On the Structure of Invertible Elements in Certain Fourier-Stieltjes Algebras

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Let G be a locally compact group. The Fourier-Stieltjes and Fourier algebras, $B(G)$ and $A(G)$ are defined by Eymard to act as dual objects of the measure and group algebras, $M(G)$ and $L^1(G)$, in a sense generalizing Pontryagin duality from the theory of abelian locally compact groups. Hence there is a natural expectation that properties of the latter two algebras ought to be reflected in the former.

Joseph L. Taylor wrote a series of ten papers within the span of 1965 - 1972 studying the structure of convolution measure algebras $\mathfrak{M}$ (important examples are $M(G)$, measure algebra and $L^1(G)$, group algebra for an locally compact abelian group G) and characterizes the invertible elements in a measure algebra of a locally compact abelian group. Several results indicated that the spectrum Δ of $M(G)$ is quite complicated, as is the problem of deciding when $\mu \in M(G)$ is invertible. He proved that the spectrum of any abelian measure algebra can be represented as the space $\hat{S}$ of semicharacters on some compact semigroup S. However, for $M(G)$ we have no concrete description of the corresponding S or $\hat{S}$. Though this failed to give a really satisfactory description of the spectrum of $M(G)$, using this description and certain semigroup techniques, he proved a theorem which significantly simplifies the problem of deciding when a measure in $M(G)$ is invertible. Basically, this theorem asserts if μ^{-1} exists it must lie in a certain "small" subalgebra of $M(G)$. This reduces the invertibility problem in $M(G)$ to the same problem in an algebra which is far less complicated than $M(G)$. He does so by identifying the closed linear span of all maximal subalgebras which are isometrically isomorphic to a group algebras of some locally compact abelian group for his convolution measure algebras. He calls this the *spine* of $\mathfrak{M}$.

In 2007, Nico Spronk and Monica Illie develop the non-commutative dual analogue of the spine of an abelian measure algebra, provide an explicit dual description of how the above could be realised in abelian Fourier-Stieltjes algebra and conjecture if the invertible elements in general $B(G)$ admit such a dual characterisation. In this thesis, we bring together a class of interesting examples within Lie groups, in particular, many totally minimal groups, classical motion groups and the $ax + b$-group that witness the desired characterisation of invertible elements in their Fourier-Stieltjes algebra.

Acknowledgements

I would like to express my gratitude to my supervisors Nico Spronk and Brian Forrest. During my initial years in Canada, while I was being accustomed to a new and competitive culture as an international student, I felt distressed most of the time: by events, by people, by questions, and even by the creative possibilities set in motion to push forward my research. Whenever Brian or Nico encounters me in such situations, they often stimulate a live demonstration of scenarios or lead me through thought processes by asking simple questions. Thereby they let certain moments act as a pure mirror in which one could see themselves. Thus, acknowledging revelations that arose out of the totality of a moment and make me feel home where distress can go *rehabilitate*. Their robust response always entertains the possibility of grounding to humbleness; it is also a kind of faith or a sense that we will survive the impact of such stressful times, though not perhaps in the manner to which we accustom. This innate wisdom they bring to the fore practically has helped me develop an ability to live fully and equally in multiple contexts: from tackling stage fear to holding a professional conversation with students as a TA or an instructor and with researchers in the department as well as in the conferences. I feel extremely lucky to have my Ph.D. getting completed under their supervision.

I would like to thank my PhD advisory committee members for their timely suggestions throughout my research. I would like to thank my readers Matt Kennedy, Vern Paulsen, Richard Cleve and Jorge Galindo for the time they will spend examining this thesis. I hope this does not prove too arduous of a task.

I have made many friendships during my time in Waterloo, some of which I am certain will last a lifetime. I wish to thank these friends for being there with me and made me feel home. Finally, I am thankful to my family and friends, both old and new, for their continued support.

Dedication

To everyone with a red line in the editor used for this work.

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Chapter 1

Introduction

For a non-discrete locally compact abelian group (l.c.a) Γ with dual group G, Taylor [Tay72, Theorem 3] proved a factorization theorem for invertible measures in its measure algebra $M(\Gamma)$: for each $\mu \in M(\Gamma)^{-1}$, there are l.c.a. groups $\Gamma_{\tau_1}, \dots, \Gamma_{\tau_n}$ continuously isomorphic to Γ and measures $\nu_i \in (L^1(\Gamma_{\tau_i}) \oplus \mathbb{C}1)^{-1},\ \omega \in M(\Gamma)$, such that

$$\mu = \nu_1 * \cdots * \nu_n * \exp(\omega).$$

The measures ν_i are unique modulo $\exp(L^1(\Gamma_{\tau_i}) \oplus \mathbb{C}1)$. By calling Γ_τ continuously isomorphic to Γ, we mean that Γ_τ is equal to Γ as a group, but it is an l.c.a. group under a topology τ possibly finer than that of Γ.

The non-commutative analogues of $M(\Gamma)$ and $L^1(\Gamma)$ for general locally compact groups G are the Fourier-Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$, introduced by Eymard [Eym64]. When G is abelian with dual group Γ, the Fourier-Stieltjes transformation maps $M(\Gamma)$ isometrically onto $B(G)$ and $L^1(\Gamma)$ onto $A(G)$. Via the Fourier-Stieltjes transform for abelian G, we see that every invertible element $u \in B(G)$ is of the form $u = v_1 \cdots v_n \cdot \exp(w)$ where $v_i \in (A(G_{\sigma_i}) \oplus \mathbb{C}1)^{-1}, w \in B(G)$ and the topologies σ_i are coarser than the ambient group topology.

For a non-discrete locally compact abelian group Γ, several factors such as the Wiener-Pitt Phenomenon, the asymmetry of $M(\Gamma)$ and the existence of independent Cantor sets indicate that the spectrum of $M(\Gamma)$ is quite complicated, as is the problem of deciding when $\mu \in M(\Gamma)$ is invertible (i.e. when $\widehat{\mu}$ does not vanish on $\Delta(M(\Gamma))$). In [Tay72], with varying degrees of technical difficulty, Taylor reduces the problem of invertibility in $M(\Gamma)$ to a symmetric subalgebra $\mathcal{L}(\Gamma)$, the closed linear span of all maximal subalgebras which are isometrically isomorphic to some locally compact abelian group algebra. He calls this the *spine* of $M(\Gamma)$. The key to Taylor's approach is to notice that $\Delta(M(\Gamma))$ has a great deal of structure not generally enjoyed by maximal ideal spaces. It has a semigroup structure, an order structure, and has certain subsets on which two important topologies, the weak and the strong topology, coincide. Taylor introduced the notion of critical points, those positive elements in $\Delta(M(\Gamma))$ which cannot be weakly approximated by strictly smaller positive elements of $\Delta(M(\Gamma))$. He establishes a one-to-one correspondence between the maximal subalgebras and critical points. These critical points have to be idempotents and the groups G_{σ_i} above are exactly the maximal groups in $\Delta(M(\Gamma))$ at these critical points and consequently are locally compact abelian groups [Tay72, Proposition 2.3. (4)].

In the hope of extending Taylor's theory to the context of Fourier-Stieltjes algebras, two different approaches have been attempted so far: spectral analysis and topological analysis. In [Wal75], Walter showed that the maximal ideal space of $B(G)$ for a general locally compact group G exhibits a structure like that of an abelian measure algebra and extended the notion of critical points to $\Delta(B(G))$. Then a reasonable conjecture would be that every invertible element u in $B(G)$ is of the form $u = v_1 \cdots v_n \cdot \exp(w)$ where $w \in B(G)$, $v_i \in (A(G_z) \oplus \mathbb{C}1)^{-1}$ for some central critical element z in $\Delta(B(G))$. Here, G_z is a locally compact group about the critical point z. On the other hand, in [IS07], Ilie and Spronk develop the non-commutative dual analogue of the spine of an abelian measure algebra and [IS07, Theorem 5.1] provides an explicit description of the above topologies σ_i. Moreover, they note that it would be interesting to determine if invertible elements u in $B(G)$ are of the form $u = v_1 \cdots v_n \cdot \exp(w)$ where $w \in B(G)$, $v_i \in (A(G_{\rho_i}) \oplus \mathbb{C}1)^{-1}$ for some locally precompact topology ρ_i on G. By G_ρ, we mean the locally compact completion of G with respect to ρ.

We prove that the above conjectures hold for Fourier-Stieltjes algebra that satisfy some additional conditions. These conditions are satisfied for interesting examples within the family of Lie groups, in particular, many connected totally minimal groups. Moreover, elementary arguments based on the Arens-Royden theorem immediately allow for a deeper analysis of invertible elements in motion groups and the $ax + b$-group. We note that these examples cover those examples for which [IS07] have constructed the spine.

We briefly describe the structure of this thesis. In Chapter 2, we establish notation and recall some of the necessary background. In Chapter 3, we observe a consequence (Corollary 3.1.2) of the Arens-Royden Theorem ([Are63], [Roy63]). In Section 3.2, as an application of Corollary 3.1.2, we obtain that the abstract index group of the regular Rajchman algebra $B_0(G)$ is exactly the abstract index group of the Fourier algebra $A(G)$ (Remark 3.2.2). Here $B_0(G)$ is the set of all functions in $B(G)$ that vanish at infinity. We end this section with another observation, which gives the desired characterization of the invertible elements in the case of Euclidean motion groups, p-adic motion groups and the $ax+b$ group. In Chapter 4, we prove our main result (Theorem 4.3.7). In Section 4.4, we consider several classes of locally compact groups G that include many totally minimal groups, connected semisimple groups with finite centre and extensions of compact analytic group by a nilpotent analytic group. We deduce from [May97b], [May97a], [May99], [Cow79a], [LM81] that their Fourier-Stieltjes algebras are symmetric, with their Gelfand spectrum being a semilattice of groups and satisfies the hypothesis of Theorem 4.3.7. Chapter 5 deals with proving the Theorem 4.3.7 through means of cohomological calculations. We also note that the paper [Tha21] correspond to chapters 3 and 4.

Chapter 2

Background and Literature

The present chapter contains the background necessary for this thesis. We assume a basic knowledge of harmonic analysis, Banach algebras, C^*-algebras and von Neumann algebras throughout this thesis. The books [Dix77] and [Fol95] of Diximier and Folland sufficiently cover these topics for the purpose of this thesis. Before proceeding to the body of thesis, we first recall some of the basics in these areas (parts of which are covered in the books listed above). We do not give individual references for each of the results listed in this section but instead provide at the beginning of each subsection a list of references in which all the results may be found. Additional background will be introduced as needed throughout this thesis.

2.1 Locally Compact Groups

Definition 2.1.1. *A topological space G is said to be a topological group if it is a group and the map $(x, y) \to xy^{-1}$ is continuous from $G \times G$ into G.*

A locally compact group is a topological group which is also a locally compact Hausdorff space. Following are some examples of locally compact group.

Example 2.1.2. *1. $\mathbb{R}^n$ is a locally compact abelian group, $n \geq 1$.*

2. $\mathbb{T}^n$ is a compact abelian group, $n \geq 1$.

3. $GL(n,\mathbb{R})$, the group of all $n \times n$ invertible matrices with real entries, is a locally compact group as it is an open subset of $\mathbb{R}^{n^2}$.

4. Heisenberg group $G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ is noncompact and nonabelian.

5. $\mathbb{F}_2$, the free group generated by a, b, with $ab \neq ba$, is nonabelian discrete group. Any element of this group is of the form $a^m b^n a^k \ldots$ where $m, n, k \in \mathbb{Z}$.

2.2 Measure and Group Algebras

Definition 2.2.1. *Let X be a locally compact space. A Borel measure μ is said to be regular if*

1. $\mu(K) < \infty$, *for every compact set* K.

2. *For every Borel set* E, $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subseteq U\}$.

3. *If* E *is any Borel set and* $\mu(E) < \infty$, *then* $\mu(E) = \sup\{\mu(K) : K \text{ is compact and } K \subseteq E\}$.

Definition 2.2.2. *A Banach space A over $\mathbb{C}$ is called a Banach algebra if A is an algebra satisfying $\|xy\| \leq \|x\| \|y\|$, for all $x, y \in A$. The Banach algebra A is commutative if $xy = yx$ for all $x, y \in A$. The Banach algebra A has an identity if there is an element $1 \in A$ such that $\|1\| = 1$ and $1x = x1 = x$ for all $x \in A$.*

Definition 2.2.3. *Let A be an algebra over $\mathbb{C}$. An involution in A is a map $x \to x^*$ of A into itself such that*

1. $(x^*)^* = x$

2. $(\lambda x + y)^* = \overline{\lambda} x^* + y^*$

3. $(xy)^* = y^* x^*$

for any $x, y \in A$ and $\lambda \in \mathbb{C}$.

Definition 2.2.4. *A Banach algebra A over $\mathbb{C}$ is called Banach* algebra if it admits an involution $x \to x^*$ such that $\|x^*\| = \|x\|$ for all $x \in A$.*

Example 2.2.5.

1. *$(\mathbb{C}, |.|)$ with involution map $z \to \overline{z}$ is a Banach* algebra.*
2. *Let X be a locally compact space. $C_0(X)$, the algebra of complex valued continuous functions vanishing at infinity on X endowed with the map $f \to \overline{f}$ is a commutative Banach* algebra under sup norm.*

One of the milestones in the history of abstract harmonic analysis was the existence of left invariant measure on a general locally compact Hausdorff group that reads as follows:

Definition 2.2.6. *A Borel measure m on G is said to be left invariant if $m(xE) = m(E)$ for every Borel set E, and for all $x \in G$.*

Theorem 2.2.7. *(Haar, Von Neumann) Let G be a locally compact Hausdorff group. There exists a non-zero, positive, left invariant, regular Borel measure m on G. Moreover, it is unique up to a positive constant.*

Such a measure is called the *Haar* measure and the corresponding integral is called the Haar integral. Let dx always denote the Haar integral.

Example 2.2.8.

1. *If G is discrete, counting measure is the Haar measure.*
2. *On $\mathbb{R}^n$, Haar measure is the n-dimensional Lebesgue measure.*
3. *On $GL(n, \mathbb{R})$, $\frac{1}{(det\ g)^n} dg_{11} dg_{12} ... dg_{nn}$ defines a Haar measure.*
4. *The Lebesgue measure $dg = dxdydz$ is the Haar measure for the Heisenberg group given in 2.1.2.4.*

Let m be the left Haar measure on G and $L^p(G), 1 \leq p < \infty$ be the Banach space consisting of all [equivalence classes of] m-measurable functions f [up to almost everywhere pointwise equivalence] such that $\displaystyle\int_G |f(x)|^p dx$ is finite. The Banach dual of $L^1(G)$ can be identified with the space $L^\infty(G)$ of bounded, m-measurable, complex valued functions ϕ on G with the ess sup norm :

$$\|\phi\|_\infty = \text{ess} \sup_{x \in G} |\phi(x)| = \inf \{M \in \mathbb{R} : m\,(x \in G : |\phi(x)| > M) = 0\}.$$

The above correspondence between $L^\infty(G)$ and the dual $L^1(G)^*$ of $L^1(G)$ is given by associating $\phi \in L^\infty(G)$ with the functional $f \to \int \phi f dx$ $(f \in L^1(G))$. The family of m-measurable subsets of G is denoted by $\mathcal{M}(G)$.

A Haar measure need not be right invariant. For a fixed x if we define dm_x by $\langle f, dm_x \rangle = \int_G f(yx)dy$, then it is a left invariant integral. Therefore, there exists a function $x \to \Delta(x)$ called modular function satisfying the following:

$$\int_G f(yx)dy = \Delta(x^{-1}) \int_G f(y)dy.$$

The modular function of G is a continuous homomorphism G into $\mathbb{R}^+$.

A group is called unimodular if $\Delta(x) = 1$ for all $x \in G$. Any abelian or discrete group is unimodular. Any compact group is unimodular because of continuity of Δ.

Let $M(G)$ be the space of all complex, regular Borel measures on G. By the Riesz representation theorem, $M(G)$ can be identified with the dual space of $C_0(G)$. In what follows, $M(G)$ is made into a Banach algebra.

Definition 2.2.9. *Let $\mu, \nu \in M(G)$. The convolution product $\mu * \nu \in M(G)$ of μ, ν is defined through*

$$\langle f, \mu * \nu \rangle := \int_G \left(\int_G f(gh) d\mu(g) d\nu(h) \right) \quad (f \in C_0(G))$$

By Fubini's theorem, the order of integration is irrelevant. $M(G)$ is a Banach space under the total variation norm. With this convolution product, $M(G)$ is a Banach* algebra.

The involution $\mu \to \mu^*$ is given by $\mu^*(E) = \overline{\mu(E^{-1})}$ for every Borel subset E of G. If $x \in G$ then δ_x denotes the Dirac measure at x. If x, y are in G, then

$$\int \phi d(\delta_x * \delta_y) = \int \int \phi(uv) d\delta_x(u) d\delta_y(v) = \phi(xy) = \int \phi d\delta_{xy}.$$

Therefore, $\delta_x * \delta_y = \delta_{xy}$. δ_e is the multiplicative identity of $M(G)$, where e is the identity of G.

By the Radon-Nikodým theorem, $L^1(G)$ is identifiable with the closed subspace of measures in $M(G)$ that are absolutely continuous with respect to m. Thus a function $f \in L^1(G)$ corresponds to the measure $\mu_f \in M(G)$ determined by

$$\langle \phi, \mu_f \rangle = \int_G \phi(x) f(x) dx \quad (\phi \in C_0(G)).$$

We show that $L^1(G)$ is a Banach* subalgebra of $M(G)$.

For every $f, g \in L^1(G)$, since

$$\begin{aligned} \int_G \phi(x) d(f * g)(x) &= \int_G \int_G \phi(xy) f(x) g(y) dx dy \\ &= \int_G \int_G \phi(x) f(y) g(y^{-1}x) dy dx, \end{aligned}$$

we have

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy.$$

Similarly, the involution of $M(G)$ restricted to $L^1(G)$, is given by

$$f^*(x) dx = \overline{f(x^{-1})} d(x^{-1}).$$

where $d(x^{-1}) = \Delta(x^{-1}) dx$ so $f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}$ for almost every $x \in G$. Thus $L^1(G)$ forms an Banach*-algebra called group algebra. Moreover it is a two-sided ideal in $M(G)$ and closed under involution. In fact

$$\mu * g(x) = \int_G g(y^{-1}x) d\mu(y) \qquad \text{and} \qquad f * \mu(x) = \int_G \Delta(y^{-1}) f(xy^{-1}) d\mu(y).$$

G is commutative iff $L^1(G)$ is commutative. Also, $L^1(G)$ has identity if and only if G is discrete.

However, the group algebra of a general locally compact group always has a bounded approximate identity.

Definition 2.2.10. *A Banach algebra A has a bounded approximate identity if there exists a net $\{x_\alpha\}_{\alpha \in I}$ in A satisfying* $\sup_\alpha \|x_\alpha\| \leq \infty$ *and* $\lim_\alpha \|x_\alpha x - x\| = 0 = \lim_\alpha \|x x_\alpha - x\|$ *for all $x \in A$.*

Remark 2.2.11. *Let $\mathcal{V}$ be a base of compact neighbourhoods of e. Then $\left(\frac{1}{m(U)} 1_U\right)_{U \in \mathcal{V}}$ defines a bounded approximate identity with $\left\|\frac{1}{m(U)} 1_U\right\|_1 = 1$ for all $U \in \mathcal{V}$.*

2.3 Representations and Group C^*-algebras

Definition 2.3.1. *An Banach* algebra A is said to be C^*-algebra if the involution of A satisfies $\|x^* x\| = \|x\|^2$.*

Example 2.3.2. *Let $\mathcal{H}$ be a Hilbert space. Then the unital Banach algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is a C^* algebra with the operator norm and the involution is given by the adjoint map $T \to T^*$. In general, any norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.*

Definition 2.3.3. *A representation of G on a Hilbert space $\mathcal{H}$ is a homomorphism π from G into $GL(\mathcal{H})$, the group consisting of all invertible operators in $\mathcal{B}(\mathcal{H})$ such that the map $x \to \pi(x)\xi$ is continuous for every $\xi \in \mathcal{H}$.*

We say $\mathcal{H}$ is the representation space. If $\pi(x)$ is unitary for every $x \in G$, then π is called unitary representation. i.e., $\langle \pi(x)\xi, \pi(x)\eta \rangle = \langle \xi, \eta \rangle \quad \forall x \in G, \xi, \eta \in \mathcal{H}$.

A representation $(\pi, \mathcal{H})$ is said to be an irreducible representation of G, if $\mathcal{H}$ does not contain any non-trivial closed G-invariant subspace.

A representation $(\pi, \mathcal{H})$ is said to be cyclic with a cyclic vector ξ if the linear span of $\{\pi(x)\xi : x \in G\}$ is a dense subspace of $\mathcal{H}$. Any two (unitary) representations $(\pi_1, \mathcal{H}_1)$ and

$(\pi_2, \mathcal{H}_2)$ are said to be (unitarily) equivalent, written $\pi_1 \sim \pi_2$, if there exists an invertible bounded (unitary) operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$T \circ \pi_1(g) = \pi_2(g) \circ T, \quad \forall g \in G$$

Example 2.3.4. 1. *Any group G admits a trivial representation on any Hilbert space $\mathcal{H}$. For $x \in G$, define $\pi(x) = I_{\mathcal{H}}$.*

2. *For any $x \in G$, let $\lambda(x)$ denote the operator on $L^2(G)$ given by*

$$\lambda(x)f(y) = \delta_x * f(y) = f(x^{-1}y) \quad \forall\ y \in G \text{ and } \forall\ f \in L^2(G).$$

Then by invariance of measure, $\|\lambda(x)f\|_2^2 = \int_G |f(x^{-1}y)|^2 dy = \int_G |f(y)|^2 dy = \|f\|_2^2$, implying the map $x \to \lambda(x)f$ defines a unitary representation of G and is known as left regular representation of G.

Definition 2.3.5. *If A is an involutive algebra, a representation π of A on a Hilbert space $\mathcal{H}$ is a *-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$, that is, for all $x, y \in A$ and $\alpha \in \mathbb{C}$, one has $\pi(x + y) = \pi(x) + \pi(y), \pi(\alpha x) = \alpha\pi(x), \pi(xy) = \pi(x)\pi(y), \pi(x)^* = \pi(x^*)$*

Remark 2.3.6. *If A is a Banach*-algebra and B is a C^*-algebra and if $\phi : A \to B$ is a *-homomorphism then $\|\phi(x)\| \leq \|x\|$ for every $x \in A$. In particular any * representation of $L^1(G)$ is norm decreasing.*

A representation π of A is said to be *non − degenerate* if the closure of the subspace spanned by $[\pi(A)(\mathcal{H})]$ is equal to $\mathcal{H}$. The notions of irreducibility and equivalence among the representations of A are similar to as defined above Example 2.3.4.

The unitary representation π is called continuous if it is continuous with respect to the strong topology of $\mathcal{B}(\mathcal{H})$, that is, the mapping $x \to \pi(x)\xi$ is continuous whenever $\xi \in \mathcal{H}$, or equivalently, with respect to the weak topology of $\mathcal{B}(\mathcal{H})$, that is, the mapping $x \to \langle \pi(x)\xi, \eta \rangle$ is continuous whenever $\xi, \eta \in \mathcal{H}$. (because $\|\pi(x)\xi - \xi\|^2 = 2\|\xi\|^2 - 2\langle \pi(x)\xi, \xi \rangle$ and thus if $x \to e$ weakly, $\|\pi(x)\xi - \xi\| \to 0$.)

If π is a continuous unitary representation of G on the Hilbert space $\mathcal{H}$ and μ is any element in $M(G)$, we put

$$\pi(\mu) = \int_G \pi(x) d\mu(x) \in \mathcal{B}(\mathcal{H}).$$

In this way we define a representation of $M(G)$ associated to π which we denote also by π. For $\xi, \eta \in \mathcal{H}, \mu \in M(G)$,

$$\langle \pi(\mu)\xi, \eta\rangle = \int_G \langle \pi(x)\xi, \eta\rangle \, d\mu(x);$$

The restriction to $L^1(G)$ of this representation of $M(G)$ is non-degenerate. More precisely, we have the following theorem.

Theorem 2.3.7. *Suppose $(\pi, \mathcal{H})$ is a unitary representation of G. For every f in $L^1(G)$ define $\pi(f)$ on $\mathcal{H}$ by*

$$\langle \pi(f)(\xi), \eta\rangle = \int_g \langle \pi(x)\xi, \eta\rangle \, f(x)dx.$$

Then $f \to \pi(f)$ defines a non-degenerate representation of $L^1(G)$. Moreover, the above correspondence is bijective from the equivalence classes of all unitary representations of G and the equivalence classes of all non-degenerate representations of $L^1(G)$.

Remark 2.3.8. *In particular, this correspondence associates to λ the left regular representation of G , λ the left regular representation L of $L^1(G)$ defined by*

$$\lambda(f) : g \to f * g; \quad L^2(G) \to L^2(G)$$

for $f \in L^1(G)$. Notice that, if $f \in L^1(G)$ and $g, h \in L^2(G)$,

$$\int \langle \lambda(x)g, h\rangle \, f(x)dx = \int\int g(x^{-1}y)\overline{h(y)}f(x)dxdy = \int f * g(y)\overline{h(y)}dy = \langle f * g, h\rangle$$

This representation is faithful in the sense that if $\lambda(f) = 0$ then $f = 0$ in $L^1(G)$.

2.3.1 Positive Functionals & Representations

Definition 2.3.9. *Let A be an involutive algebra over $\mathbb{C}$. A linear form f on A is said to be positive if $f(x^*x) \geq 0$ for each $x \in A$. If A is a normed involutive algebra, a state of A is a continuous positive linear form f on A such that $\|f\| = 1$.*

Remark 2.3.10. *Let A be an involutive algebra and f a positive form on A. For $x, y \in A$ put $(x|y) = f(y^*x)$. Then A is a pre-Hilbert space. In particular, we have*

1. $f(y^*x) = \overline{f(x^*y)} \qquad (x \in A, y \in A)$,
2. $|f(y^*x)|^2 \leq f(x^*x)f(y^*y) \qquad (x \in A, y \in A)$.

If A admits a unit u, for every $x \in A$, $f(x^*) = \overline{f(x)}$ and $|f(x)|^2 \leq f(u)f(x^*x)$. If A is an Banach* algebra with unit u such that $\|u\| = 1$, then every positive functional f on A is continuous and $\|f\| = f(u)$.

If $(\pi, \mathcal{H})$ is a representation of the Banach*-algebra A, for every $\xi \in \mathcal{H}$, $f(x) = \langle \pi(x)\xi, \xi\rangle$ is a positive functional on A; note that, for every $x \in A$,

$$f(x^*x) = \langle \pi(x^*x)\xi, \xi\rangle = \langle \pi(x)\xi, \pi(x)\xi\rangle \geq 0.$$

We say that f is a positive functional associated to π. For any $y \in A$, $f' : x \mapsto f(y^*xy)$ is also a positive functional associated to π because $\langle \pi(y^*x^*xy)\xi, \xi\rangle = \langle \pi(xy)\xi, \pi(xy)\xi\rangle \geq 0$ whenever $x \in A$ and $\xi \in \mathcal{H}$.

Lemma 2.3.11. *Let A be a C^*-algebra. If $\mathcal{S} = \{(\rho_i, \mathcal{H}_i) : i \in I\}$ is a family of representations of A and $(\pi, \mathcal{H}_\pi)$ is a particular representation of A, then the following conditions are equivalent:*

1. $\bigcap_{i \in I} ker(\rho_i) \subset ker(\pi)$.
2. *Every positive functional on A associated to π is the w^*-limit of sums of positive functionals associated to elements of $\mathcal{S}$.*
3. *If ξ is a cyclic vector for the representation π, the positive functional $f : x \mapsto \langle \pi(x)\xi, \xi\rangle$ is the w^*-limit of sums of positive functionals associated to elements of $\mathcal{S}$.*

Definition 2.3.12. *(Weak Containment) Let A be a C^*-algebra, $\mathcal{S}$ a family of representations of A, and π a particular representation of A. If $\mathcal{S}$ and π satisfy the equivalent conditions of Lemma 2.3.11, we say that π is weakly contained in $\mathcal{S}$ denoted by $\pi \preceq \mathcal{S}$. The support of a representation π is the set of all representations of A that are weakly contained in π.*

2.3.2 Fell Topology

Recall that if $(\pi, \mathcal{H})$ is a nondegenerate representation of the Banach* algebra A, then π is said to be irreducible if the only closed subspaces of $\mathcal{H}$ that are invariant under π are $\{0\}$ and $\mathcal{H}$. The dual of A is the collection of all equivalence classes of irreducible nonzero *-representations of A. Through Lemma 2.3.11, define a topology on the dual of a C^*-algebra A by calling a nonvoid subset $\mathcal{P}$ closed if it consists of all equivalence classes of irreducible representations of A that are weakly contained in $\mathcal{P}$. This not necessarily Hausdorff topology is the *Fell Topology*.

2.3.3 Enveloping C^*-algebra

Let A be an Banach* algebra admitting approximate units and let $\mathcal{P}$ be a subset of its dual. For $x \in A$, let

$$\|x\|_{\mathcal{P}} = \sup\{\|\pi(x)\| : \pi \in \mathcal{P}\};$$

$\|.\|_{\mathcal{P}}$ is a seminorm on A such that

$$\|xy\|_{\mathcal{P}} \leq \|x\|_{\mathcal{P}} \|y\|_{\mathcal{P}}, \|x^*\|_{\mathcal{P}} = \|x\|_{\mathcal{P}}, \|x^*x\|_{\mathcal{P}} = \|x\|^2_{\mathcal{P}}.$$

Let $N_{\mathcal{P}} = \{x \in A : \|x\|_{\mathcal{P}} = 0\}$. Then $N_{\mathcal{P}}$ is a closed, two-sided and self-adjoint ideal of A. Then the map $x + N_{\mathcal{P}} \mapsto \|x\|_{\mathcal{P}}$ is a well-defined norm on the $*$-algebra $A/N_{\mathcal{P}}$. The completion of $A/N_{\mathcal{P}}$ with respect to this norm, say B, is a C^*-algebra. If $\mathcal{P}$ is the dual of A, the associated C^*-algebra is called *enveloping C^* algebra* of A. In particular, if $A = L^1(G)$ for a locally compact group G, we denote by $C^*_{\mathcal{P}}(G)$ the completion of $A/N_{\mathcal{P}}$ and by $C^*(G)$ the corresponding enveloping C^*-algebra. There exists a canonical bijection from the dual of $C^*(G)$ onto the set of equivalence classes of irreducible continuous unitary representations of G. The latter set, equipped with the topology induced from the dual of $C^*(G)$, is the *dual* $\hat{G}$ of G. If $\pi \in \hat{G}$ and $\mathcal{S} \subset \hat{G}$ we also say that π is weakly contained in $\mathcal{S}$ if π, considered as an element of the dual of $C^*(G)$, is weakly contained in $\mathcal{S}$, considered as a subset of the dual of $C^*(G)$. The map $j : (A, \|.\|) \to (B, \|.\|_{\mathcal{P}})$ given by $j(x) = x + N_{\mathcal{P}}$ is then a contractive homomorphism with dense image.

Remark 2.3.13. *In $\hat{G}$, by Lemma 2.3.11, $\pi \preceq \mathcal{S}$ if and only if there is a surjective *-homomorphism from $C^*_\pi(G)$ onto $C^*_{\mathcal{S}}(G)$. The support of π is again the set of all elements of $\hat{G}$ that are weakly contained in π. The reduced dual $\hat{G}_\lambda$ of G is the support of the left regular representation λ of G. From Remark 2.3.8, $\|.\|_\lambda$ is a norm. Hence $C^*_\lambda(G)$ is a completion of $L^1(G)$ with respect to $\|.\|_\lambda$.*

Proposition 2.3.14. *Let A, B, j be as above. For any representation π of A, there exists exactly one representation ρ of B such that $\pi = \rho \circ j$, in which case $\rho(B)$ is the C^*-algebra generated by $\pi(A)$. The map $\pi \mapsto \rho$ is a bijection of the set of representations of A onto the set of representations of B. Moreover, π is nondegenerate if and only if ρ is nondegenerate.*

Proof. Let π be a representation of A. Then π vanishes on $N_{\mathcal{P}}$ and hence induces a representation π' of $A/N_{\mathcal{P}}$ given by $\pi'(x + N_{\mathcal{P}}) = \pi(x)$ thereby $\|\pi'(x + N_{\mathcal{P}})\| = \|\pi(x)\| \leq \|x\|_{\mathcal{P}} = \|x + N_{\mathcal{P}}\|$. Since $A/N_{\mathcal{P}}$ is dense in B, π' extends to a unique representation ρ on B such that $\pi = \rho \circ j$. This immediately implies that $\pi(A)$ is dense in $\rho(B)$, and since $\rho(B)$ is a C^*-algebra, it is the C^*-algebra generated by $\pi(A)$. The map $\pi \mapsto \rho$ is injective, and the map $\rho \mapsto \rho \circ j$ is its inverse. Suppose that π represents A on $\mathcal{H}_\pi$, then π (resp. ρ) is nongenerate if and only if $T\xi = 0$ for all $T \in \pi(A)$ (resp. $T \in \rho(B)$) implies $\xi = 0$ for all $\xi \in \mathcal{H}_\pi$. By denseness, it is clear that nongeneracy of π is equivalent to nondegeneracy of ρ. $\square$

Corollary 2.3.15. *Any unitary representation $(\rho, \mathcal{H})$ of a locally compact group G induces a unique nondegenerate representation ρ on $C^*(G)$ over $\mathcal{H}$, such that*

$$\rho(f) = \int f(x)\rho(x)d\mu(x)$$

for all $f \in L^1(G)$. Conversely, any non-degenerate representation of $C^*(G)$ arises from a unitary representation this way.

Proof. The proof is immediate from Theorem 2.3.7 and Proposition 2.3.14. $\square$

2.4 Positive Definite Functions

Definition 2.4.1. *A continuous complex-valued function ϕ on G is said to be positive definite if, for any elements $s_1, \ldots, s_n$ of G, the matrix $(\phi(s_i^{-1}s_j))_{1\leq i,j\leq n}$ is positive semidefinite. That is, for any $s_1, \ldots, s_n \in G$ and $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$, we have*

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \phi(s_i^{-1}s_j) \geq 0.$$

Example 2.4.2. *If $(\pi, \mathcal{H})$ is any unitary representation of G then for any $\xi \in \mathcal{H}$ the positive functional associated to $\pi, f : x \mapsto \langle \pi(x)\xi, \xi\rangle$ is positive definite since for any choice of s_i and α_i,*

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \phi(s_i^{-1}s_j) = \langle \eta, \eta \rangle$$

where $\eta = \sum_i \alpha_i \pi(s_i)(\xi)$.

Remark 2.4.3. *Taking $n = 1$ and $\alpha_1 = 1$ in Definition 2.4.1, we have $\phi(e) \geq 0$. Now put $n = 2, s_1 = e, s_2 = s \in G$, Then the matrix*

$$\begin{pmatrix} \phi(e) & \phi(s) \\ \phi(s^{-1}) & \phi(e) \end{pmatrix}$$

must be positive semidefinite. This implies that

$$\phi(s^{-1}) = \overline{\phi(s)}$$

$$|\phi(s)| \leq \phi(e)$$

for every $s \in G$. In particular, ϕ is bounded, and $\|\phi\|_\infty = \phi(e)$.

Let $\mathcal{P}(G)$ denote the set of all continuous positive definite functions on G. As the Hadamard product of two positive definite matrices is positive definite, $\mathcal{P}(G)$ is closed for the pointwise multiplication. If $\phi \in \mathcal{P}(G)$ then $\overline{\phi} \in \mathcal{P}(G)$.

Remark 2.4.4. *Let $f \in L^2(G)$. If λ denotes the left regular representation of G, letting $\tilde{f}(x) = \overline{f(x^{-1})}$ for almost all $x \in G$, we have*

$$\overline{\langle \lambda(x)f, f \rangle} = \int \overline{f}(x^{-1}y)f(y)dy = \int f(y)\tilde{f}(y^{-1}x)dy = (f * \tilde{f})(x).$$

Hence $f * \tilde{f}$ is a continuous positive definite function associated to λ.

Theorem 2.4.5. *Let ϕ be a continuous function on G. Then the following are equivalent:*

1. *ϕ is positive definite.*
2. *ϕ is bounded and $\langle \phi, f^* * f \rangle \geq 0, \forall f \in C_c(G)$.*
3. *ϕ is bounded and $\langle \phi, \mu^* * \mu \rangle \geq 0, \forall \mu \in M(G)$.*

Theorem 2.4.6. *(Naimark Theorem) Let ϕ be any continuous positive definite function on G. Then there exists a cyclic representation $(\pi_\phi, \mathcal{H})$ with the cyclic vector ξ such that $\phi(x) = \langle \pi_\phi(x)\xi, \xi \rangle$ locally almost everywhere.*

Proof. Define $\langle ., . \rangle_\phi$ on $L^1(G)$ by

$$\langle f, g \rangle_\phi = \langle g^* * f, \phi \rangle = \int_G \int_G \phi(x^{-1}y)\overline{g(x)}f(y)dxdy.$$

It defines a sesquilinear form on $L^1(G)$. If $\mathcal{N} = \{f \in L^1(G) : \langle f, f \rangle_\phi = 0\}$, then by Cauchy Schwartz inequality $\mathcal{N} = \{f \in L^1(G) : \langle f, g \rangle_\phi = 0 \ \forall \ g \in L^1(G)\}$. Therefore it forms a closed subspace of $L^1(G)$. Moreover, since

$$\langle {}_xf, \ {}_xg \rangle_\phi = \langle f, g \rangle_\phi ,$$

$\mathcal{N}$ is invariant under left translation. Here ${}_xf(y) = f(xy)$ for almost every $y \in G$.

Let $\mathcal{H}_0$ denote the quotient space $L^1(G)/\mathcal{N}$. Let $\mathcal{H}$ be its completion. Define a representation π on $\mathcal{H}$ as follows. If $\tilde{f} \in L^1(G)/\mathcal{N}$, let

$$\pi(x)\tilde{f} = \overline{{}_{x^{-1}}f} =_{x^{-1}} \overline{f}.$$

Then π extends to a unitary representation on G.

If $\{f_\alpha\}$ is a bounded approximate identity of $L^1(G)$, then take a subnet if necessary to conclude that $\tilde{f}_\alpha$ converges to a vector ξ weakly in $\mathcal{H}$. Then

$$\left\langle \tilde{f}, \xi \right\rangle_\phi = \lim_\alpha \left\langle \tilde{f}, f_\alpha \right\rangle_\phi = \lim_\alpha \langle f_\alpha^* * f, \phi \rangle = \int_G f(x)\phi(x)dx.$$

Since $\left\langle \tilde{g}, \tilde{f} \right\rangle_\phi = \int_G \int_G \phi(x^{-1}y)\overline{g(x)}f(y)dxdy$ and

$$\int_G \phi(x^{-1}y)g(y)dy = \int_G \phi(y)g(xy)dy = \left\langle \pi(x^{-1}\tilde{g}), \xi \right\rangle_\phi = \langle \tilde{g}, \pi(x)\xi \rangle$$

Therefore,

$$\left\langle \tilde{g}, \tilde{f} \right\rangle_\phi = \int_G \tilde{f}(x) \langle \tilde{g}, \pi(x)\xi \rangle \, dx = \langle \tilde{g}, \pi(f)\xi \rangle_\phi .$$

Hence $[\pi(L^1(G))\xi]$ is total in $\mathcal{H}$ and so the representation is cyclic. Finally,

$$\left\langle \xi, \tilde{f} \right\rangle = \lim_\alpha \left\langle \tilde{f}_\alpha, \tilde{f} \right\rangle = \lim_\alpha \left\langle \tilde{f}_\alpha, \pi(f)\xi \right\rangle = \langle \xi, \pi(f)\xi \rangle .$$

Therefore, $\int_G f(x)\phi(x)dx = \langle \pi(f)\xi, \xi \rangle$. □

Remark 2.4.7. *If ϕ is a positive linear form on $L^1(G)$, then ϕ gets extended to a positive linear form ϕ' on $C^*(G)$. The map $\phi \to \phi'$ is bijective and $\|\phi\| = \|\phi'\|$.*

Theorem 2.4.8. *(Raikov's theorem) Let $\mathcal{P}_1(G)$ be the set of continuous positive-definite functions ϕ on G such that $\phi(e) = 1$. On $\mathcal{P}_1(G)$, the weak*-topology $\sigma(L^\infty(G), L^1(G))$ coincides with the topology of uniform convergence on compact sets.*

Theorem 2.4.9. *Let G be a locally compact group and π be a continuous unitary representation of G and $\mathcal{S}$ be a family of continuous unitary representations of G. Then the following are equivalent:*

1. $\pi \preceq \mathcal{S}$

2. *Every positive definite function ψ associated to π is limit of sum of positive definite functions associated to elements of $\mathcal{S}$ with respect to the topology of uniformly convergence on compact sets.*

If π is further assumed to be irreducible, then the above is equivalent to the following.

3. *If ξ is a cyclic vector for the representation π, the positive definite function $f : x \mapsto \langle \pi(x)\xi, \xi \rangle$ is the limit of sum of positive definite functions associated to elements of $\mathcal{S}$ with respect to the topology of uniform convergence on compact sets.*

Proof. The proof follows by Theorem 2.4.8 and Lemma 2.3.11. $\square$

2.5 Fourier-Stieltjes and Fourier algebra

In this section, we will review the Fourier A_π and Fourier-Stieltjes B_π spaces of a locally compact group, in particular, the Fourier and Fourier-Stieltjes algebras. All the results listed here can be found in [Eym64] and [Ars76].

Denote the class of continuous unitary representations of G by Σ_G. For $(\pi, \mathcal{H}_\pi), (\rho, \mathcal{H}_\rho) \in \Sigma_G$, let $\pi \oplus \rho : G \to \mathcal{U}(\mathcal{H}_\pi \oplus \mathcal{H}_\rho)$ be given by

$$\pi \oplus \rho(x)\,(\xi \oplus \eta) = \pi(x)\xi \oplus \rho(x)\eta$$

for $x \in G, \xi \in \mathcal{H}_\pi$ and $\eta \in \mathcal{H}_\rho$. Here $\xi \oplus \eta = (\xi, \eta)$. Let $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$ be the Hilbert space tensor product of $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$. Let $\pi \otimes \rho : G \to \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ be given by

$$\pi \otimes \rho(x)\,(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$$

for $x \in G$ and elementary tensor $\xi \otimes \eta$ in $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$.

Given $\pi \in \Sigma_G$ and $\xi, \eta \in \mathcal{H}_\pi$, let $\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle$ for $x \in G$. Then $\pi_{\xi,\eta} \in C_b(G)$. Let

$$F_\pi = \operatorname{span}\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}.$$

Let $B(G) = \bigcup_{\pi \in \Sigma_G} F_\pi$. For $\alpha \in \mathbb{C}$ and $\pi_{\xi,\eta}, \rho_{\zeta,\vartheta} \in B(G)$, we have

$$\alpha\pi_{\xi,\eta} = \pi_{\alpha\xi,\eta} = \pi_{\xi,\overline{\alpha}\eta}, \quad \pi_{\xi,\eta} + \rho_{\zeta,\vartheta} = (\pi \oplus \rho)_{\xi\oplus\zeta,\eta\oplus\vartheta}$$

$$\text{and } \pi_{\xi,\eta}.\rho_{\zeta,\vartheta} = (\pi \otimes \rho)_{\xi\otimes\zeta,\eta\otimes\vartheta}$$

So $B(G)$ is a subalgebra of $C_b(G)$. It is called the *Fourier-Stieltjes algebra* of G.

Since up to local almost everywhere equivalence by Theorem 2.4.5,

$$\mathcal{P}(G) = \left\{ u \in L^\infty(G) : \int_G u(s) f^* * f(s) \geq 0 \text{ for } f \in L^1(G) \right\},$$

$\mathcal{P}(G)$ is exactly the set of elements of $L^\infty(G)$ that can be extended to positive functionals on $C^*(G)$, and $C^*(G)^* \cong \text{span}\mathcal{P}(G)$ in terms of the $L^1(G) - L^\infty(G)$ dual pairing. By the Gelfand-Naimark Theorem $\mathcal{P}(G) \subset B(G)$. By the Polarization Identity, for $\pi_{\xi,\eta} \in B(G), \pi_{\xi,\eta} = \frac{1}{4}\sum_{n=0}^{3} i^n \pi_{\xi+i^n\eta,\xi+i^n\eta} \in \text{span } \mathcal{P}(G)$. Hence $C^*(G)^* \cong B(G)$.

The norm on $B(G)$ arising from this identification is given by

$$\|u\| = \inf\{\|\xi\| \, \|\eta\| : u = \pi_{\xi,\eta}\}.$$

When equipped with this norm, $B(G)$ is a commutative Banach algebra with respect to above pointwise operations.

Recall that when G is abelian with dual group $\hat{G}$, $C^*(G) = C_0(\hat{G})$ and, hence, $B(G)$ is isometrically isomorphic to the measure algebra $M(\hat{G})$ as a Banach space. In this case $B(G)$ is the set of all Fourier-Stieltjes transforms of measures in $M(\hat{G})$ and the Fourier-Stieltjes transform is an isometric Banach algebra isomorphism between $M(\hat{G})$ and $B(G)$.

For $\pi \in \Sigma_G$, let the *Fourier space* A_π and the *Fourier-Stieltjes space* B_π be defined by

$$A_\pi = \overline{F_\pi}^{\|\cdot\|} \quad \text{and} \quad B_\pi = \overline{F_\pi}^{\sigma(B(G),C^*(G))}.$$

2.5.1 The Dual of A_π

Let $\mathcal{H}$ be a Hilbert space. The *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ is the linear topology generated by the functionals $T \mapsto \langle T\xi, \eta\rangle$ for $\xi, \eta \in \mathcal{H}$. A *von Neumann algebra* is a non-degenerate *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ which is closed in the WOT.

The *strong operator topology* (SOT) on $\mathcal{B}(\mathcal{H})$ is the topology generated by the seminorms $T \to \|T\xi\|$ for ξ in $\mathcal{H}$. Given a family $\mathcal{S}$ of bounded operators on $\mathcal{H}$, its *commutant* $\mathcal{S}'$ is given by $\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for } S \in \mathcal{S}.\}$ we write the second commutant as $(S')' = S''$.

Theorem 2.5.1 (von Neumann's Double Commutant Theorem)**.** *If $\mathcal{B}$ is a non-degenerate *-subalgebra of bounded operators on a Hilbert space $\mathcal{H}$, then*

$$\overline{\mathcal{B}}^{WOT} = \overline{\mathcal{B}}^{SOT} = \mathcal{B}''.$$

If $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, let $\mathcal{S}_{sa} = \{S \in \mathcal{S} : S = S^*\}$ (self-adjoint part of $\mathcal{S}$).

Theorem 2.5.2 (Kaplansky's Density Theorem)**.** *If $\mathcal{B}$ is a *-subalgebra of bounded operators on a Hilbert space $\mathcal{H}$, and $\mathcal{M} = \overline{\mathcal{B}}^{WOT}$, then*

$$\overline{b_1(\mathcal{B})}^{SOT} = \overline{\{B \in \mathcal{B} : \|B\| \leq 1\}}^{SOT} = b_1(\mathcal{M}) \quad \textit{and} \quad \overline{b_1(\mathcal{B})_{sa}}^{SOT} = b_1(\mathcal{M})_{sa},$$

and hence

$$\overline{b_1(\mathcal{B})}^{WOT} = b_1(\mathcal{M}) \quad \textit{and} \quad \overline{b_1(\mathcal{B})_{sa}}^{WOT} = b_1(\mathcal{M})_{sa}.$$

The dual pairing of $\mathcal{H} \hat{\otimes} \overline{\mathcal{H}} - \mathcal{B}(\mathcal{H})$, where $\mathcal{H} \hat{\otimes} \overline{\mathcal{H}}$ is the projective tensor product, gives a linear topology on $\mathcal{B}(\mathcal{H})$, the *weak* topology.*

Proposition 2.5.3. *The weak* topology and WOT coincide on bounded sets in $\mathcal{B}(\mathcal{H})$. In particular, if $\mathcal{B}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$, then*

$$\overline{\mathcal{B}}^{weak^*} = \overline{\mathcal{B}}^{WOT}$$

If $\pi \in \Sigma_G$, then $\tilde{\pi}(L^1(G))$ is a non-degenerate self-adjoint subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$. Then we define

$$VN_\pi = \overline{\tilde{\pi}(L^1(G))}^{WOT} = \overline{\tilde{\pi}(L^1(G))}^{\text{weak}^*},$$

Lemma 2.5.4. $VN_\pi = \overline{\text{span}\, \pi(G)}^{WOT} = \overline{\text{span}\, \pi(G)}^{weak^*}$.

We now characterize the dual of A_π.

Theorem 2.5.5. *If $\pi \in \Sigma_G$ then:*

1. $A_\pi^* \cong VN_\pi$ *via the dual pairing* $\langle \pi_{\xi,\eta}, T \rangle = \langle T\xi, \eta \rangle$,
2. $A_\pi = \left\{ u \in B(G) : u = \sum_{i=1}^{\infty} \pi_{\xi_i,\eta_i}, \xi_i, \eta_i \in \mathcal{H}_\pi \textit{ and } \sum_{i=1}^{\infty} \|\xi_i\| \, \|\eta_i\| < \infty \right\}$,

3. *For* $u \in A_\pi$, $\|u\| = \inf\left\{\sum_{i=1}^{\infty} \|\xi_i\|\,\|\eta_i\| : u = \sum_{i=1}^{\infty} \pi_{\xi_i,\eta_i}\right\}$.

Remark 2.5.6. *For each* $s \in G$, $\langle \cdot, \pi(s)\rangle$ *is the* evaluation functional *at* s. *That is, for* $u \in A_\pi$,

$$\langle u, \pi(s)\rangle = u(s).$$

Remark 2.5.7. *If* π *is irreducible, then* $A_\pi = \mathcal{H}_\pi \hat{\otimes} \overline{\mathcal{H}_\pi}$, *which is isometrically isomorphic to the trace class operators on* $\mathcal{H}_\pi$.

Remark 2.5.8. *Using the polar decomposition of an element in the predual of a von Neumann algebra, it can be shown that given* $u \in A_\pi$, *there exist* $\{\xi_i\}_{i\in\mathbb{N}}$ *and* $\{\eta_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{H}_\pi$ *such that* $u = \sum_{i=1}^{\infty} \pi_{\xi_i,\eta_i}$ *and* $\|u\| = \sum_{i=1}^{\infty} \|\xi_i\|\,\|\eta_i\|$.

2.5.2 The Correspondence $\pi \mapsto A_\pi$

If $\{\pi_i\}_{i\in I} \subset \Sigma_G$ is a set of representations, the direct sum $\bigoplus_{i\in I} \pi_i$ is the representation on $\bigoplus_{i\in I} \mathcal{H}_i$ given by

$$\bigoplus_{i\in I} \pi_i(s)\xi = (\pi_i(s)\xi_i)_{i\in I}$$

for $s \in G$ and $\xi = (\xi_i)_{i\in I}$ in $\bigoplus_{i\in I} \mathcal{H}_i$.

Proposition 2.5.9. *If* $\pi = \bigoplus_{i\in I} \pi_i$, *then*

1. $A_\pi = \left\{u \in B(G) : u = \sum_{i\in I} u_i \text{ where } u_i \in A_{\pi_i} \text{ and } \sum_{i\in I} \|u_i\| < \infty\right\}$, *and*

2. *for* $u \in A_\pi$, $\|u\| = \inf\left\{\sum_{i\in I} \|u_i\| : u = \sum_{i\in I} u_i \text{ and } u_i \in A_{\pi_i}\right\}$

If $\pi \in \Sigma_G$, a *subrepresentation* of π is the restriction of $\pi, \pi(\cdot)|_{\mathcal{K}}$, to some π-invariant subspace of $\mathcal{H}_\pi$. If $\pi' = \pi(\cdot)|_{\mathcal{K}}$, we write $\pi' \subset \pi$. Two representations π, ρ in Σ_G are *disjoint*, written $\pi \top \rho$, if no non-trivial subrepresentation of π is equivalent to one of ρ.

Theorem 2.5.10. *If $\pi, \rho \in \Sigma_G$, then*

1. $A_\pi \cap A_\rho = \{0\}$ *if and only if* $\pi \top \rho$.
2. *In particular,* $A_{\pi\oplus\rho} = A_\pi \oplus_1 A_\rho$ *if and only if* $\pi \top \rho$.

(The symbol $\oplus_1$ indicates an "l_1" direct sum.)

Corollary 2.5.11. *If $\pi = \oplus_{i\in I}\pi_i$, and $\pi_i \top \pi_j$ for $i \neq j$, then $A_\pi = l^1 - \oplus_{i\in I} A_{\pi_i}$.*

If $\pi, \rho \in \Sigma_G$, a weak*-weak* continuous linear map $\Phi : VN_\pi \to VN_\rho$ is *intertwining* if $\Phi(\tilde{\pi}(f)) = \tilde{\rho}(f)$ for each $f \in L^1(G)$. Such a map is *-homomorphism since $\Phi|_{\tilde{\pi}(L^1(G))}$ is multiplicative, and multiplication is a separately weak* continuous map.

We say that π and ρ are $quasi-equivalent$, written $\pi \simeq \rho$, if there is an intertwining isomorphism $\Phi : VN_\pi \to VN_\rho$.

Theorem 2.5.12. $A_\pi = A_\rho$ *if and only if* $\pi \simeq \rho$.

If $\pi \in \Sigma_G$ and α is a cardinal number, let $\alpha.\pi$ be the α-*ampliation* of π. That is, if I is any index set of cardinal α, $\alpha.\pi : G \to \mathcal{U}(\mathcal{H}_\pi^{\mathcal{I}})$, is given for $s \in G$ and $\xi = (\xi_i)_{i\in I}$ in $\mathcal{H}_\pi^{\mathcal{I}}$ by

$$\alpha.\pi(s) = (\pi(s)\xi_i)_{i\in I}.$$

Remark 2.5.13. *$\pi \simeq \rho$ if and only if there exist cardinals α and β such that $\alpha.\pi \simeq \beta.\rho$. In particular, $A_\pi = A_\rho$ if and only if $\alpha.\pi \simeq \beta.\rho$ for some cardinals α and β.*

Theorem 2.5.14. *If $\pi, \rho \in \Sigma_G$, then there exist subrepresentations π', π'' of π, and subrepresentations ρ', ρ'' of ρ, such that*

$$\pi' \simeq \rho', \quad \pi' \top \pi'', \quad \rho' \top \rho'', \quad \pi' \top \rho'' \quad \textit{and} \quad \rho' \top \pi''.$$

Corollary 2.5.15. *If $\pi, \rho \in \Sigma_G$, then $A_\pi \cap A_\rho = A_{\pi'} = A_{\rho'}$ and hence, $A\rho \subset A_\pi$ if and only if ρ is quasi-equivalent to a subrepresentation of π.*

Proposition 2.5.16. *If $\pi \in \Sigma_G$, then*

1. *A_π is a subalgebra of $B(G)$ if and only if $\pi \otimes \pi$ is quasi-equivalent to a subrepresentation of π, and*
2. *A_π is an ideal of $B(G)$ if and only if $\rho \otimes \pi$ is quasi-equivalent to a subrepresentation of π, for each $\rho \in \Sigma_G$.*

2.5.3 Invariant Subspaces

Using Theorem 2.5.5, we can show that there are natural right and left actions of VN_π on A_π, making A_π a Banach VN_π-module. These actions are given for $T \in VN_\pi$ and $u \in A_\pi$, by

$$\langle T \cdot u, S\rangle = \langle u, ST\rangle \quad \text{and} \quad \langle u \cdot T, S\rangle = \langle u, TS\rangle$$

for $S \in VN_\pi$. To verify that $T \cdot u$ and $u \cdot T$ are indeed in A_π, let $u = \sum_{i=1}^{\infty} \pi_{\xi_i,\eta_i}$, and we have

$$\langle T \cdot u, S\rangle = \langle u, ST\rangle = \sum_{i=1}^{\infty} \langle ST\xi_i, \eta_i\rangle = \sum_{i=1}^{\infty} \langle \pi_{T\xi_i,\eta_i}, S\rangle = \left\langle \sum_{i=1}^{\infty} \pi_{T\xi_i,\eta_i}, S \right\rangle,$$

so $T \cdot u = \sum_{i=1}^{\infty} \pi_{T\xi_i,\eta_i}$. Similarly, we can see that $u \cdot T = \sum_{i=1}^{\infty} \pi_{\xi_i,T^*\eta_i}$.

If $s, t \in G$ and $u \in A_\pi$, then

$$\pi(s) \cdot u(t) = \langle \pi(s) \cdot u, \pi(t)\rangle = \langle u, \pi(ts)\rangle = u(ts)$$

and, similarly, $u \cdot \pi(s)(t) = u(st)$. Hence the right and left actions of VN_π on A_π, when restricted to $\pi(G)$, give, respectively, left and right translations on A_π by G. In particular, A_π is invariant under these translations.

Lemma 2.5.17. *If $\mathcal{X}$ is a closed subspace of A_π, which is invariant under left (right) translations by G, then it is invariant under the right (left) action of VN_π.*

Proof. Suppose that $\mathcal{X}$ is closed under left translations. If $u \in \mathcal{X}$, then $u \cdot \pi(s) \in \mathcal{X}$ for all $s \in G$, so $u \cdot T \in \mathcal{X}$ for all $T \in \mathrm{span}\pi(G)$. The self-adjoint algebra $\mathrm{span}\pi(G)$ is SOT dense in VN_π, by Lemma 2.5.4 and von Neumann's Double Commutant Theorem. Since $VN_\pi = \mathrm{span}(VN_\pi)_{sa}$, it is enough to show that if $T \in (VN_\pi)_{sa}$, then $u \cdot T \in \mathcal{X}$. We may suppose that $T \in b_1(VN_\pi)_{sa}$. Then, by the Kaplansky Density Theorem, there is a net $\{T_\alpha\}_{\alpha\in A}$ in $b_1(\mathrm{span}\pi(G))_{sa}$ converging to T in the SOT. If $u = \sum_{i=1}^{\infty} \pi_{\xi_i,\eta_i}$, and $\varepsilon > 0$ is given, find $n \in \mathbb{N}$ such that $\displaystyle\sum_{i=n+1}^{\infty} \|\xi_i\| \, \|\eta_i\| < \varepsilon$. Then we have that

$$\begin{aligned}
\|u \cdot T - u \cdot T_\alpha\| &= \sum_{i=1}^{\infty} \left\| \pi_{\xi_i,(T-T_\alpha)\eta_i} \right\| \\
&\leq \sum_{i=1}^{n} \|\xi_i\| \, \|(T - T_\alpha)\eta_i\| + \sum_{i=n+1}^{\infty} \|\xi_i\| \, \|T - T_\alpha\| \, \|\eta_i\| \\
&< \sum_{i=1}^{n} \sum_{i=1}^{n} \|\xi_i\| \, \|(T - T_\alpha)\eta_i\| + 2\varepsilon \xrightarrow{\alpha \in A} 2\varepsilon
\end{aligned}$$

so $u \cdot T = \lim_{\alpha \in A} u \cdot T_\alpha \in \mathcal{X}$.

It is similar to show that if $\mathcal{X}$ is closed under right translations, then it is closed under the left action from VN_π. $\square$

Theorem 2.5.18. *A closed subspace $\mathcal{X}$ of A_π is invariant under both left and right translations from G if and only if $\mathcal{X} = A_\rho$, where ρ is quasi-equivalent to a subrepresentation of π. Moreover, there exists a central projection P in VN_π such that $A_\rho = P \cdot A_\pi = \{u \in A_\pi : P \cdot u = u\}$.*

Recall $\mathcal{P}_1(G) = \{u \in \mathcal{P}(G) : \|u\| = 1\}$. In the $C^*(G) - B(G)$ dual pairing, $\mathcal{P}_1(G)$ is the *state space* of $C^*(G)$. For each $u \in \mathcal{P}_1(G)$, let $(\pi_u, \mathcal{H}_u, \xi_u)$ be the Gelfand-Naimark triple for u. Let

$$\varpi = \bigoplus_{u \in \mathcal{P}_1(G)} \pi_u.$$

Theorem 2.5.19. *Every $\pi \in \Sigma_G$ is quasi-equivalent to some subrepresentation of ϖ.*

This property motivates us to call ϖ the *universal representation* of G. It follows that $B(G) = A_\varpi$ because by definition $B(G) = \cup_{\pi \in \Sigma_G} F_\pi$, and, by the theorem above, each $F_\pi \subset A_\varpi$. It follows from Theorem 2.5.18 that any norm closed subspace of $B(G)$ which is invariant under both left and right translation by G is a Fourier space A_π for some $\pi \in \Sigma_G$.

Remark 2.5.20. *The von Neumann algebra $W^*(G) = VN_\varpi$ has the property that for any $\pi \in \Sigma_G$, there is an intertwining map $\Phi_\pi : W^*(G) \to VN_\pi$, which is uniquely determined up to spatial equivalence. Indeed, decompose π into cyclic representations, $\pi = \oplus_{i\in I}\pi_i$, each with cyclic vector ξ_i in $b_1(\mathcal{H}_{\pi_i})$. Let $\mathcal{S}_\pi = \{u \in \mathcal{P}_1(G) : u = (\pi_i)_{\xi_i,\xi_i}$ for some $i \in I\}$, for $u \in \mathcal{S}_\pi$, let $I_u = \{i \in I : (\pi_i)_{\xi_i,\xi_i} = u\}$. Then if $i, j \in I_u, (\pi_i)_{\xi_i,\xi_i} = (\pi_j)_{\xi_j,\xi_j} = u$, so $\pi_i \sim \pi_j \sim \pi_u$, by the Gelfand-Naimark Theorem. Hence $\pi = \oplus_{u\in\mathcal{S}_\pi} \oplus_{i\in I_u} \pi_i \sim \oplus_{u\in\mathcal{S}_\pi}|I_u|\cdot\pi_u$. Now let $\tilde{\Phi}_\pi : W^*(G) \to \mathcal{B}(\oplus_{u\in\mathcal{S}_\pi}\mathcal{H}_u^{|I_u|})$ be given by*

$$\tilde{\Phi}_\pi = \bigoplus_{u\in\mathcal{S}_\pi} \Psi_{|I_u|} \circ \Gamma_{\mathcal{H}_u},$$

where Ψ_α is the α-ampliation map for any cardinal α, and $\Gamma_{\mathcal{K}}$ is the compression map for a $W^*(G)$-invariant subspace $\mathcal{K}$. Then $\tilde{\Phi}_\pi(W^*(G)) \cong VN_\pi$ via a spatial isomorphism, adU, for any unitary $U : \mathcal{H}_\pi \to \oplus_{u\in\mathcal{S}_\pi}\mathcal{H}_u^{|I_u|}$, which is implementing the equivalence $\pi = \oplus_{u\in\mathcal{S}_\pi} \oplus_{i\in I_u} \pi_i \sim \oplus_{u\in\mathcal{S}_\pi}|I_u| \cdot \pi_u$. Then set $\Phi_\pi = adU \circ \tilde{\Phi}_\pi$. $\tilde{\Phi}_\pi$ is uniquely determined since it is an intertwining map and $\tilde{\varpi}(L^1(G))$ is weak dense in $W^*(G)$. $W^*(G)$ is called the* enveloping von Neumann algebra *of G.*

2.5.4 Fourier Algebra

The *Fourier algebra* $A(G)$ is the Fourier space A_λ. Observe that since $\lambda \otimes \sigma$ is unitarily equivalent to an ampliation of λ for every representation σ of G (Fell's absorption principle), $A(G)$ is an ideal of the Fourier-Stieltjes algebra. The Fourier algebra is the predual of the group von Neumann algebra $VN(G) := VN_\lambda$ and is a very important object of study in abstract harmonic analysis. When G is abelian, $A(G)$ is isometrically isomorphic as a Banach algebra to $L^1(\hat{G})$ by applying the Fourier transform to functions in $L^1(\hat{G})$. Below we list a few alternate characterizations and a couple of important theorems for the Fourier algebra.

Theorem 2.5.21. *Let G be a locally compact group.*

1. *$A(G)$ is the norm closure of $B(G) \cap C_c(G)$ in $B(G)$,*

2. *$A(G)$ is the norm closed linear span of $P(G) \cap C_c(G)$ in $B(G)$,*

3. *$A(G)$ is the norm closed linear span of $P(G) \cap L^2(G)$ in $B(G)$,*

4. *$A(G) = \{f * g^\vee : f, g \in L^2(G)\}$ and $\|u\| = \inf\{\|f\|_2 \|g\|_2 : u = f * g^\vee\}$ for each $u \in A(G)$ (where $g^\vee(s) := g(s^{-1})$). Moreover, this infimum is attained.*

Theorem 2.5.22 (Herz's restriction theorem)**.** . *Let G be a locally compact group and H a closed subgroup of G. Then $A(G)|_H = A(H)$.*

The spectrum or Gelfand space $\Delta(A(G))$ of $A(G)$ can naturally be identified with G. More precisely, the map $x \to \phi_x$, where $\phi_x(u) = u(x)$ for $u \in A(G)$, is a homeomorphism between G and $\Delta(A(G))$.

2.5.5 Almost Periodic Compactification of G

For a locally compact group G, we have a direct sum decomposition $M(G) = l^1(G) \oplus M_c(G)$, where $M_c(G)$ denotes the ideal of continuous measures in $M(G)$. Let G be a abelian group with dual group $\hat{G}$ whose Bohr compactification we denote by $b\hat{G}$; we write G_d for the group G equipped with the discrete topology. For $\mu \in M(G)$, we denote its Fourier-Stieltjes transform in $B(\hat{G})$ by $\hat{\mu}$. Then we have for $\mu \in M(G)$:

$$\begin{aligned} \mu \in l^1(G) &\iff \mu \in M(G_d) \\ &\iff \hat{\mu} \in B(\widehat{G_d}) \\ &\iff \hat{\mu} \in B(b\hat{G}) \\ &\iff \hat{\mu} \in B(\hat{G}) \text{ is almost periodic,} \end{aligned}$$

where the last equivalence holds by [Eym64, (2.27) Corollaire 4].

This suggests that the appropriate replacement for $l^1(G)$ in the Fourier-Stieltjes algebra context is $B(G) \cap AP(G)$, where $AP(G)$ denotes the algebra of all almost periodic functions on G. It is well known (see [Pal94, 3.2.16], for example) that $AP(G)$ is a commutative C^*-algebra whose Gelfand space is a compact group denoted by aG (for abelian G, we have $aG = bG$) and is called almost periodic compactification of G. Alternatively, let $\mathcal{F}$ denote the family of all finite-dimensional representations of G. Let $A_{\mathcal{F}}$ denote the closed linear span in $B(G)$ of the coefficient functions of all representations in $\mathcal{F}$. Since $\mathcal{F}$ is closed under taking tensor products, it is immediate that $A_{\mathcal{F}}$ is a Banach algebra.

Proposition 2.5.23. *Let G be a locally compact group. Then $A_{\mathcal{F}} = B(G) \cap AP(G)$, and we have a canonical isometric isomorphism between $A_{\mathcal{F}}$ and $B(aG)$.*

Chapter 3

Regularity of Rajchman Algebras and the Invertibles

Let G be a locally compact group. The *Rajchman algebra* associated with G, denoted by $B_0(G)$, is the set of elements of the Fourier-Stieltjes algebra which vanish at infinity, that is,

$$B_0(G) = B(G) \cap C_0(G).$$

Note that $B_0(G)$ is a subalgebra of $B(G)$, since both $C_0(G)$ and $B(G)$ are algebras. It is easy to see that the Rajchman algebra is indeed a Banach subalgebra of the Fourier-Stieltjes algebra which contains the Fourier algebra as a closed ideal. For a locally compact abelian group G with dual group $\widehat{G}$, let $M_0(G)$ denote the closed ideal consisting of all $\mu \in M(G)$ such that the Fourier-Stieltjes transform $\hat{\mu}$ vanishes at infinity on $\widehat{G}$. The measures in $M_0(G)$ are usually called Rajchman measures for historical reasons related to the work of Rajchman on trigonometric series and sets of uniqueness.

There are some (classes of) locally compact groups G for which $B_0(G)$ coincides with $A(G)$. In general, however, $B_0(G)$ may be considerably larger than $A(G)$. For instance, if G is abelian and noncompact, the quotient $B_0(G)/A(G)$ is far from being a radical algebra. The present investigation only concerns when $B_0(G)/A(G)$ is a radical algebra. As a consequence of the Arens-Royden theorem, we establish how invertibility of $B_0(G)$

reduces to that of $A(G)$ for such Rajchman algebras. We also describe the structure of invertible elements in the Fourier-Stieltjes algebra of classical motion groups.

3.1 A Consequence of the Arens-Royden Theorem

If A is a commutative Banach algebra with identity, we denote by A^{-1} the multiplicative group of all invertible elements of A. If $a = e^b$ for some $b \in A$, then $a \in A^{-1}$ with $a^{-1} = e^{-b}$. Hence the group $\exp A$ of all elements of A with logarithms is a subgroup of A^{-1}. If A^{-1} is given the norm topology, then it is a topological group with $\exp A$ as an open subgroup. Note that if $a = e^b \in \exp A$, then $t \to e^{tb}$ ($t \in [0, 1]$) yields an arc in $\exp(A)$ connecting a to 1. In fact, $\exp(A)$ is exactly the connected component of the identity in A^{-1}, and analytic functional calculus shows that any element within distance 1 of the identity is in this component. We denote the discrete group $A^{-1}/\exp(A)$ by $H^1(A)$.

Let $\Delta(A)$ be the Gelfand spectrum of A. We denote the group

$$H^1(C(\Delta(A))) = \frac{C(\Delta(A))^{-1}}{\exp(C(\Delta(A)))}$$

by $H^1(\Delta(A))$. The Gelfand transform $a \to \widehat{a} : A \to C(\Delta(A))$ where $\widehat{a}(\gamma) = \langle a, \gamma \rangle$ for $a \in A$ and $\gamma \in \Delta(A)$, maps A^{-1} into $C(\Delta(A))^{-1}$ and $\exp(A)$ into $\exp(C(\Delta(A)))$. In fact, for any unital Banach algebra homomorphism $\phi : A \to B$, we have $\phi A^{-1} \subset B^{-1}$ and $\phi \exp(A) \subset \exp(B)$; thus ϕ induces a map ϕ^* of $H^1(A)$ into $H^1(B)$. In particular, $a \to \widehat{a}$ induces a homomorphism of $H^1(A)$ into $H^1(\Delta(A))$.

Theorem 3.1.1 (Arens and Royden, [Are63], [Roy63])**.** *If A is a commutative Banach algebra with identity 1 and spectrum $\Delta(A)$, then the map $H^1(A) \to H^1(\Delta(A)) = H^1(C(\Delta(A)))$ induced by the Gelfand transform, is an isomorphism.*

The following result may be well known, though it is not explicitly found in the literature, so we provide a proof.

Corollary 3.1.2. *Let $A \subset B$ be a commutative Banach sub-algebra such that $\Delta(A) = \Delta(B)$ and $1 \in A$. Then $B^{-1} = \exp(B)A^{-1}$.*

Proof. Let $\Delta = \Delta(A) = \Delta(B)$. By Theorem 3.1.1, if Γ_j ($j = A, B$) denotes the Gelfand transform of j, we see that $(\Gamma_B^*)^{-1} \circ \Gamma_A^* : H^1(A) \to H^1(B)$ is an isomorphism. Since the inclusion $i : A \to B$ is a unital Banach algebra homomorphism, it induces a map $i^* : H^1(A) \to H^1(B)$ such that $\Gamma_B^* \circ i^* = \Gamma_A^*$. Now i^* is injective because $\Gamma_B^* \circ i^*$ is injective. It is enough to show that i^* is surjective. For $b \in B^{-1}$, find $d \in A^{-1}$ such that $(\Gamma_A^*)^{-1} \circ \Gamma_B^*(b \exp(B)) = (\Gamma_A^*)^{-1} (\Gamma_B(b) \exp(C(\Delta)) = d \exp A$. Since $\Gamma_B^*(d \exp B) = \Gamma_B^* \circ i^*(d \exp A) = \Gamma_A^*(d \exp A)$, we have $\Gamma_B^*(d \exp B) = \Gamma_B^*(b \exp B)$. Then the fact that Γ_B^* is an isomorphism implies that $b \exp B = d \exp B = i^*(d \exp A) \in i^*\left(H^1(A)\right)$. Therefore, i^* is surjective. Since $i^*(d \exp A) = d \exp B$ for $d \in A^{-1}$, we get the desired result. □

3.2 Regularity of Rajchman Algebras and the Invertibles

Recall that A is called *regular* if given any closed subset E of $\Delta(A)$ and $\gamma \in \Delta(A)\backslash E$, there exists $a \in A$ such that $\widehat{a} = 0$ on E and $\widehat{a}(\gamma) \neq 0$. Recall also that $A(G)$ is regular for any locally compact group G. The next theorem characterizes the regularity of $B_0(G)$ for general locally compact groups G. Note that in the case of abelian G, the algebra $B_0(G)$ is regular only if G is compact.

Theorem 3.2.1. [KLU16, Theorem 2.1] *Let A be any closed subalgebra of $B_0(G)$ containing $A(G)$. Then*

1. *G (i.e. $\Delta(A(G))$) is closed in $\Delta(A)$.*
2. *A is regular if and only if $\Delta(A) = G$.*

In particular, $B_0(G)$ is regular if and only if $\Delta(B_0(G)) = G$

Remark 3.2.2. *In Corollary 3.1.2, if $A = A(G) \oplus \mathbb{C}1$ and $B = B_0(G) \oplus \mathbb{C}1$ such that the algebra $B_0(G)$ is regular, we have that $(B_0(G) \oplus \mathbb{C}1)^{-1} / \exp(B_0(G) \oplus \mathbb{C}1)$ is isomorphic to $(A(G) \oplus \mathbb{C}1)^{-1} / \exp(A(G) \oplus \mathbb{C}1)$ and hence $(B_0(G) \oplus \mathbb{C}1)^{-1} = \exp(B_0(G) + \mathbb{C}1)(A(G) \oplus \mathbb{C}1)^{-1}$ for regular Rajchmann algebras $B_0(G)$.*

The condition $\Delta(B_0(G)) = G$ is closely related to asymptotic properties of (strongly continuous) unitary representations $(\pi, \mathcal{H}_\pi)$ of a locally compact group G and with the property of *square-integrability*. By the latter we mean that there is a dense subspace D of $\mathcal{H}_\pi$, such that for all $\xi, \eta \in D$, the matrix coefficient $\phi_{\xi\eta} : G \to \mathbb{C}, g \to \langle \pi(g)\xi, \eta \rangle$ is in $L^2(G)$. Square-integrable representations are C_0-representations, which means that all matrix coefficients of π lie in $B_0(G)$, but not vice versa. It follows from the results of Rieffel [Rie69], Duflo-Moore [DM76] and others that a representation $(\pi, \mathcal{H}_\pi)$ is square-integrable if and only if it is quasi-equivalent to a subrepresentation of the regular representation $(\lambda_G, L^2(G))$. We write $\pi \overset{q}{\leq} \lambda_G$.

The following theorem proved in [May99] shows that for certain real algebraic groups, every C_0-representation has a square integrable tensor power. In fact, recall that a connected real algebraic group G has a unique largest unipotent radical N, and decomposes as

$$G = N \rtimes_\varphi H,$$

where H is a reductive Levi-complement of N and $\varphi : H \to Aut(N)$ is a group homomorphism (see [Hoc81, Theorem VIII.4.3], for example). The groups N and H are Zariski-closed, N is simply connected with respect to the topology induced by $GL(n, \mathbb{C})$ and H acts algebraically and reductively on the Lie algebra n by the derived representation. The centralizer in G of a subset $S \in G$ is denoted by $C_G(S)$.

Theorem 3.2.3. [May99, Theorem 1.1] *Let G be a connected real algebraic group and keep the above notations. Suppose that*

1. *$C_G(H) \cap N = \{e\}$, i.e., H acts non-trivially on N.*
2. *$C_G(\mathcal{Z}) \cap H$ is compact, i.e., $\{h \in H : \varphi(h)a = a, \forall\, a \in \mathcal{Z}\}$ is compact, where $\mathcal{Z}$ is the center of N.*

Then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all representations $(\pi, \mathcal{H}_\pi)$ whose subrepresentations all have compact kernel

$$\pi^{\otimes k} \overset{q}{\leq} \lambda.$$

Several examples of groups are known with the property that for every C_0-representation $(\pi, \mathcal{H}_\pi)$, a sufficiently large tensor power is square-integrable: this follows for semisimple Kazhdan groups from the results of Cowling [Cow79b] and Moore [Moo87], for generalized motion groups from the results of Liukkonen and Mislove [LM81] and for several connected totally minimal groups from the results of Mayer [May97b, May97a, May99].

Theorem 3.2.3 bears significance to a conjecture of Figà-Talamanca and Picardello. The question of the square-integrability of tensor products is related to the relationship between $B_0(G)$ and the radicalizer $A_r(G)$ of the Fourier algebra. That is

$$A_r(G) = \{u \in B(G) : \exists k \in \mathbb{N} \text{ such that } u^k \in A(G)\}.$$

Figà-Talamanca and Picardello [FTP78] showed that $A_r(G)$ is not norm dense in $B_0(G)$ if the center of G is not compact or if G is non-compact nilpotent group. In particular, $A_r(G)$ is not norm dense in $B_0(G)$ if G is a non-compact *abelian* group. Their conjecture reads as follows:

Conjecture 3.2.4. *Let G be an analytic group with compact center and without non-compact nilpotent direct factors. Then $A_r(G)$ is dense in $B_0(G)$. Equivalently, by* [KLU16, Corollary 2.2 (i) and (ii)], *$\Delta(B_0(G)) = G$.*

The question of when C_0-representations are square-integrable has been widely studied. One of the most important results is that the left regular representation splits into irreducibles if every C_0-representation is square-integrable, i.e., $B_0(G) = A(G)$ ([FT77], [BT79]). The converse, though not true in general [BT78], holds for many semidirect product groups, including the affine group and p-adic motion groups. This follows from ([RS07], Proposition 2.1). In [Knu17a], Knudby proves the following result:

Theorem 3.2.5. [Knu17a, Theorem 4] *Let G be a second countable locally compact group. Then $B_0(G) = A(G)$ provided that G satisfies the following two conditions:*

- *G is of type I*
- *there is a non-compact closed subgroup H of G such that every irreducible representation of G is either trivial on H or is a subrepresentation of the left regular representation.*

Knudby proves that $B_0(P) = A(P)$ for the minimal parabolic subgroup of several simple Lie groups. In [Knu17b], it is shown that there are uncountably many (non-isomorphic) second countable locally compact groups G such that $B_0(G) = A(G)$ and G has no non-trivial compact subgroups. Knudby also proves $B_0(G) = A(G)$ by weakening the above conditions: G is type I and there is a countable family H of non-compact closed subgroups G such that each irreducible unitary representation of G is either trivial on some $H \in \mathrm{H}$ or is a subrepresentation of the left regular representation of G. The advantage of this condition is that it is preserved under direct products. Khalil [Kha74] showed that the $ax+b$ group, which is non-compact and solvable, satisfies $A(G) = B_0(G)$ and by the above mentioned result, if G is direct product of the $ax+b$ group with itself, then $B_0(G) = A(G)$.

Remark 3.2.6. *For a unital Banach algebra $A = I \oplus B$ where I is an ideal and B is an closed sub-algebra, we have $A^{-1} = (I \oplus \mathbb{C}1)^{-1}B^{-1}$ because if $(u+v)(u'+v') = 1$, then $0 = uu' + vu' + uv'$ $(\in I)$ and $vv' = 1$. So $u+v = (uv^{-1}+1)v$ with $(uv^{-1}+1)^{-1} = u'v+1 \in I \oplus \mathbb{C}1$.*

The above remark allows us to immediately describe the structure of invertible elements in the following examples.

3.2.1 Euclidean Motion Groups

Let $G = \mathbb{R}^d \rtimes SO(d), d \geq 2$, where $SO(d)$ acts on $\mathbb{R}^d$ by rotation.

Theorem 3.2.7. [KLU16, Theorem 4.1]

1. *$B(G) = B_0(G) \oplus B(SO(d)) \circ q$, where $q : G \to SO(d)$ denotes the quotient homomorphism.*

2. *$B_0(G) \neq A(G)$, but $B_0(G) = A_r(G) = \{u \in B(G) \mid u^{4d-3} \in A(G)\}$.*

3. *$\Delta(B_0(G)) = G$ (i.e. $B_0(G)$ is regular) and $\Delta(B(G)) = G \cup SO(d)$.*

4. *$B_0(G)|_{\mathbb{R}^d} \neq B_0(\mathbb{R}^d)$.*

By Remark 3.2.6 and 3.1.2, we have

$$\begin{aligned} B(G)^{-1} &= (B_0(G) \oplus \mathbb{C}1)^{-1} B(SO(d))^{-1} \circ q \\ &= \exp(B_0(G) \oplus \mathbb{C}1)(A(G) \oplus \mathbb{C}1)^{-1}(A(SO(d)))^{-1} \circ q \end{aligned}$$

In this case, the almost periodic compactification of G is $SO(d)$.

3.2.2 $n^{\mathbf{th}}$ Rigid p-adic Motion like Groups

Let $G = A \rtimes K$ where

1. K is a compact group acting on an abelian group A, with each of the groups separable, and
2. the dual space $\widehat{G}$ is countable and decomposes as $\widehat{K} \circ q \sqcup \{\lambda_k\}_{k=1}^{\infty}$ where $\widehat{K}$ is the discrete dual space of K, $q : G \to K$ is the quotient map, and each λ_k is a subrepresentation of the left regular representation.

Then by [RS07, Proposition 2.1], we have $B(G) = A(K) \circ q \oplus^{l_1} A(G)$. By Remark 3.2.6, we have

$$B(G)^{-1} = (A(G) \oplus \mathbb{C}1)^{-1}(A(K)^{-1} \circ q).$$

Examples of such groups are the n^{th} rigid p-adic motion group $G_{p,n} := \mathbb{Q}_p^n \rtimes GL(n, \mathbb{O}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{O}_p := \{r \in \mathbb{Q}_p : |r|_p \leq 1\}$ is the p-adic integers, which is a compact open subring of $\mathbb{Q}_p$, and the compact group $GL(n, \mathbb{O}_p)$, which is the multiplicative group of $n \times n$ matrices with entries in $\mathbb{O}_p$ and determinant of valuation 1 and, acts on the vector space $\mathbb{Q}_p^n$ by matrix multiplication. For $n = 1$, this is the p-adic motion group $\mathbb{Q}_p \rtimes \mathbb{T}_p$, where $\mathbb{T}_p = \{r \in \mathbb{Q}_p : |r|_p = 1\}$. The above decomposition for $B(G_{p,1})$ was proven independently in [Wal76] and [Mau77]. In this case, the almost periodic compactification of G is K.

3.2.3 The $ax+b$ Group

Let $G = \{(a,b) : a,b \in \mathbb{R}, a > 0\}$ with multiplication $(a,b)(c,d) = (ac, ad+b)$. If $j : G \to \mathbb{R}$ is the homomorphism given by $j(a,b) = \log a$, then j is continuous with $\ker j = \{(1,b) : b \in \mathbb{R}\} \cong \mathbb{R}$. In [Kha74, Théorème 7], it is shown that

$$B(G) = (B(\mathbb{R}) \circ j) \oplus^{l_1} A(G).$$

By Remark 3.2.6,

$$B(G)^{-1} = (A(G) \oplus \mathbb{C}1)^{-1}(B(\mathbb{R})^{-1} \circ j).$$

The locally compact groups continuously isomorphic to $\mathbb{R}$ are just $\mathbb{R}$ and $\mathbb{R}_d$ (reals with the discrete topology). Hence by [Tay72, Theorem 3], each $\mu \in M(\mathbb{R})^{-1}$ has the form $\mu = \nu_1 * \nu_2 * e^w$ for $\nu_1 \in (L^1(\mathbb{R}) \oplus \mathbb{C}1)^{-1}, \nu_2 \in L^1(\mathbb{R}_d)$ and $w \in M(\mathbb{R})$. Via the Fourier-Stieltjes transform, this factorization of invertible elements in $B(\mathbb{R})$ has the following form: each $u \in B(\mathbb{R})^{-1}$ has the form $u = v_1 v_2 e^w$ for $v_1 \in (A(\mathbb{R}) \oplus \mathbb{C}1)^{-1}, v_2 \in A(\mathbb{R}^{ap})^{-1}$ and $w \in B(\mathbb{R})$, Here $\mathbb{R}^{ap}$ is the almost periodic compactification of $\mathbb{R}$, which is also the Pontraygin dual group of $\mathbb{R}_d$. Hence for every $u \in B(G)^{-1}$, we have $u = (v_1 \circ j) \cdot (v_2 \circ j) \cdot v_3 e^w$ for $v_1 \in (A(\mathbb{R}) \oplus \mathbb{C}1)^{-1}, v_2 \in A(\mathbb{R}^{ap})^{-1}, v_3 \in (A(G) \oplus \mathbb{C}1)^{-1}$ and $w \in B(G)$. In this case,the almost periodic compactification of G is $\mathbb{R}^{ap}$.

Chapter 4

The Structure of Invertibles in Certain Fourier-Stieltjes Algebra

Joseph L. Taylor wrote a series of ten papers [Tay73] within the span of 1965 - 1972 studying the structure of convolution measure algebras (important examples are $M(G)$, the measure algebra and $L^1(G)$, the group algebra for an locally compact abelian group G) and characterizes the invertible and idempotent elements in a measure algebra of a locally compact abelian group. He proves there is a compact, abelian topological semigroup S associated to every commutative convolution measure algebra $\mathfrak{M}$ called the structure semigroup of $\mathfrak{M}$. Then one could identify $\widehat{S}$, the space of all continuous, non-zero, bounded semicharacters on S with the Gelfand spectrum Δ of $\mathfrak{M}$. Several special algebraic and topological properties of $\widehat{S}$ were studied and certain subsets of it were identified where the different topologies coincide. One of those subsets is the maximal group in $\widehat{S}$ containing an idempotent $h = h^2 \in \widehat{S}$, $\Gamma_h = \{f \in \widehat{S} : |f| = h\}$. In particular, he considered those elements in $\widehat{S}_+ = \{f \in \widehat{S} : f \geq 0\}$ that cannot be approximated from below by strictly smaller elements of $\widehat{S}_+$, which he called critical points. These critical points of $\widehat{S}_+$ play a major role in determining the cohomology of $\widehat{S}$ and arise from subalgebras of $\mathfrak{M}$ which are isomorphic to group algebras. In fact, if $h \in \widehat{S}_+$ is a critical point, then its maximal group Γ_h is a locally compact topological group in the weak topology. He characterizes the cohomology of $\widehat{S}$ in terms of the cohomology of Γ_h as h varies over the critical points. Then he establishes a one-one correspondence between (maximal) group algebras L_h contained in $\mathfrak{M}$ and critical points of $\widehat{S}_+$. He calls the closed linear span $\mathfrak{M}_0$ in $\mathfrak{M}$ of the group algebras in $\mathfrak{M}$ the *spine* of a convolution measure algebra $\mathfrak{M}$ and shows that the spine is exactly the direct sum of the maximal group algebras of $\mathfrak{M}$. Then he applies results of cohomology (using the Arens-Royden theorem) to get two factorization results, one of which is essentially Cohen's idempotent theorem and the following one which provides the motivation for this research project.

Theorem 4.0.1. [Tay73, Corollory 8.1.6] *If $\mu \in \mathfrak{M}^{-1}$, then there are distinct maximal group algebras $L_{h_1}, \cdots, L_{h_n}$, elements $\nu_i \in (L_{h_i} \oplus \mathbb{C}1)^{-1}$, for $i = 1, \cdots, n$, and an element $\omega \in \mathfrak{M}$, such that $\mu = \nu_1 * \nu_2 * \cdots * \nu_n * e^{\omega}$. Furthermore, each ν_i is unique modulo* $\exp\ (L_{h_i} \oplus \mathbb{C}1)^{-1}$.

In the case where $\mathfrak{M} = M(G)$, with G a l.c.a, each maximal group algebra in $M(G)$ is $L^1(G_\tau)$ for some l.c.a G_τ continuously isomorphic to G; that is, G_τ is G as a group, but is an l.c.a group under a topology τ possibly finer than that of G [Tay73, Theorem 8.2.4]. The above results rely heavily on specialized machinery from the study of convolution measure algebras and the proof of the above factorization theorem uses a considerable amount of sheaf theory and algebraic topology. In order to wade through a discouraging amount of machinery, for the community of harmonic analysts who wishes to understand these results, in [Tay72], Taylor brought together some of the above results and proved them with a minimum of machinery with particular attention restricted to abelian $M(G)$. He uses generalized characters to describe the maximal ideal space of $M(G)$ rather than using the structure semigroup description and was able to eliminate the sheaf theory and algebraic topology. The present chapter is concerned with the possibility of adapting the theory to the case of certain Fourier-Stieltjes algebra based on [Tay72], [Wal75] and the next chapter investigates the same using the cohomological calculations as done in [Tay73].

A space S endowed with a Hausdorff topology τ is said to be a semitopological semigroup if it has an associative multiplication such that the mappings $s \mapsto st$ and $s \mapsto ts$ are continuous with respect to τ for all $t \in S$.

The Ellis-Lawson theorem states that a subgroup of a compact semitopological semigroup S is topological [BJM89, 1.4.4]. Furthermore, in case that S contains a dense subgroup, every separately continuous action of S on a compact Hausdorff space X is jointly continuous at every point (s, x_0), where $s \in S$ and $x_0 \in X$ with $S \cdot x_0 = X$ [Rup84, II.4.11]. Every compact semitopological semigroup S contains an idempotent [Rup84, I.2.1] and a unique minimal ideal $K(S)$[Rup84, I.3.11]. We denote the set of idempotents in S with $J(S)$. For any $v \in J(S)$ there exists exactly one maximal subgroup $H(v)$ containing v, namely the union of all such subgroups [BJM89, 1.1.11], i.e., $H(v) = \{s \in S : \text{ there exists } t \in S \text{ such that } st = v = ts\}$. Furthermore, it can be noted that in the case S is a self-adjoint subsemigroup of the unit ball of a von Neumann algebra that each such v is a projection, and $H(v)$ is a group of partial isometries with common domain and range projection v.

There exists the following partial ordering on $J(S)$ provided that $J(S)$ is abelian:

$$v \leq w :\iff wv = v \iff v \cdot S \subseteq w \cdot S.$$

In this way, $J(S)$ becomes a complete lattice if S is a compact semitopological monoid [Rup84, I.2.13].

The semigroup S is called a semilattice of groups if S is the union of its maximal subgroups and $J(S)$ is abelian.

The study of invertibles in Fourier-Stieltjes algebras of classical motion groups instigated the search for a reasonable structure theory of Fourier-Stieljes algebra with its spectrum being a semilattice of groups. Motivated by examples of such Fourier-Stieltjes algebras (Section 4.4), we present our main theorem (Theorem 4.3.7) in this chapter. Before that let's recall how Walter extended the notion of critical points [Tay72, Definition 2.1] to the spectrum of Fourier-Stieltjes algebra. For the rest of this thesis, let Δ denote the spectrum of Fourier-Stieltjes algebra of an underlying locally compact group G.

4.1 The Spectrum of $B(G)$

In this section, we recall certain facts about the spectrum of $B(G)$ from Martin Walter's work [Wal72, Wal75]. If $s \in \Delta := \Delta(B(G))$ there are naturally associated two endomorphisms of $B(G)$, $\gamma_s : b \in B(G) \to s.b \in B(G)$ and $\delta_s : b \in B(G) \to b.s \in B(G)$, where $\langle s.b, x\rangle = \langle b, xs\rangle$ and $\langle b.s, x\rangle = \langle b, sx\rangle$ for all $x \in W^*(G) = B(G)'$.

Let Δ_+ and Δ_p denote the positive Hermitian elements, and self-adjoint idempotents in Δ, respectively. If G is not compact, then Δ contains partial isometries and projections different from e, the identity of G. We remark that any continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G extends uniquely to a normal (i.e. $\sigma(W^*(G), B(G))$ continuous) representation of $W^*(G)$. In what follows, $z' \leq z$ means $z'z = z'$.

Theorem 4.1.1. *Given $G, B(G), W^*(G)$ and ω_G the universal representation of G, then*

1. [Wal72, Theorem 1] $\Delta = \{s \in W^*(G) - \{0\} : \pi_1 \otimes \pi_2(s) = \pi_1(s) \otimes \pi_2(s)$ *where* π_1 *and* π_2 *vary over all of* $\Sigma(G)\}$, *and*
$$\Delta = \{s \in W^*(G) - \{0\} : \omega_G \otimes \omega_G(s) = \omega_G(s) \otimes \omega_G(s)\}.$$
2. *Δ is a σ-weakly compact, self-adjoint subset of $W^*(G)_1 = \{s \in W^*(G) : \|s\|_{W^*(G)} = 1\}$ and $x, y \in \Delta \Rightarrow xy \in \Delta \cup \{0\}$.*
3. [Wal75, Theorem 2]*If $z_F = \sup\{z[\pi] : z[\pi] =$ support in $W^*(G)$ of finite dimensional (unitary) representation $\pi\}$, then z_F is a central projection in $W^*(G)$, and $z_F \in \Delta_+$. Moreover if $s \in \Delta_+$, we have $z_F s = z_F$, i.e., $z_F \leq s$.*
4. [Wal75, Corollary p. 271] *Δ is a weakly compact semitopological *-semigroup.*
5. [Wal75, Corollary p. 269] *If $s \in \Delta_+$, then $s^z \in \Delta$ for all complex z with $\Re z > 0$.*

Remark 4.1.2. *In* [Sto20]*, Stokke provides an alternate proof of the fact that if A is closed unital translation invariant subalgebra of $B(G)$, then the spectrum of A, $\Delta(A)$, is a compact semitopological *-semigroup using Arens-product formulation of the product in $VN_\pi = A_\pi^*$, where $A = A_\pi$ for some $\pi \in \Sigma_G$.*

4.2 Topologies on Δ

We must discuss the strong and weak topologies on Δ. We have that Δ is compact, the involution * is continuous, and multiplication is separately continuous in the weak (weak operator) topology on Δ. Also, the weak topology is coarser than any of the following strong topologies. Due to possible non-abelianess of G, one can consider four strong topologies on Δ: the strong operator topology; the left-strong topology, i.e., $s \to s_0$ in Δ if and only if $\|\gamma_s(b) - \gamma_{s_0}(b)\| \to 0$ for each $b \in B(G)$; the right-strong topology, i.e., $s \to s_0$ in Δ if and only if $\delta_s(b) - \delta_{s_0}(b) \to 0$ for each $b \in B(G)$; and the *-strong topology, i.e., $s \to s_0$ in Δ provided both $s \to s_0$ and $s^* \to s_0^*$ in the strong operator topology. Multiplication in Δ is jointly continuous in all the strong topologies, whereas the involution is continuous in the *-strong topology. Now note that Δ is not a topological semigroup (in general) in the weak topology, but it is a topological semigroup in the strong topology. But Δ is not (in general) compact in the strong topology. The following calculation,

$$|\hat{u}(s^*s) - \hat{u}(s_0^*s_0)| \leq |\widehat{s.u}(s^*) - \widehat{s_0.u}(s^*)| + |\widehat{s_0.u}(s^*) - \widehat{s_0.u}(s_0^*)|$$

for $u \in B(G), s, s_0 \in \Delta$ shows that the map $s \in \Delta \to s^*s \in \Delta_+$(resp., $ss^* \in \Delta_+$) is continuous from the left-strong (resp., right-strong) topology to the weak topology since * is weakly continuous. Fortunately for our later arguments, the strong topologies and weak topologies agree on certain kind of subsets of Δ.

Proposition 4.2.1. [Wal75, Proposition 4]

1. *If $\{s_\alpha\}$ is a net in Δ, $s \in \Delta$, and $s_\alpha^* s_\alpha \leq s^*s$ (resp. $s_\alpha s_\alpha^* \leq ss^*$) for all α, then $s_\alpha \to s$ left-strongly (resp., right-strongly) if and only if $s_\alpha \to s$ weakly;*
2. *the weak and left-strong (resp., right-strong) topologies agree on set of the form $\{s \in \Delta : s^*s = t : t \in \Delta_+\}$ (resp., $\{s \in \Delta : ss^* = t, t \in \Delta_+\}$);*
3. *The weak and left-(or right) strong topologies agree on any subset of Δ_+ which is totally ordered.*

Note that Δ_+ is closed in both the weak and strong topologies, as is each of its subsets of the form $\{t \in \Delta_+ : s_1 \leq t \leq s_2\}$, determined by $s_1, s_2 \in \Delta_+$. On the other hand,

Proposition 4.2.2. [Wal75, Proposition 5] *If $S \subset \Delta_+$ is strongly closed, then S contains minimal and maximal elements.*

From what follows one can realize that if S is both open and closed in the strong topology, then a minimal element of S must have a very special form.

4.3 Ideals, Groups and Critical Points

Since Δ is a semigroup, we may consider its ideals, idempotents, and maximal groups. The ideals have an interesting characterization in terms of the order relation

Proposition 4.3.1. *1.* [Wal75, Proposition 2] *If $s \in \Delta$, then the principal left and right ideals determined by s have following forms:*

$$\Delta s = \{t \in \Delta : t^*t \leq s^*s\}; \quad s\Delta = \{t \in \Delta : tt^* \leq ss^*\}.$$

2. [Wal75, Proposition 3] *$I \subset \Delta$ is a left-ideal if and only if $s \in I, t \in \Delta,$ and $t^*t \leq s^*s$ imply $t \in I$. A corresponding statement holds for right ideals.*

Note that a principal right (left)-ideal $s\Delta(\Delta s)$ is necessarily weakly closed (hence, strongly closed), since it is the image of the compact set Δ under the continuous map $t \to st$ $(t \to ts)$.

Let us fix some notations, $G_{r,\gamma} = \{s \in \Delta : s^*s = r\}$, $G_{r,\delta} = \{s \in \Delta : ss^* = r\}$, $G_r = G_{r,\gamma} \cap G_{r,\delta}$, where $r \in \Delta_p$.

Proposition 4.3.2. [Wal75, Proposition 6]

1. *G_r is a topological group with * for inverse, r for identity, and the right or left-strong topology or the weak topology – all of which coincide on G_r. G_r is *-strongly closed in Δ.*

2. *$G_{r,\gamma} \subset \Delta r, G_{r,\delta} \subset r\Delta$ and the following inclusions hold: $\Delta r\Delta \supset r\Delta \cup \Delta r \supset r\Delta \cap \Delta r = r\Delta r \supset G_r$.*

Now consider the following conditions for $s \in \Delta_+$.

Condition (A) : There does not exist a net $\{s_\alpha\} \subset \Delta_+$ satisfying $s_\alpha \lneq s$ and $\lim_\alpha s_\alpha = s$.

Condition (B) : There does not exist a net $\{s_\alpha\} \subset \Delta_+$ satisfying $s_\alpha^2 \lneq s^2$(which implies $s_\alpha \lneq s$) and $\lim_\alpha s_\alpha = s$.

Note that in both conditions weak and strong limits are equivalent. Also Condition (A) implies Condition (B). Both conditions imply that $s \in \Delta_p$, since if $s^2 \neq s$ then $s^r \lneq s$ (resp., $s^{2r} \lneq s^2$) for $r > 1$ but with $\lim_{r\to 1+} s^r = s$(resp. $\lim_{r\to 1+} s^{2r} = s^2$). If s is central, then s satisfies condition (A) if and only if it satisfies condition (B), since if s_α and s commute, $0 \leq s_\alpha \leq s$ is equivalent to $0 \leq s_\alpha^2 \leq s^2$. Indeed, $0 \leq s_\alpha \leq s$ implies that $s_\alpha^2 = s_\alpha^{1/2} s_\alpha s_\alpha^{1/2} \leq s_\alpha^{1/2} s s_\alpha^{1/2} = s s_\alpha$. So $0 \leq (s - s_\alpha)^2 = s^2 + s_\alpha^2 - 2 s s_\alpha \leq s^2 - s_\alpha^2$ and we are done, since for positive operators $s_\alpha, s, s_\alpha^2 \leq s^2$ always gives $s_\alpha \leq s$. However, most importantly if s satisfies $s^2 = s \geq 0$, then we have that $0 \leq s_\alpha \leq s^2 = s$ holds if and only if $0 \leq s_\alpha^2 \leq s^2 = s$. Indeed, $0 \leq s_\alpha \leq s^2 = s$ implies that $(e - s)s_\alpha = s_\alpha(e - s) = 0$, because $0 \leq (e - s)s_\alpha(e - s) \leq (e - s)s(e - s) = 0$ which further simplifies to $(s_\alpha^{1/2}(e - s))^*(s_\alpha^{1/2}(e - s)) = 0$. Hence $s s_\alpha = s_\alpha s = s_\alpha$, and we are done. Thus we have the following generalization of the notion of critical point introduced by Taylor.

Definition 4.3.3. [Wal75, pp. 275] *If $r \in \Delta_+$ satisfies condition (A), or equivalently condition (B), then r is called a critical point of Δ_+.*

We observe that r is critical if and only if r is weakly isolated in

$$(r\Delta)_+ = \{t \in \Delta_+ : t^2 \leq r\} = \{t \in \Delta_+ : t \leq r\} = r\Delta_+ r = (r\Delta r)_+.$$

Proposition 4.3.4. [Wal75, Proposition 7] *If $r \in \Delta_+$ then the following statements are equivalent:*

1. *r is critical;*
2. *$G_{r,\gamma}$ is left-strongly (weakly) open in Δr;*
3. *$G_{r,\delta}$ is right-strongly (weakly) open in $r\Delta$;*
4. *G_r is strongly (weakly) open in $r\Delta r$;*
5. *r is a minimal element of a strongly open and closed subset of Δ_+.*

Corollary 4.3.5. [Wal75, Corollary p. 276] *If r is critical, $G_{r,\delta}, G_{r,\gamma}$ are locally compact spaces, and G_r is a locally compact topological group.*

By Theorem 3, z_F is a critical point. If z is a central critical point then we have a continuous homomorphism $\theta_z : g \in G \to gz \in G_z$. Moreover, the map $j_{\theta_z} : u \in B(G_z) \to u \circ \theta_z \in B(G)$ is a linear contraction.

Proposition 4.3.6. [Wal75, Proposition 8] *G_{z_F} is the almost periodic compactification of G, and j_{θ_z} is an isometry of $B(G_{z_F})$ onto $B(G) \cap AP(G) = z_F.B(G)$, where $AP(G)$ denotes the almost periodic functions on G.*

Given a central critical point z, then the closure of $\theta_z(G)$, call it G_{θ_z}, in G_z is a locally compact group, and $j_{\theta_z} : B(G_{\theta_z}) \to B(G)$ is an isometric isomorphism onto a closed, bi-translation invariant subalgebra of $B(G)$. Also, the inclusion $i : G_{\theta_z} \to G_z$ induces a norm-continuous homomorphism $j_i : B(G_z) \to B(G_{\theta_z})$ with the property that $j_i(A(G_z)) = A(G_{\theta_z})$. M.E. Walter asks: is $G_{\theta_z} = G_z$? By proposition 4.3.6 the answer is yes if $z = z_F$.

In what follows we can realize that for certain locally compact groups G (Section 4.4), $G_{\theta_z} = G_z$, and we characterize the invertible elements of certain $B(G)$.

The following theorem is an adaptation of [Tay72, Proposition 4.5].

Theorem 4.3.7. [Tha21, Theorem 5.1] *Let G be locally compact group such that the Fourier-Stieltjes algebra $B(G)$ with the maximal ideal space $\Delta(B(G))$ admits the following decomposition:*

1. $B(G) = \bigoplus^{l_1} \left\{ \overline{A_r(G_z)} \;\middle|\; z \in \Delta(B(G))_+ \right\}$,
2. *each* $\overline{A_r(G_z)}$ *is a closed ideal in* $\bigoplus^{l_1} \left\{ \overline{A_r(G_{z'})} \;\middle|\; z' \leq z \right\}$.
3. $\Delta(B(G)) = \bigcup \{ G_z \mid z \in \Delta(B(G))_+ \}$ *where* $G_z = \{ s \in \Delta(B(G)) : s^* s = s s^* = z \}$.

Then for any $u \in B(G)^{-1}$, we have that

$$u = v_1 \cdot v_2 \cdots v_n \cdot \exp(w)$$

with each $v_i \in (A(G_{z_i}) \oplus \mathbb{C}1)^{-1}$ for some $z_i \in \Delta(B(G))_+$ and $w \in B(G)$. The elements v_i are unique modulo $\exp(A(G_{z_i}) \oplus \mathbb{C}1)$.

Proof. The union of the finite sums of these subalgebras, $\overline{A_r(G_z)}$ for $z \in \Delta_+$, is dense in $B(G)$. So given $u \in B(G)^{-1}$, find ψ in one of those finite sums such that $\|u-\psi\| \leq \|u^{-1}\|^{-1}$. Then $\psi \in B(G)^{-1}$ and $u = \psi * \exp(\rho)$ for some $\rho \in B(G)$. Hence, we may assume $u \in A(G_{z_F}) + \overline{A_r(G_{z_1})} + \overline{A_r(G_{z_2})} + \cdots + \overline{A_r(G_{z_n})}$ for some $z_1, z_2, \ldots, z_n \in \Delta_+$. We can also assume that the almost periodic component of u is 1, because if $u \in B(G)^{-1}$, then $z_F \cdot u \in A(G_{z_F})^{-1}$ and we can replace u by $(z_F \cdot u)^{-1}u$.

Let $u = 1 + \psi_{z_1} + \psi_{z_2} + \cdots + \psi_{z_n}$, where each $\psi_{z_i} \in \overline{A_r(G_{z_i})}$ and z_1 can be chosen minimal among z_i. Now $z_1 \cdot u = 1 + \psi_{z_1}$ is invertible in $\overline{A_r(G_{z_1})} \oplus \mathbb{C}1$, and $(1+\psi_{z_1})^{-1}(1+\psi_{z_1}+\psi_{z_2}+\cdots+\psi_{z_n}) = 1+\psi'_{z_2}+\psi'_{z_3}+\cdots+\psi'_{z_n}$ where $\psi'_{z_j} = (1+\psi_{z_1})^{-1}\psi_{z_j}$ for $j = 2, \ldots, n$ and $\psi'_{z_j} \perp \overline{A_r(G_{z_1})}$. By means of induction, we obtain the following factorization: *if* $u \in B(G)^{-1}$, *then* $u = \psi_1 \cdot \psi_2 \cdots \psi_n \cdot \exp(\rho)$ *with* $\rho \in B(G)$ *and each* $\psi_i \in (\overline{A_r(G_{z_i})} \oplus \mathbb{C}1)^{-1}$ *for some* $z_i \in \Delta_+$.

By Remark 3.2.2, each ψ_i is of the form $v_i \cdot \exp(\rho_i)$, where $v_i \in (A(G_{z_i}) \oplus \mathbb{C}1)^{-1}$ and $\rho_i \in (\overline{A_r(G_{z_i})} \oplus \mathbb{C}1)$. Now let $w = \rho_1 + \cdots + \rho_n + \rho$. □

4.4 Examples

In this section, we describe some classes of locally compact groups G for which the spectrum of their Fourier-Stieltjes algebras is a semilattice of groups and satisfies the hypotheses of our main theorem 4.3.7.

4.4.1 Compact Extensions of Nilpotent Groups

Suppose G is a connected lie group which has a closed normal nilpotent subgroup M such that G/M is compact and let $\mathbb{T}^p$ be the maximal torus in the center of the connected component $N = M_0$ of M. For each subgroup H with $\mathbb{T}^p \subset H \subset G$, let $\overline{H} = H/\mathbb{T}^p$. Consider the following conditions:

1. $\overline{G} = \overline{N} \rtimes K$, where K is a compact analytic group, and

2. the action of K on the Lie algebra $L(\overline{N})$ has no nonzero fixed points.

Let X be the set of (necessarily closed) G-normal analytic subgroups of N which contain $\mathbb{T}^p$, together with the trivial subgroup $\{e\}$. We consider $\mathbb{T}^p \notin X$. Liukkonen and Mislove [LM81, Section 3] proved that for such groups $B(G)$ is symmetric if and only if the above conditions hold and their analysis is detailed enough to yield the following decomposition of $B(G)$ and description of its maximal ideal space Δ :

1. $B(G) = \bigoplus^{l_1} \left\{ \overline{A_r(G/V)} \middle| V \in X \right\}$
2. $\overline{A_r(G/V_1)} \cdot \overline{A_r(G/V_2)} \subset \overline{A_r(G/V_1 \cap V_2)}$.
3. each $\overline{A_r(G/V)}$ is a closed ideal in $\bigoplus^{l_1} \left\{ \overline{A_r(G/W)} \middle| W \supseteqq V \right\}$.

Theorem 4.4.1. [LM81, Theorem 3.1] *Suppose G is an analytic group such that $G/\mathbb{T}^p = N/\mathbb{T}^p \rtimes K$, where K is compact analytic, N is nilpotent analytic, and $\mathbb{T}^p$ is the maximal torus in $Z(N)$. Suppose the action of K on $L(N/\mathbb{T}^p)$ has no nonzero fixed points. Then the maximal ideal space Δ of $B(G)$ satisfies*

$$\Delta = \{G/V : V \in X\}.$$

Multiplication in Δ is given by $g_1V_1g_2V_2 = g_1g_2V_1V_2$. Convergence may be described as follows: if $g_nV_n \to gV$ in Δ, then eventually $V_n \subseteqq V$ and $g_nV \to gV$ in G/V. Conversely, given a sequence $\{g_nV_n\} \subset \Delta$, there is a unique smallest dimensional V such that $V_n \subset V$ eventually and $\{g_nV_n\}$ is eventually bounded in G/V; if $g_nV \to gV$ in G/V, then $g_nV_n \to gV$ in Δ.

4.4.2 Connected Semisimple Lie Groups with Finite Center

Let G be a connected semisimple Lie group with finite centre. Then we may decompose $G = (G_0 \times G_1 \times \cdots G_n)/C$ where G_0 is compact analytic, all of $G_1, \ldots, G_n$ are non-compact and simple analytic and C is a finite central group of the product. Now let $\mathcal{S}$ be the set of groups $G_F = \prod_{k \in F} G_k/C_F$ where F is a (possibly empty) subset of $\{1, \ldots, n\}$ and $C_F = C \cap \prod_{k \in F} G_k$.

Theorem 4.4.2. [Cow79a, pp. 90] *Let G be a connected semisimple Lie group with finite centre and let $\mathcal{S}$ be as described above. Then the maximal ideal space Δ of $B(G)$ is*

$$\Delta = \bigcup_{S \in \mathcal{S}} G/S,$$

G is dense in Δ and $B(G)$ is symmetric.

4.4.3 Totally Minimal Groups

We call a group G *totally minimal* if for any closed normal subgroup N, the factor group G/N admits no Hausdorff group topology which is strictly coarser than the quotient topology.

Theorem 4.4.3. [May97a, Theorem 2.5] *For a connected locally compact group G, the following are equivalent:*

1. *G is totally minimal;*
2. *there exists a compact normal subgroup $K \lhd G$ such that the factor group satisfies*

   $$G/K = N \rtimes H,$$

   where N is a simply connected nilpotent Lie group, H is connected linear reductive and H operates on N without nontrivial fixed points.

Consider a topological commensurability relation for a closed normal subgroup : $S \sim S'$ if and only if $S/(S \cap S')$ and $S'/(S \cap S')$ are compact. We may consider certain minimal representative of each commensurability class:

$$\underline{S} = \bigcap \{S' : S' \text{ is closed normal subgroup of } G \text{ with } S \sim S'\}.$$

Let $\mathcal{N}(G)$ denote the collection of these minimal non-commensurable representatives.

Given G, K, N and H be as in the theorem above, let

$$\mathcal{N}_H(N) = \{S \subseteq N : S \text{ is a connected normal } H\text{-invariant subgroup}\}.$$

Now decompose $H = (H_0 \times H_1 \times \cdots \times H_n)/C$ where H_0 is compact linear group and all $H_1, \ldots, H_n$ are non-compact simple linear Lie groups, and C is a finite central subgroup of the product. We have that $\mathcal{N}(H)$ consists of groups $H_F = \prod_{k \in F} H_k / C_F$ where F is a (possibly empty) subset of $\{1, \ldots, n\}$ and $C_F = C \cap \prod_{k \in F} H_k$. Now as in Remark 2.2 of [Spr20], we get the following structure of $B(G)$ which is a reformulation of Mayer [May97b, Theorem 15]:

$$B(G) = \bigoplus_{\substack{S \in \mathcal{N}_H(N) \\ F \subseteq \{1,\ldots,n\}}}^{l_1} B_0(G/(S \rtimes H_F))$$

Remark 4.4.4. *By Theorem 4.4.1, if G a is linear connected totally minimal with H compact, then the maximal ideal space of $B(G)$ is a semilattice of groups.*

Remark 4.4.5. *By Theorem 4.4.2 and* [May99, Corollary 2.12], *if H is a connected linear reductive group, then the maximal ideal space Δ of $B(H)$ is*

$$\Delta = \bigcup_{H_F \in \mathcal{N}(H)} H/H_F$$

and $B(H)$ is symmetric.

Remark 4.4.6. *If G is a linear connected totally minimal group with N abelian, so are its quotients. By Theorem 4.4.3, $G = V \rtimes H$ for a vector group V. By* [May99, Example 2.9], *$B_0(G) = \overline{A_r(G)}$ and hence $B_0(G/S) = \overline{A_r(G/S)}$ for any closed normal subgroup S of G. Therefore in this case, we get that $B(G)$ is symmetric and its maximal ideal space*

$$\Delta = \{G/(S \rtimes H_F) : S \in \mathcal{N}_H(V), F \subseteq \{1, \ldots, n\}\}.$$

Chapter 5

Cohomology of $\Delta(B(G))$

The present chapter investigates the cohomology of $\Delta(B(G))$ satisfying certain conditions. These cohomology calculations provide another means to realise the invertible elements of Fourier-Stieltjes algebra of locally compact groups discussed in Section 4.4 that satisfies the hypothesis of Theorem 4.3.7. As these calculations might provide a deeper insight into the problem of determining invertible elements in $B(G)$, we include in our thesis. These calculations are adapted from [Tay73, Chapter 5,6].

Let G be a locally compact group such that the spectrum Δ of $B(G)$ satisfies the following conditions:

1. Every left ideal in Δ is also a right ideal;
2. $\Delta_+ = \Delta_p$ is abelian, hence forms a complete lattice [Rup84, I.2.13]; This assumption that $\Delta_+ = \Delta_p$ is very strong.
3. Every ideal A in Δ is weakly closed and $A = \bigcup_{r \in A \cap \Delta_+} \Delta r$;
4. Every weakly convergent net in Δ_+ is eventually constant.

Remark 5.0.1. *Note that any analytic group G mentioned in section 4.4 with a description of the maximal ideal space of its $B(G)$ satisfies the above conditions. By the convergence* *discussed in Theorem 4.4.1, every weakly convergent net in Δ_+ is eventually constant. Hence each element in Δ_+ of $B(G)$ is a critical point. Since the ideals I in Δ are characterized by the order relations of elements in Δ_+ (Proposition 4.3.1.2), every left-ideal is also a right-ideal and every ideal is some union of groups G/V. Hence all ideals are weakly closed(hence strongly closed) in Δ by the same convergence because if $G/V \in I$, then $G/W \in I$ for all $W \supseteqq V(\because eW \leq eV)$. Note that $s(\in G/V) \in I$ if and only if $s^*s = eV \in I$. We suspect if conditions in the hypothesis of Theorem 4.3.7 imply the above conditions $(1)-(4)$ or vice versa, in general. If we include a condition in the hypothesis of Theorem 4.3.7 that there are only finite length of idempotent chains in Δ, we might be able to adapt the ideas of Theorem 4.4.1 to get the above conditions.*

5.1 A Covering Lemma

Although Δ may not be compact in the left-strong topology(l.s.t), there is a form of the covering property which holds in $(\Delta, l.s.t)$ and substitutes for compactness in certain key arguments in computing the cohomology of Δ. If $r \in \Delta_+$, we denote by I_r the closed ideal $r\Delta(= \Delta r)$. If $u \in P(G)$, the set of all continuous positive definite functions on G and $\varepsilon > 0$ then $I_r(u, \varepsilon)$ will be the set $\{s \in \Delta : \|s.u - sr.u\| \leq \varepsilon\}$.

Note that $I_r(u, \varepsilon)$ is strongly closed and contains a strong neighborhood, $\{s \in \Delta : \|s.u - sr.u\| < \varepsilon\}$, of I_r. In fact, since $r^2 = r$, the set I_r is exactly the set $\{s \in \Delta : s = sr\}$ and, thus, I_r is the intersection of the sets $I_r(u, \varepsilon)$ as u runs over $P(G)$ and ε runs over the positive reals.

Proposition 5.1.1. *Each of the sets $I_r(u, \varepsilon)$ is a weakly closed ideal. Furthermore, every weak neighborhood of I_r contains $I_r(u, \varepsilon)$ for some u and ε.*

Proof. If $s \in I_r(u, \varepsilon)$ and $t \in \Delta$, then $\|ts.u - tsr.u\| \leq \|t\|\|s.u - sr.u\| \leq \|s.u - sr.u\| \leq \varepsilon$ and $ts \in I_r(u, \varepsilon)$. Thus, $I_r(u, \varepsilon)$ is an ideal and hence, weakly closed.

The sets $I_r(u, \varepsilon)$, for fixed r, are directed downward under inclusion. In fact, $I_r(u, \varepsilon) \cap I_r(v, \varepsilon)$ contains $I_r(u + v, \varepsilon)$. Since $\bigcap_{u,\varepsilon} I_r(u, \varepsilon) = I_r$ and each $I_r(u, \varepsilon)$ is weakly compact, every weak neighborhood of I_r must contain some $I_r(u, \varepsilon)$. $\square$

Lemma 5.1.2. *Let A be a weakly closed ideal in Δ, $R \subset \Delta_p = \Delta_+$, $\{u_r : r \in R\} \subset \mathcal{P}(G)$, and $\{\varepsilon_r : r \in R\}$ a collection of positive reals. If $A \subset \bigcup\{I_r : r \in R\}$, then A is contained in the union of some finite subcollection of the sets $I_r(u_r, \varepsilon_r)$.*

Proof. Since $s \in I_r(u_r, \varepsilon_r)$ if and only if $s^*s \in I_r(u_r, \varepsilon_r)$, it suffices to prove that $\Delta_+ \cap A$ is covered by finitely many of these sets. Suppose this is not true. Let $\mathfrak{A}$ be the directed set consisting of all finite subsets of R ordered by inclusion. By assumption, we could choose a net $\{s_\alpha\} \subset \Delta_+ \cap A$ with the property that for each $\alpha = \{r_1, \cdots, r_n\} \in \mathfrak{A}$, $s_\alpha \in A \backslash \left(\bigcup_{i=1}^n I_{r_i}(u_{r_i}, \varepsilon_{r_i})\right)$. Since $\Delta_+ \cap A$ is weakly compact, we could choose this net to be convergent in $\Delta_+ \cap A$. Thus, to prove the lemma, we need to only show that a convergent net in $\Delta_+ \cap A$ is eventually in some $I_r(u_r, \varepsilon_r)$. But convergent nets in Δ_+ are eventually constant.

Thus, let $\{s_\alpha\} \subset \Delta_+ \cap A$ converge to $s \in \Delta_+ \cap A$ and choose $r \in R$ such that $s \in I_r$ and $s_\alpha \in I_r(u_r, \varepsilon_r)$ eventually, and the proof is complete. $\square$

5.2 Axioms of Cohomology

Although next three sections are devoted to characterizing the cohomology of Δ, no background in algebraic topology is needed. Our methods and proofs are adapted from Chapter 6 of Taylor's book [Tay73] and can be applied to any tuple of functors from compact spaces to abelian groups which satisfies certain conditions.

5.2.1 Cohomology Functors

We shall be dealing with contravariant functors H from pairs $Y \subset X$ of compact Hausdorff spaces to abelian groups. In essence, H assigns to each such pair $Y \subset X$ an abelian group $H(X, Y)$ and to each continuous map $\phi : (X, Y) \to (X', Y')$ ($\phi : X \to X'$ maps Y into Y') a group homomorphism $H(\phi) : H(X', Y') \to H(X, Y)$. Furthermore, H assigns the identity homomorphism to the identity map id: $(X, Y) \to (X, Y)$ and H respects composition ($H(\phi \circ \psi) = H(\psi) \circ H(\phi)$). Denote $H(\phi)$ simply by ϕ^*.

For such a functor and a compact space X, we denote the group $H(X, \emptyset)$ by $H(X)$.

5.2.2 Excision

If $Y \subset X$ is a pair of compact Hausdorff spaces, we obtain a new pair $Y/Y \subset X/Y$ by identifying all points of Y. The space X/Y can be thought of as the one point compactification of $X \backslash Y$ with Y/Y as the point at infinity. The natural map $P : X \to X/Y$ is continuous, injective on $X \backslash Y$ and maps Y to the point Y/Y.

If H is a functor on pairs $Y \subset X$, then $P : (X, Y) \to (X/Y, Y/Y)$ induces a map $P^* : H(X/Y, Y, Y) \to H(X, Y)$. We say that H is excisive if P^* is an isomorphism for any pair $Y \subset X$.

5.2.3 Continuity

If $K \subset X$ are compact spaces then the injection $i : K \to X$ induces a map $i^* : H(X) \to H(K)$. For $a \in H(X)$, we shall often refer to $i^* a$ as the restriction of a to K. The map i^* will be called the restriction map and denoted by q_K.

We shall call H continuous if whenever K is the intersection of a family of compact sets K_α, with $\{K_\alpha\}$ directed downward, then

1. each element of $H(K)$ is the restriction of some element of $H(K_\alpha)$ for some α, and
2. whenever $a \in H(X)$ satisfies $q_K a = 0$, it also satisfies $q_{K_\alpha} a = 0$ for some α.

The continuity condition says that $H(K)$ is the inductive limit of the directed system of groups $\{H(K_\alpha)\}$.

5.2.4 Homotopy

Suppose we have a parameter space Ω, which is a Hausdorff topological space, and a family $\{\phi_\lambda\}_{\lambda \in \Omega}$ of maps from the compact space X to the compact space Y. The family is called continuous if the map $(x, \lambda) \to \phi_\lambda(x) : X \times \Omega \to Y$ is continuous.

Given such a family, there is corresponding family of homomorphisms $\phi_\lambda^* : H(Y) \to H(X)$. We say H satisfies a strong version of the homotopy axiom if any continuous family $\{\phi_\lambda\}$ have the property that $\phi_\lambda^* a$ is locally constant for each $a \in H(Y)$. If Ω is connected, this implies that ϕ_λ^* is constant.

5.2.5 Connected Sequences, Exactness

If H is a functor as above, and $Y \subset X$ are compact, then the injections $i : (Y, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, Y)$ have the property that the composition $j \circ i : (Y, \emptyset) \to (X, Y)$ factors as $(Y, \emptyset) \to (Y, Y) \to (X, Y)$. If we assume that $H(Y, Y) = 0$ for each Y, then it follows that the composition $i^* \circ j^*$ in

$$H(X, Y) \xrightarrow{j^*} H(X) \xrightarrow{i^*} H(Y)$$

yields zero.

Now suppose we have a sequence $\{H^p\}_{p \in I}$ ($I = \{0, 1, \cdots, n\}$ or $I = \{0, 1, \cdots\}$) of such functors, and for each nonzero $p \in I$ and pair (X, Y) a map $\delta^{p-1} : H^{p-1}(Y) \to H^p(X, Y)$. We then call the system $\{H^p, \delta^p\}$ a connected sequence. We shall call it exact if for each pair (X, Y) the sequence

$$\begin{gathered} 0 \longrightarrow H^0(X, Y) \xrightarrow{j^*} H^0(X) \xrightarrow{i^*} H^0(Y) \xrightarrow{\delta^0} H^1(X, Y) \\ \xrightarrow{j^*} \cdots \longrightarrow H^{p-1}(Y) \xrightarrow{\delta^{p-1}} H^p(X, Y) \xrightarrow{j^*} H^p(X) \xrightarrow{i^*} H^p(Y) \end{gathered}$$

is exact for each $p \in I$. Note that we do not require exactness at the last state of this sequence if $I = \{0, \cdots, n\}$ is finite and $p = n$.

5.2.6 Cohomology

By a cohomology sequence we mean an exact connected sequence $\{H^p\}_{p \in I}$, for which each H^p satisfies excision, continuity, and homotopy.

5.3 The Cohomology of Δ

We use results listed in the previous chapter about the axioms of cohomology to compute the groups $H^p(\Delta)$ for any cohomology sequence $\{H^p, \delta^p\}$.

5.3.1 An Application of Homotopy

The map $(s,t) \in (\Delta, weak) \times (\Delta, l.s.t) \to st \in (\Delta, weak)$ is continuous because

$$|\hat{u}(st) - \hat{u}(s_1t_1)| \leq |\hat{u}(st) - \hat{u}(s_1t)| + |\hat{u}(s_1t) - \hat{u}(s_1t_1)| \leq |\widehat{t.u}(s-s_1)| + |(\widehat{t.u} - \widehat{t_1.u})(s_1)|$$

for any $u \in B(G), s, t, s_1, t_1 \in \Delta$. Thus, we may define a continuous family of maps $\{\phi_r\}_{r\in(\Delta,l.s.t)}$ from $(\Delta, weak)$ to $(\Delta, weak)$ by $\phi_r(s) = sr$. We shall actually only be interested in these maps for $r \in \Delta_+$.

Now if $A \subset \Delta$ is a weakly closed(hence, compact) ideal, then the maps ϕ_r, for $r \in \Delta_+$ leave A invariant. Hence, we may also consider $\{\phi_r\}_{r\in\Delta}$ to be a continuous family of maps of A into A. Recall that all ideals in Δ are two-sided.

For each closed ideal A, the map $\phi_r^* : H^p(A) \to H^p(A)$ induced on the cohomology by ϕ_r will be denoted by π_r. For an ideal $B \subset A$, the restriction map $H^p(A) \to H^p(B)$ induced by the inclusion $B \to A$ will be denoted by q_B.

Proposition 5.3.1. *For closed ideals $B \subset A \subset \Delta$, the family $\{\pi_r\}_{r\in\Delta_+}$ is a family of endomorphisms of $H^p(A)(H^p(B))$ with the following properties:*

1. *$\pi_r \circ \pi_s = \pi_{rs}$ for $r, s \in \Delta_+$;*
2. *for each $a \in H^p(A), r \to \pi_r a$ is locally constant on $(\Delta_+, l.s.t)$;*
3. *each π_r commutes with the restriction map q_B;*
4. *each π_r is a projection ($\pi_r^2 = \pi_r$) onto a subgroup of $H^p(A)$; furthermore, the restriction map $q_{Ar} : H^p(A) \to H^p(Ar)$ maps this subgroup isomorphically onto $H^p(Ar)$.*

Proof. Part (1) follows from the fact that $\phi_{rs} = \phi_s \circ \phi_r$. Part (2) is a consequence of the homotopy axiom. Part (3) is due to the commutativity of the diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & A \\ \downarrow{\scriptstyle \phi_r} & & \downarrow{\scriptstyle \phi_r} \\ B & \xrightarrow{i} & A \end{array}$$

where i is the injection.

Since $\Delta_+ = \Delta_p$, it is clear from part (1) that $\pi_r^2 = \pi_r$. Let $\rho_r : H^p(Ar) \to H^p(A)$ denote the map induced on cohomology by $s \in A \to sr \in Ar$. Then $\pi_r = \rho_r \circ q_{Ar}$, since ϕ_r is $s \to sr : A \to Ar$, followed by $i : Ar \to A$. On the other hand, since r is idempotent, $q_{Ar} \circ \rho_r = \mathrm{id} : H^p(Ar) \to H^p(Ar)$, since $s \to sr$ is the identity on Ar. It follows that ρ_r is a right inverse for q_{Ar} with image equal to $\mathrm{Im}(\pi_r)$. Hence $q_{Ar} : H^p(A) \to H^p(Ar)$ is an isomorphism of $\mathrm{Im}(\pi_r) \subset H^p(A)$ onto $H^p(Ar)$. □

The following main theorem on ideals of this section has three interrelated parts which will be proved by induction on the dimension.

Theorem 5.3.2. *Let $\{H^p, \delta^p\}$ be cohomology sequence and $A \subset \Delta$ a weakly closed ideal. Then:*

1. *If $A = A_1 \cup A_2 \cup \cdots \cup A_n$ with each A_i a closed ideal and if $a \in H^p(A)$ satisfies $q_{A_i} a = 0$ for each i, then $a = 0$;*
2. *each $a \in H^p(A)$ is generated by the elements $\pi_r a$ for $r \in A \cap \Delta_+$;*
3. *the restriction $q_A : H^p(\Delta) \to H^p(A)$ is surjective.*

The proof will proceed as follows: we first show that for a given p, (1) implies (2) and (2) implies (3). We then show that if (3) holds for $p - 1$ (or if $p = 0$) then (a) holds for p. Then the theorem follows by induction.

Lemma 5.3.3. *If* (1) *of Theorem 5.3.2 holds for a given p, then so does* (2)*.*

Proof. If $r \in A \cap \Delta_+$, then $r = r^2$. Note that, for $r \in A \cap \Delta_+$, $s \to sr$ is the identity on $Ar = \Delta r$. It follows that $q_{Ar}\pi_r a = \pi_r q_{Ar} a = q_{Ar} a$ (Proposition 5.3.1(3)) and hence, that $q_{Ar}(a - \pi_r a) = 0$. By the continuity axiom (5.2.3) and Proposition 5.1.1, for each $r \in A \cap \Delta_+$, there is an ideal of the form $I_r(u_r, \varepsilon_r)$ such that $q_{K_{u_r,\varepsilon_r}}(a - \pi_r a) = 0$(with the understanding that $-$ is inverse operation in corresponding group) where $K_{u_r,\varepsilon_r} = A \cap I_r(u_r, \varepsilon_r)$. Since $A = \bigcup_{r \in A \cap \Delta_+} \Delta r$, by Lemma 5.1.2, there are finitely many of these ideals, $I_{r_1}(u_{r_1}, \varepsilon_{r_1}), \cdots, I_{r_n}(u_{r_n}, \varepsilon_{r_n})$, which cover A. Set $A_i = A \cap I_{r_i}(u_{r_i}, \varepsilon_{r_i})$. Then, for each i, $q_{A_i}(1 - \pi_{r_i})a = 0$.

Now let b be the element

$$b = \left[1 - \prod_{i=1}^{n}(1 - \pi_{r_i})\right] a$$

and by expanding the product we see that b is an alternating sum of elements of the form $\pi_r a$ with r a product of elements of the set $A \cap \Delta_+$. Since $A \cap \Delta_+$ is closed under products, b is in the subgroup of $H^p(A)$ generated by elements $\pi_r a$ with $a \in A \cap \Delta_+$. Furthermore,

$$q_{A_j}(a - b) = q_{A_j} \prod_{i=1}^{n}(1 - \pi_{r_i}) = 0,$$

since $q_{A_j}(1 - \pi_{r_j})a = 0$ and each π_{r_j} commutes with the restriction map r_{A_j} (Proposition 5.3.1(3)). Thus, if (1) of Theorem 5.3.2 holds, then $a = b$. □

Lemma 5.3.4. *If* (2) *of Theorem* 5.3.2 *holds for a given* p*, then so does* (3).

Proof. By Proposition 5.3.1(4), q_{Ar} is an isomorphism onto $H^p(Ar)$ when restricted to the image of π_r in either $H^p(\Delta)$ or $H^p(A)$ because $\Delta r = Ar$ if $r \in A \cap \Delta_+$. Thus, q_A must map the image of π_r in $H^p(\Delta)$ onto the image of π_r in $H^p(A)$ because of the following commutative diagram,

$$\begin{array}{ccc} \pi_r(H^p(\Delta)) & \xrightarrow{q_A} & \pi_r(H^p(A)) \\ \downarrow{\scriptstyle q_{Ar}} & & \downarrow{\scriptstyle q_{Ar}} \\ H^p(\Delta r) & \xrightarrow{id} & H^p(Ar) \end{array}$$

Now by the hypothesis, since $H^p(A)$ is generated by elements $\pi_r a$ for $r \in A \cap \Delta_+$, we have that $q_A : H^p(\Delta) \to H^p(A)$ is surjective. $\square$

Remark 5.3.5. *If $r \in A \cap \Delta_+$, then $q_A : \pi_r(H^p(\Delta)) \to \pi_r(H^p(A))$ is an isomorphism whenever $A \subset \Delta$ is a weakly closed ideal.*

Lemma 5.3.6. *If $p = 0$ or if $p > 0$ and (3) of Theorem 5.3.2 holds for $p - 1$, then (1) holds for p.*

Proof. Under the hypothesis of the lemma, it implies that for a pair $B \subset A$ of closed ideals, the sequence

$$0 \longrightarrow H^p(A, B) \xrightarrow{j^*} H^p(A) \xrightarrow{q_B} H^p(B)$$

is exact. For $p = 0$ this follows from the exactness axiom (5.2.5); for $p > 0$ it follows from the fact that $H^{p-1}(A) \to H^{p-1}(B)$ is surjective (which follows from the hypothesis for $p - 1$) and the exactness axiom (5.2.5) implies $\ker \delta^{p-1} = H^{p-1}(B)$.

Now it is sufficient to prove (1) in the case $n = 2$, since the general case then follows by induction (the union of finitely many closed ideals is also a closed ideal). Hence, let $A = B \cup C$ with B, C closed subideals of A, and consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^p(A, B) \arrow[r, "j^*"] \arrow[d, "q_C"] & H^p(A) \arrow[r, "q_B"] \arrow[d, "q_C"] & H^p(B) \arrow[d, "q_{B\cap C}"] \\
0 \arrow[r] & H^p(C, B \cap C) \arrow[r, "j^*"] & H^p(C) \arrow[r, "q_{B\cap C}"] & H^p(B \cap C).
\end{tikzcd}$$

If $a \in H^p(A)$ and $q_B a = q_C a = 0$, then $a = j^* b$ for a unique $b \in H^p(A, B)$. Furthermore, $j^* q_c b = q_C j^* b = q_C a = 0$. It follows from the exactness of bottom row that $q_C b = 0$. Now from the following commutative diagram,

$$\begin{tikzcd}
H^p(A/B, B/B) \arrow[r, "id"] \arrow[d, "P^*"] & H^p(C/B \cap C, B/B \cap C) \arrow[d, "P^*"] \\
H^p(A, B) \arrow[r, "q_C"] & H^p(C, B \cap C)
\end{tikzcd}$$

and from the excision axiom (5.2.2) it follows that q_C is an isomorphism from $H^p(A, B)$ to $H^p(C, B \cap C)$ because $A \backslash B = C \backslash (B \cap C)$. Hence, $b = 0$ and $a = 0$ since j^* is injective. $\square$

This completes the proof of Theorem 5.3.2.

Remark 5.3.7. *From the above proof, we have that for any pair $B \subset A$ of weakly closed ideals and for any p, the sequence*

$$0 \longrightarrow H^p(A,B) \xrightarrow{j^*} H^p(A) \xrightarrow{q_B} H^p(B)$$

is exact.

5.4 Cohomology and Critical points

5.4.1 The Subgroups J_r^p

For each critical point $r \in \Delta_+$, we set $A_r = r\Delta = \Delta r = \{s \in \Delta : s^*s = ss^* \le r\}$ and $B_r = \{s \in \Delta : s^*s < r\}$. Note that A_r and B_r are weakly closed ideals. The maximal group at $r, G_r = \{s \in \Delta : s^*s = ss^* = r\}$, is $A_r \backslash B_r$.

We denote the subgroup $\mathrm{Im}(\pi_r) \cap \ker(q_{B_r})$ by J_r^p.

Proposition 5.4.1. *The group J_r^p consists of those $a \in H^p(\Delta)$ for which $\pi_t a = a$ if $t \ge r$ and $\pi_t a = 0$ otherwise.*

Proof. If $t \ge r$ then $tr = r$ and $\pi_t \circ \pi_r = \pi_r$(Proposition 5.3.1(1)); hence, $\pi_t a = a$ for all $a \in \mathrm{Im}(\pi_r)$. If $t \not\ge r$, then $tr < r$ and thus, $tr \in B_r$. By Remark 5.3.5, since q_{B_r} is an isomorphism on $\mathrm{Im}(\pi_{tr})$ and $q_{B_r}\pi_{tr}a = q_{B_r}\pi_t a = \pi_t q_{B_r} a = 0$ for $a \in J_r^p$, we have that π_t vanishes on J_r^p. Conversely, if $\pi_t a = 0$ for all $t \not\ge r$, then $\pi_t q_{B_r} a = 0$ for all $t \in B_r \cap \Delta_+$. Using Theorem 5.3.2(2), we can conclude that $a \in \ker(q_{B_r})$. □

Now let $\mu_r : \Delta \to G_r \cup \{\infty\}$ is the map $s \in \Delta \to sr \in A_r$ followed by the identification map $A_r \to A_r/B_r = G_r \cup \{\infty\}$. Recall that G_r is a locally compact group and so $G_r \cup \{\infty\}$ is its one-point compactification. The injection $i : \{\infty\} \to G_r \cup \{\infty\}$ has a left inverse $k : G_r \cup \{\infty\} \to \{\infty\}$ which sends every point to infinity. Hence, for any cohomology sequence $\{H^p, \delta^p\}$, it follows that $i^* : H^p(G_r \cup \{\infty\}) \to H^p(\{\infty\})$ has a right inverse k^*.

Thus, $\ker(i^*)$ is a direct summand of $H^p(G_r \cup \{\infty\})$ and, by Remark 5.3.7, is isomorphic to $H^p(G_r \cup \{\infty\}, \{\infty\})$.

We denote $\ker(i^*)$ by $H^p_c(G_r)$, and call it the pth cohomology group of G_r with compact supports.

Proposition 5.4.2. *The map $\mu_r^* : H^p(G_r \cup \{\infty\}) \to H^p(\Delta)$, when restricted to $H^p_c(G_r)$, is an isomorphism onto J^p_r.*

Proof. The restriction map $q_{A_r} : H^p(\Delta) \to H^p(A_r)$ maps J^p_r isomorphically onto the kernel of $q_{B_r} : H^p(A_r) \to H^p(B_r)$ because q_{A_r} restricted to $\mathrm{Im}(\pi_r)$ is an isomorphism, $q_{A_r}(J^p_r) \subset (\ker q_{B_r})$ and $q^{-1}_{A_r}(\ker q_{B_r}) \subset J^p_r$. By Remark 5.3.7 , this is the image in $H^p(A_r)$ of $j^* : H^p(A_r, B_r) \to H^p(A_r)$ and j^* is injective. The proposition now follows from the excision axiom since $A_r/B_r = G_r \cup \{\infty\}$ and the fact that $H^p_c(G_r)$ is isomorphic to $H^p(G_r \cup \{\infty\}, \{\infty\})$. □

5.4.2 Computation of $H^p(\Delta)$

The theorem that follows will allow us to compute $H^p(\Delta)$ in terms of the subgroups J^p_r or the maximal groups of Δ $(G_r, r \in \Delta_+)$.

Theorem 5.4.3. *For each p, $H^p(\Delta)$ is the direct sum of its subgroups J^p_r as r ranges over the critical points of Δ_+.*

Since we are working under the assumption that every weakly convergent net in Δ_+ is eventually constant, each element in Δ_+ is a critical point.

We first prove that the J^p_r's generate $H^p(\Delta)$ and then that they are independent.

Lemma 5.4.4. *The groups J^p_r(r a critical point) generate $H^p(\Delta)$.*

Proof. For a given $a \in H^p(\Delta)$, consider the set Ω of all $t \in \Delta_+$ for which $\pi_t a$ is not an element of the subgroup K^p of $H^p(\Delta)$ generated by the groups J^p_r. Since any subset of Δ_+ is weakly closed and open, the set Ω is both open and closed set in $(\Delta_+, l.s.t)$. We claim that it is empty. In fact, if $e \notin \Omega$ then $a = \pi_e a$ is in the subgroup K^p.

Suppose Ω is not empty. Then by Proposition 4.2.2, it has a minimal element r, which must be a critical point by Proposition 4.3.4 or the fact that all elements of Δ_+ are critical points. Thus, $\pi_r a \notin K^p$, but $\pi_t a \in K^p$ for all $t \in \Delta_+$ with $t < r$.

By Theorem 5.3.2(2), $q_{B_r} a = c$ is generated by elements of the form $\pi_t c$ for $t \in B_r \cap \Delta_+$, i.e., for $t \in \Delta_+$ with $t < r$. Hence, we conclude that $q_{B_r} a = q_{B_r} b$ for some $b \in K^p$ since $q_{B_r} : H^p(\Delta) \to H^p(B_r)$ is surjective (Theorem 5.3.2(3)). But then $q_{B_r}(a - b) = 0$ and hence $\pi_r(a - b) \in J^p_r$. But by Proposition 5.4.1, $\pi_r a \in K^p$ as $\pi_r b \in K^p$. This contradicts the choice of r and proves that $\Omega = \emptyset$. Hence $K^p = H^p(\Delta)$. $\square$

Lemma 5.4.5. *Let $r_1, \cdots, r_n$ be a set of distinct element in Δ_+, and a_i a non-zero element of $J^p_{r_i}$ for each i. If $\sum_{i=1}^k a_i = 0$, then $a_1 = a_2 = \cdots = a_n = 0$.*

Proof. Suppose the contrary and let k be the smallest integer for which $\sum_{i=1}^k a_i = 0$ with $0 \neq a_i \in J^p_{r_i}$ for distinct $r_1, \cdots, r_n$. Now let Ω be the set of all $t \in \Delta_+$ for which $\pi_t a_1 = a_1$. This set is open and closed in $(\Delta_+, l.s.t)$ and, hence has a minimal element r which is a critical point (Proposition 4.2.2). Thus $\pi_r a_1 = a_1$ and $\pi_t a_1 = 0$ for $t < r$. But this implies that for each i, $\pi_r a_i = a_i$ and $\pi_t a_i = 0$ for $t \ngeq r$ by the choice of k, because $\pi_t a_i = a_i$ or $\pi_t a_i = 0$. It follows that each $a_i \in J^p_r$. However, Proposition 5.4.1 says that J^p_r and $J^p_{r_i}$ meet only in zero if $r \neq r_i$. Thus contradicts the hypothesis that the r_i's were distinct. $\square$

This completes the proof of Theorem 5.4.3.

5.5 Invertibility of Elements in $B(G)$

Given a Banach algebra A with identity and a closed subalgebra B(which may not contain the identity), the restriction map $f \in \Delta(A) \to f_{|_B} \in \Delta(B) \cup \{0\}$ is a continuous map from the spectrum of A to the one-point compactification of the spectrum of B. Hence, it induces a homomorphism $H^p(\Delta(B) \cup \{0\}) \to H^p(\Delta(A))$ of cohomology for every cohomology sequence. We shall call this the canonical map.

5.5.1 Cohomology Sequence $\{H^0, H^1; \delta\}$

The following functors are very important for the structure of any semisimple Banach algebra. We will describe a terminating cohomology sequence which will be studied on the maximal ideal space of $B(G)$.

If (X, Y) is a pair of compact Hausdorff spaces with $Y \supset X$, we define $C(X, Y)$ to be the additive group of all continuous complex valued functions on X which vanish on Y. Similarly $C^{-1}(X, Y)$ is the multiplicative group of nonvanishing functions on X which are identically one on Y. If ex: $C(X, Y) \to C^{-1}(X, Y)$ is the homomorphism defined by $\text{ex}(f) = \exp(2\pi i f)$, then we define $H^0(X, Y)$ to be the kernel of ex and $H^1(X, Y)$ to be its cokernel.

Note that $H^0(X, Y)$ is the group of continuous integer valued functions on X which vanish on Y. In particular, $H^0(X) = H^0(X, \emptyset)$ is the group of all continuous integer valued functions on X and is the group generated by the idempotents of $C(X)$.

Similarly, $H^1(X, Y)$ is the factor group of $C^{-1}(X, Y)$ modulo its subgroup consisting of elements which have continuous logarithms vanishing on Y. Thus, $H^1(X) = H^1(X, \emptyset)$ is the group $C(X)^{-1}/\exp(C(X))$.

Define a map $\delta : H^0(Y) \to H^1(X, Y)$ as follows: If $f \in H^0(Y) \subset C(Y)$, we let $\tilde{f} \in C(X)$ be a continuous extension of f. Then we let δf be the equivalence class of $\exp(2\pi i \tilde{f}) \in C^{-1}(X, Y)$ in the factor group $H^1(X, Y)$. Note that δf is independent of the choice of $\tilde{f}$ since the difference of two such extensions will be an element of $C(X, Y)$ and the image of $C(X, Y)$ under ex is the kernel of $C^{-1}(X, Y) \to H^1(X, Y)$.

The following theorem is proved in Taylor's book [Tay73, Theorem 6.2.7] which we will state without any proof.

Theorem 5.5.1. *The connected sequence $\{H^0, H^1; \delta\}$ is a cohomology sequence in the sense of 5.2.6.*

The group $H^1_C(G_z)$ is just the subgroup of $H^1(G_z \cup \{\infty\})$ determined by elements of $C(G_z \cup \{\infty\})^{-1}$ which have the value one at infinity. However, if $f \in C(G_z \cup \{\infty\})^{-1}$ and $w = \ln f(\infty)$, $g = fe^{-w}$, then f and g determine the same equivalence class in $H^1(G_z \cup$

$\{\infty\}$). Thus, $H^1_c(G_z) = H^1(G_z \cup \{\infty\})$. Then via the Gelfand transform, $H^1_c(A(G_z)) := H^1(A(G_z) + \mathbb{C}1)$ and $H^1_c(G_z)$ are isomorphic.

Then Theorem 5.4.3 and Proposition 5.4.2 can be restated as follows.

Theorem 5.5.2. *Let G be a locally compact group such that the spectrum Δ of $B(G)$ satisfies the following conditions:*

1. *Every left ideal in Δ is also a right ideal;*
2. *$\Delta_+ = \Delta_p$ is abelian.*
3. *Every ideal A in Δ is weakly closed and $A = \bigcup_{r \in A \cap \Delta_+} \Delta r$;*
4. *Every weakly convergent net in Δ_+ is eventually constant.*
5. *For each $z \in \Delta_+$, there is an Banach subalgebra $A_z \subset B(G)$ such that A_z is isometrically isomorphic to an Banach subalgebra $\mathfrak{A}$ with $A(G_z) \subset \mathfrak{A} \subset \overline{A_r(G_z)}$.*

Then for each $A_z \subset B(G)$ with spectrum G_z, the canonical map of $H^1_c(G_z)$ into $H^1(\Delta)$ is injective. Furthermore, $H^1(\Delta)$ is the direct sum of the images of these as maps as z ranges over Δ_+.

Proof. The restriction map $f \to f_{|A_z}$ maps the spectrum of $B(G)$, Δ onto $G_z \cup \{\infty\}$. Since $\{H^0, H^1\}$ is a cohomology sequence(Theorem 5.5.1), it follows that the corresponding map $H^1_c(G_z) \to H^1(\Delta)$ of cohomology is injective and $H^1(\Delta)$ is the direct sum of the images as z ranges over Δ_+. The theorem now follows from the fact that the diagram

$$\begin{array}{ccc} H^1_c(A_z) & \longrightarrow & H^1(B(G)) \\ \big\downarrow & & \big\downarrow \\ H^1_c(G_z) & \longrightarrow & H^1(\Delta) \end{array}$$

is commutative and the vertical maps are isomorphisms. $\square$

Corollary 5.5.3. *Let G be a locally compact group such that the spectrum Δ of $B(G)$ satisfies the following conditions:*

1. *Every left ideal in Δ is also a right ideal;*

2. $\Delta_+ = \Delta_p$ *is abelian;*

3. *Every ideal A in Δ is weakly closed and* $A = \bigcup_{r \in A \cap \Delta_+} \Delta r$*;*

4. *Every weakly convergent net in Δ_+ is eventually constant;*

5. *For each $z \in \Delta_+$, there is an Banach subalgebra $A_z \subset B(G)$ such that A_z is isometrically isomorphic to an Banach subalgebra $\mathfrak{A}$ with* $A(G_z) \subset \mathfrak{A} \subset \overline{A_r(G_z)}$.

Then for any $u \in B(G)^{-1}$, we have that

$$u = v_1 \cdot v_2 \cdots v_n \cdot \exp(w)$$

with each $v_i \in (A_{z_i} \oplus \mathbb{C}1)^{-1}$ for some $z_i \in \Delta(B(G))_+$ and $w \in B(G)$. The elements v_i are unique modulo $\exp(A_{z_i} \oplus \mathbb{C}1)$.

Chapter 6

A Future Venture

On one hand, we have Walter's theory of critical points and these critical points give certain locally compact completions. It seems plausible that centrality of critical points may arise naturally. The complete analysis of a central critical point $z \in \Delta_+$ depends ultimately on the resolution of the following question:

Does the algebra of functions $z \cdot B(G)$ contain an element of $A(G_z)$?

The affirmative answer to this question in case G is abelian was furnished by Taylor [Tay73, Tay72] with elaborate machinery and considerable work as mentioned earlier in this thesis. For the groups satisfying the hypothesis of Theorem 4.3.7, $z \cdot B(G) = \bigoplus^{l_1} \left\{ \overline{A_r(G_{z'})} \;\middle|\; z' \leq z \right\}$.

The very examples of group G that satisfies the hypothesis of Theorem 4.3.7 have proved their worth in the study of operator amenability of Fourier-Stieltjes algebra [Spr20] and in the characterization of completely positive (completely contractive) homomorphisms from certain closed, translation-invariant unital subalgebra A of $B(G)$ into $B(H)$ [Sto20].

Now let's recall a result of Ruppert: Every idempotent e in S in a semitopological compactification (S, ϕ) of a locally compact connected group G is central, i.e. $es = se$ for all $s \in S$ and the idempotents of S form a complete meet-continuous lattice. Moreover, every left or right ideal is also a two-sided ideal. The cohomological calculations are feasible in our cases because the spectrum of Fourier-Stieltjes algebras of groups G in section 4.4 coincides with the Eberlein compactification of the underlying group, which is the spectrum of the uniform closure of $B(G)$; so it witnesses the Ruppert's result. In [Tay72], Taylor provided means to go around the cohomological calculations and achieved the result for l.c. abelian groups, we feel it is early days, to know with certainty if these kind of calculations be helpful for our purpose. The abelian and other known cases suggest that the critical points all seem to lie in the Eberlein compactification. In general if this were true, one can utilize the Galois connection carried out in Spronk's paper [Spr18, Section 5] with the hope to achieve the same results for many connected l.c. groups. So on the other hand, according to Nico's theory, certain locally compact completions will give rise to central idempotents in the Eberlein compactification. We wonder if the procedure (i.e., translation of the Tomita-Takesaki theory into the special context of group theory) mentioned in Walter's paper [Wal75, Remark pp. 277 - 281], combined with certain possible characterisations of faces corresponding to the above critical points in enveloping group von Neumann algebras will be helpful in this prospect.

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