Dual Conditions for Local Transverse Feedback Linearization

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***Abstract*—Given a control-affine system and a controlled invariant submanifold, the local transverse feedback linearization problem is to determine whether or not the system is locally feedback equivalant to a system whose dynamics transversal to the submanifold are linear and controllable. In this paper we present necessary and sufficient conditions for a single-input system to be locally transversally feedback linearizable to a given submanifold that dualize, in an algebraic sense, previously published conditions. These dual conditions are of interest in their own right and represent a first step towards a Gardner-Shadwick like algorithm for local transverse feedback linearization.**

I. Introduction

Gardner and Shadwick introduced their G.S. algorithm in [1] for solving the exact feedback linearization problem for multi-input systems. The algorithm treats a control system as an exterior differential system or, more precisely, a Pfaffian system [2]. The appeal of the GS algorithm, as opposed to the "vector field" approach presented in, for instance, [3], is that it yields the differentials whose intergrals equal the virtual output function that has the correct vector relative degree.

Transverse feedback linearization (TFL) refers to feedback linearizing that portion of a control system's dynamics which governs the transversal motion to a given submanifold of its state-space. While the necessary and sufficient conditions in [4] are checkable, they are not constructive in the sense that they do not provide a method for obtaining the linearizing output function. This motivates the construction of "dual" conditions. The necessary and sufficient dual conditions we present are also checkable but, in addition, they suggest a procedure for obtaining the one-form to be integrated to produce the "transversal output" needed for transverse feedback linearization. In this sense, the conditions presented in this paper precede the development of a GS-like algorithm for transverse feedback linearization.

As pointed out by Gardner and Shadwick, the framework of Pfaffian systems has a long history in mathematics. The paper [5] contains a well-written and entertaining account of the origins and development of the so-called Problem of Pfaff. The Pfaffian approach to feedback equivalence problems has been used by various authors of which we mention some essential contributions [6], [7], [8], [9], [10], [11]. More recently, Shöberl and Shclacher established a constructive approach to generating a triangular decomposition of a nonlinear control system [12].

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A. Notation

If n is a natural number, then $\mathbb{N}_n := \{1, \ldots, n\}$ and $\mathbb{Z}_n := \{0, \ldots, n-1\}$. If M is a smooth m-dimensional manifold then TM and $T^\star M$ denote, respectively, its tangent and cotangent bundle. The set of all smooth vector fields on M is denoted $\mathfrak{T}(M)$. The Lie bracket of $f, g \in \mathfrak{T}(M)$ is written $[f, g]$. For iterated Lie brackets we use the standard notation $\mathrm{ad}_f^0 g = g$, $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$, for $k \geq 1$.

A point $p \in M$ is called a regular point of a smooth distribution $\Delta : M \to TM$ if there is a neighbourhood U of p such that $\dim(\Delta(q))$ is constant for all $q \in U$. In this case we say that Δ is regular at p and nonsingular in U. The involutive closure of Δ is denoted $\mathrm{inv}(\Delta)$ and its annihilator $\mathrm{ann}(\Delta) : M \to T^\star M$ is defined pointwise by $\mathrm{ann}(\Delta)(p) = \{\omega(p) \in T_p^\star M : \omega(v) = 0,\ \forall v(p) \in \Delta(p)\}$. Conversely, if $\Omega : M \to T^\star M$ is a codistribution then $\mathrm{Ker}(\Omega)(p) = \{v(p) \in T_p M : \omega(v) = 0,\ \forall \omega \in \Omega(p)\}$. If Δ and Ω are constant dimensional, then $\mathrm{ann}(\Delta)$, respectively, $\mathrm{Ker}(\Omega)$ are constant dimensional and $\Delta = \mathrm{Ker}(\mathrm{ann}(\Delta))$, $\Omega = \mathrm{ann}(\mathrm{Ker}(\Omega))$.

If Δ_1, Δ_2 are distributions on M, then their sum $\Delta_1 + \Delta_2$ is defined pointwise by $(\Delta_1 + \Delta_2)(p) = \Delta_1(p) + \Delta_2(p)$ and $[\Delta_1, \Delta_2] = \{[v_1, v_2] \in \mathfrak{T}(M) : v_i \in \Delta_i, i \in \mathbb{N}_2\}$.

The vector space of smooth k-forms on an m-dimensional manifold M is denoted by $\mathcal{A}^k(M)$ and $\mathcal{A}^\star(M) := \bigoplus_{k=0}^m \mathcal{A}^k(M)$. The exterior (wedge) product of a k-form $\omega \in \mathcal{A}^k(M)$ with an l-form $\eta \in \mathcal{A}^l(M)$ is a $(k+l)$-form. An exterior ideal in $\mathcal{A}^\star(M)$ is a linear subspace $\mathcal{I} \subseteq \mathcal{A}^\star(M)$ that is closed under wedge products with arbitrary elements of $\mathcal{A}^\star(M)$. If $\mathcal{I} \subseteq \mathcal{A}^\star(M)$ is an exterior ideal, then $\mathcal{I}^k$ denotes the homegenous component of k-forms in $\mathcal{I}$.

For a k-form $\omega \in \mathcal{A}^k(M)$, its exterior derivative is denoted $\mathrm{d}\omega \in \mathcal{A}^{k+1}(M)$. Given an exterior ideal $\mathcal{I}$, its derived ideal is $\delta\mathcal{I} = \{\omega \in \mathcal{I} : \mathrm{d}\omega \equiv 0 \mod \mathcal{I}\}$. Continuing this way, the ideals $\delta^k\mathcal{I}$ determine a nested sequence of codistributions

$$\mathcal{I}^1 \supseteq \delta\mathcal{I}^1 \supseteq \cdots \supseteq \delta^N\mathcal{I}^1. \quad (1)$$

If the dimension of the subspace spanned by each $\delta^k\mathcal{I}^1$ is constant, then this construction terminates for some finite integer N. The relation (1) is called the derived flag of $\mathcal{I}$ while N is its derived length.

II. Local Transverse Feedback Linearization

Consider a single-input control system modelled by

$$\dot{x} = f(x) + g(x)u \quad (2)$$

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where $f, g : \mathbb{R}^n \to T\mathbb{R}^n$ are smooth. Let $\Gamma \subset \mathbb{R}^n$ be a closed, connected, embedded submanifold of $\mathbb{R}^n$ that is controlled invariant for the system (2). Let $n^\star := \dim(\Gamma) \in \mathbb{N}_{n-1}$.

Given (2), the set Γ and a point $x_0 \in \Gamma$, the problem of local transverse feedback linearization for (2) is as follows [4]: Find, if possible, a diffeomorphism

$$\begin{aligned} \Xi : U &\to \Xi(U) \subset (\Gamma \cap U) \times \mathbb{R}^{n-n^\star} \\ x &\mapsto (\eta, \xi) \end{aligned} \tag{3}$$

where U is a neighbourhood of x_0, such that

(i) The restriction of Ξ to $\Gamma \cap U$ is

$$\Xi|_{\Gamma \cap U} : x \mapsto (\eta, 0).$$

(ii) The dynamics of system (2) in (η, ξ)-coordinates reads

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + b(a_1(\eta, \xi) + a_2(\eta, \xi)u), \end{aligned} \tag{4}$$

where the pair (A, b) is in Brunovský normal form (one chain of integrators) and $a_2(\eta, \xi) \neq 0$ in $\Xi(U)$.

If such a diffeomorphism exists, then the smooth feedback $u = -a_1(\eta, \xi)/a_2(\eta, \xi) + v/a_2(\eta, \xi)$ yields a system of the form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + bv \end{aligned} \tag{5}$$

and we say that system (2) has been locally transversely feedback linearized with respect to the set Γ. The feedback equivalence problem between (2) and (4) (equivalently (5)) can be stated in terms of the existence of a "virtual" output function yielding a well-defined relative degree, a point of view championed by Isidori [3]. Accordingly, in [4, Theorem 3.1] it was shown that local transverse feedback linearization is possible if, and only if, there exists a smooth function $\alpha : U \subseteq \mathbb{R}^n \to \mathbb{R}$, defined in an open and connected set $U \subseteq \mathbb{R}^n$ containing x_0, such that (i) $\Gamma \cap U \subset \alpha^{-1}(0)$ and (ii) α yields a relative degree of $n - n^\star$ at x_0. Such an output function is called a local transverse output of (2) with respect to Γ. This discussion shows that local transverse feedback linearization is equivalent to the following problem.

Problem 1. (Local Zero Dynamics Assignment) Given a controlled invariant manifold Γ find an output function yielding a well-defined relative degree whose associated zero dynamics manifold locally coincides with Γ. $\triangle$

Checkable necessary and sufficient conditions for the existence of a transverse output were given in [13] for the single-input case and in [4] for the multi-input case. In order to state these conditions, define the distributions

$$(\forall i \in \mathbb{Z}_{n-n^\star})\ G_i := \operatorname{span}\left\{g, \operatorname{ad}_f g, \ldots, \operatorname{ad}_f^i g\right\}. \tag{6}$$

Theorem II.1 ([4, Theorem 3.2])**.** *Suppose that $x_0 \in \Gamma$ is a regular point of $\operatorname{inv}(G_i)$, $i \in \mathbb{Z}_{n-n^\star-1}$. Then (2) is locally transversally feedback linearizable at x_0 if, and only if,*

(a) $\dim(T_{x_0}\Gamma + G_{n-n^\star-1}(x_0)) = n$,

and there exists an open neighbourhood U of x_0 in $\mathbb{R}^n$ such that, for all $i \in \mathbb{Z}_{n-n^\star-1}$, and all $x \in \Gamma \cap U$

(b) $\dim(T_x\Gamma + G_i(x)) = \dim(T_x\Gamma + \operatorname{inv}(G_i)(x)) = n^\star + i + 1$.

The assumptions of Theorem II.1 are checkable, however its proof does not provide a constructive procedure for finding the transversal output. The next result relates transverse feedback linearization to partial feedback linearization but it isnt a viable solution to the local transverse feedback linearization problem because its assumptions are not checkable. On the other hand, the theorem provides guidelines for finding the transversal output function.

Theorem II.2 ([4, Theorem 3.5])**.** *Suppose that $x_0 \in \Gamma$ is a regular point of $\operatorname{inv}(G_i)$, $i \in \mathbb{Z}_{n-n^\star-1}$. Then (2) is locally transversally feedback linearizable at x_0 if, and only if, there exists an open neighbourhood U of x_0 and a smooth, involutive, and nonsingular distribution Δ on U such that,*

(i) $\Delta|_\Gamma = T\Gamma$,

(ii) *Δ is locally controlled invariant,*

(iii) $(\forall x \in \Gamma \cap U)\ \dim(T_x\Gamma + G_{n-n^\star-1}(x)) = n$,

(iv) *$\Delta + G_i$ is nonsingular and involutive on U.*

The procedure based on Theorems II.1 and II.2 for computing a transversal output requires the designer to first check the conditions of Theorem II.1. If the conditions hold, then one must find a (non-unique) locally controlled invariant distribution Δ, which is guaranteed to exist, and use this distribution together with the distributions G_i, to determine the exact one forms corresponding to the linearizing output.

The primary objective of this paper, inspired by [1], is to dualize the conditions of Theorem II.1 for local transverse feedback linearization.

III. Statement of Main Result

Associated to the nonlinear control system (2) is a Pfaffian system of rank n given by

$$\Sigma = \{\mathrm{d}x_i - (f_i + g_i u)\,\mathrm{d}t, i \in \mathbb{N}_n\} \tag{7}$$

on the manifold $M := \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. The first two coordinates are time and control so that points on M have the form $p = (t, u, x)$. Let $\omega_i := \mathrm{d}x_i - (f_i(x) + g_i(x)u)\,\mathrm{d}t$ for $i \in \mathbb{N}_n$ and let $\mathcal{I} = \{\omega_1, \ldots, \omega_n\}$ denote the exterior ideal generated by the ω_i. This ideal induces higher order derived ideals and a derived flag (1). To the $n^\star$-dimensional controlled invariant submanifold $\Gamma \subset \mathbb{R}^n$ we associate a family of embedded submanifolds $\hat{\Gamma} := \{t_0\} \times \mathbb{R} \times \Gamma \subset M$ where the time t_0 is fixed, but arbitrary.

Definition III.1. *Let $\mathcal{I} \subseteq \mathcal{A}^\star(M)$ be an exterior ideal and let $\hat{\Gamma} \subseteq M$ be an embedded submanifold. The **transversal codistribution** induced by $\mathcal{I}$ with respect to $\hat{\Gamma}$ is the subpace $\mathcal{I}^1_⋔ \subseteq \mathcal{I}^1$ which satisfies, for all $p \in \hat{\Gamma}$,*

$$\mathcal{I}^1_⋔(p) = \mathcal{I}^1(p) \cap \operatorname{ann}\left(T_p\hat{\Gamma}\right).$$

*A point $p \in \hat{\Gamma}$ is a **regular point** of $\mathcal{I}^1_{\pitchfork}$ if there exists a neighbourhood U in M such that $\dim(\mathcal{I}^1_{\pitchfork}(q))$ is constant for all $q \in U \cap \hat{\Gamma}$.*

Of course, Defintion III.1 applies equally well when the role of $\mathcal{I}$ is played by a derived exterior ideal $\delta^k\mathcal{I}$. In that case we will write

$$\delta^k\mathcal{I}^1_{\pitchfork}(p) = \delta^k\mathcal{I}^1(p) \cap \operatorname{ann}\left(T_p\hat{\Gamma}\right).$$

Lemma III.2. *Let $\mathcal{I} \subseteq \mathcal{A}^\star(M)$ be an ideal which is simply generated by smooth, linearly independent one-forms, let $\hat{\Gamma} \subset M$ be an embedded submanifold and let $p \in \hat{\Gamma}$ be a regular point of $\mathcal{I}^1_{\pitchfork}$ with $\dim(\mathcal{I}^1_{\pitchfork}(p)) = r$. Then there exists an open neighbourhood U of p and a set $\{\omega_1, \ldots, \omega_r\}$ of smooth, linearly independent, one-forms defined on U with the property that*

(i) $(\forall q \in U)\ \operatorname{span}\{\omega_1(q), \ldots, \omega_r(q)\} \subseteq \mathcal{I}^1(q)$,
(ii) $\left(\forall q \in U \cap \hat{\Gamma}\right)\ \mathcal{I}^1_{\pitchfork}(q) = \operatorname{span}\{\omega_1(q), \ldots, \omega_r(q)\}$.

Due to space limitations, the proof of Lemma III.2 is omitted. Next we define an exterior ideal whose generators are the one forms in the transversal codistribution.

Definition III.3. *The **transversal exterior ideal** induced by $\delta^k\mathcal{I}$ with respect to the set $\hat{\Gamma}$ is the simply generated exterior ideal whose generators are the one-forms in $\delta^k\mathcal{I}^1$.*

If $p \in \hat{\Gamma}$ is a regular point of both codistributions $\delta^k\mathcal{I}^1$ and $\delta^k\mathcal{I}^1_{\pitchfork}$, then by Lemma III.2 there exists an open set U containing p and one-forms $\omega_i \in \mathcal{A}^1(U)$ such that $\delta^k\mathcal{I}^1_{\pitchfork} = \operatorname{span}\{\omega_1, \ldots, \omega_r\}$. In this case, using the characterization in Definition $\delta^k\mathcal{I}_{\pitchfork} = \{\omega_1, \ldots, \omega_r\}$.

At times we will need to consider exterior ideals that have been augmented with the form $\mathrm{d}t$. More specifically, if $\delta^k\mathcal{I} \subseteq \mathcal{A}^\star(M)$ is a simply generated ideal, then $\delta^k\hat{\mathcal{I}} := \left\{\delta^k\mathcal{I}^1, \mathrm{d}t\right\}$ denotes the exterior ideal that is simply generated by the one forms in $\delta^k\mathcal{I}$ and the one form $\mathrm{d}t$.

We now state the main result of this paper. It is the dual of the necessary and sufficient conditions for local transverse feedback linearization in Theorem II.1.

Theorem III.4 (Main result)**.** *Let $\mathcal{I}$ be the exterior ideal simply generated by the Pfaffian system (7). Let $p = (t_0, u_0, x_0) \in \hat{\Gamma}$ be a regular point of each codistribution $\delta^k\mathcal{I}^1$, $\delta^k\mathcal{I}^1_{\pitchfork}$, $k \in \mathbb{Z}_{n-n^\star+1}$. Then system (2) is locally transverse feedback linearizable at $x_0 \in \hat{\Gamma}$ if, and only if,*

(a) $\delta^{n-n^*}\mathcal{I}_{\pitchfork}(p) = \{0\}$,
(b) *$\delta^k\hat{\mathcal{I}}_{\pitchfork}$ is a differential ideal for $k \in \mathbb{N}_{n-n^\star-1}$.*

Remark III.5. *Condition (a) is a controllability condition. It guarantees that the largest integrable subsystem contained in $\mathcal{I}$ produces a foliation of the state space which is transversal to the target set $\hat{\Gamma}$. Condition (b) is an involutivity condition.*

IV. Supporting Results

In order to prove Theorem III.4, various preliminary supporting results are required. We start by lifting vector fields in $\mathfrak{T}(\mathbb{R}^n)$ to $\mathfrak{T}(M)$.

Definition IV.1. *Let $M = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, let $f \in \mathfrak{T}(\mathbb{R}^n)$ and let $\pi : M \to \mathbb{R}^n$ be the submersion defined by $(t, u, x) \mapsto x$. The **lift** of f to the manifold M is the unique vector field $X \in \mathfrak{T}(M)$ which is π-related to f and tangent to every submanifold $\{t_0\} \times \{u_0\} \times \mathbb{R}^n$.*

Applying the procedure from Definition IV.1 to the local generators of a smooth distribution $D : \mathbb{R}^n \to T\mathbb{R}^n$, we obtain a lifted distribution on $\mathcal{D} : M \to TM$ on M which satisfies, for all $p \in M$, $\pi_*\mathcal{D}(p) = D(\pi(p))$.

Unless otherwise stated, distributions on M are denoted by script letters while distributions on $\mathbb{R}^n$ are denoted by roman letters. Applying these constructions to the distributions G_i defined in (6) we obtain the lifted distributions $\mathcal{G}_i : M \to TM$. Additionally, we need the following distributions on $\mathbb{R}^n$

$$S_0 := G_0, \tag{8a}$$
$$S_i := S_{i-1} + [S_{i-1}, S_{i-1}] + G_i,\ i \in \mathbb{N}, \tag{8b}$$

as well as the distributions

$$\mathcal{D}_0 := \operatorname{span}\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{n}(f_i + g_i u)\frac{\partial}{\partial x_i}, \frac{\partial}{\partial u}\right\}, \tag{9a}$$
$$\mathcal{D}_i := \mathcal{D}_{i-1} + [\mathcal{D}_{i-1}, \mathcal{D}_{i-1}],\ i \in \mathbb{N}, \tag{9b}$$

defined on M. Observe that $\mathcal{D}_0(p)$ has dimension 2 everywhere. Moreover, if $\mathcal{I}$ is the the exterior ideal generated by the Pfaffian system (7), then at each $p \in M$, it holds that $\mathcal{D}_0(p) = \operatorname{Ker}(\mathcal{I}^1(p))$. This implies that, for $k \geq 1$, $\mathcal{D}_k(p) = \operatorname{Ker}(\delta^k\mathcal{I}^1(p))$. The proof of our main result uses the relationship between the lift $\mathcal{S}_i$ of (8) and the distribution (9) described in the next Lemma.

Lemma IV.2. *For all $i \geq 1$, $\mathcal{D}_i = \mathcal{D}_0 + \mathcal{S}_{i-1}$.*

Proof. The proof is by induction. Simple calculations give that $\mathcal{G}_0 = [\mathcal{D}_0, \mathcal{D}_0]$ and $\mathcal{D}_1 = \mathcal{D}_0 + \mathcal{G}_0$. Since $\mathcal{S}_0 = \mathcal{G}_0$ the base case holds. Now, by way of induction, suppose for some $i \geq 1$,

$$\mathcal{D}_i = \mathcal{D}_0 + \mathcal{S}_{i-1}. \tag{10}$$

Using the definition of $\mathcal{D}_{i+1}$, (10) and bilinearity

$$\begin{aligned}
\mathcal{D}_{i+1} &= \mathcal{D}_0 + \mathcal{S}_{i-1} + [\mathcal{D}_0 + \mathcal{S}_{i-1}, \mathcal{D}_0 + \mathcal{S}_{i-1}] \\
&= \mathcal{D}_0 + \mathcal{S}_{i-1} + [\mathcal{D}_0, \mathcal{D}_0] + [\mathcal{D}_0, \mathcal{S}_{i-1}] + [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}] \\
&= \mathcal{D}_0 + \mathcal{S}_{i-1} + [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}] + [\mathcal{D}_0, \mathcal{S}_{i-1}]\,.
\end{aligned} \tag{11}$$

We've used $[\mathcal{D}_0, \mathcal{D}_0] = \mathcal{S}_0 \subseteq \mathcal{S}_{i-1}$ above. It follows from (8b) that $\mathcal{D}_0 + \mathcal{S}_{i-1} + [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}] \subseteq \mathcal{D}_0 + \mathcal{S}_i$ Then (11) becomes

$$\mathcal{D}_{i+1} \subseteq \mathcal{D}_0 + \mathcal{S}_i + [\mathcal{D}_0, \mathcal{S}_{i-1}]\,.$$

Proposition A.2 already gives $[\mathcal{D}_0, \mathcal{S}_{i-1}] \subseteq \mathcal{S}_i$. As a result we have shown that $\mathcal{D}_{i+1} \subseteq \mathcal{D}_0 + \mathcal{S}_i$. Conversely, using (8b)

$$\mathcal{D}_0 + \mathcal{S}_i = \mathcal{D}_0 + \mathcal{S}_{i-1} + [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}] + \mathcal{G}_i.$$

By the inductive hypothesis (10)

$$\mathcal{D}_0 + \mathcal{S}_i \subseteq \mathcal{D}_i + [\mathcal{D}_i, \mathcal{D}_i] + \mathcal{G}_i = \mathcal{D}_{i+1} + \mathcal{G}_i.$$

The result follows by Proposition A.3. ∎

Next we characterize the structure of the transversal codistributions induced by $\mathcal{I}$ with respect to $\hat{\Gamma}$.

Lemma IV.3. *Let $p \in \hat{\Gamma}$ be a regular point of the codistributions $\delta^k \mathcal{I}^1_{\pitchfork}$ for all $k \geq 1$. Then there exists an open set $U \subset M$ containing p such that for all $q \in U \cap \hat{\Gamma}$,*

$$\mathrm{Ker}(\delta^k \mathcal{I}^1_{\pitchfork}(q)) = \mathcal{D}_0(q) + \mathcal{S}_{k-1}(q) + T_q\hat{\Gamma}.$$

Proof. Let $p \in \hat{\Gamma}$ be a regular point of the codistributions $\delta^k \mathcal{I}^1_{\pitchfork}$ for $k \geq 1$. Let $U \subset M$ be the open set containing p for which the codistributions have constant dimension. Then for any $q \in U \cap \hat{\Gamma}$,

$$\begin{aligned}\mathrm{Ker}(\delta^k \mathcal{I}^1_{\pitchfork}(q)) &= \mathrm{Ker}\left(\delta^k \mathcal{I}^1(q) \cap \mathrm{ann}\left(T_q\hat{\Gamma}\right)\right) \\ &= \mathrm{Ker}(\delta^k \mathcal{I}^1(q)) + T_q\hat{\Gamma}.\end{aligned}$$

We have that $\mathrm{Ker}(\delta^k \mathcal{I}^1(q)) = \mathcal{D}_k(q)$ so,

$$\mathrm{Ker}(\delta^k \mathcal{I}^1_{\pitchfork}(q)) = \mathcal{D}_k(q) + T_q\hat{\Gamma}.$$

The result follows by applying Lemma IV.2. □

Corollary IV.4. *Let $p \in \hat{\Gamma}$ be a regular point of the codistributions $\delta^k \mathcal{I}^1_{\pitchfork}$ for all $k \geq 1$. Then there exists an open set $U \subset M$ containing p such that for all $q \in U \cap \hat{\Gamma}$,*

$$\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)) = \mathcal{S}_{k-1}(q) + T_q\hat{\Gamma}.$$

Proof. Let $p \in \hat{\Gamma}$ be a regular point of the codistributions $\delta^k \mathcal{I}^1_{\pitchfork}$ for $k \geq 1$. Let $U \subset M$ be the open set containing p for which the codistributions have constant dimension. Then for any $q \in U \cap \hat{\Gamma}$,

$$\begin{aligned}\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)) &= \mathrm{Ker}(\delta^k \mathcal{I}^1_{\pitchfork}(q)) \cap \mathrm{Ker}(\mathrm{span}\left\{\mathrm{dt}\right\}(q)) \\ &= \left(\mathcal{S}_{k-1}(q) + T_q\hat{\Gamma}\right) \cap \mathrm{Ker}(\mathrm{span}\left\{\mathrm{dt}\right\}(q)).\end{aligned}$$

By the way we lift the set Γ to $\hat{\Gamma}$, no element of $T_q\hat{\Gamma}$ has a component in $\frac{\partial}{\partial t}$. As a result, $T_q\hat{\Gamma} \subseteq \mathrm{Ker}(\mathrm{span}\left\{\mathrm{dt}\right\}(q))$. This implies,

$$\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)) = \mathcal{S}_{k-1}(q) \cap \mathrm{Ker}(\mathrm{span}\left\{\mathrm{dt}\right\}(q)) + T_q\hat{\Gamma}.$$

By the choice of lift used for distributions, we have that $\mathcal{S}_{k-1}(q) \subseteq \mathrm{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$. So we may conclude $\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)) = \mathcal{S}_{k-1}(q) + T_q\hat{\Gamma}$. □

We need two more technical propositions before proving the main result.

Proposition IV.5. *For all $i \geq 0$, $\mathcal{S}_i + [\mathcal{S}_i, \mathcal{S}_i] \subseteq \mathrm{inv}(\mathcal{G}_i)$.*

The proof for Proposition IV.5 is omitted. It follows an inductive proof that uses the definition of $\mathcal{S}_i$.

Proposition IV.6. *Let $\hat{\Gamma}$ be a closed, embedded submanifold of M and let $p \in \hat{\Gamma}$. If for all $j \geq 0$,*

$$\mathcal{S}_j(p) + T_p\hat{\Gamma} = \mathrm{inv}(\mathcal{G}_j(p)) + T_p\hat{\Gamma}, \tag{12}$$

then for all $k \geq 0$,

$$\mathcal{G}_k(p) + T_p\hat{\Gamma} = \mathrm{inv}(\mathcal{G}_k(p)) + T_p\hat{\Gamma}. \tag{13}$$

The proof of Proposition IV.6 is in the appendix.

V. Proof of Main Result

We now prove Theorem III.4. Conditions (a) and (b) will be shown to be equivalent to single-input variant of Theorem 3.2 in [4]. Suppose conditions (a) and (b) of Theorem III.4. Let U be an open set of M containing p on which $\delta^k \mathcal{I}^1$ and $\delta^k \mathcal{I}^1_{\pitchfork}$ are constant dimensional. Then, by Corollary IV.4 and Proposition IV.5, for all $q \in U \cap \hat{\Gamma}$,

$$\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)) = \mathcal{S}_{k-1}(q) + T_q\hat{\Gamma} \subseteq \mathrm{inv}(\mathcal{G}_{k-1}(q)) + T_q\hat{\Gamma}.$$

Since $\delta^k \hat{\mathcal{I}}_{\pitchfork}$ is a differential ideal, the distribution annihilated by $\delta^k \hat{\mathcal{I}}_{\pitchfork}$ is involutive. Therefore, since $\mathcal{G}_{k-1}(q) \subseteq \mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q))$, we may conclude that $\mathrm{inv}(\mathcal{G}_{k-1}(q)) \subseteq \mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)$. In particular we find,

$$\mathrm{inv}(\mathcal{G}_{k-1}(q)) + T_q\hat{\Gamma} \subseteq \mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}(q)).$$

As a result, for $q \in U \cap \hat{\Gamma}$ and for all $k \geq 1$,

$$\mathrm{inv}(\mathcal{G}_{k-1}(q)) + T_q\hat{\Gamma} = \mathcal{S}_{k-1}(q)) + T_q\hat{\Gamma}.$$

We can see now that the premise of Proposition IV.6 is satisfied so we may conclude that for all $j \geq 0$,

$$\mathrm{inv}(\mathcal{G}_j(q)) + T_q\hat{\Gamma} = \mathcal{G}_j(q) + T_q\hat{\Gamma}.$$

But this is just condition (b) of Theorem II.1 on the lifted manifold M. Projecting down to $\mathbb{R}^n$ using the embedding from Definition IV.1 we conclude that condition (b) of Theorem II.1 holds.

We turn to the controllability condition, condition (a) of Theorem II.1. Observe by Corollary IV.4,

$$\begin{aligned}\{0\} \times \mathbb{R} \times \mathbb{R}^n &\cong \mathrm{Ker}(\delta^{n-n^\star} \hat{\mathcal{I}}_{\pitchfork}(p)) \\ &= \mathcal{S}_{n-n^\star-1}(p)) + T_p\hat{\Gamma}.\end{aligned}$$

Then using the property proven above,

$$\begin{aligned}\{0\} \times \mathbb{R} \times \mathbb{R}^n &= \mathrm{inv}(\mathcal{G}_{n-n^\star-1}(p)) + T_p\hat{\Gamma} \\ &= \mathcal{G}_{n-n^\star-1}(p) + T_p\hat{\Gamma}.\end{aligned}$$

Projecting down to $\mathbb{R}^n$ once again we obtain

$$\mathbb{R}^n \cong +T_{x_0}\Gamma + G_{n-n^\star-1}(x_0).$$

And so both conditions of Theorem II.1 hold.

Conversely, suppose the conditions of Theorem II.1. Then there exists a distribution Δ defined on some open set $U \subset \mathbb{R}^n$ containing x_0 such that the conditions of Theorem II.2 are satisfied. We use Δ to define a distribution $\mathcal{V}\colon W \to TW$ on an open set of M containing $p = (t_0, 0, x_0)$ in a manner such that the conditions of Theorem II.2 remain true on W using $\mathcal{G}_k$ in place of G_k. In particular, we define

$$\mathcal{V} := \mathrm{span}\left\{\frac{\partial}{\partial u}\right\} + \mathcal{C},$$

where $\mathcal{C}$ is the lift of Δ into M. Define $W := \mathbb{R} \times \mathbb{R} \times U$ which is an open set on M containing p. Let $q \in W \cap \hat{\Gamma}$. Now use Corollary IV.4 to get,

$$\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork})(q) = \mathcal{S}_{k-1}(q) + T_q\hat{\Gamma}.$$

Then apply Proposition IV.5 and use the properties of $\mathcal{V}$,

$$\begin{aligned}\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork})(q) &\subseteq \mathrm{inv}(\mathcal{G}_{k-1}(q)) + T_q\hat{\Gamma}\\ &= \mathrm{inv}(\mathcal{G}_{k-1}(q)) + \mathcal{V}\\ &= \mathcal{G}_{k-1}(q) + \mathcal{V}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}) &\subseteq \mathcal{G}_{k-1} + \mathcal{V}\\ &\subseteq \mathcal{S}_{k-1} + T_x\hat{\Gamma}\\ &= \mathrm{Ker}(\delta^k \hat{\mathcal{I}}^1_{\pitchfork}).\end{aligned}$$

Therefore $\delta^k \hat{\mathcal{I}}^1_{\pitchfork} = \mathrm{ann}(\mathcal{G}_{k-1} + \mathcal{V})$ is a differential ideal (by the involutivity of the distribution) and condition (b) holds. Condition (a) holds since on $\hat{\Gamma}$ we have by condition (iii) of Theorem II.2,

$$\{0\} \times \mathbb{R} \times \mathbb{R}^n = \mathcal{G}_{n-n^*-1}(p) + T_p\hat{\Gamma}$$

By involutivity,

$$\begin{aligned}\{0\} \times \mathbb{R} \times \mathbb{R}^n &= \mathrm{inv}(\mathcal{G}_{n-n^*-1}(p)) + T_p\hat{\Gamma}\\ &= \mathcal{S}_{n-n^*-1}(p) + T_p\hat{\Gamma}\\ &= \mathrm{Ker}(\delta^{n-n^*}\mathcal{I}^1_{\pitchfork})(p).\end{aligned}$$

As a result condition (a) holds and so Theorem III.4 is implied by Theorem II.1.

VI. Illustrative Example

Consider the kinematic model of a rear-wheel drive car-like robot with a fixed forward speed. This is a system of the form (2) with $n = 4$ and $f(x) = (\cos(x_3), \sin(x_3), \tan(x_4), 0)^\intercal$, $g(x) = (0,0,0,1)$. Here the steering angle x_4 is confined to the open interval $(-\pi/2, -\pi/2)$. Setting $M = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^4$, the ideal generated by the Pfaffian system associated with the car-like robot is $\mathcal{I} = \{\omega_1, \omega_2, \omega_3, \omega_4\}_{\mathsf{alg}}$ where $\omega_1 = \mathrm{d}x_1 - \cos(x_3)\mathrm{d}t$, $\omega_2 = \mathrm{d}x_2 - \sin(x_3)\mathrm{d}t$, $\omega_3 = \mathrm{d}x_3 - \tan(x_3)\mathrm{d}t$, and $\omega_4 = \mathrm{d}x_4 - u\mathrm{d}t$. It was established in [14], in the context of path following, that this system is locally transversally feedback linearizable in a neighbourhood of any point of the set $\Gamma = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 = 1, x_1\cos x_3 + x_2 \sin x_3 = 0, 1 + \tan(x_4) = 0\}$ with dimension $n^\star = 1$. We show that, as expected, our dual conditions confirm this result.

Since $\hat{\Gamma} = \{0\} \times \mathbb{R} \times \Gamma$ is the zero level set of a smooth function, $\mathrm{ann}(T_p\hat{\Gamma})$ is spanned by the differential of the function. In particular $\mathrm{ann}(T_p\hat{\Gamma})$ is the image of

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & x_1 & x_2 & 0 & 0\\ 0 & 0 & \cos(x_3) & \sin(x_3) & x_2\cos(x_3) - x_1\sin(x_3) & 0\\ 0 & 0 & 0 & 0 & 0 & \sec^2(x_4).\end{bmatrix}$$

The transversal ideals are found – using MATLAB – by intersecting the derived flags of $\mathcal{I}_\Sigma$ with $\mathrm{ann}(T_p\hat{\Gamma})$. The basis is adapted so that $\mathcal{I}_\pitchfork = \{\omega_1, \omega_2, \omega_3\}_{\mathsf{alg}}$, $\delta^1\mathcal{I}_\pitchfork = \{\omega_1, \omega_2\}_{\mathsf{alg}}$, $\delta^2\mathcal{I}_\pitchfork = \{\omega_1\}_{\mathsf{alg}}$ and $\delta^3\mathcal{I}_\pitchfork = \{0\}$ where

$$\begin{aligned}\omega_1 &:= -(x_1\cos x_3 - x_2 \sin x_3)\mathrm{d}t + x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2,\\ \omega_2 &:= [-\tan(x_4)(x_2\cos(x_3) - x_1\sin(x_3)) - 1]\,\mathrm{d}t\\ &\quad + cos(x3)\mathrm{d}x_1 + sin(x3)\mathrm{d}x_2\\ &\quad + [x_2\cos(x_3) - x_1\sin(x_3)]\,\mathrm{d}x_3\\ \omega_3 &:= [\tan(x_4)^2(x_1\cos(x_3) + x_2\sin(x_3))\\ &\qquad - u(\tan(x_4)^2 + 1)(x_2\cos(x_3)\\ &\qquad - x_1\sin(x_3))]\mathrm{d}t\\ &\quad + \tan(x_4)\sin(x_3)\mathrm{d}x_1 + \tan(x_4)\cos(x_3)\mathrm{d}x_2\\ &\quad - [\tan(x_4)(x_1\cos(x_3) + x_2\sin(x_3))]\mathrm{d}x_3\\ &\quad + [(\tan(x_4)^2 + 1)(x_2\cos(x_3) - x_1\sin(x_3))]\mathrm{d}x_4\end{aligned}$$

Since $n - n^\star = 3$, we immediately see that condition (a) of Theorem III.4 holds at every point of $\hat{\Gamma}$.

To check condition (b) of Theorem III.4, we need to check the above transversal ideals $\mathcal{I}_\pitchfork$, $\delta\mathcal{I}_\pitchfork$ and $\delta^2\mathcal{I}_\pitchfork$, when augmented with $\mathrm{d}t$, are differential ideals. This can be done using the properties of wedge products in combination with Frobenius' Theorem [2, Proposition 1.1, Theorem 1.1]. Due to space limitations the explicit expressions are not shown, but it is straightforward to check that they are closed.

Since both conditions of Theorem III.4 are satisfied we conclude that car-like robot is locally transverse feedback linearizable around any point on Γ as expected.

Appendix

Proposition A.1. *For all $i \geq 0$, $[\mathcal{D}_0, \mathcal{G}_i] \subseteq \mathcal{S}_{i+1}$.*

Proof. Since $\mathcal{G}_i$ is tangent to every submanifold of the form $\{t\} \times \{u\} \times \mathbb{R}^n$, every vector fields in $\mathcal{G}_i$ commutes with $\frac{\partial}{\partial u}$. In light of the definition of $\mathcal{D}_0$ in (9a), this means we need only consider Lie brackets between vector fields in $\mathcal{G}_i$ and $\frac{\partial}{\partial t} + \sum_{i=1}^n (f_i + g_i u)\frac{\partial}{\partial x_i} \in \mathcal{D}_0$.

For $i = 0$, we note that $\mathcal{G}_0$ is spanned by $\sum_{i=0}^n g_i(x)\frac{\partial}{\partial x_i}$ so $[\mathcal{D}_0, \mathcal{G}_0]$ is spanned by,

$$\left[\sum_{i=0}^n f_i \frac{\partial}{\partial x_i}, \sum_{i=0}^n g_i \frac{\partial}{\partial x_i}\right].$$

which is the lift of $\mathrm{ad}_f g$. This vector field belongs to $\mathcal{G}_1 \subseteq \mathcal{S}_1$.

Suppose, by way of induction that, for some $i \geq 1$, $[\mathcal{D}_0, \mathcal{G}_{i-1}] \subseteq \mathcal{S}_i$ and consider $[\mathcal{D}_0, \mathcal{G}_i]$. By the inductive hypothesis we have that it suffices to show that

$$\left[\frac{\partial}{\partial t} + \sum_{j=0}^n (f_j + g_j u)\frac{\partial}{\partial x_j}, \sum_{j=0}^n (\mathrm{ad}_f^i g)_j \frac{\partial}{\partial x_j}\right].$$

belongs to $\mathcal{S}_{i+1}$. By linearity of the brackets, the above equals

$$\sum_{j=0}^n (\mathrm{ad}_f^{i+1} g)_j \frac{\partial}{\partial x_j} + \left[\sum_{j=0}^n g_j \frac{\partial}{\partial x_j}, \sum_{j=0}^n (\mathrm{ad}_f^i g)_j \frac{\partial}{\partial x_j}\right] u.$$

The former term is in $\mathcal{G}_{i+1}$ which is a subset of $\mathcal{S}_{i+1}$. The latter term belongs to $[\mathcal{G}_i, \mathcal{G}_i] \subseteq [\mathcal{S}_i, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1}$. □

Proposition A.2. *For all $i \geq 0$, $[\mathcal{D}_0, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1}$.*

Proof. The base case follows from applying Proposition A.1 and the fact that $\mathcal{S}_0 = \mathcal{G}_0$. By way of induction, suppose that for some $i \geq 1$,

$$[\mathcal{D}_0, \mathcal{S}_{i-1}] \subseteq \mathcal{S}_i \tag{14}$$

Consider $[\mathcal{D}_0, \mathcal{S}_i]$. Use (8b) on $[\mathcal{D}_0, \mathcal{S}_i]$ and linearity to find,

$$\begin{aligned}[\mathcal{D}_0, \mathcal{S}_i] &= [\mathcal{D}_0, \mathcal{G}_i + \mathcal{S}_{i-1} + [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}]] \\ &= [\mathcal{D}_0, \mathcal{G}_i] + [\mathcal{D}_0, \mathcal{S}_{i-1}] + [\mathcal{D}_0, [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}]] .\end{aligned}$$

By Proposition A.1, $[\mathcal{D}_0, \mathcal{G}_i] \subseteq \mathcal{S}_{i+1}$, and so,

$$[\mathcal{D}_0, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1} + [\mathcal{D}_0, \mathcal{S}_{i-1}] + [\mathcal{D}_0, [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}]] .$$

By applying the inductive hypothesis (14),

$$[\mathcal{D}_0, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1} + [\mathcal{D}_0, [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}]] . \tag{15}$$

By the Jacobi identity,

$$\begin{aligned}[\mathcal{D}_0, [\mathcal{S}_{i-1}, \mathcal{S}_{i-1}]] &= [\mathcal{S}_{i-1}, [\mathcal{D}_0, \mathcal{S}_{i-1}]] + [\mathcal{S}_{i-1}, [\mathcal{S}_{i-1}, \mathcal{D}_0]] \\ &= [\mathcal{S}_{i-1}, [\mathcal{D}_0, \mathcal{S}_{i-1}]] .\end{aligned}$$

Therefore (15) becomes,

$$[\mathcal{D}_0, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1} + [\mathcal{S}_{i-1}, [\mathcal{D}_0, \mathcal{S}_{i-1}]] .$$

It follows from the inductive hypothesis (14) that,

$$[\mathcal{D}_0, \mathcal{S}_i] \subseteq \mathcal{S}_{i+1} + [\mathcal{S}_{i-1}, \mathcal{S}_i] .$$

The result follows by the definition of $\mathcal{S}_{i+1}$. □

Proposition A.3. *For all $i \geq 0$, $\mathcal{G}_i \subseteq \mathcal{D}_{i+1}$.*

Proof. By definition, $\mathcal{D}_1 = \mathcal{D}_0 + [\mathcal{D}_0, \mathcal{D}_0]$ and $[\mathcal{D}_0, \mathcal{D}_0]$ is spanned by,

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{n} (f_i + g_i u) \frac{\partial}{\partial x_i}, \frac{\partial}{\partial u}\right] = \sum_{i=1}^{n} -g_i \frac{\partial}{\partial x_i}.$$

Therefore $\mathcal{G}_0 = [\mathcal{D}_0, \mathcal{D}_0] \subset \mathcal{D}_1$.

Suppose now, by way of induction, that for some $i \geq 0$, $\mathcal{G}_i \subseteq \mathcal{D}_{i+1}$. The distribution $\mathcal{G}_{i+1}$ equals the direct sum of $\mathcal{G}_i$ and the lifted vector field $\mathrm{ad}_f^{i+1} g.$. We know by the inductive hypothesis and by (9b) that $\mathcal{G}_i \subseteq \mathcal{D}_{i+1} \subseteq \mathcal{D}_{i+2}$. As a result, it suffices to check that the lift of $\mathrm{ad}_f^{i+1} g$ belongs to $\mathcal{D}_{i+2}$. To this end, observe that the lift of $\mathrm{ad}_f^{i} g$ and $\mathcal{D}_0$ is spanned by,

$$\sum_{j=0}^{n} (\mathrm{ad}_f^{i+1} g)_j \frac{\partial}{\partial x_j} + \left[\sum_{j=0}^{n} g_j \frac{\partial}{\partial x_j}, \sum_{j=0}^{n} (\mathrm{ad}_f^{i} g)_j \frac{\partial}{\partial x_j}\right] u. \tag{16}$$

The vector field (16) belongs to $\mathcal{D}_{i+2}$ by the induction hypothesis. Furthermore, the latter term of (16) is also in $\mathcal{D}_{i+2}$ since it belongs to $[\mathcal{G}_i, \mathcal{G}_i]$ which, again by the inductive hypothesis, belongs to $\mathcal{D}_{i+2}$. It follows that the former term is in $\mathcal{D}_{i+2}$. Hence $\mathcal{G}_{i+1} \subseteq \mathcal{D}_{i+2}$. □

Proof of Proposition IV.6. For $k = 0$ equality holds by definition since $\mathcal{S}_0 = \mathcal{G}_0$.

Now by way of induction suppose that (13) holds for some $k \geq 0$. By hypothesis we have that, for all $j \geq 0$, the equality (12) holds. Now consider $\mathrm{inv}(\mathcal{G}_{k+1}(p)) + T_p\Gamma$. By (12),

$$\mathrm{inv}(\mathcal{G}_{k+1}(p)) + T_p\Gamma \subseteq \mathcal{S}_{k+1}(p) + T_p\Gamma.$$

Use (8b) to expand $\mathcal{S}_{k+1}$,

$$\mathrm{inv}(\mathcal{G}_{k+1}) + T_p\Gamma \subseteq \mathcal{G}_{k+1} + \mathcal{S}_k + [\mathcal{S}_k, \mathcal{S}_k] + T_p\Gamma.$$

Apply Proposition IV.5,

$$\mathrm{inv}(\mathcal{G}_{k+1}) + T_p\Gamma \subseteq \mathcal{G}_{k+1} + \mathrm{inv}(\mathcal{G}_k) + T_p\Gamma.$$

Invoke the inductive hypothesis,

$$\mathrm{inv}(\mathcal{G}_{k+1}) + T_p\Gamma \subseteq \mathcal{G}_{k+1} + \mathcal{G}_k + T_p\Gamma,$$

and finally use the fact that $\mathcal{G}_k \subseteq \mathcal{G}_{k+1}$ to arrive at,

$$\mathrm{inv}(\mathcal{G}_{k+1}) + T_p\Gamma \subseteq \mathcal{G}_{k+1} + T_p\Gamma.$$

The reverse containment follows from the definition of the involutive closure. □

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