Extended thermodynamics of Taub-NUT and Einstein-scalar spacetimes

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

Chapter 3 in deals with the thermodynamics of spacetimes with scalar fields. The first section consists of material that I authored [1] and was in part inspired by a project suggested by Robert Myers. The second section contains results that I omitted in this publication but are included here for completeness. The third section of Chapter 3 is heavily influenced by Robert Myers' project suggestion and is included as an extension of previous results. It is in preparation for publication.

Chapter 4 contains an overview of five articles on the thermodynamics of Lorentzian Taub-NUT spacetimes [2–6] published in collaboration with Finnian Gray, Robie Hennigar, Tales Perche, and David Kubiznak. All authors collaborated equally for the elaboration of these manuscripts.

Appendices A, B, and C contain Mathematica notebooks of which I am the sole author. Appendix D has an unpublished auxiliary proof of which I am the only author, but it was delegated to me as an appendix for our article [3] by David Kubiznak. This appendix, however, was not included in the final version of the article. Appendix E contains some insights about the thermodynamics of Kerr-NUT AdS spacetimes, which are a combined work of Finnian Gray, David Kubiznak, Robie Hennigar, and me.

I am grateful to David Kubiznak and Robert Mann for generously allowing for the free use of figures 2.2 and 2.3 from their article [7], where they appear as figures 8 and 9.

Abstract

The main objective of this thesis is to summarize recent developments on the thermodynamics of both AdS Einstein-scalar spacetimes and Lorentzian Taub-NUT spacetimes. We summarize and apply well-known techniques in black hole thermodynamics to derive the extended first law of hairy AdS black holes. In particular, we provide expressions for the thermodynamic volume. For solutions in which the scalar field decays quickly enough, we find a relation between this volume and the integral of the potential behind the horizon. In a more general case, we show that the thermodynamic volume acquires additional terms proportional to the trace of the holographic stress tensor. We speculate that this result holds in even greater generality since previous evidence suggests that the same terms appear in spacetimes with non-AdS boundary conditions.

We also study the extended thermodynamics of Lorentzian Taub-NUTs in the presence of a Misner string. We review why these spacetimes are not necessarily pathological, meaning that it is reasonable to study their thermodynamics. Introducing a new charge and potential allows us to establish a consistent first law of thermodynamics for the Taub-NUT-AdS spacetime when the horizon and the NUT charge are allowed to vary independently. Additionally, we give a geometric prescription to calculate the thermodynamic charges and potentials. We then extend this first law to AdS NUTty dyons, flat Kerr-NUT, and a novel non-linear theory of electrodynamics. We propose different physical interpretations for the new terms introduced in the first law and pave the way for future research in this area.

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Table of Contents

List of Figures **xi**

1 Introduction **1**

1.1 The laws of black hole mechanics 3

1.2 Gravity, Thermodynamics, and the AdS-CFT correspondence 7

1.3 Thesis Outline 11

2 Techniques in Black Hole Thermodynamics **13**

2.1 Asymptotic charges 13

2.1.1 Komar integration 15

2.1.2 Conformal method 16

2.1.3 Holographic methods 18

2.2 Thermodynamics with Lambda 20

2.2.1 The Smarr relation and the extended first law 21

2.2.2 Examples of the extended first law 22

2.2.3 Black hole chemistry and thermodynamic phase structure 27

2.3 Hamiltonian perturbations 32

3 Thermodynamics of Hairy AdS Black Holes **35**

3.1 Planar black holes 36

3.1.1 Asymptotic behaviour and ADM charges 36

3.1.2 Smarr relation and the scaling argument 38
3.1.3 Komar Integration 40
3.1.4 Extended First Law 44
3.1.5 Explicit examples 46
3.2 Hyperbolic Hairy Black Holes 50
3.2.1 ADM charges and Komar integration 50
3.2.2 Thermodynamic Volume 52
3.2.3 The MTZ black hole 53
3.3 Holographic thermodynamics of hairy black branes 54
3.3.1 Asymptotic behaviour and holographic charges 54
3.3.2 Smarr relation and extended thermodynamics 57

4 Thermodynamics of Lorentzian Taub-NUT spacetimes 59
4.1 Taub-NUT spacetimes 60
4.1.1 Generalities 60
4.1.2 Euclidean Taub-NUTs and their thermodynamics 63
4.1.3 Rehabilitating Lorentzian Taub-NUTs 66
4.2 The first law of flat Taub-NUTs 70
4.2.1 Misner charge and Misner potential 70
4.2.2 Rotating Taub-NUTs 73
4.2.3 NUTty dyons 75
4.2.4 Rotating dyonic Taub-NUT 79
4.3 The extended first law of AdS Taub-NUTs 85
4.3.1 Extended thermodynamics of AdS Taub-NUTs 85
4.3.2 AdS NUTty dyon and phase transitions 87
4.3.3 Beyond linear electrodynamics 94
4.4 Interpretation of results 99

5 Summary and discussion 101

References 104

APPENDICES 114

A Conformal Mass 115

B Free Energy 119

C Hamiltonian Perturbations 126

D Kerr NUT geodesics 129

E Kerr NUT AdS 132

List of Figures

2.1 **Hawking-Page phase transition.** The dotted black line represents the regime of small black holes; the continuous black line is the regime of large black holes. The red line is thermal AdS. As we see, there is a first-order phase transition from thermal AdS to large black holes. 27

2.2 **Gibbs free energy for charged AdS black holes.** [7] We see that at pressures below the critical pressure, the swallowtail behaviour appears. It corresponds to a first-order phase transition from small black holes to large black holes as the temperature increases. 30

2.3 **Coexistence line for charged black holes.** [7] Below the critical point, there exists a phase transition between small black holes and large black holes. The phase transition disappears when the transition temperature is larger than T_c. 31

4.1 **Misner tubes:** The boundary of the Taub-NUT spacetime has, in addition to the horizon and S_∞ components, two tube-shaped boundary components surrounding each of the Misner strings. We define the Misner charges $N_\pm$ as the Komar integral component over such tubes. 72

4.2 **Free energy of rotating NUTty dyon**. Plotted against the temperature for different values of J and $Q = 1$. This behaviour is the same that is observed for charged rotating black holes. 84

4.3 **Uncharged Gibbs free energy.** Free energy vs. temperature for an uncharged AdS Taub-NUT with $n = 1$ for two values of the pressure. The blue lines indicate solutions of negative mass. There is no phase transition since a “pure NUT” does not correspond to a pure radiation phase. 90

4.4 **Swallowtail Behaviour.** We see that for different values of ℓ, there the Gibbs free energy exhibits different behaviour. In this plot, $Q = 1$ and $n = 0.2$. At $\ell = \ell_c$, corresponding to the critical pressure, the swallowtail behaviour starts to appear. The red dot labels the critical point. 91

4.5 **Branch of small Taub-NUTs.** The lower straight branch is a branch of small Taub-NUTs with $r_+ < n$. The blue parts represent regions with $M < 0$. In (a), where $\ell = 50$, the small Taub-NUT directly transitions to a much larger one. In (b), where $\ell = 25$, the Taub-NUTs transitions first to a medium Taub-NUT and then to a larger one. 92

4.6 **Phase Diagram**. Here, $Q = 0.98$ and $n = 1$. The phase diagram allows us to see in more detail the possible phase transitions. The continuous lines represent first order phase transitions; the dotted lines represent zeroth order transitions. In addition to the changes of phase that we saw in 4.5, we see that there is a small range of pressure where the transition from small to large Taub-NUT is of zeroth order. 93

Chapter 1

Introduction

More than one hundred years after Einstein's theory of general relativity withstood its first test, our understanding of gravity is still nowhere near complete. General relativity has been astonishingly successful not only in its description of the motion of gravitating bodies but also in its predictions of novel physical phenomena such as the existence of gravitational waves and the formation of black holes. It is remarkable that the simple (yet difficult to solve) Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

can provide us with such a precise description of gravitational interactions. Why, then, do physicists claim that the theory is incomplete? At large scales, there is some experimental evidence against it. For instance, galaxies do not rotate as predicted by the theory, and cosmological observations suggest that the universe expands much faster than anticipated. These phenomena suggests that either relativity must need modification, or that we need to add new matter fields to the standard model. General relativity mathematically predicts its own demise as well. In some cases where the gravitational interaction is very strong, the spacetime curvature can increase without limit allowing for spacetime singularities. Such singularities should not be allowed and indeed, we expect that at small scales, quantum effects would prevent them from existing at all. Unfortunately, after several attempts for more than half a century, physicists have not been able to reconcile the laws of quantum field theory and general relativity, and the handful of successful attempts could only ever be experimentally confirmed using technology we might never achieve.

One strategy to tackle the problem of gravitational singularities is to carefully study some idealized spacetimes that contain them. This is, for examples, the case of black hole spacetimes. These contain event horizons, behind which the singularity remains hidden to outside observers[1]. Very roughly speaking, a black hole spacetime is a Lorentzian manifold in which not all null rays can escape to null infinity. More specifically [8], there is a subset $\mathcal{B}$ of points in the manifold such that, if future-directed null rays originate from such points, the rays will never reach null infinity: they cannot keep going forever. In all physically plausible cases, the black hole interior $\mathcal{B}$ contains a singularity: a region where the curvature is infinite. The boundary between $\mathcal{B}$ and its complement is known as the event horizon.

To give a precise mathematical definition of null infinity, we need to work with asymptotically flat spacetimes, which in essence means that the curvature vanishes at distances far from a certain region (in our case, the black hole region). If we additionally assume that our black hole spacetime is stationary[2], so that it admits an asymptotically timelike Killing vector, then we can use one of Hawking's most famous results [9] to conclude that the spacetime must be axially symmetric or static. Such an axially symmetric black hole spacetime has two killing vectors ∂_t and ∂_ϕ and there exists a linear combination

$$\xi = \partial_t + \Omega_H \partial_\phi \tag{1.2}$$

that is null on the event horizon. The quantity Ω_H can be interpreted as the horizon angular velocity of a rotating black hole, which is zero when the spacetime is static. The vector ξ is said to be the generator of the event horizon; being a Killing vector, the horizon is, in turn, said to be a Killing horizon.

The generating Killing vector turns out to be useful to study black hole spacetimes. It allows one to calculate the energy and angular momentum content of a spacetime. This procedure will be discussed in detail later; but we note now that the Killing generator can be used to calculate the surface gravity κ of the black hole, as measured by a static asymptotic observer. It is calculated via

$$\kappa^2 = -\frac{1}{2}\nabla^\mu \xi^\nu \nabla_\mu \xi_\nu, \tag{1.3}$$

evaluated at the horizon; indeed κ itself can be interpreted as the force needed at infinity to keep a test mass at the horizon.

Additionally, we can add some charge to the black hole; hence the stress tensor in Einstein's field equations will not be trivial anymore. The assumption of stationarity makes

[1] Whether singularities always remain hidden in this way is still unclear, and it is the statement of the strong cosmic censorship conjecture.

[2] This is precisely the approximation that one would have to give up if we were to model more realistic black holes formed from gravitational collapse.

the charged black hole axisymmetric, and as a consequence, this most general spacetime can be described using very few degrees of freedom: the energy content M, the total charge Q, and the angular momentum J. This result is known as the no-hair theorem [10–12]. Another surprising insight is that these quantities, together with the black hole surface area A, satisfy relations similar to those that thermodynamic variables do in a particular choice of ensemble. This gives us some hope that we could, in principle, find the laws of statistical mechanics satisfied by the gravitational degrees of freedom, thus avoiding this reprieve of the ultraviolet catastrophe that happens at the singularity. This, of course, needs a hitherto incomplete theory of quantum gravity, but we can still gain intuition by studying these analogous laws of thermodynamics. We will start by stating and explaining these laws, and by subsequently understanding their physical interpretation.

1.1 The laws of black hole mechanics

The discipline known as black hole thermodynamics started around 1972 when Hawking proved that the laws of general relativity alone imply that the surface area of a black hole event horizon cannot decrease in any physical process [13]. For example, in a two black hole merger, the total area of the merged black hole must be equal to or larger than the sum of the initial black hole areas. Consequently, the energy production by the merger is limited by this law, in a similar way that the second law of thermodynamics limits the efficiency of thermodynamic processes. For this reason, this area theorem, previously stated by Floyd and Penrose for some particular black hole spacetimes [14], hints at a relation between the horizon area and the entropy of the black hole. Through a series of thought experiments involving matter falling into a black hole, and relying on the no-hair theorem and Hawking's result, Bekenstein argued that the entropy S of a black hole must be proportional to its surface area A [15]:

$$S \propto A. \tag{1.4}$$

However, these thought experiments were not enough for Bekenstein to get the correct constant of proportionality, and it was only through dimensional analysis that Planck's constant was found to be needed in such a prefactor, implying that the relation can only be explained using quantum mechanics.

Shortly after, Bardeen, Carter, and Hawking published their four laws of black hole mechanics [16]. Of particular interest to us is the first law. It was already known that the most general asymptotically flat black hole solutions could be fully described by the mass

M, electric charge Q, and angular momentum J. The first law provides a relation between infinitesimal variations of these parameters in the form

$$dM = \frac{\kappa}{8\pi G} dA + \phi dQ + \Omega_H dJ. \tag{1.5}$$

Here, ϕ is the electric potential at the horizon. This relation is reminiscent of the first law of thermodynamics, where the conservation of energy takes the form

$$dU = TdS + pdV + \sum \mu_i dN_i, \tag{1.6}$$

where U, T, and S are the internal energy, temperature, and entropy of the system, N_i are the thermodynamic variables of our ensemble, and μ_i are the conjugate potentials. This first law is emphasized here because deriving it for more general cases will be the main objective of this thesis. The four laws of black hole mechanics that the famous article proved are the following:

1. The surface gravity is constant on the black hole horizon.
2. The variations of the energy M, area A, charge Q, and angular momentum J are related via the formula (1.5).
3. For any classical physical process, the total surface area of the event horizon is never decreasing
$$\delta A \geq 0. \tag{1.7}$$
4. It is impossible to decrease the surface gravity of a black hole to zero in finitely many steps.

While these are indeed very similar to the four laws of thermodynamics, there is no a priori reason to assign a physical thermodynamic significance to them. We can heed Bekenstein's suggestion and identify the area with the entropy, but nothing so far remotely suggests that surface gravity could play the role of the temperature. Moreover, the first law, as stated, does not refer to physical processes but is merely a relation between parameters in the family of black hole solutions. However, Wald showed that this law could be made to refer to physical processes involving black holes absorbing charged matter, and in this case, it is known as the "physical process first law" [17]. Also, in a similar fashion to thermodynamics, the first law can be derived from a relation between the internal energy

M and the extensive thermodynamic variables A, Q, and J. Such an expression is called a Smarr relation [18], and in the case at hand it reads

$$M(S, Q, J) = 2\kappa A + \phi Q + 2\Omega_H J. \tag{1.8}$$

This relation, which is easy to verify, will become a powerful tool to guess the first law in more general cases so that all that remains is to prove it explicitly.

These laws of black hole mechanics have been extended beyond four-dimensional asymptotically flat black hole spacetimes. One of the main challenges in this task is to find prescriptions to calculate the extensive quantities such as internal energy and angular momentum. These will be described in Chapter 2; by using them we can successfully establish the first law for a wide variety of spacetimes and even for generalized theories of gravity. A class of spacetimes for which we have robust tools to calculate these variables are the asymptotically anti-de Sitter spacetimes, the main topic of the present thesis.

Two years later, Hawking published the key paper that would revolutionize black hole physics and provide insight into the four laws above [19]. To do this, Hawking developed the formalism of quantum field theory in curved spacetimes. The main idea is to expand a massless and spinless quantum field in terms of its modes and quantize. However, the mode expansion uses a preferred time coordinate: the proper time of the observer. In general relativity, all observers are on the same standing, which means that we can find different field expansions in the different frames of reference. Formally, we have two different expansions [20]:

$$\phi = \int d\omega (a_\omega f_\omega + a_\omega^\dagger f_\omega^*), \tag{1.9}$$

$$\phi = \int d\omega (b_\omega f'_\omega + b_\omega^\dagger f'^*_\omega), \tag{1.10}$$

where a and b are the annihilation operators with respect to the vacuum, which need not be the same in both frames of reference. Namely, we have two possibly different vacua $|0\rangle_a$ and $|0\rangle_b$ such that

$$a_\omega |0\rangle_a = 0, \tag{1.11}$$

$$b_\omega |0\rangle_b = 0. \tag{1.12}$$

Both the modes f_ω and f'_ω and the vacuum state depend on the frames S and S' chosen. The completeness of the modes implies that there is a linear relation between the mode functions

$$f'_\omega = \int d\eta (\alpha_{\omega\eta} f_\eta + \beta_{\omega\eta} f_\eta^*), \tag{1.13}$$

it then follows, skipping several cumbersome integrals, that if the field state is $|0\rangle_a$, the vacuum of S, then the average number of particles with frequency ω in the primed system is given by

$$N_\omega = {}_a\langle 0|\, b^\dagger_\omega b_\omega \,|0\rangle_a = \int d\eta |\beta_{\omega\eta}|^2. \tag{1.14}$$

The immediate implication of this is that, for non-trivial $\beta_{\omega\eta}$, one observer might detect particles while a different one does not. For example, integral (1.14) is non-zero when we go from Minkowski spacetime to Rindler spacetime, so an observer with acceleration a will see particles where an inertial observer does not, with expectation value

$$N = \frac{1}{e^{2\pi\omega/\hbar a} - 1}. \tag{1.15}$$

Surprisingly, this is precisely the Bose-Einstein distribution for particles at a temperature

$$kT = \frac{\hbar a}{2\pi}, \tag{1.16}$$

so that the vacuum state is a thermal state for accelerated observers; this is called the Unruh effect[3] [21].

Hawking carried out precisely this calculation, comparing the vacuum of a geodesic observer, who does not see the event horizon, with that of an asymptotic observer. The analysis is non-trivial for the case that Hawking tackled, which is that of a black hole formed by gravitational collapse. The case of an eternal black hole (maximally extended Schwarzschild spacetime) follows by using steps analogous to the Unruh effect derivation. In either case, the average number of particles detected by the observer at infinity is given by

$$N_\omega = \frac{\Gamma_\omega}{e^{2\pi\omega/\hbar\kappa} - 1}, \tag{1.17}$$

where the gray-body factor Γ_ω accounts for the scattering of particles through the collapsing star and is equal to one in the eternal black hole case. As we see, the temperature of the black hole, according to the asymptotic observer is related to the surface gravity κ via

$$kT = \frac{\hbar\kappa}{2\pi}. \tag{1.18}$$

The immediate consequence is that the four laws of thermodynamics stated above are not only formal analogies; they represent true laws of thermodynamics for black holes.

[3] Let us note here that, for experimentally achievable accelerations, this temperature is minuscule, so this effect has never been directly observed.

Combining this with Bekenstein's argument, we can identify the proportionality constant in (1.4), so that the entropy reads

$$S = \frac{kA}{4G\hbar}. \tag{1.19}$$

Hawking's argument to find the temperature of the event horizon of a black hole has been extended to de Sitter and charged black holes. However, in general, this type of argument is not easy to carry out in more complicated spacetimes since solving for the Klein-Gordon equation mode functions is a non-trivial task. Hawking's argument is not the only physical justification for assigning a temperature to black hole horizons. Unruh and DeWitt would later obtain the same result by inserting a test particle detector in curved spacetime [22, 23]. This technique gave rise to the very fruitful research branch known as relativistic quantum information, where one can use such detectors, for example, to probe the entanglement of the vacuum state in curved spacetimes (e.g. [24,25]). Further arguments in anti-de Sitter spaces are provided by the AdS-CFT correspondence, which will be the topic of the next section.

In the following, we will use natural units where $\hbar = k = 1$ (the speed of light c has been taken to be equal to 1 from the beginning of the discussion). For the time being, we do not set G to unity as is usual to leave room for some quick dimensional analysis, should it be needed.

1.2 Gravity, Thermodynamics, and the AdS-CFT correspondence

With the advent of black hole thermodynamics, there was a surge of interest in understanding the limitations that the newly acquired knowledge imposes on possible quantum theories of gravity. For example, the fact that the entropy of a black hole is proportional to its surface area tightly constrains the number of gravitational degrees of freedom that describe the interior of the black hole. This observation led to the proposal by t'Hooft and Susskind of a much more universal principle: *"in quantum gravity, the entropy of a region is, at most, equal to the surface area of the minimal sphere in which such system a is contained"* [26, 27]. While this so-called Holographic Principle was first thought to be simply a reasonable conjecture, Juan Maldacena discovered a plausible explicit realization with his formulation of the AdS-CFT correspondence [28].

The statement of the AdS-CFT correspondence conjectures the following holographic principle: the degrees of freedom of a quantum theory of gravity in anti-de Sitter spacetime can be set equal to the number of degrees of freedom of a conformal field theory defined on its conformal boundary. We will give a formal definition of asymptotically anti-de Sitter spacetime, but for now, we note that this roughly means that the metric asymptotically behaves like the anti-de Sitter metric in D spacetime dimensions:

$$ds^2 = -\left(\frac{r^2}{\ell^2}+1\right)dt^2 + \left(\frac{r^2}{\ell^2}+1\right)^{-1}dr^2 + r^2 d\Omega_{D-2}. \tag{1.20}$$

This is a solution to the Einstein field equations with negative cosmological constant Λ, which is related to the AdS radius ℓ via

$$\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}. \tag{1.21}$$

The radial coordinate can be compactified by a change of coordinates, such as taking the inverse tangent. The boundary of such a compactified spacetime is what we call the conformal boundary.

The correspondence is not merely a matching of degrees of freedom. It is much stronger in the sense that it posits a complete dynamical equivalence between theories. In this sense, there exists a mapping that provides a correspondence between quantities, such as correlation functions, defined on a quantum gravity theory in AdS spacetime in $d+1$ dimensions with quantities defined on a conformal field theory in d dimensions.

The original statement of the AdS-CFT correspondence involved considering a large number N of closely packed D3-branes in type II-B string theory and taking the low energy limit, thus decoupling them from gravity. When the string coupling g_s is large, this limit is described by type IIB superstring theory in $AdS_5 \times S^5$, while at weak couplings, it is described by $\mathcal{N}=4$ supersymmetric Yang-Mills in four dimensions, with gauge group $SU(N)$. Then Maldacena hypothesized that this description in terms of Super Yang-Mills holds for any value of the coupling, thus proposing an equivalence between a weakly coupled string theory in AdS, known as the bulk theory, and a strongly coupled gauge theory in the boundary of AdS, known as the boundary CFT. The calculation allows us to relate the string theory coupling and the anti-de Sitter radius ℓ (in units of the string length) with the rank $N-1$ of the gauge group and the Yang-Mills coupling g_{YM}:

$$4\pi g_s = g_{YM}^2, \qquad \ell^4 = 4\pi g_s N. \tag{1.22}$$

A more precise statement of the correspondence rests on the equivalence of partition functions. Since we are interested in calculating correlation functions, it is natural to deform the boundary CFT Lagrangian by adding a source $\phi_{(0)}$ coupled to an operator $\mathcal{O}$.

In the fundamental defining equation of AdS-CFT, the partition function corresponding to this deformed Lagrangian is equal to the gravitational partition function, with a field ϕ introduced on the gravity side such that its asymptotic behaviour determines the CFT source $\phi_{(0)}$. Explicitly, we have

$$\left\langle \exp\left(\int d^d x \phi_{(0)} \mathcal{O}\right)\right\rangle_{CFT} = \mathcal{Z}_{\text{string}}[\phi_{(0)}], \tag{1.23}$$

where $\mathcal{Z}_{\text{string}}$ is the string theory partition function, defined on the bulk, expressed as a functional of the boundary condition $\phi_{(0)}$ [29, 30]. The on-shell partition function is a generating function for connected correlation functions involving the operator $\mathcal{O}$. Note that both theories have been Wick rotated to their Euclidean versions. The gravitational action on the right-hand side is assumed to be renormalized appropriately; the renormalization procedure depends on the specifics of the theory and will be explained in more detail in the next chapter.

One can even go beyond AdS boundary conditions; as we will see, we have well-defined asymptotic thermodynamic charges in a class of spacetimes known as asymptotically locally AdS (AlAdS), where the topology of the boundary is not the standard one. The study of the thermodynamics of such spacetimes is one of the main topics of the present thesis. The boundary conditions can be perturbed even further to the point that neither the conformal boundary remains flat nor the theory defined therein stays conformal. Since the correspondence seems to reach way beyond conformal theories, so it has become customary to refer to it as the gauge-gravity correspondence.

Since the inception of the correspondence, sundry ways to verify the conjecture have been proposed, and it has withstood the most robust tests it has been put through. While there is no formal proof of the correspondence yet, the evidence of its validity is so overwhelming that very few physicists doubt it. So far, it has been verified that the duality can be used to calculate correlation functions [29,31], critical temperatures [32], the entanglement entropy of a region [33], conformal anomalies [34], among many other non-trivial quantities in conformal field theories. Because the applications of quantum field theories with a large coupling constant are ubiquitous in physics, most notably in quantum chromodynamics, the last two decades have seen an unexpected collaboration between branches as dissimilar as condensed matter and relativity.

Our primary interest is black hole thermodynamics and, unsurprisingly, the techniques to study black holes in asymptotically anti-de Sitter spacetimes have prolifically developed in the last two decades. The AdS-CFT correspondence and the study of black holes have established a symbiotic relationship, whereby understanding the thermodynamics of different spacetimes aids in our better understanding of the correspondence and, in turn, the duality aids in our calculation of the thermodynamic quantities, such as the total energy content and the angular momentum. The critical fact that allows for such a collaboration is that the duality goes beyond pure AdS: we can perturb the metric around AdS while keeping the boundary conditions, which in turn breaks the scale symmetry of the spacetime; this will correspond to excited states of the conformal field theory, which also break the conformal symmetry. In the case of black hole spacetimes, a black hole with temperature T is dual to a thermal state in the gauge theory at the same temperature [32].

As an example, we consider the Schwarzschild-AdS metric in five spacetime dimensions, which has spacetime interval element

$$ds^2 = -\left(k + \frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) dt^2 + \left(k + \frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 dK_3, \tag{1.24}$$

where r_0^2 is a parameter proportional to the mass of the black hole. The radial coordinate r goes to infinity as we reach the conformal boundary. Unlike the asymptotically flat case, the topology of the horizon in AdS does not have to be spheric. The horizon is spherical for $k = 1$, planar for $k = 0$ and hyperbolic for $k = -1$. The three-dimensional volume element dK_3 represents the 3-sphere, flat or hyperbolic element accordingly. The radial location r_+ of the horizon is given by the solution of the equation

$$k + \frac{r_+^2}{\ell^2} - \frac{r_0^2}{r_+^2} = 0. \tag{1.25}$$

Using the techniques introduced above and any of the techniques that we will introduce in Chapter 2, we can calculate the energy content M, temperature T and entropy S of the Schwarzschild black hole in global AdS ($k = 1$), namely

$$M = \frac{3\pi r_+^2}{8G}\left(\frac{r_+^2}{\ell^2} + 1\right), \quad T = \frac{2r_+^2 + \ell^2}{2\pi r_+^2 \ell^2}, \quad S = \frac{\pi^2 r_+^3}{2G}, \tag{1.26}$$

from which the first law $dM = TdS$ follows directly. Since the Hamiltonians of the bulk and boundary theories are identified (we use the same time coordinate), the holographic interpretation of this first law in the CFT is the actual first law of thermodynamics for the thermal state. A consequence of this identification is that we can predict the existence of a phase transition [35]: the temperature T has a minimum. Below this temperature, black holes cannot exist; instead, the solution turns out to be a thermal gas of gravitons, knows as thermal AdS spacetime. The behaviour of the free energy allows us to predict a first-order phase transition between Schwarzschild-AdS and thermal AdS. Witten interpreted this as a phase transition in the dual gauge theory reminiscent of a famous prediction in quantum chromodynamics: the confinement-deconfinement phase transition for gluons. Witten showed explicitly that, in $\mathcal{N} = 4$ Super-Yang-Mills, there exists a phase transition correspondent to this gravitational thermodynamic behaviour [32].

A final development relevant to our purposes is the prescription by Ryu and Takayanagi to calculate the entanglement entropy $S(A)$ of a region A in the boundary theory. They found, using the correspondence, that it is possible to calculate such a quantity by calculating the (properly regularized) area of the extremal surface Σ in the bulk such that its boundary matches that of the region in the gauge theory [33]. This statement is expressed by the formula

$$S(A) = \frac{Area(\Sigma)}{4G}; \qquad \partial\Sigma = \partial A. \tag{1.27}$$

This relation is quite reminiscent of the Bekenstein formula for the entropy of a black hole, and it indicates a deeper relation between the entropy content and the boundary area in quantum gravity, further evidencing the holographic principle. One can use this prescription in Schwarzschild spacetime to identify the black hole entropy with the full entropy of the boundary. The first law of black hole thermodynamics becomes a particular case of the more general first law of entanglement entropy [36,37], which pertains to these extremal surfaces.

1.3 Thesis Outline

The purpose of this thesis is to study these laws of thermodynamics in a family of black hole spacetimes beyond the constraints given by asymptotic flatness and asymptotic AdS. In many cases, these assumptions can be relaxed in favour of weaker assumptions, for which the sundry definitions needed to formalize the concept of black hole spacetimes can be accordingly modified. Consequently, these black holes will have more degrees of freedom than the energy, charge, and angular momentum.

In the next chapter, we will introduce several techniques, some based on the gauge-gravity correspondence, to calculate the energy content and other asymptotic charges for asymptotically AdS spacetimes, most of which extend without modification to AlAdS spacetimes. With these essential tools in hand, we give the extended version of the first law of black hole thermodynamics in AdS, in which the cosmological constant is allowed to vary. We also explain the formalism of Hamiltonian perturbations, which provides for a straightforward derivation of this generalized first law.

Chapter 3 is based on [1] and deals with a particular class of spacetimes: AdS black holes with scalar hair, also known as Einstein-scalar spacetimes. Since the no-hair theorem does not apply to AdS spacetimes, the presence of a scalar field can give rise to many solutions that are not simply Schwarzschild-AdS. While finding solutions to the coupled equations is difficult, we show that the first law for these spacetimes can be derived from the AdS boundary conditions. We further prove a 7-year-old conjecture that the conjugate potential to the cosmological constant is proportional to the average value of the scalar potential inside the horizon, in the case of black branes. We also propose a general prescription for the thermodynamic volume for planar and hyperbolic black holes and study the MTZ black hole. Then we allow for variations of the asymptotic scalar field behaviour to explore the fully extended thermodynamics of planar black holes.

Chapter 4 is based on [2–6]. We apply the previously developed techniques to a particular class of asymptotically locally AdS spacetimes: the Lorentzian Taub-NUT spacetimes. Once considered unphysical, recent results show that the perturbed Taub-NUT may not be pathological, thus rekindling an interest in the community to study its properties. As a part of this argument, we provide a derivation of the first law of black hole mechanics for charged and rotating Taub-NUTs, for which the introduction of a Misner charge and its conjugate Misner potential are required. Similar to the case of charged black holes in AdS, we study the phase structure of charged Taub-NUT spacetimes. We finally study some properties of Taub-NUTs coupled to non-linear theories of electrodynamics.

In the fifth and final chapter, we conclude and discuss the physical significance of our results. We spell out the limitations of our methods and pave the way for future research in the area.

Chapter 2

Techniques in Black Hole Thermodynamics

In this chapter, we review some methods to calculate the energy content of an asymptotically anti-de Sitter space and explore some techniques to derive the laws of thermodynamics, including the extended case where the cosmological constant is allowed to vary.

2.1 Asymptotic charges

In general relativity, we can associate asymptotic charges to each generator of the group of isometries of a spacetime. Our interest focuses on stationary and axisymmetric black hole spacetimes, where there exists a timelike Killing vector t^α and an angular Killing vector ϕ^α. Since these vector fields are the generators of isometries, one can use them to calculate two associated conserved quantities: the mass and the angular momentum content of the spacetime.

The standard way to find such quantities, also known as asymptotic charges, is to reformulate general relativity using the Hamiltonian formalism. As in classical mechanics, we start by finding the gravitational action, which gives the Einstein field equations at stationary points. For four-dimensional asymptotically flat spacetimes, it is given by the Einstein-Hilbert action [38]

$$S_{grav} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{|h|}(K - K_0), \tag{2.1}$$

where the boundary terms ensure well-defined boundary conditions and that the on-shell action is finite. Here, K is the extrinsic curvature of ∂M embedded in M; K_0 is also extrinsic curvature, but with ∂M embedded in flat spacetime.

To define the gravitational energy contained in a spacetime, we need to calculate the gravitational Hamiltonian. Roughly, this proceeds as follows [39]: we choose a slicing of the spacetime into spacelike hypersurfaces Σ_t with normal vector field n^α, where the subscript t represents a function that is constant in each of these surfaces. Any vector field ξ^α can be decomposed into a component parallel to n^α and components tangent to the hypersurfaces:

$$\xi^\alpha = F n^\alpha + F^a e_a^\alpha, \tag{2.2}$$

where the frame e_a is a basis of the tangent space of the hypersurfaces. Similarly, the spacetime boundary is foliated by the boundaries S_t of each Σ_t. The gravitational action can then be recast in terms of the flow of ξ^α [40]:

$$S_{grav} = \frac{1}{16\pi G}\int dt \int_{\Sigma_t} d^3x\sqrt{h}F({}^3R + K^{ab}K_{ab} - K^2) + \frac{1}{8\pi}\int dt\int_{S_t} d^2y\sqrt{\sigma}F(k - k_0), \tag{2.3}$$

where ${}^3R$ is the Ricci scalar on Σ_t, K_{ab} denotes extrinsic curvature of Σ_t, K being its trace; k is the extrinsic curvature of the boundary foliation and k_0 is the curvature of S_t embedded in flat spacetime and serves a regularizing purpose. The corresponding induced metrics are denoted by h and σ.

This decomposition of the action allows us to write a gravitational Hamiltonian by considering h_{ab} as the evolving degrees of freedom in time, with conjugate momenta $p^{ab} = \partial(\sqrt{-g}\mathcal{L}_{grav})/\partial\dot{h}_{ab}$. We can then find evolution equations for p_{ab} and h_{ab}, plus some constraint equations that come from the diffeomorphism invariance of the theory. As a result, when we evaluate the on-shell Hamiltonian, we get [38]

$$H_{on-shell} = -\frac{1}{8\pi}\int_{S_t}(F(k - k_0) - F_a(K^{ab} - Kh^{ab})r_b)d^2y, \tag{2.4}$$

where r_b is the normal to S_t seen as embedded in Σ_t. From here one can calculate the mass by choosing a slicing such that $F = 1$ and $F_a = 0$. Similarly, the angular momentum can be calculated with the choice $F = 0$ and $F^a = \phi^a$. Ideally, any prescriptions used to calculate the mass and angular momentum should reduce to this expression in the asymptotically flat limit.

This calculation assumes asymptotically flat conditions. However, we are primarily interested in calculating the energy content with asymptotically AdS conditions and in further generality. Many prescriptions can be used and generally yield the same results, even though they have not been rigorously proven to be equivalent. In the following, we will explore a few different methods to calculate the gravitational energy for asymptotically AdS spacetimes.

2.1.1 Komar integration

In asymptotically flat, stationary, and axisymmetric spacetimes, the Komar formulas [41]

$$M = -\frac{1}{8\pi G}\int_{S_t \cap i_0} \nabla^\mu t^\nu dS_{\mu\nu}, \tag{2.5}$$

$$J = \frac{1}{16\pi G}\int_{S_t \cap i_0} \nabla^\mu \phi^\nu dS_{\mu\nu}; \tag{2.6}$$

where i_0 is spacelike infinity, give the correct results for the spacetime mass and angular momentum, as they can be shown to reduce to the Hamiltonian prescription described above. These formulas allow us to prove the Smarr relation for black holes, which follows by evaluating the integral

$$I = -\frac{1}{8\pi G}\int_{\partial\Sigma_t} \nabla^\mu \xi^\nu dS_{\mu\nu} \tag{2.7}$$

and by noting that Σ_t has two components: one an spatial infinity and the other at the black hole horizon H (if present). Additionally, we note that by using Stokes' theorem, the integral can be re-expressed as

$$I = -\frac{1}{8\pi G}\int_{\Sigma_t} \nabla_\mu \nabla^\mu \xi^\nu d\Sigma_\nu. \tag{2.8}$$

Killing's identity $\nabla_\mu \nabla^\mu \xi^\nu = -R^{\mu\nu}\xi_\mu$ and Einstein's field equations in the vacuum without a cosmological constant, $R_{\mu\nu} = 0$, imply that in asymptotically flat spacetimes this integral I is zero. We can therefore equate the integrals over both components of the boundary. Namely, with the generating Killing vector (1.2), we define

$$I_\infty = M - 2\,\Omega_H J = -\frac{1}{8\pi G}\int_{S_t \cap i_0} \nabla^\mu \xi^\nu dS_{\mu\nu}, \qquad I_H = -\frac{1}{8\pi G}\int_{S_t \cap H} \nabla^\mu t^\nu dS_{\mu\nu}; \tag{2.9}$$

then we must have $I_\infty = I_H$. In full generality, it is possible to show that $I_H = 2\kappa A = 2TS$ [42], hence the equality of integrals gives us the Smarr relation

$$M = 2\,TS + 2\,\Omega_H J. \tag{2.10}$$

In asymptotically flat spacetimes, the Komar method is applicable for calculating the gravitational energy and establishing a Smarr relation between the extensive thermodynamic quantities of black holes. However, in asymptotically AdS spacetimes, it becomes less reliable for calculating the mass. Although it is possible to use it for this purpose in many scenarios, it is generally safer to use more robust methods that we will introduce in further subsections. Nevertheless, it is still possible to use a similar argument to establish a Smarr relation in AdS.

In the AdS case, the calculation above changes because the Ricci tensor is no longer zero in the vacuum. Instead, tracing out Einstein's field equations with a non-zero cosmological constant in D spacetime dimensions, one obtains

$$R_{\mu\nu} = \frac{2\Lambda}{D-2} g_{\mu\nu}. \tag{2.11}$$

Contracting this equation with the generating Killing vector, integrating over the slice Σ_t, and making use of Stokes' theorem and Killing's identity again, we deduce that [43]

$$\int_{\partial\Sigma_t} dS_{\mu\nu} \left(\nabla^\mu \xi^\nu + \frac{2\Lambda}{D-2} \omega^{\mu\nu} \right) = 0. \tag{2.12}$$

In this equation, the Killing potential $\omega^{\mu\nu}$ is such that $\nabla_\mu \omega^{\mu\nu} = \xi^\nu$; such a tensor always exists for Killing vectors. In future sections, we will often use this equation to derive Smarr relations, but we must keep in mind that this is not useful for calculating the mass of AdS spacetimes: equations (2.5) and (2.6) no longer hold since the on-shell Hamiltonian is different. Instead, the strategy to follow is to calculate the spacetime energy and angular momentum using other methods and then rewrite (2.12) in terms of these charges. We will see that the equation obtained in this way depends on the horizon topology: different Smarr relations are obtained in each case.

2.1.2 Conformal method

The conformal method to calculate the spacetime mass and angular momentum in asymptotically AdS spacetimes [44, 45] is based on Penrose's conformal completion and was guessed from dimensional, symmetry, and asymptotic behaviour considerations, as well as the need to have a conserved charged associated with a Killing vector. It makes use of the formal definition of asymptotically AdS spacetime, which is more abstract than our previous naive definition that the metric falls off to AdS at infinity. Let $(M, g_{\mu\nu})$ be a D-dimensional spacetime such that $g_{\mu\nu}$ satisfies Einstein's field equations with negative cosmological constant. The spacetime is said to be asymptotically AdS if [46]

1. It admits a cylindrical conformal completion $\bar{M} = M \cup I$, with $I = \mathbb{R} \times S^{D-2}$ and $\partial\bar{M} = I$, and I is timelike.

2. There exists a smooth metric $\bar{g}_{\mu\nu}$ on $\bar{M}$ such that $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. Here $\Omega : \bar{M} \to \mathbb{R}$ is a smooth map that vanishes on I such that $\bar{n}_\mu = \bar{\nabla}_\mu \Omega$ is non-vanishing on I.

3. The metric induced by $\bar{g}_{\mu\nu}$ on I, denoted by $\bar{h}_{ab}$, is a conformal rescaling of

$$h_{ab} = -dt^2 + d\sigma^2, \tag{2.13}$$

where $d\sigma^2$ is the metric on the unit sphere S^{D-2}.

The choice of conformal factor Ω is not unique; for this method, we will fix it by requiring that $\nabla_\mu n_\nu$ goes to zero as we approach the boundary I (there is always a choice of rescaling that makes this possible). The next step calculates the Weyl tensor for the conformally rescaled metric with this choice of Ω. We recall that, for a metric tensor $g_{\mu\nu}$, it is defined via

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(D-2)(D-1)} R g_{\mu[\rho}g_{\sigma]\nu}. \tag{2.14}$$

Note that we will need, however, the Weyl tensor $\bar{C}_{\mu\nu\rho\sigma}$ associated with $\bar{g}_{\mu\nu}$, so all the terms in the above equation would take an overbar. We further define the electric part of this Weyl tensor, which is given by

$$\bar{\mathcal{E}}_{\mu\nu} = \ell \Omega^{3-D} \bar{n}^\rho \bar{n}^\sigma \bar{C}_{\mu\rho\nu\sigma} \tag{2.15}$$

in terms of the AdS scale ℓ (1.21). With these definitions in hand, we can give the expression for the conformal (or AMD) charge associated with a Killing vector ξ^μ of M:

$$Q(\xi) = \frac{\ell}{8\pi G(D-3)} \int_{\bar{C}} \bar{\mathcal{E}}_{\mu\nu} \xi^\mu d\bar{C}^\nu, \tag{2.16}$$

where $\bar{C}$ is a constant time slice of the boundary I of the conformal completion. As we see, these charges are characterized uniquely by the asymptotic behaviour of the metric, a fact that will be manifest in all methods. The fact that they can be written in terms of the electric part of the Weyl tensor comes from a more general property of AdS spacetimes that states that, asymptotically, the metric $\bar{g}_{\mu\nu}$ can be expanded in terms of $\bar{\mathcal{E}}_{\mu\nu}$ and Ω [46].

2.1.3 Holographic methods

Free Energy

While the computation of the conformal charges is straightforward, it is rather computationally taxing, and it becomes more so as the spacetime dimensions increase, since it involves computing the Weyl tensor. Moreover, this mass does not consider the energy contained, for example, in scalar fields. To get around some of these difficulties, we can take advantage of the AdS-CFT correspondence, particularly the identification of bulk and boundary partition functions. We can rewrite equation (1.23) in the saddle-point approximation as

$$Z_{CFT} = e^{-I_{grav}}, \tag{2.17}$$

where Z_{CFT} represents the partition function of the CFT and we denote the gravitational action as I_{grav} to avoid confusing it with entropy. Therefore, we can use the formula $F = -T \log Z$ to calculate the free energy of the boundary conformal field theory.

The calculation for the free energy requires evaluating the on-shell gravitational Euclidean action I_{grav}, which in general depends on the spacetime field content. This integral is usually divergent. For this reason, it is necessary to add counterterms to renormalize the action. How do we choose such counterterms? In general, a minimal subtraction scheme is used: we introduce local boundary counterterms that depend uniquely on the intrinsic boundary geometry, such as curvature invariants. Any other regularization scheme yielding a different finite result corresponds to an alternative choice of ensemble. Thus, the Euclidean effective gravitational action can be separated into three parts [47, 48]

$$I_{grav} = I_{bulk} + I_{GHY} + I_{ct} \tag{2.18}$$

where

$$I_{bulk} = -\frac{1}{16\pi G}\int_M d^D x\sqrt{g}(R - 2\Lambda), \tag{2.19}$$

$$I_{GHY} = \frac{1}{8\pi G}\int_{\partial M} d^{D-1}x\sqrt{h}\mathcal{K}, \tag{2.20}$$

$$I_{ct} = \frac{1}{8\pi G}\int_{\partial M} d^{D-1}x\sqrt{h}\left(\frac{2(D-2)}{\ell} + \frac{\ell}{D-3}\mathcal{R}\right). \tag{2.21}$$

These counterterms are the ones necessary when $D \leq 5$, they are only valid for purely gravitational systems. Additional counterterms are required in higher dimensions and when there are other fields in the bulk. Upon finding a solution to the variational problem, we can evaluate the on-shell action and hence compute the free energy using

$$F = TI_{grav}. \tag{2.22}$$

When Wick-rotating the theory to the Euclidean sector, the time becomes a compact coordinate which is integrated from 0 to $1/T$. For this reason, the time integration does not make our on-shell action diverge.

Although we can, with the procedure above, find the free energy associated with the spacetime, it is not necessarily helpful for obtaining the internal energy of the spacetime, mainly because the ensemble is unknown for us and will depend on how we interpret the terms using the duality. But there should be a unique answer, so what is the correct ensemble to use? From the first law of thermodynamics, we expect for the entropy S that

$$S = -\left(\frac{\partial F}{\partial T}\right) \tag{2.23}$$

holding all other thermodynamic variables constant. From the holographic principle and the Ryu-Takayanagi prescription, we expect that this entropy is proportional to the surface area of the black hole. Taking this hypothesis seriously, we should choose an ensemble such that taking this derivative of the free energy with respect to the temperature gives the black hole surface area.

Brown-York Tensor

The evaluation of the on-shell action does not give a spacetime gravitational energy dual to the thermodynamic internal energy. However, the gauge gravity correspondence can be used to obtain the thermodynamic internal energy (which will be later re-interpreted as enthalpy when considering the cosmological constant as a thermodynamic variable). The idea is to use the Hilbert prescription for calculating the stress-energy tensor using the Lorentzian gravitational effective action and use the correspondence to identify this as the expectation value for the CFT stress-energy tensor. Explicitly, the proposal states that the holographic stress-energy tensor is [49, 50]

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-h}} \frac{\delta S_{eff}}{\delta h^{\mu\nu}}. \tag{2.24}$$

Of course, since the counterterms are different for every spacetime dimension, the explicit expression for the holographic stress tensor changes as well. The most common case is for the bulk to have five spacetime dimensions; in this case, the stress-energy tensor reads [50]

$$T^{\mu\nu} = \frac{1}{8\pi G}\left[K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{\ell}h^{\mu\nu} - \frac{\ell}{2}G^{\mu\nu}\right] \tag{2.25}$$

where $K^{\mu\nu}$ is the second fundamental form of the boundary, as embedded in the bulk. One can then calculate the asymptotic charges associated with the Killing vector ξ^μ, which we do by contracting the stress tensor with the Killing vector and the unit normal to the boundary, and then integrating over a spacelike slice C of the boundary, namely:

$$Q(\xi) = \int_C d^{D-2}x\sqrt{\sigma}n^\mu T_{\mu\nu}\xi^\nu, \tag{2.26}$$

where σ is the induced metric on C. We observe that the holographic stress tensor contains all the information about the conserved asymptotic charges. The results of this procedure, in general, match those of the conformal method in the pure gravity case but may differ in the case that we include additional matter fields.

2.2 Thermodynamics with Lambda

A salient feature of the first law of black hole thermodynamics is that there are no terms related to the pressure and the volume. Interestingly, one can get such terms by noticing that the quantity

$$P = -\frac{\Lambda}{8\pi G} \tag{2.27}$$

can be interpreted as vacuum pressure [43]. This insight allows us to add a VdP term to the first law of black hole mechanics. While a first impression one may have is that this is purely for academic interest, it has been proposed that there are physical mechanisms by which the cosmological constant may be a thermodynamic variable, for example, by interpreting it as the energy content of some particular gauge fields [51–53].

Another reason for the interest in such a scenario is that, as we can see from equation (1.22), the cosmological constant (related to the AdS radius via (1.21)) maps to the gauge group dimension N under the gauge gravity correspondence. Since N^2 is proportional to the number of degrees of freedom of the CFT boundary theory, one could interpret varying Λ as a variation of the number of particles or colors [54–56]. Hence, the conjugate potential would correspond to the chemical potential associated with these particles. Alternative proposals such as the volume being related to an actual CFT volume [57] or that the law characterizes a renormalization group equation have also been proposed [58]. A correct understanding of this term is vital to obtain further insights into the AdS-CFT correspondence. In the following, we will give an argument for this law from dimensional analysis, and we will derive it for some examples that will be extended in further chapters of this thesis.

2.2.1 The Smarr relation and the extended first law

One reason to prefer the first law with varying Λ over the one without is that, as is the case for asymptotically flat black holes, it follows from a scaling argument [43]. The reasoning proceeds as follows: for AdS black holes with mass M and angular momentum J, we will later show that we can use (2.12) to obtain a Smarr relation

$$(D-3)M = (D-2)TS + (D-2)\Omega J - 2PV. \tag{2.28}$$

Here, the term V suggestively represents a volume. As we will see, in the simpler non-rotating case, the term V is exactly equal to the naive geometric black hole volume

$$V_{geom} = \int_{r<r_+} d\Omega_{D-2} dr \sqrt{-g}, \tag{2.29}$$

but deviates from this quantity in general. We note that there is something very particular about the coefficients of the terms in this equation: they all correspond to the length dimensions[1] of the extensive thermodynamics variable that they precede. This fact is a consequence of the mass being a homogeneous function of the extensive variables; that being the case, this equation is equivalent to the extended first law

$$dM = TdS + \Omega dJ + VdP. \tag{2.30}$$

The assertion above follows from Euler's theorem for homogeneous functions: let $f : \mathbb{R}^n \to \mathbb{R}$ be a weighted homogeneous function, which means that for every real number λ we have

$$f(\lambda^{m_1} x_1, \ldots, \lambda^{m_n} x_n) = \lambda^m f(x_1, \ldots, x_n), \tag{2.31}$$

then the following identity for the partial derivatives of f holds:

$$mf = m_1 x_1 \left(\frac{\partial f}{\partial x_1}\right) + \ldots + m_n x_n \left(\frac{\partial f}{\partial x_n}\right). \tag{2.32}$$

[1] Modulo the gravitational constant G, which is not varied.

Therefore, if we assume that the mass M can be written as a homogeneous function of the extensive thermodynamic variables $M(S, J, P)$ where, by dimensional analysis, the weights are equal to the length dimensions of the variables, we can compare (2.28) and (2.32) and identify

$$\left(\frac{\partial M}{\partial S}\right)_{J,P} = T, \quad \left(\frac{\partial M}{\partial J}\right)_{T,P} = \Omega, \quad \left(\frac{\partial M}{\partial P}\right)_{J,T} = V \tag{2.33}$$

from which the first law (2.30) follows immediately. Such an identity cannot follow without including the cosmological constant term in the equations, and it has been shown to work in a wide variety of AdS black hole solutions. For this reason, deriving a Smarr relation between thermodynamic variables is a powerful tool to derive the extended first law of black hole mechanics in most situations or, at the very least, to guess it and subsequently verify it.

In this extended first law of thermodynamics, the physical meaning for the energy M is not that of thermodynamical internal energy. Instead, the first law suggests that this M corresponds to a gravitational enthalpy. However, the holographic meaning of the thermodynamic volume is not completely clear. Therefore, M will correspond to the enthalpy on the dual CFT only if the thermodynamic volume can be interpreted as a CFT volume.

2.2.2 Examples of the extended first law

We now exhibit some examples of the extended first law of thermodynamics which will be helpful in further chapters.

Schwarzschild-AdS black hole

The simplest example is the Schwarzschild-AdS black hole, which is given by metric (1.24) for $k = 1$. We reproduce it here in D dimensions:

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1 - \frac{r_0^{D-3}}{r^{D-3}}\right) dt^2 + \left(\frac{r^2}{\ell^2} + 1 - \frac{r_0^{D-3}}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \tag{2.34}$$

where $d\Omega_{D-2}$ is the metric of a $(D-2)$-dimensional sphere with unit radius. The prescriptions mentioned above can be used to calculate the mass, temperature, and entropy of this black hole. Using the conformal method, we find that the mass is given by

$$M = \frac{(D-2)\Omega_{D-2} r_+^{D-3}}{16\pi G}\left(1 + \frac{r_+^2}{\ell^2}\right), \tag{2.35}$$

where Ω_{D-2} is the area of a unit $(D-2)$-sphere and r_+ is the position of the black hole horizon. The temperature, entropy, and volume are given by

$$T = \frac{1}{4\pi r_+}\left((D-1)\frac{r_+^2}{\ell^2} + D - 3\right), \quad S = \frac{\Omega_{D-2} r_+^{D-2}}{4G}, \quad V = \frac{\Omega_{D-2} r_+^{D-1}}{D-1}, \tag{2.36}$$

where the volume is simply the geometric volume as defined above. In this case, it corresponds to the conjugate potential for the pressure, which we call the thermodynamic volume. In more general cases, both volumes do not coincide.

Let us observe that, if we rescale $r \to \lambda r$, $t \to \lambda t$, $r_0 \to \lambda r_0$, and $\ell \to \lambda \ell$, we obtain a rescaling of the metric. As expected, the consequence is that the mass is homogeneous under this rescaling, as are the entropy and the cosmological constant. From our expression, one can directly verify the Smarr relation

$$(D-3)M = (D-2)TS - 2PV \tag{2.37}$$

and the extended first law of thermodynamics, as expected from the scaling argument

$$dM = TdS + VdP. \tag{2.38}$$

While a first law $dM = TdS$ also holds if we leave the cosmological constant untouched, since the mass is not homogeneous as a function of only the entropy, this first law does not follow from a Smarr relation.

Hyperbolic (topological) black holes

A hyperbolic or topological black hole is the name given to the case when $k = -1$ in formula (1.24). There is, in fact, not much of a change to the formulas for the Schwarzschild spherical black hole. If ω_{D-2} represents the volume of the $(D-2)$-dimensional hyperbolic space of radius 1, the mass reads

$$M = \frac{(D-2)\omega_{D-2} r_+^{D-3}}{16\pi G}\left(\frac{r_+^2}{\ell^2} - 1\right), \tag{2.39}$$

while the temperature, entropy and geometric volume are calculated to be

$$T = \frac{1}{4\pi r_+}\left((D-1)\frac{r_+^2}{\ell^2} - (D-3)\right), \quad S = \frac{\omega_{D-2} r_+^{D-2}}{4G}, \quad V = \frac{\omega_{D-2} r_+^{D-1}}{D-1}. \tag{2.40}$$

Then a direct calculation allows us to conclude that the first law (2.38) holds for hyperbolic black holes in AdS as well [59].

Planar black holes

The planar black hole metric, also known as a black brane, corresponds to (1.24) with $k = 0$, which in D dimensions reads

$$ds^2 = -\frac{r^2}{\ell^2}\left(1 - \frac{r_+^{D-1}}{r^{D-1}}\right)dt^2 + \frac{\ell^2}{r^2}\left(1 - \frac{r_+^{D-1}}{r^{D-1}}\right)^{-1}dr^2 + \frac{r^2}{\ell^2}\delta_{ij}dx^i dx^j. \tag{2.41}$$

As the name suggests, the horizon of this black hole is an infinite plane $r = r_+$; consequently, the extensive thermodynamic quantities such as the mass and the entropy will be infinite. However, one can instead work with densities of such quantities. One way to do this is to compactify the transverse coordinates x_i via the identification $x_i \equiv x_i + L_i$ [60]. A calculation of the mass using, for example, the conformal method or the holographic stress-energy tensor yields

$$M = (D-2)\frac{r_+^{D-1}v}{16\pi G\ell^2}. \tag{2.42}$$

Here, the volume element v represents a box of sides L_i in units of ℓ, explicitly:

$$v = \frac{\prod_{i=1}^{D-2} L_i}{\ell^{D-2}}. \tag{2.43}$$

Addtionally, since this spacetime possesses symmetries in the transverse directions x_k, one can find three additional conserved quantities corresponding to the Killing vectors in these directions, which we call spacetime tensions:

$$\tau_i = -\frac{r_+^{D-1}v}{16\pi G\ell^2 L^i}. \tag{2.44}$$

We also draw attention to the fact that the trace of the holographic stress tensor vanishes

$$M + \sum_{i=1}^{D-2}\tau_i L^i = 0, \tag{2.45}$$

as is expected from a conformal fluid, which is the state of the CFT dual to the black brane.

With these quantities in hand, it is easy to establish the first law of thermodynamics for these spacetimes. We calculate the temperature, entropy, and geometric volume V, which read

$$S = \frac{r_+^{D-2}v}{4G}, \quad T = \frac{D-1}{4}\frac{r_+}{\pi\ell^2}, \quad V = \frac{vr_+^{D-1}}{D-1}. \tag{2.46}$$

In the literature, the Smarr relation for planar black holes is usually stated as

$$(D-1)M = (D-2)TS, \tag{2.47}$$

which in turn gives the usual first law $dM = TdS$. Note that this does correspond to the scaling dimensions of M if we only rescale r_+. However, an extended first law can also be obtained; this means that we need to rescale ℓ as well as L_k to rescale the metric properly, leaving the dimensionless volume v invariant. The Smarr relation then becomes

$$(D-3)M = (D-2)TS + \sum_{i=1}^{D-2} \tau_i L^i - 2P\Theta, \tag{2.48}$$

where the thermodynamic volume Θ is equal to half of the geometric volume V. By using Euler's theorem or by calculating explicitly, one obtains the extended first law

$$dM = TdS + \sum_{i=1}^{D-2} \tau_i dL^i + \Theta dP. \tag{2.49}$$

Let us note that by merely changing the topology of the black hole, the geometric volume is no longer equal to the thermodynamic volume. The change, in this case, is not too dramatic since it only acquires a factor of 1/2. The literature has been, in general, obscure about this particular case. Some references (e.g, [61]) do not define the compactifying lengths L_i, which still give a correct first law with $\Theta = V$. However, this law does not arise from a Smarr relation and relies on cancelling infinities arbitrarily.

Kerr-AdS

When we add rotation to an AdS black hole, the thermodynamic volume, defined as the potential conjugate to the spacetime pressure P, is no longer proportional to the geometric volume. To see this, let us consider the Kerr-AdS black hole in four spacetime dimensions, which is described by the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2, \tag{2.50}$$

where

$$\begin{aligned} \Delta_r &= \left(r^2+a^2\right)\left(1+r^2/\ell^2\right) - 2mr, \\ \Delta_\theta &= 1 - \left(a^2/\ell^2\right)\cos^2\theta, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \\ \Xi &= 1 - a^2/\ell^2. \end{aligned} \tag{2.51}$$

where we use $G = 1$ to simplify some expressions. The two obvious Killing vectors for this metric are ∂_t and ∂_ϕ, so one might be tempted to calculate the mass as the conformal charge associated with the former. However, in such a metric, the observer at infinity is not static, as can be shown by calculating the angular velocity at infinity:

$$\Omega_\infty = -\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r=\infty} = -\frac{a}{\ell^2}. \tag{2.52}$$

Then the mass and angular momentum can be computed from the perspective of an observer that sees a non-rotating background [62]:

$$M = Q(\partial_t - \Omega_\infty \partial_\phi) = \frac{m}{\Xi^2}, \tag{2.53}$$

$$J = Q(\partial_\phi) = aM. \tag{2.54}$$

The mass can also be found in terms of the position of the horizon r_+, which is the largest positive root of Δ_r. We compute the entropy and temperature:

$$S = \frac{\pi\left(a^2 + r_+^2\right)}{\Xi}, \quad T = \frac{r_+\left(\frac{a^2}{\ell^2} - \frac{a^2}{r_+^2} + \frac{3r_+^2}{\ell^2} + 1\right)}{4\pi\left(a^2 + r_+^2\right)}. \tag{2.55}$$

The angular velocity should be measured with respect to the background as well, thus obtaining

$$\Omega = \Omega_H - \Omega_\infty = \frac{a\left(\ell^2 + r_+^2\right)}{\ell^2\left(a^2 + r_+^2\right)}. \tag{2.56}$$

With the thermodynamics quantities defined above, one can obtain a Smarr relation from the Komar integral (2.12), using the generating Killing vector $\xi = \partial_t + \Omega_H \partial_\phi$ and grouping terms together in terms of the conserved charges. However, an easier trick is to assume that we will have a Smarr relation of the form [63]

$$M = 2TS + 2\Omega J - 2P\Theta, \tag{2.57}$$

where $P = 3/8\pi\ell^2$. The $2P\Theta$ term comes from the Komar integral, but we use the Smarr relation to guess the thermodynamic volume. Solving this equation for Θ, we obtain

$$\Theta = \frac{2\pi\ell^2\left(a^2 + r_+^2\right)\left(a^2\left(\ell^2 - r_+^2\right) + 2\ell^2 r_+^2\right)}{3r_+\left(a^2 - \ell^2\right)^2}, \tag{2.58}$$

which is the thermodynamic volume since the first law

$$dM = TdS + \Omega dJ + \Theta dP \tag{2.59}$$

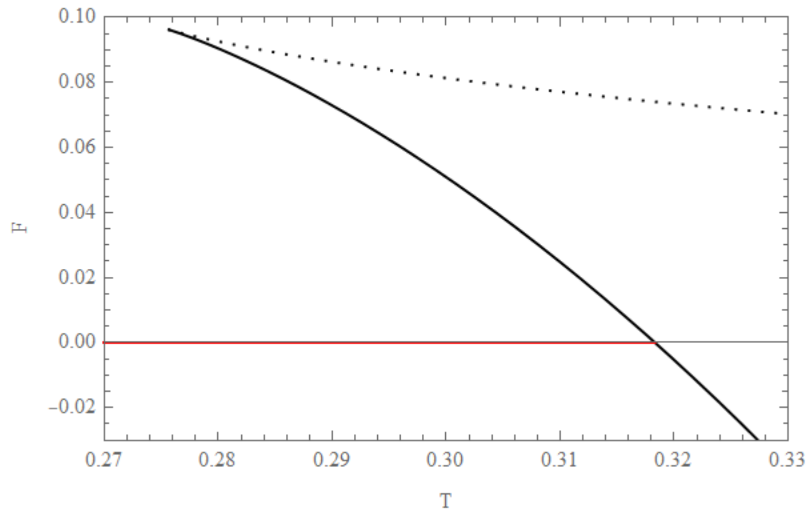


Figure 2.1: **Hawking-Page phase transition.** The dotted black line represents the regime of small black holes; the continuous black line is the regime of large black holes. The red line is thermal AdS. As we see, there is a first-order phase transition from thermal AdS to large black holes.

holds. However, we note that this thermodynamic volume is not equal to the geometric volume V; we have

$$V = \int_{r<r_+} d\Omega dr \sqrt{-g} = \frac{4\pi}{3} \frac{r_+ \left(a^2 + r_+^2\right)}{\Xi} \neq \Theta. \tag{2.60}$$

As we see, it is in general possible to derive an extended first law of thermodynamics, even in very non-trivial spacetimes. Moreover, the naive interpretation of the volume seems to beg for more questions than it answers. Understanding the thermodynamic volume, using the AdS-CFT correspondence or otherwise, is a very active area of research. For this reason, these extended laws will be a primary focus of this thesis, with the hope that these results will eventually help to elucidate the mystery behind this term in the first law.

2.2.3 Black hole chemistry and thermodynamic phase structure

We see that the extended thermodynamics of black holes is now very similar to classical thermodynamics. Can we push this analogy even further? We want to explore, for example, the phase transitions of black holes. In AdS, where we have non-trivial pressure, we will see that we obtain such phase transitions between black hole spacetimes (for a comprehensive review of this topic, see [64]). This means that a specific class of spacetimes is more stable at a certain temperature, but this preferred phase can change as we push the temperature beyond its critical value. To study this stability, we need to minimize the free energy. The Euclidean action prescription introduced in section 2.1.3 thus becomes of central importance.

Hawking-Page transition

The oldest and most famous example of a phase transition for an AdS black hole is the Hawking-Page transition [35], which is relevant for Schwarzschild-AdS black holes. Let us analyze the expression for their temperature in four spacetime dimensions in the Euclidean regime. The temperature is

$$T = \frac{1}{4\pi r_+}\left(3\frac{r_+^2}{\ell^2} + 1\right). \tag{2.61}$$

It is straightforward to show that this temperature has a minimum at $r_+ = r_0 = \ell/\sqrt{3}$, where it takes the value

$$T_0 = \frac{\sqrt{3}}{2\pi}\frac{1}{\ell}. \tag{2.62}$$

Below this temperature, black holes cannot exist since there is no solution for r_+ in terms of T; instead we have a spacetime filled with pure radiation, known as thermal AdS. However, for temperatures greater than T_0, we have two possible positive solutions for r_+, where one root will evidently be larger than the other. This means that at a certain temperature, two kinds of black holes can exist: a "large black hole" and a "small black hole".

How do we choose the stable configuration given a temperature? Drawing on our knowledge from thermodynamics, it will be the configuration for which the free energy $F = M - TS$ is minimized. Thermal AdS has zero free energy since there is not any mass nor entropy. Calculating the Euclidean action, we get

$$F = \frac{r_+\left(\ell^2 - r_+^2\right)}{4G\ell^2}. \tag{2.63}$$

This expression already tells us that black holes with $r_+ > \ell$ are preferred over thermal AdS at temperatures $T > T_0$. To visualize the phase transitions, we need to analyze the dependence of F with T. The plot is shown in figure 2.1. We observe that small black holes are never preferred and that there is a phase transition between thermal AdS and large black holes when $r_+ = \ell$, that is when

$$T = T_c := \frac{1}{\pi\ell}. \tag{2.64}$$

We can additionally determine the line in $P - T$ space where both phases are allowed to coexist. This coexistence curve is found by expressing ℓ in terms of the pressure in our expression for the critical temperature. We thus obtain

$$P = \frac{3\pi}{8}T^2. \tag{2.65}$$

We have made explicit the phase transition that we mentioned in passing in the introduction. The AdS-CFT correspondence maps it to the confinement-deconfinement transition in $\mathcal{N} = 4$ super Yang-Mills [32].

Phase transitions of charged black holes

A more interesting scenario is provided by the Reissner-Nordstrom AdS black holes, which are endowed with an electric charge. They are solutions of electromagnetism coupled to gravity as given by the action

$$S_{EM} = -\frac{1}{16\pi G}\int_M \sqrt{-g}\left(R - 2\Lambda - F^2\right). \tag{2.66}$$

Here, $F_{\mu\nu}$ is the electromagnetic stress tensor, which can be expressed as an exterior derivative $dA = F$. Then the Reissner-Nordstrom-AdS black hole is the simple solution with metric and gauge field

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \\ A &= -\frac{Q}{r}dt \end{aligned} \tag{2.67}$$

with

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \tag{2.68}$$

Let us examine the thermodynamic behaviour of these charged black holes. We will follow reference [7] closely and recommend that the reader consult this article for more details. The temperature and entropy are given by

$$T = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2}\right), \quad S = \pi r_+^2. \tag{2.69}$$

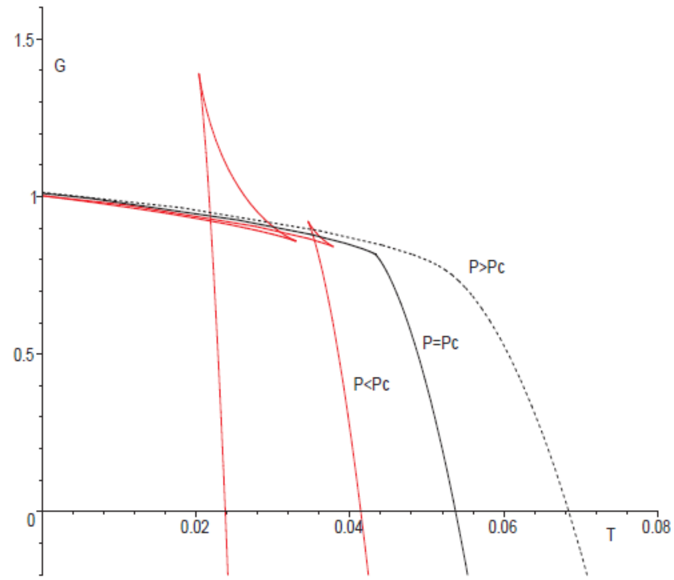


Figure 2.2: **Gibbs free energy for charged AdS black holes.** [7] We see that at pressures below the critical pressure, the swallowtail behaviour appears. It corresponds to a first-order phase transition from small black holes to large black holes as the temperature increases.

The volume turns out to be equal to the geometric volume, and if we allow for variations of the electric charge, then the conjugate potential is given by the electric potential difference between the black hole horizon and infinity. This is calculated using the timelike generating Killing vector $k = \partial_t$ as follows

$$\phi = -\left(k \cdot A|_{r=r_+} - k \cdot A|_{r=\infty} \right) = \frac{Q}{r_+}. \tag{2.70}$$

Then the following first law of thermodynamics is verified

$$dM = TdS + \phi dQ + VdP. \tag{2.71}$$

Unlike the uncharged case, this family of black holes allows for a phase transition between small and large black holes. We can see this phenomenon by expressing the temperature in terms of the pressure and the thermodynamic volume using (2.69) and

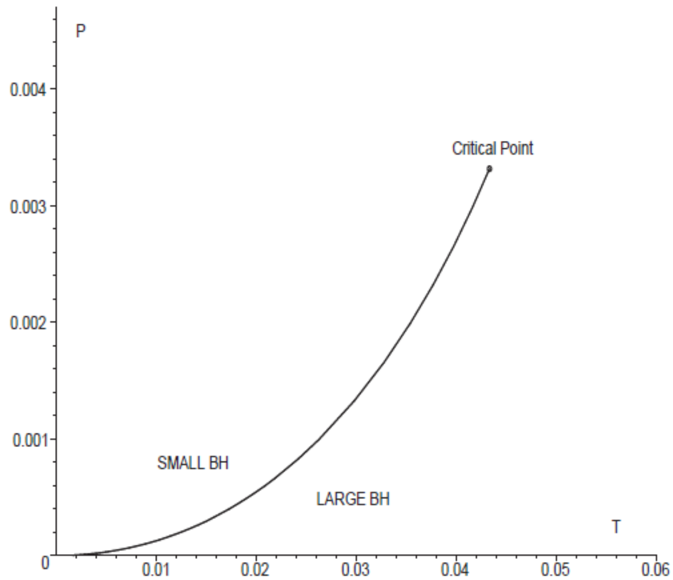


Figure 2.3: **Coexistence line for charged black holes.** [7] Below the critical point, there exists a phase transition between small black holes and large black holes. The phase transition disappears when the transition temperature is larger than T_c.

then solving for the pressure. We thus obtain the equation of state[2]

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}, \quad v = 2r_+ = 2\left(\frac{3V}{4\pi}\right)^{1/3}. \tag{2.72}$$

In a pressure vs. volume diagram, we would find that, for certain values of T, the same pressure admits two different volumes. That is, $P(v)$ is not a one-to-one function for certain values of T. However, this function becomes injective at values of the temperature $T > T_c$. This critical value can be found by imposing that the pressure have an inflection point

$$\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0, \tag{2.73}$$

which in turn yields the critical values for the pressure and volume as well:

$$T_c = \frac{\sqrt{6}}{18\pi Q}, \quad v_c = 2\sqrt{6}Q, \quad P_c = \frac{1}{96\pi Q^2}. \tag{2.74}$$

[2]It is, in fact, $v = 2r_+$ that acts as a specific volume for this equation of state. Indeed, recovering the units, we have $v = \ell_p^2 r_+$, see details on [7].

Let us now derive what these phases correspond to. As for the uncharged case, we need to compute the free energy, which is now given by

$$G = \frac{1}{4}\left(r_+ - \frac{8\pi}{3}Pr_+^3 + \frac{3Q^2}{r_+}\right). \tag{2.75}$$

Here, $G = M - TS - \phi Q$. We take Q to be fixed and express r_+ implicitly in terms of T and P. We thus observe the swallowtail behaviour shown in figure 2.2, where for $T < T_c$, there is indeed a first-order phase transition between small and large black holes. Beyond the critical point, such a phase transition disappears. This phenomenon is reminiscent of a van der Waals fluid; the analogy is made more precise by the authors of [7].

2.3 Hamiltonian perturbations

Deriving the first law can be relatively straightforward once we have an explicit solution to the Einstein field equations, as evidenced by the above examples. However, we might face the problem that we do not know the solution, but we have limited information about it, such as its asymptotic behaviour and its symmetries. In such a case, the methods introduced above are not very helpful since we might have trouble, for example, in finding the precise position of the horizon, which in turn renders it impossible to obtain explicit expressions for the entropy and area. For this reason, it is essential to have a robust method that allows us to derive the first law of thermodynamics in more general cases. The most well-known prescriptions are Wald's Noether charge method [65] and the method of Hamiltonian perturbations [66]. We will focus on the latter since it is straightforward to implement on a computer algebra system (see appendix C).

As the name suggests, this method is based on finding the on-shell Hamiltonian to draw the conserved charges from it, as seen at the beginning of this chapter. One must be careful, however, with how one regularizes it. Since we are no longer working with asymptotically flat spacetimes, it is useful to write the Hamiltonian expression for the charges when we work in a generic background. As before, we foliate the spacetime $(M, g_{\mu\nu})$ with spacelike hypersurfaces Σ_t. Let $(M_0, g^{(0)}_{\mu\nu})$ be the background spacetime and let us denote the induced background metric on some fixed Σ by $h^{(0)}_{\mu\nu}$ and the induced metric on the boundary $\partial\Sigma$ by $\sigma^{(0)}_{\mu\nu}$, following the notation at the beginning of this chapter. Then one can show that for a Killing vector that decomposes as in equation (2.2), the Hamiltonian charge associated with it is given by [42, 66]

$$Q(\xi) = \int_{\partial\Sigma_\infty} da_c B^c \tag{2.76}$$

where

$$B^c = F\left(D^c\delta h - D_b\delta h^{bc}\right) - \delta h D^c F + \delta h^{cb} D_b F + \frac{1}{|h|} F^d \left(p^{(0)ab}\delta h_{ab} h^{(0)c}{}_d + 2p^{(0)cb}\delta h_{db} - 2\delta p^c_d\right). \tag{2.77}$$

Here, the (not necessarily small) differences δ stand for the subtraction between our metric and the background metric of choice, which will be the AdS metric in many applications. The covariant derivative D is taken with respect to the induced background metric $h_{\mu\nu}$. The component $\partial\Sigma_\infty$ of the boundary that is chosen is the one at asymptotic infinity.

How can we use this to derive the first law of black hole thermodynamics? The trick is to take all the parameters in our black hole solutions, such as charge, rotation, and mass, and vary them slightly. Then we use the original metric as a background and calculate the charges of the perturbed metric with respect to this background. This procedure will give us an expression δQ for the small perturbations of the charges. Moreover, it was shown in [66] that if the (now small) metric perturbation $\delta g_{\mu\nu}$ satisfies the linearized Einstein field equation, then a Gauss-type law is satisfied

$$\int_{\partial\Sigma} da_c B^c = 0. \tag{2.78}$$

In a black hole spacetime, the boundary $\partial\Sigma$ has a component $\partial\Sigma_\infty$ at infinity and a component $\partial\Sigma_H$ at the black hole horizon. Writing down explicitly the equality, we obtain

$$\int_{\partial\Sigma_\infty} da_c B^c = \int_{\partial\Sigma_H} da_c B^c. \tag{2.79}$$

It was shown by Sudarsky and Wald [67] that, in all generality (even with the inclusion of further gauge fields), if we choose the generating Killing vector ξ to be the black hole generator, then

$$\int_{\partial\Sigma_H} da_c B^c = T\delta S. \tag{2.80}$$

Assuming this result, if we choose $\xi = \partial_t + \Omega_H\partial_\phi$ for an asymptotically flat rotating black hole, it becomes easy to see that we have the relation

$$\delta M - \Omega_H \delta J = T\delta S \tag{2.81}$$

which is the first law of black hole thermodynamics for Kerr spacetimes.

However, suppose we want to derive the extended law that allows for variations $\delta\Lambda$ of the cosmological constant. In that case, the argument above needs a slight modification since the cosmological constant appears explicitly in the gravitational Hamiltonian, as opposed to the other parameters that appear only in a particular solution. Defining $\omega^{\mu\nu}$ as in equation (2.12), Gauss' law and equation (2.78) imply that

$$D_a \left(\frac{B^a}{16\pi G} + 2\delta\Lambda\omega^{ab} n_b \right) = 0. \tag{2.82}$$

Similar arguments to the ones explained above allow for the derivation of the extended first law of black hole thermodynamics. Nevertheless, we must note that this is not necessarily a very efficient method to derive the first law. In cases where the explicit solution is known, we will prefer to deduce it from the Smarr relation since this seems to work in all known cases.

Chapter 3

Thermodynamics of Hairy AdS Black Holes

In this chapter, we explore whether the extended thermodynamics of black holes in anti-de Sitter space changes when we include scalar "hair" which, for our purposes, is nothing but a scalar field that is minimally coupled to Einstein gravity. To do this, we fix the asymptotic behaviour of the metric to be asymptotically anti-de Sitter. For simplicity, we also assume radial symmetry. The ordinary first law of thermodynamics for hairy black holes has already been studied in the literature [68, 69]; we focus on generalizing these results to the case of varying cosmological constant, represented here as value of the scalar potential around its extremum.

The bulk action (without boundary terms and counterterms) of a minimally coupled scalar field to Einstein gravity in AdS is given by

$$S_{bulk} = -\int d^D x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - 2\Lambda \mathcal{V}(\phi) \right). \tag{3.1}$$

Here, we have introduced a scalar potential of the form $V(\phi) = 2\Lambda\mathcal{V}(\phi)$, where we assume that all the dependence on the cosmological constant Λ is on the overall coefficient. While this may be a limitation, to the author's knowledge, all the explicit black hole solutions that have been found can be written in this way. Since we are assuming AdS asymptotics, we will restrict ourselves to potentials such that $\mathcal{V}(\phi_\infty) = 1/16\pi G$, where ϕ_∞ is the asymptotic value of the scalar field. Moreover, since we will search for metric solutions that only depend on r, we will assume that the scalar field ϕ is also only r-dependent. These assumptions are still quite generic since the form of the potential remains unspecified.

To derive the first law of thermodynamics for an unspecified solution, we will follow the steps below:

- Find the common asymptotic behaviour of all the solutions that respect the desired symmetries.
- Calculate the asymptotic charges, which can be expressed in terms of the integration constants that appear in the asymptotic metric.
- Derive, using equation (2.12), a Smarr relation for the solution. Hence propose the expression for the thermodynamic volume.
- Use the method of Hamiltonian perturbations to verify that the proposed thermodynamic volume does appear in the extended first law.
- Provide examples from explicit solutions found in the literature.

3.1 Planar black holes

Firstly, we will attempt to find solutions with a planar horizon topology. The most general ansatz for a planar black hole with scalar hair is characterized by three functions A, σ, h, plus the scalar field solution ϕ: [70]

$$ds^2 = \frac{r^2}{\ell^2} e^{2A(r)} \left(-h(r)dt^2 + \delta_{ij}dx^i dx^j\right) + \frac{\ell^2}{r^2}\frac{\sigma^2(r)}{h(r)}dr^2, \quad i = 2, \ldots D, \tag{3.2}$$

$$\phi = \phi(r). \tag{3.3}$$

The AdS scale ℓ is related to the cosmological constant Λ via equation (1.21); however, this will not always be true if we go beyond asymptotically AdS spacetimes, as explored in [42, 71].

3.1.1 Asymptotic behaviour and ADM charges

From the ansatz above, we can write Einstein's field equations in terms of the ansatz functions A, h, and σ. Let us note that the stress-energy tensor for the scalar field, given by

$$T^{\mu\nu} = g^{\mu\nu}\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right) + \partial^\mu\phi\partial^\nu\phi \tag{3.4}$$

is in general non-trivial. Then the coupled field equations read

$$(D-2)\,(rA'+1)\,((D-1)h\,(rA'+1)+rh')-8\pi G\left(r^2h\phi'^2-2\ell^2V(\phi)\sigma^2\right)=0, \tag{3.5}$$

$$\left(((D-1)rA'+D)-\frac{r\sigma'}{\sigma}\right)h'+rh''=0, \tag{3.6}$$

$$r\left((D-2)A''+8\pi G\phi'^2\right)+(D-2)A'-(D-2)\,(1+rA')\,\frac{\sigma'}{\sigma}=0. \tag{3.7}$$

We also have an additional Klein-Gordon equation for the scalar field. The condition that the metric is asymptotically anti-de Sitter means that we must have $A\to 0$, $h\to 1$ and $\sigma\to 1$ as we approach the conformal boundary. In general, it is not easy to obtain explicit solutions to this set of non-linear differential equations, although some have been found and we will explore them later. For now, we note that the second equation can, in fact, be easily integrated to get

$$h(r)=(D-1)\mu\int_{r_+}^{r}\frac{e^{-(D-1)A(r)}\sigma(r)}{r^D}dr, \tag{3.8}$$

where μ is a constant of integration and r_+ is the largest root of h, which we assume exists in order to have a black hole horizon. Using a Taylor expansion around infinity, we see that the asymptotic behaviour of h is

$$h\sim 1-\frac{\mu}{r^{D-1}}+\dots \tag{3.9}$$

It is also well known that, in general, the scalar field behaves asymptotically as [72]

$$\phi\sim\frac{\Phi_0}{r^\Delta}+\frac{\Phi_+}{r^{D-1-\Delta}}+\dots, \tag{3.10}$$

where Δ is related to the mass of the scalar field and corresponds to the conformal dimension of the operator dual to ϕ in the AdS-CFT correspondence. This behaviour is true as long as $2<\Delta<3$, and extra logarithmic terms will need to be added if Δ takes values at the ends of this interval. For simplicity, we will keep the asymptotic behaviour of the scalar field fixed and assume that $\Phi_0=0$, but we will relax this assumption later on. In such simple cases, the thermodynamic black hole volume is thought to be proportional to the integral of the potential behind the horizon [63]. This section aims to prove this; however, as we will see, this conjecture is no longer true without this assumption.

This knowledge of the asymptotic behaviour allows us to calculate the energy content of the space using either the Hamiltonian or conformal methods. We obtain

$$M=(D-2)\frac{\mu v}{16\pi G\ell^2}, \tag{3.11}$$

where the transverse coordinates have been compactified and v is defined as in (2.43). However, unlike the hairless case (2.42), there is no simple relation between the integration constant μ and the locus of the horizon r_+. The lesson we learned is that the mass is calculated from the $1/r^{D-1}$ coefficient in the asymptotic expansion of the blackening factor h. Similarly, we can obtain the spacetime tensions:

$$\tau_i = -\frac{\mu v}{16\pi G \ell^2 L^i}. \tag{3.12}$$

We observe that we still have $M + \sum \tau_i L^i = 0$, as expected from the dual conformal fluid. This identity will be useful when deriving the Smarr relation, but it will not hold when the asymptotic behaviour deviates from that of AdS [42], nor when we turn on a more slowly decaying scalar field.

The mass and tensions calculated above are purely gravitational. This fact is manifest since the Hamiltonian mass was calculated using the pure scalar field spacetime as background, as opposed to pure AdS; the conformal method automatically gives the gravitational energy and is not affected by the presence of the scalar. These two methods do not calculate any energy content associated with the scalar field, and they cannot. The conformal method only sees gravity, and the Hamiltonian method cannot be used when the asymptotic behaviour of the scalar field is as in equation (3.10), since then the mass is not a conserved charge[1] [73]. To calculate a mass associated with the scalar, we need to calculate the holographic stress tensor. To do this, we need to go through the procedure of holographic renormalization [74] of the bulk action and add adequate counterterms that render the on-shell Euclidean action finite. Then one can express the first law of thermodynamics in terms of the holographic mass, including variations of Φ_+ and Φ_0. Work in these lines has been done in [68]; however, an extended first law has not been derived yet.

3.1.2 Smarr relation and the scaling argument

The easy way to derive a Smarr relation is, as we mentioned before, to assume that the mass is homogeneous under a rescaling of the metric. This is not necessarily self-evident, but it is useful because, in most scenarios, it tells us the answer without doing a full calculation. Since the case of a generic hairy planar black hole is not trivial at all, we will first provide the scaling argument for the Smarr relation, but then we will derive it using the Komar integral (2.12). We will see that the results match, but the Komar integration gives us

[1]This does not matter much when $\Phi_0 = 0$. If the asymptotic scalar field decays in this way, the Hamiltonian method yields zero mass for it, as do other methods that can calculate the scalar field energy content.

additional insight into how the thermodynamic and geometric volumes of the black hole are related.

The authors of [75] derived a Smarr relation from the following scaling argument. Let us observe that the metric (3.2) is invariant under the rescaling $t \to \lambda^{-1} t$, $r \to \lambda r$, $x^i \to \lambda^{-1} x^i$; if $\mu \to \lambda^{D-1}\mu$. Here, we have not re-scaled the AdS length ℓ, so we are not deriving a Smarr relation involving Λ. Seen as a function of the entropy, the mass $M(S)$ is a homogeneous function with weights

$$M(\lambda^{D-2} S) = \lambda^{D-1} M(S), \tag{3.13}$$

from which Euler's theorem leads to

$$(D-1)M = (D-2)\left(\frac{\partial M}{\partial S}\right) S. \tag{3.14}$$

However, it is impossible to derive the first law of thermodynamics from this expression since we do not know what $\partial M/\partial S$ is. For hairy black branes, it has been shown that $dM = TdS$ [75], so we know that this law comes from the Smarr relation.

The above is not a true Smarr relation, since the coefficients preceding the thermodynamics charges do not correspond to their dimensions. To get the correct Smarr relation, we need to include the cosmological constant in the thermodynamics. In order to see what happens when we vary the cosmological constant, let us perform a different and more natural rescaling of the metric, given by $t \to \lambda t$, $r \to \lambda r$, $x^i \to \lambda x^i$ and $\ell \to \lambda \ell$. Then the metric is rescaled $ds^2 \to \lambda^2 ds^2$ only if $\mu \to \lambda^{D-1}\mu$. The compactifying lengths L^i are rescaled in the same way since the transversal coordinates are rescaled. The mass is now considered a homogeneous function of the entropy, the compactifying lengths, and the cosmological constant, such that

$$M(\lambda^{D-1} S, \lambda L^i, \lambda^{-2}\Lambda) = \lambda^{D-3} M(S, L^i, \Lambda). \tag{3.15}$$

Using Euler's theorem again, we see that

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial S}\right) S + \sum_i \left(\frac{\partial M}{\partial L^i}\right) L^i - 2\left(\frac{\partial M}{\partial \Lambda}\right)\Lambda. \tag{3.16}$$

Now, from the fact that the holographic stress-energy tensor must be traceless, which translates into equation (2.45), we see that we can obtain this extended Smarr relation from (3.14) by subtracting M two times on each side. On the right-hand side, we subtracted once in the form $M = -\sum \tau_i L^i$ and a second time in the form $M = (M/2\Lambda) \times 2\Lambda$. We thus obtain

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial S}\right) S + \sum_i \tau_i L^i - 2\left(\frac{M}{2\Lambda}\right)\Lambda. \tag{3.17}$$

Since we expect the thermodynamic volume Θ to be equal to $\partial M/\partial P$, these two last equations suggest the very simple expression

$$\Theta = -\frac{4\pi G M}{\Lambda}, \tag{3.18}$$

which is, in fact, also valid for non-hairy planar black holes. This expression turns out to be the correct thermodynamic volume; however, since this relies on assumptions such as the homogeneity of the mass, we should provide an independent derivation.

3.1.3 Komar Integration

We now proceed to derive the Smarr relation using an expression akin to (2.12), that is, a Komar integral. This procedure will help us obtain a relation between the geometric volume of the hairy planar black hole and its thermodynamic volume. A long-standing conjecture asserts that these two quantities are proportional [76]. In this section and the following, we aim to show this for the generic problem we are studying.

Equation (2.12) is no longer true since we now have a scalar field potential. However, a similar relation can be derived for this case by tracing out Einstein's field equations, with a stress-energy tensor given by (3.4). This gives us the following expression for the Ricci tensor:

$$R_{\mu\nu} = 8\pi G \left(\partial_\mu \phi \partial_\nu \phi + \frac{2}{D-2} g_{\mu\nu} V(\phi)\right), \tag{3.19}$$

where $V(\phi) = 2\Lambda\mathcal{V}(\phi)$. Contracting with a Killing vector ξ^α, and by virtue of the Killing identity, this becomes

$$\nabla_\mu \nabla^\mu \xi^\nu + \frac{16\pi G}{D-2} \xi^\nu V(\phi) = 0. \tag{3.20}$$

Since the scalar field is assumed to respect the same symmetries as the planar black hole spacetime, we can assume that the directional derivative $\xi^\mu \partial_\mu \phi$ vanishes. As a consequence, the covariant divergence $\nabla_\mu(\xi^\mu V(\phi))$ vanishes as well. Now, Poincaré's lemma implies that there exists a Killing potential, namely a 2-tensor $\omega^{\mu\nu}$ such that

$$V(\phi)\xi^\nu = \nabla_\mu (V(\phi)\omega^{\mu\nu}). \tag{3.21}$$

Thus, using Gauss' law, we can express this identity in integral form [42]

$$I = \int_{\partial\Sigma} d\sigma_{\mu\nu} \left(\nabla^\mu \xi^\nu + \frac{16\pi G}{D-2} \omega^{\mu\nu} V(\phi)\right) = 0. \tag{3.22}$$

The boundary $\partial\Sigma$ has two components: $\partial\Sigma_\infty$ and $\partial\Sigma_H$, at conformal infinity and at the black hole horizon respectively. Therefore, I can be separated into two integrals over such components. We will denote these integrals by I by I_∞ and I_H, so that (3.22) can be rewritten as

$$I_\infty - I_H = 0. \tag{3.23}$$

We now go back to the assumption of radial symmetry. For the Killing potential associated with the Killing vector ∂_t, it means that the only non-trivial components are $\omega^{rt} = -\omega^{tr}$, which only depend on the radial coordinate r. In fact, in this case, one can integrate equation (3.21) by expressing the covariant divergence in terms of ordinary derivatives

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}V(\phi)\omega^{rt}) = V(\phi) \tag{3.24}$$

so that

$$\omega^{rt} = \frac{1}{\sqrt{-g}V(\phi)}\int_\rho^r dr'\sqrt{-g}V(\phi). \tag{3.25}$$

The constant ρ comes from the integration and, as we will see, can be chosen without affecting the final expression that we get from the Komar integration. This underdetermination amounts to the fact that the Killing potential is unique up to a closed 2-form. Taking all of these observations into account, the Komar integral I now reads

$$I = \int d^{D-2}x\sqrt{-g}\nabla^r\xi^t + \frac{16\pi G}{D-2}\int d^{D-2}x\int_\rho^r dr'\sqrt{-g}V(\phi) = 0. \tag{3.26}$$

To derive the Smarr relation, we need to evaluate this integral both at infinity and at the horizon. While this does not seem to be possible in the presence of the potential, we can obtain an exact expression. To do this, we notice that $\sqrt{-g}V(\phi)$ can be expressed as a total derivative in terms of the ansatz functions. Indeed, if one solves for ϕ'^2 in equation (3.7) and then replaces the result in equation (3.5), then it is straightforward to verify that

$$W(r) \equiv \int dr\sqrt{-g}V(\phi) = -\frac{(D-2)r^{D-1}h(r)e^{(D-1)A(r)}\left(rA'(r)+1\right)}{16\pi G\ell^D\sigma(r)}. \tag{3.27}$$

We can also express the covariant derivative of the timelike Killing vector in terms of the ansatz functions, obtaining

$$\nabla^r\xi^t = \frac{r\left(2h(r)\left(rA'(r)+1\right)+rh'(r)\right)}{2\ell^2\sigma^2(r)}. \tag{3.28}$$

Thus, plugging in (3.27) and (3.28) into (3.26), we can completely integrate I as a function of r, since in expression (3.26) it is assumed that we integrate over a constant r hypersurface:

$$I = \frac{r^D e^{(D-1)A(r)} h'(r)}{2\ell^2 \sigma(r)} v - \frac{16\pi G W(\rho)}{D-2} v. \tag{3.29}$$

With this expression, we can evaluate I_∞ and I_H and subtract them. It is easy to see that the ρ dependent terms will cancel. The component at infinity can be calculated from the asymptotic behaviour of the functions A, σ, and h. It has been shown that the component at the horizon is always equal to twice the temperature times the entropy, although this is not hard to see in this particular case. Upon performing these calculations, the integrals reduce to

$$I_\infty = 8\pi G \left(\frac{D-1}{D-2} M - \frac{2W(\rho)v}{D-2} \right), \tag{3.30}$$

$$I_H = 8\pi G \left(TS - \frac{2W(\rho)v}{D-2} \right). \tag{3.31}$$

The integral of $\nabla^r \xi^t$ has a divergent term, but as we can see from equation (3.29), it is always exactly cancelled by the integral of the potential. It is from such cancellation that we obtain the mass term. Setting $I_H = I_\infty$, we obtain the Smarr relation

$$(D-1)M = (D-2)TS, \tag{3.32}$$

which is the one that is often stated in the many exact hairy planar black hole examples in the literature [77–79]. However, such a relation does not include all the conserved charges that we can calculate from the independent Killing vectors and is not useful to derive the thermodynamic volume. Since (2.45) holds, we can perform similar manipulation as we did in (3.17) when we derived the (incomplete) Smarr relation from the scaling argument and write

$$(D-3)M = (D-2)TS + \sum \tau_i L^i - 2P\Theta, \tag{3.33}$$

where Θ is given by (3.18). Therefore, the expression for the thermodynamic volume of a planar black hole is, in fact, rather simple and depends only on the black hole mass. However, we would like to know how this thermodynamic volume is related to the geometric volume. In the hairless case, the relation is simply that the former is one-half of the latter, but it remains to be seen whether this relation still holds when we introduce the scalar field. In the hairy case, we introduce the quantity Υ, which is defined as the integral of the potential inside the black hole volume $\mathcal{B}$; namely,

$$\Upsilon = \int_{\mathcal{B}} d^{D-2}x \int_0^{r_+} dr \sqrt{-g}\, \mathcal{V}(\phi) = \frac{(W(r_+) - W(0))v}{2\Lambda}. \tag{3.34}$$

We know that $W(r_+) = 0$, since it is proportional to $h(r_+)$. But we do not have much information about $W(0)$, since we did not impose any boundary conditions in the infrared (the deep black hole interior). For this reason, we need to impose some additional assumptions. The first one is that $A(r)$ is sufficiently regular at $r = 0$. The second is that

$$\sigma \sim \frac{1}{r^{\alpha}}, \quad \alpha \geq 1 - D \tag{3.35}$$

asymptotically as $r \to 0$. These two conditions together are sufficient for there to be a singularity at $r = 0$. The expression for $W(0)$ simplifies to

$$W(0) = -\left(\frac{(D-2)e^{(D-1)A(0)}}{16\pi G\ell^D}\right) \lim_{r\to 0} \frac{h(r)}{\sigma(r)/r^{D-1}}. \tag{3.36}$$

While it seems impossible to go further, thankfully we have equation (3.8), an integral formula for $h(r)$. The condition that $\alpha \geq 1 - D$ makes the limit indeterminate, since the integral for h must then diverge. Therefore, a combination of L'Hopital's theorem with the first fundamental theorem of calculus yields

$$\begin{aligned} W(0) &= -\left(\frac{(D-2)e^{(D-1)A(0)}}{16\pi G\ell^D}\right) \lim_{r\to 0} h'(r) \left(\frac{\sigma'(r)}{r^{D-1}} - (D-1)\frac{\sigma(r)}{r^D}\right)^{-1} \\ &= -\left(\frac{(D-2)(D-1)\mu e^{(D-1)A(0)}}{16\pi G\ell^D}\right) \lim_{r\to 0} \frac{e^{-(D-1)A(r)}\sigma(r)}{r^D} \left(\frac{\sigma'(r)}{r^{D-1}} - (D-1)\frac{\sigma(r)}{r^D}\right)^{-1} \\ &= \frac{M}{v} \lim_{r\to 0} \left(1 - \frac{1}{D-1}\frac{r\sigma'(r)}{\sigma(r)}\right)^{-1} = \frac{M}{v}\left(\frac{D-1}{D-1+\alpha}\right). \end{aligned} \tag{3.37}$$

Finally, we obtain for the integral of the potential

$$\Upsilon = -\left(\frac{D-1}{D-1-\alpha}\right)\frac{M}{2\Lambda}, \tag{3.38}$$

and thus we have derived a relation between the thermodynamic volume and Υ

$$\Theta = 8\pi G\left(1 + \frac{\alpha}{D-1}\right) \int_{\mathcal{B}} d^{D-2}x \int_0^{r_+} dr \mathcal{V}(\phi) = 8\pi G\left(1 + \frac{\alpha}{D-1}\right)\Upsilon. \tag{3.39}$$

As we see, this relation depends on the infrared behaviour of the metric: to find the relations between the two volumes, it will suffice to find the constant leading order exponent α of the function σ. While this is not always easy, we will see that it is, in general, much more straightforward than integrating a potential, which often has a mathematically involved form. We have therefore derived the Smarr relation from purely geometric methods, and we have managed to find a connection between the thermodynamic volume and the integral of the potential behind the horizon as an additional consequence of using such methods.

3.1.4 Extended First Law

We now proceed to use the method of Hamiltonian perturbations to derive the first law for hairy planar black holes. Here, since we have a scalar field, we are met with an additional challenge: we have to consider both the gravitational and the scalar Hamiltonian. Indeed, even though the scalar field behaviour is kept constant, the variation of the cosmological constant needs to be considered since it appears in the potential term. In this case, the corresponding Gauss law identity turns out to be [42]

$$D_a \left(\frac{B^a}{16\pi G} + 2\delta\Lambda \mathcal{V}(\phi) \omega^{ab} n_b \right) = 0. \tag{3.40}$$

In order to calculate B^a, we need to calculate the asymptotic perturbations of the metric. When we vary the parameters μ, ℓ and L^i, these read

$$\delta h_{rr} = \left(\frac{\ell}{r} \right)^2 \frac{\delta\mu}{r^{D-1}}, \tag{3.41}$$

$$\delta h_{jk} = \delta_{ij} \frac{r^2}{\ell^2} \left(2\frac{\delta L^i}{L^i} - 2\frac{\delta\ell}{\ell} \right). \tag{3.42}$$

To establish the extended first law, we need to use Gauss' law to integrate (3.40). We can thus express them as an integral over the two components $\partial\Sigma_\infty$ and $\partial\Sigma_H$ of the boundary as in the derivation of the Smarr relation. We define

$$J_\infty = \int_{\partial\Sigma_\infty} d\sigma_a \left(\frac{B^a}{16\pi G} + 2\delta\Lambda \mathcal{V}(\phi) \omega^{ab} n_b \right), \tag{3.43}$$

$$J_H = \int_{\partial\Sigma_H} d\sigma_a \left(\frac{B^a}{16\pi G} + 2\delta\Lambda \mathcal{V}(\phi) \omega^{ab} n_b \right), \tag{3.44}$$

where we must have $J_\infty = J_H$. Since we know the value of the volume integral of the potential by virtue of (3.27), it is more convenient to express this integral as a mixture of surface and volume integrals. Namely, we write

$$\int_{\partial\Sigma_\infty} d\sigma_a \left(\frac{B^a}{16\pi G} \right) + 2\delta\Lambda \int d^{D-2}x \int_{r_+}^{\infty} dr \sqrt{-g} \mathcal{V}(\phi) = \int_{\partial\Sigma_H} d\sigma_a \left(\frac{B^a}{16\pi G} \right). \tag{3.45}$$

Therefore, we need to express these integrals in terms of the variations, grouping terms together to obtain δM (since M depends on ℓ and the compactifying lengths L_i, it is not enough to group together the terms dependent on $\delta\mu$). As we have mentioned before, the right hand side of this last equation has been show to be always equal to $T\delta S$ in full generality. We then evaluate each of the terms on the left hand side:

$$\int_{\partial\Sigma_r} d\sigma_a \left(\frac{B^a}{16\pi G}\right) = -\frac{(D-2)}{8\pi G}\frac{r^{D-1}}{\ell^{D+1}}\delta\ell + \frac{M}{\ell}\delta\ell - \delta M - \sum_{k=1}^{D-2}\tau_k \delta L^k, \tag{3.46}$$

$$2\delta\Lambda \int d^{D-2}x \int_{r_+}^{\infty} dr\sqrt{-g}\mathcal{V}(\phi) = \frac{(D-2)}{8\pi G}\frac{r^{D-1}}{\ell^{D+1}}\delta\ell - \frac{2M}{\ell}\delta\ell, \tag{3.47}$$

where the integrals have been evaluated for large r. We immediately see that the divergent terms $\sim r^{D-1}$ cancel each other. Re-expressing the variations $\delta\ell$ in terms of $\delta\Lambda$, we obtain the extended first law for hairy planar black holes, which is no different from the hairless case:

$$dM = TdS + \sum_i \tau_i dL^i + \frac{M}{2\Lambda}d\Lambda. \tag{3.48}$$

Expressing this law in terms of the spacetime pressure $P = -\Lambda/8\pi G$, the law reads

$$dM = TdS + \sum_i \tau_i dL^i + \Theta dP, \tag{3.49}$$

where Θ is defined as in (3.18).

Does this mean that there is no difference whatsoever between the hairless first law and our case? There is, in fact, a change, and it is that the thermodynamic volume is no longer one-half of the geometric volume. As we already discovered, the thermodynamic volume is related to the integral of the potential behind the horizon. Considering the following straightforward generalization of the geometric volume for the hairy case

$$V = 16\pi G \int_{\mathcal{B}} d^{D-2}x \int_0^{r_+} dr\sqrt{-g}\mathcal{V}(\phi) \tag{3.50}$$

we obtain that the thermodynamic and generalized geometric volumes are proportional to each other:

$$\Theta = \frac{1}{2}\left(1 + \frac{\alpha}{D-1}\right)V; \tag{3.51}$$

where we recall that α is the leading power in the radial coordinate of the expansion of the ansatz function σ around $r = 0$. This thus proves a version of the conjecture set forth in [63], that the thermodynamic volume is related in a simple way to the integral of the potential behind the horizon in the presence of scalar hair.

3.1.5 Explicit examples

We now study the thermodynamics of two explicit AdS hairy black branes solutions that are found in the literature. We aim to verify that the extended first law of thermodynamics that we have derived is demonstrated, in addition to the relation between geometric and thermodynamic volumes.

Hypergeometric solution

We expect the solutions to the non-linear Einstein equations to have complicated forms for the potential. This expectation is justified, as shown by the hypergeometric solution [79]. The expression for the potential is

$$\begin{aligned}16\pi G V(\phi) =& 2\Lambda(\cosh\phi)^{\frac{\nu k_0^2}{D-2}}\left(1-\eta(\sinh\phi)^{\frac{D-1}{\nu}}{}_2F_1\left(\frac{D-1}{2\nu},\frac{\nu k_0^2}{4(D-2)},\frac{D+2\nu+1}{2\nu},\sinh^2\phi\right)\right)\\ &\times\left(1-\frac{\nu^2k_0^2\tanh^2\phi}{2(D-1)(D-2)}\right)+2\Lambda\eta(\cosh\phi)^{\frac{\nu k_0^2}{2(D-2)}}(\sinh\phi)^{\frac{D-1}{\nu}},\end{aligned} \tag{3.52}$$

which depends on the parameters ν, k_0 and η. For the mass of the scalar to be real and for the potential to reach a maximum at $\phi = 0$, we need to satisfy $1/2(D-1) < \nu < D-1$ [80]. We thus obtain a solution with ansatz functions $A = 0$ and

$$\sigma(r) = \left(1+\frac{q^{2\nu}}{r^{2\nu}}\right)^{-\frac{\nu k_0^2}{4(D-2)}}, \quad h(r) = 1-\frac{q^{D-1}\eta}{r^{D-1}}{}_2F_1\left(\frac{D-1}{2\nu},\frac{\nu k_0^2}{4(D-2)},\frac{D+2\nu+1}{2\nu},-\frac{q^{2\nu}}{r^{2\nu}}\right). \tag{3.53}$$

The scalar field is of the form

$$\phi(r) = \text{Arcsinh}\left(\frac{q^\nu}{r^\nu}\right). \tag{3.54}$$

The Hamiltonian charges can be calculated from an asymptotic expansion of h, and the thermodynamic volume can be computed from the fact that $\Theta = -4\pi GM/\Lambda$. These quantities read

$$M = (D-2)\frac{\eta q^{D-1}}{16\pi G\ell^2}v, \quad \tau_i = -\frac{\eta q^{D-1}}{16\pi G L^i \ell^2}v, \quad \Theta = \frac{\eta q^{D-1}}{2(D-1)}v. \tag{3.55}$$

Similarly, we can find the temperature and the entropy, with r_+ being the largest positive zero of h:

$$T = \frac{(D-1)\eta q^{D-1}}{4\pi r_+^{D-2}\ell^2}, \quad S = \frac{r_+^{D-2}}{4}v. \tag{3.56}$$

The Smarr relation follows directly from these expressions. In order to derive the first law of thermodynamics, we first observe that h depends only on the quotient q/r. Regarding q as a function of r_+, we can relate the variations using the fact that $h(r_+) = 0$, namely:

$$\delta r_+ = \frac{r_+}{q}\delta q. \tag{3.57}$$

Using this identity, the extended first law can be verified by taking the variations of the relevant thermodynamic quantities.

Next, we should verify the proposed relation between the thermodynamic volume and the integral of the potential. However, the potential is hard to integrate, and not even computer algebra systems such as Mathematica are able to provide an analytic expression. For this reason, we will choose some values for the parameters that simplify this integral. One can show that, for $D = 5$, $\nu = 2$, and $k_0 = 1$, the hypergeometric function becomes algebraic and the potential reads

$$16\pi G V(\phi(r)) = -\frac{\Lambda\left(36\eta r^{2/3}\left(q^4+r^4\right)^{5/6} - (6\eta+5)\left(5q^4+6r^4\right)\right)}{30 r^{4/3}\left(q^4+r^4\right)^{2/3}}, \tag{3.58}$$

which integrates to

$$16\pi G \int d^3x \int dr \sqrt{-g} V(\phi(r)) = -\frac{2}{5\ell^2}\left(\frac{3}{2}(6\eta+5) r^{10/3}\sqrt[6]{q^4+r^4} - 9\eta r^4\right) v. \tag{3.59}$$

We need to evaluate this indefinite integral at $r = 0$ and $r = r_+$. While it is clearly zero at $r = 0$, for the $r = r_+$ case we need to solve $h(r_+) = 0$ in terms of q. The hypergeometric function in the blackening factor also simplifies for these values of the parameters, so that one simply finds

$$q^4 = \frac{\left(5\times 6^{4/5}\sqrt[5]{\frac{5}{\eta}+6} + \left(6^{9/5}\sqrt[5]{\frac{5}{\eta}+6} - 36\right)\eta\right) r_+^4}{36\eta}. \tag{3.60}$$

From this integral, the generalized geometric volume can be integrated:

$$V = \frac{3q^4\eta}{10} v = \frac{12}{5}\Theta, \tag{3.61}$$

so these are directly proportional as expected.

Our results in the previous sections show that one could have derived this result without integrating the potential. All we need is to analyze the behaviour of $\sigma(r)$ close to $r = 0$.

For our current choice of parameter, $\sigma \sim r^{2/3}$ close to the origin, so $\alpha = -2/3$. Thus, from relation, (3.51), we immediately get this relation between the geometric and the thermodynamic volume. More generally, allowing for arbitrary values of the parameters, we see that we must have

$$\Theta = \frac{1}{2}\left(1 - \frac{\nu^2 k_0^2}{2(D-1)(D-2)}\right) V, \tag{3.62}$$

where no cumbersome integration of a hypergeometric function needed to be performed.

Double Branch Solution

The double branch solution was initially proposed in 5 dimensions [77] but was later generalized to higher dimensions [78]. There has been some interest in this solution in the context of the AdS-CFT correspondence and the dual RG flow corresponding to turning on scalar fields on the spacetime. Its extended thermodynamics has also been studied [81,82]; here, we show that the methods outlined above allow for a straightforward calculation of the volume and an easy derivation of the first law.

For simplicity, we only consider the 5-dimensional solution, although unlike the previous case, other parameters will be left free. The metric is obtained by solving for an ansatz of the form

$$ds^2 = \Omega(x)\left(-f(x)dt^2 + \frac{\eta^2}{f(x)}dx^2 + \delta_{ij}dx^i dx^j\right) \tag{3.63}$$

and potential

$$\begin{aligned} 16\pi G V(\phi) = {} & 2\Lambda\left(\frac{(9\nu^2-5)\,e^{-\phi l_\nu}}{8\nu^2}\right)\left(1 - \frac{8\mu}{(\nu^2-25)\,(9\nu^2-25)}\right) \\ & \times\left(\frac{(\nu-1)e^{-\nu\phi l_\nu}}{2(3\nu+5)} + \frac{(\nu+1)e^{\nu\phi l_\nu}}{2(3\nu-5)} + \frac{5\,(\nu^2-1)}{9\nu^2-25}\right) \\ & + 2\Lambda\mu\frac{e^{3l_\nu\phi/2}}{4\nu^3}\left[\frac{5\,(\nu^2-1)}{\nu^2-25}\left(\frac{e^{-l_\nu\nu\phi/2}}{3\nu-5} + \frac{e^{l_\nu\nu\phi/2}}{3\nu+5}\right)\right. \\ & \left. + \frac{1}{3}\left(\frac{(\nu+1)e^{-3l_\nu\nu\phi/2}}{(\nu-5)(3\nu-5)} + \frac{(\nu-1)e^{3\nu l_\nu\phi/2}}{(\nu+5)(3\nu+5)}\right)\right], \end{aligned} \tag{3.64}$$

where

$$l_\nu^{-1} = \sqrt{\frac{3\,(\nu^2-1)}{2}}. \tag{3.65}$$

This gives the following solution for the metric

$$\Omega(x) = \frac{\nu^2 x^{\nu-1}}{\eta^2 (x^\nu - 1)^2}, \tag{3.66}$$

$$\begin{aligned} f(x) = -\frac{\Lambda}{6} + \Lambda\mu\Bigg(\frac{4}{3\left(\nu^2-25\right)\left(9\nu^2-25\right)} + \frac{x^{5/2}}{12\nu^3}\Bigg(-\frac{x^{-\frac{\nu}{2}}}{\nu-5} - \frac{x^{\nu/2}}{\nu+5} \\ + \frac{x^{-\frac{3\nu}{2}}}{3(3\nu-5)} + \frac{x^{\frac{3\nu}{2}}}{3(3\nu+5)}\Bigg)\Bigg), \end{aligned} \tag{3.67}$$

and for the scalar field

$$\phi(x) = l_\nu^{-1} \log(x). \tag{3.68}$$

Let us note that this metric is not in Poincaré coordinates, and in fact, it is impossible to perform the exact coordinate transformation to express it in this way unless $\nu = 1$. However, this value of ν corresponds to the case without a scalar field. As we will see, it is possible to change coordinates locally so that we can, for example, show that the metric is asymptotically AdS and find the deep infrared behaviour, which is necessary to verify equation (3.51). As is usual, we compute the Hamiltonian charges and obtain

$$M = \left(\frac{1}{16\pi G}\right)\frac{\mu}{16\eta^4\ell^2}v, \quad \tau_i = \left(\frac{1}{16\pi G}\right)\frac{\mu}{48\eta^4\ell^2 L^i}v, \quad \Theta = \frac{1}{4}\frac{\mu}{96\eta^4}v. \tag{3.69}$$

If the locus of the horizon is $x = x_+$, we can also calculate the temperature and the entropy

$$T = \frac{\mu |x_+^\nu - 1|^3}{48\pi\eta\nu^3 x_+^{3/2(\nu-1)}\ell^2}, \quad S = \frac{\nu^3 x_+^{3/2(\nu-1)}}{4G\eta^3 |x_+^\nu - 1|^3} v. \tag{3.70}$$

Once again, with these explicit expressions at hand, it is easy to check the Smarr relation. Verifying the first law requires a little bit more care: instead of varying x_+, we take it to be fixed and instead vary the parameter η. This changes the scale and with it, we can also control the position of the horizon.

To compare the potential integral and thermodynamic volumes, we need to integrate (3.64) behind the horizon. In these coordinates, conformal infinity is located at $x = 1$, and there are two branches for the x coordinate. If the scalar is negative, x ranges from 0 to 1; otherwise, it ranges from 1 to infinity. In each case, the singularity is at zero or infinity respectively. Stable black branes correspond to the latter case, so we choose this one to work with. The integral of the potential then reads

$$16\pi G \int d^3x \int_\infty^{x_+} dr \sqrt{-g} V(\phi(x)) = -\frac{\nu+1}{3\nu+5}\frac{\mu}{4\eta^4\ell^2}v, \tag{3.71}$$

which we can use to infer the relation between the thermodynamic and the geometric volumes

$$\Theta = \frac{1}{8}\left(\frac{3\nu+5}{\nu+1}\right)V. \tag{3.72}$$

Of course, the integration of the potential is unnecessary to derive this relation. To see this, let us locally change to Poincaré coordinates close to $x=\infty$. This means that we must have $\Omega \sim r^2$ and

$$x \sim r^{-2/(1+\nu)}, \quad dx^2 \sim r^{-2(3+\nu)/(1+\nu)}. \tag{3.73}$$

Thus, for the ansatz functions in Poincaré coordinates

$$\sigma \sim r^{-\alpha} \quad \text{with} \quad \alpha = \frac{1-\nu}{1+\nu}. \tag{3.74}$$

Using this value of α in (3.51) gives back relation (3.71).

3.2 Hyperbolic Hairy Black Holes

The above arguments are not only applicable to planar black holes. This section will analyze how to derive the Smarr relation from a Komar integral for hyperbolic hairy black holes, for which there are simple explicit solutions[2]. The most well-known exact solution is the MTZ solution [83]. We develop technology similar to that of the previous section and apply it to derive the extended thermodynamics of this black hole.

3.2.1 ADM charges and Komar integration

We consider the Einstein-scalar theory (3.1) once again, this time solving for the ansatz

$$ds^2 = \Omega(r)\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma^2\right), \tag{3.75}$$

where $d\sigma^2$ is the metric of a $(D-2)$-dimensional spacetime with constant negative curvature. A caveat about this ansatz is that it is not the most general one; however, all known exact and uncharged hairy hyperbolic solutions can be written in this form [83, 84]. Here, Ω accounts for the presence of a scalar, and f is the blackening factor that indicates that

[2]The spherical case can be obtained by making trivial changes to the argument in this section.

we are solving for a black hole spacetime. We can then write the field equations in terms of Ω and f.

$$(D-2)\Omega'\left(2(D-2)f+rf'\right)+\frac{2(D-2)\Omega\left((D-3)(1+f)+rf'\right)}{r} \\ +\frac{(D-2)(D-1)rf(r)\Omega'^2}{2\Omega}+2r\Omega^2V(\phi)=rf\Omega\phi'^2, \quad (3.76)$$

$$f''=\frac{2(D-3)(1+f)}{r^2}-(D-4)\frac{f'}{r}+(D-2)\frac{\Omega'}{\Omega}\left(\frac{f}{r}-\frac{1}{2}f'\right), \quad (3.77)$$

$$\phi'^2=(D-2)\left(\frac{3}{2}\frac{\Omega'^2}{\Omega^2}-\frac{\Omega''}{\Omega}\right). \quad (3.78)$$

As before, we can derive some asymptotic behaviour for the ansatz functions, assuming asymptotically AdS behaviour; for large r we must have

$$\Omega\sim 1+O(r^{-2}), \quad (3.79)$$

$$f\sim\frac{r^2}{\ell^2}-1-\frac{2\mu}{r^{D-3}}+O\left(\frac{1}{r^{D-2}}\right). \quad (3.80)$$

Knowing the asymptotic behaviour, one can calculate the mass. Since the Hamiltonian method cannot provide us with a well-defined mass for a scalar, we take

$$ds^2=\Omega(r)\left(-\left(\frac{r^2}{\ell^2}-1\right)dt^2+\left(\frac{r^2}{\ell^2}-1\right)^{-1}dr^2+r^2d\sigma^2\right) \quad (3.81)$$

as our background metric. Using (2.76) with Killing vector ∂_t, we obtain the mass

$$M=(D-2)\frac{\mu}{8\pi G}\omega_{D-2}. \quad (3.82)$$

Similar to the planar case, we express the Komar integral as in equation (3.26). To calculate it explicitly, we need two ingredients: the covariant derivative of the timelike Killing vector and the integral of the potential. The former is

$$\nabla^r\xi^t=\frac{(\Omega f)'}{2\Omega^2}; \quad (3.83)$$

while for the latter the equations of motion allow us to express the potential as a total derivative, so that the indefinite integral is

$$\int drd\omega_{D-2}\sqrt{-g}V(\phi)=-\frac{1}{16\pi G}\left(\frac{D-2}{2}\right)\Omega^{(D-4)/2}r^{D-2}(\Omega f)'\omega_{D-2}. \quad (3.84)$$

The evaluation of the Komar integral component at infinity is then straightforward; we get

$$I_\infty = \frac{1}{16\pi G}\left(\frac{D-2}{2}\right)\Omega^{(D-2)/2}(r_+)r_+^{D-2}f'(r_+), \tag{3.85}$$

which is nothing but twice the product of the temperature and the entropy of the black hole. As we see, this calculation only amounted to the confirmation that $I_\infty = I_H = (D-2)TS$. While a Komar integral does not give us a proper Smarr relation in this case, we can still attempt to define the thermodynamic volume Θ from a proposed Smarr relation

$$(D-3)M = (D-2)TS - 2P\Theta. \tag{3.86}$$

Of course, this definition will only work if we can derive the extended first law of thermodynamics. We can use Hamiltonian perturbations once again; this is where our efforts to integrate the potential will pay off.

3.2.2 Thermodynamic Volume

Let us now use the method of Hamiltonian perturbations to derive the first law. Our starting point is equation (3.45), from which we can also derive the extended first law for hyperbolic hairy black holes. As before, we calculate:

$$\int_{\partial\Sigma_r} d\sigma_a \left(\frac{B^a}{16\pi G}\right) = -(D-2)\frac{r^{D-3}}{8\pi G}\left(r^2\frac{\delta\ell}{\ell^3} + \frac{\delta\mu}{r^{D-3}}\right), \tag{3.87}$$

$$2\delta\Lambda \int d\omega_{D-2}\int_{r_+}^{\infty}\sqrt{-g}\mathcal{V}(\phi) = \frac{(D-2)}{8\pi G}\frac{\delta\ell}{\ell^3}\left(r^{D-1} - (D-3)\mu + \frac{1}{4}\Omega^{(D-2)/2}r^{D-2}f'\bigg|_{r_+}\right). \tag{3.88}$$

Re-expressing the variations of the AdS scale ℓ in terms of the pressure, and regrouping in terms of the mass, temperature and entropy, we can express the first law as

$$dM = TdS - 1/2\left((D-3)M - (D-2)TS\right)dP, \tag{3.89}$$

which means that we can indeed define Θ from (3.86) and thus obtain the extended first law

$$dM = TdS + \Theta dP. \tag{3.90}$$

The good news is that we have a simple expression for the thermodynamic volume, and we have proven that the extended first law holds in full generality for hyperbolic hairy black holes. However, we have not been able to find an argument to relate the thermodynamic volume to the integral of the potential as we did in the planar case. The reason is that the differential equations are now more complicated, and obtaining even one of the ansatz functions as an exact integral as we did before seems to be out of the question. The example that we will study next appears to suggest a relation does hold between the thermodynamic volume and the potential integral.

3.2.3 The MTZ black hole

The prototypical hairy black hole example in asymptotically AdS spacetimes is the Martinez-Troncoso-Zanelli (MTZ) solution [83], which has a hyperbolic horizon. The extended thermodynamics of this solution has not been studied in the literature (see, however [84] for the thermodynamic analysis of a simpler solution in the case of conformal coupling in de Sitter space). The analysis above tells us that establishing the first law should be a straightforward matter. Indeed, consider the MTZ metric in AdS:

$$ds^2 = \frac{r(r+2\mu)}{(r+\mu)^2}\left[-\left(\frac{r^2}{l^2} - \left(1+\frac{\mu}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{l^2} - \left(1+\frac{\mu}{r}\right)^2\right)^{-1} dr^2 + r^2 d\omega^2\right]. \quad (3.91)$$

This metric turns out to be an exact solution to the Einstein field equations with potential and scalar field solution

$$V(\phi) = \frac{\Lambda}{8\pi G} + \frac{\Lambda}{4\pi G}\sinh^2\left(\sqrt{\frac{4\pi G}{3}}\phi\right), \quad \phi = \sqrt{\frac{3}{4\pi G}}\text{Arctanh}\left(\frac{\mu}{r+\mu}\right). \quad (3.92)$$

We can express the mass, temperature and entropy in terms of r_+, the largest positive zero of the blackening factor. We obtain

$$M = r_+\left(\frac{r_+}{\ell} - 1\right)\frac{\omega}{4\pi G}, \quad T = \frac{1}{2\pi\ell}\left(\frac{2r_+}{\ell} - 1\right), \quad S = (2r_+ - \ell)\frac{\ell\omega}{4G}. \quad (3.93)$$

Using the definition (3.86) for the thermodynamic volume, we obtain

$$\Theta = \frac{1}{3}l\omega\left(l^2 - 3lr_+ + 3r_+^2\right). \quad (3.94)$$

Then it is straightforward to verify the first law of thermodynamics $dM = TdS + \Theta dP$. Perhaps a bit more surprising is the fact that we still have a relation between the geometric and the thermodynamic volume, which begs the question on how to prove that this a general fact for spherical and hyperbolic black holes. Indeed, we observe that

$$\Theta = 16\pi G \int d\sigma \int_0^{r_+} \sqrt{-g}\mathcal{V}(\phi). \tag{3.95}$$

It could be, as in the planar case, that this equality holds in the general case up to a proportionality constant which depends on the infrared behaviour of the metric, but this remains to be investigated further.

3.3 Holographic thermodynamics of hairy black branes

In the final part of this chapter, we return to studying black branes and explore their thermodynamics when the asymptotic decay of the scalar field is slower and is allowed to vary. To do this, we need to resort to holographic methods since the Hamiltonian mass is ill-defined for self-gravitating scalar spacetimes. The AdS-CFT correspondence will enable us to assign an energy content to the hairy planar black hole spacetime that accounts for the scalar. However, this needs more care than the calculation of naive Hamiltonian charges.

3.3.1 Asymptotic behaviour and holographic charges

Since the on-shell action diverges, new terms have to be added to regularize and renormalize it. There exists a well-defined procedure to do this, known as holographic renormalization [74]. Since we are only interested in the particular case of the scalar fields, we will not derive these terms but instead cite the result [85], namely that the counterterms needed in 5 spacetime dimensions are

$$S_{ct} = -\frac{1}{16\pi G} \int d^4x\sqrt{h} \left(\frac{6}{\ell} + \frac{\ell}{2}\mathcal{R} + \frac{4-\Delta}{2}\phi^2 \right). \tag{3.96}$$

As mentioned before, in Poincaré coordinates the scalar field has asymptotic behaviour

$$\phi \sim \frac{\Phi_+}{r^\Delta} + \frac{\Phi_0}{r^{4-\Delta}} + \dots \tag{3.97}$$

For our purposes, it will be more convenient to write the metric and the above asymptotic behaviour in coordinates that go to zero as we approach the conformal boundary. The change of coordinates is given by $z = \ell^2/r$, so that close to the AdS boundary the scalar field behaves as

$$\phi \sim \phi_{(+)} z^{\Delta} + \phi_{(0)} z^{4-\Delta} + \dots \tag{3.98}$$

Since we are not going to subtract the scalar field background when calculating the mass using the holographic method, we need to know the asymptotic behaviour of the metric in more detail; that is, including the asymptotics that will not be subtracted away. Let us consider the ansatz (3.2). Then we can derive the asymptotic behaviour for each of the ansatz functions

$$\begin{aligned} &A \sim O(1/r^5), \\ &\sigma \sim 1 + \frac{b}{r^{8-2\Delta}} + \frac{c}{r^4} + \dots, \\ &h \sim 1 + \frac{\mu}{r^4} + \dots, \end{aligned} \tag{3.99}$$

where

$$b = -\frac{4\pi G}{3}(4-\Delta)\Phi_0^2, \tag{3.100}$$

$$c = -\frac{4\pi G}{3}(4-\Delta)\Delta\Phi_0\Phi_+. \tag{3.101}$$

A well-known result from the AdS-CFT correspondence is that an AdS spacetime with a scalar field is dual to a CFT perturbed by an operator $\mathcal{O}$ of scaling dimension Δ coupled to a current j [29, 31]. Namely, the CFT lagrangian is deformed according to

$$\mathcal{L} = \mathcal{L}_{\text{CFT}} + j\mathcal{O}. \tag{3.102}$$

The expectation value of $\mathcal{O}$ and the coupling j are in correspondence to the coefficients $\phi_{(0)}$ and $\phi_{(+)}$ according to

$$\langle\mathcal{O}\rangle = \frac{(\Delta-2)\ell^6}{8\pi G}\phi_{(+)}v = \frac{(\Delta-2)\ell^{6-2\Delta}}{8\pi G}\Phi_+ v, \tag{3.103}$$

$$j = \phi_{(0)} = \Phi_0 \ell^{2\Delta-8}. \tag{3.104}$$

Our aim is then to obtain a first law of thermodynamics allowing for the variations of Φ_+ and Φ_0 (or equivalently, $\langle\mathcal{O}\rangle$ and j). This scenario has been studied for spherical black holes in [68], but the extended thermodynamics with variable Λ has not been derived. This computation is our objective here.

We thus give up the Hamiltonian method to derive the masses and tensions and use the holographic stress tensor instead, as introduced in section 2.1.3. To calculate it, it is useful to expand the metric in Fefferman-Graham coordinates (which can be done in general), which are of the form [86]

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + g_{\mu\nu}(z)dx^i dx^j\right). \tag{3.105}$$

where the metric $g_{\mu\nu}$ can be expanded near the boundary $z = 0$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_k g^{(k)}_{\mu\nu} z^k. \tag{3.106}$$

Then one can show that the expectation value of the dual stress energy tensor can be computed via [72]

$$\langle T^{\mu\nu}\rangle = \lim_{z\to 0} \frac{2}{\sqrt{-g^{(0)}}} \frac{\delta S_{reg}}{\delta g^{(0)}_{\mu\nu}} \tag{3.107}$$

which for our case yields, for a boundary metric $h_{\mu\nu}$ [87]:

$$T_{\mu\nu} = \frac{1}{8\pi G}\left(K_{\mu\nu} - h_{\mu\nu}K - \frac{3}{\ell}h_{\mu\nu} + \frac{\ell}{2}G_{\mu\nu}(h) - h_{\mu\nu}\frac{(\Delta-4)}{\ell^3}\phi^2\right). \tag{3.108}$$

Aided with this expression, it is then possible to calculate the holographic mass $\mathcal{M}$, integrating the T_{00} component; and the holographic tensions $\mathcal{T}_i$, integrating the T_{ii} components ($T_{\mu\nu}$ is diagonal). We obtain

$$\mathcal{M} = \frac{v}{16\pi G}\left(\frac{3\mu}{\ell^2} - \Phi_0\Phi_+\frac{(\Delta-2)(4-\Delta)}{2\ell^2}\right) = M - \frac{\Delta_-}{4}\langle\mathcal{O}\rangle j \tag{3.109}$$

$$\mathcal{T}_i = \frac{v}{16\pi G}\left(-\frac{\mu}{L^i\ell^2} - \Phi_0\Phi_+\frac{(\Delta-2)(4-\Delta)}{2L^i\ell^2}\right) = \tau_i - \frac{\Delta_-}{4L^i}\langle\mathcal{O}\rangle j \tag{3.110}$$

where we have defined $\Delta_- := 4 - \Delta$ and M and τ_i represent the mass and the tension due to the presence of the planar black hole alone. Thus we have that

$$\mathcal{M} + \sum \mathcal{T}_i L^i = -\Delta_-\langle\mathcal{O}\rangle j = \langle T^\mu_\mu\rangle. \tag{3.111}$$

The trace of the holographic stress-energy tensor vanished for the hairless planar black hole case, but this fact is no longer true now that the scalar field is included. Conformality is broken due to the presence of the scalar field, which corresponds to the deformation (3.102) of the boundary theory.

3.3.2 Smarr relation and extended thermodynamics

We are now in a good place to establish the complete first law of thermodynamics and a Smarr relation, which include variations of the asymptotic behaviour of the scalar field. We have learned that $4M = 3TS$ in 5 dimensions, so for the holographic mass, we have

$$4\mathcal{M} = 3TS - \Delta_- \langle \mathcal{O} \rangle j. \tag{3.112}$$

This is not the Smarr relation that we need, since it does not include the Λ and L^i terms. We can obtain it, as before, by subtracting $2\mathcal{M}$ on each side and using (3.111):

$$2\mathcal{M} = 3TS + \Delta_- \langle \mathcal{O} \rangle j + \sum \mathcal{T}_i L^i - 2\Lambda\Psi, \tag{3.113}$$

where the potential Ψ reads

$$\Psi = \frac{\mathcal{M}}{2\Lambda} + \frac{\Delta_- \langle \mathcal{O} \rangle j}{2\Lambda}. \tag{3.114}$$

This means that we expect the thermodynamic volume to be

$$\Theta = -\frac{4\pi G}{\Lambda} \left(\mathcal{M} + \Delta_- \langle \mathcal{O} \rangle j \right). \tag{3.115}$$

Once again, we need to use Hamiltonian perturbations[3] to verify that this is indeed the case. Here, we have to consider the Hamiltonian of the scalar field since we are allowing for a variation $\delta\phi$ at infinity. Except for this, the method remains similar in principle; the only change is that B^c in (2.77) acquires an extra term B^c_{Matter} given by [42]

$$B^c_{\text{Matter}} = \left(D^a \phi + \frac{1}{\sqrt{|h|}} F^a p_\phi \right) \delta\phi. \tag{3.116}$$

Using this on (3.2), and taking into account that the variation of the function A is zero to order $1/r^5$, we derive

$$\begin{aligned} \int_{\partial\Sigma_r} d\sigma_a \left(\frac{B^a}{16\pi G} \right) = \frac{r^4}{16\pi G \ell^6 \sigma^2} \Big(& -6\ell h \delta\sigma - 6\delta\ell h\sigma + 3\ell\sigma\delta h + 3r\delta\ell\sigma h' \\ & + \ell r h \sigma \phi' \delta\phi - r\ell\sigma h' \sum \frac{\delta L^i}{L^i} \Big). \end{aligned} \tag{3.117}$$

[3]We emphasize here that the Hamiltonian method cannot be used to find the mass anymore, because it does not provide a proper regularization. However, it can be used to find *differences* of mass between a spacetime and a background.

This integral at infinity will have terms of order $(4-2\Delta_-)$, present in the first and last term of the right hand side, but they cancel when added together. As before, further divergent terms of order r^4 are cancelled by the integral of the potential from r_+ to infinity. There are also some clearly divergent terms in the $-6\delta\ell h\sigma$ term, which do not seem to be cancelled by anything. However, since the field now has two asymptotic modes, there are corrections to the integral of the potential:

$$2\delta\Lambda \int d^3x \int_{r_+}^{\infty} dr\mathcal{V}(\phi) = 3\frac{r^4\delta\ell}{8\pi G\ell^3}\left(1 - \frac{b}{r^{2\Delta_-}} - \frac{\mu + c}{r^4} + \dots\right) v, \tag{3.118}$$

which exactly cancel such divergences.

Since we do not have to worry about divergences anymore, we can then focus on keeping the terms of order 1 in equations (3.117) and (3.118). The sum of all such terms reads

$$\begin{aligned}\int_{\partial\Sigma_r} d\sigma_a \left(\frac{B^a}{16\pi G}\right) + 2\delta\Lambda \int d^3x \int_{r_+}^{\infty} dr\mathcal{V}(\phi) = &-\delta\mathcal{M} - \frac{M}{\ell}\delta\ell + \frac{5\Phi_0\Phi_+(\Delta-2)\Delta_-}{2\ell^6}\delta\ell \\ &- \frac{2(\Delta-2)}{\ell^2}\Phi_+\delta\Phi_0 + \sum \mathcal{T}_i\delta L^i.\end{aligned} \tag{3.119}$$

Identifying this expression with $-T\delta S$ yields the first law. It has a much nicer expression in terms of $\langle\mathcal{O}\rangle$ and j, which is in fact what we expected from the Smarr relation:

$$d\mathcal{M} = TdS + \sum \mathcal{T}_i dL^i - \langle\mathcal{O}\rangle\, dj + \Theta dP. \tag{3.120}$$

This first law that we have derived, which holds for non-integer values of Δ, and only in 5 dimensions, seems to hold in much more generality. In our setup, the theory is perturbed away from a CFT by deforming it linearly. A similar expression has been derived for spacetimes that are not asymptotically AdS; indeed, in [42], the authors show that for asymptotically dilaton spacetimes, (3.120) holds with thermodynamic volume

$$\Theta = -\frac{4\pi G}{\Lambda}\left(\mathcal{M} - \langle T^{\mu}_{\mu}\rangle\right), \tag{3.121}$$

which is precisely our result. This coincidence suggests that this expression of the thermodynamic volume is much more general. Regrettably, there are no explicit finite temperature examples in the literature that would allow us to check this law explicitly. Finding one seems to be rather difficult, so we will have to trust that our methods have given us the correct expression. Finding explicit solutions and proving that this law holds in more generality are left for further investigation.

Chapter 4

Thermodynamics of Lorentzian Taub-NUT spacetimes

This chapter is dedicated to studying the thermodynamics of a rather intricate family of spacetimes known as Taub-NUT spacetimes [88,89]. These are axially symmetric solutions to the Einstein field equations, with or without a cosmological constant. These spacetimes were considered unphysical for a long time since they are not geodesically complete and allow for closed timelike curves everywhere. However, the authors of [76] argue that these spacetimes are geodesically complete and that the absence of causality may not be robust under perturbations. Therefore, Taub-NUTs are not as pathological as previously thought. This discovery started the program of "rehabilitating" Taub-NUT spacetimes.

This redeeming result was unknown for decades, so most of the community's interest was focused on Euclidean Taub-NUTs. The principal reason for studying them lies in the AdS-CFT correspondence. Taub-NUT solutions with a negative cosmological constant are not asymptotically AdS spacetimes; instead, they define their own type of asymptotics where the spacetime boundary is not a sphere but instead has a more complex topology. It is an exciting question to probe the limits of the correspondence when the topology of the boundary changes; since quantum field theories are easier to study in Euclidean spacetimes, it is a good idea to do this using Euclidean Taub-NUTs as a prototypical example.

Since the start of the rehabilitation program, the interest in Lorentzian Taub-NUTs has grown. Riding on this wave, we contributed to the study of Lorentzian Taub-NUTs by studying their thermodynamics, which turns out to be somewhat different from their Euclidean counterparts. While these spacetimes are not technically considered black holes (since they are not asymptotically flat or AdS), there is a horizon present and, as argued in [90], the association of entropy with a horizon is quite general. As such, we can establish the first law of thermodynamics that they obey.

The first section of this chapter is intended as a succinct overview of the study of Taub-NUTs before the first step towards their rehabilitation, following references [48, 91–93] closely. We also summarize the argument that removes the pathologies of Lorentzian Taub-NUTs as presented in [76]. The bulk of the chapter is a compilation of a series of papers studying the first law for Lorentzian Taub-NUTs [2–6, 94]. We close by analyzing the physical meaning of the newly proposed thermodynamic quantities in the light of our investigations and further contributions from other authors.

4.1 Taub-NUT spacetimes

We now introduce some general facts about Taub-NUT spacetimes and explain in more depth why they were considered unphysical for a long time. We quickly review the results for Euclidean Taub-NUTs, where the thermodynamic quantities have unexpected forms, thus making Taub-NUTs live up to their fame as a "counterexample to almost anything" [95]. Then, we review the proof introduced in [76] that removes the most unphysical features of the Lorentzian Taub-NUTs.

4.1.1 Generalities

A Taub-NUT spacetime is a vacuum axisymmetric solution to Einstein's field equations in four dimensions[1] with or without a cosmological constant [97]; like the Kerr solution, its line element is of the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{4.1}$$

When the $g_{t\phi}$ component is asymptotically of the form

$$g_{t\phi} \sim 2n(\cos\theta + \sigma), \tag{4.2}$$

where n is some real parameter, we say that the metric is asymptotically Taub-NUT (AdS). There exist many solutions in this family and we will deal with plenty of them in further

[1]There are generalizations of Taub-NUT spacetimes to higher dimensions, but these will not concern us here [96].

sections. The simplest of such solutions is the Taub-NUT AdS metric, given by

$$ds^2 = -f[dt + 2n(\cos\theta + \sigma)d\phi]^2 + \frac{dr^2}{f} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.3}$$

where the function f is given by

$$f(r) = \frac{r^2 - 2mr - n^2}{r^2 + n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{\ell^2\left(r^2 + n^2\right)}, \tag{4.4}$$

n is a real parameter, and $|\sigma| < 1$. The flat Taub-NUT can be found by taking the limit $\ell \to \infty$. More commonly studied in the literature is the Euclidean Taub-NUT solution,

The Taub-NUT spacetime without a cosmological constant is not asymptotically flat; instead, it is asymptotically locally flat: the topology of the boundary is not that of a sphere. This fact allows us to bypass the no-hair theorem, which is why we can obtain axisymmetric and stationary solutions beyond Kerr-Newman. Similarly, the AdS Taub-NUT is asymptotically locally AdS, which for our purposes means that asymptotically

$$R_{\mu\nu\sigma\rho} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right), \tag{4.5}$$

or equivalently, that our definition in section 2.1.2 applies. However, the boundary has an exotic topology. What is such a topology? In general, it can be quite convoluted. In the Euclidean version of metric (4.4) with periodic time coordinate, obtained by Wick rotating $t \to it$ and $n \to in$, it is a "squashed" sphere: an S_1 fibre bundle over S_2 with first Chern number proportional to n. This weird topology need not concern us much: the fact that the Taub-NUT solution is asymptotically locally flat allows for the use of standard methods to calculate its mass. For example, nothing prevents the conformal method from giving us the correct conserved charges, so this will be our preferred prescription. Calculating with the Hamiltonian method needs more care since one does not get a finite Hamiltonian using the flat or AdS spacetimes as backgrounds [47, 98].

Physically, the NUT charge n is interpreted as a magnetic charge analog to the mass, so that the Taub-NUT is thought to be a gravitational dyon [99]. To make sense of this claim, we should recall the basics of Dirac's argument for the quantization of the electric charge [100]. The idea is that for the concept of a magnetic monopole g with a magnetic field given by

$$\vec{B} = \frac{g\hat{r}}{r^3} \tag{4.6}$$

to be compatible with the existence of a magnetic vector potential $\vec{A}$, we need to impose that the electric and magnetic charges e and g are related by

$$eg = 2\pi\hbar n, \tag{4.7}$$

where n is an integer. This constraint ensures that the wave-function is single-valued. While this quantization of charge saves the wave-function, it does not explain how we can still have $\vec{B} = \nabla \times \vec{A}$. Of course, we can have a potential $\vec{A}$ that is singular at the origin since the magnetic monopole is located there. If we go further and allow for it to be singular at the z axis, namely if we allow for a Dirac string, we can write

$$\vec{A}^{+} = \frac{g}{4\pi r} \frac{1 - \cos\theta}{\sin\theta} \hat{\phi}, \tag{4.8}$$

which gives (4.6) after taking the curl. We observe that this gauge field is singular on the negative z-axis. Therefore, we need to patch up this axis. To do this, we use

$$\vec{A}^{-} = -\frac{g}{4\pi r} \frac{1 + \cos\theta}{\sin\theta} \hat{\phi}, \tag{4.9}$$

which is in turn singular on the positive z-axis. Both gauge fields thus defined are related by a gauge transformation $\vec{A}^{+} = \vec{A}^{-} + \nabla\lambda$, where

$$\lambda = \frac{g\phi}{2\pi} \tag{4.10}$$

is a multi-valued function. This strange feature is deemed acceptable since the only physical quantity is the wave-function, which is still single-valued as long as (4.7) holds. In this way, we have removed the Dirac string, simply regarding it as a coordinate singularity, at the cost of quantizing the electric charge.

In analogy with the Dirac strings, Taub-NUT spacetimes feature singularities known as Misner strings. Indeed, we observe that the metric does not have an inverse at $\cos\theta = \pm 1$; these wire singularities can also be removed at a certain cost [97]. To do this, we change coordinates on the coordinate chart $\theta \geq \pi/2$

$$t^{+} = t + 2n(1 + \sigma)\phi \tag{4.11}$$

to obtain the metric

$$ds_{+}^{2} = -f[dt^{+} + 2n(\cos\theta - 1)d\phi]^{2} + \frac{dr^{2}}{f} + \left(r^{2} + n^{2}\right)\left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right). \tag{4.12}$$

Similarly, on the patch $\theta \leq \pi/2$, we can do the change

$$t^{-} = t - 2n(1 - \sigma)\phi \tag{4.13}$$

and express locally

$$ds_+^2 = -f[dt^- + 2n(\cos\theta + 1)d\phi]^2 + \frac{dr^2}{f} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.14}$$

On the region common to both patches, we have

$$t^+ = t^- + 4n\phi, \tag{4.15}$$

and since the ϕ coordinate has a period 2π, then the time coordinate must have a period $8\pi n$. Consequently, the price that we have to pay for the Misner strings to disappear is that we have to make the time coordinate compact, which is unphysical. However, these seem to be our only choices: either we have exposed wire singularities [101] which, to every physicist's dismay, are surrounded by closed timelike curves; or we identify the time coordinate periodically and obtain a regular Euclidean spacetime [93]. For several decades, the latter was the preferred choice as long as we used Euclidean time, with the hope of generalizing the AdS-CFT correspondence to gain insights on field theories defined on the topologically exotic boundary.

4.1.2 Euclidean Taub-NUTs and their thermodynamics

The thermodynamics of Euclidean Taub-NUTs has been a subject of much interest throughout the last half-century. With the introduction of AdS Taub-NUTs in [98], we also expect to obtain an extended first law. The simplest Euclidean Taub-NUT spacetime is obtained via the substitutions $t \to it$ and $n \to in$ on the metric (4.4), thus obtaining

$$ds^2 = f[dt + 2n(\cos\theta + \sigma)d\phi]^2 + \frac{dr^2}{f} + \left(r^2 - n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.16}$$

with

$$f(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2} - \frac{3n^4 + 6n^2r^2 - r^4}{\ell^2\left(r^2 - n^2\right)}. \tag{4.17}$$

In the case of Euclidean Taub-NUTs, we can distinguish between the Nut and Bolt spacetimes. In the former case, the function f vanishes at $r_+ = n$, which implies that the boundary fibre degenerates on a zero-dimensional set of points, called the nut. The mass, which is given in general by the parameter m, is in this case given by [47]

$$m = n - \frac{4n^3}{\ell^2}. \tag{4.18}$$

In the Bolt spacetime [102], $r_+ > n$, so that the fibre degenerates on a two-dimensional set of points. It is then possible to express the mass in terms of r_+ by solving $f(r_+) = 0$. Since the thermodynamic quantities are different in each of these cases, we will study them separately.

We finally recall that the time coordinate has a period $8\pi n$, so we identify the temperature $\beta = 1/T = 8\pi n$ for both Nut and Bolt spacetimes. However, this temperature is not the one obtained by using equation (1.18). To make this consistent, we need to identify both temperatures. Namely, we must have

$$\frac{f'(r_+)}{4\pi} = 8\pi n. \tag{4.19}$$

The consequence is that the parameters n and r_+ are no longer independent. This implicit relation is a central assumption that we will make in our derivation of the laws of thermodynamics of Euclidean Taub-NUTs.

Nut spacetime

The easiest way to derive the thermodynamic quantities for these spacetimes, which we will also use for the Lorentzian case, is to find the free energy by finding the on-shell action, as described in section 2.1.3. However, this begs the question as to whether this procedure is valid when the spacetime is not asymptotically AdS. Instead of diving into string theory to justify this, we will take a more straightforward approach: if the procedure gives us consistent thermodynamics, it serves as evidence that the correspondence might also work in greater generality.

Using the counterterms introduced in (2.21), one obtains [47, 48]:

$$F = \frac{4\pi n}{G}\left(1 - \frac{2n^2}{\ell^2}\right). \tag{4.20}$$

Recalling that, in thermodynamics, the internal energy can be calculating using

$$U = \frac{\partial \log Z}{\partial \beta}, \tag{4.21}$$

then we can find the entropy using

$$S = \beta\frac{\partial \log Z}{\partial \beta} - \log Z \tag{4.22}$$

since $\log Z = \beta F$ and $U = F + TS$. We are now in a position to calculate the entropy, which reads

$$S = \frac{4\pi n^2}{G}\left(1 - \frac{6n^2}{\ell^2}\right). \tag{4.23}$$

Let us note that since $r_+ = n$, the entropy depends only on n and ℓ. Since r_+ and n are not independent, we expect to have an extended first law of the form

$$dU = TdS + VdP \tag{4.24}$$

which can be verified simply by taking partial derivatives with respect to n and ℓ. The volume is [91]

$$V = -\frac{8\pi n^3}{3}, \tag{4.25}$$

which is derived by finding the 3-dimensional volume between $r = 0$ and $r = n$. Surprisingly, this volume is negative.

These thermodynamic quantities have some rather strange features. For example, the area that we would expect for the zero-dimensional nut is zero. If there are no horizons to assign an entropy to, where are we getting this total entropy from? The common lore [103] is that the Misner strings themselves either hide or remove some degrees of freedom, increasing or decreasing the entropy. Therefore, the entropy that enters the first law is the total entropy due to the nut and the Misner strings [48, 104]. Something similar happens in the case of the Bolts, which are our next object of study.

Bolt spacetimes

We proceed by calculating the free energy once more, which in this case is given by [47]

$$F = \frac{1}{2G\ell^2}\left(\ell^2 m + 3n^2 r_+ - r_+^3\right), \tag{4.26}$$

in terms of the position $r = r_+$ of the horizon or "bolt". Here, we insist on the fact that r_+ and n are not independent; the relation can be found by using $f'(r_+) = 4\pi/\beta$. This expression is not very illuminating in the case of AdS Taub-NUTs, although in the asymptotically flat case, it is simply $r_+ = 2n$. The entropy is calculated with similar methods:

$$S = \frac{4\pi n}{G}\left(m - 3\frac{n^2 r_+}{\ell^2} + \frac{r_+^3}{\ell^2}\right); \tag{4.27}$$

the volume can be calculated by identifying the mass m with the enthalpy $H = U + PV$, where U is calculated using $U = F + TS$, thus giving [91]

$$V = \frac{4\pi}{3}\left(r_+^3 - 3n^2 r_+\right). \tag{4.28}$$

These quantities also satisfy a first law $dU = TdS - PdV$, and again, we see that the volume can be negative. Moreover, the entropy is not area but a sum of the contributions of the bolt and the Misner strings. An argument for why this is not an issue is that this total entropy can be interpreted as a conserved charge [105], giving it the same status as the mass or the angular momentum. This insight seems to be suggestive enough to give up the Bekenstein expression for the entropy. However, as we will see in further sections, this interpretation is dependent on the choice of thermodynamic ensemble.

4.1.3 Rehabilitating Lorentzian Taub-NUTs

The Euclidean Taub-NUT is not free of problems. While it has the advantage that it can be maximally extended, such an extension is not Hausdorff [101]. Moreover, since the time coordinate is periodically identified, we have closed timelike curves everywhere. Could we give up the time identification and accept that we have a wire singularity and interpret it as a matter source of angular momentum, as suggested in [106, 107]? Should we accept that closed timelike curves are surrounding the now observable Misner string?

We have at no point proven that the Misner strings are gravitational singularities. The Ricci and Kretschmann scalars do not blow up there, so is it possible that they are just coordinate singularities? To answer this question, the authors of [76, 108] studied the geodesics that go through the string and found that they can go right through them, which is impossible if the Misner strings were unremovable singularities. One can find a similar fix for the closed timelike curves around the Misner strings: we can prove that they cannot be geodesic. Therefore, to break causality, one would need to add extra energy to the spacetime to accelerate the probe, possibly preserving causality by perturbing the spacetime[2]. We now present a summary of the proof of these two remarkable results. We warn the reader, however, that the evidence presented here only works for asymptotically flat Taub-NUTs. The rehabilitation of more general asymptotically Taub-NUT spacetimes is still the subject of current research. We offer, however, a sketch for the proof of geodesic completeness for flat Kerr-NUT spacetimes in Appendix D.

[2]This statement, however, has not been proven yet.

Geodesic completeness of the Lorentzian Taub-NUT

In this subsection and the next, we follow [76] closely but present a simplified version of some arguments. Let us consider a geodesic with tangent vector field u^μ. To find the geodesic equations, the most convenient method, in this case, is to use the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + g^{\mu\nu}\partial_\mu S \partial_\nu S = 0 \tag{4.29}$$

and identify $\partial_\mu S = u_\mu$. Since we have timelike and angular Killing vectors ∂_t and ∂_ϕ, we can write the following ansatz for Hamilton's principal function

$$S = \kappa\tau - \mathcal{E}t + R'(r) + \Theta'(\xi) + J_z\phi, \tag{4.30}$$

where $\xi = \cos\theta$; $\mathcal{E}$ represent the particle's energy and J_z stands for the z component of the angular momentum, which are preserved under geodesic motion. Plugging this ansatz into (4.29) yields the equation relating the ansatz functions R and Θ

$$(n^2+r^2)fP'^2(r) - \kappa(n^2+r^2) - \mathcal{E}^2\frac{n^2+r^2}{f} = \Theta'^2(\xi)(1-\xi^2) + \frac{(J_z+2n\mathcal{E}\xi)^2}{1-\xi^2}. \tag{4.31}$$

Since the right-hand side is only dependent on r and the left-hand side depends only on ξ, both sides must be equal to a (negative) constant $-l^2$, which is known as Carter's constant [109]. Changing the parameter to the Mino time [110] defined via $\lambda = (r^2+n^2)\tau$, the equation for ξ can be derived by using $\dot{\xi} = g^{\xi\xi}\partial_\xi S$, thus obtaining

$$\left(\frac{d\xi}{d\lambda}\right)^2 = l^2 - l^2\xi^2 - (J_z+2n\mathcal{E}\xi)^2. \tag{4.32}$$

We can also find the equation for the ϕ coordinate of the geodesic by using the fact that the quantity $\xi^\mu u_\mu$ remains constant on the geodesic when ξ^μ is a Killing vector. For $\xi^\mu = \delta^\mu_\phi$, this consant is equal to J_z; we get the equation

$$\frac{d\phi}{d\lambda} = \frac{1}{2}\left[\frac{J_z-2n\mathcal{E}}{1-\xi} + \frac{J_z+2n\mathcal{E}}{1+\xi}\right]. \tag{4.33}$$

Let us examine the behaviour of geodesics. Equation (4.32) has real solutions only if the discriminant of the right-hand side is non-negative, which means that we must have

$$J_z^2 \leq 4n^2\mathcal{E}^2 + \ell^2. \tag{4.34}$$

We would like to focus on the geodesics that intersect the Misner string. Here, $\xi^2 = 1$ which also must be an extremum of ξ. Evaluating the left hand side of (4.32) at $\xi = \pm 1$ and setting it equal to zero, we get that for such geodesics

$$J_z = \pm 2n\mathcal{E}, \tag{4.35}$$

where the plus sign is for the southern string and the minus sign is for the north string. A geodesic can intersect both strings if $\mathcal{E} = 0$, but then they turn out to be spacelike. Since both cases are symmetric, let us simply study the geodesics that cross the northern string. In this case we can solve for ξ and ϕ. With $J^2 := 4n^2\mathcal{E}^2 + l^2$ and $J_\perp^2 = J^2 - J_z^2$, we have

$$\xi(\lambda) = \frac{1}{J^2}\left(-l^2 + 4n^2\mathcal{E}^2 \cos J\lambda\right), \tag{4.36}$$

$$\cos(\phi(\lambda) - \phi_0) = \frac{J_\perp}{J_z}\tan\left(\frac{\theta}{2}\right). \tag{4.37}$$

These expressions already tell us that the Misner strings present no obstruction to geodesic motion. While it is true that ϕ may have jumps, this is not a problem since it is expected when we cross the Misner string, where the ϕ coordinate is not defined.

Absence of closed timelike geodesics

In order to argue that timelike geodesics are causal, we need to write the equation for the time coordinate, which comes from $\xi^\mu u_\mu = \mathcal{E}$ with Killing vector $\xi^\mu = \delta^\mu_t$. It reads

$$\mathcal{E} = (\dot{t} + (2n\cos\theta + \sigma)\dot{\phi})f. \tag{4.38}$$

By plugging in (4.33), we can separate t into a θ-dependent part t_θ and an r-dependent part t_r, which satisfy

$$\frac{dt_\theta}{d\lambda} = 4n^2\mathcal{E} + n\left[\frac{(\sigma+1)(J_z - 2n\mathcal{E})}{1-\cos\theta(\lambda)} + \frac{(\sigma-1)(J_z + 2n\mathcal{E})}{1+\cos\theta(\lambda)}\right], \tag{4.39}$$

$$\frac{dt_r}{d\lambda} = \mathcal{E}\frac{r^2+n^2}{f(r)}. \tag{4.40}$$

Equation (4.39) can be integrated exactly over a λ period $2\pi/J$ to yield

$$\Delta t_\theta = 2\pi n\left[\frac{4nE}{J} + (\sigma+1)\,\mathrm{sgn}\,(J_z - 2nE) + (\sigma-1)\,\mathrm{sgn}\,(J_z + 2nE)\right]. \tag{4.41}$$

We are not assuming that these cross the Misner strings so that no further simplification can be done. We want to find a lower bound for Δt_θ, so we should find the minimum negative value that the sign terms could achieve. When $|\sigma| \leq 1$, $(\sigma+1)$ is positive and $(\sigma-1)$ is negative. If the first sign is negative and the second one is positive, then we achieve the minimum value:

$$\inf_{|\sigma|\leq 1} \left\{(\sigma+1)\,\mathrm{sgn}\,(J_z - 2n\mathcal{E}) + (\sigma-1)\,\mathrm{sgn}\,(J_z + 2n\mathcal{E})\right\} = -2. \tag{4.42}$$

Therefore we can bound from below

$$\Delta t_\theta \geq 4\pi n \left(\frac{2n\mathcal{E}}{J} - 1\right). \tag{4.43}$$

The equation for the r coordinate obtained from the Hamilton-Jacobi formalism is

$$\left(\frac{dr}{d\lambda}\right)^2 + f\left[\frac{l^2}{r^2+n^2} - \kappa\right] = \mathcal{E}^2. \tag{4.44}$$

This can be used to deduce that, as long as $f > 0$ (outside the horizon, which is the region we are interested in), we have

$$\frac{\mathcal{E}^2(r^2+n^2)}{f} \geq l^2 - \kappa(r^2+n^2). \tag{4.45}$$

For positive energies, which we have when $f > 0$ by (4.38), we can bound over one period

$$\Delta t_r \geq \frac{2\pi}{\mathcal{E}J}\left[l^2 - \kappa n^2\right]. \tag{4.46}$$

Finally, we obtain

$$\Delta t = \Delta t_r + \Delta t_\theta \geq \frac{2\pi}{\mathcal{E}}\left[J - 2n\mathcal{E} - \kappa n^2/J\right], \tag{4.47}$$

which is positive for timelike geodesics ($\kappa = -1$), since $J^2 \geq 4n^2\mathcal{E}^2$. This means that we cannot have closed timelike geodesics for values of σ with modulus less than 1. Of course, this says nothing about other types of timelike curves; this is a rehabilitation step that remains to be taken. In principle, it could be possible that the energy needed to deviate from geodesic motion backreacts on the metric and renders it non-pathological. For now, we can at least rest assured that it is not entirely necessary to remove the Misner strings by periodically identifying the time coordinate. It would seem that astrophysicists' hopes to detect signatures of these strings are not completely unjustified [111, 112].

4.2 The first law of flat Taub-NUTs

We now proceed to study the first law of thermodynamics for Lorentzian Taub-NUTs. It will be different from the laws of Euclidean Taub-NUTs since we are now allowed to keep the Misner strings. Compactification of the time coordinate is unnecessary, so the temperature is not fixed by the constraint $\beta = 8\pi n$, which means that the NUT charge n and the position of the outer horizon r_+ are no longer dependent on each other.

We are looking for a full-cohomogeneity first law, which means that all the independent parameters can be varied. In the Taub-NUT-AdS solution, the independent parameters are r_+, n and ℓ, so we expect to get a law of the form

$$dM = TdS + \psi dN + VdP; \tag{4.48}$$

ψ and N are new variables that account for the NUT charge variation. In this section, our focus is on the asymptotically flat Taub-NUTs. This serves three purposes. First, this serves as a gentle introduction to the newly introduced charges since the first law and the expressions are simpler. Second, and more importantly, while we have been able to derive the first law for the Kerr-NUT solution (Taub-NUT with rotation) in the asymptotically flat case, so far it has proven rather difficult to establish a first cohomogeneity first law for Kerr-NUT-AdS. We do hope to derive this result in further work. Thirdly, the Komar integral methods are robust in the case of flat Taub-NUTs. However, this is not the situation for the AdS case. While we can modify the Komar integral accordingly, we will have to calculate the mass using the conformal or holographic methods, and sometimes rely solely on a free energy calculation.

4.2.1 Misner charge and Misner potential

Let us consider the metric (4.4) with non-trivial σ. We rewrite it in terms of the dimensionful parameter $s := n\sigma$, so that it reads

$$ds^2 = -f[dt + (2n\cos\theta + 2s)d\phi]^2 + \frac{dr^2}{f} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.49}$$

For simplicity, we will now start using natural units where $G = 1$. What is the meaning of this parameter s? To interpret it, we notice that if we change coordinates via

$$t \to t - 2s\phi \tag{4.50}$$

which corresponds to inserting a Misner string on the entire spacetime, we recover the Taub-NUT metric for $\sigma = 0$. This parameter s affects both strings and could be thus interpreted as an overall string strength. Given that it also has a physical interpretation, we will also vary it to get the first law. As argued in the previous section, to avoid pathologies, we must have $|s| \leq n$, so we will assume this bound henceforth.

To calculate the quantities involved in the first law, we need the Killing vector that generates the Taub-NUT horizon, which is $\xi = \partial_t$. We can also associate with this horizon a temperature and entropy by computing with the surface gravity and area. In terms of the locus of the horizon $r = r_+$,

$$S = \pi(r_+^2 + n^2), \quad T = \frac{1}{4\pi r_+}. \tag{4.51}$$

The mass M can be calculated using, most conveniently, using the Komar integral (2.5). We can verify this result using the conformal method for the AdS version of the Taub-NUT, and then taking the limit $\Lambda \to 0$. As expected, $M = m$. The Misner strings themselves also happen to be Killing horizons, generated by the Killing vector field

$$\xi_\pm = \partial_t - \frac{1}{2(s \pm n)}\partial_\phi, \tag{4.52}$$

where the plus sign represents the northern string and the minus sign the southern one. Based on this fact, we might attempt to associate a temperature and entropy to the Misner string. However, we have no evidence that they indeed radiate, so calling a quantity a Misner string temperature could be misleading. But we can, in any case, define the Misner potential as the surface gravity

$$\psi_\pm = \frac{\kappa_\pm}{4\pi} = \frac{1}{8\pi(n \pm s)}. \tag{4.53}$$

Since we expect to obtain a first law of the form

$$dM = TdS + \psi_+ dN_+ + \psi_- dN_-, \tag{4.54}$$

we should try to obtain a Smarr relation that will allow us to compute the Misner charges $N_\pm$. Smarr relations can be obtained from a Komar integration. In this case we have $\Lambda = 0$, so we can write (2.12) in the language of differential forms as

$$0 = \int_{\partial\Sigma} *d\xi. \tag{4.55}$$

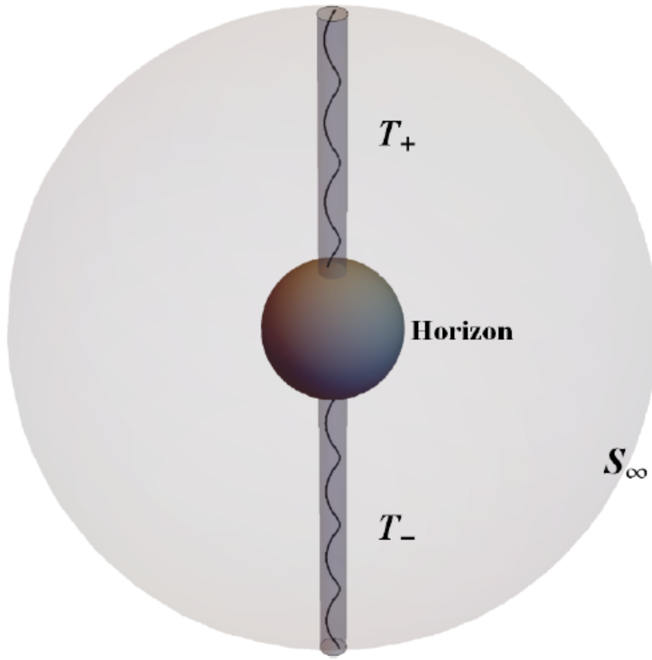


Figure 4.1: **Misner tubes:** The boundary of the Taub-NUT spacetime has, in addition to the horizon and S_∞ components, two tube-shaped boundary components surrounding each of the Misner strings. We define the Misner charges $N_\pm$ as the Komar integral component over such tubes.

How could we expect to obtain anything different from what we had without a NUT charge? The key insight is that the boundary of Σ no longer comprises a sphere at infinity and the horizon; it also includes some Misner tubes $T_\pm$, which are cylinders of infinitely small radius that enclose the Misner strings. We take these tubes to be surfaces of constant $\theta = \epsilon$ or $\pi - \epsilon$ on the northern and southern string. Thus, with signs indicating the orientation, the boundary $\partial\Sigma$ is split into

$$\partial\Sigma = T_+ + S_\infty - T_- - H. \tag{4.56}$$

With this decomposition, the Komar integral separates into four parts:

$$0 = -\frac{1}{8\pi}\left[\int_{S_\infty} \star d\xi - \int_H \star d\xi + \int_{T_+} \star d\xi - \int_{T_-} \star d\xi\right]. \tag{4.57}$$

The first term gives, in fact, the mass, and we have already seen that the second integral is $-2TS$. Therefore, it is natural to identify the integrals over the tubes as the products

$\psi_\pm N_\pm$, namely:

$$N_\pm \equiv \pm \frac{1}{16\pi\psi_\pm} \int_{T_\pm} \star d\xi. \tag{4.58}$$

Explicitly, the Misner charges $N\pm$ are

$$N_\pm = -\frac{2\pi n(s\pm n)^2}{r_+}. \tag{4.59}$$

Then the Smarr relation

$$M = 2\left(TS + \psi_+ N_+ + \psi_- N_-\right) \tag{4.60}$$

is satisfied by definition, and it is straightforward to verify that the first law (4.54) holds as well. Note that when $s = 0$, we have that $\psi_+ = \psi_-$, so with $N = N_+ + N_-$, the first law can be rewritten in the form (4.54), as originally stated in [94].

Even though we are not dealing with AdS Taub-NUTs, one can abuse the free energy prescription by taking the limit $\Lambda \to 0$ after calculating in AdS. By doing this, we obtain that the free energy $\mathcal{G}$ is

$$\mathcal{G} = \frac{M}{2}. \tag{4.61}$$

We can see from the Smarr relation that this energy is equal to

$$\mathcal{G} = M - TS - \psi_+ N_+ - \psi_- N_-, \tag{4.62}$$

which is not the Helmholtz free energy, but instead, it is the Gibbs free energy in the grand canonical ensemble. This is an essential fact that we will use to derive the first law for more general Taub-NUT spacetimes.

4.2.2 Rotating Taub-NUTs

One of such more general asymptotically Taub-NUT spacetimes is the Kerr-NUT spacetime, in which a rotation parameter is added. It represents a Kerr spacetime with an additional NUT charge. The flat Kerr-NUT metric is given by

$$\begin{aligned} ds^2 = &- \frac{\Delta}{\Sigma}\left[dt + \left(2n\cos\theta + 2s - a\sin^2\theta\right)d\phi\right]^2 + \frac{\Sigma}{\Delta}dr^2 \\ &+ \frac{\sin^2\theta}{\Sigma}\left[a dt - \left(r^2 + a^2 + n^2 - 2as\right)d\phi\right]^2 + \Sigma d\theta^2, \end{aligned} \tag{4.63}$$

where

$$\begin{aligned}\Delta &= r^2 + a^2 - 2mr - n^2, \text{ and}\\ \Sigma &= r^2 + (n + a\cos\theta)^2.\end{aligned} \tag{4.64}$$

The mass is again $M = m$, and the usual prescriptions give for the temperature, entropy, and angular velocity

$$S = \pi(r_+^2 + n^2), \quad T = \frac{n^2 + r_+^2 - a^2}{4\pi r_+ \left(a^2 - 2as + n^2 + r_+^2\right)}, \quad \Omega = \frac{a}{\left(a^2 - 2as + n^2 + r_+^2\right)}. \tag{4.65}$$

The conformal method also gives the total angular momentum

$$J = a(M - 3s). \tag{4.66}$$

However, as we will see, this is not the angular momentum that enters the first law.

We proceed to derive the Smarr relation as in the pure Taub-NUT case using the Komar integral. The Killing vector that generates the Taub-NUT horizon is $\xi = k + \Omega\eta$, where $k = \partial_t$ and $\eta = \partial_\phi$. The Komar integral is then split into

$$\begin{aligned}0 = \int_{S_\infty} \star dk - \int_H \star d\xi + \int_{T_+} \star dk - \int_{T_-} \star dk\\ + \Omega\left(\int_{S^2_\infty} \star d\eta + \int_{T_+} \star d\eta - \int_{T_-} \star d\eta\right).\end{aligned} \tag{4.67}$$

From the pure Taub-NUT case, we already know how to identify each term on the right-hand side. The integral at infinity of $\star dk$ is the Komar mass, a valid prescription to calculate the mass in asymptotically flat spacetimes. The following terms are equal to $-2TS$ and the products $\psi_\pm N_\pm$. The terms in parentheses multiplying Ω can be interpreted as angular momenta: the integral at infinity is the Komar angular momentum[3], while the integrals over the tube are the (negative) angular momentum contributions of the strings. A calculation of the sum of all three terms gives us the horizon angular momentum J_H, which is not surprising since we can write a Komar integral for $\star d\eta$ alone. Finally, we get the Smarr relation and the first law for rotating NUTs:

$$M = 2\left(TS + \Omega J_H + \psi_+ N_+ + \psi_- N_-\right), \tag{4.68}$$

[3]This is not exactly equal to the angular momentum calculated using the conformal method, they differ by a factor of $r_+ s$. Therefore, everything works out well for $s = 0$, but we need further understanding of the varying string tensions. For now, we only care about getting the correct quantities for the first law, which this procedure does accomplish.

$$dM = TdS + \Omega dJ_H + \psi_+ dN_+ + \psi_- dN_- . \tag{4.69}$$

Here, the explicit expressions for the Misner potentials are still (4.53), so they are proportional to the angular frequency of the strings. The Misner charges are

$$N_\pm \equiv \pm \frac{1}{16\pi\psi_\pm} \int_{T_\pm} \star dk = -\frac{2\pi(s \pm n)^2(n \mp a)}{r_+} \tag{4.70}$$

and the horizon angular momentum reads

$$J_H = \frac{\left(r_+^2 + a^2 + n^2 - 2as\right)(a - s)}{2r_+}. \tag{4.71}$$

As in the pure Taub-NUT case, we can calculate the free energy in the limit $\Lambda = 0$ to obtain again that $\mathcal{G} = M/2$. Alternatively, we can express the free energy in terms of T, $\psi_\pm$ and Ω, namely

$$\begin{aligned}\mathcal{G} =& \frac{T(\psi_+ - \psi_-)}{16\Omega\psi_+\psi_-} - \frac{\pi T}{2\Omega^2} \\ &+ \frac{\sqrt{(4\pi\psi_+ + \Omega)(4\pi\psi_- - \Omega)(4\pi^2T^2 + \Omega^2)}}{16\Omega^2\sqrt{\psi_+\psi_-}}.\end{aligned} \tag{4.72}$$

The Smarr relation implies that this free energy corresponds to the Gibbs free energy in the grand canonical ensemble

$$\mathcal{G} = M - TS - \Omega J_H - \psi_+ N_+ - \psi_- N_-, \tag{4.73}$$

and then we can recover the first law variables by differentiation

$$S = -\frac{\partial \mathcal{G}}{\partial T}, \quad N_\pm = -\frac{\partial \mathcal{G}}{\partial \psi_\pm}, \quad J_H = -\frac{\partial \mathcal{G}}{\partial \Omega}. \tag{4.74}$$

This alternative method gives some additional validation to our results.

4.2.3 NUTty dyons

The dyonic Taub-NUT or NUTty dyon is an electrically and magentically charged version of the Taub-NUT spacetime. As we will see, both the horizon and the Misner strings acquire charge. Being a non-vacuum solution to Einstein's field equations, we need to specify both the metric and the electromagnetic 4-vector potential. They read

$$ds^2 = -f[dt + 2n\cos\theta d\phi]^2 + \frac{dr^2}{f} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.75}$$

$$A = -h(dt + 2n\cos\theta d\phi), \tag{4.76}$$

where

$$\begin{aligned} f &= \frac{r^2 - 2mr - n^2 + 4n^2g^2 + e^2}{r^2 + n^2}, \\ h &= \frac{er}{r^2 + n^2} + \frac{g\left(r^2 - n^2\right)}{r^2 + n^2}. \end{aligned} \tag{4.77}$$

Here, we have chosen a symmetric distribution of the Misner strings for simplicity so that $s = 0$. One can then calculate the electromagnetic tensor $F = dA$. Note that both dF and $d * F$ vanish since there is no current present. Defining $G = *F$, then there must be a one-form B such that $dB = *F$. With the generator $\xi = \partial_t$ for the Taub-NUT horizon, we can calculate the conformal mass $M = m$, as well as the temperature and entropy

$$T = \frac{f'}{4\pi} = \frac{1}{4\pi r_+}\left(1 - \frac{e^2 + 4n^2g^2}{r_+^2 + n^2}\right), \tag{4.78}$$

$$S = \frac{\text{Area}}{4} = \pi\left(r_+^2 + n^2\right). \tag{4.79}$$

We now need to think about possible approaches to establishing the other thermodynamic quantities, in particular the Misner charge and Misner potential. It is natural to expect that the electric charge and potential

$$Q = \frac{1}{4\pi}\int_\Sigma *F = e, \tag{4.80}$$

$$\phi = -\left(\xi\cdot A|_{r=r_+} - \xi\cdot A|_{r=\infty}\right) = \frac{er_+ - 2gn^2}{r_+^2 + n^2}, \tag{4.81}$$

enter the first law of thermodynamics; we expect the same for the magnetic potential difference and asymptotic magnetic charge, which are given by

$$Q_m = \frac{1}{4\pi}\int_\Sigma F = -2ng, \tag{4.82}$$

$$\phi_m = -\left(\xi\cdot B|_{r=r_+} - \xi\cdot B|_{r=\infty}\right) = -\frac{n\left(2gr_+ + e\right)}{r_+^2 + n^2}, \tag{4.83}$$

and can also be calculated employing the electromagnetic duality

$$e \to -2ng, \quad 2ng \to e. \tag{4.84}$$

However, it is possible to to show [113] that a first law of the form

$$dM = TdS + \psi dN + \phi dQ + \phi_m dQ_m \tag{4.85}$$

cannot be consistently established for any choice of ψ and N.

It seems that simply trying to guess the correct thermodynamic potentials is not the correct approach, so instead, let us draw inspiration from the Smarr relation. To derive it, we will need to express the Komar integral akin to (2.12) with $\Lambda = 0$. However, since the stress-energy tensor is no longer trivial, the terms inside the Komar integral will need appropriate modification. In terms of F and its dual G, the electromagnetic stress-energy tensor reads

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + G_{\mu\rho}G_\nu{}^\rho. \tag{4.86}$$

If we contract this stress energy tensor with the generating Killing vector ξ, and since the fields are static so that the Lie derivative $\pounds_\xi A = \pounds_\xi B = 0$, we can express

$$\nabla_\mu \left(\nabla^\mu \xi^\nu + F^{\mu\nu}\Phi + G^{\mu\nu}\Phi_m\right) = 0, \tag{4.87}$$

where Φ and Φ_m are not the potential differences but the electric and magnetic potentials themselves, $\Phi = -\xi^\mu A_\mu$ and $\Phi_m = -\xi^\mu B_\mu$: they are gauge dependent quantities. We can express the Komar integral in the language of differential forms:

$$\frac{1}{8\pi}\int_{\partial\Sigma} \left(\star d\xi + 2\Phi G - 2\Phi_m F\right) = 0. \tag{4.88}$$

As before, we can separate the Komar integral into a sum over the boundary components. The mass is given by the integral of $\star d\xi$ at infinity; the same integral over the horizon gives $-2TS$. With $\psi = 1/8\pi n$, the Minser charge is taken to be the entire integral over the Misner tubes:

$$N = \frac{1}{16\pi\psi}\int_{T_+ \cup T_-} \left(\star d\xi + 2\Phi G - 2\Phi_m F\right). \tag{4.89}$$

As we see, this is a gauge-dependent quantity. As mentioned before, a gauge choice where Φ and Φ_m vanish at the horizon is not compatible with a first law, nor is the choice where both vanish at infinity. Instead, we choose gauges where $\Phi_m = 0$ at infinity and $\Phi = 0$ at the horizon. With our definitions, the Komar integral simplifies to the Smarr relation

$$M = 2TS + \phi Q + \phi_m Q_m^{(+)} + 2\psi N. \tag{4.90}$$

Here, $Q_m^{(+)}$ is the charge enclosed by the horizon. It is not equal to the asymptotic magnetic charge, which means that the Misner strings are magnetically charged. Explicitly, we have

$$Q_m^{(+)} = -\frac{2n\left(er_+ + g\left(r_+^2 - n^2\right)\right)}{r_+^2 + n^2}, \tag{4.91}$$

$$N = -\frac{4\pi n^3}{r_+}\left(1 + \frac{\left(r_+^2 - n^2\right)\left(e^2 + 4egr_+\right)}{\left(r_+^2 + n^2\right)^2} - \frac{4n^2 g^2\left(3r_+^2 + n^2\right)}{\left(r_+^2 + n^2\right)^2}\right). \tag{4.92}$$

With these choices, the first law of NUTty dyons holds:

$$dM = TdS + \phi dQ + \phi_m dQ_m^{(+)} + \psi dN. \tag{4.93}$$

Let us note that the opposite choice of gauge can be made as well so that the magnetic charge at infinity and the electric charge at the horizon would appear in this law. It is possible to choose even more general combinations of gauges, as shown in [113].

Another approach to establishing a first law for NUTty dyons is to impose $h(r_+) = 0$ to ensure regularity in the Euclidean sector at the tip of the "cigar", $r = r_+$ [114, 115]. In this case, the e and g parameters are no longer independent; we need to have

$$g = -\frac{er_+}{r_+^2 - n^2}, \quad Q = e, \quad \phi = \frac{er_+}{r_+^2 - n^2}, \tag{4.94}$$

in analogy to the case where the time periodicity constrained the mass in terms of the NUT charge. We call this case the constrained electric thermodynamics of the NUTty dyon. Prescriptions similar to the ones we just introduced give

$$T = \frac{1}{4\pi r_+}\left(1 - \frac{e^2\left(r_+^2 + n^2\right)}{\left(r_+^2 - n^2\right)^2}\right), \tag{4.95}$$

$$S = \pi\left(r_+^2 + n^2\right), \tag{4.96}$$

$$\psi = \frac{1}{8\pi n}, \quad N = -\frac{4\pi n^3}{r_+}\left(1 - \frac{e^2\left(3r_+^2 + n^2\right)}{\left(n^2 - r_+^2\right)^2}\right). \tag{4.97}$$

With these, the following constrained electric first law and Smarr relation are satisfied:

$$
\begin{aligned}
dM &= TdS + \phi dQ + \psi dN, \\
M &= 2(TS + \psi N) + \phi Q.
\end{aligned}
\tag{4.98}
$$

We can obtain a similar law in terms of the magnetic charge by solving for e in terms of g instead.

Finally let us note that by finding the free energy $\mathcal{G}$ in the limit that $\Lambda \to 0$, we obtain, in the unconstrained case:

$$
\mathcal{G} = \frac{m}{2} - \frac{r_+ e^2 \left(r_+^2 - n^2\right)}{2\left(n^2 + r_+^2\right)^2} + \frac{2n^2 r_+ \left(r_+^2 - n^2\right) g^2}{\left(r_+^2 + n^2\right)^2} + \frac{4n^2 e g r_+^2}{\left(n^2 + r_+^2\right)^2}, \tag{4.99}
$$

which can be expressed in turn as

$$
\mathcal{G} = M - TS - \phi Q - \psi N \tag{4.100}
$$

in the grand-canonical ensemble. In fact, for the $s = 0$ case, this provides an easier way to calculate N and, since Komar integrations are not really effective in the rotating AdS case, the calculation using the free energy will become our preferred prescription for finding Misner charges. We will also use it in the rotating charged case, which we study next.

4.2.4 Rotating dyonic Taub-NUT

The rotating dyonic Taub-NUT, or rotating NUTty dyon, is a Kerr-NUT spacetime endowed with electric and magnetic charges. The metric is given by

$$
ds^2 = -\frac{\Delta}{U}\left(e^1\right)^2 + \frac{U}{\Delta} dr^2 + U d\theta^2 + \frac{\sin^2\theta}{U}\left(e^2\right)^2, \tag{4.101}
$$

where the basis 1-forms e_1 and e_2 are given by

$$
e^1 = dt + \left[2n\cos\theta - a\sin^2\theta\right] d\phi, \quad e^2 = a dt - \left(r^2 + n^2 + a^2\right) d\phi, \tag{4.102}
$$

and U and Δ are the functions

$$
\Delta = a^2 + e^2 + 4g^2 n^2 - n^2 - 2mr + r^2, \tag{4.103}
$$

$$
U = r^2 + (n + a\cos\theta)^2. \tag{4.104}
$$

Note that, for simplicity, we are assuming that the strings are symmetrically distributed. The gauge vector potential is given by

$$A = -\frac{er + g\left(r^2 + a^2 - n^2\right)}{U} e^1 + \frac{ag\sin^2\theta}{U} e^2. \tag{4.105}$$

Let us now compute some of the relevant thermodynamic quantities. By using the conformal method, the mass and angular momentum are calculated to be

$$M = m = \frac{a^2 + e^2 + 4g^2n^2 - n^2 + r_+^2}{2r_+}, \quad J_\infty = Ma. \tag{4.106}$$

The Taub-NUT horizon generating killing vector is again $\xi = k + \Omega\eta$, with

$$\Omega = -\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r=r_+} = \frac{a}{r_+^2 + n^2 + a^2}, \tag{4.107}$$

and from this we can calculate the horizon entropy and the temperature

$$T = \frac{\Delta'\left(r_+\right)}{4\pi\left(r_+^2 + n^2 + a^2\right)} = \frac{1}{4\pi r_+}\frac{r_+^2 + n^2 - a^2 - e^2 - 4g^2n^2}{a^2 + n^2 + r_+^2}, \tag{4.108}$$

$$S = \frac{\text{Area}}{4} = \pi\left(r_+^2 + n^2 + a^2\right). \tag{4.109}$$

The Misner strings are also Killing horizons generated by

$$\xi_\pm = \partial_t + \Omega_\pm\partial_\varphi, \quad \Omega_\pm = \mp\frac{1}{2n}, \tag{4.110}$$

since we now have $s = 0$.

Drawing from the lessons learned in the non-rotating case, we might need to compute the electric and magnetic charges enclosed by different spheres. Because we will need this result later, we preemptively calculate electric and magnetic charges enclosed by a sphere of radius r using Gauss' law:

$$\begin{aligned} Q_e(r) &= \frac{1}{4\pi}\int_{S^2} *F = \frac{\left(a^2 + n^2 + r^2\right)\left[e\left(r^2 + a^2 - n^2\right) - 4gn^2r\right]}{\left[(a-n)^2 + r^2\right]\left[(a+n)^2 + r^2\right]}, \\ Q_m(r) &= \frac{1}{4\pi}\int_{S^2} F = -\frac{2n\left(a^2 + n^2 + r^2\right)\left[g\left(a^2 - n^2 + r^2\right) + er\right]}{\left[(a-n)^2 + r^2\right]\left[(a+n)^2 + r^2\right]}. \end{aligned} \tag{4.111}$$

In particular, we will be making use of the horizon electric and magnetic charges

$$\begin{aligned} Q_e^{(+)} &= Q_e\left(r = r_+\right) = \frac{\left(a^2+n^2+r_+^2\right)\left[e\left(r_+^2+a^2-n^2\right)-4gn^2r_+\right]}{\left[(a-n)^2+r_+^2\right]\left[(a+n)^2+r_+^2\right]}, \\ Q_m^{(+)} &= Q_m\left(r = r_+\right) = -\frac{2n\left(a^2+n^2+r_+^2\right)\left[g\left(a^2-n^2+r_+^2\right)+er_+\right]}{\left[(a-n)^2+r_+^2\right]\left[(a+n)^2+r_+^2\right]}, \end{aligned} \tag{4.112}$$

and the total charge content of the NUTty dyon

$$\begin{aligned} Q_e &= \lim_{r\to\infty} Q_e(r) = e, \\ Q_m &= \lim_{r\to\infty} Q_m(r) = -2ng. \end{aligned} \tag{4.113}$$

We can also find the electric potential difference between the horizon and infinity. This turns out to depend on θ; however, evaluating at $\theta = \pi/2$ gives

$$\phi_e = -\left(\xi^a A_a\big|_{r=r_+} - \xi^a A_a\big|_{r=\infty}\right) = \frac{er_+ - 2gn^2}{a^2+n^2+r_+^2}. \tag{4.114}$$

While this choice of angular coordinate may seem somewhat arbitrary, it yields the same result as the well-known Hawking-Ross prescription [116].

Let us now calculate the free energy in order to use the simpler method to calculate Misner charges. The Euclidean action calculation for the free energy in the limit $\Lambda \to 0$ yields

$$\mathcal{G} = \frac{m}{2} - \frac{r_+}{2}\frac{\left(r_+^2-n^2+a^2\right)\left(e^2-4g^2n^2\right)-8egn^2r_+}{\left[r_+^2+(a+n)^2\right]\left[r_+^2+(a-n)^2\right]}. \tag{4.115}$$

For a gauge choice that cancels the electric potential at the horizon and the magentic potential at infinity, we expect

$$G = G\left(T, \psi, \Omega, \phi, Q_m^{(+)}\right). \tag{4.116}$$

We could then define

$$S = -\frac{\partial\mathcal{G}}{\partial T}, \quad N = -\frac{\partial\mathcal{G}}{\partial\psi}, \quad J = -\frac{\partial\mathcal{G}}{\partial\Omega}, \quad Q = -\frac{\partial\mathcal{G}}{\partial\phi}, \quad \phi_m = \frac{\partial\mathcal{G}}{\partial Q_m^{(+)}}. \tag{4.117}$$

A cumbersome calculation using implicit differentiation shows that we get the correct entropy and electric charge with this choice of ensemble. The magnetic potential, however, is no longer related to the electric potential via the electromagnetic duality (4.84). Instead, if $\hat{\phi}_m$ represents the electric potential transformed under the duality, the magnetic potential in the rotating case is

$$\phi_m = \hat{\phi}_m + 2n\Omega^2 Q_e. \tag{4.118}$$

The explicit expressions for the Misner charge and the angular momentum are obtained

$$\begin{aligned} N =& 2n\pi\left[\frac{4e\left(er_+ - 2gn^2\right)}{a^2+n^2+r_+^2} - \frac{2n^2\left(a^2+e^2+\left(4g^2-1\right)n^2\right)}{r_+(a-n)(a+n)}\right. \\ &+\left(\frac{2n(a+n)\left(4egn(a+n) - r_+\left(e^2-4g^2n^2\right)\right)}{\left((a+n)^2+r_+^2\right)^2} + (n\leftrightarrow -n)\right) \\ &+\left(\frac{4aegn(a+n) - r_+\left(2ae^2+3e^2n+4g^2n^3\right)}{r_+^2(a+n)+(a+n)^3} + (n\leftrightarrow -n)\right)\Bigg), \end{aligned} \tag{4.119}$$

$$\begin{aligned} J =& \frac{1}{2}\left[\frac{a\left(a^2+n^2\right)\left(a^2+e^2+\left(4g^2-1\right)n^2\right)}{r_+(a-n)(a+n)} + ar_+\right. \\ &+\left(\frac{2n^2(a+n)\left(4egn(a+n) - r_+\left(e^2-4g^2n^2\right)\right)}{\left((a+n)^2+r_+^2\right)^2} + (n\leftrightarrow -n)\right) \\ &\left.+\left(\frac{n\left(4aegn(a+n) - r_+\left(2ae^2+3e^2n+4g^2n^3\right)\right)}{r_+^2(a+n)+(a+n)^3} + (n\leftrightarrow -n)\right)\right]. \end{aligned} \tag{4.120}$$

With these quantities in hand, one can check that they satisfy the Smarr relation and the first law:

$$\begin{aligned} M &= 2\left(TS + \psi N + \Omega J\right) + \phi Q + \phi_m Q_m^{(+)}, \\ dM &= TdS + \Omega dJ + \phi dQ + \phi_m dQ_m^{(+)} + \psi dN. \end{aligned} \tag{4.121}$$

The same procedure can be carried out with the choice of ensemble $\mathcal{G} = \mathcal{G}\left(T, \psi, \Omega, Q_m\right)$, so that we obtain a similar law (with a different N) where the magnetic charge is evaluated at infinity and the electric charge is evaluated at the horizon.

As in the non-rotating case, we can also formulate constrained thermodynamics. We will focus on the electric case, where we find g in terms of e. We find that

$$g = -\frac{er_+}{r_+^2 + a^2 - n^2}. \tag{4.122}$$

Then the thermodynamic relations

$$\begin{aligned} dM &= TdS + \Omega dJ + \phi dQ + \psi dN \quad \text{and} \\ M &= 2\left(TS + \psi N + \Omega J\right) + \phi Q \end{aligned} \tag{4.123}$$

hold true, with the simpler thermodynamic variables

$$\phi = \frac{er_+}{r_+^2 + a^2 - n^2} = -g, \quad Q = e, \tag{4.124}$$

$$S = \pi\left(r_+^2 + a^2 + n^2\right), \quad T = \frac{r_+^2 + n^2 - a^2 - e^2\left(1 + \frac{4n^2r_+^2}{\left(r_+^2+a^2-n^2\right)^2}\right)}{4\pi r_+\left(r_+^2 + n^2 + a^2\right)}, \tag{4.125}$$

$$J = J_0\left(1 + \frac{Q\phi}{r_+}\right), \quad J_0 = \frac{a\left(r_+^2 + n^2 + a^2\right)}{2r_+}, \tag{4.126}$$

$$\psi = \frac{1}{8\pi n}, \quad N = -\frac{4\pi n^3}{r_+}\left(1 + \frac{e^2\left(a^2 - 3r_+^2 - n^2\right)}{\left(r_+^2 + a^2 - n^2\right)^2}\right). \tag{4.127}$$

The Gibbs free energy also simplifies greatly in the case of electric constrained thermodynamics. It reads

$$\mathcal{G} = \frac{m}{2} - \frac{r_+e^2}{2\left(r_+^2 + a^2 - n^2\right)}. \tag{4.128}$$

As an additional check of our results, we can plot the free energy against the temperature in the ensemble

$$\begin{aligned} F &= F\left(T, \psi, J, Q_e\right) = G + \phi_e Q_e + \Omega_H J \\ &= \frac{1}{4r_+}\left(r_+^2 + 3a^2 - n^2 + \frac{e^2\left(3r_+^4 + n^4 + 3a^4 - 4n^2a^2 + 6r_+^2a^2\right)}{\left(r_+^2 + a^2 - n^2\right)^2}\right). \end{aligned} \tag{4.129}$$

For finite charge $Q = 1$, we observe a behaviour of the free similar to that already known of Kerr-Newmann black holes without NUT charge.

A bit disappointingly, these expressions for the Misner charge and the angular momentum are somewhat convoluted. We were already aware that the angular momentum term that appears in the first law for Kerr-NUT is not the asymptotic angular momentum. The reader may find it uncomfortable that the preferred thermodynamic quantity is the horizon angular momentum instead. Moreover, the procedure used to calculate the Gibbs free energy obscures the physical meaning of the calculated thermodynamic variables and does not even provide a geometric intuition like in the non-rotating case. It is also puzzling that the magnetic potential is not related to the horizon potential through the electromagnetic duality. One could conjecture that this is due to some interplay between the Dirac string and the rotating Misner strings, which would also explain why the laws simplify considerably as soon as we remove the Dirac strings.

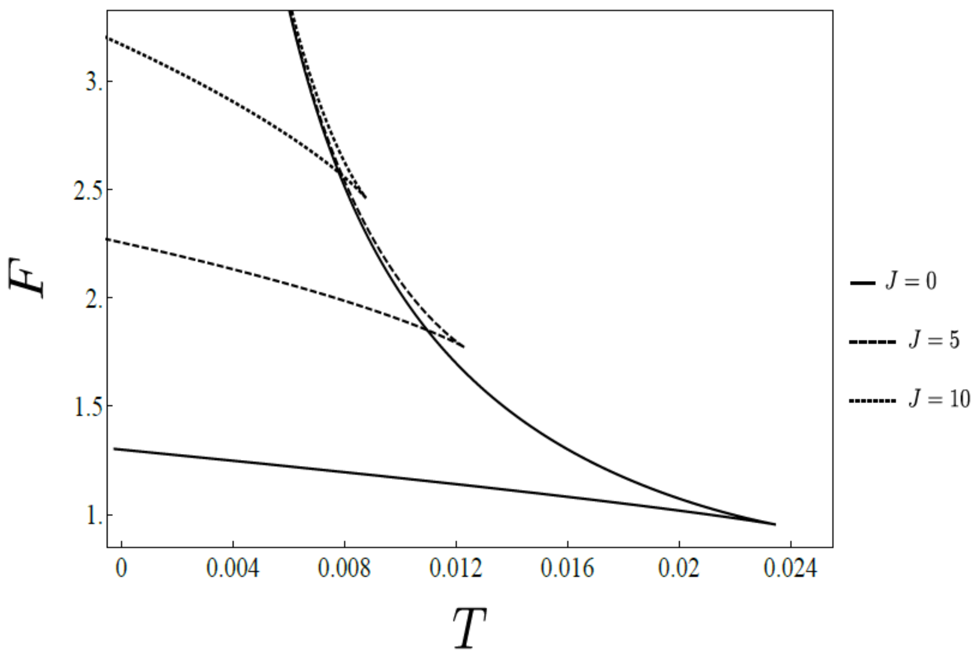


Figure 4.2: **Free energy of rotating NUTty dyon**. Plotted against the temperature for different values of J and $Q = 1$. This behaviour is the same that is observed for charged rotating black holes.

A possible way to elucidate the meaning of the terms obtained in the unconstrained thermodynamics involves using the Tomimatsu formalism, which efficiently deals with the presence of multiple horizons [117]. The mysterious Misner potentials and charges are thus interpreted and angular frequency and angular momentum of the strings. The authors of [118] proposed a Smarr relation in this interpretation, but we also found that the quantities therein satisfy a first law that is not of full cohomogeneity:

$$dM = TdS + \phi dQ^{(+)} + \Omega dJ_H + \Omega_+ d\tilde{J}_+ + \Omega_- d\tilde{J}_- + \Phi_+ dQ_+ + \Phi_- dQ_-, \qquad (4.130)$$

where $\tilde{J}\pm$ and $Q\pm$ represent the angular momentum and charge content of the Misner strings, and $\phi_\pm$ are potentials associated with the Misner strings. Since $\Omega_+ = -\Omega_-$ and $\phi_+ = \phi_- = g$ as defined therein, the terms can be grouped into a full-cohomogeneity first law which is an alternative to our first law. It is still unclear whether this could be derived from an action calculation as well.

4.3 The extended first law of AdS Taub-NUTs

Having studied the thermodynamic first laws for asymptotically flat Taub-NUTs, we would like to extend these results to anti-de Sitter space with variable cosmological constant. There are, however, a few obstacles. In the asymptotically locally flat Taub-NUTs, we could easily calculate the mass using Komar integration. Therefore, integrating over the spacetime boundaries would give us, at least in the simpler cases, a geometric prescription to calculate the thermodynamic variables.

However, this does not work as smoothly in AdS: we cannot simply use Komar integration to calculate the mass; instead, we need to use the conformal method and try to accommodate the Komar integrals conveniently, regularizing with the Killing co-potential as in equation (2.12). This approach works well in the cases that we will study in the following, and it is most likely the reason we have successfully established the first law. Komar integration stops working correctly for Kerr-NUT-AdS, so we refer the reader to Appendix E if they wish to learn more about the difficulties encountered in this scenario.

4.3.1 Extended thermodynamics of AdS Taub-NUTs

The simplest asymptotically locally AdS solution to Einstein's field equation containing a NUT charge is the pure Taub-NUT AdS metric, which has spacetime length element

$$ds^2 = -f[dt + (2n\cos\theta + s)d\phi]^2 + \frac{dr^2}{f} + \left(r^2+n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where

$$f = \frac{r^2 - 2mr - n^2}{r^2+n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{\ell^2\left(r^2+n^2\right)},$$

as already stated at the beginning of this chapter. The Taub-NUT horizon generator in this case is $k = \partial_t$ and the string generators are $\xi_\pm = \partial t - 1/2(s \pm n)\partial_\phi$. With the same prescriptions as in the flat case, we find

$$\psi_\pm = \frac{1}{8\pi(n \pm s)}, \quad T = \frac{1}{4\pi r_+}\left(1 + \frac{3\left(n^2 + r_+^2\right)}{\ell^2}\right), \quad S = \pi(r_+^2 + n^2). \tag{4.131}$$

We would like to mimic the Komar integration we carried out in the flat case; obviously, this time $\Lambda \neq 0$, so we need to use the full equation (2.12). Using differential forms for brevity, the Komar integration can be written in four dimensions as

$$\int_{\partial\Sigma} \star\left(dk + 2\Lambda\omega\right) = 0. \tag{4.132}$$

Separating the boundary into the horizon, Misner tubes, and the boundary at infinity, we can split the integral in the following manner:

$$\begin{aligned} 0 = &\int_{S_\infty} \star\left(dk + 2\Lambda\omega\right) - \int_H \star dk + \left(\int_{T_+} \star dk + 2\Lambda\Omega_+^{(r=\infty)}\right) \\ &- \left(\int_{T_-} \star dk + 2\Lambda\Omega_-^{(r=\infty)}\right) - 2\Lambda\left(\int_H \star\omega + \Omega_+^{(r=r_+)} - \Omega_-^{(r=r_+)}\right). \end{aligned} \tag{4.133}$$

Here we have defined $\Omega_\pm^{(r=r_0)}$ as the antiderivative (defined up to an additive constant)

$$\int_{T_\pm} \star\omega = \Omega_\pm^{(r=\infty)} - \Omega_\pm^{(r=r_+)}. \tag{4.134}$$

The conformal method yields a mass $M = m$, and in the Komar integration, it can be identified with

$$M = -\frac{1}{8\pi}\int_{S_\infty} \star\left(dk + 2\Lambda\omega\right). \tag{4.135}$$

We furthermore have the property of the Killing horizon generator

$$TS = -\frac{1}{16\pi}\int_H \star dk. \tag{4.136}$$

The grouping of the terms above is suggestive but performed in hindsight. Indeed, we can obtain finite Misner charges by defining

$$N_\pm := \pm\frac{1}{16\pi\psi_\pm}\left(\int_{T_\pm} \star dk + 2\Lambda\Omega_\pm^{(r=\infty)}\right). \tag{4.137}$$

The rest of the Komar integral defines the thermodynamic volume:

$$V := -\left(\int_H \star\omega + \Omega_+^{(r=r_+)} - \Omega_-^{(r=r_+)}\right), \tag{4.138}$$

so that we have finally obtained (merely by definition of the thermodynamic quantities and potentials) the Smarr relation

$$M = 2\left(TS + \psi_+ N_+ + \psi_- N_- - PV\right). \tag{4.139}$$

Let us now calculate these quantities explicitly. The copotential ω is such that $d\omega = \star k$, and it can be found to be[4]

$$\omega = -\frac{r}{3}dr \wedge (dt + 2[n\cos\theta + s]d\phi) + \frac{n}{3}\left(r^2 + n^2\right)\sin\theta d\theta \wedge d\phi. \tag{4.140}$$

Additionally, a direct calculation for $\star dk$ gives

$$\star\, dk = -\frac{2fn}{r^2 + n^2}dr \wedge (dt + 2[n\cos\theta + s]d\phi) - \sin\theta\left(r^2 + n^2\right)f' d\theta \wedge d\phi, \tag{4.141}$$

which means that we are now ready to calculate all the integrals. A straightforward computation gives

$$N_{\pm} = -\frac{2\pi n(n \pm s)^2}{r_+}\left(1 + \frac{3\left(n^2 - r_+^2\right)}{\ell^2}\right), \tag{4.142}$$

$$V = \frac{4\pi r_+^3}{3}\left(1 + \frac{3n^2}{r_+^2}\right). \tag{4.143}$$

With $P = \Lambda/8\pi$, the thermodynamic potentials and variables thus defined indeed satisfy the extended first law for AdS Taub-NUT spacetimes

$$dM = TdS + \psi_+ dN_+ + \psi_- dN_- + VdP \tag{4.144}$$

So far, this is the only method that we have found to calculate the Misner charges geometrically. However, this does little to elucidate their true physical meaning, as we discuss at the end of the chapter. This drawback might have been apparent already since we found complicated expressions for the rotating NUTty dyon, but as we will see, there are more explicit ambiguities.

4.3.2 AdS NUTty dyon and phase transitions

The thermodynamics for the AdS NUTty dyon can be derived in the same manner as in the flat case, with additional Killing potential regularizing factors. The metric and gauge field in this case are given by

$$\begin{aligned} ds^2 =& - f[dt + 2(n\cos + s)\theta d\phi]^2 + \frac{dr^2}{f} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \\ A =& - h(dt + 2(n\cos\theta + s)d\phi), \end{aligned} \tag{4.145}$$

[4]directly, using pre-defined functions in Maple.

with

$$
\begin{aligned}
f &= \frac{r^2 - 2mr - n^2 + 4n^2 g^2 + e^2}{r^2 + n^2} - \frac{3n^4 - 6n^2 r^2 - r^4}{\ell^2 \left(r^2 + n^2\right)}, \\
h &= \frac{er}{r^2 + n^2} + \frac{g\left(r^2 - n^2\right)}{r^2 + n^2}.
\end{aligned} \tag{4.146}
$$

Given that we have more experience with these calculations, we will not set $s = 0$ in this case. The flat case for $s \neq 0$ will follow by taking the limit $\ell \to \infty$ in the following.

The Killing generators for the horizon and Misner strings remain unchanged from the previous section. The entropy and Misner charges also remain the same, but the temperature becomes:

$$
T = \frac{1}{4\pi r_+}\left(1 + \frac{3\left(r_+^2 + n^2\right)}{\ell^2} - \frac{e^2 + 4n^2 g^2}{r_+^2 + n^2}\right). \tag{4.147}
$$

The Komar integral is simply the same as the flat dyonic Taub-NUT with the extra copotential term

$$
\int_{\partial\Sigma} \left(\star d\xi + 2\Phi G - 2\Phi_m F + 2\Lambda \star \omega\right) = 0. \tag{4.148}
$$

Defining $\Omega^{\pm}$ via equation (4.134), we have that the following expressions are finite, and they provide the desired definitions of the (gauge dependent) Misner charges

$$
N_{\pm} = \frac{1}{16\pi\psi_{\pm}} \int_{T_{\pm}} \left(\star d\xi + 2\Phi G - 2\Phi_m F + 2\Lambda \Omega_{\pm}^{r=\infty}\right). \tag{4.149}
$$

As before, it turns out that the simultaneous choice of gauge is constrained if we want to obtain a consistent first law of thermodynamics [113]. We choose the gauge where the electric potential vanishes at the horizon, and the magnetic potential vanishes at infinity. The expressions for $\star dk$ and ω remain unchanged from the previous section, so we can explicitly calculate the Misner charges

$$
N_{\pm} = -\frac{4\pi n (n \pm s)^2}{r_+}\left(1 + \frac{3\left(n^2 - r_+^2\right)}{\ell^2} + \frac{\left(r_+^2 - n^2\right)\left(e^2 + 4egr_+\right)}{\left(r_+^2 + n^2\right)^2} - \frac{4n^2 g^2 \left(3r_+^2 + n^2\right)}{\left(r_+^2 + n^2\right)^2}\right). \tag{4.150}
$$

The same expressions give the magnetic and electric potential and the asymptotic electric charge and horizon magnetic charge as in the flat case. With all of this in hand, the Komar integral can be expressed as the Smarr formula

$$
M = 2\left(TS - VP + \psi_+ N_+ + \psi_- N_-\right) + \phi Q + \phi_m Q_m^{(+)}, \tag{4.151}
$$

where V is defined as in (4.3.1) and has the same expression (4.143). The expected first law also holds:

$$dM = TdS + \phi dQ + \phi_m dQ_m^{(+)} + \psi_+ dN_+ + \psi_- dN_- + VdP. \tag{4.152}$$

The converse gauge choice can also be made to obtain a consistent first law with the electric charge evaluated at the horizon and the magnetic charge evaluated asymptotically. When $s = 0$, both Misner potentials are the same and they can be combined into a ψdN term, as we did in the flat case. We should finally note that the free energy obtained via the on-shell Euclidean action is equal to

$$\mathcal{G} = \frac{m}{2} - \frac{r_+\left(3n^2 + r_+^2\right)}{2\ell^2} - \frac{r_+ e^2\left(r_+^2 - n^2\right)}{2\left(n^2 + r_+^2\right)^2} + \frac{2n^2 r_+\left(r_+^2 - n^2\right)g^2}{\left(r_+^2 + n^2\right)^2} + \frac{4n^2 e g r_+^2}{\left(n^2 + r_+^2\right)^2}, \tag{4.153}$$

which in turn corresponds to the ensemble

$$\mathcal{G} = M - TS - \phi Q - \psi N \tag{4.154}$$

with fixed electric potential.

Similar to the flat case, we can obtain electric constrained thermodynamics by imposing that $h(r_+) = 0$. Then the magnetic and electric charge are related by (4.94). Since we will be doing some phase structure analysis in this scenario, let us simplify to a symmetric distribution of Misner strings $s = 0$. The entropy, Misner potential, and volume remain the same; the temperature and Misner charge now become

$$T = \frac{1}{4\pi r_+}\left(1 + \frac{3\left(r_+^2 + n^2\right)}{\ell^2} - \frac{e^2\left(r_+^2 + n^2\right)}{\left(r_+^2 - n^2\right)^2}\right), \tag{4.155}$$

$$N = -\frac{4\pi n^3}{r_+}\left(1 + \frac{3\left(n^2 - r_+^2\right)}{\ell^2} - \frac{e^2\left(3r_+^2 + n^2\right)}{\left(n^2 - r_+^2\right)^2}\right). \tag{4.156}$$

The latter can most easily be found by calculating the free energy

$$\mathcal{G} = \frac{m}{2} - \frac{r_+\left(3n^2 + r_+^2\right)}{2\ell^2} - \frac{r_+ e^2}{2\left(r_+^2 - n^2\right)} \tag{4.157}$$

and solving for N in (4.154). We thus obtain

$$\begin{aligned} dM &= TdS + \phi dQ + \psi dN + VdP, \\ M &= 2(TS - VP + \psi N) + \phi Q. \end{aligned} \tag{4.158}$$

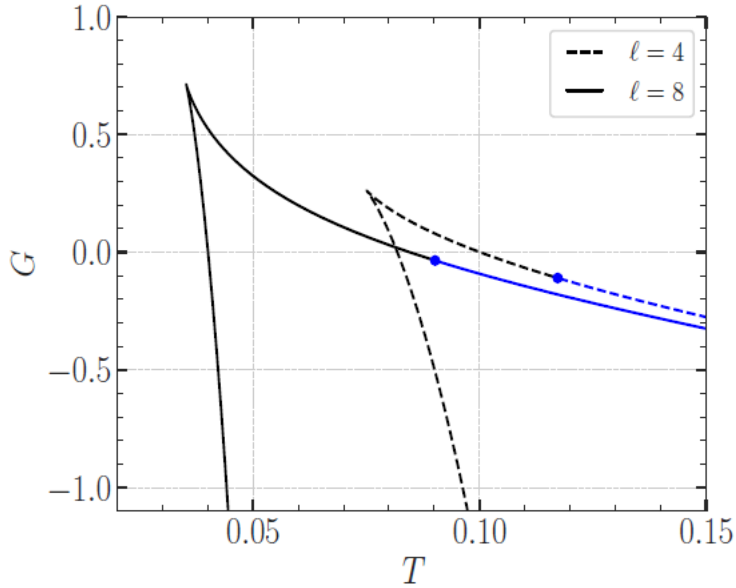


Figure 4.3: **Uncharged Gibbs free energy.** Free energy vs. temperature for an uncharged AdS Taub-NUT with $n = 1$ for two values of the pressure. The blue lines indicate solutions of negative mass. There is no phase transition since a "pure NUT" does not correspond to a pure radiation phase.

We should note that a similar law can be obtained involving only the asymptotic magnetic charge.

Let us now study the phase transitions in the purely electric case, as we did with charged AdS black holes in section 2.2.3. Since the expression of N is algebraically involved, we will focus on the ensemble given by

$$G = M - TS - \psi N, \tag{4.159}$$

where ψ is held fixed. In order to obtain the phase structure, we need to fix Q and P as well, although we will explore a few different values of these parameters to explore the difference in thermodynamic behaviour. In the uncharged case, $Q = 0$ and $n \neq 0$, the behaviour of the free energy as a function of the temperature is in essence similar to Schwarzschild-AdS (see figure 4.3). The crucial difference rests on the fact that, for $n \neq 0$, we do not find a "thermal Taub-NUT" analogous to thermal AdS, so there is no radiation phase. Additionally, Taub-NUTs have horizons even for negative masses, which correspond

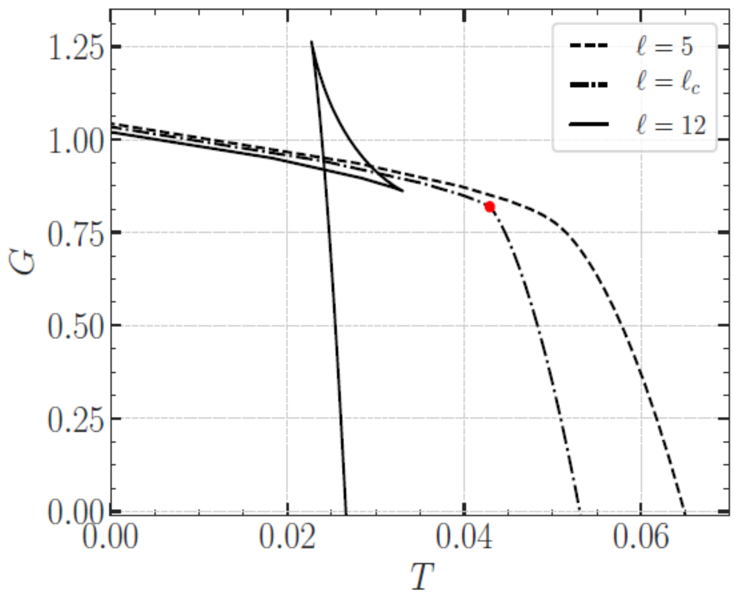


Figure 4.4: **Swallowtail Behaviour.** We see that for different values of ℓ, there the Gibbs free energy exhibits different behaviour. In this plot, $Q = 1$ and $n = 0.2$. At $\ell = \ell_c$, corresponding to the critical pressure, the swallowtail behaviour starts to appear. The red dot labels the critical point.

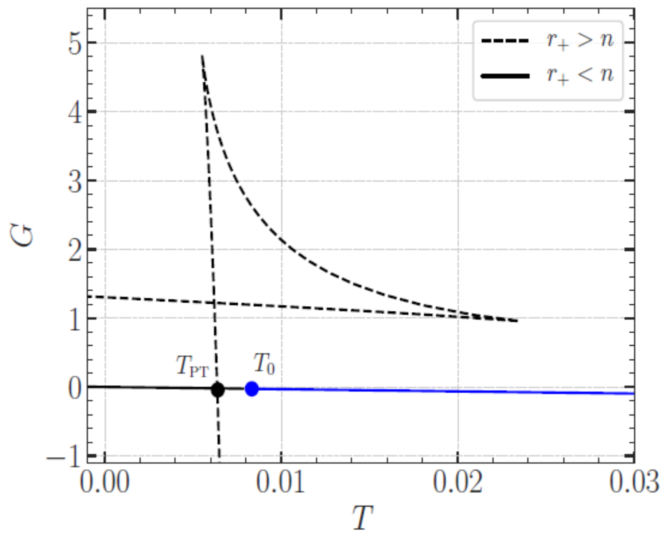
to high temperatures. One could exclude this case from the analysis if negative masses are a pathology to be avoided.

It is more interesting to study the charged case. For low enough pressures (large ℓ), we obtain a swallowtail behaviour for the free energy, in analogy to the $n = 0$ case (see figure 4.4). We can find such a critical pressure by using our expression for the temperature and solving for P in terms of T and r_+.

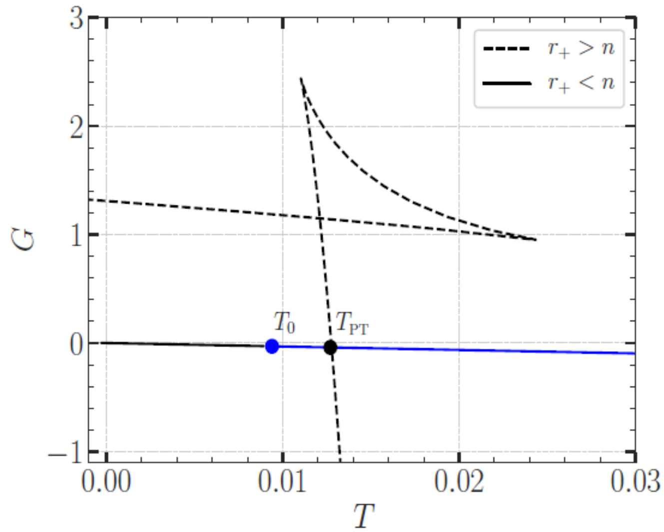
$$P = \frac{T}{2\left(r_+^2 + n^2\right)} - \frac{1}{8\pi\left(r_+^2 + n^2\right)} + \frac{Q^2}{8\pi\left(n^2 - r_+^2\right)^2} \tag{4.160}$$

In the $P - V$ diagram, the critical point corresponds to a point of inflection of P as a function of V. The fact that n is fixed allows us to solve for $r_+(V)$ from equation (4.143). Then we simply need to find the point where

$$\frac{\partial P}{\partial r_+} = 0 = \frac{\partial^2 P}{\partial r_+^2}. \tag{4.161}$$



(a)



(b)

Figure 4.5: **Branch of small Taub-NUTs.** The lower straight branch is a branch of small Taub-NUTs with $r_+ < n$. The blue parts represent regions with $M < 0$. In (a), where $\ell = 50$, the small Taub-NUT directly transitions to a much larger one. In (b), where $\ell = 25$, the Taub-NUTs transitions first to a medium Taub-NUT and then to a larger one.

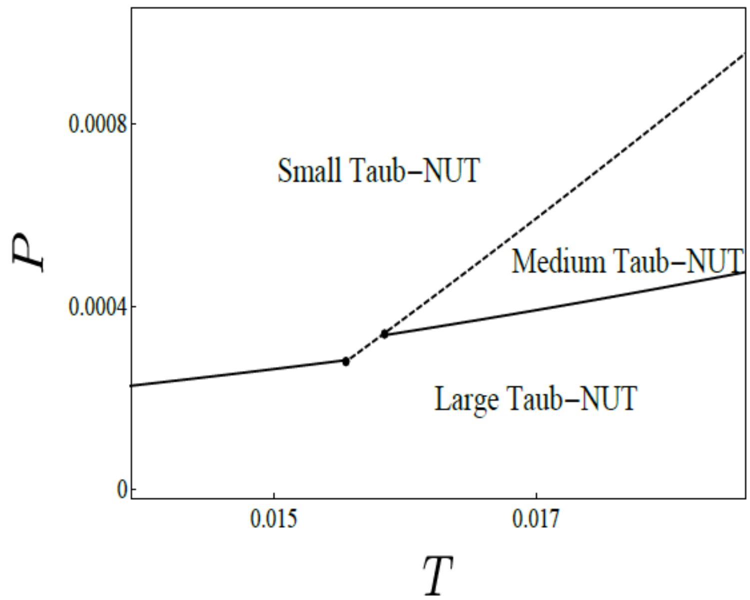


Figure 4.6: **Phase Diagram**. Here, $Q = 0.98$ and $n = 1$. The phase diagram allows us to see in more detail the possible phase transitions. The continuous lines represent first order phase transitions; the dotted lines represent zeroth order transitions. In addition to the changes of phase that we saw in 4.5, we see that there is a small range of pressure where the transition from small to large Taub-NUT is of zeroth order.

This equation cannot be solved exactly, but we may perturbatively expand when $n \leq Q$ to find the critical temperature and pressure. Defining $\tilde{n} = n/Q$:

$$T_c = T_c^{(0)} \left(1 - \frac{1}{4}\tilde{n}^2 + \frac{83}{288}\tilde{n}^4 + O\left(\tilde{n}^6\right)\right), \tag{4.162}$$

$$P_c = P_c^{(0)} \left(1 - \frac{2}{3}\tilde{n}^2 + \frac{25}{27}\tilde{n}^4 + O\left(\tilde{n}^6\right)\right), \tag{4.163}$$

with $T_c^{(0)}$ and $P_c^{(0)}$ being the critical temperature and pressure for $n = 0$, as in equation (2.74).

The addition of a NUT charge does entail some changes to the phase structure. Namely, for pressures greater than the transition pressure

$$P_t = \frac{Q^2 - n^2}{8\pi n^4}, \tag{4.164}$$

a new branch of small radius Taub-NUTs appears, which turns out to be the stable configuration (see figure 4.5). Moreover, some Taub-NUTs in this branch have a positive mass, so a priori, there is no physical reason to discard them.[5] In the regime of pressures greater than P_t, and restricting positive mass Taub-NUTs only, the phase transitions have two possible behaviours. For lower values of P, the small Taub-NUTs undergo either a zeroth-order phase transition to a larger Taub-NUT, which might be medium-sized or large, or a first-order first transition to a large black hole for even lower pressures. At higher pressures, there is a zeroth-order phase transition to medium Taub-NUTs and then a first-order phase transition to large Taub-NUTs. This is explained in more detail in figures 4.5 and 4.6.

4.3.3 Beyond linear electrodynamics

In this section, we find a new Taub-NUT solution in a newly proposed theory of electrodynamics [119], which is also the most general conformal theory of electrodynamics that is symmetric under duality rotations. We introduce the basics of this theory, obtain a Taub-NUT solution (and, by extension, simpler black hole solutions), and establish the first law of thermodynamics.

ModMax electrodynamics

ModMax electrodynamics, which stands for Modified Maxwell, is a theory of electrodynamics that is conformal and invariant under duality rotations. These conditions uniquely determine theory (more precisely, the family of theories), which turns out to be governed by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\mathcal{S}\cosh\gamma - \sqrt{\mathcal{S}^2 + \mathcal{P}^2}\sinh\gamma\right), \tag{4.165}$$

where $\gamma > 0$ and the scalars $\mathcal{S}$ and $\mathcal{P}$ are the electromagnetic invariants

$$\mathcal{S} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}, \quad \mathcal{P} = \frac{1}{2}F_{\mu\nu}(\star F)^{\mu\nu}. \tag{4.166}$$

It reduces to Maxwell when $\gamma = 0$ and, if we were to have $\mathcal{P} = 0$, it would turn into a linear theory. We aim to construct a self-gravitating solution in the presence of electromagnetic fields governed by this theory, but first, let us study some of its defining features.

[5] Some of these small Taub-NUTs violate the reverse isoperimetric inequality, outlined in [63] as a universal principle. Whether this is reason enough to deem them unphysical or is simply a consequence of Taub-NUTs being a “counterexample to almost everything” is up for discussion.

To see how this theory is invariant under $SO(2)$ duality-rotations, we define the material tensor

$$E_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2\left(\mathcal{L}_S F_{\mu\nu} + \mathcal{L}_P \star F_{\mu\nu}\right) \tag{4.167}$$

with

$$\mathcal{L}_\mathcal{S} = \frac{\partial \mathcal{L}}{\partial \mathcal{S}} = \frac{1}{2}\left(\frac{\mathcal{S}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\sinh\gamma - \cosh\gamma\right), \quad \mathcal{L}_\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{P}} = \frac{1}{2}\frac{\mathcal{P}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\sinh\gamma. \tag{4.168}$$

The vacuum field equations, derived by minimizing the action, can then be written as

$$d\star E = 0, \quad dF = 0, \tag{4.169}$$

which are invariant under

$$\begin{pmatrix} E'_{\mu\nu} \\ \star F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_{\mu\nu} \\ *F_{\mu\nu} \end{pmatrix}. \tag{4.170}$$

We can see the fact that it is conformal by computing the trace of the stress-energy tensor. By using the Hilbert prescription

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{4.171}$$

we obtain the stress-energy tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\mathcal{S}g^{\mu\nu} - 2F^{\mu\sigma}F^{\nu}_{\sigma}\right)\mathcal{L}_\mathcal{S}, \tag{4.172}$$

from where the definitions (4.166) directly imply that it must be traceless.

ModMax Taub-NUT solutions

One of the most important steps to solve Einstein's equations in this modified theory of electrodynamics has already been completed: we have an expression for the stress-energy tensor (4.172), which is our input on the right hand side of Einstein's field equations. To solve for the metric, we need an ansatz which, inspired on the Taub-NUT solutions we have studied in the previous sections, should be of the form

$$ds^2 = -f(dt + 2n\cos\theta d\phi)^2 + \frac{dr^2}{f} + \left(r^2+n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.173}$$

where f depends only on the radial coordinate r. Additionally, since $dF = 0$, there exists a 1-form potential A such that $dA = F$. To find the electromagnetic fields, it suffices to solve for A, which we assume to be of the form

$$A = h(dt + 2n\cos\theta d\phi), \tag{4.174}$$

where h is again a function of r.

With this ansatz, the electromagnetic and material tensors F and E read

$$F = -h'dt \wedge dr + 2nh'\cos\theta dr \wedge d\phi - 2nh\sin\theta d\theta \wedge d\phi, \tag{4.175}$$

$$E = h'e^{\gamma}dt \wedge dr - 2nh'e^{\gamma}\cos\theta dr \wedge d\phi + 2nhe^{-\gamma}\sin\theta d\theta \wedge d\phi. \tag{4.176}$$

Then the field equations have a unique non-trivial component which reads

$$-e^{\gamma}\left[\left(n^2 + r^2\right)h'' + 2rh'\right] - \frac{4e^{-\gamma}n^2 h}{n^2 + r^2} = 0. \tag{4.177}$$

We solve this by first defining the electric and magnetic charges Q and Q_m enclosed by a surface $\mathcal{S}$ for this theory. In analogy to Maxwell, we define

$$Q^{enc} = \frac{1}{4\pi}\int_{\mathcal{S}} *E, \quad Q_m^{enc} = \frac{1}{4\pi}\int_{\mathcal{S}} F. \tag{4.178}$$

The rationale behind this definition is that electric and magnetic charges are related via the electromagnetic duality. As in the dyonic Taub-NUT, we define e and g as the asymptotic charges, enclosed by a sphere at infinity

$$Q = e = \frac{1}{4\pi}\int_{S_\infty} *E, \quad Q_m = -2ng = \frac{1}{4\pi}\int_{S_\infty} F, \tag{4.179}$$

Imposing these conditions, we can solve for h using (4.177), in terms of e and g. One obtains

$$h = -g\cos\left[e^{-\gamma}\left(\pi - 2\arctan\frac{r}{n}\right)\right] - \frac{e}{2n}\sin\left[e^{-\gamma}\left(\pi - 2\arctan\frac{r}{n}\right)\right]. \tag{4.180}$$

From the ansatz (4.173), Einstein's field equations reduce to a single differential equation for f:

$$\frac{3}{l^2} + \frac{f\left(n^2 - r^2\right) + \left(n^2 + r^2\right)\left(rf' - 1\right)}{\left(n^2 + r^2\right)^2} - e^{\gamma}h'^2 - \frac{4e^{-\gamma}n^2h^2}{\left(n^2 + r^2\right)^2} = 0. \tag{4.181}$$

From our knowledge of h, we can directly solve this first order equation for f and obtain

$$f = \frac{e^{-\gamma}\left(e^2+4n^2g^2\right)-2mr+r^2-n^2}{r^2+n^2} - \frac{3n^4-6n^2r^2-r^4}{l^2\left(r^2+n^2\right)}, \tag{4.182}$$

which is extremely similar to the Taub-NUT AdS solution, except for a rescaling of the electric and magnetic charges. Let us note that while the metric for this new theory is, in practice, indistinguishable from the Taub-NUT solution, the electromagnetic fields behave quite differently. The meaning is a bit self-evident: we do not seem to be able to probe any deviation from Maxwell using free falling particles, but we could do it indeed by placing test particles in a gravitational field (although this method is more convoluted than simpler tests that do not involve gravity).

Thermodynamics of the ModMax Taub-NUT

Now that we have the solution, we can establish the first law of thermodynamics for this slightly different Taub-NUT spacetime. Of course, the pure gravitational quantities remain unchanged so that the expressions for the mass, horizon entropy, temperature, and Misner potential are the same as in section 4.3.2. Since the calculation of the Misner charge N does involve the electromagnetic tensors, its expression will change somewhat.

Hoping that the first law works in the same way as it does for AdS NUTty dyons, we calculate the electric and magentic potential differences, as well as the horizon magnetic charge. These read

$$Q_m^{(+)} = 2nh\left(r_+\right) \tag{4.183}$$

$$\phi = g\left(\cos\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right)-1\right) + \frac{e}{2n}\sin\left(e^{-\gamma}\left(\pi-2\arctan\frac{r_+}{n}\right)\right), \tag{4.184}$$

$$\phi_m = \frac{e}{2n}\left(\cos\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right)-1\right) - g\sin\left(e^{-\gamma}\left(\pi-2\arctan\frac{r_+}{n}\right)\right). \tag{4.185}$$

With the Euclidean action prescription, we can also calculate the free energy:

$$\begin{aligned}\mathcal{G} =& \frac{m}{2} + \frac{eg}{2}\left(1-\cos\left(2e^{-\gamma}\left(\pi-2\arctan\frac{r_+}{n}\right)\right)\right) \\ &- \frac{e^2-4n^2g^2}{8n}\sin\left(2e^{-\gamma}\left(\pi-2\arctan\frac{r_+}{n}\right)\right) \\ &- \frac{r_+\left(3n^2+r_+^2\right)}{2l^2}.\end{aligned} \tag{4.186}$$

Continuing with the analogy with the Maxwell case, we assume the ensemble

$$\mathcal{G} = \mathcal{G}\left(T,\psi,\phi,Q_m^+,P\right) = M - TS - \phi Q - \psi N, \tag{4.187}$$

from which we can find the Misner charge N, which has a very lengthy and uninteresting expression. One can similarly assume that the Smarr relation

$$M = 2TS + \phi Q + \phi_m Q_m^+ + 2\psi N - 2PV \tag{4.188}$$

is satisfied and one finds, encouragingly, that the volume is still given by expression (4.143). With these definitions, we can verify the first law

$$dM = TdS + \psi dN + \phi dQ + \phi_m dQ_m^{(+)} + VdP. \tag{4.189}$$

Furthermore, it is also possible to derive a constrained electric first law by imposing $h(r_+) = 0$. Then we can find g in terms of e via

$$g = -\frac{e}{2n}\tan\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right). \tag{4.190}$$

Using the same methods as the unconstrained thermodynamics, we can use the expression for the free energy in this constrained case, which is simpler:

$$\mathcal{G} = \mathcal{G}(T,\psi,\phi,P) = \frac{m}{2} - \frac{r_+\left(3n^2 + r_+^2\right)}{2l^2} - \frac{e}{2n}\tan\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right). \tag{4.191}$$

We can thus solve for the Misner charge

$$\begin{aligned} N = -\frac{4\pi n^3}{r_+}\Bigg(&1 + \frac{3\left(n^2 - r_+^2\right)}{l^2} \\ &- e^2\left(e^{-\gamma}\sec^2\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right)\right. \\ &\left.\left. - \frac{r_+}{2n}\tan\left(e^{-\gamma}\left(\pi - 2\arctan\frac{r_+}{n}\right)\right)\right)\Bigg). \end{aligned} \tag{4.192}$$

The volume, which can be derived from the Smarr relation, is unchanged, and we finally obtain the electric first law

$$dM = TdS + \phi dQ + \psi dN + VdP. \tag{4.193}$$

After all our hard work, we have managed to find an explicit Taub-NUT solution in ModMax electrodynamics. We have learned that our proposed law of thermodynamics for Taub-NUTs holds beyond Maxwell electrodynamics, further validating our procedures. Can we find a more complicated solution, such as Kerr black holes? So far, we have not been able to find rotating solutions. The expected form of the metric with rescaled charges does not provide a simple equation to solve for the electromagnetic gauge one-form. But as previously argued, such a solution should be gravitationally indistinguishable from Kerr-NUT AdS.

4.4 Interpretation of results

We have managed to derive the first law of thermodynamics for a large family of asymptotically Taub-NUT spacetimes, including charged Taub-NUTs and Kerr-NUT spacetimes, both in the flat and AdS cases. To do this, we introduced the novel Misner charges ψ and N (or $\psi_\pm, N_\pm$ when the Misner strengths are allowed to vary). Although we have given a geometric prescription to calculate them, the actual physical interpretation of these thermodynamic quantities associated with the Misner strings is still unclear. Let us analyze a few possibilities.

The first possibility is that the Misner potential is the temperature of the Misner strings. Since we used a prescription similar to the calculation of the temperature to calculate the Misner potentials, namely formula (4.53), it is tempting to assign the surface gravity on the Misner strings to a temperature. Thus, we would have a system out of equilibrium, which is not uncommon in spacetimes with multiple finite temperature horizons (such as de Sitter spacetime [120]). The interpretation is not without merit. As in reference [94], we can rearrange the terms in the first law to obtain

$$dM = TdS' + \psi' dN' + VdP \tag{4.194}$$

with

$$S' = \frac{\pi\left(3r_+^4 + 12n^2r_+^2 + r_+^2l^2 - n^2l^2 - 3n^4\right)}{3n^2 + l^2 + 3r_+^2} \tag{4.195}$$

and

$$\psi' = -\frac{n\left(l^2 + 3n^2 - 3r_+^2\right)}{2\left(3n^2 + l^2 + 3r_+^2\right)}, \quad N' = \frac{n}{r_+} + \frac{3n\left(n^2 + r_+^2\right)}{r_+l^2}. \tag{4.196}$$

The entropy is now not simply the area of the black hole, but is, as argued in [105], is Noether charge. The prescription set forth by Wald in [65], which asserts that the spacetime entropy is the integral of a 2-form associated with diffeomorphism invariance, backs this interpretation. If S' is not the correct entropy of the spacetime, it would be the first time this prescription fails. Another virtue of this prescription is that the potential ψ' has a smooth limit as $n \to 0$.

However, we do not have any evidence that the Misner strings radiate; moreover, we have some evidence against it:

1. The Misner strings are transparent to geodesic motion, and they do not hide any singularity. It seems unlikely that they can store any information as a black hole does.

2. The $m = 0$ case is at finite temperature and has positive entropy as well, which seems more in line with what we would expect from a system at finite temperature.

The lore that Misner strings must store entropy comes from the Euclidean Taub-NUT, but we have given these up for a time now. A second proposal, which is vouched for in this thesis, is that the ψ and N terms introduced in this chapter represent angular frequency and angular momentum of the strings. This interpretation is argued for in [118], noting that the generating Killing vector of the strings is spacelike immediately outside them. Moreover, for the dyonic Kerr Taub-NUT, formula (4.130) is obtained, which has an unambiguous physical interpretation and can be made of full cohomogeneity. Another argument in favour is that, for the Kerr-NUT spacetime, if we define

$$J_{\pm} = \mp \frac{N_{\pm}}{4\pi} \tag{4.197}$$

and set $s = 0$, then we have that

$$J = J_H + J_+ + J_-, \tag{4.198}$$

where J is the total spacetime angular momentum, found using the conformal method. Therefore, the first law can be expressed in terms of these angular momenta. Moreover, this interpretation is useful for providing the first law for Kerr-AdS with the same physical meaning, as in Appendix E. However, such a law is not of full cohomogeneity, and we have not found a way to reduce it further. Finally, a smooth limit as $n \to 0$ is not really needed, since the limit corresponds to the Misner strings vanishing, thus abruptly changing the spacetime topology as well as its boundary. Therefore, having a singular limit might not be completely unexpected. Indeed, a similar situation is present when we take the limit $r_+ \to 0$ for the temperature Schwarzschild black hole.

In the end, the interpretation of the Misner charge and potential is still up for discussion. The only way to probe the physical meaning of the new charges further is to do a calculation a-la-Hawking to determine if pure Misner string spacetimes have a temperature. In the meantime, we can rest assured that Taub-NUT spacetimes obey the first law of thermodynamics. They have been further rehabilitated.

Chapter 5

Summary and discussion

In this thesis, we have reviewed some recent progress in the thermodynamics of scalar-Einstein spaces and Taub-NUT spacetimes, providing a prescription to calculate thermodynamic quantities and, in particular, the thermodynamic volume. We verified the universality of the laws of thermodynamics with variable cosmological constant, which we have now confirmed applies in the presence of scalar matter and in Lorentzian asymptotically locally flat spacetimes.

We have provided a prescription to calculate the thermodynamic volume in both the planar and hyperbolic cases in Einstein-scalar spacetimes. It should be straightforward to generalize this to spherical black holes; we did not include this discussion here because there do not exist many examples of spherical hairy black holes in the literature. For the first time, we exhibited the extended thermodynamics explicitly for the exact planar solutions proposed in [77] and [79] and for the well-known MTZ solution. Moreover, we showed that for planar black holes with scalar fields with asymptotic behaviour $O(1/r^2)$, the thermodynamic volume is proportional to the integral of the potential behind the horizon, and we exhibited further proof that this fact holds in more generality. However, for more general scalar field behaviours, the thermodynamic volume is modified with a term proportional to the trace of the holographic stress-energy tensor. We thus conjectured that the addition of these terms holds in more generality.

The study of the extended thermodynamics of black holes with scalar hair is far from being complete. The proof that the thermodynamic and geometric volumes are proportional, at least for sufficiently quickly decaying scalar fields, has not been yet verified for hyperbolic or spherical black holes in full generality, even though it seems to be valid from the examples presented in [63]. It is possible that the methods applied for the planar case can somehow extend to different horizon topologies, but an ingenious trick might need to be discovered. It is hard, but it would also be interesting to find explicit solutions for planar black holes with scalar fields with two asymptotic modes as in equation (3.10), thus verifying the extended thermodynamics presented in section 3.3. Additionally, we should look to extend the results of [42] to other non-asymptotically AdS spacetimes, if possible, to a more general class of asymptotics. We hope this will elucidate why our expression for the thermodynamic volume for hairy AdS spacetimes matches that of non-AdS spacetimes. Further investigation in these lines may give important insights into the nature of the VdP term in the first law.

We have thoroughly studied the thermodynamics of an array of Lorentzian Taub-NUT spacetimes. We did this by introducing two new thermodynamic quantities associated with the presence of Misner strings: the Misner potential ψ, associated with the surface gravity or the angular frequency of the strings; and the Misner charge N, which might be interpreted as their angular momentum. These quantities can be calculated via a geometric prescription. However, this method is not robust enough. Namely, we have been unsuccessful in establishing the first law for Kerr-AdS Taub-NUTs. This is not surprising since Komar integrals fail to prescribe thermodynamic quantities even for Kerr-AdS black holes.

Still, one might wonder where exactly we are failing. The answer might lie in the full Plebanski-Demianski metric [121], which is the more general solution that reduces to the Taub-NUT spacetimes by setting some parameters to zero. One of the parameters that we switch off is related to acceleration. Could it be that, when we drop it, we do not entirely eliminate the acceleration? Kerr-NUT-AdS would then have a cosmic string and, to obtain the full cohomogeneity first law, its tension must also be varied [122]. It is thus possible that the extended first law that we obtain in Appendix E accounts for such variations. It is worth pursuing further research in these lines. Moreover, picking up on the fact that black holes may have an acceleration, another area for additional work is to study accelerated NUT spacetimes, which were discovered in [123] and have been thoroughly researched in [124]. However, the first law has not been established yet.

Of course, a question that still needs to be addressed is the physical meaning of the Misner charge and the Misner potential. The first law of thermodynamics alone seems to have too many equivalent expressions to provide all answers. A way forward is to replicate Hawking's procedure and determine whether the Misner strings radiate. However, we have more contemporary approaches. A common approach is to probe the thermality of spacetimes by placing Unruh-DeWitt detectors in them and calculating their excitation-de-excitation rate. Using them as probes, one can determine whether a temperature can be associated with Rindler or black hole horizons. This method has been successful in

AdS-Schwarzschild spacetimes with both static and orbiting detectors [125].

The laws of thermodynamics for Taub-NUTs that we have derived here also apply for Euclidean Taub-NUTs with a non-compactified time coordinate; to obtain it, we only need to Wick-rotate the Lorentzian first law. However, working in such a regime could be frowned upon: the reason that time is identified in Euclidean Taub-NUTs is to ensure regularity. Time compactification is thus mandatory. Nevertheless, based on more recent results, the regularity of the Euclidean solutions does not seem to be necessary to establish the first law. This fact has already been shown for de Sitter spacetimes [126] and accelerated black holes [127], where the meaning of the thermodynamic variables and potentials is more apparent. Taub-NUT spacetimes in general and the unconstrained thermodynamics of NUTty dyons are the most prominent new examples where Euclidean regularity is given up in this thesis. We believe that it is worth investigating this further, either by arguing that regularity is unnecessary or by finding further examples that violate this condition.

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APPENDICES

Appendix A

A Mathematica Notebook to Calculate the Conformal Mass

Attached here is the Mathematica notebook to calculate the conformal mass. It has the conformal factor and the metric as input, and the output is the integrand to calculate the mass. The input can be in ansatz form, or it can be an explicit function. It uses the RGTC package for Mathematica. Here, as an example, we use it to compute the mass of the double-branch solution introduced in 3.1.5.

```
Get["edcrgtccode.m"]
```

Define the coordinates forms

```
In[1]:= dt = {1, 0, 0, 0, 0};
        dr = {0, 1, 0, 0, 0};
        dx = {0, 0, 1, 0, 0};
        dy = {0, 0, 0, 1, 0};
        dz = {0, 0, 0, 0, 1};
```

Define relevant metric

```
In[6]:= xCoord = {t, r, x, y, z};

In[7]:= g = (-f[r] × Outer[Times, dt, dt] + η^2 / f[r] Outer[Times, dr, dr] +
           Outer[Times, dx, dx] + Outer[Times, dy, dy] + Outer[Times, dz, dz])
```

Out[7]= $\left\{\{-f[r], 0, 0, 0, 0\}, \left\{0, \frac{\eta^2}{f[r]}, 0, 0, 0\right\}, \{0, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 1\}\right\}$

```
In[8]:= RGtensors[g, xCoord]
```

$$\text{gdd} = \begin{pmatrix} -f[r] & 0 & 0 & 0 & 0 \\ 0 & \frac{\eta^2}{f[r]} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{LineElement} = d[x]^2 + d[y]^2 + d[z]^2 + \frac{\eta^2\, d[r]^2}{f[r]} - d[t]^2 f[r]$$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{f[r]} & 0 & 0 & 0 & 0 \\ 0 & \frac{f[r]}{\eta^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
gUU computed in 0. sec
Gamma computed in 0. sec
Riemann(dddd) computed in 0. sec
Riemann(Uddd) computed in 0.016 sec
Ricci computed in 0. sec
Weyl computed in 0.016 sec
Einstein computed in 0. sec
All tasks completed in 0.03125 seconds
```

The normal is the derivative of the inverse conformal factor

In[9]:=
```
nD = Table[D[1 / Sqrt[Ω[r]], xCoord[[i]]], {i, 1, 5}]
```

Out[9]= $\left\{0, -\frac{\Omega'[r]}{2\,\Omega[r]^{3/2}}, 0, 0, 0\right\}$

Raise the index

In[10]:=
```
nU = Raise[nD, 1]
```

Out[10]= $\left\{0, -\frac{f[r]\,\Omega'[r]}{2\,\eta^2\,\Omega[r]^{3/2}}, 0, 0, 0\right\}$

Calculate Ω bar:

In[11]:=
```
Ωbar[r] = 1 / Sqrt[Ω[r]]
```

Out[11]= $\frac{1}{\sqrt{\Omega[r]}}$

In[12]:=
```
covD[nD]
```

Out[12]= $\left\{\left\{\frac{f[r]\,f'[r]\,\Omega'[r]}{4\,\eta^2\,\Omega[r]^{3/2}}, 0, 0, 0, 0\right\},\right.$

$\left\{0, -\frac{\Omega[r]\,f'[r]\,\Omega'[r] - 3\,f[r]\,\Omega'[r]^2 + 2\,f[r]\times\Omega[r]\,\Omega''[r]}{4\,f[r]\,\Omega[r]^{5/2}}, 0, 0, 0\right\},$

$\left.\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}$

Calculate the Weyl Tensor

In[13]:=
```
WUddd = Raise[Wdddd, 1];
```

In[14]:=
```
JUdd = multiDot[WUddd, nU, {2, 1}];
```

And the electric part

In[15]:=
```
EUd = l^2 Ωbar[r]^(-2) × multiDot[JUdd, nU, {3, 1}]
```

Out[15]= $\left\{\left\{-\frac{l^2\,f[r]\,\Omega'[r]^2\,f''[r]}{16\,\eta^4\,\Omega[r]^2}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{l^2\,f[r]\,\Omega'[r]^2\,f''[r]}{48\,\eta^4\,\Omega[r]^2}, 0, 0\right\},\right.$

$\left.\left\{0, 0, 0, \frac{l^2\,f[r]\,\Omega'[r]^2\,f''[r]}{48\,\eta^4\,\Omega[r]^2}, 0\right\}, \left\{0, 0, 0, 0, \frac{l^2\,f[r]\,\Omega'[r]^2\,f''[r]}{48\,\eta^4\,\Omega[r]^2}\right\}\right\}$

Finally calculate the integrand

In[25]:=
```
i1 = 𝔼Ud[[1, 1]]
```

Out[25]=
$$-\frac{l^2 f[r]\, \Omega'[r]^2 f''[r]}{16\, \eta^4\, \Omega[r]^2}$$

In[19]:=
```
indmet = -f[r] × Outer[Times, dt, dt] +
    Outer[Times, dx, dx] + Outer[Times, dy, dy] + Outer[Times, dz, dz];

h = Transpose[Drop[Transpose[Drop[indmet, {2}]], {2}]]
```

In[26]:=
```
i2 = Sqrt[-Det[h]]
```

Out[26]=
$$\sqrt{f[r]}$$

In[27]:=
```
integrand = (l^2 f[r] Ω'[r]^2 f''[r])/(48 η^4 Ω[r]^2) Sqrt[f[r]]
```

Out[27]=
$$\frac{l^2 f[r]^{3/2}\, \Omega'[r]^2 f''[r]}{48\, \eta^4\, \Omega[r]^2}$$

Appendix B

Calculating the Free Energy using Mathematica

Here is a Mathematica notebook to calculate the free energy. It takes the metric and induced metrics as inputs, and it outputs the bulk, boundary, and counterterm parts of the action. It is made to work in 4 dimensions; in higher dimensions, the user will manually change the counterterms. It uses the RGTC package for Mathematica. Here, as an example, we use it to compute the free energy for the dyonic Taub-NUT in AdS.

In[∘]:=
```
$Assumptions = ρ > 0 && θ > 0 && θ < π && r > 0 && l > 0 && n > 0 && r > n && ρ > n;
```

Bulk Einstein Hilbert Action

Call package

```
Get["edcrgtccode.m"]
```

Define coordinates and metric, and Wick rotate parameters that must be rotated.

In[∘]:=
```
xCoord = {t, r, χ, ϕ};

f[r_] := ((r^2 - 2 M r - n^2 + 4 n^2 g^2 + e^2) - 1 / l^2 (3 n^4 - 6 n^2 r^2 - r^4)) / (r^2 + n^2);
F[r_] := ((r^2 - 2 M r - n^2 + 4 n^2 g^2 + e^2) - 1 / l^2 (3 n^4 - 6 n^2 r^2 - r^4)) / (r^2 + n^2) /.
   {n → I n, e → I e, g → I g};
```

In[∘]:=
```
-F[r] × Outer[Times, {I, 0, 0, 2 I n χ}, {I, 0, 0, 2 I n χ}] +
 1 / F[r] Outer[Times, {0, 1, 0, 0}, {0, 1, 0, 0}] +
 (r^2 - n^2) / (1 - χ^2) Outer[Times, {0, 0, 1, 0}, {0, 0, 1, 0}] +
 (r^2 - n^2) (1 - χ^2) Outer[Times, {0, 0, 0, 1}, {0, 0, 0, 1}]
```

In[∘]:=

$$G = \left\{\left\{\frac{-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}}{-n^2+r^2},\; 0,\; 0,\; \frac{2n\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi}{-n^2+r^2}\right\},\right.$$

$$\left\{0,\; \frac{-n^2+r^2}{-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}},\; 0,\; 0\right\},\; \left\{0,\; 0,\; \frac{-n^2+r^2}{1-\chi^2},\; 0\right\},$$

$$\left.\left\{\frac{2n\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi}{-n^2+r^2},\; 0,\; 0,\; \frac{4n^2\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi^2}{-n^2+r^2}+\left(-n^2+r^2\right)\left(1-\chi^2\right)\right\}\right\};$$

Compute the volume element

In[∘]:=
```
volelm = Sqrt[Det[G]] // FullSimplify
```

Out[∘]= $-n^2+r^2$

In[∘]:=
```
RGtensors[G, xCoord]
```

$$\text{gdd} = \left\{\left\{\frac{-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}}{-n^2+r^2}, 0, 0, \frac{2n\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi}{-n^2+r^2}\right\},\right.$$
$$\left\{0, \frac{-n^2+r^2}{-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}}, 0, 0\right\}, \left\{0, 0, \frac{-n^2+r^2}{1-\chi^2}, 0\right\},$$
$$\left\{\frac{2n\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi}{-n^2+r^2}, 0, 0,\right.$$
$$\left.\left.\frac{4n^2\left(-e^2+n^2+4g^2n^2-2Mr+r^2-\frac{3n^4+6n^2r^2-r^4}{l^2}\right)\chi^2}{-n^2+r^2}+\left(-n^2+r^2\right)\left(1-\chi^2\right)\right\}\right\}$$

$$\text{LineElement} = \frac{l^2\left(n^2-r^2\right)d[r]^2}{e^2l^2-l^2n^2-4g^2l^2n^2+3n^4+2l^2Mr-l^2r^2+6n^2r^2-r^4}+$$
$$\frac{\left(e^2l^2-l^2n^2-4g^2l^2n^2+3n^4+2l^2Mr-l^2r^2+6n^2r^2-r^4\right)d[t]^2}{l^2\left(n^2-r^2\right)}+$$
$$\frac{4n\left(e^2l^2-l^2n^2-4g^2l^2n^2+3n^4+2l^2Mr-l^2r^2+6n^2r^2-r^4\right)\chi\, d[t]\times d[\phi]}{l^2\left(n^2-r^2\right)}+$$
$$\frac{1}{l^2\left(n^2-r^2\right)}\left(-l^2n^4+2l^2n^2r^2-l^2r^4+4e^2l^2n^2\chi^2-3l^2n^4\chi^2-16g^2l^2n^4\chi^2+12n^6\chi^2+\right.$$
$$\left.8l^2Mn^2r\chi^2-6l^2n^2r^2\chi^2+24n^4r^2\chi^2+l^2r^4\chi^2-4n^2r^4\chi^2\right)d[\phi]^2+\frac{\left(n^2-r^2\right)d[\chi]^2}{-1+\chi^2}$$

$$\text{gUU} =$$
$$\left\{\left\{\left(-l^2n^4+2l^2n^2r^2-l^2r^4+4e^2l^2n^2\chi^2-3l^2n^4\chi^2-16g^2l^2n^4\chi^2+12n^6\chi^2+8l^2Mn^2r\chi^2-6l^2n^2r^2\chi^2+\right.\right.\right.$$
$$\left.24n^4r^2\chi^2+l^2r^4\chi^2-4n^2r^4\chi^2\right)\Big/$$
$$\left(\left(n^2-r^2\right)\left(e^2l^2-l^2n^2-4g^2l^2n^2+3n^4+2l^2Mr-l^2r^2+6n^2r^2-r^4\right)\left(-1+\chi^2\right)\right), 0, 0,$$
$$\left.-\frac{2n\chi}{\left(n^2-r^2\right)\left(-1+\chi^2\right)}\right\}, \left\{0, -\frac{-e^2l^2+l^2n^2+4g^2l^2n^2-3n^4-2l^2Mr+l^2r^2-6n^2r^2+r^4}{l^2\left(n^2-r^2\right)}, 0, 0\right\},$$
$$\left.\left\{0, 0, \frac{-1+\chi^2}{n^2-r^2}, 0\right\}, \left\{-\frac{2n\chi}{\left(n^2-r^2\right)\left(-1+\chi^2\right)}, 0, 0, \frac{1}{\left(n^2-r^2\right)\left(-1+\chi^2\right)}\right\}\right\}$$

```
gUU computed in 0.032 sec
Gamma computed in 0.047 sec
Riemann(dddd) computed in 0.125 sec
Riemann(Uddd) computed in 0.093 sec
Ricci computed in 0.032 sec
Weyl computed in 0.296 sec
Einstein computed in 0.016 sec
All tasks completed in 0.65625 seconds
```

Compute the bulk action

In[∘]:= `SEH = -1 / (4) Integrate[-6 / l^2 volelm, {r, rh, ρ}] /. {n → -I n}`

Out[∘]= $\frac{1}{4}\left(-\frac{6\,n^2\,rh}{l^2}-\frac{2\,rh^3}{l^2}+\frac{6\,n^2\,\rho}{l^2}+\frac{2\,\rho^3}{l^2}\right)$

Boundary action

Find the unit normal

In[∘]:= `normal = {0, 1 / Sqrt[F[r]], 0, 0}`

Out[∘]= $\left\{0,\ \frac{1}{\sqrt{\frac{-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}}{-n^2+r^2}}},\ 0,\ 0\right\}$

In[∘]:= `normal.gUU.normal // FullSimplify`

Out[∘]= 1

Find the extrinsic curvature

In[∘]:= `Kdd = covD[normal] // FullSimplify;`

In[∘]:= `Kext = Tr[gUU.Kdd] // FullSimplify`

Out[∘]= $\frac{-3\,n^4\,r+14\,n^2\,r^3-3\,r^5+l^2\left(-M\,n^2+e^2\,r-4\,g^2\,n^2\,r+3\,M\,r^2-2\,r^3\right)}{l\,(n-r)\,(n+r)\,\sqrt{\left(-n^2+r^2\right)\left(-e^2\,l^2-3\,n^4-6\,n^2\,r^2+r^4+l^2\left(n^2+4\,g^2\,n^2-2\,M\,r+r^2\right)\right)}}$

Input the induced metric

In[∘]:= $H=\left\{\left\{\frac{-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}}{-n^2+r^2},\ 0,\ \frac{2\,n\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi}{-n^2+r^2}\right\},\ \left\{0,\ \frac{-n^2+r^2}{1-\chi^2},\ 0\right\},\ \left\{\frac{2\,n\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi}{-n^2+r^2},\ 0,\ \frac{4\,n^2\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi^2}{-n^2+r^2}+\left(-n^2+r^2\right)\left(1-\chi^2\right)\right\}\right\};$

And calculate its determinant

In[∘]:= `Sqrt[Det[H]] // FullSimplify`

Out[∘]= $\sqrt{-\frac{(n-r)\,(n+r)\left(-e^2\,l^2-3\,n^4-6\,n^2\,r^2+r^4+l^2\left(n^2+4\,g^2\,n^2-2\,M\,r+r^2\right)\right)}{l^2}}$

Then calculate the boundary action and Wick rotate back

In[ ]:= **Series[-1 / 2 Sqrt[Det[H]] (Kext - 2 / l), {r, ∞, 1}]**

Out[ ]= $$-\frac{r^3}{2\,l^2}+\left(-\frac{1}{2}+\frac{2\,n^2}{l^2}\right)r+\frac{M}{2}+\frac{-l^4+6\,l^2\,n^2-5\,n^4}{8\,l^2\,r}+O\left[\frac{1}{r}\right]^2$$

In[ ]:= **Gh1 =** $$\left(-\frac{r^3}{2\,l^2}+\left(-\frac{1}{2}+\frac{2\,n^2}{l^2}\right)r+\frac{M}{2}\right)$$ **/. {r → ρ, n → I n}**

Out[ ]= $$\frac{M}{2}+\left(-\frac{1}{2}-\frac{2\,n^2}{l^2}\right)\rho-\frac{\rho^3}{2\,l^2}$$

Finally the counterterm action

In[ ]:= **Coord = {t, χ, ϕ};**

In[ ]:= **RGtensors[H, Coord]**

$$\text{gdd} = \begin{pmatrix} \frac{-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}}{-n^2+r^2} & 0 & \frac{2\,n\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi}{-n^2+r^2} \\ 0 & \frac{-n^2+r^2}{1-\chi^2} & 0 \\ \frac{2\,n\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi}{-n^2+r^2} & 0 & \frac{4\,n^2\left(-e^2+n^2+4\,g^2\,n^2-2\,M\,r+r^2-\frac{3\,n^4+6\,n^2\,r^2-r^4}{l^2}\right)\chi^2}{-n^2+r^2}+\left(-n^2+r^2\right)\left(1-\chi^2\right) \end{pmatrix}$$

$$\text{LineElement} = \frac{\left(e^2\,l^2-l^2\,n^2-4\,g^2\,l^2\,n^2+3\,n^4+2\,l^2\,M\,r-l^2\,r^2+6\,n^2\,r^2-r^4\right)d[t]^2}{l^2\left(n^2-r^2\right)}+$$
$$\frac{4\,n\left(e^2\,l^2-l^2\,n^2-4\,g^2\,l^2\,n^2+3\,n^4+2\,l^2\,M\,r-l^2\,r^2+6\,n^2\,r^2-r^4\right)\chi\,d[t]\times d[\phi]}{l^2\left(n^2-r^2\right)}+$$
$$\frac{1}{l^2\left(n^2-r^2\right)}\left(-l^2\,n^4+2\,l^2\,n^2\,r^2-l^2\,r^4+4\,e^2\,l^2\,n^2\,\chi^2-3\,l^2\,n^4\,\chi^2-16\,g^2\,l^2\,n^4\,\chi^2+12\,n^6\,\chi^2+\right.$$
$$\left.8\,l^2\,M\,n^2\,r\,\chi^2-6\,l^2\,n^2\,r^2\,\chi^2+24\,n^4\,r^2\,\chi^2+l^2\,r^4\,\chi^2-4\,n^2\,r^4\,\chi^2\right)d[\phi]^2+\frac{\left(n^2-r^2\right)d[\chi]^2}{-1+\chi^2}$$

$$\text{gUU} = \begin{pmatrix} \frac{-l^2\,n^4+2\,l^2\,n^2\,r^2-l^2\,r^4+4\,e^2\,l^2\,n^2\,\chi^2-3\,l^2\,n^4\,\chi^2-16\,g^2\,l^2\,n^4\,\chi^2+12\,n^6\,\chi^2+8\,l^2\,M\,n^2\,r\,\chi^2-6\,l^2\,n^2\,r^2\,\chi^2+24\,n^4\,r^2\,\chi^2+l^2\,r^4\,\chi^2-4\,n^2\,r^4\,\chi^2}{\left(n^2-r^2\right)\left(e^2\,l^2-l^2\,n^2-4\,g^2\,l^2\,n^2+3\,n^4+2\,l^2\,M\,r-l^2\,r^2+6\,n^2\,r^2-r^4\right)\left(-1+\chi^2\right)} & 0 & -\frac{2\cdots}{\left(n^2-r^2\cdots\right.} \\ 0 & \frac{-1+\chi^2}{n^2-r^2} & \\ -\frac{2\,n\,\chi}{\left(n^2-r^2\right)\left(-1+\chi^2\right)} & 0 & \frac{\cdots}{\left(n^2-r^2\right)\cdots} \end{pmatrix}$$

```
gUU computed in 0.031 sec

Gamma computed in 0.016 sec

Riemann(dddd) computed in 0.016 sec

Riemann(Uddd) computed in 0.078 sec

Ricci computed in 0.015 sec

Weyl computed in 0. sec
```

In[∘]:= **R**

Expand in series to find asymptotic behaviour

In[∘]:= $\text{Series}\left[1/2\,l/2\,\text{Sqrt}[\text{Det}[H]]\,\frac{2\left(-e^2 l^2 n^2+4 g^2 l^2 n^4-3 n^6-2 l^2 M n^2 r+3 l^2 n^2 r^2-6 n^4 r^2-l^2 r^4+n^2 r^4\right)}{l^2 (n-r)^3 (n+r)^3},\{r,\infty,2\}\right]$

Out[∘]= $\left(\frac{1}{2}-\frac{n^2}{2 l^2}\right) r+\frac{\frac{l^2}{4}-2 n^2+\frac{13 n^4}{4 l^2}}{r}+\frac{-\frac{l^2 M}{2}+\frac{3 M n^2}{2}}{r^2}+O\left[\frac{1}{r}\right]^3$

Calculate result and Wick rotate back

In[∘]:= $\text{Gh2}=\left(\frac{1}{2}-\frac{n^2}{2 l^2}\right) r\;/.\;\{r\to\rho,\,n\to I\,n\}$

Out[∘]= $\left(\frac{1}{2}+\frac{n^2}{2 l^2}\right)\rho$

Sum all terms of gravitational action

In[∘]:= **SEH + Gh1 + Gh2 // FullSimplify**

Out[∘]= $-\frac{-l^2 M+3 n^2 rh+rh^3}{2 l^2}$

Maxwell part of the action

Input gauge field

In[∘]:= **A = (e r / (r^2 + n^2) + g (r^2 - n^2) / (r^2 + n^2)) {I, 0, 0 , 2 n χ} /.
{n → I n, e → I e, g → I g} // FullSimplify**

Out[∘]= $\left\{\frac{e\,r+g\left(n^2+r^2\right)}{n^2-r^2},\,0,\,0,\,\frac{2 n\left(e\,r+g\left(n^2+r^2\right)\right)\chi}{n^2-r^2}\right\}$

Calculate F^2

In[∘]:=
```
F = Table[D[A[[j]], xCoord[[i]]] - D[A[[i]], xCoord[[j]]], {i, 1, 4}, {j, 1, 4}]
```

Out[∘]=
$$\left\{\left\{0, -\frac{e+2gr}{n^2-r^2}-\frac{2r\left(er+g\left(n^2+r^2\right)\right)}{\left(n^2-r^2\right)^2}, 0, 0\right\},\right.$$
$$\left\{\frac{e+2gr}{n^2-r^2}+\frac{2r\left(er+g\left(n^2+r^2\right)\right)}{\left(n^2-r^2\right)^2}, 0, 0, \frac{2n\,(e+2gr)\,\chi}{n^2-r^2}+\frac{4nr\left(er+g\left(n^2+r^2\right)\right)\chi}{\left(n^2-r^2\right)^2}\right\},$$
$$\left\{0, 0, 0, \frac{2n\left(er+g\left(n^2+r^2\right)\right)}{n^2-r^2}\right\},$$
$$\left.\left\{0, -\frac{2n\,(e+2gr)\,\chi}{n^2-r^2}-\frac{4nr\left(er+g\left(n^2+r^2\right)\right)\chi}{\left(n^2-r^2\right)^2}, -\frac{2n\left(er+g\left(n^2+r^2\right)\right)}{n^2-r^2}, 0\right\}\right\}$$

In[∘]:=
```
FUU = gUU.F.gUU;
```

In[∘]:=
```
F2 = Tr[FUU.F] // FullSimplify
```

Out[∘]=
$$-\frac{(e+2gn)^2}{(n-r)^4}-\frac{(e-2gn)^2}{(n+r)^4}$$

Integrate and Wick rotate back

In[∘]:=
```
1 / 4 Integrate[volelm F2, r]
```

In[∘]:=
$$\frac{r\left(8egn^2r+e^2\left(n^2+r^2\right)+4g^2n^2\left(n^2+r^2\right)\right)}{2\,(n-r)^2\,(n+r)^2}\ \text{/. \{r → rh\}}$$

In[∘]:=
$$\frac{rh\left(8egn^2rh+e^2\left(n^2+rh^2\right)+4g^2n^2\left(n^2+rh^2\right)\right)}{2\,(n-rh)^2\,(n+rh)^2}\ \text{/.}$$
```
{e → -I e, n → -I n, g → -I g} // FullSimplify
```

Out[∘]=
$$\frac{rh\,(2gn\,(-n+rh)+e\,(n+rh))\,(e\,(n-rh)+2gn\,(n+rh))}{2\left(n^2+rh^2\right)^2}$$

Appendix C

A Mathematica Notebook for Hamiltonian perturbations

The following is the Mathematica notebook used to compute Hamiltonian perturbations automatically. It takes the perturbed and background metrics as inputs and returns the first law in terms of the variations of the parameters. No additional packages are necessary. This notebook was used for the computations in Chapter 3. Shown below, as an example, is the calculation for the thermodynamics of hairy black branes as presented in 3.3.

First we define our dimensions and coordinates

```
In[ ]:= coord = {t, r, x, y, z};
n = 5;
Clear[B]
```

The following is boilerplate code. The inputs are AsMetric, which is the perturbed metric, and BMetric, the background metric. Make sure to put an ϵ symbol together with any small variation.

```
In[ ]:= AsMetric = {{- (r^2 / l^2 - 2 r^2 / l^3 ϵ δl) (h[r] + ϵ δh[r]), 0, 0, 0, 0},
    {0, (l^2 / r^2 + 2 l ϵ δl / r^2) (σ[r] + ϵ δσ[r]) / (h[r] + ϵ δh[r]), 0, 0, 0},
    {0, 0, r^2 / l^2 (1 + 2 ϵ δl1 / l1 - 2 / l ϵ δl), 0, 0},
    {0, 0, 0, r^2 / l^2 (1 + 2 ϵ δl2 / l2 - 2 / l ϵ δl), 0},
    {0, 0, 0, 0, r^2 / l^2 (1 + 2 ϵ δl3 / l3 - 2 / l ϵ δl)}};
BMetric = {{-r^2 / l^2 h[r], 0, 0, 0, 0}, {0, l^2 / r^2 σ[r] / h[r] , 0, 0, 0},
    {0, 0, r^2 / l^2, 0, 0}, {0, 0, 0, r^2 / l^2, 0}, {0, 0, 0, 0, r^2 / l^2 }};
PMetric = Simplify[AsMetric - BMetric];
IBMetric = Transpose[Drop[Transpose[Drop[BMetric, {1}]], {1}]];
IPMetric = Transpose[Drop[Transpose[Drop[PMetric, {1}]], {1}]];
InverseIBMetric = Inverse[IBMetric];
IPMetricUU = InverseIBMetric.IPMetric.Transpose[InverseIBMetric];
F = PowerExpand[Sqrt[-BMetric[[1, 1]]]];
affine = Simplify[Table[(1 / 2) * Sum[(InverseIBMetric[[i, s]]) *
        (D[IBMetric[[s, j]], coord[[k + 1]] ] +
          D[IBMetric[[s, k]], coord[[j + 1]] ] - D[IBMetric[[j, k]], coord[[s + 1]] ]),
       {s, 1, n - 1}],
     {i, 1, n - 1}, {j, 1, n - 1}, {k, 1, n - 1}] ];
CovdPert = FullSimplify[Table[Sum[D[IPMetricUU[[i, j]], coord[[j + 1]]], {j, 1, 4}] +
      Sum[Sum[affine[[i, j, k]] × IPMetricUU[[j, k]], {j, 1, 4}], {k, 1, 4}] +
      Sum[Sum[affine[[j, j, k]] × IPMetricUU[[i, k]], {j, 1, 4}], {k, 1, 4}], {i, 1, 4}]];
TraceP = Tr[InverseIBMetric.IPMetric];
b = FullSimplify[Table[
     F (Sum[InverseIBMetric[[i, j]] × D[TraceP, coord[[j + 1]]], {j, 1, n - 1}] - CovdPert[[
          i]]) - TraceP Sum[InverseIBMetric[[i, j]] × D[F, coord[[j + 1]]], {j, 1, n - 1}] +
      Sum[IPMetricUU[[i, j]] × D[F, coord[[j + 1]]], {j, 1, n - 1}], {i, 1, n - 1}]];
DetMetric = PowerExpand[Sqrt[Det[Transpose[Drop[Transpose[Drop[IBMetric, {1}]], {1}]]]]];
Bϕ[r_] := ϕ[r];
Aϕ[r_] := ϕ[r] + ϵ δϕ[r];
Pϕ[r_] := Aϕ[r] - Bϕ[r];
bMatter = Table[F (Sum [InverseIBMetric[[ i, j]] × D[Bϕ[r], coord[[j + 1]]], {j, 1, n - 1}]),
     {i, 1, n - 1}] * (Pϕ[r]);
Ac = {PowerExpand[Sqrt[BMetric[[2, 2]]]], 0, 0, 0};
Integrand = DetMetric Ac.(b + bMatter) // FullSimplify
```

The output is the variations in terms of the metric functions

Out[ ]=
```
(r^4 ∈ (-r (l l2 l3 δl1 + l1 (-3 l2 l3 δl + l l3 δl2 + l l2 δl3)) ∈ δh[r] × σ[r] h′[r] +
    l1 l2 l3 h[r]^2 (-3 (l + 2 δl ∈) δσ[r] + σ[r] (-6 δl + l r δϕ[r] ϕ′[r])) +
    h[r] × σ[r] (-r (l l2 l3 δl1 + l1 (-3 l2 l3 δl + l l3 δl2 + l l2 δl3)) h′[r] +
       l l1 l2 l3 δh[r] (3 + r ∈ δϕ[r] ϕ′[r])))) / (l^6 l1 l2 l3 (h[r] + ∈ δh[r]) σ[r]^(3/2))
```

Expanding in series in ϵ to first order gives the first law. One still has to do some work to express these variations in terms of the thermodynamic charges.

In[ ]:=
```
Series[(r^4 ∈ (-r (l l2 l3 δl1 + l1 (-3 l2 l3 δl + l l3 δl2 + l l2 δl3)) ∈ δh[r] × σ[r] h′[r] +
      l1 l2 l3 h[r]^2 (-3 (l + 2 δl ∈) δσ[r] + σ[r] (-6 δl + l r δϕ[r] ϕ′[r])) +
      h[r] × σ[r] (-r (l l2 l3 δl1 + l1 (-3 l2 l3 δl + l l3 δl2 + l l2 δl3)) h′[r] +
         l l1 l2 l3 δh[r] (3 + r ∈ δϕ[r] ϕ′[r])))) /
   (l^6 l1 l2 l3 (h[r] + ∈ δh[r]) σ[r]^(3/2)), {∈, 0, 1}]
```

Out[ ]=
```
          1
 ───────────────────────
 l^6 l1 l2 l3 h[r] σ[r]^(3/2)
  r^4 (h[r] × σ[r] (3 l l1 l2 l3 δh[r] - r (l l2 l3 δl1 + l1 (-3 l2 l3 δl + l l3 δl2 + l l2 δl3)) h′[r]) +
     l1 l2 l3 h[r]^2 (-3 l δσ[r] + σ[r] (-6 δl + l r δϕ[r] ϕ′[r]))) ∈ + O[∈]^2
```

Appendix D

Geodesic completeness of Kerr-NUT spacetimes

In this appendix, we present a proof of geodesic transparency for Kerr-NUT spacetimes. Using the Hamilton-Jacobi formalism and with the variable $\xi = \cos\theta$, one finds the following first order equations for timelike geodesics with the Mino time as the world-line parameter.

$$\frac{dt}{d\lambda} = \frac{P(r)(P(r) - aL)}{E\Delta} + \frac{a(1-\xi^2) + 2n\xi}{1-\xi^2} Q(\xi); \tag{D.1}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = P(r)^2 - \Delta(r^2 + k); \tag{D.2}$$

$$\left(\frac{d\xi}{d\lambda}\right)^2 = (1-\xi^2)(k - (n - a\xi)^2) - Q(\xi)^2; \tag{D.3}$$

$$\frac{d\varphi}{d\lambda} = \frac{a}{\Delta} P(r) + \frac{Q(\xi)}{1-\xi^2}; \tag{D.4}$$

with $P(r) = (r^2 + n^2 + a^2)E + aL$ and $Q(\xi) = L - (a(1-\xi^2) + 2n\xi)E$. The parameters a and n have been normalized by $2M$, and the constants E and L are to be interpreted as the energy and angular momentum per unit mass of the test particle and k is Carter's constant. From equation (D.3) we observe that k must be positive. Let us now assume that the geodesic passes through the north Misner string, so that $\xi = 1$ at the value $\lambda = 0$ of the parameter. This must be an extremum of ξ, so $d\xi/d\lambda$=0 implies that

$$L = 2nE. \tag{D.5}$$

The same can be done for the south Misner string, and we obtain $L = -2nE$. Given that this case is analogous to the first, we will only focus on geodesics passing through $\xi = 1$. The polynomial $Q(\xi)$ is quartic in ξ, so the existence of a zero at $\xi = 1$ implies that there is at least another real zero. This polynomial must be positive in some interval $(\xi_0, 1)$ where $-1 \leq \xi_0 \leq 1$ is another real zero of $Q(\xi)$. This requirement constrains the value of Carter's constant k in terms of the energy. Then the solution to the differential equation for ξ reads

$$\xi(\lambda) = \frac{\xi_0 - \beta + \xi_0(\beta - 1)\mathrm{sn}^2\left(\frac{1}{2}\lambda\sqrt{(1-\alpha)(\xi_0-\beta)}|\frac{(1-\beta)(\xi_0-\alpha)}{(1-\alpha)(\xi_0-\beta)}\right)}{\xi_0 - \beta + (\beta - 1)\mathrm{sn}^2\left(\frac{1}{2}\lambda\sqrt{(1-\alpha)(\xi_0-\beta)}|\frac{(1-\beta)(\xi_0-\alpha)}{(1-\alpha)(\xi_0-\beta)}\right)}, \tag{D.6}$$

where $\mathrm{sn}(\lambda, k)$ is Jacobi's sinus amplitudinis function and an integration constant was chosen so that $\xi(0) = 1$. The constants α and β are the two other zeros of $Q(\xi)$, which in principle could be complex numbers. However, the requirement that $\xi = 1$ be a local maximum for $Q(\xi)$ fixes α and β to be real and, moreover, one of them has to be greater than 1 and the other one is smaller than ξ_0. Without loss of generality, we can choose $\beta > \alpha$ so that the modulus of the sinus function remains smaller than 1. In such case, $\xi(\lambda)$ oscillates between the values ξ_0 and 1 as we would expect[1].

In order to solve for $\varphi(\lambda)$, we need to solve for the radial coordinate $r(\lambda)$ first. We assume that the particle stays in the region where $\Delta > 0$. This means that the function $r(\lambda)$ has a minimum r_0, which is a zero of $P(r)^2 - \Delta(r^2 + k)$. The solution for r is similar to that of ξ, given that the right hand side of the equation is also a polynomial of order 4 in r. Unlike the previous case, the presence of a minimum does not constrain the zeros of the polynomials to be real. We have

$$r(\lambda) = \frac{r_0 r_1 - v r_0 + r_1(v - r_0)\mathrm{sn}^2\left(\frac{1}{2}\lambda\sqrt{(r_0-u)(r_1-v)}|\frac{(r_0-v)(r_1-u)}{(r_0-u)(r_1-v)}\right)}{r_1 - v + (v - r_0)\mathrm{sn}^2\left(\frac{1}{2}\lambda\sqrt{(r_0-u)(r_1-v)}|\frac{(r_0-v)(r_1-u)}{(r_0-u)(r_1-v)}\right)}, \tag{D.7}$$

where r_1 is another real zero (not necessarily positive) and u and v are, in general, complex zeros[2]. The orbit will be bounded as long as all the zeros are real and $r_1 > r_0$, and it will be unbounded in any other case.

[1] Observe that the frequency is imaginary, which means that the sinus functions given a purely imaginary number. This functions appears squared, so this is not an issue. In the following, we let the frequency be imaginary because the expressions are simpler. The properties of the elliptic functions guarantee that the solutions remain real

[2] The sinus function can be analytically extended to the complex plane as a function of the modulus.

The angular function $\varphi(\lambda)$ depends on r and ξ, and we can write $\varphi = \varphi_r + \varphi_\xi$. The latter satisfies the differential equation

$$\frac{d\varphi_\xi}{d\lambda} = -aE + \frac{2nE}{1+\xi}. \tag{D.8}$$

This can be integrated in terms of the incomplete elliptic function of the third kind and the Jacobi amplitude function:

$$\varphi_\xi(\lambda) = \left(-aE + \frac{2nE}{1+\xi_0}\right)\lambda + \frac{1}{\omega}\left(\frac{2nE}{1+\xi_0} + nE\right)\Pi\left(\frac{(1+\xi_0)(1-\beta)}{2(\xi_0-\beta)}; \mathrm{am}(\omega\lambda|m)|m\right) \tag{D.9}$$

with modulus $m = \frac{(1-\beta)(\xi_0-\alpha)}{(1-\alpha)(\xi_0-\beta)}$ and $\omega = \sqrt{(1-\alpha)(\xi_0-\beta)}$. The solution of φ_r is more involved, but we can observe that it will have the same form as φ_ξ if we decompose the equation for $d\varphi_r/d\lambda$ into partial fractions. This assumes that Δ has two real zeros R_1 and R_2, which is guaranteed by the presence of a horizon. Upon doing this, the differential equation can be rewritten as

$$\frac{d\varphi_r}{d\lambda} = aE\left(1 + \frac{2}{R_2 - R_1}\left(\frac{n^2 - an + R_1}{r - R_1} - \frac{n^2 - an + R_2}{r - R_2}\right)\right). \tag{D.10}$$

Thus, we can obtain an analytic solution for $\varphi(r)$ in terms of the elliptic function of the third kind. Bounded orbits have a real frequency in the radial coordinate, and unbounded orbits reach $r \to \infty$ in a finite proper time. The term $aE\lambda$ is cancelled in the full function $\varphi(\lambda)$ and the periodicity of $\varphi_\xi(\lambda)$ matches that of $\xi(\lambda)$. Although we observe that $\varphi(\lambda)$ is periodically discontinuous, this is not a problem as it indicates the times at which the Misner string is crossed. The φ angle can then change abruptly, given that this coordinate is defined neither at the north pole nor the south pole. It follows that the Misner strings are transparent to geodesic motion.

Appendix E

Progress on the thermodynamics of Kerr-NUT-AdS

In this appendix, we show our advances in establishing the first law for rotating Taub-NUTs in anti-de Sitter spacetime. With $s = 0$, the metric for this spacetime is given by

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\Xi^2\rho^2}\left[dt - \left\{a\sin^2\theta + 2n(1-\cos\theta)\right\}d\phi\right]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\Xi^2\rho^2}\left[adt - \left\{r^2 + (a+n)^2\right\}\right]d\phi^2 \end{aligned} \tag{E.1}$$

where

$$\rho^2 = r^2 + (n + a\cos\theta)^2, \tag{E.2}$$

$$\Delta_r = r^2 - 2Mr + \frac{r^2\left(r^2 + 6n^2 + a^2\right)}{\ell^2} + \frac{\left(3n^2 - \ell^2\right)\left(a^2 - n^2\right)}{\ell^2}, \tag{E.3}$$

$$\Delta_\theta = 1 - \frac{a\cos\theta}{l^2}(4n + a\cos\theta), \quad \Xi = 1 - \frac{a^2}{l^2}, \Lambda = -\frac{3}{l^2}. \tag{E.4}$$

The usual methods give the following asymptotic charges

$$M = \frac{m}{\Xi^2}, \quad J = \frac{am}{\Xi^2}. \tag{E.5}$$

The temperature and entropy can also be calculated as usual:

$$T = \frac{\Delta'\left(r_+\right)}{4\pi\left(r_+^2 + a^2 + n^2 - 2as\right)}, \quad S = \frac{\pi\left(r_+^2 + a^2 + n^2 - 2as\right)}{\Xi}. \tag{E.6}$$

The horizon angular velocity and angular momentum read

$$\Omega_H = \frac{aK}{r_+^2 + n^2 + a^2}, \quad J_H = \frac{a\left(a^2 + n^2 + r_+^2\right)\left(l^2 + 3n^2 + r_+^2\right)}{2K^2 l^2 r_+}. \tag{E.7}$$

We depart from chapter 4 in that we prefer a different normalization for the Misner potentials, giving them the interpretation of angular velocity instead.

$$\Omega_\pm = 4\pi\psi_\pm = \mp\frac{\Xi}{2n} \tag{E.8}$$

Moreover, we define the Misner charges by regularizing the integral of $\star d\phi$ over the tubes instead of integrating $\star dt$. Namely

$$J_\pm = \frac{1}{16\pi}\int_{T_\pm} \star d\phi + 2\Lambda\Omega^\pm_{(r=\infty)} + \frac{L}{4\Omega_\pm}, \tag{E.9}$$

where adding a term with the length L of the Misner string render this integral finite. Explicitly:

$$J_\pm = \pm\frac{n^2\left(n\left(\ell^2 + 3n^2 - 3r_+^2\right) \mp a\left(\ell^2 + 3n^2 - r_+^2\right)\right)}{2\Xi\ell^2 r_+} \tag{E.10}$$

With these definitions, we obtain that the sum of angular momenta is the total angular momentum

$$J_H + J_+ + J_- = J. \tag{E.11}$$

This frame is rotating at infinity with $\Omega_\infty = -a/\ell^2$. Defining for all the angular velocities $\Omega^{rel} = \Omega - \Omega_\infty$, the Smarr relation and first law of thermodynamics are satisfied:

$$M = 2TS + 2\Omega_H^{rel} J_H + 2\Omega_+^{rel} J_+ + 2\Omega_-^{rel} J_- - 2PV \tag{E.12}$$

$$dM = TdS + \Omega_H^{rel} dJ_H + \Omega_+^{rel} dJ_+ + \Omega_-^{rel} dJ_- + VdP. \tag{E.13}$$

However, this first law is not of full cohomogeneity. The angular frequencies are different for each string, without there being any parameter in the metric that accounts for that difference.