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A Local Solution to the Output Regulation Problem for Sampled-Data Systems on Commutative Matrix Lie Groups

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***Abstract*— We present a smooth nonlinear control law for a kinematic plant on commutative matrix Lie groups that achieves regulation, if the state tracking and estimation errors are initialized in a suitable neighbourhood of identity. We show that in exponential coordinates, the closed-loop dynamics are linear. Our control law uses output feedback; to this end, we propose an almost-globally defined state estimator.**

I. Introduction

We examine the regulator problem for systems that evolve on matrix Lie groups, using a sampled-data feedback control configuration. The regulator problem is one of the central problems in control theory. It combines stabilization with reference tracking and disturbance rejection. The regulator problem has been analyzed in great detail in the linear case [1] and the nonlinear case, in both continuous- [2] and discrete-time [3]. More recently, the continuous-time regulator problem for systems on Lie groups has received some attention. Almost-global output regulation was achieved for a class of systems evolving on $\mathsf{SE}(n)$ by identifying a separation principle and using output feedback [4]. Local output regulation was also achieved for a class of systems evolving on general Lie groups with outputs in a homogeneous vector space and exostates evolving on a compact set [5].

There are many dynamical systems whose state spaces are naturally modelled as matrix Lie groups. Networks of oscillators can be modelled on $\mathsf{SO}(2)^n$ [6]. The group $\mathsf{SE}(3)$ captures the dynamics of rigid bodies in space, such as underwater vehicles [7] and UAVs [8]. Planar motion of robots can be modelled on $\mathsf{SE}(2)$ [9]. Quantum systems evolve on the unitary groups $\mathsf{U}(n)$ and $\mathsf{SU}(n)$ [10]. Even the noise responses of some circuits evolve on Lie groups [11].

In general, Lie groups are not vector spaces; they are smooth manifolds endowed with a group structure, which facilitates control design in global coordinates as well as local design and analysis using the rich theory of control on manifolds. The Lie structure has been leveraged, for example, for motion tracking on $\mathsf{SE}(3)$ [12], and the control of UAV [13] and spacecraft [14] orientation on $\mathsf{SO}(3)$.

We study such systems in the sampled-data configuration, i.e., a continuous-time plant and a discrete-time controller. This configuration is ubiquitous in applied control [15], where controllers are implemented on embedded devices. The main sampled-data design approach in practice is emulation. A continuous-time controller is designed for the continuous-time plant, but is implemented using an approximate discretization. If the sampling-period is sufficiently small, then the actual closed-loop system is stable. This technique has two key shortcomings [16]: 1) it may not be possible for a given approximate discretization method, e.g., Euler's method; 2) it relies on fast sampling, which may not be possible due to hardware limitations. For example, a UAV's translation relative to a reference can be computed using machine vision, where the speed of sampling is limited by the frame rate of the camera, e.g., 25 Hz [17].

For LTI systems, the continuous-time state trajectory can be solved exactly, thereby admitting a discrete-time model that matches the continuous-time behaviour at the sampling instants. This enables analysis and design to be done entirely in discrete-time, hence the controller can be implemented exactly. This technique is called direct design. Its key advantage is that it allows guarantees about performance to be made at the sampling instants, such as stability. This is not the case for most nonlinear systems, which do not admit exact discretizations. The efficacy of direct design depends on the accuracy of the discretization. In one case study, sampling periods as low as 30 μs, combined with an Euler discretization of a synchronous machine plant, yielded an ineffectual model predictive controller [15]. Right (left)-invariant systems on matrix Lie groups are an exception, in that for piecewise constant inputs, the state trajectories have exact solutions [18].

Sampled-data control of systems on Lie groups has heretofore been subject to limited formal study, including the authors' works on passivity [19] and synchronization [20] on matrix Lie groups. Controllability using multirate piecewise constant inputs was investigated for matrix Lie groups in [21] and control-affine systems whose Lie algebra of vector fields is nilpotent in [22]. Discrete-time control has also received some attention [23]. The closely related class of bilinear systems has also been studied in the discrete-time [18] and sampled-data [24] settings.

We present a control law that solves the regulator problem for sampled-data systems on commutative matrix Lie groups. If the state tracking and estimation errors are initialized in a suitable neighbourhood of identity, then regulation is achieved.

A. Notation and Terminology

For $n \in \mathbb{N}$, let $\mathbb{N}_n := \{1, \ldots, n\}$. Given a matrix $M \in \mathbb{C}^{m \times n}$, $M^\top$ is its (non-Hermitian) transpose, $\|M\|$ is

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its Frobenius norm, $M^\dagger$ is its Moore-Penrose pseudoinverse, and $\mathrm{vec}(M) := \begin{bmatrix} M_{11} & \cdots & M_{m1} & M_{21} & \cdots & M_{mn} \end{bmatrix}^\top \in \mathbb{C}^{mn}$; if $m = n$, then $\sigma(M)$ is its spectrum. Given an ordered set of matrices $\{M_1, \dots, M_q\}$, define $\bar{M} := \begin{bmatrix} \mathrm{vec}(M_1) & \cdots & \mathrm{vec}(M_q) \end{bmatrix} \in \mathbb{C}^{mn \times q}$. Let $\mathbf{0}_n \in \mathbb{R}^n$ and $\mathbf{0}_{m\times n} \in \mathbb{R}^{m \times n}$ denote the column and matrix of zeros, respectively. When a discrete-time signal appears in a continuous-time expression, it to be understood as having passed through an ideal zero order hold.

II. Sampled-Data Regulator Problem

We consider a plant modelled by the differential equation

$$\dot{X} = \left(A + \sum_{i=1}^{m} B_i u_i + \sum_{i=1}^{r_d} Q_{di} w_{di} + \sum_{i=1}^{r_c} Q_{ci} w_{ci} \right) X \quad (1)$$

with measured output

$$Y = \exp\left(C + \sum_{i=1}^{r_d} D_{di} w_{di} + \sum_{i=1}^{r_c} D_{ci} w_{ci} \right) X. \quad (2)$$

We assume, as is typical, that the exogenous signals w_d, w_c evolve according to known dynamics, modelled as

$$w_d^+ = S_d w_d, \qquad \dot{w}_c = S_c w_c. \quad (3)$$

Here, $X \in \mathsf{G}$ where $\mathsf{G} \subset \mathsf{GL}(N, \mathbb{C})$ is an n-dimensional connected matrix Lie group over the complex field $\mathbb{C}$ which includes, as a special case, real matrix Lie groups. The matrices A, B_i, Q_{ci}, Q_{di}, C, D_{di}, and D_{ci} are elements of the Lie algebra $\mathfrak{g}$ of G, which is a vector space over a field $\mathbb{F}$ equal to either $\mathbb{C}$ or $\mathbb{R}$. The control input is $u \in \mathbb{F}^m$, the discrete- and continuous-time exostates are $w_d \in \mathbb{R}^{r_d}$ and $w_c \in \mathbb{R}^{r_c}$, respectively, and $S_d \in \mathbb{R}^{r_d \times r_d}$, $S_c \in \mathbb{R}^{r_c \times r_c}$.

Assumption 1. *The Lie group* G *is commutative.* ◀

Assumption 1 is restrictive, as all commutative matrix Lie groups are diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^{n-k}$, as discussed in Section III-B. We examine this class of Lie groups to facilitate this preliminary research in this area. However, there are systems of practical interest on such Lie groups, e.g., networks of oscillators on $\mathsf{SO}(2)^n \cong \mathbb{T}^n$ [6] and single-axis rigid body manoeuvres on $\mathbb{R} \times \mathsf{SO}(2) \cong \mathbb{R} \times \mathbb{T}$ [25].

Assumption 2. *The spectra of S_d and S_c lie outside the open unit disc and in the closed right half plane, respectively.* ◀

Assumption 2 is not restrictive. If S_d or S_c has stable eigenvalues, then the dynamics of (3) can be redefined as their restriction to the unstable modal subspaces [26, Chapter 2, §3] of S_d and S_c. The dynamics on the stable modal subspaces can be ignored, since tracking or rejecting a zero signal is equivalent to stability.

Equation (1) is a kinematic model of a system evolving on a matrix Lie group G, where the output (2) models the information that is available for feedback. The exosystem (3) comprises both discrete- and continuous-time subsystems. This enables modelling of, for example, physical plants that are subject to continuous-time disturbances, but are sent reference signals from a computer. The plant is assumed to be fully actuated in the sense that $\mathrm{span}_{\mathbb{F}}\{B_1, \dots, B_m\} = \mathfrak{g}$. We justify this assumption in Section V-A. We are interested in the sampled-data control of this system in which the control law is implemented on an embedded computer, which we explicitly model using the setup in Figure 1. The blocks H and S in Figure 1 are the ideal hold and sample operators, respectively. Sample and hold are, respectively, idealized models of A/D and D/A conversion.

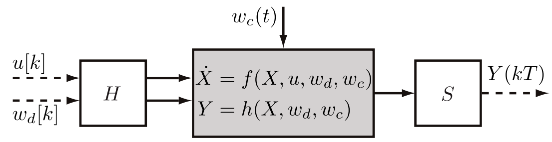

Fig. 1: Sampled-data plant on a matrix Lie group G.

Assumption 3. *The sample and hold blocks operate at the same period $T > 0$ and are synchronized.* ◀

Under Assumption 3, letting $X[k] := X(kT)$, $u[k] := u(kT)$, and $w_c[k] := w_c(kT)$, the plant (1) and exosystem (3) have exact discretizations, as we will prove in Section IV:

$$X^+ = \exp\Bigg(TA + T\sum_{i=1}^{m} B_i u_i + T\sum_{i=1}^{r_d} Q_{di} w_{di} + \sum_{i=1}^{r_c} Q_{ci} \mathrm{e}_i^\top \int_0^T e^{\tau S_c} \mathrm{d}\tau \, w_c \Bigg) X, \quad (4)$$

where e_i is the ith canonical basis vector of $\mathbb{R}^{r_c}$, and

$$\begin{bmatrix} w_d^+ \\ w_c^+ \end{bmatrix} = \underbrace{\begin{bmatrix} S_d & \mathbf{0}_{r_d \times r_c} \\ \mathbf{0}_{r_c \times r_d} & e^{TS_c} \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} w_d \\ w_c \end{bmatrix}}_{w}. \quad (5)$$

A. The Sampled-Data Regulator Problem

The goal of the regulator problem is to drive a regulation quantity to identity. Define the regulation quantity

$$Z = \exp\left(F + \sum_{i=1}^{r_d} G_{di} w_{di} + \sum_{i=1}^{r_c} G_{ci} w_{ci} \right) Y, \quad (6)$$

where $F, G_{di}, G_{ci} \in \mathfrak{g}$.

Local Output Regulation on Matrix Lie Groups : Given a plant with continuous-time dynamics (1), output (2), regulation quantity (6), sampling period $T > 0$, and exosystem (3), find, if possible, a discrete-time control law u, which depends on only the output Y, such that for all initial conditions $X(0)$, $w_c(0)$, $w_d[0]$, in a neighbourhood of the identity in $\mathsf{G} \times \mathbb{R}^{r_d} \times \mathbb{R}^{r_c}$, $Z[k] \to I_N$ as $k \to \infty$. ♦

Such a u is said to solve **the regulator problem with output information**. If instead the input u has access to X, w_d, and w_c, then u is said to solve the **the regulator problem with full information**. We will first solve the latter using a controller of the form $u := \Gamma(X, w_d, w_c) + \Psi(w_d, w_c)$. We then extend our result to design an output feedback controller using a dynamic state estimator.

III. Preliminaries

A. The matrix logarithm

Theorem III.1 ([27, Theorem 1.31]). *If $X \in \mathbb{F}^{n\times n}$ has no nonpositive real eigenvalues, then there exists a unique $A \in \mathbb{F}^{n\times n}$, whose spectrum lies in $\{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}$, such that $\exp(A) = X$.*

The matrix A in Theorem III.1 is the **principal logarithm of** X and is denoted by $\mathrm{Log}(X)$. If $\|X - I\| < 1$, then $\mathrm{Log}(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(X - I)^k$. The principal logarithm is the inverse of the matrix exponential, but only on a subset $U \subset \mathsf{GL}(N, \mathbb{C})$ containing the identity. For example, on $\mathsf{GL}(N, \mathbb{C})$, Log is well-defined on $U = \{\exp(A) \in \mathsf{GL}(N, \mathbb{C}) : \|A\| < \mathrm{Log}(2)\}$, but on $\mathsf{SO}(n)$, $U = \{R \in \mathsf{SO}(n) : -1 \notin \sigma(R)\}$, even though $\|R\| = \sqrt{n}$ for all $R \in \mathsf{SO}(n)$.

B. Exponential Coordinates and One-Parameter Subgroups

Definition III.2 (One-Parameter Subgroup). *Given a Lie group G, a one-parameter subgroup is a continuous morphism of groups $\phi : \mathbb{R} \to \mathsf{G}$.*

To generalize the concept of one-parameter subgroups to higher dimensional manifolds, we consider **generalized cylinders**. A generalized cylinder is an n-dimensional manifold that is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$. Such a diffeomorphism exists if and only if there exist n commutative and everywhere-linearly-independent vector fields on the manifold [28, §49]. If the manifold is a Lie group G, then this simplifies to the requirement that its Lie algebra $\mathfrak{g}$ have a commutative basis.

Let G be such a manifold and fix a commutative basis $H_1, \dots, H_n$ for its Lie algebra $\mathfrak{g}$. Consider the one-parameter groups $\phi_i : \mathbb{R} \to \mathsf{G}$ associated with each H_i. The image of $\phi(x_1, \dots, x_n) := \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)$ is G. Without loss of generality, let ϕ_i, $i \in \mathbb{N}_k$, $0 \le k \le n$ have nonzero kernel, and let ϕ_i, $i \in \{k+1, \dots, n\}$ have zero kernel.

Fixing such a basis, the Log map induces local coordinates on $\mathsf{G} \cap U$. Given $X \in \mathsf{G}$, by commutativity of $H_1, \dots, H_n$, $X = \exp(x_1 H_1)\cdots\exp(x_n H_n) = \exp(x_1 H_1 + \cdots + x_n H_n)$. If $X \in \mathsf{G} \cap U$, then $\mathrm{Log}(X) = x_1 H_1 + \cdots + x_n H_n$. Then, by linear independence of $H_1, \dots, H_n$, $x_1, \dots, x_n$ can be uniquely determined, yielding local coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n$. Thus, a Lie group G can be locally identified with an open subset of the vector space $\mathbb{R}^n$ containing the origin. By commutativity of G, these local coordinates coincide with the familiar exponential coordinates of both the first and second kinds [29, Remarks 5.33.1].

C. Properties of the composed flow

The map $\phi : \mathbb{R}^n \to \mathsf{G}$, defined in the previous section, is critical to our analysis throughout this paper. In this section, we establish important properties of ϕ when G is a generalized cylinder.

Proposition III.3. *If G is a generalized cylinder, then $\phi : \mathbb{R}^n \to \mathsf{G}$ is a morphism of groups.*

Proof. Let $x_i = (x_i^{(1)}, \dots, x_i^{(n)}) \in \mathbb{R}^n$ and $x_j = (x_j^{(1)}, \dots, x_j^{(n)}) \in \mathbb{R}^n$, where $\phi(x_i) = X_i$ and $\phi(x_j) = X_j$. By commutativity of $H_1, \dots, H_n$, $\phi(x_i + x_j) = \prod_{k=1}^n \exp\left((x_i^{(k)} + x_j^{(k)})H_k\right) = \prod_{k=1}^n \exp\left(x_i^{(k)} H_k\right) \prod_{k=1}^n \exp\left(x_j^{(k)} H_k\right) = \phi(x_i)\phi(x_j)$. □

To facilitate discussion and greatly simplify presentation, we shall tacitly make extensive use of the following simple lemma, whose proof follows from direct computation.

Lemma III.4. *Let $p, q \in \mathbb{N}$, $A_1, \dots, A_p \in \mathbb{C}^{q\times q}$, and $\xi \in \mathbb{C}^p$. Then $\sum_{k=1}^p A_k \xi_k = \mathrm{vec}^{-1}(\bar{A}\xi)$.*

When G is a matrix Lie group, the map ϕ is given by $\phi(x) = \exp\left(\mathrm{vec}^{-1}(\bar{H}x)\right)$.

Proposition III.5. *In a neighbourhood of identity $U \subset \mathsf{G}$, $\phi : \mathbb{R}^n \to \mathsf{G}$ has inverse*

$$\phi^{-1}(X) = \bar{H}^\dagger \mathrm{vec}(\mathrm{Log}(X)). \tag{7}$$

Proof. Since $\bar{H} : \mathbb{R}^n \to \mathrm{vec}(\mathfrak{g})$ is injective, a left inverse is $\bar{H}^\dagger : \mathrm{vec}(\mathfrak{g}) \to \mathbb{R}^n$, which is surjective. It follows from direct computation that (7) is a left inverse of ϕ.

To prove that (7) is a right inverse of ϕ, we require $\bar{H}^\dagger$ to be a right inverse of $\bar{H}$, which does not hold globally. Let $\bar{H}^{-1}\mathrm{vec}(\mathfrak{g}) := \{x \in \mathbb{R}^n : \bar{H}x \in \mathrm{vec}(\mathfrak{g})\}$ be the preimage of $\mathrm{vec}(\mathfrak{g})$ under $\bar{H}$, and let $\bar{H}^\dagger|\bar{H}^{-1}\mathrm{vec}(\mathfrak{g})$ be the restriction of $\bar{H}^\dagger$ to this subspace. Let $x \in \mathbb{R}^n$, then $\bar{H}\bar{H}^\dagger(\bar{H}x) = \bar{H}(\bar{H}^\dagger\bar{H})x = \bar{H}x$. Since two-sided inverses are unique, this implies that on the domain $\bar{H}^{-1}\mathrm{vec}(\mathfrak{g})$, the left inverse of $\bar{H}$, $\bar{H}^\dagger$, is also the right inverse. Therefore, $\bar{H}\bar{H}^\dagger|\bar{H}^{-1}\mathrm{vec}(\mathfrak{g})$ is identity. The rest follows from direct computation. □

Proposition III.6. *The map $\phi^{-1} : U \subset \mathsf{G} \to \mathbb{R}^n$ defined in (7) is a morphism of groups.*

Proof. The inverse of a morphism, if it exists, is itself a morphism [30, Chapter 1, Theorem 20]. □

IV. Local Dynamics

Proposition IV.1. *The continuous-time plant (1) has exact discretization (4).*

Proof. Consider a matrix ODE $\dot{X}(t) = M(t)X(t)$. If for all $t, t' \ge 0$, $M(t)M(t') = M(t')M(t)$, then for $t \ge 0$, $X(t) = \int_0^t M(\tau)\mathrm{d}\tau X(0)$ [31, §V]. Since $\mathfrak{g}$ is commutative, the vector field commutation property holds for all time. Solve (1) at $t = kT$ and $t = (k+1)T$. Factoring the former solution from the latter yields (4). □

Much of our analysis is facilitated by using local coordinates. Define $\widetilde{A} := \bar{H}^\dagger\bar{A}$, $\widetilde{C} := \bar{H}^\dagger\bar{C}$, $\widetilde{F} := \bar{H}^\dagger\bar{F}$, $\widetilde{Q}_d := \bar{H}^\dagger\bar{Q}_d$, and $\widetilde{Q}_c := \bar{H}^\dagger\bar{Q}_c\int_0^T e^{\tau S_c}\mathrm{d}\tau$, $\widetilde{D}_d := \bar{H}^\dagger\bar{D}_d$, $\widetilde{D}_c := \bar{H}^\dagger\bar{D}_c$, $\widetilde{G}_d := \bar{H}^\dagger\bar{G}_d$, and $\widetilde{G}_c := \bar{H}^\dagger\bar{G}_c$. For $X, Y, Z \in U$, we compute $y := \phi^{-1}(Y)$, $z := \phi^{-1}(Z)$, and the discrete-time dynamics of $x := \phi^{-1}(X)$:

$$\begin{aligned} x^+ &= T\widetilde{A} + T\widetilde{B}u + T\widetilde{Q}_d w_d + \widetilde{Q}_c w_c + x \\ y &= \widetilde{C} + \widetilde{D}_d w_d + \widetilde{D}_c w_c + x, \\ z &= (\widetilde{F} + \widetilde{C}) + (\widetilde{G}_d + \widetilde{D}_d) w_d + (\widetilde{G}_c + \widetilde{D}_c) w_c + x. \end{aligned} \tag{8}$$

V. Controllability and State Estimation

Our solution to the regulator problem uses output feedback. Thus, we require local detectability and stabilizability.

A. Controllability

We first define and characterize the notion of controllability for a discrete-time linear-affine system:

$$x^+ = Ax + Bu + c, \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x, c \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$.

Definition V.1. *System (9) is* ***controllable*** *if for all $x_0, \bar{x} \in \mathbb{R}^n$, given $x[0] = x_0$, there exist $k \in \mathbb{Z}_{\geq 0}$ and $u[0], \ldots, u[k-1] \in \mathbb{R}^m$ such that $x[k] = \bar{x}$.*

Proposition V.2. *System (9) is controllable if and only if the pair (A, B) is controllable.*

Proof. Routine calculation verifies $x[n] = A^n x[0] + \sum_{i=0}^{n-1} A^i B u[n-1-i] + \sum_{i=0}^{n-1} A^i c$. Thus, driving system (9) to $\bar{x} \in \mathbb{R}^n$ in n time steps is equivalent to driving the linear system $x^+ = Ax + Bu$ to $\bar{x} - \sum_{i=0}^{n-1} A^i c$ in n time steps. This is possible if and only if (A, B) is controllable. □

We now analyze the controllability of the sampled plant (4). Set the exostates w_d and w_c to zero. Then the local sampled dynamics (8) reduce to $x^+ = T\widetilde{A} + T\widetilde{B}u + x$.

Proposition V.3. *The sampled plant (4) is controllable if and only if $\operatorname{span}_{\mathbb{F}}\{B_1, \ldots, B_m\} = \mathfrak{g}$.*

Proof. In local coordinates (8), the state matrix is identity, so the PBH test for controllability reduces to $\operatorname{rank}(\widetilde{B}) = n$. This is satisfied if and only if $\operatorname{span}_{\mathbb{F}}\{B_1, \ldots, B_m\} = \mathfrak{g}$. □

B. State Estimation

Let $L_1 \in \mathbb{R}^{n\times n}$, $L_2 \in \mathbb{R}^{r_d\times n}$, $L_3 \in \mathbb{R}^{r_c\times n}$, $\widehat{X}, \widehat{Y} \in \mathsf{G}$, $\hat{w}_d \in \mathbb{R}^{r_d}$, $\hat{w}_c \in \mathbb{R}^{r_c}$, $e_y := \phi^{-1}(\widehat{Y}Y^{-1})$. Defining the output estimate $\widehat{Y} = \exp\left(C + \sum_{i=1}^{r_d} D_{di}\hat{w}_{di} + \sum_{i=1}^{r_c} D_{ci}\hat{w}_{ci}\right)\widehat{X}$, we propose a state estimator with dynamics:

$$\begin{aligned} \widehat{X}^+ &= \exp\Bigg(TA + T\sum_{i=1}^{m} B_i u_i + T\sum_{i=1}^{r_d} Q_{di}\hat{w}_{di} \\ &\qquad + \sum_{i=1}^{r_c} Q_{ci}\mathrm{e}_i^\top \int_0^T e^{\tau S_c}\mathrm{d}\tau\, \hat{w}_c + \operatorname{vec}^{-1}\left(\bar{H}L_1 e_y\right)\Bigg)\widehat{X} \\ \hat{w}_d^+ &= S_d\hat{w}_d + L_2 e_y \\ \hat{w}_c^+ &= e^{TS_d}\hat{w}_c + L_3 e_y. \end{aligned} \tag{10}$$

Given the output Y, we seek to choose L_1, L_2, and L_3, such that the estimation errors $E_x := \widehat{X}X^{-1} \to I_N$, $e_{w_d} := \hat{w}_d - w_d \to \mathbf{0}_{r_d}$, and $e_{w_c} := \hat{w}_c - w_c \to \mathbf{0}_{r_c}$ as $k \to \infty$.

In local coordinates, the output estimation error e_y and the dynamics of the state estimation error $e_x := \phi^{-1}(E_x)$ are $e_y = \widetilde{D}_c e_{w_c} + \widetilde{D}_d e_{w_d} + e_x$ and $e_x^+ = e_x + T\widetilde{Q}_d e_{w_d} + \widetilde{Q}_c e_{w_c} + L_1 e_y$, from which, direct calculation verifies

$$\begin{bmatrix} e_x^+ \\ e_{w_d}^+ \\ e_{w_c}^+ \end{bmatrix} = \Bigg(\overbrace{\begin{bmatrix} I_n & T\widetilde{Q}_d & \widetilde{Q}_c \\ \mathbf{0}_{r_d\times n} & S_d & \mathbf{0}_{r_d\times r_c} \\ \mathbf{0}_{r_c\times n} & \mathbf{0}_{r_c\times r_d} & e^{TS_c} \end{bmatrix}}^{A_o} + \underbrace{\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} I_n & \widetilde{D}_d & \widetilde{D}_c \end{bmatrix}}_{C_o}\Bigg) \begin{bmatrix} e_x \\ e_{w_d} \\ e_{w_c} \end{bmatrix},$$

which are the estimation error dynamics of a Luenberger observer. Thus, we directly apply linear observer theory. Since A_o is block diagonal, we have $\sigma(A_o) = \sigma(I_n) \sqcup \sigma(S_d) \sqcup \sigma(e^{TS_c})$. Thus, under Assumption 2, $\sigma(A_0)$ is entirely outside the open unit disc, so (C_o, A_o) is detectable if and only if it is observable.

Proposition V.4. *The pair (C_o, A_o) is observable if and only if the following matrix is full rank:*

$$\begin{bmatrix} \widetilde{D}_d(S_d - I_{r_d}) + T\widetilde{Q}_d & \widetilde{D}_c\left(e^{TS_c} - I_{r_c}\right) + \widetilde{Q}_c \\ \vdots & \vdots \\ \widetilde{D}_d\left(S_d^{n+r_d+r_c} - I_{r_d}\right) + T\widetilde{Q}_d \sum_{i=0}^{n+r_d+r_c-1} S_d^i & \widetilde{D}_c\left(\left(e^{TS_c}\right)^{n+r_d+r_c} - I_{r_c}\right) + \widetilde{Q}_c \sum_{i=0}^{n+r_d+r_c}\left(e^{TS_c}\right)^i \end{bmatrix}. \tag{11}$$

Proof. Given the observability matrix of (C_o, A_o), subtracting the top block row from those below yields a block triangular matrix with block diagonal elements I_n and (11). □

Corollary V.5. *If (C_o, A_o) is observable, then the following are equivalent:*

1) *there exists an L such that $A_o + LC_o$ is Schur;*
2) *the matrix (11) is full rank;*
3) *the point $(E_x, e_{w_d}, e_{w_c}) = (I_N, \mathbf{0}_{r_d}, \mathbf{0}_{r_c})$ is locally asymptotically stable.*

Thus, for suitable L and initial conditions, the state estimator dynamics (10) are well-defined for all $k \geq 0$.

VI. Solution to the Regulator Problem

In this section, we propose a solution to the regulator problem using state feedback. We also prove a separation principle, which allows the state feedback controller and state estimator to be designed independently. As with linear systems, this yields an output feedback controller that solves the regulator problem.

A. Regulation with Full Information

We design our controller in two parts: 1) a map $\Psi : \mathbb{R}^{r_d} \times \mathbb{R}^{r_c} \cong \mathbb{R}^{r_d+r_c} \to \mathbb{F}^m$ that makes the manifold $\{X \in \mathsf{G} : Z = I_N\}$ invariant; 2) a state feedback $K : \mathbb{R}^n \to \mathbb{F}^m$ that renders this manifold locally asymptotically stable.

We first find the state reference $\Pi : \mathbb{R}^{r_d} \times \mathbb{R}^{r_c} \cong \mathbb{R}^{r_d+r_c} \to \mathsf{G}$, which characterizes the trajectory of $X[k]$ yielding $Z[k] = I_N$. Substituting (2) into (6) and setting $Z = I_N$, we obtain $\Pi(w) = \exp\left(-(F+C) - \mathrm{vec}^{-1}\left(\begin{bmatrix}(\bar{G}_d + \bar{D}_d & \bar{G}_c + \bar{D}_c\end{bmatrix} w\right)\right)$, which has the local expression $\phi^{-1}(\Pi(w)) = -(\widetilde{F} + \widetilde{C}) - Gw$, where $G := \begin{bmatrix}\widetilde{G}_d + \widetilde{D}_d & \widetilde{G}_c + \widetilde{D}_c\end{bmatrix}$. For convenience, define $\pi := \phi^{-1} \circ \Pi$. We make the manifold characterized by $Z = I_N$ controlled-invariant by choosing $u = \Psi(w)$ such that if $X = \Pi(w)$, then $X^+ = \Pi(w^+)$. Setting $x^+ = \pi(w^+)$ and $x = \pi(w)$, we have $-T\widetilde{B}\Psi(w) = T\widetilde{A} + Qw + \pi(w) - \pi(w^+)$, where $Q := \begin{bmatrix}T\widetilde{Q}_d + \widetilde{Q}_c\end{bmatrix}$, which simplifies to $T\widetilde{A} + (Q + G(S - I_{r_d+r_c}))w$, which yields $\Psi(w) = -\frac{1}{T}\widetilde{B}^\dagger(T\widetilde{A} + (Q + G(S - I_{r_d+r_c}))w)$.

The map Ψ contains the term $-\widetilde{B}^\dagger\widetilde{A}$, which will cancel the affine term in the sampled plant dynamics (4). As will be seen in the proof of the following theorem, the remaining dynamics after application of Ψ are linear, thus, we augment Ψ with a stabilizing linear term.

The state-tracking error $E := X\Pi(w)^{-1}$ is of critical importance in the context of the regulator problem.

Theorem VI.1. *If* $\mathrm{span}_{\mathbb{F}}\{B_1, \ldots, B_m\} = \mathfrak{g}$*, then there exists* $K \in \mathbb{F}^{m\times n}$ *such that*

$$u = K\phi^{-1}(X\Pi(w)^{-1}) + \Psi(w) \tag{12}$$

solves the regulator problem with full information.

Proof. Using $e := \phi^{-1}(E) = x + \widetilde{F} + \widetilde{C} + Gw$,

$$\begin{aligned} e^+ &= T\widetilde{A} + T\widetilde{B}(Ke + \Psi(w)) + Qw + (e + \pi(w)) - \pi(w^+) \\ &= (I_n + T\widetilde{B}K)e. \end{aligned}$$

By Propositions V.2 and V.3, if $\mathrm{span}_{\mathbb{F}}\{B_1, \ldots, B_m\} = \mathfrak{g}$, then there exists a K such that $I_n + T\widetilde{B}K$ is Schur. □

From Theorem VI.1, it follows that there exists a positively invariant neighbourhood of the identity in G for suitable K. Thus, for $E[0]$ suitably close to the identity, the closed-loop dynamics are well-defined for all $k \geq 0$. Note that there is no restriction on the plant state X.

B. Regulation with Output Feedback

Theorem VI.2 (Separation Principle)**.** *If the control law* (12) *is implemented using the state estimates provided by the state estimator* (10)*, then the closed-loop tracking and estimation errors are locally asymptotically stable.*

Proof. Using $\hat{x} = e_x + x = e_x + e + \pi(w)$,

$$\begin{aligned} e^+ &= T\widetilde{A} + T\widetilde{B}(K(\hat{x} - \pi(\hat{w})) + \Psi(\hat{w})) + Qw + x - \pi(w^+) \\ &= (I_n + T\widetilde{B}K)e + T\widetilde{B}Ke_x \\ &\quad + T\widetilde{B}K(G - Q - G(S - I_{r_d+r_c}))e_w. \end{aligned}$$

Since $\frac{\partial}{\partial e_x}e^+ = I_N + T\widetilde{B}K$ and the state estimator dynamics (10) do not depend on u, the set of eigenvalues of the closed-loop system using output feedback is the union of the eigenvalues of the state estimator with the eigenvalues of the closed-loop system using state feedback: $\sigma(I_n + T\widetilde{B}K) \sqcup \sigma(A_o + LC_o)$. □

Corollary VI.3. *If* $\widehat{Y}$*,* $\widehat{X}$*,* $\hat{w} := \begin{bmatrix}\hat{w}_d^\top & \hat{w}_c^\top\end{bmatrix}^\top$*, are as defined in* (10)*, and* $\mathrm{span}_{\mathbb{F}}\{B_1, \ldots, B_m\} = \mathfrak{g}$*, then* $u = K\phi^{-1}(\widehat{X}\Pi(\hat{w})^{-1}) + \Psi(\hat{w})$*, where* $K \in \mathbb{F}^{m\times n}$ *satisfies* (12)*, solves the local regulator problem with output information.*

VII. Simulation on $\mathsf{G} \cong \mathsf{SO}(2) \times \mathbb{R}$

We consider the generalized cylinder $\mathsf{G} \cong \mathsf{SO}(2)\times\mathbb{R}$, with Lie algebra $\mathfrak{g}$. A commuting basis for $\mathfrak{g}$ is

$$H_1 = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad H_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

We consider a system with exosystem parameters:

$$S_d = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \qquad S_c = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

which define a ramp and sinusoid, respectively; plant parameters: $A = H_1 - H_2$, $B_1 = H_1$, $B_2 = H_2$, $Q_{d1} = H_1 + 2H_2$, $Q_{d2} = -H_1 + H_2$, $Q_{c1} = H_1 + H_2$, $Q_{c2} = H_2$; output parameters: $C = H_1 + 2H_2$, $D_{d1} = H_1 - H_2$, $D_{d2} = H_1 + 1H_2$, $D_{c1} = H_1$, $D_{c2} = 2H_1 + H_2$; and regulation quantity parameters: $F = H_1 - 3H_2$, $G_{d1} = H_1 + 2H_2$, $G_{d2} = 2H_1 + H_2$, $G_{c1} = 1H_1 - H_2$, $G_{c2} = 3H_1 + H_2$.

We solve the linear quadratic regulator problem with identity weight matrices to design both our locally stabilizing controller gain K and our state estimator gain L. This yields $K = -0.618I_2$ and

$$L_1 = \begin{bmatrix}-0.855 & 0.165\\ -0.242 & -2.43\end{bmatrix}, \qquad L_2 = \begin{bmatrix}0.0817 & -0.707\\ 0.0464 & -0.188\end{bmatrix},$$
$$L_3 = \begin{bmatrix}-0.0942 & -0.00470\\ -0.256 & 0.111\end{bmatrix}.$$

We use a sampling period of $T = 1$ and initialize with $X(0) = \exp(0.2H_1 + 0.2H_2)$, $w_d[0] = \begin{bmatrix}0 & 0.5\end{bmatrix}^\top$, $w_c(0) = \begin{bmatrix}0.3 & -0.3\end{bmatrix}^\top$, $\widehat{X}[0] = I_2$, $\hat{w}_d[0] = \mathbf{0}_2$, $\hat{w}_c[0] = \mathbf{0}_2$.

Figure 2 depicts the plant states. In Figure 3, we see that $Z[k] \to I_N$, i.e., we have solved the regulator problem. However, we see that $Z(t), t \in (kT, (k+1)T)$ does not tend to I_N – there are no guarantees regarding intersample behaviour. In Figure 2, we see that the state on $\mathsf{SO}(2)$ (locally) "wraps around" several times, which is characteristic of dynamics on a quotient space.

Since the output is a static map of the states, it too exhibits this wrapping behaviour. However, as depicted in Figure 4, the state estimates converge.

VIII. Closing Remark and Future Research

It is worth noting that all our results hold globally if G has no compact subgroups, i.e., for G commutative, $\mathsf{SO}(2)$ is not a subgroup of G. Future work includes extending our results to dynamics on nilpotent and solvable Lie groups, and allowing X, Y, and Z to be defined on distinct Lie groups.

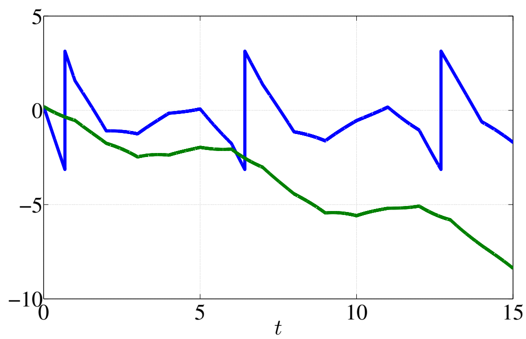

Fig. 2: Plant states $\phi^{-1}(X)$.

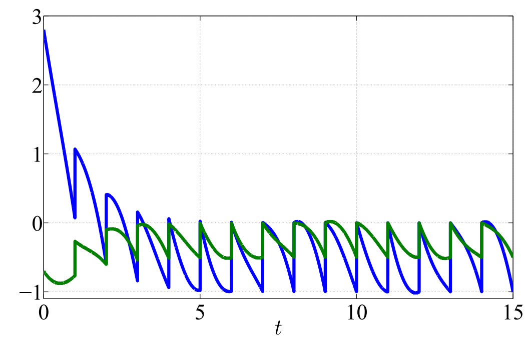

Fig. 3: Regulation quantity $\phi^{-1}(Z)$.

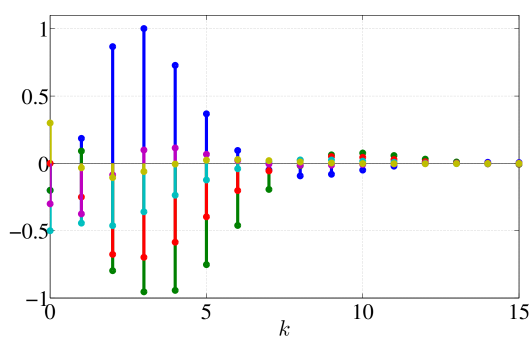

Fig. 4: Estimation errors $\phi^{-1}(\widehat{X}X^{-1})$ and $\hat{w} - w$.

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