Representations of Even-cycle and Even-cut Matroids

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contribution

Chapter 2 is based on paper [23], which is co-authored with B. Guenin and I. Pivotto. The main result Theorem 2.3.3 was originally proved by B. Guenin and I. Pivotto in [24], which is not published. Together with B. Guenin, we found a much shorter and accessible proof.

Chapters 3, 4 and 5 are based on papers [20, 21, 22, 31], which are co-authored with my advisor B. Guenin. I was responsible for the initial draft of all papers. I received help and feedback for multiple revisions.

Abstract

In this thesis, two classes of binary matroids will be discussed: even-cycle and even-cut matroids, together with problems which are related to their graphical representations. Even-cycle and even-cut matroids can be represented as signed graphs and grafts, respectively. A signed graph is a pair (G, Σ) where G is a graph and Σ is a subset of edges of G. A cycle C of G is a subset of edges of G such that every vertex of the subgraph of G induced by C has an even degree. We say that C is even in (G, Σ) if $|C \cap \Sigma|$ is even. A matroid M is an even-cycle matroid if there exists a signed graph (G, Σ) such that circuits of M precisely corresponds to inclusion-wise minimal non-empty even cycles of (G, Σ). A graft is a pair (G, T) where G is a graph and T is a subset of vertices of G such that each component of G contains an even number of vertices in T. Let U be a subset of vertices of G and let $D := \delta_G(U)$ be a cut of G. We say that D is even in (G, T) if $|U \cap T|$ is even. A matroid M is an even-cut matroid if there exists a graft (G, T) such that circuits of M corresponds to inclusion-wise minimal non-empty even cuts of (G, T).

This thesis is motivated by the following three fundamental problems for even-cycle and even-cut matroids with their graphical representations.

(a) Isomorphism problem: what is the relationship between two representations?

(b) Bounding the number of representations: how many representations can a matroid have?

(c) Recognition problem: how can we efficiently determine if a given matroid is in the class? And how can we find a representation if one exists?

These questions for even-cycle and even-cut matroids will be answered in this thesis, respectively. For Problem (a), it will be characterized when two 4-connected graphs G_1 and G_2 have a pair of signatures (Σ_1, Σ_2) such that (G_1, Σ_1) and (G_2, Σ_2) represent the same even-cycle matroids. This also characterize when G_1 and G_2 have a pair of terminal sets (T_1, T_2) such that (G_1, T_1) and (G_2, T_2) represent the same even-cut matroid. For Problem (b), we introduce another class of binary matroids, called pinch-graphic matroids, which can generate exponentially many representations even when the matroid is 3-connected. An even-cycle matroid is a pinch-graphic matroid if there exists a signed graph with a blocking pair. A blocking pair of a signed graph is a pair of vertices such that every odd cycles intersects with at least one of them. We prove that there exists a constant c such that if a matroid is even-cycle matroid that is not pinch-graphic, then the number of representations is bounded by c. An analogous result for even-cut matroids that are not duals of pinch-graphic matroids will be also proven. As an application, we construct algorithms to solve Problem (c) for even-cycle, even-cut matroids. The input matroids of these algorithms are binary, and they are given by a $(0, 1)$-matrix over the finite field GF(2). The time-complexity of these algorithms is polynomial in the size of the input matrix.

Acknowledgements

Foremost, I would like to express my deepest gratitude to my advisor Bertrand Guenin for his insight, patience, and encouragement. During my MMath and PhD programs, he has given countless pieces of advice and ideas to be a good researcher. I would like to thank him for giving me an opportunity to continue my research in this great environment. I will never forget the discussions with coffee and white boards full of graphs.

I wish to show my gratitude to the members of committee members—Jim Geelen, Joseph Cheriyan, Therese Biedl, and Gordon Royle— for their constructive feedback on my thesis.

I would also like to thank all of the faculty members at Department of Combinatorics and Optimization. Especially, I am thankful to Jim Geelen and Peter Nelson for invaluable advice on Matroid theory.

Finally, I would like to thank my family and Jihyun Kim for all their love and support.

Dedication

This thesis is dedicated to my parents and my brother.

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Chapter 1

Introduction

1.1 Matroids with graphical representations

1.1.1 Graphic and cographic matroids

Before we get into even-cycle and even-cut matroids, let us start with simplest classes of matroids with graphical representations, which are graphic and cographic matroids. Let $G := (V, E)$ be a graph with vertex set V and edge set E. In this thesis, we allow parallel edges and loops in graphs. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. For a subset W of $V(G)$ and a subset F of $E(G)$, we denote the *induced subgraph* of G by W (resp. by F) by $G[U]$ (resp. $G[F]$). A *cycle* of a graph G is a subset C of $E(G)$ such that every vertex of $G[C]$ has an even degree. A *polygon* of a graph G is an inclusion-wise minimal non-empty cycle of G. Equivalently, a non-empty subset C of $E(G)$ is a polygon if $G[C]$ is a connected 2-regular subgraph of G.

Let M be a matroid. We denote by $E(M)$ the ground set (or the edge set) of M. We say that M is *graphic* if there exists a graph G such that circuits of M precisely correspond to polygons of G. Then, G is a *graph representation* of M, and we denote this by $M = \text{cycle}(G)$. Let A be a vertex-edge incidence matrix of G. Then, A represents $\text{cycle}(G)$ over $\text{GF}(2)$, so every graphic matroid is binary.

Let G be a graph, and let $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. We denote by $G/I \setminus J$ the *minor* obtained from G by *contracting* edges in I and *deleting* edges in J. For a matroid M, we denote by $M/I \setminus J$ the *minor* obtained from M by *contracting* elements in I and *deleting* elements in J. For the contraction (resp. deletion) of one edge e, we simply write $/e$ (resp. $\setminus e$) instead of $/\{e\}$ (resp. $\setminus\{e\}$). Let $M = \text{cycle}(G)$. Then, $G/I \setminus J$ is a graph representation of $M/I \setminus J$. In particular, the class of graphic matroids is minor-closed.

For a subset W of $V(G)$, a *cut* generated by W is the set of edges which have exactly one end in W, denoted by $\delta_G(W)$. We say that W is a *shore* of the cut $\delta_G(W)$. If W contains only one vertex v, then we simply write $\delta_G(v)$ instead of $\delta_G(\{v\})$. A matroid M is *cographic* if there exists a graph G such that circuits of M precisely correspond to inclusion-wise minimal non-empty cuts of G. We say that G is a *graph representation* of M, and denote this by $M = \text{cut}(G)$. Graphic and cographic matroids are dual to each other, and thus, cographic matroids are also binary. Let $M = \text{cut}(G)$. Then, $G/I \setminus J$ is a graph representation of $M \setminus I/J$. In particular, the class of cographic matroids is minor-closed.

1.1.2 Fundamental problems

For graphic matroids, the following three fundamental questions arise:

(a) Isomorphism problem: what is the relationship between two graph representations of a graphic matroid?

(b) Bounding the number of representations: how many graph representations are there for a graphic matroid?

(c) Recognition problem: how can we efficiently determine if a given matroid is graphic? How can we find a representation if one exists?

In [55], Whitney proved a theorem that solves (a) and (b). As seen in [51], Tutte solved (c). Before we state Whitney's theorem, we need to define two operations. For a graph G, a 1-*flip* is either identifying two vertices in distinct components of G or its reverse, that is, splitting a vertex to increase the number of components by 1. Let (X, Y) be a partition of $E(G)$. We say that a vertex v is a *boundary vertex* of X in G if v is a common vertex of $G[X]$ and $G[Y]$. We denote the set of boundary vertices of X in G by $\partial_G(X)$. Note that $\partial_G(X) = \partial_G(Y)$. Suppose $\partial_G(X) = \{v_1, v_2\}$ for some distinct vertices v_1 and v_2 of G. For a positive integer k, we denote $[k] = \{1, 2, \ldots, k\}$. Let G' be the graph obtained from $G[X]$ and $G[Y]$ by identifying, for $i \in [2]$, vertex v_i of $G[X]$ and vertex v_{3-i} of $G[Y]$. We say that G' is obtained from G by a 2-*flip* on the set X. Note that 2-flips on X and Y are the same operation. Two graphs are *equivalent* if they are related by a sequence of 1-flips and 2-flips; otherwise, they are *inequivalent*. Note that this relation is indeed an equivalence relation. A single 2-flip on X is illustrated in Figure 1.1.

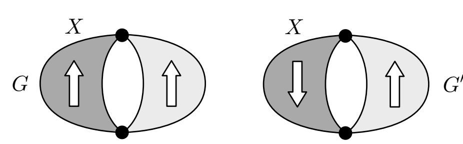

Figure 1.1: An example of a 2-flip.

Now, we are ready to state Whitney's 2-isomorphism theorem.

Theorem 1.1.1 (Whitney's 2-isomorphism Theorem)**.** *Any two graph representations of a graphic matroid are equivalent.*

Note that Theorem 1.1.1 directly solves Problem (a). It also implies that every graphic matroid has a unique equivalence class, which solves Problem (b). We denote by r_M the rank function of M and write $\mathrm{r}(M)$ for the rank of M, i.e., $\mathrm{r}(M) = \mathrm{r}_M(E(M))$. The *connectivity function* takes $X \subseteq E(M)$ as input and returns $\lambda_M(X) := \mathrm{r}_M(X) + \mathrm{r}_M(E(M) - X) - \mathrm{r}(M)$. For both r and λ, we omit the index when unambiguous. Consider $X \subseteq E(M)$ where $X \neq \emptyset$ and $X \neq E(M)$, and let k be a positive integer. Then X is *k-separating* if $\lambda(X) \leq k - 1$. It is *exactly k-separating* if equality holds. X is a *k-separation* if it is exactly k-separating and $|X|, |E(M) - X| \geq k$. A 3-separation X is *proper* if $|X|, |E(M)| \geq 4$. A matroid is 2-*connected* if it has no 1-separation; it is 3-*connected* if it 2-connected and has no 2-separation. For a graph G, if cycle(G) is 3-connected, then G is loopless and 3-connected. Thus, Theorem 1.1.1 implies that every 3-connected graphic matroid has a unique graph representation up to isolated vertices. As an application, we can construct an efficient algorithm to determine if a given matroid is graphic, which solves Problem (c). A detailed description of the algorithm will be given in Chapter 5.

1.1.3 Even-cycle and even-cut matroids

Now, we consider analogous problems for generalized classes of graphic and cographic matroids, which are even-cycle and even-cut matroids. These matroids have graphical representations, called signed graphs and grafts, respectively.

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. We say that G and Σ are the *underlying graph* and the *sign* of (G, Σ), respectively. We say that $\Gamma \subseteq E(G)$ is a *signature* of (G, Σ) if $\Sigma \,\Delta\, \Gamma := (\Sigma \cup \Gamma) - (\Sigma \cap \Gamma)$ is a cut of G. The operation that consists of replacing a signature by another signature is called *re-signing*. An edge e of G is *odd* in (G, Σ) if $e \in \Sigma$; otherwise, e is *even*. Also, a cycle C of G is *odd* in (G, Σ) if C contains an odd number of odd edges of (G, Σ), i.e., $|C \cap \Sigma|$ is odd; otherwise, C is an *even* cycle. A matroid M is an *even-cycle* matroid if there exists a signed graph (G, Σ) such that circuits of M precisely correspond to inclusion-wise minimal non-empty even cycles of (G, Σ). Then, (G, Σ) is a *signed-graph representation* of M, and we denote this by $M = \text{ecycle}(G, \Sigma)$. Let A be a binary matrix representing cycle(G) over GF(2), and let A' be a matrix obtained from A by adding a row corresponding to Σ. Then, A' represents ecycle(G, Σ) over GF(2). Thus, even-cycle matroids are binary, and they are elementary lifts of graphic matroids [40]. Every graphic matroid is an even-cycle matroid since, for a graph G, cycle$(G) = $ ecycle$(G, \emptyset)$. However, the converse is not true because the Fano matroid F_7 is not graphic while it is an even-cycle matroid as it is shown in Figure 1.2. The bold edges represent its signature in Figure 1.2. If an even-cycle matroid M has a

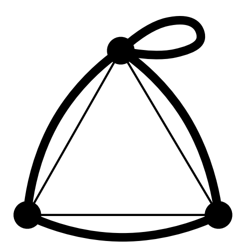

Figure 1.2: A signed-graph representation of F_7.

signed-graph representation (G, Σ) with a vertex $v \in V(G)$ such that v intersects every odd cycle, then M is graphic [40].

Consider a signed graph (G, Σ) and $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. The *minor* $(G, \Sigma)/I \setminus J$ of (G, Σ) is the signed graph defined as follows: If there exists an odd polygon of (G, Σ) contained in I, then we define $(G, \Sigma)/I \setminus J = (G/I \setminus J, \emptyset)$; otherwise, there exists a signature

Γ of (G,Σ) where $\Gamma \cap I = \emptyset$, and $(G,\Sigma)/I \setminus J = (G/I \setminus J, \Gamma - J)$. Note that minors are only defined up to re-signing. For $F \subseteq E(G)$, we denote by $(G,\Sigma)|F$ the *signed graph induced* by F, i.e., $(G,\Sigma)|F = (G,\Sigma) \setminus E(G) - F$. Consider an even-cycle matroid M with a signed-graph representation (G,Σ). Then, $(G,\Sigma)/I \setminus J$ is a signed-graph representation of $M/I \setminus J$ [40], page 21. In particular, the class of even-cycle matroids is minor-closed.

A *graft* is a pair (G,T) where G is a graph and $T \subseteq V(G)$ such that each component of G contains an even number of vertices of T. We say that G and T are the *underlying graph* and the *terminal set* of (G,T), respectively. The vertices of T are called *terminal vertices* (or simply *terminals*) of (G,T). Let $W \subseteq V(G)$ and let $D := \delta_G(W)$ be a cut of G. Then, D is *odd* if $|W \cap T|$ is odd; otherwise, D is *even*. It is well-defined because every component of G contains an even number of terminal vertices. A matroid M is an *even-cut* matroid if there exists a graft (G,T) such that circuits of M precisely correspond to inclusion-wise minimal non-empty even cuts of (G,T). Then, (G,T) is a *graft representation* of M, and we denote by $M = \text{ecut}(G,T)$. For a graph G, we denote the subset of vertices of G whose degrees are odd by $V_{odd}(G)$. A subset J of $E(G)$ is a *T-join* of a graft (G,T) if $V_{odd}(G[J]) = T$, i.e., if T is precisely the set of vertices of odd degree of the graph induced by J. Let A be a binary matrix representing $\text{cut}(G)$ over GF(2), and let A' be a matrix obtained from A by adding a row corresponding a T-join J of (G,T). Then, A' represents $\text{ecut}(G,T)$ over GF(2). Thus, even-cut matroids are binary, and they are elementary lifts of cographic matroids [40]. Every cographic matroid is an even-cut matroid since for a graph G, $\text{cut}(G) = \text{ecut}(G,\emptyset)$. However, the converse is not true, as is shown in Figure 1.3, namely, F_7 is not cographic while it is an even-cut matroid. The white vertices represent its terminals in Figure 1.3. If an even-cut matroid M has a graft representation (G,T) where $|T| \leq 2$, then M is cographic [40].

Consider a graft (G,T) and $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. The *minor* $(G,T)/I \setminus J$

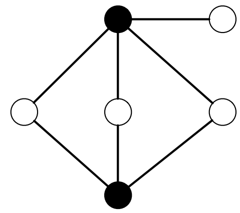

Figure 1.3: A graft representation of F_7.

is the graft defined as follows: Let $H := G/I \setminus J$. If there exists an odd cut of (G,T) contained in J, then $(G,T)/I \setminus J = (H, \emptyset)$; otherwise, there exists a T-join K of (G,T) where $K \cap J = \emptyset$, and $(G,T)/I \setminus J = (H, V_{odd}(H[K - I]))$. For $F \subseteq E(G)$, we denote by $(G,T)|F$ the *graft induced* by F, i.e., $(G,T)|F = (G,T) \setminus E(G) - F$. Consider an even-cut matroid M with a graft representation (G,T). Then, $(G,T)/I \setminus J$ is a graft representation of $M \setminus I/J$ [40], page 23. In particular, the class of even-cut matroids is minor-closed.

In this thesis, we are interested in the following problems for even-cycle and even-cut matroids, which are analogous to ones that we have seen in Section 1.1.2.

(a) Isomorphism problem: what is the relationship between two signed-graph representations (resp. graft representations) of an even-cycle (resp. even-cut) matroid?

(b) Bounding the number of representations: how many signed-graph representations (resp. graft representations) are there for an even-cycle (resp. even-cut) matroid?

(c) Recognition problem: how can we efficiently determine if a given matroid is an even-cycle (resp. even-cut) matroid? How can we find a representation if one exists?

Through Sections 1.2, 1.3, 1.4 and 1.5, we will see details for each problems.

1.1.4 More matroids with graphical representations

In this section, we review matroids with graphical representations other than even-cycle and even-cut matroids. In [56] and [57], Zaslavsky generalized graphs to biased graphs and introduced two classes of matroids that arise from biased graphs. A *biased graph* is a pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a linear class of cycles of G that satisfies the "theta property". A *theta graph* is a graph composed of three internally-disjoint paths of length at least 1 sharing their end vertices. Note that a theta graph contains exactly three polygons as subgraphs, each of which is the union of two internally-disjoint paths. We say that $(G, \mathcal{B})$ satisfies the *theta property* if, for each theta subgraph H containing two polygons of $\mathcal{B}$, the third polygon of H is also in $\mathcal{B}$. A cycle of G is called *balanced* if it is in $\mathcal{B}$; otherwise it is called *unbalanced*. The following are examples of biased graphs introduced in [56].

(a) $(G, \mathcal{C})$ where $\mathcal{C}$ is the set of all cycles of G.

(b) $(G, \emptyset)$, that is, no cycle of G is balanced.

(c) $(G, \mathcal{B})$ where G is a gain graph and $\mathcal{B}$ is the set of cycles whose gain product is 1. A *gain graph* (also called *a group-labelled graph*) is a directed graph, whose edges are labelled by elements of a group. A *gain product* of a cycle C with the given orientation is the product of group elements labelled in each edge of C according to their direction. That is, for backward edges, their inverses will be multiplied.

Zaslavsky introduced two classes of matroids that arise from biased graphs: frame and lift matroids. A *frame matroid* $\mathrm{FM}(G, \mathcal{B})$ which arises from biased graph $(G, \mathcal{B})$ is a matroid with ground set $E(G)$ such that each circuit precisely corresponds to either

(i) a balanced polygon of $(G, \mathcal{B})$;

(ii) a *tight handcuff* of $(G, \mathcal{B})$—the union of two unbalanced polygons of $(G, \mathcal{B})$ sharing exactly one common vertex;

(iii) a *loose handcuff* of $(G, \mathcal{B})$—the union of two vertex-disjoint unbalanced polygons C_1, C_2 and a path P such that for each $i \in \{1, 2\}$, the intersection of C_i and P is exactly one vertex; or

(iv) an *unbalanced* theta subgraph of $(G, \mathcal{B})$—a theta subgraph containing three distinct unbalanced polygons.

Figure 1.4 illustrates circuits of frame matroids.

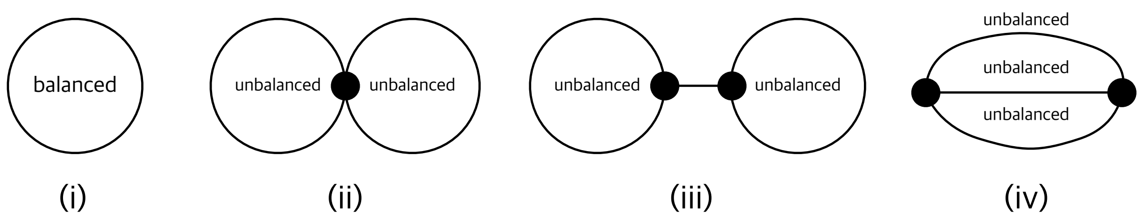

Figure 1.4: Circuits of frame matroids.

Now, consider frame matroids corresponding to biased graphs (a),(b), and (c). For (a), every polygon of G is balanced, so there are neither tight handcuffs, loose handcuffs nor unbalanced theta subgraphs. Thus, circuits of $\mathrm{FM}(G, \mathcal{C})$ precisely correspond to polygons of G, which means $\mathrm{FM}(G, \mathcal{C}) = \mathrm{cycle}(G)$. For (b), no polygon of G is balanced, so each circuit of $\mathrm{FM}(G, \emptyset)$ is either a tight handcuff, loose handcuff or unbalanced theta subgraph. This matroid is called *bicircular*. For (c) with the two-element (say, 1 and -1) group, the direction of G can be omitted, and each edge of G is labelled by 1 or -1. Then, there is no unbalanced theta subgraph, so each circuit of $\mathrm{FM}(G, \mathcal{B})$ is either a balanced polygon, tight handcuff or loose handcuff. This matroid is called *signed-graphic*. Classes of frame matroids have been researched in [2, 3, 5, 6, 35, 39].

The second class of matroids which arises from a biased graphs is lift matroids. A *lift matroid* $\mathrm{LM}(G, \mathcal{B})$ which arises from a biased graph $(G, \mathcal{B})$ is a matroid with ground set $E(G)$ such that each circuit precisely corresponds to either

(i) a balanced polygon of $(G, \mathcal{B})$;

(ii) a tight handcuff of $(G, \mathcal{B})$;

(iii) a *broken handcuff* of $(G, \mathcal{B})$ - the union of two vertex-disjoint unbalanced polygons of $(G, \mathcal{B})$; or

(iv) an unbalanced theta subgraph of $(G, \mathcal{B})$.

Figure 1.5 illustrates circuits of lift matroids.

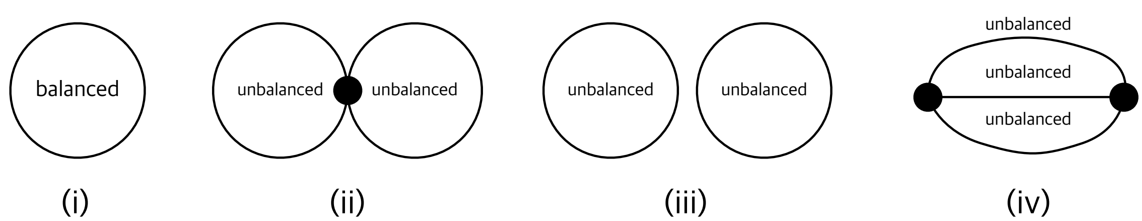

Figure 1.5: Circuits of lift matroids.

Similarly, let us consider lift matroids corresponding to biased graphs (a), (b), and (c). For (a), every polygon of G is balanced, so there are neither tight handcuffs, broken handcuffs nor unbalanced theta subgraphs. Thus, circuits of $\mathrm{LM}(G, \mathcal{C})$ precisely corresponds to polygons of G, which means that $\mathrm{LM}(G, \mathcal{C}) = \mathrm{cycle}(G)$. For (b), no polygon of G

is balanced, so each circuit of $\mathrm{LM}(G,\emptyset)$ is either a tight handcuff, broken handcuff or unbalanced theta subgraph. This matroid is called a *bicircular-lift matroid*. For (c) with the two-element (say, 1 and -1) group, there is no unbalanced theta subgraph, so each circuit of $\mathrm{LM}(G,\mathcal{B})$ is either a balanced polygon, tight handcuff, or broken handcuff. Note that these are precisely inclusion-wise minimal non-empty even cycles of (G,Σ) where Σ is the set of all edges labelled with -1. Thus, $\mathrm{LM}(G,\mathcal{B}) = \mathrm{ecycle}(G,\Sigma)$. Classes of lift matroids have been researched in [4, 5, 6]; in particular, even-cycle matroids have been researched in [19, 20, 21, 22, 25, 26, 27, 40, 47].

1.2 Isomorphism problem

1.2.1 Equivalence classes for signed graphs and grafts

In this section, we will discuss the following isomorphism problems for even-cycle and even-cut matroids.

Question 1.2.1.

(a) For an even-cycle matroid M, describe the relationship between two signed-graph representations, (G_1,Σ_1) and (G_2,Σ_2) of M.

(b) For an even-cut matroid M, describe the relationship between two graft representations, (G_1,T_1) and (G_2,T_2) of M.

Theorem 1.1.1 states that any two graph representations of a graphic matroid are equivalent. The equivalence relation can be generalized to signed graphs and grafts. Note that $\Sigma \subseteq E(G)$ is a signature of (G,Σ) if and only if $\mathrm{ecycle}(G,\Sigma) = \mathrm{ecycle}(G,\Gamma)$ [29]. Let

(G_1, Σ_1) and (G_2, Σ_2) be signed graphs such that $E(G_1) = E(G_2)$, that is, the set of edge-labels of G_1 and G_2 are the same. Then, we say that (G_1, Σ_1) and (G_2, Σ_2) are *equivalent* if they are related by a sequence of 1-flips, 2-flips, and re-signing; otherwise, they are *inequivalent*. Note that this relation is indeed an equivalence relation. The following remark in [40] implies that the isomorphism problem for even-cycle matroids in Question 1.2.1 can be easily solved when underlying graphs are equivalent.

Remark 1.2.2. *Let (G_1, Σ_1) and (G_2, Σ_2) be signed graphs such that $E(G_1) = E(G_2)$. Then, they are equivalent if and only if G_1 and G_2 are equivalent and* $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$.

Thus, it suffices to restrict our attention to the case where underlying graphs are not equivalent.

Now, we consider grafts instead of signed graphs. Let (G_1, T_1) and (G_2, T_2) be grafts such that $E(G_1) = E(G_2)$. We say (G_1, T_1) and (G_2, T_2) are *equivalent* if G_1 and G_2 are equivalent and there exists a T_1-join J of (G_1, T_1) that is a T_2-join of (G_2, T_2); otherwise, they are *inequivalent*. Note that this relation is indeed an equivalence relation. The following remark in [40] implies that the isomorphism problem for even-cut matroids in Question 1.2.1 can be easily solved when underlying graphs are equivalent.

Remark 1.2.3. *Let $(G_1, T_1), (G_2, T_2)$ be grafts such that $E(G_1) = E(G_2)$. Then, they are equivalent if and only if G_1 and G_2 are equivalent and* $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$

Similarly, Remark 1.2.3 reduces the isomorphism problem for even-cut matroids to the case of inequivalent underlying graphs.

1.2.2 The pairing theorem

In contrast to Theorem 1.1.1, even-cycle matroids (resp. even-cut matroids) can have inequivalent signed-graph representations (resp. graft representations). Consider Figure 1.6 and denote by G_1 the (edge labelled) graph on the left and by G_2 the (edge labelled) graph on the right. Let $\Sigma_1 = \{3, 5, 8, 13, 14\}$ and $\Sigma_2 = \{1, 2, 3, 4, 5\}$. Then, observe that (G_1, Σ_1)

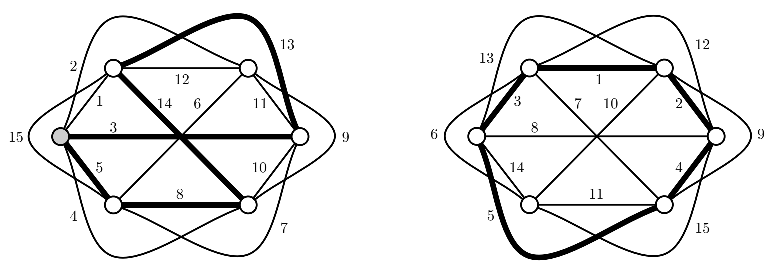

Figure 1.6: An example of inequivalent representations.

and (G_2, Σ_2) have the same set of even cycles. Hence, an answer to Question 1.2.1(a) will involve the aforementioned construction. For $i \in [2]$, let $T_i = V(G_i)$. Then, observe that (G_1, T_1) and (G_2, T_2) have the same set of even cuts. Thus, an answer to Question 1.2.1(b) will involve the aforementioned construction as well. Note that, in this example, a pair of inequivalent graphs for which we were able to construct both a pair of signed graphs with the same even-cycles and a pair of grafts with the same even cuts. This is part of a general phenomena as proven in [26], Proposition 11.

Theorem 1.2.4 (Pairing theorem). *Let G_1 and G_2 be a pair of inequivalent graphs such that $E(G_1) = E(G_2)$. Then, the following are equivalent:*

(a) there exist $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$.

(b) there exist $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$ such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$.

Moreover, if (a) and (b) occur, then Σ_1 and Σ_2 are unique up to re-signing, and T_1 and T_2 are unique.

A pair of inequivalent graphs (G_1, G_2) for which (a) and (b) hold in Theorem 1.2.4 are called *siblings*. Also, we say that (Σ_1, Σ_2) and (T_1, T_2) are the *matching-signature pair* and the *matching-terminal pair* for (G_1, G_2), respectively. Thus, Theorem 1.2.4 implies that the isomorphism problems for even-cycle and even-cut for inequivalent underlying graphs are equivalent to the following problem.

Question 1.2.5. *For siblings (G_1, G_2), describe the relationship between G_1 and G_2.*

1.2.3 Shih's theorem

In his Ph.D. thesis [47], Shih solved the following problem.

Question 1.2.6. *Let G_1, G_2 be graphs where $E(G_1) = E(G_2)$, and let $\mathcal{C}_1, \mathcal{C}_2$ be cycle spaces of G_1, G_2, respectively. Suppose that*

$$\mathcal{C}_1 \supseteq \mathcal{C}_2 \quad \& \quad \dim(\mathcal{C}_1) = \dim(\mathcal{C}_2) + 1. \tag{1.1}$$

Then, describe the relationship between G_1 and G_2.

As proven in [11], Proposition 5, the following proposition shows equivalent properties of (1.1) in Question 1.2.6.

Proposition 1.2.7. *Let G_1, G_2 be graphs where $E(G_1) = E(G_2)$, and let $\mathcal{C}_1, \mathcal{C}_2$ be cycle spaces of G_1, G_2, respectively. Then, the following are equivalent:*

(a) (1.1) *holds;*

(b) (G_1, G_2) *are siblings with a matching-signature pair* $(\Sigma, \emptyset)$*; and*

(c) (G_1, G_2) *are siblings with a matching-terminal pair* $(\emptyset, T)$*.*

In this case, we say that (G_1, G_2) are *Shih siblings*. We will state Shih's theorem in Chapter 2, which illustrate every possible construction of Shih siblings.

1.2.4 A general theorem

Let (G_1, G_2) be siblings with a matching-signature pair (Σ_1, Σ_2), and let $M := \text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. Suppose that M is graphic. Then, there exists a graph G such that $M = \text{cycle}(G) = \text{ecycle}(G, \emptyset)$. Thus, for each $i \in [2]$, (G_i, G) are Shih siblings. Then, we say that (G_1, G_2) (and also (G_2, G_1)) are *graphic siblings*. Similarly, consider siblings (G_1, G_2) with a matching-terminal pair (T_1, T_2), and let $M := \text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. Suppose that M is cographic. Then, there exists a graph H such that $M = \text{cut}(G) = \text{ecut}(G, \emptyset)$. Thus, for each $i \in [2]$, (G, G_i) are Shih siblings. Then, we say that (G_1, G_2) (and also (G_2, G_1)) are *cographic siblings*. This observation reduces Question 1.2.5 to the case when graphs are neither graphic nor cographic siblings.

Question 1.2.8. *For siblings* (G_1, G_2) *that are neither graphic nor cographic siblings, describe the relationship between* G_1 *and* G_2*.*

Recall the example of siblings in Figure 1.6. We will argue that they are neither graphic nor cographic siblings. Consider the induced signed graphs of siblings in Figure 1.6 by edge set $\{1, 2, 3, 4, 5, 6, 10, 11, 14, 15\}$, as illustrated in Figure 1.7. The bold edges represent its signature in Figure 1.7. Note that they are signed-graph representations of $\text{cut}(K_5)$, which is not graphic. Since the class of graphic matroids is minor-closed, siblings in Figure 1.6 are not graphic siblings.

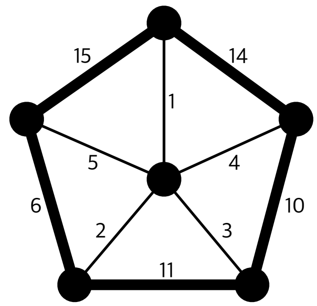

Figure 1.7: Signed-graph representations of cut(K_5).

Similarly, consider the induced grafts of siblings in Figure 1.6 by edge set $\{1, 3, 5, 8, 9, 10, 11, 12, 15\}$, as illustrated in Figure 1.8. The white vertices represent its terminals in Figure 1.8. Note that they are graft representations of cycle($K_{3,3}$), which is not cographic. Since the class of cographic matroids is minor-closed, siblings in Figure 1.6 are not cographic siblings.

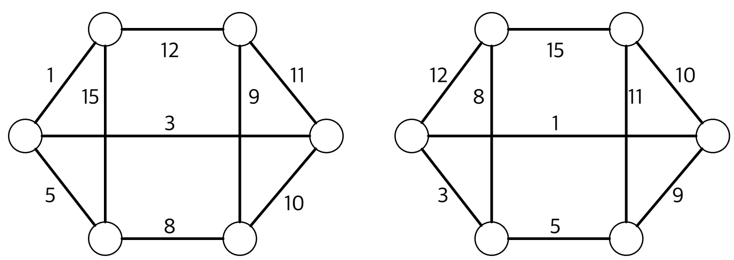

Figure 1.8: Graft representations of cycle($K_{3,3}$).

A graph is *k-connected* if it contains at least $(k+1)$ vertices, and it does not contain a set of $(k-1)$ vertices whose removal disconnects the graph. We will prove a theorem in Chapter 2 that gives a partial answer to Question 1.2.8 by assuming 4-connectivity of siblings; that is, we solve the following problem:

Question 1.2.9. *Let G_1, G_2 be 4-connected graphs such that (G_1, G_2) are siblings that are neither graphic nor cographic siblings. Describe the relationship between G_1 and G_2.*

1.3 Bounding the number of representations

1.3.1 Signed-graph representations

Note that Theorem 1.1.1 implies that every graphic matroid has a unique equivalence class. Similarly, we wonder if the number of equivalence classes of even-cycle matroids are bounded by a polynomial function of the size of the matroids. However, this is not true, even when the matroid is 3-connected [30, 40], as we illustrate next.

For a graph G and a subset F of $E(G)$, we denote by $V_G(F)$ the set of vertices of the induced graph $G[F]$. Consider a 2-connected graph H with subsets $X_1 \subset \ldots \subset X_k \subset E(H)$ $(k \geq 1)$ where, for all $i \in [k]$, $|\,\partial(X_i)| = 2$ and, for all distinct $i, j \in [k]$, $\partial(X_i) \cap \partial(X_j) = \emptyset$. Consider distinct vertices u_1, u_2, v_1, v_2 of H where $u_1, u_2 \in V_H(X_1) - \partial(X_1)$ and $v_1, v_2 \in V_H(E(H) - X_k)) - \partial(X_k)$. Let G be obtained from H by identifying u_i and v_i for $i \in [2]$. Let $\Sigma = \delta_H(u_1)\,\Delta\,\delta_H(u_2)$. We call the signed graph (G, Σ) obtained from that construction a *donut*. This construction is illustrated in Figure 1.9(i) for the case $k = 3$.

In that example, let $A = X_1$, $B = X_2 - X_1$, $C = X_3 - X_2$ and $D = E(H) - X_3$. The shaded region next to vertices $u_1 = v_1$ and $u_2 = v_2$ of G corresponds to edges in Σ. Let $M = \text{ecycle}(G, \Sigma)$. Let us now show how to construct other donuts that are also signed-graph representations of M. Let $S \subseteq [k]$, and let H' be obtained from H by doing a 2-flip on the set X_i for each $i \in S$. Let G' be obtained from H by identifying for u_i and v_{3-i} for $i \in [2]$. Then (G', Σ) is a donut that is also a signed-graph representation of M, i.e., (G, Σ) and (G', Σ') have the same even cycles [40]. This construction is illustrated in

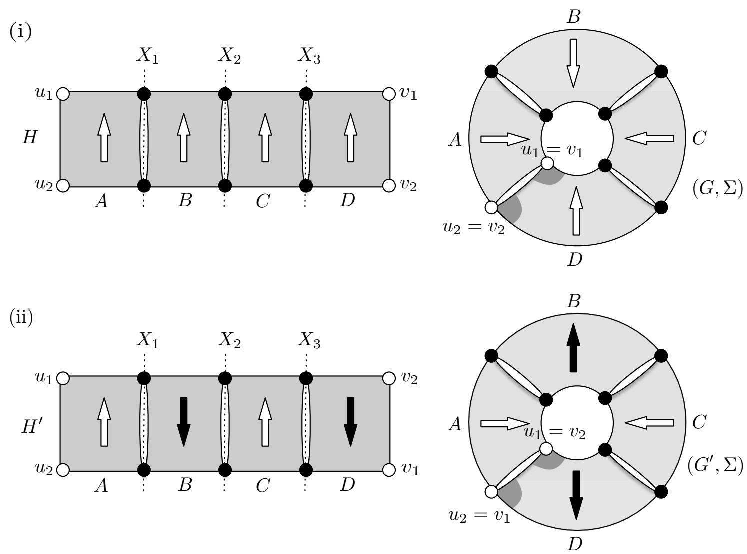

Figure 1.9: Constructing donuts.

Figure 1.9(ii). In this example, we pick $S = \{1, 2, 3\}$. There are 2^k donuts that we can obtain in that way, and they will be pairwise inequivalent for suitable choice of graph H.

Consider a signed-graph (G, Σ). A pair of vertices a, b of G is a *blocking pair* if every odd polygon of (G, Σ) uses at least one of the vertices a or b. A matroid M is *pinch-graphic* if there exists a signed-graph representation (G, Σ) of M with a blocking pair. We say that (G, Σ) is a *blocking-pair representaion* of M. Every graphic matroid is pinch-graphic and every pinch-graphic matroid is an even-cycle matroid. Moreover, the inclusions are strict. For instance, F_7^* is pinch-graphic but not graphic, and R_{10} is an even-cycle matroid that is not pinch-graphic. We saw that even-cycle matroids are elementary lifts of graphic matroids. Pinch-graphic matroids are also elementary projections of graphic matroids [40], page 30.

If a signed graph has a blocking pair, then so does every minor. In particular, the class of pinch-graphic matroids is minor-closed. Observe that all of the donuts defined in Figure 1.9 have a blocking pair. Hence, pinch-graphic matroids can have an exponential number of pairwise inequivalent blocking-pair representations. On the other hand, there is a reasonable bound for even-cycle matroids that are not pinch-graphic as the next result illustrates.

Theorem 1.3.1. *There exists a constant c such that every even-cycle matroid that is not pinch-graphic has fewer than c pairwise inequivalent signed-graph representations.*

This result will be the basis for the recognition algorithm for even-cycle matroids. We will prove Theorem 1.3.1 in Chapter 3.

1.3.2 Graft representations

In this section, we introduce an example in [30, 40] which is analogous to Figure 1.9 in Section 1.3.1. Consider a graft (G,T) where $T = \{t_1, t_2, t_3, t_4\}$. Let $P_1, \ldots, P_n$ be a partition of $E(G)$ such that, for $i \in [n]$, $\partial_G(P_i) = T$. Note that we can construct an example where $n = \mathcal{O}(|E(G)|)$ and $\mathrm{ecut}(G,T)$ is 3-connected. For every $i \in [n]$, let $G_i = G[P_i]$. Pick $I \subseteq [n]$ and for every $i \in I$, let G_i' be a graph constructed from G_i by relabelling the terminals in one of three possible ways: (i) interchange the labels of t_1 and t_2 and interchange the labels of t_3 and t_4; (ii) interchange the labels of t_1 and t_3 and interchange the labels of t_2 and t_4; or (iii) interchange the labels of t_1 and t_4 and interchange the labels of t_2 and t_3. Now, let G' be obtained by identifying vertices t_1 (resp. t_2, t_3, t_4) in each of G_i' for $i \in [n]$. We say that (G,T) and (G',T) are related by *shuffling*. We illustrate this in Figure 1.10. It can be readily checked ([40], page 28 and [25]) that $\mathrm{ecut}(G,T) = \mathrm{ecut}(G',T)$. It is now straightforward to see that we can have an exponential number of inequivalent graft representations all related by shuffling.

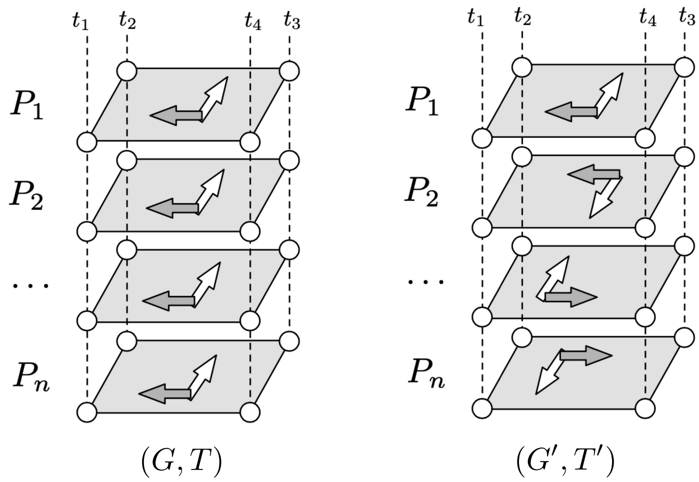

Figure 1.10: Constructing shuffles.

A matroid M is *pinch-cographic* if there is a graft representation (G, T) of M where $|T| \leq 4$. We say that (G, T) is a T_4 *representation* of the pinch-cographic matroid M. If a graft has at most four terminal vertices, then so does every minor. In particular, the class of pinch-cographic matroids is minor-closed. Observe that for the shuffling operation defined in Figure 1.10, we have four terminals, i.e., a T_4 representation. Hence, pinch-cographic matroids can have an exponential number of pairwise inequivalent T_4 representations. On the other hand, as the next result illustrates, there is a reasonable bound for even-cut matroids that are not pinch-cographic.

Theorem 1.3.2. *There exists a constant c such that every even-cut matroid that is not pinch-cographic has fewer than c pairwise inequivalent graft representations.*

This result will be the basis for the recognition algorithm for even-cut matroids. We will prove Theorem 1.3.2 in Chapter 3.

1.3.3 Blocking-pair representations

In Section 1.3.2, we showed that there exists non-graphic matroids that are 3-connected, that have an exponential number of blocking-pair representaions, and where the graph is 3-connected for each blocking-pair representaion. Thus, a stronger condition than 3-connectivity is critical in the case for pinch-graphic matroids. Recall that a matroid is 3-*connected* if it 2-connected and has no 2-separation. Let $\ell \geq 3$ be an integer, then M is $(4, \ell)$-*connected* if it is 3-connected and for every 3-separation X, $|X| \leq \ell$ or $|E(M) - X| \leq \ell$. In particular, M is *internally* 4-*connected* if it is $(4, 3)$-connected. Now, we state a theorem to bound the number of blocking-pair representaions of a pinch-graphic matroid that is not graphic under a connectivity condition.

Theorem 1.3.3. *Let M be a pinch-graphic matroid that is not graphic. If M is $(4, 5)$-connected, then the number of blocking-pair representations of M is in $\mathcal{O}(|E(M)|^4)$.*

We postpone the proof of Theorem 1.3.3 until Chapter 3. Note that we could remove the condition that M be pinch-graphic in the previous result as otherwise the number of blocking-pair representations is 0 and the result trivially holds. We kept the condition to emphasize that this is a result about pinch-graphic matroids.

1.4 Small separations in blocking-pair representations

1.4.1 Separations and sums

In Theorem 1.3.3, we assume $(4, 5)$-connectivity of a pinch-graphic matroid. Thus, we need to analyze the structure of 1-separations, 2-separations, and proper 3-separations in pinch-graphic matroids. A matroid M has a 1-separation if and only if M can be expressed as a 1-*sum*, $M_1 \oplus_1 M_2$. A 2-connected matroid has a 2-separation if and only if M can be expressed as a 2-*sum*, $M_1 \oplus_2 M_2$ [1, 8, 44]. A 3-connected binary matroid has a proper 3-separation if and only if it can be expressed as a 3-*sum*, $M_1 \oplus_3 M_2$ where $|E(M_i) - E(M_{3-i})| \geq 4$ for $i \in [2]$ [44]. (Sums will be formally defined in Section 3.1.3 and Section 4.1.1.)

1.4.2 Reducible separations

Consider a binary matroid M where $M = M_1 \oplus_k M_2$ for $k \in [3]$. If M_1 or M_2 are graphic, we say that the k-separation $X = E(M_1) - E(M_2)$ of M is *reducible*. The following result will justify the term:

Proposition 1.4.1. *Let $M = M_1 \oplus_k M_2$ for $k \in [3]$ where M_1 is graphic. If $k \in \{2, 3\}$, then assume that M is k-connected. Then, M is pinch-graphic if and only if M_2 is pinch-graphic.*

In addition, we have the following useful property,

Proposition 1.4.2. *Every 1- and 2-separation of a pinch-graphic matroid is reducible.*

Now, suppose that we wish to recognize if a binary matroid M is pinch-graphic. If M has a 1- or 2-separation X, then you may assume it is reducible; otherwise, by Proposition 1.4.2, you can deduce M is not pinch-graphic. Then, for some $k \in [2]$, you can express M as $M_1 \oplus_k M_2$ where $X = E(M_1) - E(M_2)$ and M_i is graphic for some $i \in [2]$. Finally, because of Proposition 1.4.1, it suffices to check if M_{3-i} is pinch-graphic. This allows you to reduce the recognition problem to 3-connected matroids.

1.4.3 Irreducible 3-separations

If every proper 3-separation of a pinch-graphic matroid was reducible, Proposition 1.4.1 would allow you to reduce the recognition problem to internally 4-connected matroids. Alas, it is not true. Consider the signed graphs illustrated in Figure 1.11 (i) and (ii). The shaded

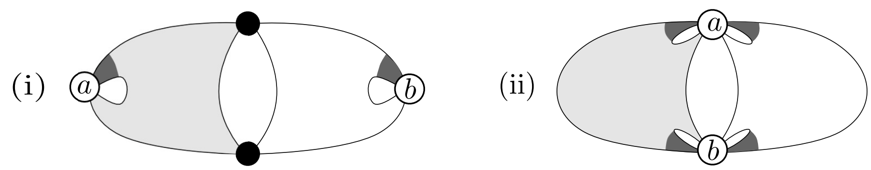

Figure 1.11: Examples of irreducible 3-separations.

region corresponds to edges X. We indicate a signature with all black edges incident to the blocking pair a, b. Then, X is a 3-separation of the corresponding pinch-graphic matroid, and X is generally not reducible. (i) is an example of a *compliant* 3-separation X, and (ii) is an example of a *recalcitrant* 3-separation X, defined below.

Given a matroid M and $X \subseteq E(M)$, denote by $\mathrm{cl}_M(X)$ the *closure* of X for matroid M. We denote by M^* the dual of M. Let M be a matroid, and let $X \subseteq E(M)$ be a proper 3-separation. Suppose $|X| \geq 5$ and there exists $e \in X$ with $e \in \mathrm{cl}_{M^*}(E(M) - X)$ and $e \in \mathrm{cl}_{M^*}(X - e)$ (here, we simply write $X - e$ instead of $X - \{e\}$). Then, observe that $X - e$ is also a proper 3-separation. We say that $X - e$ is *homologous* to X and so is any set that is obtained by repeat application of the aforementioned procedure.

Let (G, Σ) be a connected signed-graph and consider $X \subseteq E(G)$. The triple (G, Σ, X) is a *Type I* or *Type II configuration* if $|X|, |E(G) - X| \geq 4$; $G[X]$ and $G[E(G) - X]$ are both connected; and $|\,\partial_G(X)| = 2$. We denote the set of interior vertices of X in G by $\mathcal{I}_G(X)$. In addition, for Type I, there exists a blocking pair u, v where $u \in \mathcal{I}_G(X)$ and

$v \in \mathcal{I}_G(E(G) - X)$ and for Type II, $\partial_G(X)$ is a blocking pair (see Figure 1.11 (i) for a representation of a Type I configuration and Figure 1.11 (ii) for a representation of a Type II configuration). Consider a pinch-graphic matroid M with a proper 3-separation X. We say that X is *compliant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type I configuration. We say that X is *recalcitrant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type II configuration.

Here is the promised characterization, which is proven in Chapter 4.

Proposition 1.4.3. *Let M be a 3-connected pinch-graphic matroid and let X' be a proper 3-separation. Then there exists a homologous proper 3-separation X that is reducible, compliant, or recalcitrant.*

1.5 Recognition algorithm

1.5.1 Even-cycle and even-cut matroids

Tutte [51] gives a recognition algorithm for graphic matroids (as well as for cographic matroids) when the matroid is given with its binary representation. This algorithm gives a graph representation when the given matroid is graphic. If there was a polynomial-time algorithm to recognize binary matroids for general matroids described by an independence oracle, then we could extend Tutte's algorithm to general matroids. Alas, Seymour([46]) proved that such an algorithm does not exist. In the same paper, he gives a recognition algorithm for graphic matroids described by an independence oracle. For frame matroids, there are analogous results. In [2], it is proven that there is no polynomial-time recognition algorithm for bicircular matroids even if the matrix representation of a matroid is given. In [39], they give two recognition algorithms for binary signed graphic matroids: one for when a matroid is given by its binary representation, and the other for when a general matroid is described by an independence oracle. Chen and Whittle [6] proved that there is no polynomial-time algorithm to recognize frame and lift matroids described by a rank oracle. The aforementioned classes of lift matroids have not received as much attention. We will present the following algorithms in Chapter 5:

(1) Given a binary matroid M described by its $0, 1$ matrix representation A, we present an algorithm that will check if M is an even-cycle matroid in time polynomial in the number of entries of A.

(2) Given a binary matroid M described by its $0, 1$ matrix representation A, we present an algorithm that will check if M is an even-cut matroid in time polynomial in the number of entries of A.

We believe that these algorithms ought to be fast in practice but have not conducted numerical experiments. For both algorithms, the bound on the running time depends on a constant c that arises from the Matroid Minors Project and that has no explicit bound [13]. However, the algorithm does not use the value c for its computation. Rather, these algorithms rely on the existence of a polynomial algorithm to check if a binary matroid is pinch-graphic.

1.5.2 Pinch-graphic matroids

Next, we describe the relation between pinch-graphic and pinch-cographic matroids.

Proposition 1.5.1 ([40], page 26)**.** *The dual of a pinch-graphic matroid is a pinch-cographic matroid, and the dual of a pinch-cographic matroid is a pinch-graphic matroid.*

If $[I|A]$ is a $0,1$ matrix representation of a binary matroid, then $\left[A^\top|I\right]$ is a $0,1$ matrix representation of its dual. Thus a polynomial time algorithm to check if a binary matroid is pinch-graphic can be used to check in polynomial time if a binary matroid is pinch-cographic.

We present an algorithm that solves the following problem in Chapter 5,

(1) Given an internally 4-connected binary matroid M, check if M is a pinch-graphic matroid in polynomial time.

(2) Given a binary matroid M, check if M is a pinch-graphic matroid or return an internally 4-connected matroid N that is isomorphic to a minor of M such that M is pinch-graphic if and only if N is pinch-graphic in polynomial time.

By combining algorithms (1) and (2), we get a polynomial algorithm to check if a binary matroid M is pinch-graphic, and therefore, we obtain an algorithm for recognizing even-cycle and even-cut matroids. Namely, we first apply algorithm (2) and establish whether M is pinch-graphic, or we construct the matroid N and use algorithm (1) to determine whether N is pinch-graphic.

For all the aforementioned algorithms, we assume that the matroid M is given in terms of its $0,1$ matrix representation A, and, by a polynomial algorithm, we mean an algorithm that runs in time polynomial in the number of entries of A.

1.6 Organization of thesis

In Chapter 2, we state Shih's theorem and aim to prove a theorem to answer Question 1.2.9. In Chapter 3, we will give some bounds for the number of equivalence classes of graphical representations of even-cycle and even-cut matroids. We also bounds the number of blocking-pair representations of a $(4, 6)$-connected, pinch-graphic matroid that is not graphic. We will prove Theorems 1.3.1, 1.3.2 and 1.3.3 in Chapter 3. These theorems will be used to construct recognition algorithms in Chapter 5. In Chapter 4, we characterize 1-,2-, and proper 3-separations of pinch-graphic matroids, and then prove Propositions 1.4.1, 1.4.2 and 1.4.3. Furthermore, we characterize compliant and recalcitrant 3-separations. This characterization is essential to reducing the recognition algorithm for pinch-graphic matroids into the $(4, 3)$-connected case. In Chapter 5, we will describe details of algorithms to recognize even-cycle and even-cut matroids. As a subroutine, these algorithms use a recognition algorithm for pinch-graphic matroids. In Chapter 6, we discuss other open questions related to graphical representations of even-cycle and even-cut matroids.

Chapter 2

Isomorphism problem

The work in this chapter appears in [23]. Let us recall Theorem 1.1.1 and Question 1.2.1 from Section 1.2.

Theorem 1.1.1. *Any two graph representations of a graphic matroid are equivalent.*

Note that Theorem 1.1.1 also implies that any two graph representations of a cographic matroid are equivalent since, for a graph G, $\text{cycle}(G) = \text{cut}(G)^*$. In this chapter, we are interested in generalizing Theorem 1.1.1 to even-cycle and even-cut matroids, namely, we are interested in the following problems:

Question 1.2.1.

(a) For an even-cycle matroid M, describe the relationship between two signed-graph representations, (G_1, Σ_1) and (G_2, Σ_2) of M.

(b) For an even-cut matroid M, describe the relationship between two graft representations, (G_1, T_1) and (G_2, T_2) of M.

Since $\mathrm{ecycle}(G_i, \Sigma_i) = \mathrm{cycle}(G_i)$ when $\Sigma_i = \emptyset$, Question 1.2.1(a) generalizes the problem of characterizing when two graphs have the same cycles. Similarly, $\mathrm{ecut}(G_i, T_i) = \mathrm{cut}(G_i)$ when $T_i = \emptyset$. Hence, Question 1.2.1(b) generalizes the problem of characterizing when two graphs have the same cuts.

Recall that (G_1, G_2) are *siblings* if they are inequivalent and there exists a matching-signature pair (Σ_1, Σ_2) and a matching-terminal pair (T_1, T_2), that is, $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$ and $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$. As seen in Section 1.2.2, the questions in Question 1.2.1 are equivalent to each other when (G_1, G_2) are siblings, and therefore, it suffices to describe the relationship between siblings.

In Section 2.1, we review a beautiful result of Shih that solves Question 1.2.1(a) for the case when (G_1, G_2) are siblings and $\Sigma_2 = \emptyset$. Equivalently, it solves Question 1.2.1(b) for the case when (G_1, G_2) are siblings and $T_1 = \emptyset$. In this case, the siblings are called *Shih siblings*.

The main results of this chapter partially solve Question 1.2.1, namely, when G_1 and G_2 are 4-connected. We define operations that preserve even-cycles and operations that preserve even-cuts in Section 2.2. The formal statement will require a number of definitions and will be stated in Section 2.3. The proof of these results appears in Sections 2.4, 2.5 and 2.6.

2.1 Shih's theorem

2.1.1 Constructions

Before we state Shih's theorem, recall that Shih's theorem solves the following problems:

Question 1.2.6. *Let G_1, G_2 be graphs where $E(G_1) = E(G_2)$, and let $\mathcal{C}_1, \mathcal{C}_2$ be cycle spaces of G_1, G_2, respectively. Suppose that*

$$\mathcal{C}_1 \supseteq \mathcal{C}_2 \quad \& \quad \dim(\mathcal{C}_1) = \dim(\mathcal{C}_2) + 1. \tag{2.1}$$

Then, describe the relationship between G_1 and G_2.

Next, we see constructions for pairs of graphs (G_1, G_2) satisfying (2.1).

Pinching.

Consider a connected graph G_2 with distinct vertices a and b, and let G_1 be obtained from G_2 by identifying vertices a and b. Let $\mathcal{C}_1, \mathcal{C}_2$ be the set of cycles of G_1, G_2, respectively. Then clearly, every cycle of G_2 is a cycle of G_1. Moreover, as $\dim(\mathcal{C}_1) = |E(G_1)| - |V(G_1)| + 1$ and $\dim(\mathcal{C}_2) = |E(G_2)| - |V(G_2)| + 1$, we have $\dim(\mathcal{C}_1) = \dim(\mathcal{C}_2) + 1$.

Wheel pairs.

Consider graphs $R_1, \ldots, R_k$ with $k \geq 3$ and distinct vertices $x_i, y_i, z_i \in V(R_i)$ for all $i \in [k]$. Let G_1 be obtained from $R_1, \ldots, R_k$ by identifying y_i, z_{i+1}, x_{i+2} to a vertex for all $i \in [k]$ (where $k + 1 = 1$ and $k + 2 = 2$). Let G_2 be obtained from $R_1, \ldots, R_k$ by identifying $z_1, \ldots, z_k$ to a vertex and, for all $i \in [k]$, identifying y_i and x_{i+1} to a vertex (where $k+1 = 1$). We say that G_2 is a *wheel* and G_1, G_2 is a *wheel pair*. $R_1, \ldots, R_k$ are the *parts* of the wheel G_2. Moreover, the wheel pair is *proper* if we require that for all $i \in [k]$, there exists an $x_i y_i$-path of $G_2[R_i]$ that avoids z_i.

The construction is illustrated in Figure 2.1 for the case $k = 6$.

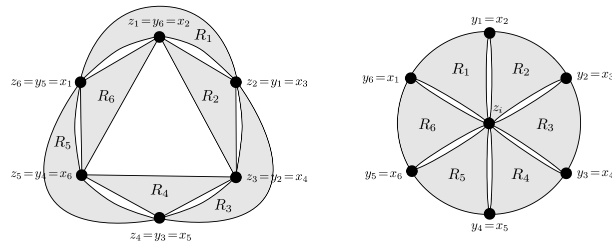

Figure 2.1: Wheel pair G_1, G_2 with G_1 on the left and G_2 on the right.

Widget pairs.

Consider graphs R_1, R_2, R_3, R_4 with distinct vertices $x_i, y_i, z_i \in V(R_i)$ for all $i \in [4]$. Let G_1 be obtained from R_1, R_2, R_3, R_4 by identifying y_1, z_2, x_3, x_4, identifying x_1, y_2, z_3, y_4, and identifying z_1, x_2, y_3, z_4. Let G_2 be obtained from R_1, R_2, R_3, R_4 by identifying z_1, z_2, z_3, identifying y_1, x_2, y_4, identifying y_2, x_3, z_4, and identifying x_1, y_3, x_4. We say that G_2 is a *widget* and G_1, G_2 is a *widget pair*. R_1, R_2, R_3, R_4 are the *parts* of the widget G_2. Moreover, the widget pair is *proper* if it is not a wheel pair. The construction is illustrated in Figure 2.2.

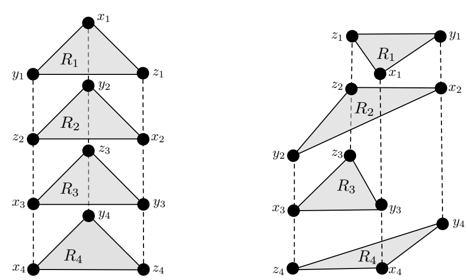

Figure 2.2: Widget pair G_1, G_2 with G_1 on the left and G_2 on the right.

2.1.2 The characterization

We are now ready to state Shih's Theorem,

Theorem 2.1.1 (Theorem 1, Chapter 2 in [47], Theorem 3 in [11])**.** *Let G_1, G_2 be siblings satisfying* (2.1)*. Then there exist G_1' equivalent to G_1 and G_2' equivalent to G_2 such that*

(a) G_1' is obtained from G_2' by identifying two distinct vertices;

(b) G_1', G_2' is a proper wheel pair; or

(c) G_1', G_2' is a proper widget pair.

2.2 Moves

2.2.1 Folding and unfolding

Let (G, Σ) be a signed graph. Recall that $v, w \in V(G)$ is a *blocking pair* if every odd cycle of (G, Σ) uses at least one of v or w. Equivalently, v, w is a blocking pair if there exists a signature Γ such that every edge of Γ has at least one end in $\{v, w\}$ [29]. Consider a graft (H, T) with terminals $T = \{t_1, t_2, t_3, t_4\}$. Let G be obtained from H by identifying vertices t_1 and t_2 and by identifying vertices t_3 and t_4. Denote by a the vertex of G corresponding to $t_1 = t_2$ and by b the vertex of G corresponding to $t_3 = t_4$. Let $\Sigma = \delta_H(t_1) \,\Delta\, \delta_H(t_3)$. Then (G, Σ) is a signed graph with blocking pair a, b. We say that (G, Σ) is obtained from (H, T) by *folding* and that (H, T) is obtained from (G, Σ) by *unfolding*. When unfolding, even edges with both ends in a, b can be chosen to have ends t_1, t_3 or t_2, t_4; and odd edges with both ends in a, b can be chosen to have ends t_1, t_4 or t_2, t_3. The construction is illustrated in Figure 2.3. The shaded regions represent the signature in Figure 2.3.

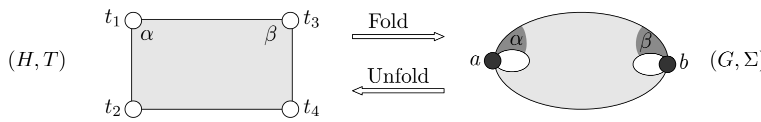

Figure 2.3: Folding and unfolding.

Proposition 2.2.1 (page 26, [40]). *Let* (G, Σ) *be a signed graph with a blocking pair and let* (H, T) *be obtained from* (G, Σ) *by unfolding. Then* $\mathrm{ecycle}(G, \Sigma) = \mathrm{ecut}(H, T)^*$*.*

2.2.2 Two operations

We now leverage Proposition 2.2.1 to construct operations that preserve even-cycles in signed graphs and that preserve even-cuts in grafts.

BP-moves

Consider a signed graph (G, Σ) that has a blocking pair u, w. There exists a signature Γ where all edges of Γ have at least one end in $\{u, w\}$. Denote by (H, T) a graft obtained from (G, Σ) by unfolding. Let (H', T') be a graft equivalent to (H, T) (see Section 1.2.1). Suppose $|T'| = 4$ and let (G', Σ') be obtained from (H', T') by folding. Then (G', Σ') is obtained from (G, Σ) by a *BP-move* (short for blocking-pair-move). We say that siblings (G_1, G_2) are *blocking-pair siblings* if there exist a matching-signature pair (Σ_1, Σ_2) such that (G_1, Σ_1) and (G_2, Σ_2) are related by a BP-move.

Next, we observe that BP-moves preserve even-cycles.

Proposition 2.2.2. *If* (G', Σ') *is obtained from* (G, Σ) *by a BP-move then* $\mathrm{ecycle}(G', \Sigma') = \mathrm{ecycle}(G, \Sigma)$.

Proof. In the aforementioned construction, Proposition 2.2.1 implies that $\mathrm{ecycle}(G, \Sigma) = \mathrm{ecut}(H, T)^*$ and $\mathrm{ecycle}(G', \Sigma') = \mathrm{ecut}(H', T')^*$. Moreover, by Remark 1.2.3, $\mathrm{ecut}(H, T) = \mathrm{ecut}(H', T')$. Thus, $\mathrm{ecycle}(G, \Sigma) = \mathrm{ecycle}(G', \Sigma')$, as required. □

T_4-moves

Consider a graft (H, T) where $|T| = 4$. Denote by (G, Σ) a signed graph obtained from (H, T) by folding. Let (G', Σ') be a signed graph equivalent to (G, Σ) (see Section 1.2.1). Suppose that (G', Σ') has a blocking pair a, b and that all odd edges are incident to at least one of $\{a, b\}$. Let (H', T') be a graft obtained from (G', Σ') by unfolding. Then we say that (H', T') is obtained from (H, T) by a *T_4-move*. We say that siblings (H_1, H_2) are T_4 *siblings* if there exist a matching-terminal pair (T_1, T_2) such that (H_1, T_1) and (H_2, T_2) are related by a T_4-move.

Next, we observe that T_4-moves preserve even-cuts.

Proposition 2.2.3. *If (H', T') is obtained from (H, T) by a T_4-move then* $\mathrm{ecut}(H', T') = \mathrm{ecut}(H, T)$.

Proof. In the aforementioned construction, Proposition 2.2.1 implies that $\mathrm{ecut}(H, T) = \mathrm{ecycle}(G, \Sigma)^*$ and $\mathrm{ecut}(H', T') = \mathrm{ecycle}(G', \Sigma')^*$. Moreover, by Remark 1.2.2, $\mathrm{ecycle}(G, \Sigma) = \mathrm{ecut}(G', \Sigma')$. Thus, $\mathrm{ecut}(H', T') = \mathrm{ecut}(H, T)$, as required. □

2.3 Main results

2.3.1 A first characterization

Recall Question 1.2.8 from Section 1.2.4. In this section, we characterize such siblings under suitable connectivity assumptions.

Question 1.2.8. *For siblings (G_1, G_2) that are neither graphic nor cographic siblings, describe the relationship between G_1 and G_2.*

We say that siblings (G_1, G_2) are 4-*connected* if both G_1 and G_2 are 4-connected. Here is our first main result:

Theorem 2.3.1. *Let (G_1, G_2) be 4-connected siblings that are neither graphic nor cographic. Denote by (Σ_1, Σ_2) the matching-signature pair and denote by (T_1, T_2) the matching-terminal pair. Then, one of the following holds:*

(a) $si(G_1)$ and $si(G_2)$ are isomorphic to subgraphs of K_6;

(b) (G_1, G_2) are blocking-pair siblings;

(c) (G_1, G_2) are T_4 siblings; or

(d) there exists $i \in [2]$ (say $i = 1$) and a graph G_1' such that (G_1, G_1') are blocking-pair siblings and (G_1', G_2) are cographic siblings.

Here $si(G_i)$ denotes the graph obtained from G_i by replacing each parallel class by a single edge. Observe that the condition that G_1, G_2 not be both isomorphic to a subgraph of K_6 is necessary because of the example in Figure 1.6. Indeed, for that example, neither (G_1, Σ_1)

nor (G_2, Σ_2) has a blocking pair, so in particular, no BP-move is possible. Moreover, in that example, $|T_1| = |T_2| = 6$, so in particular, no T_4-move is possible. In the next section, we will state a more refined version of Theorem 2.3.1 and show how it implies this result.

2.3.2 A second characterization

Before we proceed, we need to introduce a key idea. Let G_1 and G_2 be graphs where $E(G_1) = E(G_2) = E$. For a subset F of E where $|F| \geq 2$, we say that F is a *pseudo-path* of pair (G_1, G_2) if $|V_{odd}(G_i[F])| \leq 2$ for $i \in [2]$. We say that (G_1, G_2) is *closed* if, for each pseudo-path F of (G_1, G_2), either

(a) F is a cycle in both G_1 and G_2; or

(b) there exists an edge $e_F \in E$ such that $F \cup \{e_F\}$ is a cycle of both G_1 and G_2.

Proposition 2.3.2. *Let (G_1, G_2) be siblings that are not closed. Then, there exists a pair of graphs (G'_1, G'_2) where $E(G'_1) = E(G'_2)$ such that*

(a) (G'_1, G'_2) are siblings;

(b) for $i \in [2]$, $V(G_i) = V(G'_i)$; and

(c) there exists an edge $e \in E(G'_1) - E(G_1)$ such that, for $i \in [2]$, $G'_i \setminus e = G_i$.

(d) G'_1 and G'_2 have no common cycle C of size at most 2 that contains e.

Proof. Let (Σ_1, Σ_2) be the matching-signature pair for (G_1, G_2). Since (G_1, G_2) are not closed, there exists a pseudo-path F of (G_1, G_2) such that F is not a cycle in at least one of G_1 and G_2, and, for any edge $f \in E(G_1)$, $F \cup \{f\}$ is not a cycle in at least one of G_1 and G_2. For $i \in [2]$, construct G'_i as follows: if $|V_{odd}(G_i[F])| = 2$, then add an edge e joining two odd-degree vertices of G_i, and add a loop e otherwise. Note that $E(G'_1) = E(G_1) \cup \{e\}$, and $F \cup \{e\}$ is a cycle in both G_1 and G_2. By the construction, (b) and (c) hold. For (a), we define Σ'_i for $i \in [2]$ as follows: if $|F \cap \Sigma_i|$ is odd, then let $\Sigma'_i = \Sigma_i \cup \{e\}$, and let $\Sigma'_i = \Sigma_i$ otherwise. Then, by construction, $|(F \cup \{e\}) \cap \Sigma'_i|$ is even. Since $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$ and $(F \cup \{e\}) \cap \{e\}$ is an even in both (G_1, Σ_1) and (G_2, Σ_2), $\text{ecycle}(G'_1, \Sigma'_1) = \text{ecycle}(G'_2, \Sigma'_2)$, in particular, (G'_1, G'_2) are siblings. For (d), let C be a common cycle of G'_1 and G'_2 such that $e \in C$ and $|C| \leq 2$. Note that $C \neq \{e\}$; otherwise, F is a cycle in both G_1, G_2, giving a contradiction. Suppose by contradiction that $C = \{e, f\}$ for some edge $f \in E(G_1)$. Then, $F \cup \{f\}$ is a cycle in both G_1, G_2, giving a contradiction as well. □

It follows from the previous result that, for any siblings (G_1, G_2), there exists closed siblings (G'_1, G'_2) where, for $i \in [2]$, G_i is a subgraph of G'_i. Thus it will suffice to characterize closed siblings. By an *adjacent blocking pair* v, w, we mean a blocking pair v, w where v, w are joined by an edge. Here is our second main result,

Theorem 2.3.3. *Let (G_1, G_2) be 4-connected, closed siblings that are neither graphic nor cographic. Denote by (Σ_1, Σ_2) the matching-signature pair and denote by (T_1, T_2) the matching-terminal pair. Then one of the following holds:*

(a) $si(G_1)$ and $si(G_2)$ are isomorphic to K_6;

(b) (G_1, Σ_1) and (G_2, Σ_2) are blocking-pair siblings;

(c) (G_1, T_1) and (G_2, T_2) are T_4 siblings; or

(d) for some $i \in [2]$, (G_i, Σ_i) has an adjacent blocking pair.

Theorem 2.3.3 was originally proved by B. Guenin and I. Pivotto in [24], which is not published. In the following sections, we will see a much shorter and accessible proof. For both Theorem 2.3.1 and Theorem 2.3.3, we impose the condition that the siblings be 4-connected.

2.3.3 Reduction

We will show that Theorem 2.3.3 implies Theorem 2.3.1. First we require the following:

Theorem 2.3.4 ([26] Theorem 6)**.** *Let (G_1, G_2) be siblings with a matching-signature pair (Σ_1, Σ_2) and a matching-terminal pair (T_1, T_2).*

(a) If D is a an odd cut of (G_1, T_1), then D is a signature of (G_2, Σ_2).

(b) If C is an odd cycle of (G_1, Σ_1), then C is a T_2-join of (G_2, T_2).

Consider for example the siblings (G_1, G_2) in Figure 1.6 where G_1 is the graph on the left and G_2 the graph on the right. The thick edges denote the matching-signature pair (Σ_1, Σ_2) and the matching-signature pair (T_1, T_2) is the set of all vertices of G_1 and G_2, respectively. Let u denote the shaded vertex in G_1. Since $u \in T_1$, $\delta_{G_1}(u) = \{1, 2, 3, 4, 5\}$ is an odd cut of (G_1, T_1), hence a signature of (G_2, Σ_2). Since $C = \{3, 4, 10\}$ is an odd cycle of (G_1, Σ_1), C is a T_2-join of G_2.

Remark 2.3.5. *Let (G_1, G_2) be siblings with matching-terminal pairs (T_1, T_2). If $|T_1| \leq 2$ or $|T_2| \leq 2$, then (G_1, G_2) is cographic.*

Proof. We may assume $|T_1| \leq 2$. If $T_1 = \emptyset$ then let $H = G_1$ and if $|T_1| = 2$ let H be obtained from G_1 by identifying the vertices in T_1. In both cases, $\mathrm{cut}(H) = \mathrm{ecut}(G_1, T_1)$. □

Proof of Theorem 2.3.1 (assuming Theorem 2.3.3). By Proposition 2.3.2, there exist closed siblings $(\hat{G}_1, \hat{G}_2)$ such that, for $i \in [2]$, G_i is a subgraph of $\hat{G}_i$. As (G_1, G_2) are 4-connected, so are $(\hat{G}_1, \hat{G}_2)$ by Proposition 2.3.2 (b). Furthermore, as (G_1, G_2) are neither graphic nor cographic siblings, neither are $(\hat{G}_1, \hat{G}_2)$. Denote by $(\hat{\Sigma}_1, \hat{\Sigma}_2)$ the matching-signature pair of $(\hat{G}_1, \hat{G}_2)$. Note that (T_1, T_2) is the matching-terminal pair for $(\hat{G}_1, \hat{G}_2)$. We can apply Theorem 2.3.3 to siblings $(\hat{G}_1, \hat{G}_2)$. Let us review each of possible outcomes (a)-(d). For (a), each of $si(G_1)$ and $si(G_2)$ is isomorphic to subgraphs of K_6. Note that (b) and (c) are also outcomes of Theorem 2.3.1. Thus, it suffices to consider outcome (d), and we may assume it is neither outcome (a),(b), nor (c). After possibly interchanging the role of $\hat{G}_1$ and $\hat{G}_2$ we may assume that $(\hat{G}_1, \hat{\Sigma}_1)$ has an adjacent blocking pair u, w. After possibly re-signing, we may assume that edges of $\hat{\Sigma}_1$ are incident to at least one of $\{u, w\}$ and that there exists an edge $f = (u, w) \in \hat{\Sigma}_1$. Let (H, T) be obtained from $(\hat{G}_1, \hat{\Sigma}_1)$ by unfolding where $T = \{t_1, t_2, t_3, t_4\}$ and $f = (t_1, t_2)$. Let $(\hat{G}'_1, \hat{\Sigma}'_1)$ be obtained by folding (H, T) so that we have blocking pair a, b where $a = t_1 = t_2$. Then f is an odd loop of $(\hat{G}'_1, \hat{\Sigma}'_1)$. Since it is not outcome (b), $(\hat{G'}_1, \hat{G}_2)$ are siblings. Let (R_1, R_2) be the matching-terminal pair for $(\hat{G'}_1, \hat{G}_2)$. Since f is an odd loop, Theorem 2.3.4 implies $|R_2| = 2$. By Remark 2.3.5, $\hat{G}'_1, \hat{G}_2$ are cographic siblings. Let $B = E(\hat{G}_i) - E(G_i)$ and $G'_1 = \hat{G}'_1 \setminus B$. Then, (G_1, G'_1) are blocking-pair siblings, and (G'_1, G_2) are cographic siblings, as required. □

2.4 Counterexamples

We say that (G_1, G_2) are *counterexamples* if (G_1, G_2) are 4-connected, closed siblings that are neither graphic nor cographic, but none of outcomes (a)-(d) of Theorem 2.3.3 hold.

2.4.1 Properties of counterexamples

Let e, f be parallel edges in a signed graph (G, Σ). We say that $\{e, f\}$ is a *diamond* of (G, Σ) if exactly one of e, f is in Σ. A signed graph (G, Σ) is *simple* if G is loopless and, for every pair of parallel edges e and f, $\{e, f\}$ is a diamond.

Proposition 2.4.1. *If there exists a counterexample then there exists a counterexample G_1, G_2 with matching-signature pair Σ_1, Σ_2 with the property that for $i \in [2]$, (G_i, Σ_i) is simple.*

Proof. We may assume that we picked G_1 and G_2 to be a counterexample with as few edges as possible. Let (T_1, T_2) be the matching-terminal pair for (G_1, G_2). Suppose that, for some $i \in [2]$ (say $i = 1$), (G_1, Σ_1) is not simple. Then, either G_1 has a loop e or G_1 has parallel edges e, f that form an even cycle of (G_1, Σ_1). First, suppose G_1 has a loop e. Then, $e \notin \Sigma_1$; otherwise, by Theorem 2.3.4, $|T_2| = 2$, giving a contradiction. Then, $(G_1 \setminus e, G_2 \setminus e)$ are also counterexamples. Now, suppose that G_1 has parallel edges e, f that form an even cycle of (G_1, Σ_1). It follows that $\{e, f\}$ is an even cycle of (G_2, Σ_2). Moreover, as e is not a loop, e, f are parallel edges of G_2 and $G_1 \setminus e, G_2 \setminus e$ are also counterexamples. In either case, there exists a counterexample with a fewer number of edges, giving a contradiction. □

Let (G, Σ) be a signed graph. We say that $v \in V(G)$ is a *blocking vertex* if every odd polygon of (G, Σ) uses v. Equivalently, v is a blocking vertex if there exists a signature Γ such that every edge of Γ is incident to v [29].

Remark 2.4.2 ([27], Lemma 6)**.** *If an even-cycle matroid has a representation with a blocking vertex, then it is graphic.*

Proposition 2.4.3. *Let G_1, G_2 be a counterexample with matching-signature pair Σ_1, Σ_2 and matching-terminal pair T_1, T_2. Then for $i \in [2]$,* *(a)* (G_i, Σ_i) *has no blocking vertex,*

(b) $|T_i| \geq 4$.

Proof. Note that (b) follows from Remark 2.3.5. For (a), we may assume $i = 1$. If (G_1, Σ_1) has a blocking vertex v, then Remark 2.4.2 implies that ecycle(G_1, Σ_1) is graphic, contradicting that (G_1, G_2) are not graphic. □

The following is an immediate consequence of Theorem 1.1.1 and Remark 1.2.2:

Remark 2.4.4. *Let* (G, Σ) *and* (G', Σ') *be signed-graph representations of the same even-cycle matroid. Let* $C \subseteq E(G)$ *be an odd cycle of both* (G, Σ) *and* (G', Σ')*. Then,* (G, Σ) *and* (G', Σ') *are equivalent.*

Proposition 2.4.5. *Let* (G_1, G_2) *be a counterexample with a matching-terminal pair* (T_1, T_2)*. If* $|T_1| = |T_2| = 4$*, then no edge of* G_1 *has both ends in* T_i *for all* $i \in [2]$.

Proof. Suppose for a contradiction that we have edge $f \in E(G_1)$ such that $f = (t_i, t_i')$ where $t_i, t_i' \in T_i$ for all $i \in [2]$. Then, for each $i \in [2]$, let (H_i, Γ_i) be obtained from (G_i, T_i) by folding so that $t_i = t_i'$ in H_i. By Proposition 2.2.1,

$$\text{ecycle}(H_1, \Gamma_1) = \text{ecut}(G_1, T_1)^* = \text{ecut}(G_2, T_2)^* = \text{ecycle}(H_2, \Gamma_2).$$

Moreover, by construction, f is an odd loop of (H_1, Γ_1) and (H_2, Γ_2). By Remark 2.4.4, (H_1, Γ_1) and (H_1, Γ_2) are equivalent. But then, (G_1, T_1) and (G_2, T_2) are related by a T_4-move, contradicting that (G_1, G_2) are not T_4 siblings. □

The following is an immediate consequence of Theorem 1.1.1 and Remark 1.2.3:

Remark 2.4.6. *Let (G,T) and (G',T') be graft representations of the same even-cut matroid. Let $D \subseteq E(G)$ be an odd cut of both (G,T) and (G',T'). Then, (G,T) and (G',T') are equivalent.*

Proposition 2.4.7. *Let (G_1,G_2) be a counterexample with a matching-signature pair (Σ_1,Σ_2). Suppose, for $i \in [2]$, we have vertices u_i, w_i of G_i such that all edges of Σ_i are incident to at least one of u_i, w_i. Then, $\delta_{G_1}(u_1) \cap \Sigma_1 \neq \delta_{G_2}(u_2) \cap \Sigma_2$.*

Proof. Suppose that, for contradiction, we have $B := \delta_{G_1}(u_1) \cap \Sigma_1 = \delta_{G_2}(u_2) \cap \Sigma_2$. Observe that for $i \in [2]$, u_i, w_i is a blocking pair of (G_i,Σ_i). Since G_1, G_2 is a counterexample, u_i, w_i are non-adjacent. For $i \in [2]$, let (H_i,R_i) be obtained from (G_i,Σ_i) by unfolding. By Proposition 2.2.1,

$$\text{ecut}(H_1,R_1) = \text{ecycle}(G_1,\Sigma_1)^* = \text{ecycle}(G_2,\Sigma_2)^* = \text{ecut}(H_2,T_2).$$

Moreover, by construction B is an odd cut of (H_1,R_1) and (H_2,R_2). By Remark 2.4.6, (H_1,R_1) and (H_2,R_2) are equivalent. But then, (G_1,Σ_1) and (G_2,Σ_2) are related by a BP-move, contradicting that (G_1,G_2) are not blocking-pair siblings. □

Proposition 2.4.8. *Let (G_1,G_2) be counterexamples with a matching-signature pair (Σ_1,Σ_2) and matching-terminal pair (T_1,T_2) where (G_1,Σ_1) and (G_2,Σ_2) are simple. Then, the following hold for $i \in [2]$,*

(a) If (G_i,Σ_i) has a diamond $\{e,f\}$, then $\{e,f\}$ is a matching of G_{3-i} covering T_{3-i}.

(b) If $|T_i| = 4$, then there is no diamond of (G_i,Σ_i) with both ends in T_i.

(c) (G_i,Σ_i) has at most 3 distinct diamonds.

(d) If $|T_1| \leq |T_2|$, then there exists $v \in T_1$ not incident to a diamond.

Proof. For (a), since $\{e, f\}$ is an odd cycle of (G_i, Σ_i), it is a T_{3-i}-join of G_{3-i} by Theorem 2.3.4. Moreover, $|T_{3-i}| = 4$ by Proposition 2.4.3(b). For (b), suppose for a contradiction that there is a diamond $\{e, f\}$ with both ends in T_i. By (a), $|T_{3-i}| = 4$ and the ends of e are in T_{3-i}. But this contradicts Proposition 2.4.5. For (c), suppose for a contradiction that we have, for $j \in [4]$, disjoint diamonds e_j, f_j of (G_i, T_i). By (a), e_j, f_j are matchings of G_{3-i} with ends in T_{3-i}. It follows that two edges, say $e_j, e_{j'}$ of these matching are parallel in G_2. Since (G_{3-i}, Σ_{3-i}) is simple, $e_j, e_{j'}$ is a diamond of (G_2, Σ_2), contradicting (b). For (d), assume $|T_1| \leq |T_2|$. By Proposition 2.4.3(b), we have $|T_i| \geq 4$ for $i \in [2]$. We may assume (G_1, Σ_1) has a diamond. Thus, by (a), we have $|T_2| = 4$ and in particular $|T_1| = 4$. Suppose for a contradiction for every vertex v of T_1 we have diamond e_v, f_v incident to v. It follows from (c) that two of these diamonds are the same, contradicting (b). □

2.4.2 Deleting a terminal

To keep the notation light, throughout the remainder of the paper we will use the following assumptions:

(h1) (G_1, G_2) is a counterexample;

(h2) (Σ_1, Σ_2) is the matching-signature pair for (G_1, G_2);

(h3) (T_1, T_2) is the matching-terminal pair for (G_1, G_2) where $|T_1| \leq |T_2|$;

(h4) (G_1, Σ_1) and (G_2, Σ_2) are simple;

(h5) Let $\hat{v} \in T_1$ that is incident to no diamond, and, subject to that, the degree of $\hat{v}$ is minimized.

(h6) $A := \delta_{G_1}(\hat{v})$ and for $i \in [2]$, $H_i := G_i \setminus A$.

Note that if we have a counterexample then by Proposition 2.4.1 we can assume (h4). The existence of $\hat{v}$ in (h5) is guaranteed by Proposition 2.4.8. Note that H_1 is obtained from G_1 by deleting terminal $\hat{v}$. Next we identify key properties of H_1, H_2.

Proposition 2.4.9. *Assume (h1)-(h6). Then,*

(a) (H_1, H_2) *are closed siblings;*

(b) (H_1, H_2) *have matching-signature pair* $(\Sigma_1 - A, \emptyset)$*; and*

(c) (H_1, H_2) *have matching-terminal pair* $(\emptyset, T_2)$.

Proof. Since (G_1, G_2) are counterexamples, they are not cographic siblings, in particular, $T_1, T_2 \neq \emptyset$. Note that A is an odd cut of (G_1, T_1), but observe that A does not contain an odd cut A' of (G_2, T_2); otherwise, by Theorem 2.3.4, A' would be a signature of (G_1, Σ_1), and, in particular, $\hat{v}$ would be a blocking vertex, contradicting Proposition 2.4.3. Note that $\text{ecut}(G_1, T_1)/A = \text{cut}(G_1 \setminus A)$ and $\text{ecut}(G_2, T_2)/A = \text{ecut}(G_2 \setminus A, T_2)$ (see Section 1.1.3). Since $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$, we have $\text{cut}(H_1) = \text{ecut}(H_2, T_2)$. As $T_2 \neq \emptyset$, (H_1, H_2) are cographic siblings with the matching-terminal pair $(\emptyset, T_2)$. Hence, (c) holds. To show (a), it remains to prove that (H_1, H_2) are closed. Suppose otherwise. Then, there exists a pseudo-path F of (H_1, H_2) such that (i) F is not a cycle in one of H_1 and H_2, and, (ii) for any edge $f \in E(H_1)$, $F \cup \{f\}$ is not a cycle in one of H_1 and H_2. Since (G_1, G_2) is closed, there exists an edge $e \in E(G_1)$ such that $F \cup \{e\}$ is a cycle in both G_1 and G_2. Thus, $e \notin E(H_1)$ and $e \in A$. This contradicts that no edge of F is incident with $\hat{v}$. By Theorem 2.3.4, A is a signature of (G_2, Σ_2). Thus, $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, A)$. Then, $\text{ecycle}(G_1, \Sigma_1) \setminus A = \text{ecycle}(G_1 \setminus A, \Sigma_1 - A)$ and $\text{ecycle}(G_2, A) \setminus A = \text{ecycle}(G_2 \setminus A, \emptyset)$ (see Section 1.1.3). Hence, $\text{ecycle}(H_1, \Sigma_1 - A) = \text{ecycle}(H_2, \emptyset)$ which implies (b). □

2.4.3 Mates

In a connected graph G, a cut B of G determines the shore of the cut uniquely up to complementation, i.e. if we have $\delta(U) = \delta(W)$ then $U = W$ or $U = V(G) - W$.

Proposition 2.4.10. *Let G be a graph with a cut B and let H be a connected spanning subgraph of G. If $\delta_H(U) = B \cap E(H)$ then $\delta_G(U) = B$.*

Proof. Since B is a cut of G, $B = \delta_G(W)$ for some $W \subseteq E(G)$. It follows that, $B \cap E(H) = \delta_H(W)$. Since the cut $B \cap E(H)$ of H determines the shore of the cut uniquely, up to complementation, $U = W$ or $U = V(H) - W = V(G) - W$. Hence, $\delta_G(U) = \delta_G(W) = B$ as required. □

Proposition 2.4.11. *Assume (h1)-(h6) and let $B \subseteq A$ be a cut of G_1. Then,*

(a) if B is an even cut of (G_1, T_1), then $B = \emptyset$; and

(b) if B is an odd cut of (G_1, T_1), then $B = A$.

Proof. Let $G = G_1 \setminus (A - B)$. Consider first the case where $B \neq \emptyset$. Then as H_1 is connected, so is G. By construction, we have $\delta_G(\hat{v}) = B = B \cap E(G)$. Then, it follows by Proposition 2.4.10 that $\delta_{G_1}(\hat{v}) = B$. But, $A = \delta_{G_1}(\hat{v})$ by definition, so we proved that if $B \neq \emptyset$, then $B = A$. If B is an even cut of (G_1, T_1), then $B = \emptyset$ for otherwise $B = A$ but A is an odd cut of (G_1, T_1). If B is an odd cut of (G_1, T_1), then clearly $B \neq \emptyset$ and then $B = A$. □

Consider H_1, H_2 as defined in (h1)-(h6) and let $U_1 \subseteq V(H_1)$. It follows from Proposition 2.4.9 that $\delta_{H_1}(U_1) = \delta_{H_2}(U_2)$ for some U_2 where $|U_2 \cap T_2|$ is even. We say that U_2 is a *mate* of U_1.

Proposition 2.4.12. *Assume (h1)-(h6) and let $U_1 \subseteq V(H_1)$ and let U_2 be a mate of U_1. For $i \in [2]$, let $S_i = \delta_{G_i}(U_i) \cap A$. Then, the following hold:*

(a) if $|U_1 \cap T_1|$ even, then $S_1 = S_2$, and

(b) if $|U_1 \cap T_1|$ odd, then $A = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$.

Proof. By definition of mate, $\delta_{G_2}(U_2)$ is an even cut of (G_2, T_2) and hence of (G_1, T_1). Hence, $\delta_{G_1}(U_1) \,\Delta\, \delta_{G_2}(U_2) = S_1 \,\Delta\, S_2$ is a cut of (G_1, T_1) which is even if and only if $|U_1 \cap T_1|$ is even. Since $S_1 \,\Delta\, S_2 \subseteq A$, Proposition 2.4.11 implies that if $|U_1 \cap T_1|$ is even, then $S_1 \,\Delta\, S_2 = \emptyset$, proving (a), and if $|U_1 \cap T_1|$ is odd, then $S_1 \,\Delta\, S_2 = A$, proving (b). □

2.4.4 Organization of the remainder of the proof

Our proof will leverage Theorem 2.1.1 as follows,

Proposition 2.4.13. *Assume (h1)-(h6). Then there exists H_2' equivalent to H_2 such that at least one of the following holds:*

(a) H_1, H_2' is a proper widget pair.

(b) H_1, H_2' is a proper wheel pair.

(c) H_1 is obtained from H_2' by identifying two vertices.

Proof. Proposition 2.4.9 implies that H_1, H_2 satisfy (2.1). Moreover, H_1 is 3-connected since G_1 is 4-connected. The result then follows from Theorem 2.1.1 as $H_1' = H_1$ in this case. □

Outcomes (a) and (b) of the previous proposition are analyzed in Section 2.5 and outcome (c) is analyzed in Section 2.6. In all cases we will derive a contradiction.

2.5 Wheels and widgets

2.5.1 Connectivity

Consider a wheel pair (H_1, H_2) with parts $R_1, \ldots, R_k$ for $k \geq 3$. Recall that each part R_i has special vertices x_i, y_i, z_i (see Section 2.1.1). Vertices x_i, y_i, z_i for $i \in [k]$ are the *boundary vertices* of H_1 respectively H_2. Vertices of H_1 respectively H_2 that are not boundary vertices are *interior vertices*. For H_2 vertices $x_1, \ldots, x_k$ are the *rim vertices* and $z_1 = \ldots = z_k$ is the *hub*. Note, that if v is an interior vertex of H_1, H_2 then $\delta_{H_1}(v) = \delta_{H_2}(v)$. Similarly, if H_1, H_2 is a widget pair with parts R_1, R_2, R_3, R_4, then x_i, y_i, z_i for $i \in [4]$ are the *boundary vertices* and the non-boundary vertices are the *interior vertices.*

We describe the matching-terminal pairs for wheels and widgets [11].

Proposition 2.5.1. *If (H_1, H_2) is a wheel pair, then the matching-terminal pair is (T_1, T_2) where $T_1 = \emptyset$ and $T_2 = \{x_1, \ldots, x_k\}$ for even k and $T_2 = \{x_1, \ldots, x_k, z_1\}$ for odd k. If H_1, H_2 is a widget pair, then the matching-terminal pair is (T_1, T_2) where $T_1 = \emptyset$ and $T_2 = \{x_1, x_2, x_3, z_1\}$.*

Recall that, for a graph G, $X \subseteq E(G)$ is a 2-separation if $G[X]$, $G[E(G) - X]$ are connected and $\partial(X) = \{u, v\}$ for some distinct vertices u, v.

Proposition 2.5.2. *Assume (h1)-(h6). If (H_1, H_2') is a proper widget pair for some H_2' equivalent to H_2, then H_2' is 3-connected. In particular, $H_2 = H_2'$.*

Proof. Recall that H_1 is 3-connected by Proposition 2.4.9. It follows readily that H_2' is 2-connected. Furthermore, if H_2' is not 3-connected then it must have a 2-separation X. Then, X is contained in one of its part R_i, i.e. $X \subseteq E(R_i)$. This implies in turn that X is a 2-separation of H_1, giving a contradiction. $\square$

Proposition 2.5.3. *Assume (h1)-(h6). If H_1, H_2' is a proper wheel pair for some H_2' equivalent to H_2 then H_2' is 3-connected. In particular, $H_2 = H_2'$.*

Proof. Let $R_1, \ldots, R_k$ denote the parts of the wheel pair H_1, H_2' where $k \geq 3$. As the wheel pair is proper, for all $i \in [k]$, there exists an $x_i y_i$-path of $H_2'[R_i]$ that avoids z_i. Denote by C the polygon formed by the union of these paths. We call C the *rim polygon* of H_2'. Denote by h the hub $z_1 = \ldots = z_k$ of H_2'. Note that H_1 is 3-connected since G_1 is 4-connected. We leave it as an exercise to show that this implies in particular that H_2' is 2-connected.

Claim 1. *Let X be a 2-separation of H_2'. Then there does not exist $i \in [k]$ such that $X \subseteq E(R_i)$.*

Subproof. Otherwise, X is also a 2-separation of H_1, a contradiction as H_1 is 3-connected. ♢

Claim 2. *There do not exist series edges e, f, g of H_2' contained in C.*

Subproof. Otherwise, two of e, f, g will be in series in H_1, a contradiction as H_1 is 3-connected. ♢

A *spoke* of H_2' is a path Q with ends h and v where v is the only vertex of Q in the rim polygon C. Let $\mathcal{P}$ be a maximal collection of spokes that are pairwise vertex disjoint except for the hub h. Since H_2' is 2-connected, $|\mathcal{P}| \geq 2$. We will consider two cases in the proof: $|\mathcal{P}| \geq 3$ and $|\mathcal{P}| = 2$. A part R_i is *trivial* if it consists of a unique edge (which, as H_1, H_2' is a proper wheel pair, is in C).

<u>**Case 1.**</u> $|\mathcal{P}| \geq 3$.

Claim 3. *If X is a 2-separation of H_2', then (after possibly replacing X by $E(H_2') - X$) we have $X = \{e, f\}$ and for some $i \in [k]$, $e \in R_i, f \in R_{i+1}$ $(k+1 = 1)$ and $h \notin \partial_{H_2'}(X)$.*

Proof. Suppose for a contradiction that X is a 2-separation of H_2'. Then X (resp. $X - E(G)$) is not contained in a part R_i by Claim 1. Since H_1, H_2' is a proper wheel and $|\mathcal{P}| \geq 3$, we may assume (after possibly interchanging the role of X and $E(H_2') - X$ and relabeling the parts) that X is contained in the union of consecutive parts, say $R_1, R_2, \ldots, R_j$ where u is a vertex of R_1, v a vertex of R_j, $\partial_{H_2'}(X) = \{u, v\}$ and $X \cap E(R_\ell) \neq \emptyset$ for $\ell \in [j]$. By Claim 1, each part $R_2, \ldots, R_{j-1}$ is trivial, $X \cap E(R_1)$ consists of a single edge (say e), and $X \cap E(R_j)$ consists of a single edge (say f). But then, Claim 2 implies that $j = 2$, and the result follows. We illustrate this in Figure 2.4. □

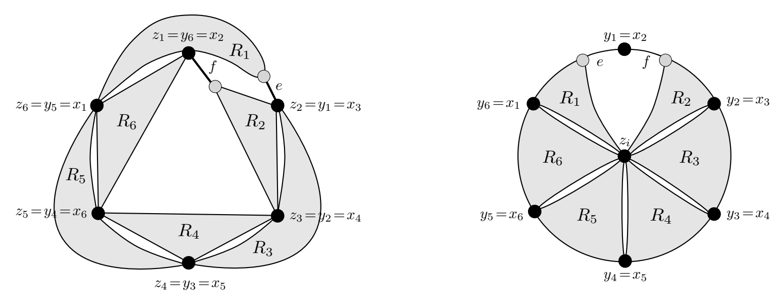

Figure 2.4: Potential 2-separation in wheel pairs.

We call a pair of edges e, f as in Claim 3 an *i-series pair*. By Claim 2 no i-series and j-series pair share an edge. It follows that H_2' is obtained from H_2 by a sequence of 2-flips on disjoint i-series pairs. To complete the proof of Case 1, it suffices to show that there is no i-series pair for any $i \in [k]$. Suppose for a contradiction we have say, a 1-series pair e, f. If R_1 consists of a single edge e then we define Q to be a $y_k z_k$-path in R_k. If e is not the only edge of R_1 then denote by u the end of e distinct from y_1 and we then define Q to be a uz_1-path in R_1 (note that Q exists as H_1 is 3-connected). Then both $Q \cup e$ and $Q \cup f$ form a path of H_1 (for a set A and element of the ground set a, we write $A \cup a$ for $A \cup \{a\}$ and write $A - a$ for $A - \{a\}$). If H_2' is obtained from H_2 without a 2-flip on $\{e, f\}$ then $Q \cup e$

is a path of H_2'. But then as H_1, H_2 is closed by Proposition 2.4.9, there exists $g \neq f$ such that $Q \cup \{e, g\}$ form a cycle of H_1, H_2', a contradiction as e, f are in series in H_2'. If H_2' is obtained from H_2 using a 2-flip on $\{e, f\}$ then $Q \cup f$ is a path of H_2'. But then as H_1, H_2 is closed there exists $g \neq e$ such that $Q \cup \{f, g\}$ form a cycle of H_1, H_2', again a contradiction.

Case 2. $|\mathcal{P}| = 2$.

Claim 4. *There exists exactly two edges e_i, e_j incident to h in H_2' with $e_i \in E(R_i)$ and $e_j \in E(R_j)$ for distinct $i, j \in [k]$.*

Subproof. Since $|\mathcal{P}| = 2$, there exists, by Menger's theorem, a pair of vertices $u_i \in V(R_i)$ and $u_j \in V(R_j)$ that separates h from C in H_2'. Thus there exists a 2-separation Y with $\partial_{H_2'}(Y) = \{u_i, u_j\}$ such that $h \in \mathcal{I}_{H_2'}(Y)$ and $C \cap Y = \emptyset$. Moreover, since h is the hub of H_2', Y can be partitioned into 2-separations Y_i, Y_j so that $\partial_{H_2'}(Y_i) = \{u_i, h\}$ and $\partial_{H_2'}(Y_j) = \{u_j, h\}$. It suffices to show now that Y_i (and by the same argument Y_j) consists of a single edge. If u_i is an interior vertex of R_i then Y_i is a 2-separation contained in R_i and it follows by Claim 1 that Y_i consists of a single edge. Thus, we may assume u_i is a rim vertex $y_i = x_{i+1}$ and Y_i is contained in $E(R_i) \cup E(R_{i+1})$. Let $Y_i' = Y_i \cap E(R_i)$ and let $Y_i'' = Y_i \cap E(R_{i+1})$. Then $\partial_{H_2'}(Y_i') = \partial_{H_2'}(Y_i'') = \{y_i, h\}$. It follows by Claim 1 that Y_i' and Y_i'' consists of parallel edges, giving a contradiction as H_2 and thus H_2' is simple. $\Diamond$

Claim 5. *We may assume $k = 5$, i.e., H_1, H_2' is a wheel pair with parts $R_1, \ldots, R_5$.*

Subproof. If $k = 3$, we can exchange the role of h with any rim vertex $y_i = x_{i+1}$. Note, not all these vertices can have degree 2, so pick as a hub such a vertex with degree at least three. It then follows from Claim 4 that we are now in Case 1. Suppose that k is even. Let $(\emptyset, T_2')$ be the matching-terminal pair for (H_1, H_2'). Then by Proposition 2.5.1, $h \notin T_2'$. Consider R_i, R_j and e_i, e_j as in Claim 4. Then $\delta_{H_2'}(h) = \{e_i, e_j\}$, is an even cut of (H_2', T_2')

and hence a cut of H_1, a contradiction as H_1 is 3-connected. Finally, Claim 2 and the fact R_i, R_j are the only non-trivial parts imply $k \leq 6$. ♢

By Claim 1 and Claim 4 there are only two non-trivial parts among $R_1, \dots, R_5$. We may assume because of Claim 2 that the non-trivial parts are R_1 and R_3. Denote by e_1 and e_3 the edges of R_1 and R_3 incident to h in H_2'. Let $R_1' = R_1 \setminus e_1$ and let $R_3' = R_3 \setminus e_3$. Denote by r_2, r_4, r_5 the unique edge in R_2, R_4, R_5 respectively. We describe H_1, H_2' in Figure 2.5. Observe that H_2' has exactly two 2-separations, which are $\{e_1, e_3\}$ and $\{r_4, r_5\}$. We also

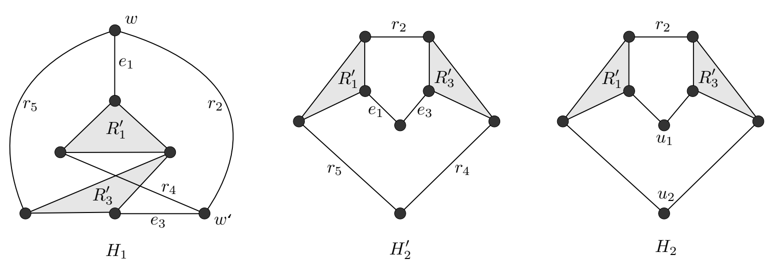

Figure 2.5: Two edges incident to the hub.

describe H_2 in Figure 2.5. Denote by w the vertex of H_1 incident to e_1, r_2, r_5 and by w' the vertex of H_1 incident to e_3, r_2, r_4. Recall that $\hat{v}$ is the vertex picked in (h5).

Claim 6. $\deg_{G_1}(\hat{v}) = 4$.

Subproof. By Proposition 2.5.1, $|T_1| \geq 6$. Proposition 2.4.8 implies that (G_1, Σ_1) has no diamond. Note, $deg_{H_1}(w) = 3$. It follows that $deg_{G_1}(w) = 4$. The result now follows from (h5). ♢

Now, let $A = \{a_1, a_2, a_3, a_4\}$. Denote by u_1 the vertex of H_2 incident to e_1, e_3 and denote by u_2 the vertex of H_2 incident to r_4, r_5. As G_2 is 4-connected, there exist two edges of A, say a_1 and a_2, incident to u_1. Since G_1, G_2 are closed, there exists an edge $\beta_1 \in E(H_1) = E(H_2)$

such that $\Delta_1 := \{a_1, a_2, \beta_1\}$ is a triangle of both H_1 and H_2. Similarly, for some $i, j \in [4]$ and $\beta_2 \in E(H_1)$, we have that a_i, a_j are incident to u_2 and $\Delta_2 := \{a_i, a_j, \beta_2\}$ is a triangle of both H_1 and H_2. Now, consider G_1. Since G_1 is 4-connected, there are two edges in A joining $\hat{v}, w$ and $\hat{v}, w'$, respectively. Moreover, for some $x \in \mathcal{I}_{H_1}(R'_1)$ and $x' \in \mathcal{I}_{H_1}(R'_3)$, there are two edges in A joining $\hat{v}, x$ and $\hat{v}, x'$, respectively; otherwise, either $E(R'_1) \cup e_1, E(R'_1) \cup r_4$, $E(R'_3) \cup e_3$, or $E(R'_3) \cup r_5$ is a 3-separation of G_1, giving a contradiction. Note that, among $\{w, w', x, x'\}$, only w and w' are adjacent. Thus, $\{i, j\} = \{1, 2\}$ and $\beta_1 = \beta_2 = r_2$. This implies that $\Delta_1 = \Delta_2$, giving a contradiction. □

2.5.2 Terminals and interior vertices

Proposition 2.5.4. *Assume (h1)-(h6). If H_1, H_2 is a proper wheel pair with parts $R_1, \ldots, R_k$ ($k \geq 3$), then every part R_i is either a complete graph spanning x_i, y_i, z_i or a single edge with ends x_i, y_i. If H_1, H_2 is a proper widget pair with parts R_1, R_2, R_3, R_4, then every part R_i is a complete graph spanning x_i, y_i, z_i.*

Proof. This follows from the fact that H_1 is 3-connected by hypothesis and that H_1, H_2 are closed siblings by Proposition 2.4.9. □

Proposition 2.5.5. *Assume (h1)-(h6). If H_1, H_2 is a proper wheel pair then every vertex x_i, $i \in [k]$ is incident to at least one edge of A in G_2.*

Proof. By Proposition 2.5.1, $x_i \in T_2$. It follows by Theorem 2.3.4 that $\Gamma = \delta_{G_2}(x_i)$ is a signature of (G_1, Σ_1). Suppose that x_i is not incident to any edge of A for some $i \in [k]$. Then $\Gamma = \delta_{G_2}(x_i) = \delta_{H_2}(x_i)$. But then observe that (G_1, Σ_1) has an adjacent blocking pair. □

For proper widget pairs or wheel pairs, Proposition 2.5.1 shows that interior vertices are not in T_2. We show that they are also not in T_1, i.e. all terminals are boundary vertices.

Proposition 2.5.6. *Assume (h1)-(h6) and that H_1, H_2 is a proper widget pair or a proper wheel pair. Let u be an interior vertex of H_1, then $u \notin T_1$.*

Proof. We have $\delta_{H_1}(u) = \delta_{H_2}(u)$. By Proposition 2.5.1, $u \notin T_2$. It follows that $\{u\} \subseteq V(H_2)$ is a mate of $\{u\} \subseteq V(H_1)$. For $i \in [2]$ let $S_i = \delta_{G_i}(u) \cap A$. Suppose for a contradiction that $u \in T_1$. Then Proposition 2.4.12 implies that $S_1 \cup S_2 = A$ and that $S_1 \cap S_2 = \emptyset$. By (h5), $|S_1| \leq 1$. Moreover, by Theorem 2.3.4, $\delta_{G_1}(u)$ is a signature of (G_2, Σ_2). Since by Proposition 2.4.3 (G_2, Σ_2) has no blocking vertex, S_1 contains a single edge, say e. Hence, e is the only edge of A incident to u in G_1 and e is the only edge of A not incident to u in G_2.

Case 1. H_1, H_2 is a widget pair.

Consider first the case where there exists an interior vertex u' of H_1 where $u' \in T_1$ and $u' \neq u$. Then proceeding as above we deduce that there exists $e' \in A$ such that e' is the only edge of A incident to u' in G_1 and e' is the only edge of A not incident to u' in G_2. It follows that in G_2, e is incident to u' but not u, e' is incident to u but not u' and every edge of $A - \{e, e'\}$ has ends u, u'. Hence, u, u' is a blocking pair of (G_2, Σ_2) since by Proposition 2.4.9 A is a signature. Thus u, u' are not adjacent in G_2. Therefore, $A = \{e, e'\}$ and u, u' are not in the same part R_i by Proposition 2.5.4. But then observe that any pair of boundary vertices of H_1 is an adjacent blocking pair of (G_1, Σ_1), a contradiction. It follows that u is the unique interior vertex of H_1 contained in T_1. Suppose now that e has an end u' in G_2 that is an interior vertex of some part R_i. As $u' \notin T_1$, it follows from Proposition 2.4.12 that e has end u' in G_1 as well. Since H_1, H_2 are closed there exist edges $f \in A$, $g \in H_1$ such that $\{e, f, g\}$ is triangle of G_1, G_2 and f is incident to a boundary vertex of R_i. But then $f \neq e$ and f is not incident to u in G_2 a contradiction. Hence, both ends of e are boundary vertices. By Proposition 2.5.4 there exists an edge f of H_2 such that e, f is a diamond of (G_1, Σ_2) which contradicts Proposition 2.4.8(b).

Case 2. H_1, H_2 is a proper wheel pair.

Let $R_1, \ldots, R_k$ be the parts of the proper wheel pair. Note that if $k = 3$, then the wheel pair is also a widget pair. Thus, we may assume $k \geq 4$. Since H_1, H_2 are closed, for all edges of $A - e$ the end distinct from $\hat{v}$ is in the same part R_j for some $j \in [k]$. Without loss of generality assume $j = 1$. By Proposition 2.5.1, $T_2 = \{x_1, \ldots, x_k\}$ when k is even and $T_2 = \{x_1, \ldots, x_k, z_1\}$ otherwise. Proposition 2.5.5 says that every vertex x_i, $i \in [k]$ is incident to at least one edge of A in G_2. Thus none of $x_3, \ldots, x_k$ are incident to edges of $A - e$ in G_2. Hence, $k = 4$ and by Proposition 2.5.5, x_3, x_4 are the ends of e in G_2. Since H_1, H_2 are complete, there exists an edge f of H_2 such that e, f is a diamond of (G_2, Σ_2) with ends in T_2, contradicting Proposition 2.4.8(b). □

We can now combine the previous propositions to get the following useful result,

Proposition 2.5.7. *Assume (h1)-(h6) and suppose that H_1, H_2 is a proper widget pair or a proper wheel pair. Let u be an interior vertex in some part R_i with a boundary vertex x_i. If there exists $e \in A$ where e is incident to u in either G_1 or G_2, then*

(a) e is incident to u in both G_1 and G_2;

(b) the other end, say w, of e distinct from u in G_2 is a boundary vertex; and

(c) there exists $f \in A$ such that for G_1 (resp. G_2) f one has end $\hat{v}$ (resp. w) and one end in x_i.

Proof. By Proposition 2.5.6, $u \notin T_1$ and by Proposition 2.5.1 $u \notin T_2$. Then (a) follows by Proposition 2.4.12. Let w denote the end of e in G_2 that is distinct from u. Then w is not an interior vertex, for otherwise by (a), e is incident to both u and w, a contradiction as e is incident to $\hat{v}$. Hence, (b) holds. Finally, (c) follows from (a) and (b) and the fact that H_1, H_2 are closed (see Proposition 2.4.9). □

2.5.3 Classification

We can now exclude the case of the widgets, namely,

Proposition 2.5.8. *Assume (h1)-(h6). Then H_1, H_2 is not a widget pair, and if it is a proper wheel pair, then it has at least 4 parts.*

Proof. Because of Proposition 2.5.7 there exists an edge of A in G_2 with both ends in the boundary of H_2. Recall that each part R_i is a complete graph. Hence, there exists f that is parallel to e. Note that e, f is a diamond of (G_2, Σ_2). If H_1, H_2 is a widget or a wheel with 3 parts, then the boundary vertices are $\{x_1, x_2, x_3, z_1\}$, and then e, f have both ends in T_2. But, this contradicts Proposition 2.4.8(b). □

Combining the previous result with Proposition 2.5.2 and Proposition 2.5.3 yields,

Proposition 2.5.9. *Assume (h1)-(h6), and suppose we have outcome (a) or (b) of Proposition 2.4.13. Then we may assume H_1, H_2 is a proper wheel pair and it has at least 4 parts.*

In light of the previous result, we define (h7).

(h7) H_1, H_2 is a proper wheel pair with $k \geq 4$ parts.

Assume now (h1)-(h7). We wish to classify proper wheel pairs H_1, H_2. First we require a definition. Denote by $R_1, \ldots, R_k$ the parts of the wheel pair H_1, H_2. The boundary vertices of H_1 are $\{z_1, \ldots, z_k\}$. Because of Proposition 2.5.4 for each $i \in [k]$ there exists an edge f_i of H_2 with ends x_i, y_i. We define the mapping $\Theta : \{z_1, \ldots, z_k\} \to E(H_2)$ where $\Theta(z_i) = f_i$.

Proposition 2.5.10. *Assume (h1)-(h7) and let S denote the boundary vertices of H_1 that are not in T_1. Then for every $v \in S$ there exists an edge $e \in A$ such that $e, \Theta(v)$ is a diamond of (G_2, Σ_2).*

Proof. We may assume that $v = z_1$. Then $\Theta(v) = f$ where f is the edge of H_2 with ends x_1, y_1. Let $U = V(R_1) - \{z_1\}$ and observe that U is a mate of $\{v\}$. Let $S_1 = \delta_{G_1}(v) \cap A$ and let $S_2 = \delta_{G_2}(U) \cap A$. By Proposition 2.4.12, we have $S_1 = S_2$. Moreover, by (h5), we know that $|S_1| \leq 1$. It follows by Proposition 2.5.5 that there exist edges $e_x, e_y \in A$ incident to x_1 and y_1 respectively in G_2. Because of Proposition 2.5.7 we may assume that both ends of e_x and e_y are in the boundary. As $|S_2| \leq 1$ it follows that $e_x = e_y$ and $e_x, \Theta(v)$ is the required diamond. □

Proposition 2.5.11. *Assume (h1)-(h7). Then $|T_1| \leq 6$.*

Proof. By Proposition 2.5.7 there exists an edge $e \in A$ that has both ends in the boundary of H_2. Let Q be an arbitrary path of H_2 that has the same ends as e. Then observe that Q is either a path of H_1 or the union of two path of H_1. Since $Q \cup e$ is an odd cycle of (G_2, Σ_2), by Theorem 2.3.4 $Q \cup e$ is a T_1-join of G_1 and the result follows. □

The next proposition reduces classifies the possible configurations for wheel pairs.

Proposition 2.5.12. *Assume (h1)-(h7) and let S denote the boundary vertices of H_1 that are not in T_1. Suppose H_1, H_2 is a proper wheel pair with $k \geq 4$ parts. Then exactly one of the following holds,* *(a)* $k = 5$, $S = \emptyset$ *and* $|T_1| = |T_2| = 6$.

(b) $k = 5$, $|S| = 2$, $|T_1| = 4$ *and* $|T_2| = 6$.

(c) $k = 6$, $|S| = 3$, $|T_1| = 4$ *and* $|T_2| = 6$.

Proof. First, we prove the following claim.

Claim 1.

(i) *For every* $v \in S$, *there exists a diamond of* (G_2, Σ_2) *where one of the edge is* $\theta(v)$.

(ii) *If* $S \neq \emptyset$ *then* $|T_1| = 4$.

(iii) $|S| \leq 3$.

(iv) *If* $S \neq \emptyset$ *then* $|T_2| \geq 6$.

(v) H_1, H_2 *is a wheel pair with* $k = |S| + |T_1| - 1$ *parts.*

Subproof. Proposition 2.5.10 implies (i). Proposition 2.4.8(a) and (i) then imply (ii). (iii) follows from Proposition 2.4.8(c) and (i). For (iv), note that, for $v \in S$, the ends of $\Theta(v)$ are contained in T_2. Hence, Proposition 2.4.8(b) and (i) imply (iv). Finally, (v) holds because the boundary vertices of H_1 are contained in $S \cup (T_1 - \hat{v})$. $\diamondsuit$

Consider first the case where $S = \emptyset$. Then, by the hypothesis and Claim 1(v), $k \geq 4$ and $k = |T_1| - 1$. It follows that $|T_1| \geq 6$ and thus by Proposition 2.5.11 that $|T_1| = 6$. Therefore, $k = 5$ and by Proposition 2.5.1 we have $|T_2| = 6$. This is outcome (a). Thus we may assume that $S \neq \emptyset$. It follows by Claim 1(ii) that $|T_1| = 4$. By Claim 1(v), $k = |S| + 3$ where $|S| \leq 3$ by Claim 1(iii). Claim 1(iv) imply that $|T_2| \geq 6$. Therefore, by Proposition 2.5.1 we must have $k \geq 5$. As $k = |S| + 3$, it follows that $|S| \in \{2, 3\}$. Since $k = |S| + 3$ and by

Proposition 2.5.1 it follows that $|T_2| = 6$. When $|S| = 2$ we have outcome (b) and when $|S| = 3$ we have outcome (c). □

2.5.4 Case analysis

Next we show that each outcome of Proposition 2.5.12 leads to a contradiction.

Proposition 2.5.13. *Case (a) in Proposition 2.5.12 does not occur.*

We illustrate this proof in Figure 2.6. G_1 and G_2 are the left and right graphs, respectively. The square vertices represent T_1 and T_2.

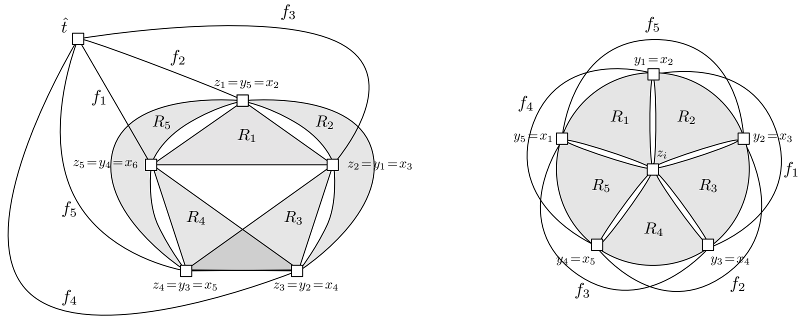

Figure 2.6: Case (a) in Proposition 2.5.12.

Proof of Proposition 2.5.13. Recall that case (a) is $k = 5$, $S = \emptyset$ and $|T_1| = |T_2| = 6$. The boundary vertices of H_2 which are distinct from $z_1 = \ldots = z_k$ are the *rim vertices*.

Claim. *If $e \in A$ then the ends of e in G_2 are non-consecutive rim vertices. In particular, $|A| \leq 5$.*

Subproof. Suppose u, v denote the ends of e in G_2. It follows from the fact that $|T_1| = 6$ and Proposition 2.4.8(a) that (G_2, Σ_2) has no diamond. Thus u, v are not consecutive rim vertices and u, v are not a rim vertex and the center vertex z_i. Therefore, by Proposition 2.5.7, u, v are all boundary vertices and the result follows. $\Diamond$

Since H_1, H_2 is a proper closed wheel pair we have, for both G_1, G_2 edges $e_i = (x_i, y_i)$ for all $i \in [5]$ where $5 + 1 = 1$. By the Claim and by Proposition 2.5.5 we may assume, up to symmetry, that we have edges $f_1, f_2, f_3 \in A$ where for G_2,

$$f_1 = (x_2, x_4), \quad f_2 = (x_3, x_5), \quad f_3 = (x_1, x_4).$$

Since $\{f_1, e_2, e_3\}$ is an odd cycle of (G_2, Σ_2) it follows that $\{f_1, e_2, e_3\}$ is a T_1-join of G_1. Therefore, $f_1 = (\hat{v}, x_1)$ in G_1. Similarly, we show that $f_2 = (\hat{v}, x_2)$ and $f_3 = (\hat{v}, x_3)$ in G_1. Denote by g the edge with ends x_4, y_4 in G_1, G_2. Note that $\{f_1, g\}$ is a path of both G_1 and G_2. It follows as G_1, G_2 are closed that there exists an edge f_4 that has ends $\hat{v}, x_4$ in G_1 and ends x_2, x_5 in G_2. Similarly, we also have an edge f_5 that has ends $\hat{v}, x_5$ in G_1 and x_1, x_3 in G_2. But then G_1, G_2 are both isomorphic to K_6, a contradiction. $\square$

Proposition 2.5.14. *Case (b) in Proposition 2.5.12 does not occur.*

We illustrate this proof in Figure 2.7. G_1 and G_2 are the left and right graphs, respectively. The square vertices represent T_1, T_2, and the shaded edges represent the signature of (G_1, Σ_1).

Proof of Proposition 2.5.14. Recall that case (b) is $k = 5$, $|S| = 2$, $|T_1| = 4$ and $|T_2| = 6$. Let v_1, v_2 denote the vertices of S. For all $i \in [5]$ denote by e_i in R_i with ends x_i, y_i. Then for $i \in [2]$, $\Theta(v_i) = e_i$. By Proposition 2.5.10 for all $i \in [2]$ there exists $f_i \in A$, for which e_i, f_i is a diamond of (G_2, Σ_2). Note that f_1, f_2 are independent edges of G_2, for otherwise as G_1, G_2 are closed, we would have an edge $f_3 \in H_2$ such that $\{f_1, f_2, f_3\}$ is a triangle, contradicting the fact that H_1, H_2 is a wheel pair. Thus we may assume that in G_1 and

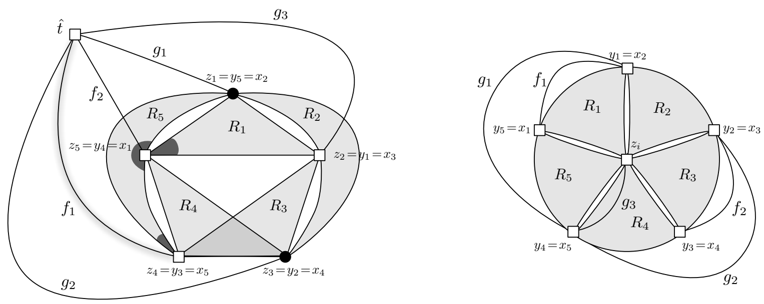

Figure 2.7: Case (b) in Proposition 2.5.12.

G_2, $e_1 = (x_1, y_1)$, $e_2 = (x_3, y_3)$. For $i \in [2]$, e_i, f_i is a diamond of (G_2, Σ_2) and hence by Theorem 2.3.4 a T_1-join of G_1. It follows that $T_1 = \{x_1, y_1 = x_3, y_3, \hat{v}\}$ and that $f_1 = (\hat{v}, y_3)$ and $f_2 = (\hat{v}, x_1)$ in G_1. Since $\{f_1, e_5\}$ is a path of G_1, G_2 there exists $g_1 \in A$ such that $\{g_1, f_1, e_5\}$ is a triangle of G_1 and G_2. Since $\{f_2, e_4\}$ is a path of G_1, G_2 there exists $g_2 \in A$ such that $\{g_2, f_2, e_4\}$ is a triangle of G_1 and G_2. Let h be the x_3, z_3 edge of G_1, G_2. Then as $\{h, g_2\}$ is a path of G_1, G_2 there exists g_3 such that $\{h, g_2, g_3\}$ is triangle of G_1, G_2. Define, $A_1 := \{f_1, f_2, g_1, g_2, g_3\}$. We have proved that $A \supseteq A_1$.

Claim 1.

(a) There is no edge $h \in A$ incident to x_1 (resp. z_5, z_4, y_3) in both G_1 and G_2.

(b) There is no edge $h \in A - \{g_1, g_2, g_3\}$ incident to any of $\{x_2, y_2, z_2\}$ in both G_1 and G_2.

Subproof. For (a), note that $f_2 \in A$ is incident to x_1 in G_1 but not in G_2. Suppose that we have $h \in A$ incident to x_1 in both G_1 and G_2. Then $f_2 \neq h$ and f_2, h are parallel in G_1. Since (G_1, Σ_1) is simple, h, f_2 is a diamond. It follows by Proposition 2.4.8(a) that $|T_2| \leq 4$, giving a contradiction. The cases for z_5, z_4, and y_3 are similar. For (b), it follows as in (a) for if we had such an edge h then g_i, h would be a diamond of (G_1, Σ_1) for some $i \in [3]$. $\diamondsuit$

Let A_2 denote the set of edges of A that have an end in the interior of R_2.

Claim 2. $A = A_1 \cup A_2$.

Subproof. Suppose for a contradiction that there exists an edge $h \in A - (A_1 \cup A_2)$. It follows from Claim 1 that h has an end that is an interior vertex u of R_i for some $i \in \{1, 3, 4, 5\}$. Let $L = \delta_{H_1}(u) = \delta_{H_2}(u)$. As $u \notin T_1$, $\delta_{G_1}(u) \supseteq L \cup h$. It follows from Proposition 2.4.10 that $\delta_{G_2}(u) = \delta_{G_1}(u)$. In particular, e is incident to u in G_2. But then, for all $i \in \{1, 3, 4, 5\}$, e together with Proposition 2.5.7 contradicts Claim 1(b). ♢

By Theorem 2.3.4, $\Gamma = \delta_{G_2}(x_1{=}y_5) \, \Delta \, \delta_{G_2}\Big(V(R_5) - \{x_5, y_5, z_5\}\Big)$ is a signature of (G_1, Σ_1) where all edges of Γ are incident to y_4 or z_4. Hence, y_4, z_4 is an adjacent blocking pair of (G_1, Σ_1), a contradiction. □

Proposition 2.5.15. *Case (c) in Proposition 2.5.12 does not occur.*

We illustrate this proof in Figure 2.8. G_1 and G_2 are the left and right graphs, respectively. The square vertices represent T_1 and T_2.

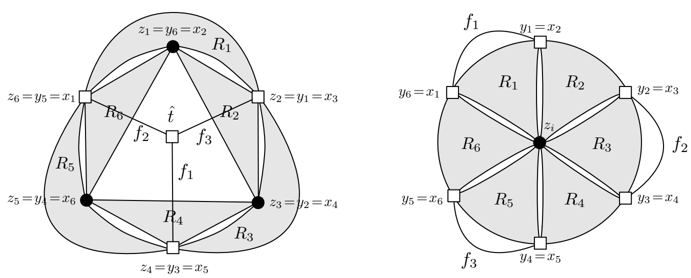

Figure 2.8: Case (c) in Proposition 2.5.12.

Proof of Proposition 2.5.15. Recall that case (c) is $k = 6$, $|S| = 3$, $|T_1| = 4$ and $|T_2| = 6$. Let v_1, v_2, v_3 denote the vertices of S. For all $i \in [3]$ denote by e_i the edge $\Theta(v_i)$. By

Proposition 2.5.10 for all $i \in [3]$ there exists $f_i \in A$, for which e_i, f_i is a diamond of (G_2, Σ_2). As in the proof of Proposition 2.5.14 we argue that f_1, f_2, f_3 are independent edges of G_2. Thus we may assume that in G_1 and G_2,

$$e_1 = (x_1, y_1), \quad e_2 = (x_3, y_3), \quad e_3 = (x_5, y_5).$$

Proposition 2.4.8(a) implies that for all $i \in [3]$, e_i, f_i is a matching of T_1. In particular, the ends of e_i are in T_1. It follows that $T_1 = \{x_1{=}y_5, y_1{=}x_3, y_3{=}x_5, \hat{v}\}$. Hence, in G_1, we have

$$f_1 = (\hat{v}, y_3), \quad f_2 = (\hat{v}, x_1), \quad f_3 = (\hat{v}, y_1).$$

Next, we show that f_1, f_2, f_3 are the only edges in A.

Claim. $A = \{f_1, f_2, f_3\}$.

Subproof. Suppose for a contradiction that there exists $g \in A - \{f_1, f_2, f_3\}$. We may assume, by Proposition 2.5.7, that g has both ends in the boundary of G_2. Up to symmetry, there are two cases to consider: (i) $g = (y_1, z_1)$ in G_2 and (ii) $g = (y_1, x_6)$ in G_2. For (i), let h be the edge of G_2 with ends x_2, y_2. Then $\{g, h\}$ is an odd cycle of (G_2, Σ_2) and hence by Theorem 2.3.4 a T_1-join of G_1. As $|T_1| = 4$ both ends of h must have ends in T_1 in G_1, giving a contradiction. For (ii), let h be the edge of G_2 with ends x_6, y_6. Then $\{g, h, e_1\}$ is an odd cycle of (G_2, Σ_2) and hence a T_1-join of G_1. As $|T_1| = 4$ and e_1, f are independent in G_1, e, f_1 need to cover 3 vertices of T_1, giving a contradiction. ◊

Let H be the graph obtained from G_1 by moving f_1 so that it has ends z_6, z_2; moving f_2 so that it has ends z_2, z_4; moving f_3 so that it has ends z_4, z_6. Then $\text{cut}(H) = \text{ecut}(G_1, T_1)$. It follows that G_1, G_2 are cographic siblings, giving a contradiction. □

2.6 The pinch case

We have now shown the following.

Proposition 2.6.1. *Assume (h1)-(h6). Then there exists H_2' equivalent to H_2 such that H_1 is obtained from H_2' by identifying two vertices.*

Proof. It suffices to show that case (a) and (b) of Proposition 2.4.13 do not occur. By Proposition 2.5.9, case (a) does not occur, and if (b) occurs, then H_1, H_2 is a proper wheel pair and it has at least 4 parts. Then one of cases (a), (b), and (c) of Proposition 2.5.12 must occur. But we showed in Proposition 2.5.13, Proposition 2.5.14, and Proposition 2.5.15 that none of these cases are possible. □

In light of the previous result, we define the following:

(h8) H_1 is obtained from H_2' by identifying distinct vertices s, t,

(h9) z denotes the vertex of H_1 corresponding to $s = t$.

Proposition 2.6.2. *Assume (h1)-(h6), (h8), and (h9).*

(a) If X is a 2-separation of H_2' then $|\mathcal{I}_{H_2'}(X) \cap \{s, t\}| = 1$ and $\partial_{H_2'}(X) \cap \{s, t\} = \emptyset$.

(b) There exists a 2-separation X of H_2'.

Proof. (a) follows from the fact that H_1 is 3-connected as G_1 is 4-connected. Suppose for a contradiction that (b) does not hold. Then, $H_2' = H_2$. It then follows that $T_2 = \{s, t\}$ since $\mathrm{cut}(H_1) = \mathrm{ecut}(H_2', \{s, t\})$, a contradiction to Proposition 2.4.3. □

In light of the previous proposition we can find, (p1) a 2-separation Y of H_2' where $s \in \mathcal{I}_{H_2'}(Y)$, $t \in \mathcal{I}_{H_2'}(E(H_2') - Y)$, and

(p2) subject to (p1) we pick Y that is inclusion-wise minimal.

Proposition 2.6.3. *Assume (h1)-(h6), (h8), and (h9), and assume Y satisfies (p1) and (p2).*
For every 2-separation X of H_2' we have $Y \subseteq X$ or $Y \subseteq E(H_2') - X$.

Proof. Otherwise $X \cap Y$ is a 2-separation of H_2' where $X \cap Y \subset Y$, contradicting (p2). ☐

We will require the following observation,

Proposition 2.6.4. *Assume (h1)-(h6), (h8), and (h9), and assume Y satisfies (p1) and (p2). Then, $T_2 \cap \mathcal{I}_{H_2'}(Y) = \{s\}$.*

Proof. Observe that $\mathrm{cut}(H_1) = \mathrm{ecut}(H_2', \{s,t\})$ as H_1 is obtained from H_2' by identifying s and t. By Proposition 2.4.9 (c), $\mathrm{cut}(H_1) = \mathrm{ecut}(H_2, T_2)$. Thus, $\mathrm{ecut}(H_2', \{s,t\}) = \mathrm{ecut}(H_2, T_2)$. By Remark 1.2.3, this implies that there exists a $\{s,t\}$-join J of $(H_2', \{s,t\})$ that is also a T_2-join of (H_2, T_2). Let P denote an st-path of H_2'. Then, P is a $\{s,t\}$-join of $(H_2', \{s,t\})$, and $J \,\Delta\, P$ is a cycle of H_2'. Thus, $J \,\Delta\, P$ is also a cycle of H_2, and $P = (P \,\Delta\, J) \,\Delta\, J$ is a T_2-join of (H_2, T_2). It follows from Proposition 2.6.3 that $H_2'[Y] = H_2[Y]$. In particular, $H_2'[P \cap Y] = H_2[P \cap Y]$, which implies $T_2 \cap \mathcal{I}_{H_2'}(Y) = \{s\}$. ☐

Proposition 2.6.5. *Assume (h1)-(h6), (h8), (h9).*
Then $z, \hat{v}$ is a blocking pair of (G_1, Σ_1). In particular, $z, \hat{v}$ are not adjacent in G_1.

Proof. By Proposition 2.6.4, $s \in T_2$. It follows from Theorem 2.3.4 that $B := \delta_{G_2}(s)$ is a signature of (G_1, Σ_1). Then observe that all edges of B are incident to z or $\hat{v}$. Finally, as G_1, G_2 are a counterexample (G_1, Σ_1) has no adjacent blocking pair. ☐

Proposition 2.6.6. *Assume (h1)-(h6), (h8), (h9).*
Then $z \notin T_1$ and there exists $\Omega \in A$ such that $\delta_{G_2}(s) \cap A = \delta_{G_2}(t) \cap A = \{\Omega\}$.

Proof. Observe that $\{s, t\}$ is a mate of $\{z\}$. Let $S_1 = \delta_{G_1}(z) \cap A$ and let $S_2 = \delta_{G_2}(\{s, t\}) \cap A$. By Proposition 2.6.5, $z, \hat{v}$ are not adjacent in G_1. Hence, $S_1 = \emptyset$. Suppose for a contradiction that $z \in T_1$. It then follows by Proposition 2.4.12 that $A = S_1 \,\Delta\, S_2 = S_2$. Hence, every edge of A has exactly one end in $\{s, t\}$ in G_2. In particular, s, t is a blocking pair of (G_2, Σ_2). Denote by A' the edges in $A \cap \delta_{G_2}(s)$. Since $s \in T_2$, Theorem 2.3.4 implies that $A' \cup \delta_{H_2}(s)$ is a signature of (G_1, Σ_1). But now a blocking pair $z, \hat{v}$ of (G_1, Σ_1), a blocking pair s, t of (G_2, Σ_2), and A' contradict Proposition 2.4.7. Thus $z \notin T_1$ and it follows from Proposition 2.4.12 that $S_2 = S_1 = \emptyset$. Hence, edges of A have both or none of their ends in $\{s, t\}$. Let $\hat{A}$ be the set of edges of A that have ends r and s in G_2. By Theorem 2.3.4 $\delta_{G_2}(r)$ is a signature of (G_1, Σ_1). Then $\hat{A} \neq \emptyset$ for otherwise z is a blocking vertex of (G_1, Σ_1), a contradiction to Proposition 2.4.3. Moreover, all edges of $\hat{A}$ have the same parity in (G_2, Σ_2) as A is a signature. Thus as (G_2, Σ_2) is simple, $\hat{A}$ contains a unique edge Ω. □

Proposition 2.6.7. *Assume (h1)-(h6), (h8), (h9) and Y satisfying (p1), (p2). Then $T_1 \cap \mathcal{I}_{H_1}(Y) = \emptyset$.*

Proof. Consider the edge Ω in Proposition 2.6.6.

Claim 1. *Ω does not have an end in $\mathcal{I}_{H_1}(Y)$.*

Subproof. Suppose for a contradiction Ω has end $u \in \mathcal{I}_{H_1}(Y)$. Proposition 2.4.9 imply that H_1, H_2 are closed. Then Proposition 2.6.3 implies that there is an edge f with ends u, z in H_1 and ends u, s in H_2. Note that Ω is then incident to f in both G_1 and G_2. Since G_1, G_2 are closed, there exists a triangle $\{f, g, \Omega\}$ of G_1, G_2. But then $z, \hat{v}$ are joined by g in G_1 contradicting Proposition 2.6.5. ◊

Let P denote an st-path of H_2. It follows by Proposition 2.6.6 that $P \cup \{\Omega\}$ is an odd cycle of (G_2, T_2). Hence, by Theorem 2.3.4, $P \cup \{\Omega\}$ is a T_1-join of G_1. Proposition 2.6.3 implies that $H_1[P] = H_2[P]$. Together with Claim 1, this completes the proof. $\square$

Assume now (h1)-(h6), (h8), (h9) and that we have Y as in (p1), (p2). We will now derive a contradiction, thereby completing the proof of Theorem 2.3.3. Denote by a, b the vertices in $\partial_{H_2}(Y)$ and let $Z := Y \cup \{\Omega\}$. Note that $s \in \mathcal{I}_{G_2}(Z)$. To derive a contradiction we will show that Z is a 3-separation of G_2. By possibly interchanging the role of s and t in the previous arguments, we may assume that $\mathcal{I}_{G_2}(E(G_2) - Z) \neq \emptyset$. We have $\partial_{G_2}(Z) \supseteq \{a, b, t\}$ and it suffices to show that equality holds. Suppose for a contradiction we have an edge $e \in A - \{\Omega\}$ that has end $u_2 \in \partial_{G_2}(Z)$. By Proposition 2.6.6, $u_2 \neq s$. We have vertex u_1 of G_1 and $\{u_2\}$ is a mate of $\{u_1\}$. By Proposition 2.6.7, $u_1 \notin T_1$ and thus by Proposition 2.4.12, e is also incident to u_1 in G_1. As H_1, H_2 are closed by Proposition 2.4.9 there exists an edge an edge f with ends u_1, z in H_1 and ends u_2, s in H_2. As G_1, G_2 are closed, there then exists an edge g such that $\{e, f, g\}$ is a triangle of G_1 and G_2. However, g has ends z and $\hat{v}$, contradicting Proposition 2.6.5. Hence, we have proved Theorem 2.3.3 that characterizes 4-connected closed siblings that are neither graphic nor cographic.

Chapter 3

Bounding the number of representations

The work in this chapter appears in [20, 21, 31]. Let us restate Theorem 1.1.1.

Theorem 1.1.1. *Any two graph representations of a graphic matroid are equivalent.*

Theorem 1.1.1 implies that every graphic matroid has a unique equivalence class of graph representations, equivalently, every cographic matroid has a unique equivalence class of graph representations. Since every graph representation of a 3-connected graphic (resp. cographic) matroid has no 2-separation, there exists at most one graph representation for a 3-connected matroid.

Alas, as seen in Section 1.3.1 and Section 1.3.2, 3-connected even-cycle (resp. even-cut) matroids can have exponentially many pair-wise inequivalent blocking pair (resp. T_4) representations. Recall that a signed-graph representation is a *blocking-pair representation* if it has a blocking pair. A graft representation is a T_4 representation if its terminal set contains at most 4 vertices.

Recall the three theorems from Section 1.3 that give a polynomial bounds for each of non-pinch-graphic even-cycle, non-pinch-cographic even-cut and $(4,5)$-connected pinch-graphic matroids, respectively.

Theorem 1.3.1. *There exists a constant c such that every even-cycle matroid that is not pinch-graphic has fewer than c pairwise inequivalent signed-graph representations.*

Theorem 1.3.2. *There exists a constant c such that every even-cut matroid that is not pinch-cographic has fewer than c pairwise inequivalent graft representations.*

Theorem 1.3.3. *Let M be a pinch-graphic matroid that is not graphic. If M is $(4,5)$-connected then the number of blocking-pair representations of M is in $\mathcal{O}(|E(M)|^4)$.*

To prove Theorem 1.3.1, we use the stabilizer theorem for even-cycle matroids proven by Guenin, Pivotto and Wollan [26], which will be stated in Section 3.1.1. Then, we characterize 1- and 2-separations in signed-graph representations of non-pinch-graphic even-cycle matroids in Section 3.1.3. In Section 3.1.4, we prove Theorem 1.3.1.

To prove Theorem 1.3.2, we use similar steps as above. We state the stabilizer theorem for even-cut matroids [25] in Section 3.2.1, and we characterize, in Section 3.2.3, 1- and 2-separations in graft representations of non-pinch-cographic even-cut matroids. As a result, we prove the theorem in Section 3.2.4.
A chain theorem obtained by combining existing results will be presented in Section 3.3.1. We review connectivity in even-cycle and even-cut matroids in Section 3.3.2. We will prove a strong version of Theorem 1.3.3 in Section 3.3.3 by using two key lemmas that are proved in Section 3.3.4.

3.1 Even-cycle matroids

3.1.1 The 3-connected case

The goal of this section is to prove Theorem 1.3.1. First, we will prove Theorem 1.3.1 for the special case where the even-cycle matroid is 3-connected.

A binary matroid is *minimally non-pinch-graphic* if it is not pinch-graphic, but every proper minor is. Minor-closed classes of binary matroids are well-quasi ordered [13]. Hence,

Theorem 3.1.1. *There exists a constant c, such that every minimally non-pinch-graphic (resp. minimally non-pinch-cographic) matroid has at most c elements.*

For a matroid N, a connected component of N is a maximal subset F of $E(N)$ such that, for every pair of edges in F, there exists a circuit of N containing both of them. We denote by $\lambda_1(N)$ the number of connected components of N. Now, N can be constructed from a collection $\Lambda_2(N)$ of 3-connected matroids by 1-sum and 2-sum. Cunningham and Edmonds [8] showed that $\Lambda_2(N)$ is unique up to isomorphism. Let $\lambda_2(N)$ be the number of matroids in $\Lambda_2(N)$.

Theorem 3.1.2 (Lemos and Oxley [34])**.** *Let N be a non-empty matroid and M be a minor-minimal 3-connected matroid having N as a minor. Then $|E(M)| - |E(N)| \leq 22\,(\lambda_1(N) - 1) + 5\,(\lambda_2(N) - 1)$.*

Note that, for an even-cycle matroid M, the set of signed-graph representations for M can be partitioned into equivalence classes.

Theorem 3.1.3 (Guenin, Pivotto, and Wollan [27])**.** *Let M be a 3-connected matroid and let N be a 3-connected minor of M that is not pinch-graphic. Then there exists a matroid*

$\tilde{N}$ isomorphic to N that is a minor of M such that for every equivalence class $\mathcal{F}$ of $\tilde{N}$, the set of extensions of $\mathcal{F}$ to M is the union of at most two equivalence classes.

Given a binary matroid M we denote by $f(M)$ the number of pairwise inequivalent signed-graph representations of M (thus M is an even-cycle matroid exactly when $f(M) \geq 1$).

Let us now restate and prove Theorem 1.3.1 assuming 3-connectivity.

Theorem 3.1.4. *There exists a constant d such that for every 3-connected even-cycle matroid M that is not pinch-graphic, $f(M) \leq d$.*

Proof. Since M is not pinch-graphic, it has a minor N that is minimally non-pinch-graphic. By Theorem 3.1.1, $|E(N)| \leq c$ for some constant c. In particular, $\lambda_1(N), \lambda_2(N) \leq c$. Let N' be a minor-minimal matroid with the following properties:

(a) N' is 3-connected,

(b) N is a minor of N' and

(c) N' is a minor of M.

Since $M = N'$ satisfies (a)-(c), N' is well-defined. By Theorem 3.1.2,

$$|E(N')| \leq c + 22(c-1) + 5(c-1) \leq 28c.$$

Thus N' has a constant number, say c', of equivalence classes. It follows by Theorem 3.1.3 that there are at most $2c'$ equivalence classes for M, i.e. $f(M) \leq 2c' =: d$ as required. □

3.1.2 A connectivity function and auxiliary graphs

Recall that, for a matroid M, the *connectivity function* takes $X \subseteq E(M)$ as input and returns $\lambda_M(X) := r_M(X) + r_M(E(M) - X) - r(M)$. In this section, we wish to specialize this function to the case of even-cycle and even-cut matroids. Given a graph H we denote by $\kappa(H)$ the number of components of H. A signed graph is *bipartite* if it has no odd cycle. Given a signed graph (G, Σ), we define

$$p[(G,\Sigma)] := \begin{cases} 0 & \text{if } (G,\Sigma) \text{ is bipartite} \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.1.5 ([27], Lemma 26)**.** *Consider an even-cycle matroid M with a non-bipartite connected signed-graph representation (G, Σ). Let X, Y be a partition of $E(M)$ where X, Y are non-empty. Then*

$$\lambda_M(X) = |\,\partial_G(X)| - \kappa(G[X]) - \kappa(G[Y]) + p[(G,\Sigma) \setminus X] + p[(G,\Sigma) \setminus Y].$$

Consider a graph G and a set $X \subseteq E(G)$ where $X \neq \emptyset$ and $X \neq E(G)$. We define the *auxiliary graph H for the pair G and X* as follows: H is bipartite with bipartition U, W where vertices in U correspond to components of $G[X]$ and vertices in W correspond to components in $G[E(G) - X]$. For every $v \in \partial_G(X)$ we have an edge e_v of H with endpoints $u \in U$ and $w \in W$ where u corresponds to the unique component of $G[X]$ containing v and w corresponds to the unique component of $G[E(G) - X]$ containing v. We give an example in Figure 3.1. For each of (i), (ii), (iii) we have the auxiliary graph H on top and G where the non-shaded region correspond to edges in X on the bottom.

Let us restate Proposition 3.1.5 in terms of auxiliary graph,

Proposition 3.1.6. *Consider an even-cycle matroid M with a non-bipartite connected signed-graph representation (G, Σ). Let X, Y be a partition of $E(M)$ where X, Y are*

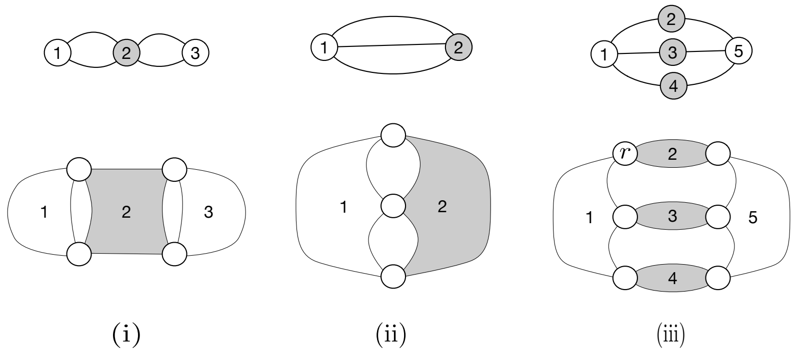

Figure 3.1: Examples of auxiliary graphs

non-empty. Denote by H the auxiliary graph for the pair G and X. Then

$$|E(H)| = |V(H)| + \lambda_M(X) - p[(G, \Sigma) \setminus X] - p[(G, \Sigma) \setminus Y] \geq |V(H)| - 1.$$

Proof. Note that $|V(H)| = \kappa(G[X]) + \kappa(G[Y])$ and $|E(H)| = |\,\partial_G(X)|$. So, Proposition 3.1.5 implies that

$$\lambda_M(X) = |E(H)| - |V(H)| + p[(G, \Sigma) \setminus X] + p[(G, \Sigma) \setminus Y].$$

Finally, as G is connected, so is H. Thus $|E(H)| \geq |V(H)| - 1$ and the result follows. □

A graph obtained from two disjoint polygons C_1, C_2 by identifying a vertex of C_1 with a vertex of C_2 is a *double ear*. Recall that a graph that consists of three internally disjoint ab-paths P_1, P_2, P_3 (all vertices except a, b have degree two) is a *theta*. The auxiliary graph in Figure 3.1 (i) is a double ear, and the auxiliary graphs in Figure 3.1 (ii), (iii) are thetas.

Remark 3.1.7. *If H is a 2-edge-connected graph where $|E(H)| = |V(H)| + 1$ then H is a theta or a double ear.*

3.1.3 1- and 2-separations in signed graphs

Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 1$. Suppose that $E_1 \cap E_2 = \emptyset$. Then, we define the 1-*sum* M of M_1, M_2, denoted by $M_1 \oplus_1 M_2$, as follows: the ground set of M is $E := E_1 \cup E_2$ and a subset C of E is a circuit of M if and only if C is either a circuit of M_1 or a circuit of M_2.

Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 3$. Suppose that $E_1 \cap E_2 = \{\Omega\}$ and that Ω is neither a loop nor a coloop of M_i for $i \in [2]$. Then, we define the *2-sum* M of M_1, M_2, denoted by $M_1 \oplus_2 M_2$, as follows: the ground set of M is $E := E_1 \,\Delta\, E_2$ and a subset C of E is a circuit of M if and only if either C is a circuit of $M_1 \setminus \Omega$ or $M_2 \setminus \Omega$, or $C = C_1 \,\Delta\, C_2$ where for $i \in [2]$, C_i is a circuit of M_i containing Ω.

Let M be a connected matroid and let X be a 2-separation of M. Then $M = M_1 \oplus_2 M_2$ for some matroids M_1, M_2 where $X = E(M_1) - E(M_2)$ [44], (2.6). We would like to describe how M_1 and M_2 arise from the separation X for the case of binary matroids.

We first require the following folklore observation,

Proposition 3.1.8. *Let M be a matroid with matrix representation A and let $X \subseteq E(M)$. We denote by $\langle X \rangle$ the vector space spanned by the columns of A indexed by X. Then*

$$\lambda_M(X) = \dim\left[\langle X \rangle \cap \langle E(M) - X \rangle\right].$$

Let M be a binary matroid with matrix representation A, and let X be a 2-separation of M. Then, $\lambda_M(X) = 1$. It follows from Proposition 3.1.8 that $\dim\left[\langle X \rangle \cap \langle E(M) - X \rangle\right] = 1$. Thus there exists a unique non-zero $0,1$ vector p for which $\langle p \rangle = \langle X \rangle \cap \langle E(M) - X \rangle$. Let A^+ be obtained from matrix A by adding column p and let N be the binary matroid represented by matrix A^+. Then N is the *completion of* M with respect to X.

Proposition 3.1.9. *Let M be a binary matroid with a 2-separation X. Let N be the completion of M with respect to X. Then $M = (N \setminus X) \oplus_2 (N \setminus E(M) - X)$.*

Proof. Let $M_1 = N \setminus X$ and $M_2 = N \setminus E(M) - X$. Denote by Ω the unique element in $E(M_1) \cap E(M_2)$. It suffices to show that the following statements are equivalent,

(1) C is a circuit of M where $C \cap X, C - X \neq \emptyset$,

(2) $C = C_1 \Delta C_2$ where for $i \in [2]$, C_i is a circuit of M_i using Ω.

Let A denote the 0, 1 matrix representation of N. Suppose (1) holds. Let $p = \sum(A_j : j \in C \cap X) = \sum(A_j : j \in C - X)$. Then $\emptyset \neq p \in \langle X \rangle \cap \langle E(M) - X \rangle$ and thus $p = A_\Omega$. Hence, $C_1 = (C \cap X) \cup \Omega$ is a circuit of M_1 and $C_2 = (C - X) \cup \Omega$ is circuit of M_2 satisfying (2). Suppose (2) holds. Then $\sum(A_j : j \in C \cap X) = \sum(A_j : j \in C - X) = A_\Omega$. Thus $\sum(A_j : j \in C) = \mathbf{0}$, i.e. C is a cycle of M. As C_1, C_2 are circuits so is C. As $A_\Omega \neq \mathbf{0}$ we have $C \cap X, C - X \neq \emptyset$, i.e. (1) holds. □

The following straightforward observation will allow us to construct completions,

Remark 3.1.10. *Let M be a binary matroid with a 2-separation X. Let N be a binary matroid where $M = N \setminus \Omega$ for some Ω that is not a loop of N. If we have cycles C and D of N where $\Omega \in C \cap D$ and $C \subseteq X \cup \Omega$, $D \subseteq (E(M) - X) \cup \Omega$ then N is the completion of M with respect to X.*

Here we apply Proposition 3.1.9 to even-cycle and even-cut matroids.

Proposition 3.1.11. *Let $M = \text{ecycle}(G, \Sigma)$ with a 2-separation X and let $Y = E(M) - X$. Suppose that $p[(G, \Sigma) \setminus X] = p[(G, \Sigma) \setminus Y] = 1$. Let (G_1, Σ_1) be obtained from $(G, \Sigma) \setminus Y$ by adding an odd loop Ω and let (G_2, Σ_2) be obtained from $(G, \Sigma) \setminus X$ by adding an odd loop Ω. Then $M = \text{ecycle}(G_1, \Sigma_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2)$.*

Proof. Let (H, Γ) be the signed-graph obtained from (G, Σ) by adding an odd loop Ω. Note, Ω is not a loop of $\text{ecycle}(H, \Gamma)$ as $\Omega \in \Gamma$. Since $(G, \Sigma) \setminus X$ and $(G, \Sigma) \setminus Y$ are non-bipartite there exists odd polygons $C_1 \subseteq Y$ and $C_2 \subseteq X$. Then $C_1 \cup \Omega$ and $C_2 \cup \Omega$ are even-cycles of (H, Γ). By Remark 3.1.10 $\text{ecycle}(H, \Gamma)$ is the completion of M with respect to X. By Proposition 3.1.9, $M = (N \setminus Y) \oplus_2 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecycle}(H, \Gamma) \setminus Y = \text{ecycle}(G_1, \Sigma_1)$ and similarly, $N \setminus X = \text{ecycle}(G_2, \Sigma_2)$. □

In Proposition 3.1.11, we say that (G, Σ) is obtained from (G_1, Σ_1) and (G_2, Σ_2) by *summing on a loop*.

Proposition 3.1.12. *Let* $M = \text{ecycle}(G, \Sigma)$ *with a* 2*-separation* X *and let* $Y = E(M) - X$*. Suppose that* $p[(G, \Sigma) \setminus Y] = 1$*,* $p[(G, \Sigma) \setminus X] = 0$ *that* $G \setminus X$*,* $G \setminus Y$ *are connected and that* $\partial_G(X) = \{a, b\}$ *where* a, b *are distinct vertices. Then we may assume, after possibly re-signing, that* $\Sigma \subseteq X$*. Let* G_1 *(resp.* G_2*) be obtained from* $G \setminus Y$ *(resp.* $G \setminus X$*) by adding edge* $\Omega = (a, b)$*. Then* $M = \text{ecycle}(G_1, \Sigma) \oplus_2 \text{cycle}(G_2)$.

Proof. As $p[(G, \Sigma) \setminus X] = 0$ we may assume $\Sigma \subseteq X$. Let (H, Σ) be the signed-graph obtained from (G, Σ) by adding edge $\Omega = (a, b)$. Note, Ω is not a loop of $\text{ecycle}(H, \Sigma)$ as it is not a loop of H. Since $(G, \Sigma) \setminus Y$ is connected and non-bipartite, there exists an $\{a, b\}$-join J of $G \setminus Y$ where $|J \cap \Sigma|$ is even. Since $G \setminus X$ is connected there exists an ab-path P of $G \setminus X$. Then $J \cup \Omega$ and $P \cup \Omega$ are even cycles of (H, Σ). It follows by Remark 3.1.10 that $\text{ecycle}(H, \Sigma)$ is the completion of M with respect to X. By Proposition 3.1.9, $M = (N \setminus Y) \oplus_2 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecycle}(H, \Sigma) \setminus Y = \text{ecycle}(G_1, \Sigma)$ and $N \setminus X = \text{ecycle}(H, \Sigma) \setminus X = \text{cycle}(G_2)$. □

In Proposition 3.1.12, we say that (G, Σ) is obtained from (G_1, Σ) and G_2 by *summing on an edge*.

3.1.4 The proof of Theorem 1.3.1

Because of Theorem 3.1.4 and the fact that every matroid M can be constructed from a collection of 3-connected matroids by 1-sums and 2-sums, it suffices to prove the following Proposition 3.1.14 and Proposition 3.1.15 to complete the proof of Theorem 1.3.1.

Consider an even-cycle matroid M and a set $X \subseteq E(G)$ where $X \neq \emptyset$ and $X \neq E(G)$. A connected signed-graph representation (G, Σ) of M is *extremal* for X if among all connected signed-graph representations of M that are equivalent to (G, Σ), the auxiliary graph for G and X has fewest number of vertices. (Note that if (G, Σ) is a signed-graph representation of M then there is an equivalent signed-graph representation that is connected.)

We leave the following as an easy exercise.

Remark 3.1.13. *Let (G, Σ) be an extremal signed-graph representation of an even-cycle matroid M for some $X \subseteq E(M)$. Then the auxiliary graph H for G and X is 2-edge-connected unless H consists of two vertices joined by a single edge.*

Recall that, for a binary matroid M, $f(M)$ is the number of pairwise inequivalent signed-graph representations of M.

Proposition 3.1.14. *If $M = M_1 \oplus_1 M_2$ for some binary matroids M_1, M_2 where M is not graphic then*

$$f(M) \leq \max\{f(M_1), f(M_2)\}.$$

Proof. Define $X = E(M_1)$ and $Y = E(M_2)$. Then $M_1 = M \setminus Y$, $M_2 = M \setminus X$ and $\lambda_M(X) = 0$. Since M is not graphic at least one of M_1 or M_2 is not graphic. Thus, we may assume that M_1 is not graphic. We may assume that $f(M) \geq 1$; otherwise, $f(M) = 0$ and the result holds. Let (G, Σ) be a signed-graph representation of M that is extremal for X.

Then, $(G, \Sigma) \setminus Y$ is a representation of M_1. In particular, as M_1 is not graphic, $(G, \Sigma) \setminus Y$ is not bipartite, i.e. $p[(G, \Sigma) \setminus Y] = 1$. Let H denote the auxiliary graph for G and X. Then by Proposition 3.1.6,

$$|E(H)| = |V(H)| - p[(G, \Sigma) \setminus X] - p[(G, \Sigma) \setminus Y] \geq |V(H)| - 1.$$

Hence, (i) $|E(H)| = |V(H)| - 1$ and (ii) $p[(G, \Sigma) \setminus X] = 0$. By (i) and Remark 3.1.13, we have $|V(H)| = 2$ and $|E(H)| = 1$, i.e. $G \setminus X$, $G \setminus Y$ are connected and share exactly one vertex in G. By (ii), we may assume $\Sigma \subseteq X$ and $M_2 = \text{cycle}(G \setminus X)$. Let (G', Σ') be any other signed-graph representation of M that is extremal for X. Then we may assume that $\Sigma' \subseteq X$ and $M_2 = \text{cycle}(G' \setminus X)$. Then $G \setminus X$ and $G' \setminus X$ are equivalent by Theorem 1.1.1. It follows that (G, Σ) and (G', Σ') are equivalent if and only if $(G, \Sigma) \setminus Y$ and $(G', \Sigma') \setminus Y$ are equivalent. Hence, $f(M) = f(M_1)$. □

Proposition 3.1.15. *If $M = M_1 \oplus_2 M_2$ for some binary matroids M_1, M_2 where M is not pinch-graphic then*

$$f(M) \leq \max\{f(M_1), f(M_2)\}.$$

Proof. Denote by Ω the unique element in $E(M_1) \cap E(M_2)$ and let $X = E(M_1) - \Omega$, $Y = E(M_2) - \Omega$. We have $\lambda_M(X) = 1$. Let (G, Σ) be a signed-graph representation of M that is extremal for X. Let H denote the auxiliary graph for G and X. Then by Proposition 3.1.6

$$|E(H)| = |V(H)| + 1 - p[(G, \Sigma) \setminus X] - p[(G, \Sigma) \setminus Y] \geq |V(H)| - 1. \tag{3.1}$$

Claim 1. $p[(G, \Sigma) \setminus X] + p[(G, \Sigma) \setminus Y] \geq 1$.

Subproof. Otherwise, by (3.1), we have $|E(H)| = |V(H)| + 1$. (G, Σ) is extremal for X. Thus, by Remark 3.1.13, H is 2-edge-connected. Remark 3.1.7 implies that H is a theta or a double ear. Consider first the case where H is a theta formed by st-paths P_1, P_2, P_3. As (G, Σ) is extremal for X and since H is bipartite, either (a) for $j \in [3]$, P_j consists of one edge, or (b) for $j \in [3]$, P_j consists of two edges. Case (a) is illustrated in Figure 3.1(ii) and case (b) is illustrated in Figure 3.1(iii). For both cases we may assume after re-signing that $\Sigma = \delta_G(r) \cap X$ where r is denoted in the figures. Hence, M is pinch-graphic, a contradiction (in fact, M is graphic). Consider the case where H is a double ear formed by polygons C_1, C_2. As (G, Σ) is extremal for X, C_1 and C_2 each consist of two parallel edges. This case is illustrated in Figure 3.1(a). We may assume after re-signing that $\Sigma = [\delta_G(r) \cup \delta_G(s)] \cap X$ where X is the non-shaded region and r, s are indicated in the figure. But then (G, Σ) has a blocking pair and M is pinch-graphic, a contradiction. $\Diamond$

Claim 2. *Suppose $p[(G, \Sigma) \setminus X] = 0$ and $p[(G, \Sigma) \setminus Y] = 1$. Then we may assume that $\Sigma \subseteq X$. Moreover, $M_1 = \mathrm{ecycle}(G_1, \Sigma)$, $M_2 = \mathrm{cycle}(G_2)$ where (G, Σ) is obtained from (G_1, Σ) and G_2 by a summing on edge Ω. In particular, M_2 is graphic.*

Subproof. By (3.1), $|E(H)| = |V(H)|$, and by Remark 3.1.13, H is a 2-connected graph with exactly one polygon. Since (G, Σ) is extremal for X we have $|V(H)| = |E(H)| = 2$, i.e. $G \setminus X$, $G \setminus Y$ are connected and share exactly two vertices, say u, v in G. Then the result holds by Proposition 3.1.12. $\Diamond$

Claim 3. *Suppose $p[(G, \Sigma) \setminus X] = p[(G, \Sigma) \setminus Y] = 1$. Then $M_1 = \mathrm{ecycle}(G_1, \Sigma_1)$, $M_2 = \mathrm{ecycle}(G_2, \Sigma_2)$ where (G, Σ) is obtained from (G_1, Σ_1) and (G_2, Σ_2) by a summing on loop Ω.*

Subproof. By (3.1), $|E(H)| = |V(H)| - 1$, and by Remark 3.1.13, we have $|V(H)| = 2$ and $|E(H)| = 1$, i.e. $G \setminus X$, $G \setminus Y$ are connected and share exactly one vertex in G. Then the result holds by Proposition 3.1.11. $\Diamond$

M_1 and M_2 are not both graphic for otherwise so would M, a contradiction. It follows from Claim 2 that we cannot have extremal representations (G, Σ) and (G', Σ') of M with $p[(G, \Sigma) \setminus X] = 0$, $p[(G, \Sigma) \setminus Y] = 1$ and $p[(G', \Sigma') \setminus X] = 1$, $p[(G', \Sigma') \setminus Y] = 0$. We may assume that M_1 is not graphic. Hence, because of Claim 1 (G, Σ) is of one of the following types,

Type 1. $p[(G, \Sigma) \setminus X] = 0$ and $p[(G, \Sigma) \setminus Y] = 1$ or

Type 2. $p[(G, \Sigma) \setminus X] = 1$ and $p[(G, \Sigma) \setminus Y] = 1$.

Let h_1 (resp. h_2) denote the number of inequivalent representations of M_1 with a non-loop Ω (resp. loop Ω). Let f_1 (resp. f_2) denote the number of inequivalent Type 1 (resp. Type 2) representations of M. Note, $f(M) = f_1 + f_2$ and $f(M_1) = h_1 + h_2$.

Claim 4. $f_1 \leq h_1$.

Subproof. Consider Type I representations (G, Σ) and (G', Σ') of M. Then (G, Σ) is obtained from (G_1, Σ) and G_2 by a summing on edge Ω and (G', Σ') is obtained from (G'_1, Σ') and G'_2 by a summing on edge Ω. As $M_2 = \text{cycle}(G_2) = \text{cycle}(G'_2)$ it follows from Theorem 1.1.1 that G_2 and G'_2 are equivalent. It follows that (G, Σ) and (G', Σ') are equivalent if and only if (G_1, Σ) and (G'_1, Σ') are equivalent. The result follows. $\diamondsuit$

Claim 5. $f_2 \leq h_2$.

Subproof. Consider Type 2 representations (G, Σ) and (G', Σ') of M. Then (G, Σ) is obtained from (G_1, Σ_1) and (G_2, Σ_2) by a summing on loop Ω and (G', Σ') is obtained from (G'_1, Σ'_1) and (G'_2, Σ'_2) by a summing on loop Ω. It follows from Remark 2.4.4 that (G_1, Σ_1) and (G'_1, Σ'_1) are equivalent and also that (G_2, Σ_2) and (G'_2, Σ'_2) are equivalent. Thus (G, Σ) and (G', Σ') are equivalent. It follows that $f_2 \leq 1$ and clearly, $f_2 = 0$ if $h_2 = 0$. $\diamondsuit$

Then $f(M) = f_1 + f_2 \leq h_1 + h_2 = f(M_1)$ as required. $\square$

3.2 Even-cut matroids

3.2.1 The 3-connected case

The goal of this section is to prove Theorem 1.3.2. First, we will prove Theorem 1.3.2 for the special case where the even-cut matroid is 3-connected.

A binary matroid is *minimally non-pinch-cographic* if it is not pinch-cographic, but every proper minor is. Note that, for an even-cut matroid M, the set of graft representations can be partitioned into equivalence classes.

Theorem 3.2.1 (Guenin, Pivotto [25], [40])**.** *Let M be a 3-connected matroid and let N be a 3-connected minor of M that is not pinch-cographic. Then there exists a matroid $\tilde{N}$ isomorphic to N that is a minor of M such that for every equivalence class $\mathcal{F}$ of $\tilde{N}$, the set of extensions of $\mathcal{F}$ to M is the union of at most two equivalence classes.*

Given a binary matroid M we denote by $g(M)$ the number of pairwise inequivalent graft representations of M (thus, M is an even-cut matroid exactly when $g(M) \geq 1$).

Theorem 3.2.2. *There exists a constant d such that for every 3-connected even-cut matroid M that is not pinch-cographic, $g(M) \leq d$.*

The proof is nearly identical to that of Theorem 3.1.4. It suffices to replace in that proof, pinch-graphic by pinch-cographic, and Theorem 3.1.3 by Theorem 3.2.1.

3.2.2 A connectivity function and auxiliary graphs

A graft is *eulerian* if it has no odd cut. Given a graft (G,T) we define,

$$q[(G,T)] := \begin{cases} 0 & \text{if } (G,T) \text{ is eulerian} \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.2.3 ([25])**.** *Consider an even-cut matroid M with a non-eulerian connected graft representation (G,T). Let X,Y be a partition of $E(M)$ where X,Y are non-empty. Then*

$$\lambda_M(X) = |\,\partial_G(X)| - \kappa(G[X]) - \kappa(G[Y]) + q[(G,T)/X] + q[(G,T)/Y].$$

Similarly, we can restate Proposition 3.2.3 in terms of the auxiliary graph.

Proposition 3.2.4. *Consider an even-cut matroid M with a non-eulerian connected graft representation (G,T). Let X,Y be a partition of $E(M)$ where X,Y are non-empty. Denote by H the auxiliary graph for the pair G and X. Then*

$$|E(H)| = |V(H)| + \lambda_M(X) - q[(G,T)/X] - q[(G,T)/Y] \geq |V(H)| - 1.$$

We omit the proof as it is similar to that of Proposition 3.1.6.

3.2.3 1- and 2-separations in grafts

In a graft (G,T) an edge e is a *pin* with *head* v if e has an endpoint $v \in T$ where v has degree 1. By *adding a pin* e to a graft (G,T) we mean adding a pendent edge $e=(u,v)$ to G (where v denotes the vertex of degree 1) and replacing the set of terminals by $T\,\Delta\{u,v\}$. Consider a graph G and let α be a subset of edges that are either incident to a fixed vertex

w or contained in loops. Let G' be obtained from G by replacing w with vertices w' and w'' such that (i) edges in $\alpha \cap \delta_G(w)$ are incident to w', (ii) edges in $\delta_G(w) - \alpha$ are incident to w'', and (iii) loops of G in α are joining w' and w''. Then G' is obtained from G by *splitting w according to α*. If in addition to (i)-(iii) we add an edge $\Omega = (w', w'')$ then the resulting graph is obtained from G by *uncontracting Ω at w according to α*. Next we state the analogue of propositions 3.1.11 and 3.1.12 for even-cuts.

Proposition 3.2.5. *Let $M = \text{ecut}(G,T)$ with a 2-separation X and let $Y = E(M) - X$. Suppose that $q[(G,T)/X] = q[(G,T)/Y] = 1$. Let (G_1,T_1) be obtained from $(G,T)/Y$ by adding a pin Ω and let (G_2,T_2) be obtained from $(G,T)/X$ by adding a pin Ω. Then $M = \text{ecut}(G_1,T_1) \oplus_2 \text{ecut}(G_2,T_2)$.*

Proof. Let (H,R) be the graft obtained from (G,T) by adding a pin Ω. Note, Ω is not a loop of $\text{ecut}(H,R)$ as Ω is a pin. There exist odd cuts C_1 and C_2 of $(G,T)/Y$ and $(G,T)/X$ respectively. Then $C_1 \cup \Omega$ and $C_2 \cup \Omega$ are even-cuts of (H,R). By Remark 3.1.10, $\text{ecut}(H,R)$ is the completion of M with respect to X. By Proposition 3.1.9, $M = (N \setminus Y) \oplus_2 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecut}(H,R) \setminus Y = \text{ecut}\left((H,R)/Y\right) = \text{ecut}(G_1,T_1)$. Similarly, $N \setminus X = \text{ecut}(G_2,T_2)$. □

In Proposition 3.2.5, we say that (G,T) is obtained from (G_1,T_1) and (G_2,T_2) by *summing on a pin*.

Proposition 3.2.6. *Let $M = \text{ecut}(G,T)$ with a 2-separation X and let $Y = E(M) - X$. Suppose that $q[(G,T)/Y] = 1$, $q[(G,T)/X] = 0$ that $G \setminus X$, $G \setminus Y$ are connected and that $\partial_G(X) = \{a,b\}$ where a,b are distinct vertices. Then, $T \subseteq V_G(X)$ and let G_1 (resp. G_2) be obtained from $G \setminus Y$ (resp. $G \setminus X$) by adding an edge Ω with endpoints a,b. Then $M = \text{ecut}(G_1,T) \oplus_2 \text{cut}(G_2)$.*

Proof. As $q[(G,T)/X] = 0$, we have $T \subseteq V_G(X)$. Let H be obtained from G by uncontracting Ω at a according to $X \cap \delta_G(a)$ where Ω has endpoints a, a' in H and a is incident to $X \cap \delta_G(a)$ and a' is incident to $\delta_G(a) \cap Y$. Since $G \setminus X$ and $G \setminus Y$ are connected Ω is not a bridge of H, in particular, Ω is not a loop of $\text{ecut}(H,T)$. Since $(G,T)/Y$ is non-eulerian, there exists an even cut $C_1 \subseteq X \cup \Omega$ of (H,T) where $\Omega \in C_1$. There exists a cut $C_2 \subseteq Y \cup \Omega$ of H where $\Omega \in C_2$. Note, C_2 is a T-even cut as $T \subseteq V_G(X)$. It follows by Remark 3.1.10 that $\text{ecut}(H,T)$ is the completion of M. By Proposition 3.1.9, $M = (N \setminus Y) \oplus_2 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecut}((H,T)/Y) = \text{ecut}(G_1,T)$ and $N \setminus X = \text{ecut}((H,T)/X) = \text{cut}(G_2)$. □

In Proposition 3.2.6, we say that (G,T) is obtained from (G_1,T_1) and G_2 by *summing on an edge*.

3.2.4 The proof of Theorem 1.3.2

Since every matroid M can be constructed from a collection of 3-connected matroids by 1-sums and 2-sums, it suffices to prove propositions 3.2.7 and 3.2.8 to complete the proof of Theorem 1.3.2.

Consider an even-cut matroid M and a set $X \subseteq E(G)$ where $X \neq \emptyset$ and $X \neq E(G)$. A connected graft representation (G,T) of M is *extremal* for X if among all connected graft representations of M that are equivalent to (G,T), the auxiliary graph for G and X has fewest number of vertices. (Note that if (G,T) is a graft representation of M then there is an equivalent graft representation that is connected.) Recall that, for a binary matroid M, $g(M)$ is the number of pairwise inequivalent graft representations of M.

Proposition 3.2.7. *If $M = M_1 \oplus_1 M_2$ for some binary matroids M_1, M_2 where M is not* *cographic then*

$$g(M) \le \max\{g(M_1), g(M_2)\}.$$

Proof. Define $X = E(M_1)$ and $Y = E(M_2)$. Then $M_1 = M \setminus Y$, $M_2 = M \setminus X$ and $\lambda_M(X) = 0$. Since M is not cographic we may assume M_1 is not cographic. We may assume that $g(M) \ge 1$; otherwise, $g(M) = 0$ and the result holds. Let (G, T) be a graft representation of M that is extremal for X. Then $(G, T)/Y$ is a representation of M_1. In particular, as M_1 is not cographic, $(G, T)/Y$ is not eulerian, i.e. $q[(G, T)/Y] = 1$. Let H denote the auxiliary graph for G and X. Then by Proposition 3.2.4,

$$|E(H)| = |V(H)| - q[(G, \Sigma)/X] - q[(G, \Sigma)/Y] \ge |V(H)| - 1.$$

Hence, (i) $|E(H)| = |V(H)| - 1$ and (ii) $q[(G, \Sigma)/X] = 0$. By (i) and since (G, T) is extremal for X, we have $|V(H)| = 2$ and $|E(H)| = 1$, i.e. G/X, G/Y are connected and share exactly one vertex in G. By (ii) we may assume $T \subseteq V_G(X)$ and $M_2 = \mathrm{cut}(G/X)$. Let (G', T') be any other graft representation of M that is extremal for X. Then we may assume that $T' \subseteq V_G(X)$ and $M_2 = \mathrm{cut}(G'/X)$. Then $\mathrm{cut}(G/X) = \mathrm{cut}(G'/X)$ or equivalently, $\mathrm{cycle}(G/X) = \mathrm{cut}(G'/X)$. It follows by Theorem 1.1.1 that G/X and G'/X are equivalent. Hence, (G, T) and (G', T') are equivalent if and only if $(G, T)/Y$ and $(G', T')/Y$ are equivalent. Thus, $g(M) \le g(M_1)$. □

Proposition 3.2.8. *If $M = M_1 \oplus_2 M_2$ for some binary matroids M_1, M_2 where M is not pinch-cographic then*

$$g(M) \le \max\{g(M_1), g(M_2)\}.$$

Proof. Denote by Ω the unique element in $E(M_1) \cap E(M_2)$ and let $X = E(M_1) - \Omega$, $Y = E(M_2) - \Omega$. We have $\lambda_M(X) = 1$. Let (G, T) be a graft representation of M that is extremal for X. Let H denote the auxiliary graph for G and X. Then by Proposition 3.2.4

$$|E(H)| = |V(H)| + 1 - q[(G,T)/X] - q[(G,T)/Y] \geq |V(H)| - 1. \tag{3.2}$$

Claim 1. $q[(G,T)/X] + q[(G,T)/Y] \geq 1$.

Subproof. Otherwise, by (3.2), we have, $|E(H)| = |V(H)| + 1$. Because (G,T) is extremal for X, H is 2-edge-connected. Remark 3.1.7 implies that H is a theta or a double ear. Consider first the case where H is a theta formed by st-paths P_1, P_2, P_3. As (G,T) is extremal for X and since H is bipartite, either (a) for $j \in [3]$, P_j consists of one edge, or (b) for $j \in [3]$, P_j consists of two edges. Case (a) is illustrated in Figure 3.1(ii) and case (b) is illustrated in Figure 3.1(iii). Since $q[(G,T)/X] = q[(G,T)/Y] = 0$ we have $T \subseteq \partial_G(X)$. As M is not pinch-cographic, $|T| \geq 6$. Thus, (a) cannot occur and if (b) occurs then $T = \partial_G(X)$. For (b), we may assume that X is the non-shaded region in the figure. But then $(G,T)/X$ has two terminals, contradicting the fact that $(G,T)/X$ is eulerian. Consider the case where H is a double ear formed by polygons C_1, C_2. As (G,T) is extremal for X, C_1 and C_2 each consist of two parallel edges. This case is illustrated in Figure 3.1(a). As in the previous case we must have $T \subseteq \partial_G(X)$. Hence, $|T| \leq 4$ and in particular, M is pinch-cographic, a contradiction. ♢

Claim 2. *Suppose $q[(G,T)/X] = 0$ and $q[(G,T)/Y] = 1$ then we may assume that $T \subseteq V_G(X)$. Moreover, $M_1 = \text{ecut}(G_1, T)$, $M_2 = \text{cut}(G_2)$ where (G,T) is obtained from (G_1, T) and G_2 by a summing on edge Ω. In particular, M_2 is cographic.*

Subproof. By (3.2), $|E(H)| = |V(H)|$, and H is a connected graph with exactly one polygon. Since (G,T) is extremal for X we have $|V(H)| = |E(H)| = 2$, i.e., $G\backslash X$, $G\backslash Y$ are connected and share exactly two vertices, say u, v in G. Then the result holds by Proposition 3.2.6. ♢

Claim 3. *Suppose* $q[(G,T)/X] = q[(G,T)/Y] = 1$. *Then* $M_1 = \text{ecut}(G_1, T_1)$, $M_2 = \text{ecut}(G_2, T_2)$ *where* (G,T) *is obtained from* (G_1, T_1) *and* (G_2, T_2) *by a summing on pin* Ω.

Subproof. By (3.2), $|E(H)| = |V(H)| - 1$ and since H is connected H is a tree. Since (G,T) is extremal for X we have $|V(H)| = 2$ and $|E(H)| = 1$, i.e. $G \setminus X$, $G \setminus Y$ are connected and share exactly one vertex in G. Then the result holds by Proposition 3.2.5. $\Diamond$

M_1 and M_2 are not both cographic for otherwise so would M, a contradiction. It follows from Claim 2 that we cannot have extremal representations (G,T) and (G',T') of M with $q[(G,T)/X] = 0$, $q[(G,T)/Y] = 1$ and $q[(G',T')/X] = 1$, $q[(G',T')/Y] = 0$. We may assume that M_1 is not cographic. Hence, because of Claim 1 (G,T) is of one of the following types,

Type 1. $q[(G,T)/X] = 0$ and $q[(G,T)/Y] = 1$ or

Type 2. $q[(G,T)/X] = 1$ and $q[(G,T)/Y] = 1$.

Let h_1 (resp. h_2) denote the number of inequivalent representations of M_1 with a non-pin Ω (resp. pin Ω). Let g_1 (resp. g_2) denote the number of inequivalent Type 1 (resp. Type 2) representations of M. Note, $g(M) = g_1 + g_2$ and $f(M_1) = h_1 + h_2$.

Claim 4. $g_1 \leq h_1$.

Subproof. Consider Type I representations (G,T) and (G',T') of M. Then (G,T) is obtained from (G_1,T) and G_2 by a summing on edge Ω and (G',T') is obtained from (G'_1,T') and G'_2 by a summing on edge Ω. As $M_2 = \text{cut}(G_2) = \text{cut}(G'_2)$, or equivalently, $\text{cycle}(G_2) = \text{cycle}(G'_2)$, it follows from Theorem 1.1.1 that G_2 and G'_2 are equivalent. Thus (G,T) and (G',T') are equivalent if and only if (G_1,T) and (G'_1,T') are equivalent. The result follows. $\Diamond$

Claim 5. $g_2 \leq h_2$.

Subproof. Consider Type 2 representations (G,T) and (G',T') of M. Then (G,T) is obtained from (G_1,T_1) and (G_2,T_2) by a summing on a pin Ω and (G',T') is obtained from (G'_1,T'_1) and (G'_2,T'_2) by a summing on pin Ω. It follows from Remark 2.4.6 that (G_1,T_1) and (G'_1,T'_1) are equivalent and also that (G_2,T_2) and (G'_2,T'_2) are equivalent. Thus (G,T) and (G',T') are equivalent. It follows that $g_2 \leq 1$ and clearly, $g_2 = 0$ if $h_2 = 0$. $\diamondsuit$

Then $g(M) = g_1 + g_2 \leq h_1 + h_2 = g(M_1)$ as required. $\square$

3.3 Pinch-graphic matroids

The goal of this section is to prove Theorem 1.3.3.

3.3.1 Chain theorems

Let M and N be matroids. M *contains an N-minor* if for some $I,J \subseteq E(M)$ where $I \cap J = \emptyset$, we have that $M/I \setminus J$ is isomorphic to N. Note, F_7 denotes the Fano matroid, $M(G)$ is the graphic matroid of graph G, and we write M^* for the dual of M. Let M be a binary matroid. We say that a sequence $M_1,\dots,M_k$ of matroids is a *good sequence* for M if

1. $M_1 = M$ and $M_k \in \{F_7, F_7^*, M(K_5)^*, M(K_{3,3})^*\}$,
2. for all $i \in [k-1]$, M_{i+1} is a single element deletion or contraction of M_i,
3. for all $i \in [k]$, M_i is $(4,6)$-connected.

Here is the key result of this section,

Proposition 3.3.1. *Let M be a binary non-graphic matroid that is $(4,5)$-connected. Then there exists a good sequence $M_1, \ldots, M_k$ for M. Moreover, if we are given M by its $0,1$ matrix representation A, then in time polynomial in the number of entries of A we can construct that good sequence.*

The proof will require the following Splitter theorems,

Theorem 3.3.2 (Seymour [44]). *Let M be a matroid that is 3-connected that is not a wheel or a whirl, and let N be a 3-connected proper minor of M. If $|E(M)| \geq 4$, then there exists $e \in E(M)$ such that $M \backslash e$ or M/e is 3-connected and contains an N-minor.*

Theorem 3.3.3 (Geelen and Zhou [15]). *Let M be a binary matroid that is $(4,5)$-connected and let N be an internally 4-connected proper minor of M. If $|E(M)| \geq 7$, then there exists either*

(a) $e \in E(M)$ such that $M \backslash e$ or M/e is $(4,5)$-connected and contains an N-minor; or

(b) $e, e' \in E(M)$ such that $M/e \backslash e'$ is $(4,5)$-connected and contains an N-minor.

To be able to find the good sequence in polynomial time we require the following results,

Proposition 3.3.4 (Cunningham [8]). *Let M be a binary matroid described by an $m \times n$ $0,1$ matrix A and let k, ℓ be fixed integers. In time polynomial in m and n, we can either find a k-separating set X where $|X|, |E(M) - X| \geq \ell$ or establish that none exists.*

Proposition 3.3.5 (Tutte [51]). *Let M be a binary matroid described by an $m \times n$ $0,1$ matrix A. In time polynomial in m and n we can check whether M is graphic.*

Note that, for both results, we have actual algorithms, not just proof of existence.

We are ready for the main proof of this section, *Proof of Proposition 3.3.1.* Since M is not graphic there exists a minor N of M which is minimally non-graphic. It follows from Tutte's characterizations of regular matroids [49] and graphic regular matroids [50, 45] that N is isomorphic to one of F_7, F_7^*, $M(K_5)^*$, or $M(K_{3,3})^*$. Let $M_1 := M$ and let $k = |E(M_1)| - |E(N)| + 1$. Let us show that there exists a good sequence by induction on k. If $k = 1$ then $M_1 = N$ and trivially M_1 is the good sequence. Thus, we may assume that $k \geq 2$ and in particular that N is a proper minor of M_1. As $N \in \{F_7, F_7^*, M(K_5)^*, M(K_{3,3})^*\}$, it is internally 4-connected and $|E(M_1)| \geq |E(N)| \geq 7$. It follows then from Theorem 3.3.3 that there exists a $(4,5)$-connected matroid $\tilde{M}$ with an N-minor where

1. $\tilde{M} = M_1 \backslash e$ or $\tilde{M} = M_1/e$ for some $e \in E(M_1)$; or

2. $\tilde{M} = M_1 \backslash e/e'$ for some distinct $e, e' \in E(M_1)$.

If (1) occurs, then we let $M_2 := \tilde{M}$. By induction there exists a good sequence $M_2, \ldots, M_k$. But then $M_1, \ldots, M_k$ is a good sequence for M as required. Hence, we may assume that (2) occurs.

Since $\tilde{M}$ is binary and non-graphic, $\tilde{M}$ is neither a wheel nor a whirl. By Theorem 3.3.2, there exists a matroid M' where either $M' = M_1 \backslash f$ or $M' = M_1/f$ for some $f \in E(M_1)$ such that M' is 3-connected and has a minor M'' isomorphic to $\tilde{M}$.

Claim. *M' is $(4,6)$-connected.*

Subproof. Suppose for a contradiction that there exists a 3-separation X of M' such that $|X|, |E(M') - X| \geq 7$. Observe that $M'' = M'/f'$ or $M'' = M' \setminus f'$ for some $f' \in E(M')$. After possibly replacing X by $E(M') - X$ we may assume that $f' \notin X$. Corollary 8.2.6 in [36] implies that $\lambda_{M''}(X) \leq \lambda_{M'}(X) = 2$. Since M'' is 3-connected (as it is isomorphic to $\tilde{M}$)

$\lambda_{M''}(X) = 2$. But $|X|, |E(M'') - X| \geq 6$ contradicts the fact that M'' is $(4,5)$-connected. $\Diamond$

Now, we let $M_2 := M'$ and $M_3 := M''$. By induction there exists a good sequence $M_3, \ldots, M_k$. But then $M_1, M_2, M_3, \ldots, M_k$ is a good sequence for M as required. We leave it as an exercise to show how to find the sequence in polynomial-time by mean of Propositions 3.3.4 and 3.3.5. $\square$

Theorem 3.3.6. *Let $M_1, \ldots, M_k$ denote $(4,6)$-connected binary matroids where for each $i \in [k-1]$, M_{i+1} is a single element contraction or deletion of M_i. Suppose that M_k is non-graphic and that $|E(M_k)| \in \mathcal{O}(1)$. Then for each $i \in [k]$, the number of blocking-pair representations of M_i is in $\mathcal{O}(|E(M_i)|^4)$.*

We will postpone the proof of this result until Section 3.3.3. Combining Proposition 3.3.1 and Theorem 3.3.6 proves Theorem 1.3.3.

Proof of Theorem 1.3.3. Note M is binary as it is a pinch-graphic matroid. It follows from Proposition 3.3.1 that M admits a good sequence $M_1, \ldots, M_k$. Then $|E(M_k)| \leq 10$, hence $|E(M_k)| \in \mathcal{O}(1)$. It follows by Theorem 3.3.6 that $|E(M)| = |E(M_1)| \in \mathcal{O}(|E(M)|^4)$, as required. $\square$

3.3.2 $(4,6)$-Connectivity

In this section, we translate our condition that an even-cycle or an even-cut matroid is $(4,6)$-connected in terms of its signed-graph or graft representation. In particular, we will see that the graph must be essentially 3-connected and that all 2-separations have bounded size.

Even-cycles matroids

Proposition 3.3.7 ([40], Proposition 2.6)**.** *Suppose that* ecycle(G,Σ) *is* 3*-connected. Then*

(a) if G has no loop then G is 2*-connected;*

(b) G has at most one loop, which is odd;

(c) if X is a 2*-separation of G then X contains an odd polygon.*

Proposition 3.3.8. *Let* $M = \text{ecycle}(G,\Sigma)$ *be* $(4,6)$*-connected. Then the following hold,*

(a) G is 2*-connected except for a unique possible odd loop;*

(b) if X is a 2*-separation of G then X contains an odd polygon;*

(c) if X is a 2*-separation of G then* $\min\{|X|, |E(G) - X|\} \leq 6$*;*

(d) (G,Σ) *has no parallel edges of the same parity.*

Proof. Since M is 3-connected, Proposition 3.3.7 implies that (a) and (b) hold. For (d), let $e, f \in E(M)$ where e and f are parallel edges of the same parities in (G,Σ). Then, e, f are parallels of M, contradicting 3-connectivity of M. For (c), suppose X is a 2-separation of G, and then, by Proposition 3.1.5 and 3-connectivity of M, X is 3-separating. It follows from $(4,6)$-connectivity of M that $\min\{|X|, |E(G) - X|\} \leq 6$. □

Even-cut matroids

Recall that an edge of a graft (H,T) is a *pin* if it has an end $v \in T$ of degree 1.

Proposition 3.3.9 ([40], Proposition 2.5)**.** *Suppose that* ecut(H,T) *is* 3*-connected. Then* *(a) if H has no bridge, then H is 2-connected;*

(b) if H has a bridge e, then e is a pin and (H, T) has at most one pin;

(c) if X is a 2-separation of H then $\mathcal{I}(X) \cap T \neq \emptyset$.

Proposition 3.3.10. *Let $M = \mathrm{ecut}(H, T)$ that is $(4, 6)$-connected. Then the following hold,*

(a) H is 2-connected except for a unique possible pin;

(b) if X is a 2-separation of H then $\mathcal{I}(X) \cap T \neq \emptyset$;

(c) if X is a 2-separation of H then $\min\{|X|, |E(H) - X|\} \leq 6$;

(d) H has no parallel edges;

(e) every even cut of (H, T) has cardinality at least 3.

Proof. Since M is 3-connected, Proposition 3.3.9 implies (a) and (b) hold. For (d), let $e, f \in E(M)$ where e and f are parallel edges of H. Then, $\{e, f\}$ is a cocycle of M, contradicting 3-connectivity of M. For (e), let $D \subseteq G$ be an even cut of (H, T) where $|D| \leq 2$. Then, D is a cycle of M, contradicting 3-connectivity of M. For (c), suppose X is a 2-separation of H, then by Proposition 3.2.3 and 3-connectivity of M, X is 3-separating. It follows that $\min\{|X|, |E(H) - X|\} \leq 6$, hence (c) holds. □

3.3.3 The proof of Theorem 3.3.6

We require the following two results which we will prove in Section 3.3.4.

Proposition 3.3.11. *Let M be a $(4,6)$-connected pinch-graphic matroid with a blocking-pair representation (G,Σ). Then the number of graphs H equivalent to G for which (H,Σ) is a blocking-pair representation of M is in $\mathcal{O}(|V(G)|)$.*

Note that this result is asymptotically tight. Indeed consider the following example. Construct a graph G as follows: pick a polygon with vertices $v_1,\dots,v_n$ and add a new vertex c and for all $i\in[n]$ add a pair of edges f_i,g_i with ends c and v_i and add a loop Ω. Let $\Sigma=\{f_i : i\in[n]\}\cup\{v_1v_2,\Omega\}$. Then $\text{ecycle}(G,\Sigma)$ is internally 4-connected. Moreover, for every $i\in[n]$ we can place the loop Ω to be incident to vertex v_i or to c. This yields $|V(G)|$ distinct equivalent signed-graphs each with a blocking-pair.

Proposition 3.3.12. *Let M be a $(4,6)$-connected non-cographic pinch-cographic matroid with a T_4-representation (H,T). Then the number of grafts equivalent to (H,T) is in $\mathcal{O}(1)$.*

Nice and special representations

We say that a graft (H,T) is *special* if $|T|=4$, properties (a)-(e) of Proposition 3.3.10 hold, and there exists an odd cut of cardinality at most 3. We say that a graft (H,T) is *nice* if $|T|=4$, properties (a)-(e) of Proposition 3.3.10 hold, and every odd cut has cardinality at least 4.

Proposition 3.3.13. *The number of special representations of a pinch-cographic $(4,6)$-connected matroid M is in $\mathcal{O}(|E(M)|^3)$.*

Proof. Let (H,T) be a special graft-representation of M. Then, for some set $B\subseteq E(H)$ with $|B|\le 3$, B is an odd cut of (H,T). By Remark 2.4.6, all special representations with a fixed odd cut B are equivalent. Hence, by Proposition 3.3.12, the number of such representations is in $\mathcal{O}(1)$. As the number of possible choices for a set $B \subseteq E(H)$ with $|B| \leq 3$ is in $\mathcal{O}(|E(M)|^3)$ the result follows. □

For a graph G and $v \in V(G)$ we denote the degree of v by $d_G(v)$.

Next we show that nice grafts are indeed nice.

Proposition 3.3.14. *If (H,T) is a nice graft, then H is 3-connected.*

Proof. Note that (H,T) satisfies properties (a)-(e) of Proposition 3.3.10.

Claim 1. *Cuts of (H,T) have cardinality at least 3, and odd cuts of (H,T) have cardinality at least 4.*

Subproof. The result follows from Proposition 3.3.10(e). ◊

In particular, (H,T) has no pin. Then by (a) H is 2-connected. Suppose for a contradiction that H is not 3-connected. Then there exists a partition X, Y of the edges of H where $H[X]$ and $H[Y]$ are connected, $\partial(X) = \{u_1, u_2\}$ for some distinct $u_1, u_2 \in V(H)$, and $\mathcal{I}(X), \mathcal{I}(Y) \neq \emptyset$. By (b), there exists $z \in \mathcal{I}(X) \cap T$ and by Claim 1, $d_H(z) \geq 4$. By (d), there are no parallel edges. Hence, z has at least 4 neighbours and $|\mathcal{I}(X)| \geq 3$. By Claim 1 vertices $v \in \mathcal{I}(X)$ satisfy $d_H(v) \geq 3$. Thus,

$$\sum_{v \in \mathcal{I}(X)} d_H(v) \geq 3|\mathcal{I}(X)| + 1. \tag{1}$$

Let L denote the edges with one end in $\mathcal{I}(X)$ and one end in $\{u_1, u_2\}$. The Claim implies that $|L| \geq 3$. Let G be the graph induced by vertices $\mathcal{I}(X)$. Then (1) implies that,

$$\sum_{v \in \mathcal{I}(X)} d_G(v) \geq 3|\mathcal{I}(X)| + 1 - |L|. \tag{2}$$

Then X consists of edges of G, edges in L and possibly an edge with ends u_1, u_2. By (2),

$$|X| \geq \frac{1}{2} \sum_{v \in \mathcal{I}(X)} d_G(v) + |L| \geq \frac{1}{2}(3|\mathcal{I}(X)| + 1 - |L|) + |L|.$$

As $|\mathcal{I}(X)|, |L| \geq 3$ we have $|X| \geq \frac{1}{2}(3 \times 3 + 1 - 3) + 3 > 6$, a contradiction to (c). ☐

Unstable sets of grafts

Proposition 3.3.15. *Suppose* ecycle(G, Σ) *is* $(4, 6)$*-connected and let* a, b *be a blocking pair of* (G, Σ)*. Then the number of signatures of* (G, Σ) *with all edges incident to* a *or* b *is in* $\mathcal{O}(1)$.

Proof. Let $\mathcal{S}$ be the set of signatures of (G, Σ) with all edges incident to a or b. We will show $|\mathcal{S}| \in \mathcal{O}(1)$. We may assume after re-signing that $\Sigma \in \mathcal{S}$. Pick $\Gamma \in \mathcal{S}$ where $\Gamma \neq \Sigma$. Since $\Sigma \,\Delta\, \Gamma$ intersects every cycle of G with even parity, it is a cut $\delta(U)$ of G. We say that Γ is *skewed* if exactly one of a, b is in U. Let $\Gamma_1, \ldots, \Gamma_\ell$ denote the signatures of $\mathcal{S} - \Sigma$ that are not skewed. Observe that if $\Gamma \in \mathcal{S}$ is skewed, then $\Gamma \,\Delta\, \delta(a)$ is not skewed, thus $\Gamma \,\Delta\, \delta(a) = \Gamma_i$ for some $i \in [\ell]$. It follows that $|\mathcal{S}| \leq 2\ell + 1$. For all $i \in [\ell]$, $\Sigma \,\Delta\, \Gamma_i = \delta(U_i)$ and we may assume that $a, b \notin U_i$. Let $H_1, \ldots, H_k$ denote the components of the graph obtained from G by deleting vertices a, b. Let $i \in [\ell]$ and $j \in [k]$. Since $\delta(U_i)$ is a cut either (i) $V(H_k) \subseteq U_i$ or (ii) $V(H_k) \cap U_i = \emptyset$. Define for $i \in [\ell]$, the set $J(i) = \{j \in [k] : V(H_k) \subseteq U_i\}$, i.e. $J(i)$ indicates what components of $H \setminus \{a, b\}$ are contained in the shore U_i of cut $\Sigma \,\Delta\, \Gamma_i$. Now observe that $\delta(U_i)$ are exactly the edges between vertices of H_j and $\{a, b\}$ for all $j \in J(i)$. It follows that $J(i)$ determines Γ_i uniquely. Hence, ℓ is bounded by the number of choices for $J(i)$ thus $\ell \leq 2^k$. However, as ecycle(G, Σ) is $(4, 6)$-connected, Proposition 3.3.8(c) implies that $k \in \mathcal{O}(1)$. Thus $|\mathcal{S}| \leq 2\ell + 1 \in \mathcal{O}(2^k) = \mathcal{O}(1)$. ☐

Let $\mathcal{S} = \{(H_i, T_i) : i \in [n]\}$ where (H_i, T_i) are nice grafts. We say that the set $\mathcal{S}$ is *unstable* if there exists a pinch-cographic matroid M and for all $i \in [n]$ there exists a graph H_i' obtained from H_i by adding an edge Ω with both ends in T_i, so that (H_i', T_i) is a representation of M. Observe that this implies that $\mathcal{S}$ are all T_4-representations of M/Ω.

Proposition 3.3.16. *Let $\mathcal{S}$ be an unstable set of representations of a matroid M. Then $|\mathcal{S}| \in \mathcal{O}(|E(M)|^3)$.*

Proof. Then $\mathcal{S} = \{(H_i, T_i) : i \in [n]\}$. For each $i \in [n]$ denote by x_i, y_i, w_i, z_i the vertices in T_i. We may assume that H_i' is obtained from H_i by adding edge Ω with ends x_i, y_i. Let G_i be obtained from H_i by identifying x_i with y_i and by identifying w_i with z_i. Denote by a_i the vertex of G_i corresponding to $x_i = y_i$ and denote by b_i the vertex of G_i corresponding to $w_i = z_i$. Let $\Sigma_i = \delta_{H_i}(x_i) \,\Delta\, \delta_{H_i}(w_i)$. Observe that (G_i, Σ_i) is obtained from (H_i, T_i) by folding and that a_i, b_i is a blocking pair of (G_i, Σ_i). Then let $\mathcal{R} = \{(G_i, \Sigma_i) : i \in [n]\}$.

Claim 1. *The signed-graphs in $\mathcal{R}$ are all pairwise equivalent.*

Subproof. Pick $i, j \in [n]$. We will show that (G_i, Σ_i) and (G_j, Σ_j) are equivalent. Since $\mathcal{S}$ is unstable, $\text{ecut}(H_i', T_i) = \text{ecut}(H_j', T_j)$. Hence, $\text{ecut}(H_i', T_i) \setminus \Omega = \text{ecut}(H_j', T_j) \setminus \Omega$. Note that

$$\text{ecut}(H_i', T_i) \setminus \Omega = \text{ecut}(H_i'/\Omega, \{w_i, z_i\}) = \text{cut}(G_i).$$

Similarly, $\text{ecut}(H_j', T_j) \setminus \Omega = \text{cut}(G_j)$. Thus $\text{cut}(G_i) = \text{cut}(G_j)$. It follows from Theorem 1.1.1 that G_i and G_j are equivalent. Furthermore, by Proposition 2.2.1, (G_i, Σ_i) and (G_j, Σ_j) are blocking-pair representations of M^*. In particular, $\text{ecycle}(G_i, \Sigma_i) = \text{ecycle}(G_j, \Sigma_j)$. Hence, (G_i, Σ_i) and (G_j, Σ_j) are equivalent as required. ◊

By Proposition 2.2.1, we know that for each $i \in [n]$, (G_i, Σ_i) is a blocking-pair representation of the $(4, 6)$-connected matroid M^*. By Claim 1, the signed graphs in $\mathcal{R}$ are equivalent.

It follows from Proposition 3.3.11, that the number of distinct graphs among $G_1, \dots, G_n$ is in $\mathcal{O}(|E(M)|)$. Let $K \subseteq [n]$ such for all $k \in K$, $G_k = G$ for some fixed graph G. There are at most $|V(G)|^2 \in \mathcal{O}(|E(M)|^2)$ distinct blocking pairs $\{a_k, b_k\}$ of (G, Σ_k) among all $k \in K$. Moreover, by Proposition 3.3.15 there are at most $\mathcal{O}(1)$ different signatures Σ_ℓ, $\ell \in K$ for a given blocking pair $\{a_k, b_k\}$ of G. It follows that $|K| \in \mathcal{O}(|E(M)|^2)$ and hence that $|\mathcal{S}| = |\mathcal{R}| \in \mathcal{O}(|E(M)|^3)$ as required. □

Proposition 3.3.17. *Let M be a binary matroid, $\Omega \in E(M)$ that is not a loop of M and let $N = M/\Omega$ where N is $(4,6)$-connected. Consider a nice representation (H, T) of N that does not extend uniquely to M. Then (H, T) extends to exactly two representations (H_1, T) and (H_2, T) of M where H_1 is obtained from H by adding edge Ω between t_1, t_2 and where H_2 is obtained from H by adding edge Ω between t_3, t_4 for some labeling t_1, t_2, t_3, t_4 of the vertices of T.*

Proof. Since Ω is not a loop of M there exists a cocircuit D of M with $\Omega \in D$. Cocircuits of even-cut matroids are polygons or inclusion-wise minimal T-joins of the graft representation [40], Remark 2.2. After possibly replacing D with $D \,\Delta\, J$ for some T-join J of H, we may assume that D is a polygon of H_1. If D is a cycle of H_2 then H_1 and H_2 have the same set of cycles, and thus by Theorem 1.1.1 are equivalent. But then as H is 3-connected by Proposition 3.3.14, so are H_1, H_2 which implies that $H_1 = H_2$, a contradiction. Thus D is a T-join of H_2. Let $P = D - \Omega$. Since D is a polygon of H_1, P is a path of H. As $T = \{t_1, t_2, t_3, t_4\}$ is the set of odd degree vertices in $P \cup \Omega$ of H_2 it follows that Ω has ends say t_3, t_4 in H_2, and that P has end t_1, t_2 in H. But then Ω has ends t_1, t_2 in H_1 and H_1, H_2 are as required. □

This last result implies immediately the following,

Proposition 3.3.18. *Let M be a binary matroid, $\Omega \in E(M)$ that is not a loop of M and let $N = M/\Omega$ where N is $(4,6)$-connected. Then the set of nice representations of N that do not extend uniquely to M form an unstable set.*

Extending graft representations

Proposition 3.3.19 ([25] Lemma 9.4)**.** *Let N be an even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft-representations of N. Let M be a matroid with a non-coloop $e \in E(M)$ for which $N = M \setminus e$. Then the set of extensions of $\mathcal{F}$ to M is a (possibly empty) equivalence class of graft-representations.*

Proposition 3.3.20 ([25] Lemma 9.12)**.** *Let N be a non-cographic, even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft-representations of N. Let M be a matroid with a non-loop $e \in E(M)$ for which $N = M/e$. Then the set of extensions of $\mathcal{F}$ to M is either a (possibly empty) equivalence class of graft-representations or the union of two equivalence classes of graft-representations.*

The proof of the main result

Proof of Theorem 3.3.6. For all $i \in [k]$ let $N_i = M_i^*$. Then $N_1, \ldots, N_k$ denote $(4,6)$-connected binary matroids where for each $i \in [k-1]$, N_{i+1} is a single element contraction or deletion of N_i and where $|E(N_k)| \in \mathcal{O}(1)$.

Claim 1.

(a) for all $i \in [k]$, every representation of N_i has at least four terminals;

(b) for all $i \in [k]$, every T_4-representation of N_i is nice or special.

Subproof. By hypothesis, M_k is not graphic, hence M_i is not graphic, or equivalently, N_i is not cographic, that is, (a) holds. Since M_i is $(4,6)$-connected, so is N_i. Then (a) and Proposition 3.3.10 implies (b). $\diamondsuit$

Denote by $f(i)$ the number of nice representations of N_i.

Claim 2. *For all $i \in [k-1]$, $f(i) \le f(i+1) + \mathcal{O}(|E(N_i)|^3)$.*

Subproof. By Claim 1, every T_4-representation of N_{i+1} is nice or special. Hence, every T_4-representation of N_{i+1} extends either: (i) a nice representation of N_{i+1}, or (ii) a special representation of N_{i+1}. A nice representation of N_i is of *Type I* if it arises as in (i) and of *Type II* if it it arises as in (ii). Note, the definition allows N_i to be of both Type I and Type II.

Case 1. $N_{i+1} = N_i \setminus \Omega$ for some $\Omega \in E(N_i)$.

Let (H',T') be a T_4-representation of N_i that extends some T_4-representation (H,T) of N_{i+1}. Then (H',T') is obtained from (H,T) by uncontracting edge Ω. If (H,T) is special it has an odd cut B with $|B| \le 3$. But then B is an odd cut of (H',T') and (H',T') is not nice. Hence, nice representations of N_i are only of Type I. Then by Proposition 3.3.14 the equivalence classes of nice representations of N_i, N_{i+1} have cardinality one. By Proposition 3.3.19, every equivalence class of graft-representations of N_{i+1} extends to one (possibly empty) equivalence class of graft-representations of N_i. It follows that every nice representation of N_{i+1} extends to at most one nice representation of N_i. Therefore, $f(i) \le f(i+1)$.

Case 2. $N_{i+1} = N_i/\Omega$ for some $\Omega \in E(N_i)$.

Let (H,T) be a representation of N_{i+1} and let (H',T') be a representation of N_i that extends (H,T). Then (H',T') is obtained from (H,T) by adding edge Ω. By Proposition 3.3.20, every equivalence class of graft-representation of N_{i+1} extends to at most two equivalence classes of graft-representations of N_i. By Proposition 3.3.14 the equivalence classes of nice representations of N_i have cardinality one. Proposition 3.3.13 implies that the number of special representations of N_{i+1} is in $\mathcal{O}(|E(N_{i+1})|^3)$. Therefore, the number of nice Type II representations of N_i is in $\mathcal{O}(|E(N_i)|^3)$. Let $\mathcal{L}$ be the set of all nice representations of N_{i+1} that extend non-uniquely to N_i. Proposition 3.3.18 implies $\mathcal{L}$ is an unstable set. Therefore by Proposition 3.3.16, $|\mathcal{L}| \in \mathcal{O}(|E(N_i)|^3)$. Thus the number of Type I nice representations of N_i is at most, $f(i+1)+|\mathcal{L}| \in f(i+1)+\mathcal{O}(|E(N_i)|^3)$. Hence, $f(i) \leq f(i+1)+\mathcal{O}(|E(N_i)|^3)$. $\lozenge$

As $|E(N_k)| \in \mathcal{O}(1)$, $f(k) \in \mathcal{O}(1)$. It then follows from Claim 1 that for all $i \in [k]$,

$$f(i) \in \mathcal{O}(|E(N_i)|^4). \tag{3.3}$$

Pick $\hat{\imath} \in [k]$. Let $\mathcal{R}$ denote the set of all T_4-representations of $N_{\hat{\imath}}$. By Claim 1 every graft in $\mathcal{R}$ is special or nice. By Proposition 3.3.13, the number of special representations of $N_{\hat{\imath}}$ is in $\mathcal{O}(|E(N_{\hat{\imath}})|^3)$. Together with (3.3) this implies that $|\mathcal{R}| \in \mathcal{O}(|E(N_{\hat{\imath}})|^4)$. Let $\mathcal{S}$ denote the set of all blocking-pair representations of $M_{\hat{\imath}}$. Pick an arbitrary representation $(G,\Sigma) \in \mathcal{S}$ and unfold it to get a graft (H,T). By Proposition 2.2.1, (H,T) is a T_4-representation of $N_{\hat{\imath}}$, i.e. $(H,T) \in \mathcal{R}$. Moreover, there are at most 12 ways of folding (H,T) to get a blocking-pair representation in $\mathcal{R}$, [1] i.e. at most 12 blocking-pair representations of $\mathcal{S}$ get mapped to the same T_4-representation of $\mathcal{R}$. It follows that $|\mathcal{S}| \leq 12|\mathcal{R}| \in \mathcal{O}(|E(N_{\hat{\imath}})|^4)$ as required. $\square$

[1]3 choices for decide which pairs of terminals get identified, and 2×2 choices for the signature.

3.3.4 Size of equivalent classes

The goal of this section is to prove Propositions 3.3.11 and 3.3.12. For the former, we consider blocking-pair representations of a $(4,6)$-connected pinch-graphic matroid, and for the latter, we consider T_4-representations of a $(4,6)$-connected pinch-cographic matroids. In both cases these representations have the property that for every 2-separation one side has cardinality at most 6. This leads to the notion of well connected-graphs that we study next.

Well connected-graphs

A graph G is *well-connected* if the following conditions hold:

(w1) $|E(G)| \geq 25$;

(w2) G is loopless and 2-connected;

(w3) for every 2-separation X of G, we have $\min\{|X|, |E(G) - X|\} \leq 6$;

(w4) parallel classes have cardinality at most two.

Let X be a 2-separation of a well-connected graph G. We say that X is *small* if $|X| \leq 6$. A small 2-separation X is *maximal* if it is inclusion maximal among all small 2-separations.

The following is the motivation for considering maximal small 2-separation,

Proposition 3.3.21. *In a well-connected graph any two maximal small* 2*-separations are disjoint. In particular, every small* 2*-separation is contained in a unique maximal small* 2*-separation.*

Before we present the proof we require some definitions. A pair of 2-separations X and Y *cross* if all of the following are non-empty,

$$X \cap Y, \quad X \cap (E(G) - Y), \quad (E(G) - X) \cap Y \quad \text{and} \quad (E(G) - X) \cap (E(G) - Y).$$

A *necklace* is a graph obtained from a polygon C with at least 4 edges by replacing each edge by a connected graph. The graphs replacing edges of C are the *beads* of the necklace.

Proof of Proposition 3.3.21. Let G be a well-connected graph G and consider an arbitrary pair of maximal small 2-separations X and Y. We need to show that $X \cap Y = \emptyset$. Let $\overline{X} = E(G) - X$ and let $\overline{Y} = E(G) - Y$. Note that since X, Y are maximal, $X \cap \overline{Y}, \overline{X} \cap Y \neq \emptyset$. Moreover, since X, Y are small and $|E(G)| \geq 25$, $X \cup Y \neq V(G)$, or equivalently $\overline{X} \cap \overline{Y} \neq \emptyset$. Thus it suffice to show that X and Y are not crossing. Suppose otherwise. As X, Y cross, one of the following cases occurs [40],

i. $\partial(X \cap Y) = \partial(X \cap \overline{Y}) = \partial(\overline{X} \cap Y) = \partial(\overline{X} \cap \overline{Y})$; or

ii. G is a necklace with beads $X \cap Y, X \cap \overline{Y}, \overline{X} \cap Y, \overline{X} \cap \overline{Y}$.

In both cases, $\overline{X} \cap \overline{Y}$ is either a 2-separation of G, a one edge set, or a set of two parallel edges. Let $Z = X \cup Y$ and observe that $E(G) - Z = \overline{X} \cap \overline{Y}$. Since $|X|, |Y| \leq 6$ and $|E(G)| \geq 25$ we have $|E(G) - Z| \geq 7$. Clearly, Z is not an edge or a pair of parallel edges as $X, Y \subseteq Z$. Thus Z is a small 2-separation. But this contradicts our assumption that X was a maximal small 2-separation. □

We are working in this paper with edge-labeled graphs. At this juncture we need to concern ourselves with vertex labels as well. Consider a pair of graphs G, G' with the same set of (labeled) edges. Then a bijection $f : V(G) \to V(G')$ is an *isomorphism* from G to G' if for every *labelled edge* e: e has ends u, v in G if and only if e has ends $f(u), f(v)$ in G'.

Proposition 3.3.22. *Consider a well-connected graph G. Let $X_1, \ldots, X_k$ denote the maximal small 2-separations of G and let $Y = E(G) - (X_1 \cup \ldots \cup X_k)$. Suppose that G' is equivalent from G. Then*

(a) $X_1, \ldots, X_k$ are precisely the maximal small 2-separations of G'; and

(b) there is an isomorphism f from $G[Y]$ to $G'[Y]$ such that for every $i \in [k]$: f maps $\partial_G(X_i)$ to $\partial_{G'}(X_i)$.

Proof. Since G is 2-connected, G and G' are related by a sequence of 2-flips. It suffices to prove (a) and (b) when G' is obtained from G by a single 2-flip on a set Z as we can then iterate the result. After possibly replacing Z by $E(G) - Z$ we may assume that Z is a small 2-separation of G. For some $j \in [k]$, X_j is the maximal small 2-separation of G containing Z. Let us prove (a). Pick $i \in [k]$. By Proposition 3.3.21 $i = j$ or $X_i \cap X_j = \emptyset$. In particular, $Z \subseteq X_i$ or $Z \cap X_i = \emptyset$ and it follows that X_i is also a 2-separation of G'. Suppose for contradiction X_i is not maximal in G'. Then there exists a maximal small 2-separation X_ℓ of G' strictly containing X_i. Note that G can be obtained from G' by a 2-flip on Z. But then by the previous argument, X_ℓ is a small 2-separation of G, a contradiction as X_i is maximal in G. Finally, note that (b) follows from the fact that $Y \cap Z \subseteq Y \cap X_j = \emptyset$. □

The proof of Proposition 3.3.12

Throughout this section M will denote a $(4, 6)$-connected pinch-cographic matroid that is not cographic. Also throughout this section (H, T) will denote a T_4-representation of M. We will need to show that the number of T_4-representations equivalent to (H, T) is in $\mathcal{O}(1)$. Note that by Proposition 3.3.9, (H, T) has at most one pin.

Claim 1. *Suppose that $e = uv$ is a pin of (H, T) where $v \in T$ has degree 1. Then* *(a)* $u \notin T$*, and*

(b) there are exactly 4 *equivalent* T_4*-representations that can be obtained from* (H, T) *by moving* e*.*

Proof. Let (H', T') be a graft equivalent to (H, T) obtained by moving the end u of e to some new vertex w. Using the same labeling of the vertices in $H \setminus e$ and $H' \setminus e$ we have $T' = T \,\Delta\{u, w\}$. In particular, $u \notin T$ for otherwise we can pick $w \in T$ and this yields $|T'| = 2$, which implies that M is cographic, a contradiction. Moreover, if $w \notin T$ then $|T'| = 6$ and (H', T') is not a T_4-representation. It follows that there are exactly 4 possible T_4-representations that can be obtained from (H, T) by moving the pin e, namely move the end u of the pin to each terminal T. □

If (H, T) has no pin then we let $(G, R) = (H, T)$. If (H, T) has pin $e = uv$ then we let $G = H/e$ and $R = T - v \cup \{w\}$ where w is the vertex of G that corresponds to edge e of H.

Claim 2. *We may assume the following hold for the graft* (G, R)*,*

(a) G *is well-connected;*

(b) for any 2*-separation* X *of* G *we have* $\mathcal{I}_G(X) \cap R \neq \emptyset$*.*

Proof. We may assume $|E(G)| \geq 25$ for otherwise trivially the number of grafts equivalent to (H, T) is in $\mathcal{O}(1)$. By Proposition 3.3.10(a), H is 2-connected except for a unique possible pin. It follows that G is 2-connected. By Proposition 3.3.10(c), if X is a 2-separation of H then $|X| \leq 6$. By Proposition 3.3.10(d), H and hence G have no parallel edges. This implies that (w1)-(w4) hold and G is well-connected, i.e. (a) holds. Finally, (b) follows from Proposition 3.3.10(b). □

Let $\mathcal{S}$ be the set of T_4-representation that are equivalent to (G, R). Note that it suffices to show that $|\mathcal{S}| \in \mathcal{O}(1)$ since by Claim 1(b), every graft in $\mathcal{S}$ corresponds to at most 4 grafts equivalent to (H, T). Let $X_1, \ldots, X_k$ denote the maximal small 2-separations of G. By Proposition 3.3.21 $X_1, \ldots, X_k$ are pairwise disjoint, in particular, for all distinct $i, j \in [k]$, $\mathcal{I}_G(X_i) \cap \mathcal{I}_G(X_j) = \emptyset$. Because of Claim 2(b), $R \cap \mathcal{I}_G(X_j) \neq \emptyset$ for all $j \in [k]$. As $|R| = 4$ it follows that $k \leq 4$. Let $Y := E(G) - (X_1 \cup \ldots \cup X_k)$. Pick an arbitrary graft (G', R') from $\mathcal{S}$. By Proposition 3.3.22 there is an isomorphism f from $G[Y]$ to $G'[Y]$. Hence, G and G' only differ in the subgraphs induced by $X_1, \ldots, X_k$. As $k \leq 4$ and $|X_i| \leq 6$, G' is obtained from G by a sequence of 2-flips that is bounded by a constant. Finally, observe that G' determines the set of terminals R' uniquely (as an R-join of G must be an R'-join of G' [40], page 11). Hence, $|\mathcal{S}| \in \mathcal{O}(1)$ as required.

The proof of Proposition 3.3.11

Throughout this section M will denote a $(4, 6)$-connected pinch-graphic matroid. Also throughout this section (H, Γ) will denote a blocking-pair representation of M. We will need to show that the number of graphs H' equivalent to H for which (H', Γ) has a blocking pair is in $\mathcal{O}(|V(H)|)$. Note that by Proposition 3.3.7, (H, T) has at most one loop (that is odd). If (H, Γ) has no loop then we let $(G, \Sigma) = (H, \Gamma)$. If (H, Γ) has loop e then we let $G = H \setminus e$ and $\Sigma = \Gamma - e$.

Claim 1. *We may assume the following hold for the signed graph* (G, Σ)*,*

(a) G *is well-connected;*

(b) if X *is a* 2*-separation of* H *then* X *contains an odd polygon.*

Proof. We may assume $|E(G)| \geq 25$ for otherwise trivially the number of graphs equivalent to H is in $\mathcal{O}(1)$. By Proposition 3.3.8(a), H is 2-connected except for a unique possible loop. It follows that G is 2-connected. By Proposition 3.3.8(c), if X is a 2-separation of H then $\min\{|X|, |E(H) - X|\} \leq 6$. By Proposition 3.3.8(d), H and hence G have no parallel edges of the same parity, in particular, every parallel class has at most two edges. This implies that (w1)-(w4) hold and G is well-connected, i.e. (a) holds. Finally, (b) follows from Proposition 3.3.8(b). □

Let $\mathcal{S}$ be the set of pairs (G', v) where

i. G' is equivalent to G;

ii. (G', Σ) has a blocking pair v, w for some $w \in V(G')$.

We claim that it suffices to show that $|\mathcal{S}| \in \mathcal{O}(|V(G)|)$. Let $\mathcal{S}'$ denote the set of graphs H' equivalent to H for which (H', Γ) has a blocking pair. If (H, Γ) has no loop, then $\mathcal{S}' = \{G : (G, v) \in \mathcal{S}\}$. If (H, Γ) has a loop e then $\mathcal{S}'$ is the set of all graphs obtained by adding loop e at vertex v for graph G for all $(G, v) \in \mathcal{S}$. In both case $|\mathcal{S}'| = |\mathcal{S}| \in \mathcal{O}(|V(H)|)$ as required.

In the proof of Proposition 3.3.12 we could bound the number of maximal small 2-separations. Alas, this is not possible in this case. Tackling this more complex situation requires additional tools.

The blocking vertex lemma

First, we observe the following claim:

Claim 2. *Let G_1, G_2 be equivalent graphs where P is a path of both G_1 and G_2. For $i \in [2]$ let H_i be the graph obtained from G_i by adding an edge e between the ends of P. Then, H_1, H_2 are equivalent.*

Proof. Note that $P \cup e$ is a polygon of H_1 and H_2. The cycle space of H_i is generated by cycles not containing e and one cycle containing e. Since $H_i \setminus e = G_i$ and G_1, G_2 are equivalent, H_1, H_2 have the same cycle space and the result follows from Theorem 1.1.1. $\square$

Recall that a vertex of a signed graph is a *blocking vertex* if it intersects every odd polygon.

Claim 3. *Let G' be equivalent to G and let X be a maximal small 2-separation of G and thus of G'. Suppose that $G[E(G) - X]$ and $G'[E(G) - X]$ are isomorphic and have the same vertex labelling. Let $\{v, w\} = \partial_G(X) = \partial_{G'}(X)$. If v is a blocking vertex of both $(G[X], \Sigma \cap X)$ and $(G'[X], \Sigma \cap X)$, then G and G' are isomorphic.*

Proof of Claim 3. Let $X' \subseteq X$ be an (inclusion-wise) minimal 2-separation of G. Consider first the case where there exists two internally disjoint paths in $G[X']$ between $\partial_G(X')$. Then $G[X']$ and $G'[X']$ are isomorphic and we may use the same vertex labelling for $\mathcal{I}_G(X')$ and $\mathcal{I}_{G'}(X')$. Let $s \in \mathcal{I}_G(X')$. By Claim 1 there exists an odd polygon $C \subseteq X'$. Since v is a blocking vertex of $(G[X], \Sigma \cap X)$ and $(G'[X], \Sigma \cap X)$, C contains v in both G and G'. It follows that there exists an sv-path P_1 contained in X' in both G and G'. Consider now the case where there does not exists two internally disjoint paths in $G[X']$ between $\partial_G(X')$. Then since X' is minimal and since v is a blocking vertex of $(G[X], \Sigma \cap X)$, $X' = \{f, g, h\}$ where $\{f, g\}$ is an odd polygon, both f, g are incident to v and $\{f, g, h\}$ is a cut. Since $\{f, g\}$ is an odd polygon, and since v is a blocking vertex of $(G'[X], \Sigma \cap X)$, f, g are incident to v in G'. Then define P_1 as the path that consists of edge f. In both cases P_1 is a path of G and G' contained in X with an end in $\mathcal{I}_G(X')$ and an end v. Let t be a vertex in

$\mathcal{I}_G(E(G) - X) = \mathcal{I}_{G'}(E(G) - X)$. Let P_2 be an vt-path in $G[E(G) - X]$ and hence of $G'[E(G) - X]$. Then $P_1 \cup P_2$ is an st-path of both G and G'. Let H, H' be the graphs obtained from G, G' by adding an edge $e_{X'}$ joining s, t. By Claim 2, H, H' are equivalent. Let J, J' be the graphs obtained from G, G' by adding repeatedly $e_{X'}$ for every minimal 2-separation $X' \subseteq X$ of G. Then, J, J' are equivalent and 3-connected. Hence, J, J' are isomorphic and thus so are G and G'. □

Denote by $X_1, \ldots, X_k$ the maximal small 2-separations of G. By Proposition 3.3.21, $X_1, \ldots, X_k$ are pairwise disjoint. Let $Y = E(G) - (X_1 \cup \ldots \cup X_k)$. Here is our key result about blocking vertices.

Claim 4. *Let G' be equivalent to G and assume because of Proposition 3.3.22 that $G[Y]$ and $G'[Y]$ have the same vertex labeling. Let $i \in [k]$ and denote by v, w the vertices in $\partial_G(X_i) = \partial_{G'}(X_i)$. If v is a blocking vertex of both $(G[X_i], \Sigma \cap X_i)$ and $(G'[X_i], \Sigma \cap X_i)$, then $G[Y \cup X_i]$ and $G'[Y \cup X_i]$ are isomorphic.*

Proof. G' is obtained from G by a sequence of 2-flips. Since $X_1, \ldots, X_k$ are disjoint we can assume that all 2-flips on sets contained in X_i are done first. Then apply Claim 3 after all these 2-flips to deduce that $G[Y \cup X_i]$ and $G'[Y \cup X_i]$ are isomorphic as required. □

Case analysis

Observe that for distinct $i, j \in [k]$, $|\partial_G(X_i) \cap \partial_G(X_j)| \leq 1$ for otherwise this would contradict the maximality of the sets X_i, X_j. We say that a vertex v of G is *special* if there exists distinct $i, j, \ell \in [k]$ such that $v \in \partial_G(X_i) \cap \partial_G(X_j) \cap \partial(X_\ell)$. Similarly, we define special vertices of G'.

Claim 5. *If v is a special vertex of G then every blocking pair of (G, Σ) contains v.*

Proof. As v is special, there exists distinct $i, j, \ell \in [k]$ such that $v \in \partial_G(X_i) \cap \partial_G(X_j) \cap \partial(X_\ell)$. By Claim 1 each of X_i, X_j, X_ℓ contains an odd polygon C_i, C_j, C_ℓ, respectively. Since $V(C_i) - v, V(C_j) - v, V(C_\ell) - v$ are pairwise disjoint the result follows. □

It follows from the previous claim that there are at most two special vertices. We will thus consider three cases, namely, (1) there are two special vertices, (2) there is exactly one special vertex and (3) there is no special vertex.

Case 1. G has exactly two special vertices, say v and w.

We will prove that $|\mathcal{S}| \in \mathcal{O}(1)$ in this case. By Claim 5, $\{v, w\}$ is the unique blocking pair of (G, Σ). Note that there are three possibility for each $i \in [k]$,

i. $\partial_G(X_i) \cap \{v, w\} = \{v\}$,

ii. $\partial_G(X_i) \cap \{v, w\} = \{w\}$, or

iii. $\partial_G(X_i) \cap \{v, w\} = \{v, w\}$.

Pick an arbitrary pair $(G', v') \in \mathcal{S}$. For cases (i) and (ii), by Claim 4, $G[Y \cup X_i]$ and $G'[Y \cup X_i]$ are isomorphic. There is at most one $i \in [k]$ for case (iii). Thus the graph G' is obtained by a sequence of 2-flips on sets contained in X_i. Since $|X_i| \leq 6$, there are $\mathcal{O}(1)$ such graphs G'. Finally, observe that v, w are also special vertices of G'. In particular, v, w is the unique blocking pair of (G', Σ). It follows that $v' = v$ or $v' = w$. Thus $|\mathcal{S}| \in \mathcal{O}(1)$ as claimed.

Case 2. G has exactly one special vertex, say v.

Pick an arbitrary pair $(G', v') \in \mathcal{S}$. Let us partition the set $[k]$ as follows,

$$A_1 = \{i \in [k] : v \notin \partial_G(X_i)\}$$
$$A_2 = \{i \in [k] : v \in \partial_G(X_i) \text{ and } v \text{ is a blocking vertex of } (G'[X_i], \Sigma \cap X_i)\}$$
$$A_3 = \{i \in [k] : v \in \partial_G(X_i) \text{ and } v \text{ is not a blocking vertex of } (G'[X_i], \Sigma \cap X_i)\}$$

Claim 6. $|A_1| \leq 2$.

Proof. By Claim 5, every blocking pair of (G, Σ) contains v. Let v, w be an arbitrary blocking pair of (G, Σ). Let $i \in A_1$. By Claim 1(b), X_i contains an odd polygon. Since $v \notin \partial_G(X_i)$ either: (i) $w \in \mathcal{I}_G(X_i)$ or (ii) $w \in \partial_G(X_i)$. If we have $i \in A_1$ with outcome (i) then there is no $j \in A_1$, $j \neq i$ with either outcomes (i) or (ii). Since v is the unique special vertex of G, there are at most two $i \in [k]$ for which outcome (ii) holds. Hence, $|A_1| \leq 2$. □

Claim 7. $|A_3| \leq 1$.

Proof. Suppose for a contradiction there exists distinct $i, j \in A_3$. As v is special, there exists ℓ with $\delta_G(X_\ell) \ni v$. Then there exists odd polygons $C_i \subseteq X_i$ and $C_j \subseteq X_j$ of G' avoiding v. Moreover, there exists an odd polygon $C_\ell \subseteq X_\ell$. But then C_i, C_j, C_ℓ are vertex disjoint in G' which contradicts the fact that (G', Σ) has a blocking pair. □

For $i \in [3]$, let $Z_i = \bigcup(X_i : i \in A_i)$. It follows by Claim 4 that $G[Y \cup Z_2]$ and $G'[Y \cup Z_2]$ are isomorphic. Consider first the case where $A_3 = \emptyset$ and let $(G', v') \in \mathcal{S}$. Then G' is obtained from G by a sequence of 2-flips that are contained in Z_1. Since G is well-connected, $|X_i| \leq 6$ for all $i \in [k]$ and by Claim 6, $|Z_1| \leq 12$. Hence, there are $\mathcal{O}(1)$ such graphs G'. Trivially, there are $|V(G)|$ choices for the vertex v', thus $|\mathcal{S}| \in \mathcal{O}(|V(G)|)$ in this case. Consider now the case where $A_3 \neq \emptyset$ and let $(G', v') \in \mathcal{S}$. Then by Claim 7 there is a unique element $\hat{\imath} \in A_3$. Then G' is obtained from G by a sequence of 2-flips that are contained in $Z_1 \cup Z_3$.

Since $|Z_1 \cup Z_3| \leq 18$ and since there are at most $|A_3| \leq |V(G)|$ choices to pick the element $\hat{\imath}$ in A_3 the number of such possible graphs G' is in $\mathcal{O}(|V(G)|)$. Observe that every blocking pair of (G', Σ) consists of v and a vertex of $G'[X_{\hat{\imath}}]$. Thus, there are $\mathcal{O}(1)$ choices for the vertex v' and $|\mathcal{S}| \in \mathcal{O}(|V(G)|)$ in this case as well.

Case 3. G has no special vertex.

Let v, w denote a blocking pair of (G, Σ). For every $i \in [k]$, X_i contains an odd polygon. It follows that for all $i \in [k]$ either (i) $\{v, w\} \cap \partial_G(X_i) \neq \emptyset$ or (ii) $\{v, w\} \cap \mathcal{I}_G(X_i) \neq \emptyset$. Since neither v nor w are special there are at most 4 elements in $[k]$ for which (i) holds. Trivially, (ii) can hold for at most 2 elements in $[k]$. Thus $k \leq 6$. Let $(G', v') \in \mathcal{S}$. Let $Z = \bigcup(X_i : i \in [k])$ then $|Z| \leq 6k = 36$. Then G' is obtained from G by a sequence of 2-flips that are contained in Z. Hence, there are $\mathcal{O}(1)$ such graphs G'. Trivially, there are $|V(G)|$ choices for the vertex v', thus $|\mathcal{S}| \in \mathcal{O}(|V(G)|)$ in this case.

In all cases, we have $|\mathcal{S}| \in \mathcal{O}(|V(G)|)$ which completes the proof of Proposition 3.3.11.

Chapter 4

Separations

The work in this chapter appears in [22, 31]. The goal of this chapter is to characterize 1-, 2- and 3-separations of pinch-graphic matroids. Let M, M_1 and M_2 be binary matroids such that $M = M_1 \oplus_k M_2$ for some $k \in [3]$. Then, we say that the k-separation $X = E(M_1) - E(M_2)$ of M is *reducible* if M_1 or M_2 are graphic. First, we restate the following propositions from Section 1.4 that characterize reducible separations.

Proposition 1.4.1. *Let $M = M_1 \oplus_k M_2$ for $k \in [3]$ where M_1 is graphic. If $k = 2$, assume that M is 2-connected, and if $k = 3$, assume that M is 3-connected. Then, M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proposition 1.4.2. *Every 1- and 2-separation of a pinch-graphic matroid is reducible.*

Propositions 1.4.1 and 1.4.2 will be proved in Sections 4.1 and 4.2, respectively. We will use these propositions to construct a recognition algorithm for pinch-graphic matroids in Chapter 5.

As seen in Section 1.4.3, there exist 3-separations that are not reducible. In Section 4.3, we will characterize non-reducible 3-separations, i.e., we will prove the following proposition:

Proposition 1.4.3. *Let M be a 3-connected pinch-graphic matroid and let X' be a proper 3-separation. Then there exists a homologous proper 3-separation X that is reducible, compliant, or recalcitrant.*

4.1 Reducible separations

4.1.1 Sums

Let us recall 1- and 2-sums from Section 3.1.3. Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 1$. Suppose that $E_1 \cap E_2 = \emptyset$. Then, we define the *1-sum* M of M_1, M_2, denoted by $M_1 \oplus_1 M_2$, as follows: the ground set of M is $E := E_1 \cup E_2$ and a subset C of E is a circuit of M if and only if C is either a circuit of M_1 or a circuit of M_2. Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 3$. Suppose that $E_1 \cap E_2 = \{\Omega\}$ and that Ω is not a loop and not a coloop of M_i for $i \in [2]$. Then, we define the *2-sum* M of M_1, M_2, denoted by $M_1 \oplus_2 M_2$, as follows: the ground set of M is $E := E_1 \Delta E_2$ and a subset C of E is a circuit of M if and only if either C is a circuit of $M_1 \setminus \Omega$ or $M_2 \setminus \Omega$, or $C = C_1 \Delta C_2$ where for $i \in [2]$, C_i is a circuit of M_i containing Ω.

Now, we define 3-sums. Let M_1, M_2 be binary matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 7$. Suppose that $E_1 \cap E_2 = D$ where $|D| = 3$ where for $i \in [2]$, D is a circuit of M_i and D contains no cocircuit of M_i. Then, we define the *3-sum* M of M_1, M_2, denoted by $M_1 \oplus_3 M_2$, as follows: the ground set of M is $E := E_1 \Delta E_2$ and a subset C of E is a circuit of M if and only if either C is a circuit of $M_1 \setminus D$ or $M_2 \setminus D$ or $C = C_1 \Delta C_2$ where for some $e \in D$ and for $i \in [2]$, C_i is a circuit of M_i with $C_i \cap D = \{e\}$. Note that 3-sums are defined for binary matroids only. If we have matroids

$M = M_1 \oplus_1 M_2$ then M_1 and M_2 are restrictions of M and in particular are minors of M. Analogous results hold for 2- and 3-sums [44]:

Proposition 4.1.1. *Suppose that $M = M_1 \oplus_2 M_2$ where M is 2-connected, or that $M = M_1 \oplus_3 M_2$ where M is 3-connected and binary. Then M_1 and M_2 are isomorphic to minors of M.*

4.1.2 Completion

Recall Propositions 3.1.8 and 3.1.9 from Section 3.1.3 and the completion with respect to a 2-separation.

Proposition 3.1.8. *Let M be a matroid with matrix representation A and let $X \subseteq E(M)$. We denote by $\langle X \rangle$ the vector space spanned by the columns of A indexed by X. Then*

$$\lambda_M(X) = \dim\left[\langle X \rangle \cap \langle E(M) - X \rangle\right].$$

Proposition 3.1.9. *Let M be a binary matroid with a 2-separation X. Let N be the completion of M with respect to X. Then $M = (N \setminus X) \oplus_2 (N \setminus E(M) - X)$.*

Similarly, we define a completion with respect to 3-separation. Let M be a binary matroid with matrix representation A and let X be a 3-separation of M. Then $\lambda_M(X) = 2$. By Proposition 3.1.8, $dim\left[\langle X \rangle \cap \langle E(M) - X \rangle\right] = 2$. Thus there exists non-zero $0, 1$ vectors p, q for which $\langle \{p, q\} \rangle = \langle X \rangle \cap \langle E(M) - X \rangle$. Let A^+ be obtained from matrix A by adding columns p, q and $r = p + q$ (where the sum is taken over the two element field). Note that the set $\{p, q, r\}$ is uniquely determined by $\langle X \rangle \cap \langle E(M) - X \rangle$. Let N be the binary matroid represented by matrix A^+. Then N is the *completion of M with respect to the 3-separation X*. Next, we explain the relevance of the notion of completion for 3-separations, which is analogous to Proposition 3.1.9.

Proposition 4.1.2. *Let M be a binary matroid with a proper 3-separation X. Let N be the completion of M with respect to X. Then $M = (N \setminus X) \oplus_3 (N \setminus E(M) - X)$.*

The proof of Proposition 4.1.2 is easy and similar to that of Proposition 3.1.9 so we shall omit it. The following straightforward observation will allow us to construct completions for 3-separations.

Remark 4.1.3. *Let M be a binary matroid with a proper 3-separation X. Let N be a binary matroid where $M = N \setminus \{\Omega_1, \Omega_2, \Omega_3\}$ and where $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. Suppose for $i \in [2]$ we have cycles C_i and D_i of N where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq (E(M) - X) \cup \Omega_i$. Then, N is the completion of M with respect to X.*

4.1.3 Examples of 3-sums

For a signed graph (G, Σ) and $X \subseteq E(G)$, we denote $(G, \Sigma)|X = (G, \Sigma) \setminus E(G) - X$.

Proposition 4.1.4. *Let $M = \mathrm{ecycle}(G, \Sigma)$ with a proper 3-separation X and let $Y = E(M) - X$. Suppose that $p[(G, \Sigma)|X] = p[(G, \Sigma)|Y] = 1$, that $G[X]$, $G[Y]$ are connected and that $\partial_G(X) = \{a, b\}$ where a, b are distinct vertices. Let (G_1, Σ_1) (resp. (G_2, Σ_2)) be obtained from $(G, \Sigma)|X$ (resp. $(G, \Sigma)|Y$) by adding an even edge $\Omega_1 = (a, b)$, an odd edge $\Omega_2 = (a, b)$ and an odd loop Ω_3. Then $M = \mathrm{ecycle}(G_1, \Sigma_1) \oplus_3 \mathrm{ecycle}(G_2, \Sigma_2)$.*

Proof. Let (H, Γ) be the signed-graph obtained from (G, Σ) by adding an even edge $\Omega_1 = (a, b)$, an odd edge $\Omega_2 = (a, b)$ and an odd loop Ω_3. Since $(G, \Sigma)|X$ is connected and non-bipartite, it has an even $\{a, b\}$-join J_1 and an odd $\{a, b\}$-join J_2 and since $(G, \Sigma)|Y$ is connected and non-bipartite, it has an even ab-join K_1 and an odd ab-join K_2. For $i \in [2]$, let $C_i = J_i \cup \Omega_i$ and let $D_i = K_i \cup \Omega_i$. Then for $i \in [2]$, C_i and D_i are cycles of $N := \text{ecycle}(H,\Gamma)$ where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq Y \cup \Omega_i$. Moreover, $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. It follows from Remark 4.1.3 that N is the completion of M with respect to X. By Proposition 4.1.2, $M = (N \setminus Y) \oplus_3 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecycle}(H,\Gamma) \setminus Y = \text{ecycle}(G_1, \Sigma_1)$ and $N \setminus X = \text{ecycle}(H,\Gamma) \setminus X = \text{ecycle}(G_2, \Sigma_2)$. □

Proposition 4.1.5. *Let $M = \text{ecycle}(G,\Sigma)$ with a proper 3-separation X and let $Y = E(M) - X$. Suppose that $p[(G,\Sigma)|X] = 1$, $p[(G,\Sigma)|Y] = 0$, that $G[X]$, $G[Y]$ are connected and that $\partial_G(X) = \{a,b,c\}$ where a,b,c are distinct vertices. Then we may assume, after possibly re-signing, that $\Sigma \subseteq X$. Let G_1 (resp. G_2) be obtained from $G[X]$ (resp. $G[Y]$) by adding edges $\Omega_1 = (a,b)$, $\Omega_2 = (b,c)$ and $\Omega_3 = (a,c)$. Then $M = \text{ecycle}(G_1,\Sigma) \oplus_3 \text{cycle}(G_2)$.*

Proof. Let H be the graph obtained from G by adding edges $\Omega_1 = (a,b)$, $\Omega_2 = (b,c)$ and $\Omega_3 = (a,c)$. Since $(G,\Sigma)|X$ is connected and non-bipartite, it has an even ab-join J_1 and an even ab-join J_2 and since $G[Y]$ is connected, it has an ab-join K_1 and an ab-join K_2. For $i \in [2]$, let $C_i = J_i \cup \Omega_i$ and let $D_i = K_i \cup \Omega_i$. Then for $i \in [2]$, C_i and D_i are cycles of $N := \text{ecycle}(H,\Sigma)$ where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq Y \cup \Omega_i$. Moreover, $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. It follows from Remark 4.1.3 that N is the completion of M with respect to X. By Proposition 4.1.2, $M = (N \setminus Y) \oplus_3 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecycle}(H,\Sigma) \setminus Y = \text{ecycle}(G_1,\Sigma)$ and $N \setminus X = \text{ecycle}(H,\Sigma) \setminus X = \text{cycle}(G_2)$. □

4.1.4 Applications

In this section we prove Proposition 1.4.1 by proving Propositions 4.1.6, 4.1.7 and 4.1.8.

Proposition 4.1.6. *Let $M = M_1 \oplus_1 M_2$ where M_1 is graphic. Then M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proof. If M is pinch-graphic so is $M_2 = M \setminus E(M_1)$ as pinch-graphic matroids form a minor closed class. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \text{ecycle}(G_2, \Sigma)$ for some signed-graph (G_2, Σ) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \text{cycle}(G_1)$ for some graph G_1. Then $M = \text{ecycle}(G, \Sigma)$ where G is the union of G_1 and G_2. As v, w is a blocking pair of (G, Σ), M is pinch-graphic. $\square$

Proposition 4.1.7. *Let $M = M_1 \oplus_2 M_2$ where M_1 is graphic and M is 2-connected. Then M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proof. If M is pinch-graphic then so is M_2 since M_2 is isomorphic to a minor of M (Proposition 4.1.1) and pinch-graphic matroids form a minor closed class. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \text{ecycle}(G_2, \Sigma_2)$ for some signed-graph (G_2, Σ_2) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \text{cycle}(G_1)$ for some graph G_1. Denote by e the unique element in $E(M_1) \cap E(M_2)$. By definition of 2-sums, e is not a loop or a co-loop of M_1 or M_2. In particular, e is not a loop of G_1 and not a bridge of G_1 or G_2.

Case 1. e is not a loop of G_2.

After possibly re-signing (G_2, Σ_2) we may assume $e \notin \Sigma_2$. Let G be obtained from G_1 and G_2 by identifying edge e and then deleting e. Let $M' = \text{ecycle}(G, \Sigma_2)$. Proposition 3.1.5 and the fact that e is not a bridge of G_1, G_2 implies that $\lambda_{M'}(X) = 1$, thus X is a 2-separation of M'. By Proposition 3.1.12 $M' = \text{cycle}(G_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_2 M_2$. Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic.

Case 2. e is a loop of G_2.

Since e is not a loop of M_2, $e \in \Sigma_2$, thus e is incident to v or w in G_2. Suppose r, s denote the ends of e in G_1. Then let $\Sigma_1 = \delta_{G_1}(r)$ and let G_1' be the graph obtained from G_1 by identifying r and s. Note that e is an odd loop of (G_1', Σ_1) with ends $r = s$. Let G be obtained from G_1 and G_2 by identifying the vertex incident to e and then deleting e. Let $M' = \text{ecycle}(G, \Sigma_1 \cup \Sigma_2 - e)$. Proposition 3.1.5, and the fact that e is not a bridge of G_1, implies that $\lambda_{M'}(X) = 1$, thus X is a 2-separation of M'. By Proposition 3.1.11, $M' = \text{ecycle}(G_1', \Sigma_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_2 M_2$. Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic. □

Proposition 4.1.8. *Let $M = M_1 \oplus_3 M_2$ where M_1 is graphic and M is 3-connected. Then M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proof. Sufficiency follows as in Proposition 4.1.7. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \text{ecycle}(G_2, \Sigma_2)$ for some signed-graph (G_2, Σ_2) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \text{cycle}(G_1)$ for some graph G_1. Denote by $D = \{e, f, g\} = E(M_1) \cap E(M_2)$. By definition of 3-sum, D is a circuit of M_1 of M_2. In particular, D is a polygon of G_1. Also by definition of 3-sum D does not contain a cocircuit of M_1 or M_2. Hence, D does not contain a cut of G_1 or G_2. After possibly interchanging the roles of e, f, g we may assume that one of Case 1 or Case 2 occurs.

Case 1. D is a polygon of G_2.

After possibly re-signing (G_2, Σ_2) we may assume that $\Sigma_2 \cap D = \emptyset$. Let G be obtained from G_1, G_2 by identifying e, identifying f, identifying g, and deleting e, f, g. Let $M' = \text{ecycle}(G, \Sigma_2)$. Proposition 3.1.5 and the fact that D does not contain a cut of G_1, G_2, implies that $\lambda_{M'}(X) = 2$, thus X is a 3-separation of M'. By Proposition 4.1.5 $M' = \text{cycle}(G_1) \oplus_3 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_3 M_2$. Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic.

Case 2. e, f are parallel and g is a loop in G_2. Moreover, $\{e, f, g\} \cap \Sigma_2 = \{f, g\}$.

Let r be the vertex of G_1 incident to g, f and let s be the vertex of G_1 incident to g, e. Then let $\Sigma_1 = \delta_{G_1}(r)$ and let G_1' be the graph obtained from G_1 by identifying r and s. Note that g is an odd loop of (G_1', Σ_1) with ends $r = s$. Since $\{e, f\}$ is an odd polygon of (G_2, Σ_2) we may assume that one of the end of e, f is vertex v of the blocking pair v, w and that g is incident to v in G_2. Let G be obtained from G_1 and G_2 by identifying vertex $r = s$ of G_1 with vertex v of G_2, by identifying the other end of e, f, and then deleting e, f, g. Let $\Gamma = (\Sigma_1 \cup \Sigma_2) - \{e, f, g\}$. Let $M' = \text{ecycle}(G, \Gamma)$. Proposition 3.1.5 and the fact that D does not contain a cut of G_1, G_2 implies that $\lambda_{M'}(X) = 2$, thus X is a 3-separation of M'. By Proposition 4.1.4 $M' = \text{ecycle}(G_1', \Sigma_1) \oplus_3 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_3 M_2$. Thus $M = M'$ and in particular (G, Γ) is a representation of M. Note that v, w is a blocking pair of (G, Γ), hence, M is pinch-graphic. □

4.2 1- and 2-separations

4.2.1 1-separations

Recall that if a bipartite signed-graph is a representation of an even-cycle matroid, then that matroid is graphic.

Proposition 4.2.1. *If X is a 1-separation of an even-cycle matroid M, then X is reducible.*

Proof. Let $M = M_1 \oplus_1 M_2$ for some M_1, M_2 where $X = E(M_1)$, and let $Y = E(M_2)$.

We may assume that M is not graphic for otherwise so is M_1 and X is reducible. Thus M has a non-bipartite representation (G, Σ). After possible 1-flips, we may assume that G is connected. Let H be the auxiliary graph for X and (G, Σ). Since $\lambda_M(X) = 0$, Proposition 3.1.6 (b) implies $|V(H)| + 0 - p[(G, \Sigma)|X] - p[(G, \Sigma)|Y] \geq |V(H)| - 1$, or equivalently, $p[(G, \Sigma)|X] + p[(G, \Sigma)|Y] \leq 1$. Thus $(G, \Sigma)|X$ or $(G, \Sigma)|Y$ is bipartite. As $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are representations of M_1 and M_2 respectively, at least one of M_1, M_2 is graphic, i.e. X is reducible. □

4.2.2 Preliminaries

It remains to prove the following analogous result for 2-separations.

Proposition 4.2.2. *If X is a 2-separation of a 2-connected pinch-graphic matroid M, then X is reducible.*

Before we can proceed with the proof we require some preliminaries. Recall that a *theta* is a graph H with two distinct vertices $r, s \in V(H)$ that consists of three internally disjoint rs-paths P_1, P_2, P_3 (all vertices of H except r, s have degree two). If $|P_1| = |P_2| = |P_3| = k$ for some integer k (where P_i are viewed as subsets of edges), then the theta graph is *k-uniform*. Consider now a graph H that is obtained from two disjoint polygons C_1, C_2 by identifying a vertex of C_1 with a vertex of C_2. Recall that H is called a *double ear*. If $|C_1| = |C_2| = k$ for some integer k (where C_i are viewed as subsets of edges) then the double ear is *k-uniform*. In Figure 3.1(i)(p. 72) top graph, we have a 2-uniform double ear, in Figure 3.1(ii) top graph, we have a 1-uniform theta graph, and in Figure 3.1(iii) top graph, we have a 2-uniform theta graph. Recall the following remark from Section 3.1.2.

Remark 3.1.7. *If H is a 2-edge-connected graph where $|E(H)| = |V(H)| + 1$ then H is a theta or a double ear.*

Consider a signed-graph (G, Σ) and vertices $v_1, v_2 \in V(G)$ where every edge of Σ is incident to either v_1 or v_2. So v_1, v_2 is a blocking pair of (G, Σ). Consider first the case where there is no odd edge between v_1 and v_2 and there is no odd loop. Let H be obtained from G by, for $i \in [2]$, splitting v_i into v'_i, v''_i such that $\delta_H(v'_i) = \delta_G(v_i) \cap \Sigma$. Then let G' be obtained from H by identifying v'_1 and v''_2 to a new vertex w_1 and by identifying v'_2 and v''_1 to a new vertex w_2. If (G, Σ) has an odd loop f, then f will have ends w_1, w_2 in G' and if (G, Σ) has an odd edge g with ends v_1, v_2 then g will be an odd loop of (G', Σ). We then say that (G', Σ) is obtained from (G, Σ) by a *Lovász-flip* on v_1, v_2 and that w_1, w_2 is the *resulting blocking pair*. Informally, G' is obtained from G by exchanging the odd edges incident to v_1 with the odd edges incident to v_2 where odd loops and odd edges between v_1 and v_2 behave like odd walks of length two. It is not difficult to see that Lovász-flips preserve even cycles [40, 11]. In Figure 4.1 we illustrate a pair of signed-graphs related by a Lovász-flip. Vertices v_1 and v_2 are indicated in white. Odd edges correspond to dashed lines. Even edges are unchanged.

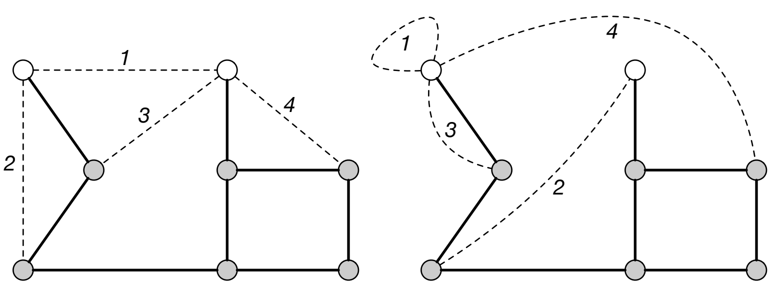

Figure 4.1: Lovász-flip.

4.2.3 2-separations

Since X is a 2-separation of M, $M = M_1 \oplus_2 M_2$ for some matroids M_1, M_2 where $X = E(M_1) - E(M_2)$. Let $Y = E(M_2) - E(M_1)$. We may assume that M_1, M_2 are not graphic; otherwise, X is reducible. Let e denote the unique element in $E(M_1) \cap E(M_2)$. Since M is an even-cycle matroid it has a representation (G, Σ). Note that (G, Σ) is not bipartite for otherwise M is graphic and then by Proposition 4.1.1 so is M_1, a contradiction. Among all possible connected representation (G, Σ) of M pick one according to the following priorities,

(m1) (G, Σ) has a blocking pair and $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are both non-bipartite,

(m2) $(G, \Sigma)|X$ is bipartite and $(G, \Sigma)|Y$ is non-bipartite and we minimize $\kappa(G[X]) + \kappa(G[Y])$.

(m3) $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are both non-bipartite and we minimize $\kappa(G[X]) + \kappa(G[Y])$.

Note for (m2) and (m3) we do not require that (G, Σ) have a blocking pair. Let H denote the auxiliary graph for X and (G, Σ). Since G is connected, so is H.

Claim 1. *(G, Σ) is not picked according to (m1).*

Subproof. Suppose otherwise. Proposition 3.1.6 implies, $|E(H)| = |V(H)| + 1 - 1 - 1 = |V(H)| - 1$. Since H is connected, H is a tree. Denote by $X_1, \ldots, X_p$ the connected components of $G[X]$ and by $Y_1, \ldots, Y_q$ the connected components of $G[Y]$. For all $j \in [p]$, $(G, \Sigma)|X_j$ is non-bipartite, for otherwise X_j is a 1-separation of M by Proposition 3.1.5, a contradiction. Similarly, $(G, \Sigma)|Y_j$ is non-bipartite for all $j \in [q]$. If for all $j \in [p]$, $(G, \Sigma)|X_j$ has a blocking vertex and for all $j \in [q]$, $(G, \Sigma)|Y_j$ has a blocking vertex then after a sequence of 1-flips there exists a blocking vertex of (G, Σ), a contradiction as Remark 2.4.2 then implies that M is graphic. Thus we may assume that $(G, \Sigma)|X_1$ has no blocking vertex and that $(G,\Sigma)|X_1$ has a blocking pair v,w of (G,Σ). Note that $p=1$ since otherwise $(G,\Sigma)|X_2$ contains an odd polygon that avoids v,w, a contradiction. After possible 1-flips we may assume that v is vertex of $G[Y_j]$ for all $j \in [q]$, in particular, $q=1$. Thus $G[X]$ and $G[Y]$ are both connected and $v \in \partial_G(X)$. Let G_1 be obtained from $G[X]$ by adding a loop e_1 incident to vertex v and let G_2 be obtained from $G[Y]$ by adding a loop e_2 incident to vertex v. Let $\Sigma_1 = (\Sigma \cap X) \cup e_1$ and $\Sigma_2 = (\Sigma \cap Y) \cup e_2$. It follows by Proposition 3.1.11 that $M = \text{ecycle}(G,\Sigma) = \text{ecycle}(G_1,\Sigma_1) \oplus_2 \text{ecycle}(G_2,\Sigma_2)$. Thus $M_2 = \text{ecycle}(G_2,\Sigma_2)$. Finally, note that (G_2,Σ_2) has a blocking vertex v. It follows from Remark 2.4.2 that M_2 is graphic, a contradiction. $\Diamond$

Claim 2. *(G,Σ) is not picked according to (m2).*

Subproof. Suppose otherwise. Since $(G,\Sigma)|X$ is bipartite, we may assume after possibly re-signing that $\Sigma \subseteq Y$. Proposition 3.1.6 implies $|E(H)| = |V(H)| + 1 - 0 - 1 = |V(H)|$. We picked (G,Σ) that minimizes $\kappa(G[X]) + \kappa(G[Y])$. Since we can rearrange G by 1-flips, H is bridgeless, hence H is a polygon. Since we can rearrange G by 2-flips, $|V(H)| = 2$, i.e. $G[X]$ and $G[Y]$ are connected and $\partial_G(X) = \{a,b\}$ for some distinct vertices a,b. Let G_1 be obtained from $G[X]$ by adding an edge e_1 between a,b and let G_2 be obtained from $G[Y]$ by adding an edge e_2 between a,b. It follows by Proposition 3.1.12 that $M = \text{ecycle}(G,\Sigma) = \text{cycle}(G_1) \oplus_2 \text{ecycle}(G_2,\Sigma)$. Hence, $M_1 = \text{cycle}(G_1)$ is graphic, a contradiction. $\Diamond$

It follows from Claim 1 and Claim 2 that $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are both bipartite. Proposition 3.1.6 implies that $|E(H)| = |V(H)| + 1 - 0 - 0 = |V(H)| + 1$. Hence, by Remark 3.1.7 H is either a theta or a double ear. By the choice (m3) and since we can rearrange G by 1-flips, H is bridgeless. (Note, that here we are free to perform 1-flips and 2-flips as we are not trying to preserve blocking pairs).

Case 1. H is a theta.

H consists of three internally disjoint path P_1, P_2, P_3. By (m3) and since we can rearrange G by 2-flips, for $j \in [3]$, $|P_j| \in [2]$. As H is bipartite, $|P_1|, |P_2|, |P_3|$ have the same parity. Hence, the theta H is 1 or 2-uniform. Consider first the case where H is 1-uniform. Then $G[X]$ and $G[Y]$ are connected and $|\partial_G(X)| = 3$. Since $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are bipartite some vertex of $\partial_G(X)$ is a blocking vertex. This implies by Remark 2.4.2 that M is graphic, a contradiction. Consider now the case where H is 2-uniform. After possibly interchanging the role of X and Y, (G, Σ) is of the form given in Figure 4.2 where X corresponds to the shaded region. Since $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are bipartite, some vertex of $\partial_G(X)$ is a blocking vertex. Again, this implies by Remark 2.4.2 that M is graphic, a contradiction.

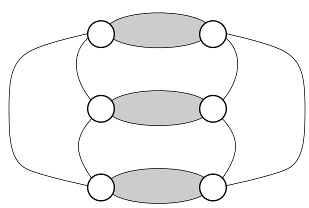

Figure 4.2: A theta.

Case 2. H is a double ear.

H consists of two polygons C_1, C_2 sharing a single vertex. Since H is bipartite $|C_1|, |C_2| \geq 2$. By (m3) and since we can rearrange G by 2-flips, $|C_1| = |C_2| = 2$, i.e. the double ear H is 2-uniform. Thus $V(H) = \{r, s, t\}$, $E(H) = \{a = rs, b = rs, c = st, d = st\}$. After possibly interchanging the role of X and Y, we may assume $X = R \cup T$ where R and T are the components of $G[X]$ corresponding to $r, t \in V(H)$, and that $Y = S$ where S is the component of $G|Y$ corresponding to $s \in V(H)$. Then $a, b, c, d \in E(H)$ correspond to vertices in $\partial_G(X)$ where, $\partial_G(R) = \{a, b\}$ and $\partial_G(T) = \{c, d\}$. We illustrate (G, Σ) in

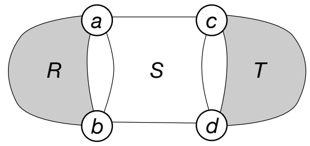

Figure 4.3: A double ear.

Figure 4.3 where X correspond to the shaded region, Since $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are bipartite after possibly re-signing we have $\Sigma = [\delta_G(a) \cap R] \cup [\delta_G(c) \cap S]$. Let (G', Σ) be obtained from a Lovász-flip on the blocking pair a, c. Then observe that $\partial_{G'}(X) = \{b, d\}$, that $(G', \Sigma)|X$ is bipartite and that $(G', \Sigma)|Y$ is non-bipartite. But then (G', Σ) is a representation as in (m2) contradicting our choice of representation.

4.3 Structure theorem for 3-separations

4.3.1 Representation of 3-connected even-cycle matroid

We will use the following multiple times,

Proposition 4.3.1. *Let M be a 3-connected even cycle matroid with representation (G, Σ). Then (G, Σ) has at most one loop, which is odd. Moreover, if we let G' be obtained from G by deleting that loop (if it exists) then G' is 2-connected and every cut of G contains at least three edges.*

Proof. Since M is 3-connected, M has no loops and parallel elements. Suppose e is a loop of G. Then $e \in \Sigma$, for otherwise e is a loop of M. Moreover, if G has distinct loops e, f then $\{e, f\}$ is a circuit of M, i.e. e, f are parallel, a contradiction. We may assume that

G' is connected by identifying components as we will show that G' is in fact 2-connected. Suppose that G' has a cut vertex v. Then there exists $X \subseteq E(G')$ such that $\partial_{G'}(X) = \{v\}$ and $G'[X]$ and $G'[E(G') - X]$ are both connected. By moving the loop e of G (if it exists) we may assume that $G[X]$ and $G[E(G) - X]$ are also connected. Then, by Proposition 3.1.5, $\lambda_M(X) = |\,\partial_G(X)| - \kappa(G[X]) - \kappa(G[E(M) - X]) + p[(G,\Sigma)|X] + p[(G,\Sigma)|E(G) - X] \leq 1 - 1 - 1 + 1 + 1 = 1$, thus X is 2-separating. If $|X| = 1$, then the unique element in X is a bridge of G, i.e. a coloop of M, a contradiction. Thus $|X| \geq 2$ and similarly, $|E(G) - X| \geq 2$ and it follows that X is a 2-separation, a contradiction. Finally, if D is a cut of G then D is cocycle of M ([40], Remark 2.1). It follows that $|D| \geq 3$. □

4.3.2 The statement

First, we recall a few definitions from Section 1.4.3. Given a matroid M and $X \subseteq E(M)$, denote by $\mathrm{cl}_M(X)$ the *closure* of X for matroid M. Let M be a matroid and let $X \subseteq E(M)$ be a proper 3-separation. Suppose $|X| \geq 5$ and suppose that there exists $e \in X$ with $e \in \mathrm{cl}_{M^*}(E(M) - X)$ and $e \in \mathrm{cl}_{M^*}(X - e)$. Then, $X - e$ is also a proper 3-separation of M. We say that $X - e$ is *homologous* to X and so is any set that is obtained by repeat application of the aforementioned procedure. Let (G, Σ) be a connected signed graph and consider $X \subseteq E(G)$. The triple (G, Σ, X) is a *Type I* or *Type II configuration* if $|X|, |E(G) - X| \geq 4$, $G[X]$, $G[E(G) - X]$ are both connected, and $|\,\partial_G(X)| = 2$. Recall that X is *compliant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type I configuration. We say that X is *recalcitrant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type II configuration. Now, let us restate Proposition 1.4.3.

Proposition 1.4.3. *Let M be a 3-connected pinch-graphic matroid and let X' be a proper 3-separation. Then there exists a homologous proper 3-separation X that is reducible,* *compliant, or recalcitrant.*

The proof of this result will share some commonality with that of Proposition 4.2.2, namely we will analyze the auxiliary graph H for a representation (G, Σ) and separation X homologous to X'. However, when $(G, \Sigma)|X$ and $(G, \Sigma)|E(M) - X$ are both bipartite, then $|E(H)| = |V(H)| + 3$ and a simple-minded analysis of all possible graphs H becomes very complicated. Instead we will prove a key property in Section 4.3.6 that will bypass most of the case analysis. The proof of Proposition 1.4.3 will be organized as follows: Section 4.3.3 presents basic results about closure, Section 4.3.4 indicates how to pick a suitable representation (G, Σ) and a separation X of M. A key property of the auxiliary graph is derived in Sections 4.3.5 and 4.3.6. Finally in Section 4.3.7, we analyze the auxiliary graph, completing the proof.

4.3.3 Closure and small separations

For 3-connected matroids we have the following characterization of homologous separations.

Proposition 4.3.2 (Lemma 3.1 [37]). *Let M be a 3-connected matroid where $|E(M)| \geq 9$ and let $X \subseteq E(M)$ be a proper 3-separation. Suppose $|X| \geq 5$ and that there exists $e \in X$ with $e \in \mathrm{cl}_{M^*}(E(M) - X)$. Then $X - e$ is homologous to X.*

Observe that we dropped one of the condition from the definition of homologous separations.

Next we describe what it means for an element to be in the co-closure of a set, for the case of even-cycle matroids.

Remark 4.3.3. *Let $M = \mathrm{ecycle}(G, \Sigma)$ and let $X \subseteq E(M)$, $e \in E(M) - X$. Then the following are equivalent,* *(a)* $e \in \mathrm{cl}_{M^*}(X)$,

(b) there exists a signature D *of* (G, Σ) *or a cut* D *of* G *where* $D - X = \{e\}$.

Proof. Clearly, $e \in \mathrm{cl}_{M^*}(X)$ if and only if there exists a cocircuit D of M with $D - X = \{e\}$. Moreover, cocircuits of M are signatures of (G, Σ) and cuts of G ([40] Remark 2.1). □

In the proof of Proposition 1.4.3, we will consider a proper 3-separations X. We will want to check if $X - e$ is an homologous proper 3-separation for some $e \notin X$. A trivial reason for this not to be the case is if $|X| = 4$. Let us study such 3-separations,

Proposition 4.3.4. *Let* M *be a* 3*-connected binary matroid with a* 3*-separation* X *where* $|X| = 4$. *Then*

(a) $r_M(X) = r_{M^*}(X) = 3$, *and*

(b) X *contain both a circuit and a cocircuit where each have at least three elements.*

Proof. Recall, that $r_{M^*}(X) = |X| - \big[r(M) - r_M(E(M) - X)\big]$.

$r_M(X) + r_{M^*}(X) = r_M(X) + |X| - r(M) + r_M(E(M) - X) = |X| + \lambda_M(X) = 4 + 2 = 6.$

Observe that $r_M(X) \geq 3$, for otherwise as $|X| = 4$, M would have a loop or a pair of parallel elements contradicting the fact that M is 3-connected. Similarly $r_{M^*}(X) \geq 3$ and thus (a) holds. Finally (a) and $|X| > 3$ implies (b). □

4.3.4 Choice of representation and separation

Throughout the proof of Proposition 1.4.3, M denotes a 3-connected pinch-graphic matroid and X' is a proper 3-separation. Since M is pinch-graphic, there exists a signed-graph (G, Σ) such that

i. (G, Σ) is a representation of M,

ii. (G, Σ) has a blocking pair, and

iii. G is connected.

We make the following *minimality assumption*. Among all choices of (G, Σ) that satisfy (i)-(iii) and among all choices of homologous separations X of X' with $Y = E(G) - X$ we pick one that minimizes,

$$\kappa(G[X]) + \kappa(G[Y]). \tag{4.1}$$

Throughout the remainder of the proof of Proposition 1.4.3, (G, Σ) will denote the representation of M and X, Y the partition selected according to (4.1). Next we give some easy properties,

Proposition 4.3.5. *We may assume that* (G, Σ) *is non-bipartite and has no blocking vertex.*

Proof. Otherwise by Remark 2.4.2, M is graphic. Express M as a 3-sum, i.e. $M = M_1 \oplus_3 M_2$ for some matroids M_1, M_2 where $X = E(M_1) - E(M_2)$. Proposition 4.1.1 implies that M_1 is isomorphic to a minor of M. Since M is graphic then so is M_1. Thus X is reducible and Proposition 1.4.3 holds. □

4.3.5 Edge condition

The following describes necessary conditions for which a signed-graph stops having a blocking pair after a 2-flip.

Proposition 4.3.6. *Let* (H, Γ) *be a signed-graph with a blocking pair* v, w *where* H *is 2-connected. Let* Z *be a 2-separation of* H *and let* $\partial_H(Z) = \{u_1, u_2\}$*. Let* H' *be obtained from* H *by a 2-flip on* Z *and assume that* (H', Γ) *does not have a blocking pair. Then, exactly one of* u_1, u_2 *is in* $\{v, w\}$ *(say* $u_1 = v$*), and there exist odd circuits* C_1, C_2, C_3 *of* (H, Γ) *where* $C_1 \subseteq Z$ *contains* u_1*, and avoids* w, u_2*;* $C_2 \subseteq E(H) - Z$ *contains* u_1*, and avoids* w, u_2*; and* C_3 *contains* w*, and avoids* u_1, u_2*.*

Proof. Label the vertices of H' so that vertices distinct from $\partial_{H'}(Z)$ have the same label as H and vertices in $H[Z]$ and $H'[Z]$ have the same label. Then $\{v, w\} \cap \partial_H(Z) \neq \emptyset$ for otherwise v, w would be a blocking pair of H' and $\{v, w\} \neq \{u_1, u_2\}$ for otherwise u_1, u_2 would be a blocking pair of H'. We may assume, $u_1 = v$. Then C_1, C_2, C_3 exist because u_1, w is a blocking pair of (H, Γ) and for C_1 we have that u_2, w is not a blocking pair of (H', Γ), for C_2 we have that u_1, w is not a blocking pair of (H', Γ), and for C_3 we have that u_1, u_2 is not a blocking pair of (H', Γ). □

We shall also require the following observation about Lovász-flip,

Remark 4.3.7. *Let* (H, Γ) *be a signed-graph with a blocking pair* v_1, v_2 *and* $\Gamma \subseteq \delta_H(v_1) \cup \delta_H(v_2)$*. Consider* $Z \subseteq E(H)$ *such that* $H[Z]$ *is connected. Suppose for* $i \in [2]$*,*

$$\delta_H(v_i) \cap Z \in \{\emptyset, \delta_H(v_i) \cap \Sigma, \delta_H(v_i) - \Sigma\}.$$

Let (H', Γ) *be obtained from* (H, Γ) *by a Lovász-flip on* v_1, v_2*. Then* $H'[Z]$ *is also connected.*

Note, it suffices to observe that if two edges of Z are incident in H then they will be incident in H'.

The next result will be key,

Proposition 4.3.8. *No component of* $G[X]$ *or* $G[Y]$ *consists of a single edge.*

Proof. Suppose otherwise, i.e. there exists a component of $G[X]$ or $G[Y]$ that consists of a single edge e. Without loss of generality, we may assume that $e \in X$. Denote by u, v a blocking pair of (G, Σ).

Claim 1. $e \in \mathrm{cl}^*(Y)$.

Subproof. First consider the case where e not a loop. Since G is connected there exists an end, say x, of e that is a vertex of $G[Y]$. Then $D = \delta_G(x)$ is a cut of G where $D - Y = \{e\}$. Hence, by Remark 4.3.3, $e \in \mathrm{cl}^*(Y)$ as required. Now consider the case where e is a loop. By Proposition 4.3.1, $e \in \Sigma$. Suppose that $(G, \Sigma)|X - e$ is non-bipartite. Then $G[X - e]$ contains one of the two vertices of the blocking pair, say u. Let G' be obtained from G by moving e to u. Then (G', Σ) and X contradict the minimality assumption (4.1). Thus $(G, \Sigma)|X - e$ is bipartite and it follows that there exists a signature Γ of (G, Σ) where $\Gamma - Y = \{e\}$. Thus by Remark 4.3.3, $e \in \mathrm{cl}^*(Y)$ as required. $\Diamond$

Claim 2. $|X| = 4$

Subproof. Suppose for a contradiction that $|X| \geq 5$. Then by Claim 1 and Proposition 4.3.2, $X - e$ is homologous to X. But then (G, Σ) and $X - e$ violate the minimality assumption (4.1). $\Diamond$

Claim 3. *Circuits of M contained in X avoid e.*

Subproof. Since M is binary, circuits and cocircuits have an even number of common elements. By Claim 1 there is a cocircuit D of M with $D - Y = \{e\}$. Let C be a circuit of M where $C \subseteq X$. Then, $C \cap D \subseteq \{e\}$. It follows that $C \cap D = \emptyset$, i.e. $e \notin C$. $\Diamond$

Denote the elements of X by e, f, g, h. By Proposition 4.3.4 there exists a circuit $C \subseteq X$ of M with $|C| \geq 3$. By Claim 3, $e \notin C$, thus $C = \{f, g, h\}$. By Proposition 4.3.4 there exists a cocircuit $D \subseteq X$ of M with $|D| \geq 3$. Since $|C \cap D|$ is even we may assume that $D = \{e, f, g\}$.

Claim 4. *e is not a loop of G.*

Subproof. Suppose for a contradiction that e is a loop. Then D is not a cut of G. It follows that $D = \{e, f, g\}$ is a signature of (G, Σ). By Proposition 4.3.1 e is the only loop of G. Thus $C = \{f, g, h\}$ is a polygon of G. Let w denote the common end of f and g. Let G' be obtained from G by moving e to w. Then (G', Σ) has a blocking vertex, contradicting Proposition 4.3.5. ◊

There are two cases for D, namely, D is a cut of G or a signature of (G, Σ). There are two cases for C, namely, C is a polygon of G or C consists of two parallel edges (exactly one of which is odd) and an odd loop. We will consider all four possible combinations. Let x, y denote the ends of edge e.

Case 1. D is a cut and C is a polygon.

Let p, q, r denote the vertices of the polygon C in G where p is incident to f, g. As C is a cycle of M, C is even in (G, Σ). By exchanging the roles of x, y if needed, there exists a non-trivial partition Y_1, Y_2 of Y such that $\partial(Y_1) = \{x, p\}$ and $\partial(Y_2) = \{y, q, r\}$. Let $Z = Y_1 \cup e$, then $\partial(Z) = \{p, y\}$. Let G' be obtained from G by a 2-flip on Z. Suppose for a contradiction that (G', Σ) has no blocking pair. Recall, that u, v denotes the blocking pair of (G, Σ). Then by Proposition 4.3.6 we may assume (i) $u = p$ or (ii) $u = y$. Moreover, if (i) occurs then we have an odd circuit $C_2 \subseteq E(G) - Z$ that uses p and avoids v. It follows that C_2 uses edges f, g. But then $C_2 - \{f, g\} \cup h$ is an odd circuit of (G, Σ) that avoids both u, v, a contradiction. If (ii) occurs we must have a circuit $C_1 \subseteq Z$ that uses y, a contradiction as e is the only edge of Z incident to y. Hence, (G', Σ) has a blocking pair and together with X it contradicts the minimality assumption (4.1).

Case 2. D is a cut and C is not a polygon.

Then as $D = \{e, f, g\}$, C consists of parallel edges f, g and loop h. Denote by p and q the ends of f, g. By exchanging the roles of x, y if needed, there exists a non-trivial partition Y_1, Y_2 of Y such that $\partial(Y_1) = \{x, p\}$ and $\partial(Y_2) = \{y, q\}$. For $i \in [2]$ let $Z_i = Y_i \cup e$ and let G^i be obtained from G by a 2-flip on Z_i. Suppose for a contradiction that neither (G^1, Σ) nor (G^2, Σ) have a blocking pair. Since (G^1, Σ) does not have a blocking pair it follows from Proposition 4.3.6 that (i) $u = p$ or (ii) $u = y$. However, (ii) does not occur for otherwise we must have an odd circuit $C_1 \subseteq Z_1$ that uses y, a contradiction as e is the only edge of Z_1 incident to y. Hence, (i) holds, i.e. $u = p$. Applying the same argument to (G^2, Σ) we deduce that $v = q$, i.e. p, q is a blocking pair of (G, Σ). Since (G^1, Σ) has no blocking pair, Proposition 4.3.6 implies that there exists an odd circuit of (G, Σ) contained in $E(G) - Z$ that uses p but avoids q, a contradiction as no such circuit exists. Hence, at least one of (G^1, Σ), (G^2, Σ) has a blocking pair and with X contradicts the minimality assumption (4.1).

Case 3. D is a signature and C is a polygon.

Let p be the vertex common to f, g in G. Then x and p is a blocking pair. Let $\Gamma = D \,\Delta\, \delta(x)$. Let (G', Γ) be obtained from (G, Γ) by a Lovász-flip on x and p. Clearly (G', Γ) has a blocking pair. Moreover, $\kappa(G'[X]) = 1$ and by applying Remark 4.3.7 to each component of $G[Y]$ we deduce $\kappa(G'[Y]) \leq \kappa(G[Y])$. Hence, (G', Σ) together with X, contradicts the minimality assumption (4.1).

Case 4. D is a signature and C is not a polygon.

Since $C = \{f, g, h\}$ is a circuit of M but not a polygon, it is the union of two odd polygons of (G, Σ). As $D = \{e, f, g\}$, one of f or g is a loop. Without loss of generality we may assume that f is a loop and thus h, g are parallel. Note that f is not a component of $G[X]$, for otherwise moving it to an end of g, h preserves blocking pairs and contradicts the minimality assumption (4.1). Thus, f, g, h are incident to a common vertex, say p. Then end x of e and p is a blocking pair. Let $\Gamma = D \,\Delta\, \delta(x)$. Let (G', Γ) be obtained from (G, Γ) by a Lovász-flip on x and p. Clearly (G', Γ) has a blocking pair. Moreover, $\kappa(G'[X]) = 1$ and by Remark 4.3.7 $\kappa(G'[Y]) \leq \kappa(G[Y])$. Hence, (G', Σ) together with X, contradicts the minimality assumption (4.1). □

4.3.6 Degree condition

Throughout the remainder of the proof of Proposition 1.4.3, we let H denote the auxiliary graph for G and X which are selected according the minimality assumption (4.1). Next are the key properties of the auxiliary graph H.

Proposition 4.3.9. *H is bridgeless, in particular every vertex has degree at least two. Moreover, if a vertex has degree exactly two then the corresponding component of $(G, \Sigma)|X$ or $(G, \Sigma)|Y$ is non-bipartite.*

Proof. Note that H is connected since G is connected. Suppose for a contradiction that H has a bridge e. Since H is connected, $H \setminus e$ has two components H_1 and H_2. Let Z be the set of edges of G in components of $G[X]$ and $G[Y]$ corresponding to the vertices of H_1. Then $\partial_G(Z) = \{v\}$ for some $v \in V(G)$. By Proposition 3.1.5, $\lambda_M(Z) = |\,\partial_G(Z)| - \kappa(G[Z]) - \kappa(G[E(M) - Z]) + p[(G, \Sigma)|Z] + p[(G, \Sigma)|E(M) - Z] \leq 1 - 1 - 1 + 1 + 1 = 1$, thus Z is 2-separating. Moreover, by Proposition 4.3.8, $|Z|, |E(M) - Z| \geq 2$, a contradiction as M is 3-connected. Let p be a degree two vertex of H and let Z denote the edges in the component of $G[X]$ or $G[Y]$ corresponding to p. Then $|\partial_G(Z)| = 2$. Proposition 4.3.8 implies that $|Z|, |E(M) - Z| \geq 2$. Since M is 3-connected, Z is not 2-separating, i.e. $2 \leq \lambda_M(Z) \leq 2 - 1 - 1 + p[(G,\Sigma)|Z] + p[(G,\Sigma)|E(M) - Z]$. Thus $p[(G,\Sigma)|Z] = 1$, i.e. $(G,\Sigma)|Z$ is non-bipartite. □

4.3.7 Proof of Proposition 1.4.3

There are three cases to consider depending on whether each of $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite (as we can interchange the role of X and Y). We consider each of these in Propositions 4.3.10, 4.3.11 and 4.3.13.

Proposition 4.3.10. *If $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are non-bipartite then X is reducible, compliant, or recalcitrant.*

Proof. Proposition 3.1.6 implies that $|E(H)| = |V(H)| + \lambda_M(X) - p[(G,\Sigma)|X] - p[(G,\Sigma)|Y] = |V(H)| + 2 - 1 - 1 = |V(H)|$. Moreover, H is connected and by Proposition 4.3.9 it is bridgeless. It follows that H is a polygon. Since every vertex of H has degree two, Proposition 4.3.9 implies that each component of $G[X]$ and $G[Y]$ is non-bipartite. It follows that each component of $G[X]$ and $G[Y]$ contains one of u, v where u, v is the blocking pair of (G,Σ). This implies that $|V(H)| \in \{2, 4\}$. Consider first the case where $|V(H)| = 4$. Then, we have proper partitions X_1, X_2 of X and Y_1, Y_2 or Y where $u \in \partial(X_1) \cap \partial(Y_1)$ and $v \in \partial(X_2) \cap \partial(Y_2)$. Let G' be obtained from G by a 2-flip of X_1, Y_2. Then (G',Σ) has a blocking pair and (G',Σ) and X contradict our minimality assumption (4.1). Hence $|V(H)| = 2$, i.e. $G[X]$ and $G[Y]$ are connected and $\partial(X) = \{a, b\}$ for some vertices a, b of G.

Let us now analyze the possible location of the blocking pair u, v. If $u \in \mathcal{I}(X)$ and $v \in \mathcal{I}(Y)$ or vice-versa then (G, Σ, X) is a Type I configuration and X is compliant. If $\{u, v\} = \{a, b\}$ then (G, Σ, X) is a Type II configuration and X is recalcitrant. Since $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are non-bipartite, we may thus assume, after possibly interchanging the role of X, Y and u, v and a, b that $u = a$ and $v \in \mathcal{I}(Y)$. After possibly re-signing, edges of Σ are incident to u or v. Let G_1 (resp. G_2) be obtained from $G[X]$ (resp. $G[Y]$) by adding parallel edges f, g between a and b and adding a loop h incident to a. Let $\Sigma_1 = \Sigma \cap X \cup \{f, h\}$ and let $\Sigma_2 = \Sigma \cap Y \cup \{f, h\}$. Then a is a blocking vertex of (G_1, Σ_1) which implies by Remark 2.4.2 that $\text{ecycle}(G_1, \Sigma_1)$ is graphic. Finally, Proposition 4.1.5 implies that $M = \text{ecycle}(G_1, \Sigma_1) \oplus_3 \text{ecycle}(G_2, \Sigma_2)$. Hence, X is reducible. □

Proposition 4.3.11. *If $(G, \Sigma)|X$ is bipartite and $(G, \Sigma)|Y$ is non-bipartite then X is reducible.*

Proof. Proposition 3.1.6 implies that $|E(H)| = |V(H)| + \lambda_M(X) - p[(G, \Sigma)|X] - p[(G, \Sigma)|Y] = |V(H)| + 2 - 0 - 1 = |V(H)| + 1$. Recall that H is connected and Proposition 4.3.9 implies that H is bridgeless. It follows from Remark 3.1.7 that H is a theta or a double ear.

Case 1. H is a theta.

The theta graph H consists of three internally disjoint paths P_1, P_2, P_3. Consider first the case where H is 1-uniform, i.e. $|P_1| = |P_2| = |P_3| = 1$. Then $G[X]$ and $G[Y]$ are connected. Proposition 4.1.5 implies that $M = \text{cycle}(G') \oplus_3 M_2$ for some graph G' and for some matroid M_2 where $E(G') = X$. But then X is reducible. Thus we may assume H is not 1-uniform, i.e. $|P_i| \geq 2$ for some $i \in [3]$. Since $(G, \Sigma)|X$ is bipartite, Proposition 4.3.9 implies that every internal vertex of P_i corresponds to a component of $G[Y]$. Thus $|P_i| = 2$ and as H is bipartite and is not 1-uniform, $|P_i| = 2$ for all $i \in [3]$. By Proposition 4.3.9 it follows that the degree two vertex v_i of P_i correspond to a component $G[Y_i]$ of $G[Y]$ where $(G, \Sigma)|Y_i$ is non-bipartite. Hence, (G, Σ) has three pairwise vertex disjoint odd circuits, a contradiction as there exists a blocking pair.

Case 2. H is a double ear.

The double ear graph H consists of two polygons C_1 and C_2 that share exactly one vertex. Proposition 4.3.9 implies that H is 2-uniform and that for $i \in [2]$ the degree two vertex v_i of C_i corresponds to a component $G[Y_i]$ of $G[Y]$ where $(G, \Sigma)|Y_i$ is non-bipartite. Thus we may assume for the blocking pair u, v of (G, Σ) that u and v are vertices of $G[Y_1]$ and $G[Y_2]$ respectively. Since $(G, \Sigma)|X$ is bipartite, we may assume $\Sigma \subseteq Y$ and that all edges of Σ are incident to u or v. Let (G', Σ) be obtained from (G, Σ) by a Lovász-flip on u, v. Then $G'[X]$ and $G'[Y]$ are connected as $G[Y_i]$ contains an odd polygon for each $i \in [2]$. Thus (G', Σ) and X contradict our minimality assumption (4.1). □

Consider a graph F. A *ear* of F is a walk P where the two endpoints of P may coincide, but every other vertex of P has degree two. An *ear decomposition* of F is a partition of its edges into a sequence of ears, such that the one or two endpoints of each ear belong to earlier ears in the sequence and such that the internal vertices of each ear do not belong to any earlier ear. Additionally, the first ear in the sequence must be a polygon.

Theorem 4.3.12 ([42])**.** *A graph is connected and bridgeless if and only if it has an ear decomposition.*

Proposition 4.3.13. *If $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are bipartite then X is reducible, compliant, or recalcitrant.*

Proof. Proposition 3.1.6 implies that $|E(H)| = |V(H)| + \lambda_M(X) - p[(G, \Sigma)|X] - p[(G, \Sigma)|Y] = |V(H)| + 2 - 0 - 0 = |V(H)| + 2$. Proposition 4.3.9 implies that the minimum degree

$\delta(H)$ of H is at least 3. $\delta(H) \geq 3$ and $|E(H)| = |V(H)| + 2$ implies by Theorem 4.3.12 that H is obtained from a polygon C by adding a sequence of two ears, say Q_1, Q_2. Let H' be the graph obtained from C by adding ear Q_1. Then H' is either a double ear or a theta (Remark 3.1.7). Moreover, $\delta(H) \geq 3$ implies that Q_2 consists of a single edge, say f. Consider first the case where H' is a double ear that consists of polygons C and C' joined at a vertex w. Then f has ends in C and C' (distinct from w). Since $\delta(H) \geq 3$, C and C' have each exactly one vertex distinct from w that is incident to f. But then H has a triangle, a contradiction as H is bipartite. Consider now the case where F is a theta that is formed by internally disjoint paths P_1, P_2, P_3. We may assume f is not incident to an internal vertex of P_1. As $\delta(H) \geq 3$, P_1 consist of a single edge. It follows that P_2 and P_3 each have an odd number of edges. As $\delta(H) \geq 3$ we can assume that P_2 has a single edge and that $|P_3| \in \{1, 3\}$.

Case 1. $|P_3| = 1$.

The ends of f correspond to the degree 3 vertices of H, i.e. H consists of four parallel edges. Then, $G[X]$ and $G[Y]$ are connected. Denote by a, b, c, d the vertices of $\partial_G(X)$. By Remark 2.4.2, there is no blocking vertex of (G, Σ). Thus we may assume, after possibly re-signing and exchanging the roles of a, b, c, d if needed, that $\Sigma = (\delta_G(a) \cap X) \cup (\delta_G(b) \cap Y)$. Let (G', Σ') be obtained from (G, Σ) by a Lovász-flip on a and b. Observe that $G'[X]$ and $G'[Y]$ remain connected, but $(G', \Sigma)|X$ and $(G', \Sigma)|Y$ are non-bipartite. Thus (G', Σ) and X satisfy the minimality assumption and thus by Proposition 4.3.10, X is reducible, compliant, or recalcitrant.

Case 2. $|P_3| = 3$.

Since $\delta(H) \geq 3$, the ends of f must correspond to internal vertices of P_3. It follows that

H is the graph obtained from a polygon with four edges by replacing edges in a matching by two parallel edges. We illustrate the auxiliary graph H in Figure 4.4 (left) with the corresponding graph G (right). Let G' be obtained from G by a 2-flip on $X_1 \cup Y_2$. Then,

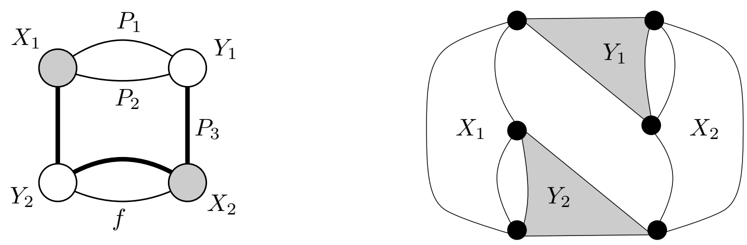

Figure 4.4: Case (2) in Proposition 4.3.13.

by Proposition 4.3.6, (G', Σ) has a blocking pair. Then (G', Σ) and X contradict our minimality assumption (4.1). □

Chapter 5

Recognition Algorithms

The work in this chapter appears in [20, 22, 21, 31]. Tutte [51] proved that one can recognize whether a binary matroid is graphic in polynomial time. Seymour [46] extended this result and showed that there exists a polynomial-time algorithm to check whether a matroid specified by an independence oracle is graphic. Thus, given a binary matroid described by its $0,1$ matrix representation, we can check in polynomial time if the matroid is graphic and we can check in polynomial time if the matroid is cographic. We prove the analogous result for even-cycle matroids, even-cut matroids, and pinch-graphic matroids. Recall from Section 1.5 that we are interested in constructing the following algorithms:

- **Algorithm (1)**: Given a binary matroid M described by its $0,1$ matrix representation A, check whether M is an even-cycle matroid, in polynomial time.
- **Algorithm (2)**: Given a binary matroid M described by its $0,1$ matrix representation A, check whether M is an even-cut matroid, in polynomial time.

- **Algorithm (3)**: Given a binary matroid M described by its $0, 1$ matrix representation A, check whether M is a pinch-graphic matroid, in polynomial time.

For each of Algorithms (1), (2), and (3), by polynomial time, we mean polynomial in the number of entries of A. We believe that these algorithms ought to be fast in practice but have not conducted numerical experiments. For Algorithms (1) and (2), the bound on the running time depends on a constant c that arises from the Matroid Minors Project and that has no explicit bound [13]. However, these algorithms do not use the value c for their computation.

Algorithms (1) and (2) rely on Algorithm (3) as a subroutine. In Section 5.1, we sketch a simple polynomial time algorithm to check whether a binary matroid is graphic. In Sections 5.3 and 5.4, we describes Algorithm (1) and (2) for even-cycle and even-cut matroids, respectively. For Algorithm (3), we construct the following algorithms:

- **Algorithm (4)**: Given an internally 4-connected binary matroid M, check whether M is a pinch-graphic matroid in polynomial time.

- **Algorithm (5)**: Given a binary matroid M, check whether M is a pinch-graphic matroid or return an internally 4-connected matroid N that is isomorphic to a minor of M such that M is pinch-graphic if and only if N is pinch-graphic, in polynomial time.

By combining Algorithms (4) and (5), we get a polynomial algorithm to check whether a binary matroid M is pinch-graphic, i.e., Algorithm (3) (thereby completing the description of the algorithm for recognizing even-cycle and even-cut matroids). Namely, we first apply Algorithm (5) and either establish whether M is pinch-graphic, or we return the matroid N. In the latter case, Algorithm (4) to determine whether N is pinch-graphic.

For all the aforementioned algorithms, we assume that the matroid M is given in terms of its $0, 1$ matrix representation A. In Sections 5.5 and 5.6, we describes Algorithm (4) and (5), respectively.

5.1 Graphic matroids

5.1.1 Reduction to the 3-connected case

Recall that a matroid M has a 1-separation if and only if M can be expressed as a 1-*sum*, $M_1 \oplus_1 M_2$. A connected matroid has a 2-separation if and only if M can be expressed as a 2-*sum*, $M_1 \oplus_2 M_2$ [1, 8, 44]. Moreover, for $k \in [2]$, $M = M_1 \oplus_k M_2$ is graphic if and only if both M_1 and M_2 are graphic ([36], Corollary 7.1.26). Assume that we know how to check whether a 3-connected binary matroid is graphic and suppose that we want to check whether an arbitrary binary matroid M is graphic. If M is 3-connected, then use the algorithm for 3-connected matroids. Otherwise find a k-separation for $k \in [2]$, express M as $M_1 \oplus_k M_2$ and recursively check whether M_1 and M_2 are both graphic. If so, then M is graphic; otherwise, M is not. We need to be able to check for the presence of 1- and 2-separations in a binary matroid in polynomial time. Cunningham and Edmonds [8] showed that the more general problem of checking whether a matroid has a k-separating set with both separators of size at least $\ell \geq k$ can be reduced to the matroid intersection problem [10, 33] and be solved in polynomial time for fixed values k and ℓ.

5.1.2 Graph representations

We say that the representation H of N *extends* to the representation G of M. Theorem 1.1.1 implies the following result:

Remark 5.1.1. *Suppose N is a 3-connected graphic matroid with a graph representation H. If N is a minor of a 3-connected matroid M, then M is graphic if and only if the representation H of N extends to a representation of M.*

5.1.3 The algorithm

A *wheel* is the graph obtained by starting with a polygon with at least three edges, adding a new vertex (the hub) and connecting every vertex of the polygon to the hub. Consider a 3-connected binary matroid M and suppose that we wish to check whether M is graphic. First, we check whether M is the graphic matroid of a wheel. Otherwise, by Tutte's Wheels-and-Whirls Theorem [53], there exists an element e such that either $N = M/e$ or $N = M \setminus e$ is 3-connected. Recursively, we check whether N is graphic. If it is not, then neither is M. Otherwise, we check whether the (unique) representation of N extends to M. If it does then M is graphic, otherwise it is not.

5.2 Decision problems on grafts and signed graphs

In this section, we describe two polynomially solvable decision problems that will be used in in the recognition algorithm for even-cycle and even-cut matroids. For a binary matroid M and $C \subseteq E(M)$, we say that C is a *cycle* (resp. *cocycle*) of M if C is a disjoint union of circuits (resp. cocircuits) of M. We will make repeated use of the following observation.

Remark 5.2.1. *Let M, N be binary matroids with the same ground set E and let $e \in E$.*

(a) If $M \setminus e = N \setminus e$ and there exists a cycle C of both M, N where $e \in C$, then $M = N$.

(b) If $M/e = N/e$ and there exists a cocycle D of both M, N where $e \in D$, then $M = N$.

Proof. (a) Let $X \subseteq E$ such that $e \in X$. Then, X is a cycle of M (resp. N) using e if and only if $X \,\Delta\, C$ is a cycle of M (resp. N) not using e. Since M and N have the same set of cycles not using e, M and N have the same set of cycles using e. Hence, M and N have the same cycles and $M = N$. (b) follows by applying (a) to the dual of M and N. □

5.2.1 A decision problem on grafts

Consider the following decision problem:

Problem 5.2.2. *Given a graft, is there an equivalent graft with at most two terminals?*

We are given a graft (G, T) and want to know if there exists an equivalent graft (G', T') where $|T'| \leq 2$. Note that the set of terminals of (G, T) is empty if and only if the T-joins of (G, T) are the cycles of G. In particular, because of Remark 1.2.3, if a graft has an empty set of terminals then so does every equivalent graft. Hence, we may assume that $T \neq \emptyset$ in Problem 5.2.2. Denote by A the matrix obtained from the vertex-edge incidence matrix of G by adding a column t that is the characteristic vector of the terminals T. Let M denote the binary matroid represented by matrix A. The matroid M is known as the *graft matroid* of (G, T). Its cycles are the cycles of G and the sets of the form $J \cup t$ where J is a T-join of G. The following result is essentially in [45].

Proposition 5.2.3. *Let (G, T) be a graft with $T \neq \emptyset$ and let M be the graft matroid of (G, T).*

(a) If M is not graphic then no graft equivalent to (G,T) has two terminals,

(b) If $M = \text{cycle}(H)$ for some H, define $G' = H \setminus t$ and denote by x, y the endpoints of edge t in H. Then $(G', \{x,y\})$ is equivalent to (G,T).

Proof. (a) Suppose there exists a graft (G', T') equivalent to (G,T) where $T' = \{x,y\}$. Let H be the graph obtained from G' by adding edge $t = (x,y)$. Since (G,T) and (G',T') are equivalent, $\text{cycle}(H) \setminus t = \text{cycle}(G') = \text{cycle}(G) = M \setminus t$. As (G',T') is a graft, there is a $\{x,y\}$-path P of G'. Thus, $P \cup t$ is a cycle of H, which is a circuit of M. Then, by Remark 5.2.1 (a), $M = \text{cycle}(H)$ and in particular M is graphic. (b) We have $M \setminus t = \text{cycle}(H) \setminus t = \text{cycle}(H \setminus t) = \text{cycle}(G')$. Moreover, the matrix representation of $M \setminus t$ is the vertex-edge incidence matrix of G, hence $M \setminus t = \text{cycle}(G)$. Thus, $\text{cycle}(G) = \text{cycle}(G')$, and by Theorem 1.1.1, G and G' are equivalent. Since $M = \text{cycle}(H)$ the cycles of M using t are of the form $J \cup t$ where J is an $\{x,y\}$-join of G'. Since M is the graft matroid of (G,T) the cycles of M using t are of the form $J \cup t$ where J is a T-join of G. Hence, T-joins in G are $\{x,y\}$-joins in G' and in particular, (G,T) and $(G', \{x,y\})$ are equivalent. □

Thus, we can use the algorithm to check if a binary matroid is graphic to solve Problem 5.2.2. We are requiring here that such an algorithm returns a graph representation in case the matroid is graphic, but this is indeed the case for [51], for instance.

5.2.2 A decision problem on signed graphs

Consider the following decision problem:

Problem 5.2.4. *Given a signed graph, is there an equivalent signed graph with a blocking vertex?*

We are given a signed graph (G, Σ) and we want to know if there exists an equivalent signed graph (G', Σ) with a blocking vertex v. We may assume that G has no loop as removing loops does not affect the answer to Problem 5.2.4. The following result is essentially in [18].

Proposition 5.2.5. *Let (G, Σ) be a loopless signed-graph, let G^+ be obtained from G by adding a loop Ω and let $M = \text{ecycle}(G^+, \Sigma \cup \Omega)$.*

a. If M is not graphic then no signed graph equivalent to (G, Σ) has a blocking vertex.

b. If $M = \text{cycle}(H)$ for some H define $G' = G/\Omega$ and denote by v the vertex of G' corresponding to Ω. Then G' is equivalent to G and v is a blocking vertex of (G', Σ).

Proof. (a) Suppose some signed-graph (G', Σ) equivalent to (G, Σ) has a blocking vertex v. Then for some signature Γ of (G', Σ) we have $\Gamma \subseteq \delta_{G'}(v)$. Let H be obtained from G' by uncontracting v according to Γ where Ω is the new edge. Then by Remark 5.2.1 (b), $M = \text{cycle}(H)$ and in particular, M is graphic. (b) We have $M/\Omega = \text{ecycle}[(G^+, \Sigma \cup \Omega)/\Omega] = \text{cycle}(G^+/\Omega) = \text{cycle}(G)$. We also have $M/\Omega = \text{cycle}(H/\Omega) = \text{cycle}(G')$. It follows by Theorem 1.1.1 that G and G' are equivalent. Let C be an odd polygon of (G, Σ). Then $C \cup \Omega$ is an even-cycle of $(G^+, \Sigma \cup \Omega)$. Thus $C \cup \Omega$ is a polygon of H and in particular, C uses vertex v of G'. Hence, v is a blocking vertex as required. □

Thus we can use the algorithm to check if a binary matroid is graphic to solve Problem 5.2.4.

5.3 Even-cycle matroids

5.3.1 Keeping track of representations

Consider an even-cycle matroid M with a representation (G, Σ). Recall that $(H, \Gamma) = (G, \Sigma)/I \setminus J$ is a representation of the minor $N = M/I \setminus J$. We say that the representation (H, Γ) of N *extends* to the representation (G, Σ) of M. Hence, every signed-graph representation of M extends some signed-graph representations of the minor N. Here we characterize when a signed-graph representation of an even-cycle matroid extends to a single element undeletion or uncontraction.

Proposition 5.3.1. *Let M be a binary matroid, let $e \in E(M)$ and let $N = M \setminus e$. Let C be a cycle of M using e and let (G, Σ) be a signed-graph representation of N. Then (G, Σ) extends to a representation (H, Γ) of M if and only if for some signature Σ' of (G, Σ) we have $\Gamma = \Sigma'$ when $|C \cap \Sigma'|$ is even and $\Gamma = \Sigma' \cup e$ otherwise, and in addition either, (i) $G[C - e]$ has no odd degree vertex in which case H is obtained from G by adding a loop e; or (ii) $G[C - e]$ has exactly two odd degree vertices v, w in which case H is obtained from G by adding an edge $e = (v, w)$.*

Proof. ($\Rightarrow$) Suppose (H, Γ) extends the representation (G, Σ) to M. Then $|C \cap \Gamma|$ is even which implies that Γ is as described. Moreover, $G = H \setminus e$ and (i) occurs if e is a loop of H and (ii) occurs if e is a not a loop of H. ($\Leftarrow$) By the choice of Γ we have $|C \cap \Gamma|$ is even. Moreover, by the construction (i) or (ii) we obtain H where C is a cycle of H. Hence, C is an even cycle of (H, Γ) and Remark 5.2.1 (a) implies that $M = \text{ecycle}(H, \Gamma)$. □

Proposition 5.3.2. *Let M be a binary matroid, let $e \in E(M)$ and let $N = M/e$ where N is non-graphic. Let D be a cocycle of M using e and let (G, Σ) be a signed-graph representation of N. Then (G, Σ) extends to a representation (H, Σ) of M if and only if either,* *(i) there exists a signature* Γ *of* $(G, D - e)$*, or*

(ii) there exists a signature Γ *of* $(G, [D - e] \,\Delta\, \Sigma)$*,*

where for (i), (ii) all edges of Γ *are incident to some vertex* v *or contained in loops and for both cases* H *is obtained from* G *by uncontracting* e *at* v *according to* Γ*.*

Proof. ($\Rightarrow$) Suppose (G, Σ) extends to a representation (H, Σ) of M. Since N is non-graphic, e is not an odd loop of (H, Σ) and since e is contained in the cocycle D, e is not a loop of M, that is, e is not an even loop of (H, Σ). Denote by w one of the ends of e in H and let $\Gamma = \delta_H(w) - e$.

Claim. Γ *is a signature of* $(G, D - e)$ *or of* $(G, [D - e] \,\Delta\, \Sigma)$*.*

Subproof. Since D is a cocycle of M, D is a cut of H or a signature of (H, Σ) [40]. Thus there exists $D' \in \{D, D \,\Delta\, \Sigma\}$ that is a cut of H with $e \in D'$. Then, Γ is a signature of $(G, D' - e)$ as $\Gamma \,\Delta (D' - e) = \delta_H(w) \,\Delta\, D'$ is a cut of H avoiding e so is a cut of G. $\Diamond$

Let v be the vertex of G obtained by contracting e from H. The edges of Γ are incident to v. Finally observe that H is obtained from G by uncontracting e at v according to Γ. ($\Leftarrow$) Suppose H is obtained from G by uncontracting e at v according to Γ. Then $\Gamma \cup e = \delta_H(w)$ where w denotes one of the endpoints of e in H. If Γ is a signature of $(G, D - e)$ then D is a cut of H. If Γ is a signature of $(G, [D - e] \,\Delta\, \Sigma)$ then D is a signature of (H, Σ). In both cases D is a cocycle of ecycle(H, Σ) [40] and by Remark 5.2.1 (b), $M =$ ecycle(H, Σ), i.e. (H, Σ) extends the representation (G, Σ) to M. $\square$

5.3.2 Equivalence classes

In our algorithm, we will keep track of signed-graph representations up to equivalence only. As a result, we shall require the following two results.

Proposition 5.3.3 ([27] Lemma 12). *Let N be an even-cycle matroid and let $\mathcal{F}$ be an equivalence class of signed-graph representations of N. Let M be a binary matroid with a non-coloop $e \in E(M)$ for which $N = M \setminus e$. Then the set extensions of $\mathcal{F}$ to M is a (possibly empty) equivalence class.*

Proposition 5.3.4 ([27] Lemma 24). *Let N be a non-graphic, even-cycle matroid and let $\mathcal{F}$ be an equivalence class of signed-graph representations of N. Let M be a binary matroid with a non-loop $e \in E(M)$ for which $N = M/e$. Then the set extensions of $\mathcal{F}$ to M is either a (possibly empty) equivalence class of representations or the union of two equivalence classes $\mathcal{F}_1$ and $\mathcal{F}_2$. Moreover, in the latter case $\mathcal{F}_1$ and $\mathcal{F}_2$ arise from respectively case (i) and (ii) in Proposition 5.3.2.*

The statements of Proposition 5.3.3 and 5.3.4 are slightly different from [27], but in the proofs, the weaker conditions are used as stated. In the previous result the "Moreover" part of the statement is not given explicitly in [27]. However, a careful reading of the proof reveals that this is what is shown.

5.3.3 Algorithm (1)

Suppose that we are given a binary matroid M by its $0, 1$ matrix representation A. We assume that we have an oracle to determine if a binary matroid given by its matrix representation is pinch-graphic. We now describe an algorithm that in oracle polynomial time in the size of A will determine whether M is an even-cycle matroid.

First we check if M is pinch-graphic. If it is, then M is an even-cycle matroid and we stop. Thus, we may assume M is not pinch-graphic. Set $N = M$. If for any $e \in E(N)$, N/e (resp. $N \setminus e$) is not pinch-graphic then replace N with N/e (resp. $N \setminus e$). When we stop we have found a minor N of M that is minimally non pinch-graphic. It follows by Theorem 3.1.1 that N has constant size and we can find all representations of N, up to equivalence, in constant time. (A finite algorithm for finding all representations of an even-cycle matroid is given in [40], page 132.) Then we construct a sequence of matroids $M = M_1, M_2, \dots, M_k = N$ where for every $i \in [k-1]$ either (i) $M_{i+1} = M_i \setminus e_i$ for some $e_i \in E(M_i)$ that is not a coloop, or (ii) $M_{i+1} = M_i/e_i$ for some $e_i \in E(M_i)$ that is not a loop. In particular, for (i) there exists a cycle of M_i using e and for (ii) a cocycle of M_i using e. For each $i \in [k]$, the set of signed-graph representations of M_i can be partitioned into equivalence classes and we denote by $\mathcal{E}_i$ a set of signed-graph representations that consist of one representative from each equivalence class. We constructed $\mathcal{E}_k$ (the set of all representations of N up to equivalence). Clearly, M is an even-cycle matroid if and only if $\mathcal{E}_1 \neq \emptyset$. Thus it suffices to show for all $i \in [k-1]$ how to construct $\mathcal{E}_i$ from $\mathcal{E}_{i+1}$. Consider an arbitrary signed graph $(G, \Sigma) \in \mathcal{E}_{i+1}$ and let $\mathcal{F}$ be the equivalence class that contains (G, Σ). Let $\mathcal{F}'$ be the set of extensions of $\mathcal{F}$ to M_i. By propositions 5.3.3 and 5.3.4, $\mathcal{F}'$ is either empty, a single equivalence class, or the union of two equivalence classes. We will show how to find representatives for each equivalence class in $\mathcal{F}'$ in polynomial time. Since, by Theorem 1.3.1 there exists a constant c such that $|\mathcal{E}_i| \leq c$ this will prove that the algorithm is polynomial.

Consider first the case where $M_{i+1} = M_i \setminus e$. By Proposition 5.3.3, $\mathcal{F}'$ consists of a single (possibly empty) equivalence class. Find a cycle C of M_i containing e. By Proposition 5.3.1, some $(G', \Sigma') \in \mathcal{F}$ extends to a representation of M_i if $G'[C - e]$ has at most two vertices of odd degree. Thus, to check for the existence of such a signed-graph (G', Σ') we define T

to be the odd degree vertices of $G[C - e]$, and then use the decision algorithm described in Section 5.2.1 to check if there exists a graft (G', T') equivalent to (G, T) where $|T'| \leq 2$. If the answer is yes, then we can extend some representation $(G', \Sigma') \in \mathcal{F}$ to M_i as described in Proposition 5.3.1. If the answer is no, then no representation $(G', \Sigma') \in \mathcal{F}$ of M_{i+1} extends to M_i and $\mathcal{F}' = \emptyset$. Consider now the case where $M_{i+1} = M_i/e$. By Proposition 5.3.4, $\mathcal{F}'$ consists of the union of at most two equivalence classes. Find a cocycle D of M_i containing e. By Proposition 5.3.2, some $(G', \Sigma') \in \mathcal{F}$ extends to a representation of M_i if either $(G', D - e)$ or $(G', [D - e] \,\Delta\, \Sigma)$ has a blocking vertex v. Then, we use the decision algorithm described in Section 5.2.2 to first (i) check if there exists G' equivalent to G such that $(G', D - e)$ has a blocking vertex v, or (ii) check if there exists G' equivalent to G such that $(G, [D - e] \,\Delta\, \Sigma)$ has a blocking vertex v. For each of (i) and (ii) if the answer is yes then we can extend some representation $(G', \Sigma) \in \mathcal{F}$ to M_i as described in Proposition 5.3.2. Moreover, by Proposition 5.3.4 if the answer is yes for both (i) and (ii) $\mathcal{F}'$ consists of the union of two equivalence classes if the answer is yes for exactly one of (i) and (ii) $\mathcal{F}'$ consists of a single equivalence class, and otherwise $\mathcal{F}' = \emptyset$.

5.4 Even-cut matroids

5.4.1 Keeping track of representations

Consider an even-cut matroid M with a graft representation (G, T). Recall that $(H, R) = (G, T)/I \setminus J$ is a representation of the minor $N = M \setminus I/J$. We say that the representation (H, R) of N *extends* to the representation (G, T) of M. Hence, every graft representation of M extends some graft representations of the minor N. Here we characterize when a graft representation of an even-cut matroid extends to a single element undeletion or uncontraction.

We present the analogue of propositions 5.3.1 and 5.3.2, namely propositions 5.4.1 and 5.4.2. We omit the proofs as they are routine and similar to those in Section 5.3.1. To clarify the statement of the next proposition, observe that cocycles of $\text{ecut}(G, T)$ are cycles of G and T-joins of (G, T) [40].

Proposition 5.4.1. *Let M be a binary matroid, let $e \in E(M)$ and let $N = M/e$ where N is not cographic. Let D be a cocycle of M using e and let (G, T) be a graft representation of N. Pick J an arbitrary T-join of G. Then (G, T) extends to a representation (H, T) of M if and only if either,*

(i) $G[D - e]$ has at most two vertices of odd degree, or

(ii) $G[J \,\Delta (D - e)]$ has at most two vertices of odd degree

and for both cases, if there are no vertices of odd degree, then H is obtained from G by adding loop e and if there are vertices of odd degree u, v, then H is obtained from G by adding edge $e = (u, v)$.

Proposition 5.4.2. *Let M be a binary matroid, let $e \in E(M)$ and let $N = M \setminus e$. Let C be a cycle of M using e, let (G, T) be a graft representation of N. Then (G, T) extends to a representation (H, R) of M if and only if there exists a signature Γ of $(G, C - e)$ where all edges of Γ are incident to some vertex v, and H is obtained from G by uncontracting e at v according to Γ where $e = (v', v'')$ in H, and $R = (T - v) \cup X$ where $X \subseteq \{v', v''\}$ and $v' \in X$ (resp. $v'' \in X$) if and only if $\delta_H(v')$ (resp. $\delta_H(v'')$) is not a cycle of M.*

5.4.2 Equivalence classes

In our algorithm we will keep track of graft representations up to equivalence only. As a result we shall require the following two results.

Proposition 5.4.3 ([25] Lemma 9.4)**.** *Let N be an even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft representations of N. Let M be a binary matroid with a non-coloop $e \in E(M)$ for which $N = M \setminus e$. Then the set extensions of $\mathcal{F}$ to M is a (possibly empty) equivalence class.*

Proposition 5.4.4 ([25] Lemma 9.12)**.** *Let N be a non-cographic, even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft representations of N. Let M be a binary matroid with a non-loop $e \in E(M)$ for which $N = M/e$. Then the set extensions of $\mathcal{F}$ to M is either a (possibly empty) equivalence class of representations or the union of two equivalence classes $\mathcal{F}_1$ and $\mathcal{F}_2$. Moreover, in the latter case $\mathcal{F}_1$ and $\mathcal{F}_2$ arise from respectively case (i) and (ii) in Proposition 5.4.1.*

The statements of Proposition 5.4.3 and 5.4.4 are slightly different from [25], but in the proofs, the weaker conditions are used as stated.

5.4.3 Algorithm (2)

Suppose that we are given a binary matroid M by its $0, 1$ matrix representation A. We assume that we have an oracle to determine if a binary matroid given by its matrix representation is pinch-graphic (or equivalently pinch-cographic). We now describe an algorithm that, in oracle polynomial time in the size of A, will determine whether M is an even-cut matroid.

First we check if M is pinch-cographic. If it is, then M is an even-cut matroid and we stop. Thus we may assume M is not pinch-cographic and proceeding as in the previous algorithm we find a minimally non-pinch-cographic minor N of M. N has constant size and we can find all representations of N, up to equivalence, in constant time. Then we construct a sequence of matroids $M = M_1, M_2, \ldots, M_k = N$ where for every $i \in [k-1]$ either (i) $M_{i+1} = M_i \setminus e_i$ for some $e_i \in E(M_i)$ that is not a coloop, or (ii) $M_{i+1} = M_i / e_i$ for some $e_i \in E(M_i)$ that is not a loop. For each $i \in [k]$, the set of graft representations of M_i can be partitioned into equivalence classes and we denote by $\mathcal{E}_i$ a set of graft representations that consist of one representative from each equivalence class. We have $\mathcal{E}_k$ and clearly M is an even-cut matroid if and only if $\mathcal{E}_1 \neq \emptyset$. Thus it suffices to show for all $i \in [k-1]$ how to construct $\mathcal{E}_i$ from $\mathcal{E}_{i+1}$. Consider an arbitrary graft $(G,T) \in \mathcal{E}_{i+1}$ and let $\mathcal{F}$ be the equivalence class that contains (G,T). Let $\mathcal{F}'$ be the set of extensions of $\mathcal{F}$ to M_i. By propositions 5.4.3 and 5.4.4, $\mathcal{F}'$ is either empty, a single equivalence class, or the union of two equivalence classes. We will show how to find representatives for each equivalence class in $\mathcal{F}'$ in polynomial time. Since, by Theorem 1.3.2 there exists a constant c such that $|\mathcal{E}_i| \leq c$ this will prove that the algorithm is polynomial.

Consider first the case where $M_{i+1} = M_i \setminus e$. By Proposition 5.4.3, $\mathcal{F}'$ consists of a single (possibly empty) equivalence class. Find a cycle C of M_i containing e. By Proposition 5.4.2, a graft (G',T') equivalent to (G,T) extends to a representation of M_i if there exists a signature Γ of $(G', C-e)$ where all edges of Γ are incident to some vertex v. We use the decision algorithm described in Section 5.2.2 to check if there exists a signed-graph equivalent to $(G, C-e)$ with a blocking vertex v. If the answer is yes then some representation $(G',T') \in \mathcal{F}$ extends to M_i as described in Proposition 5.4.2. If the answer is no then $\mathcal{F}' = \emptyset$. Consider now the case where $M_{i+1} = M_i/e$. By Proposition 5.4.4, $\mathcal{F}'$ consists of the union of at most two equivalence classes. Find a cocycle D of M_i containing e, and let J be a T-join of G. By Proposition 5.4.1, some $(G',T') \in \mathcal{F}$ extends to a representation of M_i if either, (i) $G'[D-e]$ has at most two odd degree vertices, or (ii) $G'[J \,\Delta (D-e)]$ has at most two odd degree vertices. We denote by R_1 (resp. R_2) the vertices of odd degree of $G[D-e]$ (resp. of $G[J \,\Delta (D-e)]$). For $\ell \in [2]$, we use the decision algorithm described in Section 5.2.1 to check if there exists a graft (G', R') equivalent to (G, R_ℓ) where $|R'| \leq 2$. For each $\ell \in [2]$ for which the answer is yes, some representation $(G', T') \in \mathcal{F}$ extends to M_i as described in Proposition 5.4.1. If the answer is no for $\ell \in [2]$ then $\mathcal{F}' = \emptyset$.

5.5 Internally 4-connected pinch-graphic matroids

5.5.1 Keeping track of representations

Consider an even-cycle matroid M with a representation (G, Σ). Let N be a minor of M, and let (H, Γ) be a representation of N. Observe that if (G, Σ) has a blocking pair, then so does (H, Γ). It follows in that case that (G, Σ) is a blocking-pair representation of M and that (H, Γ) is a blocking-pair representation of N. Hence, the class of pinch-graphic matroid is also minor-closed. Moreover, every blocking-pair representation of M extends some blocking-pair representation of the minor N.

Suppose that we have binary matroids M and N where $N = M/e$ or $N = M \setminus e$. We will use Propositions 5.3.1 and 5.3.2 to construct all blocking-pair representations of M from the blocking-pair representations of N. We will also require the following observation,

Remark 5.5.1. *We can check in polynomial time if a signed graph has a blocking pair.*

Proof. First observe that we can check if a signed-graph is bipartite by picking a spanning tree and checking if every fundamental polygon is even. Then we check for every pair of distinct vertices u, v if the signed graph obtained by deleting u and v is bipartite. □

5.5.2 Algorithm (4)

Recall Proposition 3.3.1 from Section 3.3.1.

Proposition 3.3.1. *Let M be a binary non-graphic matroid that is $(4,5)$-connected. Then there exists a good sequence $M_1,\ldots,M_k$ for M. Moreover, if we are given M by its $0,1$ matrix representation A, then in time polynomial in the number of entries of A we can construct that good sequence.*

The algorithm will take as input a binary matroid M that is $(4,5)$-connected, described by its $0,1$ representation A and it decides in time polynomial in the size of A if M is pinch-graphic. (As internally 4-connected matroids are $(4,5)$-connected, this yields an algorithm for checking if an internally 4-connected matroid is pinch-graphic).

First we check if M is graphic (see Proposition 3.3.5). If it is, then we can stop and M is pinch-graphic. Otherwise there exists a good sequence $M_1,\ldots,M_k$ for M by Proposition 3.3.1. Iteratively, we will construct the set $\mathcal{S}_i$ of all blocking-pair representations of M_i. Since $M_k \in \{F_7, F_7^*,\ M(K_5)^*,\ M(K_{3,3})^*\}$, we have $|E(M_k)| \le 10$ and we can find the set of blocking-pair representations $\mathcal{S}_k$ by brute force. Suppose now that for some $i \in [k]$ where $i \neq 1$ we have constructed the set $\mathcal{S}_i$. If $\mathcal{S}_i = \emptyset$ then we stop as M is not pinch-graphic since every blocking-pair representation of M should extend some representation from $\mathcal{S}_i$. Otherwise either (i) $M_{i-1} = M_i \setminus e$ or (ii) $M_{i-1} = M_i/e$ for some e. For case (i) we extend the blocking-pair representations of $\mathcal{S}_i$ to $\mathcal{S}_{i-1}$ as in Proposition 5.3.1, and for case (ii) we extend the blocking-pair representations of $\mathcal{S}_i$ to $\mathcal{S}_{i-1}$ as in Proposition 5.3.2. In both cases we use Remark 5.5.1 to only keep the blocking-pair representations. If $\mathcal{S}_1 = \emptyset$ then M is not pinch-graphic otherwise M is pinch-graphic.

Correctness is clear. Note that the algorithm runs in polynomial time in the size of A

as we can construct the good sequence in polynomial time and since by Theorem 3.3.6 each of the sets $\mathcal{S}_i$ have cardinality $\mathcal{O}(|E(M_i)|)^4$.

5.6 Pinch-graphic matroids with small separations

Recall Propositions 1.4.1 and 1.4.2 from Section 1.4.2 and Proposition 1.4.3 from Section 1.4.3.

Proposition 1.4.1. *Let $M = M_1 \oplus_k M_2$ for $k \in [3]$ where M_1 is graphic. If $k = 2$, assume that M is 2-connected, and if $k = 3$, assume that M is 3-connected. Then, M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proposition 1.4.2. *Every 1- and 2-separation of a pinch-graphic matroid is reducible.*

Proposition 1.4.3. *Let M be a 3-connected pinch-graphic matroid and let X' be a proper 3-separation. Then there exists a homologous proper 3-separation X that is reducible, compliant, or recalcitrant.*

Let M be a matroid, and let X be either a 1-separation, 2-separation or proper 3-separation of M. Then, M can be expressed by k-sum of two matroids M_1 and M_2 where $k \in [3]$. We can check if X is reducible by applying a recognition algorithm for graphic matroids to M_1 and M_2. Thus, we are interested in algorithms checking if X is compliant or recalcitrant.

5.6.1 Compliant separations

Let us motivate the term "compliant". Given a 3-connected binary matroid M described by its $0, 1$ matrix representation and given $X \subseteq E(M)$, we will show that we can check in polynomial time whether X is compliant. The key to that result is the next proposition.

Proposition 5.6.1. *Let $M = M_1 \oplus_3 M_2$ be a 3-connected binary matroid. Let $X = E(M_1) - E(M_2)$ and assume that X is not reducible. Then, the following are equivalent:*

(a) X is compliant.

(b) There exists $e \in E(M_1) \cap E(M_2)$ for which both $M_1 \setminus e$ and $M_2 \setminus e$ are graphic.

The algorithmic details on how to use this result are in Section 5.6.3.

Consider a binary matroid M with an element e and a cycle C using e. Observe that every cycle of M is either a cycle avoiding e or the symmetric difference of C and a cycle avoiding e. Thus, Remark 5.6.2 follows.

Remark 5.6.2. *Let M, N be binary matroids with the same ground set containing an element e. Then $M = N$ if and only if all the cycles avoiding e are the same in M and N and at least one cycle using e is the same in M and N.*

Proof of Proposition 5.6.1. Suppose (a) holds. Then (G, Σ, X) is a Type I configuration for some representation (G, Σ) with blocking pair u, v where $u \in \mathcal{I}_G(X)$ and v is not a vertex of $G[X]$. Let G_1 be obtained from $G[X]$ by adding a pair of parallel edges e, f between $a, b \in \partial_G(X)$ and by adding a loop g to be incident to u. Let $\Sigma_1 = \Sigma \cap X \cup \{e, g\}$. By Proposition 4.1.5, $M_1 = \text{ecycle}(G_1, \Sigma_1)$. Moreover, $M_1 \setminus e = \text{ecycle}(G_1 \setminus e, \Sigma_1 - e)$. Then $(G_1 \setminus e, \Sigma_1 - e)$ has blocking vertex u. Thus, Remark 2.4.2 implies that $M_1 \setminus e$ is graphic. By interchanging the role of X and $E(M) - X$ we similarly prove that $M_2 \setminus e$ is graphic.

Now, suppose (b) holds. Then $M_1 \setminus e = \text{cycle}(H)$ for some graph H. Let f, g be edges of $E(M_1) \cap E(M_2)$ other than e. Since M has no loop, neither does $M_1 \setminus e$. Hence, H has no loop. Let r, s denote the ends of g in H. Since f, g are not parallel, we may assume that r is not incident to f. Let G_1 be obtained from H by identifying r and s into t_1 and adding an edge e parallel to f. Let $\Sigma_1 = \delta_H(r) \cup \{e\}$. Observe that $\{e, f, g\}$ is an even cycle of (G_1, Σ_1) and a cycle of M_1. It follows from Remark 5.6.2 that $M_1 = \mathrm{ecycle}(G_1, \Sigma_1)$. Similarly, construct (G_2, Σ_2) by identifying r and s into t_2 where $M_2 = \mathrm{ecycle}(G_2, \Sigma_2)$ where e, f are parallel edges of G_2 and g is a loop G_2. Let G be obtained from G_1 and G_2 by identifying e, f and deleting e, f, g. Let $\Sigma = [\Sigma_1 \cup \Sigma_2] - \{e, g\}$. Then Proposition 4.1.4 implies that $M = \mathrm{ecycle}(G, \Sigma)$. Since X is not reducible, $t_1 \in \mathcal{I}_G(X)$ and $t_2 \in \mathcal{I}_G(E(G) - X)$. Finally, observe that (G, Σ, X) is a Type I configuration, i,.e. (a) holds. □

5.6.2 Recalcitrant separations

Consider a 3-connected binary matroid M described by its $0, 1$ matrix representation. A natural approach for algorithm (5) is to design a subroutine to check if a proper 3-separation is recalcitrant in polynomial time. However, this seems to be harder than checking if X is compliant, so instead we will either establish that X is recalcitrant or find another proper 3-separation that is reducible. We develop the necessary tools in this section, the algorithmic details will appear in Section 5.6.3. Throughout this section, M denotes a 3-connected matroid with a proper 3-separation X and $Y = E(M) - X$.

Working with the completion

Next, we relate representations of M and its completion N with respect to a recalcitrant separation.

Remark 5.6.3. *Suppose that (G, Σ, X) is a Type II configuration and that (G, Σ) is a representation of M. Let a, b denote the vertices of $\partial_G(X)$. Let (H, Γ) be the signed graph obtained from (G, Σ) by adding an odd loop e_1, an even edge e_2 with ends a, b and an odd edge e_3 with ends a, b. Then,* *(a)* $N = \mathrm{ecycle}(H, \Gamma)$ *is the completion of* M *with respect to* X*,*

(b) if $N = \mathrm{ecycle}(H', \Gamma')$ *and* $(H', \Gamma')|E(M) = (G, \Sigma)$ *then* (H, Γ) *and* (H', Γ') *are isomorphic up to moving* e_1*.*

Proof. (a) is shown in the proof of Proposition 4.1.4. (b) Let $e \in E(N) - E(M)$. Then by definition of completion there exists cycles C and D of N where $C \subseteq X \cup e$, $D \subseteq (E(M) - X) \cup e$ and $e \in C \cap D$. Thus $C \,\Delta\, D = (C - e) \cup (D - e)$ is a cycle of M and in particular an even-cycle of (G, Σ). It follows that $C - e$ and $D - e$ are either both odd cycles of (G, Σ) or both ab-joins of G. Hence, e is either an odd loop of (H', Γ') or has ends a, b in H'. As elements in $E(N) - E(M)$ form a circuit of N they form an even cycle of (H', Γ') and the result follows. □

Throughout this section N shall denote the completion of M with respect to X. Moreover, e_1, e_2, e_3 denote the elements of $E(N) - E(M)$.

The next result shows that it suffices to check whether X is recalcitrant for N.

Remark 5.6.4. *X is a recalcitrant separation of M if and only if X is a recalcitrant separation of N.*

Proof. Suppose that X is a recalcitrant separation of M. Then, there exists a representation (G, Σ) of M for which (G, Σ, X) is a Type II configuration with $\{a, b\} = \partial_G(X)$. Then, the signed graph (H, Γ) obtained from (G, Σ) as in Remark 5.6.3 is a representation of N. Since a, b is a blocking pair of (G, Σ) it is a blocking pair of (H, Γ). Thus, (H, Γ, X) is a Type II configuration and X is a recalcitrant separation of N. Suppose that X is a recalcitrant separation of N. Then there exists a representation (H, Γ) of N for which (H, Γ, X) is a Type II configuration with $\{a, b\} = \partial_H(X)$. Let $(G, \Sigma) = (H, \Gamma) \setminus E(N) - E(M)$. Then

(G, Σ) is a representation of M. As X is a proper 3-separation, $|X|, |E(G) - X| \geq 4$, moreover, as M is 3-connected, $G[X]$, $G[E(G) - X]$ are both connected and $\partial_G(X) = \{a, b\}$. Finally, as a, b is a blocking pair of (H, Γ) it is also a blocking pair of (G, Σ). Thus (G, Σ, X) is a Type II configuration and X is a recalcitrant separation of M. $\square$

Bilateral representations

A representation (H, Γ) of N is *bilateral* if, for some $\{i, j, k\} = [3]$,

i. $H[X]$ and $H[Y]$ are both connected,

ii. $\partial_H(X) = \{a, b\}$,

iii. e_i is a loop and the ends of e_j, e_k are a, b,

iv. $e_i, e_j \in \Gamma$, $e_k \notin \Gamma$.

Remark 5.6.5. *If (H, Γ, X) is a Type II configuration where $N = \text{ecycle}(H, \Gamma)$ then (H, Γ) is a bilateral representation of N. Moreover, if (H, Γ) is a bilateral representation of N with blocking pair a, b then (H, Γ, X) is a Type II configuration.*

Next we show that if X is recalcitrant, then we can construct a bilateral representation. Moreover, it suffices to find an equivalent representation for which $\partial(X)$ is a blocking pair to certify that the representation is recalcitrant.

Proposition 5.6.6. *Suppose X is a recalcitrant separation of N and pick $\{i, j, k\} = [3]$. Then,*

(a) $N/e_i = \text{cycle}(G)$ for some graph G.

Pick an arbitrary graph G' equivalent to G and let H be obtained from G' by adding loop e_i. Let Γ be a cocircuit of N using e_i. Then, the following also hold:

(b) (H,Γ) *is a representation of* N*.*

(c) There exists (H',Γ') *equivalent to* (H,Γ) *for which* (H',Γ',X) *is a Type II configuration.*

(d) (H,Γ) *is a bilateral representation of* N*.*

Before we can proceed with the proof we require a number of preliminaries.

Proposition 5.6.7. *If X is a recalcitrant separation of N then for every $i \in [3]$ there exists a Type II configuration (H,Γ,X) where (H,Γ) is a representation of N and where e_i is an odd loop.*

Proof. Since X is a recalcitrant separation of N there exists a Type II configuration (H',Γ,X) where (H',Γ) is a representation of N. Denote by a,b the vertices in $\partial_H(X)$. By Remark 5.6.3, we have that e_1,e_2,e_3 are are either loops or have ends a,b. We may assume that e_i has ends a,b. After possibly re-signing, we have Γ with $e_i \in \Gamma$ and $\Gamma \subseteq \delta_{H'}(a) \cup \delta_{H'}(b)$. Let (H,Γ) be obtained from (H',Γ) by a Lovász-flip on the blocking pair a,b. Observe that e_i is an odd loop of H. Then (H,Γ,X) is the required Type II configuration. □

We will also require the following immediate consequence of Proposition 2.4.10 (see [27], Lemma 17).

Remark 5.6.8. *Two signed-graphs with the same even cycles and a common odd cycle are equivalent.*

We are now ready for the main proof of this section.

Proof of Proposition 5.6.6. For (a), since X is a recalcitrant separation of N, it follows by Proposition 5.6.7 that there exists a Type II configuration (H', Γ', X) where $N = \text{ecycle}(H', \Gamma')$ and where e_i is an odd loop of H'. Since e_i is an odd loop, $N/e_i = \text{ecycle}(H', \Gamma')/e_i = \text{cycle}(H'/e_i)$. Then let $G = H'/e_i = H' \setminus e_i$. For (b), let $N' = \text{ecycle}(H, \Gamma)$. Recall that by Proposition 2.4.10, $\text{cycle}(G') = \text{cycle}(G)$. Then

$$N'^* \setminus e_i = (N'/e_i)^* = (\text{ecycle}(H, \Gamma)/e_i)^* = \text{cycle}(G')^* = (N/e_i)^* = N^* \setminus e_i.$$

By hypothesis Γ is a cocircuit of N. Moreover, Γ is a cocircuit of N' as Γ is a signature of (H, Γ) and signatures correspond to cocircuits. It follows from Remark 5.6.2 that $N'^* = N^*$ i.e. that $N = N' = \text{ecycle}(H, \Gamma)$. For (c), note that (b) implies that (H, Γ) and (H', Γ') have the same set of odd cycles. Moreover, e_i is an odd loop of both signed graphs. It follows from Remark 5.6.8 that (H, Γ) and (H', Γ') are equivalent, proving (c). For (d), note that Remark 5.6.5 implies that (H', Γ') is a bilateral representation of N. Remark 5.6.3 implies that e_j, e_k are joining the ends of $\delta_{H'}(X)$. Moreover, H is 2-connected by Proposition 4.3.1. We proved in (c) that H and H' are equivalent. Hence, H is obtained from H' by a sequence of 2-flips on sets $U \subseteq X$ or $U \cap X = \emptyset$. Thus (H, Γ) is also a bilateral representation of N. □

Solid separations

Proposition 5.6.6 suggests the following procedure for recognizing if X is a recalcitrant separation of N, pick $i \in [3]$ and check if N/e_i is graphic. If it is not, then X is not recalcitrant. Otherwise, $N/e_i = \text{cycle}(G')$ for some graph G', so construct a representation (H, Γ) of N as described in Proposition 5.6.6. If H is 3-connected, then $H = H'$ and it follows from Remark 5.6.5 that X is recalcitrant if and only if the ends of e_j (resp. e_k) form a blocking pair of (H, Γ). Alas H need not be 3-connected. Moving forward, our strategy will be to analyze the 2-separations of H. In this section we analyze one such type of separations.

Let (H, Γ) be a bilateral representation of N. Then $Z \subseteq E(H)$ is a *solid separation* of H if the following hold,

i. $Z \cap \{e_1, e_2, e_3\} = \emptyset$,

ii. $H[Z]$ and $H[E(H) - Z]$ are connected,

iii. $|\,\partial_H(Z)| = 2$ and $\partial_H(X) \neq \partial_H(Z)$,

iv. there exists internally disjoint path P_1, P_2 in $H[Z]$ with ends $\partial_H(Z)$ and $|P_1|, |P_2| \geq 2$.

Next we present the key result of this section.

Proposition 5.6.9. *Suppose X is a recalcitrant separation of N and (H, Γ) is a bilateral representation of N. If Z is a solid separation, then Z is a reducible separation of M.*

First we require the following observation used in [11].

Remark 5.6.10. *Let H be a graph that contains a theta subgraph consisting of internally disjoint path P_1, P_2, P_3. Let H' be a graph equivalent to H. Then P_1, P_2, P_3 are paths of H' that form a theta subgraph. Note that the order of the edges in P_i need not be the same in H and H'.*

Proof of Proposition 5.6.9. Let u, v denote the vertices in $\partial_H(Z)$. Because edge e_j (resp. e_k) has ends u, v, $Z \subset X$ or $Z \subset Y$ and we may assume the former. Since Z is solid there exists internally disjoint uv-paths P_1, P_2 in $H[Z]$. Since $H[E(H) - Z]$ is connected there exists a uv-path P_3 in $H[E(H) - Z]$. Observe that P_1, P_2, P_3 form a theta graph of H.

By Proposition 5.6.6 there exists H' equivalent to H such that (H', Γ, X) is a Type II configuration. By Remark 5.6.10 P_1, P_2, P_3 form a theta graph of H'. Denote by u', v' the ends of P_1, P_2, P_3 in H'. By Proposition 4.3.1 H, H' are 2-connected (up to a single loop). It follows that $\partial_{H'}(Z) = \{u', v'\}$ and that $H'[Z]$ connected.

Claim 1. *If u' or v' is a blocking vertex of $(H', \Gamma)|Z$ then Z is reducible for M.*

Subproof. Suppose for a contradiction that u' is blocking vertex of (H', Γ'). Let (G, Σ) be obtained from $(H', \Gamma)|Z$ by adding an odd loop f_1, an even edge f_2 with ends $\partial_{H'}(Z)$ and an odd edge f_3 with ends $\partial_{H'}(Z)$. It follows from Proposition 3.1.5 that Z is 3-separating in M. As $|P_1|, |P_2| \geq 2$, $|Z| \geq 4$. Note, $Z \subseteq X$ hence $|E(M) - Z| \geq 4$. Since M is 3-connected, Z must be exactly 3-separating and Z is a proper 3-separation of M. It follows that $M = M_1 \oplus_3 M_2$ for some matroids M_1, M_2 where $E(M_1) - E(M_2) = Z$. By Proposition 4.1.4, M_1 is isomorphic to ecycle(G, Σ). Note, u' is a blocking vertex of (G, Σ). It follows from Remark 2.4.2 that M_1 is graphic, hence by definition Z is reducible. $\diamondsuit$

Because (H', Γ, X) is a Type II configuration, the ends of e_j (resp. e_k) is a blocking pair of (H', Γ). Suppose for a contradiction, Z is not reducible for M. Then by Claim 1 neither u' nor v' is a blocking vertex of $(H', \Gamma)|Z$. It follows that u', v' must be the ends of e_j. Observe that P_1, P_2, e_2 is theta graph. Hence, by Remark 5.6.10, P_1, P_2, e_j is a theta graph of H. It follows that u, v are the ends of e_j in H. But the ends of e_j are $\partial_H(X)$, hence $\partial_H(X) = \partial_H(Z)$, contradicting the definition of solid separation. $\square$

Kernels

We still need to consider bilateral representations (H, Γ) of N that are not 3-connected but do not have a solid separation. Consider a signed-graph (H, Γ) and let $Z = \{f_1, f_2,$

$f_3\} \subseteq E(H)$ where $H[Z]$ is connected and $\partial_H(Z) = \{u, v\}$ for some $u, v \in V(H)$. Suppose that f_1, f_2 form a u, v-path, that f_3 is parallel to f_2, and that $\{f_2, f_3\}$ is an odd polygon. Then Z is a *degenerate separation* of H. We say that $(H', \Gamma)/f_1$ is obtained from (H, Γ) by *reducing* the degenerate separation Z. The graph obtained from (H, Γ) by reducing all degenerate separations is the *kernel* of (H, Γ). We leave the following as an easy exercise,

Remark 5.6.11. *Consider a signed-graph (H, Γ) with kernel (H', Γ') and $g \in E(G) \cap E(G')$. Then the ends of g form a blocking pair of (H, Γ) if and only if the ends of g form a blocking pair of (H', Γ').*

In a signed-graph a *double path* is obtained from an internally disjoint path by replacing every edge by an odd polygon of size two. The next proposition will be the key for recognizing recalcitrant separations,

Proposition 5.6.12. *Let (H, Γ) be a bilateral representation of N with loop e_i. Suppose that H has no solid separation. Then X is a recalcitrant separation of N if and only the ends of e_j (resp. e_k) form a blocking pair of the kernel of (H, Γ).*

Proof. Suppose first that the ends of e_2 form a blocking pair of the kernel of (H, Γ). It then follows that the ends of e_2 also form a blocking pair of (H, Γ) by Remark 5.6.11. Hence, (H, Γ, X) is a Type II configuration and by definition X is a recalcitrant separation of N. Suppose now that X is a recalcitrant separation of N. By Proposition 5.6.6 there exists a Type II configuration (H', Γ, X) where H and H' are equivalent. After possibly re-signing we have kernel $(\hat{H}, \Gamma)$ of (H, Γ) and kernel $(\hat{H}', \Gamma)$ of (H', Γ) where $\hat{H}$ and $\hat{H}'$ are equivalent. Suppose that Z is a (not necessarily proper) 2-separation of $\hat{H}$ with $\{u, v\} = \partial_{\hat{H}}(Z)$.

Claim 1. *Z consists of a single edge or a double path with ends u, v.*

Subproof. Let us proceed by induction on $|Z|$. Clearly, the result holds if $\mathcal{I}_{\hat{H}}(Z) = \emptyset$. Since there is no solid separation we may assume that there exists $w \in \mathcal{I}_{\hat{H}}(Z)$ and a partition Z_1, Z_2 of Z such that Z_1, Z_2 are 2-separations with $\partial_{\hat{H}}(Z_1) = \{u, w\}$ and $\partial_{\hat{H}}(Z_2) = \{v, w\}$. But then apply induction on Z_1, Z_2. For $i \in [2]$, Z_i is either a single edge or a double path. If Z_1 and Z_2 are single edges, then we have a degree two vertex contradicting Proposition 4.3.1. If Z_i is an edge and Z_{3-i} is a double path for some $i \in [2]$ then we have a trivial separation, a contradiction. If Z_1 and Z_2 are both double paths, then so is Z as required. ♢

Claim 2. *Z is a double path of length two.*

Subproof. Otherwise one of the odd polygon will not be incident to the ends of e_2 in $\hat{H}'$ contradicting the fact that the ends of e_2 form a blocking pair of $(\hat{H}', \Gamma)$. ♢

The ends of e_2 form a blocking pair of (H', Γ). It follows by Remark 5.6.11 that the ends of e_2 form a blocking pair of $(\hat{H}', \Gamma)$. Since $\hat{H}$ and $\hat{H}'$ are equivalent and because of Claim 1, the ends of e_2 form a blocking pair of $(\hat{H}, \Gamma)$ as required. □

5.6.3 Algorithm (5)

In this section, we will show that, when trying to recognize whether a matroid is pinch-graphic, we can restrict ourselves to internally 4-connected matroids. We will describe a number of procedures that take a (binary) matroid M as input. In each case, the matroid will be described by its $m \times n$, $0, 1$ matrix representation A. A procedure runs in polynomial time if its running time is bounded by a polynomial in m and n. The algorithm rely on Propositions 3.3.4 and 3.3.5 from Section 3.3.1.

Next we describe algorithms that will analyze reducible, compliant, and recalcitrant separations.

Algorithm A: reducible separations

The procedure takes as input a pair M and $X \subseteq E(M)$ where either (1) X is a 1-separation of M; (2) M is 2-connected and X is a 2-separation of M; or (3) M is 3-connected and X is a proper 3-separation of M. In polynomial time, the procedure will either indicate that either (a) X is not reducible, or (b) return a matroid N where $|E(N)| < |E(M)|$ and where N is pinch-graphic if and only if M is pinch-graphic. We proceed as follows: X is a k-separation for some $k \in [3]$. If $k = 1$, then we let $M_1 = M \setminus X$ and $M_2 = M \setminus (E(M) - X)$. If $k \in \{2, 3\}$, then we construct the completion N of M with respect to X and we let $M_1 = N \setminus X$ and $M_2 = N \setminus (E(M) - X)$. Then, $M = M_1 \oplus_k M_2$ (see Propositions 3.1.9 and 4.1.2). Then, we check whether there exists $i \in [2]$ such M_i is graphic (see Proposition 3.3.5). If not, we stop and indicate that X is not reducible. Otherwise, we stop and return $N := M_{3-i}$ that is pinch-graphic if and only if M is pinch-graphic (see Propositions 4.1.6, 4.1.7 and 4.1.8).

Algorithm B: compliant separations

The procedure takes as input a pair M and $X \subseteq E(M)$ where X is a proper 3-separation of M and M is 3-connected. In polynomial time, the procedure will indicate whether X is compliant. We proceed as follows: first we construct the completion N of M with respect to X and we let $M_1 = N \setminus X$ and $M_2 = M \setminus (E(M) - X)$. Let $\{e_1, e_2, e_3\} = E(M_1) \cap E(M_2)$. If $M_1 \setminus e$ and $M_2 \setminus e$ are both graphic for some $i \in [3]$, then we stop and indicate that X is compliant. Otherwise, we stop and indicate that X is not compliant. Correctness follows from Proposition 5.6.1.

Algorithm C: recalcitrant separations

The procedure takes as input a pair M and $X \subseteq E(M)$ where X is a proper 3-separation of M and M is 3-connected. In polynomial time, the procedure will do one of the following: (a) establish that X is recalcitrant, (b) establish that X is not recalcitrant, or (c) return a matroid N where $|E(N)| < |E(M)|$ and where N is pinch-graphic if and only if M is pinch-graphic. We proceed as follows: first, we construct the completion N of M with respect to X and we let $M_1 = N \setminus X$ and $M_2 = N \setminus (E(M) - X)$. Let $\{e_1, e_2, e_3\} = E(M_1) \cap E(M_2)$. If M/e_1 is not graphic, then we stop and indicate that X is not recalcitrant. Correctness follows from Proposition 5.6.6 (a). Otherwise, we find a graph G' for which $M/e = \text{cycle}(G')$. Construct H by adding loop e_1, and let Γ be any cocircuit of M using e_1. If (H, Γ) is not bilateral, then we stop and indicate that X is not recalcitrant. Correctness follows from Proposition 5.6.6 (c). We then check whether H has a solid separation Z. If it does, we run procedure A with input M and Z. If procedure A says that Z is not reducible, then we stop and indicate that X is not recalcitrant. Correctness follows from Proposition 5.6.9. Otherwise, we stop and return the matroid N given by procedure A. Next, we construct the kernel of (H, Γ). If the ends of e_2 (resp. e_3) of the kernel form a blocking pair, then we indicate that X is recalcitrant; otherwise, we indicate that X not recalcitrant. Correctness follows from Proposition 5.6.12.

Putting it together

The procedure takes as input a matroid M. In polynomial time it will either (a) establish that M is pinch-graphic, (b) establish that M is not pinch-graphic, or (c) construct a matroid N that is internally 4-connected where N is isomorphic to a proper minor of M and where N is pinch-graphic if and only M is pinch-graphic. Note we can check if M has a

1-, 2-, or proper 3-separation in polynomial time (see Proposition 3.3.4). Also observe that if we establish that M has a compliant or a recalcitrant separation then by definition M is pinch-graphic. Finally note that a proper 3-separation of a 3-connected binary matroid has at most 8 homologous 3-separations.

We repeat the following steps until we stop,

(1) Try to find a 1-separation or a 2-separation. If there exists such a k-separation, pick one minimizing k. Use Algorithm A to check whether such a separation X is reducible. If X is not reducible, then stop and return that M is not pinch-graphic (see Propositions 4.1.7 and 4.1.8). If X is reducible, then set $M := N$ where N is the matroid returned by Algorithm A and start again at the beginning of (1).

At this stage of the algorithm, the matroid M is 3-connected.

(2) Try to find a proper 3-separation Y. If no such separation exists, we stop and return M.

(3) For each separation X that is homologous to Y do the following:

   (3.1) Use Algorithm A to check whether X is reducible. If it is, then set $M := N$ where N is the matroid returned by Algorithm A and start again at the beginning of (1).

   (3.2) Use Algorithm B to check whether X is compliant. If it is, then stop and return that M is pinch-graphic.

   (3.3) Use Algorithm C. If the algorithm indicates that X is recalcitrant, then stop and return that M is pinch-graphic. If the algorithm returns a matroid N, then set $M := N$ and start again at the beginning of (1).

At this stage, none of the separations homologous to Y is either reducible, compliant or recalcitrant. Hence, by Proposition 1.4.3, M is not pinch-graphic.

(4) Stop and return that M is not pinch-graphic.

Chapter 6

Future works

6.1 Isomorphism problem

Consider graphs H and H' where $E(H) \cap E(H') = \{e, f, g\}$ where $\{e, f, g\}$ is a triangle of both H and H'. The graph obtained by identifying e, f, g in H and H'; and deleting $\{e, f, g\}$ is a *3-sum of H and H' on e, f, g*. 3-sums preserve the sibling property.

Proposition 6.1.1. *Let (G_1, G_2) be siblings with triangle $\{e, f, g\}$ and let H be a graph with a triangle $\{e, f, g\}$. For $i \in [2]$, let G'_i be obtained from G_i by a* 3*-sum of G_i and H on e, f, g. Then, (G'_1, G'_2) are siblings.*

Proof. Let Σ_1, Σ_2 be a matching-signature pair for G_1, G_2. Since $\{e, f, g\}$ is a circuit of both G_1 and G_2, it must be even, for otherwise would have $\text{cycle}(G_1) = \text{cycle}(G_2)$, contradicting the fact that G_1, G_2 are siblings. Hence, we may assume by re-signing that for $i \in [2]$, $\Sigma_i \cap \{e, f, g\} = \emptyset$. Let C be an even cycle of (G_i, Σ_i) for some $i \in [2]$. We claim that C is an even cycle of (G_{3-i}, Σ_{3-i}). This is obvious if $C \subseteq E(G_i)$ or if $C \subseteq E(H)$. Thus, we may assume that $C = P \cup Q$ where P is a path of H and where Q is a path of G_i and $(P \cup Q) \cap \{e, f, g\} = \emptyset$. After possibly interchanging the role of e, f, g, we may assume that P and Q have have the same ends as e. But then $P \cup \{e\}$ and $Q \cup \{e\}$ are both even cycles of (G_2, Σ_2). But then so is $P \cup Q$ and it follows that $\text{ecycle}(G_1', \Sigma_1) = \text{ecycle}(G_2', \Sigma_2)$ and in particular that (G_1', G_2') are siblings. □

Consider siblings G_1 and G_2 with matching-terminal pair T_1, T_2 where for $i \in [2]$, G_i is isomorphic to K_6 and $|T_i| = 6$. An example is given in Figure 1.6. Then repeatedly, pick a triangle $\{e, f, g\}$ of G_1 that is also a triangle in G_2 and for $i \in [2]$ replace G_i by a 3-sum of G_i and some graph H on e, f, g. The resulting pair of graphs are K_6-*siblings*. Note that they are indeed siblings by Proposition 6.1.1.

We conjecture that Theorem 2.3.1 has an analogue result for the case where the siblings are 3-connected, namely, we predict the following:

Conjecture 6.1.2. *Let (G_1, G_2) be 3-connected, closed siblings that are neither graphic nor cographic. Denote by (Σ_1, Σ_2) the matching-signature pair and denote by (T_1, T_2) the matching-terminal pair. Then, one of the following holds:*

(a) (G_1, G_2) are K_6 siblings;

(b) (G_1, G_2) are blocking-pair siblings;

(c) (G_1, G_2) are T_4 siblings;

(d) there exist $i \in [2]$ (say $i = 1$) and a graph G_1' such that (G_1, G_1') are blocking-pair siblings, and (G_1', G_2) are cographic siblings; or

(e) there exists $i \in [2]$ (say $i = 1$) and a graph G_1' such that (G_1, G_1') are T_4 siblings, and (G_1', G_2) are graphic siblings.

Note that we added outcome (e). This is necessary because of the following siblings given in Figure 6.1. Denote by G_1 and G_2 the graphs on the left and right. The white vertices represent terminals. Then, (G_1, G_2) are siblings, but none of outcomes (a)–(d)

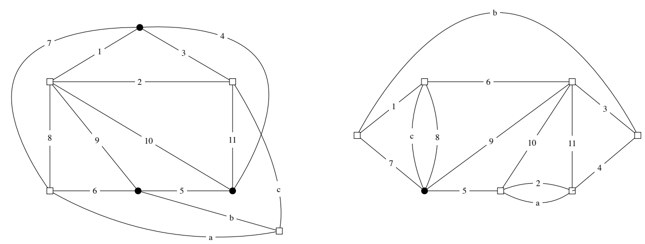

Figure 6.1: A bad example.

arise. However, there exists a graph G_1', as illustrated in Figure 6.2, where (G_1, G_1') are T_4 siblings and (G_1', G_2) are graphic siblings, which is outcome (e) in Conjecture 6.1.2. More examples appear in [23].

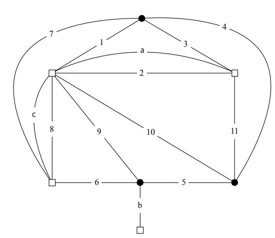

Figure 6.2: Graphic siblings after a T_4-move.

6.2 Excluded minors

6.2.1 Well-known classes

For a minor-closed class of matroids, it is often described as a list of excluded minors. An *excluded minor* is a minor-minimal matroid that is not in the class. The sets of the excluded minors of the following classes are well-known:

(a) $\{U_{2,4}\}$ for binary matroids [49],

(b) $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ for ternary matroids [48],

(c) $\{U_{2,4}, F_7, F_7^*\}$ for regular matroids [49, 17],

(d) $\{U_{2,4}, F_7, F_7^*, \mathrm{cut}(K_5), \mathrm{cut}(K_{3,3})\}$ for graphic matroids [50] and

(e) $\{U_{2,4}, F_7, F_7^*, \mathrm{cycle}(K_5), \mathrm{cycle}(K_{3,3})\}$ for cographic matroids [50]

6.2.2 Even-cycle and even-cut matroids

We are interested in analogous problems for classes of even-cycle, even-cut, pinch-graphic and pinch-cographic matroids, namely, we are interested in the following problems:

Problem 6.2.1.

(a) Describe excluded minors for even-cycle matroids,

(b) describe excluded minors for even-cut matroids,

(c) describe excluded minors for pinch-graphic matroids that are even-cycle matroids, and

(d) describe excluded minors for pinch-cographic matroids that are even-cut matroids.

Let us recall Theorem 3.1.1 from Section 3.1.1, which implies that there is a finite number of excluded minors for even-cycle matroids.

Theorem 3.1.1. *There exists a constant c, such that every minimally non-pinch-graphic (resp. minimally non-pinch-cographic) matroid has at most c elements.*

However, Pivotto and Royle [41] found more than 400 excluded minors for the class of even-cycle matroids. In [19], it is proved that, for a "highly" connected matroid M, M is an even-cycle matroid if and only if it does not have a minor isomorphic to $PG(3,2)\setminus e, L_{19}$ or L_{11}. The definitions of these matroids and similar results for even-cut, pinch-graphic and pinch-cographic matroids can be found in [19].

As shown in [40] (Lemma 7.1 and Lemma 7.2), the following hold:

Lemma 6.2.2. *Let $M = M_1 \oplus_1 M_2$ be a 1-sum of M_1 and M_2. Then, M is an excluded minor for even-cycle matroids if and only if both M_1 and M_2 are minimally non-graphic matroids.*

Lemma 6.2.3. *Let $M = M_1 \oplus_1 M_2$ be a 1-sum of M_1 and M_2. Then, M is an excluded minor for even-cut matroids if and only if both M_1 and M_2 are minimally non-cographic matroids.*

As a corollary of Propositions 1.4.1 and 4.2.1, the following can be proved.

Corollary 6.2.4. *Let $M = M_1 \oplus_1 M_2$ be a 1-sum of M_1 and M_2. Then, M is an excluded minor for pinch-graphic matroids if and only if both M_1 and M_2 are minimally non-graphic matroids.*

Corollary 6.2.5. *Let $M = M_1 \oplus_1 M_2$ be a 1-sum of M_1 and M_2. Then, M is an excluded minor for pinch-cographic matroids if and only if both M_1 and M_2 are minimally non-cographic matroids.*

As a consequence of the characterization of 1-, 2- and 3-separations of even-cycle matroids, the following problems can be considered.

Problem 6.2.6.

(a) Describe excluded minors for even-cycle matroids that contain a 2- or 3-separation,

(b) describe excluded minors for even-cut matroids that contain a 2- or 3-separation,

(c) describe excluded minors for pinch-graphic matroids that are even-cycle matroids and contain a 2- or 3-separation, and

(d) describe excluded minors for pinch-cographic matroids that are even-cut matroids and contain a 2- or 3-separation.

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List of Notations

$\partial_G(X)$ the intersection of $V_G(X)$ and $V_G(E(G) - X)$ 3

$[k]$ set $\{1, 2, \ldots, k\}$ 3

r_M the rank function of matroid M 4

$\mathrm{r}(M)$ the rank of matroid M 4

λ_M the connectivity function of matroid M 4

(G, Σ) the signed graph with graph G and sign Σ 4

$\mathrm{ecycle}(G, \Sigma)$ the even-cycle matroid which arises from signed graph (G, Σ) 5

$(G, \Sigma)/I \setminus J$ the minor of signed graph (G, Σ) obtained by contracting I and deleting J 5

$(G, \Sigma)|F$ the induced signed graph of (G, Σ) by edge subset F of G 5

(G, T) the graft with graph G and terminal set T 6

$\mathrm{ecut}(G, T)$ the even-cut matroid which arises from graft (G, T) 6

$V_{odd}(G)$ the subset of odd-degree vertices of graph G 6

$(G, T)/I \setminus J$ the minor of graft (G, T) obtained by contracting I and deleting J 6

$(G, T)|F$ the induced graft of (G, T) by edge subset F of G 7

$(G, \mathcal{B})$ the biased graph with graph G and set $\mathcal{B}$ of balanced cycles 8

$\mathrm{FM}(G, \mathcal{B})$ the frame matroid which arises from biased graph $(G, \mathcal{B})$ 8

$\mathrm{LM}(G, \mathcal{B})$ the lift matroid which arises from biased graph $(G, \mathcal{B})$ 10

$\Sigma \,\Delta\, \Gamma$ the symmetric difference of Σ and Γ 12

$V_G(F)$ the vertex set of the induced subgraph of G by edge subset F 17

$\mathrm{cl}_M(X)$ the closure of edge subset X for matroid M 23

M^* the dual of matroid M 23

$\mathcal{I}_G(X)$ the set of vertices in $V_G(X) - \partial_G(X)$ 23

$\lambda_1(N)$ the number of connected components of matroid N 69

$\Lambda_2(N)$ the collection of 3-connected matroids that construct matroid N by 1-sums and 2-sums 69

$\lambda_2(N)$ the number of matroids in $\Lambda_2(N)$ 69

$f(M)$ the number of pairwise inequivalent signed-graph representations of matroid M 70

$\kappa(H)$ the number of components of graph H 71

$M_1 \oplus_1 M_2$ the 1-sum of matroids M_1 and M_2 73

$M_1 \oplus_2 M_2$ the 2-sum of matroids M_1 and M_2 73

$g(M)$ the number of pairwise inequivalent graft representations of matroid M 80

$M_1 \oplus_3 M_2$ the 3-sum of matroids M_1 and M_2 113

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