On the Number of Trials Needed to Obtain k Consecutive Successes

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Abstract A sequence of independent Bernoulli trials, each of which is a success with probability p, is conducted. For $k \in \mathbb{Z}^+$, let X_k be the number of trials required to obtain k consecutive successes. Using techniques from elementary probability theory, we present a derivation which ultimately yields an elegant expression for the probability mass function of X_k, and is simpler in comparison to what is found in the literature. Following this, we use our derived formula to obtain explicit closed-form expressions for the complementary cumulative distribution function and the n^{th} factorial moment of X_k.

Keywords Bernoulli trials · Consecutive successes · Factorial moments · Generating function · Polynomial coefficients · Bell polynomials · Combinatorial probability

1 Introduction

We consider a well-known problem in applied probability in which independent Bernoulli trials, each having success probability $p \in (0, 1)$, are performed until k consecutive successes are achieved where $k \in \mathbb{Z}^+$. Let X_k count the number of trials needed to obtain k consecutive successes. Clearly, X_k is a discrete random variable (rv) with probability mass function (pmf) $f_k(x) = \mathbb{P}(X_k = x)$ on the support set $\{k, k+1, k+2, \ldots\}$. The distribution of X_k has been studied previously, most notably by Shane (1973), who derived the probability generating function (pgf) of X_k by developing a recursive formula for its pmf in terms of his Polynacci polynomials of order k in p. Other related papers followed, particularly those by Turner (1979), Philippou and Muwafi (1982), and Philippou et al. (1983). In the latter paper, the authors introduce a particular type of generalized geometric distribution to which the distribution of X_k belongs (not surprisingly, given the fact that X_k has a geometric

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distribution when $k = 1$). Specifically, Philippou et al. (1983) shows that

$$f_k(x) = \sum_{i_1, i_2, \ldots, i_k} \binom{i_1 + i_2 + \cdots + i_k}{i_1, i_2, \ldots, i_k} p^x \left(\frac{1-p}{p}\right)^{i_1+i_2+\cdots+i_k}, \quad x = k, k+1, k+2, \ldots, \tag{1}$$

where the above summation is over all non-negative integers $i_1, i_2, \ldots, i_k$ such that

$$i_1 + 2i_2 + \cdots + ki_k = x - k.$$

Using the above pmf, Philippou et al. (1983) also derives the associated pgf as a means of obtaining (through differentiation) the following results for the mean and variance of X_k:

$$\mathbb{E}(X_k) = \frac{1 - p^k}{(1-p)p^k}, \tag{2}$$

and

$$\mathbb{V}\mathrm{ar}(X_k) = \frac{1 - (2k+1)(1-p)p^k - p^{2k+1}}{(1-p)^2 p^{2k}}. \tag{3}$$

For the sake of completeness, we remark that (2) and (3) also agree with the results found independently by Woodside (1990).

In this paper, we present an alternative means of obtaining the pmf of X_k, one which sheds a different light on the problem and ultimately gives rise to a simpler formula for $f_k(x)$ which, unlike (1), does not involve the solutions of a diophantine equation. The approach we use is, in some sense, less specialized, and is based on a clever conditioning argument which Ross (2010, Example 3.15, p. 113) successfully employs to obtain $\mathbb{E}(X_k)$. By conditioning on the rv X_{k-1} (since one must first obtain $k-1$ consecutive successes before reaching k) and using the law of total expectation, Ross develops a recursive formula for $\mathbb{E}(X_k)$ which, when solved, agrees with (2) but does not involve any complicated sum formulas or the differentiation of a pgf. We adapt this argument to derive the pmf. An added advantage to our approach is that it leads to an equally elegant formula for the complementary cumulative distribution function (ccdf) of X_k. Finally, we conclude our paper with a novel derivation revealing an interesting relationship between the factorial moments of X_k and a sequence of polynomials with combinatorial significance, namely the exponential partial Bell polynomials.

2 Derivation of the pmf and ccdf of X_k

We adopt the approach used by Ross (2010), as described above, but this time for the pmf. In particular, conditioning on the rv X_{k-1} (for $k \geq 2$), we first obtain

$$\begin{aligned} f_k(x) &= \sum_{\ell=k-1}^{\infty} \mathbb{P}(X_k = x | X_{k-1} = \ell) f_{k-1}(\ell) \\ &= \sum_{\ell=k-1}^{x-1} \mathbb{P}(X_k = x | X_{k-1} = \ell) f_{k-1}(\ell), \end{aligned} \tag{4}$$

since $\mathbb{P}(X_k = x|X_{k-1} = \ell) = 0$ for $\ell \geq x$. Now, for $k-1 \leq \ell \leq x-1$, we condition on the outcome of the $(\ell+1)^{\text{th}}$ trial. If this trial is a success, then $X_k = \ell + 1$ with probability 1. However, if this trial is a failure, then the counter essentially resets itself following this failed trial. Altogether, letting $q = 1 - p$, this leads to

$$\mathbb{P}(X_k = x|X_{k-1} = \ell) = p\delta_{\ell+1,x} + qf_k(x - \ell - 1),$$

where $\delta_{i,j}$ denotes the Kronecker delta. Therefore, for $k \geq 2$, (4) becomes

$$\begin{aligned} f_k(x) &= \sum_{\ell=k-1}^{x-1} \big(p\delta_{\ell+1,x} + qf_k(x-\ell-1)\big) f_{k-1}(\ell) \\ &= pf_{k-1}(x-1) + q\sum_{\ell=k-1}^{x-2} f_k(x-\ell-1)f_{k-1}(\ell),\ x = k, k+1, k+2, \ldots. \end{aligned} \tag{5}$$

However, by defining $f_0(0) = 1$ and $f_0(x) = 0$ when $x \in \mathbb{Z}^+$, we note that (5) holds true even when $k = 1$. Moreover, with $f_k(x) = 0$ when $x < k$ and $f_{k-1}(\ell) = 0$ when $\ell < k - 1$, we can extend the bounds on the summation in (5) as well as the range of x to obtain

$$f_k(x) = pf_{k-1}(x-1) + q\sum_{\ell=0}^{x-1} f_k(x-1-\ell)f_{k-1}(\ell),\ \ x \in \mathbb{Z}^+. \tag{6}$$

For $k \in \mathbb{Z}^+$, let $G_k(z)$ denote the pgf of X_k, given by

$$G_k(z) = \sum_{x=0}^{\infty} f_k(x)z^x. \tag{7}$$

If we now multiply both sides of (6) by z^x and sum over $x \in \mathbb{Z}^+$, we obtain

$$\begin{aligned} \sum_{x=1}^{\infty} f_k(x)z^x &= p\sum_{x=1}^{\infty} f_{k-1}(x-1)z^x + q\sum_{x=1}^{\infty}\left(\sum_{\ell=0}^{x-1} f_k(x-1-\ell)f_{k-1}(\ell)\right)z^x \\ &= pz\sum_{x=1}^{\infty} f_{k-1}(x-1)z^{x-1} \\ &\quad + \ qz\sum_{x=1}^{\infty}\left(\sum_{\ell=0}^{x-1} f_k(x-1-\ell)f_{k-1}(\ell)\right)z^{x-1} \\ &= pz\sum_{x=0}^{\infty} f_{k-1}(x)z^x + qz\sum_{x=0}^{\infty}\left(\sum_{\ell=0}^{x} f_k(x-\ell)f_{k-1}(\ell)\right)z^x. \end{aligned} \tag{8}$$

Since $f_k(0) = 0$ for $k \in \mathbb{Z}^+$, the left-hand side of (8) equals $G_k(z)$. Moreover, the inner sum in the second expression on the right-hand side of (8) is the convolution of $f_k(x)$ and $f_{k-1}(x)$.

Applying the convolution property for generating functions (e.g., Spivey, 2019, Theorem 13, p. 122), (8) readily becomes

$$G_k(z) = pzG_{k-1}(z) + qzG_k(z)G_{k-1}(z),$$

or equivalently,

$$G_k(z) = \frac{pzG_{k-1}(z)}{1 - qzG_{k-1}(z)}, \tag{9}$$

with initial condition $G_0(z) = \sum_{x=0}^{\infty} f_0(x)z^x = f_0(0) = 1$.

We recognize the recurrence relation in (9) as a first-order rational difference equation of the form

$$G_k(z) = \frac{a(z)G_{k-1}(z) + b(z)}{c(z)G_{k-1}(z) + d(z)},$$

where $a(z) = pz$, $b(z) = 0$, $c(z) = -qz$, and $d(z) = 1$. To solve such an equation, we employ a well-known approach (e.g., Mitchell, 2000) which proceeds in the following manner. Define $\eta(z) = (pz - 1)/(qz)$ and $y_k(z) = (\eta(z) + G_k(z))^{-1}$, so that

$$G_k(z) = \frac{1}{y_k(z)} - \eta(z). \tag{10}$$

By means of this change of variable, (9) subsequently becomes

$$\begin{aligned}
\frac{1}{y_k(z)} - \eta(z) &= \frac{pz\left(\frac{1}{y_{k-1}(z)} - \eta(z)\right)}{1 - qz\left(\frac{1}{y_{k-1}(z)} - \eta(z)\right)}, \\
\frac{1}{y_k(z)} &= \frac{qpz - p(pz-1)y_{k-1}(z)}{q(pzy_{k-1}(z) - qz)} + \frac{pz-1}{qz}, \\
\frac{1}{y_k(z)} &= \frac{qpz^2 - pz(pz-1)y_{k-1}(z) + (pz-1)(pzy_{k-1}(z) - qz)}{qz(pzy_{k-1}(z) - qz)}, \\
y_k(z) &= pzy_{k-1}(z) - qz,
\end{aligned}$$

with initial condition $y_0(z) = qz/(z-1)$. Since this is a simple linear, first-order difference equation with constant coefficients, the solution is immediately given by (e.g., Elaydi, 2005, Equation 1.2.8, p. 4)

$$y_k(z) = \left(y_0(z) + \frac{qz}{1-pz}\right)(pz)^k - \frac{qz}{1-pz} = \frac{qz(1 - z + qp^kz^{k+1})}{(z-1)(1-pz)}.$$

Substituting the above equation into (10), we obtain

$$G_k(z) = \frac{(z-1)(1-pz)}{qz(1 - z + qp^kz^{k+1})} - \frac{pz-1}{qz} = \frac{(pz)^k(1-pz)}{1 - pz - qz(1 - (pz)^k)}. \tag{11}$$

We remark that (11) agrees with the result found independently by Woodside (1990, p. 605). Taking a closer look at (11), however, we find that

$$\begin{aligned} G_k(z) &= \frac{(pz)^k}{1-\frac{qz(1-(pz)^k)}{1-pz}} \\ &= (pz)^k \sum_{m=0}^{\infty} (qz)^m \left(\frac{(1-(pz)^k)}{1-pz} \right)^m \\ &= (pz)^k \sum_{m=0}^{\infty} (qz)^m \left(1 + pz + \cdots + (pz)^{k-1}\right)^m . \end{aligned} \tag{12}$$

However, we have

$$\left(1 + pz + \cdots + (pz)^{k-1}\right)^m = \sum_{j=0}^{(k-1)m} \binom{m}{j}_k (pz)^j,$$

where the coefficient of $(pz)^j$ in the above expansion, namely $\binom{m}{j}_k$, is the so-called *polynomial coefficient* (e.g., Comtet, 1974, p. 77). These coefficients generalize the binomial coefficients, and since $\binom{m}{j}_k = 0,\ \forall\ j \notin \{0, 1, \ldots, (k-1)m\}$, we can express (12) as

$$\begin{aligned} G_k(z) &= (pz)^k \sum_{m=0}^{\infty} (qz)^m \sum_{j=0}^{\infty} \binom{m}{j}_k (pz)^j \\ &= \sum_{m=0}^{\infty} (qz)^m \sum_{j=0}^{\infty} \binom{m}{j}_k (pz)^{j+k} \\ &= \sum_{m=0}^{\infty} (qz)^m \sum_{x=k+m}^{\infty} \binom{m}{x-k-m}_k (pz)^{x-m} \\ &= \sum_{m=0}^{\infty} (qz)^m \sum_{x=0}^{\infty} \binom{m}{x-k-m}_k (pz)^{x-m} \\ &= \sum_{x=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{m}{x-k-m}_k q^m p^{x-m} \right) z^x. \end{aligned} \tag{13}$$

Equating the forms of (7) and (13), we immediately obtain

$$f_k(x) = \sum_{m=0}^{\infty} \binom{m}{x-k-m}_k q^m p^{x-m},\ \ x = k, k+1, k+2, \ldots . \tag{14}$$

It is possible to tighten the summation bounds on the above expression for $f_k(x)$. To do so, the following lemma proves to be useful.

Lemma 1. Suppose $k, x \in \mathbb{Z}^+$ with $x \geq k \geq 1$. Let

$$r = \left\lfloor \frac{x-1}{k} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Then, for $m \leq r-1$,

$$\binom{m}{x-k-m}_k = 0.$$

Proof. For $x \geq k \geq 1$, we have $r \leq (x-1)/k$ by definition. Therefore, if $m \leq r-1$, then

$$\begin{aligned}
m &\leq \frac{x-1}{k} - 1, \\
km &\leq x - 1 - k, \\
km - m &\leq x - 1 - k - m, \\
(k-1)m &\leq (x-k-m) - 1 < x - k - m.
\end{aligned}$$

Since $x - k - m > (k-1)m$ and $\binom{m}{j}_k = 0 \ \forall\ j \notin \{0, 1, \ldots, (k-1)m\}$, it immediately follows that

$$\binom{m}{x-k-m}_k = 0.$$

□

We are now ready to state our main result.

Theorem 1. The pmf of the random variable X_k, where X_k is the number of trials needed to obtain k consecutive successes in a sequence of independent Bernoulli trials with success probability p, is given by

$$f_k(x) = \sum_{m=\lfloor (x-1)/k \rfloor}^{x-k} \binom{m}{x-k-m}_k q^m p^{x-m}, \ x = k, k+1, k+2, \ldots. \tag{15}$$

Proof. Apply Lemma 1 to (14) and use the fact that $\binom{m}{x-k-m}_k = 0$ for $m > x - k$. □

As an immediate consequence of Theorem 1, we are also able to obtain an elegant formula for the ccdf of X_k.

Theorem 2. The ccdf of the random variable X_k, denoted by $\bar{F}_k(x) = \mathbb{P}(X_k > x)$, is given by (for $x = k-1, k, k+1, \ldots$)

$$\bar{F}_k(x) = (1-p^k)^{x-k+2} + \sum_{m=\lfloor x/k \rfloor}^{x-k+1} q^m \sum_{y=x+1}^{k(m+1)} \binom{m}{y-k-m}_k p^{y-m}. \tag{16}$$

Proof. Using (15), we have that

$$\bar{F}_k(x) = \sum_{y=x+1}^{\infty} \sum_{m=\lfloor (y-1)/k \rfloor}^{y-k} \binom{m}{y-k-m}_k q^m p^{y-m}. \tag{17}$$

Interchanging the order of summation in (17) leads to

$$\begin{aligned}\bar{F}_k(x) &= \sum_{m=\lfloor x/k \rfloor}^{\infty} \sum_{y=\max\{x+1,k+m\}}^{km+k} \binom{m}{y-k-m}_k q^m p^{y-m} \\ &= \sum_{m=\lfloor x/k \rfloor}^{x-k+1} q^m \sum_{y=x+1}^{k(m+1)} \binom{m}{y-k-m}_k p^{y-m} + \sum_{m=x-k+2}^{\infty} q^m \sum_{y=k+m}^{k(m+1)} \binom{m}{y-k-m}_k p^{y-m}.\end{aligned} \tag{18}$$

Looking at the second term on the right-hand side of (18), note that

$$\begin{aligned}\sum_{m=x-k+2}^{\infty} q^m \sum_{y=k+m}^{k(m+1)} \binom{m}{y-k-m}_k p^{y-m} &= \sum_{m=x-k+2}^{\infty} q^m \sum_{j=0}^{(k-1)m} \binom{m}{j}_k p^{j+k} \\ &= p^k \sum_{m=x-k+2}^{\infty} q^m (1+p+\cdots+p^{k-1})^m \\ &= p^k \sum_{m=x-k+2}^{\infty} [(1-p)(1+p+\cdots+p^{k-1})]^m \\ &= p^k \sum_{m=x-k+2}^{\infty} (1-p^k)^m \\ &= (1-p^k)^{x-k+2}.\end{aligned} \tag{19}$$

Substituting (19) into (18) yields the desired formula. ☐

3 Derivation of the factorial moments of X_k

We now turn our attention to finding an expression for the n^{th} factorial moment of X_k, given by

$$\mathbb{E}(X_k^{\underline{n}}) = \mathbb{E}(X_k(X_k-1)\cdots(X_k-n+1)), \ n \in \mathbb{Z}^+.$$

Using (15), we have that

$$\mathbb{E}(X_k^{\underline{n}}) = \sum_{x=k}^{\infty} \sum_{m=\lfloor (x-1)/k \rfloor}^{x-k} x^{\underline{n}} \binom{m}{x-k-m}_k q^m p^{x-m}. \tag{20}$$

Interchanging the order of summation in (20) leads to

$$\begin{aligned}
\mathbb{E}(X_k^{\underline{n}}) &= \sum_{m=0}^{\infty}\sum_{x=m+k}^{km+k} x^{\underline{n}} \binom{m}{x-k-m}_k q^m p^{x-m} \\
&= p^n \sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^m \sum_{y=0}^{(k-1)m}\binom{m}{y}_k (y+m+k)^{\underline{n}} p^{y+m+k-n} \\
&= p^n \sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^m \frac{d^n}{dp^n}\Big(h_m(p)p^{m+k}\Big), \qquad (21)
\end{aligned}$$

where $h_m(p) = \sum_{j=0}^{(k-1)m}\binom{m}{j}_k p^j = (1+p+\cdots+p_{k-1})^m$. Applying Leibniz's product rule for differentiation (and keeping in mind that the zeroth derivative corresponds to the function itself), (21) becomes

$$\begin{aligned}
\mathbb{E}(X_k^{\underline{n}}) &= p^n \sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^m \sum_{\ell=0}^{n}\binom{n}{\ell}\frac{d^\ell}{dp^\ell}\Big(h_m(p)\Big)(m+k)^{\underline{n-\ell}} p^{m+k-(n-\ell)} \\
&= p^k \sum_{m=0}^{\infty} q^m \sum_{\ell=0}^{n}\binom{n}{\ell}(m+k)^{\underline{n-\ell}} p^\ell \frac{d^\ell}{dp^\ell}\Big(g_m(h_1(p))\Big), \qquad (22)
\end{aligned}$$

where $h_m(p) = g_m(h_1(p))$ with $g_m(u) = u^m$ and

$$h_1(p) = 1 + p + \cdots + p^{k-1} = \frac{1-p^k}{1-p}.$$

For notational convenience, let $g_m^{(i)}(u) = \frac{d^i}{du^i}(g_m(u))$ and $h_1^{(i)}(p) = \frac{d^i}{dp^i}(h_1(p))$, $i = 0, 1, 2, \ldots$. Applying Faà di Bruno's formula generalizing the chain rule to higher order derivatives (e.g., Riordan, 1946), we have that

$$\begin{aligned}
\frac{d^\ell}{dp^\ell}\Big(g_m(h_1(p))\Big) &= \sum_{j=\min\{1,\ell\}}^{\ell} g_m^{(j)}(h_1(p)) B_{\ell,j}\Big(h_1^{(1)}(p), h_1^{(2)}(p), \ldots, h_1^{(\ell-j+1)}(p)\Big) \\
&= \sum_{j=\min\{1,\ell\}}^{\ell} m^{\underline{j}} h_1(p)^{m-j} B_{\ell,j}\Big(h_1^{(1)}(p), h_1^{(2)}(p), \ldots, h_1^{(\ell-j+1)}(p)\Big), \qquad (23)
\end{aligned}$$

where $B_{\ell,j}(x_1, x_2, \ldots, x_{\ell-j+1})$ is the so-called *exponential partial Bell polynomial* with arguments ℓ and j (e.g., Comtet, 1974, pp. 133–134). Substituting (23) into (22) and noting that

$$q^m h_1(p)^{m-j} = q^j (q h_1(p))^{m-j} = q^j[(1-p)(1+p+\cdots+p^{k-1})]^{m-j} = q^j(1-p^k)^{m-j},$$

we subsequently obtain

$$\mathbb{E}(X_k^{\underline{n}}) = p^k \sum_{\ell=0}^{n} \binom{n}{\ell} p^\ell \sum_{j=\min\{1,\ell\}}^{\ell} q^j B_{\ell,j}\Big(h_1^{(1)}(p), h_1^{(2)}(p), \ldots, h_1^{(\ell-j+1)}(p)\Big)$$
$$\times \sum_{m=0}^{\infty} (m+k)^{\underline{n-\ell}} m^{\underline{j}} (1-p^k)^{m-j}. \tag{24}$$

Two remarks can be made concerning the formula given by (24). First of all, it is a straightforward application of Leibniz's product rule for differentiation which yields

$$h_1^{(i)}(p) = \frac{i!}{(1-p)^{i+1}} - i! \sum_{w=0}^{i} \binom{k}{w} \frac{p^{k-w}}{(1-p)^{i+1-w}}, \;\; i \in \mathbb{N}.$$

Note that this formula even holds true for $i \geq k$, correctly giving $h_1^{(i)}(p) = 0$ in this case. Secondly, it is possible to simplify the infinite series which appears in (24). To aid us in this regard, we provide the following lemma.

Lemma 2. If $0 < \alpha < 1$ and $x, y \in \mathbb{N}$, then

$$\kappa_i = \sum_{w=0}^{\infty} (w+x+y)^{\underline{i}} (1-\alpha)^{w+y} = \sum_{v=0}^{i} i^{\underline{v}} (x+y)^{\underline{i-v}} \frac{(1-\alpha)^{v+y}}{\alpha^{v+1}}, \;\; i \in \mathbb{N}. \tag{25}$$

Proof. We define the generating function

$$H(z) = \sum_{w=0}^{\infty} z^{w+x+y} (1-\alpha)^{w+y} = \frac{z^{x+y}(1-\alpha)^y}{1-z(1-\alpha)}, \;\; |z| < (1-\alpha)^{-1},$$

from which it immediately follows that

$$H(z)[1-(1-\alpha)z] = (1-\alpha)^y z^{x+y}. \tag{26}$$

Differentiating both sides of (26) i times (with respect to z) leads to

$$\begin{aligned} \sum_{j=0}^{i} \binom{i}{j} H^{(j)}(z) \frac{d^{i-j}}{dz^{i-j}} \Big(1-(1-\alpha)z\Big) &= (1-\alpha)^y (x+y)^{\underline{i}} z^{x+y-i}, \\ H^{(i)}(z)[1-(1-\alpha)z] - iH^{(i-1)}(z)(1-\alpha) &= (1-\alpha)^y (x+y)^{\underline{i}} z^{x+y-i}. \end{aligned}$$

Plugging in $z = 1$ into the above equation (noting that $\kappa_{i-1} = H^{(i-1)}(1)$ and $\kappa_i = H^{(i)}(1)$) yields the recursive equation

$$\alpha\kappa_i - i(1-\alpha)\kappa_{i-1} = (1-\alpha)^y (x+y)^{\underline{i}},$$

or equivalently,

$$\kappa_i = \left(\frac{1-\alpha}{\alpha}\right) i\kappa_{i-1} + \frac{(1-\alpha)^y (x+y)^{\underline{i}}}{\alpha}. \tag{27}$$

With $\kappa_0 = H^{(0)}(1) = H(1) = (1-\alpha)^y/\alpha$, the use of induction verifies that (25) is the explicit solution to (27). $\square$

If we now consider the infinite series in (24) and let $\alpha = p^k$, note that

$$\begin{aligned}\sum_{m=0}^{\infty}(m+k)^{\underline{n-\ell}}\, m^{\underline{j}}\,(1-p^k)^{m-j} &= \sum_{m=j}^{\infty}(m+k)^{\underline{n-\ell}}\, m^{\underline{j}}\,(1-\alpha)^{m-j}\\ &= \sum_{w=0}^{\infty}(w+j+k)^{\underline{n-\ell}}\,(w+j)^{\underline{j}}\,(1-\alpha)^{w}\\ &= \sum_{w=0}^{\infty}(w+j+k)^{\underline{n-\ell}}\,(-1)^j\,\frac{d^j}{d\alpha^j}\Big((1-\alpha)^{w+j}\Big)\\ &= (-1)^j\,\frac{d^j}{d\alpha^j}\left\{\sum_{w=0}^{\infty}(w+j+k)^{\underline{n-\ell}}\,(1-\alpha)^{w+j}\right\}. \qquad (28)\end{aligned}$$

However, recognizing that the expression within curly brackets in (28) is κ_i from Lemma 2 with $x = k$, $y = j$, and $i = n - \ell$, we have that

$$\begin{aligned}&\frac{d^j}{d\alpha^j}\left\{\sum_{w=0}^{\infty}(w+j+k)^{\underline{n-\ell}}\,(1-\alpha)^{w+j}\right\}\\ &= \sum_{v=0}^{n-\ell}(n-\ell)^{\underline{v}}\,(j+k)^{\underline{n-\ell-v}}\,\frac{d^j}{d\alpha^j}\left\{\frac{(1-\alpha)^{v+j}}{\alpha^{v+1}}\right\}, \qquad (29)\end{aligned}$$

where

$$\begin{aligned}&\frac{d^j}{d\alpha^j}\left\{\frac{(1-\alpha)^{v+j}}{\alpha^{v+1}}\right\}\\ &= \sum_{z=0}^{j}\binom{j}{z}\frac{d^z}{d\alpha^z}\Big((1-\alpha)^{v+j}\Big)\frac{d^{j-z}}{d\alpha^{j-z}}\Big(\alpha^{-(v+1)}\Big)\\ &= \sum_{z=0}^{j}\binom{j}{z}(-1)^z(v+j)^{\underline{z}}\,(1-\alpha)^{v+j-z}(-1)^{j-z}(v+j-z)^{\underline{j-z}}\,\alpha^{-(v+j-z+1)}\\ &= (-1)^j\,\frac{(v+j)!(1-\alpha)^v}{v!\alpha^{v+1}}\sum_{z=0}^{j}\binom{j}{z}\left(\frac{1-\alpha}{\alpha}\right)^{j-z}\\ &= (-1)^j\,\frac{(v+j)!(1-\alpha)^v}{v!\alpha^{v+1}}\left(1+\frac{1-\alpha}{\alpha}\right)^{j}\\ &= (-1)^j\alpha^{-(j+1)}\,\frac{(v+j)!}{v!}\left(\frac{1-\alpha}{\alpha}\right)^{v}. \qquad (30)\end{aligned}$$

Using the results of (29) and (30), (28) becomes

$$\begin{aligned}
&\sum_{m=0}^{\infty}(m+k)^{\underline{n-\ell}}\,m^{\underline{j}}\,(1-p^k)^{m-j} \\
&=\sum_{v=0}^{n-\ell}\frac{(n-\ell)!(j+k)!}{(n-\ell-v)!(j+k-n+\ell+v)!}p^{-(j+1)k}\frac{(v+j)!}{v!}\left(\frac{1-p^k}{p^k}\right)^{v} \\
&=(n-\ell)!\,p^{-(j+1)k}\sum_{v=0}^{n-\ell}\binom{j+k}{n-\ell-v}\frac{(v+j)!}{v!}\left(\frac{1-p^k}{p^k}\right)^{v}. \qquad (31)
\end{aligned}$$

We can now state our simplified formula for the n^{th} factorial moment of X_k.

Theorem 3. The n^{th} factorial moment of the random variable X_k, where X_k is the number of trials needed to obtain k consecutive successes in a sequence of independent Bernoulli trials with success probability p, is given by

$$\begin{aligned}
\mathbb{E}(X_k^{\underline{n}})=n!\sum_{\ell=0}^{n}\frac{p^\ell}{\ell!}\sum_{v=0}^{n-\ell}\frac{(p^{-k}-1)^v}{v!}\sum_{j=\min\{1,\ell\}}^{\ell}&\binom{j+k}{n-\ell-v}(v+j)!\left(\frac{1-p}{p^k}\right)^{j} \\
&\times B_{\ell,j}\left(h_1^{(1)}(p),h_1^{(2)}(p),\ldots,h_1^{(\ell-j+1)}(p)\right). \qquad (32)
\end{aligned}$$

Proof. Substitute (31) into (24) and simplify the resulting expression. □

We demonstrate the use of (32) by determining $\mathbb{E}(X_k^{\underline{n}})$ for $n=1,2,3,4$. This requires knowledge of the following Bell polynomials:

$$\begin{aligned}
B_{0,0}(x_1)&=1, \\
B_{1,1}(x_1)&=x_1, \\
B_{2,1}(x_1,x_2)&=x_2, & B_{2,2}(x_1)&=x_1^2, \\
B_{3,1}(x_1,x_2,x_3)&=x_3, & B_{3,2}(x_1,x_2)&=3x_1x_2, \\
B_{4,1}(x_1,x_2,x_3,x_4)&=x_4, & B_{4,2}(x_1,x_2,x_3)&=3x_2^2+4x_1x_3, \\
\\
B_{3,3}(x_1)&=x_1^3, \\
B_{4,3}(x_1,x_2)&=6x_1^2x_2, & B_{4,4}(x_1)&=x_1^4.
\end{aligned}$$

After some tedious but straightforward algebra, the first four factorial moments of X_k are:

$$\mathbb{E}(X_k^{\underline{1}})=\frac{1-p^k}{(1-p)p^k}, \qquad (33)$$

$$\mathbb{E}(X_k^{\underline{2}})=\frac{2+2(1-p)p^{2k}-2[k+2-(k+1)p]p^k}{(1-p)^2p^{2k}}, \qquad (34)$$

$$\mathbb{E}(X_k^3) = \Big\{6 - 6(1-p)^2p^{3k} + 3(k+2)[k+3-(k+1)p](1-p)p^{2k} \\ - 6[2k+3-2(k+1)p]p^k\Big\}\left[(1-p)^3p^{3k}\right]^{-1}, \tag{35}$$

$$\mathbb{E}(X_k^4) = \Big\{24+24(1-p)^3p^{4k} - 4(k+2)(k+3)[k+4-(k+1)p](1-p)^2p^{3k}+ \\ 24(2k+3)[k+2-(k+1)p](1-p)p^{2k} - 24[3k+4-3(k+1)p]p^k\Big\} \\ \times \left[(1-p)^4p^{4k}\right]^{-1}. \tag{36}$$

We remark that (33) agrees with the known result for the mean given by (2). Moreover, it is easily verified that (33) and (34) correctly combine to yield the result for variance given by (3). To the best of our knowledge, the results for (35) and (36) are new.

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