On Finding Large Cliques when χ is close to Δ

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We prove that every graph G with chromatic number $\chi(G) = \Delta(G) - 1$ and $\Delta(G) \geq 66$ contains a clique of size $\Delta(G) - 17$. Our proof closely parallels a proof from Cranston and Rabern, who showed that graphs with $\chi = \Delta$ and $\Delta \geq 13$ contain a clique of size $\Delta - 3$ [6]. Their result is the best currently known for general Δ towards the Borodin-Kostochka conjecture [1], which posits that graphs with $\chi = \Delta$ and $\Delta \geq 9$ contain a clique of size Δ. We also outline some related progress which has been made towards the conjecture.

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Thank you to Dr. Paul Buckingham, who introduced me to the subject of graph theory in one of the most enjoyable courses I've ever taken.

We are indebted to the mathematical contributions made by Dr. Daniel W. Cranston and the late Dr. Landon Rabern. Their work was of central importance to this thesis.

Dedication

This thesis is dedicated to my parents, Scott and Stella MacDonald. Thank you for always believing in me, even during those times when I can't see why.

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List of Symbols

1. $[n]$: The set $\{1, 2, 3, 4, \ldots, n\}$.
2. $act(M)$: The active component of M, see Definition 3.9.
3. $bad(n)$: The set of vertex-critical counterexamples to BK with $\Delta = n$.
4. $ce(U, v)$: The chromatic excess of v into U, see Definition 4.1.
5. $cg(v)$: The core graph of v, see Definition 4.3.
6. $clb(X)$: The club of X, see Definition 3.18.
7. $cores(v)$: The cores of v, see Definition 4.3.
8. C_n: A cycle with n vertices, or a core in a Mozhan partition.
9. $C_{n,m}$: The graph formed by expanding the vertices of C_n to cliques of size m, see Figure 1.1 for an example.
10. $deg(v), deg(G, v)$: The degree of vertex v, and the degree of vertex v in graph G, respectively.
11. $E(G)$: The edge set of G.
12. E_n: An edgeless graph (independent set) with n vertices.
13. $g(G)$: The girth of the graph G, see Definition 1.3.
14. G_t: Defined as $C_{5,t}$.
15. $G[U]$: The induced subgraph of G on the vertices U.
16. $G - v$: The induced subgraph $G[V(G) \setminus \{v\}]$.

17. $G - U$: The induced subgraph $G[V(G) \setminus U]$.

18. $G + v$: Given a parent graph H containing $V(G)$ and v, the induced subgraph $H[V(G) \cup \{v\}]$.

19. $G + U$: Given a parent graph H containing $V(G)$ and U, the induced subgraph $H[V(G) \cup U]$.

20. $G \vee H$: The join of G and H, formed by taking the disjoint union of G and H, and adding all edges between them, see Definition 1.11.

21. k: Usually used to denote the number of parts of a Mozhan partition.

22. K_n: A clique with n vertices.

23. $K_{n,m}$: A complete bipartite graph joining vertex sets of size n and m.

24. $N(v), N(G, v)$: The neighbours of vertex v, and the neighbours of vertex v in graph G: respectively.

25. m: Used to denote the minimum parameter of a Mozhan partition.

26. P_n: A path with n vertices, or a part in a Mozhan partition, see Definition 3.5.

27. $pot, pot(L), pot(\phi)$: The pool of available colours, see Definition 2.13.

28. $\mathcal{P}(S)$: The power set of S.

29. $V(G)$: The vertex set of graph G.

30. $r_1, \ldots, r_k$: The parameters of a Mozhan partition, see Definition 3.6.

31. s: Used to denote the special index of a Mozhan partition.

32. $\Delta, \Delta(G)$: Maximum degree, the maximum number of neighbours of any vertex.

33. $\chi, \chi(G)$: Chromatic number, the minimum number of colours necessary for a colouring.

34. $\omega, \omega(G)$: Clique number, the maximum size of a clique.

35. $\sigma_1(M), \sigma_2(M)$: Functions to be minimized on Mozhan partitions, see Definition 3.8.

“All these equations are like miracles. You take two numbers and when you add them, they magically become one ***new*** number! No one can say how it happens. You either believe it or you don’t.”

- Calvin, *Calvin and Hobbes*

Chapter 1

Introduction

1.1 Notation and Conventions

We will take our graphs to be finite, undirected, without loops, and without multiple edges. Throughout this document, we will use the notational conventions of Diestel [7], except that an n-**clique** (a graph with all possible edges) will be written K_n instead of K^n. The **maximum degree** of a graph G is denoted $\Delta(G)$, the **chromatic number** denoted $\chi(G)$, and the **clique number** denoted $\omega(G)$. A list of the more technical terminology may be found in the Glossary, and symbols in the List of Symbols.

1.2 Large Chromatic Number

The first bound one learns in graph colouring is the Greedy Bound.

Theorem 1.1. ***The Greedy Bound***
Let G be a graph. Then $\chi(G) \leq \Delta(G) + 1$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Let $c_1, c_2, \ldots, c_{\Delta(G)+1}$ be colours. We can colour G in n steps. At step i, colour v_i some colour c_j, such that c_j has not yet been used on any of the neighbours of v_i. There will always be an available colour c_j, as the vertex v_i has at most $\Delta(G)$ neighbours. □

Having established this relatively easy bound, one may ask if it can be improved. In a sense the answer is no. If G is a clique on n vertices then G has $\Delta(G) = n - 1$, and

$\chi(G) = n$, reaching the ceiling of the Greedy Bound and dashing our hopes for something lower. However, it turns out that for Δ sufficiently large, cliques are the *only* graphs which are tight with the Greedy Bound. This result, shown in 1941 by R. Leonard Brooks, is quite famous in graph colouring.

Theorem 1.2. ***Brooks' Theorem*** *[2]*
Let G be a connected graph with $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geq 3$. Then G is a clique of size $\Delta(G) + 1$.

The condition that $\Delta(G) \geq 3$ is necessary to exclude the case when G is an odd cycle, which needs $\Delta(G) + 1 = 3$ colours, but which is not a clique unless $|G| = 3$.

Less formally, this theorem states that if a connected graph G has chromatic number χ as large as possible, and Δ sufficiently large, then G is a large clique. If we remove the condition that G is connected, we weaken the conclusion so that some *component* of G is a large clique. This theorem has led to a rich line of inquiry concerning the appearance of large cliques in graphs with large chromatic number.

Brooks' Theorem tells us that when a graph has maximum chromatic number $\chi = \Delta + 1$, a large clique is responsible. It is natural to wonder whether large cliques are *always* responsible for large chromatic numbers in graphs, say, when $\chi = \Delta$ or $\chi = \Delta - 1$. The answers to these questions are unknown, but as we will see, there has been some progress made towards answering them.

Before we consider what happens when the chromatic number χ is close to Δ, it will be interesting to consider a theorem due to Erdős, for which we will need a quick definition.

Definition 1.3.
Let G be a graph. If G contains a cycle, the **girth** of G, denoted $g(G)$, is the minimum length of a cycle in G. If G contains no cycles, then $g(G) \coloneqq \infty$.

Theorem 1.4. ***Erdős, 1959*** *[7, p.125]*
For every integer n, there exists a graph G with $\chi(G) \geq n$ and girth $g(G) \geq n$.

Now, every clique K with at least 3 vertices has the smallest possible girth $g(K) = 3$. So in this absolute sense of "large", graphs can have large chromatic numbers, and no big cliques at all. Graphs with large girth look locally like trees, if you focus on a vertex and the vertices near to it, you find no cycles. Since all trees are 2-colourable, one might expect graphs with large girth to have small chromatic number. This theorem defies that intuition, and is commonly understood to mean that the chromatic number is a "global" property of a graph. That is, colouring one region of a graph cannot necessarily be done without considering the colourings of other regions in the same connected component. This conclusion does not look good for the search for big cliques. If graphs can have astronomical chromatic numbers and lack so much as a triangle, then big cliques cannot always be responsible for large chromatic numbers.

So in this extremely informal, intuitive realm, we have two opposing ideas at work. Brooks' Theorem suggests that chromatic numbers which are large *with respect to* Δ might be caused by large cliques. The theorem of Erdős suggests that chromatic numbers can be inflated by complicated global factors unrelated to the simple presence of a clique. We know, via Brooks', exactly what happens in the case $\chi = \Delta + 1$, so the natural next step is to ask what happens when $\chi = \Delta$. Strangely enough, in the 80 years that have passed since the proof of Brooks' Theorem, this question has not been answered. However, the following conjecture made in 1977 by Borodin and Kostochka posits the existence of another large clique. In fact, the largest that could be hoped for.

1.3 The Borodin-Kostochka Conjecture

Conjecture 1.5. ***The Borodin-Kostochka Conjecture*** *[1]*
Let G be a graph with $\chi(G) = \Delta(G)$, and $\Delta(G) \geq 9$. Then G contains a clique of size $\Delta(G)$.

As indicated by the name, the Borodin-Kostochka Conjecture has not yet been settled. However, much progress has been made towards finding the desired big cliques. The Borodin-Kostochka conjecture is of central importance to us, and for brevity it will be often written "BK".

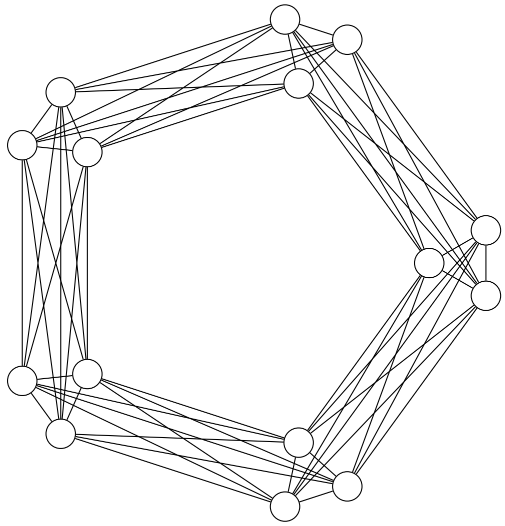

Figure 1.1: The graph $G_3 = C_{5,3}$, a cycle of five triangles.

The bound $\Delta \geq 9$ is known to be tight, as can be seen from the following example. We will denote the cycle on n vertices by C_n, and the path on n vertices by P_n, and $C_{n,m}$ will denote the graph formed by taking C_n and expanding each vertex to a clique of size m. Now, let G be the graph $C_{5,3}$ (see Figure 1.1), where each vertex of C_5 is expanded into a triangle. Then G contains 15 vertices, but the size of an independent set in G is at most 2. Thus, the chromatic number of G is at least $15/2 = 7.5$. Since $\chi(G)$ must be an integer we obtain the stronger bound $\chi(G) \geq 8$. Finally, G is a regular graph with $\Delta(G) = 8$, but $\omega(G) = 6$, so there is no 8-clique, as would be required by BK. Interestingly, not only does this graph fail to contain an 8-clique, it actually misses the mark by 2, and contains only 6-cliques.

The Borodin-Kostochka conjecture is a well-known problem which has attracted much effort (for instance, see [3, 4, 5, 6, 8, 12, 19]), which we will now outline in brief. For another summary of this history, please see Cranston and Rabern [4].

In 1977, in the same paper where they proposed their conjecture, Borodin and Kostochka themselves proved the following weakening.

Theorem 1.6. ***Graphs with $\chi = \Delta$ have Medium Cliques*** *[1]*
Let G be a graph with $\chi(G) = \Delta(G)$ and $\Delta(G) \geq 7$. Then G contains a clique of size $\lfloor(\Delta(G) + 1)/2\rfloor$.

The proof is not complicated and is illustrative of many key techniques for related arguments, including the main argument of this thesis. We will cover it in more detail in

Chapter 3, see Theorem 3.4. The result establishes some hope for our search, as it proves that graphs with $\chi = \Delta$ contain a clique which is bounded below by a linear function in Δ.

In 1980, Kostochka proved the following.

Theorem 1.7. *[12]*
Let G be a graph with $\Delta(G) \geq 29$ and $\chi(G) = \Delta(G)$. Then G contains a clique of size $\Delta(G) - 28$.

This yields a larger clique than the $\lfloor(\Delta(G) + 1)/2\rfloor$ result for every $\Delta \geq 58$. Conceptually, this result puts the finish line in view, as it allows us to focus entirely on reducing the constant 28 down to (hopefully, eventually) 0.

In 1999, Reed used probabilistic methods to show that the Borodin-Kostochka conjecture holds for all graphs with sufficiently large maximum degree. The specific bound he established was $\Delta \geq 10^{14}$, but he remarked that a more detailed analysis could potentially bring this down to 10^6 or 10^3.

Theorem 1.8. ***BK Holds Eventually*** *[19]*
Let G be a graph with $\chi(G) = \Delta(G) \geq 10^{14}$. Then G contains a clique of size $\Delta(G)$.

There are many results which prove the conjecture in the case of graphs which do not have some forbidden subgraph. In 2020 Pradhan and Gupta proved that the conjecture holds for graphs with no induced P_5 or C_4.

Theorem 1.9. ***BK Holds without*** P_5 ***and*** C_4 *[8]*
Let G be a graph with $\chi(G) = \Delta(G) \geq 9$, with no induced P_5 or C_4. Then G contains a clique of size $\Delta(G)$.

The **claw** is the graph $K_{1,3}$. In 2012, Cranston and Rabern proved that the conjecture holds for claw-free graphs.

Theorem 1.10. ***BK Holds with no Claws*** *[5]*
Let G be a graph with $\chi(G) = \Delta(G) \geq 9$, with no induced claw. Then G contains a clique of size $\Delta(G)$.

In 2015 Cranston and Rabern proved that the Borodin-Kostochka conjecture is equivalent to a statement that appears quite weak in comparison. First, we need a definition.

Definition 1.11.
Let G and H be graphs. Then the **join** of G and H, denoted $G \vee H$, is the graph formed by taking the disjoint union of G and H, and adding every edge between $V(G)$ and $V(H)$.

Similarly, when two subgraphs A and B of a parent graph G have all possible edges between them, we say that A and B are joined.

The equivalent conjecture is then:

Conjecture 1.12.
Let G be a graph with $\chi(G) = \Delta(G)$ and $\Delta(G) = 9$. Then G contains $K_3 \vee E_6$.

Chapter 2 is dedicated to discussing the proof that this conjecture is equivalent to BK.

In 2013, Cranston and Rabern proved the following.

Theorem 1.13. ***Cranston and Rabern, 2013*** *[6]*
Let G be a graph with $\chi(G) = \Delta(G)$ and $\Delta(G) \geq 13$. Then G contains a clique of size $\Delta(G) - 3$.

As of today (December 23, 2021), this is the best known result towards the Borodin-Kostochka Conjecture for general Δ. In proving this result, Cranston and Rabern have gotten tantalizingly close to the finish line, producing a clique only 3 vertices away from the target Δ, and a minimum necessary Δ only 4 away from the target of 9. The main purpose of this thesis is to demonstrate a thorough understanding of the proof of Theorem 1.13. To do so, we extend the argument of Cranston and Rabern to address the case $\chi = \Delta - 1$, and use their methods to prove the existence of a $\Delta - 17$ clique. This result is presented in Theorem 4.16.

1.4 Relation to Reed's Conjecture

Reed's conjecture is another open problem in graph colouring which has also attracted much attention, for instance, see [9, 11, 16, 20]. The statement is as follows.

Conjecture 1.14. ***Reed's Conjecture*** *[18]*
Let G be any graph. Then $\chi(G) \leq \lceil \frac{1}{2} (\Delta(G) + 1 + \omega(G)) \rceil$.

The bound in Reed's conjecture cannot be improved, as can be seen from the following family of counterexamples. For any natural t, let G_t be the graph $C_{5,t}$, an "expanded cycle" where each vertex in a copy of C_5 is expanded into a clique of size t (for an example, see Figure 1.1). Then G_t is regular, and $\Delta(G_t) = 3t - 1$. Furthermore, $\omega(G_t) = 2t$, and as the size of any independent set in G_t is at most 2, we have $\chi(G_t) \geq |G_t|/2 = 5t/2$.

Now, we compute

$$\begin{aligned} \lceil \frac{1}{2} (\Delta(G_t) + 1 + \omega(G_t)) \rceil &= \lceil \frac{1}{2} ((3t - 1) + 1 + (2t)) \rceil \\ &= \lceil \frac{5t}{2} \rceil \end{aligned}$$

but we had $\chi(G_t) \geq \frac{5t}{2}$, meaning that the proposed inequality in Reed's conjecture is best-possible.

This conjecture strengthens the Greedy Bound, and is closely related to our search for big cliques in graphs with large chromatic number. Indeed, assuming the truth of this inequality, heuristically we see that if χ has any hope of getting close to Δ it must be that ω is large.

To make things precise, suppose that we have $\omega(G) = \Delta(G) - k$. Again supposing the truth of Reed's conjecture for a moment, we obtain

$$\begin{aligned} \chi(G) &\leq \lceil \frac{1}{2} (\Delta(G) + 1 + \Delta(G) - k) \rceil \\ \implies \chi(G) &\leq \lceil \Delta(G) - \frac{1}{2} (k - 1) \rceil \end{aligned}$$

and using the fact that integers can be pulled out from the ceiling function,

$$\implies \chi(G) \leq \Delta(G) + \lceil \frac{1}{2} (1 - k) \rceil.$$

Now in the case that k is even, we get

$$\begin{aligned} \lceil \frac{1}{2} (1 - k) \rceil &= -\frac{k}{2} + 1 \\ \implies \chi(G) \leq \Delta(G) &- \frac{k}{2} + 1 \end{aligned}$$

and when k is odd, we get

$$\begin{aligned} \lceil \frac{1}{2}(1-k) \rceil &= -\frac{k}{2} + \frac{1}{2} \\ \implies \chi(G) &\leq \Delta(G) - \frac{k}{2} + \frac{1}{2}. \end{aligned}$$

Now consider what happens for various small values of k. If we take $k = -1$ (the only time k can be negative), we get $\chi(G) \leq \Delta(G) + 1$, the Greedy Bound. If we take $k = 0$ so that $\omega(G) \leq \Delta(G)$ we again get $\chi(G) \leq \Delta(G) + 1$. From Brooks' Theorem we know that $\omega(G) \leq \Delta(G) \implies \chi(G) \leq \Delta(G)$, so Reed's conjecture is weaker in this case. If we take $k = 1$ so that $\omega(G) \leq \Delta(G) - 1$, we get $\chi(G) \leq \Delta(G)$. The Borodin-Kostochka conjecture would have $\omega(G) \leq \Delta(G) - 1 \implies \chi(G) \leq \Delta(G) - 1$, so Reed's conjecture is also weaker in this case. So what *does* Reed's conjecture have to say about the case $\chi(G) = \Delta(G)$? Well, if we take $k \geq 3$, then Reed's would have $\chi(G) \leq \Delta(G) - 1$, which does not allow $\chi = \Delta$. However, $k = 2$ gives only $\chi(G) \leq \Delta(G)$, so the truth of Reed's conjecture requires a clique of size $\Delta(G) - 2$ when $\chi(G) = \Delta(G)$. This is one off of Cranston and Rabern's $\Delta - 3$ result (Theorem 1.13). Finally, in the case $\chi(G) = \Delta(G) - 1$, taking $k \geq 5$ gives $\chi \leq \Delta - 2$, a contradiction, but $k \leq 4$ requires only $\chi \leq \Delta - 1$. So our $\Delta - 17$ result (Theorem 4.16) is 13 vertices away from the clique which would be required by Reed's conjecture when $\chi = \Delta - 1$.

1.5 Related Asymptotics

We will need the following definitions.

Definition 1.15.
Let G be a graph. We say that a vertex $v \in G$ is **dominating** when $N(v) = V(G - v)$. That is, v is adjacent to all other vertices.

Definition 1.16.
Let G be a graph, and let $v \in G$ be a vertex. We say that v is **critical** when $\chi(G - v) < \chi(G)$.

Remark. If G is a graph, and $v \in G$ is critical, then $\chi(G - v) = \chi(G) - 1$. Indeed, we know that $\chi(G - v) \leq \chi(G) - 1$ by the definition of criticality. Finally, if $G - v$ can be $(\chi(G) - 2)$-coloured, then using a single additional colour for v produces a $(\chi(G) - 1)$-colouring of G, which is impossible.

Definition 1.17.
Let G be a graph. If $\chi(G) = n$, and every proper subgraph of G is $(n-1)$-colourable, then we say that G is n**-critical**.

Observe that if G is a n-critical graph, then every vertex $v \in G$ is a critical vertex. Indeed, the proper subgraph $G - v \subset G$ is $(n-1)$-colourable, and thus satisfies $\chi(G - v) < \chi(G)$.

As mentioned previously, Reed has proved BK for graphs with $\Delta \geq 10^{14}$ (Theorem 1.8). Together with Farzad and Molloy, Reed has also proved a family of related statements about graphs with $\chi = \Delta - t$ for $t = 1, 2, 3, 4, 5$ [14]. Using probabilistic methods, they proved that $\Delta - t$ critical graphs must contain a clique of size $\Delta - t$, or a clique of size $\Delta - t - l$ for some small integer l, joined to an odd cycle, or a small critical graph with no dominating vertices.

For instance, their result in the $\Delta - 1$ case is the following.

Theorem 1.18. *[14]*
There exists a number N such that, if G is a graph with $\Delta(G) > N$ and $\chi(G) \geq \Delta(G) - 1$, then G must contain one of:

1. *A $(\Delta(G) - 1)$-clique.*
2. *A $(\Delta(G) - 4)$-clique joined to C_5.*

As t increases, the list of excluded subgraphs grows longer and more complicated.

Theorem 1.19. *[14]*
There exists a number N such that, if G is a graph with $\Delta(G) > N$ and $\chi(G) \geq \Delta(G) - 2$, then G must contain one of:

1. *A $(\Delta(G) - 2)$-clique.*
2. *A $(\Delta(G) - 5)$-clique joined to C_5.*
3. *A $(\Delta(G) - 6)$-clique joined to a 4-critical graph on 7 vertices, with no dominating vertex.*

There are exactly two 4-critical graphs on 7 vertices with no dominating vertex, giving 4 possibilities. For brevity in what follows, we make the following definition.

Definition 1.20.
Suppose that G is a l-critical graph with n vertices and no dominating vertex. Then we refer to G as an (n, l)-**nucleus**.

Theorem 1.21. *[14]*
There exists a number N such that, if G is a graph with $\Delta(G) > N$ and $\chi(G) \geq \Delta(G) - 3$, then G must contain one of:

1. *A $(\Delta(G) - 3)$-clique.*
2. *A $(\Delta(G) - 6)$-clique joined to C_5.*
3. *A $(\Delta(G) - 6)$-clique joined to C_7.*
4. *A $(\Delta(G) - 7)$-clique joined to a $(7, 4)$-nucleus.*
5. *A $(\Delta(G) - 7)$-clique joined to an $(8, 4)$-nucleus.*
6. *A $(\Delta(G) - 8)$-clique joined to a $(9, 5)$-nucleus.*
7. *A $(\Delta(G) - 9)$-clique joined to a $(10, 6)$-nucleus.*

This gives exactly 26 total possibilities. Interestingly, there is only a single $(10, 6)$ nucleus.

In the case $\chi = \Delta - 4$, the list of possibilities contains 420 graphs, and in the case $\chi = \Delta - 5$, the list is only known to contain at *least* 17000 graphs. This is because a complete characterization of the possible critical subgraphs relies on a characterization of the l critical graphs on n vertices, which quickly becomes computationally expensive.

We conclude by observing that these results are promising for our search. Indeed, they seem to suggest that when χ is close to Δ, that a clique of size close to Δ must be involved in raising the chromatic number, which cannot be inflated by complex, global, structural factors alone.

1.6 Overview

The remainder of this thesis is structured as follows. Chapter 2 is dedicated to discussing the following result due to Cranston and Rabern (see Conjecture 1.12):

Theorem 1.22. ***BK is Weaker than it Appears[4]***
Suppose that every graph G with $\chi(G) = \Delta(G) = 9$ contains $K_3 \vee E_6$ as a subgraph. Then the Borodin-Kostochka conjecture is true.

We present some of the main ideas involved in this proof, including a complete description of the reduction to $\Delta = 9$, the special minimality condition introduced by the authors, and the main idea involved in excluding the $K_3 \vee E_6$ subgraph.

Chapter 3 begins the proof of our result by introducing some of the key ideas used in the proof, most importantly "Mozhan partitions" (Definition 3.5). We then go on to prove some preliminary facts about Mozhan partitions and their transformations, dubbed "moves" (Definition 3.16).

Chapter 4 builds on Chapter 3. Using the accumulated facts about Mozhan partitions, we show that it is possible to find many small cliques inside the parts of a Mozhan partition, and then show that many of these small cliques are actually joined together, producing the desired result.

In Chapter 5, we discuss future avenues for these ideas and how they might be used to further our understanding of graphs with high chromatic number.

Chapter 2

A (Seemingly) Weaker Statement

As mentioned before, the main result of this thesis (Theorem 4.16) was produced as a demonstration of the methods used by Cranston and Rabern in their $\Delta - 3$ result (Theorem 1.13), extended to the $\chi = \Delta - 1$ case. In order to more thoroughly survey the progress which has been made on BK, we will now highlight some of the main ideas used in another result proved by Cranston and Rabern, prolific as they are, in pursuit of the conjecture. This highlighting will not constitute a full proof, we only hope to acquaint the reader with the main ideas. The result is as follows.

Theorem 2.1. ***BK is Weaker than it Appears*** *[4]*
Suppose that every graph G with $\chi(G) = \Delta(G) = 9$ contains $K_3 \vee E_6$ as a subgraph. Then the Borodin-Kostochka conjecture is true.

There are two interesting details to note here. The first is that it is only necessary to consider the case $\Delta = 9$, as opposed to $\Delta \geq 9$. The second is that containing $K_3 \vee E_6$ is strictly weaker than containing a K_9, as is required by the Borodin Kostochka Conjecture. Taken together it is surprising that this statement, which appears to be a strictly weaker consequence of the conjecture, is in fact equivalent to the conjecture. We now outline how this equivalence is proved.

The proof can be split into two independent steps. The first step is to prove that it is sufficient to prove BK in the case of $\Delta = 9$, ignoring every $\Delta > 9$. The second step is to prove that the subgraph $K_3 \vee E_{\Delta-3}$ cannot appear in any minimal counterexample to BK, with respect to a carefully defined minimality condition.

2.1 Reduction to $\Delta = 9$

First, we will focus on the reduction of $\Delta \geq 9$ to $\Delta = 9$. The possibility of this reduction was known to Kostochka [12] and Catlin [3]. The reduction is actually quite straightforward, and the main instrument involved is the following lemma due to King. The lemma was proved as a tool for approaching Reed's conjecture, and is an improvement on a prior result of Rabern's [17]. It will also be instrumental in the final step of our own $\Delta - 17$ result (Theorem 4.16).

Lemma 2.2. ***King*** *[10]*
Let G be a graph with $\omega(G) > (2/3)(\Delta(G) + 1)$. Then G contains an independent set I, which contains one vertex from every maximum clique in G.

The bound in this lemma is known to be tight. Indeed, recall the graphs $G_t = C_{5,t}$ for any natural number t. These graphs satisfy $\Delta = 3t - 1$, and $\omega = 2t$, so that $\omega(G_t) = (2/3)(\Delta(G_t) + 1)$. However, any maximal independent set in G_t has size 2, and must not intersect one of the 5 maximum cliques.

Suppose for a moment that we have a counterexample G to BK with $\chi = \Delta = n$ for some $n \geq 9$. Let G' be a vertex-critical subgraph of G, so that $\chi(G') = n$. As G' is a subgraph, we have $\Delta(G') \leq \Delta(G)$. However, if we actually get that $\Delta(G') < \Delta(G)$, then we have $\Delta(G') < \Delta(G) = \chi(G) = \chi(G')$. Now, as $\Delta(G') < \chi(G')$, the Greedy Bound tells us that $\chi(G') = \Delta(G') + 1$. We therefore obtain $\Delta(G') = \Delta(G) - 1 \geq 8$. But now, Brook's Theorem kicks in, and gives us a clique K_n of size $\chi(G') = n$ in G'. But G' is a subgraph of G, and so G also contains K_n as a subgraph. This contradicts the fact that G is a counterexample to BK, and therefore does not contain a clique of size n. All together, $\chi(G') = \Delta(G') = n$, and G' is n-critical. In summary, if we can find a counterexample, we can find a vertex-critical counterexample.

For $n \geq 9$, let $bad(n)$ be the set of vertex-critical counterexamples to BK with $\Delta = n$. Formally,

Definition 2.3.

$$bad(n) := \{G \ : \ \chi(G) = \Delta(G) = n, \ \omega(G) < n, \ G \text{ is vertex-critical}\}.$$

We will now show that for $n \geq 10$, if $bad(n)$ is nonempty, then $bad(n-1)$ is nonempty, completing the reduction to $\Delta = 9$. By our previous discussion, if there is a counterexample G to BK with $\chi(G) = n$, then all vertex-critical subgraphs of G (there must be at least one) are members of $bad(n)$. Now, suppose we have a fixed number $n \geq 10$, and a counterexample graph $G \in bad(n)$.

Suppose first that $\omega(G) \leq n-2$. Now let $I \subseteq V(G)$ be a maximal independent set, and let $G' \coloneqq G - I$. As G is vertex-critical, and I is an independent set, we have $\chi(G-I) = \chi(G) - 1$. Furthermore, as I is a maximal independent set, we have $\Delta(G') \leq \Delta(G) - 1$, as for every vertex $v \in G$ of maximum degree, I must contain a vertex in $\{v\} \cup N(G, v)$. Now let G'' be a vertex-critical subgraph of G'. Observe that $\chi(G'') = \chi(G') = n - 1$. Thus, if $\Delta(G'') \leq n-2$, Brooks' Theorem and the Greedy Bound would give us that G'' and thus G contain an $n-1$ clique, contradicting our assumption on $\omega(G)$. This means that $\Delta(G'') = \chi(G'') = n - 1$. By construction G'' is vertex-critical, and all together, we get $G'' \in bad(n-1)$.

Now suppose that $\omega(G) = n - 1$. We derive

$$\begin{aligned} & 5 < \Delta \\ \implies & 2 < \Delta - 3 \\ \implies & 2\Delta + 2 < 3\Delta - 3 \\ \implies & (2/3)(\Delta+1) < \Delta - 1 \\ \implies & (2/3)(\Delta+1) < \omega \end{aligned}$$

and so we may use the King result (Lemma 2.2) to get a maximal independent set I which intersects every maximum clique in G. As G is vertex-critical and I is independent, we have $\chi(G-I) = \chi(G) - 1$. Furthermore, as I contains one vertex from every maximum clique in G, we have $\omega(G-I) = \omega(G)-1$. Once again, we have that $\Delta(G-I) \leq \Delta(G)-1$ by the maximality of I. If we have $\Delta(G-I) \leq \Delta(G)-2 = \chi(G-I)-1$, the Greedy Bound gives us $\Delta(G-I) = \chi(G-I) - 1$, and then Brooks' Theorem gives us a clique of size $\chi(G-I) = \chi(G)-1 = n-1$ in $G-I$. This contradicts the construction of I, which implied that $\omega(G-I) = \omega(G) - 1 = n - 2$. Therefore, we once again have $\Delta(G-I) = \Delta(G) - 1$. Let G' be a critical subgraph of $G - I$. Then we compress our above argument and derive:

$$\Delta(G') < \Delta(G-I) = \chi(G-I) = \chi(G') \implies \omega(G-I) \geq \chi(G-I) = n-1$$

(via Brooks' and the Greedy Bound, as above), a contradiction to $\omega(G-I) = n-2$. This gives us $\Delta(G') = \Delta(G-I)$, and so we get $G' \in bad(n-1)$. Therefore, if $bad(n)$ is nonempty for $n > 9$, then $bad(n-1)$ is also nonempty. It follows that the emptiness of $bad(9)$ implies the nonexistence of any counterexample to BK, and would thus prove the conjecture.

2.2 Mules

As mentioned before, the second step is to prove that the subgraph $K_3 \vee E_{\Delta-3}$ cannot appear in any "minimal" counterexample to BK. Before we elaborate on the meaning of "minimal" in this context, we will need the following definition.

Definition 2.4.
Let G and H be graphs. A **graph epimorphism** is a map

$$f : V(G) \to V(H)$$

such that $vw \in E(G) \implies f(v)f(w) \in E(H)$, and $f(V(G)) = V(H)$. We sometimes write $f : G \to H$.

Graph epimorphisms are graph homomorphisms which surject onto the target vertex set. Observe that as a graph epimorphism $f : G \to H$ must be a graph homomorphism, every inverse image $f^{-1}(v) \subseteq V(G)$ must be an independent set, as graph homomorphisms are not allowed to send the two endpoints of any edge to the same vertex.

Cranston and Rabern then define the following graph relation.

Definition 2.5.
Let G be a graph. We call a graph H a **child** of G when there exists a subgraph $G' \subseteq G$ and a graph epimorphism from $G' \to H$.

If we have a graph epimorphism $f : G \to H$, we can construct the graph H from G as follows. For each $v \in V(H)$, collapse the independent set $f^{-1}(v) \subseteq V(G)$ to a single vertex, call the resulting graph G'. As f is surjective, we can identify $V(G')$ with $V(H)$. Then simply add each edge of $E(H) \setminus E(G')$ to G'. Thus, the child relation has the following equivalent definition.

Definition 2.6.
Let G be a graph. Let H be a graph obtained from G by:

1. Taking a subgraph.
2. Contracting independent sets into single vertices.
3. Adding edges.

Then we call H a **child** of G.

As might be suggested by the name, the child relation is a partial order.

Lemma 2.7.
The child relation is a partial order.

Proof. First, we show reflexivity. Let G be any graph. We have the subgraph $G \subseteq G$ and the graph epimorphism $id : G \to G$. Thus, G is a child of itself, as required.

For transitivity, suppose that A is a child of B, and B is a child of C. We would like to show that A is a child of C. Let $B' \subseteq B$ be a subgraph and $f : B' \to A$ be a graph epimorphism. Likewise, let $C' \subseteq C$ be a subgraph and $g : C' \to B$ be a graph epimorphism. Consider the subgraph $C'' := C[g^{-1}(B')]$, and let g' be the restriction of g to C''. Then g' is also a graph epimorphism, and as the composition of graph epimorphisms is a graph epimorphism, we have a graph epimorphism $f \circ g' : C'' \to A$. Thus, A is a child of C, as desired.

Finally we show antisymmetry. Observe that if H is a child of G and $H \neq G$, then either H has fewer vertices than G, or more edges than G. So suppose that G is a child of H, and H is a child of G. It cannot be that $H \neq G$, as otherwise G would have fewer vertices or more edges than itself, a contradiction. This proves that the child relation is reflexive, transitive, and antisymmetric, i.e. a partial order. □

This partial order is well-founded, that is, every nonempty set S of graphs has a minimal element under the child relation. Indeed, if G is a graph, then any proper child of G must have fewer vertices or more edges. So if $G \in S$ is taken to have a minimum number of vertices, and with respect to that a maximum number of edges, then G must be minimal under the child relation.

Definition 2.8.
Let S be a nonempty set of graphs. We refer to the graphs in S which are minimal under the child relation as S**-mules**, or just “mules” if the set S is clear from context.

Cranston and Rabern did not discuss the reasoning behind this terminology, but as pointed out by P. Haxell, it likely has to do with the fact that mules are sterile and therefore cannot have any children.

2.3 A Sample of the Subgraph Reduction

The remainder of Cranston and Rabern's proof is in showing that any mule $M \in bad(n)$, $n \geq 9$ cannot contain the subgraph $K_3 \vee E_{\Delta-3}$. Thus, when $n = 9$, no mule $M \in bad(9)$ may contain $K_3 \vee E_6$, so if every graph G with $\chi(G) = \Delta(G) = 9$ contains an $K_3 \vee E_6$, then it is impossible for there to be any counterexamples to BK, and so BK is true. To yield some insight into how this result is proved, we outline the proof of a related result, that every mule in $bad(n)$, $n \geq 9$ cannot contain the subgraph $K_4 \vee E_{\Delta-4}$. The main ideas used in both arguments are the same, but turning the "4" into a "3" requires a more in-depth technical analysis. Except for the short proof of Lemma 2.17 on d_1-choosability (to be defined later), the proofs in this section are not original, and appear as shown in [4].

We will need some definitions from the domain of list colouring.

Definition 2.9.
Let G be a graph. A **list assignment** on G is a map $L : V(G) \to \mathcal{P}(\mathbb{N})$.

Definition 2.10.
Let G be a graph, and let $L : V(G) \to \mathcal{P}(\mathbb{N})$ be any list assignment on G. We say that G is **L-colourable** when there is a colouring $\phi : V(G) \to \mathbb{N}$ such that $\phi(v) \in L(v)$ for every vertex v.

Definition 2.11.
Let G be a graph, let $f : V(G) \to \mathbb{Z}$ be any function, and let L be a list assignment on G. If for all $v \in V(G)$, we have $|L(v)| = f(v)$, then we say that L is an **f-assignment**.

Definition 2.12.
Let G be a graph, and let $f : V(G) \to \mathbb{Z}$ be any function. We say that G is **f-choosable** when, for every list assignment f-assignment L, we have that G is L-colourable.

Definition 2.13.
Let G be a graph, and let L be a list assignment on G. We define the **pot** of L to be

$$pot(L) \coloneqq \bigcup_{v \in V(G)} L(v).$$

If ϕ is a colouring of G, then we define $pot(\phi)$ to be the range $\phi(V(G))$.

List colouring generalizes the notion of colouring, as each vertex is allowed to have a different set of possible colours instead of having to choose from one global set of colours. The idea of list colouring arises naturally when one considers the idea of extending partial colourings of graphs. If you have a graph G and an induced subgraph U, and you have a l-colouring ϕ of $G - U$, then you can form the list assignment

$$L_\phi : V(U) \to \mathcal{P}(\mathbb{N}),\ v \mapsto \{n \in pot(\phi)\ :\ n \text{ is not used by } \phi \text{ on } N(G - U, v)\}$$

and if U admits an L-colouring ψ, then combining ϕ and ψ gives you a l-colouring of the entire graph G. Cranston and Rabern use this idea repeatedly in their proof of the subgraph reduction.

The following well-known theorem due to Hall is useful in the context of list colouring.

Theorem 2.14. ***Hall's Marriage Theorem*** *[7, p.38]*
Let G be a bipartite graph with bipartition (A, B). Then G admits a matching covering A if and only if, for all $U \subseteq A$, we have $|U| \leq |N(U)|$.

This is because, given a graph G and a list assignment L on G, we can form a bipartite graph H between $V(G)$ and $pot(L)$, including an edge vc between a vertex and a colour exactly when $c \in L(v)$. Then, if H admits a matching covering $V(G)$, that matching can be used to construct an L-colouring of G by colouring each vertex the colour it is paired with in the matching. This idea is not very useful in the context of ordinary graph colouring, as such a matching would entail a colouring where every vertex receives a different colour, which would be possible if and only if the number of colours is at least the number of vertices of the graph, an extremely generous (and therefore dull) situation. To see these ideas in action, we will give a proof of the "Small-Pot Lemma", a useful result which allows one to restrict the size of the pot considered in list colourings. To reiterate, this proof is not new. It appears here in the same form as in [4].

Lemma 2.15. ***The Small-Pot Lemma***
Let G be a graph, and let $f : V(G) \to \{1, 2, \ldots, |G| - 1\}$ be any function. Suppose that G is not f-choosable. Then there is an f-assignment L on G where G is not L-colourable, and $|pot(L)| < |G|$.

Proof. Suppose we have such a graph G and map f, and for the sake of a contradiction, assume that G is not f-choosable, but whenever an f-assignment L has $|pot(L)| < |G|$ that G is L-colourable. Since G is not f-choosable let L be an f-assignment, such that G is not L-colourable. By assumption, it must be that $|pot(L)| \geq |G|$. For every subset $U \subseteq V(G)$, let $L(U)$ be the set $L(U) \coloneqq \bigcup_{v \in U} L(v)$. Now, let H be a bipartite graph between $V(G)$ and $pot(L)$, where there is an edge $vc \in H$ exactly when $c \in L(v)$. That is, H forms an explicit representation of the colours available to each vertex v via L. Now, if there is a matching of H covering $V(G)$, then we have an L-colouring of G. By Hall's theorem, we know that such a matching can be found exactly when each subset $U \subseteq V(G)$ satisfies $|U| \leq |N(H, U)|$. For each subset $U \subseteq V(G)$, define $h(U) \coloneqq |U| - |L(U)|$. Since G is not L-colourable, there must be a subset U with $h(U) > 0$. Choose U to maximize $h(U)$, and let C be an arbitrary set of $|G| - 1$ colours which contains $L(U)$. We will construct a new list assignment L', as follows. For $v \in U$ define $L'(v) \coloneqq L(v)$. For $v \notin U$, define $L'(v)$ to be an arbitrary subset of C of size $f(v)$. By assumption, G admits an L'-colouring, which restricts to an L-colouring ϕ on U. Now for any subset $W \subseteq V(G) \setminus U$, we claim that $|W| \leq |L(W) \setminus L(U)|$. Indeed, supposing on the contrary that $|W| > |L(W) \setminus L(U)|$, we derive:

$$\begin{aligned} h(U \cup W) &= |U \cup W| - |L(U \cup W)| \\ &= |U| + |W| - |L(U \cup W)| \text{ Because } U \text{ and } W \text{ are disjoint.} \\ &= |U| + |W| - |L(U) \cup L(W)| \\ &= |U| + |W| - |L(U) \cup (L(W) \setminus L(U))| \\ &\geq |U| + |W| - |L(U)| - |L(W) \setminus L(U)| \\ &> |U| - |L(U)| \\ &= h(U). \end{aligned}$$

This contradicts the maximality of $h(U)$. Thus, for all sets $W \subseteq G \setminus U$, we indeed have $|W| \leq |L(W) \setminus L(U)|$. But via Hall's Theorem, this means that the L-colouring ϕ of U can be extended to all of G, contradicting the fact that G is not L-colourable. □

The Small Pot Lemma is used by Cranston and Rabern to characterize the graphs of the form $K_4 \vee D$, where D is any graph, which are not d_1-choosable (Lemma 2.20). This characterization restricts the number of possible subgraphs, in mules, which could be induced by the vertices of a $K_4 \vee E_{\Delta-4}$. They then show explicitly that each of these possible induced subgraphs cannot appear, and so there cannot be any $K_4 \vee E_{\Delta-4}$ subgraph.

We will need the following definition.

Definition 2.16.
Let G be a graph, and let l be some integer. Let $f : V(G) \to \mathbb{Z}, v \mapsto deg(G, v) - l$. If G is f-choosable, then we say that G is d_l-**choosable**.

The heart of Cranston and Rabern's weakening of BK is the following.

Lemma 2.17.
Let G be a vertex-critical graph with $\chi(G) = \Delta(G)$, and $\Delta(G) \geq 9$. Let U be an nonempty induced subgraph of G. Then U is not d_1-choosable.

Proof. Suppose for the sake of a contradiction that we have such a graph G, and an induced subgraph U which is d_1-choosable. By the vertex-criticality of G, and the nonemptiness of U, let ϕ be a $(\Delta(G)-1)$-colouring of $G-U$. We construct the following list assignment

$$L_\phi : V(U) \to \mathcal{P}(\mathbb{N}), u \mapsto pot(\phi) \setminus \bigcup_{v \in N(G-U,u)} \{\phi(v)\}$$

so that each list $L_\phi(u)$ contains exactly those colours which have not been used yet on the neighbours of u. Now, as U is an induced subgraph, we have that for each vertex $u \in U$, $deg(G,u) = deg(G-U,u) + deg(U,u)$. This allows us to derive:

$$\begin{aligned} |L(u)| &\geq (\Delta(G)-1) - deg(G-U,u) \\ &= (\Delta(G)-1) - (deg(G,u) - deg(U,u)) \\ &\geq (\Delta(G)-1) - (\Delta(G) - deg(U,u)) \\ &= deg(U,u) - 1. \end{aligned}$$

But this is a problem. By assumption, U is d_1-choosable. Thus, let ψ be an L_ϕ-colouring of U. Combining ϕ and ψ therefore yields a $(\Delta(G)-1)$-colouring of G. But $\chi(G) = \Delta(G)$, so this is impossible. □

To give an example of the use of the minimality criteria of mules, we prove the following.

Lemma 2.18.
Let G be a mule in $bad(n)$ for $n \geq 4$. Suppose that H is a child of G with $\Delta(H) \leq n$. Then either H is $(n-1)$-colourable, or H contains a clique of size n.

Proof. Let H be a child of G, with $\Delta(H) \leq n$. Suppose for the sake of contradiction that both $\chi(H) \geq n$ and that $\omega(H) \leq n-1$. If $\chi(H) \geq n+1$, then the Greedy Bound and Brooks' Theorem would give us that $\omega(H) = n+1$, contradicting our assumption that H does not contain a clique this large. Likewise, it cannot be that $\Delta(H) \leq n-1$, giving us that $\chi(H) = \Delta(H) = n$. But as G is a mule, H cannot be in $bad(n)$, and so H must not be vertex-critical. Let $H' \subset H$ be an induced vertex-critical subgraph. Observe that H' is a proper child of H, obtained by removing some vertices, contracting no independent sets, and adding no edges. By construction, we have $\chi(H') = \chi(H) = n$, and $\Delta(H') \leq \Delta(H) = n$, and $\omega(H') \leq \omega(H) < n$. So it cannot be that $\Delta(H') < n$, as this would imply that $\omega(H') = n$. This means that we have $\chi(H') = \Delta(H') = n$, and $\omega(H') < n$, and we know that H' is vertex critical. Thus, $H' \in bad(n)$. But H' is a proper child of H, and therefore a proper child of G, contradicting the fact that G is a mule. □

The following lemma is an intermediate result in Cranston and Rabern's full reduction of BK. The proof of this lemma demonstrates the main ideas used in the final proof of their subgraph reduction. Namely, assume the bad subgraph exists, use d_1-choosability to restrict the possible number of induced subgraphs on those vertices, then explicitly rule out each induced subgraph. We will need two specific mules, named $M_{7,1}$ and $M_{7,2}$ (see figures 2.1 and 2.2), both of which live in $bad(7)$. The mule $M_{7,1}$ is formed by taking the disjoint union of $K_5 \vee E_3$ and K_6, and joining each vertex of the E_3 to two distinct vertices of the K_6. The mule $M_{7,2}$ is formed by removing a maximal independent set from the graph $C_{5,3}$, which was the graph demonstrating the tightness of the bound $\Delta \geq 9$ for BK.

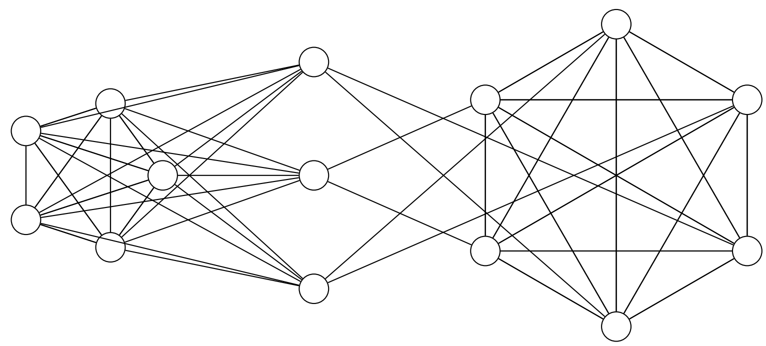

Figure 2.1: The mule $M_{7,1}$.

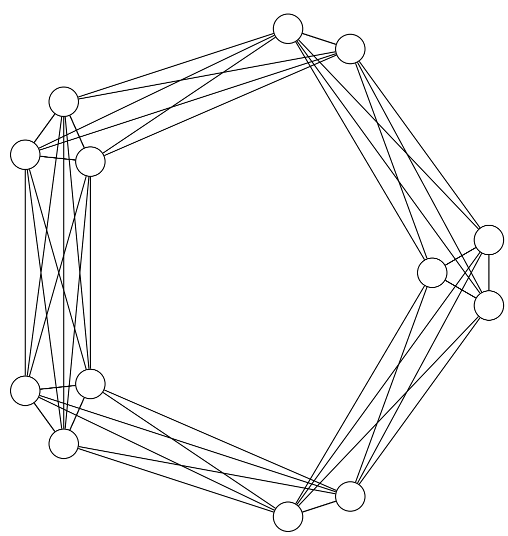

Figure 2.2: The mule $M_{7,2}$.

Lemma 2.19.
Let $n \geq 7$ and let G be a mule in bad(n) which is not $M_{7,1}$. Then G does not contain $K_4 \vee E_{n-4}$ as a subgraph.

We will need some intermediate results to prove this intermediate result.

Lemma 2.20.
Let $n \geq 4$. Suppose that the graph $K_n \vee D$ is not d_1-choosable. Then one of the following must hold:

- *D is a clique.*
- *D is a clique, minus one edge.*
- *$n = 4$ and $D = E_3$*
- *$n = 4$ and $D = K_{1,3}$*
- *$n = 5$ and $D = E_3$.*

Lemma 2.21.
For $n \geq 7$, the only mule in bad(n) containing an induced $E_3 \vee K_{n-3}$ is $M_{7,1}$.

Lemma 2.22.
Let $n \geq 7$, and let G be a mule in bad(n) which is not $M_{7,1}$ or $M_{7,2}$. Suppose that $H \subseteq G$ is an $(n-1)$-clique. Then any vertex in $G - H$ has at most 1 neighbour in H.

Lemma 2.23.
Let $n \geq 7$, and let G be a mule in bad(n) which is not $M_{7,1}$. Then G does not contain an induced $E_3 \vee K_{n-3}$.

Using these lemmas, they prove Lemma 2.19.

Proof. Let $n \geq 7$ and let G be a mule in *bad*(n) which is not $M_{7,1}$. Suppose for the sake of a contradiction that G contains $K_4 \vee E_{n-4}$ as a subgraph. Then the induced subgraph H on these vertices is of the form $H = K_4 \vee D$ where $|D| = n-4$. By Lemma 2.17, the graph $K_4 \vee D$ is not d_1-choosable, so by Lemma 2.20, D is a clique, a clique minus one edge, E_3, or $K_{1,3}$. If D is a clique, then G contains a K_n, which contradicts the fact that all graphs in *bad*(n) have $\omega < n$. If D is a clique less an edge, say an edge $xy \notin E(G)$ for $x, y \in V(D)$, then the graph $H - x$ is an $(n-1)$-clique. By Lemma 2.22, the vertex y has at most 1 neighbour in $H - y$, contradicting the fact that y is joined to $H - x$. The only remaining possibilities are that D is E_3 or $K_{1,3}$. However, observe that $K_4 \vee K_{1,3} = K_5 \vee E_3$. Thus, in either of the two remaining cases, H is an induced $E_3 \vee K_{n-4}$, which is impossible by Lemma 2.23. Thus, every possibility is impossible, and G cannot contain the subgraph $K_4 \vee E_{n-4}$. □

This proves that BK is equivalent to the "weaker" statement:

Conjecture 2.24.
Every graph G with $\chi(G) = \Delta(G) \geq 9$ contains $K_4 \vee E_{\Delta(G)-4}$ as a subgraph.

The remainder of Cranston and Rabern's reduction consists of ratcheting the "4" down to a "3". This is accomplished using very similar means. The induced subgraph on $K_3 \vee E_{\Delta-3}$ can only take a handful of forms, to prevent d_1-choosability, and each of these possible forms is proven to be impossible using specific colouring arguments. Rabern has published an extensive list of d_1-choosable graphs on his website, all of which therefore cannot be induced subgraphs in mules. For those reading a physical copy of this document, the link is: "https://landon.github.io/graphdata/borodinkostochka/offline/index.html".

This concludes our explicit tour of ideas which have been used to attack the Borodin-Kostochka conjecture. We continue on with an implicit tour, given in the form of a novel result.

Chapter 3

Extending to $\Delta - 1$

3.1 Background

Our goal is to prove the following theorem.

Theorem 3.1. ***Graphs with*** $\chi = \Delta - 1$ ***have Big Cliques***
Let G *be a graph with* $\chi(G) = \Delta(G) - 1$*, and* $\Delta(G) \geq 66$*. Then* G *contains a clique of size* $\Delta(G) - 17$*.*

To do this, we will begin by partitioning our graph into many smaller subgraphs, most of which will contain a small clique. Then we will show that many of these small cliques are fully joined to each other, producing a large clique.

For the remainder of this document, we will make frequent use of induced subgraphs. When G is a graph, and $U \subseteq V(G)$ is a set of vertices, we will sometimes refer to the set U as though it were the induced subgraph $G[U]$. Therefore we will speak of such things as $\chi(U)$, when we really mean $\chi(G[U])$. This is to avoid an overabundance of symbolism when the parent graph G is clear from context, if there is ambiguity then the full form $G[U]$ will be used instead.

We will warm up by reviewing a similar result proved by Borodin and Kostochka in [1]. The reader should observe a few key details of this proof. It begins with a graph satisfying $\chi = \Delta$. We partition our graph, maximizing the number of edges that cross between parts, then we leverage Brooks' Theorem to show that one of the parts must contain a clique. These three points, partition the graph, maximize edges between parts, find cliques within parts, will remain crucial in the proof of our main result. The formulation of this particular partition was done by Lovász [13].

Definition 3.2.
Let G be a graph. A **Lovász Partition** of G is a partition $P = (P_1, P_2)$ of $V(G)$ such that, if $v \in P_i$, then $deg(P_j, v) \geq deg(P_i, v)$ where $j \neq i$. That is, every vertex has at least as many neighbours in the other part as its own part.

Lemma 3.3. ***The Existence of Lovász Partitions***
Let G be any graph. Then G admits a Lovász partition.

Proof. For any partition $P = (P_1, P_2)$ of $V(G)$, define the function

$$f(P) := |\{vw \in E(G) \ : \ v \in P_i, \ w \in P_j, \ i \neq j\}|$$

so that f counts the number of edges which cross between P_1 and P_2. Now, f is bounded above by $|E(G)|$, so let $P = (P_1, P_2)$ be a partition of G which maximizes $f(P)$. We claim that P is a Lovász partition. Indeed, suppose that P is not a Lovász partition. We may then find $v \in P_i$ with $deg(P_i, v) > deg(P_j, v)$. Move v to P_j to obtain P'. Then $f(P') \geq f(P) + 1$, contradicting the maximality of $f(P)$. Thus, P is indeed a Lovász partition. □

This proof contains another idea which will be key for what follows. If our partition does not satisfy the property we want, we can move vertices to obtain a new partition which is "more extreme" (with respect to whatever extremal criterion we set up beforehand) and get a contradiction.

Now, we will use the Lovász partition to prove the following.

Theorem 3.4. ***Graphs with $\chi = \Delta$ have Medium Cliques*** *[1]*
Let G be a graph with $\chi(G) = \Delta(G)$ and $\Delta(G) \geq 7$. Then G contains a clique of size $\lfloor (\Delta(G) + 1)/2 \rfloor$.

Proof. As per Lemma 3.3 (existence of Lovász partitions), let $P = (A, B)$ be a Lovász partition of G. Observe that $\chi(G) \leq \chi(A) + \chi(B)$, as we can colour each part separately with distinct colours. Therefore, we must have $\chi(A) \geq \chi(G)/2$ or $\chi(B) \geq \chi(G)/2$. Observe further that $\Delta(A), \Delta(B) \leq \Delta(G)/2$ by the definition of the Lovász partition. Suppose without loss of generality that $\chi(A) \geq \chi(G)/2$.

We split the proof into two cases. In the first case, suppose that $\chi(G) = \Delta(G)$ is odd. Because both χ and Δ must always take integer values, we may strengthen our inequalities to get $\chi(A) \geq (\Delta(G) + 1)/2$ and $\Delta(A) \leq (\Delta(G) - 1)/2$. Thus, we have $\chi(A) > \Delta(A)$, and by the Greedy Bound we know that this implies $\chi(A) = \Delta(A) + 1$. We assumed that $\Delta(G) \geq 7$, giving us that

$$\Delta(A) = \chi(A) - 1 \geq (\chi(G) + 1)/2 - 1 \geq (7 + 1)/2 - 1 \geq 3$$

so by Brooks' Theorem we have that A contains a clique of size $\chi(A)$. But

$$\chi(A) \geq (\chi(G) + 1)/2 = (\Delta(G) + 1)/2 = \lfloor (\Delta(G) + 1)/2 \rfloor$$

so we have found a clique in A, and therefore G, of the necessary size.

Suppose now that $\Delta(G)$ is even. We can no longer exploit the integrality of χ and Δ to derive $\chi(A) > \Delta(A)$ as we did in the previous case. If it happens that $\chi(A) > \Delta(A)$ by chance, the Greedy Bound once again implies $\chi(A) = \Delta(A) + 1$, and we derive

$$\Delta(A) = \chi(A) - 1 \geq \chi(G)/2 - 1 \geq (8/2) - 1 = 3$$

so Brooks' Theorem tells us that A contains a clique of size

$$\chi(A) \geq \chi(G)/2 = \lfloor (\Delta(G) + 1)/2 \rfloor$$

as needed.

It only remains to check the case $\chi(A) = \Delta(A) = \chi(G)/2$, which would prevent us from using Brooks' Theorem. However, we can fix this problem. If there is a vertex $v \in A$ with $deg(A, v) = \Delta(A) = \Delta(G)/2$, then by the definition of a Lovász partition, it must be that $deg(B, v) = \Delta(G)/2$ as well. Thus we can move v from A to B to obtain another Lovász partition P'. Repeating this procedure if any more such vertices exist, we may assume without loss of generality that $\Delta(A) < \Delta(G)/2$. This is good, however, we can no longer assume that $\chi(A) \geq \chi(G)/2$, as moving vertices may have changed the chromatic number, but this is not a problem. If it happens by chance that we still have $\chi(A) \geq \chi(G)/2$, then invoking Brooks' Theorem on A gives us a clique of size $\chi(A) \geq \chi(G)/2 = \lfloor (\Delta(G) + 1)/2 \rfloor$. Otherwise, we have that $\chi(A) < \chi(G)/2$. However, now it must be that $\chi(B) \geq (\chi(G)/2) + 1$, while we still have that $\Delta(B) \leq \Delta(G)/2 = \chi(G)/2$. Therefore, Brooks' Theorem applied to B gives us the necessary clique, and we are done. □

3.2 Mozhan Partitions

We will now define a very important type of graph partition, named the "Mozhan partition" in honour of N. N. Mozhan, who introduced it [15].

Definition 3.5.
Let G be a graph. A **Mozhan partition** of G is a triple $M = (P, v, s)$ satisfying the following three properties.

1. $P = (P_1, P_2, \ldots, P_k)$ is a partition of $V(G)$.
2. $v \in P_s$ is critical for P_s.
3. $\chi(G) = \sum_{i=1}^{k} \chi(P_i)$.

Mozhan partitions come with some additional data.

Definition 3.6.
Let G be a graph, and let $M = (P, v, s)$ be a Mozhan partition of G, with k parts. The **parameters** of M are the integers $r_1, r_2, \ldots, r_k$, where $r_s \coloneqq \chi(P_s) - 1$, and for $i \neq s$, $r_i \coloneqq \chi(P_i)$.

The parameters of a Mozhan partition may seem like a superfluous definition at first, they merely hold information about the chromatic numbers of the parts. However, later on we will consider transformations between Mozhan partitions and the parameters will become an important invariant.

We will often need the minimum parameter of a Mozhan partition to satisfy some lower bound, so from now on we will use the letter m to denote the minimum parameter of a given Mozhan partition. In a similar vein, the letter k will always refer to the number of parts of a Mozhan partition, and the letter s will always refer to the index of the part in which the special vertex v lives (think "special part" for P_s).

Remark. For a graph G with a Mozhan partition $M = (P, v, s)$, we know that v is critical in P_s by definition. However, the vertex v is also critical in G. Recall that $\chi(G) = \sum_{i=1}^{k} \chi(P_i)$. The induced subgraph $P_s - v$ may be $(\chi(P_s) - 1)$-coloured. So by using distinct colours on all parts, we may $(\chi(G) - 1)$-colour $G - v$. Similarly, if a vertex is critical in any part of the Mozhan partition, it is critical in the whole graph.

In fact, criticality in one part implies criticality in all *other* parts.

Lemma 3.7. ***Local Criticality is Mobile***
Let G be a graph with a Mozhan partition M. Suppose that $w \in P_i$ is critical for P_i. Then w is critical in $P_j + w$, for all $j \neq i$.

Proof. We need to show that $\chi((P_j + w) - w) = \chi(P_j + w) - 1$. Simplifying, we must show that $\chi(P_j + w) = \chi(P_j) + 1$. Assume not. Since adding a vertex cannot decrease the chromatic number, we have $\chi(P_j + w) = \chi(P_j)$. But by the criticality of w in P_i, we have $\chi(P_i - w) = \chi(P_i) - 1$. So by moving w to P_j and using distinct colours on all parts, we may $(\chi(G) - 1)$-colour G, which is impossible. This contradiction proves the claim. □

The previous lemma demonstrates a technique that will be used often. For a graph G with a Mozhan partition $M = (P, v, s)$, we have that $\chi(G) = \sum_{i=1}^{k} \chi(P_i)$. Thus, it is impossible to move vertices between parts to decrease the chromatic number of some part without increasing the chromatic number of another. Otherwise, we could use distinct colours on all parts to colour G with fewer than $\chi(G)$ colours. Intuitively, the fact that these movements bump up the chromatic numbers of other parts implies the existence of many edges between parts, which will come in handy for finding a large clique.

We now introduce the minimality condition which has proven very useful for Mozhan partitions. It appears here in exactly the same form as in Cranston and Rabern's $\Delta - 3$ proof [6].

Definition 3.8.
Let $M = (P, v, s)$ be a Mozhan partition. We say that M is **minimum** when M minimizes the sum

$$\sigma_1(M) \coloneqq \sum_{i \neq s} |P_i|_E + |P_s - v|_E$$

and subject to that, minimizes

$$\sigma_2(M) \coloneqq deg(P_s, v) - r_s.$$

A minimum Mozhan partition first seeks to minimize the number of edges within parts, and then seeks to minimize the number of neighbours of the special vertex inside the special part. This reduction of edges is what will produce the aforementioned small cliques within parts, which will be seen shortly. Strangely, it will turn out that for a minimum Mozhan partition M we will always have $\sigma_2(M) = 0$.

Now we will give a name to the most important component within a Mozhan partition.

Definition 3.9.
Let G be a graph, and $M = (P, v, s)$ be a Mozhan partition of G. We define the **active component** of M to be the component of P_s which contains v. The active component will be denoted $act(M)$.

The following lemma essentially follows from the observation that if you add a vertex v to a graph G to produce G', and $deg(G', v) < \chi(G)$, then $\chi(G') \leq \chi(G)$. That is, the chromatic number cannot increase if you add a vertex with low degree. Indeed, if ϕ is a $\chi(G)$-colouring of G, then as $|N(G', v)| < \chi(G)$, there is some colour c, used for ϕ, but not used in $N(G', v)$, and so we can use c on v to $\chi(G)$-colour G'. Conversely, if adding a vertex increases the chromatic number, that vertex must have a lot of neighbours.

Lemma 3.10. ***Critical Vertices have Many Neighbours***
Let G be a graph, and let $v \in G$ be a critical vertex. Then there is a component C of $G - v$ with $deg(C, v) \geq \chi(G) - 1$.

Proof. Suppose not, so that for all components C of $G - v$, we have $deg(C, v) \leq \chi(G) - 2$. For each component C, let ϕ_C be a $(\chi(G) - 1)$-colouring of C, allowed by the criticality of v in G. Greedily extend each colouring ϕ_C to ϕ'_C, a $(\chi(G) - 1)$-colouring of $C + v$, allowed by the fact that $deg(C, v) \leq \chi(G) - 2 = (\chi(G) - 1) - 1$, meaning that there will be a colour left over for v. By definition there are no edges between distinct components of a graph, so we may choose these colourings such that v always receives the same colour. Combining these colourings produces a $(\chi(G) - 1)$-colouring of G, a contradiction. □

Now we will make our first real use of the fact that χ is close to Δ, to show that no vertex can have too many neighbours in all parts. Given what we have said about the utility of many edges between parts, this may seem prohibitive. However, for reasons we have not yet seen (Lemma 3.15), we would really like our vertices to have *exactly* r_i neighbours into each part P_i, so this restriction will be helpful.

Lemma 3.11. ***Every Vertex has a Part with Low Degree***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and M a Mozhan partition of G with $k \geq 3$ (recall that k denotes the number of parts). Then every vertex $w \in G$ has a part P_i with $deg(P_i, w) \leq r_i$.

Proof. Recall that $\chi(G) = \sum_{i=1}^{k} \chi(P_i) = 1 + \sum_{i=1}^{k} r_i = \Delta(G) - 1$. Assume the claim is false. We then have:

$$\begin{aligned} deg(G, w) &\leq \Delta(G) \\ &= 2 + \sum_{i=1}^{k} r_i \\ &< k + \sum_{i=1}^{k} r_i \\ &= \sum_{i=1}^{k} (r_i + 1) \\ &\leq \sum_{i=1}^{k} deg(P_i, w) \\ &= deg(G, w) \end{aligned}$$

which is a contradiction, as no number can be smaller than itself. □

Lemma 3.12. ***The Active Component is a Clique***
Let G be a graph with $\chi(G) = \Delta(G) - 1$. Let $M = (P, v, s)$ be a minimum Mozhan partition of G with $k \geq 3$. Suppose that $r_s \geq 3$. Then $act(M)$ is a clique of size $r_s + 1$ ($= \chi(P_s)$).

Proof. For brevity let $A := act(M)$. Recall that by the definition of a Mozhan partition, v is critical in P_s. Since A is the component of P_s containing v, we thus have that $\chi(A) = \chi(P_s)$. We also have $\chi(P_s) = r_s + 1$ by the definition of the parameter r_s. To show that A is a clique, we will show that $\Delta(A) \leq r_s$. Together with the Greedy Bound, this will give us that $\chi(A) = \Delta(A) + 1$. Then since odd cycles are 3-colourable, A cannot be an odd cycle, and so Brooks' Theorem will give us that A is a clique.

So suppose for a contradiction that we can find a vertex $u \in A$ with $deg(A, u) \geq r_s + 1$. Choose u to have minimum distance to v in A (in the case $u = v$, this distance is 0). Let q be a minimum path from v to u through A, say that $q = (q_1 = v, q_2, q_3, \ldots, q_n = u)$. Now, we claim that u is critical in P_s. First, since v is critical in P_s, let ϕ be an r_s-colouring of $P_s - q$. We can extend ϕ to ϕ', an r_s-colouring of $P_s - u$, by greedily colouring along $q - u$, starting at v. If $v = u$, then we have no vertices to colour, and so we are done. If $v \neq u$, then for $i < n - 1$, we have that $deg(P_s, q_i) \leq r_s$ by the minimal distance of u to v, and since q_{i+1} has not yet been coloured, there is a colour available for q_i. Then for q_{n-1}, the vertex u has been removed, and so $deg(P_s - u, q_i) \leq r_s - 1$. Therefore, there is also a colour available for q_{n-1}, and so the colouring ϕ may indeed be extended. By Lemma 3.11 (every vertex has a part with low degree), let P_i be a part of M with $deg(P_i, u) \leq r_i$. As u is critical in P_s, and local criticality is mobile (Lemma 3.7), we have $\chi(P_s - u) = \chi(P_s) - 1$, and $\chi(P_i + u) = \chi(P_i) + 1$. Thus, moving u to P_i and taking u to be the special vertex yields another Mozhan partition $M' = (P', v', s') = (P', u, i)$ of G, with the same parameters as M.

Our minimality criteria now come into play. If $u = v$, then $\sigma_1(M') = \sigma_1(M)$, as the special vertex is the same for both partitions, but we have

$$\sigma_2(M') = deg(P_{s'}, v') - r_{s'} = deg(P_i + v, v) - r_i \leq 0$$

while

$$\sigma_2(M) = deg(P_s, v) - r_s \geq 1$$

by our assumption that $deg(P_s, v) > r_s$. Therefore, $\sigma_2(M') < \sigma_2(M)$ which is impossible by the minimality of M. So it cannot be that $u = v$, which gives us in particular that $deg(P_s, v) \leq r_s$. But now we can derive

$$\sigma_1(M') = \sigma_1(M) - deg(P_s, u) + deg(P_s, v) \leq \sigma_1(M) - (r_s + 1) + r_s = \sigma_1(M) - 1.$$

So again we have contradicted the minimality of M. Therefore, our posited vertex of u of high degree cannot exist, giving us $\Delta(A) \leq r_s$. As described above, this completes the proof that A is a clique of size $r_s + 1$. □

We get the corollary:

Corollary 3.13. ***The Local Degree of the Special Vertex***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and M a minimum Mozhan partition of G with $m \geq 3$ (recall that m is the minimum parameter of M) and $k \geq 3$. Then $deg(P_s, v) = r_s$.

Remark. As promised, we now see that it is always the case that $\sigma_2(M) = 0$ when M is minimum.

3.3 Moving Between Mozhan Partitions

It will be useful to restate the idea of "moving vertices" used in the proof of Lemma 3.12 (the active component is a clique) as its own lemma. First, we will show that any vertex in the active component can serve as the special vertex.

Lemma 3.14. ***Vertices in*** $act(M)$ ***all Act the Same***
Let $M = (P, v, s)$ *be a minimum Mozhan partition with* $k \geq 3$ *and* $m \geq 3$*. Let* $u \in act(M)$ *be any vertex. Then* $M' = (P, u, s)$ *is another minimum Mozhan partition with the same parameters.*

Proof. We know that $act(M)$ is a clique (Lemma 3.12) and so $deg(P_s, v) = deg(P_s, u) = r_s$. Furthermore, as v is critical in P_s, we know that $act(M)$ is the only component C of P_s with $\chi(C) = \chi(P_s)$. Since every vertex in a clique is critical, we get that u is critical in $act(M)$ as well, and so u is critical in P_s. So, M' is indeed a Mozhan partition with the same parameters. To show minimality, we derive

$$\sigma_1(M') = \sigma_1(M) + deg(P_s, v) - deg(P_s, u) = \sigma_1(M) - r_s + r_s = \sigma_1(M)$$

and we know

$$\sigma_2(M') = deg(P_s, u) - r_s = 0$$

as is required for a minimum Mozhan partition. Therefore, M' is minimum as well. □

This puts us in a situation where, if we have a Mozhan partition $M = (P, v, s)$, and a vertex $u \in act(M)$ with degree r_i into some other part P_i, then we can move u to P_i and get another minimum Mozhan partition! After u moves, Lemma 3.12 (the active component is a clique) is applicable, and finds us a clique in P_i of size r_i. Repeated application of this principle is what allows us to find lots of small cliques in the parts of our Mozhan partition. We now make these ideas more formal.

Lemma 3.15. ***Moving Vertices***
Let G *be a graph, let* $M = (P, v, s)$ *be a Mozhan partition of* G *with* $k \geq 3$ *and* $m \geq 3$*, and let* $u \in act(M)$ *be any vertex in the active component. Suppose we have a part* $P_i \neq P_s$ *with* $deg(P_i, u) = r_i$*. Then moving* u *to* P_i *yields another Mozhan partition* M' *with the same parameters. Furthermore, if* M *is minimum, then so is* M'*.*

Proof. By Lemma 3.14 we may suppose that $v = u$. By the criticality of v in P_s, we have $\chi(P_u - v) = \chi(P_s) - 1$. By Lemma 3.7 (local criticality is mobile) the vertex v is critical in $P_i + v$, so that $\chi(P_i + v) = \chi(P_i) + 1$. Move v to P_i to obtain vertex partition P'. Then $M' = (P', v, i)$ is a Mozhan partition with the same parameters as M.

Suppose now that M is minimum. To show that M' is minimum, we first have that $\sigma_1(M') = \sigma_1(M)$, as v is the special vertex in both partitions. By our assumption that $deg(P_i, v) = r_i$, we have $deg(P_i, v) - r_i = 0$. Thus, $\sigma_2(M') = \sigma_2(M) = 0$. Therefore, M' is another minimum Mozhan partition, as desired. □

Definition 3.16.
Let G be a graph, and $M = (P, v, s)$ a Mozhan partition of G with $k \geq 3$ and $m \geq 3$. Let $u \in act(M)$ be any vertex, and suppose that we have a part P_i with $deg(P_i, u) = r_i$. By Lemma 3.15 (moving vertices), we can move u to P_i to produce another Mozhan partition $M' = (P', u, i)$, which is minimum if M is minimum. The pair (M, M') is referred to as a **move**.

Definition 3.17.
A finite list of Mozhan partitions $S = (M_1, M_2, M_3, \ldots, M_n)$ such that every pair (M_i, M_{i+1}) is a move will be referred to as a **move sequence**.

We can now outline our argument in more detail. To find a large clique, we will first start with a critical graph G satisfying $\chi(G) = \Delta(G)$. The criticality of G allows us to find a Mozhan partition. Indeed, let $v \in G$ be any vertex, and let ϕ be a $(\Delta(G) - 2)$-colouring of $G - v$. Create the parts P_i by taking the unions of colour classes of ϕ, then add v to any part P_i. The resulting vertex partition will be a Mozhan partition. We can thus find a minimum Mozhan partition, call it M. We set off a move sequence, starting at M, subject to a few criteria. Eventually the move sequence must halt, and some analysis will show that the only possible reason for this halting is the presence of a large clique. We will need a way to talk about the components within parts as they change across a move sequence, and for this purpose, we define:

Definition 3.18.
Let $S = (M_1, \ldots, M_{q+1})$ be a move sequence. For all $t \in [q]$, let v_t be the vertex which moves at time t. A **club** is a sequence

$$X = (X_1, X_2, \ldots, X_{q+1})$$

such that X_1 is a component of some part P_i in M_1, and we have

$$X_{t+1} := \begin{cases} X_t - v_t \text{ if } X_t = act(M_t) \\ X_t + v_t \text{ if } X_t = act(M_{t+1}) \\ X_t \text{ otherwise.} \end{cases}$$

That is, a club is a component within a part, as seen across a move sequence. For a component Y within a part at any time, we denote by $clb(Y)$ the club to which Y belongs.

To familiarize the reader with clubs, we remark on a few properties. First, recall that vertices only ever move out of the active component, a clique of size $r_s + 1$, into cliques of size r_i in some part P_i. Thus, for a component X_1 in part P_i in M_1 with $|X_1| < r_i$, we have $X_2 = X_1$, and $X_3 = X_2$, and so on. No vertices ever move into $clb(X_1)$, or out of $clb(X_1)$, this component remains static across time. One might imagine that X_1 has sub-critical mass, and cannot participate in the reaction of moving vertices.

Chapter 4

Finding the Large Clique

We will now use the facts we have collected about Mozhan partitions to prove some more difficult facts, which we will then use to find our large clique. As mentioned before, this will be done by first finding many small cliques, and then showing that many of them are fully joined together. What follows is a series of technical lemmas, building on what we have discussed previously, which detail exactly how to find these cliques and how to show they are fully joined.

4.1 Chromatic Excess and Cores

For what follows, it will be useful to formalize the idea of "too many edges" discussed in Chapter 3. When a vertex w has $\chi(P_i)$ neighbours into part P_i, that is good. It is easier to find cliques when we have many edges to work with, and if w is in the active component, we may even move w to P_i to get a clique in P_i to which w is joined. However, when $deg(P_i, w) > \chi(P_i)$, the "excess neighbours" end up posing a problem, as it becomes difficult to determine exactly where the neighbours of w *are*, that is, what component of P_i they lie in. We therefore wrap up this idea of "too many neighbours" in a definition.

Definition 4.1.
Let G be a graph, and let $U \subseteq G$ be an induced subgraph. Let $v \in V(G)$ be any vertex. We define the **chromatic excess** of v into U to be $ce(U, v) \coloneqq deg(U, v) - \chi(U)$.

Observe now that for a Mozhan partition $M = (P, v, s)$, the vertex v must have at least

r_i neighbours into each part P_i, as v is critical in P_i+v. However, since $\Delta(G) = 2+\sum_{i=1}^{k} r_i$, v can have at most 2 excess neighbours. This observation produces the following lemma.

Lemma 4.2. *Small Chromatic Excess*
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a Mozhan partition of G. Suppose that $w \in P_i$ is critical in P_i, and let $T \subseteq \{1, 2, \ldots, k\}$ be any subset of indices. Then $ce(\cup_{j\in T} P_j, w) \leq 2$.

Proof. Let $P_T \coloneqq \bigcup_{j\in T} P_j$. From Lemma 3.7 (local criticality is mobile), we know that w is critical in each part $P_j + w$ for $j \neq i$. Thus, from Lemma 3.10 (critical vertices have many neighbours), we have that $deg(P_j, w) \geq \chi(P_j)$ (when $j \neq i$), and that $deg(P_i, w) \geq \chi(P_i)-1$. We therefore have $deg(G-P_T, w) = deg(\cup_{l\notin T} P_l, w) \geq \left(\sum_{l\notin T} \chi(P_l)\right)-1$. Observe that by the definition of M we have

$$\Delta(G) = 1 + \chi(G) = 1 + \sum_{l=1}^{k} \chi(P_l).$$

Thus, the vertex w satisfies

$$\begin{aligned} deg(P_T, w) &= deg(G, w) - deg(G - P_T, w) \\ &\leq \Delta(G) - deg(G - P_T, w) \\ &\leq \left(1 + \sum_{l=1}^{k} \chi(P_l)\right) - \left(\left(\sum_{l\notin T} \chi(P_l)\right) - 1\right) \\ &= 2 + \sum_{l\in T} \chi(P_l) \\ &= 2 + \chi(P_T) \end{aligned}$$

Therefore, $ce(G - P_s, w) \leq 2$, as desired. □

So when it comes to the special vertex v (which is critical), we have a Goldilocks situation. Into each part P_i, the vertex v must have at least r_i neighbours into some component by criticality, but at most $r_i + 2$ neighbours since $\Delta - 2 = \sum_{i=1}^{k} r_i$. Together with the fact that $\chi(P_i) \geq 3$ (given by our assumption that $m \geq 3$), these bounds produce a single component in P_i into which v has at least $\chi(P_i)$ neighbours. We will now give a name to these noteworthy components.

Definition 4.3.
Let G be a graph, and let M be a Mozhan partition of G. Let $w \in G$ be any vertex. Suppose that for all $i \in \{1, 2, \ldots, k\}$ there is a single component C_i of P_i such that $deg(C_i, w) \geq r_i$. Then we define the **cores** of w to be $cores(w) \coloneqq \{C_1, C_2, \ldots, C_k\}$. Furthermore, we define the **core graph** of w to be the induced subgraph $cg(M, w) \coloneqq G[C_1 \cup C_2 \cup \cdots \cup C_k]$. If M is clear from context, we simply write $cg(w)$.

Lemma 4.4. ***Minimum Mozhan Partitions Create Cores***
Let G be a graph, and let $M = (P, v, s)$ be a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 3$. Then v has cores in M.

Proof. For $i \in \{1, 2, \ldots, k\}$, by Lemma 3.7 (local criticality is mobile) and Lemma 3.10 (critical vertices have many neighbours), there is a component C_i of P_i such that $deg(C_i, v) \geq r_i$. Furthermore, by Lemma 4.2 (small chromatic excess), we know that (in particular) $deg(P_i, v) \leq r_i + 2$. Since $r_i \geq m \geq 3$, the component C_i is therefore the *only* component of P_i into which v has at least r_i neighbours. □

4.2 Two Technical Lemmas

In the following lemma, it may seem strange that we have to include so many assumptions. Indeed, many of the things which we are assuming are simply properties of minimum Mozhan partitions which we have proved already. Unfortunately, the lemma will have to be used in a context where we cannot guarantee a minimum Mozhan partition (Lemma 4.12), so the necessary facts must be dragged in as assumptions.

Lemma 4.5. ***Adding a Vertex to a Core***
Let G be a graph, and let $M = (P, v, s)$ be a Mozhan partition of G with $k \geq 3$ and $m \geq 3$. Suppose that v has cores in M, call them $C_1, \ldots, C_k$. Let $w \in C_k \cap N(v)$ be any vertex with $ce(G - P_k, w) \leq 2$. Suppose that $deg(C_k, v) = r_k$, that $deg(P_k, v) \leq r_k + 2$, and that w has at least 4 neighbours in $cg(v) - C_k$. Then there is some core C_i, $i \in \{1, \ldots, k-1\}$ with $deg(C_i, w) \geq \chi(P_i)$.

Proof. For simplicity of notation, and with no loss of generality, suppose that $P_1 = P_s$. We claim that w is critical in $P_k + v$. Observe that as v is critical in P_1, and local criticality is mobile (Lemma 3.7), we have $\chi(P_k + v) = \chi(P_k) + 1$. But by assumption, we have $deg(P_k, v) \leq r_k + 2$, and $deg(C_k, v) = r_k$. Our assumption that $m \geq 3$ gives us that $r_k \geq 3$. Since $w \in C_k$, the vertex v has at most $r_k - 1$ neighbours into all components of

$P_k - w$. Thus we may extend an r_k-colouring of $P_k - w$ to $P_k - w + v$ by ensuring that there is a colour unused on the neighbours of v in each component. This gives us that $\chi(P_k + v - w) = r_k = \chi(P_k + v) - 1$, so w is critical in $P_k + v$ as desired.

Let $P_1' \coloneqq P_1 - v$, and $P_k' \coloneqq P_k + v$, and $P_i' \coloneqq P_i$ for $i \in \{2, 3, \ldots, k-1\}$. Let $C_1' \coloneqq C_1 - v$, and $C_i' \coloneqq C_i$ for $i = 2, \ldots, k-1$. The vertex w, being critical in P_k', must have at least $\chi(P_i') = r_i$ neighbours into some component of each part P_i' for $i = 1, \ldots, k-1$ by Lemma 3.7 (local criticality is mobile) and Lemma 3.10 (critical vertices have many neighbours). By our assumption on the chromatic excess of w, we have $deg(G - P_k, w) \leq \chi(G - P_k) + 2$. Observe that

$$\chi(G - P_k') = \chi(G - P_k - v) = \chi(G - P_k) - 1$$

and that since since we assumed $w \in N(v)$, we have

$$deg(G - P_k', w) = deg(G - P_k - v, w) = deg(G - P_k, w) - 1.$$

So we have the analogous statement $deg(G - P_k', w) \leq \chi(G - P_k') + 2$. Phrasing things differently, $ce(G - P_k', w) \leq 2$. Now $P_1', \ldots, P_{k-1}'$ partitions $G - P_k'$ so that $\chi(G - P_k') = \sum_{i=1}^{k-1} \chi(P_i')$. Thus, $ce(G - P_k', w) = \sum_{i=1}^{k-1} ce(P_i', w)$. But we have that $deg(C_1' \cup \cdots \cup C_{k-1}', w) \geq 3$ by our assumption that $deg(cg(v) - C_k, w) \geq 4$. Thus, there is some $i \in \{1, \ldots, k-1\}$ with $deg(C_i', w) > ce(P_i', w)$. Unpacking notation, we get $deg(C_i', w) > deg(P_i', w) - \chi(P_i')$, and rewriting, $deg(P_i', w) - deg(C_i', w) < \chi(P_i')$. But w needs at least $r_i = \chi(P_i')$ neighbours into some component of P_i', so this component must be C_i'. If $i \neq 1$, then $C_i' = C_i$, and $P_i' = P_i$, and we are done. If $i = 1$, then re-adding the vertex v gives the desired result. □

Once again, we need to include many assumptions in the statement of a lemma. Fortunately this will be the last time we need to do this.

Lemma 4.6. ***Adding a Vertex to a Big Clique***
Let G be a graph, and let $M = (P, v, s)$ be a Mozhan partition of G with $k \geq 3$ and $m \geq 5$. Suppose that v has cores in M, and let them be $C_1, \ldots, C_k$. Suppose further that $deg(P_k, v) \leq r_k + 2$, and $deg(C_k, v) = r_k$. Suppose that for all $w \in cg(v)$, say $w \in C_i$, that $ce(G - P_i, w) \leq 2$. Assume that $cg(v) - C_k$ and C_k are complete, and let $w \in C_k$ be any vertex. Finally, suppose that $deg(cg(v) - C_k, w) \geq 4$. Then w is fully joined to $cg(v) - C_k$.

Proof. By Lemma 4.5 (adding a vertex to a core), there is some core $C_i, i \in \{1, \ldots, k-1\}$ such that $deg(C_i, w) \geq \chi(P_i)$. As C_i is a clique, and thus has $|C_i| \leq \chi(P_i)$, we have that $|C_i| = \chi(P_i)$, so that w is fully joined to C_i. Now if v is not in P_i, then we can move v

to P_i to produce another Mozhan partition M'(not necessarily minimum) where all of the hypotheses still hold, so we may assume that $i = s$.

Since we assumed that the claim is false, there is $j \in \{1, \ldots, k-1\}$ such that w is not joined to C_j. So without loss of generality, suppose $P_s = P_1$, (so that w is fully joined to $act(M) = C_1$), that w is not fully joined to C_2, and $w \in C_3$.

Let $a \in act(M) - v$ be any vertex. Let $b \in C_2$ be any non-neighbour of w. Because of our assumption that such a vertex b can be found, we will be able to reduce a chromatic number by merely moving vertices around, giving us a contradiction.

Let $P_1' := P_1 - \{v, a\} + \{w, b\}$, let $P_2' := P_2 - b + a$, and let $P_3' := P_3 - w + v$.

First, we will show that P_2' is $\chi(P_2)$-colourable. Since $cg(v) - C_3$ is a clique (recall that C_k was relabelled C_3), we have that the induced subgraph $C_2 \subseteq cg(v) - C_3$ is a clique, and $deg(C_2, a) = |C_2|$. The fact that C_2 is a clique implies $|C_2| \leq \chi(P_2)$, and the fact that C_2 is a core of v gives us that $deg(C_2, v) \geq \chi(P_2)$, so all together we have $|C_2| = \chi(P_2)$. Furthermore since $a \in cg(v)$, we assumed that $ce(G - P_1, a) \leq 2$. Again by the assumption that $cg(v) - C_3$ is a clique we get that $act(M)$, being an induced subgraph of $cg(V - C_3)$, is a clique. The vertex v is critical in P_1 by the definition of the Mozhan partition, so a must be critical in P_1 as well. But by Lemmas 3.7 (local criticality is mobile) and 3.10 (critical vertices have many neighbours), the vertex a has at least $\chi(P_i)$ neighbours into all parts P_i with $i \neq 1$. Therefore, we obtain $deg(P_2, a) \leq \chi(P_2) + 2$, as a larger degree would imply $ce(G - P_1, a) \geq 3$. Thus, the vertex a as at most 2 neighbours on any component of $P_2 - b$ besides $C_2 - b$. We have that $\chi(P_2) \geq m \geq 3$, so $P_2 - b + a = P_2'$ is $\chi(P_2)$-colourable.

Analogously, we will show that P_3' is $\chi(P_3)$-colourable. We assumed that C_3 is a clique, and that $deg(C_3, v) = r_3$. We also assumed that $deg(P_3, v) \leq r_3 + 2$. Thus, the vertex v has at most 2 neighbours on any component of $P_3 - w$ besides C_3. We have that $\chi(P_3) \geq m \geq 3$, so v has at most $\chi(P_3) - 1$ neighbours in all components of $P_3 - w$, so $P_3 - w + v = P_3'$ is $\chi(P_3)$-colourable.

Finally, we will show that $\chi(P_1') \leq \chi(P_1) - 1$. To start, we will need to show that $deg(P_1, b) \leq \chi(P_1) + 2$. If we have some index $i \neq 2$ with $deg(P_i, b) < \chi(P_i)$, then $P_i + b$ can be $\chi(P_i)$-coloured. The criticality of v in P_1 gives us that $deg(P_j, v) \geq \chi(P_j)$ for all $j \neq 1$, and so the assumption that $ce(G - P_1, v) \leq 2$ implies that $ce(P_2, v) \leq 2$. So the vertex v has at most $\chi(P_2) - 1$ neighbours into all components of $P_2 - b$, meaning $P_2 - b + v$ can be $\chi(P_2)$-coloured. Moving v to P_2 and b to P_i would therefore allow us to $(\chi(G) - 1)$-colour G, a contradiction. Thus, for all $i = 1, \ldots, k,\ i \neq 2$ we have $deg(P_i, b) \geq \chi(P_i)$. Thus, to ensure the truth of our assumption $ce(G - P_2, b) \leq 2$, we must have $deg(P_1, b) \leq \chi(P_1) + 2$. Completely analogously, we have $deg(P_1, w) \leq \chi(P_1) + 2$. Then since $C_1 = act(M)$ is a clique, the component $C_1' := C_1 - \{v, a\} + \{w, b\}$ is isomorphic to $K_{\chi(P_1)}$ less the edge bw. Thus, let ϕ be a $(\chi(P_1) - 1)$-colouring of C'_1 such that b, w receive the same colour. Now, we have that $\chi(P_1) \geq 6$, as the minimum parameter of M is at least 5. Thus we can let ψ be a $(\chi(P_1) - 1)$-colouring of $P'_1 - C'_1 \subseteq P_1 - v$ such that the at most 4 vertices in $N(P'_1 - C'_1, w) \cup N(P'_1 - C'_1, b)$ do not receive the colour $\phi(w)$. Then combining ϕ with ψ yields a $(\chi(P_1) - 1)$-colouring of P'_1.

We therefore have that $\chi(P'_1+P'_2+P'_3) \leq \chi(P'_1)+\chi(P'_2)+\chi(P'_3) < \chi(P_1)+\chi(P_2)+\chi(P_3) = \chi(P_1 + P_2 + P_3)$. But as induced subgraphs of G, we have $P'_1 + P'_2 + P'_3 = P_1 + P_2 + P_3$, so this is a contradiction. □

4.3 Joining Big Cliques

We will now use the results we have collected to demonstrate that big cliques in Mozhan partitions have a tendency to be fully joined to each other.

Lemma 4.7. ***The Active Component Sticks to Big Cliques***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a Mozhan partition of G with $k \geq 3$ and $m \geq 3$. Let $v \in act(M)$ be any vertex, and $P_i \neq P_s$ be a non-special part of M. Suppose that there is some clique $C \subseteq P_i$ such that $deg(C, v) \geq 3$. Then v is fully joined to C.

Proof. By Lemma 3.7 (local criticality is mobile), the vertex v is critical in $P_i + v$. Thus by Lemma 3.10 (critical vertices have many neighbours), there is a component D of P_i such that $deg(D, v) \geq \chi(P_i)$. Combining this with Lemma 4.2 (small chromatic excess), we have $deg(P_i, v) \leq \chi(P_i) + 2$. But $\chi(P_i) \geq m \geq 3$, so the only component into which v can have $\chi(P_i)$ neighbours is C. Therefore $C = D$, so we have $deg(C, v) \geq \chi(P_i)$. But C is a clique, and so $|C| \leq \chi(P_i)$. All together, we get that v is fully joined to C, as desired. □

Lemma 4.8. ***Big Cliques Stick to the Active Component***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a Mozhan partition of G with $k \geq 3$ and $m \geq 3$. Let $P_i \neq P_s$ be some non-special part, and $C \subseteq P_i$ be some clique. Suppose that $act(M)$ is a clique, and that there is a vertex $v \in act(M)$ with at least 3 neighbours in C. Suppose further that there is $w \in C$ with at least 4 neighbours in $act(M)$. Then w is fully joined to $act(M)$.

Proof. Suppose that $w \in P_i$. By Lemma 4.7 (the active component sticks to big cliques), v is fully joined to C. The vertex w is critical in $P_i + v$, as v has $deg(P_i, v) \leq \chi(P_i) + 2$ (Lemma 4.2, small chromatic excess) and so C is the only component of P_i into which v

has at least $\chi(P_i)$ neighbours. Thus, we can move v to P_i to get another Mozhan partition $M' = (P', w, i)$. Now w has at least 3 neighbours in the clique $act(M) - v$. Using Lemma 4.7 again, we know that w is fully joined to $act(M) - v$. Thus, w is fully joined to $act(M)$, as desired. □

A consequence of all this stickiness is that two cliques cannot be joined by moves. This is because only one vertex moves per move, and so if a move were to try and join two cliques, they would already have to be so "almost joined" that they were actually joined all along. Since reversed moves are also moves, we get as a consequence that moves cannot *separate* cliques either.

Definition 4.9.
Let S be a move sequence. Let X be a club of S, and suppose that X is in part P_i. We say that X is **full** when at each time t, we have $|X_t| \geq r_i$.

Lemma 4.10. *Moves Cannot Join Big Cliques*
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 4$. Let S be a move from M to M', another minimum Mozhan partition. Let $A \subseteq P_a$ and $B \subseteq P_b$ be two cliques with at least 5 vertices, and let A', B' be their corresponding components in M'. Suppose that A' is fully joined to B'. Then A is fully joined to B.

Proof. Suppose that A is not fully joined to B, say the edge ab does not exist between $a \in A$ and $b \in B$. Either a must leave A during S, or b must leave B, otherwise the edge ab would still fail to exist between A' and B'. So without loss of generality, $A = act(M)$, and a moves during S. Observe that $A' = A - a$ is fully joined to $B' = B$. Since A' is fully joined to B', in particular, some $a' \in A'$ has three neighbours in B. Every vertex $b \in B'$ has at least 4 neighbours in A, those vertices in $A' \subseteq A$. Therefore, by Lemma 4.8 (big cliques stick to the active component), B' $(= B)$ is fully joined to A, contradicting our assumption. □

If (M, M') is a move, then (M', M) is also a move. This fact gives us the corollary:

Corollary 4.11. *Moves Cannot Separate Big Cliques*
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 4$. Let S be a move from M to M', another minimum Mozhan partition. Let $A \subseteq P_a$ and $B \subseteq P_b$ be two cliques with at least 4 vertices, and let A', B' be their corresponding components in M'. Suppose that A' is not fully joined to B'. Then A is not fully joined to B.

During any move sequence, two full clubs are therefore fully joined for all times, or not fully joined for all times. Thus, is makes sense to refer to clubs themselves as being fully joined or not fully joined.

Now we will leverage those two technical lemmas from earlier to show that if two big cliques are joined to the active component, then they are joined to each other. This will be very helpful for our final proof, where we will first find many big cliques joined to the active component.

Lemma 4.12. ***Joining Big Cliques***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and M a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 5$. Suppose that there are two cliques $A \subseteq P_a$ and $B \subseteq P_b$ with $|A| = \chi(P_a)$, $|B| = \chi(P_b)$ in different parts, both fully joined to $act(M)$. Suppose further that $deg(P_a, v) = \chi(P_a)$ and $deg(P_b, v) = \chi(P_b)$. Then A is fully joined to B.

Proof. Without loss of generality, say that $s = 1$, $a = 2$, and $b = 3$. Let $b \in B$ be any vertex. We would like to use Lemma 4.6 (adding a vertex to a big clique) to show that b is fully joined to A. To do this, observe that $M' = ((P_1, P_2, P_3), v, 1)$ is a Mozhan partition of $G' := P_1 + P_2 + P_3$. Indeed, (P_1, P_2, P_3) is a vertex partition, v is critical in P_1, and $\chi(P_1, P_2, P_3) = \chi(P_1) + \chi(P_2) + \chi(P_3)$. Furthermore, the vertex v has cores $act(M), A, B$ in P_1, P_2, P_3. The Mozhan partition M' has $k' = 3$ and $m' \geq m \geq 5$. We know that $deg(P_3, v) = r_3 \leq r_3 + 2$ by assumption. Let $C := act(M) \cup A \cup B$, so that $C = cg(M', v)$. We only need to show that for all $w \in C$, say $w \in P_i$, that $ce(G - P_i, w) \leq 2$. If $w \in act(M)$, then w is critical in P_1, and we are done by Lemma 4.2 (small chromatic excess). If $i \neq 1$, then as we assumed $deg(P_i, v) = \chi(P_i)$, moving v to P_i produces $M'' = (P'', w, i)$, a minimum Mozhan partition of G (Lemma 3.15, moving vertices) where w is critical. Once again, Lemma 4.2 (small chromatic excess) applied to M'' gives us that $ce(G - (P_i + v), w) \leq 2$. Observe that $deg(G - (P_i + v), w) = deg(G - P_i, w) - 1$, and $\chi(G - (P_i + v)) = \chi(G - P_i) - 1$. Thus, we have that $ce(G - P_i, w) \leq 2$, as needed. Thus, Lemma 4.6 (adding a vertex to a big clique) applied to M' gives us that b is fully joined to A. But $b \in B$ was arbitrary, proving that B is fully joined to A, as desired. □

4.4 Properties of Move Sequences

The next few lemmas describe some restrictions on what can happen during a move sequence. As mentioned previously, we will find our big clique by setting a move sequence in motion under some specific conditions, watching it grind to a halt, and deducing that the halt was caused by a big clique. Intuitively, the following results will be used to rule out the other possible reasons our future move sequence could stop.

We will begin with another lemma of the edge-finding type. Imagine we have a club $X = (X_1, X_2, X_3, \ldots, X_n)$, and a vertex v joined to X_1. Each time X becomes active, a vertex leaves X, and so v loses a neighbour in X. However, under certain conditions the vertex v can be "re-joined" to X_j at a later time j, and this rejuvenation of edges can be pictured as a "pit-stop".

Lemma 4.13. ***The Pit-stop Lemma***
Let G be a graph with $\chi(G) = \Delta(G) - 1$, and let M be a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 8$. Let S be a move sequence starting at M. Suppose that a club X sends a vertex to a club Y at two times a, b. Suppose further that X is active at most 4 times in $\{a+1, a+2, \ldots, b-1\}$. Let $v \in G$ be some vertex which is fully joined to X_{a+1}, where $v \in Y_b$ at time b. Then v is fully joined to X_b.

Proof. Since the minimum parameter of M is at least 8, and X_a is active at time a, we have that $|X_a| \geq 9$. Thus, $|X_{a+1}| \geq 8$. Now by assumption, the club X is active at most 4 times between a and b. Thus, $|X_{a+1} \cap X_b| \geq 8 - 4 = 4$. This means that v has at least 4 neighbours in X_b, those vertices in $X_{a+1} \cap X_b$. Let v_b be the vertex which X sends to Y at time b. Since X sends v_b to Y_b at time b, the vertex v_b has at least 3 neighbours in Y_b (indeed, it is fully joined to Y_b). Therefore, by Lemma 4.8 (big cliques stick to the active component), v is fully joined to X_b, as desired. □

Lemma 4.14. ***Clubs Don't Have Many Targets***
Let G be a graph, and let M be a minimum Mozhan partition of G. Let S be a move sequence starting at M in which no vertex moves twice. Suppose that a club X in part P_i sends two vertices v_1, v_2 to a part P_j at times t_1, t_2, and $t_1 < t_2$. Suppose that v_1 is sent to a club $Y \subseteq P_j$. Then v_2 is also sent to Y.

Proof. Suppose that v_2 is instead sent to a club $Z \subseteq P_j$ with $Y \neq Z$. Then Y_{t_2+1}, Z_{t_2+1} are distinct components of the part P_j at time $t_2 + 1$. However, as no vertex can move twice, we have that v_1, v_2 are both in the component X_1 at time $t = 1$. Since X sends a vertex during S, we know that X is a full club, so X_1 is a clique. Thus we have the edge $v_1v_2 \in E(G)$. But then, the edge v_1v_2 goes between Y_{t_2+1} and Z_{t_2+1} in part P_j at time $t_2 + 1$, contradicting the fact that they are two distinct components. This contradiction proves the claim. □

Lemma 4.15. ***Full Clubs are Not Very Active***
Let G be a graph. Let M be a minimum Mozhan partition of G with $k \geq 3$ and $m \geq 8$. Let S be a move sequence starting at M, where no vertex moves twice. Suppose that each club sends a vertex to at most 3 other parts. Finally, suppose that a club X never sends a vertex to a club Y if X and Y are fully joined. Then every club of S is active at most 7 times.

Proof. We prove the claim by contradiction. Let S be a move sequence meeting all of the hypotheses (no vertex moves twice, each club sends to at most 3 parts, clubs don't send to fully joined clubs), but violating the conclusion. Choose S to have minimal length, call that length $q+1$. Let X be a club of S which is active at least 8 times. By the minimality of S, X is active exactly 8 times. Thus, X sends out exactly 7 vertices during S (at the final time $q+1$, the component X_{q+1} is active but does not send out a vertex). Since X sends vertices to at most 3 other parts, there must be some part P_i to which X sends at least 3 vertices, as $3 \cdot 2 = 6 < 7$. Choose P_i to be the first such part. By Lemma 4.14 (clubs don't have many targets), the club X always sends vertices to the same club Y in P_i. For $i \in \{1, 2, \ldots, q\}$ let v_i be the vertex which moves at time i. Let t_1, t_2, t_3 be the first three times at which X sends a vertex to Y. Observe that between any two times t_i, t_j, the club X is active at most 4 times, as being active 5 times or more would imply that P_i was not the first part to which X sends 3 vertices. Let t_4 be the first time after t_3 at which X is active. We will show that X_{t_4} is fully joined to Y_{t_4}. Thus, X_i will be fully joined to Y_i for all times i (Lemma 4.10 and Corollary 4.11, which imply that full clubs are either always joined or always not joined). This contradicts one of our assumptions about S, namely, that a club never sends a vertex to a club to which it is fully joined.

First, we will show that $v_{t_1}, v_{t_2}, v_{t_3}$ are all fully joined to X_{t_3+1}. For this, we will make use of Lemma 4.13 (the pit-stop lemma). Since X_{t_3} is a clique, and $v_{t_3} \in X_{t_3}$, we have that v_{t_3} is fully joined to X_{t_3+1}. Likewise, v_{t_2} is fully joined to X_{t_2+1}. We also have $v_{t_2} \in Y_{t_3}$ as v_{t_2} does not move twice, and we know that the club X is active at most 4 times in $\{t_2+1, t_2+2, \ldots, t_3-1\}$, so the Pit-stop Lemma implies that v_{t_2} is fully joined to X_{t_3} and thus to X_{t_3+1}. Completely analogously, v_{t_1} is fully joined to X_{t_2}, and thus to X_{t_3}, and thus to X_{t_3+1}. Now, the vertices $v_{t_1}, v_{t_2}, v_{t_3}$ are all in Y_{t_4}. So the vertices of $X_{t_3} \cap X_{t_4}$ all have at least 3 neighbours in Y_{t_4}, and by Lemma 4.7 (the active component sticks to big cliques) applied at time t_4, they are thus fully joined to Y_{t_4}. But as $X_{t_3+1} \subseteq X_{t_4}$ since t_4 is the first time after t_3 at which X is active, we get $|X_{t_3+1} \cap X_{t_4}| = |X_{t_3+1}| \geq 7$. So then, the vertices of Y_{t_4} each have at least 4 neighbours in X_{t_4}, and by Lemma 4.8 (big cliques stick to the active component) applied at time t_4, we have that Y_{t_4} is fully joined to X_{t_4}. As mentioned earlier, this implies that the club X is fully joined to Y at all times, contradicting the fact that this means that X is not allowed to send a vertex to Y during S. $\square$

4.5 Putting it Together

Finally, we use our body of facts about Mozhan partitions and move sequences to produce our big clique.

Theorem 4.16. ***Graphs with*** $\chi = \Delta - 1$ ***have Big Cliques***
Let G *be a graph with* $\chi(G) = \Delta(G) - 1$*, and* $\Delta(G) \geq 66$*. Then* G *contains a clique of size* $\Delta(G) - 17$*.*

Proof. Suppose the theorem is false, and let G be a counterexample minimizing $|G|$. We claim that G is vertex critical. Indeed, let $v \in G$ be any vertex, and suppose that v is not critical. Then $\chi(G - v) = \chi(G)$. If $\Delta(G - v) = \Delta(G)$, then $G - v$ is a smaller counterexample. Therefore, $\Delta(G - v) \leq \Delta(G) - 1$. Now, we have that $\chi(G - v) = \chi(G) = \Delta(G) - 1 \geq \Delta(G - v)$. Therefore by Theorem 1.13, $\omega(G - v) \geq \Delta(G - v) - 3$. Furthermore we have $\Delta(G - v) \geq \chi(G - v) - 1 = \Delta(G) - 2$, and so $\omega(G) \geq \omega(G - v) \geq \Delta(G) - 5 > \Delta(G) - 17$, contradicting that G is a counterexample. So G is indeed vertex critical.

Any number $n \geq 56$ may be written as a sum of 8's and 9's. Indeed, taking $k := \lfloor n/8 \rfloor$, and letting $s := n \mod 8$, we can write $n = (k - s) \cdot 8 + s \cdot 9$ (the condition $n \geq 56$ is necessary to ensure that $k - s \geq 0$). In particular, $\Delta(G) - 2 \geq 64$ may be written as such a sum. We can construct a Mozhan partition M of G by removing any vertex $v \in G$, letting ϕ be a $(\Delta(G) - 2)$-colouring of $G - v$, taking each part P_i to be the induced subgraph on an arbitrary choice of $n \in \{8, 9\}$ colour classes of ϕ, and adding v back to any part P_i. So let M be a minimum Mozhan partition of G with parameters r_i, $i = 1, \ldots, k$ such that each r_i is in $\{8, 9\}$.

Now, let S be a move sequence of maximum length, starting at M, subject to the following three conditions.

s1: Each vertex moves at most once.

s2: Each club sends vertices to at most 3 other parts.

s3: A vertex never moves from club X to club Y if X is fully joined to Y.

Let A be the final active component of S, and let X be the club of A. By Lemma 4.15 (clubs are not very active), the club X has been active at most 7 times. The minimum parameter of M is at least 8, giving us that $|A| \geq 9$, so let $v \in A$ be a vertex which has not moved yet. Let T be the set of indices of the parts to which X has sent vertices. If there is $i \in T$ with $deg(P_i, v) = r_i$, then we can move the vertex v to P_i to produce another minimum Mozhan partition (Lemma 3.15, vertices can move). This move would not move any vertex twice (satisfying s1), and would not cause a club to send a vertex to four or more parts (satisfying s2), so it must be that v is fully joined to a full club C in P_i (violating s3). But this club C is the only club in P_i to which X can have ever sent a vertex (Lemma 4.14, clubs don't have many targets), showing that S was already in violation of condition s3, contradicting the construction of S.

Thus, every $i \in T$ has $deg(P_i, v) > r_i$. But this implies that $|T| \leq 2$ by Lemma 4.2 (small chromatic excess) applied to v. Now suppose we have any part P_i with $deg(P_i, v) = r_i$, and let d be the move obtained by sending v to P_i. The move d cannot extend S, as S is maximal, so d must violate one of the criteria by which S was constructed. The vertex v moves during d, and v has not moved yet, so s1 is satisfied. The club X has only sent vertices to two parts, so the move d will not make any club send a vertex to four or more parts, so s2 is satisfied. There is only one criterion remaining, it must be that X is fully joined to a full club in P_i. Thus, the club X is fully joined to full clubs in all parts P_i with $deg(P_i, v) = r_i$. By Lemma 4.12 (joining big cliques), these full clubs are all fully joined to each other. Together, they induce a large clique, of size $1 + \sum_{i \notin T} r_i$. If $\Delta(G) = 66$, then we can choose each r_i to be 8, and we are left with a clique of size $\Delta(G) - 1 - 16 = \Delta(G) - 17$.

Since we assumed that G is a counterexample, it must be that $\Delta(G) > 66$. Our big clique will have size at least $\Delta(G) - 1 - 9 \cdot 2 = \Delta(G) - 19$. Suppose that $\omega(G) < \Delta(G) - 17$. Recall that the King result (Lemma 2.2) allows us to find a maximal independent set intersecting all maximum cliques when $\omega > 2/3(\Delta + 1)$. We have that $\Delta - 19 > 2/3(\Delta + 1)$ when $\Delta > 59$, and we have $\Delta(G) > 66$, so the King result may be used to find such an independent set I. As I is an independent set, we have $\chi(G - I) \geq \chi(G) - 1$, and as I is maximal we have $\Delta(G - I) \leq \Delta(G) - 1$. Thus,

$$\begin{aligned}\Delta(G - I) &\geq \chi(G - I) - 1 \text{ by the Greedy Bound} \\ &\geq \chi(G) - 2 \\ &= \Delta(G) - 3.\end{aligned}$$

If $\Delta(G - I) \leq \Delta(G) - 2$, then $\chi(G - I) \geq \chi(G) - 1 = \Delta(G) - 2 \geq \Delta(G - I)$, so Theorem 1.13 gives us that $\omega(G - I) \geq \Delta(G - I) - 3 \geq \Delta(G) - 6$, contradicting the fact that G cannot contain a clique this large. Therefore we know that $\Delta(G - I) = \Delta(G) - 1$. If $\chi(G - I) = \chi(G)$, then we obtain $\Delta(G - I) = \chi(G - I)$, and again Theorem 1.13 contradicts the fact that G is a counterexample. Thus, $\chi(G - I) = \chi(G) - 1$. All together, we have that

$\chi(G-I) = \Delta(G-I) - 1$, and $\omega(G-I) = \omega(G) - 1 < \Delta(G-I) - 17$, and $\Delta(G-I) > 66$, as the case $\Delta = 66$ was already handled. We therefore get that $G - I$ is a more minimal counterexample than G, and this final contradiction proves the claim. $\square$

Chapter 5

Conclusion and Further Work

Thus, we have proven that when $\chi = \Delta - 1$, for Δ sufficiently large, all graphs must contain a $\Delta - 17$ clique. This shows that the methods used by Cranston and Rabern to prove Theorem 1.13 can be extended to the case $\Delta - 1$, at the expense of the size of the derived clique. Indeed, we believe that these results may be directly extended even further to yield a clique of size $\Delta - O(t^2)$ in the general case $\chi = \Delta - t$ so long as $\Delta \geq f(t)$, where f is some function of t. Heuristically, the loss is $O(t^2)$ because the minimum parameter of the necessary Mozhan partitions must be linear in t, and because the chromatic error of any critical vertex is also linear in t. These two error quantities multiply together (each excluded part "subtracts" some vertices from the largest clique we can find) to produce a loss that is $O(t^2)$.

The methods discussed and the results shown will be of some interest in the continued investigation into the question of what happens to ω when χ is close to Δ. In a joint work currently in progress with P. Haxell, we use Mozhan partitions and a *different* framework for move sequences to prove that graphs with $\chi = \Delta - t$ and $\Delta \geq f(t)$ (again, for some function f) contain a clique of size $\Delta - 2t^2 - 6t - 3$, generalizing the result of Cranston and Rabern (Theorem 1.13).

All of these results make progress towards resolving the conjectures of Borodin and Kostochka (Conjecture 1.5), and Reed (Conjecture 1.14). However, each result leaves a small margin of error, and it appears that some new ideas will be necessary to close the gap and prove the conjectures in full, or disprove them. Nonetheless, Mozhan partitions and move sequences will continue to be fruitful ideas and play an important role in the settling of these conjectures.

The resolution of these conjectures would go a long way towards our understanding of the chromatic number and its relation to the clique number, particularly when χ is close to Δ. Once again, we would like to observe how strange it is that we are ignorant of what happens to ω when $\chi = \Delta$, given the near century which has passed since Brook's full characterization of the case $\chi = \Delta + 1$.

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Glossary

1. Active Component: See Definition 3.9.
2. BK: The Borodin-Kostochka Conjecture, see Conjecture 1.5.
3. Child Relation: A partial order on graphs, see Definition 2.5.
4. Chromatic Excess: See Definition 4.1.
5. Claw: The graph $K_{1,3}$.
6. Clique: A graph with all possible edges.
7. Club: A component inside a part as seen across a move sequence, see Definition 3.18.
8. Core, Core Graph: See Definition 4.3.
9. Critical, n-Critical: A vertex $v \in G$ is critical when $\chi(G-v) < \chi(G)$, see Definition 1.16. A graph is n-critical when $\chi(G) = n$ and all proper subgraphs are $(n-1)$-colourable, see Definition 1.17.
10. d_l-Choosable: See definition 2.16.
11. Dominating Vertex: A vertex adjacent to all other vertices, see Definition 1.15.
12. f-Assignment: See Definition 2.11.
13. f-Choosable: See Definition 2.12.
14. Full Club: See Definition 4.9.
15. Girth: The minimum length of a cycle, if one is present, else ∞. see Definition 1.3.
16. Graph Epimorphism: A surjective graph homomorphism, see Definition 2.4.

17. Graph Homomorphism: Given graphs G and H, a map $f : V(G) \to V(H)$ such that $vw \in E(G) \implies f(v)f(w) \in E(H)$.

18. Independent Set: A set of vertices which are pairwise non-adjacent.

19. Induced Subgraph: A subgraph formed by deleting vertices.

20. Joined: Let G be a graph, and let $A, B \subseteq V(G)$. Then A is joined to B when every edge between A and B exists in G.

21. L-Colourable: See Definition 2.10.

22. List Assignment: See Definition 2.9.

23. Lovász Partition: See Definition 3.2.

24. Matching: A set of edges, no two of which share an endpoint.

25. Minimum Mozhan Partition: See Definition 3.8.

26. Move: See Definition 3.16.

27. Move Sequence: A finite sequence of moves, see Definition 3.17.

28. Mozhan Partition: See Definition 3.5.

29. Mule: A graph which is minimal with respect to the child relation, see Definition 2.8.

30. Nucleus: See Definition 1.20.

31. Parameters (of a Mozhan partition): Given a Mozhan partition $M = ((P_1, \ldots, P_k), v, s)$, the parameters are $r_s \coloneqq \chi(P_s) - 1$ and $r_i \coloneqq \chi(P_i)$ for $i \neq s$, see Definition 3.6.

32. Pot: A pool of available colours, see Definition 2.13.