_1, z_2\}$ . If  $v$  is not incident to the spar of the kite in  $H_2^z$ , then any split of  $v$  leaves a triangle in  $(H_2^z)^v$ , and since there is a triangle in  $H_1 - \{x, y\}$ , we get that  $T^3(G^v) \geq 2$  as desired. A similar argument works for the other splits, unless we split  $v$  in such a way that  $v_1$  has degree one and is incident to a spar of a kite in  $G$ .

**Case 3: Neither  $H_1$  nor  $H_2$  is  $K_4$ .**

First suppose that  $v \in \{x, y\}$  and without loss of generality, that  $v = x$ . Note that since  $T^3(H_1) \geq 2$  and  $T^3(H_2) \geq 2$ , it follows that  $T^3(H_1 - x) \geq 1$  and  $T^3(H_2^z - \{z_1, z_2\}) \geq 1$ . Hence in this case, after splitting  $v$  we have that  $T^3(G^v) \geq 2$ .

Now suppose that  $v \in V(H_1) \setminus \{x, y\}$ . If  $T^3(H_1^v) \geq 2$ , then it follows that  $T^3(H_1^v - xy) \geq 1$ . Since  $T^3(H_2^z - \{z_1, z_2\}) = 1$ , we have that  $T^3(G^v) \geq 2$ , a contradiction. Thus by induction we can assume that  $d(v) = 3$ , and  $v_1$  has degree one and is incident to the spar of a kite in  $H_1$ . Let  $K$  be this kite. If  $K$  does not contain  $x$  and  $y$  then (ii) holds in  $G^v$ , a contradiction. Otherwise  $x, y, v$  induce a triangle, and by Observation 4.2,  $H_1 - \{x, y, v\}$  contains a triangle, and  $H_2^z - \{z_1, z_2\}$  contains a triangle by Observation 4.1. Hence  $T^3(G^v) \geq 2$  as desired.

Finally suppose that  $v \in V(H_2) \setminus \{z_1, z_2\}$ . If  $T^3(H_2^z) \geq 2$ , then  $T^3((H_2^z)^v) \geq 1$ , and since  $T^3(H_1 - xy) \geq 1$ , it follows that  $T^3(G^v) \geq 2$  as desired. Therefore  $v$  has degree three, lies in a kite  $K$  in  $H_2$ ,  $d(v_1) = 1$  and is incident to the spar of the kite in  $K$ . If  $z \notin V(K)$ , then (ii) occurs in  $G^v$ . So  $z \in V(K)$ . Let  $T$  be a triangle in  $K$  which contains  $z$  and  $v$ . Then by Observation 4.2,  $H_2 - T$  contains a triangle, and as deleting any vertex in  $H_1$  leaves a triangle, it follows that  $T^3(G^v) \geq 2$ .  $\square$

We now describe the structure of the 4-Ore graphs with 3-clique packings of size three.

**Lemma 4.10.** *Let  $G$  be a 4-Ore graph with  $T^3(G) = 3$ , and let  $T$  be a triangle in  $G$ . Either  $T^3(G - T) \geq 2$ , or there exists a kite in  $G - T$ .*

*Proof.* Let  $G$  be a vertex-minimum counterexample. Since  $G \neq K_4$ ,  $G$  is the Ore composition of two 4-Ore graphs  $H_1$  and  $H_2$ . Up to relabelling, we may assume that  $H_1$  is the edge side of the composition where we delete the edge  $xy$ , and that  $H_2$  is the split side where we split the vertex  $z$  into two vertices  $z_1$  and  $z_2$ . Let  $T$  be a triangle of  $G$ . Observe that at most one of  $z_1$  and  $z_2$  is in  $V(T)$ . Additionally, notice if  $T^3(H_i) \geq 3$  for all  $i \in \{1, 2\}$ ,

then by Proposition 4.4 we have that  $T^3(G) \geq T^3(H_1) + T^3(H_2) - 2 \geq 4$ , a contradiction. We break into cases depending on which (if any) of  $H_1$  or  $H_2$  is isomorphic to  $K_4$ .

**Case 1:**  $H_1 = K_4$ .

Note that  $H_2 \neq K_4$  as otherwise  $T^3(G) = 2$ .

**Subcase 1:**  $T^3(H_2) = 3$ .

We may assume that  $T^3(H_2^z - z_1 - z_2) \leq 2$  since otherwise  $T^3(G) \geq 4$ , a contradiction. Note this implies that  $T^3(H_2^z - z_1 - z_2) = 2$ , since  $T^3(H_2 - z) \geq T^3(H_2) - 1 \geq 2$ . Hence every 3-clique packing of  $H_2$  has a triangle which contains the vertex  $z$ . If  $V(T) \subseteq V(H_1)$ , then as  $T^3(H_2^z - z_1 - z_2) = 2$ , we have  $T^3(G - T) \geq 2$  as desired. Thus we may assume  $V(T) \subseteq V(H_2^z)$ . Note that  $T$  contains one of  $z_1$  and  $z_2$ , since otherwise  $H_1 - xy$  is a kite in  $G - T$ , a contradiction. Without loss of generality, let  $z_1 \in V(T)$ . Let  $T'$  be the triangle in  $H_2$  whose vertex set is  $V(T) \setminus \{z_1\} \cup \{z\}$ . Consider  $H_2 - T'$ . By minimality, we have two possibilities: either  $T^3(H_2 - T') \geq 2$ , or  $H_2 - T'$  contains a kite. First suppose  $T^3(H_2 - T') \geq 2$ . Then since  $z \in V(T')$ , it follows that there are two vertex-disjoint triangles in  $H_2^z - T$ , as desired. Therefore  $H_2 - T'$  contains a kite,  $D$ ; but then  $D$  exists in  $G - T$ , a contradiction.

**Subcase 2 :**  $T^3(H_2) = 2$ .

Again up to relabelling  $z_1$  with  $z_2$ , we may assume  $z_1 \in V(T)$ , as otherwise the  $G - T$  contains the kite  $H_1 - e$ . By Lemma 4.8, either  $H_2 = M$ , or  $H_2$  contains two vertex-disjoint kites. First consider the case where  $V(T) \subseteq V(H_1)$ . If there is a kite in  $H_2^z$ , then  $G - T$  contains a kite and we are done. Thus we may assume that  $H_2^z$  does not contain a kite, and so that  $H_2 = M$  and  $z$  is the unique vertex of degree four in  $H_2$ . But in this case there are two vertex-disjoint triangles in  $H_2 - z$ , which implies that  $T^3(G - T) \geq 2$ , as desired.

Thus for the remainder of the analysis, we assume that  $V(T) \subseteq V(H_2^z)$ , and  $z_1 \in V(T)$ . Let us deal with the case where  $H_2 = M$  first. Suppose that  $z$  is the unique vertex of degree four in  $M$ . Since  $z_1 \in V(T)$ , we have that  $H_2^z - T$  contains a triangle not using  $z_2$ . Since  $H_1 - z_1$  also contains a triangle it follows that  $T^3(G - T) \geq 2$ , as desired. Thus  $z$  is not the unique vertex of degree four in  $M$ . If  $z$  is any of the vertices in  $M$  incident to a spar of a kite, then either  $T^3(G) = 2$ , a contradiction, or for any triangle intersecting  $z_1$  in  $H_2^z$ , there is a triangle in  $H_2^z - T - z_2$ . Thus it follows that  $T^3(G - T) \geq 2$  by using the triangle in  $H_1 - xy$  which contains  $z_2$ . If  $z$  is either of the other two vertices of degree three, we have a kite in  $H_2^z - T$ , and hence there is a kite in  $G - T$ .

Therefore  $H_2 \neq M$ , and so by Lemma 4.8 we have that  $H_2$  contains two vertex-disjoint kites  $D_1$  and  $D_2$ . Without loss of generality, we may assume  $V(D_1) \subseteq V(H_2 - z)$ . We claim that no vertex in  $D_1$  incident with the spar is contained in  $T$ . To see this, suppose not, and let  $v$  be a vertex incident with the spar of  $D_1$ . If  $v$  is in  $T$ , then since all neighbours of  $v$  are in  $D_1$  and  $z$  is in  $T$ , it follows that  $v$  is adjacent to  $z$ . But then  $z$  is in  $D_1$ , contradicting the definition of  $D_1$ . Moreover, we claim at most one vertex of  $D_1$  is contained in  $T$ . If the two vertices in  $D_1$  which are not incident to the spar of  $D_1$  are in  $T$ , then  $G$  contains a  $K_4$  subgraph, which implies  $G = K_4$ , a contradiction. It follows that we have  $T^3(H_2 - T) \geq 1$ . Hence using one of the triangles in  $H_1 - xy$ , we see that  $T^3(G - T) \geq 2$ , as desired.

**Case 2:**  $H_2 = K_4$ .

Note in this case  $H_1 \neq K_4$  as otherwise  $T^3(G) = 2$ .

**Subcase 1:**  $T^3(H_1) = 3$ .

In this case,  $T^3(H_1 - xy) = 2$  since otherwise Proposition 4.4 implies  $T^3(G) = 4$ . It follows that every 3-clique packing of  $H_1$  contains a triangle using the edge  $xy$ , and thus there are two vertex-disjoint triangles in  $H_1 - x - y$ . If  $V(T) \subseteq V(H_2^z)$ , then since  $T^3(H_1 - x - y) = 2$ , it follows that  $T^3(G - T) \geq 2$  as desired. Thus  $V(T) \subseteq V(H_1)$ . Now consider  $H_1 - T$ . By minimality, we have two possibilities: either  $T^3(H_1 - T) \geq 2$ , or  $H_1 - T$  contains a kite. If  $H_1 - T$  contains two vertex-disjoint triangles, then  $H_1 - T - xy$  contains at least one triangle. Since  $H_2 - \{z_1, z_2\}$  contains a triangle, we see that  $T^3(G - T) \geq 2$  as desired. Otherwise,  $H_1 - T$  contains a kite  $D$ . Thus either  $xy$  is the spar of  $D$ , or  $T^3(H_1 - T - xy) \geq 1$ . If  $T^3(H_1 - T - xy) \geq 1$ , then again using the triangle in  $H_2 - \{z_1, z_2\}$  we see that  $T^3(G - T) \geq 2$ . Thus  $xy$  is the spar of  $D$ . In this case,  $V(T) \subseteq V(H_1 - x - y)$  as  $x$  and  $y$  both do not lie in a triangle. But then  $G - T$  contains the kite in  $H_2^z$ .

**Subcase 2:**  $T^3(H_1) = 2$ .

Suppose first that  $H_1 = M$ . If  $xy$  is not incident to the unique vertex of degree four, then either there is a kite in  $H_1 - xy$  that does not contain  $x$  or  $y$ , or  $H_1 - xy$  contains two edge-disjoint kites. First suppose there is a kite in  $H_1 - xy$  that does not contain  $x$  or  $y$ . Observe there is a kite in  $H_2^z$ . Since either  $V(T) \subseteq V(H_1)$  or  $V(T) \subseteq V(H_2^z)$ , by the structure of the Moser spindle it follows that  $G - T$  contains a kite for any  $T$ . Now consider the case where  $H_1 - xy$  contains two edge-disjoint kites. Since  $T$  does not contain both  $x$  and  $y$ ,  $T$  contains the unique vertex of degree four in  $M$ . Otherwise,  $H_1 - xy - T$  contains a kite. But then  $H_1 - xy - T$  contains a triangle using (say)  $z_1 = x$ . As  $H_2^z - z_1$  contains a triangle, we have that  $T^3(G - T) \geq 2$ , as desired.

So we may assume that  $xy$  is incident to the unique vertex of degree four in  $H_1 = M$ . Note that in this case, either up to relabeling  $d(z_1) = 5$  and  $d(z_2) = 3$ , or  $d(z_1) = d(z_2) = 4$ . If  $d(z_1) = 5$ , then  $T$  contains  $z_1$ : otherwise,  $G - T$  contains a kite. If  $T \subseteq H_1 - xy$ , then since  $H_1 - xy$  contains a triangle disjoint from  $T$  and  $H_2 - z$  contains a triangle, it follows that  $T^3(G - T) = 2$ , as desired. If on the other hand  $T \subseteq H_2^z$ , then since  $T^3(H_1 - xy - z_1) \geq 2$ , again it follows that  $T^3(G - T) \geq 2$ . Thus we may assume  $d(z_1) = d(z_2) = 4$ . But then  $G$  contains two vertex-disjoint kites  $K_1$  and  $K_2$ , and no triangle in  $G$  intersects both  $K_1$  and  $K_2$ . Thus  $G - T$  contains a kite, as desired.

Therefore by Lemma 4.8 we may assume that  $H_1 \neq M$ , and so that  $H_1$  contains two vertex-disjoint kites  $D_1$  and  $D_2$ . Up to relabelling, let  $z_1$  be in the kite in  $H_2^z$ . First suppose  $V(T) \subseteq V(H_2^z)$ . Note there is kite in  $H_1 - xy$  not using  $z_1$ . Since  $z_1 z_2 \notin E(G)$ , we have that  $z_2 \notin V(T)$ . Thus  $H_1 - xy - T = H_1 - xy - z_1$ , and so  $G - T$  contains at least one of the kites  $D_1$  and  $D_2$ . Therefore we may assume that  $H_1 \neq M$ . Up to relabeling, let  $z_1$  be in the kite in  $H_2^z$ . First suppose  $T \subseteq H_1 - xy$ . Note then that  $z_1 \in V(T)$ , since otherwise  $G - T$  contains the kite in  $H_2^z$ . Thus  $H_1 - xy - T = H_1 - T$ , since  $xy$  is incident with a vertex in  $T$ . By Observation 4.2,  $H_1 - T$  contains a triangle. Since  $H_2 - z$  also contains a triangle, it follows that  $T^3(G - T) \geq 2$ , as desired. Thus we may assume  $T \subseteq H_2^z$ . By Lemma 4.8, since  $H_1 \neq M$ ,  $H_1$  contains two vertex-disjoint kites. But then  $H_1 - z_1 \subseteq H_1 - T$  contains a kite, as desired.



**Case 3: Neither  $H_1$  nor  $H_2$  is  $K_4$ .**

**Subcase 1:  $T^3(H_1) = 2$  and  $T^3(H_2) = 2$ .**

Note that by Lemma 4.8,  $H_1$  and  $H_2$  contains two edge-disjoint kites. If  $T \subseteq H_2^z - z_1 - z_2$ , then  $H_1 - xy$  (and therefore  $G - T$ ) contains a kite, as desired. Moreover, if  $T \subseteq H_1 - z_1 - z_2$ , then either  $H_2^z$  (and therefore  $G - T$ ) contains a kite, or  $H_2 = M$  and  $z$  is the unique vertex of degree four in  $M$ , in which case  $T^3(G - T) \geq T^3(H_2 - z) \geq 2$ . Thus we may assume that  $T$  contains one of  $z_1$  and  $z_2$ : up to relabeling, suppose  $T$  contains  $z_1$ . If  $T \subseteq H_1 - xy$ , then  $H_1 - xy - T = H - T$ . By Observation 4.2,  $H_1 - xy - T$  (and therefore  $G - T$ ) contains a triangle. Since  $T^3(H_2) = 2$ , there is a triangle in  $H_2 - z$ , and so  $T^3(G - T) \geq 2$  as desired. If, on the other hand,  $T \subseteq H_2^z$ , then since  $T^3(H_1) = 2$ , again we have  $T^3(H_1 - z_1) \geq 1$ . Since  $z_1 \in T$ , it follows that  $H_2 - T \subseteq H_2^z - z_1 - z_2$ . By Observation 4.2,  $H_1 - T$  (and thus  $H_2^z - z_1 - z_2$ ) also contains a triangle. Thus  $T^3(G - T) \geq 2$ , as desired.

**Subcase 2:  $T^3(H_1) = 3$ .**

Note  $T^3(H_2) = 2$ , since  $H_2 \neq K_4$  by assumption and as noted prior to Case 1, if  $T^3(H_1) \geq 3$ , then  $T^3(H_2) < 3$ . Suppose first that every 3-clique packing of  $H_1$  uses the edge  $xy$ . Then  $H_1 - x - y$  has a 3-clique packing of size two, and so  $T \subseteq H_1 - xy$  as otherwise  $T^3(G - T) \geq 2$  and we are done. Similarly, if there is a 3-clique packing of  $H_1$  that does not use the edge  $xy$ , then  $T^3(H_1 - xy) = 3$ , and so again  $T \subseteq H_1 - xy$ , as otherwise  $T^3(G - T) \geq 2$  and we are done (since  $T$  contains at most one of  $x$  and  $y$ ). Since  $T^3(H_2) = 2$ , it follows from Lemma 4.8 that  $H_2$  contains two edge-disjoint kites  $D_1$  and  $D_2$ . Thus either  $H_2^z$  contains a kite that does not contain  $z_1$  or  $z_2$  (and so  $G - T$  contains this kite), or  $z \in D_1 \cap D_2$ . In this case,  $H_2 = M$ , and  $z$  is the unique vertex of degree four in  $M$ . But then  $H_2^z - z_1 - z_2$  contains a 3-clique packing of size two, and since  $T \subseteq H_1 - xy$ , it follows that  $T^3(G - T) \geq 2$  as desired.

**Subcase 3:  $T^3(H_2) = 3$ .**

Then  $T^3(H_2^z - \{z_1, z_2\}) \geq 2$ . It follows that  $V(T) \subseteq V(H_2^z)$ , as otherwise  $T^3(G - T) \geq 2$ . Note that since  $H_1 \neq K_4$  by assumption,  $T^3(H_1) \neq 1$ . Furthermore, as noted prior to Case 1,  $T^3(H_1) < 3$ . Thus  $T^3(H_1) = 2$ . By Lemma 4.8, it follows that either  $H_1 = M$ , or that  $H_1$  contains two vertex-disjoint kites. Suppose first  $H_1 = M$ . Then either  $H_1 - xy - T$  contains a kite, or  $T^3(H_1 - xy - T) = 2$ , since  $T \subset H_2^z$ . (To see this, note that since  $T \subseteq H_2^z$  and  $T$  contains at most one of  $x$  and  $y$ , removing the edge  $xy$  and  $T$  from  $H_1$  amounts to deleting one edge and at most one of its incident vertices.) Thus we may assume  $H_1 \neq M$ , and so that  $H_1$  contains two vertex-disjoint kites  $D_1$  and  $D_2$ . But then  $H_1 - xy - T$  contains a kite (since removing  $xy$  and  $T$  from  $H_1$  again amounts to deleting an edge and at most one of its incident vertices, and  $D_1$  and  $D_2$  are vertex-disjoint).  $\square$

**Definition 4.11.** Let  $G$  be a 4-Ore graph with  $T^3(G) = 3$ . An edge  $f$  is *foundational* if both  $T^3(G - f) = 2$  and there is no kite in  $G - f$ .

**Lemma 4.12.** *Let  $G$  be a 4-Ore graph with  $T^3(G) = 3$ . Then there is at most one foundational edge in  $G$ . Moreover, if  $f$  is a foundational edge, then  $f$  is the spar of a kite.*

*Proof.* Suppose not and let  $G$  be a vertex-minimum counterexample. As  $G \neq K_4$ ,  $G$  is the Ore composition of two 4-Ore graphs  $H_1$  and  $H_2$ . Up to relabelling, let  $H_1$  be the edge side



of the composition where we delete the edge  $xy$  and  $H_2$  the split side of the composition where we split  $z$  into two vertices  $z_1$  and  $z_2$ , and identify  $z_1$  with  $x$  and  $z_2$  with  $y$ . Let  $f$  be an edge in  $G$ .

**Case 1:**  $H_1 = K_4$ .

Suppose  $f$  is foundational. Observe that if  $f \notin E(H_1)$ , then  $f$  is not foundational since there is a kite left over after deleting  $f$ . If  $T^3(H_2^z) = 3$ , then since  $f \in E(H_1)$  we have that  $T^3(G - f) = 3$ , a contradiction. Thus,  $T^3(H_2^z) = 2$ , and so  $T^3(H_2) \leq 3$ . Further,  $T^3(H_2) \geq 2$ , since if  $H_2 = K_4$ , then  $G = M$  and  $T^3(M) = 2$ .

If  $T^3(H_2) = 3$ , then there are two vertex-disjoint triangles in  $H_2 - z$ , say  $T_1$  and  $T_2$ . If  $f$  is not the spar of the kite  $H_1 - xy$ , then  $H_1 - xy - f$  contains a triangle, so it follows that  $T^3(G - f) \geq 3$ . Therefore in this case there is at most one foundational edge, and if there is a foundational edge, it is the spar of a kite.

Thus we may assume  $T^3(H_2) = 2$ . By Lemma 4.8, we have that  $H_2$  contains two edge-disjoint kites that share at most one vertex. If  $H_2^z$  contains a kite, then  $G - f$  contains a kite, and so  $f$  is not foundational. Thus by Lemma 4.8, the two copies are not vertex-disjoint, and so  $H_2 = M$ , and further  $z$  is the vertex of degree four in  $M$ . But for any split of  $z$  into  $z_1$  and  $z_2$ , we get that  $T^3(H_2^z - z_1 - z_2) = 2$ . Thus if  $f$  is not the spar in  $H_1 - xy$ ,  $G - f$  has a 3-clique packing of size three, a contradiction.

**Case 2:**  $H_2 = K_4$ .

By possibly relabelling, let  $z_1$  be the vertex of degree two in  $H_2^z$  resulting from the split of  $z$ . Notice that splitting  $K_4$  leaves a kite subgraph, and hence as  $f$  is foundational,  $f$  is in  $E(H_2)$ . Furthermore,  $f$  is not incident with  $z_1$  or  $z_2$ , as otherwise  $T^3(G - f) = T^3(G) = 3$ .

Note that  $T^3(H_1) = 3$ , since if  $T^3(H_1) = 2$  then by Lemma 4.8  $H_1$  contains two edge-disjoint copies of kites, and thus  $H_1 - xy$  contains at least one kite, contradicting that  $f$  is foundational. Hence  $T^3(H_1) = 3$ . Observe that  $T^3(H_1 - xy) = 2$  and there exists a 3-clique packing of  $H_1 - xy$  which does not use  $x$  or  $y$ . To see this: if  $T^3(H_1 - xy) = 3$ , then consider any 3-clique packing of  $H_1 - xy$ . This clique packing combined with the triangle in  $H_2 - z$  gives four vertex-disjoint triangles, contradicting that  $T^3(G) = 3$ . Thus every 3-clique packing of  $H_1$  uses  $xy$ , and hence there exists a 3-clique packing of  $H_1 - xy$  which does not use  $x$  or  $y$ . Therefore if  $f$  is not the spar in the kite contained in  $H_2^z$ ,  $G - f$  contains three vertex-disjoint triangles.

**Case 3: Neither  $H_1$  nor  $H_2$  is  $K_4$ .**

Note that either  $T^3(H_1) = 2$  or  $T^3(H_2) = 2$ , as otherwise  $T^3(G) \geq 4$  by Proposition 4.4.

First suppose both  $T^3(H_1) = 2$  and  $T^3(H_2) = 2$ . Then by Lemma 4.8, in both  $H_1$  and  $H_2$  there are two edge-disjoint kites which share at most one vertex. Hence there is a kite in  $H_1 - xy$ . If  $f \in E(H_2)$ , then  $G - f$  thus contains a kite, contradicting that  $f$  is foundational. Therefore  $f \in E(H_1)$ . If  $H_2^z$  contains a kite, then  $G - f$  contains a kite, and thus in this case  $G$  contains no foundational edge. It follows that the kites in  $H_2$  were not vertex-disjoint, and hence that  $H_2 = M$ , and  $z$  is the unique vertex of degree four in  $M$ . Thus  $T^3(H_2^z - \{z_1, z_2\}) = 2$ . Moreover, since  $H_1 - xy$  contains a kite, if  $f$  is not the spar of the kite in  $H_1 - xy$  then  $T^3(G - f) = 3$ , contradicting that  $f$  is foundational.

Now suppose that  $T^3(H_1) = 3$ . Since  $T^3(G) = 3$ , this implies that  $T^3(H_2) = 2$ . Then by Lemma 4.8, there are two edge-disjoint kites in  $H_2$  which share at most one vertex. If there is no kite in  $H_2^z$ , then  $H_2 = M$  and  $z$  is the unique vertex of degree four in  $M$ . But then  $T^3(H_2^z - z_1 - z_2) = 2$ , which implies that  $T^3(G) \geq 4$ , a contradiction. Thus there is a kite in  $H_2^z$ . If both of the edge-disjoint kites in  $H_2$  are in  $H_2^z$ , then  $G - f$  contains a kite for all edges  $f$ , and hence  $G$  has no foundational edge. Therefore  $H_2^z$  contains exactly one kite, and this kite does not contain either  $z_1$  nor  $z_2$ . If  $f$  does not lie in this kite, then  $G - f$  contains a kite. If  $f$  is not the spar, then  $T^3(H_2^z - f - z_1 - z_2) \geq 1$  and so  $T^3(G - f) \geq 3$ , contradicting that  $f$  is foundational. Thus there is at most one foundational edge, and it is the spar of a kite.

Now suppose that  $T^3(H_2) = 3$ . Thus  $T^3(H_2^z - z_1 - z_2) \geq 2$ . Since  $T^3(G) = 3$ , this implies that  $T^3(H_1) = 2$ . Thus  $H_1$  contains two edge-disjoint kites which share at most one vertex. Thus  $H_1 - xy$  contains a kite. If  $f$  does not lie in this kite, then  $f$  is thus not foundational. Moreover, if  $f$  is not the spar of this kite, then  $T^3(G - f) \geq 3$  by using two triangles in  $H_2 - z$  and a triangle from  $H_1 - xy - f$ . Hence there is at most one foundational edge, and it is the spar in a kite.  $\square$

**Lemma 4.13.** *Let  $G$  be 4-Ore with  $T^3(G) = 3$ . Let  $v \in V(G)$ , and let  $G^v$  be obtained from  $G$  by splitting  $v$  into two vertices of positive degree  $v_1$  and  $v_2$  with  $N(v_1) \cup N(v_2) = N(v)$  and  $N(v_1) \cap N(v_2) = \emptyset$ . Then one of the following occurs:*

- (i)  $T^3(G^v) \geq 3$ ,
- (ii)  $G^v$  contains a kite,
- (iii) there is an  $i \in \{1, 2\}$  such that  $d(v_i) = 1$ , and the edge incident to  $v_i$  is foundational in  $G$ .

*Proof.* Suppose not. Let  $v \in V(G)$ , and suppose that  $G$  is a vertex-minimum counterexample. As  $G \neq K_4$ ,  $G$  is the Ore composition of two 4-Ore graphs  $H_1$  and  $H_2$ . Up to relabelling, let  $H_1$  be the edge side of the composition where we delete the edge  $xy$  and  $H_2$  the split side of the composition where we split  $z$  into two vertices  $z_1$  and  $z_2$ , and identify  $z_1$  with  $x$  and  $z_2$  with  $y$ . Note that at least one of  $H_1$  or  $H_2$  is not  $K_4$ , as otherwise  $G = M$  and  $T^3(M) = 2$ .

**Case 1:**  $H_1 = K_4$ .

Observe that  $H_1 - xy$  is a kite, so if  $v \notin V(H_1)$ , then  $G^v$  contains a kite and so (ii) holds. Hence we assume that  $v \in V(H_1)$ .

Suppose  $T^3(H_2) = 3$ . Observe that there are two vertex-disjoint triangles in  $H_2 - z$ . If the split of  $H_1 - xy$  contains a triangle, then  $T^3(G^v) \geq 3$  and (ii) holds. Notice if we split either  $x$  or  $y$ , then  $H_1 - xy$  contains a triangle, so we can assume that  $v \in V(H_1) - \{x, y\}$ . Let  $w$  and  $v$  be the two vertices in  $V(H_1) - \{x, y\}$ . Observe there is exactly one way to split  $v$  into  $v_1$  and  $v_2$  so that there is no triangle left over in  $H_1^v - xy$ . That is, up to relabelling  $v_1$  to  $v_2$ , to have  $v_1$  adjacent to  $w$ , and  $v_2$  adjacent to  $x$  and  $y$ . To finish, notice

that the number of vertex-disjoint triangles in  $G^v$  after performing such a split is the same as the number of vertex-disjoint triangles in  $G - vw$ . Hence if  $T^3(G - vw) \geq 3$ , we have  $T^3(G^v) = 3$  and thus (i) holds. So  $T^3(G - vw) = 2$ , and further  $G^v$  has no kite subgraph, which implies that  $G - vw$  does not contain a kite. Thus  $vw$  is foundational, and so (iii) holds.

Therefore we can assume that  $T^3(H_2) = 2$ . If there is a kite in  $H_2^z - z_1 - z_2$ , then  $G$  contains two vertex-disjoint kites, and thus there is a kite in  $G^v$ . Thus by Lemma 4.8, we have that  $G = M$  and  $z$  is the unique vertex of degree four in  $M$ . Then  $T^3(H_2^z - z_1 - z_2) = 2$ . Therefore we can assume that  $v \notin \{x, y\}$  as otherwise  $T^3(G^v) \geq 3$  and (i) holds. Let  $w, v$  be the two vertices in  $H_1 - x - y$ . By the same argument as in the  $T^3(H_2) = 3$  case, there is exactly one split so that  $T^3(G^v) \leq 2$ , and in this case, we split  $v$  into two vertices  $v_1, v_2$  where without loss of generality,  $d(v_1) = 1$ , and  $v_1$  is incident to a foundational edge in  $G$ . In this case, (iii) holds.

**Case 2:**  $H_2 = K_4$ .

Throughout this case, without loss of generality let  $z_1$  have degree two in  $H_2^z$  and  $z_2$  have degree one in  $H_2^z$ . Observe that  $H_2^z - z_2$  contains a kite, so if  $v \notin V(H_2^z) - z_2$ , then  $G^v$  contains a kite and (ii) holds. If  $T^3(H_1) = 3$ , then  $T^3(H_1 - x) \geq 2$ , and it follows that if  $v = z_1$ , then  $T^3(G^v) = 3$  and (i) holds, as desired. Therefore we can assume that  $v \in V(H_2) - \{z_1, z_2\}$ . If  $v$  is incident to  $z_2$ , then there is a triangle in  $H_2^z$  after splitting  $v$ . Since  $T^3(H_1 - x) \geq 2$ , we get  $T^3(G^v) \geq 3$  and (i) holds. If  $v$  is either of the other two possible vertices, the only split which does not leave a triangle is one where up to relabelling  $v_1$  has degree one, and is incident to the spar of the kite in  $H_2^z$ . Let  $f$  be this edge. Then if  $T^3(G - f) \geq 3$ , we have  $T^3(G^v) \geq 3$ , so  $T^3(G - f) = 2$ , and  $G - f$  does not contain a kite as otherwise  $G^v$  contains a kite (satisfying (ii)), Hence  $f$  is foundational in  $G$ , and thus  $v_1$  is incident to the foundational edge in  $G$ . Thus (iii) holds, as desired.

Therefore we can assume that  $T^3(H_1) = 2$ . Then by Lemma 4.8, either  $H_1 = M$  or  $G$  contains two vertex-disjoint kites. If  $H_1 - x - y$  contains a kite, then there are two vertex-disjoint kites in  $G$ , and hence  $G^v$  contains a kite, satisfying (ii). Thus  $H_1 = M$ , and  $T^3(H_1 - xy) = 2$ . Then if we split  $z_1$ , observe we have  $T^3(G^v) \geq 3$  (thus (i) holds), and if we split  $z_2$ , we have a kite in  $G^v$  (and so (ii) holds). If  $v$  is incident to  $z_2$  in  $H_2^z$ , then observe that any split of  $v$  results in a triangle, and hence  $T^3(G^v) \geq 3$  and (i) holds. Thus  $v$  is a vertex in  $H_2^z$  incident to a spar of the kite. By the same arguments as the case when  $T^3(H_1) = 3$ , the only split of such a vertex that does not leave a triangle results in, up to relabelling,  $v_1$  having degree one and being incident to a foundational edge in  $G$ . But then (iii) holds, as desired.

**Case 3:**  $T^3(H_1) = 2$ .

By the previous cases, we may assume that  $H_2 \neq K_4$ . Thus  $T^3(H_2) \geq 2$ . By Lemma 4.8, either  $H_1 = M$  or  $H_1$  contains two vertex-disjoint kites. In either case, there is a kite subgraph in  $H_1 - xy$ . Let  $L$  be such a subgraph. Then  $v \in V(L)$ , as otherwise  $G^v$  contains a kite subgraph, satisfying (ii).

First suppose that  $T^3(H_2) = 3$ . If we split  $v$  and there is still a triangle left in  $H_1^v - xy$ , then as  $T^3(H_2 - z) \geq 2$ , we have  $T^3(G^v) \geq 3$ . Hence (i) holds. Therefore if we split  $v$ , there

is no triangle left in  $H_1^v - xy$ , and by the same arguments as in previous cases, this implies that  $v$  is incident to the spar of  $L$ , and we split  $v$  in such a way that up to relabelling  $v_1$  is incident to the spar of the kite and has degree one in  $G^v$ . Further, the spar of  $L$  is foundational, satisfying (iii); otherwise, such a split satisfies at least one of (i) and (ii).

Thus we may assume that  $T^3(H_2) = 2$ . First suppose that  $T^3(H_2 - z) = 2$ . Then if we split  $v$  in  $L$  and are left with a triangle,  $T^3(G^v) \geq 3$  and (i) holds. Thus by the same arguments as in previous cases,  $v$  is incident to the spar of  $L$ , and we split  $v$  in such a way that up to relabelling  $v_1$  is incident to the spar of the kite and has degree one in  $G^v$ . Further, the spar of  $L$  is foundational, satisfying (iii); otherwise, such a split satisfies at least one of (i) and (ii). Thus  $T^3(H_2 - z) = 1$ . By Lemma 4.8, since  $T^3(H_2) = 2$  either  $H_2 = M$  or  $H_2$  has two vertex-disjoint kites. As  $T^3(H_2 - z) = 1$ , this implies that  $z$  is incident to a spar of a kite. But then regardless of whether  $H_2 = M$  or  $H_2$  has two vertex-disjoint kites, we have that there is a kite in  $H_2^z$  that does not contain either of  $z_1$  or  $z_2$ . But then  $G^v$  contains a kite, satisfying (ii).

**Case 4:**  $T^3(H_2) = 2$ .

From the previous cases, we may assume that  $T^3(H_1) = 3$ . Then  $T^3(H_1 - xy) \geq 2$ , and it equals two only if every 3-clique packing contains a triangle which uses the edge  $xy$ . Note by Proposition 4.4, if  $T^3(H_1 - xy) \geq 3$ , then  $T^3(G) \geq 4$ , a contradiction. Hence there are two disjoint triangles in  $H_1 - xy$  which do not use  $x$  or  $y$ . Therefore  $T^3(H_2^z) = 1$ , as otherwise by Proposition 4.4,  $T^3(G) = 4$ . By appealing to Lemma 4.8, this implies that there is a kite  $L$  in  $H_2^z$  that does not contain  $z_1$  or  $z_2$ . If  $v \notin V(L)$ , then  $G^v$  contains a kite, satisfying (ii). If splitting  $v$  leaves a triangle, then  $T^3(G^v) \geq 3$  and so (i) holds. Let  $w, v$  be the two vertices of degree three in  $L$ . It follows that up to relabelling, after splitting we have  $v_1w \in E(G^v)$  and  $v_2$  is incident to the other two edges of  $v$ . Thus  $d(v_1) = 1$ . Note that if  $vw$  is not foundational, then this split satisfies one of (i) and (ii) and we are done. Hence  $vw$  is foundational, and thus (iii) holds.

**Case 5: Both  $T^3(H_1) = 3$  and  $T^3(H_2) = 3$ .**

Then by Proposition 4.4,  $T^3(G) \geq 3 + 3 - 2 = 4$ , a contradiction. □

### 4.3 Properties of graphs in $\mathcal{B}$

In this section we prove similar lemmas as in Section 4.2 except now we focus on graphs in  $\mathcal{B}$ . We recall the definition of  $\mathcal{B}$ . The graph  $T_8$  is in  $\mathcal{B}$ , and given a graph  $G \in \mathcal{B}$  and a 4-Ore graph  $H$ , the Ore composition  $G'$  of  $G$  and  $H$  is in  $\mathcal{B}$  if  $T^3(G') = 2$ . We start off by proving that the potential of graphs in  $\mathcal{B}$  is in fact  $-1$ .

Let the *Kostochka-Yancey potential* of a graph  $G$  be  $KY(G) = 5v(G) - 3e(G)$ . The following observation is immediate from the definition of Ore composition.

**Observation 4.14.** Let  $G$  be the Ore composition of two graphs  $H_1$  and  $H_2$ . Then  $v(G) = v(H_1) + v(H_2) - 1$ ,  $e(G) = e(H_1) + e(H_2) - 1$ , and  $KY(G) = KY(H_1) + KY(H_2) - 2$ .

**Corollary 4.15.** Let  $G \in \mathcal{B}$ . Then  $KY(G) = 1$ , and  $p(G) = -1$ .

*Proof.* Let  $H$  be a vertex-minimum counterexample. If  $H = T_8$ , then  $v(T_8) = 8$  and  $e(T_8) = 13$ , and thus  $5 \cdot 8 - 13 \cdot 3 = 1$ . Now suppose  $H$  is the Ore composition of two graphs  $H_1$  and  $H_2$ . Without loss of generality,  $H_1 \in \mathcal{B}$ , and  $H_2$  is 4-Ore. By minimality, it follows that  $KY(H_1) = 1$ . Note that by Theorem 1.50,  $KY(H_2) = 2$ . Then by Observation 4.14, it follows that  $KY(H) = 1 + 2 - 2 = 1$ , as desired.

Since  $KY(G) = 1$  and  $T^3(G) = 2$ , it follows that  $p(G) = -1$ , as desired.  $\square$

We overload the terminology.

**Definition 4.16.** Given a graph  $G \in \mathcal{B}$ , an edge  $e \in E(G)$  is *foundational* if  $T^3(G - e) = 1$  and there is no  $K_4 - e$  subgraph in  $G - e$ .

Note in this definition we enforce that  $G - e$  contains no  $K_4 - e$  subgraph. Such a subgraph may not be a kite.

**Lemma 4.17.** *If  $G$  is a graph in  $\mathcal{B}$ , then  $G$  contains at most one foundational edge. Further, if  $G$  is not  $T_8$  and  $G$  contains a foundational edge, then this edge is the spar of a kite.*

*Proof.* Suppose not. First suppose that  $G = T_8$ . Observe that the edge  $u_1u_2$  is the only foundational edge in  $G$ . Hence we may assume that  $G$  is the Ore composition of a 4-Ore graph  $H_1$  and a graph  $H_2$  in  $\mathcal{B}$ . Let  $f$  be an edge in  $G$ . Note that  $T^3(H_1) \leq 2$ , as otherwise  $T^3(G) \geq 3$  by Proposition 4.4.

**Case 1:**  $H_1 = K_4$ .

Suppose that  $H_1$  is the edge side where we delete the edge  $xy$ , and we split a vertex  $z$  in  $H_2$  into two vertices  $z_1$  and  $z_2$ . Note that if  $f$  is not in  $H_1 - xy$ , then  $G - f$  contains a kite. Hence  $f$  lies in  $E(H_1 - xy)$ . As  $H_1 - xy$  is a kite, if we delete any edge that is not the spar of this kite, we have  $T^3(H_1 - xy - f) \geq 1$ . Further, there is at least one triangle in  $H_2$  which does not use  $z$ , and hence  $T^3(G - f) \geq 2$ . Thus the only possible foundational edge is the spar of a kite, as desired.

Now suppose that  $H_1$  is the split side where we split  $z$  into  $z_1$  and  $z_2$ . Then  $H_1^z$  contains a kite. If  $f$  does not lie in this kite,  $G - f$  contains a kite as desired. If  $f$  is not the spar of the kite, then  $T^3(H_1^z - f) \geq 1$ , and since there is a triangle in  $H_2 - xy$ , we see that  $T^3(G - f) \geq 2$ , as desired. Hence there is at most one foundational edge, and if there is a foundational edge it is the spar of a kite.

**Case 2:**  $T^3(H_1) = 2$ .

By Lemma 4.8, either  $H_1 = M$  or  $H_1$  contains two vertex-disjoint kites. First suppose that  $H_1$  is the edge side of the composition, where we delete the edge  $xy$ . By Proposition 4.4 and the fact that  $T^2(G) = 2$ , every 3-clique packing of  $H_1$  contains a triangle using the edge  $xy$ . Thus regardless of whether  $H_1 = M$  or not, there is a kite  $L$  in  $H_1 - xy$ . Then  $f$  is in  $L$ , otherwise  $G - f$  contains a kite. If  $f$  is not the spar of the kite in  $L$ , then  $T^3(H_1 - f - xy) \geq 1$ , and since  $T^3(H_2 - z) \geq 1$ , we have that  $T^3(G - f) \geq 2$ . Therefore in this case a foundational edge is the spar of a kite.

Therefore we may assume that  $H_1$  is the split side of the composition where we split a vertex  $z$  into two vertices  $z_1$  and  $z_2$ . Again by Proposition 4.4 and the fact that  $T^2(G) = 2$ , every 3-clique packing of  $H_1$  uses  $z$ . Thus regardless of whether  $H_1 = M$  or not,  $z$  is incident to the spar of a kite in  $H_1$ , and so there is a kite  $L$  in  $H_1^z$  that does not contain either of  $z_1$  or  $z_2$ . Then  $f$  is in  $L$  otherwise  $G - f$  contains a kite. If  $f$  is not the spar of the kite in  $L$ , then  $T^3(H_1 - f - z) \geq 1$ , and thus  $T^3(G - f) \geq 2$ . Otherwise  $f$  is the spar of a kite, and is foundational (otherwise at least one of the other conditions holds).  $\square$

**Lemma 4.18.** *Let  $G \in \mathcal{B}$ , and let  $G^v$  be obtained from  $G$  by splitting a vertex  $v$  into two vertices  $v_1$  and  $v_2$ . Then at least one of the following occurs:*

(i)  $T^3(G^v) \geq 2$ ,

(ii)  $G^v$  contains a  $K_4 - e$  subgraph,

(iii) there is an  $i \in \{1, 2\}$  such that  $d(v_i) = 1$  and the edge incident to  $v_i$  is foundational in  $G$ .

*Proof.* Let  $G$  be a vertex-minimum counterexample. First consider the case where  $G = T_8$ . If we do not split one of  $u_1$  or  $u_2$  we have a  $K_4 - e$  subgraph remaining. If we split either  $u_1$  or  $u_2$  such that (iii) does not hold, then it is easy to see  $T^3(G^v) = 2$  and so (i) holds.

Therefore we can assume that  $G$  is the Ore composition of a graph  $H_1 \in \mathcal{B}$  and a 4-Ore graph  $H_2$ . If  $T^3(H_2) \geq 3$ , then by Proposition 4.4 we have that  $T^3(G) \geq 3 + 2 - 2 \geq 3$  contradicting that  $T^3(G) = 2$ . Hence  $T^3(H_2) \leq 2$ .

**Case 1:**  $H_2 = K_4$ .

Suppose first that  $H_2$  is the split side where we split a vertex  $z$  into two vertices  $z_1$  and  $z_2$ . Then  $H_2^z$  contains a kite, say  $L$ , so if  $v \notin V(L)$ , then  $G^v$  contains a kite and (ii) holds. If  $v$  is not incident to a spar of the kite, then any split of  $v$  results in a triangle, and thus  $T^3(G^v) \geq 2$  and so (i) holds. Therefore  $v$  is incident to the spar of the kite, and further the split of  $v$  must leave up to relabelling  $v_1$  with degree one and incident to the spar of the kite. This edge is foundational, otherwise (i) or (ii) occurs. Thus (iii) holds.

Now suppose that  $H_2$  is the edge side of the composition where we delete the edge  $xy$ . Then  $H_2 - xy$  is a kite. If  $v \notin V(H_2 - xy)$ , then  $G^v$  contains a kite, as desired. If  $v$  is not incident to a spar of a kite, then any split leaves a triangle in  $H_2^v - xy$ , and thus  $T^3(G^v) \geq 2$  and (i) holds. To see this, note that since  $T^3(H_1) = 2$ , it follows that  $T^3(H_1 - z) \geq 1$ . Thus  $v$  must be incident to the spar of  $H_2 - xy$ , and if  $T^3(G^v) = 1$ , then we must have split  $v$  in such a way that up to relabelling,  $d(v_1) = 1$  and is incident to the spar of the kite. Thus (iii) holds.

**Case 2:**  $H_2 \neq K_4$ .

Then  $T^3(H_2) = 2$  by Proposition 4.4. First suppose  $H_2$  is the edge side of the composition where we delete the edge  $xy$ . Then  $T^3(H_2 - xy) = 1$  as otherwise  $T^3(G) \geq 3$  by Proposition 4.4. By Lemma 4.8 either  $H_2 = M$  or there are two vertex-disjoint kites in  $H_2$ . This implies that there is a kite in  $H_2 - xy$ . Let  $L$  be this kite. If  $v \notin V(L)$ , then  $G^v$  contains a kite,



and so (ii) holds. If  $v$  is not incident to a spar of a kite, then any split leaves a triangle in  $H_2^v - xy$ , and thus  $T^3(G^v) \geq 2$  and (i) holds. As in the previous case, this follows from the fact that  $T^3(H_1 - z) \geq 1$ . Thus  $v$  must be incident to the spar of  $H_2 - xy$ , and if  $T^3(G^v) = 1$ , then we must have split  $v$  in such a way that up to relabelling,  $d(v_1) = 1$  and is incident to the spar of the kite. But then (iii) holds.

Therefore we can suppose that  $H_2$  is the split side of the composition, where we split the vertex  $z$  into two vertices  $z_1$  and  $z_2$ . By Proposition 4.4, every 3-clique packing of  $H_2$  uses the vertex  $z$ . Thus regardless of whether  $H_2 = M$  or not,  $z$  is incident to a spar of a kite in  $H_2$ . Thus there is a kite in  $H_2^z$  that does not contain either of  $z_1$  or  $z_2$ . Let  $L$  be this kite. If  $v \notin V(L)$ , then  $G^v$  contains a kite, as desired. If  $v$  is not incident to the spar of  $L$ , then any split leaves a triangle in  $(H_2^z - z_1 - z_2)^v$ , and thus  $T^3(G^v) \geq 2$ , and (ii) holds. Thus  $v$  is incident to the spar of  $L$ , and if  $T^3(G^v) = 1$ , then  $v$  was split in such a way that up to relabelling,  $d(v_1) = 1$  and  $v_1$  is incident to the spar of the kite. But then (iii) holds, as desired.  $\square$

## 4.4 Potential Method

In this section we review the potential method which will be the critical tool for the rest of the chapter.

Let  $H$  and  $G$  be graphs such that  $G$  does not admit a homomorphism to  $H$ . Let  $F$  be an induced subgraph of  $G$  such that  $F$  has a homomorphism to  $H$ . Let  $f : V(F) \rightarrow V(H)$  be a homomorphism. Let  $C_1, \dots, C_t$  be the non-empty colour classes of  $f$  (where a *colour class* is understood here to be a set of vertices in  $G$  which are mapped to the same vertex in  $H$  under  $f$ ).

The *quotient of  $G$  by  $f$*  denoted  $G_f[F]$ , is a graph with vertex set  $(V(G) \setminus V(F)) \cup \{c_i \mid 1 \leq i \leq t\}$  and edge set  $E_1 \cup E_2 \cup E_3$ , where:

- $E_1 = \{uv \mid uv \in E(G[V(G) \setminus V(F)])\}$ ;
- $E_2$  is the set of edges of the form  $c_i c_j$  where there is a  $u \in C_i$  and a  $v \in C_j$  such that  $uv \in E(G)$ ;
- $E_3$  is the set of edges of the form  $uc_i$  such that there is a  $v \in C_i$  where  $uv \in E(G)$ .

We record some easy observations which have been made before (see for example [43, 28, 29]).

**Observation 4.19.** Let  $G$  be a graph with no homomorphism to a graph  $H$ , let  $F$  be a strict induced subgraph of  $G$ , and let  $f : V(F) \rightarrow V(H)$  be a homomorphism. Then  $G \rightarrow G_f[F]$ .

*Proof.* For all  $v \in V(G) \setminus V(F)$ , let  $\phi(v) = v$ . For all  $v \in V(F)$ , let  $C_i$  be the colour class that  $v$  is in with respect to  $f$ , and define  $\phi(v) = c_i$ . We claim that  $\phi$  is a homomorphism. Consider any edge  $uv \in E(G_f[F])$ . If both of  $u$  and  $v$  are in  $V(G) \setminus V(F)$ , then  $\phi(u)\phi(v) = uv \in E(G_f[F])$ . If  $u \in V(F)$  and  $v \in V(G) \setminus V(F)$  then  $\phi(u)\phi(v) = c_i v$  where  $f(u) = i$ . Then by definition  $c_i v \in E(G_f[F])$ . Finally, if both  $u$  and  $v$  are in  $V(F)$ , then  $\phi(u)\phi(v) = c_i c_j$  where  $f(u) = i$  and  $f(v) = j$ . Then as  $f$  is a homomorphism,  $ij \in E(H)$ , and hence  $c_i c_j \in E(G_f[F])$ . Hence  $G \rightarrow G_f[F]$ .  $\square$

**Corollary 4.20.** *Let  $G$  be a graph with no homomorphism to  $H$ ,  $F$  a strict induced subgraph of  $G$ , and  $f$  a homomorphism from  $F \rightarrow H$ . Then  $G_f[F]$  does not admit a homomorphism to  $H$ . In particular, if  $G$  is  $k$ -critical, then  $G_f[F]$  contains a  $k$ -critical subgraph  $W$ , where at least one vertex of  $W$  is in  $V(G) \setminus V(F)$ .*

*Proof.* If  $G_f[F] \rightarrow H$  then as  $G \rightarrow G_f[F]$ , then composing homomorphisms implies that  $G \rightarrow H$ , a contradiction. If  $G$  is  $k$ -critical,  $G \not\rightarrow K_{k-1}$ , and hence  $G_f[F] \not\rightarrow K_{k-1}$ . Thus  $G_f[F]$  contains a  $k$ -critical subgraph  $W$ , and further at least one vertex of  $W$  is in  $V(G) \setminus V(F)$ , as otherwise  $G$  contains a  $k$ -critical subgraph as a strict subgraph, a contradiction.  $\square$

This motivates the following definitions.

**Definition 4.21.** Let  $G$  be a  $k$ -critical graph. Let  $F$  be a strict induced subgraph of  $G$ . Let  $f$  be a  $k$ -colouring of  $F$ . Let  $W$  be a  $k$ -critical subgraph of  $G_f[F]$ . Let  $X$  be the graph induced in  $W$  by the vertices which are not vertices of  $G$ . We will call  $X$  the *source*. Let  $F'$  be the subgraph of  $G$  obtained by taking the induced subgraph in  $G$ , of vertices  $x$ , where  $x \in f^{-1}(y)$  for some  $y \in W \cap X$ , and any vertex  $z \in W \setminus X$  and including all vertices of  $F$ . We say  $F'$  is the *extension* of  $W$  and  $W$  is the *extender* of  $F$ .

The following lemma has effectively been proven before (see [31]).

**Lemma 4.22.** *Let  $F$  be a strict induced subgraph of  $G$  and  $f$  a  $(k-1)$ -colouring of  $F$ . Let  $W$  be a  $k$ -critical graph of  $G_f[F]$ , and let  $F'$  be the extension of  $W$ . Let  $X$  be the source of  $f$ . Then the following hold*

- $v(F') = v(F) + v(W) - v(X)$ ,
- $e(F') \geq e(F) + e(W) - e(X)$ , and
- $T^{k-1}(F') \geq T^{k-1}(F) + T^{k-1}(W \setminus X)$ .

*Proof.* Observe that  $V(F') = V(F) \cup (V(F') \setminus V(F))$ . Additionally,  $V(W) = (V(F') \setminus V(F)) \cup X$ . Thus  $V(F') = (V(F) \cup V(W)) \setminus X$ . Thus  $v(F') = v(F) + v(W) - v(X)$ . From the above identity and the fact that the subgraphs are induced that  $e(F') \geq e(F) + e(W) - e(X)$ . Finally, consider a maximal set of  $k-1$ -cliques both  $F$  and  $W \setminus X$ , say  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Then  $\mathcal{T}_1 \cup \mathcal{T}_2$  is a set of disjoint  $(k-1)$ -cliques in  $F'$ , and has size  $T^{k-1}(F) + T^{k-1}(W \setminus X)$ , which gives the result.  $\square$



We will refer to the next lemma as the potential-extension lemma and will use it frequently.

**Lemma 4.23** (Potential-Extension Lemma). *Let  $F$  be a strict induced subgraph of  $G$  and fix a  $k$ -colouring  $\phi$  of  $F$ . With respect to  $\phi$ , let  $F'$ ,  $W$  and  $X$  be an extension, extender and source of  $F$ . Then for any positive numbers  $a, b$  and  $c$  we have*

$$p(F') \leq p(F) + p(W) - av(X) + be(X) + cT^{k-1}(W) - cT^{k-1}(W \setminus X)$$

and

$$p(F') \leq p(F) + p(W) - av(X) + be(X) + cv(X).$$

*Proof.* Observe that  $T^{k-1}(W) \leq T^{k-1}(W \setminus X) + v(X)$  since every vertex-disjoint  $k-1$ -clique either lies in  $W - X$  or uses a vertex from  $X$ . Thus we have

$$\begin{aligned} p(F') &= av(F') - be(F') - cT^{k-1}(F') \\ &\leq a(v(F) + v(W) - v(X)) - b(e(F) + e(W) - e(X)) - cT^{k-1}(F) - cT^{k-1}(W \setminus X) \\ &= p(F) + p(W) - av(X) + be(X) + cT^{k-1}(W) - cT^{k-1}(W \setminus X) \\ &\leq p(F) + p(W) - av(X) + be(X) + cv(X). \end{aligned}$$

□

## 4.5 Properties of a minimum counterexample

In this section we prove lemmas regarding the structure of a vertex-minimum counterexample. For this entire section, let  $G$  be a vertex-minimum counterexample to Theorem 1.64. Then  $p(G) \geq -1$ .

**Observation 4.24.**  $G$  is not 4-Ore.

*Proof.* Observe that if  $G$  is 4-Ore, then by Theorem 1.60,  $p(G) = 2 - T^3(G)$ . If  $T^3(G) \geq 4$ , then  $p(G) \leq -2$ . All other cases are covered as special cases of Theorem 1.64. □

We note the well-known folklore result (See Fact 12 in [29]).

**Theorem 4.25.** *If  $G$  is  $k$ -critical and has a two-vertex cut  $\{x, y\}$ , then  $G$  is the Ore composition of two graphs  $H_1$  and  $H_2$ .*

**Observation 4.26.**  $G$  is not the Ore composition of two graphs  $H_1$  and  $H_2$ . In other words,  $G$  is 3-connected.

*Proof.* Suppose not: that is, suppose  $G$  is the Ore composition of  $H_1$  and  $H_2$  with  $H_1 \cap H_2 = \{x, y\}$ . From Observation 4.14, we have

$$p(G) = \text{KY}(H_1) + \text{KY}(H_2) - 2 - T^3(G).$$

Note that if both  $H_1$  and  $H_2$  are 4-Ore, then  $G$  is 4-Ore and the claim follows from Observation 4.24.

**Case 1:**  $H_1 = K_4$ .

First suppose that  $H_2 = W_5$ . Then  $T^3(G) \geq 2$  as every split of  $W_5$  contains at least one triangle that does not contain at least one of  $x$  or  $y$ , and deleting any edge of  $W_5$  also leaves at least one triangle that does not contain at least one of  $x$  or  $y$ . Therefore we have  $p(G) \leq 2 + 0 - 2 - 2 = -2$  as desired. Thus  $H_2 \neq W_5$ . Now suppose  $H_2 \in \mathcal{B}$ . Then either  $G$  is in  $\mathcal{B}$ , in which case  $T^3(G) = 2$  and  $p(G) = 2 + 1 - 2 - 2 = -1$  as desired, or  $T^3(G) \geq 3$ , in which case  $p(G) \leq -2$  as desired. Finally we have the case where  $p(H_2) \leq -2$ . Here it follows that  $p(G) \leq p(H_2) \leq -2$  as desired.

**Case 2:**  $H_1$  is 4-Ore with  $T^3(G) = 2$ .

Then  $\text{KY}(H_1) = 2$ . First suppose that  $H_2 = W_5$ . Then as above we have  $T^3(G) \geq 2$ , and hence  $p(G) \leq -2$  as desired. Now suppose  $H_2 \in \mathcal{B}$ . If  $G$  is in  $\mathcal{B}$ , we have  $T^3(G) = 2$ , and  $p(G) = -1$  as desired. Therefore  $G$  is not in  $\mathcal{B}$  and thus  $T^3(G) \geq 3$ , but then  $p(G) \leq -2$  as desired. The last case is when  $p(H_2) \leq -2$ . In this case, by Proposition 4.4 we have  $T^3(G) \geq T^3(H_2) + T^3(H_1) - 2 \geq T^3(H_2)$ , and so  $p(G) \leq p(H_2) \leq -2$  as desired.

**Case 3:**  $H_1$  is  $W_5$ .

Suppose first  $H_2 = W_5$ . Immediately it follows that  $p(G) \leq -4$  as desired. Now suppose that  $H_2 \in \mathcal{B}$ , then  $T^3(G) \geq T^3(H_2) + T^3(W_5) - 1 \geq T^3(H_2) = 2$  so  $p(G) \leq -3$  as desired. Otherwise  $p(H_2) \leq -2$  and  $p(G) \leq p(H_2) \leq -2$  as desired.

**Case 4:**  $H_1$  is in  $\mathcal{B}$ .

If  $H_2 \in \mathcal{B}$ , we have  $p(G) \leq -2$  as desired. If  $p(H_2) \leq -2$  then  $p(G) \leq p(H_2) \leq -2$  as desired.

**Case 5:**  $p(H_1) \leq -2$ .

Note that the previous cases cover all outcomes except when  $p(H_2) \leq -2$ . In this case, it follows that  $p(G) \leq -2$  as desired.  $\square$

**Lemma 4.27.** *If  $F$  is a subgraph of  $G$  with  $v(F) < v(G)$ , then  $p(F) \geq 3$ . Further,  $p(F) \geq 4$  unless one of the following occurs:  $G \setminus F$  is a triangle of degree three vertices, or  $G \setminus F$  is a vertex of degree three, or  $G$  contains a kite.*

*Proof.* Suppose not. Let  $F$  be a counterexample that is maximal with respect to  $v(F)$  and, subject to that, with  $p(F)$  minimized. We may assume that  $F$  is an induced subgraph, as adding edges reduces the potential. Observing that  $p(K_1) = 5$ ,  $p(K_2) = 7$ ,  $p(P_2) = 9$  (where  $P_2$  is the path of length two), and  $p(K_3) = 5$ , we may assume that  $v(F) \geq 4$ . Let  $\phi$  be a 3-colouring of  $F$ , and let  $F', W, X$  be an extension, extender, and source of  $G_\phi[F]$  respectively. If  $F' \neq G$ , then by the potential-extension lemma (Lemma 4.23) we have  $p(F') \leq p(F)$ , which implies that  $F'$  is a larger counterexample. Therefore  $F' = G$ . We split into cases.

Throughout let  $f$  be a function that takes as input the number of vertices of  $X$ , and returns 5 if  $v(X) = 1$ , 7 if  $v(X) = 2$ , and 6 if  $v(X) = 3$ .

**Case 1:**  $W = K_4$ .

First suppose that  $v(X) = 1$ . Then by the potential-extension lemma,  $-1 \leq p(F) + 1 - 5$  which implies that  $p(F) \geq 3$ . If further,  $p(F) \geq 4$ , then we are done, so we can assume that  $p(F) = 3$ . Observe that  $G \setminus F$  contains three vertices, and they must induce a triangle as  $W - X$  is a triangle. Let  $T$  be the triangle in  $G \setminus F$ . Then  $-1 \leq p(G) \leq p(F) + 5 - 3e(T, F)$ , where  $e(T, F)$  is the number of edges with one endpoint in  $T$  and one endpoint in  $F$ . If  $e(T, F) \geq 4$ , then we have  $-1 \leq p(F) - 7$  so  $p(F) \geq 6$ . Hence  $e(T, F) \leq 3$ . But as  $G$  is 4-critical, the minimum degree is three, hence  $e(T, F) \geq 3$  and  $T$  is a triangle of degree three vertices.

If  $v(X) = 2$ , then  $-1 \leq p(F) + 1 - 7 + 1$  which implies that  $p(F) \geq 4$ .

If  $v(X) = 3$ , then  $-1 \leq p(F) + 1 - 6 + 1$  which gives  $p(F) \geq 3$ . If further  $p(F) \geq 4$ , then we are done. So we can assume that  $p(F) = 3$ . Note that  $G \setminus F$  is a single vertex  $v$ . We want to argue that this vertex has degree three. Note that  $-1 \leq p(G) \leq p(F) + 5 - 3d(v) = 8 - 3d(v)$ . As  $G$  is 4-critical,  $d(v) \geq 3$ , and by the inequality,  $d(v) \leq 3$ , hence  $d(v) = 3$ .

**Case 2:**  $W$  is 4-Ore with  $T^3(W) = 2$ .

If  $v(X) = 1$ , then  $-1 \leq p(F) + 0 - 5 + 1$  so  $p(F) \geq 3$ . As  $W$  is 4-Ore with  $T^3(G) = 2$ , by Lemma 4.8 either  $W$  contains two vertex-disjoint kites, or  $W = M$ . If  $W \neq M$ , then  $G$  contains a kite. If  $W = M$ , and the vertex in  $X$  is not the unique degree four vertex in the Moser spindle, then  $G$  contains a kite.

Otherwise  $X$  contains only the unique vertex of degree four in the Moser spindle, and in this case  $T^3(W - X) = 2$ , so from the potential-extension lemma we get  $-1 \leq p(F) + 0 - 5$  which implies that  $p(F) \geq 4$ . Thus it follows that either  $G$  contains a kite or  $p(F) \geq 4$ .

If  $v(X) \in \{2, 3\}$ , then  $-1 \leq p(F) - f(X) + 1$ . To see this, note that by Lemma 4.8,  $W$  contains two edge-disjoint kites that share at most one vertex. It follows from this that  $T^3(W \setminus X) \in \{1, 2\}$ . Since  $f(X) \geq 6$ , it follows that  $p(F) \geq 4$ .

**Case 3:**  $W = W_5$ .

Observe that deleting any of a vertex,  $K_2$  or triangle in  $W_5$  may result in having no triangles left over. Hence, for any  $v(X) \in \{1, 2, 3\}$ , we have  $-1 \leq p(F) - 1 - f(X) + 1$ . Thus as  $f(X) \geq 5$ ,  $p(F) \geq 4$  as desired.

**Case 4:**  $W \in \mathcal{B}$ .

Note that in this case  $T^3(W) = 2$ . If  $v(X) = 1$ , then  $T^3(W \setminus X) \geq 1$ , and so  $-1 \leq p(F) - 1 - 5 + 1$  which gives  $p(F) \geq 4$ . If  $v(X) \in \{2, 3\}$ , then  $-1 \leq p(F) - 1 - f(X) + 2$ , which gives  $p(F) \geq 4$ .

**Case 5:**  $W$  is 4-Ore with  $T^3(W) = 3$ .

If  $v(X) = 1$ , then  $T^3(W \setminus X) \geq 2$ . Thus  $-1 \leq p(F) - 1 - 5 + 1$  which gives  $p(F) \geq 4$ . If  $v(X) \in \{2, 3\}$ , then  $-1 \leq p(F) - 1 - f(X) + 2$ , which gives  $p(F) \geq 4$ . (Note that in the  $v(X) = 3$  case, we are using the fact that  $T^3(W \setminus X) \geq 1$  (see Observation 4.2)).

**Case 6:** All other cases.

If  $v(X) = 1$ , then  $-1 \leq p(F) - 2 - 5 + 1$  which gives  $p(F) \geq 4$ . If  $v(X) = 2$ , then

$-1 \leq p(F) - 2 - 7 + 2$  which gives  $p(F) \geq 6$ . If  $v(X) = 3$ , then  $-1 \leq p(F) - 2 - 6 + 3$  which gives  $p(F) \geq 4$ . This is all possible cases, so the result follows.  $\square$

We will attempt to strengthen this bound now.

**Lemma 4.28.**  *$G$  does not contain  $K_4 - e$  as a subgraph.*

*Proof.* Suppose not. Let  $F$  be a  $K_4 - e$  subgraph in  $G$  where  $e = xy$  and  $w, z$  are the two other vertices in  $F$ , where we pick  $F$  so that the number of degree three vertices in  $F$  which are degree three in  $G$  is maximized. Note that as  $G \neq K_4$ ,  $F$  is an induced subgraph. We claim that  $x$  and  $y$  have no common neighbours aside from  $w$  and  $z$ . Suppose not, and let  $u$  be a common neighbour of  $x$  and  $y$  with  $u \notin \{w, z\}$ . By 4-criticality,  $G - ux$  has a 3-colouring, say  $f$ . Then  $f(u) = f(x)$  as otherwise  $G$  has a 3-colouring. Notice in any 3-colouring of  $F$ ,  $f(x) = f(y)$ . But then  $uy \in E(G)$  and  $f(u) = f(y)$ , a contradiction. Hence  $x$  and  $y$  have no common neighbours outside  $\{w, z\}$ .

Fix any 3-colouring of  $F$ , and let  $F', W$  and  $X$  be an extension, extender, and source of  $F$ . By the potential-extension lemma, we have

$$p(F') \leq p(F) + p(W) - 5v(X) + 3e(X) + T^3(W) - T^3(W \setminus X).$$

Observe that  $p(F) = 4$ . Throughout the proof of this lemma, we let  $d$  be the vertex obtained by identifying  $x$  and  $y$ . Note that since  $W \not\subseteq G$ , it follows that  $d \in X$ .

We now split into cases depending on what graph  $W$  is.

**Case 1:**  $W = K_4$ .

First suppose  $v(X) = 3$ . In this case,  $w, z$ , and one of  $y$  and  $x$  share a common neighbour, and so  $G$  contains a  $K_4$ . This is a contradiction, as  $K_4$  is 4-critical and  $G \neq K_4$ .

Now suppose  $v(X) = 2$ . Then without loss of generality let  $X = \{z, d\}$ . Then there is a subgraph  $H$  of  $G$  where  $V(H) = V(F) \cup \{u, u'\}$  and there are edges  $u'z, u'u, u'x, uy, uz$  and  $E(F)$ . But this subgraph is  $W_5$ , so  $G = W_5$ , a contradiction.

Finally suppose that  $v(X) = 1$ . Then similarly to the above argument,  $G$  is isomorphic to the Moser spindle, and we are done.

**Case 2:**  $W$  is 4-Ore with  $T^3(W) = 2$ .

First suppose that  $v(X) = 1$ . Then it follows that  $G$  contains a subgraph  $H$  that is the Ore composition of  $W$  and  $K_4$ . Since  $H$  is 4-critical,  $G = H$ . This implies that  $G$  is 4-Ore, contradicting Observation 4.24.

Now suppose that  $v(X) = 2$ . By Lemma 4.8,  $W$  contains two edge-disjoint kites that share at most a vertex. Thus  $T^3(W \setminus X) \geq 1$ . By the potential-extension lemma, we have  $p(F') \leq 4 + 0 - 7 + 1$  which gives  $p(F') \leq -2$ . If  $F' \subset G$ , this contradicts Lemma 4.27. If  $F' = G$ , this contradicts the assumption that since  $G$  is a counterexample to Theorem 1.64,  $p(G) \geq -1$ .

Now suppose  $v(X) = 3$ . Note that by Lemma 4.8,  $W$  contains two edge-disjoint kites that share at most one vertex. Thus  $T^3(W \setminus X) \geq 1$ . By the potential-extension lemma, we

have  $p(F') \leq 4+0-6+1$  which implies that  $p(F') \leq -1$ . By Lemma 4.27, since  $F' \subseteq G$ , it follows that  $F' = G$ . Thus  $G$  is obtained from  $W$  by unidentifying  $d$  into  $x$  and  $y$ . Note as  $x$  and  $y$  have no common neighbours aside from  $w$  and  $z$ , and every vertex in  $G$  has degree at least three,  $d$  has degree at least four in  $W$ . First consider the case where  $W = M$ . Then  $d$  is the unique vertex of degree four in  $M$ . As  $G$  is obtained by unidentifying  $d$  to  $x$  and  $y$ , it follows that either  $G$  is 3-regular, in which case as  $G \neq K_4$ ,  $G$  is 3-colourable by Brook's Theorem, or  $G$  has a vertex of degree two, contradicting 4-criticality. Hence  $W \neq M$ . Therefore by Lemma 4.8,  $W$  contains two vertex-disjoint kites. As  $G$  is obtained by unidentifying  $d$ , this implies that  $G$  contains a kite. But note that both  $w$  and  $z$  have degree at least three in  $W$ , as  $W$  is 4-critical, and thus in  $G$  after unidentifying, have degree at least four. But this contradicts our choice of  $F$ , as we picked  $F$  to contain the largest number of vertices which are degree three in the  $K_4 - e$  and in  $G$ .

**Case 3:**  $W = W_5$ .

If  $v(X) = 1$ , then  $G$  is an Ore composition of  $K_4$  and  $W_5$ , a contradiction to Observation 4.26. If  $v(X) \in \{2, 3\}$ , then by the potential-extension lemma we have  $p(F') \leq 4-1-6+1$  which gives  $p(F') \leq -2$ . If  $F' \subset G$ , this contradicts Lemma 4.27. If  $F' = G$ , this contradicts the assumption that  $p(G) \geq -1$ .

**Case 4:**  $W \in \mathcal{B}$ .

If  $v(X) = 1$ , then  $G$  is the Ore composition of  $W$  and  $K_4$ , contradicting Observation 4.26. If  $v(X) \in \{2, 3\}$ , then by the potential-extension lemma we have  $p(F') \leq 4-1-6+1$  which gives  $p(F') \leq -2$ . As in Case 3, this leads to a contradiction.

**Case 5:**  $W$  is 4-Ore with  $T^3(G) = 3$ .

If  $v(X) = 1$ , then  $G$  is 4-Ore, contradicting Observation 4.24. If  $v(X) = 2$ , then  $p(F') \leq 4-1-7+2$  and  $p(F') \leq -2$ . As in Cases 3 and 4, this leads to a contradiction.

So  $v(X) = 3$ . In this case we claim  $F'$  is all of  $G$ . If not, take any 3-colouring  $\psi$  of  $F'$  (which exists by 4-criticality). As  $x$  and  $y$  get the same colour in this 3-colouring, this implies when we identify  $x$  and  $y$ , we get a 3-colouring of  $W$ , contradicting that  $W$  is 4-critical. Hence  $F' = G$ .

If  $T^3(W \setminus X) \geq 2$ , then by the potential-extension lemma we have  $p(F') \leq 4-1-6+1$ , which gives  $p(F') \leq -2$ , a contradiction.

Therefore by Lemma 4.10 it follows that  $W - X$  contains a kite.

Let  $K$  be the kite in  $W - X$ , with spar  $st$ . We claim there is at most one edge from  $F$  to  $K$ : otherwise,  $p(G[V(F) \cup V(K)]) \leq 5(8) - 3(10+2) - 2 = 2$ , contradicting Lemma 4.27. (Note trivially  $G[V(F) \cup V(K)] \neq G$ , since  $T^3(W) = 3$  but  $T^3(G[V(F) \cup V(K)]) = 2$ .) Thus at least one of  $s$  and  $t$  has degree three in  $G$ . It now suffices to argue that  $w$  and  $z$  do not have degree three in  $G$ , thus contradiction our choice of  $F$ .

To see this, note that since  $W$  is 4-critical, both  $w$  and  $z$  have degree at least three in  $W$ . But  $G$  is obtained from  $W$  by unidentifying  $d$  into the vertices  $x$  and  $y$ . As  $x$  and  $y$  share  $w$  and  $z$  as neighbours,  $w$  and  $z$  have degree at least four in  $G$ . But this contradicts that we picked  $F$  such that the number of degree three vertices in the  $K_4 - e$  subgraph which have degree three in  $G$  is maximized, a contradiction.

**Case 6: All other cases.**

In this case,  $p(W) \leq -2$ . If  $v(X) = 1$ , then  $p(F') \leq 4 - 2 - 5 + 1 \leq -2$ , a contradiction. If  $v(X) = 2$ , then  $p(F') \leq 4 - 2 - 7 + 2 \leq -3$ , a contradiction.

Lastly, assume that  $v(X) = 3$ . In this case, by a similar argument as in Case 5,  $F' = G$ , and thus  $G$  is obtained from  $W$  by splitting  $d$ . Then  $T^3(G) \geq T^3(W) - 1$ , and thus  $p(G) = p(W) + 5 - 6 + 1 \leq -2$ , a contradiction. □

Let  $D_3(G)$  be the subgraph of  $G$  induced by the vertices of degree three. Now we will build towards showing that  $D_3(G)$  is acyclic, and further if a vertex of degree three is in a triangle, then it is the only vertex of degree three in this triangle.

**Definition 4.29.** For an induced subgraph  $R$  of  $G$ , where  $R \neq G$ , we say  $u, v \in V(R)$  are an *identifiable pair* if  $R + uv$  is not 3-colourable.

**Lemma 4.30.** *If  $R$  is an induced subgraph of  $G$  with  $R \neq G$ ,  $v(R) \leq v(G) - 3$ , and such that  $G \setminus R$  is not a triangle of degree three vertices, then  $R$  has no identifiable pair.*

*Proof.* Suppose not. Let  $x$  and  $y$  be an identifiable pair in  $R$ , and consider  $R + xy$ . As  $R + xy$  is not 3-colourable by definition, there exists a 4-critical subgraph  $W$  of  $R + xy$ . Moreover, since  $T^3(W - xy) \geq T^3(W) - 1$ , we have that  $p(W - xy) \leq p(W) + 4$  (note that  $xy \in E(W)$ , as otherwise  $G$  contains a 4-critical subgraph, a contradiction). By the assumptions, Lemma 4.27 implies that  $p(W - xy) \geq 4$ . If  $p(W) \leq -1$ , then we obtain a contradiction. If  $W = K_4$ , then  $G$  has a  $K_4 - e$  subgraph, contradicting Lemma 4.28. If  $W$  is 4-Ore with  $T^3(G) = 2$ , then by Lemma 4.8,  $G$  contains a  $K_4 - e$  subgraph, again contradicting Lemma 4.28. For all other  $W$ , we have  $p(W) \leq -1$ , and thus we get a contradiction. □

For a subgraph  $H$ , let the *neighbourhood of  $H$* , denoted  $N(H)$  be the set of vertices not in  $H$  which have a neighbour in  $H$ . We will need the following well-known consequence of the Gallai-Tree Theorem [14].

**Theorem 4.31.** *Let  $C$  be a cycle of degree three vertices in a 4-critical graph. Then  $v(C)$  is odd,  $N(C)$  induces an independent set, and in any 3-colouring of  $G - C$ , all vertices in  $N(C)$  receive the same colour.*

**Corollary 4.32.** *All cycles in  $D_3(G)$  are triangles.*

*Proof.* Let  $C$  be a cycle in  $D_3(G)$  where  $v(C) \geq 5$ . If  $|N(C)| = 1$ , then since  $G$  has minimum degree three, it follows that  $G$  is isomorphic to an odd wheel. If  $G = W_5$ , then  $G$  is not a counterexample to Theorem 1.64. So we may assume that  $v(C) \geq 7$ . Note that  $v(G) = v(C) + 1$ , and  $e(G) = 2v(C)$ . So  $p(G) = 5(v(C) + 1) - 6v(C) - 1 = -v(C) + 4 \leq -3$  since  $v(C) \geq 7$ . Thus  $|N(C)| \geq 2$ . Then by Theorem 4.31, any pair of vertices in  $N(C)$  are an identifiable pair in  $G$ . This contradicts Lemma 4.30 as  $v(G - C) < v(G) - 3$ . □

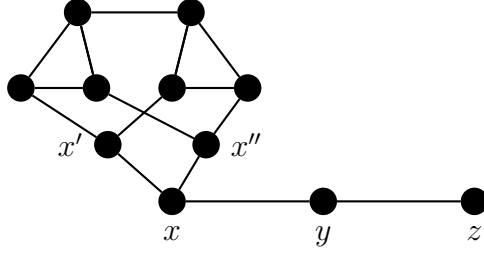


Figure 4.1: An example of the  $M$ -gadget outcome in Lemma 4.34

**Corollary 4.33.** *If  $T$  is a triangle in  $G$ , then  $V(T)$  does not contain exactly two vertices of degree three.*

*Proof.* Suppose not. Let  $x, y$  and  $z$  induce a triangle where  $x$  and  $y$  are vertices of degree three and  $z$  has degree at least four. Let  $x'$  and  $y'$  be the unique other neighbours of  $x$  and  $y$  respectively. Note  $x' \neq y'$  as otherwise  $G$  contains a subgraph isomorphic to  $K_4 - e$ , contradicting Lemma 4.28.

If  $x'y' \in E(G)$ , then any 3-colouring of  $G - \{x, y\}$  extends to a 3-colouring of  $G$ , a contradiction. In particular, every 3-colouring of  $G - \{x, y\}$  gives  $x'$  and  $y'$  the same colour and hence  $G - \{x, y\}$  contains an identifiable pair.

So consider  $G - \{x, y\} + \{x'y'\}$ . Then there is a 4-critical subgraph  $W$  containing  $x'y'$ , and  $p(W - x'y') \leq p(W) + 4$ . By the same argument as in Lemma 4.30, it suffices to show  $H := G \setminus (W - x'y')$  is not a triangle of degree three vertices, or a single vertex of degree three. Notice that  $H$  is not a vertex of degree three, as both  $x$  and  $y$  are in  $V(H)$ . We claim  $H$  is not a triangle of degree three vertices. If so, then  $z \notin V(H)$ , as  $d(z) \geq 4$ , but then as  $x, y \in V(H)$ ,  $x$  and  $y$  lie in a triangle of degree three vertices. But then as  $x, y, z$  induce a triangle,  $G$  contains a  $K_4 - e$  subgraph, again contradicting Lemma 4.28.  $\square$

An  $M$ -gadget is a graph obtained from  $M$  by first splitting the vertex  $v$  of degree four into two vertices  $v_1$  and  $v_2$  such that there is no  $K_4 - e$  in the resulting graph,  $N(v_1) \cap N(v_2) = \emptyset$ , and  $N(v) = N(v_1) \cup N(v_2)$ , and after this, adding a vertex  $v'$  adjacent to only  $v_1$  and  $v_2$ . We call  $v'$  the *end* of the  $M$  gadget.

**Lemma 4.34.** *Let  $C$  be a component of  $D_3(G)$  with  $v(C) \geq 3$ . Let  $x, y, z \in V(C)$  such that  $xy, yz \in E(G)$ , and  $y, z$  do not lie in a triangle of degree three vertices. Let  $x', x''$  be two neighbours of  $x$  which are not  $y$ . Then either  $x'x'' \in E(G)$ , or  $x, x', x''$  lie in an  $M$ -gadget with end  $x$ , and this  $M$ -gadget does not contain  $y$  or  $z$ .*

*Proof.* Suppose not. Then  $x'x'' \notin E(G)$ , so identify  $x'$  and  $x''$  into a new vertex  $x'''$ , and let  $G'$  be the resulting graph. Note neither  $x'$  nor  $x''$  is  $z$ , as otherwise  $x, y, z$  lie in a triangle of degree three vertices. Moreover, if there exists a 3-colouring of  $G'$ , then this 3-colouring readily extends to  $G$ . Hence  $G'$  is not 3-colourable. Let  $W'$  be a 4-critical subgraph of  $G'$ .



Note that  $y$  and  $z$  are not in  $W'$ , since after identifying  $x'$  and  $x''$ ,  $x$  has degree two, which implies that  $x \notin V(W')$ , and in  $G' - x$ ,  $y$  has degree two hence  $y \notin V(W')$ , and similarly this implies that  $z \notin V(W')$ .

Let  $W = W' \setminus \{x'''\} \cup \{x, x', x''\}$ . If  $T^3(W) = T^3(W') - 1$ , we have  $p(W) \leq p(W') + 10 - 6 + 1 = p(W') + 5$ , and otherwise  $p(W) \leq p(W') + 4$ . Note that  $G \setminus W$  is not a single vertex of degree three as both  $y$  and  $z$  are in  $G \setminus W$ , and  $G \setminus W$  is not a cycle of degree three vertices as  $y$  and  $z$  do not lie in a cycle of degree three vertices by assumption. Thus by Lemma 4.27, we have that  $p(W) \geq 4$ . Hence if  $p(W') \leq -2$  we get a contradiction. Observe that  $W' \neq K_4$ , as  $W$  is obtained from  $W'$  by splitting  $x'''$  into two vertices, and that would imply that  $G$  contains a  $K_4 - e$  subgraph, contradicting Lemma 4.28. If  $T^3(W') = 2$  and  $W'$  is 4-Ore, then if  $W'$  is not  $M$ ,  $G$  contains a  $K_4 - e$  subgraph. Again, this contradicts Lemma 4.28. Further, if  $W' = M$ , and the split is not on the unique vertex of degree four in such a way that does not leave a  $K_4 - e$ , then  $G$  contains a  $K_4 - e$  subgraph. Therefore,  $x$  is the end of an  $M$ -gadget.

Now consider the case where  $T^3(W') = 3$  and is 4-Ore, or  $W' \in \mathcal{B}$ . By Lemma 4.13 and Lemma 4.18 then either splitting  $x'''$  does not reduce the number of triangles, or  $G$  contains a  $K_4 - e$ , or in  $W \setminus x$  either  $x'$  or  $x''$  has degree one and is incident to a foundational edge in  $G$ . The first case gives a contradiction as then  $p(W) \leq p(W') + 4$ , and  $p(W') \leq -1$ , which contradicts that  $p(W) \geq 4$ . The second case contradicts that  $G$  has no  $K_4 - e$  subgraph. Therefore without loss of generality suppose that  $x'$  has degree one in  $W'$  after splitting  $x'''$  back into  $x'$  and  $x''$  and that the edge incident to  $x'$  is foundational in  $W'$ . Let the other endpoint of this foundational edge be  $y'$ . Now we claim that in any 3-colouring of  $W'' := W' - x''' \cup \{x''\}$ ,  $x''$  and  $y'$  get the same colour. If not, then we have a 3-colouring of  $W'$ , which contradicts that  $W'$  is 4-critical. Hence  $W''$  contains an identifiable pair. Further,  $y, z, x, x' \notin V(W'')$ . Thus we contradict by Lemma 4.30.  $\square$

See Figure 4.1 for an illustration of the  $M$ -gadget outcome.

**Corollary 4.35.**  $D_3(G)$  does not contain an induced path of length four.

*Proof.* Suppose not, and let  $v_1, v_2, v_3, v_4, v_5$  be such a path. Let  $x_3$  be the vertex not  $v_2$  and  $v_4$  which is incident to  $v_3$ . By Lemma 4.33,  $x_3v_2 \notin E(G)$  and  $x_3v_4 \notin E(G)$ . Further as the path is induced,  $v_2, v_3, v_4$  does not induce a triangle. Additionally,  $v_4$  and  $v_5$  are not in a triangle of degree three vertices as the path is induced. Hence by Lemma 4.34, applied to  $v_3, v_4, v_5$  with  $v_3$  playing the role of  $x$ ,  $v_3$  is the end of a  $M$ -gadget containing  $v_2$  and  $x_3$  but not  $v_4$  or  $v_5$ . This implies that  $v_1$  is in a triangle, say  $v_1, u_1, u_2$  and by Lemma 4.33 both  $u_1$  and  $u_2$  have degree at least four. Now we apply Lemma 4.34 to  $v_2, v_3, v_4$  with  $v_2$  playing the role of  $x$ . By similar reasoning as above,  $v_2$  is not in a triangle and  $v_3$  and  $v_4$  do not lie in a triangle of degree three vertices, and thus  $v_2$  lies in an  $M$ -gadget say  $M'$ . We claim the subgraph  $M'$  is not induced. First observe that  $v_1 \in V(M')$ , since  $v_2$  has degree three. Then it follows that  $u_1, u_2 \in V(M')$ , as  $v_1 \in V(M')$  and  $v_1$  has degree three. Further, as  $v_2$  is the degree two vertex in the  $M$ -gadget, the edge  $u_1u_2$  does not lie in  $M'$ .



Let  $H' = M' \cup \{u_1u_2\}$ . Then  $v(H') = 9$  and  $e(H') \geq 14$  so  $p(H') \leq 45 - 42 - 2 = 1$ . This contradicts Lemma 4.27.  $\square$

**Corollary 4.36.** *If  $C$  is an acyclic component of  $D_3(G)$ , then  $v(C) \leq 6$ .*

*Proof.* Let  $P$  be a longest path in a component  $C$ . First suppose that  $P$  is a path of length 3, say  $v_1, v_2, v_3, v_4$ . Then  $v_2$  and  $v_3$  are adjacent to at most one vertex not in the path, say  $v'_2$  and  $v'_3$  respectively. Suppose  $v'_2$  has degree three. Then  $v'_2$  is not adjacent to a vertex  $u \in D_3(G) \setminus \{v_2\}$ . If not, the path  $u, v'_2, v_2, v_3, v_4$  is an induced path on five vertices, contradicting Corollary 4.35. Applying a similar argument to  $v'_3$  we see that  $v(C) \leq 6$  in this case. Now suppose that  $P$  is a path of length 2, say  $v_1, v_2, v_3$ . If  $v_2$  is adjacent to a vertex of degree three, say  $v'_2$ , then  $v'_2$  is not adjacent to another vertex of degree three, as otherwise we have a path of length 3, contradicting our choice of  $P$ . Hence in this case,  $v(C) \leq 4$ . Lastly, if the longest path has length at most one, then  $v(C) \leq 2$  as desired.  $\square$

Now we build towards proving every component of  $D_3(G)$  is acyclic.

**Lemma 4.37.** *Let  $\{x, y, z\} \subseteq V(G)$  induce a triangle  $C$  of degree three vertices. Then at most one vertex in  $N(C)$  has degree three.*

*Proof.* First observe that  $|N(C)| = 3$ . If not, either  $G = K_4$ , or  $G$  is not 2-connected, which in either case gives a contradiction. Let  $x', y', z'$  be the vertices in  $N(C)$ , where  $x'$  is adjacent to  $x$ ,  $y'$  is adjacent to  $y$ , and  $z'$  is adjacent to  $z$ . Without loss of generality, suppose that  $x'$  and  $y'$  both have degree three. Note that  $yy'$  is not in a triangle of degree three vertices as otherwise  $G$  contains a  $K_4 - e$ , contradicting Lemma 4.28. Thus Lemma 4.34 applies to  $x, y, y'$ . Since  $x'z \notin E(G)$ ,  $x$  is the end of an  $M$ -gadget not containing  $y$  or  $y'$ . But now it follows that there are two vertex-disjoint triangles in  $G - x - y - z$ , and hence  $T^3(G) \geq 3$ . As  $G$  is not 4-Ore,  $KY(G) \leq 1$ , and thus  $p(G) \leq -2$ , a contradiction.  $\square$

**Lemma 4.38.**  *$D_3(G)$  is acyclic.*

*Proof.* Suppose not. Let  $T$  be a triangle in  $D_3$ . As  $G$  is 3-connected and  $G \neq K_4$ , it follows that  $|N(T)| = 3$ . By Theorem 4.31 all vertices of  $N(T)$  receive the same colour in any three colouring. Hence every pair of vertices in  $N(T)$  are an identifiable pair in  $G - T$ . Let  $R := G - T$ . Let  $x, y$  be two vertices in  $N(T)$ , such that  $y$  is adjacent to a vertex  $z$  in  $T$ . Observe that in  $R + xy$ , we have a 4-critical graph  $W$ , and since  $T^3(W - xy) \geq T^3(W) - 1$ , we have that

$$p(W - xy) \leq p(W) + 4. \quad (4.39)$$

If  $W = K_4$ , then  $G$  has a  $K_4 - e$  subgraph, contradicting Lemma 4.28. If  $W$  is 4-Ore with  $T^3(G) = 2$ , then by Lemma 4.8, again  $G$  contains a  $K_4 - e$  subgraph, contradicting Lemma 4.28. If  $p(W) \leq -2$ , then we obtain a contradiction to Lemma 4.27. Further, we can assume that  $W \neq W_5$  since otherwise  $p(W - xy) = 5(6) - 3(9) - 1 = 2$ , again contradicting Lemma 4.27. Additionally, if  $W - xy \neq R$ , then we obtain a contradiction to Lemma 4.27

when  $p(W) \leq -1$ . Thus we can assume that  $W = R + xy$  and that either  $W$  is 4-Ore with  $T^3(W) = 3$ , or  $W \in \mathcal{B}$ .

First assume that  $W$  is 4-Ore with  $T^3(W) = 3$ . Now consider splitting  $x$  into two vertices  $x_1$  and  $x_2$  such that  $d(x_1) = 1$  and  $x_1$  is only adjacent to  $y$ . Let  $W^x$  denote this graph. Note that  $W^x \subset G$  with  $x_2$  playing the role of  $x$  and  $x_1$  playing the role of  $z$ . By Lemma 4.13 either  $W^x$  has  $T^3(W^x) \geq T^3(W)$ ,  $W^x$  contains a  $K_4 - e$ , or  $xy$  is a foundational edge. If  $W^x$  has  $T^3(W^x) \geq T^3(W)$ , then since  $T^3(W^x) = T^3(W^x - x_1y) = T^3(W - xy)$  it follows that Equation 4.39 can be strengthened to  $p(W - xy) \leq p(W) + 3$ . Since  $p(W) = -1$  and  $W - xy \subset G$ , this contradicts Lemma 4.27. If  $W^x$  contains a  $K_4 - e$ , then  $G$  contains a  $K_4 - e$ , contradicting Lemma 4.28. Therefore we can assume that  $xy$  is a foundational edge, and by Lemma 4.12 such an edge is the spar of a kite. Thus in  $W - xy$ , both  $x$  and  $y$  have degree two, which implies that in  $G$ , both  $x$  and  $y$  have degree three. But this contradicts Lemma 4.37.

Therefore we can assume that  $W$  is in  $\mathcal{B}$ . Then  $xy$  is a foundational edge, as otherwise by Lemma 4.13 either  $G - xy$  contains a  $K_4 - e$  subgraph, contradicting Lemma 4.28, or as above we can strengthen Equation 4.39 and obtain a contradiction. If  $W \neq T_8$ , then by Lemma 4.17 we have that  $xy$  is the spar of a kite. Then in  $W - xy$ , both  $x$  and  $y$  have degree two, which implies that in  $G$ , both  $x$  and  $y$  have degree three, contradicting Lemma 4.37. Therefore  $W = T_8$ . As  $W = R + xy = G - T + xy$ , our entire graph is  $T_8 - u_1u_2 + T$ . In this case, we label the vertices of  $T$  by setting  $T = v_1v_2v_3v_1$ . We may assume without loss of generality,  $v_1$  is adjacent to  $u_1$ , and  $v_2$  is adjacent to  $u_2$ . Moreover, by Theorem 4.31, the neighbour of  $v_3$  outside of  $\{v_1, v_2\}$  forms an independent set with  $\{u_1, u_2\}$ . It follows that the third edge incident with  $v_3$  is incident with a vertex in  $\{u_6, u_7, u_8\}$ . It is easy to verify that the resulting graph is 3-colourable. As these are all the cases, it follows that  $D_3(G)$  is acyclic.  $\square$

From the above sequence of lemmas, we obtain the following corollary.

**Corollary 4.40.** *Every component in  $D_3(G)$  has at most six vertices.*

## 4.6 Discharging

In this section we provide the discharging argument which shows a vertex-minimum counterexample does not exist. We start off by showing that there exists a component of  $D_3(G)$  with at least three vertices.

**Lemma 4.41.** *There exists a component of  $D_3(G)$  with at least three vertices.*

*Proof.* Suppose not. Let  $F$  be the subgraph of  $G$  with  $V(F) = V(G)$  and  $E(F) = \{xy \in E(G) \mid d(x) \geq 4 \text{ and } d(y) \geq 4\}$ .

**Claim 4.42.**  *$F$  is not bipartite.*

*Proof.* Suppose not. Let  $(A, B)$  be a bipartition of  $F$ . By the assumption, every component of  $D_3(G)$  contains at most two vertices, and by Corollary 4.33, each triangle contains at most one vertex of degree three. It follows that  $D_3(G) \cup \{A\}$  is bipartite. Now colour  $D_3(G) \cup \{A\}$  with colours 1 and 2, and colour  $B = G - D_3(G) - A$  with colour 3. This is a proper 3-colouring of  $G$ , contradicting that  $G$  is 4-critical.  $\square$

Set  $ch_i(v) = d(v)$  for each vertex  $v \in V(G)$ , and have every vertex of degree at least four send  $\frac{1}{6}$  charge to each neighbour of degree three. For each  $v \in V(G)$ , let  $ch_f(v)$  denote the final charge of  $v$ . Note that all degree three vertices end up with  $\frac{10}{3}$  final charge, and if  $v$  has degree at least four, then  $ch_f(v) = \frac{10}{3}$  if and only if  $d(v) = 4$  and  $v$  is adjacent to exactly four vertices of degree three. Further, if either of those conditions do not hold, the final charge of  $v$  is at least  $\frac{10}{3} + \frac{1}{6}$ . Therefore for every edge  $e = xy \in E(F)$ , we have  $ch_f(x) \geq \frac{10}{3} + \frac{1}{6}$  and  $ch_f(y) \geq \frac{10}{3} + \frac{1}{6}$ . Thus it follows that

$$\sum_{v \in v(G)} ch_f(v) \geq \frac{10v(G)}{3} + \frac{e(F)}{3}.$$

Since  $F$  is not bipartite,  $e(F) \geq 3$ . Then we have

$$2e(G) \leq \frac{10v(G) + 3}{3}.$$

Thus it follows that

$$p(G) \leq \text{KY}(G) \leq -\frac{3}{2}.$$

Since potential is integral, we get that  $p(G) \leq -2$ , contradicting that  $G$  is a counterexample.  $\square$

Now we proceed with the main discharging argument. We assign to each vertex  $v \in V(G)$  an initial charge  $ch_i(v) = d(v)$ . We discharge in three steps: in each step, the discharging occurs instantaneously throughout the graph. The final charge will be denoted by  $ch_f$ . For  $v \in V(G)$ , let  $i_3(v)$  denote the number of neighbours of  $v$  that are isolated vertices in  $D_3(G)$ .

### Discharging Steps

1. If  $u$  is a vertex of degree at least four,  $uv$  is an edge, and  $v$  is a vertex of degree three, then  $u$  sends  $\frac{3ch_i(u)-10}{3d_3(u)}$  charge to  $v$ .
2. If  $u$  is an isolated vertex in  $D_3(G)$ ,  $u$  sends  $\frac{1}{18}$  charge to each adjacent vertex in  $G$ .
3. Let  $u$  be a vertex of degree at least four, and let  $f(u)$  be the total charge received by  $u$  in Step 2. The vertex  $u$  sends  $\frac{f(u)}{d_3(u)-i_3(u)}$  charge to each adjacent vertex of degree three that is not isolated in  $D_3(G)$ .

We will show that after discharging, the sum of the charges is at least  $v(G) \left(\frac{10}{3}\right)$ . Note that by the discharging rules, we have immediately that every vertex of degree at least four has final charge at least  $\frac{10}{3}$ . In light of this, we will focus our attention on vertices of degree three: let  $C$  be a component in  $D_3(G)$ , and let  $ch_f(C) = \sum_{v \in V(C)} ch_f(v)$ .

We note the following.

**Observation 4.43.** If  $u$  sends charge to  $v$  in Step 1, then  $u$  sends  $v$  at least  $\frac{1}{6}$  charge.

**Claim 4.44.** If  $C$  is an isolated vertex, then  $ch_f(C) \geq \frac{10}{3}$ .

*Proof.* Let  $v \in V(C)$ . Note that  $ch_i(v) = d(v) = 3$ . Since  $v$  is isolated in  $D_3(G)$ , every neighbour of  $v$  has degree at least four. Thus by Observation 4.43,  $v$  receives at least  $\frac{1}{6}$  from each of its neighbours in Step 1. Moreover,  $v$  returns at most  $\frac{1}{18}$  to each of its neighbours in Step 2. It follows that:

$$\begin{aligned} ch_f(v) &\geq 3 + 3 \left(\frac{1}{6}\right) - 3 \left(\frac{1}{18}\right) \\ &= \frac{10}{3} \end{aligned}$$

as desired. □

**Claim 4.45.** If  $C$  is a path of length one, then  $ch_f(C) \geq v(C) \left(\frac{10}{3}\right)$ .

*Proof.* Let  $v_1 v_2 \in V(C)$ . Note that  $ch_i(v_1) = ch_i(v_2) = 3$ , and that by Observation 4.43, each of  $v_1$  and  $v_2$  receive at least  $\frac{1}{6}$  from each of their neighbours of degree at least four. It follows that:

$$\begin{aligned} ch_f(C) &\geq ch_i(v_1) + 2 \left(\frac{1}{6}\right) + ch_i(v_2) + 2 \left(\frac{1}{6}\right) \\ &= 2 \left(\frac{10}{3}\right), \end{aligned}$$

as desired. □

For the remaining cases, we will make use of the following.

**Claim 4.46.** If  $v$  is a leaf in a tree  $C \subseteq D_3(G)$  with  $v(C) \geq 3$ , then  $v$  receives at least  $\frac{4}{9}$  charge from its neighbourhood during Step 1.

*Proof.* As  $v$  is a leaf in a tree with at least three vertices, there exists a path  $vuw$  in  $C$ . Let  $x$  and  $y$  be two neighbours of  $v$  which are not  $u$ . By Lemma 4.34, either  $xy \in E(G)$ , or  $x$ ,  $v$ , and  $y$  lie in an  $M$ -gadget with end  $v$  that does not contain  $u$  or  $w$ . If  $xy \in E(G)$ , then note that  $d_3(x) \leq d(x) - 1$ , and likewise  $d_3(y) \leq d(y) - 1$ . In this case, each of  $z \in \{x, y\}$

sends at least  $\frac{d(z)}{d(z)-1} - \frac{10}{3(d(z)-1)}$  to  $x$ . Since  $d(z) \geq 4$ , it follows that  $v$  receives at least  $\frac{2}{9}$  from each of  $x$  and  $y$ , and so at least  $\frac{4}{9}$  in total.

If  $xy \notin E(G)$ , then  $v, x$ , and  $y$  lie in an  $M$ -gadget with end  $v$ . Thus there exist two triangles  $T$  and  $T'$  such that  $x$  is adjacent to a vertex  $a \in V(T)$  and  $a' \in T'$ , and  $y$  is adjacent to a vertex  $b \neq a$  in  $V(T)$  and  $b' \neq a'$  in  $V(T')$ . Note by Corollary 4.33, each of  $T$  and  $T'$  contain at most one vertex of degree three. Thus at least two of  $\{a, a', b, b'\}$  have degree at least four. Without loss of generality, we may assume that either  $a$  and  $a'$  have degree at least four, or that  $a$  and  $b'$  have degree at least four. In the first case,  $x$  sends at least  $\frac{1}{3}$  to  $v$ , and  $y$  sends at least  $\frac{1}{6}$  to  $v$ . Thus  $v$  receives at least  $\frac{1}{2}$  from  $x$  and  $y$ . In the second case, each of  $x$  and  $y$  sends at least  $\frac{2}{9}$  to  $v$ , and so  $v$  receives at least  $\frac{4}{9}$ .

Thus  $v$  receives at least  $\frac{4}{9}$  charge from its neighbourhood.  $\square$

**Claim 4.47.** *If  $C = v_1v_2v_3$  is a path of length 2, then  $ch_f(C) \geq v(C) \left(\frac{10}{3}\right)$ .*

*Proof.* By Claim 4.46, each of  $v_1$  and  $v_3$  receives at least  $\frac{4}{9}$  units of charge from its neighbourhood during Step 1. Moreover, by Observation 4.43,  $v_2$  receives at least  $\frac{1}{6}$  units of charge during Step 1. Thus

$$\begin{aligned} ch_f(C) &\geq ch_i(v_1) + \frac{4}{9} + ch_i(v_2) + \frac{1}{6} + ch_i(v_3) + \frac{4}{9} \\ &= \frac{181}{18} \\ &> 3 \left(\frac{10}{3}\right), \end{aligned}$$

as desired.  $\square$

**Claim 4.48.** *If  $C$  is a star with four vertices, then  $ch_f(C) \geq 4 \left(\frac{10}{3}\right)$ .*

*Proof.* By Claim 4.46, each leaf in  $C$  receives at least  $\frac{4}{9}$  from its neighbourhood during Step 1 of the discharging process. Moreover, each  $u \in V(C)$  has  $ch_i(u) = 3$ . Thus

$$\begin{aligned} ch_f(C) &\geq \left(\frac{4}{9} + 3\right) + \left(\frac{4}{9} + 3\right) + \left(\frac{4}{9} + 3\right) + 3 \\ &= 12 + \frac{4}{3} \\ &= 4 \left(\frac{10}{3}\right), \end{aligned}$$

as desired.  $\square$

For the remaining cases, we will need the following lemma.

**Lemma 4.49.** *Let  $v$  be a leaf in a tree  $C \subseteq D_3(G)$  with  $v(C) \geq 3$ , and let  $u, w$  be the neighbours of  $v$  that are not contained in  $C$ . Suppose  $u, w$  are contained in an  $M$ -gadget with end  $v$ . At the end of the discharging process,  $v$  will have received at least  $\frac{1}{2}$  charge from its neighbours.*

*Proof.* By the structure of  $M$ -gadgets, there exist two distinct triangles  $T$  and  $T'$  such that  $u$  is adjacent to vertex  $a$  in  $T$  and  $a'$  in  $T'$ ,  $w$  is adjacent to  $b \neq a$  in  $T$  and  $b' \neq a'$  in  $T'$ . First, we note that if either  $d(u) \geq 5$  or  $d(w) \geq 5$ , we are done. To see this, suppose without loss of generality that  $d(u) \geq 5$ . Note that by Lemma 4.33, at most one vertex in  $T$  and at most one vertex in  $T'$  has degree three. If both  $b$  and  $b'$  have degree at least four, then in Step 1  $u$  sends  $v$  at least  $\frac{1}{3}$  and  $w$  sends  $v$  at least  $\frac{1}{3}$ . If  $a$  and  $a'$  have degree at least four, then in Step 1  $u$  sends  $v$  at least  $\frac{5}{9}$ . Finally, if  $a$  and  $b'$  have degree at least four, then in Step 1  $u$  sends  $v$  at least  $\frac{5}{12}$ , and  $w$  sends  $v$  at least  $\frac{2}{9}$ . In all cases,  $v$  receives at least  $\frac{1}{2}$ .

Thus we may assume that  $d(u) = d(w) = 4$ . We now break into cases depending on the degrees of  $a, b, a'$ , and  $b'$ .

**Case 1. Precisely one of  $a, b, a'$ , and  $b'$  has degree three.**

Suppose  $d(a) = 3$ . Then  $w$  is adjacent to at least two vertices of degree not equal to three, and so  $w$  sends at least  $\frac{1}{3}$  to  $v$  in Step 1. Moreover,  $u$  is adjacent to  $a'$  with  $d(a') \neq 3$ , and so  $u$  sends  $v$  at least  $\frac{2}{9}$  in Step 1. By Lemma 4.33, since  $a$  is contained in a triangle, it follows that  $a$  is isolated in  $D_3(G)$ . Thus  $a$  sends  $\frac{1}{18}$  to  $u$  in Step 2. Since  $a$  is isolated and  $d(a') \neq 3$ , we have that  $u$  sends at least  $\frac{1}{18(d(u)-2)}$  to  $v$  in Step 3. As our choice for  $a$  was arbitrary but  $d(u) = d(w) = 4$ , it follows that  $v$  receives at least

$$\frac{1}{3} + \frac{2}{9} + \frac{1}{18(4) - 36} = \frac{7}{12}$$

during discharging.

**Case 2. Either  $d(a) = d(a') = 3$ , or  $d(b) = d(b') = 3$ .**

Suppose  $d(a) = d(a') = 3$ . Then  $u$  sends  $v$  at least  $\frac{1}{6}$  in Step 1. By Lemma 4.33, neither  $b$  nor  $b'$  has degree three, and so  $w$  sends  $v$  at least  $\frac{1}{3}$  in Step 1. By Lemma 4.33, since  $a$  and  $a'$  are each contained in a triangle, it follows that both  $a$  and  $a'$  are isolated in  $D_3(G)$ . Thus each of  $a$  and  $a'$  sends  $\frac{1}{18}$  to  $u$  in Step 2, and so  $u$  sends at least  $\frac{1}{9(d(u)-2)}$  to  $v$  in Step 3.

As our choice of premise was arbitrary and  $d(u) = d(w) = 4$  by assumption, it follows that  $v$  receives at least

$$\frac{1}{6} + \frac{1}{3} + \frac{1}{9(4-2)} = \frac{5}{9}$$

during discharging.

**Case 3.  $d(c) = d(d') = 3$  for  $c \in \{a, b\}$  and  $d \in \{a, b\} \setminus \{c\}$ .**

Suppose  $d(a) = d(b') = 3$ . By Lemma 4.33, each of  $a$  and  $b'$  are isolated in  $D_3(G)$ , and so

each of  $u$  and  $w$  sends  $v$  at least  $\frac{2}{9}$  in Step 1. Moreover,  $a$  sends  $u$   $\frac{1}{18}$  charge in Step 2; similarly,  $b'$  sends  $w$   $\frac{1}{18}$  charge in Step 2. Thus  $v$  receives at least  $\frac{1}{18(d(u)-2)}$  from  $u$  in Step 3, and at least  $\frac{1}{18(d(w)-2)}$  from  $w$  in Step 3. It follows that  $v$  receives at least

$$\frac{2}{9} + \frac{2}{9} + \frac{2}{18(4-2)} = \frac{1}{2}$$

during discharging.

The result follows.  $\square$

**Claim 4.50.** *If  $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$  and  $E(C) = \{v_1v_2, v_1v_3, v_1v_4, v_4v_5\}$ , then  $ch_2(C) \geq v(C) \left(\frac{10}{3}\right)$ .*

*Proof.* Let  $u_1, u_2$  be the neighbours of  $v_2$  that are not in  $C$ . Let  $w_1, w_2$  be the neighbours of  $v_3$  not in  $C$ . Note by Lemma 4.34 applied to the path  $v_4v_1v_2$ , either  $u_1u_2 \in E(G)$  or  $u_1, u_2$ , and  $v_2$  are in an  $M$ -gadget with end  $v_2$ . By symmetry, either  $w_1w_2 \in E(G)$  or  $w_1, w_2$ , and  $v_3$  are in an  $M$ -gadget with end  $v_3$ . We will aim to show  $u_1u_2 \notin E(G)$  (and by a symmetrical argument,  $w_1w_2 \notin E(G)$ ) as otherwise we are done. To see this, suppose not. To see this, suppose not. Then  $u_1u_2 \in E(G)$ . By Lemma 4.34 applied to the path  $v_5v_4v_1$ , since  $v_2v_3 \notin E(C)$  it follows that  $v_1, v_2$ , and  $v_3$  are in an  $M$ -gadget with end  $v_1$ . Thus since  $d(v_2) = d(v_3) = 3$ , up to relabelling of  $w_1, w_2$  there exist triangles  $T_1$  and  $T_2$  with  $u_1w_1 \in T_1$  and  $u_2w_2 \in T_2$ . Note that each of  $u_1, u_2, w_1$ , and  $w_2$  has degree at least four as they are not contained in  $C$ . Since  $u_1u_2 \in E(G)$  by assumption, each of  $u_1$  and  $u_2$  sends at least  $\frac{1}{3}$  to  $v_2$  in Step 1. By Lemma 4.46, each of  $v_3$  and  $v_5$  receives at least  $\frac{4}{9}$  in Step 1. Finally,  $v_4$  receives at least  $\frac{1}{6}$  by Observation 4.43. Thus

$$\begin{aligned} ch_f(C) &\geq 5(3) + 2\left(\frac{1}{3}\right) + 2\left(\frac{4}{9}\right) + \frac{1}{6} \\ &> 5\left(\frac{10}{3}\right) \end{aligned}$$

as desired. So we may assume  $u_1u_2 \notin E(G)$ , and by symmetry  $w_1w_2 \notin E(G)$ . By Lemma 4.34, it follows that each of  $v_2$  and  $v_3$  is the end of an  $M$ -gadget with its neighbours outside  $C$ . By Lemma 4.49,  $v_2$  and  $v_3$  each receives at least  $\frac{1}{2}$  during discharging. As above,  $v_5$  receives at least  $\frac{4}{9}$  in Step 1, and  $v_4$  receives at least  $\frac{1}{6}$  by Observation 4.43. It follows that

$$\begin{aligned} ch_f(C) &\geq 5(3) + 2\left(\frac{1}{2}\right) + \frac{4}{9} + \frac{1}{6} \\ &> 5\left(\frac{10}{3}\right) \end{aligned}$$

as desired.  $\square$

**Claim 4.51.** *If  $V(C) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and  $E(C) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_5, v_2v_6\}$ , then  $ch_f(C) \geq v(C) \left(\frac{10}{3}\right)$ .*

*Proof.* Let  $u_1, u_2$  be the neighbours of  $v_3$  that are not in  $C$ . Let  $w_1, w_2$  be the neighbours of  $v_4$  not in  $C$ . Note by Lemma 4.34 applied to the path  $v_2v_1v_3$ , either  $u_1u_2 \in E(G)$  or  $u_1, u_2$ , and  $v_3$  are in an  $M$ -gadget with end  $v_3$ . By symmetry, either  $w_1w_2 \in E(G)$  or  $w_1, w_2$ , and  $v_4$  are in an  $M$ -gadget with end  $v_4$ . We will aim to show  $u_1u_2 \notin E(G)$  (and by a symmetrical argument,  $w_1w_2 \notin E(G)$ ) as otherwise we are done. To see this, suppose not. Then  $u_1u_2 \in E(G)$ . By Lemma 4.34 applied to the path  $v_5v_2v_1$ , since  $v_3v_4 \notin E(C)$  it follows that  $v_1, v_3$ , and  $v_4$  are in an  $M$ -gadget with end  $v_1$ . Thus since  $d(v_3) = d(v_4) = 3$ , up to relabelling of  $w_1, w_2$  there exist triangles  $T_1$  and  $T_2$  with  $u_1w_1 \in T_1$  and  $u_2w_2 \in T_2$ . Note that each of  $u_1, u_2, w_1$ , and  $w_2$  has degree at least four as they are not contained in  $C$ . Since  $u_1u_2 \in E(G)$  by assumption, each of  $u_1$  and  $u_2$  sends at least  $\frac{1}{3}$  to  $v_2$  in Step 1. By Lemma 4.46, each of  $v_4, v_5$ , and  $v_6$  receives at least  $\frac{4}{9}$  in Step 1. Thus

$$\begin{aligned} ch_f(C) &\geq 6(3) + 2 \left( \frac{1}{3} \right) + 3 \left( \frac{4}{9} \right) + \\ &= 6 \left( \frac{10}{3} \right), \end{aligned}$$

as desired. So we may assume  $u_1u_2 \notin E(G)$ , and by symmetry  $w_1w_2 \notin E(G)$ . By symmetry,  $v_5$  is not contained in a triangle with its neighbours outside  $C$ , and nor is  $v_6$ . By Lemma 4.34, it follows that each of  $v_3, v_4, v_5$ , and  $v_6$  is the end of an  $M$ -gadget with its neighbours outside  $C$ . By Lemma 4.49,  $v_3, v_4, v_5$ , and  $v_6$  each receive at least  $\frac{1}{2}$  during discharging. It follows that

$$\begin{aligned} ch_f(C) &\geq 6(3) + 4 \left( \frac{1}{2} \right) \\ &= 6 \left( \frac{10}{3} \right), \end{aligned}$$

as desired. □

Finally, we show the following.

**Claim 4.52.** *If  $C$  is a path of length three, then  $ch_f(C) \geq v(C) \left( \frac{10}{3} \right)$ .*

*Proof.* Let  $C$  be the path  $v_1v_2v_3v_4$ . Let  $u$  be the neighbour of  $v_2$  not contained in  $C$ . By Lemma 4.34 applied to the path  $v_4v_3v_2$ , either  $uv_1 \in E(G)$ , or  $v_1, v_2$ , and  $u$  are contained in an  $M$ -gadget with end  $v_2$ . By Lemma 4.33,  $uv_1$  is not an edge in  $E(G)$ , as otherwise  $uv_1v_2$  is a triangle containing two vertices of degree three. Thus by the structure of  $M$ -gadgets, there exist two disjoint triangles  $T = abca$  and  $T' = a'b'c'a'$  such that, up to relabelling,  $u$  is adjacent to  $a$  in  $T$  and  $a'$  in  $T'$ , and  $v_1$  is adjacent to  $b$  in  $T$  and  $b'$  in  $T'$ .

Next, note that by Lemma 4.34 applied to the path  $v_3v_2v_1$ , either  $bb' \in E(G)$ , or  $v_1$  is the end of an  $M$ -gadget with  $b$  and  $b'$ . First suppose  $bb' \in E(G)$ . In this case, note that by Lemma 4.33, at most one of  $a$  and  $c$  has degree three. Thus  $b$  does not send charge



to at least two of its neighbours in Step 1. Symmetrically, at most one of  $a'$  and  $c'$  has degree three, and so  $b'$  does not send charge to at least two of its neighbours in Step 1. Thus  $v_1$  receives at least  $\frac{1}{3}$  from each of  $b$  and  $b'$  in Step 1. By Claim 4.46,  $v_4$  receives at least  $\frac{4}{9}$  charge in Step 1. Finally, each of  $v_2$  and  $v_3$  receive at least  $\frac{1}{6}$  by Observation 4.43. It follows that

$$\begin{aligned} ch_f(C) &\geq 4(3) + 2\left(\frac{1}{3}\right) + \frac{4}{9} + 2\left(\frac{1}{6}\right) \\ &> 4\left(\frac{10}{3}\right), \end{aligned}$$

as desired. Thus we may assume that  $bb'$  is not an edge in  $G$ . But then by Lemma 4.34 applied to the path  $v_3v_2v_1$ , we have that  $v_1$ ,  $b$ , and  $b'$  are contained in an  $M$ -gadget with end  $v_1$ . By Claim 4.49,  $v_1$  thus receives at least  $\frac{1}{2}$  charge during discharging. By a perfectly symmetrical argument,  $v_4$  receives at least  $\frac{1}{2}$  charge during discharging. As above, each of  $v_2$  and  $v_3$  receive at least  $\frac{1}{6}$  by Observation 4.43. It follows that:

$$\begin{aligned} ch_f(C) &\geq 4(3) + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{6}\right) \\ &= 4\left(\frac{10}{3}\right) \end{aligned}$$

as desired. □

We are now equipped to prove Theorem 1.64.

*Proof of Theorem 1.64.* Suppose not. Let  $G$  be a vertex-minimum counterexample. It follows from Claims 4.44 through 4.52 that  $KY(G) \leq 0$ . Moreover, by Lemma 4.41,  $D_3(G)$  contains a component with at least three vertices. We break into cases depending on the structure of the components in  $D_3(G)$ .

**Case 1:**  $D_3(G)$  contains a component  $C$  with  $v(C) \geq 3$  such that  $C$  is any of the graphs described in Claims 4.50 through 4.52.

In this case,  $C$  contains a path  $P$  of length two ending with a non-leaf vertex,  $v$ . Thus, by applying Lemma 4.34 to  $P$  with  $v$  playing the role of  $x$ , we get that either  $x$  is contained in a triangle with its neighbours not on  $P$ , or that  $x$  is the end of an  $M$ -gadget,  $H$ . By Lemma 4.33,  $x$  is not contained in a triangle with another vertex in  $C$ , and so it follows that  $x$  is the end of an  $M$  gadget,  $H$ . But  $T^3(H) = 2$ , and so  $T^3(G) \geq 2$ . It follows that  $p(G) \leq KY(G) - 2 \leq -2$ , and so  $G$  is not a counterexample.

**Case 2:**  $D_3(G)$  contains no components described in Case 1, but contains a star  $H$  with four vertices.

Let  $V(H) = \{v_1, v_2, v_3, v_4\}$  and  $E(H) = \{v_4v_1, v_4v_2, v_4v_3\}$ . By applying Lemma 4.34 to each of the paths  $v_1v_4v_3$ ,  $v_1v_4v_2$ , and  $v_2v_4v_3$ , we see that  $v_1, v_2$ , and  $v_3$  are each the ends of  $M$ -gadgets, or that they are contained in triangles with their neighbours outside  $H$ . As in the above case, if  $G$  contains an  $M$ -gadget, then  $p(G) \leq -2$ , so  $G$  is not a counterexample. Thus we may assume neither  $v_1, v_2$  nor  $v_3$  is the end of an  $M$ -gadget. Let  $T_1, T_2$ , and  $T_3$  be the triangles containing  $v_1, v_2$  and  $v_3$ , respectively. By Lemma 4.33, these triangles are distinct. If  $T^3(T_1 + T_2 + T_3) \geq 2$ , then  $p(G) \leq -2$  and  $G$  is not a counterexample. Thus we may assume the triangles share some vertices. There are two cases to consider: either there exists a vertex contained in all three triangles, or this does not happen and instead every pair of triangles shares a vertex. If every pair of triangles shares a vertex, then since  $H + T_1 + T_2 + T_3$  is 4-critical,  $G = H + T_1 + T_2 + T_3$ . But then  $p(G) = 5(7) - 3(12) - 1 = -2$ , and so  $G$  is not a counterexample. Thus we may assume that  $V(T_1) \cap V(T_2) \cap V(T_3) = \{u\}$ , for some vertex  $u \in G$ . In this case, note that  $d(u) \geq 6$ . Moreover,  $u$  is adjacent to at least three vertices that are adjacent to  $H$  but not in  $H$ ; thus  $u$  neighbours at least three vertices of degree greater than three. It follows that  $u$  sends at least  $\frac{8}{9}$  charge to each of  $v_1, v_2$ , and  $v_3$  in Step 1 of the discharging process. Thus  $ch_f(H) \geq 3\left(\frac{8}{9}\right) + 4(3) = \frac{44}{3} = 4\left(\frac{10}{3}\right) + \frac{4}{3}$ . Note that every other component  $C$  in  $D_3(G)$  has final charge at least  $v(C)\left(\frac{10}{3}\right)$  and every vertex of degree at least four has final charge at least  $\frac{10}{3}$ . Thus the sum of the charges is at least  $v(G)\left(\frac{10}{3}\right) + \frac{4}{3}$ . Moreover since potential is integral, it follows that the sum of the charges is at least  $v(G)\left(\frac{10}{3}\right) + 2$ . Thus  $KY(G) \leq -3$ . Moreover, as  $G$  contains a triangle,  $T^3(G) \geq 1$ . Thus  $p(G) \leq -4$ , which contradicts the fact that  $G$  is a counterexample.

**Case 3:**  $D_3(G)$  contains no components described in Cases 1 or 2, but contains a path  $H$  of length 2.

Let  $H = v_1v_2v_3$ . Note that by Claim 4.47, the final charge of  $H$  is strictly greater than  $v(H)\left(\frac{10}{3}\right)$ . Moreover, every other component  $C$  of  $D_3(G)$  has final charge at least  $v(C)\left(\frac{10}{3}\right)$  and every vertex of degree at least four has final charge at least  $\frac{10}{3}$ . Since potential is integral, it follows that the sum of the charges is at least  $v(G)\left(\frac{10}{3}\right) + 1$ , and so  $KY(G) \leq -\frac{3}{2}$ . But since  $KY(G)$  is also integral,  $KY(G) \leq -2$ . Thus  $p(G) \leq -2$ , and so  $G$  is not a counterexample.  $\square$

# Chapter 5

## Sparse 4-critical graphs have low circular chromatic number

In this chapter we will prove Theorem 1.79, Theorem 1.82 and Theorem 1.83, as well as Observation 1.77 and Observation 5.1 We also show:

**Observation 5.1.** For any positive integers  $p, q$  satisfying  $3 \leq \frac{p}{q} < \frac{7}{2}$ , there exists a 4-critical graph with no  $(p, q)$ -colouring which satisfies

$$e(G) < \frac{27v(G) - 20}{15}.$$

We give a brief overview of the proof of Theorem 1.79 and Theorem 1.82. Let  $G$  be a vertex minimal counterexample to Theorem 1.79 or Theorem 1.82. From a fundamental result of Gallai, we know that the subgraph  $D_3(G)$  has every block isomorphic to either an odd cycle or a clique. It is easy to see that the cliques in  $D_3(G)$  have size at most three, or  $G$  is isomorphic to  $K_4$ . The first part of the proof is to show that if any block of  $D_3(G)$  is isomorphic to an odd cycle, then  $G$  is isomorphic to an odd wheel. This is done by taking an odd cycle  $C$ , deleting it, and characterizing when 3-colouring of  $G - C$  extends to a  $(7, 2)$ -colouring. In the cases where we cannot extend, the neighbours of  $C$  will form an independent set, and if  $G$  is not a wheel, we will be able to reconfigure the colouring so that it will be able to extend to a  $(7, 2)$ -colouring.

The next part of the proof is to show that assuming we have no odd cycle blocks or  $K_4$  blocks, that every component of  $D_3(G)$  is isomorphic to a path or has at most 4 vertices. This proof follows the same themes as the odd cycle reduction. We delete vertices from  $D_3(G)$  and ask when we can extend a 3-colouring to a  $(7, 2)$ -colouring, and show that unless each component is isomorphic to a path or a claw, we can always extend.

The remaining arguments are to show that a vertex minimal counterexample to Theorem 1.79 cannot contain a claw component. Then we use reconfiguration arguments to

show that vertices close to path components in  $D_3(G)$  have reasonably large degree, and finish the proof via discharging.

The chapter is structured as follows. In Section 5.1 we introduce the basics of circular colouring that will be needed for the paper. We also prove Observation 5.1 and give the examples of 4-critical graphs with no  $(7, 2)$ -colouring whose components of the Gallai Tree are isomorphic to claws and arbitrarily long paths, proving part of Theorem 1.82. In Section 5.2 we review the Hell-Nešetřil indicator construction and prove Observation 1.77. In Section 5.3, we prove that for a 4-critical graph  $G$  with no  $(7, 2)$ -colouring, if  $D_3(G)$  contains an odd cycle, then  $G$  is isomorphic to an odd wheel. In Section 5.4 we prove that for given a  $k$ -critical graph  $G$ ,  $k \geq 4$ , if  $D_{k-1}(G)$  contains a clique of size  $k - 1$ , then either  $G$  is isomorphic to  $K_k$ , or admits a  $(2k - 1, 2)$ -colouring. In Section 5.5, we prove that the Gallai Tree of 4-critical graphs with no  $(7, 2)$ -colouring can only have components isomorphic to claws, paths, or  $G$  is isomorphic to an odd wheel. In Section 5.6 we prove that path components are close to vertices of large degree. In Section 5.7 we provide the discharging argument to finish the proof.

## 5.1 Preliminaries, Sharpness and Examples

In this section we collect the basic results from colouring that we will make use of throughout the chapter. We also prove Observation 5.1. We also give some examples of 4-critical graphs with no  $(7, 2)$ -colouring.

### 5.1.1 A sparse 4-critical graph with a $(7, 2)$ -colouring

We will need to know what happens when we have a circular colouring that does not use all of the colours. For this we need the notion of lower parents.

**Definition 5.2.** Let  $p$  and  $q$  be positive integers where  $\frac{p}{q} \geq 2$ , and  $\gcd(p, q) = 1$ . The unique positive integers  $p'$  and  $q'$  where  $p' < p$  that satisfy the equation

$$pq' - qp' = 1$$

are called the *lower parents* of  $p$  and  $q$ .

We will say that two graphs  $G$  and  $H$  are *homomorphically equivalent* if  $G \rightarrow H$  and  $H \rightarrow G$ .

**Lemma 5.3** ([21] Lemma 6.6). *Let  $p$  and  $q$  be positive integers that are relatively prime, and satisfy  $\frac{p}{q} \geq 2$ . Let  $p'$  and  $q'$  be the lower parents of  $p$  and  $q$ . Then for any vertex  $x \in V(G_{p,q})$ , the graph  $G_{p,q} - x$  is homomorphically equivalent to  $G_{p',q'}$ .*

Observe that the lower parents of 7 and 2 are 3 and 1. Hence we have the following corollary:

**Corollary 5.4.** *Let  $p$  and  $q$  be relatively prime. If a graph  $G$  admits a  $(p, q)$ -colouring that does not use all  $p$  colours, then  $G$  admits a  $(p', q')$ -colouring where  $p', q'$  are the lower parents of  $p$  and  $q$ . In particular, if  $G$  admits a  $(7, 2)$ -colouring that does not use all 7 colours, then  $G$  admits a 3-colouring.*

With this we will prove that there is a graph on seven vertices that has eleven edges and circular chromatic number  $\frac{7}{2}$ .

Recall that the *Moser Spindle*, denoted  $M$ , is the unique 4-Ore graph on 7 vertices. For clarity, we have  $V(M) = \{a, b, c, d, e, f, g\}$ , and  $E(M) = \{ab, ac, af, ag, bc, bd, cd, de, ef, eg, fg\}$ .

**Observation 5.5.** The Moser Spindle has seven vertices, eleven edges, and circular chromatic number  $\frac{7}{2}$ .

*Proof.* As the Moser Spindle is 4-Ore, it is 4-critical, and hence does not have a 3-colouring. The map where we colour  $a$  with 0,  $f$  with 2,  $g$  with 4,  $e$  with 6,  $d$  with 1,  $b$  with 5,  $c$  with 3 is a  $(7, 2)$ -colouring. Finally it is easy to check that for every  $p \in \{4, 5, 6\}$  there is no integral  $q$  where  $\gcd(p, q) = 1$  such that  $3 \leq \frac{p}{q} < \frac{7}{2}$ . Therefore  $\chi_c(M) = \frac{7}{2}$ .  $\square$

An astute reader may realize that the Moser Spindle is not isomorphic to  $G_{7,2}$ , and this implies that there are strict subgraphs of  $G_{7,2}$  with circular chromatic number  $\frac{7}{2}$ . This turns out to be the case for any tuples  $(p, q)$  unless  $p = 2k + 1$  and  $q = k$ , or  $q = 1$ . Rather surprisingly, you can find a subgraph with roughly  $O(\sqrt{e(G_{p,q})})$  edges on  $p$  vertices with circular chromatic number  $\frac{p}{q}$  [52]. By appealing to the Kostochka-Yancey Theorem, the Moser Spindle has the fewest edges for a graph on 7 vertices that is also 4-critical and has circular chromatic number  $\frac{7}{2}$ . Now the sharpness claim follows immediately.

**Corollary 5.6.** *For integers  $p$  and  $q$ , satisfying  $2 \leq \frac{p}{q} < \frac{7}{2}$ , there exists a graph that is 4-critical with no  $(p, q)$ -colouring that has*

$$e(G) < \frac{27v(G) - 20}{15}.$$

*Proof.* The Moser Spindle has circular chromatic number  $\frac{7}{2}$ , seven vertices and eleven edges. Observe that  $11 < \frac{169}{15}$ .  $\square$

Of course, this is not the most satisfying sharpness example. It would be much more interesting if an infinite family were found. Nevertheless, it does show that the bound in Observation 1.77 is sharp with respect to the values of  $p$  and  $q$ .

## 5.1.2 Useful Background Lemmas

We will need one more idea from circular colouring. It is convenient to be able to talk about intervals of colours. If we have  $p$  colours, and integers  $i, j \in \{0, 1, \dots, p-1\}$  we denote  $[i, j]$  as the set of colours  $\{i, i+1, \dots, j\}$  where the values are reduced modulo  $p$ . When  $p$  is fixed, we will assume intervals are taken modulo  $p$ .

Given a graph  $G$  and a vertex  $v$ , let  $N_G(v)$  denote the neighbourhood of  $v$  in  $G$ . If there is no possibility of confusion we will just use  $N(v)$ . Observe that in any  $(p, q)$ -colouring  $f$ , for any vertex  $v$  we have

$$f(v) \in \bigcap_{u \in N_G(v)} N_{G_{p,q}}(f(u))$$

Given a graph  $G$ , and an induced subgraph  $F$  of  $G$  equipped with a  $(p, q)$ -colouring  $f$  of  $F$ , we say that the set of available colours for  $v$  in  $G$  is  $[0, p-1]$  if  $v$  has no neighbours in  $F$ , and

$$\bigcap_{u \in N_F(v)} N_{G_{p,q}}(f(u))$$

otherwise.

A very useful fact is that when  $2 < \frac{p}{q} < 4$ , the set of available colours is always an interval.

**Lemma 5.7** ([4]). *If  $\frac{p}{q} < 4$ , then for any graph  $G$ , any  $(p, q)$ -colouring of  $G$ , and any vertex  $v \in V(G)$ , the set of available colours of  $v$  is an interval.*

We will use this fact without reference. We also record some facts about  $k$ -critical graphs which we use without reference. We start off with a well known observation.

**Observation 5.8.** A  $k$ -critical graph is  $(k-1)$ -edge-connected. In particular, the minimum degree of a  $k$ -critical graph is at least  $k-1$ .

Recall that a *block* of a graph is a maximal 2-connected subgraph. The Gallai-Tree Theorem gives structure to subgraph induced by the vertices of degree  $k-1$ .

**Theorem 5.9** ([14], Gallai-Tree Theorem). *Let  $G$  be a  $k$ -critical graph, and  $B$  be the set of vertices of  $G$  with degree  $k-1$ . Then every block of  $G[B]$  is a clique or an odd cycle.*

We will call the graph  $G[B]$  the *Gallai Tree* of  $G$ . We will use Theorem 5.9 without reference. For 4-critical graphs, this implies the following.

**Corollary 5.10.** *In a 4-critical graph that is not  $K_4$ , every block of the Gallai Tree is either isomorphic to  $K_1$ , isomorphic to  $K_2$ , or an odd cycle.*

*Proof.* Clearly  $K_4$  is 4-critical, and hence the largest clique a 4-critical graph can have is  $K_4$ . The rest follows immediately from the Gallai-Tree Theorem.  $\square$

As some notation for this chapter, we use  $d_t(v)$  to denote the number of vertices in the neighbourhood of  $v$  with degree  $t$ . If  $G$  is equipped with a  $k$ -colouring, we let  $N_t(X)$  denote the set of neighbours of  $X$  coloured  $t$ . For ease throughout the chapter, when given a 3-colouring, we will always assume the colours used are from the set  $\{0, 2, 4\}$ . This is so we can extend to a  $(7, 2)$ -colouring without any cumbersome change in values.

### 5.1.3 Examples of 4-critical graphs with no $(7, 2)$ -colourings

In this section we collect the known examples from the literature of 4-critical graphs with no  $(7, 2)$ -colouring, and provide an operation which preserves the property of being 4-critical and having no  $(7, 2)$ -colouring. This operation produces to the best of the authors knowledge a new infinite family of 4-critical graphs with no  $(7, 2)$ -colouring (the family is almost assuredly not new, the main point is to show they have no  $(7, 2)$ -colouring). In particular this family demonstrates that there are 4-critical graphs with no  $(7, 2)$ -colourings with claw components or arbitrarily long paths in their Gallai Tree.

Before we describe the operation, we collect some examples from the literature. As a corollary of a theorem in [45], if the complement of a graph  $G$  is disconnected, then  $\chi_c(G) = \chi(G)$ . As the complement of an odd wheel is disconnected, we obtain our first example.

**Observation 5.11.** The odd wheel  $W_n$  has  $\chi_c(W_n) = 4$ .

Our next example uses a well known construction. Given a graph  $G$ , we let  $M(G)$  denote the *Mycielskian* of  $G$ , where  $V(M(G)) = V(G) \cup V'(G) \cup \{u\}$ ,  $V'(G) = \{x' | x \in V(G)\}$ , and  $E(M(G)) = E(G) \cup \{xy' | xy \in E(G)\} \cup \{y'u | y' \in V'(G)\}$ .

**Theorem 5.12** ([6]).  $\chi_c(M(C_{2k+1})) = 4$ , and  $M(C_{2k+1})$  is 4-critical.

Observe that the Gallai Tree of the Mycielskian of an odd cycle is a collection of isolated vertices.

We do not define the family here as they do not contain vertices of degree 3, but we note that in [51], an infinite family of 4-regular 4-critical graphs with  $\chi_c(G) = \chi(G)$  was found.

Now we describe an operation which preserves 4-criticality and not having a  $(7, 2)$ -colouring. The operation given here generalizes the operation called the “iterated Mycielskian” in [30] when restricted to 4-critical graphs.

**Definition 5.13.** Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. Let  $v \in V(G)$  such that  $N(v) = \{x, y, z\}$ . The  $C_6$ -expansion of  $G$  with respect to  $v$  is the graph  $G'$  obtained by deleting  $v$  from  $G$ , and adding four new vertices  $x', y', z', w$  with edges  $x'y, x'z, x'w, y'x, y'z, y'w, z'x, z'y$  and  $z'w$ .

**Lemma 5.14.** *Let  $G$  be the  $C_6$ -expansion of a graph  $H$  at a vertex  $v$ . Then  $G$  is 4-critical and has no  $(7, 2)$ -colouring.*

*Proof.* Let the neighbours of  $v$  in  $H$  be  $x, y, z$ , with the new vertices in  $G$  being  $x', y', z', w$  with adjacencies as in Definition 5.13. First we observe that  $\chi(G) \leq 4$ . Let  $f$  be a 4-colouring of  $H$ . Let  $f'$  be a colouring of  $G$  where for all  $t \in V(G) \setminus \{x', y', z', w\}$  let  $f'(t) = f(t)$ ,  $f'(x') = f'(y') = f'(z') = f(v)$  and give  $w$  any colour that is not  $f(v)$ . This is a 4-colouring of  $G$ , and hence  $\chi(G) \leq 4$ .

Now we prove that  $\chi_c(G) > \frac{7}{2}$ . Suppose not and let  $f$  be a  $(7, 2)$ -colouring of  $G$ . Observe that if  $f(x') = f(y') = f(z')$ , then by identifying  $x', y'$  and  $z'$  into one vertex and deleting  $w$  we obtain a  $(7, 2)$ -colouring of  $H$ , a contradiction.

Suppose that  $f(x') = 0$ . Further suppose that  $f(y') = 0$ . Then the image of the neighbourhood of  $z'$  is contained in the neighbourhood of 0 in  $G_{7,2}$ , and so we can recolour  $z'$  to 0. But then there exists a  $(7, 2)$ -colouring of  $G$  where  $x', y'$  and  $z'$  receive the same colour, a contradiction. Thus we can assume that  $f(y') \neq 0$ , and by symmetry  $f(z') \neq 0$ . If  $f(y') = 1$  and  $f(z') = 6$ , then again the image of the neighbourhood of both  $y'$  and  $z'$  is contained in the neighbourhood of 0 in  $G_{7,2}$ , and so we can recolour  $y'$  and  $z'$  to 0, and obtain a  $(7, 2)$ -colouring of  $H$ , a contradiction. A similar argument works if  $f(y') = 2$  and  $f(z') = 6$ . Now assume that  $f(y') = 2$ . Then  $f(z') \in \{0, 1, 2, 3\}$  or else there is no available colour for  $w$ . By the previous cases, it follows that  $f(z') = 3$ . But this implies that we can recolour  $z'$  and  $x'$  to 2, and again contradict that  $H$  has no  $(7, 2)$ -colouring. All other cases follow similarly, and thus  $G$  has no  $(7, 2)$ -colouring. Observe this also implies that  $\chi(G) \geq 4$ , and hence  $\chi(G) = 4$ .

Therefore to finish the proof, we just need to show that for every edge  $e \in E(G)$ ,  $G - e$  is 3-colourable. First let  $e$  be incident to  $w$ . Let  $f$  be a 3-colouring of  $H - v$ . We extend  $f$  to a 3-colouring of  $G - e$  in the following manner: as  $x'$  has only two neighbours in  $\{x, y, z\}$ , there exists a colour  $c$  which is adjacent to both  $f(y)$  and  $f(z)$  in  $G_{7,2}$ . In a similar manner, we can pick colours for  $y'$  and  $z'$ . Then  $w$  sees at most two colours, and so there is a colour available for  $w$ , and we obtain a  $(7, 2)$ -colouring of  $G$ .

Now suppose that  $e$  is incident to  $x'$  but not  $w$ . Without loss of generality, let  $e = zx'$ . Then take a 3-colouring  $f$  of  $H - v$ , and extend  $f$  by first picking a colour  $c_1$  and  $c_2$  for  $y'$  and  $z'$  respectively, and then as  $x'$  is only adjacent to  $w$  and  $z$ , we can pick  $c_1$  for  $x$  as well, and then there exists a colour for  $w$ . By symmetry we can assume that  $e$  is not incident to any of  $x', y', z'$ .

Now let  $f$  be a 3-colouring of  $H - e$ . Then extend  $f$  to a 3-colouring of  $G - e$  by colouring  $x', y', z'$  the same colour as  $v$ , and then giving  $w$  any colour left over. Thus it follows that  $G - e$  has a 3-colouring for every edge  $e$ , and thus  $G$  is 4-critical with no  $(7, 2)$ -colouring.  $\square$

**Corollary 5.15.** *There is a 4-critical graph with no  $(7, 2)$ -colouring whose Gallai Tree is a claw.*



*Proof.* Let  $H = K_4$ , and  $G$  a  $C_6$ -expansion of any vertex in  $H$ . Then by Lemma 5.14,  $G$  is 4-critical, has no  $(7, 2)$ -colouring, and it is easily seen that the Gallai Tree of  $G$  is just the claw.  $\square$

**Corollary 5.16.** *For any odd positive integer  $t$ , there is a 4-critical graph with no  $(7, 2)$ -colouring whose Gallai Tree contains a component isomorphic to a path of length  $t$ .*

*Proof.* Fix an odd positive integer  $t$ . Let  $G$  be a  $C_6$ -expansion of any of the degree 3 vertices in  $W_{t+4}$ . Then the Gallai Tree of  $G$  contains a component isomorphic to the claw, and a path of length  $t$ .  $\square$

## 5.2 Hell-Nešetřil indicator constructions

In this section we prove that the  $t = 3$  case of Question 1.74 implies that 4-critical graphs with no  $(7, 2)$ -colouring satisfy  $e(G) \geq \frac{27v(G)-20}{15}$ .

We first review the basics of the indicator construction. Let  $I$  be a graph with distinguished vertices  $x$  and  $y$ , and suppose that there is an automorphism (an *automorphism* is an from  $I \rightarrow I$ ) which maps  $x$  to  $y$ . Let  $G$  be a graph. We will say that  $G * I$  is the graph obtained by taking every edge  $uv \in E(G)$ , deleting the edge, and adding the graph  $I$  to  $G$  where we identify  $u$  with  $x$  and  $v$  with  $y$ . Observe this is well defined because there is an automorphism of  $I$  which maps  $x$  to  $y$  (it could have been defined without this condition, but for all purposes we will have this property).

**Definition 5.17.** Let  $G$ ,  $H$ , and  $I$  be graphs, where  $V(H) = V(G)$ , and  $x$  and  $y$  are distinguished vertices of  $I$ , where there is an automorphism of  $I$  which sends  $x$  to  $y$ . Suppose that for any edge  $uv \in E(G)$ , there exists a homomorphism  $f : I \rightarrow H$  such that  $f(x) = u$  and  $f(y) = v$ . Further, suppose for every homomorphism  $f : I \rightarrow H$ , we have  $f(x)f(y) \in E(G)$ . Then we say that  $(I, x, y)$  is an *indicator* for  $G$  and  $H$ .

The following is easily verified from the definition.

**Lemma 5.18** ([21], Lemma 5.5). *Suppose  $(I, x, y)$  is an indicator for  $G$  and  $H$ . Then for any graph  $K$ ,  $K \rightarrow G$  if and only if  $K * I \rightarrow H$ .*

We can use Lemma 5.18 to deduce the non-existence of homomorphisms in some instances.

**Corollary 5.19.** *Suppose that  $(I, x, y)$  is an indicator for  $G$  and  $H$ . If  $K$  is  $G$ -critical, then  $K * I \not\rightarrow H$ .*

*Proof.*  $K$  is  $G$ -critical, so  $K \not\rightarrow G$ . By Lemma 5.18, this implies that  $K * I \not\rightarrow H$ .  $\square$

This is of course not useful unless there exist indicator constructions. Here is a particularly useful class of indicators.

**Corollary 5.20** ([21], proof of Corollary 5.6). *Let  $I$  be the path of length  $k - 2$  with endpoints  $x, y$ . Then  $(I, x, y)$  is an indicator for  $K_k$  and  $C_k$ .*

This so far is not useful for critical graphs. However, one can observe that path indicators with the endpoints as the distinguished vertices preserve criticality.

**Proposition 5.21.** *Let  $(I, x, y)$  be an indicator for connected graphs  $G$  and  $H$  which contain at least one edge, where  $I$  is a path with at least one edge and  $x$  and  $y$  are the two endpoints for the path. Let  $K$  be a  $G$ -critical graph. Then  $K * I$  is  $H$ -critical.*

*Proof.* From Corollary 5.19, we have that  $K * I \not\rightarrow H$ . Now consider any edge  $e \in K * I$ . Then  $e$  is contained in a copy of  $I$ , where this copy of  $I$  replaced an edge  $e'$  in  $K$ . By  $G$ -criticality,  $K - e' \rightarrow G$ . Hence  $(K - e') * I \rightarrow H$  by Lemma 5.18.  $K * I - e$  may have uncoloured vertices, and since  $I$  is a path, at least one of these vertices has degree one. Now we can map  $K * I - e \rightarrow (K - e') * I$  by repeatedly mapping the degree one vertices in the copy of  $I$  containing  $e$  onto some vertex adjacent to their neighbour. But then as homomorphisms compose,  $K * I - e \rightarrow H$ , and hence  $K * I$  is  $H$ -critical.  $\square$

As notation throughout this chapter let  $P_n$  denote the path on  $n$  vertices. Now to finish the intended goal of the section, we prove that  $P_4$  with the endpoints as distinguished vertices is an indicator for  $G_{7,2}$  and  $C_7$ .

**Observation 5.22.** Let  $P_4$  be a path with endpoints  $x$  and  $y$ . Then  $(P_4, x, y)$  is an indicator for  $G_{7,2}$  and  $C_7$ .

*Proof.* We just need to check the possible homomorphisms of  $P_4$ . Suppose  $V(P_4) = \{x, x_1, x_2, y\}$ , with edges  $xx_1, x_1x_2, x_2y$ . As  $G_{7,2}$  is vertex transitive, it suffices to consider the case when we colour  $x$  with 0. The following are  $C_7$ -colourings of  $P_4$  which give the necessary adjacencies. The  $C_7$ -colouring of  $P_4$  where we colour  $x$  with 0,  $x_1$  with 3,  $x_2$  with 6 and  $y$  with 2. The  $C_7$ -colouring of  $P_4$  where we colour  $x$  with 0,  $x_1$  with 3,  $x_2$  with 0 and  $y$  with 3. The  $C_7$ -colouring of  $P_4$  where we colour  $x$  with 0,  $x_1$  with 4,  $x_2$  with 0, and  $y$  with 4. The  $C_7$ -colouring of  $P_4$  where we colour  $x$  with 0,  $x_1$  with 4,  $x_2$  with 1 and  $y$  with 5.

Now we just need to show the non-adjacencies. Suppose that both  $x$  and  $y$  are coloured 0. Then both of  $x_1$  and  $x_2$  would need to get a colour from  $\{3, 4\}$ , but that is impossible.

Suppose that  $y$  is coloured 1, then  $x_1$  must be coloured 4, as if it is coloured 3 we cannot colour  $x_2$  in a way that will be compatible with  $y$  being coloured 1. But if  $x_1$  is coloured 4, none of the neighbours of 4 in  $C_7$  are adjacent to 1, and  $y$  cannot be coloured 1. The analysis is the same if  $y$  is coloured 6.  $\square$

Therefore we have the following corollary

**Corollary 5.23.** *For any  $G_{7,2}$ -critical graph  $G$ , the graph  $G * P_4$  is  $C_7$ -critical.*

Now we can prove the observation.

**Observation 5.24.** If the  $t = 3$  case is true in Question 1.74, then for every  $G_{7,2}$ -critical graph  $G$ ,

$$e(G) \geq \frac{27v(G) - 20}{15}$$

*Proof.* Let  $G$  be a  $G_{7,2}$ -critical graph. By Corollary 5.23, the graph  $G * P_4$  is  $C_7$ -critical. Observe that  $e(G * P_4) = 3e(G)$  and  $v(G * P_4) = 2e(G) + v(G)$ . Appealing the  $t = 3$  case of Question 1.74, we have

$$e(G * P_4) \geq \frac{27v(G * P_4) - 20}{23}.$$

Thus

$$3e(G) \geq \frac{27(2e(G) + v(G)) - 20}{23}.$$

Rearranging we have

$$69e(G) \geq 54e(G) + 27v(G) - 20.$$

Now simplifying we have

$$e(G) \geq \frac{27v(G) - 20}{15}$$

as desired. □

If we apply the same analysis using the bound on the density of  $C_7$ -critical graphs in Theorem 1.75, we get a bound on  $G_{7,2}$ -critical graphs which to the best of my knowledge is the best known (however, this bound does not even beat the Kostochka-Yancey bound for 4-critical graphs - which suggests many improvements should be possible).

**Corollary 5.25** ([43]). *If  $G$  is a  $G_{7,2}$ -critical graph, then*

$$e(G) \geq \frac{17v(G) - 2}{11}.$$

### 5.3 Odd cycles in the Gallai-Tree

The point of this section is to prove that the class of 4-critical graphs with no  $(7, 2)$ -colouring and whose Gallai Tree contains an odd cycle is exactly the class of odd wheels.

The set up is to first prove a series of list colouring claims, which will allow us to assert that the neighbours of an odd cycle in the Gallai Tree form an independent set. If

the independent set has size one, then the graph is an odd wheel, and otherwise using a reconfiguration argument we will be able to find a  $(7, 2)$ -colouring.

We start off with some definitions. A  $k$ -list-assignment  $L$  is a function which assigns a set of at least  $k$  colours to each vertex (however, without loss of generality we will always assume that each list is size exactly  $k$ ). For a vertex  $v$ , we will denote  $L(v)$  as the *list of  $v$* . A 4-interval-list assignment  $L$  is a 4-list-assignment where for all  $v \in V(G)$ ,  $L(v) \subseteq \{0, 1, 2, 3, 4, 5, 6\}$ , and each list has size at least four, and contains an interval of size at least 4. A list assignment is *uniform* if all vertices receive the same list. An  $L$ -colouring is a proper colouring where each vertex  $v$  gets a colour from  $L(v)$ . An  $L$ - $(7, 2)$ -colouring is an  $L$ -colouring which is also a  $(7, 2)$ -colouring. Given a graph  $G$  equipped with a list assignment  $L$ , and a subgraph  $H$  of  $G$ , the list assignment induced by  $H$  is simply the list assignment  $L$  on the vertices of  $H$ . A vertex is *precoloured* if  $|L(v)| = 1$ .

### 5.3.1 4-interval-list-colouring paths

The main point of this subsection is to characterize when we can list colour paths under the assumption that the endpoints have constrained lists. In particular we will characterize when we can colour  $P_n$  when both of the endpoints have a list of size 2 that forms an interval, and the internal vertices have lists of size 4 that form intervals.

We start with an easy observation.

**Proposition 5.26.** *Let  $P_n$  be a path with endpoints  $x$  and  $y$  (possibly not distinct if  $n = 1$ ). Let  $L$  be a list assignment where  $x$  is precoloured from  $\{0, \dots, 6\}$ , and the list assignment induced on  $P_n - x$  is a 4-interval list assignment. Then  $P_n$  is  $L$ - $(7, 2)$ -colourable.*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$ , then the precolouring is an  $L$ - $(7, 2)$ -colouring. So  $n \geq 2$ . Let  $x'$  be the neighbour of  $x$ . By the pigeon hole principle,  $|L(x') \cap N_{G_{7,2}}(L(x))| \geq 1$ . Now colour  $x'$  with some colour from  $L(x') \cap N_{G_{7,2}}(L(x))$  and delete  $x$ . If  $n = 2$  then we are done, and otherwise the result follows by induction.  $\square$

We note that the above Proposition actually works even when the lists do not form intervals, as long as they are from the set  $\{0, \dots, 6\}$ .

Note that it is possible to satisfy the hypothesis of the above claim and have exactly one  $L$ - $(7, 2)$ -colouring. Now instead of precolouring one end of the path, we will restrict both endpoints of the path but not as severely.

**Lemma 5.27.** *Let  $P_n$  be a path with endpoints  $x$  and  $y$ . Suppose  $L$  is a list assignment such that  $P_n - x - y$  induces a 4-interval-list assignment,  $L(x)$  and  $L(y)$  both forming intervals modulo 7,  $|L(x)| \geq 2$  and  $|L(y)| \geq 3$ . Then there exists an  $L$ - $(7, 2)$ -colouring.*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$ , the claim is trivial. If  $n = 2$ , then let  $c \in L(x)$ . Unless  $L(y) = \{c - 1, c, c + 1\}$ , then we can colour  $x$  with  $c$  and extend the

colouring. If  $L(y) = \{c - 1, c, c + 1\}$ , colour  $x$  with a colour in  $L(x) - c$ , and extend the colouring in any fashion. Now we can assume that  $n \geq 3$ . Let  $u$  be the neighbour of  $x$ . Observe that if there is a  $c \in L(x)$  such that not all of  $\{c - 1, c, c + 1\}$  are contained in  $L(u)$ , we can colour  $x$  with  $c$ , delete  $x$  and apply induction. Without loss of generality suppose that  $L(x) = \{0, 1\}$ . Then by the above observation, unless  $L(u) = \{6, 0, 1, 2\}$ , we can colour  $x$  with either 0 or 1 and apply induction. Thus we may assume  $L(u) = \{6, 0, 1, 2\}$ . Now let  $v$  be the neighbour of  $u$  which is not  $x$ . If  $L(v)$  does not contain all of  $\{1, 2, 3\}$ , then we can colour  $x$  with 0,  $u$  with 2, and the set of available colours for  $v$  has size at least 2, so we can apply induction (or simply colour  $v$  and finish the colouring if  $n = 3$ ). Thus  $L(v)$  contains all of  $\{1, 2, 3\}$ . Therefore  $L(v)$  is one of three possible lists,  $\{0, 1, 2, 3\}$ ,  $\{1, 2, 3, 4\}$ , or  $\{1, 2, 3\}$ . Regardless of which list  $L(v)$  is, colour  $x$  with 1 and  $u$  with 6. In all cases, either we can finish the colouring of the path, or the set of available colours for  $v$  is at least 2, and we can apply induction.  $\square$

The above lemma is best possible in the sense that we cannot make both endpoints have list size 2, even if they form an interval. To see this, consider the following assignment of  $P_3$  with vertices  $x, y, z$  where we have edges  $xy$  and  $yz$ . Let  $L(x) = \{0, 1\}$ ,  $L(y) = \{5, 6, 0, 1\}$ ,  $L(z) = \{5, 6\}$ . It is easy to see there is no  $L$ -(7, 2)-colouring.

### 5.3.2 4-interval list colouring cycles

Now we will turn our focus onto proving list colouring claims of odd cycles (or in some cases cycles). We now give a definition which is cooked up just to be able to apply Proposition 5.26.

**Definition 5.28.** A 4-interval-list-assignment  $L$  of  $C_n$  is *safe* if for some edge  $uv \in E(C_n)$ , there is a colour  $c \in L(u)$  such that none of  $\{c - 1, c, c + 1\}$  reduced modulo 7 are in  $L(v)$ .

**Observation 5.29.** Every safe 4-interval-list-assignment of  $C_n$  admits an  $L$ -(7, 2)-colouring.

*Proof.* Pick an edge  $uv \in E(C_n)$  such that there is a colour  $c \in L(u)$  where none of  $\{c - 1, c, c + 1\}$  reduced modulo 7 are in  $L(v)$ . Now consider  $C_n - uv$ . Colour  $u$  with  $c$ . Then we satisfy the conditions of Proposition 5.26, so consider any  $L$ -(7, 2)-colouring ensured by the claim. By design,  $u$  gets colour  $c$ , and  $v$  gets some colour that is not  $c - 1, c$  or  $c + 1$ , and hence we have a  $L$ -(7, 2)-colouring of  $C_n$ .  $\square$

There is a harder case we can deal with.

**Definition 5.30.** A 4-interval-list-assignment  $L$  of  $C_n$  is *nearly safe* if there exists a vertex  $v$  with neighbours  $v_1, v_2$  where  $L(v_1) = L(v_2)$  and  $L(v)$  shares at most three colours with  $L(v_1)$ .

**Proposition 5.31.** *Let  $L$  be a 4-interval-list-assignment of  $C_n$  which is nearly safe. Then there exists a  $L$ -(7, 2)-colouring of  $C_n$ .*

*Proof.* Let  $v$  be a vertex with neighbours  $v_1$  and  $v_2$  where  $L(v_1) = L(v_2)$  and  $L(v)$  shares at most 3 colours with  $L(v_1)$ . Without loss of generality suppose that  $L(v) = \{0, 1, 2, 3\}$ . We consider cases.

If  $L(v)$  shares exactly 1 colour with  $L(v_1)$ , then either  $L(v_1) = \{3, 4, 5, 6\}$ , or  $L(v_1) = \{0, 6, 5, 4\}$ , which implies that  $L$  is a safe list-assignment. This case follows from Observation 5.29.

If  $L(v)$  shares exactly 2-colours with  $L(v_1)$ , then  $L(v_1) = \{2, 3, 4, 5\}$  or  $L(v_1) = \{1, 0, 6, 5\}$ . Again, either of these lists imply that  $L$  is safe, and thus the result follows from Observation 5.29.

Therefore we can assume that  $L(v_1) = \{1, 2, 3, 4\}$  or  $L(v_1) = \{6, 0, 1, 2\}$ . Without loss of generality assume that  $L(v_1) = \{1, 2, 3, 4\}$ . Now colour  $v$  with 0, and remove 1 from the lists of  $v_1$  and  $v_2$ , and delete  $v$ . The remaining path satisfies the conditions of Proposition 5.27. Therefore we can find a  $L$ -(7, 2)-colouring of the path which extends to an  $L$ -(7, 2)-colouring of the entire graph, as desired.  $\square$

Finally we can cover the remaining non-uniform cases.

**Proposition 5.32.** *Let  $L$  be a 4-interval-list-assignment of  $C_{2k+1}$  which is not uniform, not safe and not nearly safe. Then there is an  $L$ -(7, 2)-colouring of  $C_{2k+1}$ .*

*Proof.* As  $L$  is not uniform, let  $vv_1 \in E(C_{2k+1})$  such that  $L(v) \neq L(v_1)$ . As  $L$  is not safe,  $L(v)$  and  $L(v_1)$  share three colours. Without loss of generality, assume that  $L(v) = \{0, 1, 2, 3\}$ . Let  $v_2$  be the other neighbour of  $v$  that is not  $v_1$ . As  $L$  is not nearly-safe, we can assume without loss of generality that  $L(v_1) = \{6, 0, 1, 2\}$  and  $L(v_2) = \{1, 2, 3, 4\}$ . First suppose that  $v, v_1, v_2$  form a triangle. Then colour  $v$  with 0,  $v_1$  with 2 and  $v_2$  with 4.

So we can assume we have at least five vertices. Let  $v_{1,1}$  be the neighbour of  $v_1$  that is not  $v$ . Initially suppose that  $L(v_{1,1}) \neq \{1, 2, 3, 4\}$  or  $\{0, 1, 2, 3\}$ . Then colour  $v$  with 0 and  $v_1$  with 2. Then the set of available colours at  $v_2$  has size 3, and the set of available colours at  $v_{1,1}$  is at least two. Therefore by Proposition 5.27 there is an  $L$ -(7, 2)-colouring.

So we can assume that  $L(v_{1,1}) = \{1, 2, 3, 4\}$  or  $\{0, 1, 2, 3\}$ . Regardless of these two lists, colour  $v$  with 1 and colour  $v_1$  with 6. Then the set of available colours for  $v_{1,1}$  is at least 3, and the set of available colours for  $v_2$  is exactly 2. Thus by Proposition 5.27 we have a  $L$ -(7, 2)-colouring.  $\square$

Now we observe that uniform lists do not admit an  $L$ -(7, 2)-colouring of  $C_{2k+1}$ .

**Observation 5.33.** Any uniform 4-interval-list assignment  $L$  of  $C_{2k+1}$  does not admit a  $L$ -(7, 2)-list colouring.

*Proof.* Without loss of generality we can assume that  $L$  assigns the colours 0, 1, 2, and 3 to each vertex. Suppose  $f$  is an  $L$ -(7, 2)-colouring of  $C_{2k+1}$ . Then the image of  $f$  is a subgraph of the graph in  $G_{7,2}$  induced on the vertices 0, 1, 2 and 3. However, this is bipartite, which by composing homomorphisms, would imply that  $C_{2k+1}$  is bipartite, a contradiction.  $\square$

Putting it all together, we have

**Lemma 5.34.** *A 4-interval-list-assignment  $L$  of  $C_{2k+1}$  admits an  $L$ -(7, 2)-colouring if and only if  $L$  is not uniform.*

Now we can prove the main result of this section.

**Theorem 5.35.** *Let  $G$  be a 4-critical graph with no (7, 2)-colouring, and whose Gallai Tree contains an odd cycle. Then  $G$  is isomorphic to an odd wheel.*

*Proof.* Suppose we have an odd cycle  $C$  in the Gallai Tree. By 4-criticality,  $G - C$  has a 3-colouring, say  $f$ . As each vertex in  $C$  has degree three in  $G$ , each vertex in  $C$  has exactly one neighbour not in  $C$ . For every vertex  $v \in V(C)$ , let  $v'$  be the neighbour of  $v$  not in  $C$ . Assign to  $v$  the list  $N_{G7,2}(f(v'))$ .

This list assignment is 4-interval, and hence by Theorem 5.34, we may assume this list assignment is uniform. Then for all  $v \in V(C)$ , we may assume that  $f(v') = 0$ . If for every  $u, v \in V(C)$ , we have  $u' = v'$ , then  $G$  is isomorphic to an odd wheel. Thus for some  $u, v \in V(C)$ , we have  $u' \neq v'$ . Change the colour of  $u'$  to 6. Observe that this is still a (7, 2)-colouring. But now if we update the lists on  $C$ , we do not have a uniform list, and hence we can apply Theorem 5.34 to extend the (7, 2)-colouring of  $G - C$  to a (7, 2)-colouring of  $G$ , a contradiction.  $\square$

We finish this section by observing that odd wheels are not counterexamples to Theorem 1.79.

**Observation 5.36.** When  $k \geq 3$ , the graph  $W_{2k+1}$  has

$$e(W_{2k+1}) \geq \frac{17v(W_{2k+1})}{10}.$$

*Proof.* Observe that  $W_{2k+1}$  has  $e(W_{2k+1}) = 4k + 2$ , and  $v(W_{2k+1}) = 2k + 2$ . Then

$$4k + 2 \geq \frac{17(2k + 2)}{10}$$

which is equivalent to

$$40k + 20 \geq 34k + 34,$$

which simplifies to

$$6k \geq 14$$

which is true if  $k \geq 3$ .  $\square$

## 5.4 A detour to $k$ -critical graphs

In this section we observe that we can extend the ideas of the previous section to prove that every  $k$ -critical graph with no  $(2k - 1, 2)$ -colouring has no block in the Gallai Tree isomorphic to a  $K_{k-1}$ .

**Lemma 5.37.** *Let  $k \geq 2$ . Let  $L$  be a  $k$ -list-assignment of  $K_{k+1}$ . Then  $K_{k+1}$  is  $L$ -colourable unless  $L$  is uniform.*

*Proof.* We proceed by induction on  $k$ . The result is immediate when  $k = 2$ , so assume that  $k \geq 3$ . Now suppose  $L$  is not uniform. Let  $uv \in E(K_{k+1})$  such that  $L(u) \neq L(v)$ . Let  $c \in L(u) \setminus L(v)$ . Now colour  $u$  with  $c$  and remove  $c$  from the lists of the remaining vertices. As  $c \notin L(v)$ , we apply induction, so there is an  $L$ -colouring of  $K_{k+1} - u$ , and hence an  $L$ -colouring of  $K_{k+1}$ .  $\square$

We will say a list assignment  $L$  is  $(2k - 1, 2)$ -near-uniform if all vertices receive one of two possible lists, and these lists correspond to the neighbourhoods of two non-adjacent vertices in  $G_{2k-1,2}$ . Further we will assume a near-uniform list assignment is not uniform.

**Lemma 5.38.** *Let  $k \geq 4$ . Let  $L$  be a  $(2k - 1, 2)$ -near-uniform list assignment of  $K_{k-1}$ . Then there is an  $L$ - $(2k - 1, 2)$ -colouring of  $K_{k-1}$ .*

*Proof.* If  $k = 4$  this follows from Lemma 5.34. Therefore we proceed by induction and assume that  $k \geq 5$ . Without loss of generality, we may assume that the lists are the intervals  $[2, 2k - 3]$  and  $[3, 2k - 2]$ . Colour some vertex with 2. Then the new lists are  $[4, 2k - 3]$ , and  $[4, 2k - 2]$ . Viewing this new list assignment as a near uniform  $(2(k - 1) - 1, 2)$ -list assignment of  $K_{k-2}$ , we see that they correspond to the neighbourhoods of the vertices 2 and 3, and hence by induction there is an  $L$ - $(2k - 1, 2)$ -colouring of  $K_{k-1}$ .  $\square$

**Corollary 5.39.** *Suppose  $G$  is a  $k$ -critical graph which contains a  $K_{k-1}$  where for every  $v \in V(K_{k-1})$ ,  $d(v) = k - 2$ . Then either  $G$  is isomorphic to  $K_k$ , or  $G$  has a  $(2k - 1, 2)$ -colouring.*

*Proof.* Suppose that  $G$  contains a  $K_{k-1}$  where every vertex in the  $K_{k-1}$  has degree  $k - 1$ . Let  $f$  be a  $k - 1$  colouring of  $G - K_{k-1}$ , where we may assume that the  $k - 1$  colouring uses the colours  $\{0, 2, 4, \dots, 2(k - 2)\}$ . By Lemma 5.37, we can extend  $f$  to a  $k - 1$ -colouring of  $G$  unless all vertices adjacent to the  $K_{k-1}$  receive the same colour, which without loss of generality we may assume to be 0. If there is only one such vertex, then  $G$  is isomorphic to  $K_k$ . Thus there is at least two vertices. Change the colour of one of these vertices from 0 to  $2k - 2$ . This remains a  $(2k - 1, 2)$ -colouring, and now we can apply Lemma 5.38 to extend the colouring, completing the claim.  $\square$



## 5.5 Acyclic Gallai Trees- A reduction to paths

In this section we prove that if the Gallai Tree of a 4-critical graph  $G$  is acyclic, then every component of the Gallai Tree is isomorphic to a path or a claw. We also show that a vertex minimal counterexample to Theorem 1.79 has no claw component.

**Lemma 5.40.** *Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. If the Gallai Tree of  $G$  is acyclic, then every component is either isomorphic to a path, or contains at most four vertices.*

*Proof.* Let  $T$  be the Gallai Tree for  $G$ . If  $T$  has no vertex of degree three, then every component of  $T$  is a path and we are done. Consider a component of  $T$  which contains a vertex of degree three in  $T$ . Let  $x$  be such a vertex, and let  $y$  and  $z$  be two neighbours of  $x$  with degree three in  $G$ . Let  $x', y', y'', z', z''$  be the neighbours (possibly not distinct) of  $x, y, z$  respectively that are not  $x, y, z$ .

By 4-criticality, we have a 3-colouring  $f$  of  $G - \{x, y, z\}$ , and we cannot extend this 3-colouring to a  $(7, 2)$ -colouring of  $G$ . Without loss of generality we may assume that  $f(x') = 0$ . Let  $L$  be the list assignment corresponding to the set of available colours of  $x, y, z$  (with respect to a  $(7, 2)$ -colouring). Observe that  $|L(x)| = 4$ , and is an interval. If either  $|L(y)| = 4$  or  $|L(z)| = 4$ , then Proposition 5.26 applies and we have a  $(7, 2)$ -colouring of  $G$ . Thus  $f(y') \neq f(y'')$  and  $f(z') \neq f(z'')$ . If  $\{f(y'), f(y'')\} = \{f(z'), f(z'')\}$ , then colour  $y$  and  $z$  the same colour from an available colour. Then the neighbourhood of  $x$  contains at most two distinct colours, and therefore there is an available colour for  $x$ , giving a  $(7, 2)$ -colouring of  $G$ , a contradiction. So  $\{f(y'), f(y'')\} \neq \{f(z'), f(z'')\}$ . By possibly exchanging colours and relabelling, we may assume that  $f(y') = 0$  and  $f(y'') = 2$ . If  $\{f(z'), f(z'')\} = \{2, 4\}$ , then colour  $y$  with 4,  $x$  with 2, and  $z$  with 0. Therefore we can assume that  $f(z') = 4$  and  $f(z'') = 0$ .

First suppose that  $y' \neq x'$  and  $y' \neq z''$ . Then we can recolour both  $x'$  and  $z''$  to 6. Then colour  $y$  with 5,  $x$  with 3, and  $z$  with 1 to obtain a  $(7, 2)$ -colouring of  $G$ . Hence either  $y' = x'$  or  $y' = z''$ . We consider cases.

**Case 1:**  $y' = x'$

In this case,  $x'$  has degree three, and is adjacent to both  $x$  and  $y$ . Therefore we have a triangle of degree three vertices, contradicting the fact that the Gallai Tree is acyclic.

**Case 2:**  $y' = z''$

By Case 1, we may assume that  $x' \neq y'$ . First we claim that  $x'y'' \in E(G)$ . Suppose not. Then recolour all vertices in  $N_4(N_2(x'))$  to 5, recolour all vertices in  $N_2(x')$  to 3, and then recolour  $x'$  to 1. This does not change the colour of  $y''$  because  $y''x' \notin E(G)$ . Then colour  $y$  with 4,  $x$  with 6, and  $z$  with 2, which is a proper  $(7, 2)$ -colouring, a contradiction.

Now we claim that  $x'z' \in E(G)$ . Suppose not. Then recolour  $z'$  to 5 and  $x'$  to 6. This is still a proper  $(7, 2)$ -colouring as  $x'z' \notin E(G)$ . But now we can colour  $y$  with 5,  $x$  with 1, and  $z$  with 3 - a contradiction.

Now observe that if any of  $y'$ ,  $z'$  or  $y''$  have degree three, we would have an odd cycle in the Gallai Tree, contradictory to our assumption. Thus the entire component is  $x, y, z, x'$  which completes the claim.  $\square$

We claim that in a vertex minimal counterexample, the Gallai Tree contains no claw.

**Lemma 5.41.** *Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. Let  $C$  be a claw component of the Gallai Tree, where  $V(C) = \{x, y, z, x'\}$ , where  $x$  is adjacent to all of  $y, z$ , and  $x'$ . Let  $G'$  be the graph obtained by identifying all of the vertices of  $C$  into a single vertex and removing multiple edges and loops. Then  $G'$  is 4-critical and has no  $(7, 2)$ -colouring. Further,  $e(G') = e(G) - 6$ , and  $v(G') = v(G) - 3$ .*

*Proof.* We follow the proof of Lemma 5.40. Let  $V(C) = x, y, z, x'$ , where  $xy, xz, xx' \in E(G)$ . Let  $y', y''$  be the neighbours of  $y$  not in  $C$ , and  $z', z''$  be the neighbours of  $z$  not in  $C$ . Let  $f$  be a 3-colouring of  $G - \{x, y, z\}$ . Without loss of generality, we can assume that  $f(x') = 0$ , and as we cannot extend this colouring to a  $(7, 2)$ -colouring of  $G$ , up to relabelling we can assume that  $f(y') = 0$ ,  $f(y'') = 2$ ,  $f(z') = 4$  and  $f(z'') = 0$ . By the same analysis as in Lemma 5.40, we can assume that  $y' = z''$ , and that both  $x'z', x'y'' \in E(G)$ .

In the graph  $G'$ , let  $w$  denote the vertex obtained after identifying  $x, y, z, x'$ .

**Claim 5.42.** *The graph  $G'$  has a 4-colouring.*

*Proof.* Take any 3-colouring  $f$  of  $G - \{x, y, z, x'\}$ . Then we can extend  $f$  to a 4-colouring of  $G'$  by giving  $w$  any available colour (there is an available colour as  $w$  has degree three).  $\square$

**Claim 5.43.** *The graph  $G'$  has no  $(7, 2)$ -colouring.*

*Proof.* Suppose not, and let  $f$  be a  $(7, 2)$ -colouring of  $G'$ . Then consider the map  $f'$  where for all vertices  $v \in V(G) - \{x, y, z, x'\}$ ,  $f'(v) = f(v)$ , for all  $t \in \{x', y, z\}$ ,  $f'(t) = f(w)$ , and let  $f(x)$  be any colour in the set  $N_{G', 2}(f(w))$ . This is a  $(7, 2)$ -colouring of  $G$ , a contradiction.  $\square$

**Claim 5.44.** *The graph  $G'$  is 4-critical.*

*Proof.* By Claim 5.43,  $G'$  does not have a 3-colouring (as every 3-colouring can be turned into a  $(7, 2)$ -colouring), and by Claim 5.42,  $G'$  is 4-colourable, so  $\chi(G') = 4$ . Therefore it suffices to show that  $G' - e$  is 3-colourable for all edges  $e$ .

First consider deleting an edge incident to  $w$  say  $e$ . To see that  $G' - e$  has a 3-colouring, simply take any 3-colouring of  $G - \{x, y, z, x'\}$ , and there will be a colour left over for  $w$ , so we can extend the colouring (as  $w$  has degree three).

Now consider an edge  $e \in E(G')$  not incident to  $w$ . Then  $G' - e$  is 3-colourable as  $G$  is 4-critical. Let  $f$  be any 3-colouring of  $G - e$ . If  $f(x') = f(y) = f(z)$ , then the colouring  $f'$  where  $f'(w) = f(x')$  and for all  $v \in V(G') - w$ ,  $f'(v) = f(v)$  is a 3-colouring of  $G'$ . Thus

at least two colours appear on  $x', y, z$ . Note that at most 2 colours appear on  $x', y, z$  as if all three colours appeared, then  $x$  would not have a colour. Without loss of generality, suppose that  $f(x') = f(y) = 0$ , and  $f(z) = 2$ . Then  $f(x) = 4$ . Additionally  $f(z') = 4$ , and  $f(y') = 4$ . But then  $z$  is not adjacent to a vertex coloured 0, and so we can change the colour of  $z$  to 0, and apply the case where  $f(x') = f(y) = f(z)$ . Hence  $G'$  is 4-critical.  $\square$

Finally, one simply observes that  $e(G') = e(G) - 6$  and  $v(G') = v(G) - 3$ .  $\square$

**Corollary 5.45.** *In a vertex minimal counterexample to Theorem 1.79, all components of the Gallai Tree are paths.*

*Proof.* Let  $G$  be a vertex minimal counterexample to Theorem 1.79. Then  $G$  is not isomorphic to an odd wheel. Thus the Gallai Tree of  $G$  contains no odd cycles. If the Gallai Tree of  $G$  contains a cycle, then there is a component isomorphic to a  $K_4$ , and hence  $G$  is isomorphic to  $K_4$ , a contradiction. so all components of the Gallai Tree of  $G$  are either paths or claws. Suppose  $G$  contains a claw component. By Lemma 5.41, there exists a graph  $G'$  which is 4-critical and has no  $(7, 2)$ -colouring such that  $e(G') = e(G) - 6$ , and  $v(G') = v(G) - 3$ . First suppose that  $G'$  is not  $K_4$  or  $W_5$ . Then by minimality, we have that

$$e(G') \geq \frac{17v(G)}{10}.$$

Thus

$$e(G) - 6 \geq \frac{17(v(G) - 3)}{10}.$$

Rearranging we have

$$e(G) \geq \frac{17v(G) + 9}{10}$$

contradicting that  $G$  is a counterexample.

Now assume that  $G'$  is isomorphic to  $K_4$ . Then  $v(G) = 7$  and  $e(G) = 12$ , and clearly  $12 \geq \frac{119}{10}$ . Now assume that  $G'$  is isomorphic to  $W_5$ . Then  $v(G) = 9$  and  $e(G) = 16$ . Clearly  $16 \geq \frac{153}{10}$ . As these are all possibilities, the claim holds.  $\square$

## 5.6 Structure around path components

The purpose of this section is to argue that the vertices near path components must have large degree.

For any vertex  $v$ , let  $N[v]$  denote the *closed neighbourhood of  $v$* , that is the set of vertices  $N(v) \cup \{v\}$ .

**Proposition 5.46.** *Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. Then there does not exist two vertices  $u$  and  $v$  where  $d(v) = d(u) = 3$ , and  $N[v] = N[u]$ .*

*Proof.* Suppose not. Let  $x$  and  $y$  be the two neighbours of  $u$  and  $v$  which are not  $u$  or  $v$ . Note  $xy \notin E(G)$ , as otherwise  $G$  contains a  $K_4$ , and hence is isomorphic to  $K_4$ . By 4-criticality,  $G - u - v$  has a 3-colouring, say  $f$ . We consider cases.

If  $f(x) = f(y)$ , then we can extend not only to a  $(7, 2)$ -colouring, but a 3-colouring, a contradiction. So we may assume that  $f(x) \neq f(y)$ . Without loss of generality, let  $f(x) = 0$ . Suppose  $f(y) = 2$ . Now consider  $N_0(y)$ . We change the colour of every vertex in  $N_0(y)$  to 6, and then change the colour of  $y$  to 1. Let  $f'$  be the resulting  $(7, 2)$ -colouring. Now we can extend  $f'$  to a  $(7, 2)$ -colouring of  $G$  by letting  $f'(u) = 3$  and  $f'(v) = 5$ . The rest of the cases follow by exchanging colours and applying one of the above arguments if necessary.  $\square$

Now we prove the most important lemma in the section, which despite being very simple, enforces a large amount of local structure around path components.

**Lemma 5.47.** *Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. Let  $v$  be a vertex where  $N(v) = \{x, y, z\}$ . For any pair  $w, t \in \{x, y, z\}$ , either  $wt \in E(G)$ , or there is a vertex  $x_{w,t} \neq v$  where  $wx_{w,t} \in E(G)$  and  $x_{w,t}t \in E(G)$ . If  $wt \notin E(G)$ , then for any  $w', t' \in \{x, y, z\}$ ,  $x_{w,t} = x_{w',t'}$  if and only if  $\{w, t\} = \{w', t'\}$ .*

*Proof.* Suppose without loss of generality that  $xy \notin E(G)$ , and  $xy$  does not lie in a 4-cycle with  $v$ . Then by permuting colours if necessary, there is a 3-colouring of  $G - v$  such that  $f(x) = 0$ ,  $f(y) = 4$ , and  $f(z) = 2$ . Now we look at  $N_2(x)$ , and  $N_4(N_2(x))$ . Change all vertices colours in  $N_4(N_2(x))$  to 5, change all vertices colours in  $N_2(x)$  to 3 and change the colour of  $x$  to 1. Observe that the colour of  $y$  did not change. Then colour  $v$  with 6 to obtain a  $(7, 2)$ -colouring of  $G$ , a contradiction. Hence either  $xy \in E(G)$ , or there is a vertex in  $N_2(x)$  which is adjacent to  $y$ , as desired. Uniqueness comes from the fact that we can assume the vertex in a 4-cycle with  $v, x, y$  is coloured 2, and for any pair that we apply this argument to, we get a distinct colour, and hence the vertices are distinct.  $\square$

We observe that if  $x_{w,t}$  exists, it may in fact be one of  $\{x, y, z\}$ . However if say  $x_{x,z} = y$ , then  $xy \in E(G)$  and  $yz \in E(G)$ .

**Observation 5.48.** *Let  $G$  be a 4-critical graph with no  $(7, 2)$ -colouring. Let  $v$  be a vertex with  $N(v) = \{x, y, z\}$ . Suppose that  $G[\{x, y, z\}]$  contains at least two edges, with  $t \in \{x, y, z\}$  having degree two in  $G[\{x, y, z\}]$ . Then  $G[\{x, y, z\}]$  contains exactly two edges, and for  $w, r \in \{x, y, z\} - \{t\}$ ,  $N(w) \cap N(r) = \{v, t\}$ .*

*Proof.* Suppose not. Observe that  $x, y$  and  $z$  cannot induce a triangle as then we have a clique cutset in a 4-critical graph. Thus without loss of generality, let  $xy, yz \in E(G)$ . Suppose there is a vertex  $x_{x,z} \notin \{v, y\}$  such that  $xx_{x,z} \in E(G)$  and  $zx_{x,z} \in E(G)$ .

Now let  $f$  be a 3-colouring of  $G - \{x_{x,z}\}$ . Then  $f(x) = f(x_{x,z})$  otherwise we have a 3-colouring of  $G$ . Without loss of generality we may assume that  $f(x) = 0$ . Then  $f(v) \neq 0$ , so without loss of generality  $f(v) = 2$ . But then  $f(y) = f(z) = 4$ , a contradiction as  $yz \in E(G)$ .  $\square$

### 5.6.1 The case where $x, y, z$ induces one edge

For the subsection, we have a 4-critical graph  $G$  with no  $(7, 2)$ -colouring, a vertex  $v$  where  $N(v) = \{x, y, z\}$  and we will assume that  $xy \in E(G)$ ,  $yz, xz \notin E(G)$ . Thus by Lemma 5.47 there are distinct vertices  $x_{y,z}$  and  $x_{x,z}$  where  $x_{y,z}$  is adjacent to both  $y$  and  $z$ , and  $x_{x,z}$  is adjacent to both  $x$  and  $z$ . Further  $x_{y,z}$  and  $x_{x,z}$  are not in  $\{x, y, z\}$ . We will assume additionally that the Gallai Tree of  $G$  has no cycles, and hence  $G$  is not  $K_4$  or an odd wheel.

The goal of the subsection is to show that the neighbours of  $v$  have large degree. We will prove stronger claims than what is necessary to deduce Theorem 1.79, but we believe the additional claims would be useful if trying to improve the bound on Theorem 1.79. We start by proving  $d(x) \geq 4$  and  $d(y) \geq 4$ . The following observation is well known.

**Observation 5.49.** Let  $H$  be an arbitrary graph with vertices  $w, t \in V(H)$  which both have degree three. Suppose that  $w$  and  $t$  have a common neighbour  $a$ . Suppose that the other neighbour of  $w$  is  $b$ , and the other neighbour of  $t$  is  $c$ . If there is a 3-colouring  $f$  of  $H - w - t$  such that  $f(b) \neq f(c)$ , then  $G$  has a 3-colouring. Additionally, if  $H$  is any graph which is not 3-colourable, then  $bc \notin E(H)$ .

*Proof.* Let  $f$  be a 3-colouring of  $H - w - t$  so that  $f(b) \neq f(c)$ . Without loss of generality, suppose that  $f(b) = 0$  and  $f(c) = 2$ . If  $f(a) = 2$ , then colour  $w$  with 4 and  $t$  with 0. If  $f(a) = 0$ , colour  $w$  with 2 and  $t$  with 4. If  $f(a) = 4$ , then colour  $w$  with 2 and  $t$  with 0. In all cases, we get a 3-colouring of  $H$ .

Now suppose  $H$  is not 3-colourable. If  $bc \in E(H)$ , then every 3-colouring of  $H - w - t$  has  $f(b) \neq f(c)$ , and extends to a 3-colouring of  $H$ , contradicting 4-criticality.  $\square$

**Lemma 5.50.** Let  $w, t$  be two vertices in  $G$  both having degree three. Suppose that  $w$  and  $t$  share a common neighbour  $a$ . Suppose that  $b$  is the other neighbour of  $w$ , and  $c$  is the other neighbour of  $t$ . Then  $ab \in E(G)$ , and  $ac \in E(G)$ .

*Proof.* Suppose not. Without loss of generality, we can assume that  $ab \notin E(G)$ . Let  $f$  be a 3-colouring of  $G - w - t$ . Without loss of generality, we can assume that  $f(b) = 0$ , which by Observation 5.49, implies that  $f(c) = 0$ . If  $f(a) = 0$ , then colouring  $w$  with 2 and  $t$  with 4 is a 3-colouring of  $G$ .

So without loss of generality assume that  $f(a) = 2$ . Now consider  $N_2(b)$ , and  $N_4(N_2(b))$ . Change the colour of all vertices in  $N_4(N_2(b))$  to 5, and change the colour of all the vertices in  $N_2(b)$  to 3, and finally change the colour of  $b$  to 1. Now as  $ab \notin E(G)$ , we can now colour  $w$  with 6 and  $t$  with 4, contradicting that  $G$  has no  $(7, 2)$ -colouring. Thus  $ab \in E(G)$ , and by the same argument, we have that  $ac \in E(G)$ .  $\square$

**Corollary 5.51.** Both  $d(x) \geq 4$  and  $d(y) \geq 4$ .

*Proof.* Suppose towards a contradiction that  $y$  has degree three. Then by Lemma 5.50,  $xz \in E(G)$ . But we assumed at the start of the section, that the vertices  $x, y, z$  induce exactly one edge, and we now have edges  $xy$  and  $xz$ , a contradiction.  $\square$

Now we make a straightforward observation.

**Observation 5.52.** Both  $yx_{x,z} \notin E(G)$ , and  $xx_{y,z} \notin E(G)$ .

*Proof.* Suppose that  $yx_{x,z} \in E(G)$ . Then  $N(v) \subseteq N(x_{x,z})$ , which does not occur in a 4-critical graph, a contradiction. An analogous argument works for  $xx_{y,z}$ .  $\square$

**Observation 5.53.** If the Gallai Tree of  $G$  has no claw component, then one of  $z, x_{x,z}$  or  $x_{y,z}$  has degree at least 4.

*Proof.* If not, then  $v, z, x_{x,z}$  and  $x_{y,z}$  form a claw in the Gallai Tree.  $\square$

Now we want to understand what happens when  $x$  and  $y$  share a neighbour that is not  $v$ .

**Lemma 5.54.** *Suppose  $x$  and  $y$  have a common neighbour  $w$  that is not  $v$ . Then at least one of the following occurs:*

- *There is at least one  $t \in \{x, y\}$  such that  $d(t) \geq 5$ .*
- *There is at least one  $t \in \{w, z\}$  such that  $d(t) \geq 4$ .*

*Proof.* Suppose none of the above conditions occur. This implies that  $d(x) = 4$  and  $d(y) = 4$ . Let  $f$  be a 3-colouring of  $G - \{v, x, y\}$ . Without loss of generality suppose that  $f(w) = 0$ .

**Case 1: Either  $f(x_{x,z}) = 0$  or  $f(x_{y,z}) = 0$**

Without loss of generality suppose that  $f(x_{x,z}) = 0$ . Then  $f(z) \neq 0$ . Colour  $v$  with 0. There exists at least one available colour for  $y$ , so colour  $y$  with this colour, and then the neighbourhood of  $x$  sees at most two colours, and so there is a colour available for  $x$ , thus we get a 3-colouring of  $G$ . A similar argument works when  $f(x_{y,z}) = 0$ .

**Case 2:  $f(x_{x,z}) = 2$  and  $f(x_{y,z}) = 4$**

Observe in this case that  $f(z) = 0$ . We claim that either  $wx_{x,z} \in E(G)$ , or there is a vertex  $x_{w,x_{y,z}}$  coloured 2 adjacent to both  $w$  and  $x_{y,z}$ . If not, change the colour of all vertices in  $N_4(N_2(w))$  to 5, change the colour of all vertices in  $N_2(w)$  to 3, and change the colour of  $w$  to 1. Then colour  $x$  with 4,  $y$  with 6, and  $v$  with 2.

Now we claim that  $w$  is adjacent to a vertex coloured 4. If not, change the colour of  $w$  to 4. Then colour  $x$  with 0,  $y$  with 2, and  $v$  with 4. From this, we deduce that  $w$  is adjacent to a vertex coloured 4 and a vertex coloured 2, and hence  $d(w) \geq 4$ .

**Case 3:**  $f(x_{x,z}) = 2$  and  $f(x_{y,z}) = 2$

Suppose  $f(z) = 4$ . In this case we claim that both  $wx_{x,z} \in E(G)$  and  $wx_{y,z} \in E(G)$ . Suppose without loss of generality that  $wx_{x,z} \notin E(G)$ . Then change the colour of all vertices in  $N_4(N_2(w))$  to 5, change the colour of all vertices in  $N_2(w)$  to 3, and change the colour of  $w$  to 1. Then colour  $x$  with 4,  $y$  with 6 and  $v$  with 2. Thus in this case  $d(w) \geq 4$  (in fact, if this case occurs then the graph is isomorphic to a  $C_6$ -expansion of  $K_4$ )

Now suppose that  $f(z) = 0$ . If  $d(z) = 3$ , then  $z$  is not adjacent to a vertex coloured 4. Hence we can change the colour of  $z$  to 4, and apply the above argument. Thus  $d(z) \geq 4$ .  $\square$

Now we will want to understand what happens when  $x$  and  $y$  do not share a neighbour and have small degree.

**Lemma 5.55.** *Suppose  $x$  and  $y$  do not share a common neighbour other than  $v$ . Let  $x'$  and  $y'$  be the neighbours of  $x$  and  $y$  that are not  $x_{x,z}$  and  $x_{y,z}$ . Then at least one of the following occurs.*

- *There exists a  $t \in \{x, y, z\}$  such that  $d(t) \geq 5$ .*
- *The edge  $x'y' \in E(G)$ , at most one of  $x'$  or  $y'$  has degree three, and either  $d(z) \geq 4$ , or both  $x_{x,z}$  and  $x_{y,z}$  have degree at least four.*
- *The edge  $x'y' \notin E(G)$ , there exists a  $t \in \{x_{x,z}, y'\}$  such that  $d(t) \geq 4$  and a  $w \in \{x_{y,z}, x'\}$  such that  $d(w) \geq 4$ .*
- *The edge  $x'y' \notin E(G)$ ,  $d(z) \geq 4$ , and at least one of  $x_{y,z}$  or  $x_{x,z}$  have degree at least four.*

*Proof.* Suppose none of the conditions hold. In particular this implies that  $d(x) = d(y) = 4$ , and  $d(z) \leq 4$ .

Consider a 3-colouring of  $G - \{v, x, y\}$ . Without loss of generality we may assume that  $f(x') = 0$ . We consider cases.

**Case 1:**  $f(y') = 0$

Observe that if this occurs, then  $x'y' \notin E(G)$ , as  $f(x') = f(y')$ .

**Subcase 1:**  $f(x_{x,z}) = f(x_{y,z})$

If  $f(x_{x,z}) = 0$ , then as  $zx_{x,z} \in E(G)$ ,  $f(z) \neq f(x_{x,z})$ . Thus colour  $x$  and  $y$  with 2 and 4, and colour  $v$  with 0. Therefore  $f(x_{x,z}) \neq 0$ .

Now suppose that  $f(x_{x,z}) = 2$ . We claim that  $x'$  and  $y'$  are adjacent to a vertex coloured 4. Suppose not and without loss of generality suppose that  $x'$  has no neighbours coloured 4. Change the colour of  $x'$  to 4. Then colour  $x$  with 0,  $y$  with 4, and since  $f(z) \neq 2$ , there is an available colour for  $v$ , a contradiction.

**Subsubcase 1:**  $f(z) = 4$ 

We claim that  $x'x_{y,z} \in E(G)$ . If not, change the colour of all vertices in  $N_4(N_2(x'))$  to 5, change the colour of all vertices in  $N_2(x')$  to 3, and the colour of  $x'$  to 1. Then colour  $x$  with 6,  $y$  with 4, and  $v$  with 2. By an analogous argument,  $y'x_{x,z} \in E(G)$ .

Now we claim that either  $x_{x,z}$  is adjacent to a vertex coloured 0 that is not  $y'$ , or  $y'$  is adjacent to a vertex coloured 2 that is not  $x_{x,z}$ . If not, exchange the colours of  $x_{x,z}$  and  $y'$ . Then colour  $y$  with 4,  $x$  with 2, and  $v$  with 0. Thus either  $d(x_{x,z}) \geq 4$ , or  $d(y') \geq 4$ . By an analogous argument, either  $d(x_{y,z}) \geq 4$  or  $d(x') \geq 4$ , a contradiction.

**Subsubcase 2:**  $f(z) = 0$ 

Observe that if  $z$  is not adjacent to a vertex coloured 4, then we can simply change the colour of  $z$  to 4 and apply the above case analysis to conclude there is a  $t \in \{x_{x,z}, y'\}$  such that  $d(t) \geq 4$ , and a  $w \in \{x_{y,z}, x'\}$  where  $d(w) \geq 4$ . So we can assume that  $z$  is adjacent to a vertex coloured four, and hence  $d(z) \geq 4$ .

We claim that both  $x_{x,z}$  and  $x_{y,z}$  are adjacent to some vertex coloured 4 (possibly not the same vertex). Suppose  $x_{x,z}$  is not adjacent to a vertex coloured 4. Then change the colour of  $x_{x,z}$  to 4 and the colour of  $x'$  to 6. Then colour  $x$  with 1,  $y$  with 5 and  $v$  with 3. Hence both  $x_{x,z}$  and  $x_{y,z}$  are adjacent to a vertex coloured 4.

Now consider the graph induced by the colour classes 0 and 2. Let  $C$  be the component of this graph containing  $z$ . If this component is only  $z, x_{x,z}$  and  $x_{y,z}$ , then colour  $x_{x,z}$  and  $x_{y,z}$  with 0 and  $z$  with 2. Then by a previous case, we obtain a 3-colouring. Thus either  $z$  is adjacent to a vertex coloured 2 that is not  $x_{x,z}$  and  $x_{y,z}$ , or one of  $x_{x,z}$  and  $x_{y,z}$  is adjacent to a vertex coloured 0 that is not  $z$ . In the case  $z$  is adjacent to a vertex coloured 0 that is not  $z$ , then  $d(z) \geq 5$ . Otherwise, at least one of  $x_{y,z}$  or  $x_{x,z}$  has degree at least four.

We do not consider the case where  $f(x_{x,z}) = 4$  as it follows a similar analysis as above.

**Subcase 2:**  $f(x_{x,z}) \neq f(x_{y,z})$ 

First suppose that  $f(x_{x,z}) = 0$ . If  $f(x_{y,z}) = 2$ , then  $f(z) = 4$ , and we can extend to a 3-colouring by colouring  $x$  with 2,  $y$  with 4, and  $v$  with 0. A similar argument works if  $f(x_{y,z}) = 4$ . Additionally, similar arguments work if  $f(x_{y,z}) = 0$ .

Thus without loss of generality  $f(x_{x,z}) = 2$  and  $f(x_{y,z}) = 4$ . Thus  $f(z) = 0$ . In this case, change the colour of  $N_4(N_2(x'))$  to 5,  $N_2(x')$  to 3, and  $x'$  to 1. Then colour  $y$  with 2,  $x$  with 6, and  $v$  with 4.

**Case 2:**  $f(y') = 2$ **Subcase 1:**  $f(x_{x,z}) = f(x_{y,z})$ 

Suppose that  $f(x_{x,z}) = 0$ . Then  $f(z) \in \{2, 4\}$ . Colour  $y$  with 4 and  $x$  with 2, and colour  $v$  any available colour. A similar argument shows that if  $f(x_{x,y}) = 2$ , we can always extend to a 3-colouring. Therefore we can assume that  $f(x_{x,z}) = 4$ . Observe that  $f(z) \neq 4$ , and hence colour  $v$  with 4,  $x$  with 2 and  $y$  with 0, a contradiction.



**Subcase 2:**  $f(x_{x,z}) \neq f(x_{y,z})$

First suppose  $f(x_{x,z}) = 0$ . If  $f(x_{y,z}) = 2$ , then  $f(z) = 4$ , and colour  $y$  with 4,  $x$  with 2, and  $v$  with 0. A similar colouring works when  $f(x_{y,z}) = 4$ . Thus  $f(x_{x,z}) \neq 0$ , and similarly we can assume that  $f(x_{y,z}) \neq 2$ .

Now suppose  $f(x_{x,z}) = 2$ . If  $f(x_{y,z}) = 4$ , then colour  $x$  with 4,  $y$  with 0 and  $v$  with 2. Hence,  $f(x_{y,z}) = 0$  and thus  $f(z) = 4$ . Now suppose that  $x'y' \notin E(G)$ . In this case, change the colour of all vertices in  $N_4(N_2(x'))$  to 5, change the colour of all vertices in  $N_2(x')$  to 3 and  $x'$  to 1. Then colour  $x$  with 6,  $y$  with 4, and  $v$  with 2, a contradiction. Thus  $x'y' \in E(G)$ .

Similarly, if  $x_{x,z}x_{y,z} \notin E(G)$ , then we change the colour of all vertices in  $N_4(N_2(x_{y,z}))$  to 5, the colour of all vertices in  $N_2(x_{y,z})$  to 3, and the colour of  $x_{y,z}$  to 1. Then colour  $x$  with 4,  $y$  with 6 and  $v$  with 0.

Now we claim that both  $x'$  and  $y'$  are adjacent to a vertex coloured 4. If either  $x'$  or  $y'$  is not adjacent to a vertex of degree four, simply change one of the vertices to colour 4, and then extend to a 3-colouring by colouring  $x$  with 0,  $y$  with 4 and  $v$  with 2.

Now we claim that either  $x'$  is adjacent to a vertex coloured 2 which is not  $y'$ , or  $y'$  is adjacent to a vertex coloured 0 which is not  $x'$ . If not, then simply exchange the colours on  $x$  and  $y$ . But now we can extend to a 3-colouring, a contradiction.

Thus it follows that at least one of  $x'$  or  $y'$  has degree at least four, and  $x'y' \in E(G)$ .

Observe that  $x_{x,z}, x_{y,z}$  and  $z$  induce a triangle. If  $d(z) = 3$ , then note that  $G[\{z, x_{x,z}, x_{y,z}, v\}]$  induces exactly one edge, and then  $d(x_{x,z}) \geq 4$  and  $d(x_{y,z}) \geq 4$  by Observation 5.51. Otherwise  $d(z) \geq 4$ .

Lastly assume  $f(x_{x,z}) = 4$ . Then  $f(x_{y,z}) = 0$  otherwise we use a previous case. Thus  $f(z) = 2$ . Then we can extend to a 3-colouring with  $x$  coloured 2,  $y$  coloured 4, and  $v$  coloured 0.  $\square$

## 5.6.2 Long paths in the Gallai Tree

In this subsection we are going to analyze what happens around long paths in the Gallai Tree. We start off by analyzing what happens if we delete a path of three degree three vertices. As before, we assume  $G$  is a 4-critical graph with no  $(7, 2)$ -colouring, and further we assume that the Gallai Tree of  $G$  is acyclic.

**Lemma 5.56.** *Let  $x, y, z \in V(G)$  such that  $x, y$  and  $z$  have degree three, and  $xy, yz \in E(G)$ . Let  $x', x'', y', z', z''$  be the other neighbours of  $x, y, z$  respectively. Then up to relabelling the vertex labels,  $x' = z', y'x'' \in E(G)$ , and  $y'z'' \in E(G)$ . Additionally,  $x'' \neq z''$ .*

*Further, if  $x' \neq y'$ , there are two distinct vertices  $x_{x',x''}, x_{x',z''}$  not in  $\{x, y, z\}$  where  $x_{x',x''}$  is adjacent to  $x'$  and  $x''$ , and  $x_{x',z''}$  is adjacent to  $x'$  and  $z''$ .*

*Proof.* Let  $f$  be a 3-colouring of  $G - \{x, y, z\}$ . Without loss of generality we may assume that  $f(y') = 0$ . If  $f(x') = f(x'')$ , then simply give  $z$  a colour from its available colours, then give  $y$  an available colour, and finally as  $f(x') = f(x'')$ ,  $x$  has an available colour and we can extend the colouring. Hence  $f(x') \neq f(x'')$ , and similarly  $f(z') \neq f(z'')$ . If  $\{f(x'), f(x'')\} = \{f(z'), f(z'')\}$ , then give  $x$  and  $z$  the same colour, and we can extend this colouring to  $y$ . Finally, if  $\{f(x'), f(x'')\} = \{2, 4\}$ , then colour  $x$  with 0, colour  $z$  with any available colour, and we can extend the colouring to  $y$ . A similar argument works when  $\{f(z'), f(z'')\} = \{2, 4\}$ .

Thus without loss of generality we can assume that  $f(x') = 0$ ,  $f(x'') = 2$ ,  $f(z') = 0$  and  $f(z'') = 4$ . Observe this implies that  $x'' \neq z''$  as they have different colours.

**Claim 5.57.**  $z' = x'$ .

*Proof.* If not, change the colour of  $z'$  to 6 and extend the colouring by colouring  $x$  with 5,  $y$  with 3, and  $z$  with 1.  $\square$

**Claim 5.58.**  $y'x'' \in E(G)$ .

*Proof.* Suppose not. Then change the colour of all vertices in  $N_4(N_2(y'))$  to 5, change the colour of all vertices in  $N_2(y')$  to 3 and change the colour of  $y'$  to 1. If  $z'' \in N_4(N_2(y'))$ , then colour  $z$  with 3,  $y$  with 6 and  $x$  with 4. Therefore  $z'' \notin N_4(N_2(y'))$ . If  $x' \neq y'$  colour  $z$  with 2,  $y$  with 6 and  $x$  with 4. If  $x' = y'$ , colour  $z$  with 6,  $y$  with 3 and  $x$  with 6.  $\square$

**Claim 5.59.**  $y'z'' \in E(G)$ .

*Proof.* Suppose not. Then change the colour of  $z''$  to 5 and the colour of  $y'$  to 6. Then colour  $x$  with 4,  $y$  with 1, and  $z$  with 3.  $\square$

To finish the proof, suppose that  $x' \neq y'$ . If  $z'' \notin N_4(N_2(x'))$ , then simply change the colour of all vertices in  $N_4(N_2(x'))$  to 5, change the colour of all vertices in  $N_2(x')$  to 3, and change the colour of  $x'$  to 1. Then colour  $z$  with 6,  $y$  with 2, and  $x$  with 5. Thus there is a vertex  $x_{x',z''}$  which is adjacent to both  $x'$  and  $z''$ . To see there is also a vertex  $x_{x',x''}$  which is adjacent to  $x'$  and  $x''$ , simply exchange the colours 2 and 4 on all vertices, and then repeat the above argument. Distinctness follows from the fact that their colours are different.  $\square$

Observe that  $x_{x',x''}$  can simply be  $z'$  if  $x'z''$  is an edge, and  $x_{x',z''}$  there is an edge  $x'x''$ , and similarly,  $x_{x',z''}$  may just be  $z''$  if  $x''z''$ .

**Corollary 5.60.** *Let  $P$  be a path with at least three vertices where all vertices in  $P$  have degree three. Let  $V(P) = \{v_0, \dots, v_n\}$  and  $E(P) = \{v_i v_{i+1} \mid i \in \{1, \dots, n-1\}\}$ . Let  $v'_0, v''_0, v'_n, v''_n$  be the neighbours of  $v_0$  and  $v_n$  which are not in  $P$ , and let  $v'_1$  be the neighbour of  $v_1$  not in  $P$ . Then there is a  $w \in \{v'_0, v''_0\}$  and a  $t \in \{v'_n, v''_n\}$  such that  $t \neq w$ , and given a*

bipartition  $(A, B)$  of  $P \cup \{w, t\}$  where  $v_0 \in A$  and  $v_1 \in B$ , all vertices in  $B$  are adjacent to  $v'_1$ , and all vertices in  $A$  are adjacent to the vertex  $w'$  in  $\{v'_0, v''_0\} - w$ .

Further, for any  $q \in \{w', v'_1\}$  and any  $p \in \{w, t\}$ , either  $pq \in E(G)$ , or there is a vertex  $x_{p,q}$  such that  $x_{p,q}$  is adjacent to both  $p$  and  $q$ , and does not lie on  $P$ .

*Proof.* We proceed by induction on  $n$ . If  $n = 2$ , the result follows from Lemma 5.56. Now assume  $n \geq 3$ . Consider the path  $v_1, \dots, v_n$ . Let  $v_0, v'_1$  be the vertices adjacent to  $v_1$ , not in  $v_1, \dots, v_n$ , and  $v'_n, v''_n$  be the vertices adjacent to  $v_n$  not in  $P$ . Let  $v'_2$  be the vertex not in  $P$  adjacent to  $v_2$ . Apply the induction hypothesis to  $v_1, \dots, v_n$ .

Observe that  $v_0$  has degree three, so  $v_0$  is not adjacent to any vertex of degree three in  $P$  aside from  $v_1$ , as then we would have a cycle of degree three vertices. Thus when applying the induction hypothesis, we can conclude that  $v_0 = w$  (where  $w$  is defined as in the statement). Let  $(A, B)$  be a bipartition of  $P \cup \{v_0, v'_n\}$  (up to relabelling  $v'_n$  with  $v''_n$  if necessary), such that  $v_0 \in A$ . Then by induction,  $v'_1$  is adjacent to all vertices in  $B$ , and  $v'_2$  is adjacent to all vertices in  $A$ .

Now let  $v'_0, v'_2$  be the vertices adjacent to  $v_0$  which are not  $v_1$ . Now apply Lemma 5.56 to  $v_0, v_1, v_2$ , where in the context of that lemma statement,  $v_0 = x$ ,  $v_1 = y$  and  $v_2 = z$ . Then  $y' = v'_1$ . As  $v'_2$  is adjacent to both  $v_2$  and  $v_0$ ,  $v'_2 = x'$  in the lemma statement, and hence  $v'_0 = x''$  and so  $v'_1 v'_0 \in E(G)$ . Therefore, for the claim,  $v'_0 = w$  and  $v'_n = t$ . We now claim that  $v'_0 \neq v'_n$ . Suppose not. Consider  $G - \{v_0, \dots, v_n\}$ , and let  $f$  be a three colouring of this graph. If  $f(v'_n) = f(v'_2) = f(v'_1)$ , then we extend the colouring to  $v_0, \dots, v_n$  by using the other two colours in any fashion. If  $f(v'_n) = 0$  and  $f(v'_2) = f(v'_1) \neq 0$ , then the greedy algorithm starting at  $v_0$  and increasing sequentially gives a 3-colouring of  $G$ . The same strategy applies if  $f(v'_n) = f(v'_i)$  for  $i \in \{1, 2\}$  and  $f(v'_j) \neq f(v'_n)$  for  $j \neq i$ . The last case to consider is if the colours of  $v'_n, v'_2$  and  $v'_1$  are all distinct. In this case again the greedy algorithm starting at  $v_0$  and increasing sequentially through the path gives a 3-colouring of  $G$ , a contradiction.

Finally, observe that if  $v'_2 v'_0 \notin E(G)$ , then Lemma 5.56 ensures that there is a vertex  $x_{v'_2 v'_0}$  not in  $v_0, v_1, v_2$  which is adjacent to both  $v'_2, v'_0$ . Observe that  $x_{v'_2 v'_0}$  is not in  $P$ , as all vertices in  $P$  have degree three. Similarly, if  $v'_1 v'_n \notin E(G)$ , then by induction we have a vertex  $x_{v'_1 v'_n}$  adjacent to vertices  $v'_1, v'_n$  and does not belong to the path  $v_1, \dots, v_n$ , and  $x_{v'_1 v'_n}$  is not  $v_0$ , as  $v_0$  has degree three ( $v_0$  would be adjacent to  $v'_1, v_1, v'_2, v'_n$ , and as  $v'_1 v'_n \notin E(G)$ ,  $v'_1 \neq v'_2$ ).

This completes the claim. □

We can strengthen Lemma 5.56 when the path of length three is a component of the Gallai Tree and a specific outcome occurs.

**Lemma 5.61.** *Suppose the following graph  $H$  is an induced subgraph of  $G$ . Let  $V(H) = \{x, y, z, x', x'', y', z'', x_{x',x''}, x_{x',z''}\}$ . Let  $E(H) = \{xy, xx', xx'', yz, yy', zx', zz'', x''x_{x',x''}, x''y', x_{x',x''}x', x'x_{x',z''}, x_{x',z''}z'', y'z''\}$ . See Figure 5.1*

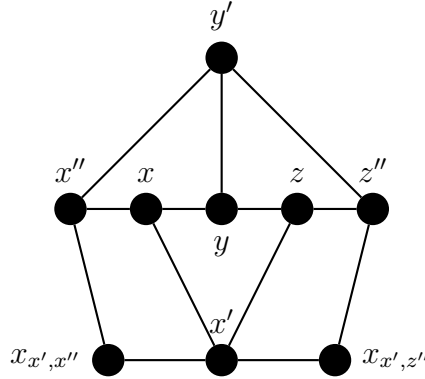


Figure 5.1: Graph from Lemma 5.61

Further suppose that all of  $x, y$  and  $z$  have degree three in  $G$ . Then at least one of the following occurs:

- $d(y') \geq 5$
- $d(x') \geq 5$
- $d(x_{x',x''}) \geq 4$
- $d(x_{x',z''}) \geq 4$
- $d(x'') \geq 5$
- $d(z'') \geq 5$ .

*Proof.* Suppose none of the conditions hold.

Let  $f$  be a 3-colouring of  $G - \{x, y, z\}$ . Without loss of generality suppose that  $f(x') = 0$ . First suppose that one of  $f(x'') = 0$ . Then colour  $z$  with any available colour,  $y$  with any available colour, and since  $f(x') = 0$  and  $f(x'') = 0$ , there is an available colour for  $x$ , a contradiction. A similar argument holds for  $f(z'')$ .

If  $f(x'') = f(z'')$ , then colour  $x$  and  $z$  the same colour, and there is an available colour for  $y$ . Thus we can assume without loss of generality that  $f(x'') = 2$  and  $f(z'') = 4$ . Then  $f(y') = 0$ ,  $f(x_{x',x''}) = 2$  and  $f(x_{x',z''}) = 4$ .

Observe that  $x', x_{x',z''}, z'', z', x'', x_{x',x''}$  are a six cycle where for any three consecutive vertices, all three colour appear. Let  $a, b \in \{x', x_{x',z''}, z'', z', x'', x_{x',x''}\}$  such that  $ab$  is an edge. We claim that either  $a$  is adjacent to a vertex not  $b$  with the same colour as  $b$ , or  $b$  is adjacent to a vertex not  $a$  with the same colour as  $a$ . To see this, if not simply exchange the colours of  $a$  and  $b$ , and by the previous case analysis (swapping colours if necessary), we can extend the colouring.

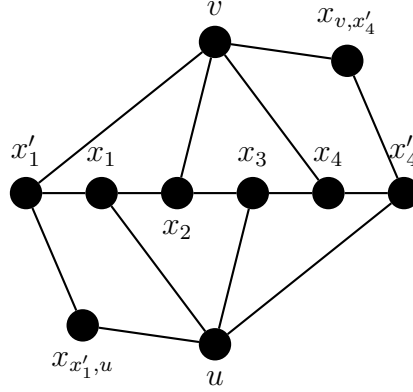


Figure 5.2: Graph from Lemma 5.62

If this observation implies that  $x'$  is adjacent to a vertex coloured 2 or 4 that is not  $x_{x',z''}$  or  $x_{x',x''}$ , then  $d(x') \geq 5$  and we are done. But then this implies that  $x_{x',z''}$  is adjacent to a vertex coloured 0 that is not  $x'$ , and thus either  $d(x',z'') \geq 4$ , or  $z''$  is adjacent to a vertex coloured 2 that is not  $x_{x',z''}$ . Similarly, this implies that either  $d(z'') \geq 5$ , or  $z''$  is adjacent to a vertex coloured 4 which is not  $z''$ . Continuing, this implies that either  $d(z') \geq 5$  or  $x''$  is adjacent to a vertex coloured 0 that is not  $z'$ . Finally, this implies that either  $d(x'') \geq 5$ , or  $d(x_{x',x''}) \geq 4$ , a contradiction in either case, and so we conclude the claim.  $\square$

**Lemma 5.62.** *Suppose the following graph  $H$  is an induced subgraph of  $G$ . Let  $V(H) = \{x_1, x_2, x_3, x_4, x'_1, x'_4, u, v, x_{x'_1,u}, x_{v,x'_4}\}$ , and  $E(H) = \{x_1x'_1, x_1u, x_1x_2, x_2x_3, x_2v, x_3u, x_3x_4, x_4x'_4, x_4v, x'_4u, x'_4x_{x'_4,v}, x'_1v, x'_1x_{x'_1,u}, vx_{x'_4,v}, x_{x'_1,u}u\}$ . See Figure 5.2.*

*Further suppose that  $x_1, x_2, x_3, x_4$  all have degree three in  $G$ . Then at least one of the following occurs:*

- $d(v) \geq 5$
- $d(u) \geq 5$
- *There is a  $t \in \{x'_1, x'_4\}$  such that  $d(t) \geq 5$  and there is a  $w \in \{x_{x'_1,u}, x_{x'_4,v}\}$  such that  $d(w) \geq 4$*
- *Both  $d(x'_1) \geq 5$  and  $d(x'_4) \geq 5$*
- *Both  $d(x_{x'_1,u}) \geq 4$  and  $d(x_{x'_4,v}) \geq 4$ .*

*Proof.* We may assume none of the conditions holds. Let  $f$  be a 3-colouring of  $G - \{x_1, x_2, x_3, x_4\}$ . Without loss of generality we may assume that  $f(v) = 0$ . We claim that  $f(u) = 0$ . Suppose not, and without loss of generality suppose that  $f(u) = 2$ . If

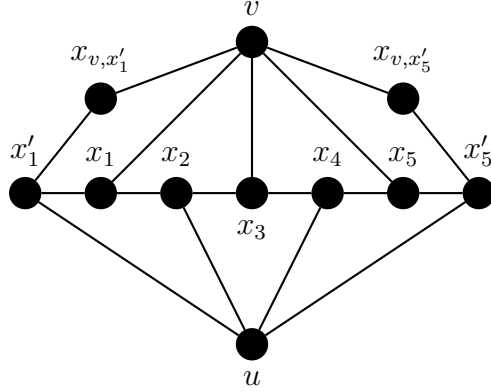


Figure 5.3: Graph from Lemma 5.63

$f(x'_1) = f(x'_4)$ , then  $f(x'_1) = 4$ , and colour  $x_1$  and  $x_3$  with 0, and  $x_2$  and  $x_4$  with 2. If  $f(x'_1) = 2$  and  $f(x'_4) = 0$ , then again colour  $x_1$  and  $x_3$  zero, and  $x_2$  and  $x_4$  two. If  $f(x'_1) = 4$ , then colour  $x_1$  and  $x_3$  zero,  $x_2$  with 2, and  $x_4$  with any available colour. All other cases follow similarly.

Hence we can assume that  $f(u) = 0$ . Now we claim that  $f(x'_1) = f(x'_4) \in \{2, 4\}$ . Suppose  $f(x'_1) \neq f(x'_4)$ . Then without loss of generality  $f(x'_1) = 2$  and  $f(x'_4) = 4$ . Then colour  $x_1$  and  $x_3$  with 4, and  $x_2$  and  $x_4$  with 2. If  $f(x'_1) = f(x'_4) = 0$ , then colour  $x_1$  and  $x_3$  with 2 and  $x_2$  and  $x_4$  with 4. Thus it follows that  $f(x'_1) = f(x'_4) \in \{2, 4\}$ . Without loss of generality we will assume that  $f(x'_1) = 4$ . Hence  $f(x_{x'_1,u}) = f(x_{v,x'_4}) = 2$ . Now the proof is analogous to Lemma 5.61. We observe that for any adjacent pair of vertices  $w, t \in \{x'_1, x'_4, v, u, x_{x'_1,u}, x_{v,x'_4}\}$ , either  $w$  is adjacent to a vertex coloured  $f(t)$  that is not  $t$ , or  $t$  is adjacent to a vertex coloured  $f(w)$  which is not  $w$ . If not, we simply exchange the colours of  $w$  and  $t$ , and remain a 3-colouring, and extend this colouring to a colouring of  $x_1, x_2, x_3, x_4$  (here this extension is possible by the same case analysis done above). But now following the same argument as Lemma 5.61 we see that one of the desired outcomes must hold.  $\square$

**Lemma 5.63.** *Suppose that  $G$  contains the following graph  $H$  as an induced subgraph. Let  $V(H) = \{x_1, x_2, x_3, x_4, x_5, x'_1, x'_5, u, v, x_{x'_1,v}, x_{v,x'_5}\}$  and  $E(H) = \{x_1x_2, x_1x'_1, x_1v, x_2u, x_2x_3, x_3v, x_3x_4, x_4u, x_4x_5, x_5x'_5, x_5v, x'_5u, x'_5x_{v,x'_5}, vx_{x'_5,v}, vx_{x'_1,v}, x'_1x_{x'_1,v}, x'_1u\}$ . See Figure 5.63. Then at least one of the following occurs:*

- $d(u) \geq 5$
- $d(v) \geq 6$
- $d(x_{x'_1,v}) \geq 4$
- $d(x_{v,x'_5}) \geq 4$

- $d(x'_1) \geq 4$
- $d(x'_5) \geq 4$ .

*Proof.* Let  $f$  be a 3-colouring of  $G - \{x_1, x_2, x_3, x_4, x_5\}$ . Without loss of generality we assume that  $f(u) = 0$ . We claim that  $f(v) = 0$ . Suppose without loss of generality  $f(v) = 2$ . Then colour  $x_1, x_3$  and  $x_5$  zero, and  $x_2$  and  $x_4$  two. Hence  $f(v) = 0$ . Now we claim that  $f(x'_1) \neq f(x'_4)$ . Suppose  $f(x'_1) = f(x'_4)$ . Then without loss of generality  $f(x'_1) = 2$ . Then colour  $x_1, x_3, x_5$  with 4, and  $x_2$  and  $x_4$  with 2. Thus without loss of generality we can assume that  $f(x'_1) = 4$  and  $f(x'_4) = 2$ . Hence  $f(x_{x'_1, v}) = 2$  and  $f(x_{v, x'_5}) = 4$ .

Then observe that for any pair of adjacent vertices  $w, t \in \{u, x'_5, x_{v, x'_5}, v, x_{x'_1, v}, x'_1\}$ , either  $w$  is adjacent to a vertex coloured  $f(t)$  that is not  $t$ , or  $t$  is adjacent to a vertex coloured  $f(w)$  that is not  $w$ . If not, we exchange the two colours and extend the colouring using the same ideas as in the above case analysis. Following a similar argument as in Lemma 5.61, we now see that at least one of the desired outcomes must follow.  $\square$

## 5.7 A basic counting argument to finish

In this section we prove Theorem 1.79. We assume that all components of the Gallai Tree are isomorphic to paths. Let  $P$  be a path component in the Gallai Tree. Let  $d_3(v)$  denote the number of neighbours of  $v$  which have degree three. Assign to each vertex  $v$  a charge of  $d(v)$ . Consider the discharging rule where each vertex  $v$  with  $d(v) \geq 4$  sends  $\frac{d(v)-3.4}{d_3(v)}$  charge to each of its neighbours of degree three. Let  $\text{ch}(v)$  denote the charge of each vertex after performing the discharging rule.

**Observation 5.64.** If  $d(v) \geq 4$ , then  $\text{ch}(v) \geq 3.4$ .

*Proof.* We have that  $\text{ch}(v) \geq d(v) - d_3(v) \left( \frac{d(v)-3.4}{d_3(v)} \right) = 3.4$ .  $\square$

Given a component  $P$  of the Gallai Tree, we let  $\text{ch}(P) = \sum_{v \in V(P)} \text{ch}(v)$ . Observe that if for every component of the Gallai Tree  $P$ , we have  $\text{ch}(P) \geq 3.4v(P)$ , then Theorem 1.79 follows. To see this we have  $2e(G) = \sum_{v \in V(G)} d(v) = \sum_{v \in V(G)} \text{ch}(v) \geq 3.4v(G)$ , and hence  $e(G) \geq \frac{17v(G)}{10}$ . We will say a component  $P$  of the Gallai Tree is *safe* if  $\text{ch}(P) \geq 3.4v(P)$ . Thus we devote the rest of the section to showing that all components of the Gallai Tree are safe.

**Proposition 5.65.** *Let  $P$  be an isolated vertex in the Gallai Tree. Then  $P$  is safe.*

*Proof.* Let  $v$  be the isolated vertex in  $P$ . Then all neighbours of  $v$  have degree at least 4. Hence  $\text{ch}(v) \geq 3 + 3 \left( \frac{4-3.4}{4} \right) \geq 3.45$ , and thus  $v$  is safe.  $\square$

**Proposition 5.66.** *Let  $P$  be isomorphic to an edge in the Gallai Tree. Then  $P$  is safe.*

*Proof.* Let  $v$  be a vertex in  $V(P)$ . Let  $x, y, z$  be the neighbours of  $v$ , and without loss of generality let  $z$  be the neighbour of  $v$  with degree three. First suppose that  $x, y, z$  are an independent set. Then by Lemma 5.47 there are distinct vertices which are not in  $\{x, y, z, v\}$ ,  $x_{y,z}$ ,  $x_{x,z}$ ,  $x_{x,y}$  which are adjacent to  $y$  and  $z$ ,  $x$  and  $z$  and  $x$  and  $y$  respectively. Then as  $P$  is isomorphic to an edge,  $d(x_{x,z}) \geq 4$  and  $d(x_{x,y}) \geq 4$ . Hence for all  $t \in \{x, y, x_{x,z}, x_{y,z}\}$ ,  $d_3(t) \leq d(t) - 1$ . Therefore  $\text{ch}(v) \geq 3 + 2(.2) = 3.4$  and  $\text{ch}(z) \geq 3 + 2(.2) = 3.4$ . Therefore it follows in this case that  $P$  is safe. Observe that if  $x, y, z$  is not an independent set but only  $xy \in E(G)$ , then the above argument still shows that  $P$  is safe, as we never considered  $x_{x,y}$  (in fact, the edge  $xy$  improves the situation).

Therefore we can assume that  $z$  is adjacent to at least one of  $x$  or  $y$ . Note that  $z$  cannot be adjacent to both  $x$  and  $y$ , as otherwise we contradict Proposition 5.46.

So without loss of generality suppose that  $yz \in E(G)$ . Then  $z$  and  $v$  share a common neighbour,  $y$ , and thus by Lemma 5.50  $yx \in E(G)$ . Additionally, the neighbour of  $z$  which is not  $y$  or  $v$ , say  $z'$  is also adjacent to  $y$ . Further  $d(z') \geq 4$ . Therefore for all  $t \in \{x, y, z'\}$ , we have  $d_3(t) \leq d(t) - 1$ , and  $d_3(y) \leq d(y) - 2$ . Hence  $\text{ch}(v) \geq 3 + .3 + .2 = 3.5$  and  $\text{ch}(z) \geq 3 + .3 + .2 = 3.5$ . Therefore in this case  $P$  is safe and thus the proposition follows.  $\square$

**Proposition 5.67.** *Let  $P$  be isomorphic to a path of length 2 in the Gallai Tree. Then  $P$  is safe.*

*Proof.* Let  $P = x, y, z$  be the path of length 2 in the Gallai Tree. Let  $x', x'', y', z', z''$  be the vertices adjacent to  $x, y, z$  respectively not on  $P$ . Then by Lemma 5.56 up to relabelling the vertices, we have that  $y'$  is adjacent to  $x''$  and  $z''$ , and  $x' = z'$ . If  $x' = y'$ , then  $d(y') \geq 5$ , and  $d_3(y') \leq d(y') - 2$ , and hence  $\frac{d(y') - 3.4}{d_3(y')} \geq \frac{8}{15}$ . Thus  $\text{ch}(P) \geq 9 + 3(\frac{8}{15}) + .4 = 11 \geq 10.2$ , and hence  $P$  is safe in this case.

Therefore  $x' \neq y'$ . If  $x'$  is adjacent to both  $x''$  and  $z''$ , then for any  $t \in \{x', x'', y', z''\}$ , we have  $d(t) \geq 4$  and  $d_3(t) \leq d(t) - 2$ . Therefore  $\text{ch}(P) \geq 9 + 5(.3) = 10.5 \geq 10.2$ , and  $P$  is safe in this case.

Therefore  $x'$  is adjacent to at most one of  $x''$  or  $z''$ . Suppose  $x'$  is not adjacent to  $z''$  but is adjacent to  $x''$ . Then there is a vertex not in  $P$ ,  $x_{x',z''}$  which is adjacent to both  $x'$  and  $z''$ . Then for  $t \in \{y, x''\}$ , we have  $d(t) \geq 4$  and  $d_3(t) \leq d(t) - 2$ . For  $w \in \{x', z''\}$ , we have  $d(w) \geq 4$  and  $d_3(w) \leq d(w) - 1$ . Hence we have  $\text{ch}(P) \geq 9 + 1.2 = 10.2$  and hence  $P$  is safe.

Thus by symmetry we may assume that  $x'$  is not adjacent to either  $x''$  or  $z''$ . Then there are distinct vertices not in  $P$ , say  $x_{x',x''}$  and  $x_{x',z''}$ , which are adjacent to  $x'$  and  $x''$ , and  $x'$  and  $z''$  respectively (and further  $x_{x',x''}, x_{x',z''} \notin \{x'', y', z'', x'\}$ ).

If  $d(x') \geq 5$ , then observe that  $\text{ch}(P) \geq 9 + .64 + .4 + .3 = 10.34 > 10.2$  and hence  $P$  is safe.



If one of  $x''$  or  $z''$  has degree five, then we have that  $\text{ch}(P) \geq 9 + .4 + .3 + 2(.15) + .2 = 10.2$ , and hence  $P$  is safe.

If  $d(y) \geq 5$ , then  $\text{ch}(P) \geq 9 + .2 + .3 + .2 + \frac{8}{15} \geq 10.23 > 10.2$ .

If either of  $d(x_{x',z''}) \geq 4$  or  $d(x_{x',x''}) \geq 4$ , then again  $\text{ch}(P) \geq 9 + 2(.3) + 3(.2) = 10.2$ , and hence  $P$  is safe.

By Lemma 5.61, at least one of the above cases occurs, and hence  $P$  is safe.  $\square$

**Proposition 5.68.** *Let  $P$  be isomorphic to a path of length 3 in the Gallai Tree. Then  $P$  is safe.*

*Proof.* Let  $P = x_1, x_2, x_3, x_4$ . By Corollary 5.60, up to relabelling the vertices, there are vertices  $x'_1, x'_4$  adjacent to  $x_1$  and  $x_4$  respectively, vertices  $u$  and  $v$  (distinct from  $x'_1$  and  $x'_4$ ), such that  $u$  is adjacent to  $x'_4, x_3, x_1$ , and  $v$  is adjacent to  $x'_1, x_2$  and  $x_4$ . If  $u = v$ , then  $d(u) \geq 6$ , and in this case it follows that  $\text{ch}(P) \geq 14.6 > 13.6$  and hence  $P$  is safe. Therefore we can assume that  $u \neq v$ .

If  $ux'_1 \in E(G)$ , then as  $d_3(u) \leq d(u) - 2$ , and  $d_3(x'_1) \leq d(x'_1) - 2$ , it follows that  $\text{ch}(P) \geq 13.6$  and hence  $P$  is safe. An analogous argument holds if  $vx'_4 \in E(G)$ .

Hence we can assume that  $ux'_1 \notin E(G)$  and  $vx'_4 \notin E(G)$ . Then there are distinct vertices  $x_{u,x'_1}$  and  $x_{v,x'_4}$  not in  $P$  that are adjacent to  $u$  and  $x'_1$  and  $v$  and  $x'_4$  respectively. If  $u$  is adjacent to one of  $x_2$  or  $x_4$ , then it follows that  $\text{ch}(P) \geq 14 > 13.6$  and hence  $P$  is safe. Similarly, if  $v$  is adjacent to either  $x_1$  or  $x_3$  we get that  $P$  is safe.

Therefore, we have a induced subgraph isomorphic to the graph in Lemma 5.62. If either  $d(v) \geq 5$  or  $d(u) \geq 5$ , then  $\text{ch}(P) \geq 14 > 13.6$  and hence  $P$  is safe.  $\square$

**Proposition 5.69.** *Let  $P$  be isomorphic to a path of length 4 in the Gallai Tree. Then  $P$  is safe.*

*Proof.* Let  $P = x_1, x_2, x_3, x_4, x_5$ . We apply Corollary 5.60 to  $P$ . Then from Corollary 5.60 there are vertices  $x'_1$  and  $x'_5$  not in  $P$  where  $x'_1$  is adjacent to  $x_1$  and  $x'_5$  is adjacent to  $x_5$  and given a bipartition  $(A, B)$  of  $P \cup \{x'_1, x'_5\}$ , there are vertices  $u$  and  $v$  so that  $u$  is adjacent to all vertices in  $A$  and  $v$  is adjacent to all vertices in  $B$ . If  $u = v$ , then  $d(u) \geq 7$ , and hence  $\text{ch}(P) \geq 15 + 5(.72) + 2(.2) = 19 > 17$ .

Therefore  $u \neq v$ . Without loss of generality we can assume that  $x'_1 \in A$ , and hence  $x'_5 \in A$ . First suppose that  $x'_1$  is adjacent to  $v$ . Then either  $d(v) \geq 5$ , or  $x'_1x'_5 \in E(G)$ . If  $x'_1x'_5 \in E(G)$ , then  $\text{ch}(P) \geq 15 + 3(.2) + .6 + 2(.3) + .3 = 17.1 > 17$  and hence  $P$  is safe in this case. If  $d(v) \geq 5$ , then  $\text{ch}(P) \geq 15 + 3(.4) + .3 + .2 + 2(.3) = 17.3 > 17$  and hence  $P$  is safe in this case.

Therefore we can assume that  $v$  is not adjacent to  $x'_1$ , and by symmetry we can assume that  $v$  is not adjacent to  $x'_5$ . Therefore there are vertices  $x_{x'_1,v}$  and  $x_{x'_5,v}$  not on  $P$  which are adjacent to  $x'_1$  and  $v$ , and  $x'_5$  and  $v$  respectively.

If  $d(v) \geq 6$ , then  $\text{ch}(P) \geq 15 + 4(\frac{13}{30}) + .4 + .6 \geq 17 + \frac{11}{15}$ , and hence  $P$  is safe in this case.

If  $d(u) \geq 5$ , then  $\text{ch}(P) \geq 15 + 2(.53) + 3(.32) + .4 = 17.42 > 17$ . Thus  $P$  is safe in this case.

Therefore we have an induced subgraph as in Lemma 5.63. If  $d(x_{x'_1,v}) \geq 4$  then  $\text{ch}(P) \geq 15 + 3(.32) + .9 + .2 = 17.06 > 17$ , and hence  $P$  is safe in this case. Similarly if  $d(x_{x'_5,v}) \geq 5$ , then  $P$  is safe. If  $d(x'_1) \geq 5$ , then  $\text{ch}(P) \geq 3(.32) + .4 + .6 + .2 = 17.16 > 17$ , and hence  $P$  is safe in this case. Similarly, if  $d(x'_5) \geq 5$  then  $P$  is safe.  $\square$

**Proposition 5.70.** *All path components in the Gallai Tree of length at least 5 are safe.*

*Proof.* Let  $P$  be a path of length at least 5 in the Gallai Tree, with endpoints  $u$  and  $v$ . We apply Corollary 5.60. Then there are vertices  $u'$  and  $v'$  adjacent to  $u$  and  $v$  respectively such that for a bipartition  $(A, B)$  of  $P \cup \{u', v'\}$ , there are vertices  $x$  and  $y$  such that  $x$  is adjacent to all vertices in  $A$  and  $y$  is adjacent to all vertices in  $B$ . Further, for any  $w \in \{u', v'\}$  and  $t \in \{x, y\}$ , if  $wt \notin E(G)$ , then there is a vertex  $x_{w,t}$  which is not in  $P$ , and adjacent to both  $w$  and  $t$ . As  $P$  has length at least five, the degree of  $x$  and  $y$  is at least five.

Observe as the length of the path is at least five, this implies that the degree of  $x$  and  $y$  is at least five. If  $x = y$ , then  $d(x) \geq 8$ ,  $\frac{d(x)-3.4}{d_3(x)} \geq \frac{23}{30}$ , and hence  $\text{ch}(P) \geq 3v(P) + \frac{23}{30}v(P) > 3.4v(P)$ . It follows that  $P$  is safe.

Thus  $x \neq y$ . If  $x$  and  $y$  have degree at least six, then for  $t \in \{x, y\}$ , we have  $\frac{d(t)-3.4}{d_3(t)} \geq \frac{13}{30}$ . Thus  $\text{ch}(P) \geq 3v(P) + \frac{13}{30}v(P) > 3.4v(P)$  and it follows that  $P$  is safe.

Thus at least one of  $x$  or  $y$  has degree five. Thus the length of  $P$  is at most six. If the length of  $P$  is exactly 6, then the vertex of degree five is adjacent to both  $u'$  and  $v'$ , and thus is adjacent to at most 3 vertices of degree three. In this case we have that  $\text{ch}(P) \geq 3v(P) + 3(\frac{8}{15}) + 4(\frac{13}{30}) > 3.4v(P)$ .

Thus the last case to consider is when the length of  $P$  is exactly 5. Observe that either the degree of  $x$  and  $y$  is greater than 6, or it is 5 and the number of neighbours of degree three is at most 4 (since  $x$  is adjacent to at least one of  $u'$  and  $v'$ , and  $y$  is adjacent to at least one of  $u'$  and  $v'$ ). As  $\frac{5-3.4}{4} \geq .4$ , it again follows that  $P$  is safe.  $\square$

Thus every component of the Gallai Tree is safe, and Theorem 1.79 follows.

# Chapter 6

## Circular clique mixing in planar graphs

This chapter is joint work with Richard Brewster.

### 6.1 Introduction

The goal of this chapter is to prove Theorem 1.96. In fact, we show this characterizes  $(p, q)$ -mixing in planar graphs when  $2 < \frac{p}{q} < 4$ . We need a litany of theorems and definitions before getting into the proof, which we provide now.

Let  $G$  be a graph and let  $f$  be a  $(p, q)$ -colouring of  $G$ . Given an edge  $uv$  of  $G$ , define the *weight of  $uv$  under  $f$*  as  $W(uv, f) = (f(v) - f(u)) \pmod{p}$ . Note there is an implied direction to  $uv$  here. Given a set of edges from  $G$ ,  $S \subseteq E(G)$ , we naturally extend the concept of weight by

$$W(S, f) = \sum_{uv \in S} W(uv, f).$$

Given a walk  $X = x_0, x_1, x_2, \dots, x_l$  in  $G$ , as a slight abuse of notation we write

$$W(X, f) = \sum_{i=1}^l ((f(x_i) - f(x_{i-1})) \pmod{p}).$$

In particular for a cycle of length  $l$ ,  $C = c_0, c_1, \dots, c_{l-1}, c_0$ ,

$$W(C, f) = \sum_{i=1}^l ((f(c_i) - f(c_{i-1})) \pmod{p}),$$

where index arithmetic is modulo  $l$ . For a cycle  $C$ , the sum telescopes and hence

$$W(C, f) = p \cdot w_f(C) \text{ for some integer } w_f(C).$$

We call  $w_f(C)$  the *wind* of  $C$  under  $f$ .

In [3], a characterization of the  $(p, q)$ -mixing problem was given.

**Theorem 6.1** ([3]). *Fix  $2 < \frac{p}{q} < 4$  and let  $G$  be a graph. Then  $G$  is not  $(p, q)$ -mixing if and only if there exists a  $(p, q)$ -colouring  $f$  of  $G$  and a cycle  $C$  in  $G$  where  $W(C, f) \neq \frac{e(C)}{2}p$ .*

Motivated by this theorem, for any cycle  $C$  of  $G$ , if  $W(C, f) \neq \frac{e(C)}{2}p$ , then we say that  $C$  is *wrapped with respect to  $f$* . An additional useful idea is that of folding.

For vertices  $x$  and  $y$ , let  $d(x, y)$  denote the distance from  $x$  to  $y$ . Given a graph  $G$ , and two vertices  $x$  and  $y$  such that  $d(x, y) = 2$ , let  $G_{xy}$  be the graph obtained by identifying  $x$  and  $y$  and calling the new vertex  $v_{xy}$ . The homomorphism  $f : G \rightarrow G_{xy}$  defined by  $f(x) = f(y) = v_{xy}$  and  $f(u) = u$  for  $u \notin \{x, y\}$  is an *elementary fold*. We say a graph  $G$  *folds* to a graph  $H$  if there is a homomorphism  $f : G \rightarrow H$  where  $f$  is a composition of elementary folds and  $f(G)$  is isomorphic to  $H$ . We call such a mapping a *folding*. For brevity, we may refer to an elementary fold as just a fold. A useful fact is that mixing is closed under folding.

**Theorem 6.2** ([3]). *Let  $p$  and  $q$  be integers where  $2 < \frac{p}{q} < 4$ . If  $G$  folds to  $H$ , and  $H$  is not  $(p, q)$ -mixing then  $G$  is not  $(p, q)$ -mixing.*

We also will need a special type of fold (in connected graphs) called a retract.

**Definition 6.3.** Let  $G$  be a graph and  $H$  an induced subgraph of  $G$ . A *retraction* of  $G$  to  $H$  is a homomorphism  $r : G \rightarrow H$  such that  $r(h) = h$  for every vertex  $h \in V(H)$ . If there exists a retraction of  $G$  to  $H$ , we say  $G$  *retracts* to  $H$ .

The next observation is well known (see [16]).

**Observation 6.4.** If  $G$  is a connected graph, and  $r : G \rightarrow H$  is a retraction, then  $r$  is a folding from  $G$  to  $H$ .

*Proof.* If  $r$  is the identity then trivially  $r$  is a folding. Otherwise,  $r$  maps a vertex  $u$  distance two away from a vertex  $v \in V(H)$ , to a vertex in  $v$ . Identify  $u$  and  $v$ , and now repeat this process. □

Finally, we need a lemma which shows that wrapped cycles behave nicely across theta subgraphs.

**Lemma 6.5** ([3]). *Let  $G$  be a graph and  $f$  a  $(p, q)$ -colouring of  $G$ . Let  $C = c_0, c_1, \dots, c_l, c_0$  be a cycle which is wrapped under  $f$ . Let  $P = p_0, p_1, \dots, p_k$  be a path whose endpoints lie on  $C$ , i.e.  $p_0 = c_s$  and  $p_k = c_t$ , and whose internal vertices do not lie on  $C$ . Then either  $C' = c_s, c_{s+1}, \dots, c_{t-1}, c_t, p_{k-1}, \dots, p_1, c_s$  or  $C'' = c_s, p_1, \dots, p_{k-1}, c_t, c_{t+1}, \dots, c_{s-1}, c_s$  is wrapped with respect to  $f$ .*

These are all the structural lemmas we need. As an overview of the proof: we first show that a graph  $G$  is  $(p, q)$ -mixing when  $2 < \frac{p}{q} < 4$  if and only if each block of  $G$  is  $(p, q)$ -mixing. We then argue that if a graph has a planar embedding with at most one face of length at least  $2k$ , where  $C_{2k}$  is the minimal even cycle which is not  $(p, q)$ -mixing, then  $G$  is  $(p, q)$ -mixing. Then we show that if  $G$  has a planar embedding with at least two  $\geq 2k$  faces and no small separating cycle, then  $G$  is not  $(p, q)$ -mixing. When  $\frac{p}{q} \geq 3$ , we can additionally assume our graph has no separating four cycles, and thus we get a full characterization in this case.

## 6.2 The proof

First we note that there is an even cycle which is not  $(p, q)$ -mixing.

**Observation 6.6.** Fix integers  $p$  and  $q$  such that  $2 < \frac{p}{q} < 4$ . If  $p$  is even, then for all even integers  $j \geq p$ ,  $C_j$  is not  $(p, q)$ -mixing. If  $p$  is odd, then for all even  $j \geq 2p$ ,  $C_{2p}$  is not  $(p, q)$ -mixing.

*Proof.* First assume that  $p$  is even. Let  $C_p = c_0, \dots, c_{p-1}, c_0$  and consider the  $(p, q)$ -colouring  $f(c_i) = iq \bmod p$ ,  $i = 0, 1, \dots, p-1$ . Then  $W(C, f) = qp < \frac{p}{2}p$  as  $\frac{p}{q} > 2$ . Hence  $C_p$  is not  $(p, q)$ -mixing. It follows by Theorem 6.2 that when  $p$  is even, for all even  $j \geq p$ ,  $C_{2j}$  is not  $(p, q)$ -mixing.

Now assume that  $p$  is odd. Let  $C_{2p} = c_0, \dots, c_{2p-1}, c_0$ . For  $i = 0, 1, \dots, p-1$ , colour  $c_i$  and  $c_{i+p}$  with  $iq \bmod p$ . Again,  $W(C, f) = 2qp < p^2$ , and thus  $C_{2p}$  is not  $(p, q)$ -mixing. Hence, for all even  $j \geq 2p$ ,  $C_j$  is not  $(p, q)$ -mixing.  $\square$

When  $3 \leq \frac{p}{q} < 4$ , we can strengthen this to saying that  $C_6$  is not  $(p, q)$ -mixing.

**Observation 6.7.** Let  $p, q$  be positive integers where  $3 \leq \frac{p}{q} < 4$ . Then  $C_6$  is not  $(p, q)$ -mixing.

*Proof.* Let  $v_1, v_2, v_3, v_4, v_5, v_6$  be the vertices of  $C_6$ , where  $v_i v_{i+1} \in E(C_6)$  for  $i \in \{1, \dots, 6\}$ . Let  $f$  be the  $(p, q)$ -colouring where  $f(v_0) = f(v_4) = 0$ ,  $f(v_1) = f(v_5) = q$ ,  $f(v_3) = f(v_6) = 2q$ . This is a proper  $(p, q)$ -colouring as  $\frac{p}{q} \geq 3$ . Observe that  $\frac{e(C)}{2}p = 3p$ . Orienting  $C$  from  $v_i$  to  $v_{i+1}$  for  $i \in \{1, 2, 3, 4, 5, 6\}$ , we have  $W(C, f) = 2p < 3p$ , and hence  $C_6$  is not  $(p, q)$ -mixing.  $\square$

Now we show we can assume graphs are to 2-connected graphs. Recall that a *block* of a graph is a maximal 2-connected component.

**Observation 6.8.** Fix integers  $p$  and  $q$  such that  $2 < \frac{p}{q} < 4$ . A graph  $G$  with a  $(p, q)$ -colouring is  $(p, q)$ -mixing if and only if every block of  $G$  is  $(p, q)$ -mixing.

*Proof.* Suppose that a block  $B$  of  $G$  is not  $(p, q)$ -mixing. By Theorem 6.1, it follows that there is a  $(p, q)$ -colouring  $f$  of  $B$  such that there is a cycle  $C$  which is wrapped with respect to  $f$ . Observe that we can extend  $f$  to a  $(p, q)$ -colouring of  $G$  by taking  $(p, q)$ -colourings of each block (which exist as  $G$  has a  $(p, q)$ -colouring), and permuting colours if necessary.

Conversely, suppose that  $G$  is not  $(p, q)$ -mixing. It follows that by Theorem 6.1 there is a  $(p, q)$ -colouring  $f$  such that there is a cycle  $C$  which is wrapped with respect to  $f$ . As a cycle is 2-connected,  $C$  lies in some block  $B$ . Then  $f$  restricted to  $B$  is a  $(p, q)$ -colouring which has a wrapped cycle, and hence  $B$  is not  $(p, q)$ -mixing.  $\square$

Therefore, we will always assume all graphs are 2-connected. Recall that if a planar graph is 2-connected, then every face is bounded by a cycle. An important fact is that for  $(p, q)$ -mixing when  $2 < \frac{p}{q} < 4$ , we may assume that there is no pair of vertices  $u$  and  $v$  such that  $uv \in E(G)$  and  $G - u - v$  disconnects  $G$ .

**Observation 6.9.** Fix integers  $p$  and  $q$  such that  $2 < \frac{p}{q} < 4$ . Let  $G$  be a graph, and suppose there are vertices  $u, v$  where  $uv \in E(G)$  and  $G - u - v$  is disconnected. Let  $G_1$  and  $G_2$  be graphs such that  $V(G_1) \cap V(G_2) = \{u, v\}$ ,  $V(G_1) \cup V(G_2) = V(G)$ , and  $E(G_1) \cup E(G_2) = E(G)$ . If  $G_1$  and  $G_2$  are  $(p, q)$ -mixing, then  $G$  is  $(p, q)$ -mixing.

*Proof.* Suppose  $G$  is not  $(p, q)$ -mixing. By Theorem 6.1 it follows there is a  $(p, q)$ -colouring  $f$  of  $G$  and a cycle  $C$  such that  $C$  is wrapped with respect to  $G$ . As  $G_1$  and  $G_2$  are  $(p, q)$ -mixing, this implies that  $V(C) \not\subseteq V(G_1)$  and  $V(C) \not\subseteq V(G_2)$ . In particular,  $u, v \in V(C)$  and  $uv \notin E(C)$ . Let  $P_1$  be path of  $C$  contained in  $G_1$ , and  $P_2$  be the path of  $C$  contained in  $G_2$ . Then  $P_1 + uv$  and  $P_2 + uv$  are cycles, and by Lemma 6.5 one of them is wrapped with respect to  $f$ . But this contradicts that  $G_1$  and  $G_2$  are  $(p, q)$ -mixing.  $\square$

We record an obvious fact which simply says that certain elementary folds preserve planarity.

**Observation 6.10.** If  $G$  is a planar graph where  $x, y, z$  are consecutive vertices on a face, where the distance from  $x$  to  $z$  is 2, then the graph obtained by folding  $x$  and  $z$  is planar.

One can see this by simply adding the edge  $xz$  inside the face and then contracting the edge and noting that planarity is closed under contraction. The next lemma is well known. A proof can be found in [5] (they claim a weaker statement, but examining the proof gives this lemma).

**Lemma 6.11.** *Let  $G$  be a connected bipartite graph, and let  $P$  be a shortest path from  $x$  to  $y$  in  $G$ . Then  $G$  retracts to  $P$ .*

Much stronger statements are known than Lemma 6.11 but for our purposes we only need Lemma 6.11.

Let  $G$  be a connected graph and  $C$  be a cycle in  $G$ . Given a plane embedding of a graph  $G$ , and a cycle  $C$ , we let the *interior of  $C$* , denoted  $G_{\text{int}}$ , be the graph induced by the vertices on the interior of  $C$  with the inclusion of the cycle  $C$ . Analogously define the *exterior of  $C$* , denoted  $G_{\text{ext}}$ . We say  $C$  is *separating* if both the interior and exterior of  $C$  contain a vertex not in  $C$ . For ease, we will say a  $\geq k$ -*face* is a face whose boundary has at least  $k$  edges. For the rest of the chapter, for every planar graph  $G$  we will associate to it a plane embedding (arbitrarily) (up to the choice of outerface). If we modify our graph and obtain a new graph via some operations, the new graph will always be planar, and the embedding of the new graph is assumed to be obtained from the embedding of  $G$ . Therefore the upcoming lemmas work for any embedding of  $G$ .

**Lemma 6.12.** *Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the minimal even cycle which is not  $(p, q)$ -mixing. Let  $G$  be a 2-connected planar bipartite graph. If  $G$  has at most one  $\geq 2k$ -face, then  $G$  is  $(p, q)$ -mixing.*

*Proof.* Let  $G$  be an edge-minimal counterexample to the claim. If  $G$  is isomorphic to a cycle, then the claim holds as the cycle has length strictly less than  $2k$  (if the cycle had length  $2k$  or more, both the unique interior face and outerface have the length of the cycle, and hence we have two  $\geq 2k$ -faces). Thus we assume that  $G$  is not isomorphic to a cycle.

If one exists, let  $f$  be the  $\geq 2k$  face, otherwise let  $f$  be an arbitrary face, and let  $C = v_0, \dots, v_{t-1}, v_0$  be the facial cycle of  $f$ . Without loss of generality, we may suppose that  $f$  is the outerface. We claim all facial cycles are chordless. Suppose a facial cycle  $C$  has a chord  $uv$ . Then  $G - u - v$  is disconnected. Let  $G_1, G_2$  be two graphs such that  $V(G_1) \cap V(G_2) = \{u, v\}$ , and  $V(G_1) \cup V(G_2) = V(G)$ . Then both of  $G_1$  and  $G_2$  have at most one  $\geq 2k$  face, and hence are both  $(p, q)$ -mixing by minimality. But then Observation 6.9 implies that  $G$  is  $(p, q)$ -mixing, a contradiction.

**Claim 6.13.** *There exists a face  $f'$ , where  $f'$  has a facial cycle  $C'$  such that  $V(C) \cap V(C')$  induces a path  $P$  of length at least one.*

*Proof.* Pick an arbitrary edge  $e = v_j v_{j+1} \in E(C)$ . Let  $f'$  and  $f$  be the two distinct faces whose boundaries contain  $e$ . Let  $C'$  be the facial cycle of  $f'$ . We may assume that  $V(C') \cap V(C)$  does not induce a path. Note  $V(C') \not\subseteq V(C)$ , as otherwise this implies that  $V(C)$  has a chord. Let  $P$  be the component of  $G[V(C') \cap V(C)]$  containing  $v_j$ . Observe that  $P$  is a path, and let  $v$  be an endpoint of  $P$ . Let  $Q$  be the path in  $C'$  starting at  $v$  and ending at a vertex  $u \in V(C)$  such that  $V(Q) \cap V(P) = \{v\}$ , and all internal vertices of  $Q$  are not in  $V(C)$ . Let  $P'$  be the path from  $v$  to  $u$  in  $C$  such that  $V(P') \cap V(P) = \{v\}$ . Then  $P' + Q$  is a cycle, say  $C''$ . If  $C''$  is a facial cycle, then this is our desired path and we are done. Otherwise,  $C''$  is a separating cycle. Without loss of generality, we can assume that the interior of  $C''$  does not contain all of  $P$ , and repeat the above argument on the interior of  $C''$ , starting with some edge on  $C$  in the interior of  $C''$ . As the graph is finite, we find a face  $f'$  whose intersection with  $f$  is exactly a path of length at least one.  $\square$



Now, up to relabelling let  $e = v_1v_2$  be on the boundary of a face  $f'$ , with facial cycle  $C'$  such that the intersection of  $V(C')$  and  $V(C)$  is a path. Let  $P$  be this path, let  $S = V(P)$ , let  $v'$  and  $v''$  be the endpoints of  $P$ , and let  $S'$  be the set of internal vertices. Consider  $G' := G - S' - \{e\}$ . (The deletion of  $e$  is only required when  $S' = \emptyset$ .) Notice as all of the vertices of  $S$  are on the boundary of  $f$ , we do not create two  $\geq 2k$  faces. We have two cases to consider.

First suppose  $G'$  is 2-connected. Then since  $G$  is an edge-minimal counterexample,  $G'$  is  $(p, q)$ -mixing. Now consider any  $(p, q)$ -colouring  $\phi$  of  $G$ . Observe that  $\phi$  restricts to a  $(p, q)$ -colouring of  $G'$ . As  $G'$  is  $(p, q)$ -mixing, for all cycles  $D$  in  $G'$ , we have  $W(D, \phi) = \frac{e(D)}{2}p$ . Thus by Lemma 6.5, it suffices to show that the winds of  $C'$  and  $C$  are  $\frac{e(C')}{2}$  and  $\frac{e(C)}{2}$  respectively. As  $e(C') < 2k$ , we have  $W(C', \phi) = \frac{e(C')}{2}p$ . So it suffices to show that  $W(C, \phi) = \frac{e(C)}{2}p$ . To see this, consider the cycle  $C''$  obtained by taking the symmetric difference of  $C$  and  $C'$ . Since  $W(C'', \phi) = \frac{e(C'')}{2}p$  and  $W(C', \phi) = \frac{e(C')}{2}p$ , by Lemma 6.5 it follows that  $W(C, \phi) = \frac{e(C)}{2}p$ .

Otherwise,  $G'$  has a cut vertex  $v$ . We are going to argue that this cannot occur. Let  $T_1, T_2, \dots, T_t$  be the components of  $G' - v$ . We claim that  $t = 2$ . As  $v$  is a cut vertex, by definition  $t \geq 2$ . Observe by planarity, all vertices in  $S'$  have degree two in  $G$ . Adding the path  $S$  to  $G'$  joins at most two of the components in  $G' - v$ . Therefore if  $t \geq 3$ ,  $G$  contains a cut vertex, a contradiction.

Thus we can decompose  $G'$  into two graphs  $G_1$  and  $G_2$ , such that  $V(G_1) \cap V(G_2) = \{v\}$ , both  $G_1$  and  $G_2$  contain at least two vertices, and up to relabelling,  $v' \in V(G_1)$ ,  $v'' \in V(G_2)$ .

First suppose that  $v = v'$ . Then as  $G'$  is simply  $G$  with  $S'$  and  $e$  deleted, there is a vertex  $x \in V(G_1) \setminus \{v'\}$  such that every path from  $x$  to  $v''$  contains  $v'$ . But this implies that  $G$  is not 2-connected, a contradiction. By a similar argument, we can assume that  $v \neq v''$ .

Now we can conclude that all paths from  $v'$  to  $v''$  in  $G'$  must have  $v$  as an internal vertex. In particular, the path from  $v''$  to  $v'$  in  $C - S' - \{e\}$  must contain  $v$ . Thus  $v \in V(C)$ . The boundary of  $f'$  contains a path from  $v'$  to  $v''$  that does not use  $S' \cup \{e\}$ . In particular, this path must contain  $v$ . Therefore  $v \in V(C) \cap V(C')$ . But now this contradicts that  $V(C) \cap V(C')$  induces a path, a contradiction.  $\square$

Now we prove a partial converse, under the assumption that we have no small separating cycles. We will need the following definition. Given a graph  $G$  with a fixed planar embedding, we call a cycle  $C$  in  $G$  *f-separating* if the face  $f$  lies in the interior  $C$  and either  $C$  is a separating cycle or  $C$  is the boundary cycle for the outerface of  $G$ .

**Lemma 6.14.** *Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the smallest even cycle which is not  $(p, q)$ -mixing. Let  $G$  be a 2-connected planar bipartite graph with no separating  $C_{2i}$ -cycle for all  $i \in \{2, \dots, k-1\}$ . If  $G$  has at least two  $\geq 2k$ -faces, then  $G$  is not  $(p, q)$ -mixing.*



**Lemma 6.15.** Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the smallest even cycle which is not  $(p, q)$ -mixing. Let  $G$  be a 2-connected bipartite graph with a planar embedding containing no separating  $C_{2i}$ -cycle for all  $i \in \{2, \dots, k-1\}$ . If  $G$  has at least two  $\geq 2k$ -faces, then  $G$  is not  $(p, q)$ -mixing.

*Proof.* Let  $f, f_o$  be two  $\geq 2k$ -faces, and suppose that the boundaries of  $f$ , and  $f_o$ , are  $C$ , and  $C_o$ , respectively. Without loss of generality, we may suppose that  $f_o$  is the outer face. A cycle  $D$  is  $f$ -separating if  $f$  lies in  $G_{\text{int}}(D)$  and either  $D$  is a separating cycle or  $D = C_o$ , the boundary of the outer face. In particular,  $C_o$  is  $f$ -separating despite the fact that  $C_o$  is not separating. As  $e(C_o) \geq 2k$  and  $G$  has no separating cycles of length less than  $2k$ ,  $G$  has no  $f$ -separating cycles of length less than  $2k$ .

If  $G$  is a cycle, then  $G$  folds to  $C_{2k}$  and  $G$  is not  $(p, q)$ -mixing by Lemma 6.2. Otherwise, we show that  $G$  folds to a bipartite graph  $G'$  on fewer vertices, such that  $G'$  contains a block with two  $\geq 2k$ -faces, the face  $f$  (from  $G$ ) and  $f'_o$  (the outface of  $B'$ ), and no  $f'$ -separating cycle of length  $2i$ ,  $i \in \{2, \dots, k-1\}$ . The result follows by induction.

Therefore assume  $G$  is not a cycle and let  $y \in V(C)$  such that  $d(y) \geq 3$ .

Let  $z$  be a neighbour of  $y$  on  $C$ , and  $a$  be a neighbour of  $y$  not in  $C$ , such that all of  $a, y$ , and  $z$  lie on a face (such a choice of  $z$  and  $a$  exists). Fold  $a$  and  $z$  and let  $G'$  be the resulting graph. As  $a, y$  and  $z$  lie on a face of  $G$ , the graph  $G'$  is planar. Moreover,  $f$  is still a face of  $G'$  as  $a \notin V(C)$ . (We may think equivalently of  $G'$  as being formed by deleting  $a$  and joining  $z$  to all the neighbours of  $a$ . As  $a \in \text{Ext}(C)$  this process leaves the face  $f$  unchanged.) Let  $B'$  be the block of  $G'$  containing  $f$ . Now we consider two cases.

**Case 1:  $B'$  has no  $f$ -separating  $C_{2i}$ -cycle for  $i \in \{2, \dots, k-1\}$ .**

The outface of  $B'$  is an  $f$ -separating cycle. Thus it has length at least  $2k$ . The outface together with  $f$  are two  $\geq 2k$ -faces of  $B'$ . By induction,  $B'$  is not  $(p, q)$ -mixing from which we conclude  $G'$  is not  $(p, q)$ -mixing by Observation 6.8, and by Lemma 6.2,  $G$  is not  $(p, q)$ -mixing.

**Case 2:  $B'$  has an  $f$ -separating cycle  $D'$  of length less than  $2k$ .**

Let  $v_{az}$  be the new vertex obtained from folding  $a$  and  $z$ . As  $G$  has no  $f$ -separating cycle of length less than  $2k$ , we have  $v_{az} \in V(D')$ . Let  $v'_{az}$  and  $v''_{az}$  be the two neighbours of  $v_{az}$  in  $D'$ . Observe that without loss of generality,  $a$  is adjacent to  $v'_{az}$  and not to  $v''_{az}$  in  $G$  and  $z$  is adjacent to  $v''_{az}$  but not to  $v'_{az}$  in  $G$ , as otherwise,  $G$  has an  $f$ -separating cycle of length less than  $2k$ . Let  $D$  be the cycle (in  $G$ ) that gave rise to  $D'$  in  $B'$ , i.e.  $D$  is the cycle obtained by replacing the path  $v'_{az}, v_{az}, v''_{az}$  in  $D'$  with the path  $v'_{az}, a, y, z, v''_{az}$ . Since  $C$  is in  $G_{\text{int}}(D)$  and  $C_o$  is in  $G_{\text{ext}}(D)$ , if  $e(D) < 2k$ , then  $G$  contains an  $f$ -separating cycle of length less than  $2k$ , a contradiction. Since  $D'$  has length less than  $2k$ , it follows that  $e(D) = 2k$  and  $e(D') = 2k - 2$ .

We claim  $D$  is an  $f$ -separating cycle. As  $e(C) \geq 2k$ ,  $e(D) = 2k$ , and  $a \in V(D) \setminus V(C)$ , there is a vertex of  $C$  in  $\text{Int}(D)$ . Thus,  $D$  is  $f$ -separating if there is a vertex in  $\text{Ext}(D)$

or  $D = C_o$  the boundary of the outerface. Suppose neither holds. Since  $e(C_o) \geq 2k$  and  $e(D) = 2k$ , it must be the case that  $V(C_o) = V(D)$ . Hence  $D$  must have a chord not belonging to  $C_o$  as  $D \neq C_o$ . However, this chord must be in the interior of  $D$  (as  $C_o$  bounds the outerface) which implies  $D$  is the sum of two shorter cycles one of which is  $f$ -separating, a contradiction.

Let  $P$  be the path of length  $2k - 1$  from  $y$  to  $z$  in  $D - zy$ . We now claim that in  $G_{\text{int}}(D) - zy$ , the path  $P$  is a shortest  $(y, z)$ -path. Suppose there is a shorter path  $P'$ . Using the fact that  $D$  bounds the outerface of  $G_{\text{int}}(D)$  and  $zy$  is an edge of  $D$ , the cycle  $P' + zy$  is  $f$ -separating in  $G$  and of length less than  $2k$ , a contradiction. Therefore, by Lemma 6.11,  $G_{\text{int}}(D) - zy$  retracts to  $P$  which implies  $G_{\text{int}}(D)$  folds to  $D$ . (The vertices  $y$  and  $z$  are fixed under the retraction.) Let  $G''$  be the resulting graph from  $G$  after folding  $G_{\text{int}}(D)$  to  $D$ . In  $G''$ ,  $D$  is the boundary of a  $\geq 2k$ -face and the outerface is a  $\geq 2k$ -face. (This includes the possibility that  $G'' = D$  is simply a cycle.) Now by induction the result follows.  $\square$

Therefore we have the following theorem:

**Theorem 6.16.** *Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the minimal non- $(p, q)$ -mixing cycle. Let  $G$  be a 2-connected bipartite planar graph with no separating  $C_{2i}$ -cycles for  $i \in \{2, \dots, k - 1\}$ . The graph  $G$  is  $(p, q)$ -mixing if and only if there is at most one  $\geq 2k$ -face.*

Now we show we can perform reductions to remove small separating cycles.

**Lemma 6.17.** *Fix  $2 < \frac{p}{q} < 4$ . Let  $C_{2k}$  be the smallest even cycle which is not  $(p, q)$ -mixing. Let  $G$  be a planar bipartite graph where  $C$  is a separating  $C_{2i}$ -cycle, for some  $i < k$ . Let  $G_1$  denote the interior of  $C$ , and  $G_2$  denote the exterior. If both  $G_1$  and  $G_2$  are  $(p, q)$ -mixing, then  $G$  is  $(p, q)$ -mixing.*

*Proof.* Suppose that  $G$  is not  $(p, q)$ -mixing. Let  $f$  be a  $G_{p,q}$ -colouring where there is a wrapped cycle  $D$ . If  $D$  contains vertices from both  $G_1$  and  $G_2$ , then  $D$  crosses the separating cycle  $C$  (note,  $D \neq C$  since  $C$  is  $(p, q)$ -mixing). Then we can (repeatedly, if required) apply Lemma 6.5 to obtain a wrapped cycle which lies completely in  $G_1$  or  $G_2$ , and thus either  $G_1$  or  $G_2$  is not  $(p, q)$ -mixing.  $\square$

Now we build towards proving the converse.

**Lemma 6.18.** *Let  $G$  be a connected bipartite planar graph and  $C$  be a facial cycle of  $G$ . There exists a bipartite planar graph  $H$  such that  $V(C) = V(H)$ ,  $C$  is a facial cycle in  $H$ , and a folding  $f : G \rightarrow H$  such that for all  $v \in V(C)$ ,  $f(v) = v$ .*

*Proof.* Let  $G$  be a vertex minimal counterexample to the lemma. Observe that the statement holds if  $V(G) = V(C)$ , as the identity map suffices. So we may assume there is a vertex in  $V(G) \setminus V(C)$ . Let  $C = v_0, \dots, v_{2k-1}, v_0$ . Observe that as  $G$  is bipartite, every face has size at least four, and hence for all vertices  $u$  there exists a vertex  $v$  with  $d(u, v) = 2$ , such that folding  $u$  and  $v$  preserves planarity.

**Claim 6.19.** *For all vertices  $u, v \in V(G) \setminus V(C)$ , if  $d(u, v) = 2$ , then folding  $u$  and  $v$  does not preserve planarity.*

*Proof.* Suppose not and let  $G_{uv}$  be the graph obtained by folding  $u$  and  $v$ . Then we do not add new chords to  $C$ , as both  $u$  and  $v$  are not in  $C$ . Thus by minimality there is a planar graph  $H$  with facial cycle  $C$ , and a homomorphism  $f : G_{uv} \rightarrow H$  such that  $f$  is the identity on  $C$ . Composing homomorphisms gives a contradiction.  $\square$

**Claim 6.20.** *All vertices in  $G$  are adjacent to a vertex in  $C$ .*

*Proof.* Suppose towards a contradiction that there is a vertex  $v \in V(G)$  which is not adjacent to any vertex in  $C$ . Then there exists a vertex  $u \in V(G)$  (necessarily on  $C$ ), such that  $d(v, u) = 2$  and that folding  $u$  and  $v$  preserves planarity. Let  $G_{uv}$  be the graph obtained by folding  $u$  and  $v$ . As  $v$  is not adjacent to a vertex in  $C$ , no new chords are added to  $C$ . As  $v(G_{uv}) < v(G)$ , by minimality, there exists a folding  $f : G_{uv} \rightarrow H$  such that  $H$  is planar,  $C$  is a facial cycle of  $H$ ,  $V(H) = V(C)$ , and  $f$  is the identity on  $C$ . Composing homomorphisms now gives a contradiction.  $\square$

**Claim 6.21.** *There are no separating cycles  $C'$  where  $C' \neq C$ .*

*Proof.* Suppose not and let  $C'$  be a separating cycle distinct from  $C$ . Let  $G'$  be subgraph of  $G$  consisting of a component of  $G - C'$  together with  $C'$ . Since  $C'$  is separating we can choose  $G'$  to be disjoint from  $C$  aside from possibly  $C$  sharing some vertices of  $C'$ . By minimality, there exists a homomorphism  $f$  from  $G'$  to a planar graph  $H'$  so that  $H'$  has facial cycle  $C'$ ,  $V(H') = V(C')$ , and  $f$  is the identity on  $C'$ . But now the resulting graph is smaller than  $G$ , and we can apply minimality again to find our desired homomorphism for  $C$ .  $\square$

**Claim 6.22.** *There does not exist a set of three vertices  $x, y, z \in V(G) \setminus V(C)$  such that  $G[\{x, y, z\}]$  induces a path of length 2.*

*Proof.* Suppose so. Without loss of generality let  $xy, yz \in E(G)$ , but  $xz \notin E(G)$ . Now by Claim 6.19 we can assume that folding  $x$  and  $z$  does not preserve planarity. Then it follows that there is a separating cycle  $C'$  which separates  $x$  from  $z$ . But this contradicts Claim 6.21.  $\square$

**Claim 6.23.** *There do not exist two adjacent vertices in  $V(G) \setminus V(C)$ .*

*Proof.* Suppose not and let  $u$  and  $v$  be adjacent vertices in  $V(G) \setminus V(C)$ . Let  $u'$  and  $v'$  be vertices in  $V(C)$  such that  $uu' \in E(G)$  and  $vv' \in E(G)$ . Note  $u' \neq v'$  as  $G$  is bipartite. Let  $P$  be a path from  $u'$  to  $v'$  in  $C$ , and consider the cycle  $C'$  formed by  $P$ ,  $u$  and  $v$ . We claim we can find a cycle  $C''$  where  $V(C'') \subseteq V(C')$  such that  $C''$  is a facial cycle. If  $u'v' \in E(G)$ , then  $C''$  is the cycle  $u, v, v', u'$ . Otherwise, without loss of generality we assume that  $C$  lies in the exterior of  $C'$ . Thus if there is a vertex not in  $C'$  in the interior of  $C'$ , then

$C'$  separates that vertex from vertices in  $V(C) \setminus V(C')$ . This contradicts Claim 6.21. If  $C'$  contains a chord  $xy$ , where  $x, y \in V(P)$ , then the path  $P'$  from  $x$  to  $y$  which is not contained in  $P$  with the chord  $xy$  is a separating cycle, contradicting Claim 6.21. If  $C'$  contains a chord  $ux$ , then let  $P''$  be the path from  $x$  to  $v'$  on  $P$ , and consider the cycle  $C''$  consisting of  $P'', u$  and  $v$  (repeating this argument, we may assume  $C'$  has no chord  $ux$ , and by symmetry no chord  $vx$ ). Therefore, it follows that we can find the desired cycle  $C''$ . Now fold  $v$  to a vertex distance two away on  $C''$ . Then  $C$  is still a facial cycle, and we obtain our desired homomorphism by minimality and composing homomorphisms.  $\square$

**Claim 6.24.** *There is at most one vertex in  $V(G) \setminus V(C)$ .*

*Proof.* Suppose not, and let  $x$  and  $z$  be a vertices in  $V(G) \setminus V(C)$ . We can assume that  $x$  has degree at least 2, otherwise we fold arbitrarily. Let  $y_1$  and  $y_2$  be two vertices in  $C$  adjacent to  $x$ . By a similar argument as in Claim 6.23, without loss of generality we may assume that  $x$  plus one of the paths from  $y_1$  to  $y_2$  form a facial cycle.

As  $G$  is bipartite,  $y_1y_2 \notin E(G)$ , and hence  $C - \{y_1, y_2\}$  has two components say  $C_1$  and  $C_2$ . Each component  $C_i$  with  $y_1, x, y_2$  forms a cycle. Now without loss of generality  $C_1 + \{y_1, x, y_2\}$  separates  $z$  from  $C_2$  contrary to Claim 6.21.  $\square$

To finish the proof of Lemma 6.18, observe our graph is  $C + x$  for some vertex  $x$ . We fold  $x$  with any vertex distance 2 from  $x$ . This preserves planarity and after folding  $C$  remains a facial cycle. The result follows by minimality.  $\square$

**Corollary 6.25.** *Let  $G$  be a 2-connected planar bipartite graph. Let  $C$  be a separating 4-cycle. For any  $p, q$  such that  $2 < \frac{p}{q} < 4$ , we have that  $G$  is  $(p, q)$ -mixing if and only if the interior and exterior of  $G$  is  $(p, q)$ -mixing.*

*Proof.* By Lemma 6.17 we may assume without loss of generality that the interior of  $G$  is not  $(p, q)$ -mixing. Then by Lemma 6.18 we can fold the exterior of  $G$  onto  $C$  in a way which the folding is a retract on  $C$ , and such that we possibly create chords. However, as  $C$  is a 4-cycle, we create no chords (as otherwise we would end up with a non-bipartite graph). Therefore  $G$  folds to a non- $(p, q)$ -mixing graph, and hence is not  $(p, q)$ -mixing.  $\square$

It follows now that we have a simple characterization for  $(p, q)$ -mixing in planar graphs when  $3 \leq \frac{p}{q} < 4$ .

# Chapter 7

## Cycles in Colour Critical Graphs

The work in this chapter is joint with Douglas B. West.

### 7.1 Extending Tuza for $k$ -colourings

In this section we prove the following theorem.

**Theorem 7.1.** *Let  $xy$  be an edge in a graph  $G$ . If  $G - xy$  is  $k$ -colourable and  $G$  is not, then  $xy$  belongs to at least  $\prod_{i=1}^{r-1} (k - i)$  cycles in  $G$  having lengths congruent to 1 modulo  $r$ , for  $2 \leq r \leq k$ .*

*Proof.* Let  $[k]$  denote  $\{1, \dots, k\}$ . Fix a proper  $k$ -colouring  $\phi$  of  $G - xy$ , using colour set  $[k]$ . We obtain one cycle through  $xy$  for each cyclic permutation of a subset of  $[k]$  containing  $\phi(x)$ .

Given such a permutation  $\sigma$ , define the  $\sigma$ -subdigraph  $D_\sigma$  of  $G$  generated by  $\phi$  to be the directed graph with vertex set  $V(G)$  such that  $uv$  is an edge in  $D_\sigma$  if and only if  $uv \in E(G)$  and  $\sigma(\phi(u)) = \phi(v)$ . Let  $H$  be the subdigraph of  $D_\sigma$  induced by all vertices reachable from  $x$  by paths in  $D_\sigma$ .

Consider the recolouring  $\phi'$  of  $G$  defined by  $\phi'(u) = \sigma(\phi(u))$  for  $u \in V(H)$ , and otherwise  $\phi'(u) = \phi(u)$ . An edge can become improperly coloured only if the colour of one endpoint is changed into the colour of the other, but then the oriented version of the edge lies in  $H$  and both endpoints change colour.

If  $y \notin V(H)$ , then  $\phi'(x) \neq \phi'(y)$ , and  $\phi'$  is a proper colouring of  $G$ , which by hypothesis does not exist. Hence  $y \in V(H)$ , meaning that  $y$  is reachable from  $x$  via a path in  $H$ .

Since paths in  $H$  follow colours according to  $\sigma$ , and  $\phi(y) = \phi(x)$ , the length of any  $x, y$ -path in  $H$  is a multiple of  $r$ , and the cycle in  $G$  completed by adding the edge  $yx$  has length congruent to 1 modulo  $r$ . Furthermore, since the colouring  $\phi$  is fixed, the resulting  $x, y$ -paths in  $G$  are distinct for distinct choices of  $\sigma$ .  $\square$

This result was motivated by a similar argument due to Brewster, McGuinness, Moore, and Noel [2], which we sketch in our language to generalize the quantitative result. For  $k > 2$ , they showed that if  $G$  is not  $k$ -colourable but  $G - xy$  is  $k$ -colourable, then  $G - xy$  contains at least  $(k - 1)!/2$  cycles whose lengths are multiples of  $k$ . The early paper of Tuza [49] notes that Toft and Tuza had observed that every graph that is not  $k$ -colourable contains a cycle whose length is divisible by  $k$ , for  $k > 2$ .

We extend the result of [2] as follows. Again fix a proper  $k$ -colouring  $\phi$  of  $G - xy$  and a cyclic permutation  $\sigma$  of a set of colours containing  $\phi(x)$ . Define the digraph  $D_\sigma$  as above. Note again that  $\phi(x) = \phi(y)$ . If  $D_\sigma$  is acyclic, then we recolour  $G$  by again changing the colour on  $v$  from  $\phi(v)$  to  $\sigma(\phi(v))$ , but now *one vertex at a time*, always changing the colour at a sink of the unchanged subgraph. At each step we have a proper  $k$ -colouring of  $G - xy$ . We do this until the colour on  $x$  or  $y$  changes, at which point we have a proper  $k$ -colouring of  $G$ . Since that does not exist,  $D_\sigma$  contains a cycle. The length of a cycle in  $D_\sigma$  must be a multiple of  $k$ . However, a cyclic permutation and its reverse will select the same cycle in  $G$ , because the corresponding digraphs are obtained from each other by reversing all the edges.

Hence we are in fact guaranteed  $(r - 1)!/2$  cycles whose length are multiples of  $r$ , for all  $r$  with  $3 \leq r \leq k$ , and none of these cycles contain  $e$ .

## 7.2 Extending Zhu for circular colouring

In this section we consider the analogous problem for  $(k, d)$ -colouring. We will only use cycles of the form  $(0, d, 2d, \dots, -d)$  and their reverse, so we get existence results rather than quantitative results. Nevertheless, the result is still sharp.

Note the proof may require many steps of recolouring to find a desired cycle. This is inherently necessary, because a  $C_{2d+1}$ -colouring of  $C_{2d-1} - xy$  may alternate 0 and  $d$  along the path.

**Theorem 7.2.** *Let  $k$  and  $d$  be relatively prime, with  $k > 2d$ , and let  $s$  be the element of  $\mathbb{Z}_k$  such that  $sd \equiv 1 \pmod k$ . Let  $xy$  be an edge in a graph  $G$ . If  $G - xy$  is  $K_{k,d}$ -colorable and  $G$  is not, then  $xy$  lies in at least one cycle in  $G$  of length congruent to  $is \pmod k$  for some  $i$  in  $\{1, \dots, d\}$ . If this does not occur with  $i \in \{1, \dots, d - 1\}$ , then  $xy$  lies in at least two cycles of length  $1 \pmod k$  and  $G - xy$  contains a cycle of length  $0 \pmod k$ .*

*Proof.* Fix a  $G_{k,d}$ -colouring  $\phi$  of  $G - xy$ . By symmetry, we may assume  $\phi(y) = 0$ . Since  $G$  is not  $G_{k,d}$ -colorable,  $\phi(x) \in \{0, \pm 1, \dots, \pm(d - 1)\}$ . Let  $\sigma$  be the cyclic permutation  $(0, d, 2d, \dots, -d)$  of colors. Define the digraph  $D_\sigma$  as in Theorem 7.1, and let  $H$  be the subdigraph of  $D_\sigma$  induced by all vertices reachable from  $x$  in  $D_\sigma$ .

Given a colouring  $\phi$  of  $G - xy$ , define  $\phi'$  by letting  $\phi'(v) = \phi(v) + 1$  for  $v \in V(H)$  and  $\phi'(v) = \phi(v)$  for  $v \notin V(H)$ . We claim that  $\phi'$  is a  $G_{k,d}$ -colouring of  $G - xy$ . First, edges

within  $H$  or in  $G - V(H)$  remain properly coloured. When  $v \in V(H)$ , the exploration of  $D_\sigma$  extends along the edge  $vw$  if  $\phi(w) - \phi(v) = d$ . Since  $\phi(w) - \phi(v) \in \{d, d+1, \dots, k-d\}$  for  $vw \in E(G)$ , having  $v \in V(H)$  and  $w \notin V(H)$  requires  $\phi(w) - \phi(v) \in \{d+1, \dots, k-d\}$ . Now  $\phi'(w) - \phi'(v) \in \{d, \dots, k-d-1\}$ , so such edges are also properly coloured in  $\phi'$ .

We will consider cases where  $\phi(x) = j$ , for  $0 \leq j \leq d-1$ . For the case  $\phi(x) = -j$  with  $1 \leq j \leq d-1$ , add  $j$  to the colour at each vertex to obtain  $\phi(x) = 0$  and  $\phi(y) = j$ , and then interchange the roles of  $x$  and  $y$  and apply the argument below.

When  $\phi(x) = j$  and  $\phi(y) = 0$ , we claim that  $G$  has a cycle through  $xy$  with length congruent to  $is$  modulo  $k$  for some  $i$  in  $\{1, \dots, d-j\}$ . We use induction on  $d-j$ .

First consider  $j = d-1$ . If  $y \notin V(H)$ , then  $\phi'$  is a  $K_{k,d}$ -colouring of  $G$ , since  $\phi'(x) = d$  and  $\phi'(y) = 0$ . Hence  $y \in V(H)$  when  $j = d-1$ . To reach value 0 from  $d-1$  along a path specified by steps of  $d$  requires  $r$  steps, where  $rd \equiv k - (d-1)$ . Multiplying both sides by  $s$  yields  $r \equiv s-1$ . Adding the edge  $xy$  thus completes a cycle of length congruent to  $s \bmod k$ , with its  $x, y$ -path stepping by  $+d$  in colours under  $\phi$ .

Now suppose  $j < d-1$ . If  $y \in V(H)$ , then an  $x, y$ -path in  $H$  has length congruent to  $(d-j)s - 1$  modulo  $k$ , since  $s$  (modulo  $k$ ) steps of  $+d$  will increase the starting value by 1. Adding  $yx$  then completes a cycle of length congruent to  $(d-j)s \bmod k$  through  $xy$  in  $G$ .

Hence we may assume  $y \notin V(H)$ . Now  $\phi'$  is a  $G_{k,d}$ -colouring of  $G - xy$  with  $\phi'(x) = j+1$  and  $\phi'(y) = 0$ . The previous argument, for  $\phi'$  with  $\phi'(x) = j+1$ , now implies that  $G$  has a cycle through  $xy$  with length congruent to  $is \bmod k$  for some  $i$  in  $\{1, \dots, d-j-1\}$ . Including  $d-j$  in the set thus covers all cases to complete the induction step.

Finally, suppose that  $G$  has no cycle through  $xy$  with length congruent to  $is \bmod k$  for  $1 \leq i \leq d-1$ . This case requires  $j = 0$ , and the argument above yields a cycle of length  $1 \bmod k$  (since  $ds \equiv 1 \bmod k$ ), with colours along its  $x, y$ -path stepping by  $+d$  under the original colouring  $\phi$ . Under  $\sigma^{-1}$ , using the same  $(k, d)$ -colouring  $\phi$  of  $G - xy$  and starting again from  $x$  yields a second cycle of length  $1 \bmod k$  through  $xy$ .

Furthermore, if  $G - e$  has no cycle of length  $0 \bmod k$ , then  $D_\sigma$  is acyclic. Working backward from sinks, we can add 1 to the colour of each reached vertex, *one vertex at a time*, always maintaining a  $(k, d)$ -colouring of  $G - xy$ , until  $x$  or  $y$  changes colour. This reduces the problem to the case  $j > 0$ . Since in this case  $G$  has no cycle of length  $is$  with  $i \in \{1, \dots, d-1\}$ , the previous arguments produce a  $(k, d)$ -colouring of  $G$ , which by hypothesis does not exist. Therefore, in fact  $G - xy$  also contains a cycle of length  $0 \bmod k$ .  $\square$

In this proof, we have not made use of cycles in  $G_{k,d}$  other than that generated by  $d$  or  $-d$ . When  $k > 2d+1$  there may be other sets of forced cycle lengths. When  $k = 2d+1$ , these two are the only permutations yielding cycles in the host graph, and that is why our sharpness examples in the introduction are for  $C_{2d+1}$ -colouring. In that case  $s = -2$ . The set of cycle lengths that cannot be avoided are the congruence classes  $-2i \bmod (2d+1)$  for  $1 \leq i \leq d$ , and if there are no cycles through  $e$  in the classes with  $1 \leq i \leq d-1$ , then we obtain two cycles with lengths  $1 \bmod (2d+1)$ .

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