

# Deformation theory of nearly $G_2$ -structures and nearly $G_2$ instantons

by

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A thesis  
presented to the University of Waterloo  
in fulfillment of the  
thesis requirement for the degree of  
Doctor of Philosophy  
in  
Pure Mathematics

Waterloo, Ontario, Canada, 2021

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## **Author's Declaration**

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

Chapter 6 is based on a paper jointly written with Shubham Dwivedi [DS20] which is to appear in the Communications in Analysis and Geometry. Chapter 7 is based on a preprint by the author [Sin21].

## Abstract

We study two different deformation theory problems on manifolds with a nearly  $G_2$ -structure. The first involves studying the deformation theory of nearly  $G_2$  manifolds. These are seven dimensional manifolds admitting real Killing spinors. We show that the infinitesimal deformations of nearly  $G_2$ -structures are obstructed in general. Explicitly, we prove that the infinitesimal deformations of the homogeneous nearly  $G_2$ -structure on the Aloff–Wallach space are all obstructed to second order. We also completely describe the de Rham cohomology of nearly  $G_2$  manifolds.

In the second problem we study the deformation theory of  $G_2$  instantons on nearly  $G_2$  manifolds. We make use of the one-to-one correspondence between nearly parallel  $G_2$ -structures and real Killing spinors to formulate the deformation theory in terms of spinors and Dirac operators. We prove that the space of infinitesimal deformations of an instanton is isomorphic to the kernel of an elliptic operator. Using this formulation we prove that abelian instantons are rigid. Then we apply our results to explicitly describe the deformation space of the canonical connection on the four normal homogeneous nearly  $G_2$  manifolds.

We also describe the infinitesimal deformation space of the  $SU(3)$  instantons on Sasaki–Einstein 7-folds which are nearly  $G_2$  manifolds with two Killing spinors. A Sasaki–Einstein structure on a 7-dimensional manifold is equivalent to a 1-parameter family of nearly  $G_2$ -structures. We show that the deformation space can be described as an eigenspace of a twisted Dirac operator.

## Acknowledgements

*The heart who gives thanks is a happy one, for we cannot feel unhappy and thankful at the same time.*

Words cannot truly describe the appreciation and admiration I have for both of my PhD supervisors, Prof. Benoit Charbonneau and Prof. Spiro Karigiannis. Their intellect, kindness, humility and interest in my research have been a great driving force throughout my PhD program. I feel so fortunate to be advised by these two amazing supervisors who not only taught me mathematics but also dictated so many life lessons that I will remember for the rest of my life. From insisting that I address them by their first name to hearing me rant for hours their patience and thoughtfulness have been very solacing. When I was full of apprehension they showed confidence in me, to which I owe them a lot. While Benoit taught me ingenuity, Spiro prepared me for vigour. Our weekly meetings always left me in high spirits. Our discussions were always very enlightening and the sparkle in their eyes on discovering a new piece of mathematics made me understand what being a mathematician actually means. They always made time for me and were constantly concerned about my professional and personal well being. As someone staying thousands of miles away from family and friends this meant a lot. They always encouraged me to discuss mathematics and meet new people and helped me both financially and logistically to attend as many conferences as I desired. They also motivated me to speak in seminars on diverse topics that really boosted my confidence and challenged me to learn new things. I have been really fortunate to be mentored by both of them as their complementary styles of advising broadened my perspective. I can only hope to achieve such an admirable relation with my student if I ever get a chance to mentor one. I would also like to thank Benoit's and Spiro's loving families. Their wives and kids were always very welcoming. Our conversations were full of happy anecdotes and memories that filled my heart with pleasure. This thesis was unthinkable without their guidance and I cannot thank them enough for that.

I would also like to express my thanks to Doug Park, Ruxandra Moraru, Gonçalo Oliveira, Florian Girelli to be on my thesis committee. Their inputs and suggestions have improved the quality of my thesis to a great extent. I am especially thankful to Ruxandra for our frequent conversations both on and besides mathematics in the department hallway.

The Department of Pure Mathematics at the University of Waterloo is one of the most cheerful group of people. The lively atmosphere and routine activities always kept me amused. I am indebted to the department secretaries Lis, Pavlina, Nancy, and Jackie who are very helpful and took care of all the administrative stuff, while I could completely focus on my research. I would also like to thank Barbara Csima, David McKinnon and Nico Spronk for addressing all my concerns and helping me throughout this journey.

Even the most difficult paths seem easy if there are good friends to walk with you so thanks to Shubham, Satish, Jitendra and Mani for making me feel at home away from home. I will also cherish our fun and food filled gatherings. A very special thanks to my friend and collaborator Shubham who has been a constant source of inspiration and motivation. From teaching me the concept to witnessing it evolve his support was much needed for the completion of this thesis. I would also like to thank my ladies Hongdi, Diana and Farida for all our stimulating conversations and of course the tattles. Thanks for listening and understanding. A huge shout out to Zack and Patrick for lifting up my spirits. Back in India I would like to thank Dharita, Reema, Amitha, Sukriti, Prabhash, Rahar and Pathak with whom I have completed more than 10 years of friendship but they always find novel ways to refresh me. Distance from all of you have only made the heart grow fonder.

Saving the best for the last I would like to thank my amazing family. In the pursuit of my degree I have often ignored my duties towards them and they never complaint about it. They always cheered for my successes and helped me overcome my failures. My grandfather never failed to appreciate me and inspired me to do more. My father, Dr. Vijai Singhal always supported me against all odds and my mother, Mrs. Laxmi Singhal never let the distance (which always troubled her) come in the way of her love and care. I know how much trouble they had to go through because of my unconventional choices but still in their eyes I only see pride and joy. I felt a great sense of relief since my brother, Saurabh and my sister in law, Neha were there to take care of everything in my absence. Thank you for constantly updating me on family chatter and making me feel I am right at home. A very huge thanks to Avi who most importantly kept me sane during the writing of this thesis. Your charming face, laughter and mischief rejuvenates me like none other. I am forever indebted to my family for their unconditional love and support.

## **Dedication**

The thesis is dedicated to my grandmother and maternal grandparents. Your presence was always felt through your blessings.

The thesis is also dedicated to all the Covid warriors and frontline workers who helped in many ways to manage this catastrophe.



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# Chapter 1

## Introduction

In this thesis we primarily focus on 7-dimensional manifolds with (non-parallel) Killing spinors. A *Killing spinor* is a section  $\eta$  of the spinor bundle  $\mathcal{S}$  of a Riemannian spin manifold  $(M^n, g)$  for which there exists a non-zero constant  $\lambda$  such that for all vector field  $X$ , the covariant derivative  $\nabla_X \eta$  (the lift of the Levi-Civita connection) and the Clifford multiplication  $X \cdot \eta$  are related by the equation

$$\nabla_X \eta = \lambda X \cdot \eta.$$

Solutions of this equation occur quite naturally in differential geometry as well as in mathematical physics. On a compact Riemannian spin manifold with non-negative scalar curvature  $R$  there is a lower bound for the first eigenvalue of the Dirac operator involving  $R$ , and eigenspinors to this lower bound are Killing spinors (see [Hij86]). Furthermore, Killing spinors are special solutions of the so-called twistor equation and in the case of a compact manifold they generate - up to a conformal change of the metric - all solutions of the twistor equation. The construction of models in supergravity also depends on Riemannian manifolds with Killing spinors. Such manifolds are Einstein with scalar curvature  $R = 4n(n-1)\lambda^2$  (see [BFGK91, Sec 1.5, Thm 8]). For some Lie group  $G \subset \text{SO}(n)$  they admit a  $G$ -structure which is the reduction of the structure group of the frame bundle from  $\text{SO}(n)$  to  $G$ . The  $G$ -structure is however not torsion-free since the holonomy group of the Levi-Civita connection is not a subgroup of  $G$ . Nevertheless, manifolds with real Killing spinors are closely related to manifolds with torsion-free  $G$ -structures since the cone metric over such manifolds has special holonomy (see [Bär93]). For example the cone over a 6-dimensional nearly Kähler manifold has a torsion-free  $G_2$ -structure and the cone over a nearly  $G_2$  manifold has a torsion-free  $\text{Spin}(7)$ -structure.

In dimension 7, manifolds with real unit Killing spinor come equipped with a particular type of  $G_2$ -structure and are known as nearly  $G_2$  manifolds. Our investigation of these nearly  $G_2$  manifolds is two-fold. First we set up the deformation theory of nearly  $G_2$ -structures. We describe the first and second order deformations of nearly  $G_2$ -structures and use them to prove the rigidity of the infinitesimal deformations of a nearly  $G_2$ -structure on the Aloff–Wallach space  $X_{1,1}$ . This work is done in a joint paper with Shubham Dwivedi [DS20]. Similar results were earlier proved by [Fos17] for nearly Kähler 6-manifolds.

The second route we embark upon is to understand the gauge theory on nearly  $G_2$  manifolds. Over a manifold equipped with a Killing spinor  $\eta$ , we consider connections  $\nabla$  of curvature  $F_\nabla$  that satisfy  $F_\nabla \cdot \eta = 0$ . Those connections are called *instantons* and are, as proved in [HN12] solutions to the Yang–Mills equation (which itself is described in more details in Chapter 5). Harland–Nölle (still in [HN12]) also constructed a distinguished connection, the *canonical connection* on the tangent bundle which is an analog to the Levi-Civita connection on manifolds with torsion-free  $G$ -structure. One can define the canonical connection for any manifold with a  $G$ -structure such that  $G \subset \mathrm{SO}(n)$ . Thus we have the splitting  $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$  and the canonical connection is the restriction of the Levi-Civita connection on  $\mathfrak{g}$ . The canonical connection is a  $G$ -instanton and has holonomy contained in  $G$ . Thus a better understanding of the canonical connection is a crucial ingredient in the geometry of manifolds with real Killing spinors. The deformation space of the canonical connection in the nearly Kähler 6-dimensional case was studied in [CH16] and in the nearly  $G_2$  case by the author in [Sin21]. This last paper is the topic of Chapter 7 where we study the deformation space of instantons on manifolds with a nearly  $G_2$ -structure.

Gauge theory and instantons have been a very active area of research for many years. In gauge theory one often tries to understand the connections on vector and principal bundles over manifolds. In search of the “best” connection on a vector or principal bundle that is, the one for which the curvature is closest to zero, we come across instantons. These instantons are critical points of the Yang–Mills functional and in some cases minimize the functional. One of the successful applications of gauge theory comes from the seminal work of Donaldson [Don83] where he showed that the moduli space of anti-self-dual (ASD) instantons can be used to assign numerical invariants to smooth 4-manifolds. In a hope to achieve a similar feat in dimension 7 on manifolds with a  $G_2$  structure in [DT98] and [DS11] it was suggested that counting  $G_2$  instantons “appropriately” might yield a geometric invariant of  $G_2$  manifolds.

For a Riemannian manifold  $(M, g)$  the holonomy group  $\mathrm{Hol}(g)$  is the group generated by all parallel transports for the Levi-Civita connection along closed null-homotopic loops on  $M$ . In 1955, Marcel Berger gave his famous Holonomy Theorem [Ber55] which states that for an irreducible non-symmetric Riemannian manifold, there are finitely many possible

holonomy groups listed in Table 1.1. Here by definition  $\mathrm{Sp}(q) \cdot \mathrm{Sp}(1) = \frac{\mathrm{Sp}(q) \times \mathrm{Sp}(1)}{\mathbb{Z}_2}$  for the obvious  $\mathbb{Z}_2$  generated by the action of  $(-1, -1)$ .

$\dim(M)$	$\mathrm{Hol}(g)$	Nomenclature
$n$	$\mathrm{SO}(n)$	Oriented Riemannian
$2m$	$\mathrm{U}(m)$	Kähler
$2m$	$\mathrm{SU}(m)$	Calabi–Yau
$4q$	$\mathrm{Sp}(q)$	Hyper-Kähler
$4q$	$\mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$	Quaternionic-Kähler
7	$\mathrm{G}_2$	$\mathrm{G}_2$ holonomy
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$ holonomy

Table 1.1: Berger’s holonomy theorem

Berger’s Holonomy Theorem only lists the possible holonomy groups, but it was initially not known which of these groups appear as holonomy groups of Riemannian manifolds. In particular, no Riemannian manifolds with holonomy group  $\mathrm{G}_2$  were known until 1984 when Bryant constructed the first local examples [Bry87] followed by complete non-compact examples in [BS89] and finally compact examples in [Joy96]. It is now known that all of these possibilities can occur as holonomy groups of manifolds.

The holonomy groups  $\mathrm{U}(m)$ ,  $\mathrm{SU}(m)$ ,  $\mathrm{Sp}(q)$ ,  $\mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$ ,  $\mathrm{G}_2$ ,  $\mathrm{Spin}(7)$  are called *special* holonomies and the groups  $\mathrm{G}_2$  and  $\mathrm{Spin}(7)$  are also referred as *exceptional* holonomies due to the nature of their Lie groups.

Another way to think about manifolds with special holonomy is *via*  $G$ -structures. A reduction of a principal  $G$ -bundle  $P \rightarrow M$  to a subgroup  $H \subset G$  is a principal  $H$ -bundle  $Q \rightarrow M$  together with a smooth map  $\nu: Q \rightarrow P$  which covers the identity on  $M$  and is  $H$ -equivariant. In terms of transition maps, a  $G$ -bundle can be reduced to  $H$  if and only if the transition maps can be taken to have values in  $H$ .

**Definition 1.0.1.** For a Lie group  $G \subset \mathrm{GL}(n, \mathbb{R})$  a  $G$ -structure on  $M$  is a reduction of the structure group of the frame bundle  $\mathcal{F}(M)$  from  $\mathrm{GL}(n)$  to  $G$ . It is a principal subbundle

$\mathcal{P}_G$  of the frame bundle  $\mathcal{F}(M)$  satisfying

$$\begin{array}{ccc}
 G & \hookrightarrow & \mathrm{GL}(n) \\
 \downarrow & & \downarrow \\
 \mathcal{P}_G & \hookrightarrow & \mathcal{F}(M) \\
 \downarrow & \swarrow & \\
 M & & 
 \end{array}$$

If  $G \subset \mathrm{O}(n)$ , the  $G$ -structure has an underlying Riemannian metric, which we denote by  $g$ . An orientation along with a compatible metric  $g$  is called an  $\mathrm{SO}(n)$ -structure on  $M$  which is equivalent to a reduction of the structure group of  $\mathcal{F}(M)$  from  $\mathrm{GL}(n, \mathbb{R})$  to  $\mathrm{SO}(n)$ . Another example is an almost Hermitian structure on  $M$  which is defined by an almost complex structure  $J$  such that  $g(JX, JY) = g(X, Y)$  for all  $X, Y \in \Gamma(TM)$ . It is easy to check that such a structure exists only if  $n = 2m$  that is  $n$  is even. An almost Hermitian structure on  $M$  is a reduction of the structure group of the frame bundle from  $\mathrm{GL}(2m, \mathbb{R})$  to  $\mathrm{U}(m)$ . We call a connection  $\nabla$  on  $TM$  compatible with the  $G$ -structure  $P$ , if the corresponding connection on  $\mathcal{F}(M)$  reduces to  $P$ . For a fixed connection  $\nabla$  on  $\mathcal{F}(M)$  there is a compatible  $G$ -structure  $P$  if and only if  $\mathrm{Hol}(\nabla) \subset G$  (for a proof of this fact refer to [Joy07, Proposition 2.6.3]).

Manifolds with special algebraic structures have gained immense popularity in mathematics since they serve as the primary and sometimes the only known examples of manifolds with some desired geometric properties such as Ricci-flatness or positive Einstein. In fact, all currently known examples of irreducible and non-symmetric compact Ricci-flat manifolds have special holonomy. In particular,  $G_2$  manifolds are always Ricci-flat. On the physics side these manifolds are studied in much detail due to their importance in supergravity and superstring theory, in particular for finding solutions which preserve some supersymmetry.

We start the discussion by setting up the groundwork and notation in Chapter 2 by discussing some well known standard material on manifolds with  $G_2$ -structures. We define a  $G_2$ -structure on a 7-dimensional manifold  $M$ . We also see how the  $G_2$ -structure induces a decomposition of differential forms on  $M$ . Then we see a classification of  $G_2$ -structures into 16 types based on their torsion form.

In Chapter 3 we discuss an alternative and equivalent way to define a  $G_2$ -structure using spinors. We begin by giving a brief introduction to spin geometry and then discuss the one-to-one correspondence between a  $G_2$ -structure form and a real unit spinor. In

many instances thinking of a  $G_2$ -structure as a unit spinor makes computation easier as we see in Chapters 6, 7, and 8 where we use this equivalence to study the deformations of nearly  $G_2$ -structures and  $G_2, SU(3)$  instantons.

From Chapter 4 we leave the realm of general  $G_2$ -structures and focus on manifolds with nearly  $G_2$ -structure. These manifolds do not have reduced holonomy and in fact their holonomy group is the full  $SO(7)$ . With respect to the spinorial description discussed in Chapter 3 nearly  $G_2$ -structure is equivalent to the presence of a unit Killing spinor  $\eta$  on the spinor bundle. The spinor bundle  $\mathcal{S}$  in this case decomposes as

$$\mathcal{S} = \Lambda^0\eta \oplus \Lambda^1 \cdot \eta \cong \Lambda^0(T^*M) \oplus \Lambda^1(T^*M).$$

In some situations to make the statement cleaner we use the notation

$$S_0 := \Lambda^0\eta, \quad S_1 := \Lambda^1 \cdot \eta.$$

These manifolds are positive Einstein (see [HN12]) and are thus an important class of manifolds to study. Nearly  $G_2$  manifolds have a lot in common with the 6-dimensional nearly Kähler manifolds and thus earlier works on 6-dimensional nearly Kähler manifolds motivates the study of these nearly  $G_2$  manifolds. Another motivation comes from the fact that the cone over these manifolds have holonomy equal to either  $Spin(7)$ ,  $SU(4)$ , or  $Sp(2)$  depending on the dimension of the space of Killing spinors on  $M$ . This property makes these spaces particularly important in the construction and understanding of manifolds with torsion-free  $Spin(7)$ -structures.

In Chapter 5 we begin the discussion of gauge theory on  $G_2$  manifolds. We define what is the 7-dimensional analogue of the self-dual connections on 4-manifolds. We also study many equivalent definitions to define a  $G_2$  instanton. For a 7-dimensional manifold  $M^7$  with a  $G_2$ -structure  $\varphi$  and a unit spinor  $\sigma$  associated to  $\varphi$ , a connection  $A$  on  $M$  is a  $G_2$  *instanton* if its curvature  $F_A$  satisfies the algebraic condition

$$F_A \wedge \varphi = *_\varphi F_A.$$

The above condition is equivalent to  $F_A \cdot \sigma = 0$  as shown in Chapter 5. When the  $G_2$ -structure is parallel these instantons clearly solve the Yang–Mills equation  $d_\nabla^* F = 0$ . The analogous result was proved in the nearly  $G_2$  case by Harland–Nölle [HN12]. They showed that the instantons on manifolds with real Killing spinors solve the Yang–Mills equation which makes the study of instantons on nearly  $G_2$  manifolds important from the point of view of gauge theory in higher dimensions. However  $G_2$  instantons in the parallel case are the minimizers of the Yang–Mills functional which is not necessarily true for the nearly parallel case, as proved by Ball–Oliveira in [BO19].

The new results in this thesis are regarding the deformation theory of nearly  $G_2$ -structures,  $G_2$  instantons on nearly  $G_2$  manifolds, and  $SU(3)$  instantons on Sasaki–Einstein 7-folds.

In Chapter 6, we study the deformation theory of nearly  $G_2$  manifolds. The infinitesimal deformations of nearly  $G_2$  manifolds were studied by Alexandrov–Simmelmann in [AS12] where they identified the space of infinitesimal deformations with an eigenspace of the Laplacian acting on co-closed 3-forms on  $M$  of type  $\Omega_{27}^3$ . We address the question of whether nearly  $G_2$  manifolds have obstructed or unobstructed deformations, i.e., whether infinitesimal deformations can be integrated to genuine deformations. This could potentially give new examples of nearly  $G_2$  manifolds. Another application of studying the deformation theory of nearly  $G_2$  manifolds can be to develop the deformation theory of  $Spin(7)$  *conifolds* which are asymptotically conical and conically singular  $Spin(7)$  manifolds, similar to the theory developed by Karigiannis–Lotay [KL20] for  $G_2$  conifolds. Lehmann [Leh21] studies the deformation theory of asymptotically conical  $Spin(7)$ –manifolds.

The study of deformation theory of special algebraic structures is not new. Deformations of Einstein metrics were studied by Koiso where he showed [Koi82, Theorem 6.12] that the infinitesimal deformations of Einstein metrics is in general obstructed, by exhibiting certain Einstein symmetric spaces which admit non-trivial infinitesimal Einstein deformations which cannot be integrated to second order. The deformation theory of nearly Kähler structures on homogeneous 6-manifolds was studied by Moroianu–Nagy–Simmelmann in [MNS08]. They identified the space of infinitesimal deformations with an eigenspace of the Laplacian acting on co-closed primitive  $(1, 1)$ -forms. Using this, they proved that the nearly Kähler structures on  $\mathbb{C}P^3$  and  $S^3 \times S^3$  are rigid and the flag manifold  $\mathbb{F}_3$  admits an 8-dimensional space of infinitesimal deformations. Later, Foscolo proved [Fos17, Theorem 5.3] that the infinitesimal deformations of the flag manifold  $\mathbb{F}_3$  are all obstructed. We follow a strategy similar to [Fos17]. After introducing a modified Dirac operator on nearly  $G_2$  manifolds, we use its properties and the Hodge decomposition theorem to completely describe the cohomology of a complete nearly  $G_2$  manifold. We prove our first two main results of the paper which characterize harmonic forms. These are the following.

**Theorem 4.3.7.** *Let  $(M, \varphi, \psi)$  be a compact nearly  $G_2$  manifold. Then every harmonic 4-form lies in  $\Omega_{27}^4$ . Equivalently, every harmonic 3-form lies in  $\Omega_{27}^3$ .*

**Theorem 4.3.8** *Let  $(M, \varphi, \psi)$  be a compact nearly  $G_2$  manifold. Then every harmonic 2-form lies in  $\Omega_{14}^2$ . Equivalently, every harmonic 5-form lies in  $\Omega_{14}^5$ .*

We note that Theorem 4.3.8 was originally proved by Ball–Oliveira [BO19, Remark 15]. We give a different proof in this paper.



We use the properties of the modified Dirac operator, explicitly we use Proposition 4.3.6, to prove a slice theorem for the action of the diffeomorphism group on the space of nearly  $G_2$ -structures on  $M$  in Proposition 6.1.1. Using this, in Theorem 6.1.2 we obtain a new proof of the identification of the space of infinitesimal nearly  $G_2$  deformations with an eigenspace of the Laplacian acting on co-closed 3-forms of type  $\Omega_{27}^3$ , a result originally due to Alexandrov–Simmelmann [AS12].

To study higher order deformations of nearly  $G_2$  manifolds, we use the point view of Hitchin [Hit01] where he interprets nearly  $G_2$ -structures as constrained critical points of a functional defined on the space  $\Omega^3 \times \Omega_{\text{exact}}^4$ . This approach is inspired from the work of Foscolo [Fos17] where he used similar ideas to study second order deformations of nearly Kähler structures on 6-manifolds. The advantage of this approach is that it allows us to view the nearly  $G_2$  equation (4.1.1) as the vanishing of a smooth map (cf. equation (6.1.10)) on  $\Omega_{+, \text{exact}}^4$ , the space of exact *positive* 4-forms on  $M$ , given by

$$\Phi : \Omega_{+, \text{exact}}^4 \times \Gamma(TM) \longrightarrow \Omega_{\text{exact}}^4.$$

Thus the obstructions on the first order deformations of a nearly  $G_2$ -structure to be integrated to higher order deformations can be characterized by  $\text{Im}(D\Phi)$  which we do in Proposition 6.1.5.

Finally, we use the general deformation theory of nearly  $G_2$ -structures developed in the first part of the paper to study the infinitesimal deformations of the Aloff–Wallach space  $\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$ . It was expected in [Fos17] that the infinitesimal deformations of the Aloff–Wallach space might be obstructed to higher orders. In §6.3 we confirm this expectation. More precisely, we prove the following.

**Theorem 6.3.1.** *The infinitesimal deformations of the homogeneous nearly  $G_2$ -structure on the Aloff–Wallach space  $X_{1,1} \cong \frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$  are all obstructed.*

In Chapter 7, we investigate the infinitesimal deformation space of  $G_2$  instantons on nearly  $G_2$  manifolds by applying a similar approach to [CH16]. In [CH16] Charbonneau–Harland studied the infinitesimal deformation space of irreducible instantons with semi-simple structure group on nearly Kähler 6-manifolds by identifying it with the eigenspace of a Dirac operator. A significant difference between nearly  $G_2$  manifolds and the 6-dimensional nearly Kähler manifolds is that the Killing spinors  $\eta$  and  $\text{vol} \cdot \eta$  are linearly dependent in the former and independent in the latter case. This prevents us from having a result like [CH16, Proposition 4(iii)] where one can relate the  $\lambda^2$ -eigenspace of the square of the Dirac operator to the  $\lambda$ -eigenspace of the Dirac operator which makes the computation

of the infinitesimal deformation space much more convenient. In fact we show that such a relation does not exist in the nearly  $G_2$  case by explicitly computing the kernel of the elliptic operator for the homogeneous nearly  $G_2$  manifolds.

We prove the following main theorems for a nearly  $G_2$  instanton  $A$  on a principal bundle  $\mathcal{P}$  with curvature  $F_A$ . Let  $EM$  be a vector bundle associated to  $\mathcal{P}$  and the Dirac operator  $D^{-1,A}$  is as defined in (7.1.2).

**Theorem 7.1.2.** *The space of infinitesimal deformations of a  $G_2$  instanton  $A$  on a principal bundle  $\mathcal{P}$  over a nearly  $G_2$  manifold is isomorphic to the kernel of the elliptic operator*

$$\left(D^{-1,A} + 2 \text{Id}\right): \Gamma(S_1 \otimes Ad_{\mathcal{P}}) \rightarrow \Gamma(S_1 \otimes Ad_{\mathcal{P}}).$$

**Theorem 7.1.7.** *Any  $G_2$  instanton  $A$  on a principal  $G$ -bundle over a compact nearly  $G_2$  manifold  $M$  is rigid if*

- (i) *the structure group  $G$  is abelian, or*
- (ii) *all the eigenvalues of the operator*

$$\begin{aligned} L_A: \Lambda^1 \otimes Ad_{\mathcal{P}} &\rightarrow \Lambda^1 \otimes Ad_{\mathcal{P}} \\ w &\mapsto -2w \lrcorner F_A \end{aligned}$$

*are greater than  $-\frac{28}{5}$ .*

A similar result as above has been proved in [BO19, Proposition 8] when the structure group is abelian or the eigenvalues are less than 6. The proof of the upper bound in [BO19] uses the Weitzenböck formula on the connection associated to the Levi-Civita connection and  $A$ . Our proof of the lower bound on the eigenvalue uses the Schrödinger–Lichnerowicz formula for the family of Dirac operators constructed in Chapter 7.

We describe the infinitesimal deformation space of the canonical connection on all the homogeneous nearly  $G_2$  manifolds whose nearly  $G_2$  metric is normal, that is it is a scalar multiple of the Killing form. By considering the actions of the Lie groups  $H$  and  $G_2$  on  $G/H$  we can view the canonical connection as an  $H$ -connection or a  $G_2$ -connection. We compute its infinitesimal deformation spaces in both of these cases. The results are recorded in Theorem 7.2. It would be interesting to see if these infinitesimal deformations are genuine. As of now, the author is unaware of any known family of nearly  $G_2$  instantons

for which the infinitesimal deformations are the ones found in Theorem 7.2. We remark that we still do not know if any of the deformations we found are genuine and can be integrated to generate new examples of  $G_2$  instantons but one can try to see if any of them are obstructed to second order by using some ideas from Chapter 6.

In Chapter 8 we discuss the infinitesimal deformation space of Sasaki instantons on 7-dimensional Sasaki–Einstein manifolds. These manifolds are orientable and spin and the cone over them has an  $SU(4)$ -structure. The Sasaki–Einstein manifolds come equipped with a contact structure defined by a Reeb vector field  $\xi$ , the 1-form dual to  $\xi$  denoted by  $\eta$  and  $\Phi \in \Gamma(\text{End}(TM))$ . There are two Killing spinors on  $M$  which correspond to two orthogonal nearly parallel  $G_2$ -structures  $\varphi_1, \varphi_2$  associated to unit real Killing spinors  $\mu_1, \mu_2$  respectively. These two  $G_2$ -structures are related by the equation

$$\varphi_2 = -\varphi_1 + 2(\xi \lrcorner \varphi_1) \wedge \eta.$$

One can show that a connection  $A$  on a principal bundle  $\mathcal{P}$  over  $M$  is a Sasaki instanton if and only if it is a  $G_2$  instanton with respect to both  $\varphi_1$  and  $\varphi_2$ . Using this characterization we can define the infinitesimal deformation space of a Sasaki instanton on  $M$  as the kernel of a Dirac operator similar to the nearly parallel case. As in the case of proper nearly  $G_2$  structures where the dimension of the space of Killing spinors is 1, we have the decomposition of the spinor bundle  $\mathcal{S} = S_0 \oplus S_1$ . The decomposition depends on the choice of the Killing spinor but since  $\mu_2 = \xi \cdot \mu_1$  both decompositions are isomorphic. Thus we get the following theorem.

**Theorem 8.2.2** *The space of infinitesimal deformations of a Sasaki instanton  $A$  on a principal bundle  $\mathcal{P}$  over a 7-dimensional Sasakian manifold  $M$  is isomorphic to the kernel of the operator*

$$\left( D^{0,A} + \frac{5}{2} \text{Id} \right) : \Gamma(S_1 \otimes Ad_{\mathcal{P}}) \rightarrow \Gamma(S_1 \otimes Ad_{\mathcal{P}}).$$

Again, one can define a canonical connection on a Sasakian manifold, and it is a Sasaki instanton. In [HN12] it was shown that the characteristic homogeneous connection on a homogeneous 7-dimensional Sasakian manifold is the same as this canonical connection. Thus the above theorem can be used to describe the deformation space of the characteristic homogeneous connection for the homogeneous 7-dimensional Sasaki manifolds.

## 1.1 Notations and conventions

Throughout the thesis, unless otherwise stated, we compute in a local orthonormal frame, so all indices are subscripts and any repeated indices are summed over all values from 1 to 7. Our convention for labelling the Riemann curvature tensor is

$$R_{ijkl}e_m = (\nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[e_i, e_j]}) e_k,$$

in terms of the orthonormal frame  $\{e_1, e_2, \dots, e_7\}$ . With this convention, the Ricci tensor is  $R_{jk} = R_{ijkl}$ , and the Ricci identity is

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = -R_{ijkl} X_l. \quad (1.1.1)$$

We use the metric to identify the vector fields and 1-forms by the musical isomorphisms. As such, throughout the thesis, we use them interchangeably without mention.

We have the following contraction identities between  $\varphi$  and  $\psi$ , whose proofs can be found in [Kar09].

$$\varphi_{ijk} \varphi_{abk} = g_{ia} g_{jb} - g_{ib} g_{ja} - \psi_{ijab}, \quad (1.1.2)$$

$$\varphi_{ijk} \varphi_{ajk} = 6g_{ia} \quad (1.1.3)$$

and

$$\varphi_{ijk} \psi_{abck} = g_{ia} \varphi_{jbc} + g_{ib} \varphi_{ajc} + g_{ic} \varphi_{abj} - g_{ja} \varphi_{ibc} - g_{jb} \varphi_{aic} - g_{jc} \varphi_{abi}, \quad (1.1.4)$$

$$\varphi_{ijk} \psi_{abjk} = -4\varphi_{iab}, \quad (1.1.5)$$

$$\psi_{ijkl} \psi_{abkl} = 4g_{ia} g_{jb} - 4g_{ib} g_{ja} - 2\psi_{ijab} \quad (1.1.6)$$

$$\psi_{ijkl} \psi_{ajkl} = 24g_{ia}. \quad (1.1.7)$$

We denote the Lie algebra associated to any Lie group  $G$  by the corresponding gothic letter  $\mathfrak{g}$ .

# Chapter 2

## Preliminaries on $G_2$ geometry

This chapter comprises of an introduction to  $G_2$  geometry. We start the section by defining  $G_2$ -structures on a seven dimensional manifold and also discuss the decomposition of the space of differential forms on such a manifold in §2.1. We define and describe the *torsion* of a  $G_2$ -structure and also discuss how this gives rise to the 16 distinct classes of  $G_2$ -structure are used throughout the thesis. Most of the material in this chapter can be found in [Joy00]. Some other references are [Bry06], [Kar10], [Kar20], [Dwi20].

### 2.1 Manifolds with $G_2$ -structure

Let  $M^7$  be a smooth 7-dimensional manifold. A  $G_2$ -structure on  $M$  is a reduction of the structure group of the frame bundle from  $GL(7, \mathbb{R})$  to the Lie group  $G_2 \subset SO(7)$ . Such a structure exists on  $M$  if and only if the manifold is orientable and spinnable, conditions which are equivalent to the vanishing of the first and second Stiefel–Whitney classes. The exceptional Lie group  $G_2$  can be defined as the stabiliser of a special 3-form on  $\mathbb{R}^7$ .

Let  $V = \mathbb{R}^7$ . The group  $GL(7, \mathbb{R})$  acts on  $\Lambda^i(V^*)$  for all  $i = 0, \dots, 7$ . Let  $\{e^i, i = 1 \dots 7\}$  be the standard basis of  $V^*$ . We write  $e^{i_1 i_2 \dots i_p}$  for  $e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p}$ .

**Definition 2.1.1.** Let

$$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437}. \quad (2.1.1)$$

Then the stabilizer of  $\varphi_0$  under the action of  $GL(7, \mathbb{R})$  on  $\Lambda^3(V^*)$  is the Lie group  $G_2$ , that is

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0\}. \quad (2.1.2)$$

The 3-form  $\varphi_0$  is called the *standard  $G_2$  form*.

The Lie group  $G_2$  is compact, connected, simply-connected, simple and 14-dimensional. It also preserves the Euclidean metric

$$g_0 = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2$$

and the volume form

$$\text{vol}_0 = e^{1234567}$$

and thus it follows that  $G_2 \subset \text{SO}(7)$ . Since  $G_2$  preserves both the metric and orientation it also preserves the Hodge star  $*_{\varphi_0}$  and thus the 4-form  $\psi_0 = *_{\varphi_0}\varphi_0$  which is explicitly given by

$$\psi_0 = *_{\varphi_0}\varphi_0 = e^{4567} - e^{4523} - e^{4163} - e^{4127} - e^{2637} - e^{1537} - e^{1526}. \quad (2.1.3)$$

**Remark 2.1.1.** Using the expression of  $\varphi_0$  and  $g_0$  we can check that for all  $i, j \in \{1, \dots, 7\}$

$$(e_i \lrcorner \varphi_0) \wedge (e_j \lrcorner \varphi_0) \wedge \varphi_0 = -6(g_0)_{ij} \text{vol}_0.$$

Now let  $M$  be a 7-dimensional oriented manifold. By using the standard  $G_2$  form in (2.1.1) for any point  $p \in M$  we can define a subset  $\Lambda_+^3(M)_p$  of the space  $\Lambda^3 T_p^* M$  by

$$\Lambda_+^3(M)_p = \{\varphi_p \in \Lambda^3 T_p^* M \mid \exists \text{ isomorphism } \rho: T_p M \rightarrow \mathbb{R}^7, \rho^* \varphi_0 = \varphi_p\}.$$

Since  $G_2$  preserves  $\varphi_0$ , we have the isomorphism  $\Lambda_+^3(M)_p \cong \text{GL}(7, \mathbb{R})/G_2$ , which implies that  $\dim(\Lambda_+^3(M)_p) = \dim(\text{GL}(7, \mathbb{R})) - \dim(G_2) = 49 - 14 = 35 = \dim(\Lambda^3 T_p^* M)$ . Therefore  $\Lambda_+^3(M)_p$  is an open subset of  $\Lambda^3 T_p^* M$

Define  $\Lambda_+^3(M) = \cup_{p \in M} \Lambda_+^3(M)_p$  to be the bundle over  $M$  with fibre  $\Lambda_+^3(M)_p$  at each  $p \in M$ . A 3-form  $\varphi$  is *positive* if for each  $p \in M$   $\varphi_p \in \Lambda_+^3(M)_p$  that is, there is a linear isomorphism between  $T_p M$  and  $\mathbb{R}^7$  identifying  $\varphi_p$  to  $\varphi_0$  of (2.1.1).

Similarly one can define the set  $\Lambda_+^4(M)_p$  to be the set of 4-forms  $\psi_p \in \Lambda^4 T_p^* M$  such that there is an isomorphism identifying  $\psi_p$  to  $\psi_0$  of (2.1.3). By duality one can see that  $\Lambda_+^4(M)$  is an open subbundle of  $\Lambda^4 T^* M$  with fibre  $\text{GL}(7, \mathbb{R})/G_2$  and the sections of  $\Lambda_+^4(M)$  are the *positive* 4-forms. One can define the fibre bundles  $\Lambda_+^3, \Lambda_+^4$  on any 7-dimensional manifold but they only admit global sections if  $w_1 = w_2 = 0$ .

On the other hand the frame bundle  $\mathcal{F}(M)$  over  $M$  is the principal  $\mathrm{GL}(7, \mathbb{R})$  bundle over  $M$  whose fibre at each point  $p \in M$  is the set of isomorphisms from  $T_p M$  to  $\mathbb{R}^7$ . Thus for a positive 3-form  $\varphi$  we can define a subbundle  $P$  of  $\mathcal{F}(M)$  for which the fibre at each point  $p \in M$  is the set of isomorphisms from  $T_p M$  to  $\mathbb{R}^7$  which identify  $\varphi_p$  and  $\varphi_0$ . The fibre of  $P$  is  $G_2$  and thus by Definition 1.0.1,  $P$  defines a  $G_2$ -structure on  $M$ .

Conversely given a  $G_2$ -structure  $P$  on  $M$  one can define a 3-form  $\varphi$ , a 4-form  $\psi$ , a metric  $g$  at each point corresponding to the standard 3-form  $\varphi_0$ , the standard 4-form  $\psi_0$  and the metric  $g_0$  respectively. The *positivity* of the 3-form implies that  $P$  has to be oriented and thus there is volume form associated to every such  $G_2$ -structure.

Hence we get that there is a one-to-one correspondence between positive 3-forms and oriented  $G_2$ -structures. From the point of view of differential geometry, a  $G_2$ -structure on  $M$  is equivalently defined by a 3-form  $\varphi$  on  $M$  that satisfies a certain pointwise algebraic non-degeneracy condition. Such a 3-form nonlinearly induces a Riemannian metric  $g_\varphi$  and an orientation  $\mathrm{vol}_\varphi$  on  $M$  and hence a Hodge star operator  $*_\varphi$ . We denote the Hodge dual 4-form  $*_\varphi \varphi$  by  $\psi$ . Pointwise we have  $|\varphi| = |\psi| = 7$ , where the norm is taken with respect to the metric induced by  $\varphi$ . We can define the metric  $g_\varphi$  and the volume form  $\mathrm{vol}_\varphi$  explicitly. The expressions are non-linear in  $\varphi$ . For defining the metric  $g_\varphi$  we define a  $\Lambda^7 T^* M$  valued bi-linear form  $B_\varphi$  on  $M$ . Let  $x^1, x^2, \dots, x^7$  be local coordinates on an open set  $U \subset M$ . For  $i, j \in \{1, 2, \dots, 7\}$

$$(B_\varphi)_{ij} dx^1 \wedge dx^2 \wedge \dots \wedge dx^7 = \left( \frac{\partial}{\partial x^i} \lrcorner \varphi \right) \wedge \left( \frac{\partial}{\partial x^j} \lrcorner \varphi \right) \wedge \varphi. \quad (2.1.4)$$

Since the 2-forms  $\left( \frac{\partial}{\partial x^i} \lrcorner \varphi \right)$  and  $\left( \frac{\partial}{\partial x^j} \lrcorner \varphi \right)$  commute,  $B_\varphi$  is symmetric. Using Remark 2.1.1 we can see that the 3-form  $\varphi$  is a  $G_2$ -structure if and only if

$$(B_\varphi)_{ij} = -6(g_\varphi)_{ij} \sqrt{\det g_\varphi}.$$

This implies  $\det(B_\varphi) = (-6\sqrt{\det g_\varphi})^7 \det g_\varphi = -6^7 (\sqrt{\det g_\varphi})^9$ . Thus the explicit expression for  $g_\varphi$  is given by

$$g_\varphi = -\frac{B_\varphi}{6\sqrt{\det g_\varphi}} = \frac{B_\varphi}{6^{\frac{2}{9}} (\det B_\varphi)^{\frac{1}{9}}}. \quad (2.1.5)$$

The volume  $\mathrm{vol}_\varphi$  is given by

$$\mathrm{vol}_\varphi = \sqrt{\det g_\varphi} dx^1 \wedge dx^2 \wedge \dots \wedge dx^7. \quad (2.1.6)$$

Using the above expressions for the metric  $g_\varphi$  (2.1.5) and orientation  $\text{vol}_\varphi$  (2.1.6) we can explicitly compute the Hodge star  $*_\varphi$  and thus the 4-form  $*_\varphi\varphi = \psi$ .

**Remark 2.1.2.** If the 3-form  $\varphi$  defines a  $G_2$ -structure on  $M$  then the metric  $g_\varphi$  obtained by the above procedure in (2.1.5) is to be Riemannian,  $\det B_\varphi$  must be non-vanishing, and  $g_\varphi$  must be positive definite everywhere in  $U$ . This provides another way to check that the space of positive 3-forms is open as both of these conditions are open.

**Definition 2.1.2.** A 7-dimensional manifold  $M$  together with a  $G_2$ -structure  $\varphi \in \Lambda_+^3(M)$  on  $M$  is a  $G_2$  manifold.

**Remark 2.1.3.** The  $G_2$ -structure on  $M$  is torsion-free if it is parallel with respect to the Levi-Civita connection. Thus for torsion-free  $G_2$ -structure  $\text{Hol}(g_\varphi) \subset G_2$ . Many authors use the term  $G_2$  manifold to denote torsion-free  $G_2$  manifold but in this thesis we use the term  $G_2$  manifold to denote a manifold with a  $G_2$ -structure of any torsion.

From now on we denote a  $G_2$ -structure on  $M$  by the tuple  $(\varphi, g)$  where  $\varphi$  is the positive 3-form and  $g$  is the associated metric. We also drop the subscript  $\varphi$  from  $g, \psi, *, \text{vol}$  if there is no risk of confusion.

**Example 2.1.4.**  $(\mathbb{R}^7, \varphi_0)$  is a  $G_2$  manifold.

**Example 2.1.5.** Let  $(S, \omega_1, \omega_2, \omega_3)$  be a hyperkähler surface and denote by  $(\delta_1, \delta_2, \delta_3)$  a parallel orthonormal frame on  $\mathbb{R}^3$ . Then

$$\varphi := \delta_1 \wedge \delta_2 \wedge \delta_3 - \delta_1 \wedge \omega_1 - \delta_2 \wedge \omega_2 - \delta_3 \wedge \omega_3$$

defines a torsion-free  $G_2$ -structure on  $\mathbb{R}^3 \times S$ . The metric and orientation induced by  $\varphi$  agree with the standard metric and orientation on  $\mathbb{R}^3 \times S$ .

In [BS89] Bryant–Salamon found explicit torsion-free  $G_2$ -structures on  $\Lambda^+S^4, \Lambda^+\mathbb{C}P^2$  and on  $\mathbb{R}^4 \times S^3$ , the total space of the spin bundle on  $S^3$ , whose associated metrics are complete and have  $\text{Hol}(g_\varphi) = G_2$ . Other examples of torsion-free  $G_2$  manifolds can be constructed as a cone over nearly Kähler 6-manifolds. Compact examples of torsion-free  $G_2$  manifolds were constructed by Joyce via a generalised Kummer construction introduced in [Joy96], Joyce–Karigiannis by gluing families of Eguchi–Hanson spaces in [JK21], the twisted connected sum construction pioneered by Kovalev [Kov03] and extended by Corti–Haskins–Nordström–Pacini [CHNP15]. The work of Corti–Haskins–Nordström–Pacini in particular provides us with an ample supply of examples of torsion-free  $G_2$ -manifolds



## 2.2 Decomposition of space of forms

The Lie group  $G_2$  acts as the subgroup of  $SO(7)$  on the exterior powers of the standard representation of  $SO(7)$  on  $\mathbb{R}^7$ . The action on  $\Lambda^p \mathbb{R}^7$  is reducible For  $2 \leq p \leq 5$ . We denote by  $\Omega_l^k$  the irreducible subspace of  $\Omega^k$  with pointwise dimension  $l$ . Given a  $G_2$ -structure  $\varphi$ , we have the following description of the space of forms :

$$\Omega_7^2 = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Omega^2(M) \mid *(\varphi \wedge \beta) = -2\beta\}, \quad (2.2.1)$$

$$\Omega_{14}^2 = \{\beta \in \Omega^2(M) \mid \beta \wedge \psi = 0\} = \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = \beta\}. \quad (2.2.2)$$

$$\Omega_1^3 = \{f\varphi \mid f \in C^\infty(M)\}, \quad (2.2.3)$$

$$\Omega_7^3 = \{X \lrcorner \psi \mid X \in \Gamma(TM)\} = \{*(\alpha \wedge \varphi) \mid \alpha \in \Omega^1\}, \quad (2.2.4)$$

$$\Omega_{27}^3 = \{\eta \in \Omega^3 \mid \eta \wedge \varphi = 0 = \eta \wedge \psi\}. \quad (2.2.5)$$

Thus a  $G_2$ -structure on  $M$  induces a splitting of the spaces of differential forms on  $M$  into irreducible  $G_2$  representations.

$$\begin{aligned} \Omega^2(M) &= \Omega_7^2(M) \oplus \Omega_{14}^2(M), \\ \Omega^3(M) &= \Omega_1^3(M) \oplus \Omega_7^3(M) \oplus \Omega_{27}^3(M), \\ \Omega^4(M) &= \Omega_1^4(M) \oplus \Omega_7^4(M) \oplus \Omega_{27}^4(M), \\ \Omega^5(M) &= \Omega_7^5(M) \oplus \Omega_{14}^5(M), \end{aligned} \quad (2.2.6)$$

Since  $\varphi, \psi$  and  $*$  are all  $G_2$  invariant it is easy to see that the spaces defined above are all  $G_2$  representations. In a local orthonormal frame, the above conditions can be re-written as

$$\beta \in \Omega_7^2 \iff \beta_{ij}\psi_{abij} = -4\beta_{ab}, \quad (2.2.7)$$

$$\beta \in \Omega_{14}^2 \iff \beta_{ij}\psi_{abij} = 2\beta_{ab} \iff \beta_{ij}\varphi_{ijk} = 0. \quad (2.2.8)$$

Moreover, the space  $\Omega_{27}^3$  is isomorphic to the space of sections of  $S_0^2(T^*M)$ , the traceless symmetric 2-tensors on  $M$ , where the isomorphism  $i_\varphi$  is given explicitly as

$$\eta = \frac{1}{6}\eta_{ijk}dx^i \wedge dx^j \wedge dx^k \in \Omega_{27}^3 \xleftrightarrow{i_\varphi} h_{ab}dx^a dx^b \in C^\infty(S_0^2(T^*M)) \quad (2.2.9)$$

$$\text{where } \eta_{ijk} = h_{ip}\varphi_{pjk} + h_{jp}\varphi_{ipk} + h_{kp}\varphi_{ijp}.$$

The description of  $\Omega_1^4, \Omega_7^4, \Omega_{27}^4, \Omega_7^5$  and  $\Omega_{14}^5$  are obtained by taking the Hodge star of (2.2.3), (2.2.4), (2.2.5), (2.2.1) and (2.2.2) respectively.

**Remark 2.2.1.** Some authors prefer to use the opposite orientation than we do for the orientation induced by  $\varphi$  (for example in [Bry06] and [Joy00]). This changes the sign of the Hodge star  $*$  and the eigenvalues  $(-2, +1)$  in (2.2.1) and (2.2.2) are replaced by  $(+2, -1)$ .

## 2.3 Torsion of $G_2$ -structure

For any subgroup  $G \subset SO(n)$ , there is a unique  $G$ -equivariant splitting  $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$  obtained by using the standard  $O(n)$ -invariant inner product on  $\mathfrak{so}(n)$ .

For any  $G$ -structure  $P$  over  $M$ , one has the associated orthonormal frame bundle  $\mathcal{P} = P \cdot O(n)$ . One can then pull back the Levi-Civita connection  $\nabla$  on  $P$  to  $\mathcal{P}$  and decompose it uniquely in the form  $\nabla = \theta + \tau$  where  $\theta$  takes values in  $\mathfrak{g}$  and  $\tau$  takes values in  $\mathfrak{g}^\perp \cong \mathfrak{so}(n)/\mathfrak{g}$ . The 1-form  $\theta$  defines a natural connection on  $\mathcal{P}$ . For the standard representation of  $\mathfrak{g}$  on  $\mathbb{R}^n$  and  $\rho: G \rightarrow \text{End}((\mathfrak{g}^\perp \otimes \mathbb{R}^n))$  being the product representation of  $\mathbb{R}^n$  and the adjoint representation  $\mathfrak{g}$ , the 1-form  $\tau$  represents a section  $T$  of the associated torsion bundle  $\mathcal{P} \times_\rho (\mathfrak{g}^\perp \otimes \mathbb{R}^n)$ .

For a  $G_2$ -structure the torsion function  $\tau$  takes values in  $\mathfrak{g}_2^\perp \otimes \mathbb{R}^7$ . Since the Lie algebra  $\mathfrak{g}_2$  has rank 2 any irreducible representation of  $\mathfrak{g}_2$  can be represented by  $\mathcal{V}_{p,q}$  where the pair of integers  $(p, q)$  corresponds to the highest weight of the representation. The torsion space  $\mathfrak{g}_2^\perp \otimes \mathbb{R}^7 \cong \mathbb{R}^7 \otimes \mathbb{R}^7$  has the irreducible  $G_2$  splitting

$$\begin{aligned} \mathfrak{g}_2^\perp \otimes \mathbb{R}^7 &\cong \mathbb{R}^7 \otimes \mathbb{R}^7 \cong \mathcal{V}_{1,0} \oplus \mathcal{V}_{1,0} \cong \mathcal{V}_{0,0} \oplus \mathcal{V}_{1,0} \oplus \mathcal{V}_{0,1} \oplus \mathcal{V}_{2,0} \\ &\cong \mathbb{R} \oplus \mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus S_0^2(\mathbb{R}^7) \\ &\cong \Lambda^0 \oplus \Lambda^1 \oplus \Lambda_{14}^2 \oplus \Lambda_{27}^3. \end{aligned} \tag{2.3.1}$$

Here  $S_0^2(\mathbb{R}^7)$  denotes symmetric, traceless endomorphisms of  $\mathbb{R}^7$ . A differential geometric way to define the torsion of a  $G_2$ -structure  $\varphi$  is by considering the tensor  $\nabla\varphi \in \Gamma(T^*M \otimes \Lambda^3 T^*M)$  where  $\nabla$  is the Levi-Civita connection associated to  $\varphi$ . For any vector field  $X$ ,  $\nabla_X\varphi$  lies only in the  $\Omega_7^3$  component of  $\Omega^3$  as proved in [Kar09, Lemma 2.24]. Thus  $\nabla\varphi \in \Gamma(T^*M \otimes \Lambda_7^3 T^*M)$  and in fact given a  $G_2$ -structure  $\varphi$  on  $M$ , we can decompose  $d\varphi$  and  $d\psi$  according to (2.2.6) to define all of the irreducible constituents of the torsion function  $\tau$ . This defines the *torsion forms*, which are unique differential forms  $\tau_0 \in \Omega^0(M)$ ,  $\tau_1 \in \Omega^1(M)$ ,  $\tau_2 \in \Omega_{14}^2(M)$  and  $\tau_3 \in \Omega_{27}^3(M)$  such that (see [Kar09])

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3, \tag{2.3.2}$$

$$d\psi = 4\tau_1 \wedge \psi + *_\varphi\tau_2. \tag{2.3.3}$$

The vanishing (or non-vanishing) of the  $\tau_i$ 's gives rise to the 16 classes of  $G_2$ -structures (see [FG82]). We note four of the more interesting classes of  $G_2$ -structures in Table 2.1.

<b>Torsion form</b>	<b>Defining equation</b>	<b><math>G_2</math>-structure</b>
$\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$	$\nabla\varphi = 0 \iff d\varphi = d\psi = 0$	<i>torsion-free</i>
$\tau_0 = \tau_1 = \tau_3 = 0, \tau_2 \neq 0$	$d\varphi = 0$	<i>closed</i>
$\tau_1 = \tau_2 = 0, \tau_0$ or $\tau_3 \neq 0$	$d^*\varphi = 0$	<i>co-closed</i>
$\tau_1 = \tau_2 = \tau_3 = 0, \tau_0 \neq 0$	$d\varphi = \tau_0\psi$	<i>nearly parallel</i>

Table 2.1: Some classes of  $G_2$ -structures

In this thesis we are mostly interested in nearly parallel  $G_2$ -structures that is those for which  $\tau_0$  is the only non-vanishing torsion form. More on nearly parallel  $G_2$ -structures is discussed in Chapter 4.

The *full torsion tensor*  $T$  of a  $G_2$ -structure is a 2-tensor satisfying

$$\nabla_i\varphi_{jkl} = T_{im}\psi_{mjkl}, \quad (2.3.4)$$

$$T_{lm} = \frac{1}{24}(\nabla_l\varphi_{abc})\psi_{mabc}, \quad (2.3.5)$$

$$\nabla_m\psi_{ijkl} = -T_{mi}\varphi_{jkl} + T_{mj}\varphi_{ikl} - T_{mk}\varphi_{ijl} + T_{ml}\varphi_{ijk}. \quad (2.3.6)$$

The full torsion  $T$  is related to the torsion forms by (see [Kar09])

$$T_{lm} = \frac{\tau_0}{4}g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}. \quad (2.3.7)$$

**Remark 2.3.1.** The space  $\Omega_7^2$  is isomorphic to the space of vector fields and hence to the space of 1-forms. Thus in (2.3.7), we are viewing  $\tau_1$  as an element of  $\Omega_7^2$  which justifies the expression  $(\tau_1)_{lm}$ .

**Relation between the curvature and torsion of a  $G_2$ -structure:** Given a  $G_2$ -structure  $\varphi$  with torsion  $T_{lm}$ , we have the expressions for the Ricci curvature  $R_{ij}$  and the scalar curvature  $R$  of its associated metric  $g$  which can be found in [Bry06] or [Kar09]. If we denote by  $|C|^2 = C_{ij}C_{kl}g^{ik}g^{jl}$  the matrix norm we have that

$$R_{jk} = (\nabla_i T_{jm} - \nabla_j T_{im})\varphi_{mki} - T_{jl}T_{lk} + \text{tr}(T)T_{jk} - T_{jb}T_{lp}\psi_{lpbk}, \quad (2.3.8)$$

$$R = -12\nabla_i(\tau_1)_i + \frac{21}{8}\tau_0^2 - |\tau_3|^2 + 5|\tau_1|^2 - \frac{1}{4}|\tau_2|^2. \quad (2.3.9)$$

From here it is immediate that torsion-free  $G_2$ -structures are *Ricci flat* and nearly parallel  $G_2$ -structures are *positive Einstein*.

**Some useful identities** We use the following identities many times in the thesis. They are all proved in [Kar05, Lemma 2.2.1 and Lemma 2.2.3] and we collect them here for the convenience of the reader. First, we note that if  $\alpha$  is a  $k$ -form and  $w$  is a vector field then

$$*(w \lrcorner \alpha) = (-1)^{k+1}(w \wedge * \alpha), \quad (2.3.10)$$

$$*(w \wedge \alpha) = (-1)^k(w \lrcorner * \alpha). \quad (2.3.11)$$

If  $\alpha$  is a 1-form then we have the following identities

$$*(\varphi \wedge *(\varphi \wedge \alpha)) = -4\alpha, \quad (2.3.12)$$

$$\psi \wedge *(\varphi \wedge \alpha) = 0, \quad (2.3.13)$$

$$*(\psi \wedge *(\psi \wedge \alpha)) = 3\alpha, \quad (2.3.14)$$

$$\varphi \wedge *(\psi \wedge \alpha) = -2(\psi \wedge \alpha). \quad (2.3.15)$$

Suppose  $w$  is a vector field then we have the following identities

$$\varphi \wedge (w \lrcorner \psi) = -4 * w, \quad (2.3.16)$$

$$\psi \wedge (w \lrcorner \psi) = 0, \quad (2.3.17)$$

$$\psi \wedge (w \lrcorner \varphi) = 3 * w, \quad (2.3.18)$$

$$\varphi \wedge (w \lrcorner \varphi) = -2 * (w \lrcorner \varphi). \quad (2.3.19)$$

Let  $\Theta: \Omega_+^3 \rightarrow \Omega_+^4$  be the non-linear map which associates to any  $G_2$ -structure  $\varphi$ , the dual 4-form  $\psi = \Theta(\varphi) = *\varphi$  with respect to the metric  $g$ . We note that  $\Theta^{-1}: \Omega_+^4 \rightarrow \Omega_+^3$  is defined only when we fix the orientation on  $M$ . See [Hit01, §8] for more details. We need the following result from [Joy00, Proposition 10.3.5].

**Proposition 2.3.2.** *Let  $\varphi$  be a  $G_2$ -structure on  $M$  with  $\psi = *\varphi$ . Let  $\xi$  be a 3-form which has sufficiently small pointwise norm with respect to  $g$  so that  $\varphi + \xi$  is still a positive 3-form and  $\eta$  be a 4-form with small enough pointwise norm so that  $\psi + \eta$  is a positive 4-form. Then*

(1) the image of  $\xi$  under the linearization of  $\Theta$  at  $\varphi$  is

$$\hat{\Theta}(\xi) = *_{\varphi} \left( \frac{4}{3} \pi_1(\xi) + \pi_7(\xi) - \pi_{27}(\xi) \right). \quad (2.3.20)$$

(2) the image of  $\eta$  under the linearization of  $\Theta^{-1}$  at  $\psi$  is

$$\widehat{\Theta^{-1}}(\eta) = *_{\varphi} \left( \frac{3}{4} \pi_1(\eta) + \pi_7(\eta) - \pi_{27}(\eta) \right). \quad (2.3.21)$$

# Chapter 3

## Spinorial description of $G_2$ -structures

In this chapter we discuss an alternative and equivalent way to define  $G_2$ -structures using spin geometry. In some cases it gives a much neater and easier framework to work with  $G_2$ -structures that we introduced in Chapter 2.

### 3.1 Brief introduction to spin geometry

Some general references of what follows and much more about spin geometry are [LM89] and [Har90].

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{R}$  or  $\mathbb{C}$ , let  $q$  be any positive definite quadratic form on  $V$ . Given a pair  $(V, q)$  we can construct the Clifford algebra  $Cl(V, q)$  as the quotient of the tensor algebra  $\bigoplus_{r=0}^{\infty} \bigotimes^r V$  by the ideal generated by  $v \otimes v + q(v)$  as  $v$  ranges over  $V$ . The associated graded algebra  $Cl(V, q)$  is isomorphic as vector space to the exterior algebra  $\Lambda^*V$  (see [LM89, Proposition 1.2] for a proof of this). For any representation  $(\rho, W)$  of the Clifford algebra  $Cl(V, q)$  one can define the *Clifford multiplication* between  $\alpha$  an element of  $Cl(V, q)$  and  $w$  an element of  $W$  by  $\alpha \cdot w = \rho(\alpha)w$ .

Let us now assume  $V = \mathbb{R}^n$  with  $q$  being the quadratic form induced by the standard inner product on  $\mathbb{R}^n$ . The Clifford algebra is denoted by  $Cl_n$  here. We have the following useful proposition.

**Proposition 3.1.1.** *With respect to the canonical isomorphism  $Cl_n \cong \Lambda^*\mathbb{R}^n$ , Clifford multiplication between  $v \in \mathbb{R}^n$  and  $\alpha \in Cl_n$  is given by*

$$v \cdot \alpha = v \wedge \alpha - v \lrcorner \alpha.$$

In the above proposition  $v$  can be thought of as a 1-form and  $\alpha$  can be any arbitrary  $p$ -form on  $\mathbb{R}^n$ .

The Clifford algebras  $Cl_n$  have been classified, and this classification reduces them (up to isomorphism) to familiar matrix algebras over  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . We denote by  $K(n)$  the matrix algebra of  $n \times n$  matrix over the field  $K = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ . We describe the classification result for  $n \leq 8$  in the following table. For  $n > 8$ ,  $Cl_n$  can be computed using the isomorphism  $Cl_{n+8} \cong Cl_n \otimes Cl_8$  (see [LM89, Theorem 4.3]).

$n$	1	2	3	4	5	6	7	8
$Cl_n$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

Now we shall define the spin groups and the associated spin representations. The spin group  $\text{Spin}(n)$  is defined as the universal cover of  $\text{SO}(n)$  if  $n \geq 3$ . For  $n \geq 3$ ,  $\text{SO}(n)$  is a connected Lie group with  $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$ . Thus it has a double cover which is the Lie group  $\text{Spin}(n)$ . The Lie group  $\text{Spin}(n)$  is a compact, simply connected Lie group. The covering map  $\pi: \text{Spin}(n) \rightarrow \text{SO}(n)$  is a Lie group homomorphism. The group  $\text{Spin}(n)$  can also be realised as a subset of  $Cl_n$  as follows.

Let  $V^\times = \{v \in V \mid q(v) \neq 0\}$  and define

$$P(V, q) = \{v_1 \cdots v_r \in Cl(V, q) \mid v_1, \dots, v_r \text{ is a finite sequence in } V^\times\}$$

$$\text{Pin}(V, q) = \{v_1 \cdots v_r \in P(V, q) \mid q(v_j) = 1 \text{ for all } j\}.$$

Then the Lie group  $\text{Spin}(V, q)$  is defined as

$$\text{Spin}(V, q) = \{v_1 \cdots v_r \in \text{Pin}(V, q) \mid r \text{ is even}\}.$$

Thus  $\text{Spin}(V, q) \subset Cl(V, q)$ . We define

$$S = \{v \in V \mid q(v) = 1\}.$$

**Definition 3.1.2.** The spin representation  $\Delta^n$  is the representation of  $\text{Spin}(n)$  on  $S$  obtained by the restriction of the standard representation of  $Cl_n$  on  $S$ .

It has the following properties:

- i)  $\Delta^{2m}$  is a complex representation of  $\text{Spin}(2m)$ , with complex dimension  $2^m$ . For  $n = 2m$  we can define the complex volume form,  $\text{vol}_{\mathbb{C}} = i^m \text{vol}$  and we have that  $\text{vol}_{\mathbb{C}}^2 = -1$  (see [LM89, Section 1.5]). We define  $\Delta_{\pm}^{2m} = (1 \pm \text{vol}_{\mathbb{C}}) \cdot \Delta^{2m}$  and thus  $\Delta^{2m}$  splits into a direct sum  $\Delta^{2m} = \Delta_+^{2m} \oplus \Delta_-^{2m}$ . Thus  $\Delta_{\pm}^{2m}$  are irreducible representations of  $\text{Spin}(2m)$  with complex dimension  $2^{m-1}$ .

- ii)  $\Delta^{2m+1}$  is a complex representation of  $\text{Spin}(2m+1)$ , with complex dimension  $2^m$ . It is irreducible.
- iii) When  $n = 8k - 1, 8k,$  or  $8k + 1$ , for a real representation  $\Delta_{\mathbb{R}}^n$  of  $\text{Spin}(n)$  we have that  $\Delta^n = \Delta_{\mathbb{R}}^n \otimes_{\mathbb{R}} \mathbb{C}$ .

**Spinor bundle over a Riemannian manifold:** Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. As we have seen in Chapter 1 a choice of an orientation and metric on  $M$  is in one-to-one correspondence with a  $\text{SO}(n)$ -structure  $P_{\text{SO}(n)}$  on  $M$ . Similarly a choice of a spin structure on  $M$  is equivalent to a  $\text{Spin}(n)$ -structure. Since  $\text{Spin}(n)$  is the double cover of  $\text{SO}(n)$  we have the universal covering homomorphism  $\xi_0: \text{Spin}(n) \rightarrow \text{SO}(n)$  with kernel  $\mathbb{Z}_2$ .

**Definition 3.1.3.** A spin structure on  $M$  is a principal  $\text{Spin}(n)$ -bundle  $P_{\text{Spin}(n)}$  over  $M$  together with the 2-sheeted covering

$$\xi: P_{\text{Spin}(n)} \rightarrow P_{\text{SO}(n)},$$

such that  $\xi(pg) = \xi(p)\xi_0(g)$  for all  $p \in P_{\text{Spin}(n)}$  and  $g \in \text{Spin}(n)$ .

The bundle  $P_{\text{Spin}(n)}$  can be regarded as a double cover of  $P_{\text{SO}(n)}$ , and  $\xi$  as the covering map. Just as we need the first Stiefel–Whitney class  $w_1(M)$  to vanish for a manifold to be orientable, spin structures do not exist on every manifold. In fact, an oriented Riemannian manifold  $M$  admits a spin structure if and only if the second Stiefel–Whitney class  $w_2(M) = 0$ . We call  $M$  a spin manifold (or *spinnable*) if  $w_2(M) = 0$ , that is, if  $M$  admits a spin structure. The family of spin structures on  $M$  is parametrized by  $H^1(M, \mathbb{Z}_2)$  and thus a spin structure on  $M$  may not be unique. The space  $H^1(M, \mathbb{Z}_2)$  is finite if  $M$  is compact, and zero if  $M$  is simply connected. Therefore on a simply connected spin manifold there is a unique spin structure.

Let  $(M, g)$  be an oriented spin Riemannian  $n$ -manifold. Given a spin structure  $(P_{\text{Spin}(n)}, \xi)$  on  $M$  we can define the (complex) spin bundle  $\mathcal{S} \rightarrow M$  to be the associated  $\text{Spin}(n)$ -bundle  $\mathcal{S} = P_{\text{Spin}(n)} \times_{\text{Spin}(n)} \Delta^n$ . The bundle  $\mathcal{S}$  is a complex vector bundle over  $M$ , with fibre at each point being the spin representation  $\Delta^n$ . Sections of  $\mathcal{S}$  are called *spinors*. If  $n = 2m$ , then  $\Delta^n$  splits as  $\Delta^n = \Delta_+^n \oplus \Delta_-^n$ , and so  $\mathcal{S}$  also splits as  $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ . The vector subbundles  $\mathcal{S}_{\pm}$  of  $\mathcal{S}$  has fibre  $\Delta_{\pm}^n$  which we introduced above. Sections of  $\mathcal{S}_+, \mathcal{S}_-$  are called positive and negative spinors respectively. As we saw before in dimensions  $8k - 1, 8k,$  and  $8k + 1$  along with the complex spin representation  $\Delta^n$  there is a real spin representation denoted by  $\Delta_{\mathbb{R}}^n$ . In this case one can define the real spin bundle  $\mathcal{S}_R = P_{\text{Spin}(n)} \times_{\text{Spin}(n)} \Delta_{\mathbb{R}}^n$ . We shall always work with real spinors, unless we explicitly say otherwise.



The Levi-Civita connection  $\nabla$  of  $g$  on  $P_{\text{SO}(n)}$  can be lifted locally to a connection on  $P_{\text{Spin}(n)}$  via the local isomorphism  $\xi: P_{\text{Spin}(n)} \rightarrow P_{\text{SO}(n)}$ . Thus,  $P_{\text{Spin}(n)}$  also carries a natural connection which induces a covariant derivative  $\nabla^S: \Gamma(\mathcal{S}) \rightarrow \Gamma(T^*M \otimes \mathcal{S})$  on  $\mathcal{S}$ , called the *spin connection*.

On the other hand the Clifford multiplication defines a natural linear map from  $T^*M \otimes \mathcal{S}$  to  $\mathcal{S}$ . Composing this map with  $\nabla^S$  gives a first-order, linear partial differential operator  $D: \Gamma(\mathcal{S}) \rightarrow \Gamma(\mathcal{S})$  called the *Dirac operator*.

**Definition 3.1.4.** For a spinor  $\alpha$ , the Dirac operator associated to a connection  $\nabla^S$  on  $\mathcal{S}(M)$  is given in a local orthonormal frame  $e_i$  of  $\Gamma(TM)$  by

$$D\alpha = e_i \cdot \nabla_{e_i}^S \alpha. \quad (3.1.1)$$

It is straightforward to check that the Dirac operator is self-adjoint and elliptic. In even dimensions, it splits as a sum  $D = D_+ \oplus D_-$ , where  $D_+$  maps  $\Gamma(\mathcal{S}_+) \rightarrow \Gamma(\mathcal{S}_-)$  and  $D_-$  maps  $\Gamma(\mathcal{S}_-) \rightarrow \Gamma(\mathcal{S}_+)$ . Here  $D_{\pm}$  are both first-order linear elliptic operators,  $D_-$  is the formal adjoint of  $D_+$ , and vice versa. The result of changing the orientation of  $M$  is, in even dimensions, to exchange  $\mathcal{S}_+$  and  $\mathcal{S}_-$ , and  $D_+$  and  $D_-$ . In odd dimensions however there is no such splitting of the Dirac operator and the Clifford multiplication by the volume form preserves the Dirac operator.

## 3.2 Correspondence between a $G_2$ -structure and a unit spinor

From now on let  $(M^7, \varphi)$  be a manifold with  $G_2$ -structure. We saw in the previous section that on spin manifolds we can define a spinor bundle. Now we show that  $M$  is a spin manifold. There is a one-to-one correspondence between isometric  $G_2$ -structures on  $M$  and real unit spinors up to sign on  $\mathcal{S}(M)$ . The following proposition states that the condition of  $M$  being spin (or equivalently  $w_2(M) = 0$ ) is necessary and sufficient for the existence of a  $G_2$ -structure on  $M$ . We sketch the proof here, see [LM89, Theorem 10.6] for a detailed exposition.

**Proposition 3.2.1.** *A 7-dimensional manifold  $M$  carries a  $G_2$ -structure if and only if it is a spin manifold.*

*Proof.* Suppose  $M$  has a  $G_2$ -structure  $P$ . Since  $G_2$  is connected and simply connected the embedding  $\iota: G_2 \hookrightarrow \text{SO}(7)$  lifts to a homomorphism  $\tilde{\iota}: G_2 \hookrightarrow \text{Spin}(7)$  between the universal

covers of  $G_2$  and  $SO(7)$ . Using the embedding  $\tilde{t}$  we may define  $P_{\text{Spin}(n)} = P \times_{G_2} \text{Spin}(7)$  which is a double cover of  $P$  with fibre  $\text{Spin}(7)$ . Thus  $M$  is spin.

For the converse suppose  $M$  is spin. Then the fibre of the spinor bundle  $\mathcal{S}(M)$  at each point has dimension  $2^3 = 8 > \dim(M)$ . Thus there exists a nowhere vanishing cross-section  $\sigma$  of  $\mathcal{S}$ . Then one can show that the subgroup of  $\text{Spin}(7)$  preserving  $\sigma$  is isomorphic to  $G_2$  and we can identify the sphere of unit spinors in  $\mathcal{S}(M)$  with  $P_{\text{Spin}(n)}/G_2$ . Thus the unit real spinor  $\sigma/\|\sigma\|$  defines a  $G_2$  reduction.  $\square$

For a 7-dimensional Riemannian manifold  $M$  with a  $G_2$ -structure  $\varphi$ , the spinor bundle  $\mathcal{S}$  is a rank 8 real vector bundle over  $M$ . At each point  $p \in M$ , we can identify the fiber of  $\mathcal{S}$  with  $\mathbb{R} \oplus T_p M \cong \mathbb{R} \oplus \mathbb{R}^7 \cong \text{Re}(\mathbb{O}) \oplus \text{Im}(\mathbb{O}) = \mathbb{O}$ . Thus  $\mathcal{S}$  is isomorphic to the bundle  $\mathbb{R} \oplus TM = \Lambda^0 \oplus \Lambda^1$ . The real Clifford algebra in dimension 7 is isomorphic to  $\text{End}(\mathbb{R}^8) \oplus \text{End}(\mathbb{R}^8)$ . There are two inequivalent irreducible 8-dimensional  $\text{Spin}(8)$ -representations  $S_+, S_-$  of  $Cl_7$  defined via the left and right representation respectively. See [LM89, Proposition 5.9, Ch1] for a proof. There is a natural way to define  $S_{\pm}$  via the octonions  $\mathbb{O}$ . An element in  $\mathbb{O}$  can be defined via a pair of quaternions  $a, b$ . The octonion multiplication of  $p = (a, b), q = (c, d) \in \mathbb{O}$  is given by

$$p \cdot q = (a, b) \cdot (c, d) = (ac - \bar{d}b, da + b\bar{c}). \quad (3.2.1)$$

The multiplication so defined is neither commutative nor associative. The conjugate of  $p = (a, b) \in \mathbb{O}$  is defined by  $\bar{p} = (\bar{a}, -b)$ . We can set

$$\text{Re}(p) = \frac{1}{2}(p + \bar{p}), \quad \text{Im}(p) = \frac{1}{2}(p - \bar{p}).$$

An inner product on  $\mathbb{O}$  can be defined by  $\langle p, q \rangle = \text{Re}(p \cdot \bar{q})$ . We now consider  $\mathbb{R}^7 = \text{Im}\mathbb{O}$  and  $\mathbb{R}^8 = \mathbb{O}$  with the induced inner product. For any  $v \in \text{Im}\mathbb{O}$  we define a linear endomorphism  $\lambda_v$  of  $\mathbb{R}^8$  by setting  $\lambda_v(x) = v \cdot x$  for  $x \in \mathbb{O} = \mathbb{R}^8$ . This endomorphism can be extended to a representation of  $Cl_7$  on  $\mathbb{R}^8$  which is irreducible for dimensional reasons. The other representation can be generated by  $\rho_v(x) = -v \cdot x$ . These two representations are equivalent when restricted to  $\text{Spin}(7)$  but are inequivalent on  $\text{Spin}(8) \cong Cl_7$  (refer to [LM89, Section 1.8] for more details).

In this thesis, we choose  $\mathbb{R}^7$  to act on  $\mathbb{R}^8$  by  $\rho_v(x) = -v \cdot x$ . Let  $E_{ij}$  denote the standard basis of  $\mathfrak{so}(8)$ . Then our choice for the real representation of the Clifford algebra on  $\mathcal{S}$  can be generated by the following action of  $\mathbb{R}^7$  on  $\mathcal{S}$ :

$$e_1 \mapsto E_{18} + E_{27} - E_{36} - E_{45}, \quad e_2 \mapsto -E_{17} + E_{28} + E_{35} - E_{46},$$

$$\begin{aligned}
e_3 &\mapsto -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &\mapsto -E_{15} - E_{26} - E_{37} - E_{48}, \\
e_5 &\mapsto -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &\mapsto E_{14} - E_{23} - E_{58} + E_{67}, \\
e_7 &\mapsto E_{12} - E_{34} - E_{56} + E_{78}.
\end{aligned}$$

Under the isomorphism  $\mathcal{S} \otimes \mathcal{S} = \mathcal{S} \otimes \mathcal{S}^* = \Lambda^* T^* M = \bigoplus_{k=0}^7 \Lambda^k(T^* M)$ , the ‘‘square’’ of a unit spinor decomposes as follows ([LM89, Theorem 10.19]).

**Theorem 3.2.2.** *For any unit spinor  $\sigma \in \Gamma(\mathcal{S})$ , there exists a  $G_2$ -structure  $\varphi$  and the dual 4-form  $\psi$  such that*

$$\sigma \otimes \sigma = 1 + \varphi + \psi + \text{vol}.$$

The globally defined unit spinor  $\sigma$  induces a  $G_2$ -structure  $\varphi$  and the cross product  $\times$  by

$$\varphi(X, Y, Z) := (X \cdot Y \cdot Z \cdot \sigma, \sigma) =: g(X \times Y, Z).$$

The above theorem establishes a relation between  $G_2$  structures and unit real spinor. With respect to the local orthonormal frame  $\{e_1, \dots, e_7\}$ , we have  $(X \times Y)_l = X_i Y_j \varphi_{ijl}$ . We have an orthogonal decomposition of the spinor bundle

$$\mathcal{S} = (\Lambda^0 T M \cdot \sigma) \oplus (\Lambda^1 T M \cdot \sigma) \cong \Lambda^0 T M \oplus \Lambda^1 T M.$$

Under this isomorphism any spinor  $s = f\sigma + \alpha \cdot \sigma \in \mathcal{S}$  can be written as  $s = (f, \alpha) \in \Lambda^0 \oplus \Lambda^1$ .

As shown in [Kar10] the Clifford multiplication of a 1-form  $Y$  and a spinor  $(f, Z)$  is the octonionic product of an imaginary octonion and an octonion and is thus given by

$$Y \cdot (f, Z) = -(-\langle Y, Z \rangle, fY + Y \times Z). \quad (3.2.2)$$

Note that the product defined above differs from [Kar10] by a negative sign due to our choice of the representation of  $Cl_7$  on  $\mathcal{S}$ . We define the Clifford multiplication of any  $p$ -form

$\beta = \frac{1}{p!} \beta_{i_1 \dots i_p} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p}$  with a spinor by

$$\beta \cdot (f, X) = \frac{1}{p!} \beta_{i_1 \dots i_p} (e_{i_1} \cdot (e_{i_2} \cdot \dots \cdot (e_{i_p} \cdot (f, X)) \dots)).$$

We record an identity for Clifford algebras for later use and refer the reader to [LM89, Proposition 3.8, Ch1] for the proof.

**Proposition 3.2.1.** For  $\alpha \in \Lambda^p(M)$ , we have

$$\sum_j e_j \cdot \alpha \cdot e_j = (-1)^{p+1}(n-2p) \alpha.$$

For the vector bundle  $\mathcal{S}$  the fibre over each point is a  $G_2$ -representation. Since  $G_2$  preserves the 3-form  $\varphi$  the map  $\mu \mapsto \varphi \cdot \mu$  from  $\mathcal{S}$  to  $\mathcal{S}$  acts by scalar identity on irreducible  $G_2$  representations by Schur's lemma. The same argument holds for the 4-form  $\psi$ .

**Lemma 3.2.3.** *The subbundles of  $\mathcal{S}$  isomorphic to  $\Lambda^0$  and  $\Lambda^1$  are eigenspaces of the operations of Clifford multiplication by  $\varphi$  and  $\psi$ . The associated eigenvalues are*

	$\Lambda^0$	$\Lambda^1$
$\varphi$	7	-1
$\psi$	7	-1.

*Proof.* For the sub-bundles  $\Lambda^0, \Lambda^1 \subset \mathcal{S}$ , the fibre at each point are irreducible  $G_2$ -representations and thus are eigenspaces of the operators defined by the Clifford multiplication by  $\varphi, \psi$  respectively. By Schur's Lemma there exist real constants  $\lambda_0, \lambda_1, \mu_0, \mu_1$  such that for all  $f \in \Lambda^0, \alpha \in \Lambda^1$

$$\begin{aligned} \varphi \cdot f &= \lambda_0 f, & \varphi \cdot \alpha &= \lambda_1 \alpha, \\ \psi \cdot f &= \mu_0 f, & \psi \cdot \alpha &= \mu_1 \alpha. \end{aligned}$$

Proposition 3.2.1 then implies  $\sum_i e_i \cdot \varphi \cdot e_i = \varphi$  and  $\sum_i e_i \cdot \psi \cdot e_i = \psi$  thus

$$\begin{aligned} \lambda_0 f &= \varphi \cdot f = \sum_{i=1}^7 e_i \cdot \varphi \cdot e_i \cdot f, \\ \mu_0 f &= \psi \cdot f = \sum_{i=1}^7 e_i \cdot \psi \cdot e_i \cdot f. \end{aligned}$$

Using the fact that  $e_i \cdot f \in \Lambda^1$  and summing over  $i$  we get

$$\lambda_0 + 7\lambda_1 = 0, \tag{R1}$$

$$\mu_0 + 7\mu_1 = 0. \tag{R2}$$

We find the eigenvalues corresponding to  $\Lambda^0$  by explicit calculations and use relations (R1) and (R2) to show the result for  $\Lambda^1$ . Let  $(f, 0) \in \Lambda^0$  be a spinor. In the local

orthonormal frame  $e_1, \dots, e_7$ , we write  $\varphi = \frac{1}{6}\varphi_{ijk}e_i \wedge e_j \wedge e_k$ , with  $\varphi_{ijk}$  skew-symmetric in each pair of indices. Using (3.2.2) we get that

$$\begin{aligned}\varphi \cdot (f, 0) &= \frac{1}{6}\varphi_{ijk}e_i \cdot (e_j \cdot (e_k \cdot (f, 0))) = -\frac{1}{6}\varphi_{ijk}e_i \cdot (e_j \cdot (0, fe_k)) \\ &= \frac{1}{6}\varphi_{ijk}e_i \cdot (-f\delta_{kj}, f\varphi_{jkt}e_t) \\ &= -\frac{1}{6}\varphi_{ijk}(-f\varphi_{ijk}, -f\delta_{kj}e_i + f\varphi_{jkt}\varphi_{itp}e_p).\end{aligned}$$

By using the skew-symmetry of  $\varphi$  and the contraction identities  $\varphi_{ijk}\varphi_{ijl} = 6\delta_{kl}$ ,  $\varphi_{ijk}\varphi_{ijk} = 42$  (see [Kar09]), we get

$$\varphi \cdot (f, 0) = \frac{1}{6}(42f, -6f\delta_{it}\varphi_{itp}e_p) = (7f, 0).$$

Similarly, in the above local orthonormal frame,  $\psi = \frac{1}{24}\psi_{ijkl}e_i \wedge e_j \wedge e_k \wedge e_l$  and using (3.2.2) we get

$$\begin{aligned}\psi \cdot (f, 0) &= \frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (e_k \cdot (e_l \cdot (f, 0)))) = -\frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (e_k \cdot (0, fel))) \\ &= \frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (-f\delta_{kl}, f\varphi_{klp}e_p)) \\ &= -\frac{1}{24}\psi_{ijkl}e_i \cdot (-f\varphi_{klp}\delta_{jp}, -f\delta_{kl}e_j + f\varphi_{klp}\varphi_{jpt}e_t) \\ &= \frac{1}{24}\psi_{ijkl}(f\delta_{kl}\delta_{ij} - f\varphi_{klp}\varphi_{jpt}\delta_{it}, -f\varphi_{klp}\delta_{jp}e_i - f\delta_{kl}\varphi_{ijs}e_s + f\varphi_{klp}\varphi_{jpt}\varphi_{itr}e_r).\end{aligned}$$

Here we can use the skew-symmetry of  $\psi$ , the contraction identity  $\psi_{ijkl}\varphi_{klp} = -4\varphi_{ijp}$  along with the contraction identities of  $\varphi$  mentioned before to obtain

$$\psi \cdot (f, 0) = \frac{1}{24}(24\delta_{il}\delta_{il}f, 0) = \frac{1}{24}(24 \cdot 7f, 0) = (7f, 0).$$

Substituting  $\lambda_0 = 7$  and  $\mu_0 = 7$  in relations (R1), (R2) respectively proves the desired result.  $\square$

Let  $\nabla$  be the Levi-Civita connection on  $M$  induced by the  $G_2$ -structure  $\varphi$ . We denote the lift of  $\nabla$  on  $\mathcal{S}$  by  $\nabla$  as well. Motivated by the fact that  $\Lambda^1 TM \cdot \sigma = \sigma^\perp$  we have the following definition [ACFH15, Definition 4.2]

**Definition 3.2.4.** There exists an endomorphism  $S$  of  $TM^7$  such that for every tangent vector  $X$  on  $M$ ,

$$\nabla_X \sigma = S(X) \cdot \sigma. \quad (3.2.3)$$

The endomorphism  $S$  is called the *intrinsic endomorphism* of  $(M, g, \varphi)$ .

We have the following lemma relating the intrinsic torsion of the  $G_2$ -structure  $\varphi$  with the intrinsic endomorphism  $S$ . We have the following lemma (see [ACFH15, Lemma 4.3] for a proof).

**Lemma 3.2.5.** *The intrinsic endomorphism  $S$  satisfies*

$$\nabla_V \varphi(X, Y, Z) = 2\psi(S(V), X, Y, Z). \quad (3.2.4)$$

$G_2$ -structure	Spinorial equations	Description of $S$
torsion-free	$\nabla_X \sigma = 0$	$S \equiv 0$
Closed	$\nabla_{X \times Y} \sigma = Y \nabla_X \sigma - X \nabla_Y \sigma + 2g(Y, S(X))\sigma$	$S \in \mathfrak{g}_2$
Co-closed	$(X \nabla_Y \sigma, \sigma) = (Y \nabla_X \sigma, \sigma)$	$S \in \mathbb{R} \oplus S_0^2 \mathbb{R}^7$
Nearly parallel	$\nabla_X \sigma = \lambda X \cdot \sigma$	$S = \lambda \text{id}$

Table 3.1:  $G_2$  classes in terms of intrinsic torsion

An easy calculation in local coordinates shows that  $X \cdot \sigma = -\frac{1}{3}(X \lrcorner \varphi) \cdot \sigma$  for  $X \in \Gamma(TM)$ . Thus the intrinsic torsion  $\nabla_X \varphi = S(X) \cdot \sigma = -\frac{1}{3}(S(X) \lrcorner \varphi) \cdot \sigma$ . This identity gives another way to prove the pointwise decomposition of torsion forms as in (2.3.1) since

$$\text{End}(\mathbb{R}^7) \cong \mathbb{R} \oplus \mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus S_0^2 \mathbb{R}^7.$$

Thus we can define the classes of  $G_2$ -structures by defining the intrinsic torsion  $S$ . We do so for the classes mentioned in Table 2.1. See [ACFH15, Table 4.1] for a description of all the 16 classes of  $G_2$ -structure in terms of the intrinsic endomorphism.

From Table 3.1 we can see that if the  $G_2$ -structure is torsion-free, that is  $d\varphi = d\psi = 0$ , then the unit spinor  $\sigma$  is parallel. Since  $\sigma$  is  $G_2$  invariant this reduces the holonomy to  $G_2$ .

# Chapter 4

## Nearly $G_2$ manifolds

### 4.1 Manifolds with nearly parallel $G_2$ -structure

In this chapter we discuss one of the sixteen classes of  $G_2$ -structures known as the nearly parallel. In later chapters we focus solely on manifolds with these special  $G_2$ -structures so here we note some important properties and identities related to these manifolds.

Let  $(M^7, \varphi, \sigma, g)$  be a manifold with a  $G_2$ -structure where  $\varphi$  denoted the positive 3-form and  $\sigma$  is the associated real unit spinor on the spinor bundle over  $M$ . From Tables 2.1 and 3.1 we can see that for nearly parallel  $G_2$ -structures the only non-vanishing torsion form is  $\tau_0$ , thus in this case we see from (2.3.7) that  $T_{ij} = \frac{\tau_0}{4}g_{ij}$ . If  $\tau_0$  is the only non-vanishing torsion component, (2.3.2) and (2.3.3) imply

$$d\varphi = \tau_0\psi \quad \text{and} \quad d\psi = 0. \quad (4.1.1)$$

From the spinorial point of view the above equation implies that the intrinsic endomorphism  $S$  defined in Chapter 3 by  $\nabla_X\sigma = S(X) \cdot \sigma$  is given by  $S = \lambda\text{id}$  for some non-zero scalar  $\lambda$ . We recall that  $\nabla$  in the equation here is the lift of the Levi-Civita connection associated to  $g$ .

**Definition 4.1.1.** A real spinor  $\sigma \in \Gamma(\mathcal{S})$  is a *Killing spinor* if for some non-zero  $\delta \in \mathbb{R}$ , we have for all  $X \in \Gamma(TM)$  that

$$\nabla_X\sigma = \delta X \cdot \sigma. \quad (4.1.2)$$

The scalar  $\delta$  is known as the *Killing constant* for the Killing spinor  $\sigma$ .

For a manifold  $M$  with a nearly  $G_2$ -structure  $\varphi$ , using (2.3.8) and (2.3.9) we see that the Ricci curvature  $\text{Ric}$  and the scalar curvature  $R$  are related to  $\tau_0$  by

$$R_{ij} = \frac{3}{8}\tau_0^2 g_{ij}, \quad (4.1.3)$$

$$R = \frac{21}{8}\tau_0^2. \quad (4.1.4)$$

Given a nearly parallel  $G_2$ -structure  $\varphi$  that satisfies (4.1.1) there exists a real Killing spinor  $\eta$  that satisfies (4.1.2) with  $\delta = -\frac{1}{2}\sqrt{\frac{R}{42}}$  and vice-versa (see [BFGK91, Theorem 6, Chapter 4.4]). Thus for a nearly  $G_2$  manifold, (4.1.4) implies that  $\delta = -\frac{\tau_0}{8}$ . Switching  $-\frac{\tau_0}{8}$  to  $\frac{\tau_0}{8}$  corresponds to changing the orientation of  $M$ . See [BFGK91, Section 4.4] and [Bär93] for more details.

The nearly parallel  $G_2$ -structures inducing the given metric and spin structure are in bijective correspondence with the projectivization of the space of Killing spinors on the real spinor bundle. Thus there is a one-to-one correspondence between nearly parallel  $G_2$ -structures and unit real Killing spinors on  $M$ . Summing up we have the following definition.

**Definition 4.1.2.** Let  $\varphi$  be a  $G_2$ -structure with associated real spinor  $\eta$ . The  $G_2$ -structure is a **nearly parallel  $G_2$  structure** if  $\tau_0$  is the only non-vanishing component of the torsion, that is

$$d\varphi = \tau_0\psi \quad \text{and} \quad d\psi = 0. \quad (4.1.5)$$

Equivalently with respect to the associated real unit spinor  $\eta$  the  $G_2$ -structure is nearly parallel if

$$\nabla_X \eta = -\frac{\tau_0}{8} X \cdot \eta. \quad (4.1.6)$$

**Remark 4.1.3.** If  $\varphi$  is a nearly  $G_2$ -structure on  $M$  then since  $d\varphi = \tau_0\psi$ , we get  $d\tau_0 \wedge \psi = 0$  and hence  $d\tau_0 = 0$ , as wedge product with  $\psi$  is an isomorphism from  $\Omega_7^1(M)$  to  $\Omega_7^5(M)$ . Thus  $\tau_0$  is a constant, if  $M$  is connected.

Using (4.1.6) and Remark 4.1.3 we can see that the unit spinor associated to a nearly  $G_2$ -structure is a *Killing spinor* with Killing constant  $-\frac{\tau_0}{8}$ .

Manifolds with nearly parallel  $G_2$ -structures have several nice properties which can be found in detail in [BFGK91]. In particular they are positive Einstein.



One can alter the constant  $\tau_0$  by rescaling the metric which changes its magnitude and readjusting the orientation which alters the sign of  $\tau_0$ . In this thesis we use  $\tau_0 = 4$ . The particular choice of  $\tau_0$  is made since the unit 7-sphere  $S^7$  has scalar curvature 42 and thus  $\tau_0 = 4$ . With this choice of  $\tau_0$  our nearly  $G_2$ -structure  $\varphi$  and Killing spinor  $\eta$  satisfies the equations

$$\begin{aligned} d\varphi &= 4\psi, \\ \nabla_X \eta &= -\frac{1}{2}X \cdot \eta. \end{aligned} \tag{4.1.7}$$

For this choice of  $\tau_0$ , (4.1.3) and (4.1.4) force the Ricci curvature and the scalar curvature to be  $6g$  and 42 respectively.

**Remark 4.1.1.** Since nearly  $G_2$  manifolds are positive Einstein ( $\text{Ric}_g = 6g$ ) it follows from *Myers's theorem* that complete nearly  $G_2$  manifolds are compact.

Friedrich–Kath–Moroianu–Semmelmann in [FKMS97] showed that, excluding the case of the round 7-dimensional sphere, there are three types of nearly parallel  $G_2$ -structures depending on the dimension of the space  $K\mathcal{S}$  of all Killing spinors. The dimension of  $K\mathcal{S}$  is bounded above by 3, giving rise to the three different types:

1.  $\dim(K\mathcal{S}) = 1$ : type 1 or *proper nearly  $G_2$  manifolds*,
2.  $\dim(K\mathcal{S}) = 2$ : type 2 or *Sasaki-Einstein manifolds*,
3.  $\dim(K\mathcal{S}) = 3$ : type 3 or *3-Sasakian manifolds*.

The cones over these manifolds have holonomy contained in  $\text{Spin}(7)$ , specifically  $\text{Spin}(7)$ ,  $\text{SU}(4)$ ,  $\text{Sp}(2)$ , when the nearly parallel  $G_2$ -structure on the link is of type 1, 2, 3 respectively. This property makes these spaces particularly important in the construction and understanding of manifolds with torsion-free  $\text{Spin}(7)$ -structures.

**Remark 4.1.2.** Throughout the thesis we abuse notation by denoting the space of Killing spinors at each point over the manifold and the subset of Killing spinors in  $\Gamma(\mathcal{S})$  by the same notation  $K\mathcal{S}$ .

A common feature between nearly Kähler 6-manifolds and manifolds with nearly parallel  $G_2$ -structures is the presence of a unique canonical connection  $\nabla^{can}$  with totally skew-symmetric torsion defined below. The Killing spinor  $\eta$  is parallel with respect to this connection and thus we have  $\text{Hol}(\nabla^{can}) \subset G_2$ . It was proved by Cleyton–Swann in [CS04, Theorem 6.3] that a  $G$ -irreducible Riemannian manifold  $(M, g)$  with an invariant skew-symmetric non-vanishing intrinsic torsion falls in one of the following categories:

1. it is locally isometric to a non-symmetric isotropy irreducible homogeneous space, or,
2. it is a nearly Kähler 6-manifold, or,
3. it is a nearly  $G_2$  manifold.

For the nearly  $G_2$  manifold  $(M, \varphi)$  we define a 1-parameter family of connections on  $TM$  that include the canonical connection  $\nabla^{can}$ . Let  $t \in \mathbb{R}$  and let  $\nabla^t$  be the 1-parameter family of connections on  $TM$  defined for all  $X, Y, Z \in \Gamma(TM)$  by

$$g(\nabla_X^t Y, Z) = g(\nabla_X^{LC} Y, Z) + \frac{t}{3}\varphi(X, Y, Z). \quad (4.1.8)$$

Let  $T^t$  be the torsion (1, 2)-tensor of  $\nabla^t$ . Since the connection  $\nabla^{LC}$  is torsion-free

$$\begin{aligned} g(X, T^t(Y, Z)) &= g(X, \nabla_Y^t Z) - g(X, \nabla_Z^t Y) - g(X, [Y, Z]) \\ &= g(\nabla_Y^{LC} Z, X) + \frac{t}{3}\varphi(Y, Z, X) - g(\nabla_Z^{LC} Y, X) - \frac{t}{3}\varphi(Z, Y, X) \\ &\quad - g(X, [Y, Z]) \\ &= \frac{2t}{3}\varphi(X, Y, Z). \end{aligned}$$

Therefore the torsion tensor  $T^t$  is given by

$$T^t(X, Y) = \frac{2t}{3}\varphi(X, Y, \cdot) \quad (4.1.9)$$

which is proportional to  $\varphi$  and is thus totally skew-symmetric.

By [LM89, Theorem 4.14], the lift of the connection  $\nabla^t$  on the spinor bundle, also denoted by  $\nabla^t$ , acts on sections  $\mu$  of  $\mathcal{S}$  as

$$\nabla_X^t \mu = \nabla_X^{LC} \mu + \frac{t}{6}(i_X \varphi) \cdot \mu. \quad (4.1.10)$$

The space of real Killing spinors is isomorphic to  $\Lambda^0$ . Thus for a Killing spinor  $\eta$  it follows from (4.1.7) and Lemma 3.2.3 that for any vector field  $X$  since  $X \cdot \varphi + \varphi \cdot X = -2 i_X \varphi$ ,

$$\nabla_X^t \eta = \nabla_X^0 \eta + \frac{t}{6}(i_X \varphi) \cdot \eta$$

$$\begin{aligned}
&= -\frac{1}{2}X \cdot \eta - \frac{t}{12}(X \cdot \varphi + \varphi \cdot X) \cdot \eta \\
&= -\frac{1}{2}X \cdot \eta - \frac{t}{12}(7X \cdot \eta - X \cdot \eta) \\
&= -\frac{t+1}{2}X \cdot \eta.
\end{aligned}$$

Therefore  $\eta$  is parallel with respect to the connection  $\nabla^{-1}$ . The connection  $\nabla^{-1}$  thus has holonomy group contained in  $G_2$  with totally skew-symmetric torsion and is therefore the canonical connection on the nearly  $G_2$  manifold  $M$  described in [CS04].

**Definition 4.1.4.** The canonical connection on nearly  $G_2$  manifold  $(M, \varphi)$  is defined by

$$\nabla_X^{LC} Y - \frac{1}{3}\varphi(X, Y, \cdot). \quad (4.1.11)$$

Owing to its position in the 1-parameter family of connections defined in (4.1.8) we denote this by  $\nabla^{-1}$  throughout the thesis.

**Proposition 4.1.3.** The Ricci tensor  $\text{Ric}^t$  of the connection  $\nabla^t$  is given by

$$\text{Ric}^t = \left(6 - \frac{2t^2}{3}\right)g.$$

*Proof.* By using the expression of the Ricci tensor for a connection with a totally skew-symmetric torsion from [FI02], we have

$$\text{Ric}^t(X, Y) = \text{Ric}^0(X, Y) - \frac{t}{3}d^*\varphi(X, Y) - \frac{2t^2}{9}g(i_X\varphi, i_Y\varphi)$$

The Ricci tensor for the Levi-Civita connection is given by  $\text{Ric}^0 = 6g$ . Since  $d\psi = 0$ ,  $\varphi$  is co-closed and the second term in the above expression vanishes. The third term can be calculated in a local orthonormal frame  $e_1, \dots, e_7$  using the contraction identity  $\varphi_{ijk}\varphi_{ijl} = 6\delta_{kl}$  as follows:

$$\begin{aligned}
g(i_X\varphi, i_Y\varphi) &= \frac{1}{4} \sum_{i,j,k,\alpha,\beta,\gamma} X_k Y_\gamma \varphi_{ijk} \varphi_{\alpha\beta\gamma} g(e_i \wedge e_j, e_\alpha \wedge e_\beta) \\
&= \frac{1}{4} \sum_{i,j,k,\gamma} X_k Y_\gamma (\varphi_{ijk} \varphi_{ij\gamma} - \varphi_{ijk} \varphi_{ji\gamma}) \\
&= 3 \sum_{k,\gamma} X_k Y_\gamma \delta_{k\gamma} = 3g(X, Y).
\end{aligned}$$

Summing up all the terms together give the desired identity for  $\text{Ric}^t$ . □

## 4.2 Some first order differential operators

In this section, we discuss various first order differential operators on a manifold with a nearly  $G_2$ -structure and prove some identities involving them.

For  $f \in C^\infty(M)$ , we have the vector field  $\text{grad } f$  given by

$$(\text{grad } f)_k = \nabla_k f$$

and for any vector field  $X$  we have the divergence of  $X$  which is a function  $\text{div } X = \nabla_k X_k$ . On a manifold with a  $G_2$ -structure  $\varphi$ , for a vector field  $X \in \Gamma(TM)$ , we define the *curl* of  $X$  as

$$(\text{curl } X)_k = \nabla_i X_j \varphi_{ijk} \tag{4.2.1}$$

which can also be written as

$$(\text{curl } X) = *(dX \wedge \psi) \tag{4.2.2}$$

and so up to  $G_2$ -equivariant isomorphisms, the vector field  $\text{curl } X$  is the projection of the 2-form  $dX$  onto the  $\Omega_7^2$  component. In fact, we have the following.

**Proposition 4.2.1.** *Let  $X$  be a vector field on  $M$ . The  $\Omega_7^2$  component of  $dX$  is given by*

$$\pi_7(dX) = \frac{1}{3}(\text{curl } X) \lrcorner \varphi = \frac{1}{3} * (\text{curl } X \wedge \psi). \tag{4.2.3}$$

*Proof.* We know that  $\pi_7(dX) = W \lrcorner \varphi$  for some vector field  $W$ . Using (2.3.18) we compute

$$\text{curl } X = *(dX \wedge \psi) = *(\pi_7(dX) \wedge \psi) = *((W \lrcorner \varphi) \wedge \psi) = 3W$$

which gives (4.2.3). □

In the next proposition we state and prove various relations among the first order differential operators described above. We prove the results for *any*  $G_2$ -structure and state the results for nearly  $G_2$ -structures. These formulas are generalizations of the formulas first proved for torsion-free  $G_2$ -structures by Karigiannis [Kar10, Proposition 4.4].

**Remark 4.2.2.** For fixed  $i, j$ , the Riemann curvature tensor  $R_{ijkl}$  is skew-symmetric in  $k$  and  $l$  and hence

$$R_{ijkl} = (\pi_7(\text{Rm}))_{ijkl} + (\pi_{14}(\text{Rm}))_{ijkl}.$$

Explicitly,

$$(\pi_7(\text{Rm}))_{ijkl} = \frac{1}{3}R_{ijkl} - \frac{1}{6}R_{abkl}\psi_{abij}, \quad (\pi_{14}(\text{Rm}))_{ijkl} = \frac{2}{3}R_{ijkl} + \frac{1}{6}R_{abkl}\psi_{abij}.$$

Moreover, from [Kar09, eq. (4.17)], we have

$$(\pi_7(\text{Rm}))_{ijkl} = (\pi_7(\text{Rm}))_{ij}^m \varphi_{mkl} \quad \text{where} \quad \pi_7(\text{Rm})_{ij}^m = \frac{1}{6}R_{ijkl}\varphi_{klm}. \quad (4.2.4)$$

**Proposition 4.2.3.** *Let  $f \in C^\infty(M)$  and  $X$  be a vector field on  $M$  with a  $G_2$ -structure  $\varphi$ . Then*

$$\text{curl}(\text{grad } f) = 0, \quad (4.2.5)$$

$$\text{div}(\text{curl } X) = (\pi_7(\text{Rm}))_{ij}^j X_l - \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij}), \quad (4.2.6)$$

$$\begin{aligned} \text{curl}(\text{curl } X)_l &= \nabla_l(\text{div } X) + R_{lm}X_m - \Delta X_l + (\text{curl } X)_m T_{ml} - (\nabla_l X_i - \nabla_i X_l)(\tau_1)_{ms}\varphi_{msi} \\ &\quad - \text{tr } T(\text{curl } X)_l - \nabla_i X_j T_{is}\varphi_{jst} - \nabla_i X_j T_{js}\varphi_{sil}. \end{aligned} \quad (4.2.7)$$

*Proof.* We compute

$$\text{curl}(\text{grad } f) = \nabla_i(\nabla_j f)\varphi_{ijk} = 0$$

as  $\varphi$  is skew-symmetric, thus proving (4.2.5).

For (4.2.6) we use the Ricci identity (1.1.1) to get

$$\begin{aligned} \text{div}(\text{curl } X) &= \nabla_k(\nabla_i X_j \varphi_{ijk}) \\ &= \nabla_k \nabla_i X_j \varphi_{ijk} + \nabla_i X_j \nabla_k \varphi_{ijk} \\ &= \frac{1}{2}(\nabla_k \nabla_i X_j - \nabla_i \nabla_k X_j)\varphi_{ijk} + \nabla_i X_j T_{km}\psi_{mijk} \quad (\text{by (2.3.4), (2.2.7), (2.2.8)}) \\ &= -\frac{1}{2}R_{kijl}X_l \varphi_{ijk} - \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij}) \quad (\text{by (2.3.7)}) \\ &= 3(\pi_7(\text{Rm}))_{ij}^j X_l - \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij}) \quad (\text{by (4.2.4)}). \end{aligned}$$

We have also used the fact that the symmetric part of  $T$  vanishes when contracted with  $\psi$ .

Finally we use the contraction identities (1.1.2) and (1.1.4) and the Ricci identity (1.1.1) to compute

$$\begin{aligned}
(\operatorname{curl}(\operatorname{curl} X))_l &= \nabla_m(\nabla_i X_j \varphi_{ijk}) \varphi_{mkl} \\
&= (\nabla_m \nabla_i X_j \varphi_{ijk} + \nabla_i X_j T_{ms} \psi_{sijk}) \varphi_{lmk} \\
&= \nabla_m \nabla_i X_j (g_{il} g_{jm} - g_{im} g_{jl} - \psi_{ijlm}) \\
&\quad - \nabla_i X_j T_{ms} (g_{ms} \varphi_{lij} + g_{mi} \varphi_{slj} + g_{mj} \varphi_{sil} - g_{ls} \varphi_{mij} - g_{li} \varphi_{smj} - g_{lj} \varphi_{sim}) \\
&= \nabla_j \nabla_l X_j - \Delta X_l - \frac{1}{2} (\nabla_m \nabla_i X_j - \nabla_i \nabla_m X_j) \psi_{ijlm} - \operatorname{tr} T \nabla_i X_j \varphi_{ijl} - \nabla_i X_j T_{is} \varphi_{slj} \\
&\quad - \nabla_i X_m T_{ms} \varphi_{sil} + \nabla_i X_j T_{ml} \varphi_{mij} + \nabla_l X_j T_{ms} \varphi_{smj} + \nabla_i X_l T_{ms} \varphi_{msi}.
\end{aligned}$$

Using the fact that  $R_{abcd} \psi_{abck} = 0$  and (2.2.8) we get that

$$\begin{aligned}
(\operatorname{curl}(\operatorname{curl} X))_l &= \nabla_l(\operatorname{div} X) + R_{lm} X_m - \Delta X_l - \operatorname{tr} T(\operatorname{curl} X)_l - \nabla_i X_j T_{is} \varphi_{jsl} - \nabla_i X_m T_{ms} \varphi_{sil} \\
&\quad + (\operatorname{curl} X)_m T_{ml} - \nabla_l X_j (\tau_1)_{ms} \varphi_{msj} + \nabla_i X_l (\tau_1)_{ms} \varphi_{msi}.
\end{aligned}$$

We can rearrange the above equation to get

$$\begin{aligned}
(\operatorname{curl}(\operatorname{curl} X))_l &= \nabla_l(\operatorname{div} X) + R_{lm} X_m - \Delta X_l + (\operatorname{curl} X)_m T_{ml} + (\nabla_i X_l - \nabla_l X_i) (\tau_1)_{ms} \varphi_{msi} \\
&\quad - \operatorname{tr} T(\operatorname{curl} X)_l - \nabla_i X_j T_{is} \varphi_{jsl} - \nabla_i X_j T_{js} \varphi_{sil}.
\end{aligned}$$

□

For a nearly  $G_2$ -structure we have  $T_{ij} = \frac{\tau_0}{4} g_{ij}$  and  $R_{ij} = \frac{3\tau_0^2}{8} g_{ij}$ . Moreover from [Kar09, eq. (4.18)],

$$(\pi_7(\operatorname{Rm}))_{jl}^j = -\nabla_l(\operatorname{tr} T) + \nabla_j(T_{lj}) + T_{la} T_{jb} \varphi_{abj} = 0.$$

Thus using the Weitzenböck formula for  $X$ ,  $\nabla^* \nabla X_l = -\nabla_j \nabla_j X_l = (\Delta_d X)_l - R_{il} X_i$ , we get the following.

**Corollary 4.2.4.** *Let  $f \in C^\infty(M)$  and  $X$  be a vector field on  $M$  with a nearly  $G_2$ -structure  $\varphi$ . Then*

$$\operatorname{curl}(\operatorname{grad} f) = 0, \tag{4.2.8}$$

$$\operatorname{div}(\operatorname{curl} X) = 0, \tag{4.2.9}$$

$$\operatorname{curl}(\operatorname{curl} X) = \operatorname{grad}(\operatorname{div} X) - \Delta X + \frac{3\tau_0^2}{8} X - \tau_0(\operatorname{curl} X), \tag{4.2.10}$$

$$= \Delta_d X + \operatorname{grad}(\operatorname{div} X) - \tau_0(\operatorname{curl} X). \tag{4.2.11}$$

### 4.2.1 Identities for 2-forms and 3-forms

In this subsection, we prove some identities for 2-forms and 3-forms on a manifold with a nearly  $G_2$ -structure. These identities are used several times in the thesis.

**Lemma 4.2.5.** *Let  $(M, \varphi)$  be a manifold with a  $G_2$ -structure. If  $\beta = \beta_7 + \beta_{14}$  is a 2-form then*

$$(1) \quad *(\beta \wedge \varphi) = -2\beta_7 + \beta_{14}.$$

$$(2) \quad *(\beta \wedge \beta \wedge \varphi) = -2|\beta_7|^2 + |\beta_{14}|^2.$$

*Proof.* The identity in (1) follows from (2.2.1) and (2.2.2). For (2) we note that for 7-dimensional manifolds  $*^2(\alpha) = \alpha$  for a  $k$ -form  $\alpha$ , so

$$\beta \wedge \beta \wedge \varphi = \beta \wedge *^2(\beta \wedge \varphi) = \beta \wedge *(-2\beta_7 + \beta_{14})$$

and the decomposition of 2-forms is orthogonal.  $\square$

**Lemma 4.2.6.** *Let  $(M, \varphi)$  be a manifold with a  $G_2$ -structure. Let  $\sigma = f\varphi + \sigma_7 + \sigma_{27}$  be a 3-form on  $M$  and let  $\sigma_7 = X \lrcorner \psi$  for some vector field  $X$  on  $M$ . Then*

$$(1) \quad *(\sigma \wedge \varphi) = 4X.$$

$$(2) \quad *(\sigma \wedge \psi) = 7f.$$

*Proof.* For (1), using the fact that  $\Omega_1^3 \oplus \Omega_{27}^3$  lies in the kernel of wedge product with  $\varphi$  and (2.3.16), we have

$$\begin{aligned} *(\sigma \wedge \varphi) &= *((f\varphi + \sigma_7 + \sigma_{27}) \wedge \varphi) = *(\sigma_7 \wedge \varphi) = *((X \lrcorner * \varphi) \wedge \varphi) \\ &= 4X. \end{aligned} \tag{4.2.12}$$

For (2) we note that  $\Omega_7^3 \oplus \Omega_{27}^3$  lies in the kernel of wedge product with  $\psi$  and  $\varphi \wedge \psi = 7 \text{vol}$ .  $\square$

Next, we explicitly derive the expressions for the exterior derivative and the divergence of various components of 2-forms and 3-forms on a manifold with a nearly  $G_2$ -structure. Some of these identities are new, at least in the present form and we believe that they are useful in other contexts as well.

**Lemma 4.2.7.** *Suppose  $(M, \varphi)$  is a manifold with a nearly  $G_2$ -structure. Let  $f \in C^\infty(M)$ ,  $\beta \in \Omega_{14}^2$  and  $X \in \Gamma(TM)$ . Then*

$$(1) \quad d(f\varphi) = df \wedge \varphi + \tau_0 f \psi,$$

$$(2) \quad d^*(f\varphi) = -(df) \lrcorner \varphi,$$

$$(3) \quad d\beta = \frac{1}{4} * (d^*\beta \wedge \varphi) + \pi_{27}(d\beta),$$

$$(4) \quad d(X \lrcorner \varphi) = -\frac{3}{7}(d^*X)\varphi + \frac{1}{2} * \left( \left( \frac{3\tau_0}{2}X + \text{curl } X \right) \wedge \varphi \right) + i_\varphi \left( \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) + \frac{1}{7}(d^*X)g_{ij} \right),$$

$$(5) \quad d^*(X \lrcorner \varphi) = \text{curl } X,$$

$$(6) \quad d(X \lrcorner \psi) = -\frac{4}{7}(d^*X)\psi + \left( \frac{1}{2} \text{curl } X - \frac{\tau_0}{4}X \right) \wedge \varphi - *i_\varphi \left( \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) + \frac{1}{7}(d^*X)g_{ij} \right).$$

*Proof.* Using (4.1.1) we have

$$\begin{aligned} d(f\varphi) &= df \wedge \varphi + f d\varphi \\ &= df \wedge \varphi + \tau_0 f \psi \end{aligned}$$

which proves (1). For part (2) we compute

$$d^*(f\varphi) = - * d * (f\varphi) = - * d(f * \varphi) = - * (df \wedge * \varphi) = -df \lrcorner \varphi$$

as  $d\psi = 0$ .

We prove part (3). Since  $d\beta$  is a 3-form, we have

$$d\beta = \pi_1(d\beta) + \pi_7(d\beta) + \pi_{27}(d\beta). \quad (4.2.13)$$

We compute each term on the right hand side of (4.2.13). We repeatedly use the identities (2.3.10)–(2.3.19). Suppose

$$\pi_1(d\beta) = a\varphi$$

for some  $a \in C^\infty(M)$ . Since  $\Omega_7^3 \oplus \Omega_{27}^3$  lies in the kernel of wedge product with  $\psi$  and  $\beta \wedge \psi = 0$  for  $\beta \in \Omega_{14}^2$ , we have

$$0 = d(\beta \wedge \psi) = d\beta \wedge \psi = \pi_1(d\beta) \wedge \psi = 7a \text{ vol}$$



and hence

$$\pi_1(d\beta) = 0.$$

Suppose  $\pi_7(d\beta) = X \lrcorner \psi$  for  $X \in \Gamma(TM)$ . Using (2.2.2) and Lemma 4.2.6 (1), we have

$$d^*\beta = *d*(\beta) = -*d(\beta \wedge \varphi) = -(d\beta \wedge \varphi) - \tau_0*(\beta \wedge \psi) = -4X.$$

Thus

$$\pi_7(d\beta) = -\frac{1}{4}d^*\beta \lrcorner \psi = \frac{1}{4}*(d^*\beta \wedge \varphi),$$

which proves (3).

Since  $d(X \lrcorner \varphi)$  is a 3-form, so we write

$$d(X \lrcorner \varphi) = \pi_1(d(X \lrcorner \varphi)) + \pi_7(d(X \lrcorner \varphi)) + \pi_{27}(d(X \lrcorner \varphi)) \quad (4.2.14)$$

and calculate each term on the right hand side. As before, assume

$$\pi_1(d(X \lrcorner \varphi)) = a\varphi$$

for some  $a \in C^\infty(M)$ . Then

$$d((X \lrcorner \varphi) \wedge \psi) = \pi_1(d(X \lrcorner \varphi)) \wedge \psi = 7a \text{ vol}$$

and hence  $7a = *d((X \lrcorner \varphi) \wedge \psi) = *d(3 * X)$ . So we get that

$$a = \frac{3}{7} * d * X = -\frac{3}{7} d^* X.$$

Assume that

$$\pi_7(d(X \lrcorner \varphi)) = Y \lrcorner \psi$$

for some  $Y \in \Gamma(TM)$ . Using the fact that  $\Omega_1^3 \oplus \Omega_{27}^3$  lies in the kernel of wedge product with  $\varphi$  we get

$$\begin{aligned} d((X \lrcorner \varphi) \wedge \varphi) &= d(X \lrcorner \varphi) \wedge \varphi + (X \lrcorner \varphi) \wedge d\varphi \\ &= \pi_7(d(X \lrcorner \varphi)) \wedge \varphi + \tau_0(X \lrcorner \varphi) \wedge \psi = (Y \lrcorner \psi) \wedge \varphi + 3\tau_0 * X. \end{aligned}$$

So we get

$$4 * Y + 3\tau_0 * X = d((X \lrcorner \varphi) \wedge \varphi) = d(-2 * (X \lrcorner \varphi)) = -2d(X \wedge \psi) = -2(dX) \wedge \psi$$

which gives

$$Y = \frac{1}{2} \left( - * ((dX) \wedge \psi) - \frac{3\tau_0}{2} X \right) = -\frac{1}{2} \left( \text{curl } X + \frac{3\tau_0}{2} X \right)$$

and hence

$$\pi_7(d(X \lrcorner \varphi)) = \frac{1}{2} * \left( \left( \text{curl } X + \frac{3\tau_0}{2} X \right) \wedge \varphi \right).$$

Recall the map  $i_\varphi$  from (2.2.9). To calculate  $\pi_{27}(d(X \lrcorner \varphi))$  we have

$$\begin{aligned} & d(X \lrcorner \varphi)_{imn} \varphi_{jmn} + d(X \lrcorner \varphi)_{jmn} \varphi_{imn} \\ &= \left[ \frac{-3}{7} (d^* X) \varphi_{imn} - \frac{1}{2} \left( \left( \text{curl } X + \frac{3\tau_0}{2} X \right) \lrcorner \psi \right)_{imn} + i(h_0)_{imn} \right] \varphi_{jmn} \\ & \quad + \left[ \frac{-3}{7} (d^* X) \varphi_{jmn} - \frac{1}{2} \left( \left( \text{curl } X + \frac{3\tau_0}{2} X \right) \lrcorner \psi \right)_{jmn} + i(h_0)_{jmn} \right] \varphi_{imn} \\ &= -\frac{36}{7} (d^* X) g_{ij} + 8(h_0)_{ij} - \frac{1}{2} \left( \text{curl } X + \frac{3\tau_0}{2} X \right)_s \psi_{simn} \varphi_{jmn} \end{aligned} \quad (4.2.15)$$

$$\begin{aligned} & -\frac{1}{2} \left( \text{curl } X + \frac{3\tau_0}{2} X \right)_s \psi_{sjmn} \varphi_{imn} \\ &= -\frac{36}{7} (d^* X) g_{ij} + 8(h_0)_{ij}. \end{aligned} \quad (4.2.16)$$

We calculate the left hand side of (4.2.16). Using (2.3.4) and (4.1.1) we have

$$\begin{aligned} d(X \lrcorner \varphi)_{imn} \varphi_{jmn} + d(X \lrcorner \varphi)_{jmn} \varphi_{imn} &= (\nabla_i (X_l \varphi_{lmn}) - \nabla_m (X_l \varphi_{lin}) + \nabla_n (X_l \varphi_{lim})) \varphi_{jmn} \\ & \quad + (\nabla_j (X_l \varphi_{lmn}) - \nabla_m (X_l \varphi_{ljn}) + \nabla_n (X_l \varphi_{ljm})) \varphi_{imn} \\ &= (\nabla_i X_l \varphi_{lmn} - \nabla_m X_l \varphi_{lin} + \nabla_n X_l \varphi_{lim}) \varphi_{jmn} \\ & \quad + \frac{\tau_0}{4} (X_l \psi_{ilmn} - X_l \psi_{mlin} + X_l \psi_{nljm}) \varphi_{jmn} \\ & \quad + (\nabla_j X_l \varphi_{lmn} - \nabla_m X_l \varphi_{ljn} + \nabla_n X_l \varphi_{ljm}) \varphi_{imn} \\ & \quad + \frac{\tau_0}{4} (X_l \psi_{jlmn} - X_l \psi_{mljn} + X_l \psi_{nljm}) \varphi_{imn}. \end{aligned}$$

So

$$\begin{aligned}
d(X \lrcorner \varphi)_{imn} \varphi_{jmn} + d(X \lrcorner \varphi)_{jmn} \varphi_{imn} &= (\nabla_i X_l \varphi_{lmn} \varphi_{jmn} - 2\nabla_m X_l \varphi_{lin} \varphi_{jmn}) \\
&\quad + \frac{\tau_0}{4} (X_l \psi_{ilmn} - X_l \psi_{mlin} + X_l \psi_{nljm}) \varphi_{jmn} \\
&\quad (\nabla_j X_l \varphi_{lmn} \varphi_{imn} - 2\nabla_m X_l \varphi_{ljn} \varphi_{imn}) \\
&\quad + \frac{\tau_0}{4} (X_l \psi_{jlmn} - X_l \psi_{mljn} + X_l \psi_{nljm}) \varphi_{imn}.
\end{aligned}$$

We use the contraction identities (1.1.2), (1.1.3) and (1.1.4) to get

$$\begin{aligned}
d(X \lrcorner \varphi)_{imn} \varphi_{jmn} + d(X \lrcorner \varphi)_{jmn} \varphi_{imn} &= 4\nabla_i X_j + 4\nabla_j X_i + 4(\operatorname{div} X)g_{ij} \\
&\quad + \frac{\tau_0}{4} (-4X_l \varphi_{ilj} + 4X_l \varphi_{lij} + 4X_l \varphi_{lji}) \\
&\quad + \frac{\tau_0}{4} (-4X_l \varphi_{jli} + 4X_l \varphi_{lji} + 4X_l \varphi_{lji}) \\
&= 4\nabla_i X_j + 4\nabla_j X_i - 4(d^* X)g_{ij}
\end{aligned}$$

and so from (4.2.16) we get

$$-\frac{36}{7}(d^* X)g_{ij} + 8(h_0)_{ij} = 4\nabla_i X_j + 4\nabla_j X_i - 4(d^* X)g_{ij}$$

and thus

$$(h_0)_{ij} = \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) + \frac{1}{7}(d^* X)g_{ij}$$

which completes the proof of (4).

We obtain (5) by

$$d^*(X \lrcorner \varphi) = *d*(X \lrcorner \varphi) = *d(X \wedge \psi) = *(dX \wedge \psi) = \operatorname{curl} X.$$

To prove part (6), we notice that since  $d\psi = 0$ ,  $d(X \lrcorner \psi) = \mathcal{L}_X \psi$  which is the image of  $\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + \tau_0 X \lrcorner \psi$  under the linearization of the map  $\Theta$ . We then use part (4) of the lemma and (2.3.20) to get part (6).  $\square$

We use the following important lemma on several occasions.

**Lemma 4.2.8.** *Let  $\varphi$  be a nearly  $G_2$ -structure on  $M$  and  $\alpha$  be a 3-form so that*

$$\alpha = f\varphi + *(X \wedge \varphi) + \alpha_0$$

where  $\alpha_0 \in \Omega_{27}^3$  with  $\alpha_0 = i_\varphi(h)$  where  $h$  is a symmetric traceless 2-tensor. Then

$$\pi_1(d\alpha) = \left( \tau_0 f + \frac{4}{7} d^* X \right) \psi, \quad (4.2.17)$$

$$\pi_7(d\alpha) = \left( df + \frac{\tau_0}{4} X - \frac{1}{2} \operatorname{curl} X - \frac{1}{2} \operatorname{div} h \right) \wedge \varphi, \quad (4.2.18)$$

$$\pi_7(d^* \alpha) = * \left( (-df + \tau_0 X + \frac{2}{3} \operatorname{curl} X - \frac{2}{3} \operatorname{div} h) \wedge \psi \right). \quad (4.2.19)$$

*Proof.* We note that  $*\alpha = f\psi + (X \wedge \varphi) + *\alpha_0$  and since  $\varphi$  is a nearly  $G_2$ -structure hence

$$d\alpha = df \wedge \varphi + \tau_0 f \psi + d*(X \wedge \varphi) + d\alpha_0 \quad (4.2.20)$$

and

$$d^* \alpha = -*d*\alpha = -* (df \wedge \psi) - *d(X \wedge \varphi) + d^* \alpha_0. \quad (4.2.21)$$

Now  $\pi_1(d\alpha) = \lambda\psi$  for some  $\lambda \in C^\infty(M)$ . We use Lemma 4.2.7 (6) to get,

$$\begin{aligned} 7\lambda &= \langle \lambda\psi, \psi \rangle = \langle \pi_1(d\alpha), \psi \rangle = \langle d\alpha, \psi \rangle \\ &= \langle df \wedge \varphi + \tau_0 f \psi + d*(X \wedge \varphi) + d\alpha_0, \psi \rangle \\ &= \langle df \wedge \varphi, \psi \rangle + 7\tau_0 f + 4d^* X + \langle d\alpha_0, \psi \rangle. \end{aligned} \quad (4.2.22)$$

The first term on the right hand side of (4.2.22) is 0 as  $df \wedge \varphi \in \Omega_7^4$  and  $\psi \in \Omega_1^4$ . The last term is also 0 as from (2.2.5)

$$\langle d\alpha_0, \psi \rangle \operatorname{vol} = d\alpha_0 \wedge \varphi = d(\alpha_0 \wedge \varphi) + \tau_0 \alpha_0 \wedge \psi = 0.$$

Thus we get that

$$7\lambda = 7\tau_0 f + 4d^* X \quad \implies \quad \lambda = \tau_0 f + \frac{4}{7} d^* X$$

which gives (4.2.17).

To derive (4.2.18) and (4.2.19), we need to contract  $\alpha_0 \in \Omega_{27}^3$  with  $\varphi$  on two indices and with  $\psi$  on three indices. Using (2.2.9) and the contraction identities (1.1.2) and (1.1.5), a short computation gives

$$\alpha_{0ijk} \varphi_{ajk} = 4h_{ia}, \quad (4.2.23)$$

$$\alpha_{0ijk}\psi_{aijk} = 0. \quad (4.2.24)$$

Suppose  $\pi_7(d\alpha) = Y \wedge \varphi$  for some 1-form  $Y$ . Note that for an arbitrary 1-form  $Z$  we have

$$\begin{aligned} \langle Y \wedge \varphi, Z \wedge \varphi \rangle \text{vol} &= Y \wedge \varphi \wedge *(Z \wedge \varphi) \\ &= -Y \wedge \varphi \wedge (Z \lrcorner \psi) = 4Y \wedge *Z \\ &= 4\langle Y, Z \rangle \text{vol}. \end{aligned}$$

So from (4.2.20) we have

$$\begin{aligned} 4\langle Y, Z \rangle &= \langle Y \wedge \varphi, Z \wedge \varphi \rangle = \langle \pi_7(d\alpha), Z \wedge \varphi \rangle = \langle d\alpha, Z \wedge \varphi \rangle \\ &= \langle df \wedge \varphi + \tau_0 f \psi + d*(X \wedge \varphi) + d\alpha_0, Z \wedge \varphi \rangle \\ &= 4\langle df, Z \rangle + \langle d*(X \wedge \varphi), Z \wedge \varphi \rangle + \langle d\alpha_0, Z \wedge \varphi \rangle. \end{aligned} \quad (4.2.25)$$

We first use Lemma 4.2.7 (6) to calculate the second term on the right hand side of (4.2.25). We have

$$\langle d*(X \wedge \varphi), Z \wedge \varphi \rangle = \left\langle \left(-\frac{1}{2} \text{curl } X + \frac{\tau_0}{4} X\right) \wedge \varphi, Z \wedge \varphi \right\rangle = \langle -2 \text{curl } X + \tau_0 X, Z \rangle$$

So in (4.2.25), we have

$$4\langle Y, Z \rangle = \langle 4df + \tau_0 X - 2 \text{curl } X, Z \rangle + \langle d\alpha_0, Z \wedge \varphi \rangle. \quad (4.2.26)$$

We compute in local coordinates

$$\begin{aligned} \langle d\alpha_0, Z \wedge \varphi \rangle &= \frac{1}{24} (d\alpha_0)_{ijkl} (Z \wedge \varphi)_{ijkl} \\ &= \frac{1}{24} (\nabla_i \alpha_{0jkl} - \nabla_j \alpha_{0ikl} + \nabla_k \alpha_{0ijl} - \nabla_l \alpha_{0ijk}) (Z \wedge \varphi)_{ijkl} \\ &= \frac{1}{6} (\nabla_i \alpha_{0jkl}) (Z_i \varphi_{jkl} - Z_j \varphi_{ikl} - Z_k \varphi_{jil} - Z_l \varphi_{jki}) \\ &= \frac{1}{6} (Z_i \nabla_i \alpha_{0jkl} \varphi_{jkl} - 3Z_j \nabla_i \alpha_{0jkl} \varphi_{ikl}) \\ &= \frac{1}{6} (Z_i \nabla_i (\alpha_{0jkl} \varphi_{jkl}) - \frac{\tau_0}{4} Z_i \alpha_{0jkl} \psi_{ijkl} - 3Z_j \nabla_i (\alpha_{0jkl} \varphi_{ikl}) + \frac{3\tau_0}{4} Z_j \alpha_{0jkl} \psi_{iikl}). \end{aligned}$$

We now use (4.2.23), (4.2.24) and the fact that  $h$  is traceless to get

$$\langle d\alpha_0, Z \wedge \varphi \rangle = \frac{1}{6} (Z_i \nabla_i (4 \text{tr } h) - 0 - 3Z_j \nabla_i (4h_{ji}))$$

$$= -2\langle \operatorname{div} h, Z \rangle.$$

Thus from (4.2.26) we get

$$\langle Y, Z \rangle = \left\langle df + \frac{\tau_0}{4}X - \frac{1}{2}\operatorname{curl} X - \frac{1}{2}\operatorname{div} h, Z \right\rangle$$

and since  $Z$  is arbitrary, we get

$$Y = df + \frac{\tau_0}{4}X - \frac{1}{2}\operatorname{curl} X - \frac{1}{2}\operatorname{div} h$$

which establishes (4.2.18).

Next, we see from (4.2.21) and (2.2.1) that

$$\begin{aligned} d^*\alpha &= -*(df \wedge \psi) - *(dX \wedge \varphi) + *\tau_0(X \wedge \psi) + d^*\alpha_0 \\ &= -*(df - \tau_0 X \wedge \psi) + 2\pi_7(dX) - \pi_{14}(dX) + d^*\alpha_0 \end{aligned}$$

which on using (4.2.3) becomes

$$d^*\alpha = -*\left(\left(df - \tau_0 X - \frac{2}{3}\operatorname{curl} X\right) \wedge \psi\right) - \pi_{14}(dX) + d^*\alpha_0. \quad (4.2.27)$$

Suppose  $\pi_7(d^*\alpha) = *(W \wedge \psi)$  for some 1-form  $W$ . For any 1-form  $Z$  we note that

$$\begin{aligned} \langle *(W \wedge \psi), *(Z \wedge \psi) \rangle \operatorname{vol} &= *(W \wedge \psi) \wedge Z \wedge \psi \\ &= *(W \wedge \psi) \wedge \psi \wedge Z = 3*W \wedge Z = 3\langle W, Z \rangle \operatorname{vol}. \end{aligned}$$

Thus using (4.2.27) and the orthogonality of the spaces  $\Omega_7^2$  and  $\Omega_{14}^2$ , we have

$$\begin{aligned} 3\langle W, Z \rangle &= \langle *(W \wedge \psi), *(Z \wedge \psi) \rangle = \langle \pi_7(d^*\alpha), *(Z \wedge \psi) \rangle = \langle d^*\alpha, *(Z \wedge \psi) \rangle \\ &= \langle -*\left(\left(df - \tau_0 X - \frac{2}{3}\operatorname{curl} X\right) \wedge \psi\right) - \pi_{14}(dX) + d^*\alpha_0, *(Z \wedge \psi) \rangle \\ &= \langle -3df + 3\tau_0 X + 2\operatorname{curl} X, Z \rangle + \langle d^*\alpha_0, *(Z \wedge \psi) \rangle. \end{aligned} \quad (4.2.28)$$

Using (4.2.23) and (4.2.24), we compute the last term on the right hand side of (4.2.28), in local coordinates. We have

$$\begin{aligned} \langle d^*\alpha_0, *(Z \wedge \psi) \rangle &= \langle d^*\alpha_0, Z \lrcorner \varphi \rangle = \frac{1}{2}(d^*\alpha_0)_{ij} Z_m \varphi_{mij} = -\frac{1}{2} \nabla_p(\alpha_{0pij}) Z_m \varphi_{mij} \\ &= -\frac{1}{2} Z_m (\nabla_p(\alpha_{0pij} \varphi_{mij}) - \frac{\tau_0}{4} \alpha_{0pij} \psi_{pmij}) \end{aligned}$$

$$= -\frac{1}{2}Z_m(4\nabla_p h_{pm} - 0) = -2\langle \operatorname{div} h, Z \rangle$$

and hence we get

$$\langle W, Z \rangle = \left\langle -df + \tau_0 X + \frac{2}{3} \operatorname{curl} X - \frac{2}{3} \operatorname{div} h, Z \right\rangle.$$

Since  $Z$  is arbitrary we get

$$W = -df + \tau_0 X + \frac{2}{3} \operatorname{curl} X - \frac{2}{3} \operatorname{div} h$$

which gives (4.2.19). □

**Remark 4.2.9.** The main point of the previous lemma is to exhibit a relation between  $\pi_7(d\alpha_0)$  and  $\pi_7(d^*\alpha_0)$ . Such a relation is expected because of the form of the linearization of the map  $\Theta$ . More precisely, from (2.3.20), applying the linearization of  $\Theta$  to Lie derivatives, we have  $\pi_{27}(\mathcal{L}_X \psi) = - * \pi_{27}(\mathcal{L}_X \varphi)$ ,  $\langle d\alpha_0, Z \wedge \varphi \rangle_{L^2} = -\langle \alpha_0, * \mathcal{L}_X \psi \rangle_{L^2}$  and  $\langle d^* \alpha_0, Z \lrcorner \varphi \rangle_{L^2} = \langle \alpha_0, \mathcal{L}_X \varphi \rangle_{L^2}$ . The computations in local coordinates were done to relate  $\pi_7(d\alpha_0)$  and  $\pi_7(d^*\alpha_0)$  to the divergence of the symmetric 2-tensor  $h$ .

**Remark 4.2.10.** The previous lemma generalizes Proposition 2.17 from [KL20] where the  $G_2$ -structure was assumed to be torsion-free ( $\tau_0 = 0$ ).

We have the following corollary of Lemma 4.2.8.

**Corollary 4.2.11.** *Let  $\varphi$  be a nearly  $G_2$ -structure and let  $\eta \in \Omega_{27}^3$ . Then*

- (1) *If  $\eta$  is closed then  $d^*\eta \in \Omega_{14}^2$ .*
- (2) *If  $\eta$  is co-closed then  $d\eta \in \Omega_{27}^4$ .*

*Proof.* In the notation of Lemma 4.2.8 we get that  $f = X = 0$  and  $\sigma = \eta$ . Thus we get that

$$\pi_7(d\eta) = 0 \quad \iff \quad \pi_7(d^*\eta) = 0$$

as from Lemma 4.2.8, both conditions are equivalent to  $\operatorname{div} h = 0$ . Now if  $d\eta = 0$  then  $\pi_7(d^*\eta) = 0$  and hence  $d^*\eta \in \Omega_{14}^2$ . If  $d^*\eta = 0$  then  $\pi_7(d\eta) = 0$ . Also, since  $f = X = 0$ , we know from (4.2.17) that  $\pi_1(d\eta) = 0$ . So  $d\eta \in \Omega_{27}^4$ . □

We also have a result similar to Lemma 4.2.8 for 4-forms which we state below. The proof follows from the proof of Lemma 4.2.8 by taking  $\zeta = *\sigma$  and noting that  $*i_\varphi(h) = -i_\varphi(h)$ . We expect that both Lemma 4.2.8 and Lemma 4.2.12 are useful in other contexts as well.

**Lemma 4.2.12.** *Let  $\varphi$  be a nearly  $G_2$ -structure on  $M$  and  $\zeta$  be a 4-form on  $M$  so that*

$$\zeta = f\psi + X \wedge \varphi + \zeta_0$$

where  $X \in \Omega^1(M)$  and  $\zeta_0 \in \Omega_{27}^4$  with  $\zeta_0 = *i_\varphi(h)$  where  $h$  is a symmetric traceless 2-tensor. Then

$$\pi_7(d\zeta) = W \wedge \psi \quad \text{where} \quad W = df - \tau_0 X - \frac{2}{3} \operatorname{curl} X - \frac{2}{3} \operatorname{div} h, \quad (4.2.29)$$

$$\pi_1(d^*\zeta) = \left( \tau_0 f + \frac{4}{7} d^* X \right) \varphi, \quad (4.2.30)$$

$$\pi_7(d^*\zeta) = Y \lrcorner \psi \quad \text{where} \quad Y = -df + \frac{1}{2} \operatorname{curl} X - \frac{\tau_0}{4} X - \frac{1}{2} \operatorname{div} h. \quad (4.2.31)$$

We get the following corollary.

**Corollary 4.2.13.** *Let  $\varphi$  be a nearly  $G_2$ -structure on  $M$  and let  $\zeta_0 \in \Omega_{27}^4$ . Then*

1. *If  $d\zeta_0 = 0$  then  $d^*\zeta_0 \in \Omega_{27}^3$ .*
2. *If  $d^*\zeta_0 = 0$  then  $d\zeta_0 \in \Omega_{14}^5$ .*

## 4.3 Hodge theory of nearly $G_2$ manifolds

### 4.3.1 Dirac operators on nearly $G_2$ manifolds

We begin this section by defining the Dirac operator on  $(M, \varphi)$  with a nearly  $G_2$ -structure. We then define a *modified* Dirac operator which is more suitable for our purposes. In Chapter 6, we study deformations of nearly  $G_2$ -structures through nearly  $G_2$ -structures  $\varphi_t$ . Since the underlying metric of any nearly  $G_2$ -structure is positive Einstein, the family of metrics  $g_t$  corresponding to  $\varphi_t$  is positive Einstein and so by [Bes87, Corollary 2.12], the scalar curvature  $R_t$  is constant in  $t$ . Thus, by (4.1.4),  $\tau_0$  is constant through the deformation. Henceforth, we assume that  $\tau_0 = 4$ . *The results presented here do not depend on the value of  $\tau_0$  chosen.*



A unit spinor  $\eta$  on a nearly  $G_2$  manifold  $M$  satisfies (4.1.7). Thus, given that  $e_i \cdot e_i = -1$ , we have

$$\not{D}(f\eta) = \sum_{i=1}^7 e_i \cdot \nabla_{e_i}(f\eta) = \nabla f \cdot \eta + \frac{7}{2}f\eta.$$

Also by Proposition 3.1.1,

$$\begin{aligned} \not{D}(X \cdot \eta) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}(X \cdot \eta) = \sum_{i=1}^7 (e_i \cdot \nabla_{e_i} X \cdot \eta + e_i \cdot X \cdot \nabla_{e_i} \eta) \\ &= (dX) \cdot \eta + (d^* X)\eta + \sum_{i=1}^7 e_i \cdot X \cdot \nabla_{e_i} \eta \end{aligned}$$

which, on using  $X \cdot e_i + e_i \cdot X = -2\langle X, e_i \rangle$  and (4.1.7), becomes

$$\begin{aligned} \not{D}(X \cdot \eta) &= (dX) \cdot \eta + (d^* X)\eta - \sum_{i=1}^7 (X \cdot e_i \cdot \nabla_{e_i} \eta + 2\langle X, e_i \rangle \nabla_{e_i} \eta) \\ &= (dX) \cdot \eta + (d^* X)\eta - \frac{7}{2}X \cdot \eta + X \cdot \eta = (dX) \cdot \eta + (d^* X)\eta - \frac{5}{2}X \cdot \eta. \end{aligned}$$

Thus we get

$$\not{D}(f\eta + X \cdot \eta) = \left(\frac{7}{2}f + d^* X\right)\eta + \left(\nabla f + dX - \frac{5}{2}X\right) \cdot \eta. \quad (4.3.1)$$

Now  $dX$  is a 2-form, hence  $dX = \pi_7(dX) + \pi_{14}(dX)$ . Since the Lie group  $G_2$  preserves the nearly  $G_2$ -structure  $\varphi$ , it preserves the real Killing spinor  $\eta$  induced by  $\varphi$ . Thus, since  $\Lambda_{14}^2(M) \cong \mathfrak{g}_2$ , we have  $\pi_{14}(dX) \cdot \eta = 0$ . Also, we know from (4.2.2) that  $\pi_7(dX) = \frac{1}{3}(\text{curl } X) \lrcorner \varphi$  and it follows from the definition of the Clifford multiplication, for instance as in [Kar10, §4.2], that  $(Y \lrcorner \varphi) \cdot \eta = 3Y \cdot \eta$  for any  $Y \in \Gamma(TM)$ . Thus we get that

$$\not{D}(f\eta + X \cdot \eta) = \left(\frac{7}{2}f + d^* X\right)\eta + \left(\nabla f + \text{curl } X - \frac{5}{2}X\right) \cdot \eta$$

which we write as

$$\not{D}(f, X) = \left(\frac{7}{2}f + d^* X, \nabla f + \text{curl } X - \frac{5}{2}X\right). \quad (4.3.2)$$

**Proposition 4.3.1.** *The Dirac operator  $\mathbb{D}$  is a first-order differential operator on  $\mathcal{S}(M)$  defined as follows. Let  $s = (f, X) \in \Gamma(\mathcal{S}(M))$ . Then*

$$\mathbb{D}(f, X) = \left( \frac{7}{2}f + d^*X, \nabla f + \text{curl } X - \frac{5}{2}X \right). \quad (4.3.3)$$

*The Dirac operator is formally self-adjoint, that is,  $\mathbb{D}^* = \mathbb{D}$  and is also an elliptic operator.*

Consider the *Dirac Laplacian*  $\mathbb{D}^2$ . We relate it to the Hodge Laplacian in the following.

**Proposition 4.3.2.** *Let  $s = (f, X)$  be a section of the spinor bundle  $\mathcal{S}(M)$ . Then*

$$\mathbb{D}^2(f, X) = \left( \Delta_d f + \frac{49}{4}f + d^*X, \Delta_d X + \text{curl } X + \frac{25}{4}X + \nabla f \right). \quad (4.3.4)$$

*Thus  $\mathbb{D}^2$  is equal to the Hodge Laplacian up to lower order terms.*

*Proof.* Using Corollary 4.2.4, we calculate

$$\begin{aligned} \mathbb{D}^2(f, X) &= \left( \frac{7}{2} \left( \frac{7}{2}f + d^*X \right) + d^* \left( \nabla f + \text{curl } X - \frac{5}{2}X \right), \right. \\ &\quad \left. d \left( \frac{7}{2}f + d^*X \right) + \text{curl} \left( \nabla f + \text{curl } X - \frac{5}{2}X \right) - \frac{5}{2} \left( \nabla f + \text{curl } X - \frac{5}{2}X \right) \right) \\ &= \left( \Delta_d f + \frac{49}{4}f + d^*X, \Delta_d X + \text{curl } X + \frac{25}{4}X + \nabla f \right) \end{aligned}$$

which proves (4.3.4). □

We need a modification of the Dirac operator defined above. The spinor bundle  $\mathcal{S}(M)$  is isomorphic to  $\Lambda_1^0 \oplus \Lambda_7^1$  and hence, via a  $G_2$ -equivariant isomorphism, it is also isomorphic to  $\Lambda_1^3 \oplus \Lambda_7^3$ . We define the **modified Dirac operator**, which we denote by  $D$ , as follows. Consider the map

$$\begin{aligned} D : \Omega_1^0 \oplus \Omega_7^1 &\longrightarrow \Omega_1^3 \oplus \Omega_7^3 \\ (f, X) &\mapsto \frac{1}{2} * d(f\varphi) + \pi_{1 \oplus 7}(d(X \lrcorner \varphi)). \end{aligned}$$

Using Lemma 4.2.7 (4) with  $\tau_0 = 4$ , we get

$$D(f, X) = \left( 2f - \frac{3}{7}d^*X, \frac{1}{2}df + 6X + \text{curl } X \right). \quad (4.3.5)$$

**Remark 4.3.3.** This operator  $D$  was defined in [KL20] and denoted in that earlier work by  $\check{D}$ .

We find the kernel of  $D$ . Let  $(f, X) \in \Omega^0 \oplus \Omega^1$  be in the kernel of  $D$ . Then

$$\begin{aligned} 2f - \frac{3}{7}d^*X &= 0, \\ \frac{1}{2}df + 6X + \text{curl } X &= 0. \end{aligned}$$

Taking  $d^*$  of the second equation and using the first equation and equation (4.2.9), we get

$$\Delta f = d^*df = -2d^*\text{curl } X - 12d^*X = -56f.$$

Since  $\Delta$  is a non-negative operator,  $f = 0$ . For  $X$ , we have

$$d^*X = 0 \quad \text{and} \quad \text{curl } X = -6X.$$

We want to prove that  $X$  is a Killing vector field. Let  $dX = Y \lrcorner \varphi + \pi_{14}(dX)$ . Then

$$\begin{aligned} dX \wedge \psi &= (Y \lrcorner \varphi) \wedge \psi \\ &= 3 * Y. \end{aligned}$$

Therefore  $\pi_7(dX) = \frac{1}{3} * (dX \wedge \psi) \lrcorner \varphi = \frac{1}{3}(\text{curl } X) \lrcorner \varphi = -2X \lrcorner \varphi$ . From Lemma 4.2.5 (2), we have

$$\begin{aligned} \int_M dX \wedge dX \wedge \varphi &= -2\|2X \lrcorner \varphi\|^2 + \|\pi_{14}(dX)\|^2 = -8\langle X \lrcorner \varphi, X \lrcorner \varphi \rangle + \|\pi_{14}(dX)\|^2 \\ &= -8\langle X, *((X \lrcorner \varphi) \wedge \psi) \rangle + \|\pi_{14}(dX)\|^2 = -24\|X\|^2 + \|\pi_{14}(dX)\|^2. \end{aligned}$$

On the other hand, since  $M$  is compact, using integration by parts we have

$$\int_M dX \wedge dX \wedge \varphi = \int_M X \wedge dX \wedge d\varphi = 4 \int_M X \wedge dX \wedge \psi = 4 \int_M X \wedge (-6 * X) = -24\|X\|^2.$$

Therefore,  $\pi_{14}(dX) = 0$  and  $dX = \pi_7(dX) = -2X \lrcorner \varphi$ . Now using Lemma 4.2.7 (4), along with the fact that  $X \in \ker D$ , i.e.,  $d^*X = 0$  and  $\text{curl } X = -6X$ , we get

$$0 = d(dX) = d(-2X \lrcorner \varphi) = -i_\varphi(\mathcal{L}_X g),$$

and hence  $X$  is a Killing vector field. Therefore  $\ker D$  is isomorphic to the set of Killing vector fields  $X$  such that  $\text{curl } X = -6X$ . We denote  $\ker D$  by  $\mathcal{K}$ , that is,

$$\ker D = \mathcal{K} = \{X \in \Gamma(TM) \mid \mathcal{L}_X g = 0 \text{ and } \text{curl } X = -6X\}. \quad (4.3.6)$$

**Remark 4.3.4.** Note that since  $\text{Ric}_g = 6g$  for  $\tau_0 = 4$  the above can also be proved using the identity  $\Delta X = d^*dX = -2d^*(X \lrcorner \varphi) = 12X$  and  $\Delta X = 2\text{Ric}(X)$  if and only if  $X$  is a Killing vector field.

**Remark 4.3.5.** If we also want the vector field  $X \in \mathcal{K}$  to preserve the  $G_2$ -structure, then

$$\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner d\varphi = 4X \lrcorner \psi = 0,$$

but since  $\Omega^1 \cong \Omega_7^4$ , we get  $X = 0$ . Hence the only vector fields in  $\mathcal{K}$  that preserve the  $G_2$ -structure are trivial. Note that when  $\varphi$  is a nearly  $G_2$ -structure of type-1, that is  $\dim(K\mathcal{K}) = 1$ , every Killing vector field preserves the  $G_2$ -structure and hence  $\mathcal{K} = \{0\}$ .

The motivation for defining the modified Dirac operator can be understood from the following.

Consider the following operator

$$\begin{aligned} D^+ : \Omega_1^3 \oplus \Omega_7^5 &\rightarrow \Omega_{1\oplus 7}^4 \\ (f\varphi, X \wedge \psi) &\mapsto \pi_{1\oplus 7}(d(f\varphi) + d^*(X \wedge \psi)). \end{aligned}$$

From previous calculations and Lemma 4.2.7 we know that

$$\begin{aligned} d(f\varphi) &= df \wedge \varphi + 4f\psi \in \Omega_{1\oplus 7}^4, \\ \pi_{1\oplus 7}(d^*(X \wedge \psi)) &= \frac{3}{7}(d^*X)\psi - \frac{1}{2}(\text{curl } X + 6X) \wedge \varphi. \end{aligned}$$

Thus using the isomorphism  $\Omega^0 \oplus \Omega^1 \cong \Omega_{1\oplus 7}^4$  we have

$$D^+(f\varphi, X \wedge \psi) = \left( 4f + \frac{3}{7}(d^*X), df - \frac{1}{2}(\text{curl } X + 6X) \right).$$

Doing a similar calculation as we did for  $\ker D$ , we observe that if  $(f, X) \in \ker D^+$ , then

$$\Delta f = -28f, \quad \text{curl } X = -6X \quad \implies \quad f = 0 = d^*X \quad \text{hence } X \in \mathcal{K}$$

and so  $\ker D^+ = \ker D$ . Since  $\Omega_1^3 \oplus \Omega_7^5 \cong \Omega_{1\oplus 7}^4$  and  $D, D^+$  are self-adjoint operators, we have the following identification

$$\begin{aligned} \Omega_{1\oplus 7}^4 &= \text{Im } D^+ \oplus \ker D^+ = \text{Im } D^+ \oplus \ker D \\ &= d\Omega_1^3 \oplus \pi_{1\oplus 7}(d^*\Omega_7^5) \oplus \{X \wedge \varphi | X \in \mathcal{K}\}. \end{aligned} \tag{4.3.7}$$

This is used in the following important result.

**Proposition 4.3.6.** *Let  $(M, \varphi, \psi)$  be a nearly  $G_2$  manifold. Then*

1.  $\Omega^4 = \{X \wedge \varphi | X \in \mathcal{K}\} \oplus d\Omega_1^3 \oplus d^*\Omega_7^5 \oplus \Omega_{27}^4$ , and
2. we have an  $L^2$ -orthogonal decomposition  $\Omega_{\text{exact}}^4 = \{X \wedge \varphi | X \in \mathcal{K}\} \oplus d\Omega_1^3 \oplus \Omega_{27,\text{exact}}^4$ .

*Proof.* The first part of the proposition follows from the decomposition of  $\Omega_{1 \oplus 7}^4$  in equation (4.3.7).

For the second part we note that the space  $d^*\Omega_7^5$  is  $L^2$ -orthogonal to exact 4-forms. To prove the  $L^2$ -orthogonality of the remaining summands we proceed term by term. Let  $X \in \mathcal{K}$ ,  $d(f\varphi) \in d\Omega_1^3$  and  $\gamma \in \Omega_{27}^4$ , such that  $d\alpha = X \wedge \varphi + d(f\varphi) + \beta$  for some exact 4-form  $d\alpha$ . Using the pointwise orthogonality of  $\Omega_1^4$  and  $\Omega_7^4$ , we have

$$\begin{aligned} \langle X \wedge \varphi, d(f\varphi) \rangle_{L^2} &= \langle X \wedge \varphi, df \wedge \varphi + 4f\psi \rangle_{L^2} \\ &= \langle X \wedge \varphi, df \wedge \varphi \rangle_{L^2} \\ &= 4\langle X, df \rangle_{L^2} = 4\langle d^*X, f \rangle_{L^2} = 0. \end{aligned}$$

Note that since  $X \in \mathcal{K}$ , Lemma 4.2.7 (6) implies that  $X \wedge \varphi = d\left(-\frac{1}{4}X \lrcorner \psi\right)$ , and hence is exact. Thus,  $\beta \in \Omega_{27,\text{exact}}^4$ . Let  $\beta = d\alpha_0$ . The  $L^2$ -orthogonality of  $\Omega_{27}^4$  and  $\Omega_1^4$ , along with the identity  $\varphi \wedge *d\alpha = 0$  implies

$$\begin{aligned} \langle d\alpha_0, d(f\varphi) \rangle_{L^2} &= \langle d\alpha_0, df \wedge \varphi + 4f\psi \rangle_{L^2} \\ &= \langle d\alpha_0, df \wedge \varphi \rangle_{L^2} + \langle d\alpha_0, 4f\psi \rangle_{L^2} = 0. \end{aligned}$$

The orthogonality of  $X \wedge \varphi$  and  $d\alpha_0$  follows from the  $L^2$ -orthogonality of  $\Omega_7^4$  and  $\Omega_{27}^4$ .  $\square$

Thus, from the previous proposition, we know that any 4-form  $\alpha$  on a nearly  $G_2$  manifold can be written as  $\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0$ , for some  $X \in \mathcal{K}$ ,  $f \in C^\infty(M)$ ,  $Y \in \Gamma(TM)$  and  $\alpha_0 \in \Omega_{27}^4$ .

**Remark 4.3.1.** Since for  $Y \in \mathcal{K}$ ,  $d^*(Y \wedge \psi) = 0$ , one can choose  $Y \in \mathcal{K}^{\perp L^2}$  in the previous proposition.

Thus for every 4-form  $\alpha$  there exists unique  $X \in \mathcal{K}$ ,  $Y \in \mathcal{K}^{\perp L^2}$ ,  $f \in C^\infty(M)$  and  $\alpha_0 \in \Omega_{27}^4$  such that

$$\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0.$$

### 4.3.2 Harmonic 2-forms and 3-forms on nearly $G_2$ manifolds

The above decomposition of 4-forms has a very useful application in determining the cohomology of nearly  $G_2$  manifolds. We first note that since nearly  $G_2$  manifolds are positive Einstein, it follows from the Bochner formula and Myers's theorem that any harmonic 1-form is 0 and hence  $\mathcal{H}^1(M) = \mathcal{H}^6(M) = 0$ . The next two theorems describe the degree 3, 4 and degree 2 and 5 cohomology of a nearly  $G_2$  manifold.

**Theorem 4.3.7.** *Let  $(M, \varphi, \psi)$  be a compact nearly  $G_2$  manifold. Then every harmonic 4-form lies in  $\Omega_{27}^4$ . Equivalently, every harmonic 3-form lies in  $\Omega_{27}^3$ .*

*Proof.* Let  $\alpha$  be a harmonic 4-form that is  $d\alpha = d^*\alpha = 0$ . From Lemma 4.2.12 there exists  $X \in \mathcal{K}$ ,  $f \in C^\infty(M)$ ,  $Y \in \mathcal{K}^{\perp L^2}$  and  $\alpha_0 \in \Omega_{27}^4$  such that

$$\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0.$$

Since  $X \in \mathcal{K}$  and hence  $6X = \text{curl } X$ , by Lemma 4.2.7 (6),  $d^*(X \wedge \varphi) = 4X \lrcorner \psi \in \Omega_7^3$  and since  $d(f\varphi) = df \wedge \varphi + 4f\psi \in \Omega_{1\oplus 7}^4$ , we have

$$\begin{aligned} 0 = \langle \alpha, d(f\varphi) \rangle_{L^2} &= \langle X \wedge \varphi, d(f\varphi) \rangle_{L^2} + \|d(f\varphi)\|_{L^2}^2 + \langle d^*(Y \wedge \psi), d(f\varphi) \rangle_{L^2} + \langle \alpha_0, d(f\varphi) \rangle_{L^2} \\ &= \langle d^*(X \wedge \varphi), f\varphi \rangle_{L^2} + \|d(f\varphi)\|_{L^2}^2 \\ &= \|d(f\varphi)\|_{L^2}^2. \end{aligned}$$

Thus  $d(f\varphi) = 0$  and hence  $f = 0$ .

Now,  $0 = d^*\alpha = d^*(X \wedge \varphi) + d^*\alpha_0 = 4X \lrcorner \psi + d^*\alpha_0$ . Using the identity,  $(X \lrcorner \psi) \wedge \varphi = 4 * X$  we have

$$\begin{aligned} \|d^*\alpha_0\|_{L^2}^2 &= 16 \langle X \lrcorner \psi, X \lrcorner \psi \rangle_{L^2} \\ &= 16 \langle X, *((X \lrcorner \psi) \wedge \varphi) \rangle_{L^2} = 64 \|X\|_{L^2}^2. \end{aligned}$$

On the other hand, again by Lemma 4.2.7 (6)

$$\begin{aligned} \|d^*\alpha_0\|_{L^2}^2 &= \langle d^*\alpha_0, d^*\alpha_0 \rangle_{L^2} \\ &= -4 \langle d^*\alpha_0, X \lrcorner \psi \rangle_{L^2} \\ &= -4 \langle \alpha_0, d(X \lrcorner \psi) \rangle_{L^2} = 16 \langle \alpha_0, X \wedge \varphi \rangle_{L^2} = 0, \end{aligned}$$

which implies  $X = 0$ . So  $\alpha = d^*(Y \wedge \psi) + \alpha_0$ .

Since  $d^*\alpha_0 = 0$ , applying Corollary 4.2.13 on  $\alpha_0$  implies  $d\alpha_0 \in \Omega_{14}^5$ . This identity together with the closedness of  $\alpha$  gives us

$$\begin{aligned} 0 &= \langle \alpha, d^*(Y \wedge \psi) \rangle_{L^2} = \|d^*(Y \wedge \psi)\|_{L^2}^2 + \langle \alpha_0, d^*(Y \wedge \psi) \rangle_{L^2} \\ &= \|d^*(Y \wedge \psi)\|_{L^2}^2 + \langle d\alpha_0, Y \wedge \psi \rangle_{L^2} = \|d^*(Y \wedge \psi)\|_{L^2}^2. \end{aligned}$$

as  $Y \wedge \psi \in \Omega_7^5$ . Hence  $d^*(Y \wedge \psi) = 0$  or equivalently  $Y \in \mathcal{K}$ , thus  $Y = 0$  which implies that  $\alpha = \alpha_0$  which completes the proof of the theorem.  $\square$

We also describe the degree 2 (and hence degree 5) cohomology on nearly  $G_2$  manifolds below. In combination with Theorem 4.3.7, this completely describes the cohomology of a nearly  $G_2$  manifold.

**Theorem 4.3.8.** *Let  $(M, \varphi, \psi)$  be a compact nearly  $G_2$  manifold with  $\tau_0 = 4$ . Let  $\beta$  be a 2-form with*

$$\beta = \beta_7 + \beta_{14} = (X \lrcorner \varphi) + \beta_{14} \quad \text{for some } X \in \Gamma(TM).$$

*If  $\beta$  is harmonic then  $\beta \in \Omega_{14}^2$ .*

*Proof.* Suppose  $\beta \in \Omega^2(M)$  is harmonic. Then  $d\beta = d^*\beta = 0$  and since  $d$  and  $d^*$  are linear, we have

$$d\beta_7 + d\beta_{14} = 0, \quad d^*\beta_7 + d^*\beta_{14} = 0$$

which on using Lemma 4.2.7 (3), (4) and (5) imply

$$\begin{aligned} & -\frac{3}{7}(d^*X)\varphi + \frac{1}{2} * ((6X + \text{curl } X) \wedge \varphi) + i_\varphi \left( \frac{1}{2}(\mathcal{L}_X g) + \frac{1}{7}(d^*X)g \right) \\ & + \frac{1}{4} * (d^*\beta_{14} \wedge \varphi) + \pi_{27}(d\beta_{14}) = 0 \end{aligned}$$

and

$$d^*\beta_{14} = \text{curl } X.$$

Thus we get

$$-\frac{3}{7}(d^*X)\varphi + \frac{1}{2} * ((6X + \text{curl } X + \frac{1}{2} \text{curl } X) \wedge \varphi)$$

$$+ i_\varphi \left( \frac{1}{2}(\mathcal{L}_X g) + \frac{1}{7}(d^* X)g \right) + \pi_{27}(d\beta_{14}) = 0$$

and so

$$d^* X = 0, \quad \text{curl } X = -4X \quad \text{and} \quad \frac{1}{2}(\mathcal{L}_X g) + \pi_{27}(d\beta_{14}) = 0. \quad (4.3.8)$$

Now  $\text{curl } X = -4X$ , so taking curl of both sides and using (4.2.11) with  $d^* X = 0$ , we get

$$\Delta_d X - 4 \text{curl } X = -4 \text{curl } X \quad \implies \quad \Delta_d X = 0.$$

Thus  $X$  is harmonic. Since nearly  $G_2$  manifolds are positive Einstein, it follows from Bochner formula and Myers's theorem that  $X = 0$ . Hence  $\beta = \beta_{14} \in \Omega_{14}^2$ .  $\square$

**Remark 4.3.9.** Theorem 4.3.8 was also proved in a very different way in [BO19, Remark 15]. The theorem has the following interesting interpretation in the context of  $G_2$  instantons on a nearly  $G_2$  manifold, as already described in [BO19, Corollary 14]. For any  $\alpha \in H^2(M, \mathbb{Z})$ , by Theorem 4.3.8, there is a unique  $G_2$ -instanton on a complex line bundle  $L$  with  $c_1(L) = \alpha$ .

**Remark 4.3.10.** Theorem 4.3.7 also follows from the description of nearly  $G_2$  manifolds using Killing spinors which is based on an old result of Hijazi saying that the Clifford product of a harmonic form and a Killing spinor vanishes. The methods can be used to investigate the cohomology class for any degree and on manifolds with *any*  $G_2$ -structure (not necessarily nearly  $G_2$ ) with suitable modifications.

## 4.4 Classification of homogeneous nearly $G_2$ manifolds

In [FKMS97] Friedrich–Kath–Moroianu–Simmelmann classify all the compact, simply connected homogeneous nearly  $G_2$  manifolds. Any homogeneous nearly  $G_2$  manifold is one of the six manifolds listed in Table 4.1. We describe the homogeneous structure on each of these spaces.

$(S^7, g_{\text{round}}) = \text{Spin}(7)/G_2,$	$(S^7, g_{\text{squashed}}) = \frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)},$	$\text{SO}(5)/\text{SO}(3),$
$M(3, 2) = \frac{\text{SU}(3) \times \text{SU}(2)}{\text{U}(1) \times \text{SU}(2)},$	$N(k, l) = \text{SU}(3)/S_{k,l}^1,$	$Q(1, 1, 1) = \text{SU}(2)^3/\text{U}(1)^2.$

Table 4.1: Homogeneous nearly  $G_2$  manifolds



- In the round  $S^7$  the embedding of  $G_2$  in  $\text{Spin}(7)$  is obtained by lifting the standard embedding of  $G_2$  into  $\text{SO}(7)$ .
- For the squashed metric on  $S^7$ , the two copies of  $\text{Sp}(1)$  in  $\text{Sp}(2) \times \text{Sp}(1)$  denoted by  $\text{Sp}(1)_u$  and  $\text{Sp}(1)_d$  [AS12] are

$$\text{Sp}(1)_u := \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) : a \in \text{Sp}(1) \right\}, \quad \text{Sp}(1)_d := \left\{ \left( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, a \right) : a \in \text{Sp}(1) \right\}.$$

- For the Berger space  $\frac{\text{SO}(5)}{\text{SO}(3)}$ , the Lie group  $\text{SO}(3)$  is embedded into  $\text{SO}(5)$  via the 5 dimensional irreducible representation of  $\text{SO}(3)$  on  $\text{Sym}_0^2(\mathbb{R}^3)$ .
- For  $\frac{\text{SU}(3) \times \text{SU}(2)}{\text{U}(1) \times \text{SU}(2)}$  the embedding of  $\text{SU}(2)$  (denoted by  $\text{SU}(2)_d$ ) and  $\text{U}(1)$  in  $\text{SU}(2) \times \text{SU}(2)$  is defined as [AS12]

$$\text{SU}(2)_d := \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \right) : a \in \text{SU}(2) \right\}, \quad \text{U}(1) := \left\{ \left( \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}, 1 \right) : \theta \in \mathbb{R} \right\}$$

- For the Aloff–Wallach spaces  $N_{k,l}$  the labels  $k, l$  are coprime positive integers and the embedding of  $S_{k,l}^1 = \text{U}(1)_{k,l}$  in  $\text{SU}(3)$  is described by

$$S_{k,l}^1 := \left\{ \left( \begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{-i(k+l)\theta} \end{pmatrix}, \theta \in \mathbb{R} \right) \right\}.$$

- For  $Q(1,1,1)$ , we denote the two copies of  $\text{U}(1)$  inside  $\text{SU}(2)^3$  as  $\text{U}(1)_u, \text{U}(1)_d$  with respective embeddings given by

$$\begin{aligned} \text{U}(1)_u &= \text{Span} \left\{ \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \text{I}_2 \right), \theta \in \mathbb{R} \right\}, \\ \text{U}(1)_d &= \text{Span} \left\{ \left( \text{I}_2, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \right), \theta \in \mathbb{R} \right\}. \end{aligned}$$

The first four homogeneous spaces are normal that is, for those the nearly  $G_2$  metric  $g$  on  $G/H$  is related to the Killing form  $B$  of  $G$  by  $g = -\frac{3}{40}B$ . The choice of the scalar constant  $\frac{3}{40}$  is based on our convention  $\tau_0 = 4$ . The general formula for the constant was derived in [AS12, Lemma 7.1]. For the remaining two homogeneous spaces the nearly  $G_2$  metric is not a scalar multiple of the Killing form of  $G$  (see [Wil99]).

$(S^7, g_{ground}) \cong \text{Spin}(7)/G_2,$	$(S^7, g_{squashed}) \cong \frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)},$
$\text{SO}(5)/\text{SO}(3),$	$M(3, 2) \cong \frac{\text{SU}(3) \times \text{SU}(2)}{\text{U}(1) \times \text{SU}(2)}.$

Table 4.2: Normal homogeneous nearly  $G_2$  manifolds

Let  $\mathfrak{m}$  be the orthogonal complement of the Lie algebra  $\mathfrak{h}$  of  $H$  in  $\mathfrak{g}$  with respect to  $g$ . Then  $\mathfrak{m}$  is invariant under the adjoint action of  $\mathfrak{h}$ . That is,  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ , and thus all the homogeneous spaces listed in Table 4.1 are naturally reductive. The reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  equips the principal  $H$ -bundle  $G \rightarrow G/H$  with a  $G$ -invariant connection whose horizontal spaces are the left translates of  $\mathfrak{m}$ . This connection is known as the characteristic homogeneous connection. On homogeneous nearly  $G_2$  manifolds the characteristic homogeneous connection has holonomy contained in  $G_2$ . If we denote by  $Z_{\mathfrak{m}}$  the projection of  $Z \in \mathfrak{g}$  on  $\mathfrak{m}$ , the torsion tensor  $T$  for any  $X, Y \in \mathfrak{m}$  is given by

$$T(X, Y) = -[X, Y]_{\mathfrak{m}},$$

and is totally skew-symmetric. Thus by the uniqueness result in [CS04] it is the canonical connection with respect to the nearly  $G_2$ -structure on  $G/H$  [HN12].

# Chapter 5

## Gauge theory on manifolds with $G_2$ -structure

### 5.1 Background on gauge theory

Gauge theory is the study of bundles (vector and principal) and objects on them. It has led to some remarkable results in both mathematics and physics over the last four decades. However, most of the effort has concentrated on gauge theory in dimensions  $d = 2, 3, 4$ . Moreover, in the past few years a great deal of work in this area evolved around intricate interactions between such theories.

In four dimensions, since  $*^2 = \text{id}$  on 2-forms, one can decompose  $F_A$  into the  $\pm 1$  eigenspaces of the Hodge star operator denoted by  $F_A^+$  and  $F_A^-$  respectively. The component  $F_A^+$  is self-dual (SD) whereas  $F_A^-$  is anti-self-dual (ASD). A connection  $A$  is called an instanton if  $F_A$  is self-dual or anti-self-dual. In higher dimensions, the instanton equation on  $M^n$  can be introduced as follows. Assuming there is a closed  $(n - 4)$ -form  $\Omega$  on  $M$ , a connection  $A$  is an instanton if the curvature  $F_A$  satisfies the equation

$$F_A = *(F_A \wedge \Omega).$$

In this chapter we review the framework for studying gauge theory on the 7 dimensional manifolds with a  $G_2$ -structure. On a  $G_2$  manifold  $(M, \varphi)$  the instanton equation becomes

$$F_A = *(F_A \wedge \varphi).$$

There are many equivalent ways to define a  $G_2$  instanton which we describe in Section 5.2. More specifically we see that the canonical connection we defined in (4.1.11) on homogeneous nearly  $G_2$  manifolds satisfies the  $G_2$  instanton condition. With our convention the  $G_2$  instantons are analogues of the SD connections in 4 dimensions.

Gauge theories in lower dimensions have been hugely successful in producing the well-known theories of Casson in dimension 2, Floer in dimension 3, and Donaldson theory in 4 dimensions. In 4-dimensional geometry the moduli space of anti-self-dual (ASD) instantons was used to assign numerical invariants to smooth 4-manifolds. In 1983, Donaldson [Don83] gave strong restrictions on the intersection form of a differentiable 4-manifold using gauge theory. It is conjectured that one may employ similar methods to construct numerical invariants of manifolds with special holonomy such as  $G_2$  and  $\text{Spin}(7)$ , a hope that was expressed in [DT98], [DS11].

Although the quest for producing invariants has deeply motivated the study of gauge theory in dimension greater than 4, gauge theory in higher dimensions has long been a subject of interest for both mathematicians and physicists. One of motivations to consider higher dimensional instantons comes from recent developments in string dualities and M-theory, where one can obtain low energy effective (supersymmetric) gauge theories in various dimensions. The main motivation for studying ASD connections in 4 dimensions came from the fact that they are minimizers for the Yang–Mills functional,  $YM(A) = \|F_A\|^2$ . The Yang–Mills connections which locally minimize the functional are known as *stable* Yang–Mills connection. In higher dimensions significant results regarding the instability of Yang–Mills connections came from Bourguignon–Lawson–Simons for spheres of dimension greater than four in 1971 [BLS79] and by Bourguignon–Lawson on quotients of  $S^n$  and more general homogeneous spaces in 1981 [BL81]. In [Ste10] these results were extended and generalized significantly to several situations. Huang in [Hua17] showed that energy minimizing Yang–Mills connections on a compact torsion-free  $G_2$  manifold are  $G_2$  instantons under certain assumptions. The  $G_2$  and  $\text{Spin}(7)$  instanton equations on  $\mathbb{R}^7$  and  $\mathbb{R}^8$  respectively were introduced by Corrigan, Devchand, Fairlie and Nuyts in [CDFN83] and for a general Riemannian manifold by Carrion in [Rey98]. Since then much research has been done on gauge theory in higher dimensions and many examples of instantons in higher dimensions have been generated.

- $\text{Spin}(7)$  instantons on  $\mathbb{R}^8$  were first constructed by Fubini–Nicolai [FN85], and Fairlie–Nuyts [FN84] around the same time derived spherically symmetric solutions of the  $\text{SO}(7)$  and  $\text{SO}(8)$  Yang–Mills equations in an eight-dimensional Euclidean space.
- In [Wal13] Walpuski introduced a method to construct  $G_2$  instantons on  $G_2$  manifolds arising from Joyce’s generalised Kummer construction.

- In [SW15] Sá Earp–Walpuski developed a method to construct  $G_2$  instantons on special types of  $G_2$  manifolds known as the *twisted connected sums* constructed by Kovalev in [Kov03]. Later in [Wal16] Walpuski used the algorithm to construct an example of a  $G_2$  instanton on a special type of twisted connected sum  $G_2$  manifold.
- Clarke in [Cla14] gave a construction of  $G_2$  and  $\text{Spin}(7)$  instantons on exceptional holonomy manifolds constructed by Bryant and Salamon.
- On a  $G_2$  manifold  $(M, \varphi, \psi)$  a  $G_2$  monopole is defined by a pair  $(A, \Phi)$  of a connection  $A$  and Higgs field  $\Phi \in \Omega^0(M)$  which satisfies the gauge theoretic equation

$$F_A \wedge \psi = * \nabla_A \Phi.$$

When  $\Phi = 0$  the monopole equation reduces to the  $G_2$  instanton equation. Oliveira in [Oli14b] studied monopoles on torsion-free  $G_2$  manifolds and as a special case found explicit irreducible  $G_2$  instantons on  $\Lambda_2^-(S^4)$  and on  $\Lambda_2^-(\mathbb{C}\mathbb{P}^2)$ .

- Lotay–Oliveira in [LO18] constructed, and in some cases classified, symmetric instantons with gauge groups  $U(1)$  and  $SU(2)$  on various complete and non-compact  $G_2$  manifolds admitting actions of  $SU(2)^2$ .
- In [MNE21] Ménet–Sá Earp–Nordström constructed examples of  $G_2$  instantons on twisted connected sum  $G_2$  manifolds.
- In [BO19] Ball–Oliveira proved the existence of nearly  $G_2$  instantons on certain Aloff–Wallach spaces and classified invariant  $G_2$  instantons on these spaces with gauge group  $U(1)$  and  $SO(3)$ .
- Waldron [Wal20] proved that the pullback of the standard instanton on  $S^7$  obtained from ASD instantons on the 4-sphere via the quaternionic Hopf fibration lies in a smooth, complete, 15-dimensional family of  $G_2$  instantons.

In [DT98] Donaldson and Thomas suggested that invariants of manifolds with special holonomy may be defined using instanton equations in higher dimensions. Later in [DS11] some of the technicalities of defining such invariants were studied. They also observed that the naive count of  $G_2$  instantons on a compact  $G_2$  manifold cannot produce a deformation-invariant number but rather this number jumps in a finite number of points as one changes the  $G_2$  metric in a one parameter family. A partial result in overcoming this difficulty was presented in [Hay12] and [HW15] where it was suggested that a suitable combination

of counts of  $G_2$  instantons and solutions of some gauge theoretic equations on associative submanifolds yield an invariant of  $G_2$ -manifolds.

Due to these suggestions of potentially defining an invariant, progress in gauge theory on 7 dimensional manifolds with a  $G_2$ -structure is important in both mathematics and physics. For a possible counting invariant the study of the moduli space of  $G_2$  instantons is of utmost importance. In this thesis we are mostly concerned with gauge theory on complete nearly  $G_2$  manifolds. Such manifolds are compact by Myers's theorem. In Chapter 7 we discuss the infinitesimal deformations of the canonical connection on homogeneous nearly  $G_2$  manifolds. The result gives us an idea of when one can possibly deform the canonical connection to get a family of  $G_2$  instantons. Since the cone over nearly  $G_2$  manifolds has holonomy  $\text{Spin}(7)$ , the deformation theory of instantons on nearly  $G_2$  manifolds is directly related to  $\text{Spin}(7)$  instantons.

We begin with a brief description of gauge theory in general and more specifically in the  $G_2$  setting. Let  $(M^n, g)$  be a Riemannian manifold, let  $\mathcal{P} \rightarrow M$  be a principal  $G$ -bundle, and let  $A$  be a connection on  $\mathcal{P}$ . We denote by  $\text{Ad}_{\mathcal{P}}$  the adjoint bundle associated to  $\mathcal{P}$ , that is  $\text{Ad}_{\mathcal{P}} = \mathcal{P} \times_{\text{Ad}} \mathfrak{g}$ . Since the difference of two connections is a Lie algebra  $\mathfrak{g}$ -valued 1-form the space of connections  $\mathcal{A}$  on  $\mathcal{P}$  is an affine space isomorphic to  $\Gamma(T^*M \otimes \text{Ad}_{\mathcal{P}})$ . For decomposable  $\mathfrak{g}$ -valued forms  $\alpha \otimes a, \beta \otimes b \in \Lambda^* \otimes \mathfrak{g}$ , let

$$[(\alpha \otimes a) \wedge (\beta \otimes b)] = \alpha \wedge \beta \otimes [a, b]$$

Let  $A$  be a connection 1-form on  $\mathcal{P}$  and  $\nabla_A = d + A$  be the covariant derivative in a local trivialization. Then the curvature  $F_A$  associated to  $A$  is given by

$$F_A = dA + \frac{1}{2}[A \wedge A]. \tag{5.1.1}$$

Thus  $F_A \in \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}})$  is a  $\text{Ad}_{\mathcal{P}}$ -valued 2-form known as the curvature 2-form. The gauge group  $\mathcal{G}$  of  $\mathcal{P}$  is the group of vertical bundle automorphisms of  $\mathcal{P}$ . It acts on the space of connections  $A$  via the action

$$g \cdot A = gAg^{-1} - (dg)g^{-1}.$$

One can easily check that  $F_{g \cdot A} = gF_Ag^{-1}$ . For a connection  $A$  on  $\mathcal{P}$ , the flatness condition

$$F_A = 0$$

is clearly gauge-invariant. In gauge theory we deal with connections satisfying some gauge-invariant condition on the curvature. Thus for a systematic study of gauge invariant

equations it is more appropriate to work with the moduli space of connections modulo the gauge group  $\mathcal{M} = \mathcal{A}/\mathcal{G}$ .

When the structure group  $G$  is compact and semisimple the Killing form on  $G$  defines an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . Using this inner product one can define the wedge product between a  $\mathfrak{g}$ -valued  $p$ -form  $\alpha$  and  $q$ -form  $\beta$  by wedging the differential forms part and taking the inner product of the Lie algebra part thus  $\langle \alpha \wedge \beta \rangle \in \Omega^{p+q}(M)$ . Using this we can define the Yang–Mills energy functional on  $\mathcal{A}$  as

$$YM(A) = \int_M |F_A|^2 \text{vol} = \int_M \langle F_A \wedge *F_A \rangle. \quad (5.1.2)$$

Since  $YM$  is clearly gauge invariant it descends to a functional on  $\mathcal{M}$ . The Euler-Lagrange equation of the Yang–Mills functional is the Yang–Mills equation

$$d_A^* F_A = 0. \quad (5.1.3)$$

The connections that solve the Yang–Mills equation are known as *Yang–Mills connections*. Flat connections are trivially Yang–Mills connections.

In dimensions  $n > 4$  Tian, in [Tia00] showed that the presence of a closed  $(n - 4)$ -form  $\Omega$  on  $M$  gives a criterion for finding classes of solutions to (5.1.3), which suggests a generalisation to the concept of instanton. Tian introduces a very general type of anti-self-duality equation to be studied for connections  $A$  on an  $n$ -manifold  $M$ . Tian’s equation is a first order equation (algebraic in the curvature  $F_A$ ), dependent on a closed  $(n - 4)$ -form  $\Omega$  on  $M$ , which implies the second order Yang–Mills equations. For suitable choices of  $\Omega$  these  $\Omega$ -ASD equations include the standard ASD equations, the Hermitian–Yang–Mills (HYM) equations, and the higher-dimensional equations of [DT98].

One can also define the analogue of the Chern–Simons theory originally introduced in 3-dimensions. On a bundle  $\mathcal{P}$  over a compact 3-manifold  $Y$  the Chern–Simons functional is a circle-valued real function on the moduli space  $\mathcal{M}$ . It has the remarkable property that its critical points are precisely the flat connections on  $\mathcal{P}$  modulo gauge. Recall that  $\mathcal{A}$  is an affine space modelled on  $\Omega^1(Y, \text{Ad}_{\mathcal{P}})$ . So, if we fix a reference connection  $A_0 \in \mathcal{A}$ , we can write

$$\mathcal{A} = A_0 + \Omega^1(Y, \text{Ad}_{\mathcal{P}}).$$

In particular the total space of the tangent bundle  $T\mathcal{A} = \mathcal{A} \times \Omega^1(Y, \text{Ad}_{\mathcal{P}})$  and thus vectors in  $T\mathcal{A}$  can be locally thought of as  $\mathfrak{g}$ -valued 1-forms on  $M$ . Thus associated to a connection  $A$  we can define a 1-form  $\rho_A$  on  $\mathcal{A}$  by

$$\rho_A(a) := \int_Y \text{tr}(F_A \wedge a).$$

Since for  $b \in \Omega^1(Y, \mathfrak{g})$ ,  $F_{A+b} = F_A + d_A b + b \wedge b$  we have the first order difference

$$\rho_{A+b}(a) - \rho_A(a) = \int_Y \text{tr}(d_A b \wedge a) + \mathcal{O}(|a| \cdot |b|^2).$$

Moreover it is easy to verify that the above difference is symmetric in  $a, b$  thus one can prove that  $\rho$  is closed. See [Sá 09, Section 1.3.2] for a complete proof. Since  $A$  is affine and hence contractible the closedness of  $\rho$  implies the existence of a functional  $\nu$  such that  $\rho = d\nu$ . One can find  $\nu$  explicitly by integrating  $\rho$  over paths  $A(t) = A_0 + ta$ , from  $A_0$  to any  $A = A_0 + a$ . On  $[A] = [A_0 + a] \in \mathcal{M}$

$$\nu([A(t)]) = \frac{1}{2} \int_Y \text{tr} \left( dA(t) \wedge A(t) + \frac{2}{3} A(t) \wedge A(t) \wedge A(t) \right). \quad (5.1.4)$$

The  $\nu$  defined above is the *Chern–Simons functional*. Since  $\partial T = \emptyset$  the derivative of  $\nu$  at  $t = 0$  is given by,

$$\left. \frac{d}{dt} \right|_{t=0} \nu([A(t)]) = \int_Y \text{tr}(a \wedge F_{A_0}).$$

Thus the critical points of the Chern–Simons functional in dimension 3 are the flat connections on  $\mathcal{P} \rightarrow Y$ .

In the rest of the chapter we review some standard material on  $G_2$  gauge theory. Since  $G_2$  manifolds are spinnable the theory can be formulated in terms of spin geometry as well.

## 5.2 $G_2$ Gauge theory

Let  $(M^7, \varphi, \sigma)$  be a manifold with a  $G_2$ -structure where  $\varphi$  is the positive 3 form and  $\sigma$  is a unit real spinor associated to  $\varphi$ . As before we denote by  $g$  the induced metric,  $*$  the induced Hodge star and  $\psi = *\varphi$  the dual 4 form. Let  $\mathcal{P} \rightarrow M$  be a principal  $K$ -bundle. We denote by  $\text{Ad}_{\mathcal{P}}$  the adjoint bundle associated to  $\mathcal{P}$ . Let  $A$  be a connection 1-form on  $\mathcal{P}$  and  $F_A \in \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}})$  be the curvature 2-form associated to  $A$  as defined in (5.1.1).

**Definition 5.2.1.** A connection  $A$  on  $\mathcal{P}$  is a  $G_2$  instanton if it satisfies

$$F_A \wedge \varphi = *F_A. \quad (5.2.1)$$



The  $G_2$  instanton condition (5.2.1) can be thought of as a 7-dimensional version of the self-duality (or anti-self-duality) condition familiar from dimension four. When the  $G_2$ -structure is torsion-free and  $A$  is a  $G_2$  instanton, we can use the Bianchi identity  $d_A F_A = 0$  and the condition  $d\varphi = 0$  to get  $d_A^* F_A = *d_A *F_A = *d_A(F_A \wedge \varphi) = *(d_A F_A \wedge \varphi + F_A \wedge d\varphi) = 0$ . Thus on a torsion-free  $G_2$  manifold  $G_2$  instantons are Yang–Mills.

For a manifold with a  $G_2$ -structure, using the 2-form decomposition (2.2.1), (2.2.2) and their  $L^2$  orthogonality the values of  $YM(A)$  can be related to a functional  $\kappa_{\mathcal{P}}(A)$  as follows:

$$\begin{aligned} \kappa_{\mathcal{P}}(A) &= \int_M \text{tr}(F_A^2) \wedge \varphi \\ &= \int_M \langle F_A \wedge F_A \wedge \varphi \rangle \\ &= \langle F_A, -2 * \pi_7(F_A) + * \pi_{14}(F_A) \rangle_{L^2} \\ &= -2 \|\pi_7(F_A)\|^2 + \|\pi_{14}(F_A)\|^2. \end{aligned}$$

Thus we have

$$YM(A) = \|\pi_7(F_A)\|^2 + \|\pi_{14}(F_A)\|^2 = 3\|\pi_7(F_A)\|^2 + \kappa_{\mathcal{P}}(A) = \frac{1}{2}(3\|\pi_{14}(F_A)\|^2 - \kappa_{\mathcal{P}}(A)).$$

**Remark 5.2.1.** If we denote by  $p_1(\mathcal{P})$  the first Pontryagin class of  $\mathcal{P}$  then for  $\varphi$  closed  $8\pi^2 \kappa_{\mathcal{P}}(A) = \langle p_1(\mathcal{P}) \cup [\varphi], [M] \rangle$  is a topological invariant independent of  $A$  depending only on the principal bundle. It is then clear that  $YM$  attains its minimum at a connection whose curvature is either in  $\Lambda_7^2$  or  $\Lambda_{14}^2$ . Moreover, since  $YM \geq 0$ , the sign of  $\kappa_{\mathcal{P}}$  obstructs the existence of one type or the other. Fixing  $\kappa_{\mathcal{P}} \geq 0$  we can say that  $G_2$  instantons minimize the Yang–Mills energy. These facts motivate our interest in the  $G_2$  instanton equation.

There are many ways to define the general instanton condition on  $A$ . If  $(M, g)$  is equipped with a  $G$ -structure such that  $G \subset O(n)$ , there is a subbundle  $\mathfrak{g}(T^*M) \subset \Lambda^2 T^*M$  whose fibre is isomorphic to  $\mathfrak{g} = \text{Lie}(G)$ . The connection  $A$  is an instanton if the 2-form part of  $F_A$  belongs to  $\mathfrak{g}(T^*M)$ . In global terms,  $A$  is an instanton if

$$F_A \in \Gamma(\mathfrak{g}(T^*M) \otimes \text{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}}).$$

Note that in dimension 7 if  $M$  is equipped with a  $G_2$ -structure then this condition implies that  $A$  is an instanton if the 2-form part of  $F_A \in \mathfrak{g}_2(T^*M) = \Gamma(\Lambda_{14}^2)$ .

The second definition of an instanton is a special case of the first when the Lie algebra  $\mathfrak{g}$  is simple. Its quadratic Casimir is a  $G$ -invariant element of  $\mathfrak{g} \otimes \mathfrak{g}$  which may be identified

with a section of  $\Lambda^2 \otimes \Lambda^2$  and thus with a 4-form  $Q$  by taking a wedge product. Since this  $Q$  is  $G$ -invariant the operator  $u \rightarrow *(*Q \wedge u)$  acting on 2-forms commutes with the action of  $G$  and hence by Schur's Lemma the irreducible representations of  $G$  in  $\Lambda^2$  are eigenspaces of the operator. Then  $F_A$  is an instanton if

$$*(*Q \wedge F_A) = \nu F_A. \quad (5.2.2)$$

for some  $\nu \in \mathbb{R}$ . In dimension 7 it turns out that  $Q = \psi$  (see [HN12]) and the above condition is equivalent to  $F_A \in \Gamma(\Lambda_{14}^2)$  when  $\nu = 1$ .

Finally if  $M$  is a spin manifold, and the spinor bundle admits one or more non-vanishing spinors  $\sigma$ , then  $A$  is an instanton if

$$F_A \cdot \sigma = 0.$$

When  $M$  has a  $G_2$ -structure and  $\sigma$  is the corresponding spinor then  $\sigma$  is preserved by the action of the Lie group  $G_2$ . Thus  $F_A \cdot \sigma = 0$  if and only if  $A$  is a  $G_2$  instanton. An interested reader can read further on these definitions and their relations in [HN12].

We remark that for an instanton  $A$  on a manifold with a  $G_2$ -structure  $\varphi$  all the above definitions are equivalent. They all imply that the curvature  $F_A$  associated to  $A$  lies in  $\Gamma(\Lambda_{14}^2)$  and thus satisfies *all* of these equivalent conditions:

$$\begin{aligned} F_A \cdot \eta &= 0, \\ F_A \wedge \varphi &= *F, \\ F_A \wedge \psi &= 0, \\ F_A \lrcorner \varphi &= 0. \end{aligned} \quad (5.2.3)$$

From now on in this thesis we use these instanton conditions interchangeably according to the context without further specification. Note that the above definitions are valid for any general  $G_2$ -structure and not only for nearly parallel ones.

On a manifold with real Killing spinors it was shown in [HN12] that instantons solve the Yang–Mills equation. In the case of a nearly  $G_2$  instanton we can prove this fact by direct computation. For an instanton  $A$ , (5.2.3) and the Bianchi identity implies

$$\begin{aligned} (d^A)^* F_A &= *d^A * F_A \\ &= *d^A(\varphi \wedge F_A) \\ &= 4 *(\psi \wedge F_A) = 0. \end{aligned}$$

Thus on closed and nearly  $G_2$  manifolds,  $G_2$  instantons are Yang–Mills. It is then a natural question to ask if they are local minimizers (maximizers) of the Yang–Mills functional. However the  $G_2$  instantons corresponding to a nearly  $G_2$ -structure are not the minimizers of the Yang–Mills functional. In [BO19] Ball–Oliveira answered the question in the negative by providing an example of a nearly parallel  $G_2$ -structure, together with a  $G_2$  instanton which is unstable as a Yang–Mills connection.

Moreover, when the  $G_2$ -structure  $\varphi$  is closed (not necessarily co-closed), it follows from Remark 5.2.1 that  $G_2$  instantons are in fact absolute minima of the Yang–Mills functional.

**Example 5.2.2.** A flat connection is trivially a  $G_2$  instanton.

**Example 5.2.3.** If  $\varphi$  is torsion-free then the Levi-Civita connection associated to  $\varphi$  is a  $G_2$  instanton. To see that, observe that at each point since  $\text{Hol}(g) \subset G_2$  we can think of the Riemannian curvature tensor  $R$  as an element of  $S^2 \mathfrak{g}_2 \subset \Lambda^2 \otimes \mathfrak{gl}(7)$ . But then it follows from (2.2.2) that  $*(R \wedge \varphi) = R$ .

**Example 5.2.4.** As proved by [HN12] the canonical connection on nearly  $G_2$  manifolds defined in Chapter 4 is a  $G_2$  instanton. In Chapter 7 we describe the infinitesimal deformation space of this  $G_2$  instanton for the four normal homogeneous nearly  $G_2$  metrics.

In the presence of a suitable closed  $(n - 3)$ -form a theory similar to the Chern–Simons theory in 3-dimensions can be formulated in higher dimensions [DT98], [Tho97]. Our case of interest is  $n = 7$ . When the  $G_2$ -structure is co-closed that is the 4-form  $\psi$  is closed one can define a Chern–Simons type functional on the space of connection similar to (5.1.4). If we choose  $A = A_0 + a$  for a  $\mathfrak{g}$ -valued 1-form  $a$  and fixed connection  $A_0$  for which  $F_{A_0} \in \Omega_{14}^2(M, \mathfrak{g})$  we have

$$\nu(A) = \frac{1}{2} \int_Y \text{tr} \left( d_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \psi.$$

On a manifold with a co-closed  $G_2$ -structure the Chern–Simons functional  $\nu$  arises analogously to the 3 dimensional case. Here the 1-form  $\rho$  on the space of connection  $\mathcal{A}$  is defined by

$$\rho_A(a) := \int_Y \text{tr}(F_A \wedge a) \wedge \psi.$$

When the  $G_2$ -structure is co-closed it follows that  $\rho$  is closed and the arguments of the 3-dimensional Chern–Simons theory holds true here as well. In particular the functional  $\nu$  descends to the moduli space  $\mathcal{M}$ .

The possibility of using Chern-Simons theory on co-closed  $G_2$ -structures which includes the nearly  $G_2$  case amplifies our hope of being able to define invariants for  $G_2$  manifolds similar to the Jones polynomial on knots by Witten [Wit89] and the Casson invariant on Calabi–Yau 3-folds by Thomas [Tho97]. Recently, Ball–Oliveira in [BO19] have constructed homogeneous  $G_2$  instantons on a special class of nearly  $G_2$  manifolds known as the Aloff–Wallach spaces. They were able to distinguish between nearly parallel  $G_2$ -structures on the same Aloff–Wallach space using gauge theoretic methods.

One can study the higher order deformations of instantons on nearly Kähler 6-manifolds and nearly  $G_2$  manifolds to check whether the infinitesimal deformations found in [Sin21] are genuine. The cones over many known manifolds with real Killing spinors admit smooth resolutions, so an obvious next step is to consider instantons on these resolutions. The instantons on the cone over a Sasaki–Einstein manifold were studied in [Pac11]. The information about a smooth family of instantons on the link can be used to construct new examples of instantons on these resolutions.

In [Wan18] Wang showed that with respect to any  $G_2$ -structure defined near the origin in  $\mathbb{R}^7$  and for any Hermitian Yang–Mills connection  $A_O$  on  $S^6$  there exists a locally defined  $G_2$  monopole asymptotic to  $A_O$  at the origin. This result is particularly interesting as it produces concrete local examples of singular  $G_2$  monopoles tangent to the canonical connection on  $S^6$ . As of now no such result is known in the case of the canonical connection on  $S^7$  and it would be interesting to see if one could construct local examples of singular Spin(7) instantons on the cone over  $S^7$  that would be asymptotic to it.

# Chapter 6

## Deformation theory of nearly $G_2$ manifolds

Let  $(M, \varphi, \psi)$  be a nearly  $G_2$  manifold with a nearly  $G_2$ -structure  $(\varphi, \psi)$ . We are interested in studying the deformation problem of  $(\varphi, \psi)$  in the space of nearly  $G_2$ -structures. The infinitesimal version of this problem was settled by Alexandrov and Semmelmann in [AS12]. We obtain new proofs of some of their results using the results proved in the previous sections.

**Remark 6.0.1.** This chapter is based on the article [DS20] co-authored by the author. However to be coherent with the rest of the thesis we use the eigenvalue convention  $(-2, +1)$  for  $\Omega_7^2, \Omega_{14}^2$  in this chapter as opposed to the  $(+2, -1)$  convention in the article. This does not effect the results.

Let  $\mathcal{P}$  be the space of  $G_2$ -structures on  $M$ , that is, the set of all  $(\varphi, \psi) \in \Omega_+^3 \times \Omega_+^4$  with  $\Theta(\varphi) = \psi$ . Given a point  $\mathfrak{p} = (\varphi, \psi) \in \mathcal{P}$  we define the tangent space  $T_{\mathfrak{p}}\mathcal{P}$ .

**Lemma 6.0.1.** *The tangent space  $T_{\mathfrak{p}}\mathcal{P}$  is the set of all  $(\xi, \eta) \in \Omega^3(M) \times \Omega^4(M)$  such that*

$$\begin{aligned}\xi &= 3f\varphi - X \lrcorner \psi + \gamma \\ \eta &= 4f\psi + X \wedge \varphi - *\gamma\end{aligned}$$

for some  $f \in \Omega^0(M)$ ,  $X \in \Gamma(TM)$  and  $\gamma \in \Omega_{27}^3$ .

*Proof.* The proof immediately follows from equations (2.3.20) and (2.3.21) from Proposition 2.3.2.  $\square$

## 6.1 Infinitesimal deformations

We want to study deformations of a given nearly  $G_2$ -structure  $\varphi$  on a compact manifold  $M$  by nearly  $G_2$ -structures  $\varphi_t$ . We are only interested in deformations of the nearly  $G_2$ -structures modulo the action of the group  $\text{Diff}_0(M)$  where  $\text{Diff}_0(M)$  denotes the space of diffeomorphisms of  $M$  which are isotopic to the identity. We first use Lemma 4.2.12 to find a slice for the action of diffeomorphism group on  $\mathcal{P}$  which is used to find the space of infinitesimal nearly  $G_2$  deformations, a result originally due Alexandrov–Simmelmann [AS12].

For the purposes of doing analysis, we consider the Hölder space  $\mathcal{P}^{k,\alpha}$  of  $G_2$ -structures on  $M$  such that  $\varphi$  and  $\psi$  are of class  $C^{k,\alpha}$ ,  $k \geq 1$ ,  $\alpha \in (0, 1)$ . Let  $\mathfrak{p} = (\varphi, \psi) \in \mathcal{P}^{k,\alpha}$  be a nearly  $G_2$ -structure such that the induced metric is not isometric to round  $S^7$ . According to a result of Friedrich [Fri06], all nearly parallel  $G_2$ -structures on  $S^7$  which induce the standard metric are conjugated under the action of the isometry group. Thus neither  $S^7$  nor its quotients admit  $G_2$  deformations. Therefore from now on we shall exclude from our considerations the case of nearly parallel  $G_2$  manifolds with constant curvature. Denote the orbit of  $\mathfrak{p}$  under the action of  $\text{Diff}_0^{k+1,\alpha}(M) - C^{k+1,\alpha}$  diffeomorphisms isotopic to the identity, by  $\mathcal{O}_{\mathfrak{p}}$ . The tangent space  $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$  is the space of Lie derivatives  $\mathcal{L}_X(\varphi, \psi)$  for  $X \in \Gamma(TM)$ . We are interested in finding a complement  $\mathcal{C}$  of  $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$  in  $T_{\mathfrak{p}}\mathcal{P}$ .

If  $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$  then for a unique  $X \in \mathcal{K}$ ,  $f \in \Omega^0(M)$ ,  $Y \in \mathcal{K}^{\perp L^2}$ , and  $\eta_0 \in \Omega_{27}^4$  using Remark 4.3.1 and Proposition 4.3.6 (1), we can write

$$\eta = X \wedge \varphi + df \wedge \varphi + 4f\psi + d^*(Y \wedge \psi) + \eta_0.$$

From Lemma 4.2.7 (4), (6) we know that

$$\begin{aligned} d^*(Y \wedge \psi) &= - * d * (Y \wedge \psi) = - * d(Y \lrcorner \varphi) \\ &= \frac{3}{7}(d^*Y)\psi - (3Y + \frac{1}{2} \text{curl } Y) \wedge \varphi - *i_{\varphi} \left( \frac{1}{2}(\nabla_i Y_j + \nabla_j Y_i) + \frac{1}{7}(d^*Y)g_{ij} \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_Y \psi &= d(Y \lrcorner \psi) \\ &= -\frac{4}{7}d^*Y\psi - \left( -\frac{1}{2} \text{curl } Y + Y \right) \wedge \varphi - *i_{\varphi} \left( \frac{1}{2}(\nabla_i Y_j + \nabla_j Y_i) + \frac{1}{7}(d^*Y)g_{ij} \right). \end{aligned}$$

Thus, we get that

$$d^*(Y \wedge \psi) = (d^*Y)\psi - (\text{curl } Y + 2Y) \wedge \varphi + \mathcal{L}_Y \psi.$$

Thus up to an element in  $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$  we get that

$$\eta = \left(4f + d^*Y\right)\psi + (X + df - \text{curl } Y - 2Y) \wedge \varphi + \eta_0 \quad (6.1.1)$$

and hence from Lemma 6.0.1

$$\xi = \left(3f + \frac{3}{4}d^*Y\right)\varphi - (X + df - \text{curl } Y - 2Y) \lrcorner \psi - *\eta_0. \quad (6.1.2)$$

Now, given that  $X \in \mathcal{K}$ , from Lemma 4.2.7 (6) and  $\text{curl } X = -6X$  we see that

$$\mathcal{L}_{-\frac{X}{4}}\psi = d\left(-\frac{X}{4} \lrcorner \psi\right) = X \wedge \varphi \quad (6.1.3)$$

and hence

$$\eta = \mathcal{L}_{-\frac{X}{4}}\psi + d(f\varphi) + d^*(Y \wedge \psi) + \eta_0$$

which implies that up to an element in  $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$  combined with the above observation, we can write

$$\eta = \left(4f + d^*Y\right)\psi + (df - \text{curl } Y - 2Y) \wedge \varphi + \eta_0 \quad (6.1.4)$$

which implies that

$$\xi = \left(3f + \frac{3}{4}d^*Y\right)\varphi - (df - \text{curl } Y - 2Y) \lrcorner \psi - *\eta_0 \quad (6.1.5)$$

and hence we get a splitting  $T_{\mathfrak{p}}\mathcal{P} = T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} \oplus \mathcal{C}$  where  $\mathcal{C} \cong \Omega^0(M) \times \mathcal{K}^{\perp L^2} \times \Omega_{27}^4$  which consists of  $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$  of the form (6.1.5) and (6.1.4) respectively. This gives a choice of slice. In fact, as discussed in [Nor08, pg. 49 & Theorem 3.1.4] we have the following.

**Proposition 6.1.1.** *There exists an open neighbourhood  $U$  of  $\mathcal{C}$  of the origin such that the “exponentiation” of  $U$  is a slice for the action of  $\text{Diff}_0^{k+1, \alpha}(M)$  on a sufficiently small neighbourhood of  $\mathfrak{p} \in \mathcal{P}^{k, \alpha}$ .*

With this choice of slice we determine the infinitesimal deformations of the nearly  $G_2$ -structure  $\mathfrak{p}$  which gives a new proof of a result of Alexandrov–Simmelmann [AS12, Theorem 3.5].

**Theorem 6.1.2.** *Let  $(M, \varphi, \psi)$  be a complete nearly  $G_2$  manifold, not isometric to the round  $S^7$ . Then the infinitesimal deformations of the nearly  $G_2$ -structure are in one-to-one correspondence with  $(X, \xi_0) \in \mathcal{K} \times \Omega_{27}^3$  with*

$$*d\xi_0 = -4\xi_0 \quad \text{and} \quad \Delta X = 12X. \quad (6.1.6)$$

Hence  $\xi_0$  is co-closed as well. Moreover,  $\Delta_d \xi_0 = 16\xi_0$ .

*Proof.* Let  $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$  be an infinitesimal nearly  $G_2$  deformation of a  $G_2$ -structure  $\mathfrak{p} \in \mathcal{P}$ . So  $\eta$  must be exact and hence from Lemma 4.2.12 (2), we can remove the  $d^*(Y \wedge \psi)$  term, in which case (6.1.1) and (6.1.2) become

$$\eta = 4f\psi + (X + df) \wedge \varphi + \eta_0 \quad \text{and} \quad \xi = 3f\varphi - (X + df) \lrcorner \psi - *\eta_0. \quad (6.1.7)$$

Moreover, for infinitesimal nearly  $G_2$  deformations we must have

$$d\xi = 4\eta$$

and hence (6.1.7) along with the fact that  $\pi_1(d*\eta_0) = 0$  imply

$$4f\psi + (4X + df) \wedge \varphi + 4\eta_0 + d((X + df) \lrcorner \psi) + d*\eta_0 = 0.$$

Using Lemma 4.2.7 (6) for the fourth term above and taking inner product with  $\psi$  gives

$$28f - 4d^*(X + df) = 0.$$

But since  $X \in \mathcal{K} \implies d^*X = 0$  and hence we get  $\Delta f = 7f$ . Since  $M$  is not isometric to round  $S^7$ , Obata's theorem then implies that  $f = 0$ , and with  $\xi_0 = *\eta_0$ ,

$$\eta = X \wedge \varphi + \eta_0 \quad \text{and} \quad \xi = -X \lrcorner \psi - *\eta_0 \quad (6.1.8)$$

which proves the one-to-one correspondence between the infinitesimal nearly  $G_2$  deformations and  $\mathcal{K} \times \Omega_{27}^3$ . Since  $\text{Ric} = 6g$  and  $X$  is a Killing vector field, we have  $\Delta X = 12X$  which is the second part of (6.1.6). Since  $\eta_0$  is exact,  $d\eta_0 = 0$ . From (6.1.8), (6.1.3), and the fact that  $d\xi = 4\eta$ , we get

$$d*\eta_0 = -4\eta_0$$

and hence

$$*d\xi_0 = -4\xi_0.$$

Taking  $d^*$  of both sides give  $d^*\xi_0 = 0$ . Moreover,

$$\Delta_d \xi_0 = d^*d\xi_0 = -4d^*\xi_0 = -4*(d\xi_0) = 16\xi_0$$

which completes the proof of the theorem.  $\square$



**Remark 6.1.3.** From (6.1.3) and Theorem 6.1.2 we see that the infinitesimal deformations of a nearly  $G_2$ -structure *modulo diffeomorphisms* are in one-to-one correspondence with  $\xi_0 \in \Omega_{27}^3$  such that  $*d\xi_0 = -4\xi_0$ .

Motivated from the study of deformations of nearly Kähler 6-manifolds by Foscolo [Fos17, §4] where he used observations of Hitchin [Hit01], we also want to interpret the nearly  $G_2$  condition (4.1.1) as the vanishing of a smooth map on the space of exact positive 4-forms. Moreover, in order to study the second order deformations, it will be convenient to enlarge the space by introducing a vector field as an additional parameter which is natural when one considers the action of the diffeomorphism group. We elaborate on this below.

Let  $\psi = d\alpha$  be an exact positive 4-form, not necessarily satisfying the nearly  $G_2$  condition. Let  $\eta \in \Omega_{\text{exact}}^4$  be a first order deformation of  $\psi$ . Given an orientation on  $M$ , Hitchin in [Hit01] defined a volume functional for the exact 4-form  $\rho = d\gamma$  given by

$$V(\rho) = \int_M * \rho \wedge \rho,$$

and a quadratic form

$$W(\rho, \rho') = \int_M \gamma \wedge \rho' = \int_M \rho \wedge \gamma',$$

where  $\rho = d\gamma$  and  $\rho' = d\gamma'$  are exact 4-forms. We denote  $W(\rho, \rho)$  by  $W(\rho)$ . When  $M$  is compact, Hitchin proves [Hit01, Theorem 5] that a stable 4-form (which is the same as a positive 4-form in our case)  $\eta \in \Omega_{\text{exact}}^4(M)$  is a critical point of the volume functional  $V$  subject to the constraint  $W(\eta) = \text{constant}$  if and only if  $\eta$  defines a nearly  $G_2$ -structure. The linearization of the volume functional at  $\psi$  is given by

$$\begin{aligned} dV(\eta) &= \left. \frac{d}{dt} \right|_{t=0} V(\psi + t\eta) = \int_M \varphi \wedge \eta + \int_M * \eta \wedge \psi \\ &= 2 \int_M \varphi \wedge \eta. \end{aligned}$$

For the linearization of the quadratic form, suppose  $\psi$  is exact with  $\psi = d\alpha$ . We use integration by parts to get

$$dW(\eta) = \left. \frac{d}{dt} \right|_{t=0} W(\psi + t\eta) = \int_M \alpha \wedge \eta + \int_M \gamma \wedge \psi$$

$$= 2 \int_M \alpha \wedge \eta.$$

Let us define an energy functional  $E$  on exact 4-forms by

$$E(\rho) := \frac{1}{2}(V(\rho) - 4W(\rho)).$$

Then from above calculations

$$dE(\eta) = \int_M (\varphi - 4\alpha) \wedge \eta = \int_M d(\varphi - 4\alpha) \wedge \eta.$$

Therefore  $\psi = d\alpha$  is a critical point of  $E$  if and only if  $dE(\eta) = 0$  for every  $\eta \in \Omega_{\text{exact}}^4$ . That is if and only if

$$d\varphi - 4d\alpha = d\varphi - 4\psi = 0.$$

Hence the critical points of the functional  $E$  on  $\Omega_{+, \text{exact}}^4$  are nearly  $G_2$ -structures. Since the energy functional  $E$  is diffeomorphism invariant, we can introduce an extra vector field, as  $dE$  vanishes in the direction of Lie derivatives. Thus  $\psi$  being a stable exact 4-form can be given by the formula

$$\psi = \frac{1}{4}d(\varphi - *d(Z \lrcorner \psi))$$

for some  $Z \in \Gamma(TM)$ . We use these observations to write the nearly  $G_2$  condition (4.1.1) as the vanishing of a smooth map. Let us denote by  $\widehat{\mathcal{P}}$  the space of stable 3-forms and stable exact 4-forms, i.e.,  $(\varphi, \psi) \in \Omega_+^3 \times \Omega_{+, \text{exact}}^4$ . We have the following

**Proposition 6.1.4.** *Suppose for some vector field  $Z$  and for the Hodge star  $*$  with respect to a fixed background metric if  $(\varphi, \psi) \in \widehat{\mathcal{P}}$  satisfies*

$$d\varphi - 4\psi = d * d(Z \lrcorner \psi). \tag{6.1.9}$$

*Then  $(\varphi, \psi)$  is a nearly  $G_2$ -structure.*

*Proof.* We prove that  $d(Z \lrcorner \psi) = 0$ . We note from (2.3.13) that

$$(Z \lrcorner \psi) \wedge \psi = 0$$

So from (6.1.9) we get that

$$\|d(Z \lrcorner \psi)\|_{L^2}^2 = \langle d(Z \lrcorner \psi), d(Z \lrcorner \psi) \rangle_{L^2}$$

$$\begin{aligned}
&= \langle (Z \lrcorner \psi), *d * d(Z \lrcorner \psi) \rangle_{L^2} \\
&= \langle (Z \lrcorner \psi), *(d\varphi - 4\psi) \rangle_{L^2} \\
&= \int_M (Z \lrcorner \psi) \wedge (d\varphi - 4\psi) = \int_M (Z \lrcorner \psi) \wedge d\varphi.
\end{aligned}$$

Since  $\varphi$  is a  $G_2$ -structure and  $d\psi = 0$  from (6.1.9), we know from (2.3.3) that  $\tau_1 = 0$  and hence  $d\varphi$  has no component in  $\Omega_7^4$ . Thus

$$\langle (Z \lrcorner \psi), *d\varphi \rangle = 0$$

which implies that

$$\|d(Z \lrcorner \psi)\|_{L^2}^2 = \int_M (Z \lrcorner \psi) \wedge d\varphi = 0$$

which proves the proposition.  $\square$

Suppose we want to describe the local moduli space of nearly  $G_2$ -structures on a manifold  $M$ . If  $\mathcal{NP}$  denotes the space of nearly  $G_2$ -structures on  $M$  then the local moduli space is  $\mathcal{M} = \mathcal{NP}/\text{Diff}_0(M)$ . A natural way to study this problem is to view the nearly  $G_2$ -structures on  $M$  as the zero locus of an appropriate function, find the linearization of the function and prove its surjectivity, so that an Implicit Function Theorem argument describes  $\mathcal{M}$ .

Now let  $(\varphi, \psi)$  be a nearly  $G_2$ -structure on  $M$ . Let  $U \subset \Omega_{+, \text{exact}}^4$  be a small neighborhood of the 4-form  $\psi$ . Since the condition of being stable is open we can assume the existence of such a neighborhood. Thus for  $\eta \in \Omega_{\text{exact}}^4$  with sufficiently small norm with respect to the metric induced by  $\varphi$ ,  $\tilde{\psi} = \psi + \eta$  is also a stable exact 4-form. Let  $\tilde{*}$  be the Hodge star operator induced from  $\tilde{\varphi}$ . From Proposition 6.1.4 the pair of stable forms  $(\tilde{\varphi}, \tilde{\psi})$  defines a nearly  $G_2$ -structure if there exists a  $Z \in \Gamma(TM)$  such that

$$d\tilde{\varphi} - 4\tilde{\psi} = d\tilde{*}d(Z \lrcorner \tilde{\psi}).$$

This condition is equivalent to the vanishing of the map

$$\begin{aligned}
\Phi : U \times \Gamma(TM) &\rightarrow \Omega_{\text{exact}}^4 \\
(\tilde{\psi}, Z) &\mapsto d\tilde{*}\tilde{\psi} - 4\tilde{\psi} - d\tilde{*}d(Z \lrcorner \tilde{\psi}).
\end{aligned} \tag{6.1.10}$$

Thus, the nearly  $G_2$ -structures are the zero locus of the map  $\Phi$  modulo diffeomorphisms.

Let  $\xi$  be the dual of  $\eta$  under Hitchin's duality map  $\Theta$  as in Proposition 2.3.2. The kernel of the linearization of the map  $\Phi$  at the point  $(\psi, 0)$  is given by those  $(\eta, Z)$  satisfying

$$d\xi - 4\eta = d * d(Z \lrcorner \psi).$$

Thus the obstructions on the first order deformations of the nearly  $G_2$ -structure  $(\varphi, \psi)$  are given by  $\text{Im}(D\Phi)$  which is characterized in the following proposition, whose proof is inspired from a similar theorem in the nearly Kähler case by Foscolo [Fos17, Proposition 4.5].

**Proposition 6.1.5.** *Let  $(\varphi, \psi)$  be a nearly  $G_2$ -structure and  $(\xi, \eta) \in \Omega^3 \times \Omega_{\text{exact}}^4$  be a first order deformation in  $\mathcal{P}$ . Then  $\alpha \in \Omega_{\text{exact}}^4$  lies in the image of  $D\Phi$  if and only if*

$$\langle d^* \alpha - 4 * \alpha, \chi \rangle_{L^2} = 0$$

for all co-closed  $\chi \in \Omega_{27}^3$  such that  $\Delta \chi = 16\chi$ .

*Proof.* From Proposition 4.3.6 (2), there exists  $X \in \mathcal{K}$ ,  $f \in C^\infty(M)$ , and  $\eta_0 \in \Omega_{27, \text{exact}}^4$  such that

$$\begin{aligned} \eta &= X \wedge \varphi + d(f\varphi) + \eta_0 \\ &= d\left(-\frac{1}{4}X \lrcorner \psi + f\varphi\right) + \eta_0 \end{aligned}$$

and from Lemma 6.0.1, the 3-form

$$\xi = 3f\varphi - (df + X) \lrcorner \psi - *\eta_0.$$

By Proposition 4.3.6,  $\alpha = Y \wedge \varphi + d(h\varphi) + \alpha_0$  for some  $Y \in \mathcal{K}$ ,  $h \in C^\infty(M)$ ,  $\alpha_0 \in \Omega_{27, \text{exact}}^4$ . Such an  $\alpha$  lies in the image of  $D\Phi$  if

$$d\xi - 4\eta - d * d(Z \lrcorner \psi) = \alpha = d\left(-\frac{1}{4}Y \lrcorner \psi + h\varphi\right) + \alpha_0.$$

From Lemma 4.2.7 (5)

$$\begin{aligned} d^*(Z \wedge \psi) &= - * d(Z \lrcorner \varphi) \\ &= \frac{3}{7}(d^*Z)\psi - \frac{1}{2}\left(6Z + \text{curl } Z\right) \wedge \varphi - *i_\varphi\left(\frac{1}{2}(\nabla_i Z_j + \nabla_j Z_i) + \frac{1}{7}(d^*Z)g_{ij}\right). \end{aligned}$$

Comparing the last term in the above expression with that of  $d(Z \lrcorner \psi)$  in Lemma 4.2.7 we get

$$d(Z \lrcorner \psi) = \frac{1}{7}(d^*Z)\psi + (2Z + \text{curl } Z) \wedge \varphi + d^*(Z \wedge \psi).$$

Using these expressions for  $\xi, \eta$  and  $d(Z \lrcorner \psi)$  we get

$$d\xi - 4\eta - d * d(Z \lrcorner \psi) = d((-f - \frac{1}{7}d^*Z)\varphi - (df - 2Z - \text{curl } Z) \lrcorner \psi) - d * \eta_0 - 4\eta_0.$$

By Corollary 4.2.13 since  $\eta_0 \in \Omega_{\text{exact}}^4$ ,  $d^*\eta_0 \in \Omega_{27}^3$ . Thus, for finding  $\text{Im}(D\Phi)$ , we need to solve the equations

$$\begin{aligned} f + \frac{1}{7}d^*Z &= -h, \\ df - 2Z - \text{curl } Z &= \frac{1}{4}Y, \\ -d * \eta_0 - 4\eta_0 &= \alpha_0. \end{aligned} \tag{6.1.11}$$

Let  $\alpha_0 = 0$ . Then by Implicit Function Theorem, a solution of the first pair of equations always exists if the operator

$$\begin{aligned} \tilde{D} : \Omega^0 \times \Omega^1 &\rightarrow \Omega^0 \times \Omega^1 \\ (f, Z) &\mapsto \left( f + \frac{1}{7}d^*Z, df - 2Z - \text{curl } Z \right) \end{aligned}$$

is invertible in a small neighborhood of its zero locus. Since  $\tilde{D}$  differs from the modified Dirac operator  $D$  in (4.3.5) only by self-adjoint zeroth-order term, it is self-adjoint and hence  $\ker(\tilde{D}) = \text{coker}(\tilde{D})$ . A pair  $(f, Z)$  is in the kernel of the operator  $D$  if and only if

$$\begin{aligned} f + \frac{1}{7}d^*Z &= 0 \\ df - 2Z - \text{curl } Z &= 0. \end{aligned}$$

Applying the operator  $d^*$  on the second equation and using the fact that  $d^*(\text{curl } Z) = 0$  gives

$$0 = d^*df - 2d^*Z = d^*df + 14f.$$

Thus  $f = 0$  as  $\Delta$  is a non-negative operator. The second equation then becomes

$$\text{curl } Z = d^*(Z \lrcorner \varphi) = *(dZ \wedge \psi) = -2Z$$

and Proposition 4.2.1 implies that  $dZ = -\frac{2}{3}Z \lrcorner \varphi + \pi_{14}(dZ)$ . Using Lemma 4.2.5 (2) we get that

$$\begin{aligned} \int_M dZ \wedge dZ \wedge \varphi &= -\frac{8}{9}\|Z \lrcorner \varphi\|^2 + \|\pi_{14}(dZ)\|^2 \\ &= -\frac{8}{3}\|Z\|^2 + \|\pi_{14}(dZ)\|^2. \end{aligned}$$

On the other hand

$$\int_M dZ \wedge dZ \wedge \varphi = 4 \int_M Z \wedge dZ \wedge \psi = -8\|Z\|^2.$$

Combining these two equations we get  $\frac{16}{3}\|Z\|^2 = -\|\pi_{14}(dZ)\|^2$  and hence  $Z = 0$  as well. Thus  $\ker(\tilde{D}) = \text{coker}(\tilde{D}) = 0$  and  $\tilde{D}$  is invertible, so we can always solve the first pair of equations in (6.1.11). Thus there are no restrictions on  $Y$ ,  $h$  to be in the image of  $D\Phi$ . Moreover if  $\alpha_0 \neq 0$  satisfies the third equation in (6.1.11) then

$$\begin{aligned} *\alpha_0 &= -d^*\eta_0 - 4*\eta_0, \\ d^*\alpha_0 &= -d^*d*\eta_0 - 4d^*\eta_0, \end{aligned}$$

which on using the fact that  $*\eta_0$  is co-closed implies

$$d^*\alpha_0 - 4*\alpha_0 = 16*\eta_0 - d^*d*\eta_0 = 16*\eta_0 - \Delta_d*\eta_0.$$

Setting  $\hat{\eta}_0 = -*\eta_0$  in the above equation we get

$$(\Delta - 16)\hat{\eta}_0 = d^*\alpha_0 - 4*\alpha_0.$$

Let us denote by  $\bar{\Delta}$  the restriction of the operator  $(\Delta - 16 \text{ id})$  to the space of co-closed 3-forms in  $\Omega_{27,\text{exact}}^3$ . Since  $\bar{\Delta}$  is a self adjoint elliptic linear operator,  $d^*\alpha - 4*\alpha_0$  will be in the image of  $\bar{\Delta}$  if and only if  $d^*\alpha - 4*\alpha_0$  is  $L^2$  orthogonal to  $\ker(\bar{\Delta}^*) = \ker(\bar{\Delta})$ .

Thus  $\alpha_0 \in \Omega_{27,\text{exact}}^4$  is a solution to the equation (6.1.11) (3) if and only if

$$\langle d^*\alpha_0 - 4*\alpha_0, \xi_0 \rangle_{L^2} = 0$$

for all co-closed  $\xi_0 \in \Omega_{27}^3$  such that  $\Delta\xi = 16\xi$ . To complete the proof of the proposition we now only need to prove the  $L^2$ -orthogonality condition for  $\alpha$ . But observe that since  $Y \in \mathcal{K}$ ,  $\mathcal{L}_X g = d^*Y = 0$  and  $\text{curl } Y = -6Y$  thus from Lemma 4.2.7 (6) we have

$$d^*\alpha = d^*(Y \wedge \varphi) + d^*d(h\varphi) + d^*\alpha_0 = -4Y \lrcorner \psi + d^*d(h\varphi) + d^*\alpha_0,$$

and so  $d^*\alpha - 4*\alpha = d^*d(h\varphi) - 4*d(h\varphi) + d^*\alpha_0 - 4*\alpha_0$ . Since  $\xi$  is co-closed, from Corollary 4.2.11 we have  $d\xi \in \Omega_{27}^4$  and thus

$$\langle d^*d(h\varphi), \xi \rangle_{L^2} = \langle d(h\varphi), d\xi \rangle_{L^2} = 0.$$

Similarly

$$\langle *d(h\varphi), \xi \rangle_{L^2} = \langle d^*(h\psi), \xi \rangle_{L^2} = \langle h\psi, d\xi \rangle_{L^2} = 0$$

which completes the proof of the proposition.  $\square$

**Remark 6.1.6.** Proposition 6.1.5 puts a very strong restriction on the first order deformations of a nearly  $G_2$ -structure to be unobstructed.

## 6.2 Second order deformations

Following the work of Koiso [Koi82] on deformations of Einstein metrics and the work of Foscolo [Fos17] on the second order deformations of nearly Kähler structures on 6-manifolds, we define the notion of second order deformations of nearly  $G_2$ -structures.

**Definition 6.2.1.** Given a nearly  $G_2$ -structure  $(\varphi_0, \psi_0)$  and an infinitesimal deformation  $(\xi_1, \eta_1)$ , a second order deformation of  $(\varphi_0, \psi_0)$  in the direction of  $(\xi_1, \eta_1)$  is a pair  $(\xi_2, \eta_2) \in \Omega^3 \times \Omega^4$  such that

$$\varphi = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}\xi_2, \quad \psi = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2$$

is a nearly  $G_2$ -structure up to terms of order  $O(\epsilon^2)$ . An infinitesimal deformation  $(\xi_1, \eta_1)$  is said to be *obstructed to second order* if there exists no second order deformation in its direction.

**Remark 6.2.2.** Second order deformations are the same as the second derivative of a curve of nearly  $G_2$ -structures on a manifold  $M$ .

**Remark 6.2.3.** In a similar way, we can define higher order deformations of a nearly  $G_2$ -structure.

Following the discussion in the previous section and in particular Proposition 6.1.4, in order to find second order deformations of a given nearly  $G_2$ -structure  $(\varphi_0, \psi_0)$ , we look for formal power series defining a positive *exact* 4-form

$$\psi_\epsilon = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2 + \dots$$

where  $\eta_i \in \Omega_{\text{exact}}^4$ , and a vector field

$$Z_\epsilon = \epsilon Z_1 + \frac{\epsilon^2}{2}Z_2 + \dots$$

which satisfy (6.1.9). That is

$$d\varphi_\epsilon - 4\psi_\epsilon = d * d(Z_\epsilon \lrcorner \psi_\epsilon) \quad (6.2.1)$$

where  $\varphi_\epsilon$  is the dual of  $\psi_\epsilon$ . Note that the Hodge star  $*$  is taken with respect to  $\varphi_\epsilon$ .

Since we are interested in second order deformations, given an infinitesimal nearly  $G_2$  deformation  $(\xi_1, \eta_1)$ , we set  $Z_1 = 0$  and look for  $\eta_2 \in \Omega_{\text{exact}}^4$  such that (6.2.1) is satisfied up to terms of  $O(\epsilon^3)$ . Explicitly, we write

$$\varphi_\epsilon = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}(\widehat{\eta}_2 - Q_3(\eta_1))$$

where  $\widehat{\eta}_2$  denotes the linearization of Hitchin's duality map  $\Theta$  for stable forms in Proposition 2.3.2 and  $Q_3(\eta_1)$  is the quadratic term of Hitchin's duality map. Since we want solutions to (6.2.1) up to second order, we look for  $\eta_2$  such that

$$d\widehat{\eta}_2 - 4\eta_2 = d(Q_3(\eta_1)) + d * d(Z_2 \lrcorner \psi_0) \quad (6.2.2)$$

as  $Z_1 = 0$  and  $Z_2 \lrcorner \psi_0$  is the only second order term in  $Z_\epsilon \lrcorner \psi_\epsilon$ . We know from Proposition 6.1.5 that there are obstructions to finding second order deformations and hence in solving the above equation. We want to establish a one-to-one correspondence between second order deformations of a nearly  $G_2$ -structure and solutions to (6.2.2). We do this in the following lemma.

**Lemma 6.2.4.** *Suppose  $\eta_2$  is a solution of (6.2.2). Then  $d(Z_2 \lrcorner \psi_0) = 0$  and  $(\widehat{\eta}_2 - Q_3(\eta_1), \eta_2)$  defines a second order deformation of  $(\varphi_0, \psi_0)$  in the direction of  $(\xi_1, \eta_1)$  in the sense of Definition 6.2.1. Conversely, every second order deformation  $(\xi_2, \eta_2)$  in the direction of  $(\xi_1, \eta_1)$  is a solution to (6.2.2).*



*Proof.* We start with

$$\begin{aligned} \|d(Z_2 \lrcorner \psi_0)\|_{L^2}^2 &= \langle Z_2 \lrcorner \psi_0, d^* d(Z_2 \lrcorner \psi_0) \rangle_{L^2} \\ &= \langle Z_2 \lrcorner \psi_0, *d * d(Z_2 \lrcorner \psi_0) \rangle_{L^2} \\ &= \langle Z_2 \lrcorner \psi_0, *(d\widehat{\eta}_2 - 4\eta_2 - dQ_3(\eta_1)) \rangle_{L^2}. \end{aligned}$$

Since  $d\psi_\epsilon = O(\epsilon^3)$ , from (2.3.2) and (2.3.3) we see that for any vector field  $Y$ ,  $\int d\varphi_\epsilon \wedge (Y \lrcorner \psi_\epsilon) = O(\epsilon^3)$ . Thus the terms which are  $O(\epsilon^2)$  in  $\int d\varphi_\epsilon \wedge (Y \lrcorner \psi_\epsilon)$  vanish, that is

$$\int d\varphi_0 \wedge (Y \lrcorner \eta_2) + d\xi_1 \wedge (Y \lrcorner \eta_1) + d(\widehat{\eta}_2 - Q_3(\eta_1)) \wedge (Y \lrcorner \psi_0) = 0.$$

Now since  $d\varphi_0 = 4\psi_0$  and  $(Y \lrcorner \eta_2) \wedge \psi_0 = -(Y \lrcorner \psi_0) \wedge \eta_2$  we get that  $d\varphi_0 \wedge (Y \lrcorner \eta_2) = -4\eta_2 \wedge (Y \lrcorner \psi_0)$ . Also using the fact that  $d\xi_1 = 4\eta_1$  we get that  $d\xi_1 \wedge (Y \lrcorner \eta_1) = 4\eta_1 \wedge (Y \lrcorner \eta_1) = 2Y \lrcorner (\eta_1 \wedge \eta_1) = 0$  since  $\eta_1 \wedge \eta_1 = 0$ , being an 8-form on a seven dimensional manifold. Thus, we get the following equation,

$$\int d(\widehat{\eta}_2 - Q_3(\eta_1)) \wedge (Y \lrcorner \psi_0) - 4\eta_2 \wedge (Y \lrcorner \psi_0) = 0$$

Taking  $Y = Z_2$  proves that  $d(Z_2 \lrcorner \psi_0) = 0$ . From (6.2.2) we get that

$$d(\widehat{\eta}_2 - Q_3(\eta_1)) = 4\eta_2$$

which proves that  $((\widehat{\eta}_2 - Q_3(\eta_1), \eta_2))$  is a second order deformation of  $(\varphi_0, \psi_0)$  in the direction of  $(\xi_1, \eta_1)$  in the sense of Definition 6.2.1. Conversely, suppose that  $(\xi_2, \eta_2) = (\widehat{\eta}_2 - Q_3(\eta_1), \eta_2)$  is a second order deformation of  $(\varphi_0, \psi_0)$ . Then  $d\xi_2 = 4\eta_2$ .  $\square$

From the previous proposition and Proposition 6.1.5 we have that if  $(\xi_2, \eta_2)$  is a second order deformation of the nearly  $G_2$ -structure  $(\varphi_0, \psi_0)$  in the sense of Definition 6.2.1 then for all  $\chi \in \Omega_{27}^3$  such that  $d^*\chi = 0$  and  $\Delta\chi = 16\chi$ , we have

$$\langle d^* dQ_3(\eta_1) - 4 * dQ_3(\eta_1), \chi \rangle_{L^2} = 0. \quad (6.2.3)$$

**Lemma 6.2.5.** *The infinitesimal deformation  $(\xi_1, \eta_1)$  of  $(\varphi, \psi)$  is unobstructed to second order if and only if for all co-closed  $\chi \in \Omega_{27}^3$  such that  $d\chi = -4 * \chi$*

$$\langle Q_3(\eta_1), \chi \rangle_{L^2} = 0.$$

*Proof.* Equation (6.2.3) can be further simplified as follows:

$$\begin{aligned}
0 &= \langle d^* dQ_3(\eta_1) - 4 * dQ_3(\eta_1), \chi \rangle_{L^2} \\
&= \langle Q_3(\eta_1), d^* d\chi \rangle_{L^2} - 4 \langle Q_3(\eta_1), d^* * \chi \rangle_{L^2} \\
&= \langle Q_3(\eta_1), \Delta\chi \rangle_{L^2} - 4 \langle Q_3(\eta_1), *d\chi \rangle_{L^2} \\
&= -4 \langle *Q_3(\eta_1), d\chi - 4 * \chi \rangle_{L^2}.
\end{aligned}$$

Thus (6.2.3) is equivalent to

$$\langle *Q_3(\eta_1), d\chi - 4 * \chi \rangle_{L^2} = 0. \quad (6.2.4)$$

Since  $\{\chi \in \Omega_{27}^3 \mid d^* \chi = 0, \Delta\chi = 16\xi\} = \{\chi \in \Omega_{27}^3 \mid d\chi = 4 * \chi\} \oplus \{\chi \in \Omega_{27}^3 \mid d\chi = -4 * \chi\}$  and the inner product in (6.2.4) vanishes if  $d\chi = 4 * \chi$ , we only need to consider the subspace where  $d\chi = -4 * \chi$ . Substituting  $d\chi = 4 * \chi$  in (6.2.4) we get the desired result.  $\square$

By Theorem 6.1.2 the space  $\{\chi \in \Omega_{27}^3 \mid d\chi = -4 * \chi\}$  is isomorphic to the space of infinitesimal deformation of  $(\varphi_0, \psi_0)$ . Hence we can conclude that  $(\hat{\eta}_2 - Q_3(\eta_1), \eta_2)$  is a second order deformation of  $(\varphi_0, \psi_0)$  in the direction of  $(\xi_1, \eta_1)$  if and only if for any infinitesimal deformation  $\chi$  of  $(\varphi_0, \psi_0)$

$$\langle Q_3(\eta_1), \chi \rangle_{L^2} = 0.$$

## 6.3 Deformations on the Aloff-Wallach space

In [AS12, Prop. 8.3] Alexandrov–Semmelmann established that the space of infinitesimal deformations of the nearly  $G_2$ -structure on the Aloff–Wallach space  $X_{1,1} \cong \frac{\mathrm{SU}(3) \times \mathrm{SU}(2)}{\mathrm{SU}(2) \times \mathrm{U}(1)}$  is an eight dimensional space isomorphic to  $\mathfrak{su}(3)$ , the Lie algebra of  $\mathrm{SU}(3)$ . The rest of the chapter is devoted to prove that these deformations are obstructed to second order. We skip some details in the proof of Theorem 6.3.1 but provide enough details that an interested reader can reproduce them on their own.

The embedding of  $\mathfrak{su}(2)$  and  $\mathfrak{u}(1)$  in  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ , which we denote by  $\mathfrak{su}(2)_d$  and  $\mathfrak{u}(1)$ , following [AS12], is given by

$$\mathfrak{su}(2)_d = \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a \right) \mid a \in \mathfrak{su}(2) \right\},$$

$$\mathfrak{u}(1) = \text{span}\{C\} = \text{span} \left\{ \left( \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0 \right) \right\}.$$

The Lie algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  splits as

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}$$

where  $\mathfrak{m}$  is the 7-dimensional orthogonal complement of  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  with respect to  $B$ , the Killing form of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ . The normal nearly  $G_2$  metric on  $X_{1,1}$  is then given by  $-\frac{3}{40}B$  where the constant  $-\frac{3}{40}$  comes from our choice of  $\tau_0 = 4$ . If we denote by  $W$  the standard 2-dimensional complex irreducible representation of  $SU(2)$  and by  $F(k)$  the 1-dimensional complex irreducible representation of  $U(1)$  with highest weight  $k$ , then as an  $SU(2) \times U(1)$ -representation

$$\mathfrak{su}(3)_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3) \oplus \mathbb{C}.$$

Let  $\{e_i\}_{i=1}^7$  be an orthonormal basis of  $\mathfrak{m}$  with respect to the nearly  $G_2$  metric. If we define  $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ ,  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ , we can take

$$e_1 := \frac{1}{3} \left( \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, -3I \right), \quad e_2 := \frac{1}{3} \left( \begin{pmatrix} 2J & 0 \\ 0 & 0 \end{pmatrix}, -3J \right), \quad e_3 := \frac{1}{3} \left( \begin{pmatrix} 2K & 0 \\ 0 & 0 \end{pmatrix}, -3K \right),$$

$$e_4 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, 0 \right), \quad e_5 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 \end{pmatrix}, 0 \right),$$

$$e_6 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, 0 \right), \quad e_7 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}, 0 \right).$$

This basis is orthonormal with respect to the metric  $g = -\frac{3}{40}B$ . We use the shorthand  $e^{i_1 i_2 \dots i_n}$  to denote the  $n$ -form  $e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_n}$ . The nearly  $G_2$ -structure  $\varphi$  is given by

$$\varphi = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}.$$

As an  $SU(2) \times U(1)$  representation,  $\mathfrak{m}_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3)$  where

$$S^2W = \text{Span}\{e^1, e^2, e^3\},$$

$$\begin{aligned}
WF(3) &= \text{Span}\{e^4 - ie^5, e^6 - ie^7\}, \\
WF(-3) &= \text{Span}\{e^4 + ie^5, e^6 + ie^7\}.
\end{aligned}$$

By Theorem 6.1.2, the space of first order deformations is given by  $\{\xi \in \Omega_{27}^3 \mid d\xi = -4 * \xi\}$ . In this example, it was found to be isomorphic to  $\mathfrak{su}(3)$ . As an  $SU(2) \times U(1)$  representation,  $\mathfrak{su}(3)$  is isomorphic to the span of  $\{C, e_1, \dots, e_7\}$ . The  $SU(2) \times U(1)$ -invariant homomorphism from  $\mathfrak{su}(3)$  to  $\Omega_{27}^3(X_{1,1})$  is given by  $\text{Span}\{A\}$  where

$$\begin{aligned}
A(C) &= \varphi - 7e^{123}, & A(e_1) &= \frac{5}{3}(e^{145} + e^{167}), \\
A(e_2) &= \frac{5}{3}(e^{245} + e^{267}), & A(e_3) &= \frac{5}{3}(e^{345} + e^{367}), \\
A(e_4) &= \frac{5}{9}(3e^{467} + e^{137} + e^{126} + e^{234}), & A(e_5) &= \frac{5}{9}(3e^{567} + e^{235} - e^{136} + e^{127}), \\
A(e_6) &= \frac{5}{9}(3e^{456} - e^{236} - e^{135} + e^{124}), & A(e_7) &= \frac{5}{9}(3e^{457} - e^{237} + e^{125} + e^{134}).
\end{aligned}$$

Let us fix an  $\alpha \in \mathfrak{su}(3)$ . The adjoint action of  $h = (h_1, h_2) \in SU(3) \times SU(2)$  is given by

$$h^{-1}\alpha h = h_1^{-1}\alpha h_1 = \begin{pmatrix} iv_1 & x_1 + ix_2 & x_3 + ix_4 \\ -x_1 + ix_2 & iv_2 & x_5 + ix_6 \\ -x_3 + ix_4 & -x_5 + ix_6 & -i(v_1 + v_2) \end{pmatrix}$$

where  $v_1, v_2, x_1, x_2, x_3, x_4, x_5, x_6$  are functions on  $X_{1,1}$ .

The infinitesimal deformation  $\xi_\alpha$  associated to  $\alpha$  such that  $d\xi_\alpha = -4 * \xi_\alpha$  is given by

$$\xi_\alpha = \frac{v_1 + v_2}{2}A(C) + \frac{v_1 - v_2}{2}A(e_1) + \sum_{i=1}^6 x_i A(e_{i+1}).$$

We can now compute the 4-form  $\eta_\alpha$  by using the relation  $d\xi_\alpha = 4\eta_\alpha = -4 * \xi_\alpha$ . In order to show that the infinitesimal deformation  $(\xi_\alpha, \eta_\alpha)$  associated to  $\alpha$  is obstructed to second order, we need to compute the quadratic term  $Q_3(\eta_\alpha)$  as discussed in equation (6.2.3) and find an element  $\beta \in \mathfrak{su}(3)$  for which the  $L^2$ -inner product is non-zero.

To compute  $Q_3(\eta_\alpha)$ , one can use the algorithm for stable 4-forms on manifolds with  $G_2$ -structures as discussed in [Hit01]. Using the fact that  $\xi_\alpha = - * \eta_\alpha$ , one can easily show that for some non-zero constant  $c_1$ ,  $Q_3(\eta_\alpha) = c_1 * Q_4(\xi_\alpha)$  where  $Q_4(\xi_\alpha)$  is the quadratic

term associated to  $\xi_\alpha$ . Thus, we instead compute  $Q_4(\xi_\alpha)$  and show that the inner product  $\langle *Q_4(\xi_\alpha), \xi_\alpha \rangle_{L^2} \neq 0$  to prove obstructedness.

Consider  $\varphi_t = \varphi + t\xi_\alpha$  to be a positive 3-form for small  $t$ . We denote the metric and the volume form induced by  $\varphi_t$  by  $g_t$  and  $\sqrt{\det g_t}$  respectively. We have a Taylor series expansion

$$g_t = g_0 + tg_1 + t^2g_2 + O(t^3).$$

Then one can define the symmetric bi-linear form  $B_t$  by

$$(B_t)_{ij} = ((e_i \lrcorner \varphi_t) \wedge (e_j \lrcorner \varphi_t) \wedge \varphi_t)(e_1, \dots, e_7).$$

The zero order term of  $B_t$ , denoted by  $B_0$  is given by

$$(B_0)_{ij} = ((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi)(e_1, \dots, e_7) = -6\delta_{ij}.$$

Similarly, one can compute the linear term  $(B_1)_{ij} = 3((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \xi_\alpha)(e_1, \dots, e_7)$  and the quadratic term  $(B_2)_{ij} = 3((e_i \lrcorner \xi_\alpha) \wedge (e_j \lrcorner \xi_\alpha) \wedge \varphi)(e_1, \dots, e_7)$ . The metric is then defined using the relation (see for example, [Kar09])

$$(B_t)_{ij} = -6(g_t)_{ij} \sqrt{\det g_t}.$$

The linear term in  $\sqrt{\det g_t}$  is proportional to  $\varphi \wedge \eta_\alpha + \psi \wedge \xi_\alpha$  and thus vanishes since  $(\xi_\alpha, \eta_\alpha) \in \Omega_{27}^3 \times \Omega_{27}^4$ . Using the above formula we get that

$$\sqrt{\det g_t} = 1 + At^2 + O(t^3),$$

where  $A$  is a quadratic polynomial in  $v_1, v_2$  and  $x_i, i = 1 \dots 6$ . Using the Taylor series expansion of  $g_t$  and  $\sqrt{\det g_t}$ , we can compute the Taylor series expansion of the Hodge star associated to  $\varphi_t$ ,  $*_t = *_0 + t*_1 + t^2*_2 + O(t^3)$ . The Hodge star operator  $*_t$  can be computed using the formula

$$*_t(e^{i_1 i_2 \dots i_k}) = \frac{\sqrt{\det g_t}}{(7-k)!} g_t^{i_1 j_1} \dots g_t^{i_k j_k} \epsilon_{j_1 \dots j_7} e^{j_{k+1} \dots j_7}.$$

The quadratic term  $Q_4(\xi_\alpha)$  is then given by

$$Q_4(\xi_\alpha) = *_2\varphi + *_1\xi_\alpha.$$

In the present case, for a general element  $\alpha \in \mathfrak{su}(3)$ , the quadratic term turns out to be very complicated and is not very enlightening. We define a cubic polynomial on  $X_{1,1}$  by

$$f_\alpha([h]) = \langle *Q_4(\xi_\alpha), \xi_\alpha \rangle_{L^2}.$$

Note that  $f_\alpha$  is cubic in  $\alpha$  since  $Q_4(\xi_\alpha)$  and  $\xi_\alpha$  are quadratic and linear in  $\alpha$  respectively. This cubic polynomial can be lifted to a polynomial  $P$  on the Lie group  $SU(3) \times SU(2)$  by

$$f_\alpha([h]) = P(h^{-1}\alpha h).$$

This lift enables us to calculate the average of  $P$  on  $SU(3) \times SU(2)$  by using the Peter–Weyl theorem. To express the polynomial  $P$  in a compact form, we set  $z_1 = x_2 + ix_1$ ,  $z_2 = x_4 - ix_3$ ,  $z_3 = x_6 + ix_5$ . Then the cubic polynomial  $P$  is given by

$$\begin{aligned} P(h^{-1}\alpha h) = & -\frac{97}{6}(v_1^2 v_2 + v_2^2 v_1) + \frac{25}{9}\text{Re}(z_1 z_2 z_3) - \frac{29}{6}(v_1^3 + v_2^3) + \frac{5}{3}(v_1 + v_2)|z_1|^2 \\ & + \frac{37}{18}(v_1|z_3|^3 + v_2|z_2|^2) + \frac{31}{9}(v_1|z_2|^3 + v_2|z_3|^2). \end{aligned} \quad (6.3.1)$$

The next step in proving obstructedness is to show that the average value of  $P$  on  $SU(3) \times SU(2)$  is non-zero. For this, we appeal to the Peter–Weyl theorem. The Peter–Weyl theorem states that for any compact Lie group  $G$ , we have

$$L^2(G) = \bigoplus_{V_\gamma \in G_{irr}} \text{Hom}(V_\gamma, G) \otimes V_\gamma$$

where  $G_{irr}$  denotes the set of all non-isomorphic irreducible representations of  $G$ .

The cubic polynomial  $P$  lies in the  $SU(3) \times SU(2)$  representation  $\text{Sym}^3 \mathfrak{su}(3)$ . The average value of the function  $P(g^{-1}\xi g)$  on  $SU(3) \times SU(2)$  is the same as the average value of  $R(h^{-1}\alpha h)$  where  $R$  is the projection of  $P$  to the invariant polynomials. This is because  $(P - R)(h^{-1}\alpha h)$  lies in the non-trivial part of the Peter–Weyl decomposition and has an average value of zero. The unique trivial sub-representation of  $\text{Sym}^3 \mathfrak{su}(3)$  is generated by the determinant polynomial  $i \det$  on  $\mathfrak{su}(3)$  which is given by

$$i \det(g^{-1}\alpha g) = -(v_1 v_2^2 + v_2 v_1^2) + (v_1 + v_2)|z_1|^2 - (v_1|z_3|^2 + v_2|z_2|^2) + 2\text{Re}(z_1 z_2 z_3).$$

The average value of the polynomial  $P$  can be computed by computing the inner product of  $P$  with  $i \det$ . On  $\mathfrak{su}(3)$ , since the Killing form  $B$  is non-degenerate,  $g = -\frac{1}{12}B$  defines

an inner product on  $\mathfrak{su}(3)$ . The inner product  $g$  induces an inner product on  $\text{Sym}^3 \mathfrak{su}(3)$  in the natural way. All the computations that follow are done using  $g$ .

If  $E_{ij}$  denotes the matrix with 1 as the  $(i, j)$ -th entry and zero elsewhere, then the subspace of  $\mathfrak{su}(3)$  generated by  $\{E_{ij} - E_{ji} + i(E_{ij} + E_{ji}) \mid i, j = 1, 2, 3, i \neq j\}$  is orthogonal to  $\text{Span}\{E_{11} - iE_{33}, E_{22} - iE_{33}\}$ . Moreover  $E_{ij} - E_{ji} + i(E_{ij} + E_{ji})$ ,  $i, j = 1, 2, 3, i \neq j$  are also orthogonal to each other. Thus the only non-trivial terms occurring in the inner product of  $P$  and  $i \det$  are,

$$\begin{aligned} \|v_1^2 v_2 + v_2^2 v_1\|^2 &= \frac{1}{3}, & \|\text{Re}(z_1 z_2 z_3)\|^2 &= \frac{2}{3}, & \langle v_1^3 + v_2^3, v_1^2 v_2 + v_2^2 v_1 \rangle &= -\frac{1}{4}, \\ \|(v_1 + v_2)|z_1|^2\|^2 &= 1, & \|v_1|z_3|^2 + v_2|z_2|^2\|^2 &= \frac{4}{3}, & \langle v_1|z_2|^2 + v_2|z_3|^2, v_1|z_3|^2 + v_2|z_2|^2 \rangle &= -\frac{1}{3}. \end{aligned}$$

From (6.3.1) and the above computations we have that

$$\langle P, i \det \rangle = \frac{97}{6} \left( \frac{1}{3} \right) + \frac{50}{9} \left( \frac{2}{3} \right) + \frac{29}{6} \left( -\frac{1}{4} \right) + \frac{5}{3}(1) - \frac{37}{18} \left( \frac{4}{3} \right) - \frac{31}{9} \left( -\frac{1}{3} \right) = \frac{191}{24} \neq 0.$$

Thus we get the following theorem.

**Theorem 6.3.1.** *The infinitesimal deformations of the homogeneous nearly  $G_2$ -structure on the Aloff–Wallach space  $X_{1,1} \cong \frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$  are all obstructed.*

# Chapter 7

## Deformations of $G_2$ instantons on nearly $G_2$ manifolds

### 7.1 Infinitesimal deformation of instantons

Let  $M^7$  be a nearly  $G_2$  manifold. We denote by  $\varphi$  and  $\eta$  the  $G_2$ -structure 3-form on  $M$  and the corresponding unit Killing spinor respectively. We are interested in studying the infinitesimal deformations of nearly  $G_2$  instantons on  $M$ . We define a  $G_2$  instanton as a connection on  $M$  satisfying any of the equivalent conditions in (5.2.3). For defining the infinitesimal deformation of an instanton we will use the spinorial definition of a  $G_2$  instanton, that is  $F_A \cdot \eta = 0$ .

Since an infinitesimal deformation of a connection  $A$  represents an infinitesimal change in  $A$  it is a section of  $T^*M \otimes Ad_{\mathcal{P}}$ . If  $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$  is an infinitesimal deformation of  $A$ , the corresponding change in the curvature  $F_A$  up to first order is given by  $d^A \epsilon$ . A standard gauge fixing condition on this perturbation is given by  $(d^A)^* \epsilon = 0$ . So in total the pair of equations whose solutions define an infinitesimal deformation of an instanton  $A$  is given by

$$(d^A \epsilon) \cdot \eta = 0, \quad (d^A)^* \epsilon = 0. \tag{7.1.1}$$

The 1-parameter family of connections on the spinor bundle  $\mathcal{S}$  defined in (4.1.10) and the connection  $A$  on  $\mathcal{P}$  can be used to construct a 1-parameter family of connections on the associated vector bundle  $\mathcal{S} \otimes Ad_{\mathcal{P}}$ . We denote by  $\nabla^{t,A}$ , the connection induced by  $\nabla^t$



and  $A$  for all  $t \in \mathbb{R}$  respectively. We denote by  $D^{t,A}$  the Dirac operator associated to  $\nabla^{t,A}$  denoted by

$$D^{t,A} = D^A + \frac{t}{2}\varphi \cdot .$$

The following proposition associates the solutions to (7.1.1) to a particular eigenspace of  $D^{t,A}$  for each  $t$ . The proposition was proved in [Fri12] for  $t = 0$ .

**Proposition 7.1.1.** Let  $\epsilon$  be a section of  $T^*M \otimes Ad_{\mathcal{P}}$ , and let  $D^{t,A}$  be the Dirac operator constructed from the connections  $\nabla^{t,A}$  for  $t \in \mathbb{R}$ . Then  $\epsilon$  solves (7.1.1) if and only if

$$D^{t,A}(\epsilon \cdot \eta) = -\frac{t+5}{2}\epsilon \cdot \eta. \quad (7.1.2)$$

*Proof.* Let  $\{e_a, a = 1 \dots 7\}$  be a local orthonormal frame for  $T^*M$ . Then

$$\begin{aligned} D^{0,A}(\epsilon \cdot \eta) &= e_a \cdot \nabla_a^0(\epsilon \cdot \eta) \\ &= (e_a \cdot \nabla_a^0 \epsilon) \cdot \eta + e_a \cdot \epsilon \cdot \nabla_a^0 \eta \\ &= (d^A \epsilon + (d^A)^* \epsilon) \cdot \eta + e_a \cdot \epsilon \cdot \nabla_a^0 \eta. \end{aligned}$$

Applying Proposition 3.2.1 to the 1-form part of  $\epsilon$  we get  $e_a \cdot \epsilon \cdot e_a \cdot \eta = 5\epsilon \cdot \eta$ . So if  $\eta$  is a real Killing spinor then (4.1.7) together with the above identity imply

$$\begin{aligned} D^{0,A}(\epsilon \cdot \eta) &= (d^A \epsilon + (d^A)^* \epsilon) \cdot \eta - \frac{1}{2}e_a \cdot \epsilon \cdot e_a \cdot \eta \\ &= (d^A \epsilon + (d^A)^* \epsilon - \frac{5}{2}\epsilon) \cdot \eta. \end{aligned}$$

It follows from (4.1.10) and the identity  $\sum_a e_a \cdot i_a \varphi = 3\varphi$  that

$$D^{t,A} = D^{0,A} + \frac{t}{2}\varphi \cdot .$$

Since  $\epsilon \cdot \eta \in \Lambda^1 \cdot \eta$ , by Lemma 3.2.3 we have

$$D^{t,A}(\epsilon \cdot \eta) = \left( d^A \epsilon + (d^A)^* \epsilon + \frac{-t-5}{2}\epsilon \right) \cdot \eta.$$

The equation  $D^{t,A}(\epsilon \cdot \eta) = -\frac{t+5}{2}\epsilon \cdot \eta$  is thus equivalent to  $(d^A \epsilon + (d^A)^* \epsilon) \cdot \eta = 0$ , which in turn is equivalent to the pair of equations  $(d^A \epsilon) \cdot \eta = 0, (d^A)^* \epsilon = 0$  since these two components live in complementary subspaces.  $\square$

Since  $\eta$  is parallel with respect to  $\nabla^{-1}$  we can view  $D^{-1,A}$  as an operator on  $\Lambda^1 \otimes Ad_{\mathcal{P}}$  defined by  $D^{-1,A}(\epsilon \cdot \eta) = (D^{-1,A}\epsilon) \cdot \eta$ . The following theorem is an immediate consequence of the above proposition.

**Theorem 7.1.2.** The space of infinitesimal deformations of a  $G_2$  instanton  $A$  on a principal bundle  $\mathcal{P}$  over a nearly  $G_2$  manifold  $M$  is isomorphic to the kernel of the operator

$$\left(D^{-1,A} + 2 \text{Id}\right): \Gamma(\Lambda^1 \otimes Ad_{\mathcal{P}}) \rightarrow \Gamma(\Lambda^1 \otimes Ad_{\mathcal{P}}). \quad (7.1.3)$$

**Remark 7.1.3.** By Proposition 7.1.1, the  $-\frac{t+5}{2}$  eigenspace of the operator  $D^{t,A}$  on  $\Lambda^1 \cdot \eta \otimes Ad_{\mathcal{P}}$  is isomorphic to the infinitesimal deformation space of the instanton  $A$  for all  $t \in \mathbb{R}$  and all these eigenspaces are thus isomorphic to each other. In particular

$$\ker(D^{-1/3,A} + \frac{7}{3}\text{id}) \cong \ker(D^{-1,A} + 2\text{id}). \quad (7.1.4)$$

We can obtain an expression for the square of the Dirac operators constructed above using the Schrödinger–Lichnerowicz formula in the case of skew-symmetric torsion obtained by Agricola–Friedrich in [AF04]. The proof adapted to our setting is presented to keep the discussion self contained.

**Proposition 7.1.4.** Let  $EM$  be a vector bundle associated to  $\mathcal{P}$  and  $\mu \in \Gamma(\mathcal{S} \otimes EM)$ . Let  $A$  be any connection on  $\mathcal{P}$ . Then for all  $t \in \mathbb{R}$ ,

$$(D^{t/3,A})^2 \mu = (\nabla^{t,A})^* \nabla^{t,A} \mu + \frac{1}{4} \text{Scal}_g \mu + \frac{t}{6} d\varphi \cdot \mu - \frac{t^2}{18} \|\varphi\|^2 \mu + F \cdot \mu. \quad (7.1.5)$$

*Proof.* Let  $\{e_1, \dots, e_7\}$  be an orthonormal frame for the tangent bundle. As before we obtain

$$D^{t,A} \mu = (D^{0,A} + \frac{t}{2} \varphi \cdot) \mu.$$

Squaring both sides we obtain,

$$\begin{aligned} (D^{t/3,A})^2 \mu &= \left(D^{0,A} + \frac{t}{6} \varphi \cdot\right)^2 \mu \\ &= (D^{0,A})^2 \mu + \frac{t}{6} (D^{0,A}(\varphi \cdot \mu) + \varphi \cdot D^{0,A} \mu) + \frac{t^2}{36} \varphi \cdot \varphi \cdot \mu. \end{aligned}$$

The first term of the above expression is given by the Schrödinger–Lichnerowicz formula

$$(D^{0,A})^2\mu = (\nabla^{0,A})^*\nabla^{0,A}\mu + \frac{1}{4}\text{Scal}_g\mu + F \cdot \mu. \quad (\text{E1})$$

The anti-commutator in the second term is given by

$$\begin{aligned} D^{0,A}(\varphi \cdot \mu) + \varphi \cdot D^{0,A}\mu &= e_a \cdot \nabla_a^{0,A}(\varphi \cdot \mu) + \varphi \cdot e_a \cdot \nabla_a^{0,A}\mu \\ &= (e_a \cdot \nabla_a^{0,A}\varphi) \cdot \mu + (e_a \cdot \varphi + \varphi \cdot e_a) \cdot \nabla_a^{0,A}\mu \\ &= d\varphi \cdot \mu + d^*\varphi \cdot \mu - 2(e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu \end{aligned} \quad (\text{E2})$$

but since  $M$  is nearly  $G_2$ ,  $\varphi$  is coclosed, therefore

$$D^{0,A}(\varphi \cdot \mu) + \varphi \cdot D^{0,A}\mu = d\varphi \cdot \mu - 2(e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu$$

For the 3-form  $\varphi$ ,  $\varphi \cdot \varphi = \|\varphi\|^2 - (e_a \lrcorner \varphi) \wedge (e_a \lrcorner \varphi)$  and  $(e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi) = -3\|\varphi\|^2 + (e_a \lrcorner \varphi) \wedge (e_a \lrcorner \varphi)$  which imply

$$\begin{aligned} \varphi \cdot \varphi \cdot \mu &= \|\varphi\|^2\mu - (e_a \lrcorner \varphi) \wedge (e_a \lrcorner \varphi) \cdot \mu, \\ &= \|\varphi\|^2\mu - ((e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi) + 3\|\varphi\|^2) \cdot \mu \\ &= -2\|\varphi\|^2\mu - (e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi) \cdot \mu. \end{aligned} \quad (\text{E3})$$

At the center of a normal frame,

$$\begin{aligned} (\nabla^{t,A})^*\nabla^{t,A}\mu &= -(\nabla_a^{0,A} + \frac{t}{6}(e_a \lrcorner \varphi))(\nabla_a^{0,A} + \frac{t}{6}(e_a \lrcorner \varphi))\mu \\ &= -\nabla_a^{0,A}\nabla_a^{0,A}\mu - \frac{t}{6}(e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu - \frac{t}{6}\nabla_a^{0,A}((e_a \lrcorner \varphi) \cdot \mu) \\ &\quad - \frac{t^2}{36}(e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi) \cdot \mu \\ &= (\nabla_a^{0,A})^*\nabla_a^{0,A}\mu - \frac{t}{6}(e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu - \frac{t}{6}(-d^*\varphi \cdot \mu + (e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu) \\ &\quad - \frac{t^2}{36}((e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi)) \cdot \mu. \end{aligned}$$

Again using the fact that  $d^*\varphi = 0$  we get

$$(\nabla_a^{0,A})^*\nabla_a^{0,A}\mu = (\nabla^{t,A})^*\nabla^{t,A}\mu + \frac{t}{3}(e_a \lrcorner \varphi) \cdot \nabla_a^{0,A}\mu + \frac{t^2}{36}((e_a \lrcorner \varphi) \cdot (e_a \lrcorner \varphi)) \cdot \mu. \quad (\text{E4})$$

Substituting the three terms in the expression of  $(D^{t/3,A})^2\mu$  using (E1), (E2), (E3), (E4) we get the result.  $\square$

When the connection  $A$  is an instanton on a nearly  $G_2$  manifold the expression for  $(D^{t/3,A})^2$  can be simplified further. For the  $G_2$ -structure  $\varphi$ ,  $\|\varphi\|^2 = 7$  and under our choice of convention  $d\varphi = 4\psi$  and  $\text{Scal}_g = 42$ . Thus we can calculate the action of  $(D^{t/3,A})^2$  on spinors in  $\Lambda^0\eta$  and  $\Lambda^1 \cdot \eta$  as follows.

Let  $\eta$  be a real Killing spinor then Lemma 3.2.3 implies  $\psi \cdot \eta = 7\eta$  and  $F_A \cdot \eta = 0$  by (5.2.3). Thus by above proposition we obtain,

$$(D^{t/3,A})^2\eta = (\nabla^{t,A})^*\nabla^{t,A}\eta - \frac{7}{18}(t^2 - 12t - 27)\eta. \quad (7.1.6)$$

Now suppose  $\epsilon$  is an infinitesimal deformation of  $A$ . Then  $\epsilon \cdot \eta \in \Gamma(\Lambda^1 M \otimes EM)$ . From Lemma 3.2.3 we know that  $\psi \cdot \epsilon \cdot \eta = -\epsilon \cdot \eta$  and since  $F \cdot \eta = 0$ ,  $F \cdot \epsilon \cdot \eta = (F \cdot \epsilon + \epsilon \cdot F) \cdot \eta = -2(\epsilon \lrcorner F) \cdot \eta$ . Thus by above proposition

$$(D^{t/3,A})^2(\epsilon \cdot \eta) = (\nabla^{t,A})^*\nabla^{t,A}(\epsilon \cdot \eta) - \frac{1}{18}(7t^2 + 12t - 189)\epsilon \cdot \eta - 2(\epsilon \lrcorner F) \cdot \eta. \quad (7.1.7)$$

In the special case when the bundle  $EM$  is equal to  $Ad_{\mathcal{P}}$ , the holonomy group  $H \subset G$  of the connection  $A$  acts on the Lie algebra  $\mathfrak{g}$  of  $G$ . Let us denote by  $\mathfrak{g}_0 \subset \mathfrak{g}$  the subspace on which  $H$  acts trivially. Let  $\mathfrak{g}_1$  be the orthogonal subspace of  $\mathfrak{g}_0$  with respect to the Killing form of  $G$ . The corresponding splitting of the adjoint bundle is given by  $Ad_{\mathcal{P}} = L_0 \oplus L_1$ . By Proposition 7.1.4  $(D^{-1/3,A})^2$  is self adjoint and hence respects the decomposition

$$\mathcal{S} \otimes Ad_{\mathcal{P}} = (\Lambda^1 M \otimes L_0) \oplus (\Lambda^1 M \otimes L_1) \oplus (\Lambda^0 M \otimes L_0) \oplus (\Lambda^0 M \otimes L_1).$$

We use the shorthand  $\Lambda^i L_j$  for  $\Lambda^i M \otimes L_j$  where  $i, j = 0, 1$ . For compact  $M$  we have the following proposition.

**Proposition 7.1.5.** Let  $A$  be a  $G_2$  instanton on a principal  $G$ -bundle  $\mathcal{P}$  with holonomy group  $H$  and suppose  $Ad_{\mathcal{P}}$  splits as above. Then

- (i)  $\ker((D^{-1/3,A})^2 - \frac{49}{9}\text{id}) = \ker((D^{-1/3,A})^2 - \frac{49}{9}\text{id}) \cap (\Lambda^1 L_1 \oplus \Lambda^0 L_0)$ .
- (ii)  $\ker((D^{-1/3,A})^2 - \frac{49}{9}\text{id}) \cap \Lambda^1 L_1 = \left( \ker(D^{-1/3,A} + \frac{7}{3}\text{id}) \oplus \ker(D^{-1/3,A} - \frac{7}{3}\text{id}) \right) \cap \Lambda^1 L_1$ .

*Proof.* To prove (i) we need to show that  $\ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\text{id}) \cap (\Lambda^0 L_1 \oplus \Lambda^1 L_0)$  is trivial.

1. Let  $\mu \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2 \text{id}) \cap \Lambda^0 L_1$ . Thus we have by (7.1.6) ,

$$\begin{aligned} 0 &= \int_M (\mu, (D^{-1/3,A})^2 - \left(\frac{7}{3}\right)^2 \mu) \\ &= \int_M (\mu, (\nabla^{-1,A})^* \nabla^{-1,A} \mu + \left(\frac{49}{9} - \left(\frac{7}{3}\right)^2\right) \mu) \\ &= \int_M \|\nabla^{-1,A} \mu\|^2. \end{aligned}$$

But since the action of the holonomy group of  $A$  fixes no non-trivial elements in  $\mathfrak{g}_1$  and the holonomy group of  $\nabla^{-1}$  acts trivially on  $\Lambda^0$  we get  $\mu = 0$ .

2. Let  $\epsilon \cdot \eta \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2 \text{id}) \cap \Lambda^1 L_0$ . By the definition of  $L_0$  the curvature  $F_A$  acts trivially on  $\epsilon \cdot \eta$  in (7.1.7) and we get,

$$\begin{aligned} 0 &= \int_M (\epsilon \cdot \eta, (D^{-1/3,A})^2 - \left(\frac{7}{3}\right)^2 \epsilon \cdot \eta) \\ &= \int_M (\epsilon \cdot \eta, (\nabla^{-1})^* \nabla^{-1}(\epsilon \cdot \eta) + \left(\frac{97}{9} - \left(\frac{7}{3}\right)^2\right) \epsilon \cdot \eta) \\ &= \int_M \|\nabla^{-1}(\epsilon \cdot \eta)\|^2 + \frac{48}{9} \int_M \|\epsilon \cdot \eta\|^2 \end{aligned}$$

hence  $\epsilon \cdot \eta = 0$ .

For proving (ii) we already know that  $(\ker((D^{-1/3,A}) + \frac{7}{3}) \oplus \ker((D^{-1/3,A}) - \frac{7}{3})) \cap \Lambda^1 L_1 \subset \ker((D^{-1/3,A})^2 - \frac{49}{9} \text{id}) \cap \Lambda^1 L_1$ . The reverse inclusion can be seen using the fact that since  $D^{-1/3,A}$  and  $(D^{-1/3,A})^2$  commute they have the same eigenvectors. Moreover since  $D^{-1/3,A}$  is self adjoint,  $\epsilon \cdot \mu \in \ker((D^{-1/3,A})^2 - \frac{49}{9} \text{id}) \cap \Lambda^1 L_1$  implies  $\|D^{-1/3,A} \epsilon \cdot \mu\| = \frac{7}{3} \|\epsilon \cdot \mu\|$  thus the corresponding eigenvalues of  $D^{-1/3,A}$  can only be  $\pm \frac{7}{3}$ .  $\square$

**Remark 7.1.6.** Note that part (i) for the above proposition holds only for  $D^{-1/3,A}$  and not for any other  $D^{t,A}$  where  $t \neq -1/3$  since the proof explicitly uses the fact that  $\eta$  is parallel with respect to  $\nabla^{-1}$ . But since  $D^{t,A}$  is self adjoint for all  $t \in \mathbb{R}$ , for any  $\lambda \in \mathbb{R}$  we have the following decomposition

$$\ker \{(D^{t,A})^2 - \lambda^2 \text{id}\} \cap \Lambda^1 Ad_{\mathcal{P}} = (\ker \{D^{t,A} - \lambda \text{id}\} \oplus \ker \{D^{t,A} + \lambda \text{id}\}) \cap \Lambda^1 Ad_{\mathcal{P}}.$$

The above proposition has the following important consequence. If the structure group  $G$  is abelian,  $H$  acts as identity on the whole of  $\mathfrak{g}$  which means  $\mathfrak{g}_1 = 0$  and  $L_1$  is trivial.

Thus by Remark 7.1.3 the space of infinitesimal deformations of the  $G_2$  instanton  $A$  which is isomorphic to  $\ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 Ad_{\mathcal{P}} = \ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 L_1$  is zero dimensional.

For  $\omega = \alpha \otimes a \in \Lambda^1 \otimes Ad_{\mathcal{P}}$  and  $F_A = \beta \otimes b \in \Lambda^2 \otimes Ad_{\mathcal{P}}$ , the  $Ad_{\mathcal{P}}$  valued 1-form  $\omega \lrcorner F_A$  is defined by

$$\omega \lrcorner F_A = \alpha \lrcorner \beta \otimes [a, b].$$

In [BO19, Proposition 24] it was proved that the  $G_2$  instanton  $A$  is rigid if all the eigenvalues of the operator

$$\begin{aligned} L_A: \Lambda^1 \otimes Ad_{\mathcal{P}} &\rightarrow \Lambda^1 \otimes Ad_{\mathcal{P}} \\ w &\mapsto -2w \lrcorner F_A \end{aligned}$$

are smaller than 6. We prove a lower bound for the eigenvalue as follows. Let  $\lambda$  be the smallest eigenvalue of  $L_A$ . If  $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$  is an infinitesimal deformation of  $A$  then from (7.1.7) and Theorem 7.1.2 we know that

$$(\nabla^{t,A})^* \nabla^{t,A} \epsilon \cdot \eta = \left( \frac{5t^2}{12} + \frac{3t}{2} - \frac{17}{4} \right) \epsilon \cdot \eta - L_A(\epsilon) \cdot \eta.$$

Taking the inner product with  $\epsilon \cdot \eta$  on both sides we get that if  $\lambda > \min \left\{ \frac{5t^2 + 18t - 51}{12} \mid t \in \mathbb{R} \right\} = -\frac{28}{5}$  then  $\epsilon = 0$  is the only solution. Thus we get the following result.

**Theorem 7.1.7.** Any  $G_2$  instanton  $A$  on a principal  $G$ -bundle over a compact nearly  $G_2$  manifold  $M$  is rigid if

- (i) the structure group  $G$  is abelian, or
- (ii) the eigenvalues of the operator  $L_A$  are either all greater than  $-\frac{28}{5}$  or all smaller than 6.

Some immediate consequences of Theorem 7.1.7 are that the flat instantons are rigid.

### 7.1.1 Infinitesimal deformations of the canonical connection

Let  $M = G/H$  be a homogeneous manifold. Consider the principal  $H$ -bundle  $G \rightarrow M$ . If  $(V, \rho)$  is an  $H$ -representation then the space of smooth sections  $\Gamma(G \times_{\rho} V)$  of the associated vector bundle  $G \times_{\rho} V$  is isomorphic to the space  $C^{\infty}(G, V)_H$  of  $H$ -equivariant smooth functions  $G \rightarrow V$ . We denote by  $l_k$  the left multiplication by  $k \in G$ . The space  $C^{\infty}(G, V)_H$

carries the left regular  $G$ -representation  $\rho_L$  defined by  $\rho_L(g)(f) = g.f = f \circ l_{g^{-1}}$  which is also known as the induced  $G$ -representation  $\text{Ind}_H^G V$ .

For any connection  $A$  on  $G$  the covariant derivative associated to  $A$  on any bundle associated to  $A$  is denoted by  $\nabla^A$ . Let  $s \in \Gamma(G \times_\rho V)$  and  $f_s: G \rightarrow V$  be the  $G$ -equivariant function given by  $s(gH) = [g, f_s(g)]$ . If we denote by  $X_h$  the horizontal lift of  $X \in \Gamma(TM)$  via  $A$ , then  $\nabla^A$  acts on  $s$  as

$$(\nabla_X^A s)(gH) = (g, X_h(f_s)(g)).$$

For the canonical connection on  $G \rightarrow M$ ,  $X_h = X$  for every vector field. Thus the covariant derivative  $\nabla^{can}$  is given by

$$(\nabla_X^{can} s)(gH) = (g, X(f_s)(g)).$$

By the Peter–Weyl Theorem [Kna86, Theorem 1.12] the space of sections can also be formulated as follows. If we denote by  $G_{irr}$  the set of equivalence classes of irreducible  $H$ -representations then

$$\Gamma(G \times_\rho V) = \overline{\bigoplus_{W \in G_{irr}} \text{Hom}(W, V)_H \otimes W}.$$

The embedding  $\text{Hom}(W, V)_H \otimes W$  into  $C^\infty(G, V)_H = \Gamma(G \times_\rho V)$  is given by sending  $(\phi, w)$  to the function  $f_{(\phi, w)}$  which for an irreducible  $G$ -representation  $(W, \tau)$  is defined by  $f_{(\phi, w)}(g) = \phi(\tau(g^{-1})w)$ . Thus  $(\phi, w)$  defines a section  $s_{(\phi, w)}(gH) = [g, f_{(\phi, w)}(g)]$  which we denote by  $(\phi, w)$  as well.

**Claim:** The left  $G$ -action is given by  $g.f_{(\phi, w)} = f_{(\phi, \tau(g)w)}$ .

*Proof.* Let  $k \in G$ . Then since  $g.f = f \circ l_{g^{-1}}$  and  $f_{(\phi, w)}(g) = \phi(\tau(g^{-1})w)$  we have

$$\begin{aligned} (g.f_{(\phi, w)})(k) &= f_{(\phi, w)}(g^{-1}k) = \phi(\tau((g^{-1}k)^{-1})w) \\ &= \phi(\tau(k^{-1})\tau(g)w) = f_{(\phi, \tau(g)w)}(k). \end{aligned}$$

The proof of the claim is now complete. □

We can compute the covariant derivative on  $s_{(\phi, w)} \in \text{Hom}(W, V)_H \otimes W \subset \Gamma(G \times_\rho V)$  by

$$\nabla_X^{can} s_{(\phi, w)}(gH) = X(f_{(\phi, w)})(g) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tX}g)$$

$$\begin{aligned}
&= \frac{d}{dt} \Big|_{t=0} (f_{(\phi,w)} \circ l_{e^{tX}})(g) = \frac{d}{dt} \Big|_{t=0} (e^{-tX} \cdot f)(g) \\
&= \frac{d}{dt} \Big|_{t=0} f_{(\phi, \tau(e^{-tX})w)} = -f_{(\phi, \tau_*(X)w)}(gH).
\end{aligned}$$

The above can be written as

$$\nabla_X^{can}(\phi, w) = -(\phi, \tau_*(X)w). \quad (7.1.8)$$

Thus we get that for the canonical connection the covariant derivative of a section  $s \in \Gamma(G \times_\rho V)$  with respect to some  $X \in \mathfrak{m}$  translates into the derivative  $X(f_s)$ , which is minus the differential of the left-regular representation  $(\rho_L)_*(X)(f_s)$ , see [MS10].

Let  $\{a_i, i = 1 \dots n\}$  be an orthonormal basis of  $\mathfrak{g}$  with respect to  $g = -\frac{3}{40}B$  where  $B$  is the Killing form on  $\mathfrak{g}$  then the Casimir element  $\text{Cas}_{\mathfrak{g}} \in \text{Sym}^2(\mathfrak{g})$  is defined by  $\sum_{i=1}^{\dim G} a_i \otimes a_i$ . On any  $\mathfrak{g}$  representation  $(V, \mu)$  we can define the Casimir invariant  $\mu(\text{Cas}_{\mathfrak{g}}) \in \text{gl}(V)$  by

$$\mu(\text{Cas}_{\mathfrak{g}}) = \sum_{i=1}^n \mu(a_i)^2.$$

For the reductive homogeneous spaces  $G/H$  let  $\{a_i, i = 1 \dots \dim(H)\}$  and  $\{a_i, i = \dim(H) + 1 \dots \dim(G)\}$  be the basis of  $\mathfrak{h}$  and  $\mathfrak{m}$  respectively. If we define  $\text{Cas}_{\mathfrak{h}} = \sum_{i=1}^{\dim(H)} a_i \otimes a_i$  and  $\text{Cas}_{\mathfrak{m}} = \sum_{i=\dim(H)+1}^{\dim(G)} a_i \otimes a_i$  we can decompose  $\text{Cas}_{\mathfrak{g}}$  as

$$\text{Cas}_{\mathfrak{g}} = \text{Cas}_{\mathfrak{h}} + \text{Cas}_{\mathfrak{m}}.$$

Note that  $\text{Cas}_{\mathfrak{m}}$  is just used for notational convenience as  $\mathfrak{m}$  may not be a Lie algebra a priori. Also in  $\text{Cas}_{\mathfrak{h}}$  the trace is taken over  $H$ .

**Remark 7.1.8.** If one uses the metric  $-cB$  instead of  $-B$  then the Casimir operator is divided by the scalar  $c$ .

The adjoint representation  $\text{ad}: H \rightarrow \text{GL}(\mathfrak{m})$  gives rise to the associated vector bundle  $G \times_{\text{ad}} \mathfrak{m}$  on  $G/H$ . Similarly since  $G/H$  has a nearly  $G_2$ -structure we have the adjoint action of  $G_2$  on  $\mathfrak{m}$  which we again denote by  $\text{ad}$  and the isotropy homomorphism  $\lambda: H \rightarrow G_2$  which we can use to construct the associated vector bundle  $G \times_{\text{ad} \circ \lambda} \mathfrak{m}$ . The canonical connection is a connection on both  $G \times_{\text{ad}} \mathfrak{m}$  and  $G \times_{\text{ad} \circ \lambda} \mathfrak{m}$  with structure group  $H$  and  $G_2$  respectively. Therefore it is natural to study the infinitesimal deformation space of the canonical connection in both these situations. Since  $H \subset G_2$ , the deformation space as an  $H$ -connection is a subset of the deformation space as a  $G_2$ -connection.



We can completely describe the deformation space when the structure group is  $H$  but for structure group  $G_2$  we can only find the deformation space for the normal homogeneous nearly  $G_2$  manifolds listed in Table 4.2 since our methods do not work for non-normal homogeneous metrics. However since  $H$  is abelian in both of the non-normal cases Theorem 7.1.7 tells us that the canonical connection is rigid as an  $H$ -connection. But we cannot say anything about the deformation space for the structure group  $G_2$  in those two cases.

Thus the only cases left to consider are listed in Table 4.2. The remainder of this chapter is devoted to computing the infinitesimal deformation space of the canonical connection with the structure group  $H$  and  $G_2$  for the homogeneous spaces listed in Table 4.2.

To study the deformation space of the canonical connection  $\nabla^{can}$  on these homogeneous spaces we rewrite the Schrödinger–Lichnerowicz formula (7.1.7) in terms of the Casimir operator of  $\mathfrak{h}$  and  $\mathfrak{g}$  and then use the Frobenius reciprocity formula to compute the deformation space of the canonical connection in each case. Let  $F$  be the curvature associated to  $\nabla^{can}$  then the operator  $-2\epsilon \lrcorner F$  can be reformulated in terms of  $\text{Cas}_{\mathfrak{h}}$  by doing similar calculations as in [CH16, Lemma 4] which gives

$$-2\epsilon \lrcorner F = (\rho_{\mathfrak{m}^*}(\text{Cas}_{\mathfrak{h}}) \otimes 1_E + 1_{\mathfrak{m}^*} \otimes \rho_E(\text{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^* \otimes E}(\text{Cas}_{\mathfrak{h}}))\epsilon. \quad (7.1.9)$$

Let  $(E, \rho_E)$  be an  $H$ -representation. We denote the tensor product of representations on  $\mathfrak{m}^*$  and  $E$  by  $\rho_{\mathfrak{m}^* \otimes E}$ . For every  $t \in \mathbb{R}$ ,  $D^{t,A}$  denotes the Dirac operator on  $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E) \otimes \mathcal{S}$  associated to the connection  $\nabla^A$  and  $\nabla^t$  on  $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E)$  and  $\mathcal{S}$  respectively. For  $D^{-1/3,can}$  we record the following proposition. From now on we use the same symbol to denote the Lie group representation and the associated Lie algebra representation wherever there is no confusion.

**Proposition 7.1.9.** Let  $\nabla^{can}$  be the canonical connection on a homogeneous nearly  $G_2$  manifold  $M = G/H$ . Let  $(E, \rho_E)$  be an  $H$ -representation and  $\epsilon$  be a smooth section of  $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E)$ . Then

$$(D^{-1/3,can})^2 \epsilon \cdot \eta = (-\rho_L(\text{Cas}_{\mathfrak{g}}) + \rho_E(\text{Cas}_{\mathfrak{h}}))\epsilon + \frac{49}{9}\epsilon \cdot \eta. \quad (7.1.10)$$

*Proof.* We begin by analyzing the rough Laplacian term in the Schrödinger–Lichnerowicz formula for  $(D^{-1/3,can})^2 \epsilon \cdot \eta$  from (7.1.7) and then substitute the  $F$ -dependent term from (7.1.9) in the same. We denote by  $\rho_L$  the left regular representation of  $G$ . From above calculations we know that at the center of a normal orthonormal frame  $\{e_i, i = 1 \dots 7\}$  of  $\mathfrak{m}$  with respect to  $g = -\frac{3}{40}B$ ,

$$(\nabla^{-1,can})^* \nabla^{-1,can} = -\nabla_{e_i}^{-1,can} \nabla_{e_i}^{-1,can} = -\rho_L(e_i)^2 = -\rho_L(\text{Cas}_{\mathfrak{m}}).$$

Since  $\text{Res}_G^H \rho_L = \text{Res}_G^H \text{Ind}_H^G(m^* \otimes E) \cong m^* \otimes E$  we have that  $\rho_{\mathfrak{m}^* \otimes E}(\text{Cas}_{\mathfrak{h}}) = \rho_L(\text{Cas}_{\mathfrak{h}})$ . Also  $\rho_{\mathfrak{m}^*}(e_i)^2 = \rho_{\mathfrak{m}^*}(\text{Cas}_{\mathfrak{h}})$  acts as  $-\text{Ric}$  of the canonical connection on 1-forms which is equal to  $-\frac{16}{3}\text{id}$  from Proposition 4.1.3. Substituting all the terms in (7.1.7) for  $t = -1$  we get

$$\begin{aligned} (D^{-1/3, \text{can}})^2 \epsilon \cdot \eta &= (-\rho_L(\text{Cas}_{\mathfrak{m}})\epsilon + \frac{97}{9}\epsilon + (\rho_{\mathfrak{m}^*}(\text{Cas}_{\mathfrak{h}}) \otimes 1_E + 1_{\mathfrak{m}^*} \otimes \rho_E(\text{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^* \otimes E}(\text{Cas}_{\mathfrak{h}}))\epsilon) \cdot \eta \\ &= (-\rho_L(\text{Cas}_{\mathfrak{m}}) + \rho_L(\text{Cas}_{\mathfrak{h}}))\epsilon + \left(\frac{97}{9} - \frac{16}{3}\right)\epsilon + \rho_E(\text{Cas}_{\mathfrak{h}})\epsilon \cdot \eta \\ &= ((-\rho_L(\text{Cas}_{\mathfrak{g}})\epsilon + \rho_E(\text{Cas}_{\mathfrak{h}})\epsilon + \frac{49}{9}\epsilon) \cdot \eta \end{aligned}$$

which completes the proof.  $\square$

Since all the homogeneous spaces considered in Table 4.2 are naturally reductive and  $H \subset G_2$ , there is an adjoint action of  $H$  on  $\mathfrak{m}, \mathfrak{h}$  and  $\mathfrak{g}_2$  and thus  $H$ -representations on  $\mathfrak{m}^* \otimes \mathfrak{h}$  and  $\mathfrak{m}^* \otimes \mathfrak{g}$  which we denote by  $\rho_{\mathfrak{m}^* \otimes \mathfrak{h}}, \rho_{\mathfrak{m}^* \otimes \mathfrak{g}_2}$ . The corresponding Lie algebra representations are denoted similarly. The infinitesimal deformation space of the instanton  $\nabla^{\text{can}}$  is a subspace of  $\Gamma(\mathfrak{m}^* \otimes E)$  where  $E$  can be either  $\mathfrak{h}$  or  $\mathfrak{g}_2$ .

**Lemma 7.1.1.** *If  $\epsilon$  is an infinitesimal deformation of  $\nabla^{\text{can}}$  on the bundle  $\mathfrak{m}^* \otimes E$  over  $G/H$  then*

$$\rho_E(\text{Cas}_{\mathfrak{h}})\epsilon = \rho_L(\text{Cas}_{\mathfrak{g}})\epsilon$$

where the trace in both the Casimirs is taken over  $G$ .

*Proof.* The proof follows from Propositions 7.1.1 and 7.1.9.  $\square$

Using Lemma 7.1.1 we can reformulate the infinitesimal deformation space of the canonical connection. Since the Casimir operator acts as scalar multiple of the identity on irreducible representations we can solve Lemma 7.1.1 for irreducible subrepresentations of  $L$ . From Theorem 7.1.2 the deformations of the canonical connection are the  $-2$  eigenfunctions  $\epsilon \cdot \eta$  of  $D^{-1, \text{can}}$ . To explicitly compute the deformation space first we need to find the solutions for Lemma 7.1.1 which by above proposition is identical to the space of  $\frac{49}{9}$  eigenfunctions  $\epsilon \cdot \eta$  of  $(D^{-1/3, \text{can}})^2$ . For  $\alpha \in \Lambda^1 \text{Ad}_{\mathcal{P}}$  by Lemma 3.2.3  $D^{t, A} \alpha \cdot \eta = D^{0, A} \alpha \cdot \eta + \frac{t}{2} \varphi \cdot \alpha \cdot \eta = D^{0, A} \alpha \cdot \eta - \frac{t}{2} \alpha \cdot \eta$ . Therefore the  $\pm \frac{7}{3}$  eigenfunctions  $\epsilon \cdot \eta$  of

$D^{-1/3,can}$  correspond to the  $-2$  and  $\frac{8}{3}$  eigenfunction of  $D^{-1,A}$  respectively. By Proposition 7.1.5 we have the following decomposition

$$\begin{aligned} \ker\left((D^{-1/3,can})^2 - \frac{49}{9}\text{id}\right) \cap \Gamma(\mathfrak{m}^* \otimes E) &= \ker(D^{-1,can} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E) \\ &\quad \ker(D^{-1,can} - \frac{8}{3}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E). \end{aligned} \quad (7.1.11)$$

The first summand on the right hand side is isomorphic to the space of infinitesimal deformations of  $\nabla^{can}$  by Theorem 7.1.2. So in the second step we check which of the subspaces in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap (\Gamma(\mathfrak{m}^* \otimes E) \cdot \eta)$  lie in the  $-2$  eigenspace of  $D^{-1,can}$ .

The Killing spinor  $\eta$  is parallel with respect to  $\nabla^{-1}$  therefore by the definition of the Dirac operator and Proposition 7.1.5 we can restrict  $D^{-1,can}$  and  $(D^{-1/3,can})^2$  to operators from  $\Gamma(\mathfrak{m}^* \otimes E) \rightarrow \Gamma(\mathfrak{m}^* \otimes E)$ . On a homogeneous space we can explicitly compute the canonical connection as we describe below.

**Step 1:** Calculating  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$  :

Let  $E_{\mathbb{C}} = \bigoplus_{i=1}^n V_i$  be the decomposition of  $E_{\mathbb{C}}$  into complex irreducible  $H$ -representations. For each  $V_i$  we find all the complex irreducible  $G$ -representations  $W_{i,j}, j = 1 \dots n_i$ , that satisfy the equation

$$\rho_{V_i}(\text{Cas}_{\mathfrak{h}}) = \rho_{W_{i,j}}(\text{Cas}_{\mathfrak{g}}).$$

In order to see whether  $W_{i,j} \subset \text{Ind}_H^G(\mathfrak{m}^* \otimes E)_{\mathbb{C}}$  we find the multiplicity  $m_{i,j}$  of  $W_{i,j}$  in  $\text{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes V_i)$ . Because of Schur's Lemma this multiplicity is given by  $\dim(\text{Hom}(W_{i,j}, \mathfrak{m}_{\mathbb{C}}^* \otimes V_i)_H)$ . Repeating this process for all the  $i, j$ 's and summing over all irreducible  $G$ -representations  $W_{i,j}$  along with their multiplicity we get,

$$\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)_{\mathbb{C}} \cong \bigoplus_{i=1}^n \left( \bigoplus_{j=1}^{n_i} m_{i,j} W_{i,j} \right). \quad (7.1.12)$$

**Step 2:** Calculating  $\ker(D^{-1,can} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$  :

To figure out which of the  $W_{i,j}$ 's found in Step 1 are in the  $\ker(D^{-1,can} + 2\text{id})$  we need to calculate the covariant derivative  $\nabla^{can}$  on  $\text{Hom}(W_{i,j}, \mathfrak{m}_{\mathbb{C}}^* \otimes V_i)_H \otimes W_{i,j} \subseteq \Gamma(\mathfrak{m}^* \otimes E)_{\mathbb{C}}$ .

If  $(W, \tau)$  is an irreducible  $G$ -subrepresentation of  $\text{Ind}_H^G(\mathfrak{m}^* \otimes E)$  then  $\text{Hom}(W, \mathfrak{m}^* \otimes E)_H$  is non-trivial. By Schur's Lemma the dimension of  $\text{Hom}(W, \mathfrak{m}^* \otimes E)_H$  is the number of common irreducible  $H$ -subrepresentations in  $\text{Res}_G^H W$  and  $\mathfrak{m}^* \otimes E$ . Let  $W_{\alpha}$  be such a

common irreducible  $H$ -representation. We denote by  $V|_U$  the subspace of  $V$  isomorphic to  $U$  then  $\text{Hom}(W|_{W_\alpha}, (\mathfrak{m}^* \otimes E)|_{W_\alpha}) = \text{Span}\{\phi_\alpha\}$ . Let  $\tau_*$  be the Lie algebra  $\mathfrak{g}$  representation associated to the  $G$ -representation  $(W, \tau)$  then for  $X \in \Gamma(TM)$  and  $(\phi = \sum c_\alpha \phi_\alpha, w) \in \text{Hom}(W, \mathfrak{m}^* \otimes E)_H \otimes W$ , (7.1.8)

$$\nabla_X^{can}(\phi, w)(eH) = -\phi(\tau_*(X)w) \in \mathfrak{m}^* \otimes E.$$

Using this we can calculate the Dirac operator at  $eH$  by

$$D^{-1,can}(\phi_\alpha, w)(eH) = -\sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\phi_\alpha, w)(eH) = -\sum_{i=1}^7 e_i \cdot \phi_\alpha(\tau_*(e_i)w). \quad (7.1.13)$$

The above method can be extended by linearity to compute the Dirac operator on  $\Gamma(\mathfrak{m}^* \otimes E)$ . Note that we have omitted the Killing spinor  $\eta$  since it is parallel with respect to  $\nabla^{-1}$  so does not effect the eigenspace.

In the following sections we implement the above procedure on each of the four homogeneous spaces.

**Remark 7.1.10.** In a nearly Kähler 6-manifold whose structure is defined by a real Killing spinor  $\eta$ , the spinor  $\text{vol} \cdot \eta$  is another independent real Killing spinor. Any Dirac operator  $\mathcal{D}$  anti-commutes with the Clifford multiplication by  $\text{vol}$  that is  $\mathcal{D}\text{vol} = -\text{vol} \cdot \mathcal{D}$ , hence for all  $\lambda \in \mathbb{R}$  we have  $\ker(\mathcal{D} - \lambda \text{id}) \cong \ker(\mathcal{D} + \lambda \text{id})$ . Therefore  $\ker(\mathcal{D}^2 - \lambda^2 \text{id}) \cong 2 \ker(\mathcal{D} \pm \lambda \text{id})$  and one can compute the  $\lambda$  eigenspace of  $\mathcal{D}$  by computing the  $\lambda^2$  eigenspace of  $\mathcal{D}^2$  as done in [CH16, Proposition 4]. In the case of nearly  $G_2$  manifolds  $\mathcal{D}$  and the 7-dimensional  $\text{vol}$  commute and thus we do not have such an isomorphism between the  $\pm\lambda$  eigenspaces of the Dirac operator. In fact there is no such automatic relation between  $\ker(\mathcal{D}^2 - \lambda^2 \text{id})$  and  $\ker(\mathcal{D} + \lambda \text{id})$  as §7.3 reveals.

**Remark 7.1.11.** The Dirac operator is always self-adjoint therefore the above method of finding a particular eigenspace of a Dirac operator  $D$  can be used more generally in any bundle associated to the spinor bundle over a homogeneous spin manifold. Often times it is easier to find the eigenspaces of the square of the Dirac operator  $D^2$  similar to the case in hand. Once we know the  $\lambda^2$ -eigenspace of  $D^2$  we can apply  $D$  on them to see which of them lie in the  $\lambda$  or  $-\lambda$ -eigenspace of  $D$ .

## 7.2 Eigenspaces of the square of the Dirac operator

In this section we follow *Step 1* of the above procedure. To see which of the irreducible representations of  $G$  satisfy Lemma 7.1.1, we need to compute the Casimir operator on

complex irreducible representations. Given any irreducible representation  $\rho_\lambda$  with highest weight  $\lambda$  we use the Freudenthal formula to compute  $\rho_\lambda(\text{Cas}_{\mathfrak{g}})$ . We drop the constant  $\frac{40}{3}$  in our definition of Casimir operator for this section as it does not play any role in comparing the Casimir operators. Let  $\mu = \frac{1}{2}$ (sum of the positive roots of  $\mathfrak{g}$ ) then the Freudenthal formula states that

$$\rho_\lambda(\text{Cas}_{\mathfrak{g}}) = B(\lambda, \lambda) + 2B(\mu, \lambda). \quad (7.2.1)$$

We compute the deformation space of the canonical connection for  $E = \mathfrak{h}$  and  $E = \mathfrak{g}_2$  as described earlier. In all the examples listed below, Case 1 is for  $E = \mathfrak{h}$  and Case 2 is for  $E = \mathfrak{g}_2$ .

Spin(7)/G<sub>2</sub>

For this space,  $H = G_2$  so there is only one case to consider.

The adjoint representation  $\mathfrak{g}_2$  is the unique 14-dimensional irreducible representation of  $G_2$ . The complex irreducible representations of  $G_2$  are identified with respect to their highest weights of the form  $(p, q) \in \mathbb{Z}_{\geq 0}^2$  and are denoted by  $V_{(p,q)}$ . Here  $V_{(1,0)}$  is the 7-dimensional standard  $G_2$ -representation and  $V_{(0,1)}$  is the 14-dimensional adjoint representation. The reductive splitting of the Lie algebra is given by

$$\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}.$$

We have the following isomorphisms of  $G_2$  representations,

$$\begin{aligned} \mathfrak{h}_{\mathbb{C}} &= (\mathfrak{g}_2)_{\mathbb{C}} \cong V_{(0,1)}, \\ \mathfrak{m}_{\mathbb{C}} &\cong V_{(1,0)}. \end{aligned}$$

The isomorphism  $\mathfrak{spin}(7) \cong \mathfrak{so}(7)$  implies that the eigenvalues of their Casimir operators on irreducible representations are equal. For  $\mathfrak{so}(7)$ , let  $E_{ij}$  be the  $7 \times 7$  skew-symmetric matrix with 1 at the  $(i, j)$ th entry and 0 elsewhere. We define  $H_1 = E_{45} - E_{23}$ ,  $H_2 = E_{67} - E_{45}$  and  $H_3 = E_{45}$ . A Cartan subalgebra for  $\mathfrak{so}(7)$  is given by  $\text{Span}\{H_i, i = 1, 2, 3\}$ . A set of simple roots  $\{\alpha_i, i = 1, 2, 3\}$  is given by

$$\alpha_1 = \begin{bmatrix} i \\ -2i \\ i \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ i \\ -i \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}.$$

The Cartan matrix  $C$  of  $\mathfrak{so}(7)$  which is given by

$$C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

Then one can compute the simple co-roots  $F_i$ s by  $\alpha_i(F_j) = C_{ij}$  which give  $F_1 = iH_2$ ,  $F_2 = -iH_1 + 2iH_3$  and  $F_3 = -2iH_2 - 2iH_3$ . The set of fundamental weights is dual to the set of the simple co-roots. We denote the fundamental weights in decreasing order by  $\lambda_1, \lambda_2$  and  $\lambda_3$  which are dual to  $F_3, F_1, F_2$  respectively. We can compute easily that

$$\begin{bmatrix} B(H_1, H_1) & B(H_1, H_2) & B(H_1, H_3) \\ B(H_2, H_1) & B(H_2, H_2) & B(H_2, H_3) \\ B(H_3, H_1) & B(H_3, H_2) & B(H_3, H_3) \end{bmatrix} = \begin{bmatrix} -20 & 10 & -10 \\ 10 & -20 & 10 \\ -10 & 10 & -10 \end{bmatrix}$$

which implies

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) & B(\lambda_1, \lambda_3) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) & B(\lambda_2, \lambda_3) \\ B(\lambda_3, \lambda_1) & B(\lambda_3, \lambda_2) & B(\lambda_3, \lambda_3) \end{bmatrix} = \begin{bmatrix} 3/40 & 1/10 & 1/20 \\ 1/10 & 1/5 & 1/10 \\ 1/20 & 1/10 & 1/10 \end{bmatrix}.$$

Since half the sum of positive roots is given by  $\lambda_1 + \lambda_2 + \lambda_3$  in [Hum78, Section 13.3] therefore by (7.2.1) on an irreducible  $SO(7)$ -representation  $V_{(m_1, m_2, m_3)}$  with highest weight  $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$ ,  $m_1, m_2, m_3 \geq 0$  we have

$$\rho_\lambda(\text{Cas}_{\mathfrak{so}(7)}) = \frac{1}{40}(3m_1^2 + 8m_2^2 + 4m_3^2 + 8m_1m_2 + 4m_1m_3 + 8m_2m_3 + 18m_1 + 32m_2 + 20m_3).$$

Now we compute the eigenvalues of the Casimir operator for the irreducible representations of  $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ . A Cartan subalgebra of  $\mathfrak{g}_2$  is given by  $\text{Span}\{H_1, H_2\}$ . Here a pair of simple roots  $\beta_1, \beta_2$  is given by

$$\beta_1 = \begin{bmatrix} i \\ -2i \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

and the Cartan matrix  $\tilde{C}$  for  $\mathfrak{g}_2$  is given by

$$\tilde{C} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Let  $\mu_1, \mu_2$  be the fundamental weights in decreasing order then their duals with respect to  $B$  are  $-iH_1 - 2iH_2, iH_2$  respectively and one can compute

$$\begin{bmatrix} B(\mu_1, \mu_1) & B(\mu_1, \mu_2) \\ B(\mu_2, \mu_1) & B(\mu_2, \mu_2) \end{bmatrix} = \begin{bmatrix} 1/15 & 1/10 \\ 1/10 & 1/5 \end{bmatrix}.$$

Again half the sum of the positive roots is given by  $\mu_1 + \mu_2$ . Using these values in the Freudenthal formula for an irreducible  $G_2$ -representation  $V_{(p,q)}$  with highest weight  $p\mu_1 + q\mu_2$  we have

$$\rho_{(p,q)}(\text{Cas}_{\mathfrak{g}_2}) = \frac{1}{15}(p^2 + 3q^2 + 3pq + 5p + 9q).$$

*Case 1:  $E = \mathfrak{g}_2$*

The adjoint representation  $(\mathfrak{g}_2)_{\mathbb{C}} \cong V_{(0,1)}$ . From above

$$\rho_{(0,1)}(\text{Cas}_{\mathfrak{g}_2}) = \frac{4}{5}.$$

Substituting the above found values into Lemma 7.1.1 we get that  $V_{(m_1, m_2, m_3)}$  can be an infinitesimal deformation space for the canonical connection if

$$\frac{1}{40}(3m_1^2 + 8m_2^2 + 4m_3^2 + 8m_1m_2 + 4m_1m_3 + 8m_2m_3 + 18m_1 + 32m_2 + 20m_3) = \frac{4}{5}.$$

But since there are no positive integral solutions of this equation there are no deformations of the canonical connection on  $\text{Spin}(7)/G_2$ .

$\text{SO}(5)/\text{SO}(3)$

The complex irreducible  $\text{SO}(5)$ -representations are characterized by highest weights  $(m_1, m_2) \in \mathbb{Z}_{\geq 0}$ . The complex irreducible representations of  $\text{SO}(3)$  are given by  $S^k \mathbb{C}^2$  which is a  $\binom{2+k-1}{k} = k+1$  dimensional space. The 3-dimensional adjoint representation  $\mathfrak{so}(3)_{\mathbb{C}}$  and the 7-dimensional representation  $\mathfrak{m}_{\mathbb{C}}$  are irreducible  $\text{SO}(3)$ -representations therefore

$$\begin{aligned} \mathfrak{m}_{\mathbb{C}} &\cong S^6 \mathbb{C}^2, \\ \mathfrak{so}(3)_{\mathbb{C}} &\cong S^2 \mathbb{C}^2. \end{aligned}$$

A Cartan subalgebra of  $\mathfrak{so}(5)$  is given by  $\text{Span}\{H_1, H_2\}$  where  $H_1 = E_{12}, H_2 = E_{34}$  where  $E_{ij}$  is the  $5 \times 5$  skew-symmetric matrix with 1 at the  $(i, j)$ th position and 0 elsewhere.

With respect to the Killing form  $B$  on  $\mathfrak{so}(5)$ ,  $H_1$  is orthogonal to  $H_2$  with  $B(H_i, H_i) = -6$  for  $i = 1, 2$ . Let  $\lambda_1, \lambda_2$  be the fundamental weights whose duals are  $i(H_1 - H_2), 2iH_2$  respectively then half the sum of positive roots is given by  $\lambda_1 + \lambda_2$ . Doing similar computations as above we get

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) \end{bmatrix} = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/12 \end{bmatrix}.$$

Using (7.2.1) for the eigenvalues of the Casimir operator for irreducible representation  $V_{(m_1, m_2)}$  of  $\text{SO}(5)$  with highest weight  $m_1\lambda_1 + m_2\lambda_2$  for  $m_1, m_2 \geq 0$  we get,

$$\rho_{(m_1, m_2)}(\text{Cas}_{\mathfrak{so}(5)}) = \frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2).$$

Under the embedding of  $\mathfrak{so}(3)$  in  $\mathfrak{so}(5)$  the Cartan subalgebra of  $\mathfrak{so}(3)$  is given by  $\text{Span}\{2H_1 + H_2\}$ . Here the Cartan subalgebra is 1-dimensional and the fundamental weight  $\mu_1$  is dual to  $4iH_1 + 2iH_2$ . Using  $B(H_i, H_i) = -6$  one can compute that  $B(4H_1 + 2H_2, 4H_1 + 2H_2) = -120$  the eigenvalue of the Casimir operator on the irreducible representation  $S^q\mathbb{C}^2$  of  $\mathfrak{so}(3)$  is given by

$$\rho_q(\text{Cas}_{\mathfrak{so}(3)}) = \frac{1}{120}(q^2 + 2q).$$

*Case 1:  $E = \mathfrak{so}(3)$*

The adjoint representation of  $\mathfrak{so}(3)_{\mathbb{C}}$  is an irreducible  $\mathfrak{so}(3)$  representation with highest weight 2. Thus

$$\rho_E(\text{Cas}_{\mathfrak{so}(3)}) = \rho_2(\text{Cas}_{\mathfrak{so}(3)}) = \frac{1}{15}.$$

We need to find irreducible representations  $V_{(m_1, m_2)}$  of  $\mathfrak{so}(5)$  that satisfy Lemma 7.1.1 which requires

$$\frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = \frac{1}{15}.$$

But since there are no integral solutions for the equation the deformation space is trivial in this case.

*Case 2:  $E = \mathfrak{g}_2$*



The adjoint representation of  $(\mathfrak{g}_2)_{\mathbb{C}}$  splits as an  $\mathfrak{so}(3)$  representation into  $S^2\mathbb{C}^2 \oplus S^{10}\mathbb{C}^2$ . The first component in the splitting has already been studied in case 1 and hence has no contribution to the deformation space. For the second component

$$\rho_{10}(\text{Cas}_{\mathfrak{so}(3)}) = 1.$$

Thus we need to find  $\mathfrak{so}(5)$  representations  $V_{(m_1, m_2)}$  such that

$$\frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = 1,$$

which has one integral solution namely  $m_1 = 0, m_2 = 2$ . Thus  $V_{(0,2)} \cong \mathfrak{so}(5)_{\mathbb{C}}$  is the only  $\text{SO}(5)$ -representation for which  $\text{Cas}_{\mathfrak{g}}$  has eigenvalue 1. As  $\mathfrak{so}(3)$  representations

$$V_{(0,2)} \cong S^2\mathbb{C}^2 \oplus S^6\mathbb{C}^2,$$

$$\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2 \cong \bigoplus_{k=2}^8 S^{2k}\mathbb{C}^2.$$

Thus  $V_{(0,2)}$  and  $\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2$  have 1 common irreducible  $\mathfrak{so}(3)$  representation namely  $S^6\mathbb{C}^2$ . Thus  $V_{(0,2)}$  occurs in  $\text{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2)$  with multiplicity 1. Therefore in this case

$$(\ker((D^{-1/3, \text{can}})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2))_{\mathbb{C}} \cong V_{(0,2)}.$$

$$\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$$

The Lie algebra  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  decomposes as

$$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d \oplus \mathfrak{m}$$

and the embeddings  $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$  are given by

$$\mathfrak{sp}(1)_u = \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) : a \in \mathfrak{sp}(1) \right\}, \quad \mathfrak{sp}(1)_d = \left\{ \left( \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, a \right) : a \in \mathfrak{sp}(1) \right\}$$

where we follow the notations used in [AS12]. Let  $H_1 = (E_1, 0), H_2 = (E_2, 0)$  and  $H_3 = (0, E_3)$  then a Cartan subalgebra of  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  is given by  $\text{Span}\{H_1, H_2, H_3\}$  where

$$E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If  $B$  denote the Killing form of  $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$  we can compute that  $H_i$ s are orthogonal with respect to  $B$  and  $B(H_i, H_i) = -12$  for  $i = 1, 2$  and  $B(H_3, H_3) = -8$ . The fundamental weights  $\lambda_1, \lambda_2, \lambda_3$  are dual to  $i(H_1 - H_2), iH_1, iH_3$  respectively and half the sum of positive roots is given by  $\lambda_1 + \lambda_2 + \lambda_3$ . By identical calculations as in other cases we get

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) & B(\lambda_1, \lambda_3) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) & B(\lambda_2, \lambda_3) \\ B(\lambda_3, \lambda_1) & B(\lambda_3, \lambda_2) & B(\lambda_3, \lambda_3) \end{bmatrix} = \begin{bmatrix} 1/12 & 1/12 & 0 \\ 1/12 & 1/6 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}.$$

Applying the Freudenthal formula (7.2.1) we get that the Casimir operator of  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  acts on the irreducible representations  $V_{(m_1, m_2, l)}$  with highest weight  $m_1\lambda_1 + m_2\lambda_2 + l\lambda_3, m_1, m_2, l \geq 0$  with the eigenvalue

$$\rho_{(m_1, m_2, l)}(\mathrm{Cas}_{\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)}) = \frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l).$$

Under the embedding given above a Cartan subalgebra of  $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$  is given by  $\mathrm{Span}\{H_1\}$  and  $\mathrm{Span}\{(E_2, E_3)\}$  respectively. Let  $P, Q$  be the standard 2-dimensional representation of  $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$  respectively. Then the unique  $(n+1)$ -dimensional irreducible  $\mathfrak{sp}(1)_u$  (respectively  $\mathfrak{sp}(1)_d$ ) representation is given by  $S^n P$  (respectively  $S^n Q$ ). From previous calculations we have  $B(H_1, H_1) = -12$  thus the eigenvalue of  $\mathrm{Cas}_{\mathfrak{sp}(1)_u}$  on  $S^n P$  is given by

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_u}) = \frac{1}{12}(n^2 + 2n).$$

Similarly with the help of previous work one can calculate  $B((E_2, E_3), (E_2, E_3)) = -20$ . Thus  $\mathrm{Cas}_{\mathfrak{sp}(1)_d}$  acts on  $S^n Q$  as the scalar multiple of

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_d}) = \frac{1}{20}(n^2 + 2n).$$

The adjoint representation  $\mathfrak{sp}(1)$  is an irreducible 3-dimensional  $\mathfrak{sp}(1)$  representation and hence we have the following decompositions into  $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$  representations

$$(\mathfrak{sp}(1)_u)_{\mathbb{C}} \cong S^2 P, \quad (\mathfrak{sp}(1)_d)_{\mathbb{C}} \cong S^2 Q, \quad \mathfrak{m}_{\mathbb{C}} \cong S^2 Q \oplus PQ$$

where  $PQ$  denotes the tensor product of  $P$  and  $Q$  and we omitted the tensor product sign for clarity and continue to do so.

*Case 1:  $E = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$*

We need to find the irreducible  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  representations  $V_{(m_1, m_2, l)}$  that satisfy Lemma 7.1.1 for each irreducible component of  $\mathfrak{h}_{\mathbb{C}}$  that is  $(\mathfrak{sp}(1)_u)_{\mathbb{C}}$  and  $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$ . For  $\mathfrak{sp}(1)_u$  this equation takes the form

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{12}.$$

The integral solution  $(m_1, m_2, l)$  for this equation is  $(0, 1, 0)$ . Thus the only irreducible  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  representations for which  $\text{Cas}_{\mathfrak{g}}$  has eigenvalue  $\frac{2}{3}$  is  $V_{(0,1,0)}$ . As  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$ -representations we have the following decomposition

$$\begin{aligned} V_{(0,1,0)} &\cong PQ \oplus \mathbb{C}, \\ (\mathfrak{sp}(1)_u \otimes \mathfrak{m})_{\mathbb{C}} &\cong S^2PS^2Q \oplus S^3PQ \oplus PQ. \end{aligned}$$

The irreducible  $\text{Sp}(1) \times \text{Sp}(1)$  representation in  $(\mathfrak{sp}(1)_u \otimes \mathfrak{m})_{\mathbb{C}}$  common with  $V_{(0,1,0)}$  is  $PQ$  with multiplicity 1. Thus  $V_{(0,1,0)}$  occurs in  $\text{Ind}_H^G(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$  with multiplicity 1. Therefore the solutions to Lemma 7.1.1 in  $\Gamma(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$  is the 5-dimensional complex  $\text{Sp}(2) \times \text{Sp}(1)$  representation  $V_{(0,1,0)}$ .

For the next irreducible  $\mathfrak{h}_{\mathbb{C}}$  component  $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$  Lemma 7.1.1 for  $V_{(m_1, m_2, l)}$  becomes

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{20},$$

which has no integral solutions and thus it has no contribution to the deformation space.

Thus from Proposition 7.1.9 we conclude that when the structure group is  $\text{Sp}(1)_u \times \text{Sp}(1)_d$  we have

$$(\ker((D^{-1/3, \text{can}})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d)_{\mathbb{C}}) \cong (V_{(0,1,0)}).$$

*Case 2:  $E = (\mathfrak{g}_2)_{\mathbb{C}}$*

The adjoint representation of  $\mathfrak{g}_2$  decomposes into irreducible  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}} = S^2P \oplus S^2Q \oplus PS^3Q.$$

We have already seen the contribution of the first two irreducible components in the summation. For the third component

$$\rho_{1,3}(\text{Cas}_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}) = 1,$$

so here we need to find the  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  representations  $V_{(m_1, m_2, l)}$  such that

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = 1.$$

The  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ -representations that satisfy Lemma 7.1.1 are  $V_{(2,0,0)}$  and  $V_{(0,0,2)}$ , which decompose into  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  representations as

$$V_{(2,0,0)} \cong \mathfrak{sp}(2)_{\mathbb{C}} \cong S^2P \oplus S^2Q \oplus PQ, \quad V_{(0,0,2)} \cong (\mathfrak{sp}(1)_d)_{\mathbb{C}} \cong S^2Q.$$

Moreover

$$PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^* \cong S^2PS^4Q \oplus S^2PS^2Q \oplus P(S^5Q \oplus S^3Q \oplus Q) \oplus S^4Q \oplus S^2Q.$$

Thus  $V_{(2,0,0)}$  and  $PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^*$  have two common irreducible representations  $PQ, S^2Q$  and  $V_{(0,0,2)}$  and  $PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^*$  have one common irreducible representation  $S^2Q$ . So by Frobenius reciprocity  $V_{(2,0,0)}$  and  $V_{(0,0,2)}$  lie in  $\text{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)$  with multiplicity 2, 1 respectively. Thus the solution of Lemma 7.1.1 in  $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  is the 28 dimensional  $\text{Sp}(2) \times \text{Sp}(1)$  complex representation  $2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}$ . So again by Proposition 7.1.9 we conclude that when the structure group is  $G_2$  we have

$$\ker((D^{-1/3, \text{can}})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}} \cong 2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}.$$

$$\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$$

The embeddings of  $\mathfrak{su}(2)$  and  $\mathfrak{u}(1)$  in  $\mathfrak{su}(3) \times \mathfrak{su}(2)$  which we denote by  $\mathfrak{su}(2)_d$  and  $\mathfrak{u}(1)$  following [AS12] in  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  are given by

$$\mathfrak{su}(2)_d = \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a \right) : a \in \mathfrak{su}(2) \right\}, \quad \mathfrak{u}(1) = \text{span} \left\{ \left( \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0 \right) \right\}.$$

A Cartan subalgebra of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  is given by  $\text{span}\{H_1 = (E_1, 0), H_2 = (E_2, 0), H_3 = (0, E_3)\}$  where

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can check that the  $H_i$ s are orthogonal with respect to the Killing form  $B$  on  $SU(3) \times SU(2)$ . As earlier we denote by  $\lambda_1, \lambda_2, \lambda_3$  the fundamental weights which are dual to  $\frac{i}{2}(H_1 - H_2), \frac{i}{2}(H_1 + H_2), iH_3$  respectively. By direct computations we get

$$\begin{bmatrix} B(H_1, H_1) & B(H_1, H_2) & B(H_1, H_3) \\ B(H_2, H_1) & B(H_2, H_2) & B(H_2, H_3) \\ B(H_3, H_1) & B(H_3, H_2) & B(H_3, H_3) \end{bmatrix} = \begin{bmatrix} -12 & 0 & 0 \\ 0 & -36 & 0 \\ 0 & 0 & -8 \end{bmatrix},$$

therefore

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) & B(\lambda_1, \lambda_3) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) & B(\lambda_2, \lambda_3) \\ B(\lambda_3, \lambda_1) & B(\lambda_3, \lambda_2) & B(\lambda_3, \lambda_3) \end{bmatrix} = \begin{bmatrix} 1/9 & 1/18 & 0 \\ 1/18 & 1/9 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}.$$

Half the sum of the positive roots is  $\lambda_1 + \lambda_2 + \lambda_3$  and thus by Freudenthal formula (7.2.1) for a  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  representation  $V_{(m_1, m_2, l)}$  with highest weight  $m_1\lambda_1 + m_2\lambda_2 + l\lambda_3$  where  $m_1, m_2, l \geq 0$

$$\rho_{m_1, m_2, l}(\text{Cas}_{\mathfrak{su}(3) \oplus \mathfrak{su}(2)}) = \frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l).$$

Using the embeddings of  $\mathfrak{su}(2)$  and  $\mathfrak{u}(1)$  given above we see that Cartan subalgebras of  $\mathfrak{su}(2)$  and  $\mathfrak{u}(1)$  in  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  are given by  $\text{span}\{(E_1, E_3)\}$  and  $\text{span}\{H_2\}$  respectively. By calculations completely analogous to the previous case we then get that if we represent the irreducible  $(n+1)$ -dimensional  $\mathfrak{su}(2)_d$  representations by  $S^n W$  where  $W$  is the standard  $\mathfrak{su}(2)_d$  representation and the 1-dimensional  $\mathfrak{u}(1)$  representation with highest weight  $k$  by  $F(k)$  we get by the Freudenthal formula (7.2.1)

$$\begin{aligned} \rho_n(\text{Cas}_{\mathfrak{su}(2)_d}) &= \frac{1}{20}(n^2 + 2n), \\ \rho_k(\text{Cas}_{\mathfrak{u}(1)}) &= \frac{1}{36}k^2. \end{aligned}$$

As  $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$  representations the 7-dimensional space  $\mathfrak{m}_{\mathbb{C}}$  decomposes as

$$\mathfrak{m}_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3),$$

whereas the 3-dimensional adjoint representation of  $(\mathfrak{su}(2)_d)_{\mathbb{C}}$  is irreducible and hence is isomorphic to  $S^2W$ .

*Case 1:*  $E = \mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$

The adjoint representation  $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$  splits as irreducible  $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$  representations as follows:

$$(\mathfrak{su}(2)_d \oplus \mathfrak{u}(1))_{\mathbb{C}} \cong S^2W \oplus \mathbb{C}.$$

Since  $U(1)$  is abelian we know by Theorem 7.1.7 that the component  $\mathfrak{u}(1)$  is abelian and thus gives rise to no deformations of the canonical connection. Therefore we only need to check for deformations corresponding to  $S^2W$ . For that we need to look for representations  $V_{(m_1, m_2, l)}$  such that

$$\frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{20},$$

which as seen before has no integral solutions.

Hence the canonical connection admits no deformations in this case.

*Case 2:  $E = \mathfrak{g}_2$*

The adjoint representation  $(\mathfrak{g}_2)_{\mathbb{C}}$  splits as  $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$  representation as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}} = S^3WF(3) \oplus S^3WF(-3) \oplus S^2W \oplus F(6) \oplus F(-6) \oplus \mathbb{C}.$$

We need to follow the same procedure as above for each of the components. For each component we need to find the  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  representation  $V_{(m_1, m_2, l)}$  that satisfies Lemma 7.1.1. We have already solved this for  $S^2W \oplus \mathbb{C}$  so we just need to compute it for the rest.

From above calculations  $\rho_{S^3WF(3)}(\text{Cas}_{\mathfrak{h}}) = 1$  therefore  $V_{(m_1, m_2, l)}$  should satisfy

$$\frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l) = 1.$$

The only possible solutions are  $V_{(0,0,2)}, V_{(1,1,0)}$ . As  $\mathfrak{su}(2) \otimes \mathfrak{u}(1)$  representations  $V_{(0,0,2)} \cong S^2W$  and  $V_{(1,1,0)} \cong \mathfrak{su}(3)_{\mathbb{C}}$ . Further one can compute

$$\begin{aligned} V_{(0,0,2)} &\cong \mathfrak{su}(2)_{\mathbb{C}} \cong S^2W, \\ V_{(1,1,0)} &\cong \mathfrak{su}(3)_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3) \oplus \mathbb{C}, \\ S^3WF(3) \otimes \mathfrak{m}_{\mathbb{C}}^* &\cong (S^5W \oplus S^3W \oplus W)F(3) \oplus (S^4W \oplus S^2W)F(6) \oplus S^4W \oplus S^2W. \end{aligned}$$

Thus  $V_{(0,0,2)}$  and  $S^3WF(3) \otimes \mathfrak{m}_{\mathbb{C}}^*$  has one common component  $S^2W$  with multiplicity 1 and  $V_{(1,1,0)}$  and  $S^3WF(3) \otimes \mathfrak{m}_{\mathbb{C}}^*$  has two common components  $S^2W, WF(3)$  both with multiplicity 1 each. So by Frobenius reciprocity  $\text{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3))$  contains a copy of  $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$ .

The representation  $S^3WF(-3)$  is the dual of the representation  $S^3WF(3)$  and since  $SU(2) \otimes U(1)$  representations are isomorphic to their duals the result for this case is same as the above and  $\text{Ind}_H^G(\mathfrak{m}_\mathbb{C}^* \otimes S^3WF(-3))$  also contains a copy of  $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$ .

For the  $\mathfrak{u}(1)$  representation  $F(6)$ ,  $\rho_6(\text{Cas}_{\mathfrak{u}(1)}) = 1$ . Thus again the only solutions are  $V_{(0,0,2)}, V_{(1,1,0)}$  by the previous case. The  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  representation  $F(6) \otimes \mathfrak{m}_\mathbb{C}^*$  has the following decomposition

$$F(6) \otimes \mathfrak{m}_\mathbb{C}^* \cong S^2WF(6) \oplus WF(9) \oplus WF(3),$$

thus  $V_{(0,0,2)}$  is not contained in  $\text{Ind}_H^G(\mathfrak{m}_\mathbb{C}^* \otimes F(6))$  but  $V_{(1,1,0)}$  is with multiplicity 1. Since  $F(-6) \cong F(6)^*$  this case is similar to the above case.

Summing up all the parts together we get that when the structure group is  $G_2$  we have

$$\ker((D^{-1/3,can})^2 - \frac{49}{9}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_\mathbb{C} \cong 2(V_{(0,0,2)} \oplus 3V_{(1,1,0)}).$$

Table 7.1 lists the  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$  when  $E = \mathfrak{h}$  and  $E = \mathfrak{g}_2$  for all the homogeneous spaces listed in Table 4.2. Note that for the remaining two homogeneous spaces  $N_{k,l}, k \neq l$  and  $SU(2)^3/U(1)^2$  our methods does not apply when  $E = \mathfrak{g}_2$  although since  $H$  is abelian for both of them there are no deformations for the  $E = \mathfrak{h}$  case. The space  $V^{(0,1)}$  listed in Table 7.1 denotes the unique irreducible 5-dimensional complex representation of  $\mathfrak{sp}(2)$ .

$G/H$	Structure group	
	$H$	$G_2$
$\text{Spin}(7)/G_2$	0	0
$\text{SO}(5)/\text{SO}(3)$	0	$\mathfrak{so}(5)$
$\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$2\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus V_{\mathbb{R}}^{(0,1)}$
$\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$	0	$2\mathfrak{su}(2) \oplus 6\mathfrak{su}(3)$
$N_{k,l}$	0	unknown
$\text{SU}(2)^3/\text{U}(1)^2$	0	unknown

Table 7.1:  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$

### 7.3 Eigenspaces of the Dirac operator

All the  $G$ -representations listed in Table 7.1 lie in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$  which by (7.1.11) is equal to  $(\ker(D^{-1,can} + 2\text{id}) \oplus \ker(D^{-1,can} - \frac{8}{3}\text{id})) \cap \Gamma(\mathfrak{m}^* \otimes E)$ . Since the canonical connection is translation invariant it takes an irreducible  $G$ -representation to itself. Hence the irreducible subspaces found in Table 7.1 lie in either  $\ker(D^{-1,can} - \frac{8}{3}\text{id})$  or  $\ker(D^{-1,can} + 2\text{id})$  where the subspaces in the latter space constitute the infinitesimal deformations of the canonical connection by Theorem 7.1.2. Thus now it remains to identify which of the subspaces in Table 7.1 lies in  $\ker(D^{-1,can} + 2\text{id})$  for each of the homogeneous spaces. For all the homogeneous spaces  $G/H$  in Table 4.2 the metric corresponding to the nearly  $G_2$ -structure  $\varphi$  is given by  $-\frac{3}{40}B$  where  $B$  is the Killing form of  $G$ . For 1-forms  $X, Y$  the Clifford product between  $X$  and  $Y \cdot \eta$  is given by

$$X \cdot Y \cdot \eta = \langle X, Y \rangle \eta - \varphi(X, Y, \cdot) \cdot \eta. \quad (7.3.1)$$

Thus we have all the ingredients in (7.1.13) to calculate the action of the Dirac operator  $D^{-1,can}$  on each irreducible subspace in Table 7.1.

SO(5)/SO(3)

From the previous section we know that there are no deformation of the canonical connection when the structure group is SO(3). For the structure group  $G_2$  we calculated that the smooth sections of  $G \times_{\rho_{\mathfrak{m}^* \otimes \mathfrak{g}_2}} (\mathfrak{m}^* \otimes \mathfrak{g}_2)$  in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cong V_{(0,2)} \cong \mathfrak{so}(5)_{\mathbb{C}}$ . If we denote by  $E_{ij}$  the skew-symmetric matrix with 1 at  $(i, j)$ ,  $-1$  at  $(j, i)$  and 0 elsewhere and define

$$\begin{aligned} e_1 &:= \frac{2}{3}(E_{12} - 2E_{34}), & e_2 &:= \frac{2}{3}(\sqrt{2}E_{45} - \frac{\sqrt{3}}{\sqrt{2}}(E_{23} - E_{14})), \\ e_3 &:= \frac{2\sqrt{5}}{3}E_{25}, & e_4 &:= \frac{2}{3}(\sqrt{2}E_{35} - \frac{\sqrt{3}}{\sqrt{2}}(E_{13} + E_{24})), \\ e_5 &:= \frac{\sqrt{10}}{3}(E_{24} - E_{13}), & e_6 &:= -\frac{\sqrt{10}}{3}(E_{23} + E_{14}), & e_7 &:= \frac{2\sqrt{5}}{3}E_{15}, \end{aligned}$$

then  $\{e_i, i = 1 \dots 7\}$  defines a basis of  $\mathfrak{m}^*$  which is orthonormal with respect to the metric  $-\frac{3}{40}B$ . With respect to this basis the nearly  $G_2$  structure  $\varphi$  is given by

$$\varphi = e_{124} + e_{137} + e_{156} + e_{235} + e_{267} + e_{346} + e_{457}.$$



We have seen that for  $\mathrm{SO}(5)/\mathrm{SO}(3)$  the canonical connection has no deformation as an  $\mathrm{SO}(3)$  connection. Now we need to check whether the  $\mathrm{SO}(5)$ -representation  $V_{(0,2)}$  lies in the  $\ker(D^{-1,\mathrm{can}} - \frac{8}{3}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  or  $\ker(D^{-1,\mathrm{can}} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ . As seen before the common irreducible  $\mathfrak{so}(3)$  representation in  $V_{(0,2)}|_{\mathfrak{so}(3)}$  and  $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  is  $S^6\mathbb{C}^2 \cong \mathfrak{m}_{\mathbb{C}}^*$ . We denote the 1-dimensional space  $\mathrm{Hom}(V_{(0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}(\alpha)$ . Let  $\mu_i, i = 1 \dots 11$  be a basis of the 11-dimensional subspace of  $(\mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to the  $\mathfrak{so}(3)$  representation  $S^{10}\mathbb{C}^2$ . Then the subspace of  $\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2 \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $S^6\mathbb{C}^2$  is given by  $\mathrm{Span}\{v_i, i = 1 \dots 7\}$  where

$$\begin{aligned}
v_1 &= -\frac{e_2}{14} \otimes (5(\mu_1 - \mu_7) + 3\sqrt{15}\mu_9) + e_3 \otimes (\mu_5 + \mu_{11}) - \frac{e_4}{14} \otimes (5\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)) \\
&\quad + e_5 \otimes (\mu_3 - \mu_4) + e_6 \otimes \mu_9 + e_7 \otimes (\mu_6 - \mu_{10}), \\
v_2 &= e_1 \otimes \mu_9 + e_2 \otimes (-2\mu_5 + \mu_4) - \frac{e_3}{28} \otimes (47\mu_1 + 37\mu_7 + 3\sqrt{5}\mu_9) - e_4 \otimes (\mu_6 + 2\mu_{10}) \\
&\quad - \frac{e_5}{14} \otimes \mu_8 + \frac{e_7}{28} \otimes (-37\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)), \\
v_3 &= -\frac{e_1}{2} \otimes (\mu_3 - \mu_4) + \frac{e_2}{2} \otimes (2\mu_6 + \mu_{10}) + \frac{e_3}{56} \otimes (47\mu_2 + 3\sqrt{5}(\mu_3 + \mu_4)) - \frac{e_4}{2} \otimes (\mu_5 - 2\mu_{11}) \\
&\quad - \frac{e_6}{28} \otimes \mu_8 + \frac{e_7}{56} \otimes (-37\mu_1 + 6\sqrt{15}\mu_9), \\
v_4 &= -\frac{e_1}{28} \otimes (5\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)) + \frac{5e_2}{28} \otimes \mu_8 - \frac{e_3}{56} \otimes (3\sqrt{15}\mu_2 + 41\mu_3 + 13\mu_4) \\
&\quad - \frac{e_5}{2} \otimes (\mu_5 - 2\mu_{11}) + \frac{e_6}{2} \otimes (\mu_6 + 2\mu_{10}) + \frac{e_7}{56} \otimes (3\sqrt{15}(\mu_1 - \mu_7) + 41\mu_9), \\
v_5 &= e_1 \otimes (\mu_5 + \mu_{11}) - \frac{e_2}{28} \otimes (3\sqrt{15}(\mu_1 - \mu_7) + 13\mu_9) + \frac{e_4}{28} \otimes (3\sqrt{15}\mu_2 + 41\mu_3 + 13\mu_4) \\
&\quad + \frac{e_5}{28} \otimes (47\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)) + \frac{e_6}{28} \otimes (47\mu_1 + 37\mu_7 + 3\sqrt{15}\mu_9) + \frac{2e_7}{28} \otimes \mu_8, \\
v_6 &= e_1 \otimes (-\mu_6 + \mu_{10}) + \frac{e_2}{28} \otimes (3\sqrt{15}\mu_2 + 13\mu_3 + 41\mu_4) + \frac{2e_3}{7} \otimes \mu_8 + \frac{e_4}{28} \otimes (3\sqrt{15}(\mu_1 - \mu_7) \\
&\quad + 41\mu_9) + \frac{e_5}{28} \otimes (37\mu_1 + 47\mu_7 - 3\sqrt{15}\mu_9) + \frac{e_6}{28} \otimes (-37\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)), \\
v_7 &= \frac{e_1}{14} \otimes (5(\mu_1 - \mu_7) + 3\sqrt{15}\mu_9) - \frac{e_3}{28} \otimes (3\sqrt{15}(\mu_1 - \mu_7) + 13\mu_9) + \frac{5e_4}{14} \otimes \mu_8 \\
&\quad - 2e_5 \otimes (\mu_6 + \mu_{10}) + e_6 \otimes (-2\mu_5 + \mu_{11}) - \frac{e_7}{28} \otimes (3\sqrt{15}\mu_2 + 13\mu_3 + 41\mu_4).
\end{aligned}$$

The subspace of  $V_{(0,2)}$  isomorphic to  $S^6\mathbb{C}^2$  is  $\mathrm{Span}_{\mathbb{C}}\{e_i, i = 1 \dots 7\}$  and the  $\mathrm{SO}(3)$  equivariant homomorphism  $\alpha$  between  $V_{(0,2)}$  and  $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  is given by

$$\alpha(e_1) = v_1, \quad \alpha(e_2) = v_7, \quad \alpha(e_3) = -v_5,$$

$$\alpha(e_4) = -2v_4, \quad \alpha(e_5) = 2v_3, \quad \alpha(e_6) = -v_2, \quad \alpha(e_7) = v_6.$$

Any section of the bundle associated to  $\mathfrak{m}^* \otimes \mathfrak{g}_2$  in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id})$  can be represented by  $(\alpha, v)$  for some  $v \in V_{(0,2)}|_{S^6\mathbb{C}^2} \cong \mathfrak{m}_{\mathbb{C}}^*$ . The action of the canonical connection on such a section is then given by  $\nabla_X^{-1,can}(\alpha, v)(eH) = -\alpha([X, v])$  where the Lie bracket is in  $\mathfrak{so}(5)$ . We can now calculate the action of the Dirac operator,  $D^{-1,can}$  on  $(\alpha, e_1) \cdot \eta$  at the point  $eH$  as follows. We omit the  $\cdot \eta$  from the computations to reduce notational clutter and continue to do so in every case.

$$\begin{aligned} D^{-1,can}(\alpha, e_1)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\alpha, e_1)(eH) \\ &= -\frac{2}{3}(e_2 \cdot \alpha(e_4) + e_3 \cdot \alpha(e_7) + e_4 \cdot \alpha(-e_2) + e_5 \cdot \alpha(e_6) \\ &\quad + e_6 \cdot \alpha(-e_5) + e_7 \cdot \alpha(-e_3)) \\ &= \frac{2}{3}(2e_2 \cdot v_4 - e_3 \cdot v_6 + e_4 \cdot v_7 + e_5 \cdot v_2 + 2e_6 \cdot v_3 - e_7 \cdot v_5) \\ &= \frac{2}{3}(-3v_1) \cdot \eta = -2\alpha(e_1). \end{aligned}$$

Thus by the translation invariance of the canonical connection

$$V_{(0,2)} \subseteq \ker(D^{-1,can} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}.$$

$$\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$$

From the previous section we know that for  $E = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$  we have

$$\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)_{\mathbb{C}} \cong V_{(0,1,0)}.$$

Let  $\{e_i, i = 1 \dots 7\}$  be an orthonormal basis of  $\mathfrak{m}^*$  with respect to the metric  $-\frac{3}{40}B$  given by

$$\begin{aligned} e_1 &:= \frac{1}{3} \left( \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i \right), & e_2 &:= \frac{1}{3} \left( \begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, -3j \right), & e_3 &:= \frac{1}{3} \left( \begin{pmatrix} 0 & 0 \\ 0 & 2k \end{pmatrix}, -3k \right), \\ e_4 &:= \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right), & e_5 &:= \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0 \right), & e_6 &:= \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, 0 \right), \end{aligned}$$

$$e_7 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, 0 \right).$$

With respect to this basis the nearly  $G_2$  form is given by

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356},$$

From Table 7.1 we know that as an  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$  connection the deformation space of the canonical connection is an irreducible subrepresentation of  $V_{(0,1,0)}$  and is thus trivial or  $(V_{(0,1,0)})_{\mathbb{R}}$ . We need to check whether this space lies in the  $-2$  eigenspace of  $D^{-1,A}$

The  $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ -representation  $V_{(0,1,0)}$  is 5 dimensional. We need to find the space  $\mathrm{Hom}(V_{(0,1,0)}, (\mathfrak{m}^* \otimes (\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d))_{\mathbb{C}})_{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$ . The common irreducible  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$  representations in  $V_{(0,1,0)}$  and  $(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$  is  $PQ$ . Let  $S^2P = \mathrm{Span}\{I, J, K\}$  then the subspace of  $(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$  isomorphic to the space  $PQ$  is given by  $\mathrm{Span}_{\mathbb{C}}\{v_1, v_2, v_3, v_4\}$  where

$$\begin{aligned} v_1 &= e_5 \otimes I + e_6 \otimes J + e_7 \otimes K, & v_2 &= -e_4 \otimes I + e_7 \otimes J - e_6 \otimes K, \\ v_3 &= -e_7 \otimes I - e_4 \otimes J + e_5 \otimes K, & v_4 &= e_6 \otimes I - e_5 \otimes J - e_4 \otimes K. \end{aligned}$$

Let the subspace of  $V_{(0,1,0)}$  isomorphic to  $PQ$  be given by  $\mathrm{Span}\{w_1, w_2, w_3, w_4\}$  and the homomorphism space  $\mathrm{Hom}(V_{(0,1,0)}, (\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}) = \mathrm{Span}(\beta)$  where  $\beta$  is defined by

$$\begin{aligned} w_1 &\mapsto v_3 + iv_4, & w_2 &\mapsto v_1 - iv_2, \\ w_3 &\mapsto v_1 + iv_2, & w_4 &\mapsto v_3 - iv_4. \end{aligned}$$

Using this isomorphism one can compute that the only non-trivial  $\mathfrak{gl}(V_{(0,1,0)}|_{PQ})$  elements with respect to the basis  $\{w_1, w_2, w_3, w_4\}$  are

$$\tau_*(e_1) = \frac{2}{3} \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad \tau_*(e_2) = \frac{2}{3} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \tau_*(e_3) = \frac{2}{3} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}.$$

Also by the definition of the canonical connection,  $\nabla_X^{-1,can}(\beta, w)(eH) = -\beta(\tau_*(X)w)$ . Thus we can calculate

$$(D^{-1,can}(\beta, w_1))(eH) = \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\beta, w_1)(eH) = -\sum_{i=1}^7 e_i \cdot \beta((\tau_*(e_i)w_1)|_{PQ})$$

$$\begin{aligned}
&= -(e_1 \cdot \beta(\frac{2}{3}iw_1) + e_2 \cdot \beta(\frac{2}{3}w_2) + e_3 \cdot \beta(\frac{2}{3}iw_2)) \\
&= -\frac{2}{3}(ie_1 \cdot (v_3 + iv_4) + e_2 \cdot (v_1 - iv_2) + ie_3 \cdot (v_1 - iv_2)) \\
&= -\frac{2}{3}(3(v_3 + iv_4)) = -2\beta(w_1).
\end{aligned}$$

Thus we have shown that  $V_{(0,1,0)}$  lies in the  $\ker(D^{-1,can} + 2\text{id})$ .

For  $E = \mathfrak{g}_2$  the subspace of  $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)$  in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id})$  is isomorphic to the  $\text{Sp}(1) \times \text{Sp}(1)$  representation  $2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}$ . We have already dealt with the space  $V_{(0,1,0)}$ . The remaining spaces are  $2V_{(2,0,0)} \cong 2\mathfrak{sp}(2)$  and  $V_{(0,0,2)} \cong \mathfrak{sp}(1)$ . The two copies of  $V_{(2,0,0)}$  arise from  $\text{Hom}(V_{(2,0,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)_{\text{Sp}(1) \times \text{Sp}(1)}$  and the one copy of  $V_{(0,0,2)}$  arises from  $\text{Hom}(V_{(0,0,2)}, \mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)_{\text{Sp}(1) \times \text{Sp}(1)}$ . Thus we have two cases:

**Case: 1-** $\text{Hom}(V_{(0,0,2)}, \mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)_{\text{Sp}(1) \times \text{Sp}(1)} \otimes V_{(0,0,2)}$

Let  $\{w_1, w_2, w_3\}$  be the standard basis of  $V_{(0,0,2)} \cong \mathfrak{sp}(1)_{\mathbb{C}}$  then the non-trivial actions of  $\mathfrak{m}$  on  $\mathfrak{sp}(1)_{\mathbb{C}}$  are given by

$$[e_1, \cdot] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad [e_2, \cdot] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad [e_3, \cdot] = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let  $\{\mu_i, i = 1 \dots 8\}$  be a basis of the  $\text{Sp}(1)_u \times \text{Sp}(1)_d$  subrepresentation of  $(\mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $PS^3Q$ . The 1-dimensional space  $\text{Hom}(V_{(0,0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \text{Span}\{\phi\}$  where  $\phi$  maps

$$\begin{aligned}
w_1 &\mapsto e_4 \otimes (\mu_5 - \mu_2) + e_5 \otimes (\mu_1 + \mu_6) + e_6 \otimes (\mu_4 - \mu_7) - e_7 \otimes (\mu_3 + \mu_8), \\
w_2 &\mapsto e_4 \otimes (\mu_3 - 2\mu_8) - e_5 \otimes (\mu_4 + 2\mu_7) + e_6 \otimes (\mu_1 - 2\mu_6) - e_7 \otimes (\mu_2 + 2\mu_5), \\
w_3 &\mapsto -e_4 \otimes (2\mu_4 + \mu_7) + e_5 \otimes (\mu_8 - 2\mu_3) - e_6 \otimes (2\mu_2 + \mu_5) + e_7 \otimes (\mu_6 - 2\mu_1).
\end{aligned}$$

The connection  $\nabla_X^{-1,can}(\phi, w) = -\phi([X, w])$  for  $w \in \mathfrak{sp}(1)$  where the Lie bracket is in the Lie algebra  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ . Thus we can calculate

$$\begin{aligned}
D^{-1,can}(\phi, w_1)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\phi, w_1)(eH) = -\sum_{i=1}^7 e_i \cdot \phi([e_i, w_1]) \\
&= -(e_2 \cdot \phi(-2w_3) + e_3 \cdot \phi(2w_2)) \\
&= -2(e_4 \otimes (\mu_5 - \mu_2) + e_5 \otimes (\mu_1 + \mu_6) + e_6 \otimes (\mu_4 - \mu_7) - e_7 \otimes (\mu_3 + \mu_8)) \\
&= -2\phi(w_1).
\end{aligned}$$

Hence again by translation invariance of  $\nabla^{-1,can}$ , we have

$$V_{(0,0,2)} \subseteq \ker(D^{-1,can} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}.$$

**Case: 2-Hom** $(V_{(2,0,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)_{\text{Sp}(1) \times \text{Sp}(1)} \otimes V_{(2,0,0)}$

The  $\text{Sp}(2) \times \text{Sp}(1)$ -representation  $V_{(2,0,0)} \cong \mathfrak{sp}(2)_{\mathbb{C}} \cong S^2P \oplus S^2Q \oplus PQ$ . The subspace of  $(\mathfrak{sp}(2))_{\mathbb{C}}$  isomorphic to  $S^2Q, PQ$  is given by  $\text{Span}_{\mathbb{C}}\{e_1, e_2, e_3\}, \text{Span}_{\mathbb{C}}\{e_4, e_5, e_6, e_7\}$  respectively. As before the basis of  $PS^3Q \subset (\mathfrak{g}_2)_{\mathbb{C}}$  is denoted by  $\{\mu_1, \mu_2, \dots, \mu_8\}$  and the subspace of  $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $S^2Q$  is given by  $\text{Span}\{\phi(w_1), \phi(w_2), \phi(w_3)\}$  defined above. The subspace of  $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $PQ$  is given by  $\text{Span}\{v_1, v_2, v_3, v_4\}$  where

$$\begin{aligned} v_1 &= e_1 \otimes (\mu_1 + \mu_6) - e_2 \otimes (\mu_4 + 2\mu_7) - e_3 \otimes (2\mu_3 - \mu_8), \\ v_2 &= e_1 \otimes (\mu_2 - \mu_5) - e_2 \otimes (\mu_3 - 2\mu_8) + e_3 \otimes (2\mu_4 + \mu_7), \\ v_3 &= -e_1 \otimes (\mu_3 + \mu_8) - e_2 \otimes (\mu_2 + 2\mu_5) - e_3 \otimes (2\mu_1 - \mu_6), \\ v_4 &= -e_1 \otimes (\mu_4 - \mu_7) - e_2 \otimes (\mu_1 - 2\mu_6) + e_3 \otimes (2\mu_2 + \mu_5). \end{aligned}$$

Let  $\{A_1, A_2\}$  be a basis of the 2-dimensional space  $\text{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\text{Sp}(1) \times \text{Sp}(1)_a}$  and let  $A = c_1A_1 + c_2A_2$  for some real constants  $c_1, c_2$ . We denote  $\phi(w_i)$  by  $w_i$  for clarity then we have that

$$\begin{aligned} A(e_1) &= c_1w_1, & A(e_2) &= c_1w_2, & A(e_3) &= c_1w_3 \\ A(e_4) &= -c_2v_2, & A(e_5) &= c_2v_1, & A(e_6) &= -c_2v_4, & A(e_7) &= c_2v_3 \end{aligned}$$

and  $A_1, A_2$  acts trivially on  $S^2P$ .

Let  $s_{(A,w)} \in \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  be the section corresponding to  $(A, w) \in \text{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\text{Sp}(1) \times \text{Sp}(1)} \otimes \mathfrak{sp}(2)$  then  $\nabla_X^{-1,can}(A, w) = -A(\text{ad}(X)w) = A([X, w])$  where the Lie bracket is in the Lie algebra  $\mathfrak{sp}(2)$ . Using this action of  $\nabla^{-1,can}$  we can calculate

$$\begin{aligned} (D^{-1,can}(A, e_1))(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(A, e_1)(eH) = - \sum_{i=1}^7 e_i \cdot A([e_i, e_1]) \\ &= - \frac{2}{3}(-e_2 \cdot A(e_3) + e_3 \cdot A(e_2) + e_4 \cdot A(e_5) - e_5 \cdot A(e_4) \\ &\quad + e_6 \cdot A(e_7) - e_7 \cdot A(6)) \\ &= - \frac{2}{3}(c_1(-e_2 \cdot w_3 + e_3 \cdot w_2) + c_2(e_4 \cdot v_1 - e_5 \cdot (-v_2) + e_6 \cdot v_3 \\ &\quad - e_7 \cdot (-v_4))) \end{aligned}$$

$$= \frac{4c_1 - 6c_2}{3} w_1 = \frac{4c_1 - 6c_2}{3} A_1(e_1).$$

By doing similar computations we get that

$$\begin{aligned} (D^{-1,can}(A, f_i))(eH) &= 0, \quad i = 1, 2, 3, \\ (D^{-1,can}(A, e_i))(eH) &= \frac{4c_1 - 6c_2}{3} A_1(e_i), \quad i = 1, 2, 3, \\ (D^{-1,can}(A, e_i))(eH) &= -\frac{20c_1 + 6c_2}{9} A_2(e_i), \quad i = 4, 5, 6, 7. \end{aligned}$$

Therefore the subspace of  $\text{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\text{Sp}(1) \times \text{Sp}(1)}$  in the  $\ker(D^{-1,can} + 2\text{id})$  is given by the condition  $c_2 = \frac{5}{3}c_1$  and is thus 1-dimensional. Therefore  $V_{(2,0,0)}$  occurs in the  $\ker(D^{-1,can} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  with multiplicity 1.

**Remark 7.3.1.** We can immediately see from above that the only other possible eigenvalue for which  $\mathfrak{sp}(2)$  is an eigenspace of  $D^{-1,can}$  is  $-\frac{8}{3}$  for  $c_2 = -\frac{2}{3}c_1$ . This shows that not all spaces in  $\ker((D^{-1/3,can})^2 - \frac{49}{9}\text{id})$  are in  $\ker(D^{-1,can} + 2\text{id})$ .

$$\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$$

As before let  $\{e_i, i = 1 \dots 7\}$  be an orthonormal basis of  $\mathfrak{m}^*$  with respect to  $g$ . If we define  $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$  we have

$$\begin{aligned} e_1 &:= \frac{1}{3} \left( \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, -3I \right), \quad e_2 := \frac{1}{3} \left( \begin{pmatrix} 2J & 0 \\ 0 & 0 \end{pmatrix}, -3J \right), \quad e_3 := \frac{1}{3} \left( \begin{pmatrix} 2K & 0 \\ 0 & 0 \end{pmatrix}, -3K \right), \\ e_4 &:= \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, 0 \right), \quad e_5 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 \end{pmatrix}, 0 \right), \\ e_6 &:= \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, 0 \right), \quad e_7 := \frac{\sqrt{5}}{3} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}, 0 \right). \end{aligned}$$

With respect to this basis the nearly  $G_2$ -structure  $\varphi$  is given by

$$\varphi = e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356}.$$

As an  $SU(2) \times U(1)$  representation,  $\mathfrak{m}_{\mathbb{C}}^* \cong S^2W \oplus WF(3) \oplus WF(-3)$  where

$$\begin{aligned} S^2W &= \text{Span}\{e_1, e_2, e_3\}, \\ WF(3) &= \text{Span}\{e_4 - ie_5, e_6 - ie_7\}, \quad WF(-3) = \text{Span}\{e_4 + ie_5, e_6 + ie_7\}. \end{aligned}$$

From our previous work we know that the canonical connection has no deformations as an  $SU(2) \times U(1)$  connection so we only have to consider the case  $E = \mathfrak{g}_2$ .

As an  $SU(2) \times U(1)$  representation,  $(\mathfrak{g}_2)_{\mathbb{C}} \cong S^3W(F(3) \oplus F(-3)) \oplus S^2W \oplus F(6) \oplus F(-6)$ . We have already seen that  $S^2W$  gives rise to no deformations. From previous calculations we know that

$$\begin{aligned} \ker((D^{-1/3, can})^2 - \frac{49}{9}\text{id}) \cap \Gamma(\mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(\pm 3)) &\cong V_{(0,0,2)} \oplus 2V_{(1,1,0)} \\ &\cong (\mathfrak{su}(2))_{\mathbb{C}} \oplus 2(\mathfrak{su}(3))_{\mathbb{C}} \end{aligned}$$

and

$$\Gamma(\mathfrak{m}_{\mathbb{C}}^* \otimes F(\pm 6)) \cap \ker((D^{-1/3, can})^2 - \frac{49}{9}\text{id}) \cong V_{(1,1,0)}.$$

Therefore there are 6 subspaces of  $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)$  to consider here.

**Case: 1-Hom** $(V_{(0,0,2)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3))_{SU(2) \times U(1)} \otimes V_{(0,0,2)}$

We denote by  $\{\mu_i, i = 1 \dots 4\}$  a basis of  $S^3WF(3)$ . Let  $f_i, i = 1 \dots 3$  be the standard basis of  $\mathfrak{su}(2)$  such that  $[f_1, f_2] = -2f_3, [f_1, f_3] = 2f_2, [f_2, f_3] = -2f_1$ . Then the subspace of  $WF(-3) \otimes S^3WF(3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $(\mathfrak{su}(2))_{\mathbb{C}}$  is given by  $\text{Span}\{v_1, v_2, v_3\}$  where

$$\begin{aligned} v_1 &= \frac{3i}{4}(e_4 + ie_5) \otimes \mu_1 + (e_6 + ie_7) \otimes (\frac{5i}{4}\mu_2 + \mu_4), \\ v_2 &= (e_4 + ie_5) \otimes (-i\mu_2 + \mu_4) + (e_6 + ie_7) \otimes (-i\mu_1 - \mu_3), \\ v_3 &= (e_4 + ie_5) \otimes (-\frac{5i}{4}\mu_1 + \mu_3) - \frac{3i}{4}(e_6 + ie_7) \otimes \mu_2 \end{aligned}$$

and the space  $\text{Hom}(V_{(0,0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \text{Span}\{\gamma^A\}$  where  $\gamma^A$  is defined by

$$\gamma^A(f_1) = v_2, \quad \gamma^A(f_2) = i(v_1 - v_3), \quad \gamma^A(f_3) = -2(v_1 + v_3).$$

For  $i = 1, 2, 3$ , since  $e_i = (\frac{2}{3}f_i, -f_i)$  we have  $[e_i, v] = -[f_i, v]$  for all  $v \in \mathfrak{su}(2)$ . The action is trivial for  $i = 4 \dots 7$  since  $[e_i, f_j] \notin \text{Span}\{f_1, f_2, f_3\}$ . We can thus calculate

$$D^{-1, can}(\gamma^A, f_1)(eH) = \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1, can}(\gamma^A, f_1)(eH)$$

$$\begin{aligned}
&= e_2 \cdot \gamma^A(2f_3) - e_3 \cdot \gamma^A(2f_2) \\
&= -(4e_2 \cdot (v_1 + v_3) + 2ie_3 \cdot (v_1 - v_3)) \\
&= -2v_2 = -2 \gamma^A(f_1).
\end{aligned}$$

Hence

$$\text{Hom}(V_{(0,0,2)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3))|_{\text{Sp}(1) \times \text{Sp}(1)} \otimes V_{(0,0,2)} \subseteq \ker(D^{-1,can} + 2\text{id}).$$

Case: 2-Hom( $V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3)$ ) $_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)}$

Let a basis of the subspace of  $V_{(1,1,0)} \cong (\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $S^2W \cong (\mathfrak{su}(2))_{\mathbb{C}}$  be given by

$$p_1 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad p_2 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad p_3 := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

where  $I, J, K$  are defined previously. Then  $[p_1, p_2] = -2p_3$ ,  $[p_1, p_3] = 2p_2$ ,  $[p_2, p_3] = -2p_1$ . The basis of  $\mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3) \subset \mathfrak{m}_{\mathbb{C}}^* \otimes \mathfrak{g}_2$  isomorphic to  $S^2W$  is given by  $\text{Span}\{w_1, w_2, w_3\}$  where

$$\begin{aligned}
w_1 &= (e_4 + ie_5) \otimes \frac{\mu_2 + i\mu_3}{2} + (e_6 + ie_7) \otimes \frac{\mu_1 - i\mu_4}{2}, \\
w_2 &= (e_4 + ie_5) \otimes \frac{\mu_4 - 2i\mu_1}{2} + (e_6 + ie_7) \otimes \frac{\mu_3 - 2i\mu_2}{2}, \\
w_3 &= -(e_4 + ie_5) \otimes \frac{\mu_1 + 2i\mu_4}{2} + (e_6 + ie_7) \otimes \frac{\mu_2 - 2i\mu_3}{2}.
\end{aligned}$$

Since  $(\mathfrak{su}(3))_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathbb{C}$ , the subspace of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $WF(3)$  is given by  $\text{Span}_{\mathbb{C}}\{e_4 - ie_5, e_6 - ie_7\}$ . The subspace of  $S^2W \otimes S^3WF(3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $WF(3)$  is given by  $\text{Span}\{u_1, u_2\}$  where

$$\begin{aligned}
u_1 &= ie_1 \otimes \frac{\mu_2 + i\mu_3}{2} + e_2 \otimes \frac{2\mu_1 + i\mu_4}{2} - ie_3 \otimes \frac{\mu_1 + 2i\mu_4}{2}, \\
u_2 &= ie_1 \otimes \frac{\mu_1 - i\mu_4}{2} + e_2 \otimes \frac{2\mu_2 - i\mu_3}{2} + ie_3 \otimes \frac{\mu_2 - 2i\mu_3}{2}
\end{aligned}$$

If we denote the space  $\text{Hom}(V^{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3))$  and  $\text{Hom}(V^{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(3))$  by  $\text{Span}\{A_1\}$ ,  $\text{Span}\{A_2\}$  respectively then

$$A_1(p_i) = w_i, \quad i = 1, 2, 3,$$



$$A_2(e_4 - ie_5) = u_1, \quad A_2(e_6 - ie_7) = u_2.$$

Define  $A = c_1 A_1 + c_2 A_2$  for some constants  $c_1, c_2$ . We need to find the conditions on  $c_1, c_2$  such that  $(A, w) \in \Gamma(\mathfrak{m}^* \otimes S^3 WF(3)) \cap \ker(D^{-1,can} + 2\text{id})$  for all  $w \in \mathfrak{su}(3)$ .

Let  $s_{(A,w)}$  be the section corresponding to  $(A, w)$ . Then for any vector field  $X$ ,  $\nabla_X^{-1,can}(A, w) = -A(\text{ad}(X)w) = A([X, w])$  where the Lie bracket is in the Lie algebra  $\mathfrak{su}(3)$ . Using this action of  $\nabla^{-1,can}$  we can calculate

$$\begin{aligned} D^{-1,can}(A, p_1)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(A, p_1)(eH) \\ &= -\left(\frac{2}{3}(-e_2 \cdot A(2p_3) + e_3 \cdot A(2p_2))e_4 \cdot A(-e_5) + e_5 \cdot A(e_4) \right. \\ &\quad \left. + e_6 \cdot A(e_7) + e_7 \cdot A(e_6)\right) \\ &= -\frac{2c_1}{3}(-e_2 \cdot w_1 + e_3 \cdot w_2) - c_2(-e_4 \cdot i\frac{u_1}{2} + e_5 \cdot \frac{u_1}{2} + e_6 \cdot i\frac{u_2}{2} - e_7 \cdot \frac{u_2}{2}) \\ &= \frac{4c_1 + 3ic_2}{3}w_1 = \frac{4c_1 + 3ic_2}{3}A_1(e_1). \end{aligned}$$

The operator  $D^{-1,can}$  acts trivially on the subspaces of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $\mathbb{C}$  and  $WF(-3)$ . On the remaining subspaces we can compute the action of the Dirac operator as

$$\begin{aligned} D^{-1,can}(A, p_1)(eH) &= \frac{4c_1 + 3ic_2}{3}A_1(e_i), \quad i = 1, 2, 3, \\ D^{-1,can}(A, e_4 - ie_5)(eH) &= \frac{20c_1 - 3ic_2}{9}A_2(e_4 - ie_5), \\ D^{-1,can}(A, e_6 - ie_7)(eH) &= \frac{20c_1 - 3ic_2}{9}A_2(e_6 - ie_7). \end{aligned}$$

Thus for any  $w \in (\mathfrak{su}(3))_{\mathbb{C}}$ ,  $(A, w) \in \ker(D^{-1,can} + 2\text{id})$  if and only if  $c_2 = \frac{10i}{3}c_1$ . Thus only one copy of  $\mathfrak{su}(3)$  lies in  $\ker(D^{-1,can} + 2\text{id})$ .

Note that similarly to Remark 7.3.1 here also for  $c_2 = -\frac{4i}{3}c_1$ ,  $(A, w) \in \ker(D^{-1,can} - \frac{8}{3}\text{id})$ .

**Case: 3-Hom** $(V_{(0,0,2)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3 WF(-3))_{\text{Sp}(1) \times \text{Sp}(1)} \otimes V_{(0,0,2)}$

Let  $f_i, i = 1 \dots 3$  be as before and denote by  $\{\nu_i, i = 1 \dots 4\}$  a basis of  $S^3 WF(-3)$ . Then the subspace of  $WF(3) \otimes S^3 WF(-3)$  isomorphic to  $S^2 W$  is given by  $\text{Span}\{w_1, w_2, w_3\}$

where

$$\begin{aligned} w_1 &= (e_4 - ie_5) \otimes \left(\frac{-3i}{4}\nu_1\right) + (e_6 - ie_7) \otimes \left(\frac{-5i}{4}\nu_2 + \nu_4\right), \\ w_2 &= (e_4 - ie_5) \otimes (i\nu_2 + \nu_4) + (e_6 - ie_7) \otimes (i\nu_1 - \nu_3), \\ w_3 &= (e_4 - ie_5) \otimes \left(\frac{5i}{4}\nu_1 + \nu_3\right) + (e_6 - ie_7) \otimes \left(\frac{3i}{4}\nu_2\right) \end{aligned}$$

and the space  $\text{Hom}(V_{(0,0,2)}, (\mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(-3))) = \text{Span}\{\gamma^B\}$  where  $\gamma^B$  is defined by

$$\gamma^B(f_1) = \frac{i}{2}w_2, \quad \gamma^B(f_2) = \frac{1}{2}(w_1 - w_3), \quad \gamma^B(f_3) = -i(w_1 + w_3).$$

The action of  $e_i, i = 1 \dots 7$  on  $f_j, j = 1 \dots 3$  is the same as Case 1 and thus we can calculate  $D^{-1, \text{can}}(\gamma^B, f_1)$  as

$$\begin{aligned} D^{-1, \text{can}}(\gamma^B, f_1)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1, \text{can}}(\gamma^B, f_1)(eH) \\ &= e_2 \cdot \gamma^B(2f_3) - e_3 \cdot \gamma^B(2f_2) \\ &= -2ie_2 \cdot (w_1 + w_3) - e_3 \cdot (w_1 - w_3) \\ &= -iw_2 = -2\gamma^B(f_1). \end{aligned}$$

Thus  $V_{(0,0,2)} \subseteq \ker(D^{-1, \text{can}} + 2\text{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ .

**Case: 4- $\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(-3))_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)}$**

As above in Case 2, let a basis of the subspace of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $S^2W \cong \mathfrak{su}(2)$  be given by  $\text{Span}\{p_1, p_2, p_3\}$ . The basis of  $\mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $S^2W$  is given by  $\text{Span}\{w_1, w_2, w_3\}$  where

$$\begin{aligned} w_1 &= (e_4 - ie_5) \otimes \frac{\nu_2 - i\nu_3}{2} + (e_6 - ie_7) \otimes \frac{\nu_1 + i\nu_4}{2}, \\ w_2 &= (e_4 - ie_5) \otimes \frac{\nu_4 + 2i\nu_1}{2} + (e_6 - ie_7) \otimes \frac{\nu_3 + 2i\nu_2}{2}, \\ w_3 &= -(e_4 - ie_5) \otimes \frac{\nu_1 - 2i\nu_4}{2} + (e_6 - ie_7) \otimes \frac{\nu_2 + 2i\nu_3}{2}. \end{aligned}$$

The subspace of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $WF(-3)$  is given by  $\text{Span}\{e_4 + ie_5, e_6 + ie_7\}$ . The subspace of  $S^2W \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$  isomorphic to  $WF(-3)$  is given by  $\text{Span}_{\mathbb{C}}\{u_1, u_2\}$  where

$$\begin{aligned}
u_1 &= -ie_1 \otimes \frac{\nu_2 - i\nu_3}{2} + e_2 \otimes \frac{2\nu_1 - i\nu_4}{2} + ie_3 \otimes \frac{\nu_1 - 2i\nu_4}{2}, \\
u_2 &= -ie_1 \otimes \frac{\nu_1 + i\nu_4}{2} + e_2 \otimes \frac{2\nu_2 + i\nu_3}{2} - ie_3 \otimes \frac{\nu_2 + 2i\nu_3}{2}.
\end{aligned}$$

Again if we denote the spaces  $\text{Hom}(V(1, 1, 0), \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(-3))$  and  $\text{Hom}(V^{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes S^3WF(-3))$  by  $\text{Span}\{B_1\}$ ,  $\text{Span}\{B_2\}$  respectively then

$$\begin{aligned}
B_1(p_i) &= w_i, \quad i = 1, 2, 3, \\
B_2(e_4 + ie_5) &= u_1, \quad B_2(e_6 + ie_7) = u_2.
\end{aligned}$$

Again as before we need to find the conditions on  $c_1, c_2$  such that  $(B = c_1B_1 + c_2B_2, w) \in \ker(D^{-1,can} + 2\text{id})$  for all  $w \in (\mathfrak{su}(3))_{\mathbb{C}}$ . By similar computations as Case 2, we can calculate,

$$\begin{aligned}
D^{-1,can}(B, p_1)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(B, p_1)(eH) \\
&= -\left(\frac{2}{3}(-e_2 \cdot B(2p_3) + e_3 \cdot B(2p_2)) + e_4 \cdot B(-e_5) + e_5 \cdot B(e_4) \right. \\
&\quad \left. + e_6 \cdot B(e_7) + e_7 \cdot B(e_6)\right) \\
&= -\frac{2c_1}{3}(-e_2 \cdot w_1 + e_3 \cdot w_2) - c_2(-e_4 \cdot i\frac{u_1}{2} + e_5 \cdot \frac{u_1}{2} + e_6 \cdot i\frac{u_2}{2} - e_7 \cdot \frac{u_2}{2}) \\
&= \frac{4c_1 - 3ic_2}{3}w_1 = \frac{4c_1 - 3ic_2}{3}B_1(e_1).
\end{aligned}$$

Once can check that  $D^{-1,can}$  acts trivially on the subspaces of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $\mathbb{C}, WF(3)$  and

$$\begin{aligned}
D^{-1,can}(A, p_1)(eH) &= \frac{4c_1 - 3ic_2}{3}B_1(e_i), \quad i = 1, 2, 3, \\
D^{-1,can}(A, e_4 + ie_5)(eH) &= \frac{20c_1 + 3ic_2}{9}B_2(e_4 + ie_5), \\
D^{-1,can}(A, e_6 + ie_7)(eH) &= \frac{20c_1 + 3ic_2}{9}B_2(e_6 + ie_7).
\end{aligned}$$

Thus for all  $w \in (\mathfrak{su}(3))_{\mathbb{C}}$ ,  $(B, w) \in \ker(D^{-1,can} + 2\text{id})$  if and only if  $c_2 = -\frac{10i}{3}c_1$  which proves that only one copy of  $\mathfrak{su}(3)$  lies in  $\ker(D^{-1,can} + 2\text{id})$  in this case as well. It immediately follows from the given action that for  $c_2 = \frac{4i}{3}c_1$ ,  $(B, w) \in \ker(D^{-1,can} - \frac{8}{3}\text{id})$ .

**Case: 5- $\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(6))_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)}$**

From before we know that the subspace of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $WF(3)$  is given by  $\text{Span}\{e_4 - ie_5, e_6 - ie_7\}$ . if we denote by  $\mu$  a basis vector for the 1-dimensional representation  $F(6)$ , the subspace of  $\mathfrak{m}_{\mathbb{C}}^* \otimes F(6)$  isomorphic to  $WF(3)$  is given by  $\text{Span}_{\mathbb{C}}\{(e_4 + ie_5) \otimes \mu, (e_6 + ie_7) \otimes \mu\}$ . Let  $\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(6)) = \text{Span}\{\alpha\}$ . We can define  $\alpha$  as follows,

$$\alpha(e_4 - ie_5) = (e_6 + ie_7) \otimes \mu, \quad \alpha(e_6 - ie_7) = -(e_4 + ie_5) \otimes \mu.$$

Since  $V_{(1,1,0)}$  is isomorphic to the adjoint representation  $(\mathfrak{su}(3))_{\mathbb{C}}$ ,  $\nabla_X^{-1,can}(\alpha, v)(eH) = -\alpha([X, v])$  where  $X \in \mathfrak{m}$ ,  $v \in WF(3) \subset \mathfrak{su}(3)$ . Thus we can compute

$$\begin{aligned} D^{-1,can}(\alpha, e_4 - ie_5)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\alpha, e_4 - ie_5)(eH) \\ &= -(e_1 \cdot \alpha(\frac{2i}{3}(e_4 - ie_5)) + e_2 \cdot \alpha(\frac{2}{3}(e_6 - ie_7)) + e_3 \cdot \alpha(\frac{2i}{3}(e_6 - ie_7))) \\ &= -\frac{2}{3}(ie_1 \cdot (e_6 + ie_7) \otimes \mu - e_2 \cdot (e_4 + ie_5) \otimes \mu - ie_3 \cdot (e_4 + ie_5) \otimes \mu) \\ &= -2(e_6 + ie_7) \otimes \mu = -2\alpha(e_4 - ie_5). \end{aligned}$$

Therefore

$$\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(6))_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)} \subset \ker(D^{-1,can} + 2\text{id})$$

and thus lies in the deformation space.

**Case: 6-Hom** $(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(-6))_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)}$

The subspace of  $(\mathfrak{su}(3))_{\mathbb{C}}$  isomorphic to  $WF(-3)$  is given by  $\text{Span}_{\mathbb{C}}\{e_4 + ie_5, e_6 + ie_7\}$ . We denote  $F(-6) = \text{Span}\{\nu\}$ . Then  $\mathfrak{m}_{\mathbb{C}}^* \otimes F(-6)$  isomorphic to  $WF(-3)$  is given by  $\text{Span}\{(e_4 - ie_5) \otimes \nu, (e_6 - ie_7) \otimes \nu\}$ . Let  $\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(-6)) = \text{Span}\{\beta\}$  then

$$\beta(e_4 + ie_5) = -(e_6 - ie_7) \otimes \nu, \quad \beta(e_6 + ie_7) = (e_4 - ie_5) \otimes \nu.$$

Since  $V_{(1,1,0)} \cong (\mathfrak{su}(3))_{\mathbb{C}}$ ,  $\nabla_X^{-1,can}(\beta, v)(eH) = -\beta([X, v])$  where  $X \in \mathfrak{m}$ ,  $v \in WF(-3) \subset (\mathfrak{su}(3))_{\mathbb{C}}$ . Thus we can compute

$$\begin{aligned} D^{-1,can}(\beta, e_4 + ie_5)(eH) &= \sum_{i=1}^7 e_i \cdot \nabla_{e_i}^{-1,can}(\beta, e_4 + ie_5)(eH) \\ &= -(e_1 \cdot \beta(\frac{-2i}{3}(e_4 + ie_5)) + e_2 \cdot \beta(\frac{2}{3}(e_6 + ie_7))) \end{aligned}$$

$$\begin{aligned}
& + e_3 \cdot \beta\left(\frac{-2i}{3}(e_6 + ie_7)\right) \\
& = -\frac{2}{3}(ie_1 \cdot (e_6 - ie_7) \otimes \nu + e_2 \cdot (e_4 - ie_5) \otimes \nu - ie_3 \cdot (e_4 - ie_5) \otimes \nu) \\
& = 2(e_6 - ie_7) \otimes \nu = -2\beta(e_4 + ie_5),
\end{aligned}$$

which by translation invariance of  $D^{-1,can}$  shows that

$$\text{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(-6))_{\text{SU}(2) \times \text{U}(1)} \otimes V_{(1,1,0)} \subset \ker(D^{-1,can} + 2\text{id}).$$

## 7.4 Describing the deformation space

The above computations describe the infinitesimal deformation space of the canonical connection for the four normal homogeneous spaces considered in Table 4.2. We studied the deformation space of the canonical connection on  $G/H$  as a connection on  $G \times_{\text{ad}} \mathfrak{m}$  and  $G \times_{\text{ado}\lambda} \mathfrak{m}$  where  $\lambda$  is the isotropy homomorphism. The structure group of the canonical connection is  $H$  on  $G \times_{\text{ad}} \mathfrak{m}$  and  $G_2$  on  $G \times_{\text{ado}\lambda} \mathfrak{m}$ . Thus pointwise the spaces of infinitesimal deformations are isomorphic to subspaces of  $\mathfrak{m}^* \otimes \mathfrak{h}$  and  $\mathfrak{m}^* \otimes \mathfrak{g}_2$  respectively.

As an  $H$ -connection we were able to obtain the deformation space for the canonical connection for all the homogeneous spaces listed in Table 4.1. We infer that the squashed 7-sphere given by  $\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$  is the only case where the canonical connection is not rigid as an  $H$ -connection.

As a  $G_2$ -connection we can only compute the deformation space for the four normal homogeneous spaces. The canonical connection has a non-trivial infinitesimal deformation space except for the case of round  $S^7$ .

Summing up all the results found above we get the following theorem.

**Theorem 7.4.1.** The infinitesimal deformation space for the canonical connection on the four normal homogeneous nearly  $G_2$  spaces  $G/H$  when the structure group is  $H$  or  $G_2$  is isomorphic to the  $G$ -representations given in Table 7.2. The space  $V^{(0,1)}$  in Table 7.2 is the unique 5-dimensional complex irreducible  $\text{Sp}(2)$ -representation.

$G/H$	Structure group	
	$H$	$G_2$
$\text{Spin}(7)/G_2$	0	0
$\text{SO}(5)/\text{SO}(3)$	0	$\mathfrak{so}(5)$
$\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus V_{\mathbb{R}}^{(0,1)}$
$\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$	0	$2\mathfrak{su}(2) \oplus 4\mathfrak{su}(3)$

Table 7.2: Deformation space of the canonical connection

We now describe some of the deformation spaces obtained in Theorem 7.4.1.

Let  $M$  be a nearly  $G_2$  manifold. We first observe that for the structure group  $G_2$  the space of non-trivial deformations in Theorem 7.4.1 are either isomorphic to or contains as a subrepresentation one or multiple copies of the Lie algebra  $\mathfrak{g}$  of the automorphism group  $G$ . A vector field  $X$  on  $M$  preserves the  $G_2$ -structure  $\varphi$  if  $\mathcal{L}_X\varphi = 0$ . We denote by  $\mathcal{X}$  the space of vector fields on  $M$  preserving the  $G_2$ -structure. Since the  $G_2$ -structure on  $G/H$  is  $G$  invariant, the space  $\mathfrak{g}$  is contained in  $\mathcal{X}$ . Note that if  $X \in \mathcal{X}$  then  $\mathcal{L}_X\psi = \mathcal{L}_Xg = 0$ .

Given a parallel section in  $\Gamma(\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}})$ , one can define an operator that associates to each vector field in  $\mathcal{X}$  an infinitesimal deformation of a  $G_2$  instanton on  $M$ . Such an operator was defined in [CH16] where a similar situation arises when one computes the deformation space of the canonical connection on the homogeneous 6-dimensional nearly Kähler manifolds.

The next proposition asserts that if we fix a section  $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}})$ , then for any vector field  $X \in \mathcal{X}$  on  $M$  the  $\text{Ad}_{\mathcal{P}}$  valued 1-form  $\epsilon_X = i_X\xi \in \Gamma(T^*M \otimes \text{Ad}_{\mathcal{P}})$  defines an infinitesimal deformation of the nearly  $G_2$  instanton  $A$  in the sense of (7.1.1). The proof of the proposition follows verbatim from the proof of [CH16, Proposition 9] and is hence omitted.

**Proposition 7.4.2.** Let  $A$  be an instanton on a principal  $G$ -bundle  $\mathcal{P}$  over a nearly  $G_2$  manifold  $M$ . Let  $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}})$  such that  $\nabla^{-1,A}\xi = 0$ . Then for any  $X \in \mathcal{X}$ ,  $\epsilon_X = i_X\xi \in \Gamma(T^*M \otimes \text{Ad}_{\mathcal{P}})$  satisfies the linearised instanton condition

$$d^A\epsilon_X \cdot \eta = 0.$$

The above proposition implies that for each  $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}})$  such that  $\nabla^{-1,A}\xi = 0$ , there is a copy of  $\mathfrak{g}$  in the deformation space of  $A$ . Thus the multiplicity of  $\mathfrak{g}$  in the deformation space can be found by identifying the parallel sections of  $\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}}$ . On  $G/H$ , when we see  $\mathcal{P}$  as a  $G_2$ -bundle, every parallel section of  $\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}}$  corresponds to an  $H$ -invariant element of the  $H$ -representation  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  (since  $\text{Ad}_{\mathcal{P}} \cong \mathfrak{g}_2$ ) and vice-versa. The number of linearly independent  $H$ -invariant elements of  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  is equal to the multiplicity of the trivial  $H$ -representation in  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ .

Observe that since  $A$  is a  $G_2$  instanton, the curvature  $F_A \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \text{Ad}_{\mathcal{P}})$ . When  $A = \nabla^{can}$  is the canonical connection on  $G/H$  and  $F$  is the curvature,  $\nabla^{-1,can}F = 0$  since  $\text{Hol}(\nabla^{can}) \subseteq G_2$ . Hence by Proposition 7.4.2 for every  $X \in \mathcal{X}$ ,  $\epsilon_X = i_X F$  defines an infinitesimal deformation of  $A = \nabla^{can}$ . Using the Bianchi identity and the definition of  $\epsilon_X$  we have that

$$\begin{aligned} d^A \epsilon_X &= d\epsilon_X + [A, \epsilon_X] \\ &= \mathcal{L}_X F - i_X dF + [A, \epsilon_X] \\ &= \mathcal{L}_X F + i_X [A, F] + [A, \epsilon_X] \\ &= \mathcal{L}_X F + [i_X A, F]. \end{aligned}$$

Since under the action of a gauge transformation  $\phi$ , the curvature  $F$  transforms by  $\phi F \phi^{-1}$ , for all  $X \in \mathcal{X}$  there exists an infinitesimal gauge transformation  $\phi_X$  such that

$$\mathcal{L}_X F = [\phi_X, F].$$

Also  $i_X A$  defines an infinitesimal gauge transformation, hence  $[\phi_X + i_X A, F]$  is an action of an infinitesimal gauge transformation on  $F$ . Thus for all  $X \in \mathcal{X}$  the deformations  $i_X F$  arise from gauge transformations and hence do not descend to the moduli space.

Thus for finding the multiplicity of  $\mathfrak{g}$  in the deformation space (modulo gauge transformations) of the canonical connection on  $G/H$ , we need to find the number of trivial sub-representations of  $H$  in  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  apart from the one that corresponds to  $F$ . In all the cases we consider, the trivial  $H$ -representation occurs with multiplicity one in the subrepresentation  $\mathfrak{h} \otimes \mathfrak{h}$  of  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ . The trivial representation coming from  $\mathfrak{h} \otimes \mathfrak{h}$  corresponds to the  $H$ -invariant element  $F$ . We deal with the four normal homogeneous spaces one by one. The notation for the irreducible  $H$ -representations in all the cases is the same as used in §7.2.

– Spin(7)/ $G_2$

Since  $H = G_2$ , in this case  $\mathfrak{g}_2$  is the irreducible adjoint representation. There is only one trivial  $\mathfrak{g}_2$ -subrepresentation of  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  which corresponds to  $F$ . Hence  $\mathfrak{g} = \mathfrak{spin}(7)$  does not occur in the deformation space as proved in Theorem 7.4.1.

–  $\mathrm{SO}(5)/\mathrm{SO}(3)$

In this case, as an  $\mathfrak{so}(3)$  representation,  $\mathfrak{g}_2$  decomposes into two irreducible  $\mathfrak{so}(3)$ -representations, the adjoint representation  $S^2\mathbb{C}^2$ , and the 11-dimensional representation  $S^{10}\mathbb{C}^2$ . Thus as  $\mathfrak{so}(3)$ -representation

$$\mathfrak{g}_2 \otimes \mathfrak{g}_2 = (S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2) \oplus 2(S^2\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2) \oplus (S^{10}\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2).$$

There are two trivial components occurring in the above decomposition from  $S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2$  and  $S^{10}\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2$  respectively but since the component coming from  $S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2$  corresponds to  $F$ , up to gauge transformations the deformation space of the canonical connection on  $\mathrm{SO}(5)/\mathrm{SO}(3)$  contains only one copy of  $\mathfrak{g} = \mathfrak{so}(5)$  as shown in Theorem 7.4.1.

–  $\mathrm{Sp}(2) \times \mathrm{Sp}(1)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$

As an  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ -representation,

$$\mathfrak{g}_2 = S^2P \oplus S^2Q \oplus PS^3Q.$$

The trivial  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$  components of  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  coming from  $S^2P \otimes S^2P$  and  $S^2Q \otimes S^2Q$  correspond to  $F$  and thus can be ignored. The only trivial component that corresponds to an infinitesimal deformation modulo gauge transformations comes from  $PS^3Q \otimes PS^3Q$ , hence again  $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  appears with multiplicity 1 in the deformation space which is consistent with our findings in Theorem 7.4.1.

–  $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$

The decomposition of  $\mathfrak{g}_2$  as an  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ -representation is given by

$$\mathfrak{g}_2 = S^2W \oplus \mathbb{C} \oplus S^3WF(3) \oplus S^3WF(-3) \oplus F(6) \oplus F(-6).$$

The first two components in the above decomposition correspond to  $\mathfrak{h}$  hence the only trivial  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ -subrepresentations of  $\mathfrak{g}_2 \otimes \mathfrak{g}_2$  that correspond to non-trivial deformations come from the spaces  $S^2WF(3) \otimes S^2WF(-3)$  and  $F(6) \otimes F(-6)$ . Hence as proved in Theorem 7.4.1 the space  $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$  occurs in the deformation space with multiplicity 2.



The only deformation spaces left to be considered in Table 7.2 are the  $\mathrm{Sp}(2)$ -representation  $V_{(0,1)}^{\mathbb{R}}$  for the squashed 7-sphere and 2 copies of the  $\mathrm{SU}(3)$ -representation  $\mathfrak{su}(3)$  on the Aloff–Wallach space  $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$ .

On the squashed 7-sphere the canonical connection splits into two connections with  $\mathrm{Hol} = \mathrm{Sp}(1)_{\mathrm{u}}$  and  $\mathrm{Sp}(1)_{\mathrm{d}}$  respectively. From §7.2 the deformations only come from the  $\mathrm{Sp}(1)_{\mathrm{u}}$  part which we suspect is the pullback of the standard instanton on  $S^4$  via the quaternionic Hopf fibration. The standard instanton on  $S^4$  is the unique  $\mathrm{Sp}(2)$ -invariant ASD connection on  $S^4$  with charge 1. Let  $\mathcal{M}$  be the moduli space of charge-1 instantons on  $S^4$  with structure group  $\mathrm{SU}(2)$ . Then, there is a diffeomorphism from  $\mathcal{M}$  to  $B^5 \subset \mathbb{R}^5$  which to an instanton associates its center. The standard instanton on  $S^4$  is the charge-1 instanton that corresponds to the center of the ball, that is to  $0 \in B^5$ , and is the unique homogeneous charge-1 instanton. As the name suggests, the homogeneous charge-1 instanton is invariant with respect to the  $\mathrm{Sp}(2)$ -action. The pullback of the homogeneous charge-1 instanton to the squashed  $S^7$  is a  $\mathrm{G}_2$ -instanton (see [BO19], [Cla14]). As shown in [AHS78] the moduli space of the standard instanton on  $S^4$  can be identified as a topological space and as a differentiable manifold with  $\mathbb{R}^+ \times \mathbb{H}$  (see [DK90, sec 4.1]). If the  $\mathrm{Sp}(1)$  part of the canonical connection is the pullback of the standard instanton, then the deformation space of the canonical connection on the squashed 7-sphere must contain the deformation space of the standard ASD instanton on  $S^4$  and hence be at least 5-dimensional. From Table 7.2 we know that the moduli space of the deformations of the canonical connection on the squashed 7-sphere is exactly 5-dimensional and hence all the deformations of the canonical connection would come from the deformations of the standard ASD connection and would thus be integrable.

As of the deformation subspace  $2\mathfrak{su}(3)$  of the canonical connection on  $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$  with structure group  $\mathrm{G}_2$ , the author is unaware of any such explicit description. It would be interesting to see whether these deformations are genuine.

## Chapter 8

# Deformations of $SU(3)$ instantons on Sasaki–Einstein 7-folds

Let  $(M^{2m+1}, g)$  be an odd-dimensional Riemannian manifold. Let  $r$  be a coordinate on  $\mathbb{R}^{>0}$ . The cone metric on  $M \times \mathbb{R}^{>0}$  is given by

$$dr^2 + r^2g.$$

The manifold  $M$  equipped with a 1-form  $\theta$  is contact if and only if the 2-form

$$\omega := t^2d\theta + 2tdt \wedge \theta$$

on its cone is symplectic. A contact Riemannian manifold is Sasakian if its Riemannian cone with the cone metric is a Kähler manifold with the Kähler form given by  $\omega$ .

If the manifold is also an Einstein manifold, is it called a Sasaki–Einstein manifold. Sasakian geometry is the odd-dimensional analogue of Kähler geometry. Indeed, just as Kähler geometry is the natural intersection of complex, symplectic and Riemannian geometry, so Sasakian geometry is the natural intersection of contact and Riemannian geometry.

In this chapter we describe the infinitesimal deformations of instantons on Sasaki–Einstein 7-dimensional manifolds. These manifolds are nearly  $G_2$  manifolds of type 2 as discussed in Chapter 4. They are spin manifolds with two linearly independent unit real Killing spinors. Thus we can employ the spinorial techniques used to describe the deformation space of  $G_2$  instantons on nearly  $G_2$  manifolds to describe the deformation space of Sasakian instantons on Sasaki–Einstein 7-dimensional manifolds. We begin by describing these manifolds in some detail. We also see how the Sasaki–Einstein structure is equivalent to a 1-parameter family of nearly  $G_2$ -structures.

## 8.1 Preliminaries on Sasaki–Einstein structure

We begin with an introduction to Sasakian geometry. Many of the results here are well known in the literature and are just included to make the discussion self contained. The reader is referred to [BFGK91] for a complete description of these manifolds from a spinorial point of view.

Let  $(M^{2m+1}, g)$  be a Riemannian manifold with a Sasakian structure. A Sasakian structure on  $M$  is defined by a reduction of the frame bundle to the structure group  $SU(m) \subset SO(2m+1)$ . We say that the Sasakian structure on  $M$  is **normal** if the almost complex structure induced by the Sasakian structure on the cone over  $M$  is integrable. Notice that the Sasakian manifold  $(M, g)$  is naturally isometrically embedded into the cone via the inclusion  $M = \{r = 1\} = \{1\} \times M \subset C(M)$ . There is also a canonical projection  $p: C(M) \rightarrow M$  which forgets the  $r$  coordinate. Being Kähler, the cone  $(C(M), \bar{g})$  is equipped with an integrable complex structure  $J$  and a Kähler 2-form  $\omega$ , both of which are parallel with respect to the Levi-Civita connection  $\bar{\nabla}$  of  $\bar{g}$ . The Kähler structure of  $(C(M), \bar{g})$ , combined with its cone structure, induces the Sasakian structure on  $M = \{1\} \times M \subset C(M)$ .

For vector fields  $X, Y$  on  $M$  appropriately interpreted also as vector fields on  $C(M)$ , and  $\bar{\nabla}$ , the Levi-Civita connection of  $g$  we have the following relations,

$$\begin{aligned} \bar{\nabla}_{r\partial_r}(r\partial_r) &= r\partial_r, & \bar{\nabla}_X(r\partial_r) &= \bar{\nabla}_{r\partial_r}X = X, \\ \bar{\nabla}_X Y &= \nabla_X Y - g(X, Y)r\partial_r. \end{aligned} \tag{8.1.1}$$

The canonical vector field  $r\partial_r$  is known as the *homothetic or Euler vector field*. Using the relations (8.1.1), together with the fact that  $J$  is parallel,  $\bar{\nabla}J = 0$ , it follows that  $r\partial_r$  is real holomorphic,  $\mathcal{L}_{r\partial_r}J = 0$ . It is then natural to define the characteristic or Reeb vector field

$$\xi = J(r\partial_r). \tag{8.1.2}$$

From the definition one can show that  $\xi$  is real holomorphic and Killing,  $\mathcal{L}_\xi \bar{g} = 0$ . Moreover,  $\xi$  is clearly tangent to surfaces of constant  $r$  and has square length  $\bar{g}(\xi, \xi) = r^2$ . Let  $d^c = -J \circ d$  denote the composition of exterior derivative with the action of  $J$  on 1-forms, and  $\partial, \bar{\partial}$  be the usual Dolbeault operators, with  $d = \partial + \bar{\partial}$ . We may express the dual 1-form to  $\xi$  by

$$\theta = d^c \log r = i(\bar{\partial} - \partial) \log r. \tag{8.1.3}$$

It follows straightforwardly from the definition that

$$\theta(\xi) = 1, \quad i_\xi d\theta = 0.$$

Since  $\theta$  is the dual form of  $\xi$ , for any vector field  $X$  on  $C(M)$  it follows from (8.1.2) that

$$\theta(X) = \frac{1}{r^2} \bar{g}(\xi, X) = \frac{1}{r^2} \bar{g}(J(r\partial_r), X).$$

The Kähler 2-form  $\omega$  on  $C(M)$  is then given by

$$\omega = \frac{1}{2} d(r^2 \theta) = \frac{1}{2} i\partial\bar{\partial} r^2. \quad (8.1.4)$$

The 1-form  $\theta$  restricts to a 1-form  $\theta|_M$  on  $M$ . Since  $\mathcal{L}_{r\partial_r} \theta = 0$  we have that  $\theta = p^*(\theta|_M)$ . We denote both  $\theta$  and  $\theta|_M$  by  $\theta$  when there is no confusion. The same holds for  $\xi$  as it is the dual of  $\theta$ .

Since the Kähler 2-form  $\omega$  is symplectic, it follows from (8.1.4) that the top degree form  $\theta \wedge (d\theta)^m$  on  $M$  is nowhere zero and hence is a volume form on  $M$ . By definition, this makes  $\theta$  a contact 1-form on  $M$ .

Define a section  $\Phi$  of  $\text{End}(TM)$  such that

$$\Phi|_{\ker \theta} = J|_{\ker \theta}, \quad \Phi|_{(\ker \theta)^\perp} = 0$$

Using  $J^2 = -\text{id}$  one shows that

$$\Phi^2 = -\text{id} + \theta \otimes \xi \quad (8.1.5)$$

and the fact that the cone metric  $\bar{g}$  is Hermitian implies for any vector fields  $X, Y$  on  $M$

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \theta(X)\theta(Y). \quad (8.1.6)$$

The triple  $(\theta, \xi, \Phi)$ , with  $\theta$  a contact 1-form with Reeb vector field  $\xi$  and  $\Phi$  a section of  $\text{End}(TM)$  satisfying (8.1.5), is known as an **almost contact structure**. An almost contact structure  $(\theta, \xi, \Phi)$  together with a metric  $g$  satisfying (8.1.6) is known as a **metric contact structure**. Sasakian manifolds are thus special types of metric contact structures, as introduced by Sasaki in [Sas60].

To summarize, one can define a contact structure on  $M^{2m+1}$  by the tuple  $(\xi, \theta, \Phi)$  such that

- $\xi \in \Gamma(TM)$  is a non-vanishing smooth unit vector field on  $M$  known as the *characteristic or Reeb vector field*,
- $\theta \in \Lambda^1(M)$  is the dual 1-form of  $\xi$ ,
- $\Phi$  is a section of  $\text{End}(TM)$  such that  $\Phi^2 = -1 + \theta \otimes \xi$ ,
- $\theta \wedge (d\theta)^m$  is a nowhere vanishing top form and is a volume form,
- $\theta \lrcorner d\theta = 0$ ,
- $C(M)$ , the cone over  $M$ , is a Calabi–Yau  $(m + 1)$  manifold.

Given a contact structure  $(\xi, \theta, \Phi)$  on  $M$  we define a section  $J$  of the endomorphism bundle of the tangent bundle  $TC(M)$  of the cone by

$$JY = \Phi(Y) + \theta(Y)r\partial_r, \quad J(r\partial_r) = \xi.$$

**Definition 8.1.1.** A contact Riemannian manifold  $(M^n, g)$  is Sasakian if and only if  $C(M) = \mathbb{R}^{>0} \times M$  along with the cone metric  $\bar{g} = dr^2 + r^2g$  and the almost complex structure  $J$  defined above is Kähler, that is  $\text{Hol}(\bar{g}) \subseteq U(m)$ .

The above definition implies that  $\dim C(M) = n + 1$  must be even and  $M$  has to be odd dimensional.

Such a manifold is orientable and spin and its spinor bundle has two linearly independent Killing spinors  $\eta_1, \eta_2 = \xi \cdot \eta_1$ . The subgroup  $\text{SU}(m) \subset \text{Spin}(2m + 1)$  fixes  $\nu_1$  and  $\nu_2$  thus  $M$  admits an  $\text{SU}(m)$  structure.

As proved in [HN12] there are special 3, 4-forms on  $M$  defined using the Killing spinors. In terms of the Sasakian structure as defined above with  $\theta$  the contact 1-form and  $\omega$  as defined in (8.1.4), we have

$$\begin{aligned} P &= \theta \wedge \omega, \\ Q &= \frac{1}{2}\omega \wedge \omega. \end{aligned}$$

In a local orthonormal frame  $e_1, \dots, e_{2m+1}$ , we can choose

$$\theta = e_1, \tag{8.1.7}$$

$$\omega = e_{23} - e_{45} - \dots - e_{2m2m+1}, \tag{8.1.8}$$

and it can be seen using (8.1.4) that

$$dP = 4Q, \quad d * Q = (2m - 2) * P. \quad (8.1.9)$$

We now discuss the case of interest, that is when  $\dim(M) = 7$ . In this case we have a relation between Sasaki–Einstein and nearly parallel  $G_2$ -structures which allows us to use techniques from Chapter 7 to define the infinitesimal deformation space of Sasaki-instantons on  $M^7$ .

### 8.1.1 Sasaki–Einstein 7-folds

When the dimension of the Sasaki–Einstein manifold  $M^{2m+1}$  is 7 (or  $m = 3$ ), one can relate (8.1.9) to the nearly  $G_2$  equations (4.1.1) with  $P = \varphi$  and  $Q = *_\varphi \varphi = \psi$ . Let  $K\mathcal{S}$  be the space of Killing spinors on  $M$ . Recall from §4.1 that when  $M^7$  has a nearly parallel  $G_2$ -structure,  $K\mathcal{S}$  is non-trivial with  $1 \leq \dim(K\mathcal{S}) \leq 3$  unless  $M = S^7$ . In fact we have the following proposition, the proof of which can be found in [AS12, Proposition 4.1].

**Proposition 8.1.1.** Let  $(M, g)$  be a compact nearly parallel  $G_2$  manifold. Then:

- (1) If  $\dim K\mathcal{S} \geq 2$ , then every choice of a two-dimensional subspace of  $K\mathcal{S}$  gives a Sasaki–Einstein structure compatible on  $M$  with  $g$ .
- (2) If  $\dim K\mathcal{S} \geq 3$ , then  $(M, g)$  then every choice of a three-dimensional subspace of  $K\mathcal{S}$  gives three compatible orthogonal Sasakian structures or equivalently a 3-Sasakian structure on  $M$  compatible with  $g$ .

Thus a 7-dimensional Sasaki–Einstein manifold  $M$  admits a nearly parallel  $G_2$ -structure of *type 2* that is the space of real Killing spinors is 2-dimensional at each point. Then the given metric and orientation are induced by a Sasaki–Einstein structure but not by a 3-Sasakian structure, that is the holonomy group of  $C(M)$  is contained in  $SU(4)$  but not in  $Sp(2)$ .

**Remark 8.1.2.** For simplicity, we write everything as if  $\dim(K\mathcal{S}) = 2$ . When  $\dim(K\mathcal{S}) \geq 3$ , any choice of 2-dimensional subspace of  $K\mathcal{S}$  gives a Sasaki–Einstein structure. In the rest of this chapter,  $K\mathcal{S}$  should be thought as this chosen 2-dimensional subspace although all the results hold for any 2-dimensional subspace of  $K\mathcal{S}$ .

Suppose that the holonomy group of the cone  $C(M)$  is equal to  $SU(4)$  (which means that  $M$  is Sasaki–Einstein but not 3-Sasakian). Then on  $M^7$  there is a one-to-one correspondence between Sasaki–Einstein structures and 1-parameter families of nearly parallel  $G_2$ -structures. Let  $\omega \in \Omega^2(C(M))$  be the Kähler form and  $\Omega \in \Omega^4(C(M))$  be the complex volume form on the cone. Then the space of parallel 4-forms on  $C(M)$  is spanned by  $\omega \wedge \omega$ ,  $\operatorname{Re}(\Omega)$ , and  $\operatorname{Im}(\Omega)$ . For any  $c_0, c_1, c_2 \in \mathbb{R}$ , we define the 3-form  $\varphi$  on  $M$  by

$$\varphi := \partial_r \lrcorner \left( \frac{1}{2} c_0 \omega \wedge \omega + c_1 \operatorname{Re}(\Omega) + c_2 \operatorname{Im}(\Omega) \right).$$

Using (8.1.4) we can rewrite the above equation in terms of  $\theta, \omega$ , and the horizontal volume form  $\Psi = \partial_r \lrcorner \Omega$ ,

$$\varphi = c_0 \theta \wedge \omega + c_1 \operatorname{Re}(\Psi) + c_2 \operatorname{Im}(\Psi). \quad (8.1.10)$$

The Sasakian structure on  $M$  is induced from the metric  $g_\varphi$  and cross product  $\times_\varphi$  by

$$\Phi(X) = \xi \times_\varphi X, \quad \theta(X) = g_\varphi(\xi, X). \quad (8.1.11)$$

There are many  $G_2$ -structures inducing the above metric and cross product. We can find these  $G_2$ -structures using the Kähler structure on the cone. The  $G_2$ -structure defined in (8.1.10) induces the metric and cross product in (8.1.11) if and only if

$$c_0 = -1, \quad c_1^2 + c_2^2 = 1.$$

Thus we get a 1-parameter family of nearly parallel  $G_2$ -structures on  $M$  associated to the Sasakian structure given by

$$\varphi_t = -\theta \wedge \omega + \cos(t) \operatorname{Re}(\Psi) + \sin(t) \operatorname{Im}(\Psi). \quad (8.1.12)$$

Since there are two independent Killing spinors on  $M$  there are 2 nearly parallel  $G_2$ -structures  $\varphi_1, \varphi_2$  associated to  $\eta_1, \eta_2$  respectively. These two  $G_2$ -structures are related by the equation (see [AF10])

$$\varphi_2 = -\varphi_1 + 2(\xi \lrcorner \varphi_1) \wedge \theta. \quad (8.1.13)$$

Thus, in this case the 3-forms inducing the given metric, orientation, and spin structure are parametrized by  $\mathbb{R}P^1$  (see [FKMS97]).

Let  $\eta_1, \eta_2$  be the Killing spinors corresponding to the nearly  $G_2$ -structures  $\varphi_1, \varphi_2$  respectively.

We choose  $e_1, \dots, e_7$  to be an orthonormal frame at  $p \in M$  with respect to  $\varphi_1$  and choose  $\xi = e_1$ .

With respect to the decomposition  $\mathcal{S} \cong (\Lambda^0 \eta_1) \oplus (\Lambda^1 \cdot \eta_1)$  with respect to  $\varphi_1$  we can identify  $\eta_1 = (1, 0)$  and  $\eta_2 = e_1 \cdot \eta_1 = (0, -e_1)$ . Note that if we chose to work with the decomposition with respect to  $\eta_2$  instead we get  $\eta_2 = (1, 0)$  and  $\eta_1 = -e_1 \cdot \eta_2 = (0, e_1)$ . Thus under the isomorphism  $\mathcal{S} \cong \Lambda^0(M) \oplus \Lambda^1(M)$  the choice of the Killing spinor is irrelevant.

At the point  $p$  we have

$$(\varphi_1)_p = e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356}.$$

By (8.1.13) we get

$$\begin{aligned} (\varphi_2)_p &= -(e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356}) + 2(e_{23} - e_{45} - e_{67}) \wedge e_1 \\ &= e_{123} - e_{145} - e_{167} + e_{246} - e_{257} + e_{347} + e_{356}. \end{aligned}$$

Following the convention in (8.1.7), (8.1.8) we can see that

$$P = e_{123} - e_{145} - e_{167}, \tag{8.1.14}$$

$$Q = -e_{2345} - e_{2367} + e_{4567}. \tag{8.1.15}$$

Each fibre of the bundle  $\mathcal{S}$  for the Sasaki–Einstein 7-fold is an  $SU(3)$ -representation. As before we denote by  $K\mathcal{S}$  the 2-dimensional subspace of Killing spinors. Let  $\{\eta_1, \eta_2 = \xi \cdot \eta_1\}$  be a basis of  $K\mathcal{S} \subset \Gamma(\mathcal{S})$ . Since  $\eta_2 = \xi \cdot \eta_1$  at each point  $p \in M$  we have

$$\langle \eta_1(p), \eta_2(p) \rangle = \langle \eta_1(p), (\xi \cdot \eta_1)(p) \rangle = \langle -(\xi \cdot \eta_1)(p), \eta_1(p) \rangle = 0.$$

Thus for all  $p \in M$ ,  $\text{Span}\{\eta_1(p), \eta_2(p)\} \subset \mathcal{S}_p$  is 2-dimensional. We denote by  $K_{\mathcal{S}}$  the rank 2 vector subbundle of  $\mathcal{S}$  with fibre  $\text{Span}\{\eta_1(p), \eta_2(p)\}$  for all  $p \in M$  and by  $K_{\mathcal{S}}^{\perp}$  the rank 6 subbundle of  $\mathcal{S}$  whose fibre at each  $p \in M$  is  $\text{Span}\{(v \cdot \eta_1)(p), v \in \xi^{\perp}\}$ . Since the fibres of  $K_{\mathcal{S}}$  and  $K_{\mathcal{S}}^{\perp}$  are orthogonal at each point we have the decomposition

$$\mathcal{S} = K_{\mathcal{S}} \oplus K_{\mathcal{S}}^{\perp}.$$

We define

$$K\mathcal{S}^{\perp} := \Gamma(K_{\mathcal{S}}^{\perp}).$$

Observe that the Sasakian structure on  $M$  preserves the subspaces  $K\mathcal{S}$  and  $K\mathcal{S}^{\perp}$  thus by Schur's lemma there exist real constants  $\lambda_0, \lambda_1, \nu_0, \nu_1$  such that for  $\eta \in K\mathcal{S}$  and  $\alpha \in K\mathcal{S}^{\perp}$ ,

$$P \cdot \eta = \lambda_0 \eta, \quad P \cdot \alpha = \lambda_1 \alpha,$$



$$Q \cdot \eta = \nu_0 \eta, \quad Q \cdot \alpha = \nu_1 \alpha.$$

We can compute  $\lambda_0, \lambda_1, \nu_0, \nu_1$  by explicit computation using the Clifford product in (3.2.2) analogous to Lemma 3.2.3. A more elegant way to compute these eigenvalues is using Lemma 3.2.3 and (8.1.13) which we present below.

**Lemma 8.1.2.** *The subspaces  $K\mathcal{S}$  and  $K\mathcal{S}^\perp$  of  $\Gamma(\mathcal{S})$  are eigenspaces for the operations of Clifford multiplication by  $P$  and  $Q$ . The associated eigenvalues are*

	$K\mathcal{S}$	$K\mathcal{S}^\perp$
$P$	3	-1
$Q$	3	-1.

*Proof.* From (8.1.13) and (8.1.9) we know that

$$P = \frac{1}{2}(\varphi_1 + \varphi_2).$$

Since  $\eta_2 = \xi \cdot \eta_1$ , using Lemma 3.2.3 we have

$$\begin{aligned} P \cdot \eta_2 &= \frac{1}{2}(\varphi_1 \cdot \eta_2 + \varphi_2 \cdot \eta_2) \\ &= \frac{1}{2}(\varphi_1 \cdot \eta_2 + \varphi_2 \cdot \xi \cdot \eta_1) \\ &= \frac{1}{2}(7\eta_2 - \xi \cdot \eta_1) = 3\eta_1. \end{aligned}$$

Similarly one can use  $\eta_1 = -\xi \cdot \eta_2$  to show that  $P \cdot \eta_1 = 3\eta_1$ . For  $\alpha \in K\mathcal{S}^\perp$ ,  $\alpha = X \cdot \eta$  for some  $X \in \text{Span}\{\xi\}^\perp$  and  $\eta \in K\mathcal{S}$  thus again by Lemma 3.2.3

$$\begin{aligned} P \cdot X \cdot \eta &= \frac{1}{2}(\varphi_1 \cdot X \cdot \eta + \varphi_2 \cdot X \cdot \eta) \\ &= \frac{1}{2}(-X \cdot \eta - X \cdot \eta) = -X \cdot \eta. \end{aligned}$$

For the 4-form  $Q$  we can compute the eigenvalues similarly. Recall that we denote by  $\psi_i = *_i \varphi_i$ . Taking the Hodge star of (8.1.13) we get

$$Q = \frac{1}{2}(\psi_1 + \psi_2).$$

Now the result follows from the fact that  $\varphi_i$  and  $\psi_i$  acts on the space of spinors with the same eigenvalues as shown in Lemma 3.2.3.  $\square$

Let  $t \in \mathbb{R}$ . We define the 1-parameter family of connections on  $TM$  given by

$$\nabla_X^t Y = \begin{cases} \nabla_X^{LC} Y + \frac{t}{3} P(X, Y, \cdot), & \text{if } X \in \text{Span}\{\xi\}, \\ \nabla_X^{LC} Y + t P(X, Y, \cdot), & \text{if } X \in \text{Span}\{\xi\}^\perp. \end{cases} \quad (8.1.16)$$

It lifts to the following 1-parameter family of connections on  $\mathcal{S}$ . Let  $\mu \in \Gamma(\mathcal{S})$ . Then

$$\nabla_X^t \mu = \begin{cases} \nabla_X^{LC} \mu + \frac{t}{6} i_X P \cdot \mu, & \text{if } X \in \text{Span}\{\xi\}, \\ \nabla_X^{LC} \mu + \frac{t}{2} i_X P \cdot \mu, & \text{if } X \in \text{Span}\{\xi\}^\perp. \end{cases} \quad (8.1.17)$$

The torsion of  $\nabla^t$  for each  $t$  is proportional to  $P$  and hence totally skew symmetric. By the above definition if  $\eta \in K\mathcal{S}$  and  $X \in \text{Span}\{\xi\}$  then by Lemma 8.1.2 we have

$$\begin{aligned} \nabla_X^t \eta &= -\frac{1}{2} X \cdot \eta - \frac{t}{12} (X \cdot P + P \cdot X) \cdot \eta \\ &= -\frac{1}{2} X \cdot \eta - \frac{t}{12} (6X \cdot \eta) \\ &= -\frac{(t+1)}{2} X \cdot \eta, \end{aligned}$$

and if  $X \in \text{Span}\{\xi\}^\perp$  we have

$$\begin{aligned} \nabla_X^t \eta &= -\frac{1}{2} X \cdot \eta - \frac{t}{4} (X \cdot P + P \cdot X) \cdot \eta \\ &= -\frac{1}{2} X \cdot \eta - \frac{t}{4} (2X \cdot \eta) \\ &= -\frac{(t+1)}{2} X \cdot \eta. \end{aligned}$$

Therefore for all  $X \in \Gamma(TM)$ , we have

$$\nabla_X^t \eta = -\frac{(t+1)}{2} X \cdot \eta. \quad (8.1.18)$$

Thus the Killing spinors are parallel with respect to  $\nabla^{-1}$  and  $\text{Hol}(\nabla^{-1}) \subset \text{SU}(3)$ . Hence  $\nabla^{-1}$  is the *canonical connection* as defined in [HN12].

## 8.2 Sasaki-instantons and deformations

Let  $M^{2m+1}$  be a Sasaki–Einstein manifold. Let  $\mathcal{P}$  be a principal  $G$ -bundle over  $M$  and  $A$  be a connection with curvature  $F_A$ . Using (5.2.2) and (8.1.9) for Sasaki manifolds we can say that  $A$  is a Sasaki instanton if and only if

$$\theta \wedge \omega \wedge F = *F. \quad (8.2.1)$$

From the discussion in §5.2 we can obtain equivalent definitions of Sasaki instantons.

- For a Sasaki–Einstein 7-fold  $M$  the space of Killing spinors  $K\mathcal{S}$  is at least 2-dimensional. The connection  $A$  is a Sasaki instanton if and only if for all  $\eta \in K\mathcal{S}$ , we have

$$F_A \cdot \eta = 0.$$

- Since a Sasaki–Einstein 7-fold has an  $SU(3)$  structure, the connection  $A$  is an instanton if the 2-form part of  $F_A$  lies in  $\mathfrak{su}(3)(T^*M)$  which is the subbundle of  $\Lambda^2(T^*M)$  whose fibre at each point is isomorphic to  $\mathfrak{su}(3)$ , that is

$$F_A \in \Gamma(\mathfrak{su}(3)(T^*M) \otimes \text{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \text{Ad}_{\mathcal{P}}).$$

The condition  $F_A \in \Gamma(\mathfrak{su}(3)(T^*M))$  implies

$$\theta \lrcorner F_A = \omega \lrcorner F_A = 0.$$

To summarize, a connection  $A$  on a principal  $G$  bundle  $\mathcal{P}$  over a Sasaki–Einstein manifold  $M^7$  is an  $SU(3)$  or Sasaki instanton if any of the following equivalent conditions hold:

$$\begin{aligned} \theta \wedge \omega \wedge F &= *F, \\ F_A \cdot \eta &= 0, \quad \text{for all } \eta \in K\mathcal{S} : \\ \theta \lrcorner F_A &= 0, \\ \omega \lrcorner F_A &= 0. \end{aligned} \quad (8.2.2)$$

From the above equivalent definitions we can observe a relationship between the Sasaki instantons and the  $G_2$  instantons on  $M$ .

**Lemma 8.2.1.** *A connection  $A$  on  $\mathcal{P} \rightarrow M$  is a Sasaki instanton with respect to the  $SU(3)$  structure  $(\theta, \omega)$  on  $M$  if and only if it is a  $G_2$  instanton with respect to the nearly  $G_2$ -structures  $\varphi_1$  and  $\varphi_2$  on  $M$ .*

*Proof.* Let  $F_A$  be the curvature associated to  $A$ . From the spinorial description of the instanton condition we know that  $A$  is a Sasaki instanton if and only if  $F_A \cdot \eta = 0$  for all  $\eta \in K\mathcal{S}$ . But since  $K\mathcal{S} = \text{Span}\{\eta_1, \eta_2\}$   $F_A \cdot \eta = 0$  if and only if  $F_A \cdot \eta_1 = F_A \cdot \eta_2 = 0$  which by (5.2.3)  $F_A$  implies that  $A$  is a  $G_2$  instanton with respect to both  $\varphi_1$  and  $\varphi_2$ .  $\square$

### 8.2.1 Infinitesimal deformations of $SU(3)$ instantons

Let  $M^7$  be a Sasaki–Einstein 7-manifold. We are interested in studying the infinitesimal deformation of Sasaki instantons on  $M$ . An infinitesimal deformation of a connection  $A$  represents a minuscule change in  $A$  and thus, is a section of  $\text{Ad}_{\mathcal{P}} \otimes T^*M$ . If  $\epsilon \in \Gamma(\text{Ad}_{\mathcal{P}} \otimes T^*M)$  is an infinitesimal deformation of  $A$ , the corresponding change in the curvature  $F_A$  up to first order is given by  $d^A\epsilon$ . Linearizing the instanton condition  $F_A \cdot \eta = 0$  as in (8.2.1) we get  $d^A\epsilon \cdot \eta = 0$  for all  $\eta \in K\mathcal{S}$ . A standard gauge fixing condition on this perturbation is given by  $(d^A)^*\epsilon = 0$ . So in total  $\epsilon \in \text{Ad}_{\mathcal{P}} \otimes T^*M$  is an infinitesimal deformation of a Sasaki instanton  $A$  if and only if for all  $\eta \in K\mathcal{S}$

$$d^A\epsilon \cdot \eta = 0, \quad (d^A)^*\epsilon = 0. \quad (8.2.3)$$

The 1-parameter family of connections on the spinor bundle  $\mathcal{S}$  defined in (8.1.16) and the connection  $A$  on  $\mathcal{P}$  can be used to construct a 1-parameter family of connections on the associated vector bundle  $\mathcal{S} \otimes \text{Ad}_{\mathcal{P}}$ . We denote by  $\nabla^{t,A}$  the connection associated to  $\nabla^t$  and  $A$ . The solutions to (8.2.3) can also be seen as eigenspaces of the Dirac operators associated to connections  $\nabla^{t,A}$  as shown in the following proposition.

**Proposition 8.2.1.** Let  $\epsilon$  be a section of  $\text{Ad}_{\mathcal{P}} \otimes T^*M$ , and let  $D^{t,A}$  be the Dirac operator constructed from the connections  $\nabla^{t,A}$  for  $t \in \mathbb{R}$ . Then  $\epsilon$  solves (8.2.3) if and only if for all  $t \in \mathbb{R}$  and  $\eta \in K\mathcal{S}$ ,

$$D^{t,A}(\epsilon \cdot \eta) = \begin{cases} \frac{7t-5}{2} \epsilon \cdot \eta, & \text{if } \epsilon \in \Gamma(\text{Span}\{\xi\} \otimes \text{Ad}_{\mathcal{P}}), \\ -\frac{7t+15}{6} \epsilon \cdot \eta, & \text{if } \epsilon \in \Gamma(\text{Span}\{\xi\}^\perp \otimes \text{Ad}_{\mathcal{P}}). \end{cases}$$

*Proof.* Let  $e_a$  be a local orthonormal frame for  $T^*M$ . We assume  $e_1$  to be the Reeb vector field  $\xi$ . Then

$$D^{0,A}(\epsilon \cdot \eta) = e_a \cdot \nabla_a^0(\epsilon \cdot \eta)$$

$$= (d^A \epsilon + (d^A)^* \epsilon) \cdot \eta + e_a \cdot \epsilon \cdot \nabla_a^0 \eta.$$

Applying Proposition 3.2.1 to the 1-form part of  $\epsilon$  we get,  $e_a \cdot \epsilon \cdot e_a \cdot \eta = 5\epsilon \cdot \eta$ . So if  $\eta$  is a real Killing spinor then (4.1.7) together with the above identity imply

$$\begin{aligned} D^{0,A}(\epsilon \cdot \eta) &= (d^A \epsilon + (d^A)^* \epsilon) \cdot \eta - \frac{1}{2} e_a \cdot \epsilon \cdot e_a \cdot \eta \\ &= (d^A \epsilon + (d^A)^* \epsilon - \frac{5}{2} \epsilon) \cdot \eta. \end{aligned}$$

It follows from (8.1.16) and (8.1.14) that the operator  $D^{t,A}$  is given by

$$\begin{aligned} D^{t,A}(\epsilon \cdot \eta) &= \left( D^{0,A} + \frac{t}{6} e_1 \cdot i_{e_1} P \cdot + \frac{t}{2} \sum_{a=2}^7 e_a \cdot i_{e_a} P \cdot \right) \epsilon \cdot \eta \\ &= \left( D^{0,A} + \frac{t}{6} P \cdot + \frac{t}{2} (2P \cdot) \right) \epsilon \cdot \eta \\ &= \left( D^{0,A} + \frac{7t}{6} P \cdot \right) \epsilon \cdot \eta. \end{aligned}$$

From Lemma 8.1.2 we have

$$D^{t,A}(\epsilon \cdot \eta) = \begin{cases} \left( d^A \epsilon + (d^A)^* \epsilon + \frac{7t-5}{2} \epsilon \right) \cdot \eta, & \text{if } \epsilon \in \Gamma(\text{Span}\{\xi\} \otimes \text{Ad}_{\mathcal{P}}), \\ \left( d^A \epsilon + (d^A)^* \epsilon - \frac{7t+15}{6} \epsilon \right) \cdot \eta, & \text{if } \epsilon \in \Gamma(\text{Span}\{\xi\}^\perp \otimes \text{Ad}_{\mathcal{P}}). \end{cases}$$

The result now follows because  $(d^A \epsilon + (d^A)^* \epsilon) \cdot \eta = 0$  is equivalent to the pair of equations  $d^A \epsilon \cdot \eta = 0, (d^A)^* \eta = 0$  since these two components live in complementary subspaces.  $\square$

The above proposition for  $t = 0$  proves the following result.

**Theorem 8.2.2.** The space of infinitesimal deformations of a Sasaki instanton  $A$  on a principal bundle  $\mathcal{P}$  over a 7-dimensional Sasakian manifold  $M$  is isomorphic to the kernel of the operator

$$\left( D^{0,A} + \frac{5}{2} \text{Id} \right) : \Gamma(\Lambda^1 \cdot K_{\mathcal{S}} \otimes \text{Ad}_{\mathcal{P}}) \rightarrow \Gamma(\Lambda^1 \cdot K_{\mathcal{S}} \otimes \text{Ad}_{\mathcal{P}}).$$

**Remark 8.2.3.** The above theorem is analogous to Theorem 7.1.2 proved for  $G_2$  instantons on nearly  $G_2$  manifolds.

Theorem 8.2.2 implies that the infinitesimal space of deformations of Sasaki instantons on a 7-dimensional manifold is isomorphic to an eigenspace of a twisted Dirac operator. This result can be used to further analyze the deformation space in specific examples as done by Charbonneau–Harland in [CH16] for nearly Kähler 6-manifolds and by the author in [Sin21] for  $G_2$  instantons on nearly  $G_2$  manifolds.

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# APPENDICES

# Appendix A

## Connections on homogeneous spaces

In this appendix we briefly discuss the well known theory of connections on homogeneous spaces. We present some insight on the characteristic homogeneous connection we encountered in Chapters 4 and 7. The main references for this discussion are [Bes87, Chapter 7], [KN96, Chapter 10]. Short notes can also be found in [Oli14a, Appendix B] and [Dri20, Section 3.3].

We begin by defining a homogeneous manifold.

**Definition A.0.1.** A Riemannian manifold  $(M, g)$  is homogeneous under the Lie group  $G$  (or  $G$ -homogeneous) if  $G$  is a closed subgroup of the isometry group  $\mathcal{I}(M, g)$  which acts transitively on  $M$ .

One should note that  $G$  need not be all of  $\mathcal{I}(M, g)$ . For example when  $M = \mathbb{R}^n$  and  $g$  is the Euclidean metric,  $\mathcal{I}(\mathbb{R}^n, g)$  is the group of orientation preserving and reversing motions but the proper subgroup of translations also acts transitively on the Euclidean space. We also assume that  $G$  is closed for simplicity. Although this restriction is not necessary as if  $G$  is not closed then there exists a unique subgroup  $\tilde{G}$  of  $\text{Diff}(M)$  such that for any  $G$ -invariant Riemannian metric  $g$  on  $M$ ,  $\tilde{G}$  is the closure of  $G$  in  $\mathcal{I}(M, g)$ .

For an arbitrary fixed point  $x_0 \in M$ , the closed subgroup

$$H := \{a \in G \mid ax_0 = x_0\}$$

is called the **isotropy subgroup** of  $G$  at  $x_0$ . Then  $M$  is diffeomorphic to the coset space  $G/H$  where the diffeomorphism from  $M \rightarrow G/H$  is given by

$$x = ax_0 \mapsto aH$$



for  $a \in G$ . Note that the existence of such an  $a$  is guaranteed since the action is transitive.

Conversely, if  $H$  is a closed subgroup of a Lie group  $G$ , then the coset space  $G/H$  becomes a  $C^\infty$  manifold, and  $G$  acts on  $M$  by

$$\begin{aligned} G \times G/H &\rightarrow G/H \\ (g, aH) &\mapsto (ga)H. \end{aligned}$$

The map

$$\begin{aligned} l_g: G/H &\rightarrow G/H \\ aH &\mapsto (ga)H \end{aligned}$$

is called a *left translation* by  $g$ . If  $\pi: G \rightarrow G/H$  is the quotient map and  $L_g$  is the left multiplication by  $g$  on  $G$  then

$$l_g \circ \pi = \pi \circ L_g.$$

**Remark A.0.1.** Since  $G$  is a closed subgroup of  $\mathcal{I}(M, g)$  the isotropy subgroup  $H$  at  $x \in M$  is a compact subgroup of  $\mathcal{I}_x(M, g)$ . Also  $M$  is compact if and only if  $G$  is compact.

**Examples.** The canonical sphere  $S^n$  may be viewed as a homogeneous manifold  $\text{SO}(n+1)/\text{SO}(n)$ . But in some dimensions there are other Lie groups acting transitively on some spheres classified by D. Montgomery and H. Samelson in [MS43], for example  $S^6 = \text{G}_2/\text{SU}(3)$  and  $S^7 = \text{Spin}(7)/\text{G}_2$ . Other examples include projective spaces

$$\begin{aligned} \mathbb{R}\mathbb{P}^n &= \text{SO}(n+1)/\text{O}(n), \\ \mathbb{C}\mathbb{P}^n &= \text{SU}(n+1)/S(\text{U}(1) \times \text{U}(n)), \\ \mathbb{H}\mathbb{P}^n &= \text{Sp}(n+1)/\text{Sp}(n)\text{Sp}(1). \end{aligned}$$

A Riemannian metric  $h$  on a homogeneous manifold  $M = G/H$  is said to be  *$G$ -invariant* if for all  $X, Y \in \Gamma(TM)$  and  $k \in G$ , we have

$$h((l_g)_*X, (l_g)_*Y) = h(X, Y).$$

Thus we see that if  $h$  is  $G$ -invariant then  $l_g \in \mathcal{I}(M, g)$  for all  $g \in G$ .

Since  $H \subset G$ , the Lie algebra of  $H$ , denoted by  $\mathfrak{h}$ , is a subspace of  $\text{Lie}(G) = \mathfrak{g}$ . We denote by  $[\cdot, \cdot]$  the Lie bracket on  $\mathfrak{g}$  and we denote by  $[\cdot, \cdot]_V$  the restriction of  $[\cdot, \cdot]$  on a subspace  $V$  of  $\mathfrak{g}$ .

Furthermore if  $\mathfrak{g}$  has an  $\text{Ad}(H)$ -invariant inner product  $B$  then with respect to  $B$  we have the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \quad (\text{A.0.1})$$

Here  $\mathfrak{m}$  is the orthogonal complement of  $\mathfrak{h}$  with respect to  $B$ .

**Definition A.0.2.** The homogeneous space  $M = G/H$  is reductive if

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

**Ad( $H$ )-invariant metrics on  $\mathfrak{m}$ .** Geometrically one can describe  $\mathfrak{m}$  as

$$T_{eH}M \cong T_eG / \ker(\pi_*)_e \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}.$$

Thus for any left-invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  we can define an inner product  $(\cdot, \cdot)$  on  $\mathfrak{m}$ . For  $X, Y \in T_{gH}M$  we have

$$\begin{aligned} \langle X, Y \rangle_{gH} &= \langle (l_{g^{-1}})_{*gH}X, (l_{g^{-1}})_{*gH}Y \rangle_{eH} \\ &= \langle (l_{g^{-1}})_{*gH}X, (l_{g^{-1}})_{*gH}Y \rangle. \end{aligned}$$

If  $M$  is reductive the decomposition in (A.0.1) is  $\text{Ad}(H)$ -invariant thus for all  $X \in \mathfrak{m}$  and  $h \in H$ ,

$$\pi_{*e}(\text{Ad}(h)X) = (l_h)_{*e}X$$

which proves that the inner product  $(\cdot, \cdot)$  is  $\text{Ad}(H)$ -invariant if and only if  $\langle \cdot, \cdot \rangle$  is  $G$ -invariant. Hence, we have the one-to-one correspondence between:

$$\begin{aligned} \langle \cdot, \cdot \rangle &: G\text{-invariant Riemannian metric on } G/H, \text{ and} \\ (\cdot, \cdot) &: \text{Ad}(H)\text{-invariant inner product on } \mathfrak{m}. \end{aligned}$$

**Definition A.0.3.** A reductive homogeneous space  $G/H$  with a  $G$ -invariant metric  $\langle \cdot, \cdot \rangle$  is naturally reductive if

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0$$

for any  $X, Y, Z \in \mathfrak{m}$  where  $[X, Y]_{\mathfrak{m}}$  denotes the  $\mathfrak{m}$ -component of  $[X, Y]$ .

**Remark A.0.2.** According to the direct sum decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , we have

$$[X, Y] = [X, Y]_{\mathfrak{h}} + [X, Y]_{\mathfrak{m}} + \mathfrak{m}.$$

**Definition A.0.4.** A homogeneous space  $M = G/H$  with a Riemannian metric  $\langle \cdot, \cdot \rangle$  is *normal* homogeneous if there exists a bi-invariant Riemannian metric  $(\cdot, \cdot)$  on  $G$  such that, if we identify  $m = h^\perp$  with respect to  $(\cdot, \cdot)$ , then

$$\langle \cdot, \cdot \rangle = (\cdot, \cdot)|_{\mathfrak{m} \times \mathfrak{m}}.$$

**Remark A.0.3.** If  $M = G/H$  is normal homogeneous,  $M$  is naturally reductive.

The examples of nearly  $G_2$  manifolds in Table 4.2 are all normal homogeneous manifolds.

If the Lie group  $G$  is semi-simple the Killing form  $B$  on  $\mathfrak{g}$  given by

$$B(X, Y) = \text{tr}(\text{ad}(X), \text{ad}(Y))$$

is negative-definite. Thus there is a canonical choice for a bi-invariant metric on  $\mathfrak{g}$ , given by  $-B$ . The induced metric on  $G/H$ , denoted by  $g$ , is called the **standard homogeneous metric** on  $G/H$ .

**Definition A.0.5.** Let  $Q \rightarrow G/H$  be a principal  $K$ -bundle. We say that  $Q$  is  $G$ -homogeneous if there is a lift of the natural left action of  $G$  on  $G/H$  to the total space  $Q$  which commutes with the right action of  $K$ .

Let  $Q$  be a homogeneous  $K$ -bundle over  $G/H$ . Choose a point  $q_0 \in \pi^{-1}(eH)$ , then for all  $h \in H$ , we see that  $h \cdot q_0 \in \pi^{-1}(eH)$ . Thus, for each  $h \in H$  there exists a unique  $k \in K$  such that  $h \cdot q_0 = q_0 \cdot k$ . This defines a map  $\lambda: H \rightarrow K$ . We see that this is in fact a homomorphism since  $q_0 \cdot \lambda(h_1 h_2) = h_1 h_2 \cdot q_0 = q_0 \cdot \lambda(h_1) \cdot \lambda(h_2)$ . We call  $\lambda$  the *isotropy homomorphism*. This allows one to reconstruct the bundle  $Q$  as follows.

Consider the associated bundle  $Q \times_{(H, \lambda)} K = (G \times K) / \sim$  where  $\sim$  is the equivalence relation  $(gh, k) \sim (g, \lambda(h)k)$  for all  $g \in G, h \in H$  and  $k \in K$ . Suppose now we have a homogeneous  $K$ -bundle  $Q = G \times_{(H, \lambda)} K$  and a representation  $(V, \rho)$  of  $K$ . Then the lift of the  $G$ -action to  $Q$  endows the associated bundle  $E = Q \times_{(K, \rho)} V$  with an action of  $G$ . Furthermore there is an isomorphism of homogeneous bundles  $E \cong G \times_{(H, \rho \circ \lambda)} V$ .

One turns now to the definition of invariant connections on the principal bundle  $Q = G \times_{(H, \lambda)} K$ . These are given by a left invariant connection 1-form  $A \in \Omega^1(K, \mathfrak{k})$  and classified by Wang's theorem.

**Theorem A.0.6** (Wang [Wan58]). *Let  $Q = G \times_{(H, \lambda)} K$  be a principal homogeneous  $K$ -bundle. Then  $G$ -invariant connections on  $P$  are in one-to-one correspondence with morphisms  $\Phi$  of  $H$  representations*

$$\Phi: (\mathfrak{m}, \text{Ad}) \rightarrow (\mathfrak{k}, \text{Ad} \circ \lambda).$$

**The canonical connection:** In this section we assume  $G/H$  is a reductive homogenous space, with the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

We define and describe the basic properties of the canonical connection of a reductive homogenous space. We consider  $G$  as a fiber bundle over the space  $G/H$  with structure group  $H$ . The action of  $H$  on  $G$  is right multiplication. The group  $G$  itself acts on the fiber bundle. This action clearly commutes with the projection map and the action of  $H$ .

**Definition A.0.7.** The *canonical connection* is a  $G$ -invariant connection on the principal bundle  $G$  for which the horizontal space at the identity is  $\mathfrak{m}$ .

Clearly this defines a horizontal distribution. To show that this distribution is a connection we have to show compatibility with right action:

$$\begin{aligned} (R_h)_*(L_g)_*\mathfrak{m} &= (L_g)_*(R_h)_*\mathfrak{m} \\ &= (L_g)_*(L_h)_*(L_{h^{-1}})_*(R_h)_*\mathfrak{m} \\ &= (L_{gh})_*(\text{Ad}^{G/H}(h))_*\mathfrak{m} \\ &= (L_{gh})_*\mathfrak{m}. \end{aligned}$$

**Proposition A.0.4.** The torsion  $T$  and the curvature  $F$  associated to the canonical connection are given by

1.  $T(X, Y) = -[X, Y]_{\mathfrak{m}}$ ,
2.  $F(X, Y)Z = -[[X, Y]_{\mathfrak{t}}, Z]_{\mathfrak{m}}$ .

*Proof.* 1. It suffices to prove it at the identity as  $T$  and  $[\cdot, \cdot]_{\mathfrak{m}}$  are left-invariant tensors. For a vector field  $X$  on  $G/H$  we define the map  $f_X : G \rightarrow \mathfrak{m}$  by

$$f_X(g) = L_g^{-1}(X(\pi(g))).$$

The torsion tensor  $T$  is then given by

$$T(X, Y) = (L_g)_*(X^*(f_Y) - Y^*(f_X)) - [X, Y]$$

For  $X, Y \in \mathfrak{m}$  let  $\tilde{X}, \tilde{Y}$  be right-invariant vector fields generated by  $X, Y$  in  $G$ . We have that  $[\tilde{X}, \tilde{Y}] = -[X, Y]$ . The right invariance implies  $\pi_*\tilde{X}, \pi_*\tilde{Y}$  are extensions of  $X, Y$  in  $G/H$ . We have that

$$\begin{aligned} [\pi_*\tilde{X}, \pi_*\tilde{Y}] &= \pi_*[\tilde{X}, \tilde{Y}] \\ &= \pi_*(-[X, Y]) \\ &= -[X, Y]_{\mathfrak{m}}. \end{aligned}$$

Now

$$\begin{aligned} f_{\pi_*\tilde{X}}(g) &= L_g^{-1}(\pi_*\tilde{X}(\pi(g))) \\ &= L_g^{-1}(\pi_*(R_g)_*(X)) \\ &= \pi_*(\text{ad}(g^{-1})_*X). \end{aligned}$$

At the identity,

$$\begin{aligned} Y^*(e)(f_{\pi_*\tilde{X}}(g)) &= Y(\pi_*(\text{ad}(g^{-1})_*X)) \\ &= \frac{d}{dt}(\pi_*(\text{ad}(\exp(-tY))_*X)) \\ &= \pi_*[-Y, X] = [X, Y]_{\mathfrak{m}}. \end{aligned}$$

Similarly we have  $X^*(e)(f_{\pi_*\tilde{Y}}(g)) = [Y, X]_{\mathfrak{m}}$ . Combining the previous equations we get

$$T(X, Y) = [Y, X]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - (-[X, Y]_{\mathfrak{m}}) = -[X, Y]_{\mathfrak{m}}.$$

2. The curvature tensor  $F$  associated to a connection for which  $h$  denotes the horizontal projection is given by

$$\begin{aligned} F(X, Y)Z &= (L_g)_*(X^*(Y^*(f_Z)) - Y^*(X^*(f_Z))) - h([X^*, Y^*])(f_Z) \\ &= ((L_g)_* - h)([X^*, Y^*])(f_Z). \end{aligned}$$

For the canonical connection  $h = \text{id}$  thus we have

$$\begin{aligned} F(X, Y)Z &= ([X, Y] - [X, Y]_{\mathfrak{m}})(f_Z) \\ &= [X, Y]_{\mathfrak{h}}(\pi_*(\text{ad}(g^{-1})_*Z)) \\ &= \frac{d}{dt}(\pi_*(\text{ad}(\exp(-t[X, Y]_{\mathfrak{h}}) * Z))) \\ &= \pi_*[-[X, Y]_{\mathfrak{h}}, Z] = -[X, Y]_{\mathfrak{h}}, Z]_{\mathfrak{m}}. \end{aligned}$$

□

From the above proposition it is obvious that the torsion tensor  $T$  is totally skew-symmetric. Moreover the torsion tensor  $T$  and curvature  $F$  associated to the canonical connection  $\nabla^{can}$  are both parallel with respect to the canonical connection, that is

$$\nabla^{can}T = 0, \quad \nabla^{can}F = 0.$$

In fact any  $G$ -invariant tensor is parallel with respect to  $\nabla^{can}$ . By first extending  $\nabla^{can}$  trivially to  $G \times K$  one obtains a connection  $\nabla^{can,\Lambda}$  on  $G \times_\lambda K$ . We have

$$\begin{aligned} \nabla^{can,\lambda} &= \lambda_* \nabla^{can}, \\ F_{can,\lambda} &= \lambda_* F_{can}. \end{aligned}$$

On a manifold with a nearly  $G_2$ -structure the canonical connection defines a nearly  $G_2$  instanton on the associated bundle with  $\text{Hol}(\nabla^{can,\lambda}) = H \subset G_2$ .

# Appendix B

## Maple code for Lie algebra computations

In this chapter we briefly present the Maple code we used to determine the deformation space of the canonical instanton in Chapter 7. The part of the computation done in Maple involve computing the space  $\text{Hom}(V, \mathfrak{m}^* \otimes E)_H$  where  $V$  is an irreducible  $G$ -representation satisfying  $\rho_E(\text{Cas}_{\mathfrak{h}}) = \rho_L(\text{Cas}_{\mathfrak{g}})$ . Once we have a non-trivial homomorphism from  $V \rightarrow \mathfrak{m}^* \otimes E$  we can compute the action of the Dirac operator on  $V$  and check which of the irreducible representations of  $G$  are in the  $-2$  eigenspace of the Dirac operator  $D^{-1,can}$  where  $E = \mathfrak{h}$  or  $\mathfrak{g}_2$ . See §7.3 for a complete discussion on why these are the spaces that describe deformations of the canonical connection  $\nabla^{can}$ .

We only present the Maple code for  $\frac{\text{Sp}(2) \times \text{Sp}(1)}{\text{Sp}(1) \times \text{Sp}(1)}$  as all the other cases are similar. The reason for choosing this space is the mere complexity of the case that allows us to present all the details that are missing in the other cases. The computation for  $E = \mathfrak{h}$  were done by hand so Maple was used only to handle the case  $E = \mathfrak{g}_2$ . In order to compute the action of the Dirac operator  $D^{-1,A}$  on  $\ker\{(D^{-1/3,can})^2 - (49/9)\text{id}\} = \sum_{i=1}^2 \text{Hom}(V_i, \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\text{sp}(1)_u \oplus \text{sp}(1)_d} \otimes V_i$  where  $V_1 = V_{(2,0,0)} \cong \mathfrak{sp}(2)$  and  $V_2 = V_{(0,0,2)} \cong \mathfrak{sp}(1)$  we compute the spaces  $\text{Hom}(V_{(2,0,0)}, \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\text{sp}(1)_u \oplus \text{sp}(1)_d}$ ,  $\text{Hom}(V_{(0,0,2)}, \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\text{sp}(1)_u \oplus \text{sp}(1)_d}$ . On elements of the form  $\text{Hom}(V_i, \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\text{sp}(1)_u \oplus \text{sp}(1)_d} \otimes V_i$  we can compute the action of the Dirac operator  $D^{-1,can}$  using (7.1.13).

**Remark B.0.1.** The Maple code presented below is not complete and is presented just to deliver the idea how such computations could be done in Maple. It however can be easily used to perform similar calculations in Maple with minor additions.

To make the code easy to read we present it in a tabular form where the first column describes the code and the second column represents the description of the command.

**Packages used:**

- 1 *LinearAlgebra*
- 2 *DifferentialGeometry* (See [AT12] and [AT16] for its applications and usage)

The following sub-packages of *DifferentialGeometry* package have also been used:

- LieAlgebras: for the symbolic analysis of Lie algebras.
- Tensor: provides an extensive suite of commands for computations with sections of any vector bundle.
- Tools: a small utility package

**Setting up the Lie Algebra  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ :**

We write  $\{e_i, i = 1, \dots, 13\}$  for a basis of  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d \oplus \mathfrak{m}$ . The basis is orthogonal with respect to the nearly  $G_2$  metric  $-\frac{3}{40}B$  where  $B$  is the Killing form of  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ . The elements  $\{e_1, \dots, e_6\}$  and  $\{e_7, \dots, e_{13}\}$  forms a basis of  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$  and  $\mathfrak{m}$  respectively.

Setting up the Lie algebras

<code>sp2sp1 := LieAlgebraData(basis_sp2sp1, Liesp2sp1):  DGsetup(basis_sp2sp1):</code>	define and set up $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$
<code>sp21vector := [e1, e2, e3, evalDG(3/5*(e4 + e7)),  evalDG(3/5*(e5 + e8)),  evalDG(3/5*(e6 + e9)), e10, e11, e12, e13];</code>	basis of $\mathfrak{sp}(2)$ inside $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$
<code>sp11vector := [evalDG(-3/5*(e7 - 2/3*e4)),  evalDG(3/5*(e8 - 2/3*e5)),  evalDG(-3/5*(e9 - 2/3*e6))];</code>	basis of $\mathfrak{sp}(1)$ inside $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$

Constructing the nearly  $G_2$  structure



```

m := [e7, e8, e9, e10, e11, e12, e13]

g := table([seq(seq(seq((i, j, k)
simplify(Killing(m[k], LieBracket(m[i],
m[j]))/Killing(m[k], m[k])), i = 1 .. 7), j =
1 .. 7), k = 1 .. 7)])

basis1form := [seq(cat(F, i), i = 1 .. 7)];
DGsetup([basis1form], [seq(d(basis1form[i])
= add(add(-1/2*g[j, k, i]*(basis1form[j] &w
basis1form[k]), j = 1 .. 7), k = 1 .. 7), i =
1 .. 7)], Q);

phi := evalDG(add(add(add(1/4*g[i, j,
k]*((basis1form[i] &w basis1form[j]) &w
basis1form[k]), i = 1 .. 7), j = 1 .. 7), k =
1 .. 7))

psi := ExteriorDerivative(1/4*phi)

```

The Lie algebra  $\mathfrak{g}_2$  that annihilates the above  $G_2$  structure

```

A := Matrix(7, 7, (i, j) -> a[i, j], shape =
antisymmetric);
alpha := evalDG(evalDG(add(add(A(i,
j)*(basis1form[i] &w basis1form[j]), i = 1 .. 7),
j = 1 .. 7) &w psi));

```

the basis of  $\mathfrak{m}$

Lie algebra coefficients for  $\mathfrak{m}$ ,  $[e_i, e_j] = \sum_{k=1}^7 g(i, j, k)e_k$

defining the space  $\mathfrak{m}$  which is the tangent space at identity; the exterior derivative is defined using the Maurer–Cartan equations as  $\mathfrak{m} \subset \mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$

this defines the 3-form  $\varphi$  for the nearly  $G_2$  structure; since torsion of the canonical connection  $T(X, Y) = -[X, Y]_{\mathfrak{m}} = -\frac{2}{3}\varphi(X, Y, \cdot)$  (4.1.9)

the 4-form  $\psi$  satisfies  $d\varphi = 4\psi$  (4.1.1)

since  $\Lambda_{14}^2 = \{\alpha \in \Lambda^2, \alpha \wedge \psi = 0\} \cong \mathfrak{g}_2$  solve for  $\alpha = 0$  to get a basis of  $\mathfrak{g}_2 \subset \mathfrak{so}(7)$

```

g1:=sk(1, 7) + sk(2, 4);
g2 := sk(1, 6) + sk(2, 5);
g3 := -sk(2, 6) - sk(1, 5);
g4 := -sk(2, 7) - sk(1, 4);
g5 := -sk(3, 4) - sk(1, 6);
g6 := sk(3, 5) + sk(1, 7);
g7 := sk(3, 6) + sk(1, 4);
g8 := -sk(3, 7) - sk(1, 5);
g9 := sk(4, 5) + sk(2, 3);
g10 := -sk(4, 6) - sk(1, 3);
g11 := sk(4, 7) - sk(2, 1);
g12 := sk(5, 6) + sk(1, 2);
g13 := sk(5, 7) + sk(1, 3);
g14 := sk(6, 7) + sk(2, 3);
basisg2 := [g1, g2, g3, g4, g5, g6, g7, g8, g9,
g10, g11, g12, g13, g14];
G2 := LieAlgebraData(basisg2, Lieg2);
DGsetup(G2);

```

```

basis_sp1sp1g2 := evalDG([e14 - e9, -e10 - e13,
e12 - e11, e9 + e14, e10 - e13, e11 + e12])

```

### Representation of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ on $\mathfrak{m}$

```

adm := [seq(Matrix(7, 7, (j, i) ->
Killing(LieBracket(sp1sp1[k], m[i]),
m[j])/Killing(m[j], m[j])), k = 1 .. 6)]:
DGsetup([seq(cat(x, i), i = 1 .. 7)], M):
repm := Representation(Liesp1, M, adm):

```

this is the basis of  $\mathfrak{g}_2$  we obtain and use to set up the Lie algebra  $\mathfrak{g}_2$

obtain a basis of  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  inside  $\mathfrak{g}_2$  by computing  $\text{ad}(h)_m$  for each element  $h \in \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$ ; since the adjoint action of  $H$  preserves the  $G_2$  structure  $\text{ad}(h)_m \in \mathfrak{g}_2$  for all  $h \in \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$

the 7 dimensional adjoint representation on  $\mathfrak{m}$  since  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$

```

casm := Eigenvectors(add(ScalarMultiply(1/8,
ApplyRepresentation(repm, sp1sp1[i])^2), i = 1
.. 6), output = 'list' );
for i to 2 do
if casm[i][2] = 3 then
invs2q_m := [seq(evalDG((Transpose(casm[i][3][k]))
. basis1vector_m), k = 1 .. 3)];
elif casm[i][2] = 4 then
invpq_m := [seq(evalDG((Transpose(casm[i][3][k]))
. basis1vector_m), k = 1 .. 4)];
end if; end do;

DGsetup([v1, v2, v3], M1);
s2q_m := SubRepresentation(repm, invs2q_m, M1);

DGsetup([u1, u2, u3, u4], M2);
pq_m := SubRepresentation(repm, invpq_m, M2);

```

### Adjoint $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ representation on $\mathfrak{g}_2$

```

adg2 := [seq(Adjoint(basis_sp1sp1g2[i]), i = 1 ..
6)];
DGsetup([seq(cat(y, i), i = 1 .. 14)], G);
repG2 := Representation(Liesp1, G, adg2);

casg2 := Eigenvectors(add(ScalarMultiply(1/8,
ApplyRepresentation(repG2, sp1sp1[i])^2), i = 1 ..
6), output = 'list');
for i to 2 do
if casg2[i][2] = 8 then invps3q_g2 :=
[seq(evalDG((Transpose(casg2[i][3][k]))
. basis1vector_g2), k = 1 .. 8)];
end if;
end do;

```

splitting the adjoint representation on  $\mathfrak{m}$  into irreducible representations of  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$  using the Casimir eigenvalue; each irreducible representation acts on the Casimir element as a scalar identity of an eigenvalue which along with the dimension can be used to find the highest weight of the representation

$\mathfrak{m} = PQ \oplus S^2Q$  where  $P, Q$  are standard representations of  $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$  respectively; so we just separated them using the dimensions

since  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \subset \mathfrak{g}_2$  we define the 14 dimensional adjoint representation on  $\mathfrak{g}_2$

we again use the Casimir element to decompose  $\mathfrak{g}_2$  in irreducible representation

```
DGsetup([seq(cat(z, i), i = 1 .. 8)], Z); PS3Q :=
SubRepresentation(repG2, invps3q_g2, Z);
```

we only need the 8-dimensional representation  $PS^3Q$  since  $\mathfrak{g}_2 = S^2Q \oplus S^2P \oplus PS^3Q$  where  $S^2P \oplus S^2Q$  is the  $E = \mathfrak{h}$  case already considered; see §7.3

### Product representation $\mathfrak{m} \otimes \mathfrak{g}_2$

```
tensor_rep := TensorProductOfRepresentations([repm,
PS3Q], T);
```

define the  $7 \times 14 = 56$  dimensional representation  $\mathfrak{m} \otimes \mathfrak{g}_2$ .

```
castensor := Eigenvectors(add(ScalarMultiply(1/8,
ApplyRepresentation(tensor_rep, sp1sp1[i]^2), i =
1 .. 6), output = 'list');
```

decompose  $\mathfrak{m} \otimes \mathfrak{g}_2$  into irreducible representations

```
for i to nops(castensor) do
if castensor[i][2] = 3 then invs2q_tensor :=
[seq(evalDG((Transpose(castensor[i][3][k])) .
basis1vector_T), k = 1 .. 3)];
elif castensor[i][2] = 4 then invpq_tensor :=
[seq(evalDG((Transpose(castensor[i][3][k])) .
basis1vector_T), k = 1 .. 4)];
end if; end do;
```

separate the subspaces isomorphic to  $S^2Q$  and  $PQ$

```
DGsetup([seq(cat(alpha, i), i = 1 .. 3)], T1);
s2q_T := SubRepresentation(tensor_rep,
invs2q_tensor, T1);
```

define  $S^2Q \subset \mathfrak{m} \otimes \mathfrak{g}_2$

```
DGsetup([seq(cat(beta, i), i = 1 .. 4)],
T2); pq_T := SubRepresentation(tensor_rep,
invpq_tensor, T2);
```

define  $PQ \subset \mathfrak{m} \otimes \mathfrak{g}_2$

Adjoint  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  representation on  $\mathfrak{sp}(2), \mathfrak{sp}(1)_d \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$

```
adsp2 := [seq(Matrix(10, 10, (j, i) ->
Killing(LieBracket(sp1sp1[k], sp21vector[i]),
sp21vector[j])/Killing(sp21vector[j],
sp21vector[j])), k = 1 .. 6)];
```

defining the representation matrix for the action on  $\mathfrak{sp}(2)$  as a subspace of  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$

```
DGsetup([seq(cat(r, i), i = 1 .. 10)], S);
repsp2 := Representation(Liesp1, S, adsp2);
```

defining the representation  $\mathfrak{sp}(2)$

```
s2q_sp2 := SubRepresentation(repsp2,
[seq(basis1vector_sp2[i], i = 4 .. 6)], SZ)
```

define the irreducible sub representation  $PQ$  and  $S^2Q$  of  $\mathfrak{sp}(2)$

```
pq_sp2 := SubRepresentation(repsp2,
[seq(basis1vector_sp2[i], i = 7 .. 10)], SW)
```

```
adsp1 := [seq(Matrix(3, 3, (j, i) ->
Killing(LieBracket(sp1sp1[k], sp11vector[i]),
sp11vector[j])/Killing(sp11vector[j],
sp11vector[j])), k = 1 .. 6)]
```

the irreducible adjoint  $\mathfrak{sp}(1)$  representation

### Constructing the Lie algebra homomorphisms

From §7.3 the deformation space is a subset of

$(\text{Hom}(\mathfrak{sp}(2), \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} \otimes \mathfrak{sp}(2)) \oplus (\text{Hom}(\mathfrak{sp}(1), \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} \oplus \mathfrak{sp}(1))$ . We compute  $\text{Hom}(\mathfrak{sp}(2), \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}$  and  $\text{Hom}(\mathfrak{sp}(1), \mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}$ .

```
Ahom1 := Matrix(3, 3, (i, j) -> aa[i, j]);
changebasis1 := [seq((ApplyRepresentation(s2q_T,
sp1sp1[i])) . Ahom1, i = 1 .. 6)];
changebasis2 := [seq(Ahom1 . (adsp1[i]), i = 1 ..
6)];
sol0 := solve(seq(seq(seq(changebasis1[k](i, j) =
changebasis2[k](i, j), i = 1 .. 3), j = 1 .. 3),
k = 4 .. 6));
hom0 := evalDG((subs(seq(seq(aa[i, j] = d[1], i
= 1 .. 3), j = 1 .. 3), Transpose(subs(sol0,
Ahom1)))) . (Vector(s2q_mg2)));
```

defining the homomorphism of  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  representation between the irreducible sub-representation of  $\mathfrak{m}^* \otimes \mathfrak{g}_2$  isomorphic to  $S^2Q$  and  $\mathfrak{sp}(1) \cong S^2Q$

```

Ahom := Matrix(3, 3, (i, j) -> a[i, j]);
basischange1 := [seq((ApplyRepresentation(s2q_T,
sp1sp1[i])) . Ahom, i = 1 .. 6)];
basischange2 := [seq(Ahom .
(ApplyRepresentation(s2q_sp2, sp1sp1[i])), i =
1 .. 6)];
soln1 := solve(seq(seq(seq(basischange1[k](i, j) =
basischange2[k](i, j), i = 1 .. 3), j = 1 .. 3),
k = 4 .. 6));
hom1 := evalDG((subs(seq(seq(a[i, j] = c[1], i
= 1 .. 3), j = 1 .. 3), Transpose(subs(soln1,
Ahom)))) . (Vector(s2q_mg2)))

```

defining the homomorphism of  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  representation between the irreducible sub-representations of  $\mathfrak{m}^* \otimes \mathfrak{g}_2$  and  $\mathfrak{sp}(2)$  isomorphic to  $S^2Q$

```

Bhom := Matrix(4, 4, (i, j) -> b[i, j]);
basischange3 := [seq((ApplyRepresentation(pq_T,
sp1sp1[i])) . Bhom, i = 1 .. 6)];
basischange4 := [seq(Bhom .
(ApplyRepresentation(pq_sp2, sp1sp1[i])), i = 1
.. 6)];
soln2 := solve(seq(seq(seq(basischange3[k](i, j) =
basischange4[k](i, j), i = 1 .. 4), j = 1 .. 4),
k = 1 .. 6));
hom2 := evalDG((subs(seq(seq(b[i, j] = c[2], i
= 1 .. 4), j = 1 .. 4), Transpose(subs(soln2,
Bhom)))) . (Vector(pq_mg2)));

```

defining the homomorphism of  $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$  representation between the irreducible sub-representations of  $\mathfrak{m}^* \otimes \mathfrak{g}_2$  and  $\mathfrak{sp}(2)$  isomorphic to  $PQ$