Exact Formulas for Averages of Secular Coefficients

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We study averages of secular coefficients that frequently appear in random matrix theory. We obtain exact formulas, identities and new asymptotics for these integrals as well as a technique to deal with singularities that classically occur in the study of these problems.

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Lastly, I would like to thank my parents, Ed and Sanja, and my brother, Denny, for their love and support.

Dedication

Dedicated to my parents, Ed and Sanja Medjedovic.

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0.2 List of Notation

(i) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function.

(ii) $\delta(x)$ is the Dirac delta function.

(iii) $d_k(n) = \sum_{n_1 n_2 \ldots n_k = n} 1$ is the k-fold divisor function. It is the number of ways to write n as a product of k natural numbers.

(iv) RMT is an abbreviation for Random Matrix Theory.

(v) NT is an abbreviation for Number Theory.

(vi) $SSYT$ is an abbreviation for Semi-Standard Young Tableaux. A construct in partition theory which we will properly define later.

(vii) λ is a partition. λ_i is the i^{th} part of the partition. $s(\lambda)$ is the size of the partition and λ' is the conjugate partition to λ.

(viii) $U(N), SP(2N), O(N), SO(N)$ are the unitary, symplectic, and (special) orthogonal matrix groups.

(ix) χ^G is a character on the group G.

Chapter 1

Introduction

The goal of this thesis is to study the random matrix theory analogue of moments of L-functions. In particular, we develop a theory of averages of powers of determinants over matrix groups. Certain properties of these determinants have been studied by Keating, Rogers, Roditty-Gershon and Rudnick [12], Bump and Gamburd [8], as well as one of the authors [3]. These averages have long been known to be related to conjectures for asymptotics of higher moments of the ζ function [10].

1.1 Outline

- We motivate the study of a class of functions and so called "Secular Coefficients". We begin by reviewing known results for the unitary case in the rest of the introduction. We define and generalize the set of polynomials known within the literature as $\gamma_k(c)$. We summarize all the results contained in this thesis.
- In the next section, we briefly review some symmetric function theory and partition theory. We prove a Lemma the will be invaluable in our investigation that will allow us the remove certain singularities that classically appear in the study of these averages of characteristic polynomials of random matrices.
- We apply this Lemma along with results from Bump-Gamburd [8] as well as enumerations coming from the theory of plane partitions to get exact determinant formulas for averages of determinants of random matrices. We can use these ideas to deal with

a wide case of matrix families, the classical groups. This is the main achievement of the thesis.

- We then further analyze the Unitary case, obtaining properties of lower order terms of $\gamma_k(c)$.
- We give a short proof of the unimodality of $\gamma_k(c)$, which was conjectured by Ze'ev Rudnick.
- Lastly, we succinctly summarize further relations between the Riemann ζ function and averages of functions over random matrix groups.

The motivation is that we are trying to understand moments of the zeta function. We begin with taking powers of ζ, and we have the following identity for the divisor function. Let $d_k(n)$ be the k-th divisor numbers, i.e. the Dirichlet coefficients of the k-th power of the Riemann zeta function:

$$\zeta(s)^k = \sum_1^\infty \frac{d_k(n)}{n^s}, \qquad \Re s > 1. \tag{1.1}$$

The Dirichlet coefficient $d_k(n)$ is equal to the number of ways of writing n as a product of k factors. Define

$$S_k(X) = \sum_{n \le X} d_k(n). \tag{1.2}$$

The main term in the asymptotics of $S_k(x)$ comes from the pole at $s = 1$ of $\zeta^k(s)$. Let $XP_{k-1}(\log X)$ be the residue, at $s = 1$ of $\zeta(s)^k X^s/s$, with $P_{k-1}(\log X)$ being a polynomial in $\log X$ of degree $k-1$. Then

$$S_k(X) = XP_{k-1}(\log X) + \Delta_k(X), \tag{1.3}$$

with $\Delta_k(X)$ denoting the remainder term. The k-divisor problem asserts that $\Delta_k(x) = O_k(x^{\frac{k-1}{2k}+\epsilon})$. It is this remainder term that needs to be understood further.

The behaviour of Δ_k in short intervals was studied by Keating, Rodgers, Roditty-Gershon, and Rudnick [12]. Let

$$\Delta_k(x; H) = \Delta_k(x + H) - \Delta_k(x) \tag{1.4}$$

be the remainder term for sums of d_k over the interval $[x, x + H]$.

Define

$$a_k = \prod_p \Big\{(1-\frac{1}{p})^{k^2}\sum_{j=0}^{\infty}\left(\frac{\Gamma(k+j)}{\Gamma(k)j!}\right)^2\frac{1}{p^j}\Big\}. \tag{1.5}$$

the product convergence is seen by expanding the terms with respect to p giving a product over $1-\frac{C}{p^2}+O(\frac{1}{p^3})$, where C is a constant in k. By considering the analogous problem for function fields and related random matrix theory statistics, Keating, Rodgers, Roditty-Gershon, and Rudnick conjectured [12]:

Conjecture 1. *If $0<\alpha<1-\frac{1}{k}$ is fixed, then for $H=X^\alpha$,*

$$\frac{1}{X}\int_X^{2X}\Big(\Delta_k(x,H)\Big)^2dx \sim a_k\mathcal{P}_k(\alpha)H(\log X)^{k^2-1}\ , \quad X\to\infty \tag{1.6}$$

where $P_k(\alpha)$ is given by

$$\mathcal{P}_k(\alpha)=(1-\alpha)^{k^2-1}\gamma_k(\frac{1}{1-\alpha})\ . \tag{1.7}$$

Here $\gamma_k(c)$ is a piecewise polynomial function defined in the next section. Thereby, we hope to gain a better understanding of the statistics of the k-divisor function by understanding the general theory of $\gamma_k(c)$ and related constructions.

We briefly touch on the results found by Keating et al. and how they connect not only RMT and NT, but analogous questions for function fields.

Let U be an $N\times N$ matrix. We define the *secular coefficients*, $\mathrm{Sc}_j(U)$, to be the coefficients of the characteristic polynomial of U:

$$\det(I+xU)=\sum_{j=0}^{N}\mathrm{Sc}_j(U)x^j. \tag{1.8}$$

Thus $\mathrm{Sc}_0(U)=1$, $\mathrm{Sc}_1(U)=\mathrm{tr}\,U$, $\mathrm{Sc}_N(U)=\det U$. The secular coefficients are just elementary symmetric functions in the eigenvalues of U.

Let G be one of the matrix groups $U(N), Sp(2N), SO(N)$ or $O(N)$. Working with respect to the natural Haar measure in each case, define, for $G=Sp(2N), SO(N)$, or $U(N)$,

$$I_k^G(n,N) := \int_G \sum_{\substack{j_1+\cdots+j_k=n \\ 0\le j_1,\ldots,j_k\le N}} Sc_{j_1}(U)\ldots Sc_{j_k}(U)dU. \tag{1.9}$$

Unless $G = U(N)$ where we introduce a conjugate term, squaring the integrand (otherwise the average becomes 0):

$$I_k^G(n,N) := \int_G \sum_{\substack{j_1+\cdots+j_k=n \\ 0\le j_1,\ldots,j_k\le N}} |Sc_{j_1}(U)\ldots Sc_{j_k}(U)|^2 dU. \tag{1.10}$$

The connection to function field theory needs some additional notation. Let f be a monic polynomial in $\mathbb{F}_q$ and use $d_k(f)$ to denote the number of ways to write f as $f = f_1 \ldots f_k$ with f_i monic. We assume that the index A is a monic polynomial in $\mathbb{F}_q$. Furthermore, for a monic, define

$$I(A;h) = \{f : ||f - A|| \le q^h\} \tag{1.11}$$

with $||f|| = q^{\deg(f)}$ and

$$\mathcal{N}(A;h) := \sum_{f\in I(A;h)} d_k(f) \tag{1.12}$$

to be the divisor sum in function fields. Defining the difference and variance in short intervals similarly,

$$\Delta_k(A;h) := \mathcal{N}(A;h) - q^{h+1}\binom{n+k-1}{k-1}, \tag{1.13}$$

$$\mathrm{Var}(\mathcal{N}) := \frac{1}{q^n} \sum_{\deg(A)=n} |\Delta_k(A;h)|^2. \tag{1.14}$$

We then have the following estimate of the function field variance:

Theorem 1 (KRRR). *If $0 \le h \le \min(n-5, (1-\frac{1}{k})n - 2)$, then as $q \to \infty$*

$$\mathrm{Var}(\mathcal{N}) = H \cdot I_k^G(n; n-h-2) + O\Big(\frac{H}{\sqrt{q}}\Big), \tag{1.15}$$

for $H = q^{h+1}$.

In this case H is comparable to the short interval X^a in the NT case.

The following result in this direction is the following theorem due to Keating et al [12] which gives the leading asymptotics of I_k^G in terms of $\gamma_k(c)$.

Theorem 2 (KRRR). *Let $c := m/N$. Then for $c \in [0, k]$,*

$$I_k^{U(n)}(m, N) = \gamma_k(c) N^{k^2-1} + O_k(N^{k^2-2}). \tag{1.16}$$

1.2 The polynomials $\gamma_k(c)$

The function $\gamma_k(c)$, mentioned in Conjecture 1 and Theorem 2, is defined by the following integral over a slice of the unit hyper-cube:

$$\gamma_k(c) = \frac{1}{k!\, G(1+k)^2} \int_{[0,1]^k} \delta(t_1 + \ldots + t_k - c) \prod_{i<j} (t_i - t_j)^2 \, dt_1 \ldots dt_k, \tag{1.17}$$

where G is the Barnes G-function, so that for positive integers k, $G(1+k) = 1! \cdot 2! \cdot 3! \cdots (k-1)!$.

The function $\gamma_k(c)$ is supported on $[0, k]$ and symmetric around $\frac{k}{2}$.

$$\gamma_k(c) = \gamma_k(k - c) \tag{1.18}$$

It is also known that

Theorem 3 (KRRR).

$$\gamma_k(c) = \sum_{0 \le \ell < c} \binom{k}{\ell}^2 (c - \ell)^{(k-\ell)^2 + \ell^2 - 1} g_{k,\ell}(c - \ell) \tag{1.19}$$

where $g_{k,\ell}(c-\ell)$ are polynomials in $c-\ell$. No explicit form for $g_{k,\ell}$ is currently known. Note that the above implies that on each interval $[j-1, j]$, (for integer j), $\gamma_k(c)$ is a polynomial.

While the motivation in studying $\gamma_k(c)$ from a number theoretic perspective comes primarily from the connection to divisor sums, they are of their own interest from the perspective of random matrix theory. The focus of our thesis is on the underlying random matrix theory.

1.3 Main Results

The main results of this thesis are determinant identities for the generating function of $I_k^G(n,N)$. No exact formulas for these generating functions are known in the literature. Let $G \in \{U(N), O(N), SP(2N), SO(N)\}$ be a matrix group and consider

$$P_{k,N}^G(u) = \sum_{n=0}^{\infty} u^n I_k^G(n,N).$$

Then if $G = U(N)$

Theorem 4.

$$P_{k,N}^G(u) = \frac{C_{N,k}}{(1-u)^{k^2}} \det \frac{1-u^{N+i+j-1}}{N+i+j-1}$$

with

$$C_{N,k} = \prod_{j=1}^{k} \frac{(N+k-j-1)!}{(j-1)!^2\,(N+j-1)!}.$$

If $G = SP(2N)$ then

Theorem 5.

$$P_{k,N}^G(u) = \frac{1}{(1-u^2)^{\binom{k+1}{2}}} \det_{1\le i,j\le k} \left[\binom{j-1}{i-1} u^{j-i} - \binom{2N+2k+1-j}{i-1} u^{2N+2k+2-j-i} \right].$$

And finally, if $G = O(N)$ or $G = SO(N)$ we have

Theorem 6.

$$P_{k,N}^G(u) = \frac{1}{2} \frac{1}{(1-u^2)^{\binom{k}{2}}}$$
$$\det \left[\binom{j-1}{i-1} u^{j-i} - \binom{2N+2k-1-j}{i-1} u^{2N+2k-j-i} \right]$$
$$+ \det \left[\binom{j-1}{i-1} u^{j-i} + \binom{2N+2k-1-j}{i-1} u^{2N+2k-j-i} \right].$$

and

$$P^{G}_{k,N}(u) =$$
$$\frac{1}{(1-u^2)^{\binom{k}{2}}} \det \left[\binom{j-1}{i-1} u^{j-1} + \binom{2N+2k-j-1}{i-1} u^{2N+2k-j-i} \right],$$

respectively.

The secondary results of this thesis are slightly more qualitative results. In Section 4 we prove that the lower order terms in the asymptotics for $I_k^{U(N)}$ in N have properties similar to $\gamma_k(c)$. That is to say, if $I_k^{U(N)}(cN, N) \sim \sum_{m=0} \gamma_{k,m}(c) N^{k^2-1-m}$ then:

1. $\gamma_{k,m}(c)$ is symmetric around $k/2$.
2. $\gamma_{k,m}(c)$ is supported on $[0, k]$ and on each interval $[j, j+1]$ (for j an integer) it is a polynomial.
3. Each polynomial piecewise composing $\gamma_{k,m}(c)$ is of degree at most $k^2 - m$.
4. $\gamma_{k,m}(c)$ is differentiable $k^2 - m - 2j(k-j) - 1$ times at a transition point $c = j$.

For example, $\gamma_{k,0}(c) = \gamma_k(c)$ and has exactly the above properties.

In section 5 we prove a conjecture of Ze'ev Rudnick [personal communication], that $\gamma_k(c)$ is unimodal.

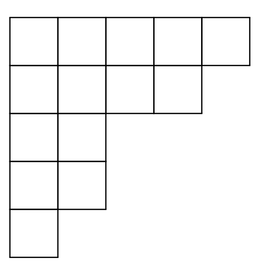
Chapter 2

Symmetric Function Theory

In this section we introduce some basics of symmetric function theory. The connection to symmetric function theory was used independently by Conrey, Farmer, Keating, Rubinstein and Snaith in CFKRS[9] as well as Bump and Gamburd in BG[8] to determine moments of characteristic polynomials of the classical compact groups. These results were used in CFKRS[9] to conjecture the asymptotics of the shifted moments of the ζ-function. We will describe the relevant symmetric function theory need for our results.

2.1 Young Diagrams

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition of n. Then $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. To each partition λ we associate to it what is known as a Ferrer's diagram. The diagram is a collection of "cells" off length λ_i across. For example the partition of 14 given by $(5, 4, 2, 2, 1)$ corresponds to Ferrer's diagram



We say a Ferrer's diagram is a semi-standard young tableau when the cells are labeled by integers less than n in such a way so that the rows are non-decreasing and the columns are increasing, starting with 1 at the top-right most cell. A young tableau for the above would be:

1	2	2	3	4
3	4	4	4	
4	5			
5	7			
7				

We say such a semi-standard young tableau, T, is of shape λ if the Ferrer's diagram of the tableau is the Ferrer's diagram for λ. In which case we write $T \sim \lambda$.

We should also introduce the Schur polynomials $s_\lambda(x_1, \ldots, x_k)$, let $\Delta(x)$ be the determinant of the Vandermonde matrix:

$$\Delta(x) = \det_{1 \le i,j \le k} x_j^{i-1} = \prod_{i \ne j} (x_i - x_j). \tag{2.1}$$

We define the Schur polynomial of λ to be

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det \begin{bmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & \cdots & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_2+k-2} & x_2^{\lambda_2+k-2} & \cdots & x_k^{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_k} & x_2^{\lambda_k} & \cdots & x_k^{\lambda_k} \end{bmatrix}}{\Delta(x)}. \tag{2.2}$$

Notice that s_λ is actually a polynomial as the determinant is 0 when $x_j = x_k$ for any j, k, canceling with the pole from the Vandermonde factor in the denominator. This definition of the Schur-functions is concise but unintuitive. An alternate definition follows.

We say T has *type* $a = (a_1, a_2, ...)$ if T has $a_i = a_i(T)$ parts equal to i. The SSYT above has type $(1, 2, 2, 5, 2, 0, 2)$. It is common to use the notational abbreviation

$$x^T = x_1^{a_1(T)} x_2^{a_2(T)} \cdots,$$

so for the example SSYT above,

$$x^T = x_1^1 x_2^2 x_3^2 x_4^5 x_5^2 x_7^2.$$

We finally come to the combinatorial definition of Schur functions.

Definition 1. *For a partition λ, the Schur function in the variables $x_1, ..., x_r$ indexed by λ is a multivariable polynomial defined by*

$$s_\lambda(x_1, ..., x_r) := \sum_T x_1^{a_1(T)} \cdots x_r^{a_r(T)},$$

where the sum is over all SSYTs T whose entries belong to the set $\{1, ..., r\}$ (i.e. $a_i(T) = 0$ for $i > r$).

For example, the SSYTs of shape $(4, 2)$ whose entries belong to the set $\{1, 2\}$ are

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

and so

$$s_{(4,2)}(x_1, x_2) = x_1^4 x_2^2 + x_1^3 x_2^3 + x_1^2 x_2^4.$$

Nota bene, the value $s_\lambda(1, \ldots, 1)$ enumerates the total number of SSYT associated to the partition λ.

2.2 Singularity Removal For Moments

Consider a polynomial $P(x)$ given by

$$P(x) = \det_{1 \le i,j \le k} \left[x_{j-1}^{a_i} \right], \tag{2.3}$$

where a_i are non-negative integers. Then,

$$P(x) = P(x_0, \ldots, x_{k-1}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=0}^{k-1} x_i^{a_{\sigma(i)}}. \tag{2.4}$$

This is an alternating polynomial and thus divisible by $\Delta(x)$. We are interested in finding $\frac{P(x)}{\Delta(x)}$ when $x_0 = x_1 = x_2 = \ldots = x_{k-1} = u$. Taking the limit as $x_1 \to x_2$, $x_2 \to x_3$, etc. and applying L' Hopital's rule gives

$$\lim_{x \to (u,\ldots,u)} \frac{P(x)}{\Delta(x)} = \frac{1}{1!2! \ldots (k-1)!} \frac{\partial^{k-1}}{\partial x_{k-1}^{k-1}} \cdots \frac{\partial^2}{\partial {x_2}^2} \frac{\partial}{\partial x_1} \Big|_{(u,\ldots,u)} P(x). \tag{2.5}$$

We expand $P(x)$ according to its definition taking derivatives and matching $i!$ with the $a_{\sigma(i)}$ terms to get binomial coefficients.

$$\lim_{x\to(u,\ldots,u)} \frac{P(x)}{\Delta(x)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=0}^{k-1} \binom{a_{\sigma(i)}}{i} x_i^{a_{\sigma(i)}-i}\big|_{(u,\ldots,u)}. \tag{2.6}$$

And we have computed the removable singularities of $\frac{P(x)}{\Delta(x)}$ to be

$$\frac{P(x)}{\Delta(x)}\big|_{(u,\ldots,u)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=0}^{k-1} \binom{a_{\sigma(i)}}{i} x_i^{a_{\sigma(i)}-i}\big|_{(u,\ldots,u)} \tag{2.7}$$

$$= \det_{1\leq i,j\leq k} \left[\binom{a_j}{i-1} x_{i-1}^{a_j-i+1}\right]\big|_{(u,\ldots,u)} \tag{2.8}$$

$$= \det_{1\leq i,j\leq k} \left[\binom{a_j}{i-1} u^{a_j-i+1}\right]. \tag{2.9}$$

We can extend this theorem slightly in the following Lemma.

Lemma 1. *Let $P(x) = \det_{1\leq i,j\leq k}[p_j(x_{i-1})]$ be an alternating polynomial where each p_j is itself a polynomial. Then*

$$\frac{P(x)}{\Delta(x)}\big|_{(u,\ldots,u)} = \det_{1\leq i,j\leq k} \left[\frac{1}{i-1!} \frac{\partial^{i-1}}{\partial u^{i-1}} p_j(u)\right]. \tag{2.10}$$

Proof. If each p_j is a monomial then the proof is detailed above. In the case that p_j are not monomials we may split up the determinant as a sum of monomials by multi-linearity and apply the above recipe on each term individually. Adding the terms together by multi-linearity again yields Lemma 1. □

This Lemma will be crucial in removing singularities that appear in expressions for averages of secular coefficients. This will allow us to get an exact formula for certain matrix theory integrals that appear in the literature.

Chapter 3

Secular Coefficients of Matrix Groups

3.1 The Unitary Group

We will apply the singularity removal technique to equation (2.9) in Autocorrelations of Random Matrix polynomials [9]. That formula is reproduced below in equation (3.3). Let $G = U(N)$ and let $U \in U(N)$. First notice the following relation

$$\det(I - xU)^k \det(I - yU^*)^k = \left(\sum_{j=1}^{N} \mathrm{Sc}_j(U)(-x)^j \right)^k \left(\sum_{i=1}^{N} \mathrm{Sc}_i(U^*)(-y)^i \right)^k \tag{3.1}$$

and integrate over the unitary group.

$$\int_G \det(I - xU)^k \det(I - yU^*)^k dU = \sum_{0 \le m \le kN} I_k^G(n, N)(xy)^n. \tag{3.2}$$

In the above equation only diagonal terms remain, i.e. the coefficients of the terms of form $x^n y^m, m \neq n$, are 0. Consider the map $U \mapsto e^{it}U$ which by the invariance of the Haar measure does not change the value of the integral. Under this map, U^* gets scaled by e^{-it}. We can absorb the e^{it} terms in x and e^{-it} in y so that the term $x^n y^m$ in the sum becomes $e^{(n-m)it} x^n y^m$. Since the integral is invariant under this transformation, the sum should be too, and so the coefficient of any term with $n \neq m$ is indeed 0.

Formula (2.9) of the Autocorrelations paper is copied below:

$$\prod_{l=m+1}^{r} w_l^N \int_{U(N)} \prod_{i=m+1}^{n} \det(I - w_i^{-1}U) \prod_{j=1}^{m} \det(I - w_j U^*) dU \tag{3.3}$$
$$= \frac{1}{\prod_{1\le \ell < q \le n}(w_q - w_\ell)} \begin{vmatrix} 1 & w_1 & w_1^2 & \cdots & w_1^{m-1} & w_1^{N+m} & w_1^{N+m+1} & \cdots & w_1^{N+n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n & w_n^2 & \cdots & w_n^{m-1} & w_n^{N+m} & w_n^{N+m+1} & \cdots & w_n^{N+n-1} \end{vmatrix}.$$

Specializing to $m = k, n = 2k$, $w_1 = w_2 = \ldots = w_k = x$ and $w_{k+1} = \ldots = w_{2k} = 1$ and removing the singularities as in Lemma 1 gives

$$I_k^{U(N)}(n, N) = [x^n] \frac{1}{(1-x)^{k^2}} \begin{vmatrix} A(x) & B(x) \\ A(1) & B(1) \end{vmatrix} \tag{3.4}$$

where

$$A_{ij}(x) = \binom{j-1}{i-1} x^{j-i} \tag{3.5}$$
$$B_{ij}(x) = \binom{N+2k+j-1}{i-1} x^{N+2k+j-i}. \tag{3.6}$$

We now are going to perform row reductions on the above. Notice $A(x)^{-1} = A(-x)$, as can be verified using the underlying binomial identity

$$\sum_{l=1}^{k} (-1)^{i+l} \binom{l-1}{i-1} \binom{j-1}{l-1} = \binom{j-1}{i-1} \sum_{l=1}^{k} (-1)^{i-l} \binom{j-i}{l-i}. \tag{3.7}$$

If $j > i$ the sum on the right is an alternating sum of the $(j-i)^{th}$ row of Pascal's triangle and vanishes. If $j < i$ the factor of $\binom{j-1}{i-1}$ infront of the sum is 0. And if $i = j$ only one term contributes to the sum, namely $l = i$, giving 1. Thus, multiplying the block matrix in 3.4 on the left by the block matrix

$$\begin{pmatrix} A(-x) & 0 \\ 0 & A(-1) \end{pmatrix} \tag{3.8}$$

gives

$$\left(\begin{array}{cc} A(-x) & 0 \\ 0 & A(-1) \end{array} \right) \left(\begin{array}{cc} A(x) & B(x) \\ A(1) & B(1) \end{array} \right) = \left(\begin{array}{cc} I & A(-x)B(x) \\ I & A(-1)B(1) \end{array} \right), \tag{3.9}$$

and then by multiplying with

$$\left(\begin{array}{cc} I & 0 \\ -I & I \end{array} \right) \tag{3.10}$$

to remove the bottom left I:

$$\left(\begin{array}{cc} I & 0 \\ -I & I \end{array} \right) \left(\begin{array}{cc} I & A(-x)B(x) \\ I & A(-1)B(1) \end{array} \right) = \left(\begin{array}{cc} I & A(-x)B(x) \\ 0 & A(-1)B(1) - A(-x)B(x) \end{array} \right). \tag{3.11}$$

These multiplications do not change the determinant as both multiplications are by triangular matrices with 1's on the diagonal. Therefore the determinant of the matrix in (3.4) equals the determinant of the lower $k \times k$ block above, i.e.

$$|A(-1)B(1) - A(-x)B(x)|_{k\times k}. \tag{3.12}$$

Next we compute the entries of the above matrix. The i, j entry is

$$\sum_{l=1}^{k} (-1)^{l-i} \binom{l-1}{i-1} \binom{N+k+j-1}{l-1} (1 - x^{N+k+j-i}) \tag{3.13}$$

but,

$$\binom{l-1}{i-1} \binom{N+k+j-1}{l-1} = \binom{N+k+j-1}{i-1} \binom{N+k+j-i}{l-i}. \tag{3.14}$$

so that equation 3.13 equals

$$\binom{N+k+j-1}{i-1} (1 - x^{N+k+j-i}) (-1)^i \sum_{l=1}^{k} (-1)^l \binom{N+k+j-i}{l-i}. \tag{3.15}$$

But the sum above equals

$$(-1)^i \sum_{l=0}^{k-i} (-1)^l \binom{N+k+j-i}{l}. \tag{3.16}$$

This is an alternating sum of the $N+k+j-i$ row of Pascal's triangle which so the above famously equals

$$(-1)^k \binom{N+k+j-i-1}{k-i}. \tag{3.17}$$

Returning to the $k \times k$ determinant we see that the i, j entry of the matrix equals

$$(-1)^{k-i} \binom{N+k+j-1}{i-1} \binom{N+k+j-i-1}{k-i} (1 - x^{N+k+j-i}). \tag{3.18}$$

This product of binomial coefficients equals

$$\binom{N+k+j-1}{i-1} \binom{N+k+j-i-1}{k-i} = \frac{(N+k+j-1)!}{(i-1)!(k-i)!(N+j-1)!(N+k+j-i)}. \tag{3.19}$$

We can thus pull out from row i of the determinant a factor of $\frac{(-1)^{k-i}}{((i-1)!(k-i)!)}$ and a factor of $\frac{(N+k+j-1)!}{(N+j-1)!}$ from column j. Therefore, the determinant in (3.4) equals, on collecting these factors,

$$\prod_{j=1}^{k} \frac{(-1)^{k-j}(N+k+j-1)!}{(j-1)!(k-j)!(N+j-1)!} \det \left[\frac{1 - x^{N+k+j-i}}{N+k+j-i} \right]_{k \times k} = \tag{3.20}$$

$$\prod_{j=1}^{k} \frac{(N+k+j-1)!}{(j-1)!^2 (N+j-1)!} \det \left[\frac{1 - x^{N+i+j-1}}{N+i+j-1} \right]_{k \times k} \tag{3.21}$$

where, in the last equality we have reversed the k rows of the matrix. We have thus arrived at the formula of Theorem 4:

$$I_k^{U(N)}(n, N) = [x^n] \frac{C_{N,k}}{(1-x)^{k^2}} \det \frac{1 - x^{N+i+j-1}}{N+i+j-1}. \tag{3.22}$$

Here $C_{N,k}$ is a constant depending only on N and k and can be given explicitly in several ways:

$$C_{N,k} = \prod_{j=1}^{k} \frac{(N+k+j-1)!}{(j-1)!^2\,(N+j-1)!} = \frac{\prod_{1 \le i,j \le k}(N+i+j-1)}{\prod_{1 \le i < j \le k}(j-i)^2} \tag{3.23}$$

$$C_{N,k} = \frac{1}{\det_{1 \le i,j \le k}[\frac{1}{N+i+j-1}]} \tag{3.24}$$

$$C_{N,k} = \frac{G(N+2k)G(N)}{G(N+k)^2 G(k)^2} \tag{3.25}$$

where $G(m) = 1!2! \ldots (m-1)!$ is the Barnes G-function.

3.2 The Symplectic Group

We move on the symplectic case now. Let $G = SP(2N)$. We begin with proposition (11) and equation (43) from Bump-Gamburd [8].

$$\int_{Sp(2N)} \prod_{i=1}^{k} \det(1 + x_i U)\, dU = (x_1 \ldots x_k)^N \chi^{Sp(2k)}_{\langle N^k \rangle}(x_1^{\pm 1}, \ldots, x_k^{\pm 1}). \tag{3.26}$$

Here $\chi^{Sp(2k)}_{\langle N^k \rangle}$ is a certain irreducible character from the representation theory of $\mathrm{GL}_n(\mathbb{C})$. A partition is said to be even if all parts of it are even. From section 7.1 of the same paper we have

$$(x_1 \ldots x_k)^N \chi^{Sp(2k)}_{<N^k>}(x_1^{\pm 1}, \ldots, x_k^{\pm 1}) = \sum_{\substack{\lambda_1 \leq 2N \\ \lambda \textbf{ even}}} s_\lambda(x_1, \ldots, x_k). \tag{3.27}$$

where the sum is taken over all even partitions.

Let $G = Sp(2N)$.

Consider the generating function

$$\sum_{n=0}^{2kN} x^n I_k^G(n, N) = \int_{Sp(2N)} \det(1 + xU)^k dU. \tag{3.28}$$

We are trying to extract the $[x^n]$ coefficient of

$$\sum_{\substack{\lambda_1 \leq 2N \\ \lambda \textbf{ even}}} s_\lambda \overbrace{(x, \ldots, x)}^{k}. \tag{3.29}$$

By the combinatorial interpretation of Schur functions the coefficient we desire is

$$\sum_{\substack{s(\lambda)=n \\ \lambda_1 \leq 2N \\ \lambda \textbf{ even}}} s_\lambda \overbrace{(1, \ldots, 1)}^{k}. \tag{3.30}$$

where $s(\lambda)$ is the size of the partition. One can see that $s_\lambda \overbrace{(1, \ldots, 1)}^{k}$ as the number of semi-standard young tableaux of type λ. Hook content formula gives $s_\lambda(1, ..., 1) = \prod_{u \in \lambda} \frac{n+c(u)}{h(u)}$

where $c(u)$ and $h(u)$ are the content and hook of a cell $u \in \lambda$.

Other identities for partitions of the form described in equation (3.29) are well-known within literature dealing with plane partitions. A famous example is the Hall-Littlewood identity [1].

$$\sum_{\lambda \textbf{ even}} s_\lambda(x_1, \ldots, x_k) = \prod_{i=1}^{k} \frac{1}{1 - x_i^2} \prod_{i<j} \frac{1}{1 - x_i x_j} \tag{3.31}$$

Note that if $n < 2N$ then the constraint from our formula drops out and the Hall-Littlewood identity allows us to immediately calculate

$$I_k^{Sp(2n)}(n, N) = \begin{cases} \binom{\frac{n}{2} + \binom{k+1}{2} - 1}{\binom{k+1}{2} - 1}, & \text{for } n \text{ even} \\ 0, & \text{otherwise} \end{cases}. \tag{3.32}$$

In other domains we must use bounded forms of the Hall-Littlewood identities. For this we use the Desarmenien-Stembridge-Proctor formula [14], [4] , [5].

$$\sum_{\substack{\lambda_1 \leq 2N \\ \lambda \text{ even}}} s_\lambda(x_1, \ldots, x_k) = \frac{1}{\Delta(x)} \prod_{i=1}^{k} \frac{1}{1 - x_i^2} \prod_{i<j} \frac{1}{1 - x_i x_j} \det_{1 \leq i,j \leq k} \left[x_i^{j-1} - x_i^{2N+2k+1-j} \right] \tag{3.33}$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. The difficulty here is singularities appear when all x_i are equal. Of course, since we are ultimately dealing with a finite sum of polynomials , these singularities must be removable.

We now apply the formula derived in Lemma 1 above to the Desarmenien-Stembridge-Proctor formula.

$$\frac{1}{\Delta(x)} \prod_{i=1}^{k} \frac{1}{1 - x_i^2} \prod_{i<j} \frac{1}{1 - x_i x_j} \det_{1 \leq i,j \leq k} \left[x_i^{j-1} - x_i^{2N+2k+1-j} \right] |_{(u,\ldots,u)} = \\ \frac{1}{(1 - u^2)^{\binom{k+1}{2}}} \frac{\det_{1 \leq i,j \leq k} \left[x_i^{j-1} - x_i^{2N+2k+1-j} \right]}{\Delta(x)} \Big|_{(u,\ldots,u)} \tag{3.34}$$

In this case, since we are not working with monomial terms anymore the determinant expression gets more complicated but we can decompose it by multi-linearity and then apply the above formula to get rid of the $\frac{1}{\Delta(x)}$, putting everything back together again with multi-linearity.

$$\frac{\det_{1\le i,j\le k}\left[x_i^{j-1}-x_i^{2N+2k+1-j}\right]}{\Delta(x)}\Big|_{(u,\dots,u)} \tag{3.35}$$

$$= \frac{1}{\Delta(x)}\sum_{\sigma\in S_n}\sum_{S\subset\{1,\dots,k\}}(-1)^{|S|}\operatorname{sgn}(\sigma)\prod_{i\in S}x_i^{2N+2k+1-\sigma(i)}\prod_{i\notin S}x_i^{\sigma(i)-1}\Big|_{(u,\dots,u)} \tag{3.36}$$

$$= \sum_{S\subset\{1,\dots,k\}}(-1)^{|S|}\sum_{\sigma\in S_n}\frac{\operatorname{sgn}(\sigma)}{\Delta(x)}\prod_{i\in S}x_i^{2N+2k+1-\sigma(i)}\prod_{i\notin S}x_i^{\sigma(i)-1}\Big|_{(u,\dots,u)} \tag{3.37}$$

$$= \det_{1\le i,j\le k}\left[\binom{j-1}{i-1}u^{j-i}-\binom{2N+2k+1-j}{i-1}u^{2N+2k+2-j-i}\right] \tag{3.38}$$

To summarize, if we let

$$P_{k,N}(u)=\sum_{n=0}^{2kN}u^n I_k^{Sp(2n)}(n,N). \tag{3.39}$$

Then we have the following formula of Theorem 5:

$$P_{k,N}(u)= \frac{1}{(1-u^2)^{\binom{k+1}{2}}}\det_{1\le i,j\le k}\left[\binom{j-1}{i-1}u^{j-i}-\binom{2N+2k+1-j}{i-1}u^{2N+2k+2-j-i}\right].$$

3.3 The Orthogonal and Special Orthogonal Group

In this section we use similar ideas to the previous section to deal with the $G=SO(2N)$ and $G=O(2N)$ case.

3.3.1 The Orthogonal Group

Let $G=O(2N)$. Our starting point is again

$$I_k^G(n,N) := \int_G \sum_{\substack{j_1+\cdots+j_k=n\\ 0\le j_1,\dots,j_k\le N}} Sc_{j_1}(U)\dots Sc_{j_k}(U)dU \tag{3.40}$$

for a matrix group G.

Consider the generating function

$$\sum_{n=0}^{2kN} x^n I_k^G(n,N) = \int_G \det(1+xU)^k dU. \tag{3.41}$$

Again, we refer to Bump-Gamburd for the first step. In equation 102, after specializing to $x_i = x_j$ for all i, j they give

$$\int_G \det(I+xU)^k dU = \sum_{\substack{\lambda_1 \le 2N \\ \lambda' \text{ even}}} s_\lambda(x,\ldots,x). \tag{3.42}$$

where λ' is the conjugate partition of λ. As before, if we want $I_k^G(n,N)$ we can isolate the x^n term of the above as

$$\sum_{\substack{s(\lambda)=n \\ \lambda_1 \le 2N \\ \lambda' \text{ even}}} s_\lambda(1,\ldots,1), \tag{3.43}$$

the total number of SSYT of partitions with even conjugate. Okada [7] gives an enumeration of such sums and we will apply our Lemma 1 to remove the singularities:

$$\sum_{\substack{\lambda_1 \le 2N \\ \lambda' \text{ even}}} s_\lambda(x_1,\ldots,x_k) = \frac{1}{2}\frac{\det(x_i^{j-1} - x_i^{2N+2k-1-j}) + \det(x_i^{j-1} + x_i^{2N+2k-1-j})}{\prod_{1\le i<j\le k}(x_i x_j - 1)(x_i - x_j)} \tag{3.44}$$

Let

$$P_{k,N}(u) = \sum_{n=0}^{2kN} u^n I_k^G(n,N)$$

be the polynomial whose coefficients enumerate the averages we are after. Setting all $x_i = u$ and using Lemma 1 the resulting sum of determinants gives the first formula of Theorem 6.

$$\begin{aligned} P_{k,N}(u) &= \frac{1}{2}\frac{1}{(1-u^2)^{\binom{k}{2}}} \\ &\left(\det\left[\binom{j-1}{i-1}u^{j-i} - \binom{2N+2k-1-j}{i-1}u^{2N+2k-j-i}\right]\right. \\ &\left.+ \det\left[\binom{j-1}{i-1}u^{j-i} + \binom{2N+2k-1-j}{i-1}u^{2N+2k-j-i}\right]\right). \end{aligned}$$

3.3.2 The Special Orthogonal Group

Let $G = SO(2N)$ and keep the same notation as the previous subsection. The special orthogonal case is a little easier to handle. Equation 71 in Bump-Gamburd gives a relation for the integral we want in terms of a matrix

$$\int_G \prod_{j=1}^{k} \det(I + x_j g) = (x_1 \dots x_k)^N \chi^{O_{2k}}_{\langle N^k \rangle}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \tag{3.45}$$

Where the character χ can be written explicitly as

$$(x_1 \dots x_k)^N \chi^{O(2k)}_{N^k}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) =$$

$$\det \begin{vmatrix} x_1^{N+k-1} + x_1^{-(N+k-1)} & x_1^{N+k-2} - x_1^{-(N+k-2)} & \cdots & x_1^{N} - x_1^{-(N)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k-1} - x_k^{-(N+k-1)} & x_k^{N+k-2} - x_k^{-(N+k-2)} & \cdots & x_k^{N} - x_k^{-(N)} \end{vmatrix} \tag{3.46}$$

$$\times \frac{(x_1 \cdots x_k)^{k+N-1}}{\prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)(x_i x_j - 1)}.$$

If we let

$$P_{k,N}(u) = \sum_{n=0}^{2kN} u^n I_k^G(n, N)$$

the consequently (after an application of Lemma 1) we obtain, the second formula in Theorem 6:

$$P_{k,N}(u) = \frac{1}{(1-u^2)^{\binom{k}{2}}} \det \left[\binom{j-1}{i-1} u^{j-1} + \binom{2N+2k-j-1}{i-1} u^{2N+2k-j-i} \right].$$

Chapter 4

Asymptotic Behavior of the Unitary Group & Lower Order Terms

4.1 Analysis by Minors

Let

$$F_{N,k}(x) := \det_{1\le i,j\le k}\left(\frac{x^{N+i+j-1}-1}{N+i+j-1}\right).$$

This is, up to sign, the determinant that occurs in Theorem 4, the unitary case, though we prefer here to write the numerator as $x^{N+i+j-1}-1$. Our goal is to get an understanding of the asymptotic behavior of this determinant so we can get higher order analogues of $\gamma_k(c)$.

We expand the above determinant as a sum of its minors. Imagine choosing sets $S, T \subset \{1, \ldots, k\}$ that denote rows/columns where we choose powers of x in our power series expansion of F and what remains is the minor S^c, T^c. Each minor is a Cauchy matrix and there are known formulas for computing these determinants. Let $s(S) = \sum_{a\in S} a$, the sum of elements of S.

$$F_{N,k}(x) = \sum_{\substack{S,T\subset\{1,\ldots,k\}\\ |S|=|T|}} (-1)^{s(S)+s(T)} \det_{i\in S, j\in T}\left(\frac{x^{N+i+j-1}}{N+i+j-1}\right) \det_{i\in S^c, j\in T^c}\left(\frac{-1}{N+i+j-1}\right). \tag{4.1}$$

The determinant on the right hand side that is dependent on x is homogeneous. A more general version of this formula can be found in [13].

$$F_{N,k}(x) =$$

$$\sum_{\substack{S,T\subset\{1,\ldots,k\}\\|S|=|T|}} (-1)^{k-|S|+s(S)+s(T)} x^{(N-1)|S|+\sum_{i\in S} i+\sum_{j\in T} j} \det_{i\in S,j\in T}\left(\frac{1}{N+i+j-1}\right) \det_{i\in S^c,j\in T^c}\left(\frac{1}{N+i+j-1}\right). \tag{4.2}$$

We now make use of Cauchy's determinant formula.

Theorem 7 (Cauchy)**.** *Let $A = \{\alpha_1, \ldots, \alpha_k\}, B = \{\beta_1, \ldots, \beta_k\}$. Then*

$$\det\left(\frac{1}{\alpha_i+\beta_j}\right) = \frac{\Delta(A)\Delta(B)}{P(A,B)},$$

where $\Delta(S) = \prod_{i<j}(s_i - s_j)$ and $P(S+T) = \prod_{s\in S,t\in T}(s+t)$.

Let $N + S$ denote the set obtained by adding the integer N to each element of S. Likewise, let $T-1$ be the set obtained by subtracting 1 from each element of T. Applying this to the product of two minors in our expression for $F_{N,k}$ with $A = N+S$ and $B = T-1$ and noticing we can factor out $C_{N,k}$, using 3.24 yields

$$\det_{i\in S,j\in T}\left(\frac{1}{N+i+j-1}\right) \det_{i\in S^c,j\in T^c}\left(\frac{1}{N+i+j-1}\right) = \frac{\Delta(S)\Delta(T)\Delta(S^c)\Delta(T^c)}{P(N+S,T-1)P((N+S)^c,(T-1)^c} \tag{4.3}$$

$$= \frac{1}{C_{N,k}} \frac{P(N+S,(T-1)^c)P\left((N+S)^c,T-1\right)}{P\left(S,-S^c\right)P\left(T,-T^c\right)}. \tag{4.4}$$

In the first equality we used $\Delta(N+S) = \Delta(S)$ and $\Delta(T-1) = \Delta(T)$ and likewise for their complements, S^c, T^c. In the second equality we factor out the $\frac{1}{C_{N,k}}$ and are left with the remaining products. To proceed multiply the polynomial $F_{N,k}(x)$ by the power series of $\frac{(-1)^k C_{N,k}}{(1-x)^{k^2}}$. The x^n coefficient of the resulting polynomial is

$$(-1)^k \sum_{m=0}^{n} C_{N,k} \binom{k^2-1+n-m}{k^2-1} [x^m] F_{N,k}(x) \tag{4.5}$$

For given k, if N is sufficiently large, notice that powers in the above polynomial cluster around jN for an integer $j \leq k$. That is, all non-zero terms in $F_{N,k}$ that involve terms x^m for $m = jN + l$ with l being an integer less than k^2. Let $jN \leq n = cN \leq (j+1)\,N$ so the above becomes

$$\sum_{j=0}^{c} \sum_{l=0}^{k^2} \binom{k^2 - 1 + (c-j)\,N - l}{k^2 - 1} \tag{4.6}$$

$$\times \sum_{\substack{S,T \subset \{1,\ldots,k\} \\ |S|=|T| \\ (N-1)|S| + \sum_{s\in S} s + \sum_{t \in T} t = jN+l}} (-1)^{|S|+s(S)+s(T)} \, \frac{P(S+N, (T-1)^c) P\left((S+N)^c, T-1\right)}{P\left(S, -S^c\right) P\left(T, -T^c\right).}$$

We can take note of the following properties from the above formula. As c passes through integers $1, 2, \ldots k$ new terms are added to the above double sum. These terms are a polynomial in $(c-j)$. Suppose we want to know the polynomials associated to the N^{k^2-m} term. This is a generalization of $\gamma_k(c)$ which occurs when $m = 1$. All terms involving $(c-j)$ to some power come from the binomial coefficient. The product of minors on the right contributes at most terms of order $N^{2j(k-j)}$. Therefore, at the transition points we are adding polynomials which have zeroes of order $k^2 - m - 2j(k-j)$ (assuming this quantity is positive), coming from the binomial coefficients in the above expression. This makes the resulting piecewise function very smooth. To be precise,

Theorem 8. *The piecewise function of polynomials giving asymptotics for the N^{k^2-m} power of N has the following properties:*

- *It is symmetric around $k/2$.*
- *It is supported on $[0, k]$ and on each interval $[j, j+1]$ (for j an integer) it is a polynomial.*
- *Each polynomial is of degree at most $k^2 - m$.*
- *It is differentiable $k^2 - m - 2j(k-j) - 1$ times at a transition point $c = j$.*

The first property is a consequence of the functional relation for $I_k^{U(N)}$. The second property comes from 4.6 and noticing that $I_k^{U(N)}$ is 0 for $c > k$. The third property comes from noticing that in the binomials in 4.6, a factor of c is paired with a factor of N always.

The fourth property comes from the previously described differentiability at 0. That is to say, if

$$I_k^{U(N)}(n,N) = \gamma_k(c)N^{k^2-1} + \gamma_{k,1}(c)N^{k^2-2} + \gamma_{k,2}(c)N^{k^2-3} + \dots,$$

then $\gamma_{k,m}(c)$ share the same properties as $\gamma_k(c)$ in the above way. All of the lower order terms in N are highly smooth symmetric piecewise polynomials on the domain $[0,k]$.

4.1.1 A recursion for $F_{N,k}(x)$

Let M be a $k \times k$ matrix, M_i^j then $(k-1)\times(k-1)$ the matrix obtained by deleting row i and column j of M and $M_{i,j}^{l,m}$ be the $(k-2)\times(k-2)$ matrix obtained from M by deleting rows i and j, and columns l and m.

The Desnanot-Jacobi identity states that

$$\det(M)\det(M_{1,k}^{1,k}) = \det(M_1^1)\det(M_k^k) - \det(M_1^k)\det(M_k^1). \tag{4.7}$$

Applying this identity to $F_{N,k}(x)$ gives

$$F_{N,k}(x) = \frac{F_{N+2,k-1}(x)F_{N,k-1}(x) - F_{N+1,k-1}(x)^2}{F_{N+2,k-2}(x)}. \tag{4.8}$$

This follows from the observation that the entries of $F_{N,k}(x)$ are of the form $\frac{X^{N+i+j-1}-1}{N+i+j-1}$, with $N+i+j-1$ increasing by 1 as we increment either i or j.

This recursion allows one to determine the polynomial $F_{N,k}(x)$ the from the polynomials for $k-1$ and $k-2$.

Chapter 5

Further Properties

5.1 Unimodality of $\gamma_k(c)$

We review some more basic properties of $\gamma_k(c)$. In the appendix we have plots of $\gamma_k(c)$ for $k = 4$. On each interval $[j-1, j]$ for $j \leq 4$, an integer, $\gamma_k(c)$ is a different polynomial. These polynomials approximate a Gaussian.

Indeed, the Gaussian behavior suggest that $\gamma_k(c)$ is unimodal. This question was raised by Rudnick during a conference a few years ago. Recently, Rogers remarked that $\gamma_k(c)$ is log-concave and outlined a proof[11]. We give a shorter proof here and show that this log-concavity implies unimodality.

The Gaussian behaviour was shown explicitly in earlier work due to Basor, Ge and Rubinstein [3], at least asymptotically around the center. The following theorem summarizes the Gaussian nature in the limiting case

Theorem 9 (Basor, Ge, Rubinstein)**.** *Let $b_k = 8(1 - 1/(4k^2))$ and $c = k/2 + o(k)$. Then*

$$\gamma_k(c) \sim \frac{G(k+1)^2}{G(2k+1)} \sqrt{\frac{b_k}{\pi}} e^{-b_k(c-k/2)^2}.$$

We move on to the proof of unimodality, log-concavity and some recurrence relations for $\gamma_k(c)$ and related functions.

Let

$$P_{\alpha,\beta,\gamma}(x) = \left(\prod_{i=1}^{k} x_i\right)^{\alpha} \left(\prod_{i=1}^{k} 1 - x_i\right)^{\beta} \left(\prod_{i \neq j} |x_i - x_j|\right)^{\gamma}. \tag{5.1}$$

We are interested in the integral

$$y_{\alpha,\beta,\gamma}(c) = \int_{C^k} \delta\left(c - \sum_{i=1}^{k} x_i\right) P_{\alpha,\beta,\gamma}(x), \tag{5.2}$$

with C^k being the unit cube and δ being the Dirac delta function which is a generalization of the integral that appears in the definition (1.15) of $\gamma_k(c)$.

Theorem 10. *The functions $y_{\alpha,\beta,\gamma}(c)$ are unimodal if $\alpha, \beta, \gamma > 1$ and real.*

We first prove unimodality is guaranteed by log-concavity. Let $f : [0,1] \to \mathbb{R}$ and assume f is bounded, continuous and log-concave. Furthermore assume f is positive on its interior. We prove that f must be unimodal.

Proof. Suppose $f'(a) = f'(b) = 0$ for some $a \neq b$ in $[0,1]$, where a is a global maximum. Since f is log-concave, $\log f$ is a concave function with vanishing derivative at a and b. Consider the line segment from $(a, \log f(a))$ to $(b, \log f(b))$. WLOG let $b < a$, so it has positive slope. Since the derivative of $\log(f)$ at b is 0 there is some neighbourhood to the right of b contained under the line segment. But this contradicts concavity. □

Now it remains to see that $y_{\alpha,\beta,\gamma}(c)$ is log-concave. Consider the domain where the integrand is non-zero, $C^k \cap H_c$ where H_c is the hyperplane $\sum_{i=1}^{k} x_i = c$ This is a convex set, it suffices to show $P_{\alpha,\beta,\gamma}(c)$ is log-concave on this set. This is because taking marginals of log-concave functions preserves log-concavity [6].

Lemma 2. *$P_{\alpha,\beta,\gamma}(x)$ is log-concave on the domain $C^K \cap H_c$.*

Proof. Since a product of log concave functions is log-concave, it suffices to prove log-concavity of each term separately. That is, we show x_i^{α}, $(1 - x_i)^{\beta}$ and $|x_i - x_j|^{\gamma}$ are log-concave. Indeed, it suffices to take the domain of integration to be $0 \leq x_i \leq x_j \leq 1$ for $i < j$ by symmetry (introducing a factor of $n!$). Taking the log of x_i^{α} gives $\alpha \log(x_i)$ which is concave on $[0,1]$. Similarly, we can substitute $u = 1 - x_i$ in the second case, and $u = |x_i - x_j| = x_j - x_i$ in the third. In each case the domain is still within $[0,1]$. □

Some Identities

We derive some general identities for the derivative of $y_{\alpha,\beta,\gamma}(c)$. Note first that

$$y_{\alpha,\beta,\gamma}(c) = y_{\beta,\alpha,\gamma}(k-c) \tag{5.3}$$

via the substitution $x_i \mapsto 1 - x_i$.

Consider the two sets $C^k \cap H_c$ and $C^k \cap H_{c+\epsilon}$. With the substitution $x_i \mapsto x_i + \frac{\epsilon}{k}$ we can get a bijection between the two sets, apart from some small section around the border.

Expanding using the definition of derivative:

$$\frac{y_{\alpha,\beta,\gamma}(c+\epsilon) - y_{\alpha,\beta,\gamma}(c)}{\epsilon}.$$

Which yields

Theorem 11.

$$y'_{k,\alpha,\beta,\gamma}(c) = \delta(\alpha) y_{k-1,\gamma,\beta,\gamma}(c) - \delta(\beta) y_{k-1,\alpha,\gamma,\gamma}(c-1) + \frac{1}{k}\int_{C^k \cap H_c} P_{k,\alpha,\beta,\gamma}(x) \left(\sum_i \frac{\alpha}{x_i} + \frac{\beta}{1-x_i} \right).$$

Here we use $\delta(\alpha)$ to denote the function that takes on the value of 1 if $\alpha = 0$ and 0 otherwise. If we instead consider the substitution $x_i \mapsto \left(1 + \frac{\epsilon}{c}\right) x_i$ which achieves a similar effect to the above we can again expand the derivative to get

Theorem 12.

$$c y'_{k,\alpha,\beta,\gamma}(c) = C_1 y_{k,\alpha,\beta,\gamma}(c) - k\delta(\beta) y_{k-1,\alpha,\gamma,\gamma}(c-1) + \beta \int_{C^k \cap H_c} \left(k - \sum_i \frac{1}{1-x_i} \right) P_{k,\alpha,\beta,\gamma}(x).$$

With $C_1 = \alpha k + \beta k + \gamma\binom{k}{2}$ being a constant in c.

Chapter 6

Conclusions

We have established determinant formulae for averages of secular coefficients. In the limit these random matrix theory averages are conjectured to behave like the number theoretic integrals over divisor sums. We also showed that the lower order terms of the random matrix theory averages have a similar behaviour to $\gamma_k(c)$. We end the thesis by raising some further questions for research.

Q1. We know that $\gamma_k(c)$ has an integral formulation as

$$\gamma_k(c) = \int_{[0,1]^k} \delta(\sum_i x_i - c) \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 dx$$

and $\gamma_k(c)$ is the highest order term (N^{k^2-1}) in the asymptotics of $I_k^G(n, N)$ with $G = U(N)$. Do there exist integral formulations of the cases when $G = O(N)$ or $G = Sp(2N)$? What about the lower order terms?

Q2. We have seen that the divisor function $d_k(n)$ in number theory gives rise to the polynomials $\gamma_k(c)$ in random matrix theory through the conjecture due to Keating et al.[12]. Is there a natural arithmetic function that gives rise to Symplectic and Orthogonal $\gamma_k(c)$? We suspect that $\chi(n)d_k(n)$, for real quadratic characters χ and $d_k(n^2)$ gives rise to Symplectic behaviour.

Q3. Since we have determinant identities for $I_k^G(n, N)$, is it possible to derive asymptotics from analyzing them? We were able to understand some properties from a general analysis in the previous section but it's not clear if these determinant identities can give asymptotics for $\gamma_k^G(c)$ and lower order terms as $k \to \infty$.

Q4. In the paper of Keating et al. a lattice point calculation for $I_k^G(n, N)$ with $G = U(N)$ is given which is then used to derive some other properties. $I_k^{U(N)}(m; N)$ is equal to the count of lattice points $x = (x_i^{(j)}) \in \mathbb{Z}^{k^2}$ satisfying the set of relations

1. $0 \leq x_i^{(j)} \leq N$ for all $1 \leq i, j \leq k$
2. $x_1^{(k)} + x_2^{(k-1)} + \cdots + x_k^{(1)} = kN - m$, and
3. $x \in A_k$,

where A_k is the collection of $k \times k$ matrices whose entries satisfy the following system of inequalities,

$$\begin{array}{ccccccc} x_1^{(1)} & \leq & x_1^{(2)} & \leq & \cdots & \leq & x_1^{(k)} \\ \vee\! | & & \vee\! | & & & & \vee\! | \\ x_2^{(1)} & \leq & x_2^{(2)} & \leq & \cdots & \leq & x_2^{(k)} \\ \vee\! | & & \vee\! | & & & & \vee\! | \\ \vdots & & \vdots & & \ddots & & \vdots \\ \vee\! | & & \vee\! | & & & & \vee\! | \\ x_k^{(1)} & \leq & x_k^{(2)} & \leq & \cdots & \leq & x_k^{(k)} \end{array}$$

Can natural lattice point counting analogues be given for $G = Sp(2N)$ or $O(N)$?

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APPENDICES

Appendix A

Tables and Plots

A.1 Tables of $\gamma_k^G(c)$

Given a matrix group and integers k, j we give the polynomial defining $\gamma_k^G(c)$ on $c \in [j-1, j]$.

A.1.1 Unitary Group

(k, j)	$(k^2-1)!\gamma_k(c)$
$(2,1)$	c^3
$(2,2)$	$(2-c)^3$
$(3,1)$	c^8
$(3,2)$	$-2c^8 + 24c^7 252c^6 + 1512c^5 4830c^4 + 8568c^3 8484c^2 + 4392c 927$
$(3,3)$	$(c-3)^8$
$(4,1)$	c^{15}
$(4,2)$	$-3c^{15} + 60c^{14} - 1680c^{13} + 29120c^{12} - 294840c^{11} + 1873872c^{10} - 7927920c^9$ $+23268960\mathrm{c}^8 - 48674340c^7 + 73653580c^6 - 80912832c^5 + 63969360c^4$ -35497280 $\mathrm{c}^3 + 13131720c^2 - 2910240c + 292464$
$(4,3)$	$3c^{15} - 120c^{14} + 3360c^{13} - 58240c^{12} + 644280c^{11} - 4948944c^{10} + 28428400c^9$ -128700000 $\mathrm{c}^8 + 470398500c^7 - 1381480100c^6 + 3179336160c^5 - 5531176560c^4$ +6950332480 $\mathrm{c}^3 - 5910494520c^2 + 3031004640c - 705916304$
$(4,4)$	$(4-c)^{15}$

A.1.2 Symplectic Group

(k,j)	$\frac{(k+2)(k-1)}{2}!\gamma_k(c)$
$(2,1)$	c^2
$(2,2)$	$(c-2)^2$
$(3,1)$	c^5
$(3,2)$	$15c^4 - 90c^3 + 190c^2 - 165c + 51$
$(3,3)$	$(3-c)^5$
$(4,1)$	c^9
$(4,2)$	$c^9 - 36c^8 + 576c^7 - 3696c^6 + 12096c^5 - 22680c^4 + 25536c^3 - 17136c^2 + 6336c - 996$
$(4,3)$	$-c^9 + 1680c^6 - 20160c^5 + 106344c^4 - 307776c^3 + 508176c^2 - 449856c + 165916$
$(4,4)$	$(4-c)^9$

A.1.3 Orthogonal Group

The orthogonal group has a slightly different form than the unitary and symplectic groups for odd k. For odd k, $\gamma_k(c)$ is supported on $[0, k-1]$. Also, when $k=2$ the scaling factor is 1 in the below table.

(k,j)	$\frac{(k+1)(k-2)}{2}!\gamma_k(c)$
$(2,1)$	1
$(2,2)$	1
$(3,1)$	c^2
$(3,2)$	c^2
$(3,3)$	0
$(4,1)$	c^5
$(4,2)$	c^5
$(4,3)$	$(4-c)^5$
$(4,4)$	$(4-c)^5$
$(5,1)$	c^9
$(5,2)$	c^9
$(5,3)$	$-c^9 + 3360c^6 - 50400c^5 + 330624c^4 - 1182720c^3 + 2396160c^2 - 2580480c + 1146880$
$(5,4)$	$-c^9 + 3360c^6 - 50400c^5 + 330624c^4 - 1182720c^3 + 2396160c^2 - 2580480c + 1146880$
$(5,5)$	0

A.2 Plots of $\gamma_k^G(c)$

To illustrate the gaussian and highly smooth nature of $\gamma_k^G(c)$ we plot it below for $k = 4$.

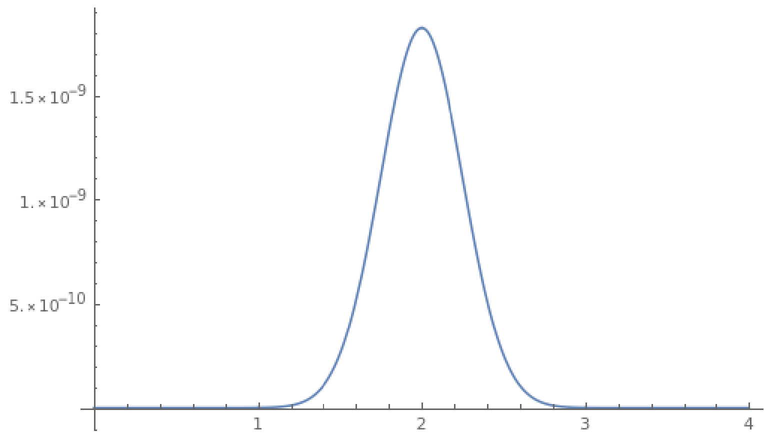

Figure A.1: $G = U(N), k = 4$

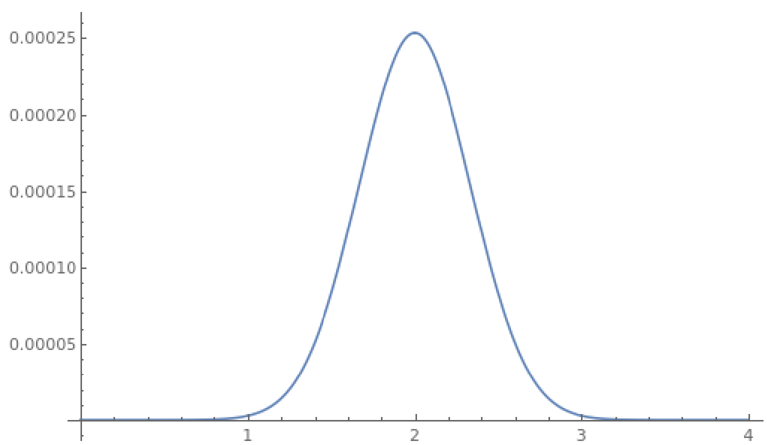

Figure A.2: $G = SP(2N), k = 4$

And for odd k in the case that $G = O(N)$:

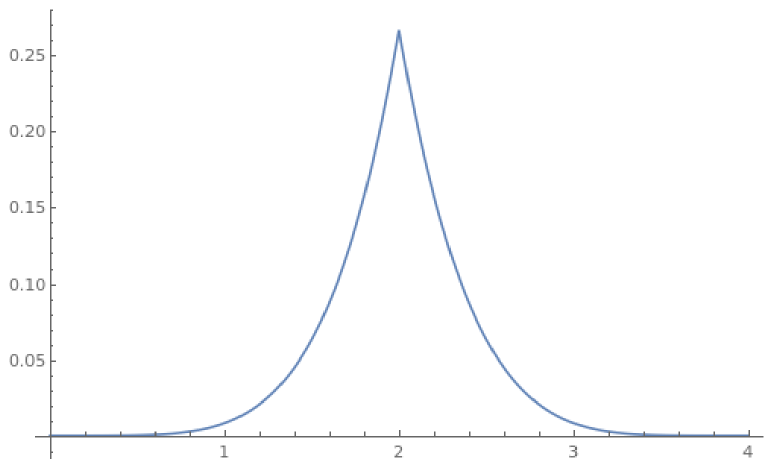

Figure A.3: $G = O(N), k = 4$

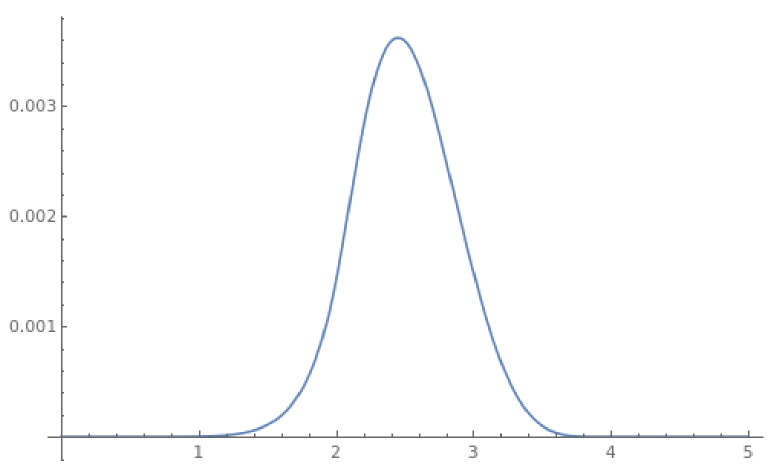

Figure A.4: $G = O(N), k = 5$