Applications of the minimal model program in arithmetic dynamics

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety defined over a number field. The dynamics of f may be studied through the dynamics of the linear action $f^*\colon \mathrm{Pic}(X)_{\mathbb{R}} \to \mathrm{Pic}(X)_{\mathbb{R}}$, which are governed by the spectral theory of f^*. Let $\lambda_1(f)$ be the spectral radius of f^*. We study $\mathbb{Q}$-divisors D with $f^*D = \lambda_1(f)D$ and $\kappa(D) = 0$ where $\kappa(D)$ is the Iitaka dimension of the divisor D. We analyze the base locus of such divisors and interpret the set of small eigenvalues in terms of the canonical heights of Jordan blocks described by Kawaguchi and Silverman. We identify a linear algebraic condition on surjective morphisms that may be useful in proving instances of the Kawaguchi-Silverman conjecture.

We prove the Kawaguchi-Silverman conjecture and verify the aforementioned linear algebraic condition holds for projective bundles $\mathbb{P}E$ over an elliptic curve C where $E = \bigoplus_{i=1}^{s} F_{r_i}$ and F_{r_i} is the semi-stable degree zero Atiyah bundle on C. This represents new progress in the last remaining case of the Kawaguchi-Silverman conjecture for projective bundles over curves.

By a result of Silverman and Kawaguchi, if $P \in X(\overline{\mathbb{Q}})$ then $\alpha_f(P) = |\mu|$ where μ is an eigenvalue of f^* acting on the Neron-Severi space of X. We give examples of abelian varieties that possess an eigenvalue μ of modulus strictly larger than one such that $\alpha_f(P) \neq |\mu|$ for all $P \in X(\overline{\mathbb{Q}})$. On the other hand, we show using the minimal model program that if X is a $\mathbb{Q}$-factorial toric variety then for every such μ we can find a point P with $\alpha_f(P) = |\mu|$.

Finally, we give a program to study the arithmetic dynamics of higher dimensional projective varieties using the minimal model program. In particular, we describe how one might use the minimal model program to determine if certain surjective morphisms have a dense set of pre-periodic points, and how to study the Medvedev-Scanlon conjecture for certain surjective endomorphisms using the minimal model program.

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Dedication

I dedicate this thesis to my brother Brooks. Forward unto dawn.

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Chapter 1

Introduction

Arithmetic dynamics is the study of the behavior of a rational map of algebraic varieties $f\colon X \dashrightarrow X$ under composition. That is, the study of the behavior of the rational maps $f^n\colon X \dashrightarrow X$ as n grows large. While it may not be obvious given this description, the tendrils of arithmetic dynamics reach deeply into both algebraic geometry and number theory. The roots of arithmetic dynamics can be seen in Newton's iterative method, and its influence seen in the work of Northcott and Tate, and later by Silverman and Zhang. The power and allure of arithmetic dynamics is partly explained by the fact that a conjecture or theorem in arithmetic geometry often has an analogous conjecture or theorem in arithmetic dynamics that tends to be *harder* then the motivating problem, due to the dynamical version being more general.

The basic method of formulating dynamical conjectures out of arithmetic conjectures comes from an analogy of Silverman described in [7]. Let $f\colon X \dashrightarrow X$ be a rational map. We call the pair (X, f) a dynamical system. Let P be a point of X such that $f^n(P)$ is defined for all $n \geq 1$. We let $\mathcal{O}_f(P) = \{P, f(P), f^2(P), \ldots\}$ be the forward orbit of P. If this orbit is finite, we call P a **pre-periodic point**. In other words, P is pre-periodic if there are integers $n \neq m$ such that $f^n(P) = f^m(P)$. If we have $f^n(P) = P$ for some $n \geq 1$ we say P is a **periodic** point. Silverman's analogy is then as follows.

$$
\begin{array}{ccc}
\text{Arithmetic Geometry} & & \text{Arithmetic Dynamics} \\
\\
\text{A variety } X & \longleftrightarrow & \text{A dynamical system } (X, f) \\
\\
\text{Rational and integral points on } X & \longleftrightarrow & \text{Rational and integral points on orbits of } (X, f) \\
\\
\text{Torsion points on abelian varieties} & \longleftrightarrow & \text{Pre-periodic and periodic points of rational map}
\end{array}
$$

The study of a dynamical system (X, f) is often very difficult, even if X is very simple. For example, let $c \in \mathbb{Q}$ and let $f_c(x) = x^2 + c$. It is unknown whether there are rational periodic points with period greater than 3. In other words, we cannot determine if there are $\alpha \in \mathbb{Q}$ with $f^{\circ n}(\alpha) = \alpha$, $f^{\circ k}(\alpha) \neq \alpha$ for $k < n$ and $n > 3$. The reason for this difficulty is that one may reinterpret this question in terms of finding rational points on certain plane curves of large genus, which is known to be a challenging problem. See [7, Section 4] for more details on this area.

In contrast with the above problem, this thesis is primarily concerned with tackling dynamical problems in dimension larger than one. Even though there are challenging unresolved problems in dimension one, progress is still possible in higher dimensions provided that one uses appropriate techniques from higher dimensional algebraic geometry. To illustrate the difference between the one-dimensional setting and the higher dimensional world consider the following. Let (X, f) be a dynamical system with X a smooth projective variety defined over $\overline{\mathbb{Q}}$. If $\dim X = 1$ then any dominant rational mapping $f\colon X \dashrightarrow X$ extends to a morphism $f\colon X \to X$. In this case then we have that X is $\mathbb{P}^1$, an elliptic curve, or a smooth curve of genus $g \geq 2$. If $X = \mathbb{P}^1$ then f is given by two homogeneous polynomials $F, G \in \overline{\mathbb{Q}}[t, s]$ without a common vanishing locus. If $X = E$ an elliptic curve, then $f = \tau_c \circ g$ where g is an isogeny of the curve. Finally, if $X = C$ a smooth curve of general type, then f is a finite order automorphism. In other words, we have at least a rough classification of the possible dynamical systems in dimension one, even if the finer details may be difficult to study. By contrast, it is in general a difficult question to understand the collection of surjective endomorphisms of a projective variety X when $\dim X \geq 2$. In the above classification, we made use of the fact that we can classify smooth projective curves by their genus. In higher dimensions such a complete classification is not available, yet there is an analog of this classification called the *minimal model program* which will be used extensively.

In this dissertation we will study higher dimensional arithmetic dynamics from the point of view of the **Kawaguchi-Silverman conjecture** (KSC), which we now explain. Let (X, f) be a dynamical system with X a normal projective variety defined over a number field K and f a *dominant* rational map. Associated to f are two numerical invariants. The **dynamical degree** is defined as

$$\lambda_1(f) := \lim_{n\to\infty}((f^n)^*H \cdot H^{\dim X-1})^{\frac{1}{n}},$$

where H is any ample divisor and $(f^n)^*H \cdot H^{\dim X-1}$ is the intersection product. This limit always converges and is independent of H. We think of the dynamical degree as a *geometric* measure of the complexity of f under iteration. There are also the upper and lower **arithmetic degrees** of a point $P \in X(\bar{K})$ such that $f^n(P)$ is defined for all $n \geq 1$. These numbers are defined by

$$\overline{\alpha}_f(P) := \limsup_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$$
$$\underline{\alpha}_f(P) := \liminf_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$$

where h_H is a height function associated to an ample divisor and $h_H^+(P) = \max\{1, h_H(P)\}$. As expected, these numbers are independent of H. The Kawaguchi-Silverman conjecture predicts a certain type of ergodic theorem relating these invariants.

Conjecture 1 ([29]). *Let X be a normal projective variety defined over a number field K and let $f\colon X \dashrightarrow X$ be a dominant rational map. Let $P \in X(\bar{K})$ such that $f^n(P)$ is well defined for all $n \geq 1$. Suppose that the orbit $\mathcal{O}_f(P)$ is Zariski dense in X. Then $\overline{\alpha}_f(P) = \underline{\alpha}_f(P) = \lambda_1(f)$. In other words the limit $\lim_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$ exists and is equal to $\lambda_1(f)$.*

When f is a dominant rational map that is not a morphism this question appears very difficult. Even for $\mathbb{P}^2$ it is still unresolved. By contrast when the map is a morphism then it is known that the limit $\lim_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$ exists. We call the limit the arithmetic degree, which is denoted $\lim_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}} := \alpha_f(P)$. The requirement of a dense forward orbit makes the conjecture interesting only when $\kappa(X) \leq 0$. When $\kappa(X) > 0$ the existence of the Iitaka fibration over a positive dimensional base makes the existence of a dense forward orbit impossible. In light of this the following conjecture was proposed, which is interesting for all Kodaira dimensions and implies Conjecture (1).

Conjecture 2 ([39, Conjecture 1.4])**.** *Let X be a normal projective variety and $f\colon X \to X$ a surjective endomorphism. Let*

$$\mathcal{S}(X,f,N) = \{P \in X(K) : [K:\mathbb{Q}] \leq N, \alpha_f(P) < \lambda_1(f)\}$$

Then $\mathcal{S}(X,f,N)$ is not Zariski dense in X.

In the morphism case a powerful tool is the following connection to linear algebra. The morphism $f\colon X \to X$ induces by pull back linear isomorphisms $f^*\colon \mathrm{Pic}(X)_{\mathbb{R}} \to \mathrm{Pic}(X)_{\mathbb{R}}$ and $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. One may profitably study the dynamics of f through the dynamics of these actions. The connection is follows,

$$\lambda_1(f) = \text{Spectral Radius}(f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}) = \{\max \mid \lambda \mid : \lambda \text{ an eigenvalue of } f^*\}\}$$

If $\lambda > 1$ is an eigenvalue of f^* then one uses the divisor classes with $f^*D \sim_{\mathbb{Q}} \lambda D$ to construct *height functions* which intertwine the arithmetic and geometry of (X,f). To contrast this with the case that f is merely a dominant rational map we note that one can still define a pull back mapping $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$, but this mapping does not reflect the dynamics of f as completely as in the morphism case. For those who know more, this is reflected in the fact that it is difficult to construct canonical height functions associated to f when f is not a morphism. In other words, Silverman's maxim that *geometry determines arithmetic* is much more tenuous and theoretical when f is not everywhere defined. As a reality check to gauge the difference in difficulty between the morphism and rational map case, note that when f is a dominant rational map the conjecture is open for $\mathbb{P}^2$. While when f is a morphism the Kawaguchi-Silverman conjecture is known in the following cases.

1. Varieties with Picard number 1, in particular for all $\mathbb{P}^n$. ([29])
2. Abelian varieties. ([28, 56])
3. Smooth projective surfaces. ([40])
4. Rationally connected varieties admitting an int-amplified endomorphism. ([42])
5. Hyper-Kahler Varieties.([34])

See ([40]) for further comments on what is known.

1.1 Statement of results and layout of the thesis.

In chapter 2 we review much of the needed geometry of the thesis including toric varieties. In chapter 3 the required arithmetic results are given. Finally in chapter 4 we combine these results and give the desired dynamical results. These chapters are expository and should be read as needed. From a dynamical perspective chapter 4 may be the most interesting and contain the newest material. The original work of the thesis is contained in chapters 5,6, and 7.

1.1.1 Results from chapter 5

In chapter 5 we study surjective endomorphisms $f: X \to X$ of a normal projective variety through the base locus of an eigendivisor D with $f^*D \equiv_{\text{lin}} \lambda_1(f)D$ and $\lambda_1(f) \in \mathbb{Z}$. When $\kappa(D) > 0$ it is known by 4.2.9 that conjecture 1 holds. We focus on the case $\kappa(D) = 0$. Our main contributions here are as follows.

Chapter 5, Theorem 1 (5.1.7.1). *The Kawaguchi-Silverman conjecture for an endomorphism f with $\lambda = \lambda_1(f) > 1$ is equivalent to $G_{f,H}$ contains no dense orbit of f.*

Here $G_{f,H}$ is an analog of the set $\mathcal{S}(X, f, N)$ in 2.

We also formulate a new property of surjective endomorphisms using the canonical heights associated to Jordan blocks.

Chapter 5, Definition (5.2.6). *Let $f: X \to X$ be a surjective endomorphism of a normal projective variety over a number field K with $\lambda = \lambda_1(f) > 1$. Let H be an integral eigendivisor and let V_H be as is (4.2.4.1). We say $f^* |_{V_H}$ has a good eigenspace if $f^* |_{V_H}$ has the following properties,*

1. *λ is the unique eigenvalue of absolute value λ of $f^* |_{V_H}$.*
2. *The multiplicity of all λ Jordan blocks of $f^* |_{V_H}$ are of multiplicity 1.*
3. *The Jordan blocks of $f^* |_{V_H}$ associated to λ can be taken to be integral nef divisor classes $D_1, ..., D_l$.*
4. *There is some $1 \leq i \leq l$ such that $\kappa(D_i) \neq 0$.*

This definition is meant to generalize 5.2.4 which is a technical definition that arises in the study of varieties admitting an int-amplified endomorphism. The benefit of this definition is that it is not as specialized to the int-amplified situation as 5.2.4. Currently there is no known example of a surjective endomorphism that does not possess a good eigenspace. As a sample result, we prove the following.

Chapter 5, Theorem 2 (5.2.13)**.** *Let X be a smooth projective variety with Picard number 1 and let E be a nef vector bundle on X with $H^0(X,E) \neq 0$ such that E is not ample and $\kappa(\mathbb{P}E, -K_{\mathbb{P}E}) \geq 0$. Suppose that there is integral ample divisor H such that $f^* |_{V_H}$ has a good eigenspace. Then the Kawaguchi-Silverman conjecture holds for X.*

We then turn to projective bundles over an elliptic curve. These are interesting because they represent the only remaining open case where the Kawaguchi-Silverman conjecture for projective bundles over curves is not known. If C is an elliptic curve, then Atiyah showed in [4] there is a unique degree zero, rank r, indecomposable vector bundle on C with a non-zero global section. Call this vector bundle F_r. We are able to prove the following results.

Chapter 5, Theorem 3 (5.3.16.2)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Then the Kawaguchi-Silverman conjecture holds for $\mathbb{P}F_r$ when $r > 2$. In particular, $\mathbb{P}F_r$ has a good eigenspace.*

Chapter 5, Theorem 4 (The reducible case, 5.3.11.1)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $E = \bigoplus_{i=1}^{s} F_{r_i}$ where $s > 1$. Then the Kawaguchi-Silverman conjecture holds for $\mathbb{P}E$. In particular, $\mathbb{P}E$ has a good eigenspace.*

Let $E = \bigoplus_{i=1}^{s} F_{r_i}$. In both of the previous two theorems we obtain the result by showing that $\dim H^0(C, \mathrm{Sym}^d E) \geq O(d)$ where we use the big O notation. In other words, the dimension of the sections grows at least linearly in d. Note that this is contrast to $\dim H^0(C, \mathrm{Sym}^d F_2) = 1$ for all $d \geq 1$.

Finally, through completely different methods we obtain the following.

Chapter 5, Theorem 5 (5.3.20.1)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let F_r be the rank r Atiyah bundle when $2 \leq r \leq 4$.*

1. *Then $\mathbb{P}F_r$ does not admit an int-amplified endomorphism.*
2. *The Kawaguchi-Silverman conjecture folds for $\mathbb{P}F_r$.*

3. *If $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$ is a surjective endomorphism then $f^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\mathrm{lin}} \mathcal{O}_{\mathbb{P}F_r}(1)$. In particular, $f^*\colon N^1(\mathbb{P}F_r)_{\mathbb{R}} \to N^1(\mathbb{P}F_r)_{\mathbb{R}}$ has 1 as an eigenvalue.*

It is our hope that the methods developed in the proof of the previous result will generalize to all r. While we have obtained the Kawaguchi-Silverman conjecture for $\mathbb{P}F_r$ when $r \geq 3$ in theorem 3, a generalization of theorem 5 would lead to a more detailed understanding of the possible surjective endomorphisms of projective bundles, extending the results of [2] and [3].

1.1.2 Results from chapter 6

In chapter 6 we consider the possible values for $\alpha_f(P)$ when $f\colon X \to X$ is a surjective endomorphism and $P \in X(\overline{\mathbb{Q}})$. Kawaguchi and Silverman showed that $\alpha_f(P) = |\mu|$ for some eigenvalue of $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. Thus, to understand the set

$$\mathcal{S}(X, f, N) = \{P \in X(K) : [K, \mathbb{Q}] \leq N, \alpha_f(P) < \lambda_1(f)\}$$

in conjecture 2 we must understand which of these arithmetic degrees actually occur. This leads to the following key notions.

Chapter 6, Definition (6.0.1)**.** *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Consider the action of f^* on $N^1(X)_{\mathbb{R}}$ and let μ be an eigenvalue of this action with $|\mu| > 1$. We call such an eigenvalue* ***potentially arithmetic****. If there is a point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) = |\mu|$ then we say that μ is* ***arithmetic****. If every potentially arithmetic eigenvalue is arithmetic, then we say that f has* ***arithmetic eigenvalues****.*

Chapter 6, Question (question 2)**.** *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Is every eigenvalue μ of f^* with $|\mu| > 1$ arithmetic?*

We then proceed to answer the question in two distinct cases.

Chapter 6, Theorem 1 (6.1.3)**.** *For each $g \in 2\mathbb{Z}_{>0}$ there is an abelian variety A of dimension g defined over a number field K with $\rho(A) = 3$ equipped with a surjective endomorphism $f\colon A \to A$ that has the following properties.*

1. *$f^*\colon N^1(A)_{\mathbb{Q}} \to N^1(A)_{\mathbb{Q}}$ has eigenvalues $a^2 > ab > b^2 > 0$ for some $a, b, c \in \mathbb{Z}$.*

2. $\alpha_f(P) = a^2$ *for all* $P \notin A(\overline{K})_{\mathrm{tors}}$. *In particular,* $\alpha_f(P) \in \{1, a^2\}$.

3. *The eigendivisors of* a^2, b^2 *are nef while the eigendivisor of* ab *is not.*

We also can show the following.

Chapter 6, Theorem 2 (6.1.4.1)**.** *For any integer* $d > 1$ *there is a smooth projective variety* X *with* $\dim X = d$ *such that there is a surjective endomorphism* $f \colon X \to X$ *with* $\lambda_1(f) > 1$ *and* f *has does not have arithmetic eigenvalues. If* $d \geq 3$ *and* $\kappa \in \{-\infty, 0, 1, \dots, d-2\}$ *then* X *may be chosen with* $\kappa(X) = \kappa$.

However, when $\mathrm{Alb}(X) = 0$ we are able to prove positive results for toric varieties. We first handle the case of *equivariant morphisms* of toric varieties. To do this we introduce the following definition in analogy with Abelian varieties.

Chapter 6, Definition (6.2.11)**.** *Let* X_Σ *be a* $\mathbb{Q}$*-factorial projective toric variety defined over* $\overline{\mathbb{Q}}$. *We say that* X_Σ *is decomposable if*

$$X_\Sigma = X_{\triangle_1} \times X_{\triangle_2}$$

with each $X_{\triangle_i}$ *a* $\mathbb{Q}$*-factorial projective toric variety of dimension at least* 1. *We say that* X_Σ *is simple if it is not decomposable.*

We also have a notion of a *dynamically simple* toric variety.

Chapter 6, Definition (6.2.7)**.** *Let* X_Σ *be a projective toric variety defined over* $\overline{\mathbb{Q}}$. *We say that* X_Σ *is* ***linearly simple*** *if* $\mathrm{Lin}(\mathrm{SEnd}_{T_\Sigma}(X_\Sigma))$ *has finite index in* $\mathbb{Z}_{\geq 0}$. *In other words, every surjective toric morphism is induced by a homomorphism of tori* $(x_1, \dots, x_n) \mapsto (x_1^d, \dots, x_n^d)$ *for some* $d > 0$ *after possibly iterating the morphism.*

We then show that these two notions are the same.

Chapter 6, Theorem 3 (6.2.15)**.** *Let* X_Σ *be a* $\mathbb{Q}$*-factorial projective toric variety defined over* $\overline{\mathbb{Q}}$. *Then* X_Σ *is linearly simple if and only if* X_Σ *is simple.*

This result can then be leveraged to give new proofs of the following results.

Chapter 6, Theorem 4. *Let* X_Σ *be a* $\mathbb{Q}$*-factorial toric variety defined over* $\overline{\mathbb{Q}}$. *Let* $f \colon X_\Sigma \to X_\Sigma$ *be an equivariant surjective endomorphism.*

1. *Then conjecture 2 is true for f.*

2. *The morphism f has arithmetic eigenvalues.*

Finally, we use the minimal model program to prove a similar result for all surjective endomorphisms of $\mathbb{Q}$-factorial toric varieties.

Chapter 6, Theorem 5 (6.2.21)**.** *Let X be $\mathbb{Q}$-factorial toric variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism. Then f has arithmetic eigenvalues.*

1.1.3 Results from chapter 7

Motivated by the use of the minimal model program in chapter 6 theorem 5 in chapter 7 we explore how we might use the minimal model program to study other questions in arithmetic dynamics.

In the discussion following 7.0.1 we outline a general program to tackle dynamical problems using the minimal model program. In section 7.1 we apply these ideas to study when a surjective morphism has a dense set of pre-periodic points. In section 7.2 we consider the Medvedev-Scanlon conjecture 5 from the perspective of the minimal model program. Finally in section 7.3 we consider automorphisms of projective varieties from this perspective. This leads to the following result.

Chapter 7, Theorem 1 (7.3.6)**.** *Let X be a normal projective variety over $\overline{\mathbb{Q}}$ with a finitely generated nef cone. Let $f \in \pi_0\mathrm{Aut}(X)$. Then $\lambda_1(f) > 1 \iff f$ has infinite order in $\pi_0\mathrm{Aut}(X)$. In particular a normal projective variety X with finitely generated nef cone has an automorphism of positive entropy if and only if $\pi_0\mathrm{Aut}(X)$ has an element of infinite order.*

While many of the results in these sections are preliminary, we hope that they will bear fruit in the future.

Chapter 2

Geometric Preliminaries

2.1 Toric Varieties

We give a brief introduction to the theory of toric varieties and convex cones. Our main references here are [19] and [12]. The basic object of study in the theory of toric varieties is a variety of the following form.

Definition 2.1.1 (Toric Varieties)**.** *Let k be a field of characteristic 0. A toric variety over k is a geometrically irreducible, normal variety X that contains a dense open algebraic torus $T \cong \mathbb{G}_m^n$. Additionally, X has a T-action which extends the action of T on itself.*

A toric morphism between toric varieties X, X' is a morphism $f\colon X \to X'$ which is torus equivariant. In other word if T is the dense torus of X and T' is the dense torus of X' then $f(T) \subseteq T'$ and $f(t \cdot x) = f(t) \cdot f(x)$ for all $t \in T$ and $x \in X$.

The theory of toric varieties is designed to describe all such varieties in terms of combinatorial data related to closed convex cones in a finite dimensional real vector space. Once this is accomplished one can develop a dictionary between properties of these varieties and the properties of the associated combinatorial data. Before we formally proceed, we sketch the idea. Fix a finitely generated monoid $P \subseteq \mathbb{Z}^n$. That is, a subset $P \subseteq \mathbb{Z}^n$ such that $0 \in P$ and that P is closed under addition and P is finitely generated. In other words there are $p_1, ..., p_s \in P$ such that every element of P is of the form $\sum_{i=1}^{s} n_i P_i$ for some $n_1, ..., n_s \in \mathbb{Z}_{\geq 0}$. Given $u \in P$ with say $u = (u_1, ..., u_n)$ we write $x^u = \prod_{i=1}^{n} x_i^{u_i} \in k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. We then define the semi-group algebra $k[P] = \{\sum_{u \in P} c_u x^u : c_u \in k,\ c_u = 0 \text{ for almost all } u \in P\}$. Addition is done in the obvious way and we multiply elements according to the multiplication in $k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. We let $X = \operatorname{Spec} k[P]$. Thus, associated to such a monoid we have constructed an affine scheme. Let $N_P = \operatorname{span}_{\mathbb{Z}} P$. Then N_P is a lattice of rank $n_P = \dim \operatorname{span}_{\mathbb{Q}}(P)$ and $\operatorname{Spec} k[N_P] \cong \operatorname{Spec} k[t_1^{\pm 1}, ..., t_{n_P}^{\pm 1}] = \mathbb{G}_m^{n_P}$. If we identify $k[P]$ as a sub-algebra of $k[t_1^{\pm 1}, ..., t_{n_P}^{\pm 1}]$ the induced morphism $\mathbb{G}_m^{n_P} \to \operatorname{Spec} k[P]$ is an open embedding induced by localization at the product $t_1 ... t_{n_P}$. Moreover, as we have a decomposition of algebras

$$k[P] = \bigoplus_{u \in P} k \cdot x^u,$$

we have that $k[P]$ has a N_P grading such that if a is in the degree u part of $k[P]$ and b is in the degree v part of $k[P]$ then ab is in the degree $u + v$ part of $k[P]$. This data is equivalent to an action of $\operatorname{Spec} k[N_P]$ on $\operatorname{Spec} k[P]$. In other words, $\operatorname{Spec} k[P]$ is acted on by the torus $\mathbb{G}_m^{n_P}$, and the action extends the usual action of $\mathbb{G}_m^{n_P}$ on itself. To sum up, given a semi-group $P \subseteq \mathbb{Z}^n$ that is finitely generated we constructed an affine variety X that contains a dense open torus $\mathbb{G}_m^{n_P}$ whose action on itself extends to all of X. These will be the affine building blocks of all toric varieties. We now turn to gluing. Now suppose that $P, P' \subseteq \mathbb{Z}^n$ are two finitely generated semi-groups with the property that there is some $u \in P$ and $u' \in P'$ such that

$$P + \mathbb{Z}_{\geq 0}(-u) = P' + \mathbb{Z}_{\geq 0}(-u') \tag{2.1}$$

Set $P'' = P + \mathbb{Z}_{\geq 0}(-u)$. Then we have open embeddings

$$\operatorname{Spec} k[P] \longleftarrow\!\supset \operatorname{Spec} k[P''] \hookrightarrow \operatorname{Spec} k[P'] \tag{2.2}$$

In other words, $\operatorname{Spec} k[P]$ and $\operatorname{Spec} k[P']$ both contain $\operatorname{Spec} k[P'']$ as a principal open subset. We will glue together all our affine toric varieties from the situation described in 2.1 and 2.2. To keep track of all of these relationships we will describe a convex geometric object called a *fan* which will keep track of all of our semi-groups. A fan will be built of convex cones. The passage to convex cones is as follows. Given P consider the cone $C(P) = \{\sum_{i=1}^{s} n_i p_i \colon n_1, ..., n_s \in \mathbb{Q}_{\geq 0}, p_1, ..., p_s \in P\}$. Set

$$\sigma_P = \{u \in \mathbb{Q}^n \colon (u, p) \geq 0 \ \forall \ p \in C(P)\} \tag{2.3}$$

where $(\cdot, \cdot)$ is the usual inner product on $\mathbb{Q}^n$. Then σ_P is a convex cone in $\mathbb{Q}^n$ and we can recover P from σ_P. The fan will now be constructed out of cones σ_P. In order to work with convex cones, we now introduce them formally.

2.1.1 Convex cones

We are interested in cones from the point of view of toric varieties as described above. However, we will also be interested in cones that arise in algebraic dynamics. In particular, given a projective variety X we will construct a number of convex cones $\mathrm{Nef}_{\mathbb{R}}(X), \mathrm{Big}_{\mathbb{R}}(X), \overline{\mathrm{NE}}(X)$ which contain geometric information about X. Moreover, in the presence of a morphism $f\colon X \to X$ we will have an induced linear map that respects the cones in question. In this way, questions about the dynamics of morphisms of X can be reduced to questions about the dynamics of linear mappings, in other words to their eigenvalues. To do this efficiently we need an efficient language of convex cones, which we now develop following [19, 1.2]. In this section we let N be a lattice. That is, a finite rank free $\mathbb{Z}$-module and we set $V = N \otimes_{\mathbb{Z}} \mathbb{R}$. The dual space $\hom_{\mathbb{Z}}(N, \mathbb{Z})$ of N will be denoted M. There is a canonical pairing $N \times M \to \mathbb{Z}$ given by $(u, n) \mapsto u(n)$. If we choose a basis for N then we may write $N \cong \mathbb{Z}^n$ and $M \cong \mathbb{Z}^n$ and $(u, n) = u \cdot n$ where $u \cdot n$ is the usual inner product on $\mathbb{R}^n$.

Definition 2.1.2 (Convex cone)**.**

1. *A convex cone in V is a subset C such that if $v_1, v_2 \in C$ then $t_1 v_1 + t_2 v_2 \in C$ for all $t_1, t_2 \in \mathbb{R}_{>0}$.*

2. *A polyhedral convex cone in V is a subset of the form*

$$C = \{\sum_{i=1}^{s} t_i v_i \colon t_1, ..., t_s \in \mathbb{R}_{>0}, v_1, ..., v_s \in V\}.$$

   We say that C is generated by $v_1, ..., v_s$.

3. *A closed polyhedral convex cone in V is a subset of the form*

$$C = \{\sum_{i=1}^{s} t_i v_i \colon t_1, ..., t_s \in \mathbb{R}_{\geq 0}, v_1, ..., v_s \in V\}.$$

   We say that C is generated by $v_1, ..., v_s$.

4. *If $v_1, ..., v_s \in N$ then we say that C is a rational polyhedral convex cone.*

5. *The dimension of a cone C is the dimension of its linear span in V.*

Example 1.

We give an example of how cones can arise in geometry. Let N be the free $\mathbb{Z}$ module generated by the line bundle $H = \mathcal{O}_{\mathbb{P}^n}(1)$. We let $V = N \otimes_{\mathbb{Z}} \mathbb{R}$, so that V is a line. Let $\mathrm{Amp}_{\mathbb{R}}(\mathbb{P}^n)$ be the cone in V generated by H. In other words

$$\mathrm{Amp}_{\mathbb{R}}(X) = \mathbb{R}_{>0}H.$$

This gives an open cone in V it is the interval $(0, \infty)$. We define the closure to be $\mathrm{Nef}_{\mathbb{R}}(\mathbb{P}^n) = \mathbb{R}_{\geq 0}H = [0, \infty)$. Notice that these cones already capture something of the geometry of X, namely that $\mathrm{Amp}_{\mathbb{R}}(X)$ represents the divisors on $\mathbb{P}^n$ with positive degree, and $\mathrm{Nef}_{\mathbb{R}}(X)$ represents the divisors on $\mathbb{P}^n$ with non-negative degree. Now consider a surjective morphism $f\colon \mathbb{P}^n \to \mathbb{P}^n$ given by $f(x_0 : \ldots : x_n) = (x_0^d : \ldots : x_n^d)$ for some $d > 0$. We have a linear action by pulling back line bundles on $\mathbb{P}^n$ with the property that $f^*H = dH$. So we have that $f^*(\mathrm{Amp}_{\mathbb{R}}(\mathbb{P}^n)) = \mathrm{Amp}_{\mathbb{R}}(\mathbb{P}^n)$ and $f^*(\mathrm{Nef}_{\mathbb{R}}(\mathbb{P}^n)) = \mathrm{Nef}_{\mathbb{R}}(\mathbb{P}^n)$.

We now may define the notion introduced in 2.3 which will be crucial in our toric dictionary.

Definition 2.1.3 (Dual cone)**.** *Let $\sigma \subseteq V$. The dual of σ is defined as*

$$\sigma^\vee = \{u \in V^* : (u, v) \geq 0 \ \forall \ v \in \sigma\}. \tag{2.4}$$

We have the following key fact.

Proposition 2.1.4 (Cone duality facts 1.2 [19])**.** *Let σ be a closed convex cone in V. Then $\sigma^\vee$ is a closed convex cone. Moreover, suppose that σ is a closed polyhedral convex cone. Then we have the following.*

1. *If $v_0 \notin \sigma$ then there is some $u \in \sigma^\vee$ with $(u, v) < 0$.*
2. *$(\sigma^\vee)^\vee = \sigma$*

Proof. We limit ourselves to the first point. We have that $\sigma^\vee$ is the collection of all $u \in V^*$ such that $(u, v) \geq 0$ for all $v \in \sigma$. Let $H_v = \{u \in V^* \colon (u, v) \geq 0\}$. Then

$$\sigma^\vee = \cap_{v \in \sigma} H_v.$$

Since H_v is closed and convex we have that $\sigma^\vee$ is closed and convex being the intersection of closed and convex sets. $\square$

Example 2.

Let σ be the closed cone generated by $(1,2)$ and $(1,0)$ in $\mathbb{R}^2$. Then the half lines $\ell_1 = \mathbb{R}_{\geq 0}(1,2)$ and $\ell_2 = \mathbb{R}_{\geq 0}(1,0)$ are intuitively faces of σ. The elements of $\sigma^\vee$ are given by vectors (a,b) with $(a,b)\cdot(1,0) = a \geq 0$ and $(a,b)\cdot(1,2) = a+2b \geq 0$. In other words, $\sigma^\vee$ is the closed cone generated by $(1,0),(2,-1)$. Notice the faces of σ can be given as follows. Let $u_1 = (0,1)$ and $u_2 = (2,-1)$. Notice we can describe the faces of σ in terms of $\sigma^\vee$. Consider the rays $u_1, u_2 \in \sigma^\vee$ which describe the faces of $\sigma^\vee$. Then

$$\ell_1 = u_1^\perp \cap \sigma \tag{2.5}$$
$$\ell_2 = u_2^\perp \cap \sigma \tag{2.6}$$

where $u^\perp = \{v \in V : (u,v) = 0\}$.

Definition 2.1.5 (Faces of cone)**.** *Let σ be a closed cone in V. A* face *of σ is a set of the form*

$$\tau = u^\perp \cap \sigma$$

where $u \in \sigma^\vee$. We call a face of dimension $\dim V - 1$ *a* facet *of σ and any 1-dimensional face a* ray*.*

We now give the basic properties of faces of convex cones.

Proposition 2.1.6 (Properties of cones 1.2 [19])**.** *Let σ be a closed polyhedral convex cone.*

1. *Any face of σ is also a closed polyhedral convex cone.*
2. *σ has finitely many faces.*
3. *The face of a face is a face.*
4. *The intersection of any faces is a face.*
5. *Any face is contained in a facet and any face contains a ray.*
6. *Any proper face is the intersection of all facets containing it.*
7. *The topological boundary of σ is the union of its facets.*

8. *If* $\dim \sigma = \dim V$ *and* $\sigma \neq V$ *and* τ *is a facet then there is a unique (up to multiplication by a positive scalar)* $u_\tau \in \sigma^\vee$ *with* $\tau = u_\tau^\perp \cap \sigma$*.*

9. *If* $\dim \sigma = \dim V$ *and* $\sigma \neq V$ *then for each facet* τ *write* $H_\tau = \{v \in V : (u_\tau, v) \geq 0\}$*. Then*
$$\sigma = \cap_{\tau \text{ a facet}} H_\tau.$$

Returning to the above example of the closed cone σ generated by $(1, 0), (1, 2)$ we now compute the faces of σ. An element of $\sigma^\vee$ is an element of the form $u = a(0, 1) + b(2, -1) = (2b, a - b)$ with $a, b \geq 0$. An element of σ is of the form $v = (x + y, 2y)$ for some $x, y \geq 0$. Then if $u \cdot v = 2(bx + ay) = 0$ we must have $a = b = 0$ or $b = 0, y = 0$ or $(a = 0, x = 0)$.

$$\begin{aligned} u^\perp \cap \sigma &= \ell_1 \text{ if } a \neq 0, b = 0 \\ u^\perp \cap \sigma &= \ell_2 \text{ if } a = 0, b \neq 0 \\ u^\perp \cap \sigma &= \{0\} \text{ if } a \neq 0, b \neq 0 \\ u^\perp \cap \sigma &= \sigma \text{ if } a = 0, b = 0. \end{aligned}$$

The faces of σ correspond to $0, \ell_1, \ell_2, \sigma$, namely the origin, the boundary rays of σ and the whole cone σ.

The above example illustrates that there should be a relationship between the faces of a closed polyhedral convex cone σ and the faces of $\sigma^\vee$.

Definition 2.1.7 (Relative interior)**.** *Let* σ *be a closed in* V*. The relative interior of* σ *is the interior of* σ *inside* $\operatorname{span}_{\mathbb{R}}(\sigma)$*. We write* $\operatorname{relint}(\sigma)$ *for the relative interior. If* σ *is a closed convex polyhedral cone then*
$$\operatorname{relint}(\sigma) = \{v \in \sigma : (u, v) > 0 \ \forall u \in \sigma^\vee - \sigma^\perp\}.$$

Given a cone σ in V it may not be the case that $\dim \sigma = \dim V$. In particular when working with the faces of σ this will always be the case when a face τ is not a facet. In such a situation it is useful to be able to consider the face τ inside the subspace of V generated by τ.

Proposition 2.1.8 (1.2 in [19])**.** *Let* σ *be a closed convex polyhedral cone in* V*. Let* τ *be a face of* σ*. Set* $\tau^* = \tau^\perp \cap \sigma^\vee$*. Then we have the following.*

1. *τ^* is a face of $\sigma^\vee$.*

2. *The correspondence $\tau \mapsto \tau^*$ gives an inclusion reversing bijection between faces of σ and faces of $\sigma^\vee$.*

3. *We have $\dim \tau + \dim \tau^* = \dim V$.*

We will also be interested in the following situation. Given a face τ of σ with $\tau = u^\perp \cap \sigma^\vee$ with $u \in \sigma^\vee$, how can we compute $\tau^\vee$?

Proposition 2.1.9 (Dual of a face)**.** *Let σ be a closed convex polyhedral cone. Let $u \in \sigma^\vee$. Set $\tau = u^\perp \cap \sigma$. Then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-u)$.*

Proof. We show that $\tau = (\sigma^\vee + \mathbb{R}_{\geq 0}(-u))$. This gives the result after taking duals. Take $w \in \tau$ and $y \in \sigma^\vee$ with $t \in \mathbb{R}_{\geq 0}$. Then $(y - tu, w) = (y, w) \geq 0$ because $(u, w) = 0$ and $y \in \sigma^\vee$. So $\tau \subseteq (\sigma^\vee + \mathbb{R}_{\geq 0}(-u))^\vee$. Conversely let $x \in (\sigma^\vee + \mathbb{R}_{\geq 0}(-u))^\vee$. Then $(w, x) \geq 0$ for all $w = y - tu$ where $y \in \sigma^\vee$ and $t \in \mathbb{R}_{\geq 0}$. Taking $t = 0$ shows that $x \in (\sigma^\vee)^\vee = \sigma$. Furthermore as $u \in \sigma^\vee$ we have $(u, x) \geq 0$. Since we have $-u \in \sigma^\vee + \mathbb{R}_{\geq 0}(-u)$ we have that $(-u, x) \geq 0$. This can only happen if $(u, x) = 0$ so $x \in u^\perp \cap \sigma = \tau$ as needed. So $(\sigma^\vee + \mathbb{R}_{\geq 0}(-u))^\vee \subseteq \tau$ and equality prevails. ☐

In the construction of affine toric varieties, it is crucial to consider the following situation. We have closed convex polyhedral cones σ, σ' with $\tau = \sigma \cap \sigma'$ being a face of both σ and σ'. In such a situation we wish for a way to compute a hyperplane equation for τ that uses the fact that τ is a face of two different cones.

Lemma 2.1.10 (1.2 in [19])**.** *Suppose we have closed convex polyhedral cones σ, σ' and that $\tau = \sigma \cap \sigma'$ is a face of both σ and σ'. Then there is some $u \in \sigma^\vee \cap (-\sigma')^\vee$ with*

$$u^\perp \cap (\sigma')^\vee = \tau = \sigma \cap u^\perp.$$

Finally, we have the following useful proposition and definition.

Proposition 2.1.11. *Let σ be a closed convex polyhedral cone. The following are equivalent.*

1. *$\sigma \cap (-\sigma) = 0$*

2. *The only linear subspace contained in σ is the zero subspace.*

3. *There is a* $u \in \sigma^\vee$ *with* $u^\perp \cap \sigma = 0$.

4. $\sigma^\vee$ *spans* V^*.

We call a cone which satisfies any of the equivalent conditions pointed.

We have the following miscellaneous results that will be used later.

Theorem 2.1.12 ([58, Theorem 4.8])**.** *Let* A *be a real* $n \times n$ *matrix. Then the following are equivalent.*

1. A *is non-zero, diagonalizable, and all eigenvalues of* A *have the same modulus, with* $\rho(A)$ *being an eigenvalue.*

2. *There is a proper cone* K *in* $\mathbb{R}^n$ *with* $A(K) \subseteq K$ *and* A *has an eigen-vector in the interior of* K.

Lemma 2.1.13. *Let* $C_1 \subseteq C_2$ *be two full dimensional pointed convex cones in* $\mathbb{R}^n$. *Suppose that* $\partial C_1 \subseteq \partial C_2$. *Then* $C_1 = C_2$.

Proof. Towards a contradiction suppose that there is a point $x \in C_2 \setminus C_1$. Let y be an interior point of C_1 and let L be the line segment between x, y with x, y excluded. Then L contains only interior points of C_2 and also a boundary point of C_1, but we assumed that the boundary of C_1 is contained in the boundary of C_2. This is a contraction so no such point x exists. □

2.1.2 Toric Varieties.

In order to define affine toric varieties we will use semi-group algebras. All of our semi-algebras will be of the following form. Fix N a lattice and $V = N \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}}$. Let σ be a pointed closed convex rational polyhedral cone in V. Then set $S_\sigma = \sigma^\vee \cap M$. Here M is the dual space $N^* = \hom_{\mathbb{Z}}(N, \mathbb{Z})$ and $M \subseteq V^*$. We will keep this notation throughout the section.

We will consider $k[S_\sigma]$ as the span of elements of the form x^u where $u \in S_\sigma$ and $x^u \cdot x^w = x^{u+w}$. We need the following to guarantee that our algebras will be finitely generated.

Lemma 2.1.14 ([19] Gordon's Lemma)**.** *Let* σ *be a closed convex rational polyhedral cone in* $N_{\mathbb{R}}$. *Then* $S_\sigma = M \cap \sigma^\vee$ *is a finitely generated semi-group.*

We can now define affine toric varieties, which will be the building blocks for all of our varieties.

Definition 2.1.15 (Affine toric variety)**.** *An affine toric variety over a field k of characteristic zero is a variety $U_\sigma = Spec\, k[S_\sigma]$ where σ is a pointed closed rational polyhedral cone.*

We now list the basic properties of affine toric varieties.

Lemma 2.1.16 (Open embedding of faces Proposition 2 and Proposition 3 of [19])**.** *Consider a closed convex rational polyhedral cone σ. Suppose that $\tau = u^\perp \cap \sigma$ with $u \in \sigma^\vee$ is a face of τ. Then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. $U_\tau \hookrightarrow U_\sigma$ is an open embedding given by localization at $x^u \in K[S_\sigma]$. Furthermore if γ is another closed convex rational polyhedral cone and $\gamma \cap \sigma = \tau$ with τ also being a face of γ. Then*

$$S_\tau = S_\sigma + S_\gamma.$$

Proof. To prove this one uses 2.1.9 and 2.1.10. □

We can now begin discussing the gluing of toric varieties.

Definition 2.1.17 (Gluing data for a toric variety)**.** *A fan Σ in $N_\mathbb{R}$ is a collection of pointed closed convex rational polyhedral cones that satisfy the following axioms.*

1. *If $\sigma \in \Sigma$ then any face of σ is also in Σ.*
2. *If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1, σ_2.*

Given a fan Σ in $N_\mathbb{R}$ we write $\mid \Sigma \mid = \bigcup_{\sigma \in \Sigma} \sigma$. Let Σ be a fan in $N_\mathbb{R}$ and Σ' a fan in $N'_\mathbb{R}$. A morphism of fans $\Sigma \to \Sigma'$ is a homomorphism $f \colon N \to N'$ such that for each cone $\sigma \in \Sigma$ there is a cone $\sigma' \in \Sigma'$ with $f_\mathbb{R}(\sigma) \subseteq \sigma'$.

Theorem 2.1.18 (Fan Construction Theorem: See 3.1.5 in [12])**.** *Let Σ be a fan in $N_\mathbb{R}$. From the data of the fan Σ one may construct a toric variety X_Σ. X_Σ has an open covering by affine toric varieties U_σ where $\sigma \in \Sigma$.*

Proof. We sketch the construction. Given $\sigma, \gamma \in \Sigma$ as Σ is a fan $\tau = \sigma \cap \gamma$ is a face of both σ, τ. Then by 2.1.16 we have that U_τ has a canonical open embedding in U_σ, U_γ. This gives gluing data for the open cover U_σ of X_Σ. □

Theorem 2.1.19 (Equivalence of constructions: See 3.1.8 and 3.3.4 in [12]). *The category of fans in a lattice N and the category of toric varieties are equivalent. Every toric variety is equivariantly isomorphic to a variety X_Σ for some fan Σ. Moreover, every toric morphism in the sense of 2.1.1 arises from a morphism of fans and vice versa.*

For us an important part of the above construction is that maps of lattices will allow us to construct self morphisms of toric varieties.

Proposition 2.1.20 (Properness criterion for toric varieties and toric morphisms: section 2.4 in [19]). *Consider toric varieties X_Σ and X'_Σ. Suppose that $\Sigma \subseteq N_\mathbb{R}$ and $\Sigma' \subseteq N'_\mathbb{R}$. Let $f\colon N \to N'$ be a morphism of fans $\Sigma \mapsto \Sigma'$.*

1. *X_Σ is proper over k if and only if $\mid \Sigma \mid = N_\mathbb{R}$.*
2. *The induced morphism $f\colon X_\Sigma \to X'_\Sigma$ is proper if and only if $f_\mathbb{R}^{-1}(\mid \Sigma' \mid) = \mid \Sigma \mid$ where $f_\mathbb{R}$ is the induced linear map of vector spaces $N_\mathbb{R} \to N'_\mathbb{R}$.*

A key proposition for us is the following, that toric varieties have many endomorphisms.

Proposition 2.1.21 (Toric Varieties have non-trivial dominant maps.). *Consider a toric variety X_Σ. Suppose that $\Sigma \subseteq N_\mathbb{R}$. Then there are infinitely many dominant morphisms $f\colon X_\Sigma \to X_\Sigma$.*

Proof. Let $f_n\colon N \to N$ be the morphism of lattices given by multiplication by n. Given a cone $\sigma \in \Sigma$ if $v \in \sigma$ then $nv \in \sigma$ as σ is a cone. So $f_n(\sigma) \subseteq \sigma$. Therefore by 2.1.19 we have that f_n induces a toric morphism $X_\Sigma \to X_\Sigma$. On the torus T of X_Σ f_n is nothing more then the homomorphism $t \mapsto t^n$ of tori. This morphism $f\colon T \to T$ where T is the torus of X_Σ is a toric morphism itself. It thus suffices to show that this map is dominant. However this is clear as image of the ring map $k[t_1^{\pm 1}, ..., t_m^{\pm 1}] \to k[t_1 \pm^1, ..., t_m^{\pm 1}]$ given by $t_i \mapsto t_i^n$ has image $k[t_1^{n \pm 1}, ..., t_m^{n \pm 1}]$. This is a torus of dimension m, as T is irreducible the closure of this torus is all of T and the map is dominant. $\square$

In general producing dominant self morphisms is not easy, this is the main reason we will be interested in toric varieties, they provide an interesting realm where there are many morphisms.

We now turn to torus orbits. These will be needed to define the singular locus of a toric variety. While all of this can be described explicitly we only give the minimum of what is needed. Notice that if X_Σ is a toric variety then we have the dense open torus T_Σ acting on X_Σ. Then we have that X_Σ can be stratified by torus orbits. In fact this stratification can be written down in terms of the fan Σ, and each torus orbit is in fact a torus itself.

Theorem 2.1.22 (Orbit Cone Correspondence 3.2.6 in [12])**.** *Let X_Σ be a toric variety defined over a field k.*

1. *There is a bijective correspondence*

$$\begin{gathered}\Sigma \to T_\Sigma \text{ orbits}\\ \text{given by } \sigma \in \Sigma \mapsto \mathcal{O}(\sigma) \text{ a torus orbit.}\end{gathered}$$

   In other words the torus orbits correspond bijectively to the cones of σ.

2. *For $\sigma \in \Sigma$ we have $\dim \mathcal{O}(\sigma) \cong \mathbb{G}_m^{\text{codim }\sigma}$.*

3. *For each cone $\sigma \in \Sigma$ we have*

$$U_\sigma = \bigsqcup_{\tau \text{ a face of } \sigma} \mathcal{O}(\tau).$$

4. *We have that τ is a face of σ if and only if*

$$\mathcal{O}(\sigma) \subseteq \overline{\mathcal{O}(\tau)}.$$

   Moreover we have that

$$\overline{\mathcal{O}(\tau)} = \bigsqcup_{\sigma \text{ has } \tau \text{ as a face.}} \mathcal{O}(\sigma).$$

We will mostly be interested in the torus closures, $V(\tau) = \overline{\mathcal{O}(\tau)}$ which can be described in a combinatorial manner. To do this we introduce a combinatorial construction called the star.

Definition 2.1.23 (Star construction)**.** *Let Σ be a fan in $N_\mathbb{R}$. Let $\tau \in \Sigma$. Set $N_\tau = \operatorname{span}_\mathbb{Z}(\sigma \cap N)$. Define $N(\tau) = N/N_\tau$ and set $p_\tau \colon N \to N(\tau)$ to be the projection. We let $\operatorname{star}_\Sigma(\tau) = \{p_\tau(\sigma) \subseteq N(\tau)_\mathbb{R} \colon \tau \text{ is a face of } \sigma\}$. Then $\operatorname{star}_\Sigma(\tau)$ is a fan in $N(\tau)_\mathbb{R}$.*

Theorem 2.1.24 (Torus closure Theorem 3.2.7 in [12])**.** *For any $\tau \in \Sigma$ we have that*

$$\overline{\mathcal{O}(\tau)} \cong X_{\operatorname{star}_\Sigma(\tau)}.$$

We finally turn to singularities of toric varieties. As we might expect at this point, these can be described combinatorically.

Definition 2.1.25 (Simplicial and Smooth cones. 1.2.16 in [12]). *Let σ be a pointed rational strongly closed convex polyhedral cone in $N_{\mathbb{R}}$.*

1. *We say σ is smooth if and only iff a generating set of σ can be extended to a $\mathbb{Z}$-basis of N.*
2. *We say that σ is simplicial if a generating set of σ may be extended to a $\mathbb{R}$-basis of $N_{\mathbb{R}}$.*

The definitions deserve there name due to the following.

Proposition 2.1.26 (Local Criterion for Smoothness 1.3.12 in [12]). *Let U_σ be an affine toric variety. Then U_σ is smooth if and only if σ is a smooth cone.*

We may think of this proposition as follows. Given a toric variety X_Σ. If there is some property P of varieties that is affine local, we can check if property P holds on the canonical open covering of X_Σ given by U_σ. In many situations the geometric property P will have a combinatorial analogue P_σ and U_σ will have P if and only if σ has P_σ. As example of this is then the following.

Theorem 2.1.27 (1.3.12 and 4.2.7 in [12]). *Let X_Σ be a toric variety.*

1. *X_Σ is smooth if and only if every cone σ in Σ is a smooth cone.*
2. *X_Σ is $\mathbb{Q}$-factorial if and only if every cone $\sigma \in \Sigma$ is simplicial.*

The smooth locus and singular locus can be described in this setting. We will use that we can describe the singular locus in terms of orbit closures.

Proposition 2.1.28 (Singular Locus of X_Σ 11.1.12 [12]). *Let X_Σ be a toric variety with Σ a fan in $N_{\mathbb{R}}$. Then*

$$(X_\Sigma)_{\text{sing}} = \bigcup_{\sigma\in\Sigma \text{ with } \sigma \text{ not smooth}} V(\sigma)$$

and

$$X_\Sigma^{\text{smooth}} = \bigcup_{\sigma\in\Sigma \text{ with } \sigma \text{ smooth}} U_\sigma.$$

2.2 Positivity in algebraic geometry

We begin by discussing divisors. An important point of view is the following. We will be using the minimal model program: in the minimal model program one can reduce to less complicated varieties by using morphisms that are defined by the contraction of a divisor. In other words to study X we try to construct a morphism $\phi\colon X \to Y$, where Y is simpler than X in some precise sense. However, even when X is smooth, Y may not be. Thus when taking this type of approach it is best to allow singular varieties from the start . When working with potentially singular varieties, it is easier to work with Cartier divisors or line bundles rather then Weil divisors. We now give a brief refresher on these notions. Our main references here will be [31] and [24].

Definition 2.2.1. *Let X be an irreducible projective variety defined over a number field K. The set of Cartier divisors is defined to be*

$$\mathrm{CaDiv}(X) = H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

where $\mathcal{M}_X$ is the constant sheaf associated to the function field $K(X)$. Concretely a Cartier divisor is given by the following data. Let $\{(U_i)_i\}$ be an open cover of X. Then a Cartier divisor is defined by the data $\{(U_i, f_i)\}$ where $f_i \in H^0(U_i, \mathcal{M}_X^*)$ and for indices i, j we have that

$$f_i \mid_{U_{ij}} = g_{ij} f_j \mid_{U_{ij}}$$

for $g_{ij} \in H^0(U_{ij}, \mathcal{O}_X^*)$, where $U_{ij} = U_i \cap U_j$. Two such sets of data say $\{(U_i, f_i)\}$ and $\{(V_j, h_j)\}$ define the same Cartier divisor if there is a common refinement $\{W_k\}$ of the open covers $\{U_i\}$, $\{V_j\}$ such that there are $w_k \in H^0(W_k, \mathcal{O}_X^*)$ with

$$f_k = w_k f'_k.$$

We write the group CaDiv(X) *additively. Given $x \in X$, if $x \in U_i$ we call f_i a local equation for D at x.*

The support *of a Cartier divisor D is the set of $x \in X$ such that if f is a local equation for D at x then f is not at unit in $\mathcal{O}_{X,x}$. In other words, f vanishes at x.*

We say that D is effective if each local equation can be taken to be in $H^0(X, \mathcal{O}_X)$ and thus write $D \geq 0$.

For each $f \in K(X)^*$ we have a Cartier divisor $\operatorname{div}(f) = \{(X, f)\}$. *We call this group of divisors the* principal divisors. *We say that D_1 is linearly equivalent to D_2 if $D_1 - D_2$ is a principle divisor. We will write*

$$D_1 \equiv_{\text{lin}} D_2$$

to denote that D_1 is linearly equivalent to D_2.

Let D be a divisor on X and let $f\colon Y \to X$ be a morphism of schemes with Y an irreducible variety defined over K. We would like to pull back Cartier divisors, and not just linear equivalence classes of Cartier divisors (this is always possible). To do so we give a criterion found in [31].

Proposition 2.2.2 (1.1.A [31] Pulling back Cartier divisors)**.** *Let X, Y be irreducible projective varieties defined over K and let $f\colon Y \to X$ be a morphism. Let $D \in$* $\operatorname{CaDiv}(X)$. *If no component of Y is mapped into the support of D then $\{(f^{-1}(U_i), f_i \circ f\}$ is a Cartier divisor on Y which is denoted f^*D.*

In this thesis we will mostly be concerned with a surjective morphism $f\colon X \to X$. As the support of a Cartier divisor is typically not all of X and all our varieties will be irreducible we will be able to pull back Cartier divisors. We will be interested in the interplay, and essential equivalence between Cartier divisors up to linear equivalence and line bundles on X.

Proposition 2.2.3 (1.1.A [31] :The map from Cartier divisors to line bundles)**.** *Let X be an irreducible projective variety defined over a number field K. Let $D \in$* $\operatorname{CaDiv}(X)$. *Choose a representation $\{(U_i, f_i)\}$. We define $\mathcal{O}_X(D)$ to be the line bundle with transition functions g_{ij}^{-1}. In other words $H^0(U_i, \mathcal{O}_X(D)) = \mathcal{O}_{U_i} f_i^{-1}$.*

We also have a natural morphism from $\operatorname{CaDiv}(X) \to Z^{n-1}(X)$ where $Z_{n-1}(X)$ is the group of Weil divisors on X and $X = \dim n$.

Definition 2.2.4 (Cycle mapping)**.** *Let X be an irreducible projective variety that is regular in codimension 1. Let $D \in$* $\operatorname{CaDiv}(X)$. *Let V be an prime Weil divisor on X, in other words, an irreducible codimension 1 closed subset of X. If $D = \{(U_i, f_i)\}$ let $V \cap U_i \neq 0$. Then we define* $\operatorname{ord}_V(D) = \operatorname{ord}_V(f_i)$. *We then define*

$$\operatorname{cyc}(D) = \sum_{V \text{ a prime Weil divisor}} \operatorname{ord}_V(D)[D].$$

We will also need the notion of the base locus, for normal projective varieties.

Definition 2.2.5. *Let X be a normal projective variety and D a divisor on X.*

1. *We let* $\mathrm{Bs}(D)$ *be the* base locus *of D. That is the set of points at which all sections of $H^0(X, D)$ vanish. Here $H^0(X, D) = H^0(X, \mathcal{O}_X(D))$.*

2. *We define the* stable base locus *to be*

$$\boldsymbol{B}(D) = \bigcap_{m \geq 1} \mathrm{Bs}(mD).$$

2.2.1 Intersection theory and Positivity

In this section we introduce the needed requirements from intersection theory. We need this to be able to define various types of positivity requirements on divisors. Our main references here are [31] and [20] and [24]

We first define rational equivalence of cycles in a variety. We will not need the full power of [20], and will attempt to work with divisors and line bundles as much as possible. However there are certain times where it will be advantageous to work with cycles that do not have codimension 1. We mostly follow [31] but will be taking material from [20] as well.

We begin by defining rational equivalence of cycles of arbitrary dimension on a variety X.

Definition 2.2.6 (Order of Vanishing: See 1.2 [20])**.** *Let X be an irreducible variety defined over a field k. Fix a variety X and let V be a closed sub-variety of codimension 1. Let $f \in K(X)$ the function field. We define*

$$\mathrm{ord}_V(f) = \mathrm{length}(\mathcal{O}_{X,V}/(f\mathcal{O}_{X,V}))$$

where $\mathcal{O}_{X,V}$ is the local ring of X along V. The length of a module is defined to be the size of a composition series for $\mathcal{O}_{X,V}/(f\mathcal{O}_{X,V})$ as a $\mathcal{O}_{X,V}$ module. See [15] chapter 2 for more details.

We now define cycles.

Definition 2.2.7 (Definition of a cycle)**.** *Let X be an irreducible variety defined over a field K. A k-cycle is a finite formal sum*

$$\sum_i n_i[V_i]$$

where $n_i \in \mathbb{Z}$ and V_i are reduced irreducible closed sub-varieties of X of dimension k. We let $Z_k(X)$ be the free abelian of all such cycles.

Now let $f \in K(X)$. Using 2.2.6 we can associate a $\dim X - 1$-cycle to f. We define

$$[\operatorname{div} f] = \sum_V \operatorname{ord}_V(f)[V]$$

where the sum is taken over all irreducible and reduced codimension 1 sub-varieties. Furthermore, one can show that this sum is finite.

Putting these concepts together we have the following.

Definition 2.2.8 (Rational equivalence of cycles. See 1.2 [20])**.** *Let $\alpha \in Z_k(X)$ as defined above. We say that α is rationally equivalent to zero if*

$$\alpha = \sum_i [\operatorname{div} f_i]$$

where $f_i \in K(V_i)^*$ with V_i a $k+1$-dimensional sub-variety of X. Furthermore the k-cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_k(X) \subseteq Z_k(X)$. The group of k-cycles modulo rational equivalence is defined to be

$$A_k(X) = Z_k(X)/\operatorname{Rat}_k(X).$$

We now begin our journey into the intersection theory of divisors. The groups $A_k(X)$ will be used to define the intersection product.

Theorem 2.2.9 ([31] Intersection Theory I)**.** *Let X be an irreducible projective variety defined over a number field. Let $D_1, ..., D_k \in \operatorname{CaDiv}(X)$ and let $V \subseteq X$ be a subvariety of pure dimension d. Then there is a class*

$$(D_1 \cdot ... \cdot D_k \cdot V) \in A_{d-k}(X).$$

In practice if $d = k$ we will often equate $(D_1 \cdot ... \cdot D_k \cdot V)$ with $\deg(D_1 \cdot ... \cdot D_k \cdot V) \in \mathbb{Z}$ and think of $(D_1 \cdot ... \cdot D_k \cdot V)$ as an integer. If $k = n$ then we will often write $(D_1 \cdot ... \cdot D_n)$ for $(D_1 \cdot ... \cdot D_n \cdot X)$.
The $d - k$ cycle $(D_1 \cdot ... \cdot D_k \cdot V)$ has the following properties.

1. $(D_1 \cdot \ldots \cdot D_k \cdot V)$ *is symmetric and multi-linear in the* D_i*.*

2. $(D_1 \cdot \ldots \cdot D_k \cdot V)$ *only depends on the linear equivalence classes of the* D_i*.*

In our study of morphisms, we will be constantly pulling back divisors and line bundles. To this end we have the following results.

Theorem 2.2.10 ([20] Pull back formula)**.** *Let* X, Y *be projective varieties. Let* $f\colon Y \to X$ *be a generically finite morphism. If* $D_1, ..., D_n \in \mathrm{CaDiv}(X)$ *then*

$$(f^*D_1 \cdot \ldots \cdot f^*D_n) = (\deg f)(D_1 \cdot \ldots \cdot D_n)$$

Theorem 2.2.11 ([20] Projection Formula I I)**.** *Let* $f\colon X \to Y$ *be a proper morphism of varieties with* D *a Cartier divisor on* Y *and* α *a* k*-cycle on* X*. Let* $[D]$ *be the linear equivalence class of* D*. Then* $f_*(f^*[D] \cdot \alpha) = (D \cdot f_*(\alpha))$*.*

We now come to an important definition that will be key in our study of dynamics of surjective endomorphisms. This is essential in the construction of the Neron-Severi group of a variety, which will be used to construct dynamical invariants from linear algebra.

Definition 2.2.12 (Numerical Equivalence)**.** *Let* X *be a irreducible projective variety defined over a number field* K*. Let* $D_1, D_2 \in \mathrm{CaDiv}(X)$*. We say that* D_1 *is numerically equivalent to* D_2 *if for all irreducible closed subsets* C *of* X *of dimension 1 we have that* $(D_1 \cdot C) = (D_2 \cdot C)$*. We write*

$$D_1 \equiv_{\mathrm{num}} D_2$$

in this situation.

This definition respects intersection products.

Lemma 2.2.13 (Intersection products are independent of numerical classes)**.** *Let* X *be a irreducible projective variety defined over a number field* K*. Let* $D_1, ..., D_k, D'_1, ..., D'_k \in \mathrm{CaDiv}(X)$*. If* $D_i \equiv_{\mathrm{num}} D'_i$ *then for all pure dimension* k *subvarieties* V *we have*

$$(D_1 \cdot ... \cdot D_K \cdot [V]) = (D'_1 \cdot ... \cdot D'_k \cdot [V]).$$

Proof. We write

$$(D_1 \cdot D_2 \cdot ... \cdot D_K \cdot [V]) - (D'_1 \cdot D_2 \cdot ... \cdot D_k \cdot [V]) = (D_1 - D'_1 \cdot D_2 \cdot ... \cdot D_k \cdot V).$$

I claim that if E is *any* numerically trivial divisor then

$$(E \cdot D_2 \cdot ... \cdot D_k \cdot [V]) = 0.$$

To see this note that $(D_2 \cdot ... \cdot D_k \cdot V)$ is represented by a 1-cycle α on X. As E is numerically trivial we have that $(E, D_2 \cdot ... \cdot D_k \cdot V) = (E \cdot \alpha) = 0$ by the functorial properties of 2.2.9 and the definition of numerically trivial. Since $D_1 - D_1'$ is numerically trivial we have that

$$(D_1 \cdot D_2 ... \cdot D_k \cdot [V]) - (D_1' \cdot D_2 ... \cdot D_k \cdot [V]) = (D_1 - D_1' \cdot D_2 ... \cdot D_k \cdot V) = 0$$

and so

$$(D_1 \cdot ... \cdot D_k \cdot [V]) = (D_1' \cdot D_2 \cdot ... \cdot D_k \cdot [V]).$$

Playing the same game with D_2, D_2' and so on gives the result. □

The key fact in the above lemma is the following. We have that $(D_2 \cdot ... \cdot D_k \cdot V)$ is represented by a 1-cycle on X. This is where we must use some of the more sophisticated results of intersection theory, namely that the an intersection class is represented by cycles.

Definition 2.2.14 (The Neron-Severi Group)**.** *Let X be an irreducible projective variety defined over a number field K. We define* $\mathrm{Num}(X)$ *to be those divisors which are numerically equivalent to the zero divisor. Clearly this is a subgroup of* $\mathrm{CaDiv}(X)$*. We define* $N^1(X) = \mathrm{CaDiv}(X)/\mathrm{Num}(X)$*.*

The Neron-Severi group is appropriate for studying the intersection theory of divisors as it eliminates the divisors which intersect every curve at zero.

Theorem 2.2.15 ([20]Theorem of the Base)**.** *Let X be an irreducible projective variety defined over a number field K. Then $N^1(X)$ is a finitely generated abelian group.*

The rank of $N^1(X)$ is an important invariant of the variety.

Definition 2.2.16 (The Picard Number)**.** *Let X be an irreducible projective variety defined over a number field K. The Picard number of X is defined to be the rank of $N^1(X)$. We will write $\rho(X)$ for the Picard number.*

2.2.2 Ample divisors

Definition 2.2.17 (Definition of very ample and ample)**.** *Let X be an irreducible projective variety defined over a number field K. Let $\mathcal{L}$ be a line bundle on X.*

1. *We say that* $\mathcal{L}$ *is* very ample *if there is a closed embedding* $\phi\colon X \to \mathbb{P}^n$ *for some* n *with* $\phi^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}$.

2. *We say that* $\mathcal{L}$ *is ample if* $\mathcal{L}^{\otimes m}$ *is very ample for some* $m > 0$.

3. *If* $D \in \mathrm{CaDiv}(X)$ *we say* D *is very ample (respectively ample) if the line bundle* $\mathcal{O}_X(D)$ *is very ample (respectively ample).*

Amplitude can be detected cohomologically. The following vanishing theorem is very useful.

Theorem 2.2.18 ([31], Cartan-Serre-Grothendieck Vanishing Theorem)**.** *Let* X *be an irreducible projective variety defined over a number field* K*. The following are equivalent. Let* $\mathcal{L}$ *be a line bundle on* X*.*

1. $\mathcal{L}$ *is ample.*

2. *For all coherent sheaves* $\mathcal{F}$ *on* X *there is an integer* $m_1(\mathcal{F})$ *such that for all* $i > 0$ *and* $m \geq m_1(\mathcal{F})$ *we have*
$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0.$$

3. *For all coherent sheaves* $\mathcal{F}$ *on* X *there is an integer* $m_1(\mathcal{F})$ *such that for all* $m \geq m_2(\mathcal{F})$ *we have that* $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ *is globally generated.*

4. *There is an integer* $m_3(\mathcal{L}) > 0$ *such that for all* $m \geq m_3(\mathcal{L})$ *we have that* $\mathcal{L}^{\otimes m}$ *is very ample.*

Lemma 2.2.19. *Let* X *be an irreducible projective variety defined over a number field* K*. Let* $\mathcal{L}$ *be a very ample line bundle on* X *and* E *a globally generated line bundle. Then* $\mathcal{L} \otimes E$ *is very ample.*

Proof. Let $\phi_{\mathcal{L}}\colon X \to \mathbb{P}^{n_{\mathcal{L}}}$ be a closed embedding with $\phi_{\mathcal{L}}^*(\mathcal{O}_{\mathbb{P}^{n_{\mathcal{L}}}}) = \mathcal{L}$. Let $\phi_E\colon X \to \mathbb{P}^{n_E}$ be a morphism with $\phi^*\mathcal{O}_{\mathbb{P}^{n_E}}(1)$ which exists as E is globally generated. Let $s\colon \mathbb{P}^{n_{\mathcal{L}}} \times \mathbb{P}^{n_E} \to \mathbb{P}^N$ be the Segre embedding. Then $s^*\mathcal{O}_{\mathbb{P}^N}(1) = p_1^*\mathcal{O}_{\mathbb{P}^{N_{\mathcal{L}}}}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{n_E}}(1)$ where $p_1\colon \mathbb{P}^{n_{\mathcal{L}}} \times \mathbb{P}^{n_E} \to \mathbb{P}^{n_{\mathcal{L}}}$ is the first projection and $p_2\colon \mathbb{P}^{n_{\mathcal{L}}} \times \mathbb{P}^{n_E} \to \mathbb{P}^{n_E}$ the second projection. On the other hand, since $\phi_{\mathcal{L}}$ is a closed embedding we have that $\phi_{\mathcal{L}} \times \phi_E$ is a closed embedding. Thus $s \circ \phi_{\mathcal{L}} \times \phi_E$ with
$$(s \circ \phi_{\mathcal{L}} \times \phi_E)^*\mathcal{O}_{\mathbb{P}^N}(1) = (\phi_{\mathcal{L}} \times \phi_E)^* p_1^*(\mathcal{O}_{\mathbb{P}^{N_{\mathcal{L}}}}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{n_E}}(1)) = \mathcal{L} \otimes E$$
as needed. $\square$

Our first application is the following crucial result.

Lemma 2.2.20. *Let X be an irreducible projective variety defined over a number field K. Let $\mathcal{L}$ be an ample line bundle on X. Let E be any other line bundle on X. Then there is a constant $m(E,\mathcal{L}) > 0$ such that for all $m \geq m(E,\mathcal{L})$ we have that $L^{\otimes m} \otimes E$ is very ample.*

Proof. Choose $m_3(\mathcal{L}) > 0$ as in 2.2.18 part 4. By 2.2.18 part 3 there is an integer $m_2(E) > 0$ such that $\mathcal{L}^{\otimes m} \otimes E$ is globally generated for all $m \geq m_2(E)$. I claim that if $m \geq m_3(\mathcal{L}) + m_2(E))$ then $\mathcal{L}^{\otimes m} \otimes E$ is very ample. Write $m = m_3 + m_2 + b$. Then

$$\mathcal{L}^{\otimes m_3+m_2+b} \otimes E = \mathcal{L}^{\otimes m_3+b} \otimes (\mathcal{L}^{\otimes m_2} \otimes E).$$

By construction $(\mathcal{L}^{\otimes m_2} \otimes E)$ is globally generated and $\mathcal{L}^{\otimes m_3+b}$ is very ample. So by 2.2.19 we have that $\mathcal{L}^{\otimes m} \otimes E$ is very ample. □

Lemma 2.2.21 (All line bundles are a difference of very ample line bundles)**.** *Let X be an irreducible projective variety defined over a number field K. Let E be any line bundle on X. Then there are very ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X with $E = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$.*

Proof. Choose any very ample line bundle $\mathcal{L}_1$. Then by 2.2.20 applied to E^{-1} we can find m with $\mathcal{L}_1^{\otimes m} \otimes E^{-1} = \mathcal{L}_2$ with $\mathcal{L}_2$ very ample. Then $\mathcal{L}_1^{\otimes m} \otimes \mathcal{L}_2^{-1} = E$ as needed. □

This will be useful in our construction of heights. We now give some basic results about endomorphisms that we will need. The proofs are omitted.

Proposition 2.2.22 (Finite ample pullbacks I)**.** *Let $f\colon X \to Y$ be a finite morphism of irreducible projective varieties defined over a number field K. Suppose that $\mathcal{L}$ is an ample line bundle on Y. Then $f^*\mathcal{L}$ is an ample line bundle on X.*

The following crucial result allows us to see how the concept of ampleness intersects with the intersection product. In the literature an ample divisor is often referred to as positive. We now make this precise.

Theorem 2.2.23 (Nakai-Moishezon-Kleiman Criterion for ampleness)**.** *Let X be an irreducible projective variety defined over a number field K. Then D is ample if and only if $(D^k \cdot V) > 0$ for all k-dimensional closed sub-varieties of X.*

The Nakai-Moishezon-Kleiman Criterion for ampleness and related theorems will often be used to show that a property of divisors is a numerical property. In other words, the property only depends on the numerical equivalence class of a divisor. Thanks to 2.2.23 we have that ampleness is a numerical property.

Proposition 2.2.24. *Let X be an irreducible variety defined over a number field K. Suppose that $D_1, D_2 \in \operatorname{CaDiv}(X)$ with $D_1 \equiv_{\text{num}} D_2$. Then D_1 is ample if and only if D_2 is ample.*

Proof. It suffices to show that if D_1 is ample then D_2 is ample when $D_1 \equiv_{\text{num}} D_2$. To this end let V be a closed sub-variety of dimension k of X. Since $(D_2^k \cdot V) = (D_1^k \cdot V)$ as $D_1 \equiv_{\text{num}} D_2$ and 2.2.13. As D_1 is ample we have by 2.2.23 that $(D_1^k \cdot V) > 0$. Putting this together gives

$$(D_2^k \cdot V) = (D_1^k \cdot V) > 0.$$

So by 2.2.23 we have that D_2 is ample as desired.

□

We also have the following which will be very useful for us. Notice that the collection of ample divisors is a cone in the general sense of 2.1.2. However it is not closed. The following result says that these cones will be preserved by pull backs in good situations.

Proposition 2.2.25 ([20]Finite ample pullbacks II)**.** *Let X, Y be irreducible projective varieties defined over a number field K. Suppose that $f \colon X \to Y$ is a finite surjective morphism. Suppose that $\mathcal{L}$ is a line bundle on Y. Suppose that $f^*\mathcal{L}$ is an ample line bundle on X. Then $\mathcal{L}$ is ample on Y.*

2.2.3 $\mathbb{Q}$-divisors

We will need to allow divisors with both rational and real coefficients. From a certain perspective this is not surprising. Ultimately we wish to study a surjective morphism

$$f \colon X \to X$$

through the induced *linear* pullback action

$$f^* \colon N^1(X) \to N^1(X).$$

As it is easier to study linear maps of vector spaces than $\mathbb{Z}$-linear mappings of finitely generated abelian groups we will tensor with $\mathbb{Q}$ or $\mathbb{R}$.

Definition 2.2.26 ($\mathbb{Q}$-divisors)**.** *Let X be an irreducible projective variety defined over a number field K. A $\mathbb{Q}$-divisor on X is an expression*

$$\sum_{i=1}^{N} q_i D_i$$

where $q_i \in \mathbb{Q}$ and $D_i \in \mathrm{CaDiv}(X)$. We write $\mathrm{CaDiv}_{\mathbb{Q}}(X)$ for the set of $\mathbb{Q}$-Cartier divisors. Clearly this is a $\mathbb{Q}$-vector space. We say that a $\mathbb{Q}$-Cartier divisor D is effective if $D = \sum_{i=1}^{N} q_i D_i$ with $q_i \geq 0$ and D_i effective. There is a natural inclusion $\mathrm{CaDiv}(X) \subseteq \mathrm{CaDiv}_{\mathbb{Q}}(X)$. Given a $\mathbb{Q}$-divisor D we say D is integral if $D \in \mathrm{CaDiv}(X)$.

There is a natural notion of intersection product on $\mathbb{Q}$-divisors.

Proposition 2.2.27 ([20] Intersecting $\mathbb{Q}$-divisors)**.** *Let X be an irreducible projective variety defined over a number field K.*

1. *Given a closed subvariety V of X of pure dimension k we have a symmetric and multilinear product*

   $$\mathrm{CaDiv}_{\mathbb{Q}}(X) \times \ldots \times \mathrm{CaDiv}_{\mathbb{Q}}(X) \to A_0(X)_{\mathbb{Q}}, (D_1, ..., D_k) \mapsto (D_1 \cdot \ldots \cdot D_k \cdot V)$$

   This is defined via extension of scalars from the previous product.

2. *Two $\mathbb{Q}$-divisors D_1, D_2 are numerically equivalent if $(D_1 \cdot C) = (D_2 \cdot C)$ for all closed irreducible curves C on X. We let $\mathrm{Num}(X)_{\mathbb{Q}}$ be the subgroup of $\mathbb{Q}$-divisors numerically equivalent to the zero divisor and let $N^1(X)_{\mathbb{Q}} = \mathrm{CaDiv}_{\mathbb{Q}}(X)/\mathrm{Num}_{\mathbb{Q}}(X)$. Moreover $N^1(X)_{\mathbb{Q}}$ is naturally isomorphic to $N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

3. *Two $\mathbb{Q}$-divisors D_1, D_2 are linearly equivalent if there is an integer r such that $rD_1, rD_2 \in \mathrm{CaDiv}(X)$ and $rD_1 - rD_2$ is a principal divisor on X. We write $D_1 \equiv_{\mathrm{lin}} D_2$ in this case.*

4. *Let Y be an irreducible projective variety defined over K and $f \colon X \to Y$ a morphism. Suppose that $D \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ and that the image of Y is not contained in a support of D. Then f^*D is defined. The linear equivalence class of f^*D is independent of the linear equivalence class of D and we may pull back $\mathbb{Q}$-linear equivalence classes.*

5. *Let Y be an irreducible projective variety defined over K and $f \colon X \to Y$ a morphism. Then there is a linear pullback morphism $f^* \colon N^1(Y)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}$.*

We can now define what it means to have an ample $\mathbb{Q}$-divisor. This is relatively easy given what we have done so far. However, more serious difficulties will arise when we deal with $\mathbb{R}$-divisors.

Definition 2.2.28 (Amplitude of $\mathbb{Q}$-divisors)**.** *Let X be an irreducible projective variety defined over a number field K. Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(X)$. We say D is ample if there is an integer r with $rD \in \mathrm{CaDiv}(X)$ and rD is ample. This is equivalent to saying that D satisfies the statement of 2.2.23.*

Using the $\mathbb{Q}$-divisor version of 2.2.23 we immediately obtain that ampleness only depends on the numerical class of a $\mathbb{Q}$-divisor D.

We thus obtain an open cone in $N^1(X)_{\mathbb{Q}}$.

Definition 2.2.29. *Let X be an irreducible projective variety defined over a number field K. We let $\mathrm{Amp}_{\mathbb{Q}}(X)$ be the subset of $N^1(X)_{\mathbb{Q}}$ consisting of ample $\mathbb{Q}$-divisors.*

Proposition 2.2.30 (The ample cone is open and full dimensional)**.** *Let X be an irreducible projective variety defined over a number field K. Then $\mathrm{Amp}_{\mathbb{Q}}(X)$ is an open cone in $N^1(X)_{\mathbb{Q}}$ of full dimension. In other words, it is subset of $N^1(X)_{\mathbb{Q}}$ closed under addition and multiplication by a positive scalar.*

Proof. It is clear from 2.2.23 that $\mathrm{Amp}_{\mathbb{Q}}(X)$ is closed under addition and positive scalar multiplication. It remains to show that it is open. It suffices to show that for any ample $\mathbb{Q}$-line bundle $\mathcal{L}$ and any line bundle E we have that $\mathcal{L} + qE$ is ample for some small rational $q > 0$. We may assume that both $\mathcal{L}$ and E are integral. Choose m so that for all $m' \geq m$ we have that $m'\mathcal{L} + E$ is very ample by 2.2.20. Then $\mathcal{L} + \dfrac{1}{m'}E$ is very ample for all $m' \geq m$ and the openness of the ample cone follows. As for being of full dimension, it suffices to show that $\mathbb{Q}$-line bundle is a difference of ample $\mathbb{Q}$-line bundles. But this follows from 2.2.21. □

2.2.4 $\mathbb{R}$-divisors.

We now turn to $\mathbb{R}$-divisors. We define an $\mathbb{R}$ divisor in a way completely analogous to a $\mathbb{Q}$-divisor. Furthermore, we may extend the intersection product in an analogous way. We thus obtain $\mathbb{R}$-vector spaces $\mathrm{CaDiv}_{\mathbb{R}}(X)$ with a subgroup $\mathrm{Num}_{\mathbb{R}}(X)$ and a vector space $\mathrm{CaDiv}_{\mathbb{R}}(X)/\mathrm{Num}_{\mathbb{R}}(X) = N^1(X)_{\mathbb{R}}$. However, subtleties arise in the definition of an ample $\mathbb{R}$-divisor. This is because given an arbitrary $\mathbb{R}$-divisor

$$D = \sum \alpha_i D_i$$

with the $\alpha_i \in \mathbb{R}$ and $D \in \mathrm{CaDiv}(X)$ we cannot always "clear denominators" and write rD as an integral divisor for some $r \in \mathbb{R}$. While there is a straightforward way to proceed using a general version of the Nakai-Moishezon-Kleiman Criterion for ampleness for $\mathbb{R}$-divisors, the proof of this statement is more difficult then other versions. One may proceed without this statement and indeed this is what is done in [31]. However as we are not developing the theory from scratch we will take this as given and indicate how one may proceed.

Definition 2.2.31 (Ample $\mathbb{R}$-divisors)**.** *Let X be an irreducible projective variety defined over a number field K. We say that D is ample if we may write*

$$D = \sum_{i=1}^{n} \alpha_i D_i$$

where the $\alpha_i \in \mathbb{R}_{>0}$ and D_i is an integral ample divisor on X.

The following variant of 2.2.23 is very convenient for us.

Theorem 2.2.32 (Nakai for $\mathbb{R}$-divisors, Campana and Peternell)**.** *Let X be an irreducible projective variety defined over a number field K. Let δ be a numerical class of an $\mathbb{R}$ divisor on X. Then δ is ample if and only if*

$$(\delta^{\dim V} \cdot V) > 0$$

for all closed $V \subseteq X$ of positive dimension.

To avoid the use of this theorem one may proceed without it and directly prove the following.

Proposition 2.2.33 (Ampleness depends only on the numerical class 1.3.13 [31])**.** *Let X be an irreducible projective variety defined over a number field K. Let $D \in \mathrm{CaDiv}_{\mathbb{R}}(X)$ with D ample. Then if $D' \in \mathrm{CaDiv}_{\mathbb{R}}(X)$ and $D \equiv_{\mathrm{num}} D'$ then D' is an ample $\mathbb{R}$-divisor.*

This statement is used to kick off the theory of $\mathbb{R}$-divisors, which can be developed to the point that a proof of 2.2.32 becomes available. The key to proving this statement is the following. If B is a numerically trivial $\mathbb{R}$-divisor, then we may write

$$B = \sum \alpha_i D_i$$

with $\alpha_i \in \mathbb{R}$ and D_i an integral divisor that is numerically trivial. To prove this one uses the following, let $B = \sum \gamma_i B_i$ with the $\gamma_i \in \mathbb{R}$ and B_i integral divisors. The condition that B is numerically trivial is then equivalent to the statement that the γ_i satisfy some linear equations with integer coefficients. Then any $\mathbb{R}$-solution is a real linear combination of integral solutions. This latter statement can be obtained by a topological comparison theorem with certain singular homology groups, or by introducing the cone of curves and proving it is finitely generated.

Theorem 2.2.34 (1.3 [31])**.** *Let X be an irreducible projective variety defined over a number field K. Let $\mathrm{Amp}_{\mathbb{R}}(X)$ be the collection of ample $\mathbb{R}$-divisors. Then $\mathrm{Amp}_{\mathbb{R}}(X)$ is a full dimensional open cone in $N^1(X)_{\mathbb{R}}$ and $N^1(X)_{\mathbb{R}} \cong N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.*

The significance of these results is that it will allow us to consider the closure $\overline{\mathrm{Amp}_{\mathbb{R}}(X)}$ in $N^1(X)_{\mathbb{R}}$. This closure is a closed convex cone in a finite dimensional vector which can be studied using techniques of linear algebra and convex analysis.

2.2.5 Nef divisors

We now begin developing the ideas discussed above by taking the closure of the ample cone of divisor classes. However, we will see that one can capture this object by working with intersections with curves, which simplifies the theory. In other words, if one is simply willing to work with limits of ample classes then one may only consider interactions with irreducible curves on your surface.

Definition 2.2.35 (Nef Divisors)**.** *Let X be an irreducible projective variety defined over a number field K. Let $D \in \mathrm{CaDiv}(X)$.*

1. *We say that D is nef if*
$$(D \cdot C) \geq 0$$
for all closed irreducible curves on X. If $\mathcal{L} = \mathcal{O}_X(D)$ then we say that $\mathcal{L}$ is nef if and only if $\mathcal{L}$ is nef.

2. *Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(X)$ or $D \in \mathrm{CaDiv}_{\mathbb{R}}(X)$. We say that D is nef if*
$$(D \cdot C) \geq 0$$
for all closed irreducible curves on X.

3. *It is clear from the definition that the property of being nef only depends on the numerical class of a divisor. We define $\mathrm{Nef}(X)$ to be the collection of nef divisor classes in $N^1(X)$. Similarly we define $\mathrm{Nef}_{\mathbb{Q}}(X)$ and $\mathrm{Nef}_{\mathbb{R}}(X)$ to be the collection of nef $\mathbb{Q}$-divisor classes in $N^1(X_{\mathbb{Q}})$ and the collection of nef $\mathbb{R}$-divisors on $N^1(X)_{\mathbb{R}}$.*

Proposition 2.2.36 (First properties of nef divisor classes). *Let X be an irreducible projective variety defined over a number field K. Then*

1. *The sum of two nef classes is nef.*
2. $\mathrm{Amp}_{\mathbb{R}}(X) \subseteq \mathrm{Nef}_{\mathbb{R}}(X)$.
3. *Let D be a real nef divisor class and H a real ample divisor class. Then $D + H$ is ample.*

Proof. Let D_1, D_2 be nef divisor classes (the coefficients are irrelevant here) and H an ample class. Let C be any irreducible closed curve on X.

1. For part 1) it suffices to compute $((D_1 + D_2) \cdot C) = (D_1 \cdot C) + (D_2 \cdot C) \geq 0 + 0$ by the linearly of the intersection product. Thus by definition $D_1 + D_2$ is nef.
2. This is immediate as $(H \cdot C) > 0$ by the definition of ample.
3. For part 3 we may compute $((D + H) \cdot C) = (D \cdot C) + (H \cdot C)$. Since $(D \cdot C) \geq 0$ and $(H \cdot C) > 0$ we have that $D + H$ is ample by Nakai's criterion.

□

We now turn to the basic properties of nef divisors under morphisms.

Proposition 2.2.37. *Let X, Y be an irreducible projective varieties defined over a number field K. Let $f : Y \to X$ be a morphism. Let $\mathcal{L}$ be a line bundle on X*

1. *If f is proper and $\mathcal{L}$ is nef then $f^*\mathcal{L}$ is nef.*
2. *If f is surjective and proper and $f^*\mathcal{L}$ is nef then $\mathcal{L}$ is nef.*
3. *If $\mathcal{L}$ is globally generated then $\mathcal{L}$ is nef.*

Proof. First suppose that $\mathcal{L}$ is nef and f is proper. Let C be a curve on Y. Then we compute

$$(f^*\mathcal{L} \cdot C) = (\mathcal{L} \cdot f_*(C))$$

using 2.2.11. Since $f_*(C)$ is a curve class on X or is zero we have that

$$(\mathcal{L} \cdot f_*(C)) \geq 0$$

as needed. Now suppose that f is surjective and proper and $f^*\mathcal{L}$ is nef. Let $C \subseteq X$ be a closed irreducible curve. Then one can find a closed irreducible curve $C' \subseteq Y$ with $f(C') = C$. Then $(\mathcal{L} \cdot C) = (\mathcal{L} \cdot f_*(C)) = (f^*\mathcal{L} \cdot C') \geq 0$ as $f^*\mathcal{L}$ is nef. Thus $\mathcal{L}$ is nef. Finally suppose that $\mathcal{L}$ is globally generated. Then pick $\phi \colon X \to \mathbb{P}^n$ with $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Note that $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample and ϕ is proper so by part 1 we have that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ is nef.

□

We now have the following fundamental result which allows us to show that the nef divisors are precisely the limits of ample divisors.

Theorem 2.2.38 (Kleiman's Theorem [31]). *Let X be an irreducible projective variety over a number field K. Let D be a nef $\mathbb{R}$ divisor. Then*

$$(D^k \cdot V) \geq 0$$

for all V irreducible and closed in X of dimension k.

We now show that the nef divisors are limits of ample ones.

Lemma 2.2.39 (Nef Cone Boundary Lemma). *Let X be an irreducible projective variety over a number field K. Let D be a nef $\mathbb{R}$ divisor. Let H be an ample $\mathbb{R}$-divisor. Then $D + H$ is ample. Conversely if H is ample and D is an $\mathbb{R}$-divisor with $D + \epsilon H$ ample for small sufficiently small ϵ then D is nef.*

Proof. Let $V \subseteq X$ be a closed irreducible subvariety of dimension k. Then $(D + H)^k = \sum_{i=0}^{k} \binom{k}{i}(D^i \cdot H^{k-i})$. Since H is ample $H^{k-i} \cdot V$ is represented by a positive $\mathbb{R}$-linear combination of cycles coming from irreducible sub-varieties. In other words

$$H^{k-i} \cdot V = \sum_j \alpha_{ij} V_{ij}$$

where $\alpha_{ij} > 0$ and V_{ij} is a $n-i$-cycle. Applying Kleiman to this we obtain that $D^i \cdot H^{k-i} \cdot V \geq 0$ for all $1 \leq i \leq k$. Taking $i = 0$ leaves $H^k \cdot V > 0$ as H is ample. In other words we have

$$((D + H)^k \cdot V) = \sum_{i=0}^{k} \binom{k}{i}((D^i \cdot H^{k-i}) \cdot V) = \sum_{i=0}^{k} \sum_j \binom{k}{i} \alpha_{ij}(D^i \cdot V_{ij}) > 0.$$

By the Nakai criterion for $\mathbb{R}$-divisors we have that $D+H$ is ample. For the second part let C be a curve. Then for small ϵ we have

$$((D+\epsilon H)\cdot C) = (D\cdot C) + \epsilon(H\cdot C) > 0$$

as we have assumed that $D+\epsilon H$ is ample for small ϵ. Since $H\cdot C>0$ we may let $\epsilon\to 0$ and obtain $(D\cdot C)\geq 0$ so that D is nef. ☐

Theorem 2.2.40 (The Nef cone is the closure of the ample cone)**.** *Let X be an irreducible projective variety over a number field K.*

1. *Then* $\mathrm{Nef}_{\mathbb{R}}(X)=\overline{\mathrm{Amp}_{\mathbb{R}}(X)}$ *and* $\mathrm{Nef}_{\mathbb{R}}(X)$ *is a closed full dimensional convex cone in* $N^1(X)_{\mathbb{R}}$.

2. *The interior of* $\mathrm{Nef}_{\mathbb{R}}(X)$ *is* $\mathrm{Amp}_{\mathbb{R}}(X)$.

Proof. By 2.2.36 it suffices to show that $\mathrm{Nef}_{\mathbb{R}}(X)=\overline{\mathrm{Amp}_{\mathbb{R}}(X)}$. Let $D\in\mathrm{Nef}_{\mathbb{R}}(X)$. Then by 2.2.5 we have that $D\in\overline{\mathrm{Amp}_{\mathbb{R}}(X)}$. Conversely suppose that $D\in\overline{\mathrm{Amp}_{\mathbb{R}}(X)}$. Then as D is on the boundary of the ample cone we may find H ample such that $D+\epsilon H$ is ample for all ϵ small enough. By 2.2.5 we have that D is nef as needed. For the second part let $D\in\mathrm{Nef}_{\mathbb{R}}(X)$ which lies in the interior. Then for all small ϵ we may find an ample H such that $D-H$ is ample. Then $(D-H)+H=D$ and D is a sum of ample classes, which means D is ample as needed. ☐

We end this section with an introduction to the cone of curves.

Definition 2.2.41 (The cone of curves)**.** *Let X be an irreducible projective variety over a number field K. Let $Z_1(X)_{\mathbb{R}}$ be the vector of all finite $\mathbb{R}$-linear combinations of irreducible closed curves on X. We say that C_1, C_2 are numerically equivalent if $(D\cdot C_1)=(D\cdot C_2)$ for all $D\in\mathrm{CaDiv}_{\mathbb{R}}(X)$. We let $N_1(X)_{\mathbb{R}}$ be the collection of numerical equivalence classes of curves. We let $\overline{\mathrm{NE}}(X)$ be the closure of the cone spanned by all classes of integral curves. We call this the closed cone of curves.*

Notice that $N_1(X)_{\mathbb{R}}$ and $N^1(X)_{\mathbb{R}}$ are dual by construction and that the intersection pairing gives a perfect pairing

$$N^1(X)_{\mathbb{R}}\times N_1(X)_{\mathbb{R}}\to\mathbb{R}.$$

In fact, this duality takes into account the nef cone in the following way.

Theorem 2.2.42 (Dual of the Nef cone)**.** *We have that*

$$\overline{\mathrm{NE}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} : (D \cdot \gamma) \geq 0 \forall\ D \in \mathrm{Nef}_{\mathbb{R}}(X).\}.$$

We will need to say something about a relatively ample mapping in certain dynamical situations. A particular example will be the case of projective bundles. We take as a definition the version which is most useful for us, and do not use the definition of f-ample which is found in other sources.

Definition 2.2.43 (Relative Ampleness 1.7 [31])**.** *Let $f : X \to T$ be a proper morphism of irreducible projective varieties over a number field K. Let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is f-ample or relatively ample (if f is clear from the context) if for all $t \in T$ we have that $\mathcal{L} \mid X_t$ is ample. Here X_t is the fiber above T.*

Proposition 2.2.44. *Let $f : X \to T$ be a proper morphism of irreducible projective varieties over a number field K. Let $\mathcal{L}$ be a line bundle on X. Then the following are equivalent.*

1. *$\mathcal{L}$ is f-ample.*
2. *For all $V \subseteq X$ with $f(V) = t$ where t is a closed point of T we have that $(\mathcal{L}^{\dim V} \cdot V) > 0$.*
3. *Let A be ample on T. Then $\mathcal{L} \otimes f^*(A^{\otimes m})$ is ample for all m sufficiently large.*

Proof. Suppose that $\mathcal{L}$ is f-ample. Then by definition we have that (1) $\iff$ (2) by 2.2.23. For the equivalence of (1) and (3) see [31, 1.7.10]. $\square$

2.2.6 Iitaka Dimensions of line bundles.

Definition 2.2.45 (Semigroup of a line bundle)**.** *Let X be a normal geometrically irreducible projective variety defined over a number field K. Let $\mathcal{L}$ be a line bundle on X. We define*

$$N(\mathcal{L}) = \{m \in \mathbb{Z}_{m \geq 0} : H^0(X, \mathcal{L}^{\otimes m}) \neq 0\}.$$

If $N(\mathcal{L}) \neq 0$ we set $e(L)$ to be the gcd of all elements in $N(\mathcal{L})$ and call this the exponent of $\mathcal{L}$. Given $m \in N(\mathcal{L})$ we let ϕ_m be the rational mapping

$$\phi_m = \phi_{|m\mathcal{L}|} : X \to \mathbb{P}H^0(X, m\mathcal{L}).$$

We now come to a crucial definition which gives an asymptpic criterion for the dimensions of the images of X under the morphisms $\phi_{|m\mathcal{L}|}$ when $m \in N(\mathcal{L})$. This will have a profound effect when we deal with height functions.

Definition 2.2.46 (Iitaka dimension.)**.** *Let X be a normal geometrically irreducible projective variety defined over a number field K. Let $\mathcal{L}$ be a line bundle on X. We define*

$$\kappa(\mathcal{L}) = \max_{m \in N(\mathcal{L})} \{\dim \phi_m(X)\}.$$

If $N(\mathcal{L}) = 0$ we set $\kappa(\mathcal{L}) = -\infty$. We call this number the Iitaka dimension of $\mathcal{L}$. We have immediately that

$$-\infty \leq \kappa(\mathcal{L}) \leq \dim X.$$

If X is smooth then we set

$$\kappa(X) = \kappa(\omega_X)$$

where ω_X is the canonical line bundle on X and call this the Kodaira dimension *of X. If X is singular we define the Kodaira dimension of X to be the Kodaira dimension of a smooth birational model. This will turn out to be a birational invariant, and is well defined in light of the theorem about resolution of singularities.*

To illustrate why we use normality we give an example of what can go wrong. The idea is that pulling back from something non-normal can have unexpected behavior. Let X be the plane curve $zy^2 = x^3$ in $\mathbb{P}^2$. The line bundles on X of degree zero correspond to points $P \neq O$ where O is the singularity of X. By Riemann-Roch for singular curves we have that $\chi(\mathcal{L}) = \deg D + p_a(X) - 1 = 0$ where $p_a(X) = \dim H^1(X, O_X)$. Since $\deg D = 0$ we have that $\chi(\mathcal{L}) = p_a(X) - 1$. But $p_a(X) = 1$ by the arithmetic genus formula for singular curves, for example [23, 4.1.18]. Thus we have that $\deg D - p_a(X) - 1 = \deg D$ for any line bundle D. Letting ω_X be the canonical sheaf on X we have that

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X - \mathcal{L}) = \deg \mathcal{L} + p_a(X) - 1 = \deg \mathcal{L} = 0.$$

However as $X \subseteq \mathbb{P}^2$ we may apply the adjunction formula to obtain

$$\omega_X = \mathcal{O}_X(-3 + \deg X) = \mathcal{O}_X(0) = \mathcal{O}_X$$

as X is a degree three plane cubic. Therefore from

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X - \mathcal{L}) = 0$$

we have that

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^0(X, -\mathcal{L}).$$

It follows immediately that $\dim_k H^0(X, \mathcal{L}) = 0$ whenever $\mathcal{L}$ is non-trivial. On the other hand, we have the normalization $\nu\colon \mathbb{P}^1 \to X$ and $\deg \nu^*\mathcal{L} = \deg \mathcal{L} = 0$. Thus $\nu^*\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}$ for any degree zero line bundle on X. Thus $H^0(X\nu^*(n\mathcal{L})) = 1 > H^0(X, n\mathcal{L}) = 0$ for all $n > 0$ and $\mathcal{L}$ degree zero on X which is non-torsion. In other words, the theory is not birationally invariant if we allow non-normal objects.

Definition 2.2.47. *Let X, Y be an irreducible projective variety defined over $\bar{\mathbb{Q}}$ and let $f\colon X \to Y$ be a surjective projective morphism. We say that f is an algebraic fiber space if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Lemma 2.2.48 (Properties of fiber spaces)**.** *Let X, Y be an irreducible projective variety defined over $\bar{\mathbb{Q}}$ and let $f\colon X \to Y$ be a surjective projective morphism.*

1. *If f is an algebraic fiber space then f has connected fibers.*
2. *If Y is normal and f has connected fibers then f is an algebraic fiber space.*
3. *Suppose that f is an algebraic fiber space. Let $\mathcal{L}$ be a line bundle on Y. Then $H^0(X, f^*\mathcal{L}) = H^0(Y, \mathcal{L})$.*
4. *If f is an algebraic fiber space then the induced morphism $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is an injection.*

Proof. We refer to the discussion following [31, 2.1.11] for the first two statements. Suppose now that f Then we have that $H^0(X, f^*\mathcal{L}) = H^0(Y, f_*f^*\mathcal{L})$. By the projection formula we have that $f_*(f^*\mathcal{L} \otimes \mathcal{O}_Y) = \mathcal{L} \otimes f_*\mathcal{O}_X = \mathcal{L}$ as f is an algebraic fiber space. Now suppose that $f^*\mathcal{L} \cong \mathcal{O}_X$. So $H^0(Y, \mathcal{L}) = H^0(X, \mathcal{O}_X) \neq 0$. On the other hand we have $f^*\mathcal{L}^{-1} = \mathcal{O}_X$ as we so the same argument gives $H^0(Y, \mathcal{L}^{-1}) = H^0(X, \mathcal{O}_X) \neq 0$. Thus $\mathcal{L}$ is trivial as $\mathcal{L}$ and $\mathcal{L}^{-1}$ both have a non-zero section. $\square$

Definition 2.2.49. *Let X be an irreducible projective variety defined over $\bar{Q}$. Let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is semi-ample if for some $m > 0$ we have that $\mathcal{L}^{\otimes m}$ is globally generated.*

The key property of the stable base locus is the following.

Lemma 2.2.50 (Realizability of the stable base locus 2.1.21 [31])**.** *There are integers m_0, n_0 such that*

$$\boldsymbol{B}(D) = \mathrm{Bs}(km_0D)$$

for all $k \geq n_0$.

We will use a theorem of Zariski and Fujita later. One can formulate a theory of ampleness on a possibly reduced projective variety with multiple components, see for example [31, Section 1.2]. Such a notion is implicit in the following result.

Theorem 2.2.51 (Theorem 2 [14])**.** *Let X be an irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let $\mathcal{L}$ be a line bundle on X. Suppose that $V \subseteq H^0(X, \mathcal{L})$ is a sub-vector space. Let B be the base scheme of V. In other words, the scheme defined by the ideal sheaf of $\mathcal{O}_X$ spanned by the sections of V. If $\mathcal{L}\mid_B$ is ample then $\mathcal{L}$ is semi-ample.*

Corollary 2.2.51.1 ([14])**.** *Let X be an irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let $\mathcal{L}$ be a line bundle on X. Suppose that $V \subseteq H^0(X, \mathcal{L})$ is a sub-vector space. Let $\mathrm{Bs}(V)$ be the base locus of V with its canonical reduced subscheme structure. If $\mathcal{L}\mid_{\mathrm{Bs}(V)}$ is ample then $\mathcal{L}$ is semi-ample.*

Proof. Let $b(\mid V \mid)$ be the base scheme of V and $\mathrm{Bs}(V)$ the base locus with its canonical closed subscheme structure. We have that $b(\mid V \mid)_{\mathrm{red}} = \mathrm{Bs}(V)$. By [31, 1.2.16] we have that if $\mathcal{L}\mid_{b(\mid V\mid)}$ is ample if and only if $\mathcal{L}\mid_{\mathrm{Bs}(V)}$ is ample and then the result follows by 2.2.51. □

We will now introduce a new class of divisors, namely big divisors. These will be divisors with large Iitaka dimension. The significance of such divisors is that they will give a closed cone in $N^1(X)_{\mathbb{R}}$ which contains $\mathrm{Nef}(X)_{\mathbb{R}}$. It will be useful to have an alternative characterization of the Iitaka dimension.

Proposition 2.2.52 (Asymptotic Iitaka definition)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let $\mathcal{L}$ be a line bundle on X. Suppose that $\kappa = \kappa(X, \mathcal{L})$. Then there are constants $a, A \in \mathbb{R}_{>0}$ such that for all large $m \in N(X, \mathcal{L})$ we have*

$$am^{\kappa} \leq \dim H^0(X, m\mathcal{L}) \leq Am^{\kappa}.$$

Definition 2.2.53 (Big Divisors)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. We say that a line bundle $\mathcal{L}$ is* big *if $\kappa(X, \mathcal{L}) = \dim X$.*

Using the alternative version of the Iitaka dimension one has the following useful property.

Lemma 2.2.54 (Asymptotic criterion for bigness)**.** *Let X be a normal irreducible projective variety of dimension n defined over $\bar{\mathbb{Q}}$. A line bundle $\mathcal{L}$ is big if and only if there is some constant $C > 0$ such that*

$$\dim_K H^0(X, m\mathcal{L}) \geq Cm^n$$

for all large $m \in N(X, \mathcal{L})$.

Notice that if $\mathcal{L}$ is an ample line bundle then $\kappa(X, \mathcal{L}) = \dim X$ as $m\mathcal{L}$ is very ample for some $m > 0$ and so ϕ_m is a closed embedding. On the other hand, a big divisor need not be ample. Here we follow [31, 2.2.4]. Let $S = \{[0:0:1], [0:1:1]\} \subseteq \mathbb{P}^2$. Let $\pi\colon X \to \mathbb{P}^2$ be the blow up of S. Let $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and let E be the exceptional divisor of the blow up. Let L be the line through $[0:1:0], [0:0:1]$. Set C to be the proper transform of L and let $D = dH - rE$. Then $H^0(X, mD) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(md) \otimes I_S^{mr})$ using that

$$\pi_*(mdH - mrE) = \mathcal{O}_{\mathbb{P}^2}(md) \otimes \pi_*(-mrE) = \mathcal{O}_{\mathbb{P}^2}(md) \otimes I_S^{mr}$$

where I_S is the ideal sheaf of S. Thus a global section of mD corresponds to a homogeneous polynomial of degree md that vanishes to order at least mr at S. Taking $d = 3$ and $r = 2$. We can compute the global sections using toric geometry in this case as we have blow up torus invariant points. The global sections of $H^0(X, D)$ correspond to the lattice points of the triangle with vertices $(0,0), (3,0), (0,2)$ by a standard toric computation which we omit. There are 6 lattice points so $\dim H^0(X, mD) \geq (2m) \cdot (3m) = 6m^2$. Thus by the alternative characterization of the Iitaka dimension we have that D is big. On the other hand $C \cdot H = 1$ and $C\dot{E} = 2$ so that

$$(C \cdot D) = 3(C \cdot H) - 2(C \cdot E) = -1.$$

Thus D is not Nef, and is so an example of a big but not nef divisor. We have the following alternative characterization of bigness.

Proposition 2.2.55. *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is big if and only if $\phi_m\colon X \dashrightarrow \mathbb{P}H^0(X, m\mathcal{L})$ is birational unto its image for some $m > 0$.*

We wish to prove that being big only depends on the numerical class of D. However we do not have a Nakai type criterion to apply directly, and so must work harder.

Proposition 2.2.56 (Kodaira's Lemma 2.2.6 in [31])**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Suppose that D is a big divisor and F an arbitrary effective divisor. Then $H^0(X, mD - F) \neq 0$ for all large enough $m \in N(X, D)$.*

Proof. We sketch the idea of the proof: We have an exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

By our assumption that D is big we have that $\dim H^0(X, \mathcal{O}_X(mD)) \geq cm^n$ for some constant C where $n = \dim X$. On the other hand, as F is a scheme of dimension $n-1$ $\mathcal{O}_F(mD) = O(m^{n-1})$ as $m \to \infty$. The previous statement can be made completely rigorous, but we omit those details. Thus we must have that $\mathcal{O}_X(mD - F)$ must be non-zero by the exactness of the above sequence. $\square$

Corollary 2.2.56.1 (Characterization of Big Divisors)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let D be a divisor on X. The following are equivalent.*

1. *D is big.*

2. *For any ample integral divisor A there is some $m > 0$ and an effective divisor N on X such that $mD \equiv_{\text{lin}} A + N$.*

3. *There is an ample integral divisor A there is some $m > 0$ and an effective divisor N on X such that $mD \equiv_{\text{lin}} A + N$.*

4. *There is an ample divisor A and a positive integer $m > 0$ and an effective divisor N such that $mD \equiv_{\text{num}} A + N$.*

Proof. Assume that D is big. Let A be any ample divisor. Choose a large $r > 0$ such that $rA \equiv_{\text{lin}} H_r$ and $(r+1)A \equiv_{\text{lin}} H_{r+1}$ with H_r, H_{r+1} effective. Then apply 2.2.56 to obtain m such that $mD - H_{r+1} = N'$ with N' effective. Then

$$mD \equiv_{\text{lin}} rA + A + N'.$$

Set $N = N' + H_r$ so that N is effective and

$$mD \equiv_{\text{lin}} A + N.$$

Clearly (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Now suppose that (4) holds. So we have that $A + N \equiv_{\text{num}} mD$. So we have that $mD - N$ is ample as ampleness is a numerical property. So we may assume that $mD \equiv_{\text{lin}} A + N$ with A ample and N effective. So choose m' with $m'A$ very ample. Thus we have that

$$mm'D \equiv_{\text{lin}} m'A + m'N.$$

Then we have $\kappa(X, D) \geq \kappa(X, m'A) = \dim X$ which shows that D is big.

□

Corollary 2.2.56.2 (Numerical properties of bigness)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let D be a divisor on X.*

1. *Suppose that $D \equiv_{\text{num}} D'$ and D is big. Then D' is big.*

2. *Let D_1, D_2 be big divisors. Then $nD_1 + mD_2$ is big for all integers $n, m > 0$.*

Definition 2.2.57 (Big $\mathbb{R}$ and $\mathbb{Q}$ -divisors.)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$.*

1. *Let D be a $\mathbb{Q}$-divisor. We say that D is big if mD is big where $m \in \mathbb{Z}_{\geq 0}$ and mD is a big integral divisor.*
2. *Let D be a $\mathbb{R}$-divisor on X. We say that D is big if we may write $D = \sum_{\alpha_i} D_i$ where $\alpha_i \in \mathbb{R}_{>0}$ and each D_i is an integral big divisor.*

We can now prove the analogue of the earlier results about big divisors. As is typical, the result about $\mathbb{R}$-divisors is slightly more difficult to prove.

Proposition 2.2.58 (Formal Properties of Big $\mathbb{R}$-Divisors)**.** *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let D, D' be $\mathbb{R}$-divisors.*

1. *Suppose $D \equiv_{\mathrm{num}} D'$. Then D is big if and only if D' is big.*
2. *D is big if and only if $D \equiv_{\mathrm{num}} A + N$ where A is an ample $\mathbb{R}$-divisor and N is an effective $\mathbb{R}$-divisor.*

Proof. For (1) we must show that if D is big and B is numerically trivial then $D + B$ is also big. Write $D = \sum_j c_j D_j$ with D_j integral big divisors and $c_j \in \mathbb{R}_{>0}$. It suffices to show that $D_1 + c_1^{-1} B$ is big. In other words that $D + B$ is big where D is an integral big divisor. We now assume the results discussed in the remark following 2.2.33. More precisely, we are using freely that we may write $B = \sum_{i=1}^{N} \alpha_i B_i$ with B_i a numerically trivial divisor and $\alpha_i \in \mathbb{R}$. Now induct on N. Assume $N = 1$. Then we must show that $D + rB$ is big where D, B are integral, D is big and B is numerically trivial. Choose $r_1 < r < r_2$ with r_1, r_2 rational. Then pick t in $[0, 1]$ with $tr_1 + (1 - t)r_2 = r$. Then $t(D + r_1 B) + (1 - t)(D + r_2 B) = D + rB$ and $(D + r_1 B), (D + r_2 B)$ are ample by the numerical nature of ampleness for $\mathbb{Q}$-divisors. Now let $N > 1$. Then we must show that $D + \sum_{i=1}^{N-1} \alpha_i B_i + \alpha_N B_N$ is big. By induction $D + \sum_{i=1}^{N-1} \alpha_i B_i$ is big and so we may write $D + \sum_{i=1}^{N-1} \alpha_i B_i = \sum \beta_j D'_j$ with D_j integral big divisors. By the base case $\beta_1 D'_1 + \alpha_N B_N$ is big as needed. We now turn to (2). Let D be a big divisor. Letting $D = \sum_j c_j D_j$ with D_j integral big divisors and $c_j \in \mathbb{R}_{>0}$ we may write each $D_j \equiv_{\mathrm{num}} A_j + N_j$ where N_j is effective and A_j is ample. Then put $A = \sum_j c_j D_j$ and $N = \sum c_j N_j$. So A is an ample $\mathbb{R}$-divisor and N is a effective $\mathbb{R}$-divisor and $D \equiv A + N$ as needed. Conversely suppose that $D \equiv_{\mathrm{num}} A + N$ with A ample and N effective. Write $A = \sum c_j A_j$ with $c_j > 0$ and A_j an ample integral divisor and $N = \sum b_i N_i$ with $b_i > 0$ and N_i integral. If $c_1 D_1 + N$ is big then $(c_1 D_1 + N) + \sum_{i \geq 2} c_i D_i$ is also big as the sum of a big divisor and an ample divisor is big. Thus we may show that $A + \sum_{i=1}^{N} b_i N_i$ is big where $b_i > 0$ and the N_i are integral and effective with A is an integral and ample divisor. As before we induct on N. The base case is to show that $A + rN$ is big where $r > 0$ is real. Choose $r_1 < r < r_2$ with r_1, r_2 rational. Then pick t in $[0,1]$ with $tr_1 + (1-t)r_2 = r$. Then $t(A + r_1 N) + (1-t)(A + r_2 N) = A + rN$. Since $A + r_i N$ is a big $\mathbb{Q}$ divisor we have that $A + rN$ is big. Now let $N > 1$. Then by induction $A + \sum_{i=1}^{N-1}$ is big and arguing as before we may directly reduce to the base case. □

Corollary 2.2.58.1 (Bigness is an open condition). *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let D be a big $\mathbb{R}$-divisor. Let $E_1, ..., E_r$ be arbitrary $\mathbb{R}$-divisors. Then $D + \sum_{i=1}^{r} \epsilon_i E_i$ is big for all ϵ_i with $\mid \epsilon_i \mid$ small enough.*

Proof. This follows from 2.2.58 and the open nature of amplitude. □

Definition 2.2.59 (The big and pseudoeffective cones). *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Let $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We let $\mathrm{Big}_R(X)$ be the collection of big R divisors in $N^1(X)_R$. We let $\overline{\mathrm{Eff}}_R(X)$ be the closure of the cone generated by the classes of effective R divisors. We call $\overline{\mathrm{Eff}}_{\mathbb{R}}(X)$ the pseudoeffective cone.*

Theorem 2.2.60 (Properties of the big cone). *Let X be a normal irreducible projective variety defined over $\bar{\mathbb{Q}}$. Then*

1. $\mathrm{Big}_{\mathbb{R}}(X) = \mathrm{int}(\overline{\mathrm{Eff}}_{\mathbb{R}}(X))$.
2. $\overline{\mathrm{Eff}}_{\mathbb{R}}(X) = \overline{\mathrm{Big}(X)}$.

Proof. The big cone is open by 2.2.58.1. By (2) of 2.2.58 we have that $\mathrm{Big}_{\mathbb{R}}(X) \subseteq \overline{\mathrm{Eff}}(X)$. Now let $D \in \overline{\mathrm{Eff}}(X)$. Then we may find D_k effective $\mathbb{R}$ divisors converging to D. Choose an ample divisor A and set $B_k = D_K + \frac{1}{k}A$. Then B_k is big by 2.2.58 and converges to D. So $D \in \overline{\mathrm{Big}_{\mathbb{R}}(X)}$ and we have that $\overline{\mathrm{Eff}}_{\mathbb{R}}(X) \subseteq \overline{\mathrm{Big}_{\mathbb{R}}(X)}$ as needed. As bigness is open we have that $\mathrm{Big}_{\mathbb{R}}(X) \subseteq \mathrm{int}(\overline{\mathrm{Eff}}_{\mathbb{R}}(X))$. Conversely suppose that $D \in \mathrm{int}(\overline{\mathrm{Eff}}_{\mathbb{R}}(X))$. Then we can find an ample H such that $D - H \equiv_{\mathrm{num}} N$ where N is effective. So $D = N + H$ is big by 2.2.58. □

2.3 Minimal Model Program

We now begin discussing the minimal model program or MMP. The overall goal of the minimal model program is as follows. Given a variety X defined over an algebraically closed field of characteristic zero. Find a *minimal model* or simplest possible birational model of X. The idea being, that it may be intractable to study X itself, but if we allow ourselves to work up to birational equivalence then we can transfer questions to the minimal model which may be simpler to work with. This begs the question, what is a minimal model? Our philosophy is that all general invariants of a general variety X can be constructed from the cotangent bundle Ω_X by standard geometric operations such as determinants, duals ect. Before giving definitions we consider a motivating case, namely that of curves. Our main source here is [36].

Example 3 (Classification of curves)**.** Let X be an irreducible curve defined over $\overline{\mathbb{Q}}$. If X is singular we have a normalization morphism

$$\nu\colon \tilde{X} \to X$$

where $\tilde{X}$ is a smooth curve and ν is a birational morphism. In fact $\tilde{X}$ is the unique smooth curve which is birational to X. Therefore we will take X to be our simplest possible birational model of X. Having chosen our simplest possible birational model, it remains to classify them in some way. Let X be a smooth projective curve defined over $\overline{\mathbb{Q}}$. Consider the canonical divisor represented by $K_X = \Omega_X$ on X. The we have the following well known trichotomy.

1. $\deg K_X < 0 \iff H^0(X, mK_X) = \{0\}\ \forall\ m > 0 \iff -K_X$ is ample.
2. $\deg K_X = 0 \iff H^0(X, mK_X) = \overline{\mathbb{Q}}\ \forall\ m > 0 \iff K_X = 0$.
3. $\deg K_X > 0 \iff H^0(X, mK_X) \geq mC\ \forall\ m > 0$ and some constant $C > 0 \iff K_X$ is ample.

In the latter two cases we see that K_X is nef, while in the first case we have that $-K_X$ is ample. We consider these two types of behavior as distinct and different from one another. In fact the first type of behavior will be distinctive to varieties with $\kappa(X) = -\infty$.

We now can define a minimal model.

Definition 2.3.1. *Let X be a projective variety defined over $\overline{\mathbb{Q}}$ such that $K_X \in \mathrm{CaDiv}(X)_{\mathbb{Q}}$. We say that X is a minimal model if K_X is nef.*

In the setting of 3 we see that cases of (2) + (3) are a minimal models. In particular, if X is a curve and $\kappa(X) \geq 0$ then X has a smooth minimal model. If $\kappa(X) < 0$ then X does not have a minimal model.

Remark 2.3.2. We point out a crucial difference between the curve case and the higher dimensional case for dynamics. For curves if we have a dominant morphism $f\colon X \to X$ where X is a curve over $\overline{\mathbb{Q}}$, then by the universal property of normalization we have that there is a dominant morphism $\tilde{f}\colon \tilde{X} \to \tilde{X}$ of normalizations that is compatible with f. If X is a surface and $\mu\colon X' \to X$ is a resolution of singularities then a dominant morphism $f\colon X \to X$ does not obviously lift to a dominant morphism $f'\colon X' \to X'$.

Example 4 (A morphism that does not ascend along a blow up)**.** Let $X = \mathbb{P}^2$ and let $f\colon X \to X$ be the morphism given by $f([x_0 : x_1 : x_2]) = [x_0^2 : x_1^2 : x_2^2]$. Now consider the blow up of $9 \geq d > 3$ very general points $\mu\colon X' \to X$. Then X' is a smooth Del-Pezzo surface which is non-toric. If $f'\colon X' \to X'$ is a surjective endomorphism with $\mu \circ f' = f \circ \mu$ then f' has degree 2 and so is not an automorphism. Furthermore, $\kappa(X') = -\infty$ and X' admits no non-constant morphism to an elliptic curve. By 4.0.1 X' must be toric, which is a contradiction. So f does not lift to X' in a compatible manner.

Thus one must contend with the possibility of working with singular varieties when studying dynamics, at least if we wish to remain in the world of morphisms and not dominant rational mappings. We will see that in fact this will be forced upon us by the minimal model program.

In higher dimensions we cannot expect a unique smooth model lying in a birational equivalence class. Indeed by blowing up points on a variety we obtain many different birationally equivalent smooth models. Intuitively, such an example is not as simple as possible, since it was obtained by blowing up points, and could thus be simplified by blowing down these curves. We will take this as motivation; we will attempt to simplify our varieties by contracting curves. Given a variety X we let $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ be the closed cone of curves, dual to the nef cone $\mathrm{Nef}(X)_{\mathbb{R}}$. To simplify X we may try and simplify $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ (equivalently $\mathrm{Nef}(X)_{\mathbb{R}}$). In other words, we will attempt to remove line bundles from X. We will obtain this by looking for morphisms

$$\pi\colon X \to Y \text{ such that there is a curve } C \subseteq X \text{ and } \pi(C) = \mathrm{pt}.$$

There are 3 types of morphisms that we will consider *contractions*. Before going further let us give examples of two of the kind of contractions we will deal with. The third being more difficult and less amenable to easy examples.

1. Let $\mu\colon X' \to X$ be the blow up of a point p on a smooth surface with exceptional divisor E. Then $\mu(E) = p$. So μ contracts the curve E and simplifies X'. In this case μ is an isomorphism in codimension 1. This is an example of a *divisorial contraction* because the exceptional locus of the morphism is a divisor.

2. Let $\pi\colon \mathbb{P}E \to \mathbb{P}^1$ be the bundle projection of a rank 2 vector bundle on $\mathbb{P}^1$. Fix a point $p \in \mathbb{P}^1$ and let $C = \pi^{-1}(p) \cong \mathbb{P}^1$. Then $\pi(C) = p$ and π contracts (many) curves. Note that each such curve is a fiber of π and all such fibers are numerically equivalent. We call this a fibering type contraction.

While the above examples are suggestive, there will be divisorial contractions that are not a blow up with a smooth center, and fibering contractions that are not projective bundles.

The minimal model program will be an algorithm which will attempt to build a minimal model out of our starting variety X. To carry out this program we will need the following ingredients.

Ingredients of the minimal model program for K_X. We will require the following.

1. A category $\mathfrak{C}$ of irreducible $\mathbb{Q}$-factorial projective varieties which contains all smooth projective varieties. The following operations will be required to preserve $\mathfrak{C}$.

2. **The cone theorem for $\mathfrak{C}$, [36, 7-2-1]:** Given $X \in \mathfrak{C}$ we have a decomposition

$$\overline{\mathrm{NE}}(X)_{\mathbb{R}} = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i R_i \tag{2.7}$$

   where the sum is indexed by some (potentially infinite or empty) subset of $\mathbb{Z}_{\geq 0}$, $R_i = \mathbb{R}_{\geq 0}[l_i] \subseteq \overline{\mathrm{NE}}(X)_{\mathbb{R}}$ and $[l_i]$ is the numerical class of an irreducible curve in X and $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$ is the subspace of $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ which have non-negative intersection with K_X. We have and $(K_X \cdot l_i) < 0$ for all l_i. The R_i are called *extremal rays.*

3. **The contraction theorem for extremal rays, [36, 8-1-3] :** For each extremal ray R in (2) there is a well behaved contraction morphism

$$\phi_R\colon X \to Y \tag{2.8}$$

   which have the following three types.

(a) $\phi_R\colon X \to Y$ is birational and the exceptional locus $\mathrm{exc}(\phi_R)$ has codimension 1 and $\rho(X) - 1 = \rho(Y)$. In this case we demand that $Y \in \mathfrak{C}$ and call ϕ_R or R divisorial because the exceptional locus is a divisor. An example of a divisorial contraction is the blow up of a point on a smooth projective variety.

(b) $\phi_R\colon X \to Y$ is birational and $\rho(X)-1 = \rho(Y)$ and the exceptional locus $\mathrm{exc}(\phi_R)$ has codimension ≥ 2. In this case we demand the existence of a projective variety $X^+ \in \mathfrak{C}$ with a contraction $\phi_R^+\colon X^+ \to Y$ such that $\mathrm{exc}(\phi_R^+)$ has codimension at least 2; this is all constructed from ϕ_R in a unique way. Furthermore we require that the birational mapping

$$\psi_R = (\phi_R^+)^{-1} \circ \phi_R\colon X \dashrightarrow X^+$$

has exceptional locus $\mathrm{exc}(\phi_R)$ and $\rho(X) = \rho(X^+)$. We call ϕ_R a small contraction or flipping type contraction and X^+ the associated flip.

(c) If $\phi_R\colon X \to Y$ is not birational then we require it to be a Mori fiber space. In other words we require

i. That Y is a normal projective variety with $\dim X > \dim Y$ and ϕ_R has connected fibers and $\rho(X) - 1 = \rho(Y)$.

ii. Every curve C in a fiber of ϕ_R is numerically proportional. Moreover $(K_X \cdot C) < 0$.

We call ϕ_R a fibering or fiber type contraction and X a Mori-fiber space. An example of a Mori-fiber space and a fibering type contraction is a projective bundle $\mathbb{P}E \to Y$ over a normal variety Y.

4. **Termination of flips conjecture [36, 3-1-14]:**

There is no infinite sequence of flips

$$\begin{tikzcd}
X_1 \arrow[rr, dotted, "\psi_1"] \arrow[dr, "\phi_1"'] & & X_2 \arrow[rr, dotted, "\psi_2"] \arrow[dl, "\phi_1^+"] \arrow[dr, "\phi_2"'] & & X_3 \arrow[rr, dotted, "\psi_3"] \arrow[dl, "\phi_2^+"] \arrow[dr, "\phi_3"'] & & X_4 \arrow[rr, dotted] \arrow[dl, "\phi_3^+"] \arrow[dr] & & \ldots \\
& Y_1 & & Y_2 & & Y_3 & & \ldots &
\end{tikzcd}$$

This is still conjectural in full generality dimension greater than 3.

We now have the following algorithm, called the minimal model program from K_X.

The minimal model program for K_X (3-1-15 in [36]). *Let* $X \in \mathfrak{C}$.
Step 1: *If* K_X *is nef, stop and output* X.
Step 2:*Since* K_X *is not nef we apply the cone theorem (2) from 2.3 to obtain a decomposition*

$$\overline{\mathrm{NE}}(X)_{\mathbb{R}} = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i R_i \tag{2.9}$$

with at least one extremal ray. Choose an extremal ray $R = R_i$ *for some* i. *Note that by definition* $(K_X \cdot R) < 0$.
Step 3: *Apply the contraction theorem ((3) in 2.3) to* R *to obtain an extremal contraction* $\phi_R \colon X \to Y$.
Step 4: *If* ϕ_R *is a divisorial contraction ((a) in 2.3) then* $Y \in \mathfrak{C}$. *Replace* X *with* Y *and return to Step 1.*
Step 5: *If* ϕ_R *is flipping contraction ((b) in 2.3) replace* X *with the flip* $X^+ \in \mathfrak{C}$ *and return to Step 1.*
Step 6: *If* ϕ_R *is a fibering type contraction ((c) in 2.3) return* X. *In other words if* $\dim Y < \dim X$ *we return* X.

If we assume that the assumptions of 2.3 (including the termination of flips) hold, then the minimal model program as described above will always terminate in a minimal model or a Mori fiber space. To see this note that every divisorial contraction decreases the Picard number of X by one. By termination of flips a flipping step only occurs finitely many times and the Picard number does not change in a flipping step. Therefore either a minimal model is reached or a fibering contraction is reached or the Picard number is one at some stage and no more flipping steps occur. If the Picard number is one then X (the current variety of interest) is either a minimal model and we are done. Otherwise the final contraction must be a Mori fiber space. Given a variety $X \in \mathfrak{C}$ we will say that an MMP for X is a series of dominant birational mappings

$$X = X_1 \overset{f_1}{\dashrightarrow} X_2 \overset{f_2}{\dashrightarrow} \dots \overset{f_{r-1}}{\dashrightarrow} X_r \tag{2.10}$$

where X_r is either a minimal model of a Mori fiber space and each f_i is either a divisorial contraction or a flipping morphism $X \dashrightarrow X^+$. We now describe the category $\mathfrak{C}$.

Terminal singularities and the category $\mathfrak{C}$.

When carrying out the minimal model program in dimension ≥ 3 we must allow singular varieties to enter the picture. For surfaces this problem can be worked around. This is strictly necessary as there are examples of smooth 3-folds which do not have a smooth minimal model. From the perspective of the algorithm outlined earlier, when one obtains a divisorial contraction $\phi\colon X \to Y$ the target variety may no longer be smooth, and in fact it may be impossible to find any divisorial contraction with smooth target. More precisely we have the following phenomenon.

Example 5 (Singularities are necessary: Obstacle 3-1-2 and example 3-1-3 of [36])**.** There is a smooth projective three-fold X defined over $\overline{\mathbb{Q}}$ such that if $(K_X \cdot C) < 0$ then any contraction that contracts C results in a singular three-fold. In fact one may construct X as follows. Let A be an abelian three-fold and let $Y = A/\pm 1$ be the quotient by the involution on A. Then Y is a singular surface with precisely 2^6 singularities. Let X be the blow up of Y at the singularities. Then X is a smooth threefold. Furthermore, the K_X negative curves are precisely those that lie in some exceptional divisor E_i for the blow up $X \to Y$ and each of these blow down to a singular three-fold.

Thus to carry out the minimal model program we must allow certain *mild* singularities to enter the picture.

Definition 2.3.3 (Terminal singularities, 4-1-1 in [36])**.** *Let X be a normal projective variety. We say that X has at worst terminal singularities if the following are true.*

1. *K_X is $\mathbb{Q}$-Cartier.*
2. *There a projective birational morphism*

$$\mu\colon V \to X$$

   with V smooth such that

$$K_V = \mu^* K_X + \sum_{i=1}^{r} a_i E_i \tag{2.11}$$

   where the E_i are the irreducible components of the exceptional locus of μ and the $a_i > 0$.

The following result tells us why this definition is useful.

Theorem 2.3.4 (Terminal singularities are necessary to run the MMP, 4-1-3 in [36])**.** *$\mathbb{Q}$-factorial and terminal singularities may be characterized as the singularities that appear when running the minimal model program with a starting variety that is smooth. In other words the category $\mathfrak{C}$ of normal projective varieties with at worst $\mathbb{Q}$-factorial and terminal* *singularities is the smallest category containing smooth projective varieties for which the MMP works. Furthermore, every terminal singularity appears in a run of the MMP starting with a smooth projective variety.*

From now on unless otherwise mentioned we work in the category $\mathfrak{C}$ of $\mathbb{Q}$-factorial projective varieties with at worst terminal singularities, as it is the natural place to work with the MMP with respect to K_X negative contractions. There are other more general ways to run the MMP, such as in a log category. In such a setting one modifies the type of singularities appropriately.

Flips and the contraction theorem

We start with a precise definition of an extremal contraction.

Definition 2.3.5 (K_X-negative extremal contractions, 8-1-1 in [36])**.** *Let X be a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Then $\phi\colon X \to Y$ is an extremal contraction with respect to K_X if:*

1. *ϕ is not an isomorphism.*
2. *For a curve $C \subseteq X$ we have that*

$$\phi(C) = \text{pt} \Rightarrow (K_X \cdot C) < 0. \tag{2.12}$$

3. *All curves contracted by ϕ are numerically proportional. That is*

$$\phi(D) = \phi(C) = \text{pt} \Rightarrow D \equiv_{\text{num}} qC \text{ for some } q \in \mathbb{Q}_{>0}. \tag{2.13}$$

4. *ϕ has connected fibers, is surjective and Y is normal and projective.*

We have the following characterization.

Theorem 2.3.6 (The contraction theorem, 8-1-3 in [36])**.** *Let X be a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let R_l be an extremal ray of $\overline{\text{NE}}(X)_{K_X<0}$ where $R_l = \overline{\text{NE}}(X)_{\mathbb{R}} \cap L^{\perp}$ where $L \in \text{Nef}(X)$. Then there is a contraction morphism*

$$\text{cont}_{R_l} = \phi_{R_l}\colon X \to Y \tag{2.14}$$

with the following properties.

1. cont_{R_l} *is not an isomorphism and a surjection.*
2. *For any curve* $C \subseteq X$ *then we have*

$$\mathrm{cont}_{R_l}(C) = \mathrm{pt} \Rightarrow (K_X \cdot C) < 0.$$

3. *For any curve* $C \subseteq X$ *we have*

$$C \in R_l \iff \mathrm{cont}_{R_l}(C) = \mathrm{pt}. \tag{2.15}$$

4. Y *is normal and projective and* cont_{R_l} *is surjective with connected fibers.*
5. $\mathrm{cont}_{R_l}\mathcal{O}_X = \mathcal{O}_Y$.
6. $L = \mathrm{cont}^*_{R_l} H$ *where* H *is an ample Cartier divisor on* Y.
7. *For any Cartier divisor* D *on* X *we have that*

$$D = \mathrm{cont}^*_{R_l} W \iff (D \cdot C) = 0 \forall\ C \in R_l.$$

   We have exact sequences

$$0 \to N^1(Y) \to N^1(X) \to N^1(X/Y) \to 0 \tag{2.16}$$
$$0 \to N_1(X/Y) \to N_1(X) \to N_1(Y) \to 0 \tag{2.17}$$

8. *We have that* $\rho(X) = \rho(Y) - 1$.

Properties (1),(2),(3) characterize the extremal contractions with respect to K_X *in the sense that every extremal contraction with respect to* K_X *arises in this way.*

We have the following results which give us the specific properties of contractions.

Proposition 2.3.7 (Properties of divisorial contractions)**.** *Let* X *be a normal* $\mathbb{Q}$*-factorial variety with at worst terminal singularities. Let* R *be an extremal ray of* $\overline{\mathrm{NE}}(X)_{K_X<0}$ *where* $R = \overline{\mathrm{NE}}(X)_{\mathbb{R}} \cap L^{\perp}$ *where* $L \in \mathrm{Nef}(X)$. *Let* $\phi_R \colon X \to Y$ *be the associated extremal contraction of 2.3.6. Suppose that the exceptional locus of* ϕ *has codimension 1. Then we call* ϕ *a divisorial contraction. Furthermore* ϕ *has the following properties:*

1. *The exceptional locus* E *of* ϕ *is a single irreducible divisor.*

2. *Y is a normal projective variety with at worst $\mathbb{Q}$-factorial terminal singularities.*

3. *$\rho(Y) = \rho(X) - 1$.*

We now turn to the flip.

Definition 2.3.8 (Defining flips, 9-1-1 in [36])**.** *Let $\phi\colon X \to Y$ be a small contraction where X is normal $\mathbb{Q}$-factorial with at worst terminal singularities. Then ϕ satisfies the following properties.*

1. *Y is normal and projective.*

2. *$-K_X$ is ϕ ample.*

3. *All the curves in the fibers of ϕ are numerically proportional.*

A morphism $\phi^+\colon X^+ \to Y$ is called a flip (if it exists) if X^+ is normal with at worst $\mathbb{Q}$-factorial terminal singularities and has the following properties.

1. *ϕ^+ is a small contraction.*

2. *K_{X^+} is ϕ^+ ample.*

3. *All the curves in the fibers of ϕ^+ are numerically proportional.*

As stated the flip is not guaranteed to exist. However, they are known to exist in dimension 3 and 4 due to Mori and Shokurov. The existence of flips in all dimensions was decided by Birkar, Cascini, Hacon, and McKernan in 2010.

Proposition 2.3.9 (Properties of flips, 9-1-2 in [36])**.** *Let $\phi\colon X \to Y$ be a small contraction where X is normal $\mathbb{Q}$-factorial with at worst terminal singularities. Then a flip exists if and only if the local canonical ring*

$$\bigoplus_{d\geq 0} \phi_* \mathcal{O}_X(dK_X) \tag{2.18}$$

is finitely generated as an $\mathcal{O}_Y$ algebra. In this case the flip X^+ is unique and given by

$$X^+ = \mathit{Proj} \bigoplus_{d\geq 0} \phi_* \mathcal{O}_X(dK_X) \to Y. \tag{2.19}$$

The proposition tells us that a flip is uniquely determined by the small contraction ϕ.

We also have the following which describes the properties of the target of a small contraction.

Proposition 2.3.10 (Target of a small contraction, 8-2-2 in [36]). *Let $\phi\colon X \to Y$ be a small contraction where X is normal $\mathbb{Q}$-factorial with at worst terminal singularities. Then K_Y is not $\mathbb{Q}$ Cartier. In particular Y is not $\mathbb{Q}$-Cartier and does not have terminal singularities.*

We finally have the following proposition involving the properties of a fibering type contraction.

Proposition 2.3.11 (The target of a fibering type contraction is $\mathbb{Q}$-factorial, 8-2-3 in [36]). *Let $\phi\colon X \to Y$ be a fibering contraction where X is normal $\mathbb{Q}$-factorial with at worst terminal singularities. Then Y is normal and $\mathbb{Q}$-factorial.*

We finally end this section with some information about the geometry of the exceptional locus of a contraction.

Theorem 2.3.12 (The exceptional locus is uniruled, 10-3-3 in [36]). *Let $\phi\colon X \to Y$ be an extremal contraction where X is normal $\mathbb{Q}$-factorial with at worst terminal singularities. Then every component of $\operatorname{exc}(\phi)$ is uniruled.*

Remark 2.3.13 (Singularity assumptions). While there are contraction theorems for varieties with other types of singularities, we will stick to the case of terminal singularities. Whenever we are dealing with an extremal contraction will will assume that the source variety in question is $\mathbb{Q}$-factorial and has at worst terminal singularities unless the variety is a projective toric variety. In the toric setting one has extra combinatorial tools that can be used. See the following section for what we will use.

We close this with a result for toric varieties that we need.

2.3.1 The toric minimal model program

The minimal model program for toric varieties can be entirely completed. In other words all statements in 2.3 can be completed via combinatorial arguments, and the termination of flips may be established. See [36, Chapter 14] or [12, 15.5]. Because of the access to combinatorial arguments, or perhaps due tot he rigid structure of toric varieties, we have that much more powerful statements are available when dealing with toric varieties. As an example we have the following powerful result.

Theorem 2.3.14 (Toric contraction theorem, 15.4.1 in [12])**.** *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety. Let $R \in \overline{\mathrm{NE}}(X_\Sigma)_\mathbb{R}$ be an extremal ray. Then R is contractible by a toric morphism. In other words there is a contraction morphism $\phi_R \colon X_\Sigma \to Y_{\Sigma_0}$. Moreover, the construction has the following features.*

1. *ϕ_R is a surjective morphism with connected fibers and Y_{Σ_0} is a normal projective toric variety. It is $\mathbb{Q}$-factorial when ϕ_R is a divisorial contraction or a fibering contraction.*
2. *If $\phi_R(C)$ is a point then $C \in R$.*
3. *We may write $R = D^\perp \cap \overline{\mathrm{NE}}(X_\Sigma)_\mathbb{R}$ where D lies in the relative interior of some facet F of $\mathrm{Nef}(X)_\mathbb{R}$. Then $(D' \cdot C) = 0 \iff D' \in F$.*

Proof. The first two features follow from detailed combinatorial arguments that would take us to far afield. See [12, Chapter 15] for the details. For the last part notice $\mathrm{Nef}(X)_\mathbb{R}$ is finitely generated and rational for any projective toric variety. In particular we have that if $D_0 \in F$ then $(D_0 \cdot C) = 0$. This is because if F has rays $D_1, ..., D_s$ then $D = \sum_{i=1}^s a_i D_i$ (this is what it means for D to be in the relative interior of the facet!) where $a_i > 0$. Therefore $(D_i \cdot C) = 0$ for all i as each D_i is nef and $(D \cdot C) = 0$. Since the D_i generate F we have that all elements of F intersect C trivially. Finally note that by the duality of cones that $F = C^\perp \cap \mathrm{Nef}(X)_\mathbb{R}$, where C is any non-zero element in R. So $(D' \cdot C) = 0 \iff D' \in F$. $\square$

Chapter 3

Arithmetic preliminaries.

Here we will begin building up the arithmetic tools required to attack the Kawaguchi-Silverman conjecture. Our main references here will be [24] and [8]. First and foremost, we need a theory of heights. The theory of heights on projective algebraic variety is a method of turning geometry into arithmetic in the following manner. Given a line bundle $\mathcal{L}$ on an irreducible projective variety X defined over a number field K one constructs a *height function*

$$h_{\mathcal{L}} \colon X(\overline{K}) \to \mathbb{R}.$$

This construction is called *Weil's height machine*. It's main feature is that the geometric properties of the line bundle $\mathcal{L}$ are reflected in the arithmetic properties of the function $h_{\mathcal{L}}$. Roughly speaking, $h_{\mathcal{L}}$ measures the arithmetic complexity of a point $P \in X(\overline{K})$ with respect to the line bundle $\mathcal{L}$. The prototypical example of this is as follows. Take $X = \mathbb{P}^1_{\mathbb{Q}}$ and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}(1)$. A point $P \in \mathbb{P}^1(\mathbb{Q})$ can be written $P = [a : b]$ where $a, b \in \mathbb{Z}$ are coprime. Then

$$h_{\mathcal{O}_{\mathbb{P}^1}(1)}([a : b]) = \log \max\{\mid a \mid, \mid b \mid\}.$$

In other words, the arithmetic complexity of the point P is measured by the size of the integers required to represent P in integral homogeneous coordinates. The full theory of heights generalizes this example in two ways: First by allowing P to lie in some number field K, and second by allowing P to lie in some projective variety X rather then in $\mathbb{P}^1$. In 3.1 we introduce this theory by first introducing absolute values and heights in projective spaces in 3.1.1 and then introducing Weil's height machine in 3.1.2.

Ultimately, we are interested in the study of endomorphisms of projective varieties. Abelian varieties provide a second large class of varieties that possess many surjective endomorphisms. An Abelian variety A has a commutative group structure, so given any point P on A and an integer n one can consider $n \cdot P = \sum_{i=1}^{n} P$. As the structure of the surjective endomorphisms on an Abelian variety has been well studied, they provide another testing ground for questions regarding the dynamics of surjective endomorphisms. In 3.2 we introduce this theory.

3.1 Heights

Arithmetic dynamics lies in the intersection of algebraic geometry and number theory. Therefore, to work efficiently, one must have methods that turns geometry into arithmetic and vice versa. The appropriate theory is the theory of heights. Our plan of attack is as follows:

1. Define a height function $h_{\mathbb{P}^n}$ on $\mathbb{P}^n(\overline{\mathbb{Q}})$. The construction here is done in 3.1.1 and requires the theory of places of a number field.

2. Let X be a projective variety and $\mathcal{L}$ a very ample line bundle on X. Then there is a closed embedding $\phi_{\mathcal{L}} \colon X \to \mathbb{P}^n$ such that $\phi_{\mathcal{L}}^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}$. We define

$$h_{\mathcal{L}} = h_{\mathbb{P}^n} \circ \phi_{\mathcal{L}}.$$

   The construction here is more geometric, and uses basic facts about the theory of line bundles on a projective variety X and the construction of $h_{\mathbb{P}^n}$.

3. Finally we construct all other functions by taking appropriate integral linear combinations of the above height functions. This is carried out in 3.1.2.

One may think of the above construction as follows. We define certain natural height functions in (1) and (2) above, and then extend by all possible integral linear combinations. It turns out, that all possible integral linear combinations of the natural height functions arise in a geometric manner. There are other constructions of heights on projective varieties different from the one outlined above, but this seems to be the most convenient for our purposes. Our references here will be [24] and [8].

3.1.1 Absolute Values and Heights

Here we let K be a number field, that is a finite field extension of $\mathbb{Q}$.

Definition 3.1.1. *An absolute value on K is a function $|\cdot|: K \to \mathbb{R}_{\geq 0}$ satisfying the following axioms.*

1. $|x| = 0 \iff x = 0$.
2. $|xy| = |x| \cdot |y|$.
3. $|x+y| \leq |x| + |y|$.

An absolute value on K introduces a metric space structure on K using the distance function $d(x, y) = |x - y|$. A natural way to keep track of absolute values is thus through their associated topology.

Definition 3.1.2. *Two absolute values on K are said to be equivalent if they define the same topology. An equivalence class of absolute values on K is called a place of K. Given a place ν we will denote $|\cdot|_\nu$ for the associated absolute value and define $\nu(x) = \log|x|_\nu$.*

Proposition 3.1.3 (1.2.3 [8])**.** *Two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are equivalent if and only if there is a positive real number s such that $|\cdot|_1^s = |\cdot|_2$.*

Definition 3.1.4. *Fix an absolute value $|\cdot|$ on K We say $|\cdot|$ is non-Archimedean if*

$$|x+y| \leq \max\{|x|, |y|\}.$$

If $|\cdot|$ is not non-Archimedean we call $|\cdot|$ Archimedean. We say that a place ν is non-Archimedean if any of the absolute values in the associated equivalence class is non-Archimedean and call it Archimedean otherwise.

To proceed we need to make certain normalization choices. This can be done using a theorem of Ostrowski which allows us to determine a canonical choice of places for $\mathbb{Q}$.

Definition 3.1.5. *Fix a prime number $p \in \mathbb{Z}$. Given an integer a we can uniquely write $a = p^\alpha b$ where b is coprime to p. We define $\mathrm{ord}_p(a) = \alpha$ and define $\mathrm{ord}_p(\frac{a}{b}) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$. We define an absolute value $|\cdot|_p$ on $\mathbb{Q}$ by the formula*

$$\left|\frac{a}{b}\right|_p = p^{-\mathrm{ord}_p(\frac{a}{b})}.$$

Equivalently, any $\frac{a}{b} \in \mathbb{Q}$ *can be uniquely written in the form*

$$\frac{a}{b} = p^{\alpha}\frac{c}{d}$$

where c, d *are co-prime to* p*, We define*

$$\mid \frac{a}{b} \mid_p = \mid p^{\alpha}\frac{c}{d} \mid_p = p^{-\alpha}.$$

Definition 3.1.6. *For each number field* K *we let* M_K *be the set of absolute values of* K *consisting all absolute values of* K *such that* $|\cdot|_\nu$ *restricts to either the Euclidean absolute value on* $\mathbb{Q}$ *or a* p*-adic absolute value in the sense of 3.1.5. We call* M_K *the set of standard absolute values on* K*. When needed we will let* M_K^0 *be the sub-set of* M_K *consisting of those absolute values* $|\cdot|_\nu$ *that restricts to a* p*-adic absolute value. We call* M_K^0 *the finite absolute values. We let* M_K^∞ *the set of absolute values which restrict to the Euclidean absolute value on* $\mathbb{Q}$*. We call these the infinite absolute values.*

We now have enough to define heights on projective space.

Definition 3.1.7 (Relative Height on Projective Space)**.** *Fix a number field* K*. Let* $x = (x_0 : ... : x_n) \in \mathbb{P}^n(K)$*. We define the multiplicative height*

$$H_K(x) = \prod_{\nu \in M_K} \max_{0 \le i \le n} \{|x_i|_\nu\}^{[K_\nu:\mathbb{Q}_\nu]}.$$

We also define the logarithmic height

$$h_K(x) = \log H_K(x).$$

Proposition 3.1.8 (B.2.1)**.** *[24]*

1. *The height* H_K *is well defined and* $H_K(P) \ge 1$ *for all* $P \in \mathbb{P}^n(K)$*.*
2. *Let* $L \mid K$ *be a finite field extension. Let* $P \in K$*. Then*

$$H_L(P) = H_K(P)^{[L:K]}$$

Definition 3.1.9 (Absolute Heights)**.** *The absolute height on* $\mathbb{P}^n(\overline{\mathbb{Q}})$ *is the function*

$$H(P) = H_K(P)^{\frac{1}{[K:\mathbb{Q}]}}$$

where K *is any number field with* $P \in K$*. We define* $h(P) = \log H(P)$ *and call this the absolute logarithmic height. By 3.1.8 this is a well defined function.*

We think of these height functions as measuring the arithmetic complexity of a point $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$. We now illustrate this for $\mathbb{P}^n(\mathbb{Q})$.

Proposition 3.1.10. *Let $P \in \mathbb{P}^n(\mathbb{Q})$ with $P = [x_0 : ... : x_n]$ with the coordinates zero or coprime integers. Then*

$$H(P) = \max_{i=0}^{n}\{\mid x_i \mid\}.$$

Proof. We have that $\mid x_i \mid_p \leq 1$ for any finite prime p. Furthermore, for each prime p there is some x_i such that p does not divide x_i as the x_i are pairwise coprime. It follows that $\max_{i=0}^{n} \mid x_i \mid_p = 1$. Thus

$$H(P) = \max_{i=0}^{n}\{\mid x_i \mid\} \cdot \prod_{p \text{ prime}} \max_{i=0}^{n}\{\mid x_i \mid_p\} = \max_{i=0}^{n}\{\mid x_i \mid\} \cdot \prod_{p \text{ prime}} 1 = \max_{i=0}^{n}\{\mid x_i \mid\}.$$

□

Thus a point in $\mathbb{P}^n(\mathbb{Q})$ has large height when one of the coordinates has a large size. We are measuring the arithmetic complexity of P by the size of the homogenous coordinates required in a coprime integral representation. We see immediate as well that the set

$$\{P \in \mathbb{P}^n(\mathbb{Q}) : H(P) \leq B\}$$

is a finite set. This can be strengthened and is a key property of heights.

Theorem 3.1.11 ([24] B.2.3. The Northcott Property)**.** *Fix real numbers $B, D \geq 0$. The set*

$$\{P \in \mathbb{P}^n(\overline{\mathbb{Q}})\colon H(P) \leq B, [\mathbb{Q}(P) : \mathbb{Q}] \leq D\}$$

is finite.

3.1.2 Weil Heights

In this section we describe how to construct height functions on projective varieties. Our functions will only be unique up to a bounded function. Therefore, we will be using the big O notation.

The goal here is to take a Cartier divisor D (equivalently a line bundle $\mathcal{L}$) and define a *height function* $h_D\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ (respectively $h_{\mathcal{L}}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$). Our strategy will be as follows: First on $\mathbb{P}^n$ we will define $h_{\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)} = h$ where h is the absolute height on projective space. Then given a closed embedding $\phi\colon X \to \mathbb{P}^n$ we obtain a very ample

Cartier divisor D on X by the pull back $D = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. The height function associated to D is defined to be $h_{X,D} = h_{\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)} \circ \phi = h \circ \phi$. Finally, given any Cartier divisor D we have that $D = H_1 - H_2$ where H_1, H_2 are very ample. Then we define

$$h_{X,D} = h_{X,H_1} - h_{X,H_2}.$$

In the above construction we have made various choice. For example when D is very ample there is not a unique morphism $f^*\mathcal{O}_{\mathbb{P}^n}(1) = D$. Furthermore, the decomposition $D = H_1 - H_2$ is not unique. However, different choices lead to functions which differ by a bounded function, hence these constructions are well defined up to $O(1)$. We now formalize these ideas.

Theorem 3.1.12 ([24] B.3.2. Weil's Height Machine)**.** *Let X be a projective variety defined over a number field K and let $\mathrm{CaDiv}(X)$ be the set of Cartier divisors on X. There is a function*

$$h\colon \mathrm{CaDiv}(X) \to \{\text{functions } X(\overline{K}) \to \mathbb{R}\}$$

with the following properties.

1. *Let $H \subseteq \mathbb{P}^n$ be the Cartier divisor associated to a hyperplane. Then $h_{\mathbb{P}^n,H}(P) = h(P) + O(1)$ where $h(P)$ is the absolute logarithmic height on projective space.*

2. *Let $\phi\colon X \to Y$ be a morphism of varieties defined over K. Let $D \in \mathrm{CaDiv}(Y)$. Then*

$$h_{X,\phi^*D}(P) = h_{Y,D}(\phi(P)) + O(1).$$

3. *Let $D, E \in \mathrm{CaDiv}(X)$. Then*

$$h_{X,D+E} = h_{X,D} + h_{X,E} + O(1).$$

4. *Let $D, E \in \mathrm{CaDiv}(X)$ with E linearly equivalent to D. Then*

$$h_{X,D} = h_{D,E} + O(1).$$

5. *Let $D \in \mathrm{CaDiv}(X)$ be effective. Let B be the associated base locus of D. Then*

$$h_{X,D}(P) \geq O(1)$$

for all $P \in (X - B)(\overline{K})$.

6. *Let $D \in \mathrm{CaDiv}(X)$ be ample. Then for every finite extension $L \mid K$ and all positive constants B we have that*

$$\{P \in X(L) : h_{X,D}(P) \leq B\}$$

is a finite set.

7. *The height functions $h_{X,D}$ are determined up to $O(1)$ by the properties (1),(2) (3) and for embeddings $X \hookrightarrow \mathbb{P}^n$.*

Proof. We only sketch the construction. Let D be a very ample divisor on X. Then by definition there is a closed embedding $\phi\colon X \to \mathbb{P}^n$ with $\phi^* H = D$ where H is a hyperplane in $\mathbb{P}^n$. We define $h_{X,D} = h \circ \phi$. Now let D be arbitrary. Then we can find very ample divisors D_1, D_2 with $D = \frac{1}{m}(D_1 - D_2)$ by 2.2.20. We define

$$h_{X,D} = \frac{1}{m}(h_{X,D_1} - h_{X,D_2}).$$

One then verifies that these functions satisfy the requirements of the theorem. $\square$

We think of the height machine in the following manner. Given a closed embedding the $\phi\colon X \to \mathbb{P}^n$ with $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) = D$ the height $h_{X,D}$ measures the arithmetic complexity of a point P *with respect to the embedding* ϕ. We also note that we may extend the height machine to $\mathbb{R}$-cartier divisors in the obvious way.

3.2 Abelian Varieties

Here we follow [24] for the basics on abelian varieties defined over a field k is characteristic zero.

Definition 3.2.1 (Definition of an Abelian Variety)**.** *Let k be a field. An abelian variety over k is a projective algebraic group defined over k.*

To get the theory started we have the following.

Lemma 3.2.2 (Rigidity Lemma: A.7.11 in [24])**.** *Let X be an irreducible reduced projective variety defined over k, and let Y, Z be arbitrary varieties. Suppose that*

$$f\colon X \times Y \to Z$$

is a morphism defined over k and that there is a k-point y_0 such that $f\colon X \times y_0 \to Z$ is constant. Then if y is any other point we have that $f\colon X \times y \to Z$ is constant. In particular if there is a point $x \in X$ such that in addition we have that $f(x \times Y) \to Z$ is constant then f is constant.

Proof. Let U be an affine open neighborhood of $f(X \times y_0) = z_0$. We have that $V = f^{-1}(Z - U)$ is closed in $X \times Y$. Since X is projective it is proper and the base change of $X \to \operatorname{Spec} k$ along $Y \to \operatorname{Spec} k$ is $p_2\colon X \times Y \to Y$. The properness of $X \times Y \to Y$ tells us that $p_2(V) = W$ is closed in Y. Notice that $y_0 \notin W$ as otherwise for some $x \in X$ we have $(x, y_0) \in V$ or that $f(x, y_0) \in X - U$ which is a contradiction by the definition of V. Now let $y \notin W$. Then $f(X \times y) \subseteq U$. In other words, we have a morphism $f\colon X \times y \to U$ where U is affine. This means $f(X \times y)$ is constant. Since X is irreducible we have that $f(X \times y) = w_0$ for some fixed w_0. Then as W is dense in Y we have that $f\colon X \times (Y - W) \to Z$ is constant. Thus f is constant as it is constant on a dense open set. The final part is clear. □

We have the following nice corollary.

Corollary 3.2.2.1. *Let A be an abelian variety and let G be an algebraic group. Let $f\colon A \to G$ be a morphism such that $f(e_A) = e_G$ where e_A is the identity element on A and e_B is the identity element on G. Then f is a group homomorphism.*

Proof. Let $g\colon A \times A \to G$ be defined as

$$g(x, y) = f(x \cdot_A y) \cdot_B f(y)^{-1} \cdot_G f(x)^{-1}.$$

Then $g(e_A \times A) = e_G$ and $g(A \times e_A) = e_G$. Thus by 3.2.2 we have that g is constant. Since $g(e_A, e_A) = e_G$ we have that $g(x, y) = e_G$. In other words, we have that

$$f(x \cdot_A y) \cdot_G f(y)^{-1} \cdot_G f(x)^{-1} = e_G$$

or that

$$f(x \cdot_A y) = f(x) \cdot_G f(y).$$

□

We can use this result to characterize morphisms between abelian varieties.

Corollary 3.2.2.2. *Let $f\colon A \to B$ be a morphism of abelian varieties defined over a field k. Then $f = t_c \circ \phi$ where $t_c\colon B \to B$ is translation morphism by $c = f(e_A) \in B$ and ϕ is a group homomorphism.*

Proof. Let $c = f(e_A)$. Let $f' = t_{c^{-1}} \circ f$. Then $f'(e_A) = e_B$ by construction. Thus $f' = \phi$ where ϕ is a group homomorphism. Then we have that $f = t_c \circ \phi$ as needed. □

From this we are able to justify the definition of an abelian variety.

Corollary 3.2.2.3 (Abelian varieties deserve their name)**.** *Let A be an abelian variety. Then the group law on A is commutative.*

Proof. Consider the morphism $i\colon A \to A$ that sends $x \mapsto x^{-1}$. Since $i(e_A) = e_A$ we have by 3.2.2.1 that i is a group homomorphism. But this means that A is abelian. □

This circle of ideas can be used to proof the following.

Proposition 3.2.3 (Rational Maps to an abelian variety: See 1.7.15 in [24])**.** *Let A be an abelian variety and X a smooth variety. Then any rational map $f\colon X \to A$ extends to a morphism.*

Given an abelian variety A, we wish to analyze the dynamics of a morphism $f\colon A \to A$. The prototypical example is given by multiplication by n $[n]\colon A \to A$. To analyze such a morphism one might pass to the $\mathbb{C}$ points of A and apply analytic results. We take a more algebraic approach following [24].

Theorem 3.2.4 (Theorem of the cube: (Mumford))**.** *Let X_1, X_2, X_3 be projective varieties and $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$. Let D be a divisor on $X_1 \times X_2 \times X_3$. Suppose that restriction of D to each of*

$$x_1 \times X_2 \times X_3, X_1 \times x_2 \times X_3, X_1 \times X_2 \times x_3$$

is linearly equivalent to 0. Then so is D.

With this result in hand, we have the following version for abelian varieties.

Theorem 3.2.5 (Theorem of the cube for abelian varieties. A.7.2.1 [24])**.** *Let A be an abelian variety. Let $I \subseteq \{1, 2, 3\}$. We define $s_I(x_1, x_2, x_3) = \sum_{i \in I} x_i$ on $A \times A \times A$. Define*

$$\mathrm{Cube}(D) = s_{123}^* D - (s_{12}^* D + s_{13}^* D + s_{23}^* D) + s_1^* D + s_2^* D + s_3^* D$$

Then $\mathrm{cube}(D) \equiv_{\mathrm{lin}} 0$.

Proof. We apply 3.2.4. Let $X_i = A$ and $x_i = 0$ where 0 is the zero element of A. Let $i\colon A \times A \to A \times A \times A$ be the morphism $i(x_1, x_2) = (x_1, x_2, 0)$. Therefore the restriction $\mathrm{Cube}(D) \mid A \times A \times 0$ is the same as

$$(s_{123} \circ i)^*D - ((s_{12} \circ i)^*D + (s_{13} \circ i)^*D + (s_{23} \circ i)^*D) + (s_1 \circ i)^*D + (s_2 \circ i)^*D + (s_3 \circ i)^*D.$$

However, we may compute all the terms in the above sequence. Namely we think of i as "deleting the index 3" to obtain the following.

$$\begin{gathered} s_{123} \circ i = s_{12} \circ i, \;\; s_{23} \circ i = s_2 \circ i \\ s_{13} \circ i = s_1 \circ i, \;\; s_3 \circ i = 0 \end{gathered}$$

Putting this back into the original equation gives

$$(s_{12} \circ i)^*D - (s_{12} \circ i)^*D - (s_1 \circ i)^*D - (s_2 \circ i)^*D) + (s_1 \circ i)^*D + (s_2 \circ i)^*D = 0.$$

A completely symmetric argument shows that the restriction of D to $A \times 0 \times A$ and $A \times A \times 0$ is also trivial. So by 3.2.4 we have that D is trivial as desired. □

The corollary we will use is the following.

Corollary 3.2.5.1. *Let V be any variety and A an abelian variety. and $f_1, f_2, f_3\colon V \to A$ morphisms. Let $D \in \mathrm{Pic}(A)$. For $I \subseteq \{1, 2, 3\}$ set $f_I = \sum_{i \in I} f_i$.*

Then

$$\mathrm{cube}(f_1, f_2, f_3)(D) = f_{123}^*D - (f_{12}^*D + f_{13}^*D + f_{23}^*D) + f_1^*D + f_2^*D + f_3^*D \equiv_{\mathrm{lin}} 0.$$

Proof. To compute the above pull back we consider $F\colon V \to A \times A \times A$ given by (f_1, f_2, f_3). Then $f_I = s_I \circ F$ as in the proof of 3.2.5. So by3.2.5 we see that $\mathrm{cube}(f_1, f_2, f_3)(D) \equiv_{\mathrm{lin}} F^*\mathrm{cube}(D) \equiv_{\mathrm{lin}} F^*0 = 0$ as needed. □

Our main application of this is the following.

Corollary 3.2.5.2 (Mumford's Formula)**.** *Let A be an abelian variety and $D \in Div(A)$. Let $n \in \mathbb{Z}$. Then*

$$[n]^*D \equiv_{\mathrm{lin}} \frac{n^2 + n}{2} D + \frac{n^2 - n}{2} [-1]^*D.$$

In particular if $[-1]^*D = D$ we have $[n]^*D = n^2 D$ and if $[-1]^*D = -D$ we have $[n]^*D = nD$.

Proof. We only sketch the proof and leaves the details to [24, A.7.2.5]. Apply 3.2.5.1 with $f_1 = [n], f_2 = [-1], f_3 = [1]$. Then we obtain after rearranging that

$$[n+1]^*D + [n-1]^*D - 2[n]^*D = D + [-1]^*D.$$

One may now proceed by induction using the fact we can verify by hand the formula for $n = 1, 0, -1$. Alternatively one can show that any function $f\colon \mathbb{Z} \to G$ where G is an abelian group with the property that $f(n+1)+f(n-1)-2f(n)$ is constant can be written as

$$f(n) = \frac{n^2+n}{2}f(1) + \frac{n^2-n}{2}f(-1) - (n^2-1)f(0).$$

See [24, A.7.2.5] for the details. In our case as $f(n) = [n]^*D$ we have $f(0) = 0$ we obtain the desired result. □

We can now compute the torsion of an abelian variety and the degree of the morphism $[n]$. To do this we introduce a crucial notion. That of an isogeny of algebraic groups.

Definition 3.2.6. *Let A, B be algebraic groups. An isogeny $f\colon A \to B$ is a surjective homomorphism with finite kernel.*

Theorem 3.2.7. *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$. Then $[n]$ is an isogeny of degree n^{2g}. Consequently $\ker[n] = A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ where $g = \dim A$.*

Proof. Let $m\colon A \times A \to A$ be the addition morphism. The is an induced morphism of tangent spaces

$$m_0\colon T_0(A) \times T_0(A) \to T_0(A)$$

which given by addition of vectors. As the multiplication by n mapping is defined in terms of m we have that the map on tangent spaces $[n]_0\colon T_0(A) \to T_0(A)$ is multiplication by n. Thus we have that $[n]_0$ is an isomorphism on tangent spaces and so $\dim([n]A) = \dim A$. It follows that $[n]$ is surjective as the image has the same dimension as the target. By 4.1.3 we have that $[n]$ is a finite surjective morphism. In particular the kernel is finite so $[n]$ is an isogeny. To compute the degree let D' be an ample divisor on A. Set $D = D' + [-1]^*D'$. Notice that $[-1]^*D = D$ so by 3.2.5.2 we have that $[n]^*D = n^2D$. We have that the top intersection product

$$([n]^*D)^g = (n^2D)^g = n^{2g}D^g.$$

However by 2.2.10 that

$$n^{2g}D^g = (([n]^*D)^g) = (\deg[n])D^g.$$

As D is ample we have that $D^g \neq 0$ and so $n^{2g} = \deg[n]$ as needed. Since we are working over $\bar{\mathbb{Q}}$ and $[n]$ induces an isomorphism of tangent spaces it is etale. In other words the fiber over 0 has precisely degree $[n]$ elements which is n^{2g}. Therefore $\#A[n] = n^{2g}$. To compute the group structure we note that $A[n]$ is a finite abelian group of order n^{2g}. Now let $m \mid n$. Then $(A[n])[m] = A[m]$ and so $\#(A[n])[m] = (\mathbb{Z}/m\mathbb{Z})^{2g}$. An argument from elementary group theory shows that such a group is cyclic, completing the argument.

□

Definition 3.2.8. *Let $c \in \operatorname{Pic}(A)$. Consider the map $\Phi_c \colon A \to \operatorname{Pic}(A)$ defined by $\phi_c(a) = t_a^*(c) - c$. If $D \in \operatorname{CaDiv}(A)$ we let ϕ_D be the morphism Φ_c where c is the class of D in $\operatorname{Pic}(A)$. We let $K(c)$ or $K(D)$ be the kernel of the associated morphism.*

Theorem 3.2.9 (Theorem of the square)**.** *Let A be an abelian variety and let $a, b \in A$. Let t_a be the translation by a mapping. Then*

$$t_{a+b}^*(D) + D \equiv_{\text{lin}} t_a^*(D) + t_b^*(D).$$

So $\Phi_c \colon A \to \operatorname{Pic}(A)$ is a homomorphism.

Proof. Apply 3.2.5.1 to $f = \mathrm{Id}, g = a, h = b$ where g, h are the constant maps. □

Theorem 3.2.10 (Ampleness criterion: A.7.2.10 in [24])**.** *Let D be an effective divisor on an abelian variety A. Then $\mid 2D \mid$ is base point free. Furthermore, the following are equivalent.*

1. *D is ample.*
2. *The group $K(D)$ is finite.*
3. *The stabilizer $G(D) = \{a \in A \mid t_a^*(D) = D\}$ is finite.*
4. *The morphism $\psi_{2D} \colon A \to \mathbb{P}H^0(A, 2D)$ is finite.*

Proof. By the theorem of the square we have that $2D \equiv_{\text{lin}} t_x^*(D) + t_{-x}^*(D)$ for any $x \in A$. Pick $y \in A$ with $t_a^*(D)$ being free at y. Then $2D$ is free at y by the above computation. The fact that (4) ⇒ (1) is a standard fact about ample divisors and (2) ⇒ (3) by definition. We sketch the idea of (4) ⇒ (2). Let $b \in K(D)^0$ the connected component of the identity in $K(D)$. Then if $\psi_{2d} \colon A \to \mathbb{P}H^0(A.2D)$ is the morphism induced by $2D$ we have that $\psi_{2D} \circ t_b$

differs from ψ_{2D} by a linear automorphism of $\mathbb{P}H^0(A.2D)$. This is because $t_b^*(2D) \equiv_{\text{lin}} 2D$ as $b \in K(D)$. Thus we obtain a morphism

$$\Theta\colon K(D)^0 \to \mathrm{PGL}(\mathbb{P}H^0(A.2D)).$$

However, $K(D)^0$ is a closed subgroup of a projective group and so is projective. On the other hand $\mathrm{PGL}(\mathbb{P}H^0(A, 2D))$ is affine. This forces the morphism to be constant. It follows that ψ_{2D} is constant on $K(D)^0$. Because we have assumed that ψ_{2D} is finite, so is $K(D)^0$. Since $K(D)$ is composed of a finite number of translates of the identity component, $K(D)$ is finite as needed. We now turn to (1) $\Rightarrow$ (3). Take D ample and $a \in A$ with $a \notin D$. Consider $a + G(D) = V$. I claim that $V \cap D = \emptyset$. Towards a contradiction set $x \in V \cap D$. Then $x = a+g$ for some $g \in G(D)$. $a = x-g \in D-g$. Since $t_{-g}^* * (D) = D$ as $g \in G(D)$ we have that $D - g = D$ and so $a \in D$ which is a contradiction. So $V \cap D = \emptyset$. However this means that $(D \cdot V) = 0$. Since D is ample it must be the case that $\dim V = 0$. So $G(D)$ is finite as needed. So we have shown (1) $\iff$ (4) and (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2) $\Rightarrow$ (3). It remains to check (3) $\Rightarrow$ (2) which we omit. ☐

These results are already interesting from a dynamical perspective.

Definition 3.2.11 (Pic$^0(A)$ for an abelian varieties.)**.** *We define* $\mathrm{Pic}^0(A) = \{c \in \mathrm{Pic}(A) : t_a^*(c) = c \forall\ a \in A\}$.

The key part of this result is (3).

We now give a small application of our intersection theory work on cones.

Proposition 3.2.12 (Classification of Pic$^0(A)$. See a.7.3.2 in [24].)**.** *Let* $c \in \mathrm{Pic}(A)$ *where* A *is an abelian variety. Then* $c \in \mathrm{Pic}^0(A) \iff [-1]^*c = -c$.

Proof. By 3.2.28 we have that $\mathrm{Pic}^0(A)$ is precisely the collection of numerically trivial divisors. Suppose that $[-1]^*c = -c$. Then we have that

$$[2]^*c = 2c$$

by 3.2.5.2. On the other hand, choosing D to be a symmetric and ample divisor we have that

$$[2]^*D = 4D$$

by 3.2.5.2. Consider the action of $[2]^*$ on $N^1(A)_{\mathbb{R}}$. We have just shown that $[2]^*$ has an ample eigenvector D. So by 2.1.12 applied to the ample cone in $N^1(A)_{\mathbb{R}}$ we have that $[2]^*$ has eigenvalues all of the same modulus. So $c \equiv_{\text{num}} 0$ as c is an eigenvalue with a smaller modulus. We refer to [24] for the reverse inclusion.

☐

The following will be useful to relate numerical equivalence to other forms of equivalence.

Theorem 3.2.13 (A.7.3.1 in [24])**.** *Let A be an abelian variety and $c \in \mathrm{Pic}(A)$.*

1. *The homomorphism $\Phi_c \colon A \to \mathrm{Pic}(A)$ has image in $\mathrm{Pic}^0(A)$.*
2. *If $nc \in \mathrm{Pic}^0(A)$ for some $n > 0$ then $c \in \mathrm{Pic}^0(A)$.*
3. *If c is ample then Φ_c is surjective with finite kernel.*

In particular, we will later define a relation on A as follows. We say that $\mathcal{L} \in \mathrm{Pic}(A)$ is algebraically equivalent to $\mathcal{O}_X$ if $\mathcal{L}^{\otimes m} \in \mathrm{Pic}^0(A)$ for some $m > 0$. This forms a subgroup of $\mathrm{Pic}(A)$. By the above this is precisely $\mathrm{Pic}^0(A)$. On the other hand, this relation can be shown to be equivalent to being numerically equivalent to 0. In other words, $\mathrm{Pic}^0(A)$ will turn out to be precisely the line bundles numerically equivalent to $\mathcal{O}_X$.

Definition 3.2.14 (The dual abelian variety)**.** *Let $A, \hat{A}$ be abelian varieties. We say that $\hat{A}$ is the dual abelian variety if there is a line bundle $\mathcal{P}$ on $A \times \hat{A}$ with the following properties. For any $a \in A$ let $i_a \colon \hat{A} \to A \times \hat{A}$ be the map $i_a(x) = (a, x)$. Similarly let define $i_{\hat{a}} \colon A \to A \times \hat{A}$.*

1. *For any $\hat{a} \in \hat{A}$ the morphism*
$$\hat{A} \to \mathrm{Pic}^0(A),\ \hat{a} \mapsto i_{\hat{a}}^*(\mathcal{P})$$
is an isomorphism.
2. *For any $a \in A$ the morphism*
$$a \to \mathrm{Pic}^0(\hat{A}),\ a \mapsto i_a^*(\mathcal{P})$$
is an isomorphism.

We call $\mathcal{P}$ the Poincare class.

Theorem 3.2.15 (Existence of the dual abelian variety. See A.7.3.4 in [24].)**.** *The dual abelian variety $\hat{A}$ exists and together with $\mathcal{P}$ it is unique up to isomorphism. Furthermore $\mathcal{P}$ is even.*

In other words, the Poincare class gives an isomorphism between A and $\mathrm{Pic}^0(A)$. Thus if c is an ample class. Then by 3.2.13 we have that $\Phi_c\colon A \to \mathrm{Pic}^0(A)$ is surjective with finite kernel, in other words Φ_c is an isogeny. These types of isogenies are special.

Definition 3.2.16 (Polarization)**.** *Let A be an abelian variety. An isogeny $\Phi_c\colon A \to \hat{A}$ induced by an ample divisor is called a polarization. If Φ_c is an isomorphism we say it is a principle polarization. If A admits a principal polarization then we say that A is principally polarized.*

This definition will be useful for us once we begin our discussion of the endomorphism group of an abelian variety.

3.2.1 Picard Schemes and the Albanese variety

Now that we have touched upon abelian varieties, we discuss the Albanese variety, the Picard scheme, and the connection with numerical equivalence. Given a normal integral projective variety X we would like to construct an abelian variety $\mathrm{Alb}(X)$ and a morphism $\alpha\colon X \to \mathrm{Alb}(X)$ which is universal for maps from X to an abelian variety. In other words, if $g\colon X \to A$ is a morphism to an abelian variety A from X then there is a unique morphism $\tilde{g}\colon \mathrm{Alb}(X) \to A$ filling in the following diagram.

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, "g"] & \\ \mathrm{Alb}(X) \arrow[r, "\exists!\ \tilde{g}"'] & A \end{tikzcd}$$

Moreover we should have that $\dim \mathrm{Alb}(X) = h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$ and $\mathrm{Alb}(X)$ is *dual* to $\mathrm{Pic}^0(X)$ where $\mathrm{Pic}^0(X)$ is the identity component of $\mathrm{Pic}(X)$. In fact, we should have that

$$(\mathrm{Pic}(X)/\mathrm{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong N^1(X)_{\mathbb{Q}} \tag{3.1}$$

naturally. The connection between the left hand side of 3.1 and the Neron-Severi group arises as follows. The identity component of $\mathrm{Pic}(X)$ can be characterized in terms of an equivalence relation called algebraic equivalence. Namely $\mathrm{Pic}^0(X)$ consists of those line bundles algebraically equivalent to $\mathcal{O}_X$. Up to torsion algebraic and numerical equivalence coincide and we obtain the desired equality. Here we very briefly explain the construction of the Picard scheme. We eschew the proofs as it would take us to far afield. Our reference will be [30].

Definition 3.2.17 (Absolute Picard Functor)**.** *Consider the category of locally Noetherian schemes over a base scheme S. That is all schemes $X \to S$ along with morphisms over S. Define*

$$\mathrm{Pic}_X(T) = \mathrm{Pic}(X_T)$$

where $X_T = X \times_S T$ whenever $T \to S$ is a locally notherian scheme.

While this definition is natural, it has the problem that this is not a sheaf in the Zariski topology. Instead, we work with a relative Picard functor.

Definition 3.2.18 (Relative Picard functor)**.** *Define*

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$$

where $\mathrm{Pic}(T) = p_2^*(\mathrm{Pic}(T))$ and $p_2 \colon X \times_S T \to T$ is the canonical morphism.

To obtain the most general results, once must allow various different topologies to work in when considering the relative Picard functor. The three main choices are the Zariski topology, the etale topology and the fppf topology, though we will not explicitly engage with these notions to any real extent. The main result we are interested in is the following.

Theorem 3.2.19 (Existence of the Picard scheme 4.8 in [30])**.** *Suppose that $X \to S$ is projective Zariski locally over S and is flat with integral geometric fibers. Then $\mathrm{Pic}_{X/S}$ is representable in the etale topology.*

In particular, take $S = \mathrm{Spec}\, k$ and let X/Stob be a projective variety that is geometrically integral. Then we have that $\mathrm{Pic}_{X/\mathrm{Spec}\, k}(\mathrm{idSpec}\, k \to \mathrm{Spec}\, k) = \mathrm{Pic}(X)$.

Definition 3.2.20 (The connected component of the identity. 5.4 in [30])**.** *Let X/S be in the situation of 3.2.19. We let $\mathrm{Pic}^0_{X/S}$ be the connected component of the identity of $\mathrm{Pic}_{X/S}$.*

Theorem 3.2.21 (The connected component of the identity is projective)**.** *Let X/S be as in 3.2.19. Then $\mathrm{Pic}^0_{X/S}$ exists and is quasi-projective. If X/S is geometrically normal then $\mathrm{Pic}^0_{X/S}$ is projective and an open and closed subgroup of $\mathrm{Pic}_{X/S}$ of finite type.*

The connected component of the identity can be characterized by an equivalence relation known as algebraic equivalence.

Definition 3.2.22 (Algebraic equivalence definition 5.9 [30])**.** *Let X/S be as in 3.2.19. Let $\mathcal{L}, \mathcal{N} \in \operatorname{Pic}(X)$. We say that $\mathcal{L}$ is* algebraically equivalent *to $\mathcal{N}$ and write*

$$\mathcal{L} \equiv_{\text{alg}} \mathcal{N}$$

if there are schemes T_i/S for some $1 \leq n$ and all $1 \leq i \leq n$ with geometric points $s_i, t_i \in T_i$ with the same residue field along with line bundles $\mathcal{M}_i$ on X_{T_i} such that

$$\mathcal{L}_{T_1,s_1} \cong \mathcal{M}_{1,s_1}, \mathcal{M}_{1,t_1} \cong \mathcal{M}_{2,s_2}, \mathcal{M}_{2,t_2} \cong \mathcal{M}_{3,s_3} \ldots \tag{3.2}$$

$$\mathcal{M}_{n-1,t_{n-1}} \cong \mathcal{M}_{n,s_n}, \mathcal{M}_{n,t_n} \cong \mathcal{N}_{T_n,t_n}. \tag{3.3}$$

Proposition 3.2.23. *Let X/S be as in 3.2.19. Then $\operatorname{Pic}^0_{X/S}$ are those line bundles algebraically equivalent to $\mathcal{O}_X$.*

Our main result is as follows.

Theorem 3.2.24 (The tangent space of the identity. 5.11 and 5.14 in [30])**.** *Let X/S be as in 3.2.19. Let $T_0\operatorname{Pic}_{X/S}$ be the tangent space at the identity of $\operatorname{Pic}_{X/S}$. Then*

$$T_0\operatorname{Pic}_{X/S} = H^1(X, \mathcal{O}_X).$$

Furthermore if $\operatorname{char}(k) = 0$ then $\operatorname{Pic}_{X/S}$ is smooth of dimension $\dim H^1(X, \mathcal{O}_X)$.

Corollary 3.2.24.1 (Dimension of Pic^0.)**.** *Let X/S be as in 3.2.19 and let k be a field of characteristic 0. Then $\operatorname{Pic}^0_{X/S}$ is an abelian variety and $\dim \operatorname{Pic}^0_{X/S} = \dim H^1(X, \mathcal{O}_X)$.*

Proof. By 3.2.24 we have that $\operatorname{Pic}_{X/S}$ is smooth of dimension $\dim H^1(X, \mathcal{O}_X)$. Furthermore, as $\operatorname{Pic}_{X/S}$ is a group variety it is covered by translates of $\operatorname{Pic}^0_{X/S}$, which by 3.2.21 is open and closed of finite type. We have that $\dim \operatorname{Pic}_{X/S} = \dim \operatorname{Pic}^0_{X/S}$ and that $\operatorname{Pic}^0_{X/S}$ is smooth. Since it is projective by 3.2.21 it is an abelian variety by since it is finite type, and so an algebraic group. □

We may now construct the Albanese variety.

Theorem 3.2.25 (The Albanese Variety. 5.25 in [30].)**.** *Let X/S be as in 3.2.19 with $k = \overline{k}$ with X normal, integral and projective. Set $P_X = \operatorname{Pic}^0_{X/S}$ an abelian variety. Set $A_X = \operatorname{Pic}^0_{P/S}$ the dual variety. Then there is a morphism $\alpha_X \colon X \to A_X$ with the property that if $b \colon X \to B$ where B is an abelian variety, then there is a unique morphism $\hat{b} \colon A_X \to B$ such that $\hat{b} \circ \alpha_X = b$. We call A_X the Albanese variety of X and write $A_X = \operatorname{Alb}(X)$.*

We now consider the torsion elements of $\mathrm{Pic}_{X/S}/\mathrm{Pic}^0_{X/S}$ and the connection to the Neron-Severi group.

Theorem 3.2.26 (Pic^0 torsion coincides with numerical equivalence. 6.3 in [30].)**.** *Let S be the spectrum of an algebraically closed field in characteristic zero. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then $m\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$ for some $m \geq 1$ if and only if $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_X$.*

Theorem 3.2.27 (Finiteness of torsion. 6.17 in [30])**.** *Let X/S be as in 3.2.26. Then $\mathrm{Pic}_{X/S}/\mathrm{Pic}^0_{X/S}$ has a finite torsion subgroup.*

Putting this all together we have the following.

Corollary 3.2.27.1 (Connection between the Albanese and Picard Groups)**.** *Let S be the spectrum of an algebraically closed field in characteristic zero. Let X/S be a normal, integral projective variety. Then*

$$(\mathrm{Pic}_{X/S}(S)/\mathrm{Pic}^0_{X/S}(S)) \otimes_{\mathbb{Z}} \mathbb{Q} = (\mathrm{Pic}(X)/\mathrm{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong N^1(X)_{\mathbb{Q}}.$$

In particular, if $\mathrm{Alb}(X) = \{0\}$ then linear equivalence and numerical equivalence coincide for $\mathbb{Q}$ divisors on X.

We will also need the following which says the dual abelian variety $\hat{A}$ constructed earlier agrees with $\mathrm{Pic}^0_{A/k}$ when A is an abelian variety.

Lemma 3.2.28 (Consistency of constructions)**.** *Let A be an abelian variety defined over an algebraically closed field k. Then $\mathrm{Pic}^0_{A/k} = \hat{A}$ where $\hat{A}$ is the dual abelian variety.*

Proof. Let $\mathrm{Pic}^0(A)$ be as in 3.2.11. Then $\mathrm{Pic}^0(A)$ is a subgroup of $\mathrm{Pic}(A)$ and contains $\mathcal{O}_X$. Furthermore, $\mathrm{Pic}^0(A)$ is connected being an abelian variety itself. Thus $\mathrm{Pic}^0(A)$ is the connected component of the identity and so coincides with $\mathrm{Pic}^0_{A/k}$ as needed.

□

3.2.2 The endomorphism ring of an abelian variety.

Definition 3.2.29. *Fix two abelian varieties X, Y defined over $\overline{\mathbb{Q}}$. We let $\hom(X, Y)$ be the collection of group homomorphisms $X \to Y$ and set $\mathrm{End}(X) = \hom(X, X)$. We let $\hom^0(X, Y) = \hom(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ and similarly $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We note that $\mathrm{End}(X)^0$ is a possibly non-commutative $\mathbb{Q}$-algebra under the usual addition and composition actions.*

We now define a simple abelian variety. These are abelian varieties with no non-trivial abelian sub varieties.

Definition 3.2.30 (Simple abelian variety. See corollary 1 of [46].)**.** *A simple abelian variety is an abelian variety whose only abelian sub varieties are $\{0\}$ and A itself.*

Let A, B be abelian varieties and $f\colon A \to B$ a homomorphism. Then we have a action by pull-back $f^*\colon \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A)$ which gives a canonical morphism $\hat{f}\colon \hat{B} \to \hat{A}$. This morphism is called the dual morphism.

Lemma 3.2.31 (Inverse isogenies. See page 169 in [46].)**.** *Let A, B be abelian varieties and $f\colon A \to B$ an isogeny. Then there is an isogeny $g\colon B \to A$ such that $g \circ f = [n]_A$ and $f \circ g = [n]_B$ for some $n > 0$.*

Proof. We sketch the proof. To make this completely rigorous one must develop the theory of quotients of abelian varieties by finite subgroups which we omit. We have a surjective endomorphism $f\colon A \to B$ with finite kernel $\ker(f) = K$. Then $B \cong A/K$. So we may replace B with A/K and consider f with the quotient map $f(a) = a + K$. Since K is a finite subgroup we can find $n > 0$ such that $n \cdot K = \{0\}$ and $K \subseteq \ker([n_X])$. Then we have a morphism $g\colon A/K \to A$ given by $g(a+K) = na$. In fact g is an isogeny as it has a finite kernel given by $\ker([n]_X)/K$ and is clearly surjective. So $g \circ f = [n_X]$. On the other hand, we may compute

$$(f \circ g)(a + K) = f(na) = na + K = n(a + K) = [n]_{A/K}(a + K).$$

So that $f \circ g = [n]_Y$ as needed. □

Definition 3.2.32 (The category of abelian varieties up to isogeny.)**.** *Consider the category with objects abelian varieties over a common field k and morphisms given by group homomorphisms over k. Let X, Y be abelian varieties. We say X is isogenous to Y if there is an isogeny $X \to Y$. By 3.2.31 we see that this is an equivalence relation.*

Theorem 3.2.33 (The decomposition theorem)**.** *Let A be an abelian variety defined over an algebraically closed field k. Then A is isogenous to an abelian variety*

$$\prod_{i=1}^{s} A_i^{n_i}$$

where the A_i are simple pairwise non-isogenous abelian varieties. Furthermore, this decomposition is unique up to isogeny.

Corollary 3.2.33.1 (Structure of endomorphisms when X is simple. Corollary 2 of [46])**.** *Let A be an abelian variety defined over an algebraically closed field k which is simple. Then $\operatorname{End}^0(X)$ is a division ring.*

Proof. Given $f\colon X \to X$ if f is non-zero then its kernel must be finite as X is simple. Furthermore, the image must be X as otherwise the image would be a non-trivial abelian subvariety. Thus f is an isogeny. By 3.2.31 we have that there is an isogeny $g\colon X \to X$ with $f \circ g = [n]_X = g \circ f$. Thus $\frac{1}{n}g$ is an inverse for f in $\operatorname{End}(X)^0$. □

We now define an involution on an abelian variety X which will be useful to interpret the Neron-Severi group of abelian varieties.

Definition 3.2.34 (The Rosati Involution)**.** *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$. Let $L \in \operatorname{Pic}(A)$ be an ample line bundle. Then we have an isogeny*

$$\phi_L\colon A \to \hat{A} = \operatorname{Pic}^0(A)$$

given by

$$\phi_L(a) = t_a^* L - L.$$

The Rosati involution associated to L is a function

$$\operatorname{End}(X)^0 \to \operatorname{End}(X)^0, \varphi \mapsto \varphi' = \phi_L^{-1} \circ \hat{\varphi} \circ \phi_L$$

where $\hat{\varphi}$ is the dual isogeny.

Proposition 3.2.35 (Properties of the Rosati involution. Page 189-190 [46].)**.** *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$. Let $L \in \operatorname{Pic}(A)$ be an ample line bundle. The Rosati involution associated to L has the following properties. Let $\varphi, \theta \in \operatorname{End}(X)^0$.*

1. *If $\varphi \in \operatorname{End}(X)^0, a \in \mathbb{Q}$ then $(a\varphi)' = a\varphi'$.*
2. *$(\varphi + \theta)' = \varphi' + \theta'$.*
3. *$\varphi\theta)' = \theta'\varphi'$.*
4. *$(\varphi)'' = \varphi$.*
5. *We may identify $N^1(A)_{\mathbb{Q}}$ with the subspace of $\operatorname{End}^0(X)$ fixed by the Rosati involution.*

It is this last property which will be crucial for us. The idea is as follows. We have a morphism

$$(\mathrm{Pic}(X))_{\mathbb{Q}} \to \mathrm{End}(X)^0, D \mapsto \phi_D = \phi_L^{-1} \circ \Phi_D$$

where $\phi_D\colon X \to \mathrm{Pic}^0(X)$ is the morphism $\phi_D(x) = t_x^* D - D$. Notice that if $D \in \mathrm{Pic}^0(X)$ then $t_a^* D - D$ is the trivial line bundle by definition so that this morphism descends to

$$\Phi\colon (\mathrm{Pic}(X)/\mathrm{Pic}^0(X))_{\mathbb{Q}} \to \mathrm{End}(X)^0, D \mapsto \phi_D = \phi_L^{-1} \circ \phi_D. \tag{3.4}$$

If we have an isogeny $f\colon A \to A$ we will be interested in studying the dynamics of f and so its action on the Neron-Severi group. This can be accomplished as follows.

Theorem 3.2.36. *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$. Fix an ample divisor H on A. Then*

1. *The image of $N^1(A)_{\mathbb{Q}}$ is precisely the set of $\alpha \in \mathrm{End}(A)_{\mathbb{Q}}$ with $\alpha' = \alpha$. That is the Neron-Severi space may be identified with the space of endomorphisms fixed by the Rosati involution.*
2. *([28, Lemma 24]) Let $f\colon A \to A$ be an isogeny and $D \in N^1(A)_{\mathbb{R}}$. Then*

$$\Phi_{f^*D} = f' \circ \Phi_D \circ f.$$

Finally, we will need to know something about the structure of endomorphism rings of abelian varieties. This work is classical. The Rosati involution satisfies good properties, to the point that a classification theorem exists.

Theorem 3.2.37 ([1], page 201 [46])**.** *Let A be a g-dimensional abelian variety over an algebraically closed field k with a chosen polarization. Let $D = \mathrm{End}(A)_{\mathbb{Q}}$ and $'$ the associated Rosati involution. Then D is a simple $\mathbb{Q}$-algebra, of finite dimension with center K a number field. Set $K_0 = \{x \in K : x' = x\}$. Then the pair $(D, ')$ is one of the following four types. Set*

$$e = [K : \mathbb{Q}], e_0 = [K_0 : \mathbb{Q}], m^2 = [D : K]$$

and let ρ be the Picard number of A.

1. *$K_0 = K = D$ with $' = \mathrm{id}_D$ with K a totally real number field. In this case $\rho = e$ and $e \mid g$.*

2. *$[D:K]=4$ with $K=K_0$ with K a totally real number field. D is a quaternion algebra over K. Moreover for each real embedding $\sigma\colon K\hookrightarrow\mathbb{R}$ we have $D\otimes_{K,\sigma}\mathbb{R}\cong M_2(\mathbb{R})$ and $D\otimes_{\mathbb{Q}}\mathbb{R}\cong\prod_{\sigma\colon K\hookrightarrow\mathbb{R}}M_2(\mathbb{R})$. The isomorphism can be chosen so that the involution can be taken to be $(M_1,...,M_e)'=(M_1^t,...,M_e^t)$.*

   We have $\rho=3e$ and $2e\mid g$.

3. *$[D,K]=4$ with $K=K_0$ with K a totally real number field. D is a quaternion algebra over K. Moreover for each real embedding $\sigma\colon K\hookrightarrow\mathbb{R}$ we have $D\otimes_{K,\sigma}\mathbb{R}\cong\mathbb{H}$ and $D\otimes_{\mathbb{Q}}\mathbb{R}\cong\prod_{\sigma\colon K\hookrightarrow\mathbb{R}}\mathbb{H}$. The isomorphism can be chosen so that the involution can be taken to be $(a_1,...,a_e)'=(\bar{a}_1,...,\bar{a}_e)$.*

   We have $\rho=e$ and $2e\mid g$.

4. *$[D:K]=m^2$ with K_0 is a totally real number field, and K is a totally imaginary quadratic extension of K_0.*

$$D\otimes_{\mathbb{Q}}\mathbb{R}\cong\prod_{\sigma\colon K_0\hookrightarrow\mathbb{R}}M_m(\mathbb{C})$$

   The isomorphism can be chosen so that the involution can be taken to be $(M_1,...,M_{e_0})'=(\bar{M}_1^t,...,\bar{M}_{e_0}^t)$. We have $\rho=e_0m^2$ and $e_0m^2\mid g$.

3.3 Projective bundles over elliptic curves

Before developing the theory, we explain our interest. When studying the Kawaguchi-Silverman conjecture we will be interested in the following situation. Let $f\colon X\to X$ be a surjective endomorphism of varieties. To study f we study the linear action $f^*\colon N^1(X)_{\mathbb{R}}\to N^1(X)_{\mathbb{R}}$. One situation of interest is the following. Suppose that D is a non-zero nef $\mathbb{Q}$-divisor and $f^*D=\lambda D$ where λ is an eigenvalue of largest possible magnitude. It is known how to study the dynamics of f if $\kappa(D)>0$. So we are interested in the case $\kappa(D)\leq 0$. Here we leave the case that $\kappa(D)=-\infty$ untouched and concentrate on the case where $\kappa(D)=0$. To construct varieties where the above dynamical phenomenon can arise, we look at a vector bundles an elliptic curve C. Given such a vector bundle say $\mathcal{F}$ we have the projective bundle $\mathbb{P}\mathcal{F}$ along with a canonical line bundle $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ on $\mathbb{P}\mathcal{F}$. We consider vector bundles on elliptic curves because Atiyah gave a robust classification of these objects, giving us many tools to work with. Furthermore, by [34] this is the last remaining case for the Kawaguchi-Silverman conjecture for projective bundles over a 1-dimensional base. The main results are summarized as below.

Theorem 3.3.1 (Atiyah [5]). *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$.*

1. *For each $r \geq 1$ there is a unique indecomposable degree 0 rank r vector bundle on C that has a non-zero global section. We call this vector bundle F_r. Furthermore, $\dim H^0(C, F_r) = 1$ ([5, Theorem 5]).*
2. *Every indecomposable degree 0 vector bundle of rank r is of the from $F_r \otimes L$ for some unique degree zero line bundle L ([5, Theorem 5]).*
3. *$F_r \otimes F_s = \bigoplus_i F_{r_i}$ for some r_i ([5, Lemma 18]).*
4. *$\det F_r = \mathcal{O}_X$ ([5, Theorem 5]).*
5. *Let L be a line bundle of degree 0. Then $F_r \otimes F_s \otimes L$ has a global section if and only if $L = \mathcal{O}_X$ ([5, Lemma 17]).*
6. *F_r is self dual ([5, Corollary 1]).*
7. *$F_r = \mathrm{Sym}^{r-1} F_2$ ([5, Theorem 9]).*

We will be forced to work with these objects explicitly, and now develop how to write down the transition functions of F_r. This follows the work of Sasha (Alexander) Zotine in his master's thesis [62].

3.3.1 Transition Functions of F_r.

Let C be an elliptic curve over $\bar{\mathbb{Q}}$ and F_r the rank r Atiyah Bundle. We describe our notation for further use.

Notation 3.3.1.1. *In this section We take C to be given by an equation $x_0 x_2^2 = x_1(x_1 - x_0)(x_1 - \lambda_0 x_0)$. Here we take O the origin to be $(0:0:1)$ because we are using the Legendre form of the curve. We can take U_0 to be the locus where $x_0 \neq 0$ and U_2 the locus where $x_2 \neq 0$. Then U_0, U_2 are an open affine cover of C.*

Zotines's idea is to study vector bundles trivialized on U_0 and U_2. Since we have a good classification of vector bundles on C due to Atiyah's classification in [5] this is possible; there is a unique indecomposable semi-stable degree zero vector bundle of rank r (namely F_r in 3.3.1) on C with a non-zero global section. Zotine constructs such a vector bundle which by the classification must be F_r. Then using further results of [5] he reconstructs all other vector bundles explicitly. The construction of these vector bundles will be through the study of the order of vanishing of functions at the origin $\mathcal{O}$.

Lemma 3.3.2 ([62]). *Let ν be the valuation at $\mathcal{O} = (0 : 0 : 1)$. Set $x = x_1/x_0$ and $y = x_2/x_0$.*

1. $\mathcal{O}_C(U_0 \cap U_2) = k[x, y^{\pm 1}]/(y^2 - x(x-1)(x-\lambda))$.
2. *We have that* $\mathcal{O}_C(U_0) \cong K[x, y]/(y^2 - x(x-1)(x-\lambda))$. *Furthermore, if* $f \in \mathcal{O}_C(U_0)$ *then* $\nu(f) \leq 0$ *and* $\nu(f) \neq 1$.
3. *We have that* $\mathcal{O}_C(U_2) = K[xy^{-1}, y^{-1}]/(y^{-1} - xy^{-1}(xy^{-1} - y^{-1})(xy^{-1} - \lambda y^{-1})$. *Furthermore,* $\mathcal{O}_C(U_2)$ *is precisely the elements* $f \in \mathcal{O}_C(U_0 \cap U_2)$ *such that* $\nu(f) \geq 0$.
4. $\mathcal{O}_C(U_2) \cap \mathcal{O}_C(U_0) = k$.

We have the following element of $\mathcal{O}_C(U_0 \cap U_2)$ that will feature prominently in the sequel.

Definition 3.3.3. *Let* $\omega = \overline{x^2 y^{-1}} \in \mathcal{O}_C(U_0 \cap U_2)$.

Theorem 3.3.4 (Main Result [62]). *Let C be an elliptic curve over $\bar{\mathbb{Q}}$ and F_r the rank r Atiyah bundle on C. Let A_r be the $r \times r$ matrix with 1 on the diagonal and ω on the upper off diagonal. For example, we have*

$$A_2 = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & \omega & 0 \\ 0 & 1 & \omega \\ 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & \omega & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & 0 & 1 & \omega \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

Then A_r defines transition functions for F_r on the open cover $\{U_0, U_2\}$. In other words let E'_r be rank r the vector bundle which is trivialized on U_0, U_2 with transition function A_r. In other words, F_r is obtained by gluing $\mathcal{O}^r_{U_0}$ and $\mathcal{O}^r_{U_2}$ via $A_r\colon (\mathcal{O}^{\oplus r}_{U_0}) \mid_{U_0 \cap U_2} \to (\mathcal{O}^{\oplus r}_{U_2}) \mid_{U_0 \cap U_2}$.

Chapter 4

Dynamical Preliminaries

Given a projective variety X defined over $\overline{\mathbb{Q}}$ (more generally any algebraically closed field) it is an old question to study the group of symmetries of X, in other words its automorphisms. One might also generalize this notion and instead of studying $\operatorname{Aut}(X)$ one might study $\operatorname{Sur}(X)$. That is, the monoid of all surjective self morphisms $X \to X$. It turns out that when $\operatorname{Sur}(X)$ is strictly larger then $\operatorname{Aut}(X)$ we should expect that X has a special geometry. Evidence for this meta-principle is the following.

Theorem 4.0.1 (Nakayama Classification of surfaces: [47])**.** *Let X be a smooth projective surface defined over $\overline{\mathbb{Q}}$. Suppose that $f\colon X \to X$ is a surjective endomorphism that is not an autmorphism.*

1. *If $\kappa(X) \geq 0$ then X is an abelian surface, a hyper-elliptic surface, or a minimal elliptic surface with $\kappa(X) = 1$ and $\chi(\mathcal{O}_X) = 0$.*
2. *If X is a ruled surface that X is one of the following.*
   - (a) *A toric surface.*
   - (b) *A $\mathbb{P}^1$-bundle over an elliptic curve.*
   - (c) *A $\mathbb{P}^1$-bundle over a smooth genus $g > 1$ curve C that is trivialized after a finite etale base change.*

Unfortunately results this strong are not known in higher dimensions. However there is some work in [18]. The main two examples of varieties that admit surjective endomorphisms that are not automorphisms are Abelian varieties and toric varieties.

4.1 Surjective Endomorphisms of projective varieties

Here we begin our development of the theory of surjective endomorphisms of projective varieties. We begin with an interesting observation.

Proposition 4.1.1 (Dominant morphism of projective varieties is surjective)**.** *Let X be irreducible projective varieties defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to Y$ be a dominant morphism. Then f is surjective.*

Proof. The morphism f factors as $X \mapsto \Gamma_f \mapsto Y$ where $\Gamma_f \subseteq X \times Y$ is the graph of f. Since the morphism $X \to \Gamma_f$ is a closed immersion and $X \times Y \to Y$ is closed as X, Y are projective we have that the image of f is closed. Since f is dominant it must be the whole of Y. Alternatively, one may use that f must be proper, and therefore is closed. □

Lemma 4.1.2. *Pushforward of curves is surjective for dominant maps. Let X, Y be an irreducible projective varieties defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to Y$ be a dominant morphism. Then $f_*\colon N_1(X)_{\mathbb{Q}} \to N_1(Y)_{\mathbb{Q}}$ is surjective.*

Proof. Let C be a closed curve on Y. Set $Z = f^{-1}C$. Then we have induced surjective morphism $f\colon Z \to C$. It suffices to show that in this situation we have that some curve C' on Z is mapped to C. To this end suppose for a contradiction that every curve C on Z was contracted. Fix a point z in Z. Then given any other point $z' \in Z$ we can find an irreducible curve C' that contains y, y'. But then $f(C') = f(y)$ and so $f(z') = f(z)$. Since z' was arbitrary we have that Z is contracted to a point, which is a contradiction. So any curve C in Y can be realized as $df_*([C'])$ for some irreducible curve C' in X. It follows that the pushforward is surjective. □

Proposition 4.1.3. *Let X be an irreducible projective varieties defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism, then f is finite.*

Proof. Consider $N_1(X)_{\mathbb{Q}}$. By 4.1.2 we have that f_* is surjective. Since $f_*\colon N_1(X)_{\mathbb{Q}} \to N_1(X)_{\mathbb{Q}}$ is surjective, it is also injective as $N_1(X)_{\mathbb{Q}}$ is a finite dimensional vector space. It follows that f_* does not contract any curve. In particular, that f has finite fibers. Since f is a proper morphism with finite fibers it is finite as needed. □

Lemma 4.1.4 (exercise 12.22 [21])**.** *Let $f\colon X \to Y$ be a dominant integral endomorphism between integral schemes. Assume Y is normal. Then f is universally open.*

Proof. We sketch the proof that f is open. Let $\operatorname{Spec} B$ be an open affine in X. Since f is finite it is affine and $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. If $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is open for all $\operatorname{Spec} B$ then f is open. So we may assume that f is induced by $f\colon \operatorname{Spec}\, A \to \operatorname{Spec}\, B$ which is dominant and finite and A, B are normal domains. So we must prove the statement for a finite extension of rings $B \subseteq A$. Let $a \in A$ be non-zero. Then we can find $b_0, ..., b_{n-1} \in B$ such that

$$b_0 + b_1 a + ... + b_{n-1} a^{n-1} + a^n = 0$$

with n minimal. Let $p = T^n + \sum_{i=0}^{n-1} b_i T^i$. We will show that $f(D(a)) = \bigcup_{i=0}^{n-1} D(b_i)$. To this end let $\mathfrak{p} \in D(a)$. Then $f(\mathfrak{p}) = \mathfrak{p} \cap B$. Since $\mathfrak{p} \in D(a)$ we have that $a \notin \mathfrak{p}$. If $\mathfrak{p} \cap B \notin D(b_i)$ for all i then we have that $b_i \in \mathfrak{p}$ for all i. Then

$$a^n = -b_i a^{n-1} - ... - b_0 \in \mathfrak{p}.$$

So $a^n \in \mathfrak{p}$ which means $a \in \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal. So $f(\mathfrak{p}) \in D(b_i)$ for some i. On the other hand, let $\mathfrak{q} \in \bigcup_{i=0}^{n-1} D(b_i)$. First suppose that $a \in B$. Then we have that a satisfies $T - a$ and so by the minimality assumption that $p(T) = T - a$. Then $a \notin \mathfrak{p}$ and $f^{-1}(D_B(a)) = D(a)$ here $D_B(a) = \{\mathfrak{q} \in \operatorname{Spec} B : a \notin \mathfrak{q}\}$.So we have the result. Therefore we may assume that $n > 1$ and $a \notin \mathfrak{q}$. Pick j with $b_j \notin \mathfrak{q}$. Since f is surjective we can find a prime ideal $\mathfrak{p} \subseteq A$ with $\mathfrak{p} \cap B = \mathfrak{q}$. If $a \in \mathfrak{p}$. I claim that $a \notin \mathfrak{p}$. Towards a contradiction suppose that $a \in \mathfrak{p}$. Then we have that

$$a^n + b_{n-1} a^{n-1} + ... + b_0 = 0$$

so that $b_0 \in \mathfrak{p} \Rightarrow b_0 \in \mathfrak{q}$. Then we have that $a(a^{n-1} + b_{n-1}a^{n-2} + ... + b1) \in \mathfrak{q}$. Since $a \notin \mathfrak{q}$ we have $a^{n-1} + b_{n-1}a^{n-2} + ... + b1 \in \mathfrak{q} \Rightarrow b_1 \in \mathfrak{q}$. Continuing on in this way we obtain that that $b_0, ..., b_{n-1} \in \mathfrak{q}$. This contradicts our choice that some $b_i \notin \mathfrak{q}$. So $\mathfrak{p}$ is in $D(a)$ and so if $\mathfrak{q} \in \bigcup_{i=0}^{n-1} D(b_i)$ then we have that if $\mathfrak{p} \cap B = \mathfrak{q}$ then there is some $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \cap B = \mathfrak{q}$. Thus $\bigcup_{i=0}^{n-1} D(b_i) \subseteq f(D(a))$. As we already showed that $\bigcup_{i=0}^{n-1} D(b_i) \supseteq f(D(a))$ we have equality. □

Given a projective variety X and a surjective endomorphism $f\colon X \to X$ unless X has some very special geometry, (such as being an abelian variety or a toric variety) it can be difficult to study both f and X itself. One way to study X is to apply the minimal model program to X and thus obtain a simpler model of X. This strategy also may be employed to study f in certain situations. Let us begin with the case of X being a Mori-fiber space. In other words there is a fibering type contraction $\pi\colon X \to Y$ where $\dim Y < \dim X$ and $\rho(Y) = \rho(X) - 1$. In this situation, f descends to Y after iterating f.

Lemma 4.1.5 ([34, Lemma 6.2])**.** *Let $\pi\colon X \to Y$ be a Mori-fiber space. Suppose that $f\colon X \to X$ is a surjective endomorphism. Then there is some iterate $f^n\colon X \to X$ and $g\colon Y \to Y$ such that*

$$\begin{tikzcd} X \arrow[r, "f^n"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes.

We see that if X is a Mori-fiber space then, then we can study f up to taking iterates by studying the induced morphism $g\colon Y \to Y$. As Y is "simpler" than X we hope that g is easier to study then X. The philosophy is that a good way to study f is to study g and the behavior of f on the fibers of π. To study birational contractions we have the following two crucial results.

Lemma 4.1.6 ([41, Lemma 3.6])**.** *Let X be a normal $\mathbb{Q}$-factorial log canonical projective variety and $f\colon X \to X$ a surjective endomorphism. Let R be a K_X negative extremal ray and $f_*R = R$. Let $\phi_R\colon X \to Y$ be the associated extremal contraction. Then there is a morphism $g\colon Y \to Y$ such that $g \circ \phi_R = \phi_R \circ f$.*

In the above lemma, log canonical is a generalization of a terminal singularity. See [36] for more on these definitions. This gives us good control over a birational morphism. However, we also need to know how to control flips. This is provided by the following crucial result of Zhang with proof suggested by N. Nakayama

Lemma 4.1.7 (Morphism extension property: lemma 6.6 of [43])**.** *Let X be a normal projective variety with at worst lc singularities. Suppose that $f\colon X \to X$ is a surjective endomorphism. Let R be a K_X negative extremal ray and ϕ_R the associated contraction. If $\phi_R\colon X \to Y$ is of flipping type and $\psi\colon X \to X^+$ is the associated flip, then the induced rational mapping $f^+\colon X^+ \dashrightarrow X^+$ extends to a morphism $f^+\colon X^+ \to X^+$. Furthermore, both f and f^+ descend to the same morphism on Y.*

To use this lemma effectively we employ the following.

Theorem 4.1.8 (Duality between pullbacks and pushforwards)**.** *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism. Then $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ is dual to $f_*\colon \overline{\mathrm{NE}}(X)_{\mathbb{R}} \to \overline{\mathrm{NE}}(X)_{\mathbb{R}}$.*

Proof. Recall that $N^1(X)_{\mathbb{R}}$ is dual to $N_1(X)_{\mathbb{R}}$. Thus if $\gamma \in N_1(X)_{\mathbb{R}}$ then γ is determined by the function

$$\theta_\gamma \colon N^1(X)_{\mathbb{R}} \to \mathbb{R}, \theta_\gamma(D) = (D \cdot \gamma).$$

The dual of $f^* \colon N^1(X)_{\mathbb{R}} \to N^1(X))\mathbb{R}$ is the mapping

$$(f^*)^\vee \colon N_1(X)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}, (f^*)^\vee(\theta_\gamma) = \theta_\gamma \circ f^*.$$

In other words we have that

$$(f^*)^\vee(\gamma)(D) = \theta_\gamma(f^*D) = (f^*D \cdot \gamma) = (D \cdot f_*\gamma)$$

Thus $(f^*)^\vee(\gamma) = f_*\gamma$ as needed.

□

Proposition 4.1.9 (Push forward preserves the closed cone of curves)**.** *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. Let $f \colon X \to X$ be a dominant morphism. Then*

$$\gamma \in \overline{\mathrm{NE}}(X)_{\mathbb{R}} \iff f_*\gamma \in \overline{\mathrm{NE}}(X)_{\mathbb{R}}.$$

That is we have a linear isomorphism $f_* \colon \overline{\mathrm{NE}}(X)_{\mathbb{R}} \to \overline{\mathrm{NE}}(X)_{\mathbb{R}}$.

Proof. Let $\gamma \in \overline{\mathrm{NE}}(X)_{\mathbb{R}}$. Note that $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)^\vee_{\mathbb{R}}$. That is $\gamma \in \overline{\mathrm{NE}}(X) \iff (\gamma \cdot D) \geq 0$ for all $D \in \mathrm{Nef}(X)_{\mathbb{R}}$. Note that $f^*D \in \mathrm{Nef}(X)_{\mathbb{R}} \iff D \in \mathrm{Nef}(X)_{\mathbb{R}}$. So we have

$$\begin{aligned}
\gamma \in \overline{\mathrm{NE}}(X)_{\mathbb{R}} &\iff (\gamma \cdot D) \geq 0 \ \forall \ D \in \mathrm{Nef}(X)_{\mathbb{R}} \\
&\iff (\gamma \cdot f^*D) \geq 0 \ \forall \ D \in \mathrm{Nef}(X)_{\mathbb{R}} \\
&\iff (f_*\gamma \cdot D) \geq 0 \ \forall \ D \in \mathrm{Nef}(X)_{\mathbb{R}} \iff f_*\gamma \in \overline{\mathrm{NE}}(X)_{\mathbb{R}}.
\end{aligned}$$

□

Thus if $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ is a finitely generated cone and R is an extremal ray generating an extremal contraction ϕ_R then we have that the linear mapping preserves $\overline{\mathrm{NE}}(X)$. Therefore for some $n > 0$ we have that $f_*^{\circ n}(R) = R$ and so we may obtain a diagram

$$\begin{tikzcd}
X \arrow[r, "f^{\circ n}"] \arrow[d, "\phi_R"'] & X \arrow[d, "\phi_R"] \\
Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

So to study f we wish to study g and proceed by induction. This breaks the dynamical study of f into three essential ingredients.

1. Completing the minimal model program. In particular the termination of flips.
2. The study of f under birational extremal contractions.
3. The study of the dynamics of Mori-fiber spaces.

The above program is was is designed to study varieties with finitely generated Nef cones. However it is possible to extend these ideas to a different setting where non-finitely generated Nef cones are allowed. These are varieties with an int-amplified endomorphisms. Note that varieties with an int-amplified endomorphism does not contain all varieties with finitely generated rational Nef cones; we will exhibit examples of varieties with finitely generated rational Nef cones that do not admit an int-amplified endomorphism.

4.1.1 Int-amplified endomorphisms

Definition 4.1.10 (Int amplified, amplified and polarized endomorphisms)**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$ and $f\colon X \to X$ a surjective endomorphism. We say that f is a* polarized *endomorphism if there is some ample $\mathbb{Q}$-Cartier divisor L on X such that $f^*L \equiv_{\text{lin}} qL$ for some $q > 1$. We say that f is* amplified *if there is a $\mathbb{Q}$-Cartier divisor L such that $f^*L - L$ is ample. We say f is an* int-amplified *endomorphism if there is some ample $\mathbb{Q}$-Cartier divisor L with $f^*L - L$ being ample.*

Int amplified endomorphisms can be characterized in terms of their eigenvalues being large.

Proposition 4.1.11 (Eigenvalues determine int-amplified endomorphisms: Theorem 3.3 [43])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism. Then f is int amplified if and only if $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ has all eigenvalues of modulus strictly larger then 1.*

From 4.1.11 we see that the composition of int-amplified endomorphisms are int-amplified. We naturally obtain a sub-monoid of all surjective endomorphisms.

Definition 4.1.12 (The monoid of surjective endomorphisms)**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. We let $\mathrm{SEnd}(X)$ be the monoid of surjective endomorphisms of X. We let $\mathrm{IAmp}(X)$ be the collection of int-amplified endomorphisms of X.*

The idea behind int-amplified endomorphisms is that if $\mathrm{IAmp}(X) \neq \emptyset$ then we have a sub-monoid $\mathrm{IAmp}(X) \subseteq \mathrm{SEnd}(X)$ that can be used to study $\mathrm{SEnd}(X)$. We will see that the existence of $f \in \mathrm{IAmp}(X)$ has consequences for the birational geometry of X.

Definition 4.1.13. *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism. We say that a subset S of X is f^{-1}-periodic if there is some $r \geq 1$ such that $f^{-r}(S) = S$.*

The way we use int-amplified endomorphisms to study the geometry of X is through the contractible rays of $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$. In particular, the dynamics of an int-amplified endomorphism of X are a potent tool to study the birational geometry of X. Our first glimpse of this is the following. We now explain this approach.

Lemma 4.1.14 (Lemma 4.3 [45])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism that is int-amplified. Let C be an irreducible curve on X such that C generates an extremal ray of $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ that is contractible. Let R_C be the contractible extremal ray, and ϕ_{R_C} the associated extremal contraction. Let $h \in \mathrm{SEnd}(X)$. Then*

1. *Let E be the exceptional locus of ϕ_{R_C} and let E' be a component of E. Then $h^i(E)$ and $h^i(E')$ are f^{-1}-periodic for any $i \in \mathbb{Z}$.*

2. *E and E' are h^{-1}-periodic.*

So the exceptional locus of a extremal contraction are strongly periodic for an int-amplified endomorphism. Moreover, the collection of such sub-varieties is finite when f is a int-amplified.

Lemma 4.1.15 (Corollary 3.8 [45])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism that is int-amplified. Then there are finitely many (not necessarily closed) sub-varieties Z of X such that Z is f^{-1}-periodic. In other words there are only finitely many sub-varieties Z of X such that $f^{-r}(Z) = Z$ for some integer $r > 0$.*

Therefore, the number of extremal contractions is finite. Thus the existence of an int-amplified endomorphism forces X to have finitely many extremal contractions. One now proves the following.

Theorem 4.1.16 (Meng-Zhang in [45]: Theorem 4.5)**.** *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety. Let $\mathcal{R}_{\mathrm{cont}}$ be the collection of contractible extremal rays of X. Then*

1. *$\mathcal{R}_{\mathrm{cont}}$ is a finite set.*

2. *The set*
$$\widetilde{\mathcal{R}_{\mathrm{cont}}} = \{h^i(R_C)\colon R_C \in \mathcal{R}_{\mathrm{cont}}, h \in \mathrm{SEnd}(X), i \in \mathbb{Z}\}$$
is finite. In fact
$$\widetilde{\mathcal{R}_{\mathrm{cont}}} = \widetilde{\mathcal{R}_{\mathrm{cont}}}^0 = \{h_*(R_C)\colon R_C \in \mathcal{R}_{\mathrm{cont}}, h \in \mathrm{SEnd}(X)\}.$$

3. *There is a monoid action of* $\mathrm{SEnd}(X)$ *on* $\widetilde{\mathcal{R}_{\mathrm{cont}}}$ *where* $\mathrm{SEnd}(X)$ *acts by permutations.*

In the above theorem the construction of $\widetilde{\mathcal{R}_{\mathrm{cont}}}$ may seem unmotivated. This is introduced to deal with the fact that if R_C is an extremal ray corresponding to a contraction then while $h_*(R_C)$ is still extremal, it may not be a contraction. Thus $\mathrm{SEnd}(X)$ cannot possible act on the set of contractible extremal rays. Thus we introduce this set $\widetilde{\mathcal{R}_{\mathrm{cont}}}$ which by design contains all possible pullbacks and push forwards of contractible extremal rays and $\mathrm{SEnd}(X)$ may act on it.

We sketch the idea of the proof. We have by 4.1.15 that the set of f^{-1}-periodic subvarieties are finite. By 4.1.14 we have that the exceptional locus of an extremal contraction is f^{-1}-invariant. So the number of possible exceptional loci are finite which gives that there are finitely many extremal contractions. One then shows $\widetilde{\mathcal{R}_{\mathrm{cont}}}^0$ is finite and has the desired monoid action. Then using the finiteness of
$$\widetilde{\mathcal{R}_{\mathrm{cont}}}^0$$
one deduces that $\widetilde{\mathcal{R}_{\mathrm{cont}}} = \widetilde{\mathcal{R}_{\mathrm{cont}}}^0$. The key tools in the argument are once again the finiteness properties of f^{-1}-periodic subsets of a given int-amplified endomorphism.

We now summarize the main properties of varieties with an int-amplified endomorphism.

Theorem 4.1.17 (Lemma 5.2 [38]). *Let X, Y be a normal projective varieties. Let $f\colon X \to X$ and $g\colon Y \to Y$ be surjective endomorphisms.*

1. *If $\pi\colon X \to Y$ is a surjective endomorphism and f is int-amplified with $g \circ \pi = \pi \circ f$ then g is int-amplified.*

2. *If $\dim X = \dim Y$ and $\pi\colon X \dashrightarrow Y$ is a dominant rational map with $g \circ \pi = \pi \circ f$ then g is int-amplified.*

3. *If X is $\mathbb{Q}$-factorial and f is int amplified then $-K_X \equiv_{\mathrm{num}} E$ where E is an effective $\mathbb{Q}$- divisor. In particular if $\mathrm{Alb}(X) = 0$ then $\kappa(-K_X) \geq 0$.*

Definition 4.1.18 (The equivariant MMP:Meng-Zhang in [45])**.** *Consider a sequence of dominant rational maps*

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \cdots \dashrightarrow X_r \tag{4.1}$$

such that each X_i is a normal projective variety. Let $f = f_1 \colon X_1 \to X_1$ be a surjective endomorphism. We say that 4.1 is f-equivariant if there are surjective endomorphisms $f_i \colon X_i \to X_i$ such that $g_i \circ f_{i+1} = f_i \circ g_i$ for all i, where $g_i \colon X_i \dashrightarrow X_{i+1}$ is the dominant rational mapping of 4.1.

Theorem 4.1.19 (The equivariant MMP of Meng-Zhang: Theorem [38])**.** *Let X be a $\mathbb{Q}$-factorial projective variety defined over $\overline{\mathbb{Q}}$ with at worst terminal singularities admitting an int-amplified endomorphism.*

1. *There are only finitely many K_X negative extremal rays of X. Moreover if $f \colon X \to X$ is a surjective endomorphism then there is some $n \in \mathbb{Z}_{>0}$ such that $f_*^n \colon \overline{\mathrm{NE}}(X)_{\mathbb{R}} \to \overline{\mathrm{NE}}(X)_{\mathbb{R}}$ fixes every K_X negative extremal ray.Let R be any extremal K_X negative extremal ray with contraction $\phi_R \colon X \to Y_R$. Then there is a surjective endomorphism $g_R \colon Y_R \to Y_R$ such that $g_R \circ \phi_R = \phi_R \circ f^n$. Moreover if R is a flip and ψ_R^+ is the associated birational mapping $X \to X_R^+$ then the induced rational mapping $f_R^+ \colon X_R^+ \dashleftarrow X_R^+$ extends to a morphism $f_R^+ \colon X_R^+ \to X_R^+$.*

2. *Then for any surjective morphism $f \colon X \to X$ there is some n and a f^n equivariant MMP for f^n g given by*

   $$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \cdots \dashrightarrow X_r.$$

   Let $g_i \colon X_i \dashrightarrow X_{i+1}$. Then we have that.

   (a) *Each g_i is a contraction of a K_X negative extremal ray.*

   (b) *X_r is a Q-Abelian variety. Note that X_r might be a point. In fact there is there is a surjective endomorphism $h \colon A \to X_r$ where h is a finite and A is an abelian variety. Moreover there is a surjective endomorphism $w \colon A \to A$ such that $w \circ h = f_r \circ w$. The existence of h is the definition of Q-Abelian, the theorem provides that the morphism g_r commutes with a morphism of the covering abelian variety. In fact this holds for any surjective endomorphism of a Q-Abelian variety.*

3. *Let $f\colon X \to X$ be a surjective endomorphism. Let*

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \cdots \dashrightarrow X_r$$

be any MMP where the $g_i\colon X_i \to X_{i+1}$ are divisorial or fibering contractions. Then there is some n such that there are surjective endomorphisms $f_i\colon X_i \to X_i$ making the MMP f-equivariant.

4.1.2 Dynamical degrees

To effectively study a surjective endomorphism of X we wish to assign a numerical notion of complexity of f under iteration. Our such notion will be the following.

Definition 4.1.20 (The dynamical degree.)**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism. The first dynamical degree of f is defined to be*

$$\lambda_1(f) = \limsup_{n\to\infty} \rho((f^{\circ n})^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}})).$$

Here $\rho((f^{\circ n})^*)$ is the spectral radius of the pull back action on $N^1(X)_{\mathbb{R}}$.

The dynamical degree can also be defined for arbitrary dominant rational maps provided that X is normal, this is to make sure that the pull back maps behave as expected. This definition is often not used in practice. We have the following.

Proposition 4.1.21 (Properties of the dynamical degree: section 1 of [27] and corollary 18 in [29])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism. Then we have the following.*

1. *Let H be an ample divisor. Then*

$$\lambda_1(f) = \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X - 1})^{\frac{1}{n}}.$$

2. *$\lambda_1(f) = \rho(f^*)$. That is, the dynamical degree of f is the spectral radius of the action of f^* on $N^1(X)_{\mathbb{R}}$.*

Often we will attempt to reduce a question about the dynamical degree to a related variety by some sort of fibration.

Definition 4.1.22 (The relative dynamical degree. [34, Definition 2.1])**.** *Suppose that we have a commuting diagram of normal projective varieties defined over* $\bar{\mathbb{Q}}$

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

where f, g, π are all surjective morphisms. The first relative dynamical degree of f with respect to π is defined to be

$$\lambda_1(f\mid_\pi) = \lim_{n\to\infty}((f^n)^* H_X \cdot (\pi^* H_Y)^{\dim Y} \cdot H_X^{\dim X - \dim Y - 1})^{\frac{1}{n}}$$

where H_X, H_Y are ample divisors on X and Y respectively.

One may also look at [34] for further references involving this notion, for example [13].

4.2 Arithmetic dynamics

4.2.1 Canonical height functions and arithmetic degrees.

We have developed a numerical invariant associated to a surjective endomorphism, the first dynamical degree. The construction of the first dynamical degree is purely geometric, and could be defined over any algebraically closed field of characteristic zero. We now develop an arithmetic notion of the complexity of f that depends on arithmetic.

Definition 4.2.1 (The arithmetic degree)**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism. Fix an ample divisor H on X. Given a point $P \in X(\bar{\mathbb{Q}})$ such that we define the arithmetic degrees*

$$\overline{\alpha_f}(P) := \limsup_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$$

and the lower arithmetic degree

$$\underline{\alpha_f}(P) := \liminf_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$$

where h_H is a choice of height function for H, and $h_H^+(P) = \max\{1, h_H(P)\}$. We think of the lower and upper arithmetic degrees as arithmetic measures of the complexity of f.

These arithmetic degrees are much more interesting when f is a dominant rational mapping rather then a morphism. We have the following.

Proposition 4.2.2 (Properties of the arithmetic degree: [29] and [28])**.** *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a dominant morphism.*

1. *The upper and lower arithmetic degrees are independent of the choice of H and the height function h_H.*
2. *The limit $\lim_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}$ exists.*
3. *For all $P \in X(\overline{\mathbb{Q}})$ we have that*
$$\alpha_f(P) = |\lambda|$$
for some eigenvalue λ of f^* acting on $N^1(X)_{\mathbb{R}}$. We can also take λ to be an eigenvalue of f^* acting on $\mathrm{Pic}(X)$.
4. $\alpha_f(P) \leq \lambda_1(f)$.

To study the Kawaguchi-Silverman conjecture we require new tools to deal with how a height function changes value on an orbit. Recall that height functions provide a way to turn geometric relationships involving divisors into arithmetic relationships involving height functions. Given a surjective endomorphism, $f\colon X \to X$ we will study the dynamics of f through the dynamics of its pull-back action on $N^1(X)_{\mathbb{R}}$. The dynamics of a linear mapping is captured by its eigenvalues, so we are interested in those D such that $f^*D \equiv_{\mathrm{lin}} \lambda D$ or $F^*D \equiv_{\mathrm{num}} \lambda D$ for some $\lambda \neq 0$. Associated to the divisor D is a height function h_D. However, we have some extra data, namely that D is preserved by f. This allows us to find a specific height function (rather then just an equivalence class of height functions) to study.

Theorem 4.2.3 ([29, Theorem 5],[10, Theorem 1.1])**.** *Let X be a normal projective variety and $f\colon X \to X$ a surjective endomorphism. Let $D \in \mathrm{CDiv}_{\mathbb{R}}(X)$ and suppose that $f^*D \equiv_{\mathrm{Num}} \lambda D$ for some $\lambda > \sqrt{\lambda_1(f)}$. Then*

1. *For all $P \in X(\bar{\mathbb{Q}})$ the limit $\hat{h}_D(P) = \lim_{n\to\infty} \dfrac{h_D(f^n(P))}{\lambda^n}$ exists for any choice of height function h_D associated to D.*
2. *We have that $\hat{h}_D(f^n(P)) = \lambda^n \hat{h}_D(P)$ and that $\hat{h}_D = h_D + O(\sqrt{h_X^+}\,)$ where h_X is any ample height on X and $h_X^+ = \max\{1, h_X\}$. If $f^*D \sim_{\mathbb{Q}} \lambda D$ then $\hat{h}_D = h_D + O(1)$.*

3. *If* $\hat{h}_D(P) \neq 0$ *then* $\alpha_f(P) \geq \lambda$.

4. *If* $\lambda = \lambda_1(f)$ *and* $\hat{h}_D(P) \neq 0$ *then* $\alpha_f(P) = \lambda_1(f)$.

5. *If we are working over a number field and* D *is ample then* $\hat{h}_D(P) = 0 \iff P$ *is pre-periodic for* f.

We will need the following generalization to Jordan blocks, however in certain cases (4.2.3) is sufficient and is the prototypical result.

Theorem 4.2.4 ([28, Theorem 13]). *Let* X *be a normal projective variety over* $\bar{\mathbb{Q}}$ *and let* $f\colon X \to X$ *be a surjective endomorphism. Let* $\lambda \in \mathbb{C}$ *with* $|\lambda| > 1$. *Let* $D_0, ..., D_p \in \mathrm{Div}(X)_{\mathbb{C}}$ *with* $f^*D_0 \sim \lambda D_0$ *and for* $i \geq 1$ *we have* $f^*D_i \sim \lambda D_i + D_{i-1}$. *We say that the* D_i *are in Jordan block form. For each* D_i *choose a Weil height* h_{D_i}. *Then we have the following.*

1. *For each* i *there are canonical height functions* $\hat{h}_{D_i}\colon X(\mathbb{C}) \to \mathbb{C}$ *such that* $\hat{h}_{D_i} = h_{D_i} + O(1)$ *and* $\hat{h}_{D_i} \circ f = \lambda \hat{h}_{D_i} + \hat{h}_{D_{i-1}}$ *where we set* $\hat{h}_{D_{-1}} = 0$.

2. *We have the following recursive formula.*

$$\hat{h}_{D_k}(x) = \lim_{n\to\infty} (\lambda^{-n} h_{D_k}(f^n(x)) - \sum_{i=1}^{k} \binom{n}{i} \lambda^{-i} \hat{h}_{k-i}(x)).$$

To use the above result we follow the ideas of Kawaguchi and Silverman in ([28, Section 4]). We will often use the following notation.

Notation 4.2.4.1. *Choose an ample divisor* $H \in \mathrm{Div}(X)$.

1. *We define* V_H *to be the vector space spanned by* $(f^n)^*H$ *for all* $n \geq 0$. *By ([28]) this is a finite dimensional space.*

2. *Notice that by construction we have a linear mapping* $f^*\colon V_H \to V_H$. *We will be interested in the eigenvalues of this linear mapping.*

3. *Let* $\lambda_1, ..., \lambda_\sigma, \mu_{\sigma+1}, ..., \mu_d$ *be the eigenvalues of* $f^* |_{V_H}$ *ordered such that*

$$|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_\sigma| > 1 \geq |\mu_{\sigma+1}| \geq ... \geq |\mu_d|.$$

   We define $l = l_H$ *to be the number of eigenvectors* λ_i *such that* $|\lambda_i| = \lambda_1(f)$. *In particular we have that*

$$|\lambda_i| = \lambda_1(f) \iff 1 \leq i \leq l.$$

4. *By possibly extending scalars we can find a Jordan form for f^* on V_H which means we find divisors $D_1, ..., D_p$ with $f^*D_i \sim \lambda_i D_i$ or $f^*D_i \sim \lambda_i D_i + D_{i-1}$. We say the choice of divisors D_i is a Jordan form for f^*.*

With this notation we have the following key result.

Theorem 4.2.5 ([28, Section 4])**.** *With the above notation in place let $P \in X(\bar{\mathbb{Q}})$.*

1. *Then $\alpha_f(P) = 1$ or $\alpha_f(P) =\mid \lambda_i \mid$. More precisely suppose that $\hat{h}_{D_i}(P) \neq 0$ for some $1 \leq i \leq \sigma$. Let k be the smallest index with $\hat{h}_{D_k}(P) \neq 0$. Then $\alpha_f(P) =\mid \lambda_k \mid$. On the other hand if $\hat{h}_{D_k}(P) = 0$ for all $1 \leq k \leq \sigma$ then $\alpha_f(P) = 1$.*

2. *In particular if $\mid \lambda_i \mid= \lambda_1(f)$ for $i = 1, ..., l$ and $\mid \lambda_{l+1} \mid< \lambda_1(f)$ then $\alpha_f(P) = \lambda_1(f) \iff \hat{h}_{D_i}(P) \neq 0$ for some $1 \leq i \leq l$.*

4.2.2 The Kawaguchi-Silverman and sAND conjectures

The Kawaguchi-Silverman conjecture is a type of Ergodic theorem in arithmetic dynamics that predicts a relationship between two numerical invariants of a dominant rational map. For simplicity we describe the conjecture for surjective endomorphisms, which is our main focus. Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety defined over $\bar{\mathbb{Q}}$. Associated to f are two basic numerical invariants: The *dynamical degree* is defined as

$$\lambda_1(f) = \lim_{n\to\infty}((f^n)^*H \cdot H^{\dim X-1})^{\frac{1}{n}}, \tag{4.2}$$

where H is any ample divisor. We think of the dynamical degree as a *geometric*/spacial measure of the complexity of f under composition. There is also the arithmetic degree of a point $P \in X(\bar{\mathbb{Q}})$

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{\frac{1}{n}}, \tag{4.3}$$

where h_X is any ample height function on X and $h_X^+ = \max\{1, h_X(P)\}$. We think of the arithmetic degree as a measure of the *arithmetic* complexity of the f-orbit of P, or a discrete time dependent measure of the complexity of f. The arithmetic degree measures the rate of growth of the arithmetic complexity of the forward orbit $P, f^1(P), f^2(P), ...f^n(P), ...$ as n grows arbitrarily large.

We can now state the Kawaguchi-Silverman conjecture for a surjective endomorphism.

Conjecture 3 ([29]). *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Let $P \in X(\bar{\mathbb{Q}})$ and suppose that the orbit $\mathcal{O}_f(P) := \{P, f(P), f^2(P), ...\}$ is Zariski dense in X. Then $\alpha_f(P) = \lambda_1(f)$.*

The conjecture states that when f *mixes* the point P around X very generally, then the space notion of complexity equals the time notion of complexity. See [29, 28, 27, 40, 34] for some recent work in this area.

The requirement of a dense forward orbit makes the conjecture interesting only when $\kappa(X) \leq 0$. When $\kappa(X) > 0$ the existence of the Iitaka fibration over a positive dimensional base makes the existence of a dense forward orbit impossible. In light of this the following conjecture was proposed, which is interesting for all Kodaira dimensions and implies Conjecture (3).

Conjecture 4 (The sAND conjecture [39, Conjecture 1.4]). *Let X be a normal projective variety and $f\colon X \to X$ a surjective endomorphism. Let*

$$\mathcal{S}(X, f, N) = \{P \in X(K) : [K : \mathbb{Q}] \leq N, \alpha_f(P) < \lambda_1(f)\}$$

Then $\mathcal{S}(X, f, N)$ is not Zariski dense in X.

Here sAND means small arithmetic non-density. We immediately obtain the following.

Proposition 4.2.6. *Let X be normal projective variety and $f\colon X \to X$ a surjective endomorphism. Suppose that conjecture 4 is true for surjective endomorphisms of X. If $P \in X(\overline{\mathbb{Q}})$ and $\mathcal{O}_f(P)$ has a Zariski dense orbit then $\alpha_f(P) = \lambda_1(f)$*

Proof. We have that $\alpha_f(f^n(P)) = \alpha_f(P)$ for any $n \geq 0$. Towards a contradiction suppose that $\alpha_f(P) < \lambda_1(f)$. Choose N so that $P \in K$ where K is a number field of degree at most N. Then

$$\mathcal{O}_f(P) \subseteq \mathcal{S}(X, f, N) = \{Q \in X(K) : [K, \mathbb{Q}] \leq N, \alpha_f(Q) < \lambda_1(f)\}$$

and consequently $\mathcal{S}(X, f, N)$ is Zariski dense in X contradicting our assumption. So $\alpha_f(P) = \lambda_1(f)$ as needed. □

Thus the Kawaguchi-Silverman conjecture is implied by the sAND conjecture. The sAND stands for small arithmetic degrees are non dense.

We collect various useful results. We will make use of the fact that the set of points in X with $\alpha_f(P) = \lambda_1(f)$ is Zariski dense.

Theorem 4.2.7 ([54, Theorem 1.8]). *Let X be a projective variety defined over $\bar{\mathbb{Q}}$ and $f\colon X \to X$ be a surjective endomorphism with $\lambda_1(f) > 1$. Then the set of points $P \in X(\bar{\mathbb{Q}})$ with $\alpha_f(P) = \lambda_1(f)$ is Zariski dense.*

We often will use the following fundamental theorem.

Theorem 4.2.8 ([29, Corollary 27][37, Theorem 1.4]). *Let X be a normal projective variety defined over a number field K and $f\colon X \to X$ a surjective endomorphism. If $P \in X(\bar{K})$ then $\alpha_f(P) \leq \lambda_1(f)$.*

We have the following key lemma.

Proposition 4.2.9 (Proposition 3.6, [38]). *Let X be a normal projective variety and $f\colon X \to X$ a surjective endomorphism with $\lambda_1(f) > 1$. Suppose that there is an non-trivial integral $\mathbb{Q}$-cartier divisor D with $f^*D = \lambda_1(f)D$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ with $\kappa(D) > 0$. Then the Kawaguchi-Silverman conjecture holds for f.*

The following result is crucial in the study of surjective morphisms of Mori-fiber spaces.

Theorem 4.2.10 (Product theorem for the relative dynamical degree. [34, Theorem 2.2]). *Suppose that we have a commuting diagram of normal projective varieties defined over $\bar{\mathbb{Q}}$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where f, g, π are all surjective morphisms. Then

$$\lambda_1(f) = \max\{\lambda_1(f\mid_{\pi}), \lambda_1(g)\}$$

We obtain immediately

Corollary 4.2.10.1 ([34, Definition 2.7]). *Suppose that we have a commuting diagram of normal projective varieties defined over $\bar{\mathbb{Q}}$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where f, g, π are all surjective morphisms. Suppose that the Kawaguchi-Silverman conjecture holds for g and $\lambda_1(f) = \lambda_1(g)$. Then the Kawaguchi-Silverman conjecture holds for f.

Proof. Let $P \in X(\bar{\mathbb{Q}})$ be a point with dense f-orbit. Then $\pi(P)$ has a dense g orbit and we have that $\alpha_f(P) \geq \alpha_g(\pi(P)) = \lambda_1(g) = \lambda_1(f)$ by the Kawaguchi-Silverman conjecture for g and our assumption on the dynamical degree. Since we know that $\alpha_f(P) \leq \lambda_1(f)$ the result follows. $\square$

Chapter 5

Good Eigenspaces

Let X be a sufficiently nice projective variety defined over $\overline{\mathbb{Q}}$. Suppose that $f\colon X \to X$ is a surjective endomorphism. Let $P \in X(\overline{\mathbb{Q}})$ with $\mathcal{O}_f(P)$ being Zariski dense in X. We would like to estimate the arithmetic degree $\alpha_f(P)$. One way to do this is as follows. Choose an ample divisor H and form the subspace $V_H \subseteq \mathrm{Pic}(X)_{\mathbb{C}}$ as in 4.2.4.1. After finding a Jordan form for f^* acting on V_H we have eigenvalues

$$\lambda_1, ..., \lambda_\sigma, \lambda_{\sigma+1}, ..., \lambda_\rho$$

with $|\lambda_i| \geq |\lambda_{i+1}|$ and $|\lambda_i| > 1$ for $i \leq \sigma$ and $|\lambda_j| \leq 1$ for $j \geq \sigma + 1$. Associated to each $1 \leq i \leq \sigma$ there is a canonical height function

$$\hat{h}_{D_i}\colon X(\mathbb{C}) \to \mathbb{C}$$

with the property that

$$\hat{h}_{D_i}(P) \neq 0 \Rightarrow \alpha_f(P) \geq |\lambda_i|.$$

Thus to obtain lower bounds for the canonical height it suffices to show that certain Jordan block canonical heights do not vanish when P has a Zariski dense orbit under f. To do this, we will mostly work with λ_i such that $|\lambda_i| = \lambda_1(f)$ and that $\lambda_1(f) \in \mathbb{Z}$. More specifically we will desire that there is a $\lambda_1(f)$-Jordan block whose divisors D_i are integral. When the divisors D_i are integral and $\kappa(D_i) > 0$ then the base locus of the D_i can be used to show that $\hat{h}_{D_i}(P) \neq 0$ when P has a Zariski dense orbit. This approach to estimating the Kawaguchi-Silverman conjecture is discussed and expounded upon in 5.1. In 5.2.3 we specialize to the case of varieties with Picard number two and in 5.3 we further specialize to the case of projective bundles over an elliptic curve.

5.1 Invariance of base locus

One of our goals in this thesis is to study which divisors D can occur as an integral eigendivisor D. That is divisors D for which $f^*D \equiv_{\mathrm{lin}} \lambda D$ where D is non-zero and $\lambda = \lambda_1(f) \in \mathbb{Z}$. In this study we encounter the following trichotomy.

1. We could have $\kappa(D) > 0$. Then by taking a large multiple of D we have rational mapping
$$\phi_{|mD|} \colon X \dashrightarrow \mathbb{P}H^0(X, mD)$$
whose image has dimension $0 < \kappa(D) \leq \dim X$. Let Y be the image of $\phi_{|mD|}$. Matsuzawa exploited this situation as follows in [38]. Resolve the indeterminacy of the rational map $\phi_{|mD|}$ by blowing up the base locus B of mD. We obtain a diagram
$$\begin{tikzcd} Z \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ X \arrow[r, dotted, "\phi_{|mD|}"'] & Y \end{tikzcd}$$
Matsuzawa constructs a **finite** set $S = S(D) \subseteq Y$ such that
$$\{P \in X(\overline{\mathbb{Q}}) \colon \hat{h}_D(P) = 0\} \subseteq \pi(E) \cup \pi(\varphi^{-1}(S))$$
where E is an exceptional divisor of the blow up. Since S is finite and $\dim Y > 0$ we have that $\varphi^{-1}(S)$ is a proper closed subset of Z. Since π is birational we have that $\pi(\varphi^{-1}(S))$ is a proper closed subset of X. On the other hand as E is exceptional we have that $\pi(E)$ is a proper closed subset of X. Finally as X is irreducible we have that $\pi(E) \cup \pi(\varphi^{-1}(S))$ is a proper closed subset of X and consequently $\{P \in X(\overline{\mathbb{Q}}) \colon \hat{h}_D(P) = 0\}$ is not Zariski dense. The Kawaguchi-Silverman conjecture now follows easily.

2. If $\kappa(D) = 0$ then the above method breaks down: In this case $\dim Y = 0$ and in fact is a point as otherwise X would not be irreducible. Thus the mapping $\phi_{|mD|}$ extends to a morphism $X \to \mathrm{pt}$ and there is no rational map to resolve. One can still blow up the base scheme of D but we do not have a morphism to a positive dimensional variety to work with as we do above, so new ideas required.

3. If $\kappa(D) = -\infty$ then the method completely breaks down as there is no base locus to work with. As above new ideas are required.

We will concentrate on the second situation in this thesis. We begin by analyzing the base locus of an eigendivisor and show that it is invariant under f.

5.1.1 Eigendivisors with $\kappa(D) = 0$

Let X be a normal projective variety and suppose that $f\colon X \to X$ is a surjective endomorphism. Suppose that we have a divisor D with D not linearly equivalent to 0 with $f^*D \equiv_{\text{lin}} \lambda D$ in the Picard group of X. When $\kappa(D) > 0$ we have the following motivating result described in passing above.

Proposition 5.1.1 ([38, Proposition 3.5])**.** *Let X be a normal projective variety defined over a number field K. Let $f\colon X \to X$ be a surjective morphism defined over K. Let D be a $\mathbb{Q}$-divisor on X with $f^*D \equiv_{\text{lin}} \lambda_1(f)D$ with $\lambda_1(f) > 1$ and $\kappa(D) > 0$. Fix positive constants A, B. Then the set $\{P \in X(L) : [L : K] \leq A, \hat{h}_D(P) \leq B\}$ is not Zariski dense in X.*

A natural weakening of Proposition 5.1.1 is to allow $\kappa(D) = 0$.

Lemma 5.1.2. *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$ and let $f\colon X \to X$ be a finite surjective endomorphism. Take D be a non-principal integral divisor with $f^*D \equiv_{\text{lin}} \lambda D$ for some integral $\lambda > 1$. Suppose that*

1. *$\boldsymbol{B}(D) = \mathrm{Bs}(mD)$ for all $m \geq 1$*
2. *$H^0(X, mD) \cong H^0(X, D)$ for all $m \geq 1$.*

Then $f^{-1}(\boldsymbol{B}(D)) = \boldsymbol{B}(D)$.

Proof.

$$f^*\colon H^0(X, D) \to H^0(X, \lambda D)$$

is an isomorphism. Thus for all $s' \in H^0(X, \lambda D)$ we have $s' = f^*s$ for some $s \in H^0(X, D)$. It follows that $P \in \mathrm{Bs}(\lambda D) = \boldsymbol{B}(D)$ as needed. □

Proposition 5.1.3. *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Let D be a non-principal divisor with $f^*D \sim_{\mathbb{Q}} \lambda D$ for some integral $\lambda > 1$. Suppose that $\kappa(D) = 0$. Then $f^{-1}(\boldsymbol{B}(D)) = \boldsymbol{B}(D)$.*

Proof. Since f^* induces an injection on the group of sections

$$\dim H^0(X, D) = \dim f^*H^0(X, D) \leq \dim H^0(X, \lambda D).$$

As $\kappa(D) = 0$ there is some m_2 such that

$$\dim H^0(X, mD) = \dim H^0(X, m_2 D)$$

for all $m \geq m_2$. We may choose a large integer $m_1 \geq m_2$ such that

$$\mathrm{Bs}(km_1 D) = \boldsymbol{B}(D)$$

for all $k \geq 1$ by ([31, 2.1.21]). In conclusion we have verified the needed hypothesis to apply Lemma 5.1.2. □

Corollary 5.1.3.1. *Let X be a normal projective variety and let $f\colon X \to X$ be a surjective endomorphism. Let D be a non-principal divisor with $f^*D \sim_{\mathbb{Q}} \lambda D$ for some integral $\lambda > 1$. Suppose that $\kappa(D) = 0$. Then there is a integral multiple D' of D such that*

$$f^{-1}(\mathrm{Bs}(D')) = \mathrm{Bs}(D').$$

Suppose that f was not an automorphism. We have $f\colon \mathrm{Bs}(D) \to \mathrm{Bs}(D)$. Since f is surjective the restriction is surjective. After possibility iterating f we may assume that f fixes the components of $\mathrm{Bs}(D)$. This puts some restrictions on the possibilities of $\mathrm{Bs}(D)$. For example, it must admit a surjective endomorphism which places structural constraints on the base locus. For example if $\mathrm{Bs}(D)$ is one dimensional then its components must all be curves of genus 0 or 1. As a general type curve does not admit a surjective endomorphism that is not an automorphism.

5.1.2 Numerical vs Linear equivalence

A basic problem to be overcome is the following. Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X. Let $\lambda_1(f) = \lambda > 1$. Then f^* acting on $N^1(X)_{\mathbb{R}}$ has λ as an eigenvalue of maximal absolute value. However, we would like to have λ as an eigenvalue of largest absolute value when acting on $\mathrm{Pic}(X)_{\mathbb{R}}$. This is almost true in the following sense.

Proposition 5.1.4. *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X. Let $\lambda_1(f) = \lambda > 1$. Then there is an eigenvalue λ' for f^* acting on $\mathrm{Pic}(X)_{\mathbb{R}}$ such that $\mid \lambda' \mid = \lambda$.*

Proof. Choose an ample divisor H for X. Following (4.2.4.1) we have a finite dimensional vector space V_H. Let $E_1, ..., E_p$ be a Jordan form for f^* after possibly extending scalars. By Theorem 4.2.7 we can find a point P such that $\alpha_f(P) = \lambda$. However by Theorem 4.2.5 we have that $\lambda = \alpha_f(P) = \mid \lambda_i \mid$. □

Corollary 5.1.4.1. *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X. Let $\lambda_1(f) = \lambda > 1$. Suppose that λ is the unique eigenvalue of f^* acting on $N^1(X)_{\mathbb{R}}$ of largest absolute value. Then λ appears as an eigenvalue of f^* acting on* $\mathrm{Pic}(X)_{\mathbb{R}}$

Proof. By Proposition 5.1.4 we have that there is an eigenvalue λ' of f^* acting on $\mathrm{Pic}(X)_{\mathbb{R}}$ of absolute value λ. Thus λ' is an eigenvalue of absolute value λ for the action of f^* on $N^1(X)_{\mathbb{R}}$. However by assumption this means have $\lambda' = \lambda$. $\square$

The above situation happens in practice.

Theorem 5.1.5 ([59, Theorem 6.1])**.** *Let $f\colon X \to X$ be a surjective endomorphism of smooth projective varieties. Assume that $\lambda_1(f)^2 > \lambda_2(f)$. Then $\lambda_1(f)$ is a simple eigenvalue of f^* and is the only eigenvalue of modulus greater than $\sqrt{\lambda_2(f)}$.*

Here $\lambda_2(f)$ is a numerical invariant of f related to how f interacts with codimension 2 sub-varieties. So we should expect that it is often the case that there is a unique eigenvalue of largest absolute value and thus can find an eigendivisor D for $\lambda_1(f)$ for linear equivalence.

5.1.3 Analysis of Jordan Blocks

Here we begin exploring our earlier results about eigendivisors with $\kappa(D) = 0$ in the context of Kawaguchi-Silverman and the sAND conjectures. Our main tools will be ([28, Section 4]) and ([54, Theorem 1.8]) Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety over a number field K. Suppose that $\lambda_1(f) = \lambda > 1$. Choose an ample divisor H of X and follow (4.2.4.1). Let $d_H = d = \dim V_H$. After possibly extending scalars let $E_1, ..., E_d$ be divisors such that the E_i are a basis for d and the associated matrix of f^* acting on V_H is in Jordan form. Since $\lambda > 1$ recall that we have $l = l_H$ such that for $i \leq l$ we have $\mid \lambda_i \mid = \lambda$ and $\mid \lambda_{l+1} \mid < \lambda$.

Definition 5.1.6. *Using (4.2.4.1) define*

$$G_{f,H} = G = \{P \in X(K) : \hat{h}_{\lambda_i}(P) = 0 \text{ for } 1 \leq i \leq l\}.$$

This is the set of points of small height, relevant for the sAND conjecture, see conjecture 4.

Proposition 5.1.7. *Let $G_{f,H}$ be as in Definition 5.1.6. Suppose that $G_{f,H}$ is not dense. Then the Kawaguchi-Silverman conjecture holds for f.*

Proof. Let P be a point with a dense f orbit. I claim that $P \notin G$. Towards a contradiction suppose that $P \in G$. We show by induction that $f^n(P) \in G$ for all n contradicting that G is not dense in X. We have that $\hat{h}_{\lambda_1}(P) = 0$. This gives

$$\hat{h}_{\lambda_1}(f^n(P)) = \lambda_1^n \hat{h}_{\lambda_1}(P) = 0.$$

Thus for $i \leq l$ we may assume that for all $j < i$ and all n we have that $\hat{h}_{\lambda_j}(f^n(P)) = 0$. By assumption we have

$$\hat{h}_{\lambda_i}(P) = 0$$

and so for $n > 1$ we have

$$\hat{h}_{\lambda_i}(f^n(P)) = \lambda_i \hat{h}_{\lambda_i}(f^{n-1}(P)) + \hat{h}_{\lambda_{i-1}}(f^{n-1}(P)) = \lambda_i \hat{h}_{\lambda_i}(f^{n-1}(P))$$

by induction. Since we can repeat this process we have that

$$\hat{h}_{\lambda_i}(f^n(P)) = 0.$$

Thus if $P \in G$ then $\mathcal{O}_f(P) \subseteq G$ contradicting that G is not Zariski dense. Thus we have that if $\mathcal{O}_f(P)$ is dense then $P \notin G$. Then for some $i \leq l$ we have that $\hat{h}_{\lambda_i}(P) \neq 0$. Then ([28, Section 4]) shows that $\alpha_f(P) = |\lambda_i| = \lambda$ as needed completing the proof. □

The above lemma shows that the Kawaguchi-Silverman conjecture for morphisms is true provided one shows that the canonical Jordan block heights cannot cut out a Zariski dense set. We will use this principle to prove Kawaguchi-Silverman in some cases. We obtain the following rephrasing of the Kawaguchi-Silverman conjecture.

Corollary 5.1.7.1. *The Kawaguchi-Silverman conjecture for an endomorphism f with $\lambda = \lambda_1(f) > 1$ is equivalent to $G_{f,H}$ contains no dense orbit of f.*

Proof. Suppose the Kawaguchi-Silverman conjecture holds for f. If f has no dense forward orbit then trivially $G_{f,H}$ contains no forward orbit. Otherwise let $\mathcal{O}_f(P)$ be dense. Then by Kawaguchi-Silverman $\alpha_f(P) = \lambda$. By ([28, Section 4]) we have that $\alpha_f(P) = \lambda \iff \hat{h}_{\lambda_i}(P) \neq 0$ for some $1 \leq i \leq l$. So in particular, $P \notin G_{f,H}$ and so $\mathcal{O}_f(P)$ is not contained in $G_{f,H}$. On the other hand suppose that $G_{f,H}$ contains no dense orbits. Let $\mathcal{O}_f(P)$ be a dense orbit, by assumption $\mathcal{O}_f(P)$ is not contained in $G_{f,H}$. Arguing as in Proposition 5.1.7 we have that $P \notin G_{f,H}$ which means $\hat{h}_{\lambda_i}(P) \neq 0$ for some $1 \leq i \leq l$ which means $\alpha_f(P) = \lambda$. □

So we may think of the Kawaguchi-Silverman conjecture as a statement about the structure of the set $G_{f,H}$. In fact, this set $G_{f,H}$ does not depend on H and has been studied recently in ([39]). Notice that $G_{f,H} = \{P \in X(K) : \alpha_f(P) < \lambda\}$, which is the set of points of small arithmetic degree. The following is a slight refinement of Theorem 4.2.7.

Proposition 5.1.8. *Let $f\colon X \to X$ be an endomorphism of a normal projective variety over a number field K with $\lambda_1(f) > 1$. Let $H \in \mathrm{Div}(X)$ be ample and take V_H as in (4.2.4.1). Let $E_1, ..., E_\rho$ be a basis of $f^* |_{V_H}$ in Jordan block form. Define*

$$\mathcal{B} = \{P \in X(\bar{K}) : \hat{h}_{E_i}(P) \neq 0 \text{ for some } 1 \leq i \leq l\}.$$

Then $\mathcal{B}$ is dense in X.

Proof. Suppose that $\mathcal{B} \subseteq Y$ where Y is closed. Then choose $P \notin Y$ with $\alpha_f(P) = \lambda_1(f)$ using Theorem 4.2.7. Since $P \notin \mathcal{B}$ we have that

$$\hat{h}_{E_i}(P) = 0$$

for all $1 \leq i \leq l$. If for some $l+1 \leq i \leq \sigma$ we have $\hat{h}_{E_i}(P) \neq 0$ then by choosing i minimal and applying ([28, Section 4]) we have

$$\alpha_f(P) = |\lambda_i| < \lambda$$

a contradiction. So we have that for all $i \leq \sigma$ we have $\hat{h}_{E_i}(P) = 0$ which by ([28, Section 4]) gives $\alpha_f(P) = 1$ which is again impossible. □

Proposition 5.1.9. *Let $f\colon X \to X$ be an endomorphism of a normal projective variety over a number field K and use the notation of Proposition 5.1.8. Suppose that for some $1 \leq i \leq l$ we have that E_i is a $\mathbb{Q}$-divisor class with $\kappa(E_i) > 0$. Then Kawaguchi-Silverman and the sAND conjecture hold for f.*

Proof. The sAND conjecture is the same as $G_{f,H}$ as defined in Definition 5.1.6 not being dense. Notice that

$$G_{f,H} \subseteq \{P \in X(K) : \hat{h}_{E_i}(P) = 0\}$$

by the definition of $G_{f,H}$. By ([38, Proposition 3.5]) we have that $\{P \in X(K) : \hat{h}_{E_i}(P) = 0\}$ is not Zariski dense and the result follows. □

5.2 Some applications of eigendivisors with few sections

5.2.1 The case of a finitely generated nef cone

Let X be a normal projective variety defined over a number field K with a finitely generated (not necessarily rational) nef cone, and Picard number $\rho(X) = \rho$. Suppose that $f\colon X \to X$ is a surjective endomorphism. Suppose that the nef cone of X has rays $v_1, ..., v_s$. Since the action of f^* is a linear isomorphism at the level of vector spaces that preserves the nef cone it preserves the boundary and so acts as a permutation on the rays. After iterating f^* we may assume that f^* fixes all of the rays. Since the nef cone is a full dimensional pointed cone, we have that $s \geq \rho$, and our assumption means that

$$f^* v_i = \lambda_i v_i.$$

In particular, by taking a linearly independent set of rays say $v_1, ..., v_\rho$ we have that the action of f^* on $N^1(X)_{\mathbb{R}}$ is diagonalizable over $\mathbb{R}$. Furthermore if μ is any eigenvalue of f^* then $\mu = \lambda_i$ for some i and the μ eigenspace has a basis

$$\{v_j : \lambda_j = \mu\}.$$

To see why let w be any μ eigenvector. Then we can write

$$w = \sum_{i=1}^{\rho} t_i v_i \text{ and } \sum_{i=0}^{\rho} \mu t_i v_i = \mu w = f^* w = \sum_{i=1}^{\rho} \lambda_i t_i v_i$$

So

$$\mu t_i = \lambda_i t_i$$

Since some $t_i \neq 0$ we see that $\lambda_i = \mu$ for some i and that $t_j \neq 0 \Rightarrow \lambda_j = \mu$.

Definition 5.2.1. *Let $T\colon V \to V$ be an invertible linear transformation of a finite dimensional real vector space that is diagonalizable. Let C be a full dimensional pointed closed cone in V with $T(C) = C$ and the rays of V contain a basis of eigenvectors for T. Then given an eigenvalue λ we say C separates the λ-eigenspace if there exists λ-eigenvectors v, w with v and w not lying on a common face.*

Proposition 5.2.2. *Let X be a normal projective variety defined over a number field K with a finitely generated (not necessarily rational) nef cone. Suppose that we are given a surjective endomorphism $f\colon X \to X$. Suppose that f^* preserves the rays of the nef cone* $\mathrm{Nef}(X)_{\mathbb{R}}$ *and has positive eigenvalues. (This can always be achieved after iterating f) Then* $\mathrm{Nef}(X)_{\mathbb{R}}$ *separates a λ-eigenspace for some eigenvalue λ of f^* if and only if f^* is a dilation*

Proof. If f^* is a dilation by λ then it certainly separates a λ-eigenspace. Now suppose that f^* separates a λ-eigenspace for some eigenvalue λ. There are λ-eigenvectors v, w that do not lie on the same face of $\mathrm{Nef}(X)_{\mathbb{R}}$. Thus we have that $v + w$ is a λ-eigenvector on the interior of the nef cone. By Theorem 2.1.12 we have that all eigenvalues of f^* have the same modulus. Since f^* has real eigenvalues that are positive, all the eigenvalues coincide and f is a dilation by λ as needed. □

We see that the obstruction to Kawaguchi-Silverman when the nef cone is finitely generated is that the eigenvectors of f^* accumulate on a single facet of the nef cone. The philosophy is that as the nef cone of a variety gets more complicated, it becomes more and difficult for an endomorphism to preserve the nef cone unless the endomorphism is something like a dilation. We make this notion precise in the context of Kawaguchi-Silverman.

Theorem 5.2.3. *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$. Suppose that X has a finitely generated nef cone and $\rho = \rho(X)$ is the Picard number of X. Let s be the number of rays of* $\mathrm{Nef}(X)$*. Let $f\colon X \to X$ be a surjective endomorphism and let t be the number of distinct eigenvalues of f^* and let q be the maximal number of rays in a facet of* $\mathrm{Nef}(X)$*. If $s \geq tq + 1$ then some iterate f^n has $(f^n)^*$ acting by a dilation and in particular Kawaguchi-Silverman and the sAND conjectures hold for X. In particular if $\rho = 3$ then if $s \geq 3(3-1)+1 = 7$ the conclusion holds.*

Proof. After iterating f we may assume that f^* fixes the rays of the nef cone and has positive eigenvalues. Let λ be an eigenvalue of f^* acting on $N^1(X)_{\mathbb{R}}$. Towards a contradiction suppose that f^* is not acting by a dilation. Suppose that an eigenvalue λ has eigenvector v_λ on a facet F of $\mathrm{Nef}(X)$. If λ appears as an eigenvalue of a ray on a different facet F' of $\mathrm{Nef}(X)$ we have that λ appears as the eigenvalue of a ray in $F \cap F'$, otherwise Proposition 5.2.2 implies that f^* acts by a λ dilation contradicting our assumptions. Thus there are at most q rays that are an eigenvector for λ. Since there are t eigenvalues we have that there are at most tq rays, a contradiction. If $\rho = 3$ then a facet is 2 dimensional cone, and thus has 2 rays. So $q = 2$ and $t \leq 3$. □

5.2.2 Varieties with a good eigenspace

We recall the following situation which is the crux of the Kawaguchi-Silverman and sAND conjecture for varieties admitting an int-amplified endomorphism.

Definition 5.2.4 ([44])**.** *Let X be an n-dimensional normal $\mathbb{Q}$-factorial projective variety with at worst klt singularities that admits an int-amplified endomorphism. Let $f\colon X \to X$ be a surjective endomorphism. The following situation is called case* TIR_n*. We assume the following conditions.*

1. *The anti canonical class $-K_X$ is nef and not big with $\kappa(X, -K_X) = 0$.*
2. *$f^*D = \lambda_1(f)D$ for some effective irreducible $\mathbb{Q}$-divisor D with D linearly equivalent to $-K_X$ and $\kappa(D) = 0$.*
3. *The support of the ramification divisor of f is D.*
4. *There is an f-invariant Mori fiber space*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

   with $\lambda_1(f) > \lambda_1(g)$.

5. $\dim X \geq \dim Y + 2 \geq 3$.

Question ([44, Question 1.8])**.** *Does case* TIR_n *ever occur?*

The interest in this technical case is that it is the remaining obstacle to performing an f invariant minimal model program to prove the Kawaguchi-Silverman and sAND conjectures. Notice that in this case, it follows that the eigenspace of $\lambda_1(f)$ is 1-dimensional by the condition $\lambda_1(g) < \lambda_1(f)$.

Theorem 5.2.5 (Theorem 1.7,[44])**.** *Let X be a normal $\mathbb{Q}$-factorial projective variety with at worst klt singularities that admits an int-amplified endomorphism. If the Kawaguchi-Silverman conjecture holds for the varieties appearing in case* TIR_n *(perhaps vacuously) then the Kawaguchi-Silverman conjecture holds for X.*

We now give a variant of TIR_n that takes into account the possibility of a $\lambda_1(f)$ eigenspace of dimension greater then 1.

Definition 5.2.6. *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety over a number field K with $\lambda = \lambda_1(f) > 1$. Let H be an integral eigendivisor and let V_H be as is (4.2.4.1). We say $f^* |_{V_H}$ has a good eigenspace if $f^* |_{V_H}$ has the following properties,*

1. *λ is the unique eigenvalue of absolute value λ of $f^* \mid_{V_H}$.*

2. *All λ Jordan blocks of $f^* \mid_{V_H}$ are of multiplicity 1.*

3. *The Jordan blocks of $f^* \mid_{V_H}$ associated to λ can be taken to be integral nef divisor classes $D_1, ..., D_l$.*

4. *There is some $1 \leq i \leq l$ such that $\kappa(D_i) \neq 0$.*

The benefit of this definition is that it is relatively simple, but has interesting consequences. It is our hope that the definition will motivate additional research into which varieties have surjective endomorphisms with a good eigenspace.

Proposition 5.2.7. *Assume that every surjective endomorphism $f\colon X \to X$ that satisfies (1)-(3) of 5.2.6 also satisfies (4) of 5.2.6. Then* TIR_n *does not occur for varieties with surjective endomorphism.*

Proof. Towards a contradiction suppose that case TIR_n does occur. In other words suppose that we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

satisfying the assumptions of 5.2.4. Let H be an ample integral eigendivisor. Let $\lambda' \neq \lambda$ be an eigenvalue of $f^*\colon V_H \to V_H$ with $|\lambda'| = \lambda_1(f) = \lambda$. Then by 5.2.9 we have that λ' has an eigendivisor $D \in V_H$ that is not numerically trivial. Then we have that $f^*D \equiv_{\text{num}} \lambda' D$ and so λ' is also an eigenvalue of $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. Since ϕ is a Mori-Fiber space we have that

$$\dim N^1(Y)_{\mathbb{R}} = \dim N^1(X)_{\mathbb{R}} - 1. \tag{5.1}$$

As $\lambda' \neq \lambda$ and λ is an eigenvalue of $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ we have that λ' is an eigenvalue of $g^*\colon N^1(Y)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$. Then $\lambda_1(g) \geq |\lambda'| = \lambda_1(f)$ contradicting (4) of definition 5.2.4. Thus λ is the unique eigenvalue of V_H of magnitude λ and (1) of 5.2.6 is satisfied. Since ϕ is a Mori-fiber space we have that (2) and (3) of 5.2.6 are satisfied because of equation 5.1. Thus by assumption we have that $f^* \mid_{V_H}$ has a good eigenspace. Let D be as in 5.2.4 (2). By 5.2.10 we have that $D \in V_H$. Thus D itself is a basis for the λ-Jordan block of f^* acting on V_H. Then by 5.2.6 (4) we have that $\kappa(D) > 0$. However this contradicts (2) of 5.2.4. So case TIR_n does not occur. $\square$

We see that the existence of a good eigenspace can be seen as an extension of case TIR_n. However, it is potentially more general as the true strength of case TIR_n relies on running a minimal model program for varieties with int-amplified endomorphisms. Yet the existence of a good eigenspace is potentially useful even when no int-amplified morphisms are present, and thus the full power of the minimal model program may not be available.

Proposition 5.2.8. *Let X be a normal projective variety with finitely generated rational nef cone. Suppose that for every nef divisor class D in $\mathrm{Pic}(X)_{\mathbb{Q}}$ there is an integer $m \geq 1$ such mD is effective. If $f\colon X \to X$ is a surjective endomorphism, and H is an integral ample divisor such that $f^* |_{V_H}$ has a good eigenspace, then the Kawaguchi-Silverman conjecture holds for f.*

Proof. If $\lambda_1(f) = 1$ the result follows, so assume $\lambda > 1$. After iterating f^* we may assume that f^* fixes the rays of the nef cone. Furthermore we have that the eigenvalues of f^* must be rational and thus integers since they are also algebraic integers. Let $D_1, ..., D_l$ be a basis of eigendivisors associated to $\lambda_1(f)$ for the action of $f^* |_{V_H}$. As $f^* |_{V_H}$ has a good eigenspace we may take the D_i nef and $\kappa(D_i) \geq 0$ by our assumption on X. Since $f^* |_{V_H}$ has a good eigenspace for some j we have $\kappa(D_j) \neq 0$. As $\kappa(D_j) \geq 0$ we have $\kappa(D_j) > 0$ and the result follows from Proposition 4.2.9. □

The proof of Theorem 5.2.8 illustrates the basic idea behind the assumption of a good eigenspace. We attempt to put ourselves in a situation where we know $\kappa(D_i) \geq 0$ and then use the good eigenspace assumption to conclude that $\kappa(D) > 0$.

Question 1. Let X be a normal projective variety defined over a number field and $f\colon X \to X$ a surjective endomorphism.

- What conditions on X guarantee that there is an integral ample divisor H such that $f^* |_{V_H}$ has a good eigenspace.
- Can one find interesting examples where f has no good eigenspace for any ample integral H.
- If X has a finitely generated rational nef cone, is it the case that f has a good eigenspace for some ample integral H.

We now show that one can get information about TIR_n from the assumption of having a good eigenspace. This material may be well known and follows from ([53, Remark 5.9 and 5.10]) in the smooth case, but we provide arguments for completeness as we need to move beyond the smooth setting.

Proposition 5.2.9. *Let $f\colon X \to X$ be a surjective endomorphism with X a normal projective variety defined over $\bar{\mathbb{Q}}$ with surjective Albanese map. Suppose that $f^*A \sim_{\mathbb{R}} \lambda A$ for some non-zero $A \in \mathrm{Pic}^0(X)_{\mathbb{R}}$. Then*

$$\lambda \leq \sqrt{\lambda_1(f)}$$

Proof. Let $\pi\colon X \to \mathrm{Alb}(X)$ be the projection. Recall that π^* induces an isomorphism

$$\pi^*\colon \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$$

on the level of $\mathbb{Z}$-modules and so also as $\mathbb{R}$-vector spaces. Thus we have that $A = \pi^*B$ for some $B \in \mathrm{Pic}^0(\mathrm{Alb}(X))$. We also have a commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathrm{Alb}(X) \arrow[r, "g"'] & \mathrm{Alb}(X) \end{tikzcd}$$

by the universal property of the Albanese variety. Note that the Albanese morphism is surjective. Thus we have that $f^*\pi^*B = \lambda\pi^*B = \pi^*g^*B$. Since π^* is an isomorphism on Pic^0 we have that $g^*B = \lambda B$. We have thus reduced to the case that X is smooth (and in fact an abelian variety). By ([53, Remarks 5.8 and 5.9]) we have that

$$\sqrt{\lambda_1(f)} \geq \sqrt{\lambda_1(g)} \geq \lambda$$

as needed.

□

Corollary 5.2.9.1. *Let $f\colon X \to X$ be a surjective endomorphism with X a normal projective variety defined over $\bar{\mathbb{Q}}$ with surjective Albanese map. Suppose that μ is an eigenvalue of f^* acting on $\mathrm{Pic}(X)_{\mathbb{R}}$ with $|\mu| > \sqrt{\lambda_1(f)}$. Suppose that $D_1, D_2 \in \mathrm{Pic}(X)_{\mathbb{R}}$ are non-zero with $f^*D_i \sim_{\mathbb{R}} \mu D_i$. If $D_1 \equiv_{\mathrm{Num}} D_2$ then we have that $D_1 \sim_{\mathbb{R}} D_2$*

Proof. By assumption we have that $D_1 = D_2 + A_0$ for some $A_0 \in \mathrm{Pic}^0(X)_{\mathbb{R}}$. Then $A_0 = D_1 - D_2$. So $f^*A_0 = \mu A_0$. Since $\mid \mu \mid > \sqrt{\lambda_1(f)}$ we have that $A_0 = 0$ by Proposition 5.2.9. □

We now obtain the following technical result that will be needed later.

Lemma 5.2.10. *Let $f\colon X \to X$ be a surjective endomorphism with X a normal projective variety defined over $\bar{\mathbb{Q}}$ with surjective Albanese map. Suppose that $\lambda_1(f) = \lambda > 1$. Assume the λ-eigenspace of f^* acting on $N^1(X)_{\mathbb{R}}$ is 1 dimensional and that λ appears with multiplicity 1 for the action of f^* on $N^1(X)_{\mathbb{R}}$. Suppose further that there is a unique eigenvalue of largest possible magnitude for the action on $N^1(X)_{\mathbb{R}}$. Suppose that D is a non-zero element of $\operatorname{Pic}(X)_{\mathbb{R}}$ with $f^*D \sim_{\mathbb{R}} \lambda D$. Let H be any integral ample divisor on X and let V_H be as in (4.2.4.1). Then $D \in V_H$.*

Proof. Let $D_1, .., D_p$ be a Jordan form for f^* acting on V_H after extending scalars. Then arguing as in Corollary 5.1.4.1 there is some eigenvalue λ' of f^* on V_H of magnitude λ. Since λ' is also an eigenvalue of f^* acting on $N^1(X)_{\mathbb{R}}$ we have that $\lambda' = \lambda$ by assumption. Let D' be a λ-eigenvalue in V_H which exists by the above argument. Since λ is real we may take D' real and thus have that the numerical class of D' is a λ-eigenvector for f^* on $N^1(X)_{\mathbb{R}}$. Thus $aD' \equiv_{\text{Num}} D$ since both are eigenvectors for the action of f^* on $N^1(X)_{\mathbb{R}}$ and the eigenspace is 1 dimensional. Now Corollary 5.2.9.1 says that $aD' = D$ and so $D \in V_H$ as needed. □

The above result tells us that the sub-spaces V_H in certain situations have base points, which tells us that there are strong restrictions on the subspaces V_H.

Proposition 5.2.11. *Let $f\colon X \to X$ be a surjective endomorphism with X a normal projective variety defined over $\bar{\mathbb{Q}}$ with surjective Albanese map. Let H be an integral ample divisor on X and assume the notation of (4.2.4.1). Let $\lambda = \lambda_1(f) > 1$. Suppose that f^* acting on $N^1(X)_{\mathbb{R}}$ has a 1 dimensional eigenspace and a size 1 Jordan block with a unique eigenvalue of largest size. Then f^* acting on V_H has a size 1 Jordan block and a 1 dimensional eigenspace for the unique largest eigenvalue.*

Proof. After extending scalars to say $D_1, ..., D_p$ assume that the action of f^* on V_H is in Jordan form. As in Lemma 5.2.10 λ appears as an eigenvalue, and no other eigenvalues have magnitude λ. Since we are dealing with a real eigenvalue, the Jordan blocks associated to λ are real. Suppose that $D_1, ..., D_l$ is such a Jordan block. I claim that $l = 1$ and that all Jordan blocks have multiplicity 1. If not we have $f^*D_2 = \lambda D_2 + D_1$ in $\operatorname{Pic}(X)_{\mathbb{R}}$ and that $D_2 = aD_1 + A_0$ for some $A_0 \in \operatorname{Pic}^0(X)_{\mathbb{R}}$ because there is a unique Jordan block of size 1 for the action of f^* on $N^1(X)_{\mathbb{R}}$. Thus $D_i \equiv_{\text{Num}} aD_1$ for some non-zero scalar a. It follows that $f^*A_0 = \lambda D_2 + D_1 - a\lambda D_1 = \lambda A_0 + D_1$. So $f^*A_0 - \lambda A_0 = D_1$ which means $D_1 \in \operatorname{Pic}^0(X)_{\mathbb{R}}$ and this is a contradiction. So all Jordan blocks have size 1. Now let $D_1, ..., D_s$ be an eigen basis for f^* acting on V_H. The same argument given earlier implies that if D_i, D_j are λ-eigenvalues then they are numerically scalar multiples of one another, and so must be scalar multiples in V_H, a contradiction. It follows that f^* acting on V_H has 1-dimensional λ eigenspace with Jordan blocks of size 1. $\square$

Lemma 5.2.12. *Let X, Y be normal projective varieties defined over $\bar{\mathbb{Q}}$. Let $f\colon X \to X$ a surjective endomorphism and suppose that the Albanese morphism is surjective. Suppose that $\pi\colon X \to Y$ is a surjective endomorphism, and that we have a commuting diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

We assume that $\rho(X) = \rho(Y) + 1$ and that $\lambda_1(f) > \lambda_1(g)$. Suppose that there is an ample integral divisor H with $f^* |_{V_H}$ having a good eigenspace. Then if $f^*D = \lambda_1(f)D$ in $\mathrm{Pic}(X)_{\mathbb{R}}$ for some nef $\mathbb{Q}$-Cartier divisor. Then $\lambda_1(f)$ is an integer and $\kappa(D) \neq 0$.

Proof. Since we have assumed that $\lambda_1(g) < \lambda_1(f) = \lambda$ we have that λ is not a eigenvalue of g^* acting on $N^1(Y)_{\mathbb{R}}$. Furthermore, there is no eigenvalue of magnitude λ for g^*. Then by the assumption on the Picard number we have that $\det f^* = \lambda \det g^*$. Since $\det f^*, \det g^*$ are integers we have that λ is a rational number and thus an integer. Then we have that $D \in V_H$ by Lemma 5.2.10. Since λ has multiplicity 1 for the action of f^* on $N^1(X)_{\mathbb{R}}$ Proposition 5.2.11 tells us the same holds for f^* acting on V_H and that this eigenspace is one dimensional. The assumption that f has a good eigenspace now tells us that $\kappa(D) \neq 0$. $\square$

Ideally one would then apply Lemma 5.2.12 to case TIR_n, where in that case we have that $\kappa(D) \geq 0$ and the conclusion is then that $\kappa(D) > 0$. The following result gives another illustration of how one might reduce to the good eigenspace case.

Theorem 5.2.13. *Let X be a smooth projective variety with Picard number 1 and let E be a nef vector bundle on X with $H^0(X, E) \neq 0$ such that E is not ample and $\kappa(\mathbb{P}E, -K_{\mathbb{P}E}) \geq 0$. Suppose that there is integral ample divisor H such that $f^* |_{V_H}$ has a good eigenspace. Then the Kawaguchi-Silverman conjecture holds for X.*

Proof. We may assume that the pseudo-effective and nef cones coincide, otherwise we may iterate f and apply Theorem 2.1.12 to obtain that f^* will act by a dilation. Put $\mathcal{L} = \mathcal{O}_{\mathbb{P}E}(1)$. By assumption $\mathcal{L}$ is nef, but not ample. Therefore $\mathcal{L}$ generates a ray of $\mathrm{Nef}(\mathbb{P}E)$ as $N^1(\mathbb{P}E)_{\mathbb{R}}$ is a 2-dimensional vector space and the nef cone is full dimensional. Let $f\colon \mathbb{P}E \to \mathbb{P}E$ be a surjective endomorphism and put $\pi\colon \mathbb{P}E \to X$ for the bundle projection. After iterating f we may also assume that there is some surjective $g\colon X \to X$

with $\pi \circ f = g \circ \pi$. We may assume that $\lambda = \lambda_1(f) > \lambda_1(g) \geq 1$ by Corollary 4.2.10.1. We have that $f^*\mathcal{L} \equiv_{\text{Num}} \lambda\mathcal{L}$. If we do not have that $K_X \equiv_{\text{num}} 0$ then either K_X is ample or $-K_X$ is ample because X has Picard number 1. If K_X is ample, then X is of general type. It is well known that in this case g is a finite order automorphism. Since a dense orbit for f implies a dense orbit for g we see that no point of $\mathbb{P}E$ has a dense orbit and the Kawaguchi-Silverman conjecture for X is trivially true. Otherwise we may assume that $-K_X$ is ample and so X is Fano. In this case $\text{Alb}(X) = 0$. Then $\text{Alb}(\mathbb{P}E) = 0$ and so $f^*\mathcal{L} \sim_{\mathbb{Q}} \lambda\mathcal{L}$. Note that we may assume that $V_H = N^1(X)_{\mathbb{R}}$. Otherwise V_H is one dimensional and so f^* has an ample eigenvector. In which case the Kawaguchi-Silverman conjecture is true. Since $\kappa(\mathcal{L}) \geq 0$ and $\mathcal{L}$ spans the λ-eigenspace of f^* by assumption we have that $\mathcal{L} \in V_H$ as V_H contains a λ-eigendivisor. By assumption we have that $f^* |_{V_H}$ has a good Jordan block and so $\kappa(\mathcal{L}) \neq 0$. As $\kappa(\mathcal{L}) \geq 0$ we have that $\kappa(\mathcal{L}) > 0$ and we obtain the Kawaguchi-Silverman conjecture by Proposition 4.2.9. So we may assume that $K_X \equiv_{\text{num}} 0$. As $\kappa(-K_{\mathbb{P}E}) \geq 0$ we see that $-K_{\mathbb{P}E}$ is pseudo-effective. If $-K_{\mathbb{P}E}$ is big then it is also ample as we have assumed the nef cone and pseudo-effective cone coincide. Therefore we may argue as above to obtain the desired result. Thus we may assume that $-K_{\mathbb{P}E}$ generates a boundary ray of the pseudo-effective cone which we have assumed is the nef cone. Therefore $f^*(-K_{\mathbb{P}E}) \equiv_{\text{Num}} \lambda(-K_{\mathbb{P}E})$. Suppose first that $\kappa(-K_{\mathbb{P}E}) > 0$. Then by [44, Proposition 1.6] the Kawaguchi-Silverman conjecture holds for f. Assume that $\kappa(-K_{\mathbb{P}E}) = 0$. The adjunction formula tells us that

$$f^*(-K_{\mathbb{P}E}) = -K_{\mathbb{P}E} + R$$

where R is some effective divisor on X. We now use the argument found in [44, Proposition 9.2 (2)]. By [44, Proposition 6.1] applied to $-K_{\mathbb{P}E}$ we have that $f^{-1}(\text{supp}(-K_{\mathbb{P}E})) = \text{supp}(-K_{\mathbb{P}E})$. Write $-K_{\mathbb{P}E} = \sum_i a_i D_i$ where the D_i are prime divisors and the $a_i > 0$. Then after iterating f we have that $f^{-1}(\text{supp}(D_i)) = \text{supp}(D_i)$. Since D_i is a prime divisor this tells us that $f^*D_i = \mu_i D_i$ for some number μ_i. On the other hand. Since $-K_{\mathbb{P}E}$ is not the pull back of some line bundle on X we have that $-K_{\mathbb{P}E} \equiv_{\text{Num}} r\mathcal{L}$ where r is the rank of E. Thus we have that $D_i \equiv_{\text{Num}} b_i\mathcal{L}$ as $\mathcal{L}$ is extremal. Since we have that $f^*D_i \equiv_{\text{Num}} \lambda_i D_i$ we have that each $\mu_i = \lambda$. In conclusion we may write $-K_{\mathbb{P}E} = D$ where D is some effective $\mathbb{Q}$-Cartier divisor with $f^*D = \lambda D$ and $\kappa(D) \geq 0$. As we have assumed the existence of a good eigenspace we may apply Lemma 5.2.12 to obtain $\kappa(D) \neq 0$ and so $\kappa(D) > 0$. Therefore by Proposition 4.2.9 the Kawaguchi-Silverman conjecture follows. $\square$

The case of projective bundles over a Fano variety with Picard number one was treated by different methods in [35].

5.2.3 The case of Picard number 2

We restrict to the case of a variety with Picard number 2 and prove some basic results. In addition we will see how the requirement of a good eigenspace arises naturally. For completeness we prove some easy results in this case which may be well known. An important fact that we will use repeatedly is that $\mathrm{Nef}(X)_{\mathbb{R}}$ is *finitely generated* when $\rho(X) = 2$. We remind the reader of our assumptions regarding the singularities of the sources of contraction morphisms given in Remark 2.3.13; for simplicity, we will always assume that the source of a contraction morphism has at worst terminal singularities in this section. When no contraction morphism is present, we will relax these assumptions.

Proposition 5.2.14. *Let X be a normal projective variety of Picard number 2. Let $f\colon X \to X$ be a surjective endomorphism. Suppose that the eigenspace of $\lambda_1(f)$ is 2-dimensional. Then the Kawaguchi-Silverman Conjecture holds for f.*

Proof. Note that f^* permutes the boundary of the nef cone of X. So in particular, the boundary must be eigendivisors. It follows that f^* acts diagonally on $N^1(X)_{\mathbb{R}}$. In particular f^* has an ample eigendivisor and the result follows. □

In fact in the setting of Picard number 2 we may assume that the nef cone and Pseudo-effective cone coincide.

Proposition 5.2.15. *Let X be a normal projective variety of Picard number 2. Suppose that $\mathrm{Nef}(X) \neq \mathrm{PEff}(X)$. Let $f\colon X \to X$ be a surjective endomorphism. Then f^2 acts by a dilation on $N^1(X)_{\mathbb{R}}$. In particular Kawaguchi-Silverman and the sAND conjectures hold for X.*

Proof. Since $\rho(X) = 2$ we may replace f with f^2 and assume that f^* fixes the rays of the nef cone and has positive eigenvalues. As we have assumed that the pseudo-effective cone $\mathrm{PEff}(X)$ is strictly larger then X, there is a boundary ray D of $\mathrm{Nef}(X)$ that lies in the interior of $\mathrm{PEff}(X)$. Thus f^* has an eigendivisor in the interior of $\mathrm{PEff}(X)$ a full-dimensional pointed cone. By Theorem 2.1.12 we see that f^* acts by a dilation as needed. □

From the perspective of the minimal model program Proposition 5.2.15 suggests that it suffices to consider the varieties which arise at the final stage of the minimal model program. Keeping in mind Remark 2.3.13 we let X be a projective normal $\mathbb{Q}$-factorial variety with terminal singularities defined over $\bar{\mathbb{Q}}$ with $\rho(X) = 2$. Suppose that $\mathrm{Nef}(X) = \mathrm{PEff}(X)$.

1. If K_X is nef then there are no K_X negative extremal contractions and the minimal model program tells us that X is a minimal model.

2. If K_X is not nef then there is at least one K_X negative extremal contraction. The assumption that $\text{Nef}(X) = \text{PEff}(X)$ tells us that this extremal contraction is of fibering type. Thus X is a Mori-fiber space.

As we want to deal with integral eigendivisors we need to work with varieties whose nef cone contains a rational ray on the boundary. The following elementary computation shows that in the presence of a surjective endomorphism with distinct eigenvalues, the boundary is rational, or completely irrational. That is, we cannot have an irrational ray and a rational ray.

Proposition 5.2.16. *Let X be a normal projective variety of Picard number 2 defined over a number field K. Let the eigenvalues of f^* acting on $N^1(X)_{\mathbb{Q}}$ be λ, μ with $\lambda \neq \mu$. Suppose that μ is an integer and that μ has an integral eigendivisor in $N^1(X)_{\mathbb{Q}}$. Then λ is an integer and λ has an integral divisor class in $N^1(X)_{\mathbb{Q}}$.*

Proof. First note that

$$\lambda\mu = \deg f^* = d \in \mathbb{Z}$$

So in particular as μ is an integer, we have that λ is rational. On the other hand, λ is an algebraic integer, so it is an integer. Now choose an integral ample divisor L on X. Let

1. $f^*L = aL + bH$

2. Set $D = uL + vH$ and $f^*D = \lambda D$.

Then we have that

$$f^*(uL + vH) = uaL + ub + \mu vH = uaL + (ub + \mu v)H = \lambda uL + \lambda vH$$

Putting this together gives and using that $u \neq 0$ by our assumptions gives

$$ua = \lambda u, ub + \mu v = \lambda v$$

As f^* is defined over $\mathbb{Z}$ and L, H are integral we have that b is an integer. We have that the λ eigenspace is given by

$$uL + \frac{ub}{\lambda - \mu}H$$

Taking $u = \lambda - \mu$ we obtain an integral λ eigendivisor

$$(\lambda - \mu)L + bH$$

$\square$

As an aside, this tells us that we can gain some information about X from the existence of a surjective endomorphism f.

Corollary 5.2.16.1. *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$ with Picard number 2. Suppose that X admits surjective endomorphism that is not an automorphism. Then both rays of the nef cone are irrational or both rays are rational.*

We have the following simple result.

Proposition 5.2.17. *Let X be a normal projective variety defined over a number field K. Let $f\colon X \to X$ be a surjective endomorphism with $\lambda_1(f) = \lambda > 1$. Suppose $f^*D_\lambda = \lambda D_\lambda$ with D_λ a non-trivial divisor class. Suppose $f^*D_\mu = \mu D_\mu$ where $0 < \mu < 1$ and $D_\lambda + D_\mu$ is ample. Then Kawaguchi-Silverman holds for f.*

Proof. By assumption $H = D_\lambda + D_\mu$ is ample. Now let P be a point with a dense forward orbit. Then since $\mu < 1$ we have that $-C < h_{D_\mu}(f^n(P)) < C$ for some positive constant C and all n. This follows from

$$\mu h_{D_\mu}(P) - C' < h_{D_\mu}(f(P)) < \mu h_{D_\mu}(P) + C'$$

for some C' applied iteratively. Now assume for a contradiction that the canonical height function $\hat{h}_{D_\lambda}(P) = 0$. Then we have that the ample height h_H is bounded on $\mathcal{O}_f(P)$ as

$$h_H(f^n(P)) = \hat{h}_{D_\lambda}(f^n(P)) + h_{D_\mu}(f^n(P)) = \lambda^n \hat{h}_{D_\lambda}(P) + h_{D_\mu}(f^n(P)) < C$$

Since H is an ample height we have that $\mathcal{O}_f(P)$ is a finite set contradicting that the orbit is dense. The result follows. $\square$

Proposition 5.2.18. *Let X be a normal projective variety defined over a number field K. Suppose that X is of Picard number 2. Let $f\colon X \to X$ be a non-int amplified morphism with $\lambda_1(f)$ not an integer. Then Kawaguchi-Silverman holds for f.*

Proof. After iterating f we may assume that the eigenvalues of f^* are positive. Suppose $\lambda = \lambda_1(f) > 1$. If f^* is assumed to be non-int amplified with eigenvalues λ, μ then $\mu \leq 1$. If $\mu = 1$ then λ is also an integer and the characteristic polynomial for f splits over $\mathbb{Z}$. It follows that f^* is diagonalizable over $\mathbb{Q}$ and so f^* would have integral eigendivisors. Thus we have that $\mu < 1$. Let $f^*D_\lambda = \lambda D_\lambda$ and $f^*D_\mu = \mu D_\mu$ with D_λ, D_μ nef divisor classes with $D_\lambda + D_\mu$ ample. Then $H = D_\lambda + D_\mu$ is ample. By Proposition 5.2.17 the result follows. $\square$

Suppose now that $\kappa(X) < 0$ and $\lambda_1(f) = \lambda > 1$. Our strategy here is as follows. Given a surjective endomorphism we may write

$$f^*K_X + R = K_X$$

where R is some effective $\mathbb{Q}$-divisor that is supported on on the ramification locus. For this to make sense we need to assume that X is $\mathbb{Q}$-Cartier at least so the canonical divisor is a $\mathbb{Q}$-divisor. First suppose that f is etale, then f is unramified and R is trivial. Then by adjunction $f^*K_X = K_X$. If $K_X \equiv_{\text{num}} 0$ then K_X is nef. On the other hand $\kappa(X) < 0$ so we obtain a contradiction to the abundence conjecture. As this is expected to be false we ignore this case. In other words we are assuming that $\kappa(X) < 0$ and K_X is not nef. On the other hand, as K_X is not numerically trivial and $f^*K_X = K_X$ we have that $-K_X$ is an eigendivisor for f^*. Since we have have assumed $\lambda_1(f) > 1$ we have that f^* has two distinct eigenvalues 1 and $\lambda > 1$. Furthermore, as f^* preserves the nef cone the one 1 and λ eigenspaces can be generated by nef and not ample eigendivisors. Since $f^*(-K_X) = -K_X$ and K_X is assumed to be not nef we may take $-K_X$ nef.

Proposition 5.2.19. *Let X be a normal projective variety defined over $\bar{\mathbb{Q}}$ that is $\mathbb{Q}$-factorial with at worst terminal singularities and of Picard number 2. Let $\kappa(X) < 0$. Let $f\colon X \to X$ be an unramified surjection. Assume that K_X is not nef. (For example assume the abundance conjecture) then Kawaguchi Silverman holds for f.*

Proof. We may assume that $\text{Nef}(X) = \text{PEff}(X)$ by Proposition 5.2.15. Since f is unramified we have that the action of f^* has two eigenvalues 1 and λ and we may assume that $\lambda > 1$. Since K_X is not nef we have that $-K_X$ is nef, and since $f^*(-K_X) = -K_X$ we may assume that $-K_X$ is not ample, as otherwise f^* preserves an element in the interior of the big cone which by our assumptions is the ample cone which would imply that $\lambda_1(f) = 1$ in this case. Thus we may assume that $-K_X$ is nef but not big. Let D_λ be a numerical class corresponding to the λ-eigenvalue. Since $-K_X$ is nef and X has at wors terminal singularities we can contract one of the rays of the cone of curves via say $\phi\colon X \to Y$. Since we cannot contract the ray generated by $-K_X$ we have contracted the ray associated to

D_λ. Since we have assumed that D_λ is not big, the contraction is of fibering type. After iterating f we obtain a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where $\phi \colon X \to Y$ is a Mori fiber space. In this case Y has Picard number 1 and $\phi^* H = D_\lambda$ for some ample $H \in N^1(Y)_{\mathbb{R}}$. It follows that $\lambda_1(f) = \lambda_1(g)$. Furthermore, as the Kawaguchi-Silverman conjecture is known for varieties of Picard number one we may apply Corollary 4.2.10.1 to obtain the Kawaguchi-Silverman conjecture for f. $\square$

We may now assume that f is not etale. So $f^* K_X + R = K_X$ where R is a non-zero effective R divisor. We deal with the case $\mathrm{Alb}(X) = 0$. In this example we see through a basic computation how the good eigenspace condition (and thus case TIR_n) may arises naturally.

Proposition 5.2.20. *Let X be a normal $\mathbb{Q}$-factorial projective variety defined over a number field K and $\rho(X) = 2$, $\kappa(X) < 0, \mathrm{Alb}(X) = 0$. Let $f \colon X \to X$ be a surjective ramified endomorphism that is not int-amplified. In other words the eigenvalues of f^* are λ, μ with $|\mu| \leq 1$. Suppose that f has a good eigenspace. Then the Kawaguchi-Silverman conjecture is true for f.*

Proof. We may iterate f and assume that λ, μ are positive and that the pseudoeffective and nef cones coincide. Suppose first that K_X and R are linearly independent. Write $f^* R = aK_X + bR$ for some a, b. The matrix of f^* with respect to the basis K_X, R is then

$$\begin{bmatrix} 1 & a \\ -1 & b \end{bmatrix}.$$

By assumption we have that $\mu \leq 1$. If $\mu < 1$ then Proposition 5.2.17 tells us that the Kawaguchi-Silverman conjecture is true for f. So we take $\mu = 1$. Now let H be an ample divisor and form the subspace V_H as in 4.2.4.1. We may assume that $V_H = \mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$ as otherwise f has an ample eigendivisor. Then $b + 1 = \lambda + 1$ and $b + a = \lambda$ by taking the trace and determinant of the above matrix. So $b = \lambda$ and $a = 0$. So in particular, the λ-eigenspace contains R. Since R is also nef and the λ eigenspace is 1-dimensional we have that $\kappa(R) > 0$ by the assumption that f has a good eigenspace. By Proposition 4.2.9 the Kawaguchi-Silverman conjecture is true for f. Now suppose that $R = aK_X$ for some a. Since R is effective and K_X is not we have that $a < 0$ or that $b(-K_X) = R$ for some

$b > 0$. Then $-K_X$ is effective and $f^*K_X = K_X - R = K_X + bK_X = (b+1)K_X$. Thus we have that $f^*(-K_X) = (b+1)(-K_X)$. Since $b+1 > 1$ we have that $\lambda = b+1$. Therefore as above R generates the λ-eigenspace and we have the same conclusion. □

5.3 Projective Bundles over elliptic curves

Projective bundles are one way to construct examples of varieties with Picard number 2. Therefore, they provide a examples of the varieties studied in 5.2.3. Let X be a smooth projective variety of Picard number one and let E be a vector bundle on X. The bundle surjection $\pi\colon \mathbb{P}E \to X$ gives $\mathbb{P}E$ the structure of a Mori-fiber space over X. Suppose that $f\colon \mathbb{P}E \to \mathbb{P}E$ is a surjective endomorphism. By 4.1.5 after replacing f with f^n we may assume that we have a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

Then we may attempt to study f through g. First we note that by the product formula for dynamical degrees 4.2.10 we have that

$$\lambda_1(f) = \max\{\lambda_1(f\mid_\pi), \lambda_1(g)\}.$$

Furthermore, if $\lambda_1(f\mid_\pi) \leq \lambda_1(g)$ then $\lambda_1(f) = \lambda_1(g)$ and the Kawaguchi-Silverman conjecture is true by 4.2.10.1. So we are primarily interested in the case that $\lambda_1(f\mid_\pi) > \lambda_1(g)$. In this case we have that $f^*\colon N^1(\mathbb{P}E)_\mathbb{R} \to N^1(\mathbb{P}E)_\mathbb{R}$ has a unique $\lambda_1(f)$-eigendivisor. Let V_H be as in 4.2.4.1. By 5.2.11 we have that V_H has a 1-dimensional $\lambda_1(f)$ eigendivisor. On the other hand as X has Picard number 1 we have that g^* has an ample integral eigendivisor. Therefore $f^*\colon N^1(\mathbb{P}E)_\mathbb{R} \to N^1(\mathbb{P}E)_\mathbb{R}$ has integral eigenvalues by 5.2.16. In particular $\lambda_1(f)$ is an integer. So we are well equipped to study the Kawaguchi-Silverman conjecture from the point of view of 5.2.6. We will focus on the simplest situation where X is a smooth projective curve C. In this case C is either $\mathbb{P}^1$, an elliptic curve, or a genus $g > 1$ curve.

Theorem 5.3.1 (Kawaguchi-Silverman for projective bundles over non-elliptic curves: [34])**.** *If C is a smooth projective curve of genus $g \neq 1$ then the Kawaguchi-Silverman conjecture holds for all projective bundles $\mathbb{P}E \to C$.*

Proof. If $g = g(C) = 0$ then $C = \mathbb{P}^1$. The Kawaguchi-Silverman conjecture for $\mathbb{P}E \to \mathbb{P}^1$ is then proven in [34]. If $g > 1$ let $f\colon \mathbb{P}E \to \mathbb{P}E$ be a surjective endomorphism. Then by

4.1.5 for some $n > 0$ we have a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f^n"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "h"'] & C \end{tikzcd}$$

Since C is a genus $g > 1$ curve it is well known only surjective automorphisms of C are finite order automorphisms. Therefore, h is a finite order automorphism. Thus f has no point P with Zariski dense orbit, otherwise f^n would have a Zariski dense orbit; if f^n has a point with Zariski dense orbit then h does as well, which is impossible as h is a finite order automorphism. Thus f has no point with Zariski dense orbit and consequently the Kawaguchi-Silverman conjecture is trivially true. □

We may now specialize to the case of elliptic curves. We first reduce to semi-stable degree zero vector bundles on elliptic curves using [34]. This allows us to use Atiyah's pioneering work on vector bundles on elliptic curves [5]. Using the results of [5] (of which we remind the reader in 3.3.1) we prove some needed results about these bundles. The key result is that if $f\colon \mathbb{P}E \to \mathbb{P}E$ is a surjective endomorphism over an elliptic curve then $f^*\mathcal{O}_{\mathbb{P}E}(1) = \mathcal{O}_{\mathbb{P}E}(d)$ for some d. This is 5.3.10. Then in 5.3.1 we turn to the Kawaguchi-Silverman for certain degree zero semi-stable bundles.

The following result is what allows us to reduce to the case semi-stable degree 0 vector bundles on elliptic curves. For a refresher on semi-stability on vector bundles see [22] and [32, 6.4.A]. We have the following key result. Recall that on an elliptic curve C defined over $\overline{\mathbb{Q}}$ there is a unique indecomposable rank r vector bundle of degree zero with a non-zero global section. We denote this vector bundle by F_r.

Lemma 5.3.2 (Page 3 of [60])**.** *Let E be an indecomposable vector bundle over an elliptic curve C. Then E is semi-stable. In particular, F_r is semi-stable of degree zero.*

Theorem 5.3.3. *[34, Corollary 6.8] Let C be a smooth curve. Then the following are equivalent.*

1. *The Kawaguchi-Silverman conjecture holds for surjective endomorphisms of projective bundles $\mathbb{P}E$ where E is a vector bundle on C.*

2. *The Kawaguchi-Silverman conjecture holds for surjective endomorphisms of projective bundles $\mathbb{P}E$ where E is a semi-stable degree 0 vector bundle on C.*

Thus it suffices to consider semi-stable and degree zero vector bundles on C.

Proposition 5.3.4. *Let E be a semi-stable degree zero vector bundle on an elliptic curve C.*

1. *We may write $E = \bigoplus_{k=0}^{N} (F_{r_k} \otimes L_k)$ where F_{r_k} are the rank r_k Atiyah bundles and L_k is a degree zero vector bundle.*
2. *If $E = \bigoplus_{k=0}^{N} (F_{r_k} \otimes L_k)$ with L_i degree zero line bundles with $L_0 = \mathcal{O}_C$ then $\kappa(-K_{\mathbb{P}E}) \geq 0$.*
3. *The line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is nef but not ample on $\mathbb{P}E$. In other worlds E is a nef but not ample vector bundle on C.*

Proof. As E is semi-stable of degree 0 every sub-sheaf has degree at most 0. Let $W_0, ..., W_N$ be the components of E. If $\deg W_i < 0$ then because $\deg E = \sum_{k=0}^{N} \deg W_i = 0$ there must be some W_i with $\deg W_i > 0$ which would destabilize E. Thus every indecomposable component is degree zero. It follows from the definition of semi-stable that each irreducible component is also semi-stable. By Theorem 3.3.1 we have that $W_i = F_{r_i} \otimes L_i$ for some degree 0 line bundle as needed. We now turn to part 2. We have from the Euler exact sequence that

$$-K_{\mathbb{P}E} = \mathcal{O}_{\mathbb{P}E}(r) - \det \pi^* E$$

where r is the rank of E. Then

$$\begin{aligned} H^0(\mathbb{P}E, -K_{\mathbb{P}E}) &= H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r) \otimes \det(E^{-1})) \\ &= H^0(C, \pi_*(\mathcal{O}_{\mathbb{P}E}(r)) \otimes \det(H^{-1})) = H^0(C, \mathrm{Sym}^r(E) \otimes \det(E^{-1})). \end{aligned}$$

We have

$$\mathrm{Sym}^r(E) = \bigoplus_{i_0 + ... + i_N = r} \bigotimes_{k=0}^{N} \mathrm{Sym}^{i_k}(F_{r_k}) \otimes L_k^{\otimes i_k}$$

from our assumed decomposition of $E = \bigoplus_{k=0}^{N} F_{r_k} \otimes L_k$. Therefore

$$\mathrm{Sym}^r(E) \otimes \det E^{-1} = \bigoplus_{i_0 + ... + i_N = r} \bigotimes_{k=0}^{N} \mathrm{Sym}^{i_k}(F_{r_k}) \otimes L_k^{\otimes i_k} \otimes \det E^{-1}.$$

Let $j_0 = r - N$ and $j_k = 1$ for $k = 1, ..., N$. Then $\sum_{k=0}^{N} j_k = r - N + N = r$. Therefore

$$\bigotimes_{k=0}^{N} \mathrm{Sym}^{j_k}(F_{r_k}) \otimes L_k^{\otimes j_k} \otimes \det E^{-1}$$

is a direct summand of $\mathrm{Sym}^r(E) \otimes \det E^{-1}$. Since $\det E = \sum_{k=0}^N L_k$ and $L_0 = \mathcal{O}_C$ we have that

$$\bigotimes_{k=0}^{N} \mathrm{Sym}^{j_k}(F_{r_k}) \otimes L_k^{\otimes j_k} \otimes \det E^{-1} = \bigotimes_{k=0}^{N} \mathrm{Sym}^{j_k}(F_{r_k})$$

since $\sum_{k=0}^N j_k L_k = \sum_{k=1}^N j_k L_k = \sum_{k=1}^N L_k$ using $L_0 = \mathcal{O}_C$ and the construction of the j_k. We have shown that $\bigotimes_{k=0}^N \mathrm{Sym}^{j_k}(F_{r_k})$ (which has a global section by Proposition 5.3.5) is a direct summand of $\mathrm{Sym}^r(E) \otimes \det E^{-1}$. Therefore $-K_{\mathbb{P}E}$ has a non-zero global section and $\kappa(-K_{\mathbb{P}E}) \geq 0$ as required. For part (3) note that by [32, 6.2.12] that extensions of nef vector bundles are nef. We have that there is a canonical short exact sequence

$$0 \to \mathcal{O}_C \to F_r \to F_{r-1} \to 0.$$

Since $F_1 = \mathcal{O}_C$ we have that F_2 is nef being the extension of nef line bundles. By induction we have that F_r is nef for all $r \geq 2$. On the other hand any degree 0 line bundle on a curve is nef, therefore by [32, 6.2.12] that $F_r \otimes L$ is nef for any degree 0 line bundle on C. By [32, 6.2.12] we have that $E = \bigoplus_{k=0}^N (F_{r_k} \otimes L_k)$ is nef being a direct sum of nef bundles. We now argue that E is not ample. By [22, 1.3] we have that $E = \bigoplus_{k=0}^N (F_{r_k} \otimes L_k)$ is ample if and only if $\deg(F_{r_i} \otimes L_i) > 0$ for all i. Since $\deg(F_{r_i} \otimes L_i) = 0$ we conclude that E is nef but not ample.

□

In conclusion we have shown that $\kappa(-K_{\mathbb{P}E}) \geq 0$ when E is a semi-stable degree zero vector bundle on C. Recall that by [44, Theorem 6.2] this is a necessary condition for $\mathbb{P}E$ to have an int-amplified endomorphism. It seems interesting to find cases when this is sufficient as well. We will see that it is not sufficient. To proceed we will need to work with the linear action of f^* on $\mathrm{Pic}(\mathbb{P}E)$ without passing to the quotient by $\mathrm{Pic}^0(\mathbb{P}E)$. More precisely, suppose that we have a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

We have that $\mathrm{Nef}(\mathbb{P}E)$ is generated by $\mathcal{O}_{\mathbb{P}E}(1)$ and $\pi^* H$ where H is some ample integral divisor on C by 5.3.4 and the fact that $\mathbb{P}E$ has Picard number two. Therefore we have that

$$f^*\mathcal{O}_{\mathbb{P}E}(1) \equiv_{\mathrm{num}} \lambda \mathcal{O}_{\mathbb{P}E}(1) \tag{5.2}$$

for some $\lambda \in \mathbb{Z}$. We would like to change numerical equivalence in Equation 5.2 to *linear* equivalence. This is because we desire to study f through the Iitaka dimension of its eigendivisors (and ultimately apply 4.2.9) while the Iitaka dimension is not preserved by numerical equivalence. Note that 5.2.11 implies that we can always find *some* nef divisor D with $f^*D \equiv_{\text{lin}} \lambda D$, we will want to compute $\kappa(D)$. If $D = \mathcal{O}_{\mathbb{P}E}(1) \otimes A$ for some non-trivial degree zero divisor A then this computation seems more complicated. However, we will show that $A = \mathcal{O}_C$ and thus we will be in the simpler situation of computing $\kappa(\mathcal{O}_{\mathbb{P}E}(1))$. We now begin proving a series of results that will culminate in 5.3.10.

Proposition 5.3.5. *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$ and let F_r be the rank r Atiyah bundle on C. Then*

$$\mathrm{Sym}^d(F_r) = \bigoplus_{r_i} F_{r_i}$$

for some integers r_i.

Proof. Recall that we have a canonical surjection giving rise to an exact sequence

$$\psi\colon F_r^{\otimes d} \to \mathrm{Sym}^d(F_r) \to 0.$$

By Theorem 3.3.1 we have that $F_r^{\otimes d} = \bigoplus_i F_{l_i}$ for some integers l_i. The semi-stability of F_{r_i} gives that $\deg \mathrm{Sym}^d(F_r) \geq 0$. By taking the dual endomorphism we have an exact sequence

$$0 \to (\mathrm{Sym}^d F_r)^* \to (F_r^{\otimes d}).$$

Since we are in characteristic 0 and $F_r^* = F_r$ we have that $(\mathrm{Sym}^d F_r)^* = \mathrm{Sym}^d F_r^* = \mathrm{Sym}^d F_r$ and $(F_r^{\otimes d})^* = F_r^{\otimes d}$. Thus $\mathrm{Sym}^d F_r$ is a sub-bundle of $F_r^{\otimes d}$. Since $F_r^{\otimes d}$ is semi-stable of degree zero (being the tensor product of semi-stable vector bundles) we have that $\deg \mathrm{Sym}^d F_r \leq 0$. So $\deg \mathrm{Sym}^d F_r = 0$, and consequently $\mathrm{Sym}^d F_r$ is semi-stable of degree 0. It follows that each of its summands is also semi-stable of degree zero. From Theorem 3.3.1 we have that

$$\mathrm{Sym}^d F_r = \bigoplus_j F_{r_j} \otimes L_j$$

where L_j is some degree 0 line bundle. Now we have that the sheaf $\mathcal{H}om(F, G) \cong F^\vee \otimes G$ whenever F, G are vector bundles. On the other hand $H^0(C, \mathcal{H}om(F, G)) = \hom(F, G)$. Applying this in our situation gives

$$\hom(F_r^{\otimes d}, \mathrm{Sym}^d F_r) = \hom(\oplus_i F_{l_i}, \oplus_j F_{r_j} \otimes L_j) = \oplus_{i,j} \hom(F_{l_i}, F_{r_j} \otimes L_j)$$

Now we have that $\hom(F_{l_i}, F_{r_j} \otimes L_j) = H^0(C, F_{l_i}^* \otimes F_{r_j} \otimes L_j) = H^0(C, F_{l_i} \otimes F_{r_j} \otimes L_j)$ as the Atiyah bundle is self dual. Suppose that $L_j \neq \mathcal{O}_C$ for some j. Then as L_j is of degree 0 we have that $H^0(F_{l_i} \otimes F_{r_j} \otimes L_j) = 0$ by Theorem 3.3.1. As $\psi\colon F_r^{\otimes d} \to \mathrm{Sym}^d(F_r) \to 0$ arises as an element of $\hom(F_r^{\otimes d}, \mathrm{Sym}^d(F_r)) = H^0(C, \mathcal{H}om(F_r^{\otimes d}, \mathrm{Sym}^d(F_r))$ we have a decomposition

$$\psi = \bigoplus \psi_{ij}$$

where $\psi_{ij}\colon F_{l_i} \to F_{r_j} \otimes L_j$. The above computation shows that if $L_{j_0} \neq \mathcal{O}_C$ for some fixed j_0 then $\psi_{ij_0} = 0$ for all i. This contradicts the assumption that ψ is surjective. This is because locally the image of $\psi = \bigoplus \psi_{ij}$ must generate $F_{r_{j_0}} \otimes L_{j_0}$. If each $\psi_{ij_0} = 0$ then this image is always zero locally, while $F_{r_{j_0}} \otimes L_{j_0}$ is non-zero locally. So we have that each $L_j = \mathcal{O}_C$ and the claim follows.

□

As a corollary we may extend the above result to direct sums of Atiyah bundles.

Corollary 5.3.5.1. *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$ and let F_r be the rank r Atiyah bundle on C. Let $E = \bigoplus_{i=1}^s F_{r_i}$. Then*

$$\mathrm{Sym}^d(E) = \bigoplus_{j=1}^N F_{w_j}$$

for some integers w_j.

Proof. We have that

$$\mathrm{Sym}^d(\bigoplus_{i=1}^s F_{r_i}) \cong \bigoplus_{t_1+\ldots+t_s=d} \bigotimes_{j=1}^s \mathrm{Sym}^{t_j}(F_{r_j})$$

where the $t_i \geq 0$. Consider $t_1, \ldots, t_s \in \mathbb{Z}_{\geq 0}$ such that $t_1 + \ldots + t_s = d$ By 5.3.5 we have that $\mathrm{Sym}^{t_j}(F_{r_j}) = \bigoplus_{g_{jk}} F_{g_{jk}}$. Then we have

$$\bigotimes_{j=1}^s \mathrm{Sym}^{t_j}(F_{r_j}) = \bigotimes_{j=1}^s \left(\bigoplus_{g_{jk}} F_{g_{jk}} \right)$$

By 3.3.1 we have that the tensor product of two Atiyah bundles is a direct sum of Atiyah bundles, and so by expanding we have that

$$\bigotimes_{j=1}^s \mathrm{Sym}^{t_j}(F_{r_j}) = \bigotimes_{j=1}^s \left(\bigoplus_{g_{jk}} F_{g_{jk}} \right) = \bigoplus_{l} F_{v_l}.$$

The result now follows by taking the direct sum of all of the above terms. □

Corollary 5.3.5.2. *Let C be an elliptic curve defined over a number field. Let E be a vector bundle on C. Then the Kawaguchi-Silverman conjecture holds for surjective endomorphisms of $\mathbb{P}E$ that admit a good eigenspace.*

Proof. By Theorem 5.3.3 we may assume that E is semi-stable of degree zero. Using Proposition 5.3.4 we may write $E = \bigoplus_{k=0}^{N} F_{r_k} \otimes L_K$ where the L_k are degree zero line bundles. After twisting by $L_0^{\otimes -1}$ we may assume that $L_0 = \mathcal{O}_C$. Therefore $\kappa(-\mathbb{P}_E) \geq 0$ by Proposition 5.3.4. By Theorem 5.2.13 the Kawaguchi-Silverman conjecture is true for f as we have assumed the existence of a good eigenspace. □

Lemma 5.3.6 (Isogenies preserve the Atiyah bundle. Corollary 2.1 [49])**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $g\colon C \to C$ be an isogeny. Then $g^*F_r \cong F_r$ where F_r is the rank r Atiyah bundle.*

Lemma 5.3.7 (Pull back by automorphisms preserve exact sequences.)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$ and let α be an automorphism of C. Suppose that*

$$0 \to E \to F \to V \to 0$$

is an exact sequence of sheaves on C. Then

$$0 \to \alpha^*E \to \alpha^*F \to \alpha^*V \to 0$$

is exact.

Proof. Automorphisms are flat and so preserve exact sequences. □

Lemma 5.3.8 (Automorphisms preserve degree.)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$ and let α be an automorphism of C. If E is semi-stable of degree zero, then so is α^*E.*

Proof. α^* preserves subsheaves by 5.3.7 and preserves degree. So if α^*E has a destabilizing subsheaf then so does E. □

Proposition 5.3.9 (Surjective endomorphisms preserve the Atiyah bundle.)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$ and let $f\colon C \to C$ be a surjective endomorphism. Then $f^*F_r \cong F_r$.*

Proof. Any surjective endomorphism of C can be written $f = \tau_c \circ g$ where g is a homomorphism and τ_c is translation by some element of C by 3.2.2.2. In fact g must be an isogeny. To see this note that since f is surjective we have that g is surjective. On the other hand, since g is surjective and finite its kernel must be finite as well. We have that $\tau_c^* F_r$ is irreducible and semi-stable of degree zero by 5.3.7 and 5.3.8 It also has a non-zero section because F_r does, and the pull back is an isomorphism on sections. Since $\tau_c^* F_r$ is irreducible, semi-stable of degree zero, with a non-zero section we have by Atiyah's classification Theorem 3.3.1 that $\tau_c^* F_r \cong F_r$. Then by 5.3.6 we have $g^* F_r \cong F_r$ as g is an isogeny. Since $f^* F_r = (\tau_c \circ g)^* \cong g^* \tau_c^* F_r$ we have the result. $\square$

Write $E = \bigoplus_i F_{r_i}$. Recall that the existence of a commutative square

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd} \tag{5.3}$$

is equivalent to the data of a surjective morphism

$$\theta \colon \pi^* g^* E \to \mathcal{L} \to 0 \tag{5.4}$$

where $\mathcal{L}$ is a line bundle on $\mathbb{P}E$. This is the content of [23, 7.12]. Given this data we have that $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}E}(1)$. Note that $\pi^* g^* E \cong \pi^* E$ by 5.3.9. On the other hand, given a surjective endomorphism $f \colon \mathbb{P}E \to \mathbb{P}E$ there is some $n \geq 1$ and a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f^n"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

Therefore, to construct a surjective endomorphism of a projective bundle we may instead construct a surjection of sheaves

$$\pi^* g^* E \to \mathcal{L} \to 0 \tag{5.5}$$

for some surjective endomorphism $g \colon C \to C$. By 5.3.4 we have that E is a nef but not ample vector bundle and the nef cone of $\mathbb{P}E$ is generated by $\mathcal{O}_{\mathbb{P}E}(1)$ and $\pi^* H$ where H is any ample line bundle on C. This is because $\mathbb{P}E$ has Picard number 2 and $\mathcal{O}_{\mathbb{P}E}(1), \pi^* H$ are two nef but not ample line bundles on $\mathbb{P}E$. It follows that they generate the nef cone. We then have that $f^* \mathcal{O}_{\mathbb{P}E}(1) \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}E}(\lambda)$ for some $\lambda > 0$ and $f^* \pi^* H \equiv_{\text{num}} d\pi^* H$ for some integer $d > 0$. In particular $\mathcal{L} = \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^* A$ where A is some degree zero line bundle on C and $\lambda > 0$. It turns out that the situation is simplified by assuming that each L_i is trivial.

Lemma 5.3.10. *Let C be an elliptic curve over $\bar{\mathbb{Q}}$. Let $E = \bigoplus_{i=1}^{s} F_{r_i}$. Suppose that we have a diagram*

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where g is a surjective endomorphism and $f^*\mathcal{O}_{\mathbb{P}E}(1) = \mathcal{O}_{\mathbb{P}E}(d) \otimes \pi^*A$ where $A \in \mathrm{Pic}^0(C)$. Then $A = \mathcal{O}_C$.

Proof. The surjection

$$\theta \colon \pi^*E \to \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^*A \to 0$$

arises as an element of $H^0(\mathbb{P}F, \hom(\pi^*E, \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^*A))$. Note that

$$\hom(\pi^*E, \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^*A) \cong H^0(\mathbb{P}E, (\pi^*E)^\vee \otimes \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^*A).$$

Since each F_{r_i} is self dual we have $E^\vee \cong E$. Thus we are interested in the global sections of

$$\pi^* \left(\bigoplus_{i=1}^{s} F_{r_i} \otimes A \right) \otimes \mathcal{O}_{\mathbb{P}E}(\lambda).$$

By the push-pull formula these are the same as

$$\bigoplus_{i=1}^{s} F_{r_i} \otimes A \otimes \mathrm{sym}^\lambda(E) = A \otimes \left(\bigoplus_{i=1}^{s} F_{r_i} \otimes \mathrm{sym}^\lambda(E) \right). \tag{5.6}$$

For this to have a non-zero section we must have $A = \mathcal{O}_C$ by Atiyah's classification. This is because by 5.3.5.1 we have

$$\mathrm{sym}^\lambda(E) = \bigoplus_{i=1}^{t} F_{w_i}$$

for some w_i. So

$$F_{r_j} \otimes \mathrm{sym}^\lambda(E) = \bigoplus_{i=1}^{t} F_{r_j} \otimes F_{w_i}.$$

By 3.3.1 we have

$$\bigoplus_{i=1}^{t} F_{r_j} \otimes F_{w_i} = \bigoplus_{t_p} F_{t_p}$$

for some integers t_p. So we have that

$$A \otimes \left(\bigoplus_{i=1}^{s} F_{r_i} \otimes \mathrm{sym}^{\lambda}(E) \right) = \bigoplus_{t_p} F_{t_p} \otimes A$$

where A is a line bundle of degree 0. For this to have a global section we need $F_{t_p} \otimes A$ to have a non-zero global section for some t_p. Since A is a degree zero line bundle on C Atiyah's classification shows that $E_{t_p} \otimes A$ has a non-zero global section if and only if $A = \mathcal{O}_C$ as needed. □

Example 6. The argument given here will fail for reducible semi-stable bundles of degree zero. In the above proof we show that

$$\hom(\pi^* E, \mathcal{O}_{\mathbb{P}E}(\lambda) \otimes \pi^* A)$$

does not have a non-zero global section when $A \neq \mathcal{O}_C$. For example, let B be a non-trivial degree zero line bundle on C. Set $E = B \oplus F_r$. Then this is a semi stable degree zero vector bundle. The case of $r = 2$ is interesting for the following reason. Later we will prove the Kawaguchi-Silverman conjecture for $\mathbb{P}F_r$. Therefore, the only remaining case for rank 3 vector bundles on an elliptic curve is $E = B \oplus F_2$ where B is some degree 0 line bundle on C. For simplicity take $\lambda = r = 2$. Consider

$$\hom(\pi^* E\, \mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* A) = H^0(\mathbb{P}E, (\pi^* E)^{\vee} \otimes \mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* A) \tag{5.7}$$

$$= H^0(C, \mathrm{Sym}^2(E) \otimes E^{\vee} \otimes A). \tag{5.8}$$

We compute

$$\mathrm{Sym}^2(B \oplus F_2) = \mathrm{Sym}^2 B \oplus \left(\mathrm{Sym}^1(B) \otimes \mathrm{Sym}^1(F_2)\right) \oplus \mathrm{Sym}^2(F_2) = B^{\otimes 2} \oplus (B \otimes F_2) \oplus F_3.$$
$$(B \oplus F_2)^{\vee} = B^{-1} \oplus F_2.$$

as $\mathrm{Sym}^{r-1} F_2 = F_r$ by 3.3.1 and F_r is self dual. Then

$$\begin{aligned}
&\mathrm{Sym}^2(B \oplus F_2) \otimes (B \oplus F_2)\\
&= \left(B^{\otimes 2} \oplus (B \otimes F_2) \oplus F_3\right) \otimes \left(B^{-1} \oplus F_2\right)\\
&= B \oplus F_2 \oplus \left(B^{-1} \otimes F_3\right) \oplus \left(B^{\otimes 2} \otimes F_2\right) \oplus \left(B \otimes F_2^{\otimes 2}\right) \oplus (F_3 \otimes F_2)
\end{aligned}$$

Write $F_2^{\otimes 2} = \bigoplus_i F_{r_i}$ and $F_3 \otimes F_2 = \bigoplus_j F_{t_j}$ we have

$$\begin{aligned}
&A \otimes \mathrm{Sym}^2(B \oplus F_2) \otimes (B \oplus F_2)\\
&= (A \otimes B) \oplus (A \otimes F_2) \oplus \left(A \otimes B^{-1} \otimes F_3\right)\\
&\oplus \left(A \otimes B^{\otimes 2} \otimes F_2\right) \oplus \left(A \otimes B \otimes \bigoplus_i F_{r_i}\right) \oplus \left(A \otimes \bigoplus_j F_{t_j}\right)
\end{aligned}$$

Applying Atiyah's classification we obtain that this has a non-zero global section if and only if

$$A \in \{\mathcal{O}_C, B, B^{-1}, B^{-2}\}. \tag{5.9}$$

In conclusion we have that if there is a surjective endomorphism $f\colon \mathbb{P}(B \oplus F_r) \to \mathbb{P}(B \oplus F_r)$ with $f^*\mathcal{O}_{\mathbb{P}(B\oplus F_r)}(1) \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(B\oplus F_r)}(2)$ then

$$f^*\mathcal{O}_{\mathbb{P}(B\oplus F_r)}(1) \equiv_{\text{lin}} \mathcal{O}_{\mathbb{P}(B\oplus F_r)}(2) \otimes A$$

where A must satisfy equation 5.9. We will see in the sequel that it would be interesting to show that $A = \mathcal{O}_C$ as in 5.3.10.

5.3.1 The Kawaguchi-Silverman conjecture for Atiyah bundles

In this section we will prove the Kawaguchi-Silverman conjecture for the projective bundles $\mathbb{P}E$ where $E = \bigoplus_{i=1}^{s} F_{r_i}$. When E has rank 2 this was proven in [40]. We will then prove the same result for $E = \mathbb{P}F_r$ when $2 \leq r \leq 4$ by different methods. We call these two different approaches method one and method two. In 5.3.1 we carry out method one. We begin with the observation that if we can show that $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) > 0$ then the Kawaguchi-Silverman conjecture will follow. See the discussion below the diagram 5.10 for the details. In 5.3.11 we show that $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) > 0$ when $s > 1$. This leaves the case where $s = 1$ which is more difficult. Interestingly, method one 5.3.1 illustrates that the vector bundles F_r with $r > 2$ have different behavior compared to F_2. In particular $\kappa(\mathcal{O}_{\mathbb{P}F_2}(1)) = 0$ while $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) > 0$ when $r > 2$. Our proof will be inductive in the following sense. If $r \geq 3$ and $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) > 0$ then this will imply that $\kappa(\mathcal{O}_{\mathbb{P}F_{r+1}}(1)) > 0$. Because $\kappa(\mathcal{O}_{\mathbb{P}F_2}(1)) = 0$ the base case will be of great import. We will use the theory of Schur functors to decompose the bundles $\mathrm{Sym}^d F_3$. It will be essential that we may use a result of Atiyah to write $F_3 = \mathrm{Sym}^2 F_2$. One reason for this is the following: In general it is difficult to understand the decomposition of $\mathrm{Sym}^m \mathrm{Sym}^n E$ in a uniform way even when E is a finite dimensional vector space. According to [61, Page 63] this is because understanding the iterated symmetric product $\mathrm{Sym}^m \mathrm{Sym}^n E$ is equivalent to understanding the ring of invariants of $\mathrm{SL}(E)$ acting on $\mathrm{Sym}^n E$ which is known to be exceedingly difficult. However, when $n = 2$ this problem is tractable which allows us to obtain positive results.

In method two 5.3.1 will study the possible surjective endomorphisms of $\mathbb{P}F_r$ and prove the Kawaguchi-Silverman conjecture for $r = 2, 3, 4$ by explicit methods. In [2] it was proven that if E is a rank two vector bundle on a smooth projective variety X, then the existence of a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

with f, g surjective and $\lambda_1(f\ |_\pi) > 1$ implies that $E \cong L_1 \oplus L_2$ for some line bundles L_1, L_2 on X. We will give a new proof of this result for F_2, F_3, F_4 by direct computation. In other words we will show that if we have a diagram

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

with f, g surjective and $r \leq 4$ then $\lambda_1(f\ |_\pi) = 1$. This extends [2] in the sense that we expect that the existence of a surjective endomorphism f^n with $\lambda_1(f^n\ |_\pi) > 1$ to imply that F_r splits as a direct sum of line bundles. Since F_r does not split as a sum of line bundles we expect no such surjective mapping to exist. However, previous methods of verifying this expectation only exist in certain cases and did not apply to $\mathbb{P}F_r$ when $r > 2$. As a corollary we have a different proof of the Kawaguchi-Silverman conjecture for $\mathbb{P}F_2, \mathbb{P}F_3, \mathbb{P}F_4$. It is our hope that the techniques developed in method two will allow a full proof for all $r \geq 2$ in the future.

To compare and contrast the two approaches we note that each method provides a genuinely different proof of the Kawaguchi-Silverman conjecture, and provides new information about vector bundles on an elliptic curve. Suppose that we have a surjective endomorphism $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$. After iterating f we have a diagram

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f^n"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd} \tag{5.10}$$

By 5.3.10 we have that $(f^n)^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\text{lin}} \mathcal{O}_{\mathbb{P}F_r}(\lambda_1(f^n\ |_\pi)$. If $\lambda_1(f^n\ |_\pi) \leq \lambda_1(g)$ then by the product formula for dynamical degrees we have that

$$\lambda_1(f^n) = \max\{\lambda_1(g), \lambda_1(f^n\ |_\pi)\} = \lambda_1(g). \tag{5.11}$$

Thus if P has a dense orbit under f then it has a dense orbit under f^n and $\pi(P)$ has a dense g orbit. We then have

$$\alpha_{f^n}(P) \geq \alpha_g(\pi(P)) = \lambda_1(g) = \lambda_1(f^n) = \lambda_1(f)^n.$$

The fundamental inequality for dynamical and arithmetic degrees shows that

$$\alpha_{f^n}(P) \leq \lambda_1(f^n).$$

Therefore we obtain that

$$\alpha_f(P)^n = \alpha_{f^n}(P) = \lambda_1(f^n) = \lambda_1(f)^n$$

and taking n^{th} roots gives $\alpha_f(P) = \lambda_1(f)$ as needed. Therefore we may assume that

$$\lambda_1(f\mid_\pi) > \lambda_1(g)$$

and consequently that $(f^n)^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\text{lin}} \lambda_1(f^n)\mathcal{O}_{\mathbb{P}F_r}(1)$.

The first method relies on showing that $\dim H^0(C, \operatorname{Sym}^d F_r) \geq O(d)$ when $r \geq 3$ where we use the big O notation. In this case $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) > 0$. We now appeal to 4.2.9 to obtain the Kawaguchi-Silverman conjecture. This approach does *not* apply to F_2 since $\kappa(\mathcal{O}_{\mathbb{P}F_2}(1)) = 0$. By contrast method two hopefully applies for all $r \geq 2$, though currently we only have a proof for $r = 2, 3, 4$. In method two we show that $\lambda_1(f^n\mid_\pi) = 1$ and so we always have that $\lambda_1(f^n) = \lambda_1(g)$ and the analysis described after equation 5.11 applies. In conclusion method one gives us information about the rate of growth of the sections of the vector bundles $\operatorname{Sym}^d F_r$ in terms of d while method two shows that $\lambda_1(f^n) = \lambda_1(g)$, and so we may prove the Kawaguchi-Silverman conjecture independent of any knowledge about $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1))$.

Theorem 5.3.11 (The reducible case)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $E = \bigoplus_{i=1}^s F_{r_i}$ where $s > 1$. Then $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) \geq s - 1$.*

Proof. Let $r = \operatorname{rank}(E) = \sum_{j=1}^s r_j$. Given a vector $\vec{i} \in \mathbb{Z}^s$ we let $i_1, ..., i_s$ be the coordinates of $\vec{i}$. Let $I_d = \{\vec{i} \in \mathbb{Z}^s_{\geq 0} : \sum_{j=1}^s i_j = d\}$ We have that $H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(d)) = H^0(C, \operatorname{Sym}^d(E))$. On the other hand we have

$$\operatorname{Sym}^d(\bigoplus_{i=1}^s F_{r_i}) = \bigoplus_{\vec{i}\in I_d} \bigotimes_{j=1}^s \operatorname{Sym}^{i_j}(F_{r_i}) = \bigoplus_{\vec{i}\in I_d} \left(\bigoplus_t F_{r_{\vec{i},t}} \right)$$

for some integers $r_{\vec{i},t}$. Note that for $\vec{i} \in I_d$

$$\dim H^0(C, \bigotimes_{j=1}^{s} \mathrm{Sym}^{i_j}(F_{r_j})) = \dim H^0(C, \bigoplus_t F_{r_{\vec{i},t}}) \geq 1.$$

We conclude that

$$\dim H^0(C, \mathrm{Sym}^d(E)) \geq \# I_d.$$

On the other hand, $\# I_d$ is the number of distinct integral solutions to $x_1 + ... + x_s = d$ with $x_i \geq 0$. There are precisely

$$\binom{s+d-1}{s-1} = \frac{(s+d-1)(s+d-2)...(d+1)}{(s-1)!}$$

solutions to $x_1 + ... + x_s = d$ with $x_i \geq 0$. We have

$$\binom{s+d-1}{s-1} = \frac{(s+d-1)(s+d-2)...(d+1)}{(s-1)!} = \prod_{i=1}^{s-1} \frac{d+s-i}{s-i} \geq \frac{d^{s-1}}{(s-1)!}.$$

So

$$\dim H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(d)) = \dim H^0(C, \mathrm{Sym}^d E) \geq \frac{d^{s-1}}{(s-1)!}$$

which means $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) \geq s-1$ as needed. □

Corollary 5.3.11.1. *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $E = \bigoplus_{i=1}^{s} F_{r_i}$ where $s > 1$. Then the Kawaguchi-Silverman conjecture holds for $\mathbb{P}E$. In particular, $\mathbb{P}E$ has a good eigenspace.*

Proof. Let $f\colon \mathbb{P}E \to \mathbb{P}E$ be a surjective endomorphism. After replacing f with some f^n we may assume that we have a diagram

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where g is a surjective endomorphism. By 5.3.10 we have that $f^*\mathcal{O}_{\mathbb{P}E}(1) \equiv_{\mathrm{lin}} \mathcal{O}_{\mathbb{P}E}(\lambda)$. So the eigendivisors of f^* correspond to $\mathcal{O}(1), \pi^*H$ where H is some ample divisor on C. If $\lambda_1(f) = \lambda_1(g)$ then the Kawaguchi-Silverman conjecture is true for f by 4.2.10.1. we may assume that $\lambda_1(f) = \lambda$ and that $\mathcal{O}_{\mathbb{P}E}(1)$ is a $\lambda_1(f)$-eigendivisor for f^*. By 5.3.11 we have that $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) > 0$ and so by 4.2.9 the Kawaguchi-Silverman conjecture holds for f. We have that f has a good eigenspace because either $\mathcal{O}_{\mathbb{P}E}(1)$ generates a $\lambda_1(f)$-eigenspace or π^*H does and in both of these cases we have positive Iitaka dimension. □

Method one: Using Schur functors

In this section we set $E = F_r$ when $r \geq 2$. First take $r = 2$. Then given a surjective endomorphism $f\colon \mathbb{P}F_2 \to \mathbb{P}F_2$ we have that $f^*(\mathcal{O}_{\mathbb{P}F_2}(1)) = \mathcal{O}_{\mathbb{P}F_2}(\lambda)$ by 5.3.10. Then we have that

$$H^0(\mathbb{P}F_2, \mathcal{O}_{\mathbb{P}E}(d)) = H^0(C, \mathrm{Sym}^d F_2) = H^0(C, F_{d+1})$$

by 3.3.1. Since we know that $\dim H^0(C, F_{d+1}) = 1$ we obtain that $\kappa(\mathcal{O}_{\mathbb{P}F_2}(1)) = 0$. This suggests that $\mathbb{P}F_r$ for $r > 1$ may give interesting dynamics because at a first glance it may seem plausible that $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) = 0$ for all $r \geq 1$. However further thought suggests that this is not the case. To compute $H^0(\mathbb{P}F_r, \mathcal{O}_{\mathbb{P}F_r}(d))$ we must compute $\mathrm{Sym}^d F_r = \mathrm{Sym}^d(\mathrm{Sym}^{r-1} F_2)$. The desire to understand an expression of the form

$$\mathrm{Sym}^n(\mathrm{Sym}^m V)$$

for some linear algebraic object V is an old one related to classical invariant theory, and remains a difficult problem in full generality. However, we expect a plethysm to govern how such an expression can be simplified. Precisely, representation theory suggest that we should have a decomposition

$$\mathrm{Sym}^d(\mathrm{Sym}^{r-1} F_2) = \bigoplus_{\mu} \mathbb{S}_\mu(F_2)^{\otimes M_{d,r-1,\mu}}$$

where $\mathbb{S}_\mu$ is a *Schur* functor. For example, the functors $\mathrm{Sym}^d(\bullet)$ and $\wedge^d(\bullet)$ are Schur functors. In general, for any vector bundle F we have a canonical and functorial injection of vector bundles

$$0 \to \mathbb{S}_\mu(F) \to F^{\otimes n}$$

for some n. Since F_r is self dual will mean that $\mathbb{S}_\mu(F_r)$ is also self dual. The existence of the above injection combined with the fact that $\mathbb{S}_\mu(F_r)$ is self dual will then imply that we have a decomposition

$$\mathbb{S}_\mu(F_r) = \bigoplus_{i=1}^{t} F_{w_i}$$

for some Atiyah bundles. Thus for each index μ we expect that $\dim H^0(C, \mathbb{S}_\mu(F_r)) > 0$ and so the dimension of the global sections of $\mathrm{Sym}^d(\mathrm{Sym}^{r-1} F_2)$ will be governed by the decomposition $\bigoplus_\mu \mathbb{S}_\mu(F_2)^{\otimes M_{d,r-1,\mu}}$. In particular, in order for $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) = 0$, we require that the decomposition

$$\mathrm{Sym}^d(\mathrm{Sym}^{r-1} F_2) = \bigoplus_{\mu} \mathbb{S}_\mu(F_2)^{\otimes M_{d,r-1,\mu}}$$

must have a bounded number of non-zero factors μ that is independent of d. We show that this is not the case.

Definition 5.3.12 (Schur Functors). *Let V be a k-dimensional vector space and λ a partition of n for some integer $n \geq 1$. That is λ is an ordered list of positive integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 1$ with $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$. The Young-Tableaux associated to λ is an array of n squares where the first row has λ_1 squares, the second has λ_2 squares and so on. We label the squares $1, ..., n$ writing left to right row by row. Now let R be any commutative ring with unity and E a finite rank free bundle on R. By [61, 2.1] there is a free R-module $\mathbb{S}_\lambda(E)$ which is naturally in a functorial way a $\mathrm{GL}(E)$-module. Now let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$ and E a rank r vector bundle on X. Let $\{U_\alpha\}$ be an open affine cover trivializing E. Set $U_{\alpha\beta} = U_\alpha \cap U_\beta$. We have transition functions*

$$\varphi_{\alpha\beta} \colon \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \to \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$$

with $\varphi_{\alpha\beta} \in \mathrm{GL}(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r})$. As the construction of Schur functors is functorial we have transition functions

$$\mathbb{S}_\lambda(\varphi_{\alpha\beta}) \colon \mathbb{S}_\lambda(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}) \to \mathbb{S}_\lambda(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r})$$

with $\mathbb{S}_\lambda(\varphi_{\alpha\beta}) \in \mathrm{GL}(\mathcal{O}_{U_{\alpha\beta}}^{\oplus N_{r,\lambda}})$. We thus obtain a vector bundle $\mathbb{S}_\lambda(E)$ on X which is $\mathbb{S}_\lambda(\mathcal{O}_{U_\alpha})$ on the affine open U_α.

Remark. Because the Schur functor $\mathbb{S}_\lambda$ is functorial, the above definition is independent of the choice of affine open cover in 5.3.12 in the sense that a different cover will give an isomorphic vector bundle.

It is a well known folklore result that any functorial relationship among Schur functors is preserved in the setting of vector bundles. We verify the relation that will be needed for us. Recall the following definition of the conjugate or dual partition.

Definition 5.3.13 (Conjugate partition). *To do this we use the notion of a dual partition. We define a partition λ^* by the rule*

$$\lambda_i^* = \#\{1 \leq j \leq k \colon \lambda_j \geq i\}.$$

This is the partition obtained by reflecting the Young diagram of λ along the line $y = -x$.

Lemma 5.3.14. *Let C be a smooth elliptic curve. Then*

$$\mathrm{Sym}^d \mathrm{Sym}^2(F_2) = \bigoplus_{|\lambda|=2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2).$$

Proof. By the proof of [61, 2.3.8] we have an *equality*

$$\mathrm{Sym}^d\mathrm{Sym}^2(E) = \bigoplus_{|\lambda|=2d, \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(E).$$

over any commutative ring R. Taking R to be the function field of C allows us to identify $\mathrm{Sym}^d\mathrm{Sym}^2(\mathcal{O}_{U_{\alpha\beta}}^{\oplus 2})$ with $\bigoplus_{|\lambda|=2d, \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(\mathcal{O}_{U_{\alpha\beta}}^{\oplus 2})$ inside the function field of C. In other words the isomorphisms

$$\theta_{\alpha\beta}\colon \mathrm{Sym}^d\mathrm{Sym}^2(\mathcal{O}_{U_{\alpha\beta}}^{\oplus 2}) \to \bigoplus_{|\lambda|=2d, \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(\mathcal{O}_{U_{\alpha\beta}}^{\oplus 2})$$

are all compatible because they can be taken to be an equality. Therefore these isomorphisms glue together to give an isomorphism

$$\mathrm{Sym}^d\mathrm{Sym}^2(F_2) \cong \bigoplus_{|\lambda|=2d, \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2)$$

as needed. Now notice that the rank of $\mathbb{S}_\lambda(F)$ is the dimension of $\mathbb{S}_\lambda(F(p))$ where $F(p)$ is any geometric fiber of F. This is because locally on an affine open U we have that $\mathbb{S}_\lambda(F) = \mathbb{S}_\lambda(\mathcal{O}_U^{\oplus \mathrm{rank}\, F})$. It suffices to now show that if V is a two dimensional vector space and λ is a partition of $2d$ with λ_i^* even for all i that if some $\lambda_i > 2$ then $\mathbb{S}_\lambda(V) = 0$. Recall that by [61, 2.1.4] that a basis of $\mathbb{S}_\lambda(V)$ is given by all standard Young tableau of shape λ on $[1, 2]$. Recall that a standard Young tableau of shape λ on $[1, 2]$ is a labeling of the squares of the Young tableau of λ with integers $1, 2$ such that across rows (left to right) the numbers are *increasing* and down columns the numbers are *non-decreasing*. If some $\lambda_i > 2$ then there is no way to label the i^{th} row of the Young tableau associated to λ left to right with $1, 2$ that gives an increasing sequence. In particular $\mathbb{S}_\lambda(V)$ has basis the empty set and so $\mathbb{S}_\lambda(V) = 0$ as claimed. Thus we obtain that

$$\mathrm{Sym}^d\mathrm{Sym}^2(F_2) = \bigoplus_{|\lambda|=2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2)$$

as needed. ∎

We will also need the following.

Lemma 5.3.15. *Let C be an elliptic curve defined over $\overline{\mathbb{Q}}$. Fix an integer $d \geq 1$. Let λ be a partition with $|\lambda| = 2d, \lambda_i \leq 2$ and λ_i^* is even for all i. Then the following are true.*

1. rank $\mathbb{S}_\lambda(F_2) \geq 1$.

2. $\mathbb{S}_\lambda(F_2) = \bigoplus_j F_{r_j}$ *for some integers* r_j. *In other words,* $\mathbb{S}_\lambda(F_2)$ *is a direct sum of Atiyah bundles.*

Proof. To show that $\mathbb{S}_\lambda(F_2)$ has rank at least one it suffices to show that there is at least one standard Young tableau of shape λ on $[1,2]$ for any partition λ with $|\lambda| = 2d, \lambda_i \leq 2$ and λ_i^* is even for all i. This is because these tableau index a basis for $S_\lambda(V)$ by [61, 2.1.4]. Since each $\lambda_i \leq 2$ we may label every row with two boxes as $(1,2)$ and every single box 1. This gives a standard Young tableau and so $\mathbb{S}_\lambda(V)$ has rank at least one. For part (2) note that by 5.3.14 we have a decomposition

$$\mathrm{Sym}^d\mathrm{Sym}^2(F_2) = \bigoplus_{|\lambda|=2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2).$$

Therefore for any $\mathbb{S}_\lambda(F_2)$ appearing in this decomposition we have that each indecomposable summand of $\mathbb{S}_\lambda(F_2)$ is an indecomposable summand of $\mathrm{Sym}^d\mathrm{Sym}^2(F_2)$. By two applications of 5.3.5.1 we see that $\mathrm{Sym}^d\mathrm{Sym}^2(F_2) = \bigoplus_j F_{l_j}$ for some Atiyah bundles F_{l_j}. As each F_{l_j} is indecomposable we have that every indecomposable summand of $\mathrm{Sym}^d\mathrm{Sym}^2(F_2)$ is an Atiyah bundle F_{r_j} for some r_j. We conclude that $\mathbb{S}_\lambda(F_2) = \bigoplus_j F_{r_j}$ for some r_j by the uniqueness of the decomposition into indecomposable summands. $\square$

We now combine these results to obtain the following key result.

Corollary 5.3.15.1. *Let* C *be an elliptic curve defined over* $\overline{\mathbb{Q}}$. *Fix an integer* $d \geq 2$. *Then*

$$\dim H^0(C, \mathrm{Sym}^d(F_3)) \geq \frac{d-1}{2}.$$

In particular, $\kappa(\mathcal{O}_{\mathbb{P}F_3}(1)) > 0$.

Proof. By 5.3.14 we may write

$$\mathrm{Sym}^d\mathrm{Sym}^2(F_2) = \bigoplus_{|\lambda|=2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2).$$

By 5.3.15 we have that for all such λ appearing in the above decomposition that we have

$$\mathbb{S}_\lambda(F_2) = \bigoplus_{j=1}^{s} F_{r_j}$$

where $s \geq 1$. Therefore $\dim H^0(C, \mathbb{S}_\lambda(F_2)) \geq 1$. In other words, we have a lower bound

$$\dim H^0(C, \mathrm{Sym}^d \mathrm{Sym}^2(F_2)) \geq \#\{\lambda \text{ a partition of } 2d : \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i\}.$$

Put $\mathcal{S} = \{\lambda \text{ a partition of } 2d : \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i\}$. If $\lambda \in \mathcal{S}$ the condition that λ_i^* is even means that every column of the Young diagram associated to λ has an even number of boxes. This is because the Young diagram of λ^* is the reflection of the Young diagram of λ reflected through the line $y = -x$. So columns of λ become rows of the dual partition λ^*. As each $\lambda_i \leq 2$ we have that λ is a sequence of say w rows of the form $\square\square$ and p rows of the form $\square$. In other words, λ is a collection of w rows which consist of two boxes and p rows that consist of one box such that $2w + p = 2d$. Since λ has at most two columns that must have an even number of boxes this data must satisfy that w is even and $p + w$ is also even. We now check directly that there are at least $\dfrac{d-1}{2}$ such partitions. First let d be even. Then for each even integer w such that $0 \leq w \leq d$ let $\lambda_w = (2^w, 1^{2d-2w})$ be the partition with first w parts 2 and all remaining parts 1. This gives $\lambda_w \in \mathcal{S}$. So there are at least $\dfrac{d}{2} + 1$ elements in $\mathcal{S}$ when d is even. When d is odd for each even integer $0 \leq w \leq d-1$ we have a partition $(2^w, 1^{2d-2w})$ and $\lambda_w \in \mathcal{S}$ So there are at least $\dfrac{d-1}{2} + 1$ such partitions. We now have

$$\begin{aligned}\dim H^0(C, \mathrm{Sym}^d \mathrm{Sym}^2(F_2)) &\geq \#\{\lambda \text{ a partition of } 2d : \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i\} && (5.12)\\ &\geq \frac{d-1}{2}. && (5.13)\end{aligned}$$

Finally we have that

$$\dim H^0(\mathbb{P}F_3, \mathcal{O}_{\mathbb{P}F_3}(d)) = \dim H^0(C, \mathrm{Sym}^d F_3).$$

As $\mathrm{Sym}^d F_3 = \mathrm{Sym}^d \mathrm{Sym}^2 F_2$ by 3.3.1 we have that

$$\dim H^0(\mathbb{P}F_3, \mathcal{O}_{\mathbb{P}F_3}(d)) = \dim H^0(C, \mathrm{Sym}^d \mathrm{Sym}^2(F_2)) \geq \frac{d-1}{2}$$

This implies that $\kappa(\mathcal{O}_{\mathbb{P}F_3}(1)) > 0$ as needed. $\square$

Theorem 5.3.16 ($\dim H^0(C, \mathrm{Sym}^d F_r)$ grows at least linearly in d.)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $r \geq 3$. Then*

$$\dim H^0(C, \mathrm{Sym}^d(F_r)) \geq \frac{d-1}{2}.$$

Proof. We induct on $r \geq 3$. The base case is 5.3.15.1.

$$\dim H^0(C, \mathrm{Sym}^d(F_3)) = \dim H^0(C, \mathrm{Sym}^2(\mathrm{Sym}^d(F_2)) \geq \frac{d-1}{2}.$$

Now suppose that the claim is true for some $r > 3$. We have a canonical exact sequence

$$0 \to \mathcal{O}_C \to F_r \to F_{r-1} \to 0$$

constructed by Atiyah. Applying [57, 01CF] there is an exact sequence

$$\mathrm{Sym}^{d-1}(F_r) \to \mathrm{Sym}^d(F_r) \to \mathrm{Sym}^d(F_{r-1}) \to 0.$$

Let $\mathcal{G}_{r,d} = \ker(\mathrm{Sym}^d(F_r) \to \mathrm{Sym}^d(F_{r-1}))$. Then we have a short exact sequence

$$0 \to \mathcal{G}_{r,d} \to \mathrm{Sym}^d(F_r) \to \mathrm{Sym}^d(F_{r-1}) \to 0.$$

Taking the dual sequence gives

$$0 \to (\mathrm{Sym}^d F_{r-1})^\vee \to (\mathrm{Sym}^d F_r)^\vee) \to \mathcal{G}_{r,d}^\vee \to 0.$$

Since we are in characteristic zero we have $(\mathrm{Sym}^d F_r)^\vee = \mathrm{Sym}^d(F_r^\vee)$ and since the Atiyah bundle is self dual we have $\mathrm{Sym}^d(F_r^\vee) = \mathrm{Sym}^d(F_r)$ for any r. Thus, we have a short exact sequence

$$0 \to \mathrm{Sym}^d F_{r-1} \to \mathrm{Sym}^d F_r \to \mathcal{G}_{r,d}^\vee \to 0.$$

As taking global sections is left exact, we have that

$$0 \to H^0(C, \mathrm{Sym}^d F_{r-1}) \to H^0(C, \mathrm{Sym}^d F_r)$$

is exact. Thus, we have that

$$\dim H^0(C, \mathrm{Sym}^d F_r) \geq H^0(C, \mathrm{Sym}^d F_{r-1}) \geq \frac{d-1}{2}$$

where the last inequality is by induction. □

Corollary 5.3.16.1 ($\mathcal{O}_{\mathbb{P}F_r}(1)$ is positive)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let $r \geq 3$. Then $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) \geq 1$.*

Proof. We have that $H^0(\mathbb{P}F_r, \mathcal{O}_{\mathbb{P}F_r}(d)) = H^0(C, \mathrm{Sym}^d F_r)$. Thus

$$\dim H^0(\mathbb{P}F_r, \mathcal{O}_{\mathbb{P}F_r}(d)) = \dim H^0(C, \mathrm{Sym}^d F_r) \geq \frac{d-1}{2}$$

by 5.3.16. This gives that $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) > 0$. □

Recall that $\mathrm{Sym}^d F_2 = F_{d+1}$ so that $\kappa(\mathcal{O}_{\mathbb{P}F_2}(1)) = 0$. Therefore $\dim H^0(C, \mathrm{Sym}^d F_2) = 1$ for all $d \geq 2$. On the other hand we just saw in 5.3.16.1 that $\dim H^0(C, \mathrm{Sym}^d F_r) \geq \frac{d-1}{2}$ when $r > 2$. This result is somewhat surprising but admits the following partial explanation. The global sections of $\mathrm{Sym}^d F_2$ are computed by a single symmetric power. We do not have a universal functorial way to decompose $\mathrm{Sym}^d V$ when V is a two dimensional vector space. In particular, the single symmetric product $\mathrm{Sym}^d F_2$ is *indecomposable* as a vector bundle. On the other hand, in 5.3.16 we saw that once we have a lower bound on $H^0(C, \mathrm{Sym}^d F_r)$ we obtain a lower bound on $H^0(C, \mathrm{Sym}^d F_{r'})$ for all $r' \geq r$. So it suffices to work with $r = 3$. In this case the global sections are computed by the double symmetric power $\mathrm{Sym}^d(\mathrm{Sym}^2 F_2)$. This is one of the few examples where we can compute a plethsym

$$\mathrm{Sym}^d(\mathrm{Sym}^{r-1} F_2) = \bigoplus_{\mu} \mathbb{S}_\mu(F_2)^{n_\mu}$$

explicitly. In 5.3.14 we saw that

$$\mathrm{Sym}^d \mathrm{Sym}^2(F_2) = \bigoplus_{|\lambda|=2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i} \mathbb{S}_\lambda(F_2)$$

so $\mathrm{Sym}^d \mathrm{Sym}^2 F_2$ is always *decomposable* as a vector bundle. Furthermore we can bound the number of global sections below by the number of partitions satisfying

$$|\lambda| = 2d, \lambda_i \leq 2 \text{ and } \lambda_i^* \text{ is even for all } i.$$

As the number of these partitions grows linearly in d we have the desired result.

We obtain as a corollary:

Corollary 5.3.16.2. *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Then the Kawaguchi-Silverman conjecture holds for $\mathbb{P}F_r$ when $r > 2$. In particular, $\mathbb{P}F_r$ has a good eigenspace.*

Proof. Let $f \colon \mathbb{P}F_r \to \mathbb{P}F_r$ be a surjective endomorphism. After replacing f with f^n we may assume the existence of

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where g is a surjective endomorphism. By 5.3.10 we have that $f^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\text{lin}} \mathcal{O}_{\mathbb{P}E}(\lambda)$. So the eigendivisors of f^* correspond to $\mathcal{O}_{\mathbb{P}F_r}(1), \pi^*H$ where H is some ample divisor on C.

If $\lambda_1(f) = \lambda_1(g)$ then the Kawaguchi-Silverman conjecture is true for f by 4.2.10.1. We may assume that $\lambda_1(f) = \lambda$ and that $\mathcal{O}_{\mathbb{P}F_r}(1)$ is a $\lambda_1(f)$-eigendivisor for f^*. By 5.3.16 we have that $\kappa(\mathcal{O}_{\mathbb{P}F_r}(1)) > 0$ and so by 4.2.9 the Kawaguchi-Silverman conjecture holds for f. We have that f has a good eigenspace because either $\mathcal{O}_{\mathbb{P}F_r}(1)$ generates an eigenspace or π^*H does and in both of these cases we have positive Iitaka dimension. □

Method two: Restricting the relative dynamical degree

We now turn to different more elementary methods. We will attempt to prove results about the *structure* of the monoid of surjective endomorphisms $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$. Recall that by [44, Theorem 6.2] that if $\mathbb{P}F_r$ admits an int-amplified endomorphism, that $\kappa(-K_{\mathbb{P}E}) \geq 0$. On the other hand by 5.3.4 we already know $\kappa(-K_{\mathbb{P}E}) \geq 0$. Thus the projective bundles $\mathbb{P}F_r$ are a potentially good source of examples to probe to what extent $\kappa(-K_{\mathbb{P}E}) \geq 0$ is a sufficient condition to have an int-amplified endomorphism.

For additional motivation consider the following results.

Theorem 5.3.17 ([2] and [3]). *Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$. Let E be a vector bundle on X. Suppose that we have a diagram*

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}E \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

with f, g surjective endomorphisms. Suppose that for a general fiber $\pi^{-1}(p)$ we have that

$$f_p\colon \pi^{-1}(p) \to \pi^{-1}(g(p))$$

has degree strictly greater than one.

1. *If $\operatorname{rank}(E) = 2$ then E splits as a direct sum of line bundles after a finite surjective base change. Moreover if E is semi-stable then the covering can be chosen to be smooth and unramified over X.*

2. *Suppose that $H^1(X, L) = 0$ for all line bundles L on X and that X is simply connected. Then E splits as a direct sum of line bundles.*

Notice that if we take $X = C$ an elliptic curve and $E = F_r$ with $r > 2$ then $\mathbb{P}F_r$ does not satisfy (1) or (2) of 5.3.17. For (2) note that an elliptic curve does not have trivial fundamental group, and the cohomology of all line bundles does not vanish on an elliptic curve. In particular $\dim H^1(C, \mathcal{O}_C) = 1$. Therefore, the varieties $\mathbb{P}F_r$ provide a good testing ground to see how 5.3.17 generalizes. For example the naive generalization of (1) in 5.3.17 states that $\mathbb{P}F_r$ does not admit a surjective endomorphism that is degree greater than one on the fibers.

We will analyze the surjective endomorphisms of $\mathbb{P}F_r$. We use the notation of 3.3.1.1. To recall the relevant details we have an elliptic curve C in Legendre form with an affine open cover U_0, U_2. Furthermore, we have the valuation $\nu_O = \nu$ which gives the order of vanishing at the origin. Let us recall the following facts described in 3.3.1 that will be used repeatedly.

1. There is an element $\omega \in \mathcal{O}_C(U_0 \cap U_2)$ with $\nu(\omega) = -1$.
2. If $f \in \mathcal{O}_C(U_0)$ then $\nu(f) \leq 0$ and $\nu(f) \neq -1$.
3. If $g \in \mathcal{O}_C(U_2)$ then $\nu(g) \geq 0$.
4. We have that $\mathcal{O}_C(U_0) \cap \mathcal{O}_C(U_2) = \bar{\mathbb{Q}}$.

Recall that by 3.3.4 we have that F_r is trivialized on U_0, U_2. Let $V = U_0 \cap U_2$. Since the U_i trivialize $\mathbb{P}F_r$ we have that $\pi^{-1}(U_i) = \mathbb{P}^{r-1}_{U_i}$. Now suppose that we have a commutative square

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where g is a surjective endomorphism of C. By 5.3.10 we have that $f^*\mathcal{O}_{\mathbb{P}F_r} = \mathcal{O}_{\mathbb{P}F_r}(d)$ for some $d \geq 1$. Due to the discussion following 5.4 we have a surjection

$$\theta \colon \pi^* F_r \to \mathcal{O}_{\mathbb{P}F_r}(d) \to 0. \tag{5.14}$$

Now restrict to the open cover given by U_0, U_2 to obtain

$$\theta_i \colon \mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_{U_i}} \to \mathcal{O}_{\mathbb{P}^{r-1}_{U_i}}(d) \to 0 \tag{5.15}$$

for $i = 0, 2$. Let A_r be the transition matrix for F_r with respect to this cover as constructed by 3.3.4. Then this data satisfies the following compatibility condition. Over the overlap $V = U_0 \cap U_2$, $\mathcal{O}_{\mathbb{P}^{r-1}_{U_0}}(d)$ is glued to $\mathcal{O}_{\mathbb{P}^{r-1}_{U_2}}(d)$ via the mapping

$$s(t_0, ..., t_{r-1}) \mapsto s(A_r \cdot \vec{t}).$$

where $\vec{t} = [t_0, ..., t_{r-1}]$. In other words

$$s(t_0, ..., t_{r-1}) \mapsto s(t_0 + \omega t_1, ..., t_{r-2} + \omega t_{r-1}, t_{r-1}). \tag{5.16}$$

We call this function $\psi_{r,d}(s) := s(A_r \cdot \vec{t})$. On the other hand, over the overlap V we have that $\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_{U_0}}$ is glued to $\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_{U_2}}$ via the transition matrix A_r. Therefore, the compatibility condition for gluing a surjective endomorphism is then that the following diagram commutes and has exact rows

$$\begin{tikzcd}
\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_V} \arrow[r, "\theta_0"] \arrow[d, "A_r"'] & \mathcal{O}_{\mathbb{P}^{r-1}_V}(d) \arrow[r] \arrow[d, "\psi_{r,d}"] & 0 \\
\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_V} \arrow[r, "\theta_2"] & \mathcal{O}_{\mathbb{P}^{r-1}_V}(d) \arrow[r] & 0
\end{tikzcd} \tag{5.17}$$

In other words, we have

$$\psi_{r,d} \circ \theta_0 = \theta_2 \circ A. \tag{5.18}$$

where $\psi_{r,d}(s) = s(A_r \cdot \vec{t})$ where s is a degree d polynomial with coefficients in $\mathcal{O}_C(U_0 \cap U_2)$. Let $e_1, ..., e_r$ be the standard basis of $\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_{U_0}}$ and $w_1, .., w_r$ the standard basis for $\mathcal{O}^{\oplus r}_{\mathbb{P}^{r-1}_{U_2}}$. We have that

$$A_r e_1 = w_1 \text{ and } A_r e_i = \omega w_{i-1} + w_i. \tag{5.19}$$

Set $s_i = \theta_0(e_i)$ and $g_i = \theta_2(w_i)$. Note that we may identify $H^0(\mathbb{P}^{r-1}_{U_i}, \mathcal{O}_{\mathbb{P}^{r-1}_{U_i}}(d))$ with the homogeneous polynomials of degree d with coefficients in $\mathcal{O}_C(U_i)$. We have that

$$s_i \in \mathcal{O}_{U_0}[t_0, ..., t_{r-1}]_d \text{ and } g_i \in \mathcal{O}_{U_2}[t_0, ..., t_{r-1}]_d \tag{5.20}$$

where $\mathcal{O}_{U_i}[t_0, ..., t_{r-1}]_d$ is the collection of degree d polynomials in $\mathcal{O}_{U_i}[t_0, ..., t_{r-1}]$.

Remark 5.3.18. The crucial idea is that the s_i have coefficients in $\mathcal{O}_C(U_0)$ and **not in** $\mathcal{O}_C(V)$, and similarly for the g_i. In particular the s_i and g_i are constrained by the algebraic facts described in 5.3.1.

The morphism θ_0 being surjective is equivalent to $s_1, ..., s_r$ having no common zero and similarly θ_2 is surjective precisely when $g_1, ..., g_r$ have no common zero. In conclusion the existence of the surjective endomorphism 5.14 is equivalent to the data of degree d homogeneous polynomials $s_1, ..., s_r \in \mathcal{O}_C(U_0)[t_0, ..., t_{r-1}]$ and degree d homogeneous polynomials $g_1, ..., g_r \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$ such that the collection $\{s_i\}$ have no common zero, and that the collection $\{g_i\}$ have no common zero and that the s_i, g_i satisfy the compatibility condition described by 5.17 or 5.18.

In conclusion we have proven the following.

Lemma 5.3.19. *Let C be an elliptic curve defined over $\overline{\mathbb{Q}}$. Let F_r be the rank r Atiyah bundle on C. Let $g\colon C \to C$ be a surjective endomorphism. Then there is a commutative square*

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

with $f^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\mathrm{lin}} \mathcal{O}_{\mathbb{P}F_r}(d)$ for some $d > 1$ if and only if the following data exist.

1. *Polynomials $s_1, \ldots, s_r \in \mathcal{O}_C(U_0)[t_0, \ldots, t_{r-1}]$ homogeneous of degree d and $g_1, \ldots, g_r \in \mathcal{O}_C(U_2)[t_0, \ldots, t_{r-1}]$ homogeneous of degree d.*
2. $s_1(A_r \cdot \vec{t}) = s_1(t_0 + \omega t_1, t_1 + \omega t_2, \ldots, \omega t_{r-2} + t_{r-1}, t_{r-1}) = g_1(t_0, \ldots, t_{r-1})$.
3. *For each $2 \leq i \leq r$ we have*

   $$s_i(A_r \cdot \vec{t}) = s_i(t_0 + \omega t_1, t_1 + \omega t_2, \ldots, \omega t_{r-2} + t_{r-1}, t_{r-1}) = \omega g_{i-1}(t_0, \ldots, t_{r-1}) + g_i(t_0, \ldots, t_{r-1})$$

4. *$s_1, \ldots, s_r$ do not have common zero in U_0 and $g_1, \ldots, g_r$ do not have a common zero in U_2.*

Our goal is now to prove the following.

Lemma 5.3.20 (Non-existence of surjections.)**.** *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Suppose that $d > 1$. Let F_r be the rank r Atiyah bundle when $2 \leq r \leq 4$. Then there is no surjection of sheaves*

$$\theta\colon \pi^*F_r \to \mathcal{O}_{\mathbb{P}F_r}(d) \to 0.$$

In particular, there is not surjective endomorphism $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$ with $f^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\mathrm{lin}} \mathcal{O}_{\mathbb{P}F_r}(d)$ with $d > 1$.

Corollary 5.3.20.1. *Let C be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let F_r be the rank r Atiyah bundle when $2 \leq r \leq 4$. Then $\mathbb{P}F_r$ does not admit an int-amplified endomorphism.*

Proof. Let $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$ be a surjective endomorphism. After iterating f we have a diagram

$$\begin{tikzcd} \mathbb{P}F_r \arrow[r, "f^n"] \arrow[d, "\pi"'] & \mathbb{P}F_r \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

with g surjective. By 5.3.10 we have that $(f^n)^*(\mathcal{O}_{\mathbb{P}F_r}(1)) \equiv_{\text{lin}} \mathcal{O}_{\mathbb{P}F_r}(d)$ for some $d \geq 1$. By 5.3.20 we have that $d = 1$. Thus $(f^n)^*$ has an eigenvalue of absolute value one. Since the eigenvalues of $(f^n)^*$ are n^{th} powers of eigenvalues of f^* we have that f^* has an eigenvalue of modulus 1 and so by 4.1.11 we have that f is not int-amplified as needed. $\square$

Our strategy is to show that no collection of polynomials s_i, g_i that satisfy the requirements of 5.3.19. It is our hope that this method will give a full proof for general r in the future. Given a polynomial $s \in \mathcal{O}_{U_i}[t_0, ..., t_{r-1}]_d$ and a monomial $t_0^{a_0}...t_{r-1}^{a_{r-1}}$ we will let

$$s[t_0^{a_0}...t_{r-1}^{a_{r-1}}] = \text{ coefficient of } t_0^{a_0}...t_{r-1}^{a_{r-1}} \text{ in } s. \tag{5.21}$$

As a warm up, we first do the case of $r = 2$. This can also be achieved by results of [2].

The argument when $r = 2$

Towards a contradiction suppose that there is a surjection

$$\theta\colon \pi^* F_2 \to \mathcal{O}_{\mathbb{P}F_2}(d) \to 0$$

with $d > 1$. Let $s = \sum_{i=0}^{d} a_i t_0^i t_1^{d-i} \in \mathcal{O}_{U_0}[t_0, t_1]$. Then

$$\psi_{2,s}(s) = \sum_{i=0}^{d} a_i (t_0 + \omega t_1)^i t_1^{d-i}.$$

Expanding with the binomial theorem gives

$$\sum_{i=0}^{d} \sum_{j=0}^{i} \binom{i}{j} a_i \omega^{i-j} t_0^j t_1^{d-j}$$

The coefficient of t_0^p with $0 \leq p \leq d$ in the above is obtained by summing

$$a_q \binom{q}{p} \omega^{q-p}$$

over all q with $p \leq q$. Thus we have that

$$\sum_{i=0}^{d} a_i (t_0 + \omega t_1)^i t_1^{d-i} = \sum_{p=0}^{d} t_0^p t_1^{d-p} \left(\sum_{p \leq q \leq d} \binom{q}{p} a_q \omega^{q-p} \right) = \sum_{p=0}^{d} t_0^p t_1^{d-p} A_{2,p}.$$

Let e_1, e_2 be the standard basis for $\mathcal{O}_{\mathbb{P}^1_{U_0}}$ and w_1, w_2 the standard basis for $\mathcal{O}_{\mathbb{P}^1_{U_2}}$. Let θ_0 be the surjection $\theta_0 \colon \mathcal{O}_{\mathbb{P}^1_{U_0}} \to \mathcal{O}_{\mathbb{P}^1_{U_0}}(d) \to 0$ and θ_2 be the surjection $\theta_2 \colon \mathcal{O}_{\mathbb{P}^1_{U_2}} \to \mathcal{O}_{\mathbb{P}^1_{U_2}}(d) \to 0$ Set $\theta_0(e_k) = s_k = \sum_{i=0}^{d} a_{ki} t_0^i t_1^{d-i}$ for $k = 1, 2$.

$$A_2(e_1) = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$$

The compatibility condition tells us that

$$\theta_2(w_1) = g_1 = \psi_{2,d}(s_1) = \sum_{p=0}^{d} t_0^p t_1^{d-p} \left(\sum_{p \leq q \leq d} \binom{q}{p} a_{1q} \omega^{q-p} \right).$$

Since $g_1 \in \mathcal{O}_{U_2}[t_0, t_1]_d$ we must have that the coefficients of $\psi_{2,d}$ must in in $\mathcal{O}_C(U_2)$. In other words, we have that

$$\sum_{p \leq q \leq d} \binom{q}{p} a_{1q} \omega^{q-p} \in \mathcal{O}_C(U_2)$$

for each p. Write

$$\psi_{2,d}(s_1)[t_0^p t_1^{d-p}] = \sum_{p \leq q \leq d} \binom{q}{p} a_{1,q} \omega^{q-p}.$$

Then $\psi_{2,d}(s_1)[t_0^d] = a_{1,d} \in \mathcal{O}_C(U_2)$. But $a_{1,d} \in \mathcal{O}_C(U_0)$ as well so by 5.3.1 part (3) we have $a_{1,d} \in \bar{\mathbb{Q}}$. So $\nu(a_{1,d}) \in \{0, \infty\}$. Towards a contradiction suppose that $\nu(a_{1,d}) = 0$, that is $a_{1,d} \neq 0$. Then

$$\psi_{2,d}(s_1)[t_0^{d-1} t_1] = \binom{d-1}{d-1} a_{1,d-1} \omega^{d-1-(d-1)} + \binom{d}{d-1} a_{1,d} \omega^{d-(d-1)} = a_{1,d-1} + a_{1,d} \omega.$$

By 3.3.2 we have $\nu(a_{1,d-1}) \leq 0$ as $a_{1,d-1} \in \mathcal{O}_C(U_0)$. Also $\nu(a_{1,d}\omega) = \nu(\omega) = -1$ by 5.3.1 part (1). Thus $\nu(a_{1,d-1}) \neq \nu(a_{1,d-1}\omega)$ and so

$$\nu(a_{1,d-1} + \omega a_{1,d}) = \min\{\nu(a_{1,d-1}), -1\} \leq -1.$$

But $\psi_{2,d}(s_1)[t_0^{d-1}t_1] \in \mathcal{O}_C(U_2)$ which means by 3.3.2 that $\nu(\psi_{2,d}(s_1)[t_0^{d-1}t_1]) \geq 0$ which gives a contradiction. So we have that $a_{,1d} = 0$ and $\psi_{2,d}(s_1)[t_0^{d-1}t_1] = a_{1,d-1}$ Since $a_{1,d-1} \in \mathcal{O}_C(U_0)$ and $\psi_{2,d}(s_1)[t_0^{d-1}t_1] \in \mathcal{O}_C(U_2)$ by 5.3.1 part (3) we have $a_{1,d-1} \in \bar{\mathbb{Q}}$. Now suppose that for some $k \leq d$ that we have $a_{1,d} = \ldots = a_{1,d-k} = 0$ with $d-k \geq 2$ and $a_{1,d-k-1} \in \bar{\mathbb{Q}}$. Then using the fact that $a_{1,d-j} = 0$ for $j = 0, ..., k$ we have

$$\psi_{2,d}(s_1)[t_0^{d-k-2}t_1^{k+2}] = \sum_{d-k-2 \leq q \leq d} \binom{q}{d-k-2} a_{1,q}\omega^{q-d+k+2} \tag{5.22}$$

$$= \binom{d-k-2}{d-k-2} a_{1,d-k-2}\omega^{d-k-2-d+2+k} + \binom{d-k-1}{d-k-2} a_{1,d-k-1}\omega^{d-k-1-d+k+2} \tag{5.23}$$

$$= a_{1,d-k-2} + a_{1,d-k-1}\omega. \tag{5.24}$$

As $a_{1,d-k-1} \in \bar{\mathbb{Q}}$ the argument that shows $a_{1,d} = 0$ and $a_{1,d-1} \in \bar{\mathbb{Q}}$ shows that $a_{1,d-k-1} = 0$ and $a_{1,d-k-2} \in \bar{\mathbb{Q}}$. We may continue in this way until we reach $a_{1,d} = a_{1,d-1} = ... = a_{1,1} = 0$. Then

$$\psi_{2,d}(s_1)[t_1^d] = \sum_{k=0}^{d} a_{1,k}\binom{d}{0}\omega^k = a_{1,0}$$

Thus $a_{1,0} \in \mathcal{O}_C(U_0) \cap \mathcal{O}_C(U_2) = \bar{\mathbb{Q}}$ and is non-zero. We have shown that $s_1 = a_{1,0}t_1^d = \alpha t_1^d$ with $\alpha \in \bar{\mathbb{Q}}^*$. Now we have that

$$A_2(e_2) = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \omega \\ 1 \end{bmatrix} = \omega w_1 + w_2$$

In other words, the compatibility condition says that

$$\sum_{p=0}^{d} t_0^p t_1^{d-p} \left(\sum_{p \leq q \leq d} \binom{q}{p} a_{2,q}\omega^{q-p} \right) = \omega\theta_2(w_1) + \theta_2(w_2).$$

Our first computation shows that $\theta_2(w_1) = \alpha t_1^d$ so we have that

$$\sum_{p=0}^{d} t_0^p t_1^{d-p} \left(\sum_{p \leq q \leq d} \binom{q}{p} a_{2,q}\omega^{q-p} \right) = \omega\alpha t_1^d + \theta_2(v_2).$$

This tells us that that for $p > 0$ we have that

$$\theta_2(w_2)[t_0^p t_1^{d-p}] = \psi_{2,s}(s_2)[t_0^p t_1^{d-p}].$$

Since $d > 0$ we may make the same argument as earlier with $a_{2,d}, a_{2,d-1}$ as with $a_{1,d}, a_{1,d-1}$ and conclude that $a_{2,d} = a_{2,d-1} = ... = a_{2,2} = 0$. Thus $s_2 = a_{2,0}t_1^d + a_{2,1}t_0t_1^{d-1}$. However, this is a contradiction as then s_2 and s_1 both vanish at the point $[1 : 0]$ contradicting the surjectivity of the morphism θ_0. We now conclude that no surjective endomorphism

$$\theta\colon \pi^*F_2 \to \mathcal{O}_{\mathbb{P}F_2}(d) \to 0$$

exists when $d > 1$. In particular, this shows that if

$$f\colon \mathbb{P}F_2 \to \mathbb{P}F_2$$

is a surjective endomorphism, then $f^*\mathcal{O}_{\mathbb{P}F_2}(1) \equiv_{\text{lin}} \mathcal{O}_{\mathbb{P}F_2}(1)$. Therefore $\mathbb{P}F_2$ does not admit an int-amplified endomorphism, any surjective endomorphism of $\mathbb{P}F_2$ admits a good eigenspace, and the Kawaguchi-Silverman conjecture holds for projective bundles over $\mathbb{P}F_2$.

The case of rank 3 and 4 bundles.

Here we assume that $2 < r$. Now suppose that there is a surjection

$$\theta\colon \pi^*F_r \to \mathcal{O}_{\mathbb{P}F_r}(d) \to 0$$

with $d > 1$. By 5.3.19 we have degree d polynomials $s_1, ..., s_r \in \mathcal{O}_C(U_0)[t_0, ..., t_{r-1}]$ and degree d polynomials $g_1, ..., g_r \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$. We let $\psi = \psi_{r,d}$ be the mapping

$$\psi(f(t_0, ..., t_{r-1})) = f(t_0 + \omega t_1, t_1 + \omega t_2, ..., t_{r-2} + \omega t_{r-1}, t_{r-1}). \tag{5.25}$$

For ease of notation given $\vec{i} \in \mathbb{Z}^r_{\geq 0}$ we write $t^{\vec{i}} = t_0^{i_1}...t_{r-1}^{i_{r-1}}$ and let $e_0, ..., e_{r-1}$ be a basis for $\mathbb{Z}^r$. We now formula the following lemma which we will use repeatedly.

Lemma 5.3.21 (The reduction tool)**.** *Suppose that $a, b \in \mathcal{O}_C(U_0)$ and $b \in \bar{\mathbb{Q}}$. If $a + b\omega \in \mathcal{O}_C(U_2)$ then $b = 0$ and $a \in \bar{\mathbb{Q}}$.*

Proof. Towards a contradiction suppose that $b \neq 0$. Since $b \neq 0$ we have that $\nu(b) = 0$. This gives $\nu(b\omega) = \nu(b) + \nu(\omega) = -1$. On the other hand we have that $\nu(a) \leq 0$ and $\nu(a) \neq -1$ by 3.3.2. We have

$$\nu(a + b\omega) = \min\{\nu(a), -1\} \leq -1 < 0$$

as ν is a valuation. However as $a + \omega b \in \mathcal{O}_C(U_2)$ by assumption. By 3.3.2 we have $\nu(a + \omega b) \geq 0$, a contradiction. Now suppose that $b = 0$. Then $a \in \mathcal{O}_C(U_2) \cap \mathcal{O}_C(U_0) = \bar{\mathbb{Q}}$ where the last equality is by 3.3.2. □

We now begin with our first easy reduction.

Lemma 5.3.22 (Easy vanishing step)**.** *If*

$$s = \sum_{\vec{i} \in \mathbb{Z}_{\geq 0}^r} a_{\vec{i}} t^{\vec{i}} \in \mathcal{O}_C(U_0)[t_0, ..., t_{r-1}]$$

is homogeneous of degree d and $\psi(s) \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$ then

$$s[t_0^d] = s[t_0^{d-1} t_i] = 0$$

for $1 \leq i \leq r-2$. Furthermore $a_{(d-1)e_0+e_{r-1}} \in \bar{\mathbb{Q}}$.

Proof. First notice that $\psi(s)[t_0^d] = a_{de_0} = s[t_0^d]$ so $a_{de_0} \in \mathcal{O}_C(U_0) \cap \mathcal{O}_C(U_2) = \bar{\mathbb{Q}}$. Now let $1 \leq i \leq r-1$. The coefficients of $\psi(s)[t_0^{d-1}t_i]$ must come from a term of the form $a_{de_0}(t_0 + \omega t_1)^d$ or $a_{(d-1)e_0+e_i}(t_0 + t_1\omega)^{d-1}(t_i + t_{i+1}\omega)$ or $a_{(d-1)e_0+e_{i-1}}(t_0 + t_1\omega)^{d-1}(t_{i-1} + t_i\omega)$ or $a_{(d-1)e_0+e_{r-1}}(t_0 + t_1\omega)^{d-1}t_{r-1}$. In particular we have that

$$\psi(s)[t_0^{d-1}t_1] = a_{de_0}\binom{d}{d-1}\omega + a_{(d-1)e_0+e_1} \tag{5.26}$$

$$\psi(s)[t_0^{d-1}t_i] = a_{(d-1)e_0+e_{i-1}}\omega + a_{(d-1)e_0+e_i} \text{ for } 2 \leq i \leq r-1. \tag{5.27}$$

Note that $\psi(s)[t_0^{d-1}t_1] \in \mathcal{O}_C(U_2)$ by assumption. Since $a_{de_0} \in \bar{\mathbb{Q}}$ we have by 5.3.21 that $a_{de_0} = 0$ and $a_{(d-1)e_0+e_1} \in \bar{\mathbb{Q}}$. We now induct on $1 \leq i \leq r-2$. When $i = 1$, we have that

$$\psi(s)[t_0^{d-1}t_2] = a_{(d-1)e_0+e_1}\omega + a_{(d-1)e_0+e_2} \in \mathcal{O}_C(U_2)$$

and $a_{(d-1)e_0+e_1} \in \bar{\mathbb{Q}}$. So by 5.3.21 we have that $a_{(d-1)e_0+e_1} = 0$ and $a_{(d-1)e_0+e_2} \in \overline{\mathbb{Q}}$. Now assume that for some $1 \leq i \leq r-2$ that $a_{(d-1)+e_{i-1}} = 0$ and $e_{(d-1)e_0+e_i} \in \overline{\mathbb{Q}}$. Then we have that

$$\psi(s)[t_0^{d-1}t_{i+1}] = a_{(d-1)e_0+e_i}\omega + a_{(d-1)e_0+e_{i+1}} \in \mathcal{O}_C(U_2)$$

which means by 5.3.21 that $a_{(d-1)e_0+e_i} = 0$ and $a_{(d-1)e_0+e_{i+1}} \in \bar{\mathbb{Q}}$. So by induction we have the desired result. ☐

The significance of this result is the following.

Lemma 5.3.23 (Cascade step)**.** *Let $s \in \mathcal{O}_C(U_0)[t_0, ..., t_{r-1}]$ and $g_1, g_2 \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$ be homogeneous polynomials of degree d that satisfy*

$$\omega g_1 + g_2 = \psi(s). \tag{5.28}$$

Suppose that for some $1 \leq k \leq r-1$ *we have that*

$$g_1[t_0^d] = g_1[t_0^{d-1}t_i] = 0 \text{ for } 0 \leq i \leq k. \tag{5.29}$$

Then

$$s[t_0^d] = s[t_0^{d-1}t_i] = 0 \text{ for } 0 \leq i \leq k-1. \tag{5.30}$$

Proof. By 5.28 we have that

$$\psi(s)[t^{\vec{i}}] = \omega g_1[t^{\vec{i}}] + g_2[t^{\vec{i}}]. \tag{5.31}$$

By assumption we have that $g_1[t_0^d] = g_1[t_0^{d-1}t_i] = 0$ for $1 \leq i \leq k$. So we obtain that

$$\psi(s)[t_0^d] = g_2[t_0^d] \text{ and } \psi(s)[t_0^{d-1}t_i] = g_2[t_0^{d-1}t_i] \text{ for } 1 \leq i \leq k. \tag{5.32}$$

Since $s \in \mathcal{O}_C(U_0)[t_0, ..., t_{r-1}]$ and $g_2 \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$ the exact argument given in 5.3.22 shows that

$$s[t_0^d] = s[t_0^{d-1}t_i] = 0 \text{ for } 0 \leq i \leq k-1. \tag{5.33}$$

$\square$

Corollary 5.3.23.1 (Consequences of the cascade step)**.** *Use the notation just following 5.3.1. Suppose that*

$$s_1[t_0^d] = s_1[t_0^{d-1}t_i] = 0 \tag{5.34}$$

for $1 \leq i \leq r-1$*. Then* $s_1, ..., s_r$ *have a common zero contradicting the assumptions of 5.3.1. Consequently no surjective homomorphism* $f\colon \mathbb{P}F_r \to \mathbb{P}F_r$ *exists with* $f^*\mathcal{O}_{\mathbb{P}F_r}(1) \equiv_{\mathrm{lin}} \mathcal{O}_{\mathbb{P}F_r}(d)$ *for* $d > 1$*.*

Proof. Notice that 5.19 gives us that

$$g_1 = \psi(s_1) \text{ and } \omega g_{i-1} + g_i = \psi(s_i) \text{ for } 2 \leq i. \tag{5.35}$$

Using 5.34 and 5.35 we may apply 5.3.23 to s_1 and obtain that

$$s_2[t_0^d] = s_2[t_0^{d-1}t_1] = ... = s_2[t_0^{d-1}t_{r-2}] = 0. \tag{5.36}$$

Using this and 5.34 once more we may apply 5.3.23 a total of r times to obtain that

$$s_i[t_0^d] = 0$$

for $1 \leq i \leq r$. In other words that $s_i([1 : 0... : 0]) = 0$ for each $s_1, ..., s_r$ as claimed. $\square$

By 5.3.22 and 5.3.23.1 we have to reduced to showing that $s_1[t_0^{d-1}t_{r-1}] = 0$. This is the crux of the argument. All the results stated up until this point hold for $r \geq 3$. We now make the assumption that $3 \leq r \leq 4$.

Lemma 5.3.24 (The crux)**.** *Let $3 \leq r \leq 4$. Suppose that $s \in \mathcal{O}_C(U_0)[t_0, ..., t_r]$ be homogeneous of degree d. Suppose that $\psi(s) \in \mathcal{O}_C(U_2)[t_0, ..., t_{r-1}]$. Then*

$$s[t_0^d] = s[t_0^{d-1}t_i] = 0$$

for $1 \leq i \leq r-1$.

Proof. We write $s = \sum_{\vec{i} \in \mathbb{Z}^r_{\geq 0}} a_{\vec{i}} t^{\vec{i}}$. By 5.3.22 we have that

$$s[t_0^d] = s[t_0^{d-1}t_i] = 0$$

for $1 \leq i \leq r-2$ and $a_{(d-1)e_0+e_{r-1}} \in \overline{\mathbb{Q}}$. We now begin an analysis of higher order terms. We study $\psi(s)[t_0^{d-2}t_1t_i]$ for $1 \leq i \leq r-1$. First we consider $i = 2$. The contributing terms to $\psi(s)[t_0^{d-2}t_1t_2]$ are

1. $a_{(d-2)e_0+e_1+e_2}(t_0 + \omega t_1)^{d-2}(t_1 + \omega t_2)(t_2 + \omega t_3)$
2. $a_{(d-2)e_0+2e_1}(t_0 + \omega t_1)^{d-2}(t_1 + \omega t_2)^2$
3. $a_{(d-1)e_0+e_1}(t_0 + \omega t_1)^{d-1}(t_1 + \omega t_2)$
4. $a_{(d-1)e_0+e_2}(t_0 + \omega t_1)^{d-1}(t_2 + \omega t_3)$ when $r > 3$.

By 5.3.22 we have that the term from (c) are zero, and when $r > 3$ the term from (d) is also zero. We then have

$$\psi(s)[t_0^{d-2}t_1t_2] = a_{(d-2)e_0+e_1+e_2} + 2\omega a_{(d-2)e_0+2e_1}. \tag{5.37}$$

1. Let $r = 3$. After iterating the morphism f we may assume that $d > 3$. We examine $\psi(s)[t_0^{d-2}t_1t_{r-1}]$. The contributing terms are

   (a) $a_{(d-2)e_0+2e_1}(t_0 + \omega t_1)^{d-2}(t_1 + \omega t_2)^2$

   (b) $a_{(d-2)e_0+e_1+e_2}(t_0 + \omega t_1)^{d-2}(t_1 + \omega t_2)t_2$

   (c) $a_{(d-1)e_0+e_1}(t_0 + \omega t_1)^{d-1}(t_1 + \omega t_2)$

   (d) $a_{(d-1)e_0+e_2}(t_0 + \omega t_1)^{d-1}t_2$

By 5.3.22 we have that (c) above is zero. So

$$\psi(s)[t_0t_1t_{r-1}] = 2\omega a_{(d-2)e_0+2e_1} + \omega\binom{d-1}{d-2}a_{(d-1)e_0+e_2} + a_{(d-2)e_0+e_1+e_2}. \tag{5.38}$$

If we can show that $a_{(d-2)e_0+2e_1} = 0$ then we may apply 5.3.21 and obtain $a_{(d-1)e_0+e_2} = 0$ as needed.

We now consider $\psi(s)[t_0^{d-2}t_1^2]$. The contributing terms of $\psi(s)$ are

(a) $a_{(d-2)e_0+2e_1}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)^2$

(b) $a_{(d-1)e_0+e_1}(t_0+\omega t_1)^{d-1}(t_1+\omega t_2)$

(c) $a_{de_0}(t_0+\omega t_1)^d$.

By 5.3.22 we have that (c) and (b) are zero which gives that

$$\psi(s)[t_0^{d-2}t_1^2] = a_{(d-2)e_0+2e_1}. \tag{5.39}$$

Since $\psi(s) \in \mathcal{O}_C(U_2)$ and $a_{(d-2)e_0+2e_1} \in \mathcal{O}_C(U_0)$ we have $a_{(d-2)e_0+2e_1} \in \overline{\mathbb{Q}}$ by 3.3.2. We now compute $\psi(s)[t_0^{d-3}t_1^3]$. The relevant terms in $\psi(s)$ are

(a) $a_{(d-3)e_0+3e_1}(t_0+\omega t_1)^{d-3}(t_1+\omega t_2)^3$

(b) $a_{(d-2)e_0+2e_1}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)^2$

(c) $a_{(d-1)e_0+e_1}(t_0+\omega t_1)^{d-1}(t_1+\omega t_2)$

(d) $a_{de_0}(t_0+\omega t_1)^d$.

The terms $(c) = (d) = 0$ by 5.3.22 so we obtain

$$\psi(s)[t_0^{d-3}t_1^3] = a_{(d-3)e_0+3e_1} + 2\omega a_{(d-2)e_0+2e_1}. \tag{5.40}$$

Since $a_{(d-2)e_0+2e_1} \in \overline{\mathbb{Q}}$ by the sentence after equation 5.39 we may apply 5.3.21 to obtain $a_{(d-2)e_0+2e_1} = 0$. Returning to equation 5.38 we have

$$\psi(s)[t_0t_1t_{r-1}] = 2\omega a_{(d-2)e_0+2e_1} + \omega\binom{d-1}{d-2}a_{(d-1)e_0+e_2} + a_{(d-2)e_0+e_1+e_2} \tag{5.41}$$

$$= \omega\binom{d-1}{d-2}a_{(d-1)e_0+e_2} + a_{(d-2)e_0+e_1+e_2}. \tag{5.42}$$

Since $a_{(d-1)e_0+e_2} \in \overline{\mathbb{Q}}$ we may apply 5.3.21 to obtain $a_{(d-1)e_0+e_2} = 0$ as needed.

2. Now suppose that $r = 4$. We as above examine $\psi(s)[t_0^{d-2}t_1t_{r-1}]$. The contributing terms are

   (a) $a_{(d-2)e_0+e_1+e_{r-1}}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)t_{r-1}$

   (b) $a_{(d-2)e_0+e_1+e_{r-2}}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)(t_{r-2}+\omega t_{r-1})$

   (c) $a_{(d-1)e_0+e_{r-1}}(t_0+\omega t_1)^{d-1}(t_{r-1})$

   (d) $a_{(d-1)e_0+e_{r-2}}(t_0+\omega t_1)^{d-1}(t_{r-2}+\omega t_{r-1})$.

   By 5.3.22 we have that (d) above is zero. So

$$\psi(s)[t_0^{d-2}t_1t_{r-1}] = a_{(d-2)e_0+e_1+e_{r-1}} + \omega a_{(d-2)e_0+e_1+e_{r-2}} + \binom{d-1}{d-2}\omega a_{(d-1)e_0+e_{r-1}}. \quad (5.43)$$

   Notice that if we can show $a_{(d-2)e_0+e_1+e_{r-2}} = 0$ then we may apply 5.3.21 to obtain that $a_{(d-1)e_0+e_{r-1}} = 0$ and we win. Now consider $\psi(s)[t_0^{d-2}t_1^2]$. The contributing terms of $\psi(s)$ are

   (a) $a_{(d-2)e_0+2e_1}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)^2$

   (b) $a_{(d-1)e_0+e_1}(t_0+\omega t_1)^{d-1}(t_1+\omega t_2)$

   (c) $a_{de_0}(t_0+\omega t_1)^d$

   By 5.3.22 we have that (c) and (b) are zero which gives that

$$\psi(s)[t_0^{d-2}t_1^2] = a_{(d-2)e_0+2e_1}. \quad (5.44)$$

   Since $\psi(s) \in \mathcal{O}_C(U_2)$ and $a_{(d-2)e_0+2e_1} \in \mathcal{O}_C(U_0)$ we have $a_{(d-2)e_0+2e_1} \in \overline{\mathbb{Q}}$ by 3.3.2. Now we have by 5.37 that

$$\psi(s)[t_0^{d-2}t_1t_2] = a_{(d-2)e_0+e_1+e_2} + 2\omega a_{(d-2)e_0+2e_1}.$$

   As $a_{(d-2)e_0+2e_1} \in \overline{\mathbb{Q}}$ we have by equation 5.3.21 that $a_{(d-2)e_0+2e_1} = 0$ and $a_{(d-2)e_0+e_1+e_2} \in \overline{\mathbb{Q}}$. We now compute $\psi(s)[t_0^{d-2}t_2^2]$. The relevant terms are

   (a) $a_{(d-2)e_0+2e_2}(t_0+\omega t_1)^{d-2}(t_2+\omega t_3)^2$

   (b) $a_{(d-2)e_0+2e_1}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)^2$

   (c) $a_{(d-2)e_0+e_1+e_2}(t_0+\omega t_1)^{d-2}(t_1+\omega t_2)(t_2+\omega t_3)$

We just showed that $a_{(d-2)e_0+2e_1} = 0$. So we have that

$$\psi(s)[t_0^{d-2}t_2^2] = a_{(d-2)e_0+e_1+e_2}\omega + a_{(d-2)e_0+2e_2}.$$

Since we just showed that $a_{(d-2)e_0+e_1+e_2} \in \overline{\mathbb{Q}}$ we may apply 5.3.21 we have $a_{(d-2)e_0+e_1+e_2} = 0$. Returning to 5.43 we obtain

$$\psi(s)[t_0^{d-2}t_1t_3] = a_{(d-2)e_0+e_1+e_3} + \omega a_{(d-2)e_0+e_1+e_2} + \binom{d-1}{d-2}\omega a_{(d-1)e_0+e_3} \tag{5.45}$$

$$= a_{(d-2)e_0+e_1+e_3} + \binom{d-1}{d-2}\omega a_{(d-1)e_0+e_3}. \tag{5.46}$$

We may thus apply 5.3.21 a final time and obtain the result.

This completes the proof of the result for the case $r = 3, 4$.

□

This completes the proof of 5.3.20. It seems possible that these methods could be extended to arbitrary r giving a generalization of 5.3.17 to all $\mathbb{P}F_r$. These methods may also be useful in extending the Kawaguchi-Silverman conjecture to bundles of the form $E = \bigoplus_{i=1}^s F_{r_i} \otimes L_i$ where the L_i are degree zero line bundles on C where the L_i may not be all trivial.

Chapter 6

Arithmetic eigenvalues.

Let X be a projective variety defined over $\overline{\mathbb{Q}}$, always assumed to be irreducible unless otherwise stated. Given a surjective endomorphism $f\colon X \to X$ and $P \in X(\overline{\mathbb{Q}})$ we have the *arithmetic degree* defined in 4.2.1,

$$\alpha_f(P) = \lim_{n\to\infty} h_H^+(f^n(P))^{\frac{1}{n}}. \tag{6.1}$$

Recall that by 4.2.2 we have that $\alpha_f(P) = |\mu|$ for some eigenvalue of $f^*\colon \mathrm{Nef}(X)_{\mathbb{R}} \to \mathrm{Nef}(X)_{\mathbb{R}}$. It is natural to wonder which eigenvalues of $f^*\colon \mathrm{Nef}(X)_{\mathbb{R}} \to \mathrm{Nef}(X)_{\mathbb{R}}$ arise as a limit of heights in an orbit of f. In the Kawaguchi-Silverman conjecture one wishes to show that if P has a dense orbit under f, then $\alpha_f(P) = \lambda_1(f)$. Therefore, any potential counterexample to the Kawaguchi-Silverman conjecture gives an eigenvalue μ of $f^*\colon \mathrm{Nef}(X)_{\mathbb{R}} \to \mathrm{Nef}(X)_{\mathbb{R}}$ such that $\alpha_f(P) = |\mu| < \lambda_1(f)$. From this perspective it is clear that the potential values of $\alpha_f(P)$ deserve to be studied.

In this chapter we prove that when $\mathrm{Alb}(X) \neq 0$ there can be eigenvalues of arbitrarily large size which are not the limit of heights. See 6.1.3 and example 8 for the main constructions. We then turn to when $\mathrm{Alb}(X) = 0$. Our primary example will be of toric varieties. In 6.2.1 we study morphisms of toric varieties which are *equivariant* with respect to the torus action. We give a classification result for such morphisms in 6.2.15 and use this result to give new proofs of the Kawaguchi-Silverman conjecture and the sAND conjecture for equivariant morphisms in 6.2.16. We further use the classification described by 6.2.15 to show that the absolute value of every eigenvalue of an equivariant surjective endomorphism of a toric variety is an arithmetic degree in 6.2.17. We then turn to non-equivariant morphisms and prove an analogous result in 6.2.21 using the minimal model program for toric varieties. Finally, in 6.3 we discuss how the minimal model program can be used to study questions of this type in the context of varieties admitting an int-amplified endomorphism. We have the following key definition.

Definition 6.0.1. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Consider the action of f^* on $N^1(X)_{\mathbb{R}}$ and let μ be an eigenvalue of this action with $|\mu| > 1$. We call such an eigenvalue a* ***potential arithmetic degree*** *or* ***potentially arithmetic****. If there is a point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) = |\mu|$ then we call μ an* ***arithmetic degree*** *or say μ is* ***realizable as an arithmetic degree*** *or that μ is* ***arithmetic****. If every potentially arithmetic eigenvalue is arithmetic then we say that f has* ***arithmetic eigenvalues****.*

We choose to exclude the points with $\alpha_f(P) = 1$ because this typically occurs even when 1 is not an eigenvalue for the action of f^* on $\mathrm{Nef}(X)_{\mathbb{R}}$. We note that the set of points with $\alpha_f(P) = 1$ is extremely interesting from the point of view of the Kawaguchi-Silverman conjecture and deserves further study. We will be concerned with the following question.

Question 2. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Is every potential arithmetic degree of f realizable as an arithmetic degree? In other words, is every eigenvalue μ of f^* with $|\mu| > 1$ arithmetic?*

Not every eigenvalue will be an arithmetic degree. If f is an automorphism with eigenvalue $\lambda > 1$ then f^* may have an eigenvalue $\frac{1}{\lambda} < 1$. Since $\alpha_f(P) \geq 1$ for all P we see that such eigenvalues are not arithmetic in this sense. On the other hand, recall that in [40] it was proven that $\lambda_1(f)$ is arithmetic. Our question is therefore most meaningful for eigenvalues μ with $|\mu| < \lambda_1(f)$. Recall that the sAND conjecture (conjecture 4) predicts that the set of points with $\alpha_f(P) < \lambda_1(f)$ is not Zariski dense. We saw in 4.2.6 that conjecture 4 implies the Kawaguchi-Silverman conjecture. Therefore, from the perspective of the Kawaguchi-Silverman conjecture it is important to understand the set of points with $\alpha_f(P) < \lambda_1(f)$. From this perspective question 2 asks which values can actually appear as $\alpha_f(P)$. We now collect some results that will be used later.

Proposition 6.0.2. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. Suppose that X admits a surjective endomorphism $f\colon X \to X$. If $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ acts by multiplication by a scalar, then f has arithmetic eigenvalues.*

Proof. Suppose that the action of f^* on $N^1(X)_{\mathbb{R}}$ is given by multiplication by λ. Since f^* is defined over $\mathbb{Z}$ we have that $\lambda \in \mathbb{Z}$. If $|\lambda| = 1$ then f has no potential arithmetic degrees as $\alpha_f(P) = 1$ for all $P \in X(\overline{\mathbb{Q}})$ by 4.2.8. Now assume that $|\lambda| > 1$. By 4.2.7 there is a point P with $\alpha_f(P) = |\lambda|$. Therefore, all potential arithmetic degrees are realized. $\square$

Dynamics studies the behavior of maps under iteration; it is valuable to know if a property of maps is preserved by iteration.

Proposition 6.0.3. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism.*

1. *If γ is a potential arithmetic degree of f^m then $\gamma = \mu^m$ where μ a potential arithmetic degree of f.*
2. *γ is realizable as an arithmetic degree $\iff$ μ is realizable as an arithmetic degree.*

If f^m has arithmetic eigenvalues then f has arithmetic eigenvalues.

Proof. The eigenvalues of $(f^m)^*$ are m^{th} powers of the eigenvalues of f^*. So $\gamma = \mu^m$ for some eigenvalue of f^*. Now let $\alpha_{f^m}(P) = |\gamma| = |\mu|^m$. Then we have $\alpha_f(P)^m = \alpha_{f^m}(P) = |\mu|^m$ and taking m^{th} roots gives the desired result. Now suppose that every potential arithmetic degree of f^m is an arithmetic degree. Let μ be a potential arithmetic degree of f. Then μ^m is a potential arithmetic degree of f^m and the result follows from the above calculation. $\square$

The result 6.0.3 is pleasing because it shows that question 2 is equivalent for all iterations of f.

6.1 Realizability for abelian varieties.

In this section we consider question 2 when $f\colon A \to A$ is an isogeny of an abelian variety defined over $\overline{\mathbb{Q}}$. We give a negative answer to question 2 and give an algebraic interpretation of dynamical questions in terms of the endomorphism algebra of A. To understand f^* acting on $N^1(A)_{\mathbb{R}}$ it suffices to understand the eigenvalues of the twisted conjugation action of f on the fixed points of the Rosati involution. Here we review the relevant facts from 3.2.2, in particular 3.2.36.

$$N^1(A)_{\mathbb{R}} = \mathrm{End}(A)_{\mathbb{R}}^{\mathrm{Sym}} = \{\alpha \in \mathrm{End}(A)_{\mathbb{R}} : \alpha' = \alpha\}. \tag{6.2}$$

$$\theta_f\colon \mathrm{End}(A)_{\mathbb{R}}^{\mathrm{Sym}} \to \mathrm{End}(A)_{\mathbb{R}}^{\mathrm{Sym}}, \alpha \mapsto f' \circ \alpha \circ f \text{ describes the action of } f^*\colon N^1(A)_{\mathbb{R}} \to N^1(A)_{\mathbb{R}}. \tag{6.3}$$

The eigenvalues of f^* acting on $N^1(X)_{\mathbb{R}}$ are thus interpreted as those $\alpha \in \mathrm{End}(A)_{\mathbb{R}}^{\mathrm{Sym}}$ such that $\theta_f\alpha = \lambda\alpha$. The benefit of interpreting the dynamics of f^* in this light is that we have a good understanding of the possible endomorphism groups of abelian varieties, at least after tensoring with $\mathbb{Q}$. This will allow us to get a refined notion of the possible dynamics and to prove that the existence of endomorphisms of abelian varieties with prescribed properties. Here we will use 3.2.37. We will need the following results of Kawaguchi and Silverman.

Theorem 6.1.1 ([28, Theorem 29]). *Let $A/\overline{\mathbb{Q}}$ be an abelian variety. Let $D \in \mathrm{Div}_{\mathbb{R}}(A)$ be a real nef divisor class with $\hat{q}_{A,D}$ the quadratic part of the canonical height associated to D.*

1. *There is a unique proper abelian subvariety $B_D \subseteq A$ such that*

$$\{x \in A(\overline{\mathbb{Q}}) : \hat{q}_{A,D}(x) = 0\} = B_D(\overline{\mathbb{Q}}) + A(\overline{\mathbb{Q}})_{\mathrm{tors}}.$$

2. *Suppose further that $f\colon A \to A$ is an isogeny defined over $\overline{\mathbb{Q}}$ with $f^*D = \lambda_1(f)D$ in $N^1(A)_{\mathbb{R}}$. Then $\hat{q}_{A,D}(P) \geq 0$ for all $P \in A(\overline{\mathbb{Q}})$ and $\hat{q}_{A,D}(P) > 0 \Rightarrow \alpha_f(P) = \lambda_1(f)$.*

We now have the following easy consequence of our set up. Recall that the Picard number of an abelian variety is related to its *type*. See theorem 3.2.37 for the details.

Proposition 6.1.2. *Let A be a geometrically simple abelian variety of type I or III. Then any isogeny f acts on $N^1(A)_{\mathbb{R}}$ by scalar multiplication. Furthermore, if $\lambda_1(f) > 1$ then any such action is polarized and f has arithmetic eigenvalues.*

Proof. In type I we have that the endomorphism ring of A is commutative and so the action of f^* on $N^1(A)_{\mathbb{R}}$ is given by

$$\alpha \mapsto f' \circ \alpha \circ f = (f' \circ f)\alpha$$

by Theorem 3.2.36. As $f' \circ f$ is a real number, f^* acts by dilation on $N^1(X)_{\mathbb{R}}$. In type III we have that D is a quaternion algebra and the Rosati involution is the standard involution. Thus the points α with $\alpha' = \alpha$ is precisely the center K. The same argument then tells us that f^* acts by a scalar multiplication on the Neron-Severi space. Now suppose that $\lambda_1(f) > 1$. Then if f^* acts by multiplication by a scalar μ we have that $|\mu| = \lambda_1(f) > 1$. Thus f^* is polarized. By Proposition 6.0.2 all potential arithmetic degrees are realized. □

In conclusion, the dynamics of a surjective endomorphism of a simple abelian variety of type I or III is easy to understand because there is an ample canonical height function. The other remaining cases are more complicated. The additional cases give rise to isogenies with eigenvalues which are not realizable as arithmetic degrees.

Theorem 6.1.3. *For each $g \in 2\mathbb{Z}_{>0}$ there is a simple abelian variety A of dimension g defined over a number field K with $\rho(A) = 3$ equipped with a surjective endomorphism $f : A \to A$ that has the following properties.*

1. *$f^* : N^1(A)_{\mathbb{Q}} \to N^1(A)_{\mathbb{Q}}$ has eigenvalues $a^2 > ab > b^2 > 0$ for some $a, b \in \mathbb{Z}$.*
2. *$\alpha_f(P) = a^2$ for all $P \notin A(\overline{K})_{\text{tors}}$. In particular, $\alpha_f(P) \in \{1, a^2\}$.*
3. *The eigendivisors of a^2, b^2 are nef while the eigendivisor of ab is not.*

Proof. Let A be a simple abelian variety with $\text{End}(A)_{\mathbb{Q}} = M_2(\mathbb{Q})$ with involution given by the transpose. Such abelian varieties exists by general results of (Oort/Shimura). See for example [52] and [51]. Consider an endomorphism of the form $f = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b \in \mathbb{Z}$ and $a > b$. In this case the Neron-Severi group is of rank 3. One checks directly that the eigenvectors of the twisted conjugation action are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with eigenvalues a^2, b^2, ab respectively. As A is simple, the set of points where $\alpha_f(P) < \lambda_1(f) = a^2$ is given by the torsion points by Theorem 6.1.1, which all have arithmetic degree 1. Thus $\alpha_f(P) = a^2$ or 1 and never can b^2 or ab when $b \neq 1$. It is known (see the proof of [28, Proposition 26]) that a symmetric matrix representing a divisor is ample if and only if it has positive eigenvalues, and it is nef if and only if it has non-negative eigenvalues. We see that a^2, b^2 have nef eigendivisors but ab does not as -1 is an eigenvalue of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\square$

We can give some geometric insight into this situation by considering Abelian surfaces.

Example 7. Let A be a simple abelian surface, Let $f : A \to A$ be a surjective endomorphism. Suppose that μ is an eigenvalue of f^* acting on $N^1(A)_{\mathbb{R}}$ space with $1 < |\mu| < \lambda_1(f)$. If there is a point P with $\alpha_f(P) = |\mu|$ we must have that $\overline{\mathcal{O}_f(P)} = V$ is a curve on A. This is because if $V = A$ then by the Kawaguchi-Silverman conjecture for abelian varieties we have $\alpha_f(P) = \lambda_1(f)$. On the other hand if V is zero dimensional then $\alpha_f(P) = 1$. So V must be a curve on A. After iterating f we may assume that V is irreducible and we obtain by restriction $f : V \to V$. Thus V is an irreducible curve on A which means that the genus of its normalization is at least 2. Since we have $f : V \to V$ we obtain a surjective morphism $\tilde{f}$ on the normalization of V. Since $f|_V$ has a dense orbit by construction, $\tilde{f}$ has a dense orbit. However, this is impossible, since a genus $g > 2$ smooth curve has no surjective endomorphism that is not an automorphism, and the automorphism group of such a curve is finite. We see that V is either zero dimensional, or all of A. In other words, every point is either pre-periodic or has a dense orbit. Thus to construct an endomorphism with non-realizable eigenvalues, it suffices to find an endomorphism of a simple abelian surface that admits a surjective endomorphism with two integral eigenvalues of different size.

We now turn to non-simple abelian varieties.

Theorem 6.1.4. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ where the A_i are simple pairwise non-isogenous abelian varieties such that if $g_i \colon A_i \to A_i$ is an isogeny then all potential arithmetic degrees of g_i are realizable as an arithmetic degree. If $f \colon A \to A$ is an isogeny then every potential arithmetic degree of f is realizable as an arithmetic degree.*

Proof. Let $f \colon A \to A$ be an isogeny. Write $f = f_1 + f_2 + \cdots + f_n$ where $f_i \colon A \to A_i$. Fix an integer $j \neq i$. Let $p_j \colon A_j \hookrightarrow A$ be the morphism that sends $a \mapsto (a_i)$ where $a_i = O_{A_i}$ for $i \neq j$ and $a_j = a$. So for example $p_1 \colon A_1 \to A$ is the canonical map $a \mapsto (a, O_{A_2}, \ldots, 0_{A_n})$. Then $h_{ji} = f_i \circ p_j \colon A_j \to A_i$ is a homomorphism of simple abelian varieties. Since A_j is simple the kernel is either finite or all of A_j. Towards a contradiction suppose that the kernel was finite. Then the image is either all of A_i or is O_{A_i}. If the image was all of A_i then h_{ji} is a surjective homomorphism with finite kernel, meaning it is an isogeny. This is a contradiction as A_i and A_j are not isogenous. So the image is O_{A_i}. This contradicts that the kernel is finite. So h_{ji} is the constant mapping to O_{A_i}. Since we can write $f_i(a_1, ..., a_n) = h_{1i}(a_1) + \cdots + h_{ni}(a_n)$ we have that $f_i(a_1, ..., a_n) = h_{ii}(a_i)$. Consequently we have that $f_i \colon A \to A_i$ is induced by some isogeny $h_i \colon A_i \to A_i$. Thus we have that $f = h_1 + \cdots + h_n$. In this case we have that since the A_i are pairwise all non-isogenous and simple that

$$\rho(A) = \sum_{i=1}^{n} \rho(A_i)$$

by [26, 2.3]. Therefore, we have that $N^1(A) = \prod_{i=1}^{n} \pi_i^* N^1(A_i)$ where $\pi_i \colon A \to A_i$ is the projection. Since $f = h_1 + \cdots + h_n$ we have that the eigenvalues of f^* are all of the form $h_i^* \pi_i^* H = \mu H$ where H is some class on A_i. Thus the potential arithmetic degrees of f are the potential arithmetic degrees of the h_i. Suppose that μ_i is a potential arithmetic degree of h_i. Then there is a point Q_i in A_i such that $\alpha_{h_i}(Q_i) = |\mu_i|$ by assumption. Then set $P = (P_1, \ldots P_n)$ with $P_j = O_{A_i}$ if $j \neq i$ and $P_i = Q_i$. We have that

$$\alpha_f(P) = \max_{w=1}^{n} \{\alpha_{h_w}(P_w)\} = \alpha_{h_i}(Q_i) = |\mu_i|.$$

As every potential arithmetic degree of f arises in this manner we have the desired result. $\square$

On the other hand, if we allow powers of a simple abelian variety then potential arithmetic degrees may be not be realizable.

Example 8. Let $A = E \times E$ where E does not have complex multiplication. Then $\rho(A) = 3$ by [26, 2.6] and $\mathrm{End}(A)_{\mathbb{Q}} = M_2(\mathbb{Q})$. Consider the isogeny $f(P, Q) = (aP, bQ)$ where $a, b > 0$ are integers. Note that f corresponds to the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ as in Theorem 6.1.3. It is clear that f^* has eigenvectors a^2, b^2 as $[n]\colon E \to E$ acts on $N^1(E)_{\mathbb{R}}$ by multiplication by n^2. It remains to find the third eigenvalue. We have that $N^1(E \times E)_{\mathbb{R}}$ can be identified with symmetric matrices and f^* acts by

$$A \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^t A \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

The final eigenvalue of f^* is then represented by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = ab \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We see that that $f^*\colon N^1(A)_{\mathbb{R}} \to N^1(A)_{\mathbb{R}}$ has 3 eigenvalues, a^2, ab, b^2. Note that ab does not have a nef eigendivisor, as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has a negative eigenvalue. Let $\hat{h}_E$ be the canonical height on E. Then we have that

$$\hat{h}_A(P, Q) = \hat{h}_E(P) + \hat{h}_E(Q)$$

is an ample height on A. Then

$$\hat{h}_A(f^n(P, Q)) = \hat{h}_A(a^n P, b^n Q) = a^{2n} \hat{h}_E(P) + b^{2n} \hat{h}_E(Q).$$

Thus we see that

$$\lim_{n \to \infty} \hat{h}_A^+(f^n(P, Q))^{1/n} \in \{a^2, b^2, 1\}.$$

In fact all of the above values are realizable, by taking $(P, O), (O, P)$ and (O, O) where O is the identity element of E and P is a point with $\hat{h}_E(P) \neq 0$. First take $a > b > 1$. In this case we see that a^2 has eigenvector $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and b^2 has eigenvector $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ which are both nef as these matrices have non-negative eigenvalues. On the other hand the non-realizable eigenvalue ab has a non-nef eigenvector $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. On the other hand if we take $a > 1, b = 1$ then we obtain examples of a surjective endomorphism with eigenvalues $a^2, a, 1$ but a is not realizable.

The above example is illustrative in the following sense. Let X, Y be normal projective $\mathbb{Q}$-factorial varieties. Suppose that $f\colon X \to X$ and $g\colon Y \to Y$ are surjective endomorphisms. If $\rho(X \times Y) > \rho(X) \times \rho(Y)$ then $(f \times g)^*$ will have mixed eigendivisors that do not arise as the pull back of some divisor on X or Y. On the other hand,

$$\alpha_{f\times g}(P, Q) = \max\{\alpha_f(P), \alpha_g(Q)\}.$$

If these mixed eigendivisors have mixed eigenvalues that do not appear as eigenvalues on X and Y then they will not be realizable. In terms of abelian varieties, the Picard number of A^k is always strictly larger then the Picard number of A when $k > 1$ and A is simple by [26, 2.4]. Thus we might always expect that A^k always has an isogeny with unrealizable potential arithmetic degree. Indeed, this is the case for squares of elliptic curves without CM by 8.

Corollary 6.1.4.1. *For any integer $d > 1$ there is a smooth projective variety X with $\dim X = d$ such that there is a surjective endomorphism $f\colon X \to X$ with $\lambda_1(f) > 1$ and f has does not have arithmetic eigenvalues. If $d \geq 3$ and $\kappa \in \{-\infty, 0, 1, \ldots, d-2\}$ then X may be chosen with $\kappa(X) = \kappa$. If $\kappa \in \{-\infty, 0\}$ then the surjective endomorphism may be taken to be int-amplified.*

Proof. Let E be a fixed elliptic curve without CM. If $d = 2$ then take $X = E \times E$. By example 8 there is an isogeny with positive dynamical degree but with non-arithmetic eigenvalues. Now let $d \geq 3$ and $\kappa \in \{-\infty, 0, 1, \ldots, d-2\}$. If $\kappa = -\infty$ then take $X = (\mathbb{P}^1)^{d-2} \times E^2$. Let f be as in 8 with eigenvalues $a^2 > ab > b^2 > 1$ and ab not arithmetic. Let $h\colon (\mathbb{P}^1)^{d-2} \to (\mathbb{P}^1)^{d-2}$ be any surjective endomorphism of the form $h_1 \times \cdots \times h_{d-2}$ with $\lambda_1(h_i) \neq ab$ for all $1 \leq i \leq d-2$. Then given any point $P = (Q, R) \in X$ where $Q = (Q_1, \ldots, Q_{d-2})$ we have that

$$\alpha_{h\times f}(P) = \max\{\alpha_h(Q), \alpha_f(R)\}.$$

Since

$$\alpha_h(Q) = \max_{i=1}^{d-2}\{\alpha_{h_i}(Q_i)\} \in \{1, \lambda_1(h_1), \ldots, \lambda_1(h_{d-2})\}$$

and
$$\alpha_f(R) \in \{1, a^2, b^2\}$$
we have that
$$\alpha_{h\times f}(P) \in \{1, \lambda_1(h_1), \ldots, \lambda_1(h_{d-2}), b^2, a^2\}.$$
By construction $ab \notin \{1, \lambda_1(h_1), \ldots, \lambda_1(h_{d-2}), b^2, a^2\}$ and so ab is not an arithmetic degree of $h \times f$. On the other hand ab is an eigenvalue of $(h \times f)^*$ as it is an eigenvalue of f^*. If we choose $\lambda_1(h_i) > 1$ for all i then the endomorphism is int-amplified as all the eigenvalues are strictly larger then 1. Now let $\kappa \geq 0$. If $\kappa = 0$ take $X = E^d$. Let f be as above and define $g =: E^d \to E^d$ as $g = (g_1, ..., g_{d-2}, f)\colon E^{d-2} \times E^2 \to E^{d-2} \times E^2$ where the g_i are surjective endomorphisms with $\lambda_1(g_i) \neq ab$. The argument given above gives that ab is an eigenvalue of g^* but not arithmetic. If we take $\lambda_1(g_i) > 1$ for all i we once more obtain that g is int-amplified. Finally take $\kappa > 0$. Set $X = C^\kappa \times E^{d-\kappa}$ where C is any smooth curve of genus at least 2. Since $\kappa \leq d-2$ we may write $X = C^\kappa \times E^{d-\kappa-2} \times E^2$. Now let $g\colon X \to X$ be the morphism which is the identity on the first $d-2$ factors and f on E^2. That is
$$g = \text{identity} \times f\colon (C^\kappa \times E^{d-\kappa-2}) \times E^2 \to (C^\kappa \times E^{d-\kappa-2}) \times E^2.$$
Let $\pi\colon (C^\kappa \times E^{d-\kappa-2}) \times E^2 \to E^2$ be the projection. Then given any point $P \in X(\overline{\mathbb{Q}})$ we have
$$\alpha_g(P) = \alpha_f(\pi(P)) \in \{a^2, b^2, 1\}.$$
As above we obtain that ab is a non-arithmetic eigenvalue of g^* as needed. $\square$

Remark 6.1.5. In 6.1.4.1 the examples of potential arithmetic degrees which are not realizable occur when the potential arithmetic degree does not have a nef eigendivisor. This observation leads to the following question.

Question 3. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Let μ be a potential arithmetic eigenvalue of f with a nef eigendivisor in $N^1(X)_\mathbb{R}$. Then is μ arithmetic?*

In light of the above discussion, one might wonder if eigenvalues of f^* that are constructed in some geometrically meaningful sense are realizable. Indeed, $\lambda_1(f)$ has a geometric realization and is arithmetic. From the point of view of question 3 it also has a nef eigendivisor.

Question 4. *Let X, Y be smooth projective varieties over $\overline{\mathbb{Q}}$. Consider a diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

with f, g, π being surjective. Then is $\lambda_1(f\mid_\pi)$ realizable as an arithmetic degree?

6.2 Realizability when $\mathrm{Alb}(X) = 0$ and $\kappa(X) = -\infty$.

In light of 6.1.4.1 and its proof, to construct varieties with endomorphisms that possess arithmetic eigenvalues we must eliminate the possibility of morphisms to an abelian variety that has non-arithmetic eigenvalues. To ensure this we will demand that $\mathrm{Alb}(X) = 0$. We first consider the case of smooth surfaces, and then of small Picard numbers. We then piece together some small results to be used in the future.

Proposition 6.2.1. *Let X be a smooth toric surface and let $f\colon X \to X$ be a surjective endomorphism that is not an automorphism with $\lambda_1(f) > 1$. Then f^s is polarized for some s or $X = \mathbb{P}^1 \times \mathbb{P}^1$*

Proof. There is a classification of smooth toric surfaces; X is either $\mathbb{P}^2, \mathcal{H}_r$ for $r \geq 2$ (The Hirzebruch surface) or $\mathbb{P}^1 \times \mathbb{P}^1$ or a series of blow ups of torus invariant points of one of these varieties. See [12, 10.4] for the details. Since the nef cone of any toric variety is finitely generated, after iterating f we may assume that f^* fixes the rays of the nef cone. First suppose that X is a series of blow ups at torus invariant points starting at $\mathbb{P}^2$. Then we have a diagram

$$X = X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} \dots \xrightarrow{g_{r-1}} X_r = \mathbb{P}^2$$

where each g_i is a divisorial extremal contraction. By 4.1.19 there is some $n \geq 1$ such that there are surjective endomorphisms $f_i\colon X_i \to X_i$ making the above sequence f^n-equivariant. Then $f_r\colon \mathbb{P}^2 \to \mathbb{P}^2$ has an ample eigendivisor $H_r = \mathcal{O}_{\mathbb{P}^2}(1)$. Since $g_{r-1}\colon X_{r-1} \to \mathbb{P}^2$ is a blow up we have $H_{r-1} = g_{r-1}^* H_r$ is a nef and big divisor that is not ample. Furthermore since $g_{r-1} \circ f_{r-1} = f_r \circ g_{r-1}$ we have that $f_{r-1}^* g_{r-1}^* H_r = g_{r-1}^* f_r^* H_r$ and so f_{r-1} has a nef and big eigendivisor. By induction we have that f_0 has a nef and big eigendivisor. So by 2.1.12 applied to the big cone and its closure we have that f is polarized as $\lambda_1(f) > 1$. One can check that $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a point is isomorphic to $\mathbb{P}^2$ blown up at two points. Therefore the previous argument applies. If $X = \mathcal{H}_r$ then one may directly check that $\mathcal{H}_r$ has a nef and big divisor that is not ample. In other words $\mathcal{H}_r$ has effective divisors that are not nef. Since $\mathcal{H}_r$ has Picard number two it must be an eigendivisor. The argument just given gives for $\mathbb{P}^2$ now applies to show that series of blow ups starting at $\mathcal{H}_r$ has a nef and big eigendivisor and so is polarized. $\square$

The above result is sharp in the sense that if f is a degree $d > 1$ morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a degree $d' > 1$ morphism with $d \neq d'$ then $f \times g\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a surjective morphism with $\lambda_1(f \times g) > 1$ but $f \times g$ is not polarized.

Proposition 6.2.2. *Let X be a smooth projective surface defined over number field with* $\mathrm{Alb}(X) = 0$ *and $\kappa(X) \leq 0$. Let $f\colon X \to X$ be a surjective endomorphism. Then f has arithmetic eigenvalues.*

Proof. If $\lambda_1(f) = 1$ then there is nothing to prove, so we may assume that $\lambda_1(f) > 1$. First suppose that $f\colon X \to X$ is an automorphism. Then by [11, 2.4.3] we have that $\lambda_1(f)$ is the only potential arithmetic degree and by Theorem 4.2.7 it is realizable. So we may assume that f is not an automorphism. Then by [17, Page 1] we have that X is a smooth toric surface. If X is not $\mathbb{P}^1 \times \mathbb{P}^1$ then f is polarized by Proposition 6.2.1 and every arithmetic degree is realizable. We may thus assume that $X = \mathbb{P}^1 \times \mathbb{P}^1$. After iterating f we may by 4.1.19 applied to the two fibering contractions of $\mathbb{P}^1 \times \mathbb{P}^1$ assume that $f = g_1 \times g_2\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ with $g_i\colon \mathbb{P}^1 \to \mathbb{P}^1$ a degree d_i morphism. Without loss of generality assume that $d_1 > 1$. Choose a point $P \in \mathbb{P}^1$ such that $\alpha_{g_1}(P) = d_1$ and $P_2 \in \mathbb{P}^1$ a pre-periodic point for g_2. Then $\alpha_{g_1 \times g_2}((P_1, P_2)) = d_1$ as needed. ☐

The last remaining case when $\mathrm{Alb}(X) = 0$ according to [17] is when $\kappa(X) = 1$ which we do not treat as to not go too far afield into the world of elliptic surfaces. After developing some theory we will return to the case of certain singular surfaces.

Proposition 6.2.3. *Let X be a projective normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that we have the Picard number $\rho(X)$ is 2. If X admits an extremal contraction that is not of fibering type then $(f^2)^*$ acts by scalar multiplication. In particular, if X is not a Mori fiber space and not minimal then given any surjective endomorphism f of X we have that $(f^2)^*$ X acts on $N^1(X)_{\mathbb{R}}$ by scalar multiplication.*

Proof. By hypothesis there is an extremal ray R of X an a contraction $\phi\colon X \to Y$ of birational type. Let H be an ample divisor on Y. Then $\phi^* Y$ is a big and nef divisor on X that is not ample. In particular, it must lie on the boundary of $\mathrm{Nef}(X)$. So $\mathrm{Nef}(X)$ has a ray which is big. Then $(f^*)^2$ has a nef and big eigenvector. By 2.1.12 we have that $(f^*)^2$ acts by scalar multiplication. ☐

We now describe a method of producing arithmetic degrees.

Proposition 6.2.4. *Let X be a projective normal $\mathbb{Q}$-factorial variety with at worst terminal singularities and* $\mathrm{Alb}(X) = 0$*. Suppose we have*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where ϕ is a fibering extremal contraction and f, g surjective endomorphisms. Let $\mu_1, ..., \mu_{\rho-1}$ be the eigenvalues of g^* acting on $N^1(Y)_{\mathbb{R}}$. Suppose that λ is an eigenvalue of f^* that is not an eigenvalue of g^* and that D is a non-zero nef divisor class with $f^*D \equiv_{\text{lin}} \lambda D$. In addition suppose that $|\lambda| > 1$ and D does not lie in the image of $\phi^*(N^1(Y)_{\mathbb{R}})$. If there is a point $Q \in Y(\overline{\mathbb{Q}})$ with $g(Q) = Q$ then there is a point $P \in \pi^{-1}(Q)$ with $\alpha_f(P) = \lambda$.

Proof. Since ϕ is fibering we have $\rho(X) - 1 = \rho(Y)$ where $\rho(X)$ is the Picard number of X. We may choose an ample divisor H on Y with $A = D + \phi^*H$ being ample on X. Otherwise, there would be a full dimensional subset of the boundary of $\mathrm{Nef}(X)$ which is impossible. Set $F = \pi^{-1}(Q)$. Let $\hat{h}_D$ be the canonical height function of D (see 4.2.3) defined as

$$\hat{h}_D(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} h_D(f^n(x)).$$

Notice that

$$D + \pi^*H \mid_F = D \mid_F$$

since the restriction of π^*H to a fiber is zero. Since $D + \pi^*H$ is ample the restriction is as well. Choose a height function h_H for H. We have that

$$\hat{h}_D + h_H \circ \phi$$

is a height function for $D + \pi^*H$ and the restriction to F is a height function for $(D + \pi^*H) \mid_F$. Since the fiber is contracted there is a point $P \in F$ with $\hat{h}_D(P) \neq 0$. As otherwise the ample height function $\hat{h}_D + h_H \circ \phi$ is constant, which is absurd as the fiber is not finite. So we have found $P \in F$ with $\hat{h}_D(P) \neq 0$. Then we compute

$$h_A(f^n(P)) = \lambda^n \hat{h}_D(P) + h_H(Q)$$

which tells us that $\alpha_f(P) = \lambda$ as desired as A is an ample height function, and the arithmetic degree is independent of the chosen height function. $\square$

Proposition 6.2.5. *Let X be a projective normal $\mathbb{Q}$-factorial variety with at worst terminal singularities and $\mathrm{Alb}(X) = 0$. Suppose that we have a diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where ϕ is a birational and extremal contraction. Let λ be a potential arithmetic degree of f that is not a potential arithmetic degree of g. Suppose that there is a non-trivial nef divisor D with $f^*D = \lambda D$ for linear equivalence. Let E be the exceptional locus of ϕ and let $Z = \phi(E)$. Suppose that there is a point $Q \in Z$ with $g(Q) = Q$. Then λ is an arithmetic degree.

Proof. We may choose an ample divisor H on Y with $A = D + \phi^*H$ being ample on X. Set $F = \phi^{-1}(Q)$. Let $\hat{h}_D$ be the canonical height function of D

$$D + \pi^*H \mid_F = D \mid_F$$

since the restriction of π^*H to a fiber is zero. Since $D + \pi^*H$ is ample the restriction to the fiber is as well. As we argued above there is a point $P \in F$ with $\hat{h}_D(P) \neq 0$. So we have found $P \in F$ with $\hat{h}_D(P) \neq 0$. As before we compute

$$h_A(f^n(P)) = \lambda^n \hat{h}_D(P) + h_H(Q)$$

which tells us that $\alpha_f(P) = \lambda$ as desired. $\square$

The above result tells us that the new eigenvalue introduced by a fibering type contraction is always achieved provided the base morphism has a fixed point and that we have a nef eigendivisor.

Remark 6.2.6. One runs into the following problem when trying to run a minimal model type program to obtain realizability results. Suppose that μ is a potential arithmetic degree, and $1 < |\mu| < \lambda_1(f)$. To find a point P with $\alpha_f(P) = |\mu|$ we must have that $\overline{\mathcal{O}_f(P)} = V_P$ is a proper sub variety, and $\alpha_{f|_{V_P}}(P) = |\mu|$. By construction V_P has a dense orbit given by $\mathcal{O}_f(P)$. Thus the Kawaguchi-Silverman conjecture suggests that $\lambda_1(f \mid_{V_P}) = |\mu|$. So to realize potential arithmetic degrees one must find invariant sub varieties where $\lambda_1(f \mid_V) < \lambda_1(f)$. However in very general situations it seems difficult to find invariant sub varieties. Even in the case of a fibering type contraction there are potential difficulties. Suppose that $\phi\colon X \to Y$ is a fibering type contraction, and that we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Suppose that every potential arithmetic degree of g is realized and that $\lambda_1(f) \geq \lambda_1(g)$. If say $\alpha_g(Q) = |\mu|$ then choose a nef eigendivisor D_λ for λ and an ample divisor H for Y then consider the ample height $h_A = \hat{h}_{D_\lambda} + h_H \circ \phi$ To construct a point P with $\alpha_f(P) = |\mu|$ we must find a point P such that $\hat{h}_{D_\lambda}(P) = 0$ and $\alpha_g(\phi(P)) = |\mu|$. A result of this type seem to require some knowledge of the set of points where the canonical height $\hat{h}_{D_\lambda}$ vanishes. This set is extremely interesting but currently still mysterious.

6.2.1 Realizability for toric varieties.

In this section we prove that every potential arithmetic degree of a surjective morphisms of $\mathbb{Q}$-factorial toric varieties is realizable as an arithmetic degree. We first consider equivariant morphisms in 6.2.1 and give a classification result for such morphisms in 6.2.15. We apply 6.2.15 to prove the sAND conjecture for equivariant morphisms of $\mathbb{Q}$-factorial toric varieties in 6.2.16 and show that any equivariant morphism of $\mathbb{Q}$-factorial toric varieties has arithmetic eigenvalues in 6.2.17. We then turn to the case of general morphisms in 6.2.1 and prove that every surjective morphism of $\mathbb{Q}$-factorial toric varieties has arithmetic eigenvalues. Our strategy will be to realize all the potential arithmetic degrees as the degree on the fiber of an extremal contraction as in 6.2.4 and 6.2.5. Notice that to apply 6.2.4 and 6.2.5 one must be able to find fixed points of morphisms on the target of an extremal contraction. Since we may freely iterate our morphism by 6.0.3 we must be able to find pre-periodic points. In this section, the relevant varieties will be $\mathbb{Q}$-factorial toric varieties, so we seek to guarantee the existence of fixed points for endomorphisms of $\mathbb{Q}$-factorial toric varieties. In what follows we will use ([36]) for the minimal model program applied to toric varieties.

Toric morphisms

Projective toric varieties provide an interesting class of varieties all of which admit surjective endomorphisms that are not automorphisms. It seems natural to study the endomorphism schemes $\mathrm{SEnd}(X_\Sigma)$ for a projective toric variety X_Σ and the collection of *equivariant* surjective endomorphisms $\operatorname{SEnd}_{T_\Sigma}(X_\Sigma)$ where T_Σ is the dense torus of Σ. Let X_Σ be a projective toric variety $\Sigma \subseteq N \cong \mathbb{Z}^r$. Then the toric surjective endomorphism of X_Σ correspond to matrices in $\mathrm{GL}(r, \mathbb{Q})$ with integer entries that preserve Σ is the sense that if $\sigma \in \Sigma$ and f is such a matrix then $f(\sigma) \subseteq \sigma' \in \Sigma$. Since $f_n = n \cdot \mathrm{Id}_r$ preserves all cones when $n > 0$ we have that

$$\mathbb{Z}_{\geq 0} \subseteq \operatorname{SEnd}_{T_\Sigma}(X_\Sigma). \tag{6.4}$$

Associated to any $f \in \operatorname{SEnd}(X_\Sigma)$ we have a natural linearization anti-homomorphism of monoids which sends $f \mapsto f^* \colon N^1(X_\Sigma)_\mathbb{Q} \to N^1(X_\Sigma)_\mathbb{Q}$. Anti-homomorphism here refers to the fact that $(f \circ g)^* = g^* f^*$. In other words, we have an anti-homomorphism which we call Lin which linearizes a morphism,

$$\operatorname{Lin} \colon \operatorname{SEnd}(X_\Sigma) \to \mathrm{GL}(N^1(X)_\mathbb{Q}), f \mapsto f^*. \tag{6.5}$$

One may check that $n \in \mathbb{Z}_{\geq 0} \subseteq \operatorname{SEnd}(X_\Sigma)$ is mapped to

$$n \cdot \mathrm{Id}_{N^1(X)_\mathbb{Q}}. \tag{6.6}$$

In other words, $\mathbb{Z}_{\geq 0} \subseteq \operatorname{SEnd}(X_\Sigma)$ is mapped to $\mathbb{Z}_{\geq 0} \subseteq N^1(X_\Sigma)_\mathbb{Q}$. This leads to the following natural question.

Question 5. Let X_Σ be a projective toric variety.

1. For which toric varieties is $\operatorname{Lin}(\operatorname{SEnd}(X_\Sigma))$ strictly larger then $\mathbb{Z}_{\geq 0}$? In other words, which toric varieties possess a surjective endomorphism which is not polarized?

2. For which toric varieties is $\operatorname{Lin}(\operatorname{SEnd}_{T_\Sigma}(X_\Sigma))$ strictly larger then $\mathbb{Z}_{\geq 0}$? In other words, which toric varieties possess a *equivariant* surjective endomorphism which is not polarized?

3. Can it be the case that $\operatorname{Lin}(\operatorname{SEnd}_{T_\Sigma}(X_\Sigma))$ is strictly smaller then $\operatorname{Lin}(\operatorname{SEnd}(X_\Sigma))$. In other words, is the linear action of surjective endomorphisms of a toric variety completely determined by the linear action of equivariant surjective endomorphisms?

Definition 6.2.7. *Let X_Σ be a projective toric variety defined over $\overline{\mathbb{Q}}$. We say that X_Σ is **linearly simple** if* $\operatorname{Lin}(\operatorname{SEnd}_{T_\Sigma}(X_\Sigma)$ *has finite index in $\mathbb{Z}_{\geq 0}$. In other words, every surjective toric morphism is induced by a homomorphism of tori $(x_1, \ldots, x_n) \mapsto (x_1^d, \ldots, x_n^d)$ for some $d > 0$ after possibly iterating the morphism.*

We think about this in the following way. A toric variety is linearly simple when the sub-group of obvious surjective endomorphisms is large in the sense that it is finite index. In terms of the action on $N^1(X)_{\mathbb{R}}$ we have the following interpretation.

Proposition 6.2.8. *Let X_Σ be a projective toric variety defined over $\overline{\mathbb{Q}}$. Then X_Σ is linearly simple if and only if for all $f \in \mathrm{Lin}(\mathrm{SEnd}_{T_\Sigma}(X_\Sigma)$ then eigenvalues of $f^* \colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ have the same magnitude.*

Proof. Suppose that f is linearly simple. Then there is some $n \geq 1$ such that $(f^n)^*$ acts on $N^1(X)_{\mathbb{R}}$ by scalar multiplication by $\lambda > 0$. Thus $(f^n)^* D \equiv_{\mathrm{lin}} \lambda D$ for divisors D. If μ is an eigenvalue of f^* then $\mu^n = \lambda$. Thus $|\mu| = |\lambda|^{\frac{1}{n}}$. Conversely suppose that if $f \in \mathrm{Lin}(\mathrm{SEnd}_{T_\Sigma}(X_\Sigma))$ then every eigenvalue of f has the same magnitude. Since the nef cone of a projective toric variety is finitely generated ([12, 6.3.20]). By the discussion in 5.2.1 we have that $(f^m)^*$ is diagonalizable with real eigenvalues $\lambda_1, \ldots, \lambda_\rho$. So $(f^{2m})^*$ has positive real eigenvalues which are all of the same magnitude. Thus all eigenvalues are the same, and since $(f^{2m})^*$ is diagonalizable we have that $(f^{2m})^*$ acts by scalar multiplication on $N^1(X)_{\mathbb{R}}$ as needed. □

For toric surfaces we already have results that can be leveraged to answer question 5.

Theorem 6.2.9. *Let X_Σ be a smooth projective toric surface.*

1. *Then $\mathrm{Lin}(\mathrm{SEnd}(X_\Sigma))$ is linearly simple if X_Σ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Furthermore $\mathbb{P}^1 \times \mathbb{P}^1$ is not linearly simple.*

2. *For all cases we have that*
$$\mathrm{Lin}(\mathrm{SEnd}(X_\Sigma))$$
is also finite index in $\mathbb{Z}_{\geq 0}$. In other words, the linear part of dynamics on a smooth toric surface is determined by equivariant morphisms.

Proof. This follows directly from 6.2.1. □

Notice that $\mathbb{P}^n$ is linearly simple for any n. We have the following basic necessary condition for X_Σ to be linearly simple.

Proposition 6.2.10. *Let X_Σ be a projective toric variety defined over $\overline{\mathbb{Q}}$. If the fan Σ has a non-trivial decomposition as $\Sigma_1 \times \Sigma_2$ then X_Σ is not linearly simple.*

Proof. If X_Σ and $X_\triangle$ are any two toric varieties then $X_\Sigma \times X_\triangle \cong X_{\Sigma\times\triangle}$ is not linearly simple. To see this note that we may take $f_n\colon X_\Sigma \to X_\Sigma$ to be the surjective endomorphism induced by multiplication by n on the lattice N_Σ containing the fan Σ and $g_m\colon X_\triangle \to X_\triangle$ the equivariant morphism induced by multiplication by $n \neq m$ on the lattice $N_\triangle$ containing the fan $\triangle$. Then $f_n \times g_m\colon X_\Sigma \times X_\triangle \to X_\Sigma \times X_\triangle$ is a surjective endomorphism that does not act by scalar multiplication on X_Σ. So X_Σ is not linearly simple. □

This leads to the following definition.

Definition 6.2.11 (Simple toric varieties.)**.** *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. We say that X_Σ is decomposable if*

$$X_\Sigma = X_{\triangle_1} \times X_{\triangle_2}$$

with each $X_{\triangle_i}$ a $\mathbb{Q}$-factorial projective toric variety of dimension at least 1. We say that X_Σ is simple if it is not decomposable.

We think of the above definition a toric analogy of the definition of a simple abelian variety, and $\mathrm{SEnd}_{T_\Sigma}(X_\Sigma)$ an analogy for the endomorphism ring of an abelian variety. The following is an immediate corollary of 6.2.9

Corollary 6.2.11.1. *Let X_Σ be a smooth projective toric surface defined over $\overline{\mathbb{Q}}$. Then X_Σ is simple if and only if it is linearly simple.*

It is natural to wonder if the analogous result holds for higher dimensional varieties. We now prove that this is indeed the case in 6.2.15.

Lemma 6.2.12. *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X_\Sigma \to X_\Sigma$ a surjective endomorphism induced by a mapping of lattices $\phi\colon N \to N$. Then ϕ is injective and if σ is a ray of Σ then $\phi(\sigma)$ is a ray of Σ.*

Proof. Because ϕ induces a surjective endomorphism of toric varieties, $\phi\colon N \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism of rational vector spaces. This is because the induced mapping $\phi^\vee\colon M \to M$ induces a map of semi group rings

$$\phi^\vee\colon \overline{\mathbb{Q}}[M] \to \overline{\mathbb{Q}}[M]$$

that induces the homomorphism of tori

$$\operatorname{Spec}\overline{\mathbb{Q}}[M] = T_\Sigma \to T_\Sigma = \operatorname{Spec}\overline{\mathbb{Q}}[M]$$

associated to $f\colon X_\Sigma \to X_\Sigma$. Since this map is surjective it is dominant and so is $T_\Sigma \to T_\Sigma$. As this is a morphism of affine schemes it is dominant if and only if the morphism of algebras

$$\phi^\vee\colon \overline{\mathbb{Q}}[M] \to \overline{\mathbb{Q}}[M]$$

is injective, this occurs precisely when $\phi^\vee\colon M \to M$ is injective. Thus on the level of vector spaces we have that $\phi^\vee$ and thus ϕ is injective and so bijective being a linear mapping between vector spaces of the same dimension. Now let σ be a ray of Σ. Then as ϕ is compatible with the fan Σ we have that $\phi(\sigma) \subseteq \tau$ where $\tau \in \Sigma$. Suppose that τ is the minimal such cone. Associated to σ, τ are torus orbits, $\mathcal{O}(\sigma), \mathcal{O}(\tau)$ and by [12, 3.3.21] we have that

$$f(\mathcal{O}(\sigma)) \subseteq \mathcal{O}(\tau). \tag{6.7}$$

Suppose that N is n-dimensional so that $\dim X_\Sigma = n$. Then if $\gamma \in \Sigma$ is of dimension k we have that $\dim \mathcal{O}(\gamma) = n - k$. As σ is a ray we have that $\dim \mathcal{O}(\sigma) = n - 1$. Since f is a finite morphism we have that $f(\mathcal{O}(\sigma))$ has dimension $n-1$ as well. So $\dim \mathcal{O}(\tau) \geq \dim f(\mathcal{O}(\sigma)) = n - 1$. So $\mathcal{O}(\tau)$ has dimension $n-1$ or dimension n. If $\mathcal{O}(\tau)$ has dimension n then τ must be zero dimensional which is impossible since ϕ is injective and so $\phi(\sigma)$ is at least one dimensional. It follows that τ is of dimension $n-1$ and so τ is a ray as needed. □

Lemma 6.2.13. *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. Let $X_\Sigma \to X_\Sigma$ be an equivariant surjective endomorphism induced by a morphism of lattices $f\colon N \to N$. Then f^n is diagonalizable for some n. Let f^n have eigenvalues $\lambda_1, ..., \lambda_s$ of multiplicities $m_1, ..., m_s$. Let E_i be the λ_i eigenspace of f. Put*

$$\Sigma_i = \Sigma \cap E_i = \{\sigma \cap E_i \colon \sigma \in \Sigma\}.$$

Then Σ_i is a complete fan in E_i and we have a decomposition

$$X_\Sigma = X_{\Sigma_1} \times \cdots \times X_{\Sigma_s}.$$

Proof. We first show that f^n is diagonalizable for some n. By 4.1.5 f maps rays to rays. Since f is also injective we have that f permutes the rays of Σ. So for some m we have that f^m fixes the rays of Σ. So we may replace f with f^m and assume that f fixes all the rays of Σ. Since there are at least $\dim X = n$ rays we have that f has at least n-eigenvectors and so a basis of eigenvectors because there is a maximal dimensional cone with a basis of $\mathbb{Q}$ eigenvectors. We conclude that f is diagonalizable. We now show that Σ_i is a fan for any i. Let $\sigma_i = \sigma \cap E_i$ for any $\sigma \in \Sigma$. We may assume that $v_1, ..., v_w$ are the ray generators of σ and $v_1, ..., v_t \in E_i$. Then I claim that

$$\sigma_i = \{\sum_{i=1}^{t} a_i v_i \colon a_i \geq 0\}. \tag{6.8}$$

It is clear that $\{\sum_{j=1}^{t} a_j v_j \colon a_j \geq 0\} \subseteq \sigma_i$. Now let $v \in \sigma_i$. Then $v = \sum_{j=1}^{w} b_j v_j$. As each v_j is a ray it is an eigenvector and so we may write $f(v_j) = \gamma_j v_j$ where $\gamma_j = \lambda_i$ for $1 \leq i \leq t$ and $\gamma_j \neq \lambda_i$ for $j > t$. We then compute

$$f(v) = \lambda_i v = \sum_{j=1}^{w} \lambda_i b_j v_j \tag{6.9}$$

$$= \sum_{j=1}^{w} b_j f(v_j) = \sum_{j=1}^{t} \lambda_i b_j v_j + \sum_{j>t} \gamma_j b_j v_j. \tag{6.10}$$

We then have that

$$0 = \sum_{j=1}^{w} \lambda_i b_j v_j - (\sum_{j=1}^{t} \lambda_i b_j v_j + \sum_{j>t} \gamma_j b_j v_j) \tag{6.11}$$

$$= \sum_{t<j\leq w} (\lambda_i - \gamma_j) b_j v_j. \tag{6.12}$$

Since we assumed that Σ was simplicial we have that $v_1, ..., v_w$ are independent. So for all $t < j \leq w$ we have that

$$(\lambda_i - \gamma_j) b_j = 0.$$

Since $\lambda - \gamma_j \neq 0$ by assumption we have $b_j = 0$ for $w \geq j > t$. Thus

$$v = \sum_{j=1}^{t} b_j v_j$$

and σ_i is spanned by the rays of σ in E_i. So σ_i is a finitely generated polyhedral cone that is strongly convex as if $\sigma \cap E_i$ contains a non-trivial linear subspace then so does σ, contradicting that $\sigma \in \Sigma$ is strongly convex. Now if τ is a face of σ then τ_i is a face of σ_i since if $\tau_i = u^{\perp} \cap \sigma$ with $u \in \sigma^{\vee}$. Then u restricted to E_i gives an element of the dual space E_i^* say u_i. We also have $u_i \in \sigma_i^{\vee}$ and

$$u_i^{\perp} \cap \sigma_i = \{v \in \sigma_i \colon (u_i, v) = 0\} = \{v \in \sigma \cap E_i : (u, v) = 0\} = \tau \cap E_i.$$

So τ_i is a face of σ_i. Now we have that since Σ is a fan that given any two cones,σ, σ' that $\sigma \cap \sigma'$ is a face of both σ and σ'. So

$$\sigma_i \cap \sigma_i' = (\sigma \cap \sigma')_i$$

is a face of σ_i and σ_i' and so Σ_i is a fan. Since Σ is complete $|\Sigma| = N_{\mathbb{R}}$ and so $|\Sigma| \cap E_i = E_i$ and Σ_i is a complete fan. Furthermore, given any cone σ we have that

$$\sigma = \bigoplus \sigma_i$$

since we showed that σ_i is generated precisely by the ray generators of σ in E_i. It follows that we have a decomposition

$$X_\Sigma \cong X_{\Sigma_1} \times \cdots \times X_{\Sigma_s}.$$

As each X_{Σ_i} is a closed sub-variety of X_Σ the X_{Σ_i} are projective toric varieties. Since given σ the ray generators of σ_i are a subset of the ray generators of σ we have that since X_Σ is $\mathbb{Q}$-factorial that σ_i has a linearly independent ray generating set as well. So each X_{Σ_i} is a $\mathbb{Q}$-factorial projective toric variety of dimension m_i and X_Σ decomposes as claimed. □

We see that the eigenspaces of a surjective toric morphism decompose the toric variety.

Lemma 6.2.14. *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. Suppose that surjective toric morphism $f\colon X_\Sigma \to X_\Sigma$ is induced by a lattice mapping $\phi\colon N \to N$. If ϕ is scalar multiplication by $n \geq 1$ then $f^*\colon N^1(X_\Sigma)_{\mathbb{R}} \to N^1(X_\Sigma)_{\mathbb{R}}$ is scalar multiplication by n.*

Proof. By [12, 4.2.8] a Cartier divisor D on a toric variety X_Σ is equivalent to a collection $(m_\sigma)_{\sigma \in \Sigma}$ where $m_\sigma \in M$ and D on U_σ has local equation χ^{-m_σ}. To pullback D we pull back the local equations and obtain that f^*D is has local equation $f^*\chi^{-m_\sigma} = \chi^{-\phi^\vee(m_\sigma)}$. Since ϕ is multiplication by a scalar n we have that ϕ is represented by a matrix of the form $n \cdot \mathrm{I}_{\dim X_\Sigma}$. Then $\phi^\vee$ is given by the transpose $(n \cdot \mathrm{I}_{\dim X_\Sigma})^t = n \cdot \mathrm{I}_{\dim X_\Sigma}$. Thus $\phi^\vee(m_\sigma) = n m_\sigma$. Consequently we have that $f^*D \equiv_{\mathrm{lin}} nD$ as required.

□

Theorem 6.2.15. *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. Then X_Σ is linearly simple if and only if X_Σ is simple.*

Proof. Suppose that X_Σ is linearly simple. Then X_Σ is simple by 6.2.10. Now suppose that X_Σ is simple. Towards a contradiction let $f\colon X_\Sigma \to X_\Sigma$ be an equivariant surjective endomorphism with f not linearly simple. Let f be induced by a lattice map ϕ. After iterating ϕ we may assume that ϕ fixes the rays of Σ and diagonalizable by 6.2.13. Since f is not linearly simple we have by 6.2.14 we have that ϕ is not multiplication $n \geq 1$. Then ϕ must have at least two distinct eigenvalues. By 6.2.13 we have that X_Σ decomposes non-trivially contradicting our assumption. □

It is natural at this point to ask, to what extent is a decomposition

$$X_\Sigma \cong X_{\Sigma_1} \times \cdots \times X_{\Sigma_r}$$

into simple toric varieties unique. We intend to return to this issue in the future. The ultimate goal being some sort of *analogy* between dynamics in the toric situation and dynamics in the abelian variety situation. These results can now be applied to the dynamics of toric morphisms. We first give a new proof of the sAND conjecture for equivariant surjective toric morphisms.

Theorem 6.2.16. *Let X_Σ be a $\mathbb{Q}$-factorial toric variety defined over $\overline{\mathbb{Q}}$ and $f\colon X_\Sigma \to X_\Sigma$ an equivariant surjective toric morphism. Then the sAND conjecture holds for f.*

Proof. Suppose that f is induced by a lattice mapping ϕ. By 6.2.12 we may assume that ϕ^m fixes the rays of X_Σ. Now write

$$X_{\Sigma_1} \times \cdots \times X_{\Sigma_r} \tag{6.13}$$

where each X_{Σ_i} is simple. Since ϕ^m fixes the rays of Σ we have that $f^m = h_1 \times \cdots \times h_r$ where $h_i\colon X_{\Sigma_i} \to X_{\Sigma_i}$ is a surjective equivariant endomorphism. Note that

$$\lambda_1(f^m) = \max_{i=1}^{r}\{\lambda_1(h_i)\}.$$

We may assume that $\lambda_1(f) > 1$ as the sAND conjecture is trivial when $\lambda_1(f) = 1$. For some i we have $\lambda_1(f^m) = \lambda_1(h_i) > 1$. Let $\pi_i\colon X_\Sigma \to X_{\Sigma_i}$ be the canonical projection. Since X_{Σ_i} is simple it is linearly simple by 6.2.15. As $\lambda_1(h_i) > 1$ we have that $h_i^*\colon N^1(X_{\Sigma_i})_\mathbb{R} \to N^1(X_{\Sigma_i})_\mathbb{R}$ is multiplication by $\lambda_1(h_i)$. Now fix a number field K over which our data is defined and choose $d \geq 1$. The sAND conjecture is equivalent to the assertion that

$$\mathcal{S}_{K,d} = \{P \in X(\overline{\mathbb{Q}})\colon [K(P) : K] \leq d, \alpha_f(P) < \lambda_1(f)\} \tag{6.14}$$

is not Zariski dense. Note that $h_i\colon X_{\Sigma_i} \to X_{\Sigma_i}$ is a surjective toric morphism and X_{Σ_i} is simple. Therefore by 6.2.15 h_i is linearly simple. As $1 < \lambda_1(f^m) = \lambda_1(h_i)$ we may assume that h_i is polarized, in particular there is an ample divisor H_i on X_{Σ_i} with $h_i^* H_i \equiv_{\text{lin}} \lambda_1(h_i) H_i$ Thus $\alpha_{h_i}(P') = \lambda_1(h_i)$ unless the canonical height $\hat{h}_{H_i}(P') = 0$. By the Northcott property for $\hat{h}_{H_i}$ there are finitely many points $P' \in X_{\Sigma_i}(\overline{\mathbb{Q}})$ with $\hat{h}_{H_i}(P') = 0$ and $[K : K(P)] \leq d$. Let $Q_{i1}, ..., Q_{is} \in X_{\Sigma_i}(\overline{\mathbb{Q}})$ be the points with vanishing canonical height just described and residue degree at most d that was just described. Note that we have

$$\alpha_{f^m}(P) = \max_{i=1}^{r} \{\alpha_{h_i}(\pi_i(P))\}.$$

Therefore, we have that $\alpha_{f^m}(P) = \lambda_1(h_i) = \lambda_1(f^m)$ except possibly on the proper Zariski closed set

$$\bigcup_{j=1}^{s} \pi_i^{-1}(Q_{ij}).$$

So we have that

$$\alpha_f(P)^m = \alpha_{f^m}(P) = \lambda_1(f^m) = \lambda_1(f)^m$$

except at

$$\bigcup_{j=1}^{s} \pi_i^{-1}(Q_{ij}).$$

Taking m^{th} roots now gives the desired result.

□

We now turn to realizability.

Theorem 6.2.17. *Let X_Σ be a $\mathbb{Q}$-factorial toric variety defined over $\overline{\mathbb{Q}}$ and $f\colon X_\Sigma \to X_\Sigma$ an equivariant surjective toric morphism. Then f has arithmetic eigenvalues.*

Proof. Suppose that f is induced by a lattice mapping ϕ. By 6.2.12 we may assume that ϕ^m fixes the rays of X_Σ. Now write

$$X_{\Sigma_1} \times \cdots \times X_{\Sigma_r} \tag{6.15}$$

where each X_{Σ_i} is simple. Since ϕ^m fixes the rays of Σ we have that $f^m = h_1 \times \cdots \times h_r$ where $h_i\colon X_{\Sigma_i} \to X_{\Sigma_i}$ is a surjective equivariant endomorphism. By 6.0.3 we may replace f with f^m and prove the result. We may assume that $f = h_1 \times \cdots \times h_r$. If $\lambda_1(f) = 1$ then there is nothing to prove. Otherwise assume that $\lambda_1(f) > 1$. Note that the Picard number of X_Σ is $d-n$ where d is the number of rays in Σ. Let d_i be the number of rays in Σ_i and n_i the dimension of X_{Σ_i}. Now note that $\pi_i^*\colon N^1(X_{\Sigma_i})_\mathbb{R} \to N^1(X_\Sigma)_\mathbb{R}$ is an injection and so the image has dimension $d_i - n_i$. We have that $\bigoplus_{i=1}^r \pi_i^* N^1(X_{\Sigma_i})_\mathbb{R} \subseteq N^1(X_\Sigma)_\mathbb{R}$ has rank

$$\sum_{i=1}^r (d_i - n_i) = d - n$$

as $\sum_{i=1}^r d_i = d$ and $\sum_{i=1}^r n_i = n = \dim X_\Sigma$. Thus

$$N^1(X_\Sigma)_\mathbb{R} = \bigoplus_{i=1}^r \pi_i^* N^1(X_{\Sigma_i})_\mathbb{R}$$

and the action of f on $N^1(X_\Sigma)_\mathbb{R}$ is given by

$$f^* \sum_{i=1}^r D_i = \sum_{i=1}^r h_i^* D_i$$

where $D_i \in \pi_i^* N^1(X_{\Sigma_i})_\mathbb{R}$. It follows that the only eigenvalues for f^* are the eigenvalues of the various h_i^*. This means that the eigenvalues of f^* are precisely the integers $n_i = \lambda_1(h_i)$. This is because $h_i\colon X_{\Sigma_i} \to X_{\Sigma_i}$ is a surjective endomorphism with X_{Σ_i} simple, by 6.2.15 we have that $h_i^*\colon N^1(X_{\Sigma_i})_\mathbb{R} \to N^1(X_{\Sigma_i})_\mathbb{R}$ acts by multiplication by $n_i = \lambda_1(h_i)$. Now consider any $n_i > 1$. Choose $P \in X_{\Sigma_i}(\overline{\mathbb{Q}})$ with $\alpha_{h_i}(P) = n_i$. Let e_j be the identity of the torus for X_{Σ_j} Let Q be the point of X_Σ whose i^{th} coordinate is P and for $j \neq i$ the j^{th} coordinate is e_j. In other words, Q is a point with $\pi_i(Q) = P$ and $\pi_j(Q) = e_j$ for $j \neq i$. Note that $h_j\colon X_{\Sigma_j} \to X_{\Sigma_j}$ is induced by a lattice homomorphism that is multiplication by a scalar. So $h_j(t_1, ..., t_s) = (t_1^{n_j}, ..., t_s^{n_j})$ on the torus of X_{Σ_j}. Thus $h_j(e_j) = e_j$. Then we have that $\alpha_{h_j}(\pi_j(Q)) = 1$ for $j \neq i$ and $\alpha_{h_i}(\pi_i(Q)) = n_i$. Thus, we have that

$$\alpha_f(P) = \max_{j=1}^r \{\alpha_{h_j}(\pi_j(P))\} = n_i$$

as needed. $\square$

In conclusion, equivariant toric surjective morphisms are built out of polarized morphisms of simple $\mathbb{Q}$-factorial toric varieties. This is reminiscent of the program to understand surjective endomorphisms of projective varieties admitting int-amplified endomorphisms. Notice that a polarized endomorphism is in fact int-amplified, so we have realized this part of the program for this special well behaved class of endomorphisms.

Non-equivariant morphisms

We now turn to the general case of non-equivariant surjective endomorphisms of $\mathbb{Q}$-factorial toric varieties.

Proposition 6.2.18. *Let X_Σ be a $\mathbb{Q}$-factorial projective toric variety with fan $\Sigma \subseteq N$ and $\tau \in \Sigma$. Then $V(\tau)$ is $\mathbb{Q}$-factorial and $\rho(V(\tau)) = \rho(X_\Sigma)$.*

Proof. $V(\tau)$ is the toric variety with fan $\mathrm{Star}(\tau)$. Let $v_1, ..., v_d$ be the rays of Σ. After reordering we have that $v_1, ..., v_t$ are the rays of τ. Given a cone $\sigma = \mathrm{Cone}(v_1, ..., v_t, v_{t+1}, ..., v_s)$ with face τ the associated cone in $\mathrm{Star}(\tau)$ is given by $\mathrm{Cone}(\bar{v}_{t+1}, ..., \bar{v}_s)$. Now suppose that $\bar{v}_{t+1}, ..., \bar{v}_s$ was not independent. Then we could find scalars not all zero with

$$a_{t+1}\bar{v}_{t+1} + ... + a_s\bar{v}_s = 0$$

which means we can find scalars $a_1, \ldots, a_t$ with

$$a_{t+1}v_{t+1} + ... + a_s v_s = a_1 v_1 + \ldots + a_t v_t$$

which contradicts σ being a simplicial cone. Thus $V(\tau)$ is simplicial. The rays of $V(\tau)$ are then $\bar{v}_{t+1}, ..., \bar{v}_d$. Since $V(\tau)$ has dimension $n - t$ and $d = n + \rho$ we have that there are $d - t = n + \rho - t = n - t + \rho$ rays. So $\rho(V(\sigma)) = \rho(X_\Sigma)$ as desired. □

Lemma 6.2.19. *Let X be a $\mathbb{Q}$-factorial projective toric variety of Picard number 1. Let $f \colon X \to X$ be a surjective endomorphism. Then f has a pre-periodic point.*

Proof. Suppose that $\lambda_1(f) > 1$. Then by a result of Fakkruddin [16, Theorem 5.1] we have that the set of pre-periodic points is dense in X. So we may assume that $\lambda_1(f) = 1$ and that f is an automorphism. We now induct on $\dim X = d$. When $d = 1$ we have that $X = \mathbb{P}^1$ and an automorphisms of $\mathbb{P}^n$ always has a fixed point given by an eigenvector for an associated matrix. Now let $d > 1$. First suppose that X is singular. Let S be the singular locus of X. Recall that we may write

$$S = \bigcup_{\sigma \in I} V(\sigma)$$

where I is the set of singular cones of the fan of Σ. The minimal singular cones thus are the components of the singular locus S and f permutes them being an automorphism. After iterating f we may assume that f fixes the components $V(\sigma)$ and thus we obtain $f^k \colon V(\sigma) \to V(\sigma)$. Now $V(\sigma)$ is a torus closure of a $\mathbb{Q}$-factorial toric variety and so is

$\mathbb{Q}$-factorial and of Picard number 1 by Proposition 6.2.18. So by induction f^k has a pre-periodic point and therefore so does f. Now assume that X is smooth. Since $\rho(X) = 1$ we have that $X = \mathbb{P}^n$ (all smooth toric varieties of Picard number 1 are projective spaces) for some n and as noted above every automorphism of $\mathbb{P}^n$ has a fixed point. ☐

Lemma 6.2.20. *Let X be a $\mathbb{Q}$-factorial projective toric variety. Let $f\colon X \to X$ be a surjective endomorphism. Then f has a pre-periodic point.*

Proof. We induct on the dimension. If $\dim X = 1$ the result follows from the result on $\mathbb{P}^1$. Otherwise first suppose that X is not a Mori-fiber space. Then X admits K_X negative birational extremal contraction. Let E be the exceptional locus, then $\dim E < \dim X$ and we have $f\colon E \to E$ after possibly iterating f. Since E is an orbit closure, by Proposition 6.2.18 that E is a $\mathbb{Q}$-factorial toric variety. By induction we have that $f\colon E \to E$ has a fixed point and we are done. Otherwise we may assume that X is a Mori-fiber space $\phi\colon X \to Y$. After iterating f we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

By induction $g\colon Y \to Y$ has a pre-periodic point. After iterating f and g we may assume that there is a point Q such that $g(Q) = Q$. Then we obtain a mapping $f\colon F \to F$ where $F = \phi^{-1}(Q)$. Here F is a $\mathbb{Q}$-factorial toric variety by [12, 15.4.5]. If $\dim F < \dim X$ then by induction there is a pre-periodic point as needed. This is because F is a normal projective variety, and the fibers of a Mori-fiber space are connected thus F is irreducible. Now $f\colon F \to F$ is finite mapping as f is a finite mapping. So the image of f is a $\dim F$-dimensional closed sub-variety, it follows that f is surjective so we may apply the inductive hypothesis. Otherwise $\dim F = \dim X$ and Y is a point. Then X has Picard number 1 and Lemma 6.2.19) gives the result. ☐

Theorem 6.2.21. *Let X be $\mathbb{Q}$-factorial toric variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective endomorphism. Then f has arithmetic eigenvalues.*

Proof. Let λ be a potential arithmetic degree of f. After replacing f with an iterate we may assume that f^* fixes all rays of the nef cone of X. Let D_λ be a nef eigendivisor for λ. Choose a facet F of $\mathrm{Nef}(X)$ that does not contain D_λ and let $\phi\colon X \to Y$ be the associated extremal contraction. First suppose that ϕ is birational. Since a toric variety admits an int-amplified endomorphism, then after iterating f by ([38, Theorem 5.3]) we have a conjugating diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Then we have that F is identified with the Nef cone of Y. Let H be an ample divisor on Y such that $A = D_\lambda + \phi^*H$ is ample. This occurs as D_λ does not appear in $\phi^*N^1(Y)_{\mathbb{R}}$ by construction. Let E be the exceptional locus of ϕ. Let $Z = \phi(E)$. Notice that we obtain maps $f \colon E \to E$ and $g \colon Z \to Z$. Using 6.2.20 we may take P to be a fixed point of f in E after potentially iterating f. Thus we have that $P = \phi(Q)$ is a fixed point of g in Z. By the argument in Proposition 6.2.5 we have that λ is realized. Now suppose that ϕ is fibering. By 6.2.20 we have that $g \colon Y \to Y$ has a fixed point (after potentially iterating all morphisms) and so using the same argument as the previous paragraph and Proposition 6.2.4 we have that λ is realizable. □

The key result here is two fold:

1. We could find pre-period points to construct good fibers.
2. we could contract all faces of the nef cone, and that every such contraction corresponds to an extremal contraction of a pair (X, D) which allowed us to conclude the result.

The basic *reason* why the theorem is true, is that potential arithmetic degrees on the fibers can be realized, and in the toric case every potential arithmetic degree appears on the fibers of some extremal contraction. Notice that this shares many details with the argument for surjective equivariant morphisms.

6.3 Realizability in the Int-amplified setting

In this section we study the realizability question in the setting of varieties admitting int-amplified endomorphisms. We first give some basic results that illustrate the points of friction in this approach.

Proposition 6.3.1. *Let X be a $\mathbb{Q}$-factorial variety with terminal singularities and finitely generated nef cone and* $\mathrm{Alb}(X) = 0$*. Suppose that $f\colon X \to X$ is a surjective endomorphism. Let $\phi\colon X \to Y$ be a birational extremal contraction. Suppose that we have a diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Let E be the exceptional locus of ϕ and let $Z = \phi(E)$. We have a second diagram given by

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\phi"'] & E \arrow[d, "\phi"] \\ Z \arrow[r, "g"'] & Z \end{tikzcd}$$

Suppose that every potential arithmetic degree of $f\,|_E\colon E \to E$ is realized as an arithmetic degree. If every potential arithmetic degree of g is realizable as an arithmetic degree and $g\colon Z \to Z$ admits a pre-periodic point. then every potential arithmetic degree of f is realizable as an arithmetic degree.

Proof. After iterating f we may assume that f^* fixes the rays of the nef cone and so every eigenvalue has a nef eigendivisor. Let μ be realizable as a potential arithmetic degree of g. Choose a point P so that $\alpha_g(P) = |\mu|$. If $g^n(P) \notin Z$ for all n then by the birational invariance of arithmetic degrees ([34, Lemma 2.4]) we have that $|\mu| = \alpha_g(P) = \alpha_f(\phi^{-1}(P))$ as needed. So we may assume that $P \in Z$. Therefore μ is a potential arithmetic degree of $g\,|_Z$ and so a potential arithmetic degree of $f\,|_E$ which by assumption is realizable. Now let λ be a potential arithmetic degree of f that is not a potential arithmetic degree of g. Then by Proposition 6.2.5 we have that λ is an arithmetic degree. □

Corollary 6.3.1.1. *In the situation of the above proposition Proposition 6.3.1 if $f\colon E \to E$ is such that f^* acts by a dilation on $N^1(E)_{\mathbb{R}}$ and every potential arithmetic degree of $g\colon Y \to Y$ is an arithmetic degree then every potential arithmetic degree of f is realized.*

Proof. This is immediate as if $f\colon E \to E$ is polarized, then every potential arithmetic degree realizable, as there is a potential arithmetic degree. □

Because of the importance we give these eigenvalues a name.

Definition 6.3.2. *Let $f\colon X \to X$ be a surjective endomorphism and let $\phi\colon X \to Y$ be a birational morphism. Let $g\colon Y \to Y$ be a surjective endomorphism with*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Let E be the exceptional locus of ϕ and let $Z = \phi(E)$. Suppose we have a second diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\phi"'] & E \arrow[d, "\phi"] \\ Z \arrow[r, "g"'] & Z \end{tikzcd}$$

We call the eigenvalues of $f\colon E \to E$ the exceptional eigenvalues of f with respect to ϕ.

The above results say that if we know what happens with the exceptional eigenvalues, then the problem of realizability is translated along a birational extremal contraction. If we imagine attempting to run the minimal model program to simplify the situation, this would correspond to a divisorial contraction. To do the minimal model program one must also consider the situation of flips. This is where we use [42, Theorem 5.3] to transfer dynamical information between the flips.

Lemma 6.3.3. *Let X be $\mathbb{Q}$-factorial projective variety with at worst terminal singularties and finitely generated nef cone. Suppose that X that admits an int-amplified endomorphism and* $\mathrm{Alb}(X) = 0$. *Take $f\colon X \to X$ a surjective morphism and let $\phi\colon X \to Y$ be a flipping extremal contraction with $\phi^+\colon X^+ \to Y$ the associated flip. Let $g\colon Y \to Y$ be a surjective endomorphism with $\phi \circ f = g \circ \phi$. We let E be the exceptional locus of ϕ. In addition assume the following.*

1. *Every potential arithmetic degree of $f\colon E \to E$ is an arithmetic degree.*

2. *Put $Z = \phi(E)$. We assume that $g\colon Z \to Z$ and that this morphism has a pre-periodic point.*

3. *We assume that every potential arithmetic degree on X^+ is an arithmetic degree.*

Then every potential arithmetic degree of X is realized as an arithmetic degree.

Proof. Let $\psi\colon X \dashrightarrow X^+$ be the associated birational map. By ([42, Theorem 5.3]) we have a diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] & X^+ \arrow[d, "f^+"] \\ X \arrow[r, "\psi"'] & X^+ \end{tikzcd}$$

where f^+ is a everywhere defined morphism. Since E has codimension two we note that f^* and $(f^+)^*$ have eigenvalues of the same magnitude, so the potential arithmetic degrees of both f and f^+ coincide. Let λ be a potential arithmetic degree of f and suppose that there is a point $P^+ \in X^+$ with $\alpha_{f^+}(P^+) = |\lambda|$. If $(f^+)^n(P^+) \notin E^+$ for all n then by the birational invariance of the arithmetic degree ([34, Lemma 2.4]) we have that $\alpha_f(\psi^{-1}(P^+)) = |\lambda|$ as needed. We may assume that for all $P \in X^+ \setminus E^+$ that $\alpha_f(P) \neq |\lambda|$. So we may assume that $P^+ \in E^+$ and so λ is a potential arithmetic degree of $f^+\colon E^+ \to E^+$. If λ is a potential arithmetic degree of $g\colon Z \to Z$ then we are done by assumption because then λ is a potential arithmetic degree of E. Therefore λ is not a potential arithmetic degree of g. By Proposition 6.2.5 we have that λ is realizable as an arithmetic degree. $\square$

The above result shows that while flipping the exceptional eigenvalues remain the same, and that they are the issue that prohibits a reduction to mori fiber spaces. We now give an example of how these ideas can be used in practice to prove realizability results. We first treat the case of rationally connected surfaces with $\mathrm{Alb}(X) = 0$ that admit an int amplified endomorphism.

Theorem 6.3.4. *Let X be a normal $\mathbb{Q}$-factorial surface with at worst terminal singularities and finitely generated nef cone that is rationally connected over $\overline{\mathbb{Q}}$ and $\mathrm{Alb}(X) = 0$ that admits an int-amplified endomorphism. Let $f\colon X \to X$ be a surjective endomorphism and λ a potential arithmetic degree of f. Then λ is realizable as an arithmetic degree.*

Proof. We proceed by induction on the Picard number $\rho = \rho(X)$. When $\rho = 1$ the result is true. When $\rho > 1$ suppose that X admits a K_X-negative extremal contraction that is divisorial. After iterating f ([42, Theorem 5.3]) we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

The exceptional locus is now an irreducible curve E that is contracted by ϕ and $Z = \phi(E)$ is a point say Q that is fixed by g. Thus by Proposition 6.2.5 we have that any potential arithmetic degree of f that is not a potential arithmetic degree of g is realized. By induction every potential arithmetic degree of g is realized and the birational invariance of the arithmetic degree gives the required result. So we may suppose that every K_X-negative extremal contraction of X is of fibering type as flips do not occur for surfaces because of codimension reasons. We may now assume the existence of a diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where ϕ is fibering and Y is a curve. By assumption we have no non-trivial morphism to an elliptic curve or an abelian variety. So we must have that Y is a $\mathbb{Q}$-factorial and normal curve that is not of general type or an elliptic curve. It follows that $Y = \mathbb{P}^1$ as a curve of general type admits a morphism to its Jacobian. The Picard number of X is now 2. If $\lambda_1(f) = \lambda_1(g)$ then $\lambda_1(f)$ can be realized as an arithmetic degree by Theorem 4.2.7. On the other hand, if μ is a second potential arithmetic degree in this situation with $|\mu| < \lambda_1(f)$ then as g has a fixed point, being an endomorphism of $\mathbb{P}^1$ and μ is realizable by 6.2.4. Thus we may assume that $\lambda_1(f) > \lambda_1(g)$. In this case [44, Theorem 5.2] gives the existence of a morphism $\psi\colon X \to \mathbb{P}^1$ and a conjugating morphism $h\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ \mathbb{P}^1 \arrow[r, "h"'] & \mathbb{P}^1 \end{tikzcd}$$

commutes and $\lambda_1(h) = \lambda_1(f)$. The argument just given shows that any other potential arithmetic degree is realizable. $\square$

Chapter 7

Using the equivariant MMP to obtain results in arithmetic dynamics

Our goal in this chapter is to describe a method of obtaining at least partial results in arithmetic dynamics for varieties that will admit an equivariant MMP. We first describe the general approach and then describe three applications of these ideas. In 7.1 we study when a morphism has a dense set of a pre-periodic points. In 7.2 we consider the Medvedev-Scanlon conjecture from the perspective of the minimal model program. Finally, in 7.3 consider the Kawaguchi-Silverman conjecture for automorphisms and prove a criterion for when a variety with a finitely generated nef cone to have an automorphism of positive entropy.

Definition 7.0.1 (Tractable minimal model programs)**.** *Let X be a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that $f\colon X \to X$ is a surjective endomorphism.*

1. *A f-equivariant minimal model program or f-equivariant MMP is a sequence*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r \tag{7.1}$$

   *where each ϕ_i is a flipping, divisorial, or fibering contraction along with **morphisms** $f_i\colon X_i \to X_i$ with $f_0 = f$ such that $f_i \circ \phi_{i+1} = \phi_{i+1} \circ f_i$.*

2. *If X_r is a Q-abelian variety then we call the MMP **tractable**. The reason for this terminology is that if X admits an int-amplified endomorphism then we can always find an MMP ending in a Q-abelian variety.*

3. *We call the MMP* ***standard*** *if $X_{r-1} \to X_r$ is a fibering contraction and for $i > r-1$ we have that $X_i \to X_{i+1}$ is birational or if $X_i \to X_{i+1}$ is always birational, and X_r is a minimal model.*

4. *We will often denote an f-equivariant MMP by $\mathcal{M}$ to denote the data of the sequence of contractions along with the conjugating morphisms.*

5. *We write $\lambda_1(f\mid_{\mathcal{M}})$ for the sequence whose i^{th} coordinate is $(\dim X_i, \lambda_1(f_i\mid_{\phi_{i+1}}))$. The purpose of this notation is to differentiate between the various types of minimal model operations that may occur.*

6. *If X_r is not zero dimensional then we say that $f_r\colon X_r \to X_r$ is a* ***primordial model*** *of f and call $\lambda_1(f_r)$ the* ***primordial dynamical degree*** *of $\mathcal{M}$. If X_r is zero dimensional we call $f_{r-1}\colon X_{r-1} \to X_{r-1}$ a* ***primordial model*** *of f and $\lambda_1(f_{r-1})$ the* ***primordial dynamical degree*** *of $\mathcal{M}$. We denote the primordial degree of $\mathcal{M}$ to be $\lambda_1^{\mathrm{pr}}(\mathcal{M})$.*

7. *We define the* ***primordial dynamical degrees*** *of the morphism f as*

$$\underline{\lambda}_1^{\mathrm{pr}}(f) = \min\{\lambda_1^{\mathrm{pr}}(\mathcal{M}) : \mathcal{M} \text{ a tractable } f \text{ equivariant MMP}\} \tag{7.2}$$
$$\overline{\lambda}_1^{\mathrm{pr}}(f) = \max\{\lambda_1^{\mathrm{pr}}(\mathcal{M}) : \mathcal{M} \text{ a tractable } f \text{ equivariant MMP}\} \tag{7.3}$$

if a tractable f-equivariant MMP exists for f and ∞ otherwise. We think of the collection of primordial models of f as its collection of ancestors. The number $\underline{\lambda}_1^{\mathrm{pr}}(f)$ measures the simplest ancestor, while $\overline{\lambda}_1^{\mathrm{pr}}(f)$ measures the most complex ancestor.

Our goal is to capture the complexity of those morphisms which are built from Q-abelian varieties. If $\mathcal{M}$ is an f-equivariant MMP as in 7.1 and X_r is zero dimensional. Then a primordial model for f is X_{r-1} which must have Picard number 1. If X_r is positive dimensional then f is built out of a surjective endomorphism of a Q-abelian variety, for example an abelian variety if there is a tractable MMP.

Example 9. Let $f_1 \times f_2\colon \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ be a surjective morphism. Since $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ admits two extremal contractions, $\pi_i\colon \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_i}$ there are two $f_1 \times f_2$ equivariant minimal model programs

$$\begin{tikzcd}
\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \arrow[r, "f_1\times f_2"] \arrow[d, "\pi_i"'] & \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \arrow[d, "\pi_i"] \\
\mathbb{P}^{n_i} \arrow[r, "f_i"'] & \mathbb{P}^{n_i}
\end{tikzcd}$$

In this case we have $\overline{\lambda}_1^{\mathrm{pr}}(f_1 \times f_2) = \max\{\deg f_1, \deg f_2\}$ and $\underline{\lambda}_1^{\mathrm{pr}}(f_1 \times f_2) = \min\{\deg f_1, \deg f_2\}$.

Example 10. Let X be a smooth variety admitting an MMP

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r$$

with X_r a Q-abelian variety. Let $f_i = \mathrm{id}_{X_i}$. Then this is a equivariant MMP for id_X and $\overline{\lambda}_1(\mathrm{id}_X) = \underline{\lambda}_1(\mathrm{id}_X) = 1$

Example 11. Let X be a simple abelian variety of Picard number 1 and $\tau_c \colon X \to X$ a translation by a non-torsion point. Then $\mathrm{id} \colon X \to X$ is the only equivariant MMP for τ_c. Thus $\overline{\lambda}_1(\tau_c) = \underline{\lambda}_1(\tau_c) = 1$.

The tractable minimal model program for a dynamical property $\mathcal{D}$. Let X be a variety defined over $\overline{\mathbb{Q}}$ with mild singularities so that some version of the minimal model program is possible. Suppose $f \colon X \to X$ is a surjective endomorphism . Consider some dynamical property of surjective endomorphisms property $\mathcal{D}$. Our goal is to check if f has $\mathcal{D}$. For example, $\mathcal{D}$ could be if f satisfies the Kawaguchi-Silverman conjecture, f has arithmetic eigenvalues, or if f has a dense set of pre-periodic points. Then we have the following program.

1. Verify that if we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

   with g surjective and ϕ a divisorial contraction then $\mathcal{D}$ holds for f if and only if it holds for g.

2. Let $\phi \colon X \to Y$ is a flipping contraction with flip $\phi^+ \colon X^+ \to Y$. If $f^+ \colon X^+ \dashrightarrow X^+$ extends to a morphism, verify that f has $\mathcal{D}$ if and only if f^+ has $\mathcal{D}$.

3. Determine a condition $F(\mathcal{D})$ such that if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

   is a diagram with ϕ fibering and that ϕ has $F(\mathcal{D})$ then f has $\mathcal{D}$ if and only if g has $\mathcal{D}$. We think of $F(\mathcal{D})$ of some formal notion that says that ϕ has well behaved fibers.

4. Verify that all primordial models have $\mathcal{D}$. In other words, show that surjective endomorphisms of Q-abelian varieties and surjective endomorphisms of Picard number 1 varieties have $\mathcal{D}$.

5. Define that a tractable MMP $\mathcal{M}$ has $F(\mathcal{D})$ if every fibering contraction in $\mathcal{M}$ has $F(\mathcal{D})$.

6. Conclude that all surjective endomorphisms that possess a tractable MMP with $F(\mathcal{D})$ has $\mathcal{D}$.

We now illustrate this idea with some examples of the program and some variants.

7.1 Pre-periodic points for varieties admitting an int-amplified endomorphism

In this section we begin to enact the tractable minimal model program outlined in 7 with $\mathcal{D}$ being the property that a surjective endomorphism has a dense set of pre-periodic points. We proved earlier that the collection of pre-periodic points of a surjective endomorphism of $\mathbb{Q}$-factorial toric varieties is non-empty 6.2.20. However it is natural to ask when is this set Zariski dense. We first handle (1) and (2) in 7.

Proposition 7.1.1. *Let X be a variety defined over $\overline{\mathbb{Q}}$ and let $f\colon X \to X$ be a surjective endomorphism. Fix $n \geq 1$. Then f has a dense set of pre-periodic points if f^n does.*

Proof. Let f^n have a dense set of pre-periodic points. Let U be an open set of X. Then there is a point $u \in U$ with $f^{an}(u) = f^{nb}(u)$. Then u is a pre-periodic point for f as well. $\square$

Proposition 7.1.2. *Let X, Y be a irreducible varieties defined over $\overline{\mathbb{Q}}$. Let $\phi\colon X \to Y$ be a birational morphism. Let $f\colon X \to X$ and $g\colon Y \to Y$ be a surjective endomorphisms. Suppose that $\phi \circ f = g \circ \phi$. Then f has a dense set of pre-periodic points if and only if g has a dense set of pre-periodic points*

Proof. Let $U \subseteq X$ be an open set of X and $V \subseteq Y$ be an open set of Y with $\phi\colon U \to V$ an isomorphism. Let $\mathcal{P}_f$ be the set of pre-periodic points of f and $\mathcal{P}_g$ the set of pre-periodic points of g. Suppose that $\mathcal{P}_f$ is dense in X. Let W be an open set in Y. Then $W \cap V$ is non-empty and open and $\phi^{-1}(W \cap V)$ is open in X. So there is a point $p \in \phi^{-1}(W \cap V)$

such that $f^n(p) = f^k(p)$. Then $\phi(f^n(p)) = g^n(\phi(p))$ and $\phi(f^k(p)) = g^k(\phi(p))$ which tells us that $q = \phi(p)$ is a pre-periodic in $W \cap V$. So $\mathcal{P}_g$ is dense in Y. Now suppose that $\mathcal{P}_g$ is dense in Y. Let W be an open set in X. Then $\phi(W \cap U)$ is an open set of V and so there is a point $q \in \phi(W \cap U)$ with $g^n(q) = g^k(q)$. Since $q = \phi(p)$ for some $p \in U \cap W$ we have $\phi(f^n(p)) = \phi(f^k(p))$. Since ϕ is an isomorphism on U we have that $f^n(p) = f^k(p)$ and so $\mathcal{P}_f$ is dense in X as needed. □

We immediately obtain the following results.

Corollary 7.1.2.1. *Let X be a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let $f\colon X \to X$ be a surjective endomorphism. Let $\phi\colon X \to Y$ be a divisorial contraction. Let $g\colon Y \to Y$ be a surjective endomorphism. Suppose that $\phi \circ f = g \circ \phi$. Then f has a dense set of pre-periodic points if and only if g has a dense set of pre-periodic points.*

Corollary 7.1.2.2. *Let X be a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let $f\colon X \to X$ be a surjective endomorphism. Let $\phi\colon X \to Y$ be a flipping contraction. Let $\psi\colon X \dashrightarrow X^+$ be the associated flipping birational mapping. Let $f^+\colon X^+ \to X^+$ be a surjective endomorphism. Suppose that $\psi \circ f = f^+ \circ \psi$. Then f has a dense set of pre-periodic points if and only if f^+ has a dense set of pre-periodic points.*

Proof. After iterating f we may assume by [42, Theorem 3.3] that we have a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\
Y \arrow[r, "g"'] & Y \\
X^+ \arrow[u, "\phi^+"] \arrow[r, "f^+"'] & X^+ \arrow[u, "\phi^+"']
\end{tikzcd}$$

If f has a dense set of pre-periodic points then so does g by Proposition 7.1.2 and therefore so does f^+ by Proposition 7.1.2 applied once more. The same argument applies if f^+ has a dense set of pre-periodic points. □

We now define $F(\mathcal{D})$. Recall that here $\mathcal{D}$ is the property that a morphism has a dense set of pre-periodic points.

Definition 7.1.3. *Suppose we are given a commuting diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where X, Y are normal projective varieties and f, g, ϕ are surjective endomorphisms. Suppose further that ϕ has connected fibers and that the general fiber is normal. We say that f has enough pre-periodic points (with respect to ϕ) if there is a non-empty open set $W \subseteq Y$ such that for all $p \in W$ with $g^n(p) = g^{n+k}(p)$ for some $n, k \in \mathbb{Z}_{\geq 0}$ we have:

1. *The induced morphism*

$$f^k : \phi^{-1}(g^n(p)) \to \phi^{-1}(g^{n+k}(p)) = \phi^{-1}(g^n(p))$$

has a dense set of pre-periodic points and $\phi^{-1}(g^n(p))$ is normal.

2. *$f^n : \phi^{-1}(p) \to \phi^{-1}(g^n(p))$ is dominant.*

Example 12. Let $f : X \to X$ be a surjective endomorphism defined over a field K. Suppose that X is a normal projective variety. Then we have

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Then f has enough pre-periodic points for the structure morphism if and only if f has a dense set of pre-periodic points.

Example 13. Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the structure morphism of a projective bundle. Suppose we have a diagram

$$\begin{tikzcd} \mathbb{P}\mathcal{E} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

Suppose that $\lambda_1(f\mid_\pi) > 1$. Then f has enough pre-periodic points with respect to π. This is because for $p \in X$ we have an induced morphism

$$f : \pi^{-1}(p) \to \pi^{-1}(g(p)).$$

The degree of f on the fibers is $\lambda_1(f\mid_\pi) > 1$. So f restricted to a fiber is a polarized endomorphism of projective space; any polarized morphism of projective space has a dense set of pre-periodic points by [16, 5.3].

We now formalize how these ideas relate to the minimal model program.

An f-equivariant MMP $\mathcal{M}$ **has enough pre-periodic points** if for all $\phi_i \colon X_{i-1} \to X_i$ of fibering type in $\mathcal{M}$ we have that f_{i-1} has enough pre-periodic points with respect to ϕ_i.

Definition 7.1.4. *Let X be a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that $f \colon X \to X$ is a surjective endomorphism. Consider a tractable f-equivariant MMP $\mathcal{M}$ given by*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r\ .$$

We say that $\mathcal{M}$ has enough pre-periodic points if for all $\phi_i \colon X_{i-1} \to X_i$ of fibering type in $\mathcal{M}$ we have that f_{i-1} has enough pre-periodic points with respect to ϕ_i.

Our strategy is to run an equivariant MMP on f to determine if f has a dense set of pre-periodic points. However, a basic issue with the above approach is morphisms with $\overline{\lambda}_1^{\mathrm{pr}}(f) = 1$. These are morphisms whose primordial ancestors all have dynamical degree 1. In other words their simplest ancestors may be akin to a translation $\tau_c \colon A \to A$ or a non-trivial isomorphism $f \colon \mathbb{P}^n \to \mathbb{P}^n$. Such a morphism is induced by a morphism of dynamical degree 1, which may not have a dense set of pre-periodic points. More generally using the notation of Definition 7.0.1 if $\phi_{i+1} \colon X_i \to X_{i+1}$ is a fibering type contraction and the general fiber of ϕ_{i+1} is a Fano variety of Picard number 1, then if $\lambda_1(f_i\mid_{\phi_{i+1}}) = 1$ we have that f_i may fail to have enough pre-periodic points with respect to ϕ_{i+1}. The definitions given above are meant to isolate precisely where these issues may arise when applying an MMP. Given an MMP $\mathcal{M}$ as in Definition 7.0.1 we see that the coordinates i of $\lambda_1(f\mid_{\mathcal{M}})$ are of the form $(\dim X_i, 1)$ with $\dim X_{i+1} < \dim X_i$ are where problems occur. Thus we see that the crucial factor to understand when to the denseness of pre-periodic points of endomorphisms is the behavior with respect to fibering type contractions.

Proposition 7.1.5. *Let $\phi \colon X \to Y$ be a fibering type extremal contraction of a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that we have a diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

with f, g surjective morphisms. Suppose that g has a dense set of pre-periodic points and that f has enough pre-periodic points with respect to ϕ. Then f has a dense set of pre-periodic points.

Proof. Let U be an open set in X. Set $W_0 = \phi(U)$. Since ϕ is a surjective morphism between normal varieties 4.1.4 says that ϕ is open. Consequently we have that W_0 is open. Let W be as in 7.1.3. Set $W_0' = W \cap W_0$. We may find $p \in W_0'$ with $g^n(p) = g^m(p)$ since g has a dense set of pre-periodic points. We may further take p general so that $\phi^{-1}(p)$ is normal and connected; the general fiber of a Mori-fiber space is normal and connected. Suppose that $n \leq m$ and that $n + k = m$. By the definition of enough pre-periodic points $\phi^{-1}(g^n(p))$ is normal and by our choice of p we have that $\phi^{-1}(p)$ is normal. Both being connected and normal they are irreducible and $f^n \colon \phi^{-1}(p) \to \phi^{-1}(g^n(p))$ is the composition $f \circ \iota \colon \phi^{-1}(p) \to \phi^{-1}(g^n(p))$ where $\iota \colon \phi^{-1}(p) \to X$ is the closed immersion. Since f^n is finite and closed immersions are finite we have that f^n is finite unto its image. Thus $f^n \colon \phi^{-1}(p) \to \phi^{-1}(g^n(p))$ is a dominant finite morphism with normal target. It follows that

$$f^n \colon \phi^{-1}(p) \to \phi^{-1}(g^n(p))$$

is an open mapping by 4.1.4. Taking $U' = U \cap \phi^{-1}(p)$ we have that $f^n(U')$ is open in $\phi^{-1}(g^n(p))$. Since f has enough pre-periodic points with respect to ϕ, the pre-periodic points of

$$f^k \colon \phi^{-1}(g^n(p)) \to \phi^{-1}(g^{n+k}(p)) = \phi^{-1}(g^n(p))$$

are dense by definition. So there is a point $q \in f^n(U')$ with $f^{lk}(q) = f^{lk+t}$ for some $t > 0$. Since $q \in f^n(U')$ we have that $q = f^n(a)$ for some $a \in U'$. Thus

$$f^{kl+n}(a) = f^{kl}(f^n(a)) = f^{kl+t}(f^n(a)) = f^{kl+n+t}(a)$$

so that a is a pre-periodic point of U'. Since $U' \subseteq U$ we have that the pre-periodic points of f are dense as claimed. □

Theorem 7.1.6. *Let X be a $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose further that X is rationally connected with* $\operatorname{Alb}(X) = 0$. *Let $f \colon X \to X$ be a surjective endomorphism. Suppose that f has a tractable MMP with enough pre-periodic points. Then f has a dense set of pre-periodic points.*

Proof. We induct on the Picard number $\rho = \rho(X)$. If $\rho(X) = 1$ then $f^n \colon X \to X$ are the only possible candidates for a MMP. By assumption for some n we have that $f^n \colon X \to X$ has a dense set of pre-periodic points. Thus f does as well. Now let $\rho(X) > 1$. After iterating f we have by assumption the existence of a tractable MMP with enough pre-periodic points. If a flipping operation appears in the MMP then we have a diagram

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, dotted, "\phi_{i+1}"'] & X_i \arrow[d, dotted, "\phi_i+1"] \\ X_{i+1} \arrow[r, "f_{i+1}"'] & X_{i+1} \end{tikzcd}$$

Then by Corollary 7.1.2.2 f_i has a dense set of pre-periodic points if and only if f_{i+1} does. Since by assumption we have a finite MMP, we eventually hit a divisorial or fibering contraction. Suppose that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

is the first non-flipping contraction in the MMP. Suppose first that ϕ is divisorial. Then $g\colon Y \to Y$ has a MMP with enough pre-periodic points by construction and by induction we have that g has a dense set of pre-periodic points. By Corollary 7.1.2.1 f does as well. On the other hand if ϕ is fibering then as above by induction we have that g has a dense set of pre-periodic points. Since by assumption f has enough pre-periodic points for ϕ we apply Corollary 7.1.5) and obtain that f has a dense set of pre-periodic points as needed. $\square$

As an application of the above ideas we analyze the behavior of the pre-periodic points of toric morphisms between $\mathbb{Q}$-factorial toric varieties from this perspective. Of course we could argue that f has a dense set of pre-periodic points using our characterization of toric morphisms.

Proposition 7.1.7. *Let X be a n-dimensional $\mathbb{Q}$-factorial projective toric variety with Picard number 1. Let $f\colon X \to X$ be a surjective toric morphism. Then some iterate of f is polarized or f is the identity.*

Proof. Since X_Σ is of Picard number one, X_Σ is a simple toric variety. By 6.2.13 every equivariant surjective endomorphism is induced by a dilation. If f is induced by $\phi\colon N \to N$ and $\phi(v) = nv$ and $n > 1$ then f is polarized by 6.2.14 and we are done. Otherwise $n = 1$ and we are done. $\square$

Proposition 7.1.8. *Let X be a $\mathbb{Q}$-factorial toric variety and $f\colon X \to X$ a surjective toric morphism. Let $\phi\colon X \to Y$ be a fibering type extremal contraction and let $g\colon Y \to Y$ be a toric morphism with $\phi \circ f = g \circ \phi$. Let T_Y be the dense torus of Y and T_X the dense torus of X. Then for $t \in T_Y$ we have that $\phi^{-1}(t) \cong X'$ for some $\mathbb{Q}$-factorial projective toric variety of Picard number 1. Moreover if $g(t) = t'$ for $t \in T_Y$ then the induced map $f\colon \phi^{-1}(t) \to \phi^{-1}(t')$ is a surjective toric morphism.*

Proof. Here we follow [48, 3.2]. Let $X = X_\Sigma$ for a fan $\Sigma \subseteq N_\mathbb{R}$. The fibering type contraction can be given by a mapping of lattices

$$0 \longrightarrow N_0 \xrightarrow{i} N \xrightarrow{\phi} N' \longrightarrow 0$$

where ϕ is the natural quotient mapping and $Y = Y_{\Sigma'}$ where Σ' is the quotient fan. Let $\Sigma_0 = \{\sigma \in \Sigma : \sigma \subseteq (N_0)_\mathbb{R}\}$. Then one can check (see for example [48, 3.2]) that $\phi^{-1}(T_{N'}) \cong T_{N'} \times X_{N_0,\Sigma_0}$. Thus the fibers above the torus are naturally toric varieties with the torus N_0 inherited from the torus action on X_Σ. Now let $t \in T_{N'}$ and consider the morphism $f\colon \phi^{-1}(t) \to \phi^{-1}(g(t))$. Since f is equivariant and by construction induces a map on Y_Σ we have that f preserves N_0 and thus for $s \in T_{N_0}$ we have that $f(s) \in T_{N_0}$. Since f is equivariant we obtain that $f\colon \phi^{-1}(t) \to \phi^{-1}(g(t))$ is as well with respect to the action of T_{N_0} and thus the action on the fibers is equivariant. $\square$

Corollary 7.1.8.1. *Let X be a $\mathbb{Q}$-factorial toric variety and $f\colon X \to X$ a surjective toric morphism. Let $\phi\colon X \to Y$ be a fibering type extremal contraction. Then we have a commuting diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where g is toric. Furthermore f has enough pre-periodic points for ϕ.

Proof. Take W to be T_Y. Then by Proposition 7.1.8 we have that f is toric on the fibers which have Picard number 1 and f has a dense set of pre-periodic points by Proposition 7.1.7. $\square$

Theorem 7.1.9. *Let X be a $\mathbb{Q}$-factorial projective toric variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \to X$ be a surjective toric morphism. Then the set of pre-periodic points of f is Zariski dense.*

Proof. Using the toric minimal model program we may choose equivariant MMP

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r = \mathrm{pt}$$

and we may take all morphisms here to be toric. Furthermore, by Corollary 7.1.8.1 every fibering type contraction has enough pre-periodic points. So by Theorem 7.1.6 we have that f has a dense set of pre-periodic points as needed. $\square$

7.2 Medvedev-Scanlon conjecture

In this section we illustrate how to apply the tractable minimal model program outlined in definition 7 to the Medvedev-Scanlon conjecture. In other words we take $\mathcal{D}$ to be the property that $f: X \to X$ has a point with a dense orbit.

Definition 7.2.1. *Let X be a projective variety and suppose that $f: X \to X$ is a surjective endomorphism. We say that f is* fiber preserving *if there is a positive dimensional variety Z and a dominant rational map $\psi: X \dashrightarrow Z$ such that $\psi \circ f = \psi$.*

The conjecture is usually stated as follows.

Conjecture 5 (The Medvedev-Scanlon conjecture)**.** *Let X be an irreducible variety defined over an algebraically closed field F of characteristic 0. Let $\phi: X \dashrightarrow X$ be a dominant rational map. If ϕ is not fiber preserving then there is a point $x \in X(F)$ with a forward dense orbit under ϕ.*

The Medvedev-Scanlon conjecture behaves well with respect to iteration.

Lemma 7.2.2 (Lemma 2.1 [6])**.** *Let X be an irreducible variety defined over an algebraically closed field F of characteristic 0. Let $\phi: X \dashrightarrow X$ be a dominant rational map. If ϕ^n is not fiber preserving for some $n \geq 1$ then ϕ is not fiber preserving.*

Definition 7.2.3. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. Let $f: X \to X$ be a dominant morphism. We say that a closed sub-variety $V \subseteq X$ is* dynamically constructible *or D-constructible by f if there is a closed irreducible sub-variety $W \subseteq X$ such that*

$$\overline{\mathcal{O}_f(W)} = V. \tag{7.4}$$

We say that W a generator for V. If V is a dynamically constructible sub-variety of X we say the dynamical difficulty *or D-difficulty of V is the number*

$$\mathrm{Difficulty}(V,f) = \min_{W \subseteq X, W \text{ a generator for } V} \dim(W). \tag{7.5}$$

*We think of the difficulty as a measure of how hard it is to dynamically construct V. If V is not D-constructible then we set the difficulty to ∞. If the D-difficulty of V is finite then we say that an irreducible sub-variety W is a **progenitor** of V if $\overline{\mathcal{O}_f(W)} = V$ and $\dim W = \mathrm{Difficulty}(V,f)$. In other words, the progenitors of V are the generators of minimal dimension.*

Example 14. *Let X be any projective variety and f a finite order automorphism of X. Then the D-difficulty of X with respect to f is $n = \dim X$.*

Example 15. *Let $X = C \times C$ where C is an elliptic curve. Let $f(x,y) = (2x,y)$. Then f has no point with dense forward orbit but if P is a non-torsion point of C then $P \times C$ is a curve with a dense orbit under f. So the D-difficulty of $C \times C$ is 1.*

With this notation we have the following rephrasing of the Kawaguchi-Silverman conjecture. Let X be a normal projective variety and $f \colon X \to X$ a surjective endomorphism. Suppose that $\mathrm{Difficulty}(X,f) = 0$. Then if P is an progenitor for X we have that $\alpha_f(P) = \lambda_1(f)$. Our idea here is to point out that the Kawaguchi-Silverman conjecture is only interesting when X can be built as an orbit closure of the forward orbit of a point of f. Now let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ equipped with a surjective endomorphism $f \colon X \to X$. The Medvedev-Scanlon conjecture is equivalent to the statement that the D-difficulty of X with respect to f is zero unless f preserves a rational fibration.

Definition 7.2.4 (Relative difficulty)**.** *Let X, Y be normal irreducible projective varieties. Let $f \colon X \to X$ be a surjective endomorphism and $\pi \colon X \to Y$ a surjective endomorphism with $g \circ \pi = \pi \circ f$. We say that f has relative D-difficulty at most k with respect to π if there is a non-empty open set $W \subseteq Y$ such that for all $y \in W(\overline{\mathbb{Q}})$ we have that*

$$V_y = \overline{\mathcal{O}_f(\pi^{-1}(y))}$$

has D-difficulty at most k. In other words for all $y \in W$ we have

$$\mathrm{Difficulty}(\mathcal{O}_f(\pi^{-1}(y)), f) \leq k. \tag{7.6}$$

*We say that the relative difficulty is k if the relatively difficulty is at most k and **not** at most $k-1$.*

Definition 7.2.5 (Dynamical set up for the Medvedev-Scanlon conjecture.)**.** *Let X be a $\mathbb{Q}$-factorial normal variety with at worst terminal singularities. Suppose that $f\colon X \to X$ is a surjective endomorphism. Consider an f-equivariant MMP*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r$$

where each ϕ_i is a flipping, divisorial, or fibering contraction along with morphisms $f_i\colon X_i \to X_i$ with $f_0 = f$ such that $f_i \circ \phi_{i+1} = \phi_{i+1} \circ f_i$. An f-equivariant MMP $\mathcal{M}$ has relative difficulty at most k if for all $\phi_i\colon X_{i-1} \to X_i$ of fibering type in $\mathcal{M}$ we have that f_{i-1} has relative difficulty at most k with respect to ϕ_i.

Lemma 7.2.6. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial normal variety with at worst terminal singularities. Let $\phi\colon X \to Y$ be a contraction morphism and $g\colon Y \to Y$ a surjective endomorphism with g surjective and $g \circ \phi = \phi \circ f$. Suppose that ϕ is birational. Then* $\mathrm{Difficulty}(X, f) = r \iff \mathrm{Difficulty}(Y, g) = r$. *In particular, f has a point with dense orbit if and only if g does.*

Proof. Let E be the exceptional locus of ϕ and let $Z = \phi(E)$. Suppose that $W \subseteq X$ is closed and irreducible and $\mathcal{O}_f(W)$ is dense in X. I claim that $\phi(W)$ is dense in Y with respect to g. Let V be open in Y. Then $\phi^{-1}(V) = U$ is open in X. So there is some $n \geq 1$ with $f^n(W) \cap U \neq \emptyset$ as $\mathcal{O}_f(W)$ is dense in X. So there is some $P \in W$ with $f^n(P) \in U$. Then $g^n(\phi(P)) \in V$. Since $\phi(P) \in \phi(W)$ we have that $\mathcal{O}_g(\phi(W))$ is dense in in Y. Conversely, suppose that $W \subseteq Y$ is closed and irreducible and $\mathcal{O}_g(W)$ is dense in Y. I claim that $\mathcal{O}_f(\phi^{-1}(W))$ is dense in X. Let U be open in X. Then $\phi(U \setminus E) = V$ is open in $Y \setminus Z$. Since W is dense in Y we have that $g^n(W) \cap V \neq \emptyset$. Therefore there is a point $Q \in W$ with $g^n(Q) \in V$. Let $P \in \phi^{-1}(Q)$. Then $f^n(P) \in \phi^{-1}(g^n(P))$. Since $g^n(P) \notin Z$ and ϕ is birational $\phi^{-1}(g^n(P))$ is a singleton (because Z is the image of the exceptional locus of ϕ). Thus $f^n(P)$ is the unique point of X with $\phi(f^n(P)) = g^n(Q)$. Since $g^n(Q) \in V$ we have that the point in the fiber $\phi^{-1}(g^n(Q))$ must be in U. Therefore $f^n(P) \in U$ and so $\phi^{-1}(W)$ is dense in X as needed. Now let W be a progenitor of X as in 7.2.3. Then $\mathcal{O}_g(\phi(W))$ is dense in Y by the above argument. Since $\mathcal{O}_f(W)$ is dense we have that W is not contained in the exceptional locus of f. Thus $\dim \phi(W) = \dim W$ as ϕ is birational and W is not contained in the exceptional locus. Thus $\mathrm{Difficulty}(X, f) \geq \mathrm{Difficulty}(Y, g)$ as Y has a generator of dimension $\mathrm{Difficulty}(X, f)$. Conversely, let W be a progenitor of Y. Then W is not contained in $Z = \phi(E)$ the image of the exceptional locus. As ϕ is birational and W is not contained in the image of exceptional locus we have $\dim W = \dim \phi^{-1}(W)$. By the above argument we have that $\phi^{-1}(W)$ has dense orbit. Thus some irreducible component of $\phi^{-1}(W)$ has a dense orbit and we obtain that $\mathrm{Difficulty}(X, f) \leq \mathrm{Difficulty}(Y, g)$ as

X has a generator of dimension $\text{Difficulty}(Y, g)$. We thus obtain the desired result that $\text{Difficulty}(X, f) = \text{Difficulty}(Y, g)$. $\square$

We see that the difficulty is preserved by a conjugating birational morphism.

Corollary 7.2.6.1. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let $\phi\colon X \to Y$ be a flipping contraction and $g\colon Y \to Y$ a surjective endomorphism with g surjective and $g \circ \phi = \phi \circ f$. Suppose that $\psi\colon X \dashrightarrow X^+$ is the canonical birational morphism to the flip of ϕ and that the birational mapping $f^+ = \phi \circ f \circ \phi^{-1}\colon X^+ \to X^+$ extends to a surjective morphism $f^+ X^+ \to X^+$. Then* $\text{Difficulty}(X, f) = r \iff \text{Difficulty}(X^+, f^+) = r$. *In particular f has a point with Zariski dense orbit if and only if f^+ does.*

Lemma 7.2.7. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let $\phi\colon X \to Y$ be a contraction morphism and $g\colon Y \to Y$ a surjective endomorphism with g surjective and $g \circ \phi = \phi \circ f$. Suppose that ϕ is of fibering type. Let $W \subseteq Y$ be closed. Then $\mathcal{O}_g(W)$ is dense in Y if and only if $\mathcal{O}_f(\phi^{-1}(W))$ is dense in X.*

Proof. Suppose that W has a dense orbit under g. Let U be a non-empty open set of X and set $V = \phi(U)$. Then V is open in Y by 4.1.4. So $g^n(W) \cap V$ is non-empty as W has a dense orbit. This intersection contains general points, so we may find $P \in W$ with $g^n(P) \in V$ and $\phi^{-1}(g^n(p))$ a normal $\mathbb{Q}$-factorial Fano variety of dimension $\dim X - \dim Y$. We can then find $Q' \in U$ with $\phi(Q') = g^n(P)$. Now consider the mapping on fibers

$$h = f^n\colon \phi^{-1}(P) \to \phi^{-1}(g^n(P)). \tag{7.7}$$

The mapping h is open since f^n is open by 4.1.4. Therefore h is dominant since $\phi^{-1}(g^n(P))$ is an irreducible variety. Since the map f^n is also closed we have that h is a closed and dominant mapping with an irreducible target. It follows that h is surjective. So there is a point $Q \in \phi^{-1}(P)$ with $h(Q) = f^n(Q) = Q'$. Since $\phi(Q) = P \in W$ we have that $Q \in \phi^{-1}(W)$ and so $f^n(\phi^{-1}(W)) \cap U \neq \emptyset$. Thus $\mathcal{O}_f(\phi^{-1}(W))$ is dense in X as needed. On the other hand suppose that $\mathcal{O}_f(\phi^{-1}(W))$ is dense in X. Let V be open and non-empty in Y. Then $\phi^{-1}(V) \cap f^n(\phi^{-1}(W)) \neq \emptyset$ as $\phi^{-1}(W)$ is dense in X. So there is a point $Q \in \phi^{-1}(W)$ with $f^n(Q) \in \phi^{-1}(V)$. Thus $g^n(\phi(Q)) \in V$. Since $\phi(Q) \in W$ we have that $g^n(W) \cap V \neq \emptyset$ and $\mathcal{O}_g(W)$ is dense in Y as needed. $\square$

Corollary 7.2.7.1. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Let $\phi\colon X \to Y$ be a contraction morphism and*

$g\colon Y \to Y$ a surjective endomorphism with g surjective and $g \circ \phi = \phi \circ f$. Suppose that ϕ is of fibering type. Then

$$\operatorname{Difficulty}(X, f) \leq \dim X - \dim Y + \operatorname{Difficulty}(Y, g)$$

Proof. Let W be a progenitor for Y as in 7.2.3. As $\mathcal{O}_g(W)$ is dense we have that the orbit of the generic point of W is dense in Y. Thus we may assume that

$$\dim \phi^{-1}(W) = \dim X - \dim Y + \dim W$$

as this is generically the case. By 7.2.7 we have that $\mathcal{O}_f(\phi^{-1}(W))$ is dense in X. Therefore one of the irreducible components of $\phi^{-1}(W)$ has a dense orbit in X. In other words

$$\operatorname{Difficulty}(X, f) \leq \dim \phi^{-1}(W) = \dim X - \dim Y + \dim W.$$

Since W is a progenitor for Y with respect to g we have that $\operatorname{Difficulty}(Y, g) = \dim W$ and consequently

$$\operatorname{Difficulty}(X, f) \leq \dim X - \dim Y + \operatorname{Difficulty}(Y, g).$$

□

We obtain the following pleasing consequence of the definition of difficulty.

Corollary 7.2.7.2. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that there is an f-equivariant MMP*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r$$

with respect to morphisms $f_i\colon X_i \to X_i$.

1. *Assume that each ϕ_i is divisorial or a flipping contraction and X_r is a minimal model. Then*
$$\operatorname{Difficulty}(X, f) = \operatorname{Difficulty}(X_r, f_r).$$
In other words, the difficulty can be computed on a minimal model.

2. *Suppose that $\phi_r\colon X_{r-1} \to X_r$ is a Mori-fiber space and each ϕ_i for $i < r$ is a divisorial contraction or a flipping contraction. Then*
$$\operatorname{Difficulty}(X, f) \leq \dim X - \dim X_r + \operatorname{Difficulty}(X_r, f_r).$$

Proof. Assume that each ϕ_i is divisorial or a flipping contraction and X_r is a minimal model. Then by 7.2.6 and 7.2.6.1 the difficulty is preserved by divisorial and flipping contractions. Thus $\text{Difficulty}(X, f) = \text{Difficulty}(X_r, f_r)$ as needed. Now suppose that $\phi_r \colon X_{r-1} \to X_r$ is a Mori-fiber space and each ϕ_i for $i < r$ is a divisorial contraction or a flipping contraction. By the above argument we have that $\text{Difficulty}(X, f) = \text{Difficulty}(X_{r-1}, f_{r-1})$. By 7.2.7.1 we have that

$$\text{Difficulty}(X_{r-1}, f_{r-1}) \leq \dim X_{r-1} - \dim X_r + \text{Difficulty}(X_r, f_r).$$

Since the ϕ_i are birational for $i < r$ we have that $\dim X = \dim X_{r-1}$ and so

$$\begin{aligned} \text{Difficulty}(X, f) &= \text{Difficulty}(X_{r-1}, f_{r-1}) \\ &\leq \dim X_{r-1} - \dim X_r + \text{Difficulty}(X_r, f_r) \\ &= \dim X - \dim X_r + \text{Difficulty}(X_r, f_r) \end{aligned}$$

as claimed. □

The above corollary shows that the notion of difficulty is reasonably well behaved in the presence of an equivariant MMP. We now return to the Medvedev-Scanlon conjecture.

Lemma 7.2.8. *Let $f \colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that f does not preserve a rational fibration. If f admits an equivariant MMP*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r$$

then $f_i \colon X_i \to X_i$ does not preserve a rational fibration

Proof. This follows from the fact that all the morphisms commutes. In particular if f_i preserves a rational fibration then we have a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, dashed] & & X \arrow[d, dashed] \\ X_i \arrow[rr, "f_i"'] \arrow[dr, dashed] & & X_i \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

This contradicts the fact that f does not preserve a rational fibration. □

Theorem 7.2.9. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities. Suppose that f admits an equivariant MMP*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_r}{\dashrightarrow} X_r$$

where ϕ_i is a divisorial or flipping contraction for $i < r-1$ and $\phi_r\colon X_{r-1} \to X_r$ is a Mori-fiber space. Suppose that ϕ_{r-1} has relative difficulty 0. If the Medvedev-Scanlon conjecture holds for X_r then it holds for X.

Proof. If f preserves a rational fibration then there is nothing to show. Suppose that f does not preserve a rational fibration. Then $f_r\colon X_r \to X_r$ does not preserve a rational fibration by 7.2.8 and the assumption that X_r satisfies the Medvedev-Scanlon conjecture we have that f_r has a point with dense forward orbit. By 7.2.7 we have that f_{r-1} has a fiber $\phi_r^{-1}(y)$ with a dense orbit. By assumption ϕ_r has relative difficulty 0 and so

$$\text{Difficulty}(\mathcal{O}_{f_{r-1}}(\phi_r^{-1}(y)), f_{r-1}) \leq 0.$$

Since the orbit is dense we have that $\text{Difficulty}(X_{r-1}, f_{r-1}) = 0$ and there is a dense orbit of a point under f_{r-1}. Applying 7.2.6 and 7.2.6.1 we obtain that f has a dense forward orbit as needed. $\square$

This leads to the following question:

Question 6. Consider a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd} \tag{7.8}$$

where f is surjective and $\phi\colon X \to Y$ a Mori-fiber space. Under what conditions do we have that the relative difficulty of f with respect to ϕ is at most zero. In other words when is it the case that orbits of fibers of ϕ under f can always be generated by a point. To study this situation fix a point $p \in Y$ and consider $Z_p = \overline{\mathcal{O}_g(p)}$ and $V_p = \overline{\mathcal{O}_f(\phi^{-1}(p))}$. Then we have a diagram

$$\begin{tikzcd} V_p \arrow[r, "f"] \arrow[d, "\phi"'] & V_p \arrow[d, "\phi"] \\ Z_p \arrow[r, "g"'] & Z_p \end{tikzcd}$$

In this situation Z_p now has a canonical dense orbit and V_p is built out of the fibers of a fibering contraction and we look for a dense orbit of a point under f. Since g cannot preserve a rational fibration, neither can $f\colon V_p \to V_p$ so the Medvedev-Scanlon conjecture predicts that we can always find a point with dense orbit for f in this situation.

We now turn to the case that X admits an int-amplified endomorphism. We give a reduction to two special cases which represent the current bottleneck for the Medvedev-Scanlon conjecture for varieties admitting an int-amplified endomorphism.

Theorem 7.2.10. *Assume the following.*

1. *The Medvedev-Scanlon conjecture holds for surjective endomorphisms of all Q-abelian varieties.*

2. *Consider a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[g]{} & Y \end{array} \tag{7.9}$$

   where we assume the following:

   (a) *X is a normal projective $\mathbb{Q}$-factorial variety with at worst terminal singularities.*

   (b) *X admits an int-amplified endomorphism.*

   (c) *$\phi\colon X \to Y$ is a Mori-fiber space and f is surjective.*

   Then f has relative difficulty 0 with respect to ϕ.

Then the Medvedev-Scanlon conjecture holds for surjective endomorphisms of $\mathbb{Q}$-factorial normal projective varieties with at worst terminal singularities that admit an int-amplified endomorphism.

Proof. Let $f\colon X \to X$ be a surjective endomorphism. Suppose that f does not preserve a rational fibration. Since X admits an int-amplified endomorphism by 4.1.19 we have an f^n equivariant

$$X = X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_r$$

where X_r is a Q-abelian variety. Since f does not preserve a rational fibration, $\kappa(X) \leq 0$ as otherwise f preserves the Iitaka fibration. Suppose first that $\kappa(X) = 0$. Since $\kappa(X) = 0$

we have that no fibering contractions occur and that $\dim X_r > 0$. Since f preserves no rational fibration neither does f_r and so as we have assumed the Medvedev-Scanlon conjecture for Q-abelian varieties we have that f_r has a point with dense orbit. As all the other contractions are birational we may apply 7.2.6 and 7.2.6.1 repeatedly to obtain that f^n and so f has a dense orbit as required. Now suppose that $\kappa(X) < 0$. We now induct on the Picard number of X. If X has Picard number 1 then the equivariant MMP exhibits X as a Mori-fiber space over a point after a finite sequence of flips. By 7.2.6.1 we may assume that f is a Mori-fiber space over a point. In other words we have a diagram

$$\begin{tikzcd} X \arrow[r, "f^n"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

where Y is a point and g is the identity. By assumption (2) we have that f has relative difficulty 0 with respect to ϕ. As any point has difficulty 0 we have that f has a dense orbit as required. Now suppose that the Picard number of X is larger then one. We have a diagram

$$\begin{tikzcd} X_1 \arrow[r, "f^n"] \arrow[d, dotted, "\phi_1"'] & X_2 \arrow[d, dotted, "\phi_1"] \\ X_2 \arrow[r, "f_2"'] & X_2 \end{tikzcd} .$$

If ϕ_1 is not a fibering type contraction then by induction we have that the Medvedev-Scanlon conjecture holds for f_2. Since ϕ is birational we have by 7.2.6 and 7.2.6.1 that f^n has a point with dense orbit and therefore so does f. We may now assume that ϕ_1 is a Mori-fiber space. Now consider the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f^n"] \arrow[d, "\phi_1"'] & X_2 \arrow[d, "\phi_1"] \\ X_2 \arrow[r, "f_2"'] & X_2 \end{tikzcd} .$$

Then X_2 is normal and $\mathbb{Q}$-factorial with at worst terminal singularities. Furthermore, X_2 admits an int-amplified endomorphism. To see this note that by assumption X admits an int-amplified endomorphism, say h. Then there is some m such that $\phi_1 \circ h^m = h_2 \circ \phi_1$ where $h_2 \colon X_2 \to X_2$ is a surjective endomorphism. Then if λ is an eigenvalue of $h_2^* \colon N^1(X_2)_{\mathbb{R}} \to N^1(X_2)_{\mathbb{R}}$ we have that λ is an eigenvalue of h^m. Recall that a surjective endomorphism is int-amplified if and only if every eigenvalue has absolute value strictly greater then one. So $|\lambda| = |\mu|^m$ where μ is an eigenvalue of h^*. Since $|\mu| > 1$ we have $|\mu|^n = |\lambda| > 1$ and h_2 is int-amplified. Therefore X_2 is a normal $\mathbb{Q}$-factorial variety with at worst terminal singularities that admits an int-amplified endomorphism. Furthermore, f_2 does not preserve a rational fibration as otherwise f would as well by 7.2.8. If $\kappa(X_2) = 0$ then by our earlier argument the Medvedev-Scanlon conjecture holds for X_2. On the other hand if $\kappa(X_2) < 0$ then X_2 satisfies the inductive hypothesis and so the Medvedev-Scanlon holds for X_2 in all cases. In particular, f_2 has a point with dense orbit. By assumption (2) we know that the difficulty of f relative to ϕ_1 is zero. Since f_2 has a point with dense orbit we have that f^n has a fiber with a dense orbit. As the relative difficulty is zero this means that f^n has a dense orbit by a point and so does f. Thus f satisfies the Medvedev-Scanlon conjecture.

□

Theorem 7.2.10 shows that a possible attack on the Medvedev-Scanlon conjecture for varieties is to first prove the conjecture for Q-abelian varieties. Then verify that Mori-fiber spaces have relative difficulty zero. While this may seem daunting, showing that Q-abelian varieties satisfy the Medvedev-Scanlon conjecture would give partial results on the difficulty of f. The proof gives the following.

Corollary 7.2.10.1. *Assume that the Medvedev-Scanlon conjecture holds for surjective endomorphisms of all Q-abelian varieties. Then the Medvedev-Scanlon conjecture holds for all $\mathbb{Q}$-factorial normal projective varieties with at worst terminal singularities that satisfy the following two conditions:*

1. *X admits an int-amplified endomorphism.*
2. *$\kappa(X) = 0$.*

7.3 Automorphisms of positive entropy and the Kawaguchi Silverman conjecture.

In this final section we illustrate how the MMP can be used to obtain results in the Kawaguchi-Silverman conjecture. We will focus on automorphisms of varieties with finitely generated Nef cones. We first discuss some easy reductions to the Kawaguchi-Silverman conjecture for automorphisms that surprisingly seem to have not appeared in the literature, but are relatively easy. Recall that for any projective variety we have an exact sequence

$$0 \to \mathrm{Aut}^0(X) \to \mathrm{Aut}(X) \to \pi_0\mathrm{Aut}(X) \to 0.$$

Here $\mathrm{Aut}^0(X)$ is the connected component of the identity element of $\mathrm{Aut}(X)$ and is a smooth connected algebraic group. See section one of [9] for an introduction to these notions. Thus $\pi_0\mathrm{Aut}(X)$ in part measures how far $\mathrm{Aut}(X)$ is from being a smooth connected algebraic group.

Lemma 7.3.1 (2.8 in [9])**.** *Let X be a projective variety. Then $\mathrm{Aut}^0(X)$ acts trivially on $N^1(X)_{\mathbb{R}}$ by pull back.*

Proof. Fix a line bundle L. Then we have a morphism

$$t_L \colon \mathrm{Aut}(X) \to \mathrm{Pic}^0(X)$$

given by

$$g \mapsto g^*L \otimes L^{-1}$$

This defines a morphism of schemes $\mathrm{Aut}(X) \to \mathrm{Pic}(X)$. Since $t_L(\mathrm{iden}_X)) = \mathcal{O}_X$ it takes the connected component of the identity of $\mathrm{Aut}(X)$ to the connected component of the identity of $\mathrm{Pic}^0(X)$. In other words we have that $g^*L \otimes L^{-1} \in \mathrm{Pic}^0(X)$ when $g \in \mathrm{Aut}^0(X)$. Since L was arbitrary we have that

$$g^*L \equiv_{\mathrm{num}} L$$

for any line bundle L. Thus $\mathrm{Aut}^0(X)$ acts trivially on $N^1(X)$. as needed. □

Corollary 7.3.1.1. *Let X be a projective variety and $g \colon X \to X$ an automorphism. Then $\lambda_1(g)$ only depends on the equivalence class of g in $\pi_0\mathrm{Aut}(X)$*

Proof. If g and g' have the same class in $\pi_0\mathrm{Aut}(X)$ then there is some $h \in \mathrm{Aut}^0(X)$ with $gh = g'$. Thus

$$(g')^* = (gh)^* = h^*g^* = g^*$$

since h^* is the identity on $N^1(X)_{\mathbb{R}}$. As $\lambda_1(f)$ is the spectral radius of the action of f^* on $N^1(X)$ we have that $\lambda_1(g) = \lambda_1(g')$. □

This raises the following question of realizability.

Question 7. Let $g \in \mathrm{Aut}(X)$ and $h \in \mathrm{Aut}^0(X)$. Now set $g' = gh$. Then the above argument shows that g^* and $(g')^*$ have the same eigenvalues when acting on $N^1(X)$. Therefore the set of potential arithmetic degrees of g and g' are the same. In general, is the set of arithmetic degrees the same? In other words if $\alpha_g(P) = |\mu|$ then is there some point Q with $\alpha_{gh}(Q) = |\mu|$?

In the setting of Question 7 we have that $\lambda_1(g) = \lambda_1(g')$. This means we will have points P, Q such that

$$\alpha_g(P) = \lambda_1(g) = \lambda_1(g') = \alpha_{g'}(Q).$$

Consequently we obtain that the *maximum arithmetic degree* of an automorphism only depends on the class of the automorphism in the component group $\pi_0\mathrm{Aut}(X)$. The question can then be reduced to the following. Suppose that g is an automorphism and $1 < \alpha_g(P) < \lambda_1(g)$ for some point $P \in X(\overline{\mathbb{Q}})$. Then for all $h \in \mathrm{Aut}^0(X)$ is there a point $Q \in X(\overline{\mathbb{Q}})$ with $\alpha_{gh}(Q) = \alpha_g(P)$. This question should have a positive answer when X is a smooth surface, for in that case the eigenvalues of automorphisms are of the form $\lambda_1(g), \lambda_1(g)^{-1}, \mu_1, \ldots \mu_s$ where $|\mu_i| = 1$ by [11, 2.4.3]. Thus the question for smooth projective surfaces is reduced to the case of the maximum arithmetic degree being an invariant of the class in the component group, which we know has a positive answer.

We obtain the following easy result that says that the Kawaguchi-Silverman conjecture for automorphisms is only meaningful for varieties with a complicated automorphism group. Recall that it is common to say that an automorphism $f\colon X \to X$ **has positive entropy** if $\lambda_1(f) > 1$.

Theorem 7.3.2. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$. If $\mathrm{Aut}(X)$ is an algebraic group then X has no automorphism with positive entropy. In particular, the Kawaguchi-Silverman conjecture is trivially true for automorphisms of X.*

Proof. If $\mathrm{Aut}(X)$ is a algebraic group then $\mathrm{Aut}(X)$ has finitely many components. Since the components of $\mathrm{Aut}(X)$ are precisely the cosets of $\mathrm{Aut}^0(X)$ we have that $\pi_0\mathrm{Aut}(X)$ must be finite. In other words given $f \in \mathrm{Aut}(X)$ we have that $f^N \in \mathrm{Aut}^0(X)$ for some N. Then we have that f^N acts trivially on $N^1(X)$ as $\mathrm{Aut}^0(X)$ acts trivially on $N^1(X)$. Since the eigenvalues of f^N are all one we have that for all eigenvalues λ of f^* acting on $N^1(X)$ we have that $\lambda^N = 1$. In other words the eigenvalues of f^* are all roots of unity and consequently we have that $\lambda_1(f) = 1$. ☐

We also have the following useful result.

Lemma 7.3.3 (2.10 in [9])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$. Let L be an ample line bundle on X. Let $\mathrm{Aut}(X, [L]_{\mathrm{Num}})$ be the subgroup of all elements $[f] \in \pi_0\mathrm{Aut}(X)$ with $f^*L \equiv_{\mathrm{num}} L$. Then $\mathrm{Aut}(X, [L]_{\mathrm{Num}})$ is finite.*

One easily obtains the following.

Corollary 7.3.3.1 (2.11 and 2.12 in [9])**.** *Let X be a projective variety defined over $\overline{\mathbb{Q}}$.*

1. *The kernel of the action of* $\pi_0\mathrm{Aut}(X)$ *on* $N^1(X)$ *is finite.*

2. *If the nef cone of* X *is finitely generated and rational then* $\mathrm{Aut}(X)$ *is an algebraic group.*

Proof. We first prove (1). Fix an ample line bundle L. If $[f] \in \pi_0\mathrm{Aut}(X)$ and $[f]$ is in the kernel of the action of $\pi_0\mathrm{Aut}(X)$ on $N^1(X)$ then $[f] \in \mathrm{Aut}(X, [L]_{\mathrm{Num}})$ which by 7.3.3 is finite as needed. Now suppose that the nef cone of X is finitely generated and rational It suffices to prove that $\pi_0\mathrm{Aut}(X)$ is finite. Fix $[f] \in \pi_0\mathrm{Aut}(X)$. Let $v_1, \ldots v_r$ be the ray generators of $\mathrm{Nef}(X)$. We have that each v_i is a primitive element of $N^1(X)$ in the sense that it is the first lattice point on the half line $\mathbb{R}_{\geq 0}v_i$. Note that f^* is an automorphism of the lattice $N^1(X)$. This is because f^* is represented by an integral matrix, and so is its inverse $(f^{-1})^*$. Thus $\det f^* \circ (f^{-1})^* = 1 = \det f^* \det(f^{-1})^*$. It follows that $\det f \pm 1$. Therefore we must have that $f^*v_i = v_j$. Thus $f^*(\sum_{i=1}^r v_i) = \sum_{i=1}^r v_i$ and consequently $[f]$ preserves the ample class $\sum_{i=1}^r v_i$. By 7.3.3 we have that $\pi_0\mathrm{Aut}(X)$ is finite as needed. □

We obtain the Kawaguchi-Silverman conjecture for automorphisms of any normal projective variety with finitely generated and rational nef cone.

Corollary 7.3.3.2. *Let* X *be a normal projective variety over* $\overline{\mathbb{Q}}$*. If* X *has a finitely generated and rational nef cone then* X *has no automorphism of positive entropy. In particular, the Kawaguchi-Silverman conjecture trivially holds for all automorphisms of* X*.*

Proof. By 7.3.3.1 we have that $\mathrm{Aut}(X)$ is an algebraic group. By 7.3.2 we have the result. □

We see that the Kawaguchi-Silverman conjecture for automorphisms of varieties with a finitely generated and rational nef cone is trivial. However, this leads to the question about varieties with finitely generated but non-rational nef cone. For example in [50] there are examples of a Hyper-Kahler with Picard number 2 and an infinite automorphism group. In this case an automorphism of positive entropy may arise.

We would like to now define a subgroup of $\mathrm{Aut}(X)$ as those automorphisms which have dynamical degree 1. However, we run into the following problem. It is possible to have invertible integer matrices A, B with $\det A = \det B = 1$ and $\rho(A) = \rho(B) = 1$ but $\rho(AB) > 1$ where ρ is the spectral radius function. Therefore it is a priori possible that there are automorphisms $f, g \in \mathrm{Aut}(X)$ with $\lambda_1(f) = \lambda_1(g) = 1$ but $\lambda_1(fg) > 1$. There is a way to avoid this issue when X has finitely generated nef cone.

Definition 7.3.4. *Let X be a projective variety over $\overline{\mathbb{Q}}$ with finitely generated nef cone. That is $\mathrm{Nef}(X)_{\mathbb{R}}$ is generated as a cone by finitely many real classes. Suppose that $\mathrm{Nef}(X)_{\mathbb{R}}$ has rays $v_1, \ldots v_r$. Then any surjective endomorphism of X permutes the rays of $\mathrm{Nef}(X)$. In particular we have a homomorphism*

$$\mathfrak{r}\colon \pi_0\mathrm{Aut}(X) \to S_r$$

where S_r is the symmetric group on r letters. Let d_1 be the size of the image of $\mathfrak{r}$. So d_1 is the smallest integer such that for all f we have that $(f^{d_1})^* v_i = \lambda_i v_i$ for all i and some real numbers λ_i. On the other hand the kernel of the action of $\pi_0\mathrm{Aut}(X)$ on $N^1(X)$ is finite by 7.3.3.1. Let d_2 be the size of this kernel and let $d = \mathrm{lcm}(d_1, d_2)$. Now define $\mathfrak{D}(X)$ to be the subgroup of $\pi_0\mathrm{Aut}(X)$ generated by all $2d^{th}$ powers. That is

$$\mathfrak{D}(X) = \langle [f^{2d}]\colon [f] \in \pi_0\mathrm{Aut}(X)\rangle \subseteq \pi_0\mathrm{Aut}(X).$$

We think of $\mathfrak{D}(X)$ as the subgroup of all classes of automorphisms $\pi_0\mathrm{Aut}(X)$ which are simultaneously diagonalizable with positive eigenvalues. The basic properties of this group are outlined below.

Proposition 7.3.5. *Let X be a projective variety over $\overline{\mathbb{Q}}$ with finitely generated nef cone.*

1. *If $f_1, f_2 \in \mathfrak{D}(X)$ then f_1^*, f_2^* are simultaneously diagonalizable.*
2. *There is an homomorphism $\mathrm{Lin}\colon \mathfrak{D}(X) \to \mathrm{diag}_{\rho(X)}(\mathbb{R}_{>0}) \cong (\mathbb{R}^*_{>0})^{\rho(X)}$ with finite kernel. Here $\rho(X)$ is the Picard number of X, $\mathrm{diag}_{\rho(X)}(\mathbb{R}_{>0})$ are diagonal $\rho(X)\times\rho(X)$ matrices with positive entries and $\mathrm{Lin}([f]) = f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$.*
3. *The kernel of $\mathrm{Lin}\colon \mathfrak{D}(X) \to \mathrm{diag}_{\rho(X)}(\mathbb{R}_{>0})$ is precisely the set of $f \in \mathfrak{D}(X)$ with $\lambda_1(f) = 1$.*

Proof. We write $f_1 = \prod_{i=1}^{s_1} g_i^{2d}$ and $f_2 = \prod_{i=1}^{s_2} h_i^{2d}$ where we use that f_1, f_2 represent classes in $\mathfrak{D}(X)$. Then $f_1^* = (g_{s_1}^{2d})^* \ldots (g_1^{2d})^*$. Since each $(g_i^{2d})^* v_j = \lambda_{ij} v_j$ we have that $f_1^* v_j = \prod_{i=1}^{s_1} \lambda_{ij} v_j$. Similarly we have $(h_i^{2d})^* v_j = \mu_{ij} v_j$ so that $f_2^* v_j = \prod_{i=1}^{s_2} \mu_{ij} v_j$. Since some sub-set of the rays is a basis of $N^1(X)_{\mathbb{R}}$ we have that f_1^* and f_2^* share a mutual basis of eigenvectors.

Now suppose that $f \in \mathfrak{D}(X)$ and $\mathrm{Lin}(f) = $ identity. Then f^* lies in the kernel of the action of $\pi_0\mathrm{Aut}(X)$ which is finite by 7.3.3.1. On the other hand, the above calculation shows that for any f_1, f_2 we have that

$$f_1^* f_2^* v_i = \left(\prod_{i=1}^{s_1} \lambda_{ij}\right) \cdot \left(\prod_{i=1}^{s_2} \mu_{ij}\right) v_i = f_2^* f_1^* v_i.$$

Since some subset of the v_i is a basis we have $f_1^* f_2^* = f_2^* f_1^*$ and so

$$\mathrm{Lin}(f_1 f_2) = (f_1 f_2)^* = f_2^* f_1^* = f_1^* f_2^* = \mathrm{Lin}(f_1)\mathrm{Lin}(f_2)$$

so Lin is a homomorphism as desired. Finally note that by the definition of d we have that for any $f \in \pi_0 \mathrm{Aut}(X)$ that $(f^d)^* v_i = \mu_i v_i$. So $(f^{2d})^* v_i = \mu_i^2 v_i$. Thus the eigenvalues of any $f \in \mathfrak{D}(X)$ are positive. Thus $\mathrm{Lin}(f)$ is a diagonal matrix with positive entries in the basis given by the rays with positive entries. Finally if $\mathrm{Lin}(f)$ is the identity then $\lambda_1(f) = 1$. Conversely let $\lambda_1(f) = 1$ with $f \in \mathfrak{D}(X)$. Let $f^* v_i = \lambda_i v_i$ with $\lambda_i > 0$. Then we have shown that f^* is diagonal with eigenvalues λ_i. Then $\det f^* = \prod_i \lambda_i = 1$. As $0 < \lambda_1(f) \leq 1$ if some $\lambda_i < 1$ then for the product to equal one we must have some $\lambda_j > 1$. It follows that $\lambda_i = 1$ for all i. Since f^* is diagonal we have that f^* is the identity. So $\ker \mathrm{Lin} = \{f \in \mathfrak{D}(X) \colon \lambda_1(f) = 1\}$ and consequently this set is finite. □

We can now give a group theoretic criterion for when a variety has an automorphism with positive entropy in terms of the component group. Let $[f] \in \pi_0 \mathrm{Aut}(X)$. It is certainly a necessary condition for $\lambda_1(f) > 1$ that $[f]$ have infinite order in $\pi_0 \mathrm{Aut}(X)$. We show that this is in fact sufficient. In other words, the obvious necessary condition is also sufficient.

Theorem 7.3.6 (Criterion for when a variety with finitely generated nef cone has an automorphism of positive entropy)**.** *Let X be a normal projective variety over $\overline{\mathbb{Q}}$ with a finitely generated nef cone. Let $f \in \pi_0 \mathrm{Aut}(X)$. Then $\lambda_1(f) > 1 \iff f$ has infinite order in $\pi_0 \mathrm{Aut}(X)$. In particular a normal projective variety X with finitely generated nef cone has an automorphism of positive entropy if and only if $\pi_0 \mathrm{Aut}(X)$ has an element of infinite order.*

Proof. If $f \colon X \to X$ is an automorphism and $\lambda_1(f) > 1$ then as $\lambda_1(f^n) = \lambda_1(f)^n$ we see that f has infinite order. On the other hand suppose that f has infinite order. Let d be as in 7.3.4. Then $f^{2d} \in \mathfrak{D}(X)$. Towards a contradiction suppose that $\lambda_1(f) = 1$. Then $\lambda_1(f^{2d}) = 1$ and so f^{2d} lies in the kernel of the action of $\pi_0 \mathrm{Aut}(X)$ by 7.3.5. By 7.3.3.1 this group has finite order and $(f^{2d})^N = f^{2dN} =$ identity contradicting that f had infinite order. □

Thus to produce examples of varieties with automorphisms of positive entropy it suffices to produce varieties with finitely generated nef cone with component group having an element of positive entropy. On the other hand, the component group is still currently a mysterious object. It was only recently shown by Lesieutre in [33] that $\pi_0 \mathrm{Aut}(X)$ can be non-finitely generated. On the other hand, it is a folklore question that asks if there exists infinite finitely presented groups with every element of finite order. See [55, Section 1] or

[25]. One might ask if such a group can arise as the automorphism group of a projective variety with a finitely generated nef cone.

Question 8. *Let X be a normal projective variety defined over $\overline{\mathbb{Q}}$ with finitely generated nef cone. Is it possible that $\pi_0 \mathrm{Aut}(X)$ is finitely presented but X has no automorphism of positive entropy? By 7.3.6 this is equivalent to asking if $\pi_0 \mathrm{Aut}(X)$ can be finitely presented with no element of infinite order.*

We finally note that it may be useful to apply this same analysis to some of the other cones sitting inside $N^1(X)_{\mathbb{R}}$ such as the closure of the big cone. Even if the nef cone is not finitely generated, perhaps one of these other cones could be and similar results could be applied.

We now turn to the Kawaguchi-Silverman conjecture for automorphisms.

Definition 7.3.7. *Let X be a $\mathbb{Q}$-factorial normal projective variety with at worst terminal singularities and finitely generated not necessarily rational nef cone.*

1. *Let $\phi\colon X \to Y$ be a small contraction and $\phi^+\colon X^+ \to Y$ an associated flip. We say that ϕ is **polyhedral** if X^+ also has finitely generated nef cone.*

2. *Suppose that X admits an MMP*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r$$

*with each $X_i \dashrightarrow X_{i+1}$ either a divisorial, flipping, or fibering contraction associated to a K_{X_i}-negative extremal ray, and either X_r is minimal or $X_{r-1} \to X_r$ is a fibering contraction. We call the MMP **polyhedral** if each flipping contraction is polyhedral.*

3. *Let $\mathscr{P}$ be the category of all $\mathbb{Q}$-factorial normal projective varieties with at worst terminal singularities and finitely generated not necessarily rational nef cone that **admit a polyhedral MMP**. We let $\mathscr{P}_{-\infty}$ be the sub-category of $\mathscr{P}$ that admit a **tractable polyhedral MMP** ending at a point. That is all varieties X which admit an MMP*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \ldots \overset{\phi_r}{\dashrightarrow} X_r$$

with each ϕ_i a divisorial, fibering, or polyhedral flipping contraction and X_r a Q-abelian variety.

We have the following easy result.

Theorem 7.3.8. *Suppose that the Kawaguchi-Silverman conjecture holds for automorphisms of minimal varieties with finitely generated nef cones. Then the Kawaguchi-Silverman conjecture for automorphisms holds for all varieties in $\mathscr{P}$.*

Proof. Let $X \in \mathscr{P}$. We induct on the Picard number $\rho(X)$. If $\rho(X) = 1$ then the Kawaguchi-Silverman conjecture is true for all automorphisms of X. So we may assume $\rho(X) > 1$. If X is minimal we are done by assumption. So assume that X is not minimal. Then as $X \in \mathscr{P}$ we have a polyhedral MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r$$

Since X is not minimal and the contractions are contractions of K_{X_i}-negative extremal rays we have that $r > 0$. Note that

$$X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots \dashrightarrow X_r$$

is a polyhedral MMP for all $i > 0$. So each X_i is in $\mathscr{P}$. This is because if $X \to Y$ is a divisorial or fibering contraction then if X has a finitely generated nef cone then so does Y because the nef cone of Y is the face of the nef cone of X. As X has a finitely generated nef cone, all its faces are also finitely generated. On the other hand since we have assumed that each flipping contraction is polyhedral, we are guaranteed that all flips X^+ that arise in the MMP also have finitely generated nef cone. Now let $f\colon X \to X$ be an automorphism. As $\mathrm{Nef}(X)_{\mathbb{R}}$ has dual cone $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ we have that the closed cone of curves is finitely generated. Then as f_* preserves the closed cone of curves we have that f_* permutes the rays of $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$. Thus for some $N > 0$ we have that f_*^N fixes the rays of $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$. Let $\phi\colon X \to X_1$ be the first contraction. Suppose that ϕ_1 is induced by the contraction of an extremal ray $\mathfrak{R}$. Suppose first that $\mathfrak{R}$ gives a divisorial contraction. Since $f_*^N\mathfrak{R} = \mathfrak{R}$ by 4.1.6 we have a diagram

$$\begin{tikzcd} X \arrow[r, "f^N"] \arrow[d, "\phi_1"'] & X \arrow[d, "\phi_1"] \\ X_1 \arrow[r, "f_1"'] & X_1 \end{tikzcd}$$

By induction the Kawaguchi-Silverman conjecture holds for f_1 and since ϕ_1 is birational it also holds for f^N and so f. Now suppose that $\mathfrak{R}$. By [34, 6.2] we have a diagram

$$\begin{tikzcd} X \arrow[r, "f^N"] \arrow[d, "\phi_1"'] & X \arrow[d, "\phi_1"] \\ X_1 \arrow[r, "f_1"'] & X_1 \end{tikzcd}$$

with f_1 an automorphism. By induction the Kawaguchi-Silverman conjecture holds for f_1 and so by [34, 6.3] the Kawaguchi-Silverman conjecture holds for f^N and so for f as well. Thus we may assume that $\mathfrak{R}$ is a flipping contraction. Since $f_*^N\mathfrak{R} = \mathfrak{R}$ we have that f^N extends to a morphism $f^+ \colon X^+ \to X^+$ that is birationally conjugate to f by 4.1.7. So the Kawaguchi-Silverman conjecture for f is equivalent to the Kawaguchi-Silverman conjecture for f^+. Now repeat the procedure for f^+. Either we eventually arrive at a divisorial or fibering contraction and apply the earlier arguments, or the MMP is a series of polyhedral flips terminating at a minimal model. By assumption the Kawaguchi-Silverman conjecture holds for minimal models and so for f^N and consequently f. □

The ideas in the proof give an argument for the triviality of the Kawaguchi-Silverman conjecture for varieties in $\mathscr{P}_{-\infty}$.

Corollary 7.3.8.1. *Let $X \in \mathscr{P}_{-\infty}$. Then X has no automorphism with positive entropy. In particular the Kawaguchi-Silverman conjecture holds and every element of $\pi_0\mathrm{Aut}(X)$ has finite order.*

Proof. Let $X \in \mathscr{P}_{-\infty}$. We induct on the Picard number $\rho(X)$. If $\rho(X) = 1$ then the Nef cone of X is finitely generated and rational. So by 7.3.3.2 we have that X has no automorphism of positive entropy. Now let $\rho(X) > 1$. A variety in $\mathscr{P}_{-\infty}$ has a tractable polyhedral MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r$$

ending in a point. Arguing as in the proof of 7.3.8 we eventually have a diagram

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\phi_i"'] & X_i \arrow[d, "\phi_i"] \\ X_{i+1} \arrow[r, "f_{i+1}"'] & X_{i+1} \end{tikzcd}$$

where f_i and f_{i+1} are automorphisms, and ϕ_i is a fibering or divisorial contraction. Moreover f_i^* and f^* have the same eigenvalues and $\rho(X) = \rho(X_i)$. Note that $\rho(X_{i+1}) = \rho(X_i) - 1 = \rho(X) - 1$. By induction we see that f_{i+1} does not have positive entropy. Let $\mu_1, ..., \mu_{\rho(X)-1}$ be the eigenvalues of f_{i+1}^*. As $\lambda_1(f_{i+1}) = 1$ we have $|\mu_k| \leq 1$ for all k. Since the diagram above commutes and $\rho(X_i) - 1 = \rho(X_{i+1})$ we have that the eigenvalues of f_i^* are the eigenvalues of f_{i+1} along with a single potentially new eigenvalue γ. It suffices to show that $|\gamma| \leq 1$. We have that

$$1 = |\det f_i^*| = |\gamma \cdot \mu_1 \ldots \mu_k| = |\lambda|$$

as needed. So X has no automorphism of positive entropy. By 7.3.6 we have that $\pi_0\mathrm{Aut}(X)$ does not contain an element of infinite order. $\square$

References

[1] A. Adrian Albert. Involutorial simple algebras and real Riemann matrices. *Ann. of Math. (2)*, 36(4):886–964, 1935.

[2] Ekaterina Amerik. On endomorphisms of projective bundles. *Manuscripta Math.*, 111(1):17–28, 2003.

[3] Ekaterina Amerik and Alexandra Kuznetsova. Endomorphisms of projective bundles over a certain class of varieties. *Bull. Korean Math. Soc.*, 54(5):1743–1755, 2017.

[4] M. Atiyah. On the Krull-Schmidt theorem with application to sheaves. *Bull. Soc. Math. France*, 84:307–317, 1956.

[5] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[6] Jason P. Bell, Dragos Ghioca, Zinovy Reichstein, and Matthew Satriano. On the Medvedev-Scanlon conjecture for minimal threefolds of nonnegative Kodaira dimension. *New York J. Math.*, 23:1185–1203, 2017.

[7] Robert Benedetto, Patrick Ingram, Rafe Jones, Michelle Manes, Joseph H. Silverman, and Thomas J. Tucker. Current trends and open problems in arithmetic dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 56(4):611–685, 2019.

[8] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[9] Michel Brion. Notes on automorphism groups of projective varieties. 2018.

[10] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.

[11] Serge Cantat. Dynamics of automorphisms of compact complex surfaces. In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pages 463–514. Princeton Univ. Press, Princeton, NJ, 2014.

[12] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[13] Tien-Cuong Dinh and Viêt-Anh Nguyên. Comparison of dynamical degrees for semi-conjugate meromorphic maps. *Comment. Math. Helv.*, 86(4):817–840, 2011.

[14] Lawrence Ein. Linear systems with removable base loci. volume 28, pages 5931–5934. 2000. Special issue in honor of Robin Hartshorne.

[15] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[16] Najmuddin Fakhruddin. Questions on self maps of algebraic varieties. *J. Ramanujan Math. Soc.*, 18(2):109–122, 2003.

[17] Yoshio Fujimoto and Noboru Nakayama. Compact complex surfaces admitting non-trivial surjective endomorphisms. *Tohoku Math. J. (2)*, 57(3):395–426, 2005.

[18] Yoshio Fujimoto and Noboru Nakayama. Endomorphisms of smooth projective 3-folds with nonnegative Kodaira dimension. II. *J. Math. Kyoto Univ.*, 47(1):79–114, 2007.

[19] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[20] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[21] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises*. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2020] ©2020. Second edition [of 2675155].

[22] Robin Hartshorne. Ample vector bundles on curves. *Nagoya Math. J.*, 43:73–89, 1971.

[23] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[24] Marc Hindry and Joseph H. Silverman. *Diophantine geometry*, volume 201 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[25] JeremyKun (https://mathoverflow.net/users/6429/jeremykun). Finitely presented infinite group with no element of infinite order? MathOverflow. URL:https://mathoverflow.net/q/78410 (version: 2011-10-18).

[26] Klaus Hulek and Roberto Laface. On the Picard numbers of Abelian varieties. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):1199–1224, 2019.

[27] Shu Kawaguchi and Joseph H. Silverman. Examples of dynamical degree equals arithmetic degree. *Michigan Math. J.*, 63(1):41–63, 2014.

[28] Shu Kawaguchi and Joseph H. Silverman. Addendum to "Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties". *Trans. Amer. Math. Soc.*, 373(3):2253, 2020.

[29] Shu Kawaguchi and Joseph H. Silverman. Erratum to: "On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties" (J. Reine Angew. Math. 713 (2016), 21–48). *J. Reine Angew. Math.*, 761:291–292, 2020.

[30] Steven L. Kleiman. The picard scheme, 2005.

[31] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[32] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[33] John Lesieutre. A projective variety with discrete, non-finitely generated automorphism group. *Invent. Math.*, 212(1):189–211, 2018.

[34] John Lesieutre and Matthew Satriano. Canonical Heights on Hyper-Kähler Varieties and the Kawaguchi–Silverman Conjecture. *Int. Math. Res. Not. IMRN*, (10):7677–7714, 2021.

[35] Sichen Li and Yohsuke Matsuzawa. A note on kawaguchi-silverman conjecture, 2020.

[36] Kenji Matsuki. *Introduction to the Mori program*. Universitext. Springer-Verlag, New York, 2002.

[37] Yohsuke Matsuzawa. On upper bounds of arithmetic degrees, 2016.

[38] Yohsuke Matsuzawa. Kawaguchi-Silverman conjecture for endomorphisms on several classes of varieties. *Adv. Math.*, 366:107086, 26, 2020.

[39] Yohsuke Matsuzawa, Sheng Meng, Takahiro Shibata, and De-Qi Zhang. Non-density of points of small arithmetic degrees, 2020.

[40] Yohsuke Matsuzawa, Kaoru Sano, and Takahiro Shibata. Arithmetic degrees and dynamical degrees of endomorphisms on surfaces. *Algebra Number Theory*, 12(7):1635–1657, 2018.

[41] Yohsuke Matsuzawa and Shou Yoshikawa. Int-amplified endomorphisms on normal projective surfaces, 2019.

[42] Yohsuke Matsuzawa and Shou Yoshikawa. Kawaguchi-silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism, 2019.

[43] Sheng Meng. Building blocks of amplified endomorphisms of normal projective varieties. *Math. Z.*, 294(3-4):1727–1747, 2020.

[44] Sheng Meng and De-Qi Zhang. Kawaguchi-silverman conjecture for surjective endomorphisms, 2019.

[45] Sheng Meng and De-Qi Zhang. Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism. *Math. Res. Lett.*, 27(2):523–549, 2020.

[46] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[47] Noboru Nakayama. Ruled surfaces with non-trivial surjective endomorphisms. *Kyushu J. Math.*, 56(2):433–446, 2002.

[48] Edilaine Ervilha Nobili. Birational geometry of toric varieties, 2012.

[49] Tadao Oda. Vector bundles on an elliptic curve. *Nagoya Math. J.*, 43:41–72, 1971.

[50] Keiji Oguiso. Automorphism groups of Calabi-Yau manifolds of Picard number 2. *J. Algebraic Geom.*, 23(4):775–795, 2014.

[51] F. Oort and M. van der Put. A construction of an abelian variety with a given endomorphism algebra. *Compositio Math.*, 67(1):103–120, 1988.

[52] Frans Oort. Endomorphism algebras of abelian varieties. In *Algebraic geometry and commutative algebra, Vol. II*, pages 469–502. Kinokuniya, Tokyo, 1988.

[53] Kaoru Sano. The canonical heights for jordan blocks of small eigenvalues, preperiodic points, and the arithmetic degrees, 2017.

[54] Kaoru Sano and Takahiro Shibata. Zariski density of points with maximal arithmetic degree, 2020.

[55] M. V. Sapir. Some group theory problems, 2007.

[56] Joseph H. Silverman. Arithmetic and dynamical degrees on abelian varieties. *J. Théor. Nombres Bordeaux*, 29(1):151–167, 2017.

[57] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2021.

[58] Bit-Shun Tam. A cone-theoretic approach to the spectral theory of positive linear operators: the finite-dimensional case. *Taiwanese J. Math.*, 5(2):207–277, 2001.

[59] Tuyen Trung Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. *J. Reine Angew. Math.*, 758:139–182, 2020.

[60] Loring W. Tu. Semistable bundles over an elliptic curve. *Adv. Math.*, 98(1):1–26, 1993.

[61] Jerzy Weyman. *Cohomology of vector bundles and syzygies*, volume 149 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.

[62] Alexander Zotine. Explicitly representing vector bundles over elliptic curves, 2017.