Minimum Number of Triangles of K_5 Descendants

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Masters of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2022

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

In the study of Quantum Field Theory and Feynman Periods, the operation of double triangle expansion plays an important role. This is largely due to double triangle expansions not affecting the maximum weight of the period. In this thesis, we take a look at the effects of double triangle expansions on K_5 graphs. More specifically, given any graph G that can be obtained through a sequence of double triangle expansions on K_5, we calculate the minimum number of triangles of any graph that can be obtained through double triangle expansions on G. While the minimum number of triangles of graphs that are obtained through double triangle expansions on K_5 is already known, this is a generalization of that. This is done by understanding the structure of graphs that come from K_5 and double triangle expansions, and how double triangle expansions relate to this structure. Commonly arising graphs are studied, and showed to be building blocks for more complicated graphs.

Acknowledgements

I would like to thank my parents for their continued support through my education. Without their support, I wouldn't have been able to get anywhere close to as far as I did. I would also like to thank all teachers and students that I have encountered on my way. The simple interactions are often the ones that make things the most enjoyable. A massive thank you also goes out to my advisor, Karen Yeats, for her overwhelming support through what was a difficult year for all of us. Finally, I'd like to thank JB, for being a constant reminder of what I'm capable of throughout my childhood, through the spirit of healthy competition.

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Chapter 1

Introduction

In this thesis, we look into the minimum number of triangles of graphs that are obtained from some number of operations, called double triangle expansions, on the graph K_5. The focus of chapter 2 will be defining and examining this operation. For now, let's back up and focus on why this is of interest.

This operation uses combinatorial techniques, as opposed to analytical ones, to predict a property known as transcendental weights of Feynman Integrals. We will not define transcendental weights directly, but those interested can see [2]. Feynman integrals arise in perturbative quantum field theory. While the integrals are normally difficult to calculate, they are closely related to Feynman periods [5]. Feynman periods are easier to calculate integrals that arise from certain graphs. The edges of the graph represent particles and the vertices represent their interactions. The focus of this thesis will be on graphs that come from ϕ^4-theory, for simplicity. These graphs are 4-regular, with potentially external edges, which naively explains the relationship to K_5. The operation which will be the focus of this thesis, the double triangle expansion, has interesting properties regarding the weight of the Feynman periods. Mainly, applying a double triangle expansion to a graph does not change the maximum weight of the period [9]. So, we focus on studying this operation to better understand the weight of these Feynman periods [8].

1.1 Graph Theory Background

A lot of the techniques and definitions used throughout this thesis do not stem from classic graph theory. We will go through some of the definitions that stem from background as opposed to things that arise directly from double triangle expansions. For definitions that arise in the paper but are not explicitly defined, see [3].

Definition 1.1 A subdivision of an edge, v_iv_j is an operation that replaces the edge v_iv_j with the edges v_iv_k and v_kv_j, where v_k is the vertex created by the subvision.

Definition 1.2 (Double Triangle Expansion) A double triangle expansion, denoted as DTE, is an operation performed on a triangle $(v_1v_2v_3)$ and an additional vertex not in the triangle, v_4, that is adjacent to v_1. Then, the operation is defined as follows:
The edge (v_1v_4) is removed.
The edge (v_2v_3) is subdivided where the new vertex is denoted as v_5.
The edge (v_1v_5) and (v_4v_5) are added.

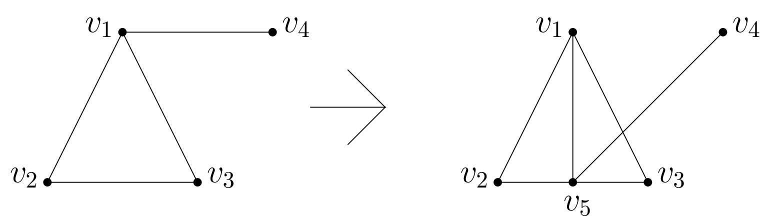

Figure 1.1: Double Triangle Expansion

A double triangle reduction (denoted DTR) is the inverse operation of a DTE. Given two triangles that share an edge $(v_1v_2v_3)$ and $(v_2v_3v_4)$, we turn these into a single triangle by removing edges v_1v_3, v_2v_3 and v_3v_4, adding the edge v_1v_4 and identifying the vertices v_2, v_3. The majority of the thesis will focus on DTE as opposed to DTR.

Definition 1.3 (Parent, Child) A graph, G, is a parent of another graph, H, if there is some DTE that transforms G into H. H is then a child of G.

Definition 1.4 (Ancestor, Descendant) A graph G is an ancestor of another graph H if there is some sequence of DTE that transforms G into H. Similarly, H is then a descendant of G.

Definition 1.5 (Circulant Graphs) Suppose we have a graph, G, on n vertices with each vertex labelled as an element of $\boldsymbol{Z}/\boldsymbol{Z}_n$ and a set C, where C is a subset of $\boldsymbol{Z}_n\backslash 0$. Then G is a circulant graph if and only if any two vertices, x, y are adjacent if and only if $x - y \in$C.[4]

Definition 1.6 $(\triangle(G))$ $\triangle(G)$ of a graph, G, is the minimum number of triangles on all graphs that can be obtained through some sequence of DTE on G.

Definition 1.7 (Spanning Tree) A subgraph T of a graph G is a spanning tree if T is a tree and $V(T) = V(G)$. [3]

Definition 1.8 (Point Count) The point count of the variety of f over a field of K is defined to be: $[f]_K = |(\bar{x} \in K^n : f(\bar{x}) = 0 \text{ in } K)|$. If K is the field with q elements we abreviate to $[f]_q$.

Definition 1.9 (4-point graph) A 4-point graph is a graph that can be obtained by removing a vertex from a 4-regular graph.

Definition 1.10 (Completion/Decompletion) A decompletion, H, of a 4-regular graph, G, is formed by removing a vertex and all incident edges from G. G is then the completion of H.

Example 1.11 K_5 is a completion of K_4. Similarly, K_4 is the only decompletion of K_5, up to ismorphism. K_5 and K_4 are 4-regular and 4-point graphs respectively. See figure 1.2.

Definition 1.12 (Kirchhoff Polynomial) Let G be a graph. The Kirchhoff Polynomial of G is defined as

$$\Psi_G = \sum_T \prod_{e \notin E(T)} a_e$$

where the sum is over the spanning trees of G and a_e are the variables of the edges. [6] [7]

So, the Kirchoff Polynomial is a polynomial in the variables that are assigned to the edges. Going through an example for the cycle of length 3, we get:

Example 1.13 Labelling the edges as a_1, a_2, a_3, we can see that each spanning tree of the $3-$cycle misses exactly one edge. So,

$$\Psi_G = a_1 + a_2 + a_3$$

Definition 1.14 (c_2-invariant) Let G be a graph with at least 3 vertices. The c_2-invariant is the sequence over the primes, where the term at prime p is given by

$$c_2^{(p)}(G) = \frac{[\Psi_G]_p}{p^2} \bmod p$$

[6, 7]

The mod p is the standard definition of mod p. That is, the c_2^p invariant for any given p is an element of $\boldsymbol{Z}/\boldsymbol{Z}_p$.

The c_2-invariant is interesting as it predicts properties of Feynman periods. Better understanding c_2 will result in a better understanding of Feynman periods. Furthermore, c_2 yields interesting sequences in p such as coefficient sequences of modular forms [6, 7].

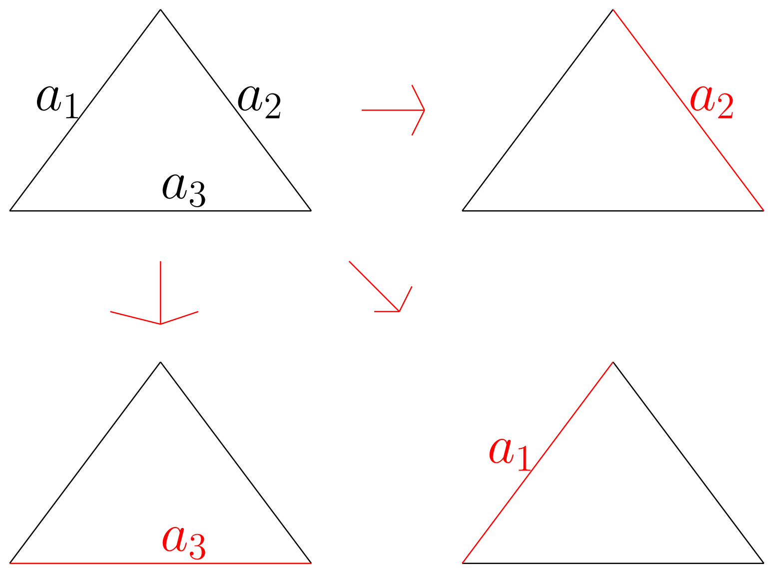

Figure 1.2: Kirchoff Polynomial For A 3-Cycle; The red edges represent the edges not in the respective spanning trees. The Kirchoff Polynomial is a sum of the indices of the red edges.

Theorem 1.15 *$c_2^p(G) = -1$ for all K_5 descendants and for all values of p*

It can be shown that $c_2^p(K_5) = -1$, for all p, and since c_2-invariant doesn't change by double triangle expansions, we have that all graphs that are obtained through a sequence of double triangle expansions on K_5 have $c_2^p(G) = -1$. [6, 7]

Conjecture 1.16 (Brown/Schnetz, Conjecture 25, [1]) *Suppose G is a 4-point graph. Then $c_2^{(p)}(G) = -1$ if and only if G is a decompletion of some K_5 descendant.*

This conjecture would hopefully be solved with a better understanding of structures of K_5 descendants and the graphs that can be obtained from a sequence of double triangle expansions of K_5. This would tell us that $c_2^p(G) = -1$ for all p is a result of some form of structure on the graph. In this thesis, we focus on a specific type of graph that can be obtained from K_5 through double triangle expansions. That is, we focus on graphs that minimize the number of triangles. While these graphs have a lot of additional structure that other graphs don't, the techniques talked about through the thesis can be applied to any graph. The goal of this thesis is not to solve the conjecture. The goal of this thesis is to provide a better understanding of K_5 descendants and the effects DTEs have on these descendants [7].

1.2 Previous Results of Double Triangle Expansions on K_5 Graphs

In previous papers on this topic, there is generally a focus on one of two questions. The first is calculating the minimum number of triangles of graphs that can be obtained from performing double triangle expansions on K_5 graphs. The minimum number of triangles has been shown to be 4, and there are many known graphs with 4 triangles that come from K_5 by double triangle expansions. This is already a slightly surprising result, as a DTE only guarantees 2 triangles in the local environment, so having a minimum number of triangles of 4 tells us that the local operation of DTE is still related to the global structure of the graph in some way. This thesis extends this question further by not looking at just K_5 descendants, but by looking at graphs that descendant from some fixed K_5 descendant. For example, let's take the closed zig-zag on 100 triangles. See figure 2.4 for the structure of closed zig-zags. Then, is there a graph with 4 triangles that can be obtained from this graph through double triangle expansions? If so, what is special about 4 triangles as opposed to a different number of triangles. If not, what is the minimum number of triangles a graph can have if we start at this 100 triangle graph?

The second question that arises, is the enumeration of the number of different types of graphs that can be obtained through double triangle expansions on K_5. The enumeration is generally done with respect to the number of vertices and the level.

Definition 1.18 The level of a graph is defined as the number of vertices minus the number of triangles.

There are explicitly known generating functions for levels 0 to 4 [7]. That is, given any number of vertices, we know how many graphs of a given level from 0 to 4, exist with that number of vertices. As the level increases, and so, the number of triangles decreases, there is an increasing number graphs that can be obtained. This thesis doesn't focus on this question, and the techniques involved are quite different. However, this thesis does dive deeply into the structure of K_5 descendants, and may be useful when dealing with enumeration of unknown levels. See [7] for detailed proofs of these generating functions.

1.3 Roadmap

Chapter 2 focuses on the structure of graphs that are obtained from K_5 through double triangle expansions (page 14). The structure of these graphs will play a major role in determining the fewest number of triangles of graphs that can be obtained from K_5. This is the main focus of the thesis.

Chapter 3 focuses on the types of double triangle expansions we will need to pay close attention to when trying to remove as many triangles as possible. This chapter highlights how double triangle expansions interact with the structure of the graphs we are working with. Although we are focusing on graphs with a minimal number of triangles, the effects of double triangle expansions studied in this chapter give insight even to graphs that don't have a small number of triangles.

Chapter 4 focuses on finding the minimum number of triangles of types of graphs that commonly arise when removing triangles through double triangle expansions. It introduces the main theorem of the paper, which deals with the number of triangles that can be removed from a zig-zag graph (page 14), which is the building block of the types of graphs we are looking at. In this chapter, we start to prove three theorems simultaneously through induction. More specifically, theorem 4.1 is shown to be true up until some natural number N. Theorem 4.7 is shown to be true up until the same N, using theorem 4.1. Theorem 4.8 is shown to be true up until the same N, using theorem 4.1 and 4.7. The three theorems all deal with the minimum number of triangles of commonly arising zig-zags.

Chapter 5 continues on with the proof of the theorem introduced in chapter 4. More specifically, theorem 4.1, 4.7 and 4.8 are shown to be true for all n, using the fact that the three theorems are true up for $n \leq N$. We finish of the proof by proving a statement about general zig-zag chains. So, while our theorems in chapter 4 dealt with the simplest zig-zag structures, we actually need to be able to deal with any zig-zag structure for this proof.

Chapter 6 takes the results from chapter 5 and generalizes them to a different form of zig-zag but equally important type of graph that arises from K_5. While the earlier chapters were dealing with open zig-zags (page 13), we now extend our theorems to closed zig-zags. Many of the techniques used in previous chapters are used here. The results in this chapter, allow us to deal with any K_5 descendant.

The conclusion recaps the main theorems of the thesis and briefly talks about future directions.

Chapter 2

Double Triangle Expansions

2.1 General Definitions

Having already defined the main operation, DTE, in the last chapter, we move onto the class of graphs that will be the focus; K_5 descendants. That is, we are focusing on all graphs that can be obtained from some sequence of DTE performed on K_5. This chapter will deal with understanding the structure of this class of graphs. That is, we will look at properties that are similar for all (or most) K_5 descendants and then use these properties in future chapters.

Perhaps the nicest property of K_5 descendants is that they are all 4-regular. Often, throughout this thesis, we will show images that aren't 4-regular. The reason for this will be due to the fact that the 4-regularity won't be important at that specific moment and so we don't include all the edges to avoid clutter.

Definition 2.1 (O and $Z_{5:}$) The unique graph obtained by performing one DTE on K_5 will be denoted O. $Z_{5:}$ is the unique graph obtained from performing one DTE on O.

O is the octahedron; see figure 2.1. The structure of the octohedron is important to this thesis but the graph appears as an exception to a lot of the following theorems. More specifically, K_5 descendants that are also $Z_{5:}$ descendants have very similar structure and the octahedron and K_5 are exceptions to this fact. The notation of $Z_{5:}$ is a special case of $Z_{n:}$, a type of graph that will be discussed shortly.

Both O and $Z_{5:}$ are unique K_5 descendants with respect to the number of vertices. That is, the choices of DTE on K_5 and O result in the same graphs since K_5 and O are vertex transitive. The graphs obtained from performing any DTE on K_5 or O are isomorphic

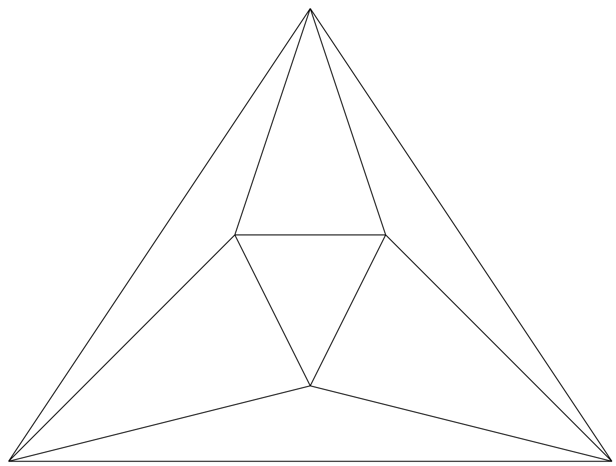

Figure 2.1: Octahedron

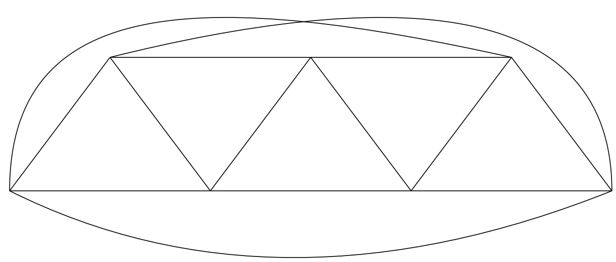

Figure 2.2: $Z_{5:}$

to any other graph obtained from performing a different DTE on K_5 or O, respectively. Starting with $Z_{5:}$, there will be multiple choices in DTE, in the sense that there are multiple non-isomorphic graphs that can be obtained from a given graph.

Another nice property of K_5 descendants is that they are all simple graphs.

Lemma 2.2 Let G be a K_5 descendant. Then G is simple.

Proof of 2.2 Suppose towards a contradiction that G is a non-simple K_5 descendant with the lowest number of vertices among all non-simple K descendants. Then, G is a child of some simple K_5 descendant, H. Denote the triangle and additional vertex that the DTE was performed on as $(v_1v_2v_3)$ and v_4, respectively. Denote the new vertex as v_5. Then, all edges of G are in H with the exception of v_1v_5, v_2v_5, v_3v_5 and v_4v_5. So, there exists two edges between v_5 and one of at least one of v_1, v_2, v_3 and v_4. But the edges incident to v_5 are only the four stated above, and v_1, v_2, v_3, v_4 are distinct. This is a contradiction, so all K_5 descendants are simple. ■

Going through some of the more obvious properties of DTE, we already showed it preserves 4-regularity. The number of vertices increases by 1. It would seem that DTE increases the number of triangles by 1. After all, we are splitting 1 triangle into 2. This, however, is not the case. In fact, DTE can actually decrease the number of triangles.

Theorem 2.3 *Let G be a K_5 descendant that is not K_5. If G' is a child of G, then the number of triangles of G' differs from the number of triangles of G by at most* 1.

We will prove the theorem shortly, but first we need some understanding of excluded subgraphs in K_5 descendants.

2.2 Excluded Subgraphs and Structure of K_5 Descendants

Theorem 2.4 *No K_5 descendant other than K_5 contains a K_4 subgraph.*

Proof of 2.4. We will show that if G' is a graph that is obtained through a DTE on some other graph G, and G contains no K_4 subgraph, then G' contains no K_4 subgraph. Let $v_5 \in G'$ be the vertex created by the DTE on G. Furthermore, denote the vertices that were incident to the edge that v_5 subdivided as v_2 and v_3,the vertex that is not in the triangle as v_4 and the vertex in the triangle that is incident to the additional edge, the edge not in the triangle, as v_1. Furthermore, denote the remaining two vertices as v_2 and v_3. This can be seen in figure 2.4. All vertices of G' that are not v_5 are also in G. Furthermore, all edges in G' that are not incident to v_5, are also in G.

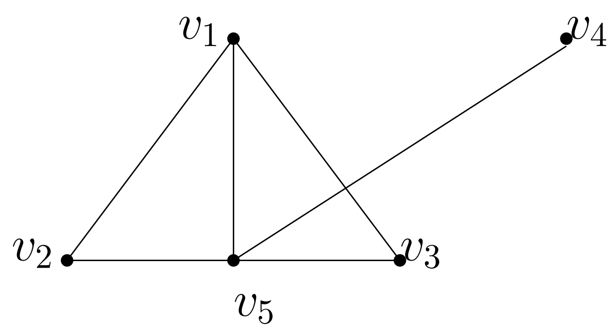

Figure 2.3: DTE

$G' \setminus v_5$ is a subgraph of G, so it has no K_4 subgraph. Suppose then, that v_5 is contained in the K_4 subgraph. Then, the K_4 subgraph contains v_5, and three of v_1, v_2, v_3, v_4. However, v_2 and v_3 are not adjacent since the unique edge they were both incident to was subdivided. Furthermore, v_1 and v_4 are not adjacent in G' since the unqiue edge they were both incident to was removed by the DTE. So, any set of size 3 of those vertices includes a pair of vertices that are not adjacent to each other. So K_4 isn't a complete graph, which is a contradiction. Therefore, G' will not contain any K_4 subgraphs if G doesn't.

O doesn't contain any K_4 subgraphs. Furthermore, since all DTE descendants of K_5 excluding K_5 are also DTE descendants of O, we conclude that all K_5 descendants other than K_5 and O don't contain any K_4 subgraphs. ■

Theorem 2.5 *No K_5 descendant other than K_5 contains a $K_{3,1,1}$ subgraph.*

Proof of 2.5 Similar to the proof above, we will show that if G' is a graph obtained through a DTE on some other graph G and G contains no $K_{3,1,1}$ subgraph, then G' contains no $K_{3,1,1}$ subgraph. Denote the newly created vertex as v_5 and the edges e_1, e_2, e_3, e_4 as the edges incident to v_5 (with no further distinction between the edges).

Suppose towards a contradiction that there exists a $K_{3,1,1}$ subgraph in G' but not in G. Denote the edge in $K_{3,1,1}$ that is contained in 3 triangles as e. Suppose e is not one of e_1, e_2, e_3, e_4 and not incident to any of these edges. Then, e is an edge in G and is still contained in 3 triangles since all edges in G' are in G other than e_1, e_2, e_3, e_4. So, there exists an edge contained in 3 triangles in G. So, G contains a $K_{3,1,1}$ subgraph. This is a contradiction.

So, suppose then that e is one of e_1, e_2, e_3, e_4. Then v_5 and one of its neighbours share 3 neighbours. But this is a contradiction, as v_5 only shares at most 2 neighbours with any of its given neighbours. Therefore, G' will not contain a $K_{3,1,1}$ subgraph.

Finally, suppose e is not one of e_1, e_2, e_3, e_4 but is adjacent to at least one of them. Then, v_5 is in the $K_{3,1,1}$ subgraph since all of e_1, e_2, e_3, e_4 are incident to v_5 and all edges incident to e are in the subgraph since the graph is 4-regular. Then, $(v_2v_5v_1)$ or $(v_3v_5v_1)$ is one of the triangles in the $K_{3,1,1}$ subgraph, along with two other triangles T_1 and T_2. Then, $(v_2v_3v_1)$, T_1 and T_2 form a $K_{3,1,1}$ in G. But this is a contradiction. Therefore, G' will not contain any $K_{3,1,1}$ subgraphs if G doesn't.

O doesn't contain a $K_{3,1,1}$ subgraph. Furthermore, since all DTE descendants of K_5 are also DTE descendants of O, we conclude that all K_5 descendants dont contain any $K_{3,1,1}$ subgraphs. ■

Now, time to go back and prove theorem 2.4, dealing with minimum number of triangles.

Proof of 2.3. Suppose G is a K_5 descendant that isn't K_5 or O and G' can be obtained by performing a double triangle expansion on triangle v_1, v_2, v_3 and vertex v_4 for vertices $v_1, v_2, v_3, v_4 \in G$. Then, all edges in G are also in G' with the exception of v_1v_4 and v_2v_3. So, all triangles in G are also in G' excluding the triangles that have v_1v_4 or v_2v_3 as an edge. v_2v_3 is always in the triangle $v_1v_2v_3$. It may also be in the triangle $v_2v_3b_1$ for some vertex b_1. v_2v_3 can't be in another triangle, $v_2v_3b_2$? If such a triangle existed then v_1, b_1, b_2, v_2, and v_3 would form a $K_{3,1,1}$ subgraph, which is excluded by a theorem 2.6. So, v_2v_3 is in either 1 or 2 triangles in G. Similarly, v_1 is adjacent to both v_2 and v_3, so $v_1v_2v_4$ and $v_1v_3v_4$ are both potential triangles. However, there may also exist another vertex, b_3, such that $v_1v_4b_3$ is a triangle in G. Similar to the case of v_2v_3, only two of these cases can occur or else we have a $K_{3,1,1}$ subgraph. So we have v_1v_4 in either $0, 1$ or 2 triangles in G. So G has either $1, 2, 3$ or 4 triangles that aren't in G'.

Now, how many triangles are in G' that aren't in G. $v_1a_1v_2$ and $v_1a_1v_3$ are two triangles that are always in G' and not in G'. If v_4 is adjacent to v_2 (or v_3) then $v_4a_1v_2$ (or $v_4a_1v_3$) are triangles in G' that aren't in G. These triangles' existence directly corresponds to the existence of $v_1v_{4v_2}$ (or $v_1v_4v_3$) in G. That is, $v_4a_1v_2$ is a triangle in G' if and only if $v_1v_2v_4$ is a triangle in G. A similar argument can be made with $v_1v_3v_4$ and $v_4a_1v_3$. So, in some sense, these triangles cancel each other out. So, G has $1, 2$ or 3 triangles that aren't in G' and that don't get replaced by a corresponding triangle in G'.

Back to G', there aren't any other triangles in G' that aren't in G. This is true since v_1a_1 is in two triangles, and can't be in any more due to $K_{3,1,1}$ being forbidden. Similarly, v_4a_1 can be in two triangles, but not a third since v_4 is not adjacent to v_1 (as that edge was removed as part of the double triangle expansion) and a_1 is adjacent to v_1. What about v_2a_1 and v_3a_1? Well, they are in the triangles $v_2a_1v_1$ and $v_3a_1v_1$ which we've already mentioned above, and they are also potentially in the triangles $v_2a_1v_4$ and $v_3a_1v_4$. However, $v_2a_1v_3$ can't be a triangle since v_2v_3 is not an edge as v_2v_3 was subdivided as part of the double triangle expansion.

So, we have exactly 2 triangles in G' that aren't in G and don't correspond to a triangle that was removed in G. Since G has $1, 2$ or 3 triangles that aren't in G', we get the number of triangles in G differs from the number of triangles in G' by at most 1. ■

Lemma 2.6 Suppose G' is obtained by performing a DTE on G. Further suppose e is an edge in G'. If e is contained in no triangles in G, then e is contained in no triangles in G'.

Proof of Lemma 2.6 We can see from the proof of theorem 2.4, that the only triangles in G' that aren't in G are the triangles $(v_1v_2a_1)$, $(v_1v_3a_1)$ and possibly the triangles $(v_1v_3b_1)$ and $(v_1v_2b_1)$. However, the edges, v_1v_2 and v_1v_3 are already in triangles in G and the edges $, v_1b_1, v_2b_1$ and v_3b_1 are only in the triangles specified above in G' if they are in triangles in G. ■

From the theorem and lemma above, we can see that DTE only affect triangles that directly contain an edge that the DTE is acting upon.

Back to theorem 2.4, we have shown that DTE can indeed decrease the number of triangles in a graph. However, it is dependent on there being other triangles near the double triangle expansion. With this in mind, we are able to introduce the main question of this paper:

Question 2.7 *Given a K_5 descendant, G, what is the minimum number of triangles that a descendant of G has?*

The minimum number of triangles of any K_5 descendant was already proven to be 4 by Yeats, Mishna and Laradji. Of course then, it is true that any descendant of K_5 [7]. There are many examples of graphs with 4 triangles. Are we done then? Well, not quite. While the minimum number of triangles that a K_5 descendant contains is 4, that does not mean that given any K_5 descendant, G, that there exists a descendant of G with 4 triangles. Perhaps, while performing DTE to get from K_5 to G, one changes K_5 in such a way that now it is no longer possible to get to a descendant of G that has 4 triangles. Is it possible for G to have a minimum number of triangles that is greater than 4?

To begin, one should consider the actual structure of K_5 descendants. We will show that the building blocks of K_5 descendants are zig-zag graphs. There are two different types of zig-zags, although they behave very similarly.

Defintion 2.8 (Closed Zig-Zag Graph) A zig-zag graph on n vertices is a circulant graph where the connection set is $-2, -1, 1, 2$. This graph is denoted $Z_{n-2:}$.

Generally, these graphs are drawn with the odd vertices in a lower row in increasing order and the even vertices in an upper row in increasing order. The unique graph obtained from any DTE on O is a closed zig-zag; $Z_{5:}$. These graphs are not the main focus of the thesis but will be touched upon in chapter 6.

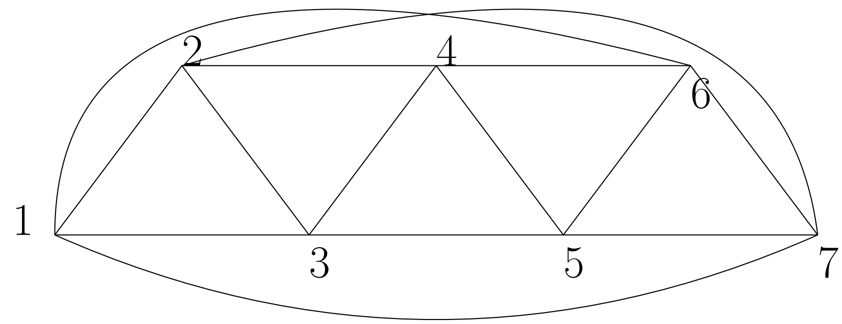

Figure 2.4: Closed Zig-Zag

Definition 2.9 (Open Zig-Zag Graph) An open zig-zag graph on n vertices is a zig-zag graph on n vertices with the edges $1n, 1(n-1), 2n$ removed. These graphs will be denoted Z_{n-2} since $n-2$ is the number of triangles in the zig-zag.

Visually, an open zig-zag graph is a zig-zag graph where the vertices don't wrap around the graph back to themselves. The majority of this thesis will focus on this type of graph due to the importance these graphs have in the structure of K_5 descendants.

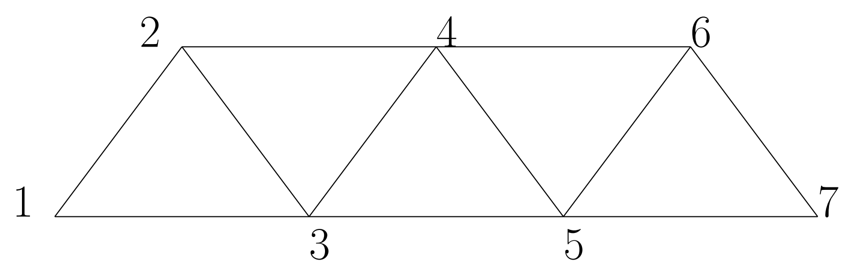

Figure 2.5: Open Zig-Zag, Z_5

Definition 2.10 (Disjoint Zig-Zags) Two zig-zags are said to be disjoint if there is no vertex that is in both zig-zags.

Definition 2.11 (Zig-Zag Chain) A zig-zag chain of length m is a graph of m open zig-zag subgraphs where each open zig-zag shares a vertex with the next zig-zag in the sequence. A zig-zag chain is closed if every zig-zag piece shares an end vertex with exactly 2 zig-zags. That is, the zig-zag chain wraps back around to the start. A zig-zag chain is open if it is not closed.

Zig-zag pieces that share a vertex will be joined by a , and if there is an edge between two zig-zag pieces then it will be denoted by a ;. The latter case will arise in chapter 4 and onward when we start destroying triangles in some zig-zags.

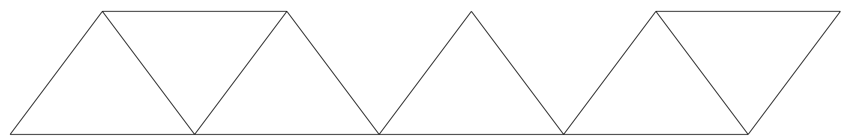

Figure 2.6: Zig-Zag Chain, $Z_{3,1,2}$

The structure of zig-zags and zig-zag chains refers to the length of the zig-zags and zig-zag chains and how they are joined. So, we will say a DTE doesn't change the structure of the zig-zags and zig-zag chains if the length of each of the zig-zags and zig-zag chains doesn't change. More specifically, if the labelled graph is different after a DTE but the unlabelled graph is the same, then we say the DTE doesn't change the structure of the DTE.

Definition 2.12 (Completed Primitive) A graph is considered completed primitive if it is internally 6-edge-connected. That is, a graph is completed primitive if the only way to disconnect the graph by removing 5 or fewer edges is to disconnect a single vertex.

Theorem 2.13 *All K_5 descendant are completed primitive.*

Proof 2.13 It follows from observation that K_5 is completed primitive [9]. Schnetz showed that DTE maintains completed primitiveness. So, all K_5 descendants are completed primitive. ■

2.3 Structure Of K_5 Descendants

In this next section, we will discuss the structure of K_5 descendants. We will show that zig-zags and zig-zag chains are the building blocks of K_5 descendants. While understanding zig-zags and zig-zag chains will not tell us everything about K_5 descendants, they play a crucial role when dealing with the minimum number of triangles of K_5 descendants.

Theorem 2.14 *All K_5 descendants, with the exception of K_5 and O, can be expressed as a edge disjoint union of graphs where each graph is either an open or closed zig-zag, zig-zag chain or an edge which is not in any triangle in the initial graph.*

Before proving this theorem, let's reiterate what it is saying. Theorem 2.14 is saying that the structure of K_5 descendants breaks down to zig-zags and zig-zag chains. While there are still further statements to make about the structure of zig-zags and zig-zag chains in K_5 descendants, this theorem is the backbone of techniques used in later chapters.

We will prove theorem 2.14 by showing that all K_5 descendants, with the exceptions of K_5 and O, have no vertex that is in 4 or more triangles or is in a subgraph shown in figure 2.7. Initially, it may not be clear why showing the above statements proves that theorem 2.14 is true. If these two statements weren't true, then there could exists subgraphs that aren't in a zig-zag or zig-zag chain structure. So, showing that these statements hold, shows that the triangles in K_5 descendants can be expressed as zig-zags or zig-zag chains.

Proof 2.14 An equivalent statement to the theorem is that all K_5 descendants, other than K_5 and O descendants have no vertex that is in 4 or more triangles or is in a subgraph shown in figure 2.8, or in a K_4 or $K_{3,1,1}$ subgraph. So, we need to show that given any vertex v_1 in some K_5 descendant, G, v_1 is in at most 3 triangles and the graph in figure 2.7 isn't a subgraph of any K_5 descendant. First, we will show figure 2.7 is indeed an excluded subgraph. Performing a double triangle reduction on 2.7, that is, reversing a DTE, we can see that this graph becomes a K_4 subgraph. But this subgraph is excluded by theorem 2.5. So figure 2.7 is a forbidden subgraph.

Now, to show that all vertices of any K_5 descendant that isn't K_5 or O is in at most 3 triangles. Recall that all K_5 descendants are 4-regular so each vertex is in at most 6 potential triangles. Suppose $v_1v_2v_3$ and $v_1v_2v_4$ are both triangles in G. Then $v_1v_3v_4$ is not a triangle in G since its existence would create a K_4 graph, which by Theorem 2.15, is not possible. So, each triple of v_2, v_3, v_4, v_5 forms at most 2 triangles with v_1. Excluding duplicates, we are left 4 potential triangles. If there are 3 or fewer triangles, we are done. If there are 4 triangles, then we have a 4 wheel and spoke. However, each of the vertices in the wheel have degree 3, so they only connect to the rest of the graph by one edge. Removing each of these edges results in disconnecting the graph by removing 4 edges. This would imply G is not internally 6-edge-connected, so this case doesn't occur. So, each vertex in G is in at most 3 triangles, and we are done. ■

Lemma 2.15 *Suppose G is a K_5 descendant. Then there exists no vertex that is in three edge disjoint triangles.*

Lemma 2.15 follows from 4-regularity. ■

Theorem 2.16 *Suppose G is a $Z_{5:}$ descendant. Then, no zig-zag piece of G is incident to more than 2 other zig-zag pieces.*

Theorem 2.16 is saying that the structure of K_5 descendants, excluding K_5 and O, are not just graphs consisting of zig-zag pieces and zig-zag chains, but they are graphs where the zig-zags only interact with at most 2 other zig-zags. Informally, this means for

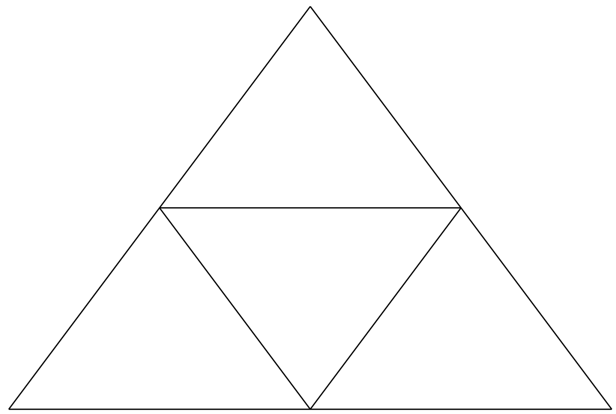

Figure 2.7: Forbidden Subgraph

any $Z_{5:}$ descendant, one draw the descendant by taking all the zig-zag and zig-zag chains and putting them on the outside outer face of the graph and not have any zig-zags going through the inner faces of the graph.

Proof of 2.16. If a zig-zag, Z_m, is incident to another zig-zag, Z_n, then there are 2 edges of Z_n that are incident to a vertex of Z_m. Since any $Z_{5:}$ descendant is 4-regular, the number of zig-zags that can be incident to a given zig-zag is at most the number of vertices in a zig-zag piece that have degree 2. However, only 2 vertices in a zig-zag piece have degree 2, so there are at most 2 zig-zag pieces incident to any zig-zag piece. ■

Theorem 2.17 *Suppose G is a K_5 descendant. Then there exists no triangle in G, where each vertex of the triangle is part of a triangle that shares no other vertices of the initial triangle (figure 2.8).*

Proof of 2.17 Suppose G is a K_5 descendant, that isn't K_5, that has the excluded structure listed above, and is minimal with respect to the number of vertices. Since G is a K_5 descendant, one can find a K_5 descendant, G^*, such that G is obtained by performing a DTE on G^*. Then, by minimality, G^* does not contain the excluded structure. But, DTEs either break up structure or split a triangle in two (see **??** for proofs of these statements); they do not cause two triangles that were not incident to each other, to become incident. So, G^* either had the excluded structure or contained a zig-zag piece which was incident to 3 zig-zags. The latter case is excluded by theorem 2.16, so we have reached a contradiction, and no such G exists. Since K_5 doesn't contain the excluded structure, we conclude theorem 2.17 is true. ■

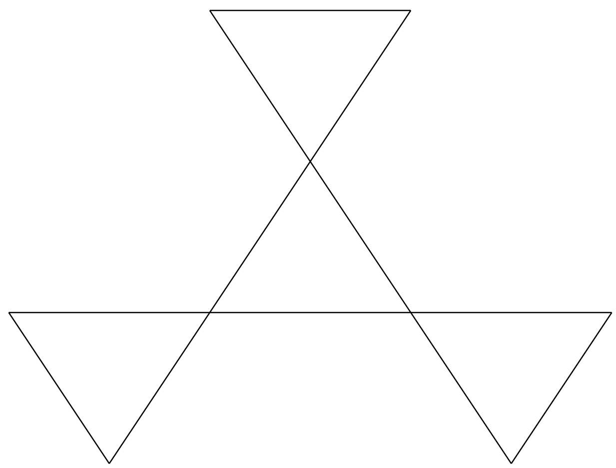

Figure 2.8: Excluded Structure

Chapter 3

Minimum Number Of Triangles

The focus of this thesis is trying to figure out the minimum number of triangles for any K_5 descendant. We will eventually show that this is indeed achievable, although there are many steps to get there. One technique that will be discussed next chapter, is showing that if we are looking for the minimum number of triangles of all descendants of some graph, we only need to look at DTE that decrease the number of triangles or DTE that don't change the number of triangles but change the structure of the zig-zag chains. This will allow us to approach descendants with a minimal of triangles with much more structured techniques; which will be shown in chapter 4.

Definition 3.1 (Increasing DTE) An increasing DTE is any DTE that increases the number of triangles of the graph by 1.

From theorem 2.3, see page 9, we know that the number of triangles changes by at most 1. The theorem also tells us when this occurs. That is, a DTE is increasing if the edge being subdivided is not part of an additional triangle and the edge being removed is not part of a triangle where a vertex of the triangle is not one of the four vertices that the DTE is initially acting upon. The reason for the condition to include a vertex not in the initial DTE is that if the removed edge is in a triangle only with vertices of the initial DTE then that triangle will be destroyed but a new one will be additionally created, resulting in no change in the number of triangles. So, given any K_5 descendant that isn't K_5 or O, one can always find an increasing DTE, since the edges in the upper and lower rows of a zig-zag are not in any triangle with vertices not in the DTE. This is also true for isolated triangles. Performing an increasing DTE does not change the structure of the zig-zag chain. It simply increases the size of whichever piece the DTE is being performed on.

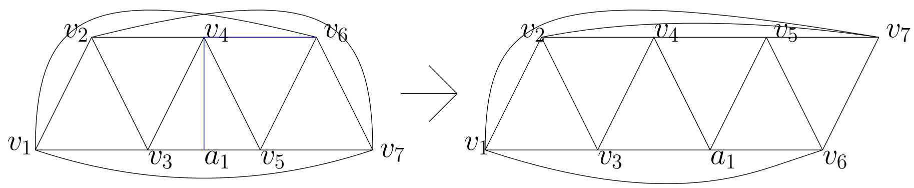

Figure 3.1: Increasing DTE

Definition 3.2 (Decreasing DTE) A decreasing DTE is any DTE that decreases the number of triangles in the graph by 1.

From theorem 2.3, we know that the number of triangles changes by at most 1. A DTE is decreasing if both the edge that is being subdivided is part of an additional triangle and the edge being removed is part of a triangle that contains a vertices that is not one of the four vertices that the DTE is initially acting upon. It is not always possible to perform a decreasing DTE on a given graph. A decreasing DTE requires other triangles to be close to the triangle that is having the DTE performed upon it.

So, performing a decreasing DTE requires the existence of at least 3 triangles. The triangle in which the DTE is being performed on, a triangle that contains the edge being subdivided and a triangle that contains the edge being removed and some vertex that isn't part of the DTE. In terms of zig-zags, this means that a decreasing DTE is performed on a $Z_{2,1}$ subgraph. This is the only type of zig-zag that a decreasing DTE can be performed on. Of course, either piece could contain more than 2 or 1 triangles, but the decreasing DTE only effects 2 triangles from one piece and one 1 triangle from the other (or all triangles could be part of one long zig-zag piece).

A decreasing DTE can not be performed on a Z_3 zig-zag (but can be for Z_n with n greater than 3). The reason for this is that there is no triangle that contains a vertex not in the initial DTE. So, even though 3 triangles are destroyed as a result of the DTE, 3 new triangles are created; the two of the split triangle, but also one where the destroyed edge was. The edge was replaced with a new edge, and the triangle still exists just with the new edge instead. As a result, we are still left with a Z_3 zig-zag.

Suppose Z_n is an open zig-zag of length greater than 3. Then, there are two choices of DTE, increasing or decreasing. Staying the same is not possible due to the fact that the decreasing and increasing DTE examples show all the cases possible. The increasing DTE results in a Z_{n+1} zig-zag. The decreasing DTE results in a zig-zag of the form $Z_{q,3,n-q-4}$. The decreasing DTE destroys some of the structure of the open zig-zag around 3 triangles that are still close together. The remaining structure on both sides remains unchanged. What happens if we then perform another decreasing DTE on the right most triangle of

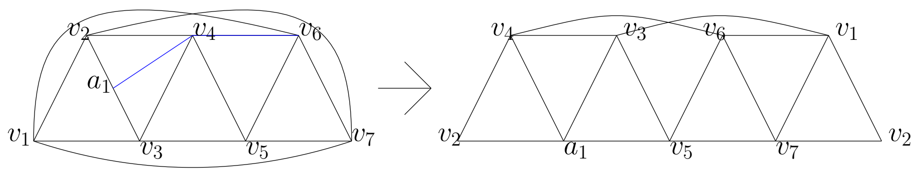

Figure 3.2: Decreasing DTE

the piece of size 3 and remove the leftmost triangle of size the zig-zag of size $n-q-4$. This will then result in the zig-zag chain being broken up completely and then being disjoint, at least this part of the zig-zag. This is shown in figure 3.3. So, a decreasing DTE within an open zig-zag results in a zig-zag chain. Furthermore, a decreasing DTE over a zig-zag chain then results in disjoint zig-zags. As one performs more and more decreasing DTEs, the structure of the zig-zags are continuously destroyed until triangles are left either isolated or isolated in pairs.

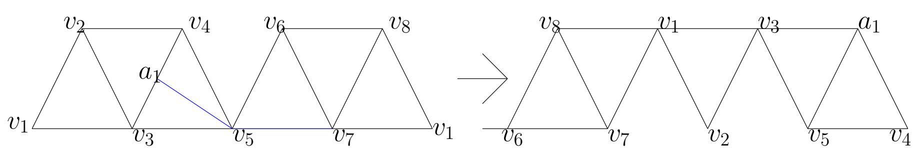

Figure 3.3: Further Decreasing DTE

Definition 3.3 (Trivial Neutral DTE) A trivial neutral DTE is a DTE that does not change the number of triangles in the graph nor does it change the structure of the zig-zags.

A trivial neutral DTE occurs when the DTE occurs on an isolated zig-zag of size 2 or size 3. The structure of the rest of the zig-zags aren't affected since the triangles in this zig-zag are too far away from other zig-zags for any DTE performed on these triangles to matter.

Definition 3.4 (Non-Trivial Neutral DTE) A non-trivial is a DTE that changes the structure of the zig-zags but doesn't change the number of triangles in the graph.

Non-trivial neutral DTE occur when either the edge being subdivided is in another triangle or the edge being removed is in a triangle that contains a vertex that isn't part of the initial DTE, but not both. This follows from the proof of theorem 2.4. There are two possible triangles that could be destroyed by a DTE, so if only one is destroyed, the result is a non-trivial neutral DTE. Non-trivial neutral DTE have fewer requirements than a decreasing DTE to occur, but still require triangles to be close to each other. So, not every graph can have a non-trivial neutral DTE performed on it.

Lemma 3.5 *Suppose Z_n and Z_m are two open zig-zags or zig-zag chains of a K_5 descendant that are disjoint and separated by at least one edge. Then no DTE affects both zig-zags.*

Proof of 3.5. Since the zig-zags are edge disjoint, any triangle in either zig-zag will not be in the other zig-zag. So, any DTE on Z_n will change the structure of triangles in Z_n along with one additional edge. But, Z_n and Z_m are separated by at least one edge, so this additional edge won't affect triangles in Z_m. So, any DTE will only affect the structure of one of the zig-zags at most. ■

Now, back to our focus on the minimum number of triangles of a K_5 descendant, we will define DTE sequences.

Definition 3.6 (DTE Sequence) Suppose G is a K_5 descendant. Then a DTE sequence, denoted $S_{G,..G^*}$, is a sequence of graphs, starting with G and ending at G^*, where each graph in the sequence can be obtained from the previous graph in the sequence by a DTE. A sequence is minimal if the number of triangles in G^* is equal to the minimum number of triangles of any descendant of G.

As discussed earlier, the building blocks of K_5 descendants are zig-zag graphs. So, it makes sense to look at these graphs first when trying to figure out the minimum number of triangles for any K_5 descendant. At first glance, this isn't much easier, as any zig-zag graph has an infinite number of DTE descendants. Fortunately, we can restrict the types of DTE operations that get us to graph with minimum number of triangles. After these restrictions, the number of relevant DTE descendants is finite and they behave a lot nicer.

The restrictions on the DTE operations are quite natural. When trying to get to a DTE descendant with a minimum number of triangles, one only needs to look at DTE descendants that are obtained through decreasing DTE or neutral DTE that change the structure of the zig-zag. In other words, we will never need to increase the number of triangles to get to a minimum number of triangles. Similarly, we won't need to perform any DTE that don't actually change the structure of the zig-zags. The next few theorems will focus on proving these facts.

Theorem 3.7 *Suppose G is a zig-zag chain and G' is a zig-zag obtained from G by applying a neutral DTE expansion that doesn't changes the structure of the zig-zag. Suppose H' is a minimal DTE descendant of G' and DTE sequence from G' to H' contains only decreasing and non-trivial neutral DTE expansions. Then there exists a minimal descendant of G, that is obtained through a non-trivial, decreasing DTE sequence, that has the same number of triangles as H'.*

Proof of 3.7 Suppose G is a zig-zag chain and G' is a zig-zag obtained from G by applying a neutral DTE expansion that doesn't changes the structure of the zig-zag. Further, suppose a minimum DTE descendant of G' is H' and is obtained by applying decreasing and non-trivial neutral DTEs on G' and obtaining through a sequence of graphs, $S_{G',G'_1,G'_2,...H'}$. Then, each graph G_i, can be partitioned into zig-zags, zig-zag chains and edges which are not in any zig-zag. Then, there exists a sequence of graphs, $S_{G,G_1,G_2,...G_n}$ where each G_i differ from G'_i only in the additional edges not in any zig-zag. That is, the zig-zags and zig-zag chains of the graphs are the same. One constructs each G_i by performing a DTE on the zig-zag chain of G_{i-1} that is the same as the DTE on the zig-zag chain of G'_{i-1}. Then H has the same number of triangles as H' since they have the same zig-zag chains.

To show that this is the minimum number of triangles a descendant of G can have, assume towards a contradiction that there exists a descendant of G with fewer triangles, say J. Further suppose that J is reached through the sequence $S_{G,J_1,J_2,...J}$. Then, applying DTE to G' in such a way that the zig-zags are affected in the same way as the sequence S_J, we get a new sequence, $G', G'_1, G'_2, ...G'^*$ where each G'_i has the same zig-zag chain structure as J_i. Then, the number of triangles of G'^* equals the number of triangles of J. But the number of triangles of J is less than the number of triangles of G', and so G'^* has fewer triangles than the minimum number of triangles that a descendant of G' can have. Thus, we have reached a contradiction, and G has a descendant obtained through non-trivial, decreasing DTE that has the same number of triangles as H'. ■

The proof of this theorem depends on the fact that the minimum number of triangles of any descendant only depends on the zig-zag structure. So, any DTE that doesn't effect the zig-zag structure doesn't effect the minimum number of triangles.

Now, to generalizing the above statement, we get the following theorem.

Theorem 3.8 *Suppose G is a zig-zag chain. Then the minimum number of triangles of a descendant of G when performing non-increasing DTE is equal to the minimum number of triangles of a descendant of G when performing only decreasing and non-trivial neutral DTE.*

Proof of 3.8 Suppose towards a contradiction, that there exists some zig-zag chain G, such that every descendant obtained through non-increasing sequences of G has a minimum descendant that must be obtained by at least one non-trivial neutral DTE. Suppose $G, G_1, ...G^*$ is such a sequence. Then, one of those graphs, say G_i, was obtained through the last non-trivial neutral DTE. By theorem 3.7, there exists a sequence without that non-trivial neutral DTE expansion that also ends with a zig-zag chain with the minimum number of triangles. Then, there is a new graph, say G_j, which was obtained through the last non-trivial neutral DTE expansion. Applying theorem 3.7 repeatedly, we eventually get a sequence of graphs from G to a graph with the same number of triangles as G^*, with no non-trivial neutral DTE expansions. But this is a contradiction. So, for any zig-zag chain G, there will always exist a non-trivial, non-increasing sequence from G to some graph G^* such that G^* has a minimum number of triangles amongst all descendants of G obtained through non-increasing sequences. ■

So, what we have just shown is that when looking for what the minimum number of triangles of a descendant of some graph, G, is, we don't need to look at descendants obtained from any trivial neutral DTE. Unfortunately, this is still leaves an infinite number of graphs to deal with. However, if we can show that increasing DTE aren't needed either, then we will have a finite number of operations to deal with since we are then left with just decreasing and non-trivial DTE.

Theorem 3.9 *Suppose G is a zig-zag chain and G' is obtained from G by applying an increasing DTE. Suppose H' is a minimum DTE descendants of G', if DTE operations are restricted to non-increasing DTE expansions. Then the minimum number of triangles of descendants of G are at most the number of triangles of H'.*

The proof for this theorem is quite detailed but the general idea is similar to the proof for theorem 3.8. The difference is that instead of being able to mimic a minimal sequence exactly, we need to be careful. The minimal sequence we will be starting with is being applied to a graph with an additional triangle. So, as we are defining our new sequence for the graph without the addition triangle, as long as we are working on a sequence of graphs that either has the same number of triangles or one more than our given sequence, there will not be a problem because there will be enough free choices.

Proof of 3.9 Suppose that there exists a non-increasing minimal DTE sequence from G' to H' through $G'_1, G'_2, ..., H'$. We will show there exists a non-increasing DTE sequence from G to some graph J (not necessarily minimal) where the number of triangles in J is the same as the number of triangles in H'. Denote this sequence as $G_1, G_2, ..., J$. The zig-zag chain of G is the same as the zig-zag chain of G' with the exception that one of the zig-zags has one addition triangle. We can see from definition 3.1 and figure 3.1, increasing DTE will not change the structure of a zig-zag chain. Now, for each G_i, perform a DTE that affects the zig-zags on G_{i-1} as the way that the DTE affected zig-zags on G'_{i-1}, to get G'_i. If the DTE is not being performed on the vertex with the additional triangle, then the zig-zag of G_{i-1} can be affected in the exact same way as the zig-zag of G'_i was affected. If the zig-zag has the additional triangle and has more than 3 triangles, then we are still able to apply the same DTE, but G'_i will have an additional triangle in one of the zig-zags near the DTE when compared to G_i. This is possible as long as there are enough triangles in the zig-zag that the DTE is being performed on. Since DTE affect at most 3 triangles, problems only arise when we are working with zig-zags of size less than 3. However, these problems don't prevent a new sequence from being created as long as one is careful with how the new sequence is being defined.

Suppose that G'_{i-1} has a zig-zag of size 3 where a DTE is being performed on and the corresponding zig-zag of G_{i-1} is only of size 2. Then, one will perform the DTE on Z_2 in G_{i-1} using an adjacent edge. The zig-zag in G'_{i-1} will become an isolated zig-zag of size 3, and since the sequence is minimal, this zig-zag will eventually become a zig-zag of size 2. We will admit this proof for now, but see figure 4.1, for a visual of why this is true. Similarly, the zig-zag in G_i will become a isolated zig-zag of size 2. If there is an edge that is not part of triangle incident to the DTE, then the zig-zag chains of G_i and G_{i-1} will be the same and the rest of the sequence can mimic the original sequence. Otherwise, the DTE performed on G'_{i-1} would have removed a triangle from G_{i-1}. So, G_i will have the same zig-zag chain as G'_i with the exception that one zig-zag will have one less triangle. But this is our initial relation between the two sequences, so this does not correct a problem.

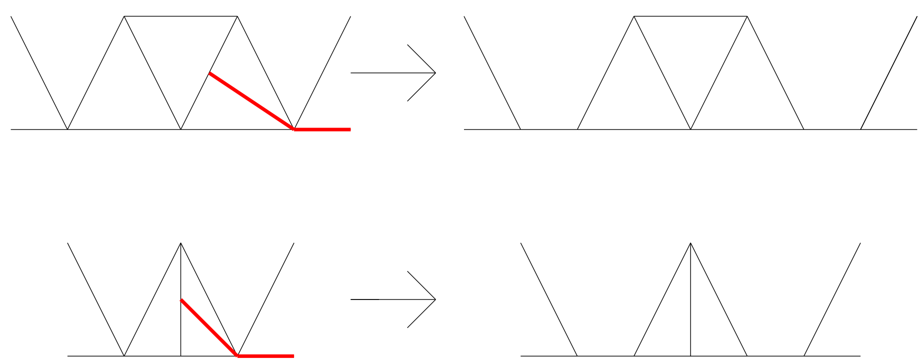

Figure 3.4: DTE On Z_3 and Z_2

Suppose that G'_{i-1} has a zig-zag of size 2 where a DTE is being performed on and the corresponding zig-zag is only of size 1. Then one will perform no DTE on Z_1 in G_{i-1} as the Z_2 in G'_{i-1}. The zig-zag in G'_{i-1} will remain a zig-zag of length 2 but be isolated from both sides. The isolation is a result of performing a DTE over a vertex in 2 triangles, as shown in figure 3.3. The zig-zag in G_{i-1} will remain a Z_1 zig-zag that may not be isolated. So, G_i and G'_i will have the same zig-zag structure with the exception of G_i having one zig-zag with an additional piece that is also potentially more connected to the other zig-zags. The extra structure in G_i isn't a problem, since at worse a DTE from G'_i may remove an additional triangle. Regardless, this is our initial relation between the two sequences, so this does not correct a problem.

The case for G'_{i-1} having a zig-zag of size 1 where the DTE is being performed and the corresponding zig-zag of G_{i-1} doesn't exist is not as straightforward. The issue in this case is one can't even attempt to perform a DTE triangle expansion on G_{i-1} because there is no triangle to perform it on. So, we will make sure to define our sequence so this case doesn't arise. That is, we will make sure in previous steps, to not create a non-isolated zig-zag of size 1 in G'_i that corresponds to a zig-zag of size 0 in G_i.

First off, notice that this case only arises when G'_{i-1} has a non-isolated zig-zag of size 1. Since these sequences have no increasing DTE, no isolated zig-zag of size 1 will have a DTE performed on it. The non-isolated zig-zag of size 1 case arises from a few different subcases:

Suppose a DTE on G'_{i-1} breaks a Z_n chain into a $Z_{n-4,3,1}$ chain, for n ≥ 7. To avoid the case mentioned above, the corresponding DTE in G_{i-1} will break the Z_{n-1} chain into a $Z_{n-5,3,1}$ chain as opposed to a $Z_{n-4,3,0}$ chain. Then we have encountered no problems with this step.

Suppose a DTE on G'_{i-1} breaks a Z_6 chain into a $Z_{1,3,1}$ chain. Then, there will be no corresponding DTE performed in G_{i-1} and G_{i-1} will be the exact same graph as G_i. Are there any problems that arise from this lack of a DTE? No, since any DTE performed on a $Z_{1,3,1}$ chain can be performed on a Z_5 chain. There is extra structure on the Z_5 chain, but this does not prevent any DTE from being performed on it. On the contrary, this could mean that there are more decreasing DTE that could be performed on G than G'. This shows why the increasing DTE could actually lead to an increase in the minimum number of triangles of a graph. Regardless, it is not at this point that we discuss that possibility, and this subcase doesn't correct any problems.

Suppose there exists a Z_2 chain in $S_{G'}$ where one triangle of it is destroyed by a DTE and the other triangle has a DTE performed on it to split it into 2 triangles. Further suppose that the corresponding zig-zag in S_G is of length 1. So, if we perform the DTE in the same order as in S_G, then we will remove the only triangle, then attempt to perform a DTE when no such triangle exists. However, since the in S_G affect different triangles, they commute. So, one will simply perform the DTE on the single triangle in S_G first, and then destroy one of the additional triangles after. This will result in the zig-zag chains in S_G and $S_{G'}$ being the same, with the exception that $S_{G'}$ has an isolated pair of triangles as opposed to an isolated triangle. From here, the sequence will follow naturally.

Suppose there exists a Z_2 chain in $S_{G'}$ where both triangles are destroyed. Further suppose that the corresponding zig-zag in S_G is of length 1. Then, the corresponding DTE of $S_{G'}$ will be performed that will first destroy the one triangle, and then have a neutral DTE. Once again, this results in the same zig-zag chains in both sequences.

Therefore, given any minimum sequence of G', we can find a corresponding sequence from G to H where H has the same number of triangles as H'. ■

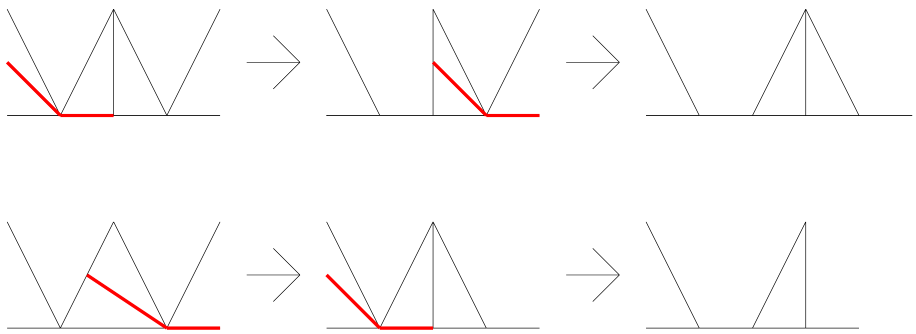

Figure 3.5: DTE on Z_2 and Z_1

Theorem 3.10 Suppose G is a zig-zag chain and $S_G = G_1, G_2, ...G^*$ is a minimal sequence of G. Then, there exists a decreasing, non-trivial neutral minimal sequence of G, S'_G.

Proof of 3.10 Suppose G_i is the last increasing or trivial neutral DTE in S_G. Then by theorem 3.9, in the case of increasing DTE, or theorem 3.8, in the case of trivial neutral DTE, there exists a sequence from G_{i-1} to a graph with the same number of triangles as G^* with no increasing or trivial neutral DTE. Using this logic repeatedly for all increasing or trivial neutral DTE in S_G we get a minimal sequence S'_G that is not increasing or trivially neutral. ∎

The focus shouldn't be what the new sequence looks like. The main takeaway from these theorems is that when looking at the minimum number of triangles that a graph has, one only needs to look at decreasing and non-trivial sequences. While there are an infinite amount of sequences that are increasing or trivially neutral, there is a finite number of sequences that are decreasing and non-trivially neutral. This is due to the fact that decreasing and non-trival neutral DTE destroy the structure of the zig-zag chains. They take a zig-zag chain and break the whole pieces into chains and break the chains into disjoint chains. So as one performs more DTE the zig-zag chain will eventually break down to zig-zags that are far apart that can't have anymore DTE of this type performed on them. This will allow techniques in the following chapters to be use-able.

3.1 Commutative Property

One of the problems that arises when dealing with zig-zag chains is that the DTE on a triangle in one zig-zag can affect triangles in a different zig-zag. This is always true for decreasing DTE. This can be a good thing though, as after this decreasing DTE, the zig-zags are disjoint. Upon first glance, it would be nice if when given a zig-zag chain, if the first thing we do is perform DTE that causes all the zig-zag pieces to be disjoint so we could just look at each zig-zag piece individually. Intuitively, an open zig-zag should be the simpliest type of zig-zag to deal with. This last statement, however, isn't true as open zig-zags breakdown into zig-zag chains. The idea of breaking up zig-zag chains is something useful though.

When looking at minimum sequences of zig-zags, some (likely the majority) of zig-zag pieces will end up being disjoint from the zig-zags that they share a vertex with initially. It would be ideal if one could do this splitting of the zig-zags at the beginning of the sequence. This brings up the question about whether DTE have the commutative property.

In general, the answer is no. DTE don't have the commutative property. Suppose we have a Z_n zig-zag. It is possible to perform a decreasing DTE and then a neutral DTE. However, it is not possible to perform a neutral DTE and then a decreasing DTE. This is true as the only DTE possible on a closed zig-zag are either a decreasing or increasing DTE due to construction. This is perhaps the easiest counter example.

Are there more specific cases where the DTE has the commutative property. Frankly, there are a lot of questions about the structure of K_5 descendants and this question is still unknown. However, one case where DTE has the commutative property is when two DTE are being applied to a different set of vertices and edges. That is, if two DTE effect completely different parts of a graph, then the order which the DTE are applied is arbitrary. More specifically, if a DTE is being applied to a specific part of a zig-zag, as long as we can guarantee that that part of the zig-zag will exist after some number of DTE, we can apply our initial DTE after the rest of the DTE. Similarly, if we know the zig-zag structure exists before applying some number of DTE, we can apply our initial DTE before some number of DTE. This is useful as it allows us to look at the DTE that split up zig-zag piece and cause them to be disjoint and do these DTE at the beginning.

Theorem 3.11 *Suppose $S_{G,G_1,\ldots G_n}$ is a DTE sequence. Denote the DTEs of this sequence as $s_1, s_2, \ldots s_n$. That is, G_1 is obtained by performing s_1 on G, G_2 is obtained by performing s_2 on G_1, etc. If the structure of the zig-zags that s_n affects in G_{n-1} exists in G, then we can create a new DTE sequence from G to G_n, defined as $S_{G,G'_1\ldots G_n}$ where the DTEs of this sequence is $s_n, s_1, \ldots s_{n-1}$.*

The proof of the theorem above was explained in the paragraph above. If the DTEs $s_1, s_2, \ldots s_n$, affect different zig-zag structures, then the order of which they are performed is trivial. ■

Chapter 4

Minimum Number of Triangles Of Common Zig-Zags

Our goal is, given a K_5 descendant, to find the minimum number of triangles of all descendants of that graph. As shown in the previous chapter, one doesn't need to pay close attention to edges that are not contained in any triangle; that is, one only needs to focus on the zig-zag structure. Since open zig-zags are the building blocks for all zig-zags, and also the simplest, finding the minimum number of triangles for these types of graphs is a good place to start. The following theorem, which is the main theorem of the thesis, states what the minimum number of triangles are for open zig-zags of any size.

Theorem 4.1 *The minimum number of triangles for a Z_n descendant is as follows:*

n	Minimum Number of triangles
$5m$	$2m$
$5m+1$	$2m+1$
$5m+2$	$2m+2$
$5m+3$	$2m+2$
$5m+4$	$2m+2$

The theorem above is stating that if the length of a zig-zag increases by 5, then the minimum number of triangles increases by 2. The base cases for $n = 1, ..., 5$ are shown in the table above. While the base cases are fairly straightforward, we do not yet have the machinery to prove the inductive step. The issue with the inductive step is that the open zig-zag pieces break down into zig-zag chains when having DTE performed on them.

So, we actually need to figure out how to deal with zig-zag chains before dealing with this inductive step.

As stated above, we need to understand zig-zag chains in order to better understand individual open zig-zags. However, zig-zag chains are made of open zig-zags. So, we have to tackle both of the problems at once. There are 3 main theorems of this chapter; each theorem dealing with a different commonly used zig-zag. The first one, is the theorem just introduced. This theorem needs an understanding of zig-zag chains before proving it. The other two theorems, theorem 4.7 and 4.8, don't need an understanding of zig-zag chains. All they need is theorem 4.1. So, we will be performing a triple induction. That is, we will show that theorem 4.1 holds up to some N. Then, we will show that theorems 4.7 and 4.8 hold true as long as theorem 4.1 is true. So, theorems 4.7 and 4.8 will also be true up to that same N. Then, finally, we will use theorems 4.7 and 4.8 to show that theorem 4.1 (and theorems 4.7 and 4.8 by extension), are true for all n.

Proof of 4.1 **Base Cases for Theorem 4.1**

The case for Z_1 is trivial as any DTE leaves us with at least 2 triangles, so the minimum number of triangles of a descendant of Z_1 is 1 by not performing any DTE on Z_1.
Using similar logic as above, Z_2 also has a minimum number of triangle among descendants as 2 by performing any number of non-increasing DTE.

The first non-trivial case, a zig-zag of length 3 can be reduced to a zig-zag of length 2. As mentioned earlier, the zig-zags talked about throughout the paper are considered to be K_5 descendants. So, they are 4-regular and thus have edges that aren't in the zig-zag that are incident to the end vertices of it. The 4-regularity ensures that one always has an adjacent edge to perform such a DTE on. Performing a DTE to remove such an edge results in a $Z_{2,1}$ chain. Then, the next DTE removes the single triangle giving us a Z_2 zig-zag. As stated above, the minimum number of triangles among descendants of this graph is 2. Furthermore, a zig-zag of length 3 can not be reduced to a single triangle as a DTE results in at minimum 2 triangles. So the minimum number of triangles among Z_3 descendants is 2.

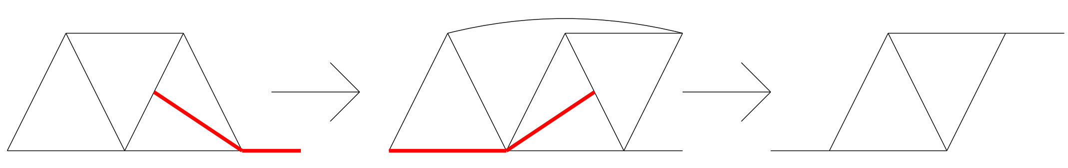

Figure 4.1: Minimal Sequence for Z_3

A zig-zag of length 4 can be reduced to a zig-zag of length 2. The first DTE removes one of the outer triangles resulting in a Z_3. Then, using the same process and logic as above, we have a minimum number of triangles as 2. As stated earlier, we can't go lower than 2, so the minimum number of triangles among Z_4 descendants is 2.

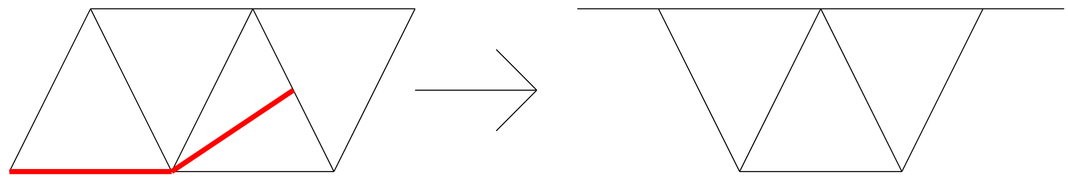

Figure 4.2: Minimal Sequence for Z_4

A zig-zag of length 5 can be reduced to a zig-zag of length 2. There are multiple choices for the first DTE but we only need to show that there is some sequence that results in a zig-zag of length 2 since we know that the minimum number of triangles can't be less than 2. The first DTE results in a $Z_{3,1}$ chain. The next DTE results in a $Z_{1,2}$ chain, which as explained in the explanation for Z_3, results in a Z_2 and thus a minimum of 2 triangles.

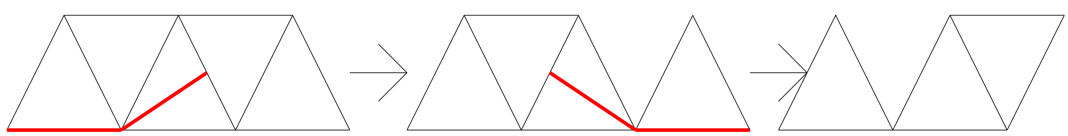

Figure 4.3: Minimal Sequence for Z_5

Inductive Step Suppose that theorem 4.1 holds for true up to some natural number $N \geq 5$.

The issue with showing this inductive step is true is that we don't have the machinery to do it quite yet. Suppose we have a zig-zag of length $N+1$. After performing a decreasing DTE, what zig-zag are we left with? Well, there are actually a lot of choices. We are left with a $Z_{q,3,N+1-q-4}$ zig-zag, with $q \in [N-3]$. Not only is that a lot of zig-zags to deal with, we don't even know how to deal with any individual one yet either. Suppose $q = 0$, then we are left with a $Z_{3,N-3}$ zig-zag. What is the minimum number of triangles of this zig-zag? This still isn't obvious. Despite starting with the most simple zig-zag, just a single zig-zag piece, one still ends up with long zig-zag chains. So, in order to prove the theorem above to know how simple zig-zags work to, we actually need to know how more complex zig-zag chains work.

The following theorems that we are going to show regarding zig-zag chains will apply to zig-zag chains where each chain is of length less than N. So, we will be able to use theorem 4.1 without having fully proved it. After developing more machinery, we will come back to prove theorem 4.1.

Suppose we have 2-piece zig-zag chain, $Z_{a,b}$ with $a, b \leq n$. What is the minimum number of triangles of this zig-zag chain. Since both pieces are of size less than n, we know the minimum number of triangles of the individual pieces. But they share a vertex, so how does this effect the minimum number of triangles? It means that a DTE can be performed on one of the pieces and remove a triangle from the other piece. Doing so, we reduce the length of one of the pieces in exchange for destroying a bit of the structure in the other zig-zag. After this DTE operation, the zig-zag pieces are now completely disjoint.

Definition 4.2 (Left DTE) We will define a left DTE as a DTE expansion that is performed on a $Z_{a,b}$ zig-zag chain that results in a $Z_{a-1;2,b-2}$ chain. That is, it is the DTE that removes a triangle from the left zig-zag piece at the expense of some of the structure on the right zig-zag piece. A sequence that contains a left DTE will be defined as a left DTE sequence.

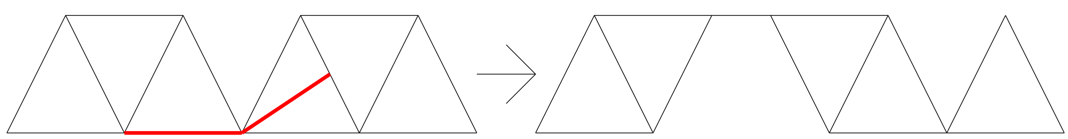

Figure 4.4: Left DTE

Definition 4.3 (Right DTE) Similarly, we will define a right DTE as a DTE expansion that is performed on a $Z_{a,b}$ zig-zag chain that results in a $Z_{a-2,2;b-1}$ chain. So it is the DTE expansion that removes a triangle from the right zig-zag piece at the expense at some of the structure of the left. A sequence that contains a right DTE will be defined as a right DTE sequence.

Definition 4.4 (Center DTE) It is also possible that a DTE sequence starting with $Z_{a,b}$ has no DTE expansions that affects triangles that started in both the zig-zag piece of size a and size b. This means that DTE expansions in each zig-zag piece destroyed triangles near the shared vertex, so no further DTE would effect both pieces. A DTE causes the

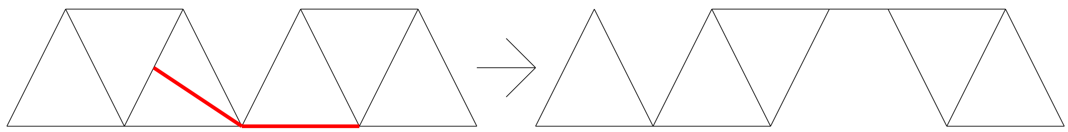

Figure 4.5: Right DTE

zig-zag pieces to be disjoint but only effects on zig-zag piece is defined as a center DTE. Sequences that contain a center DTE are called center DTE sequences.

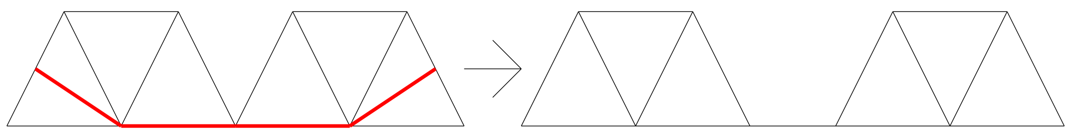

Figure 4.6: Center DTE

Definition 4.5 (Splitting DTE) Left DTE, right DTE and center DTE are defined to be the splitting DTE. That is, they are the DTE that split a zig-zag chain into disjoint zig-zag pieces.

Since each of the splitting DTE requires a zig-zag chain and results in an addition disjoint zig-zag piece, only one splitting DTE will be performed between any two zig-zag pieces. Furthermore, given any sequence with a splitting DTE, we can create a new sequence with the splitting DTE as the first DTE in the sequence. That is, we can immediately split the zig-zag pieces from each other so they are disjoint, then continue with the rest of the sequence. This is possible as the splitting DTE require triangles in both zig-zags and so we know that there is no earlier DTE that destroyed the triangles near the shared vertex.

It may not be clear how a minimal sequence is forced to have a splitting DTE. In fact, it is not forced to contain one. However, we can then create such a sequence that contains a splitting DTE. If a minimal sequence has no splitting DTE, then there is still at least a $Z_{1,1}$ at the vertex that is shared between both zig-zag pieces. But then, we can perform a right DTE (or left DTE), not change the number of triangles, and have a new minimal sequence that contains a splitting sequence. So, while we don't need to perform a splitting DTE to get to a minimal descendant, there is always some minimal sequence that contains a splitting DTE.

Theorem 4.6 $\triangle(Z_{a,b}) = \text{Min}((\triangle(Z_a) + \triangle(Z_b)), (\triangle(Z_{a-1}) + \triangle(Z_{2,b-2})), (\triangle(Z_{a-2,2}) + \triangle(Z_{b-1})))$

Proof of 4.6 The theorem above states that the minimum number of triangles in a two piece zig-zag chain is the minimum of the minimum number of triangles after performing a left, right and center DTE. For a minimal DTE sequence starting with $Z_{a,b}$, it is either a center, left or right DTE sequence. We can change the sequence so that the splitting DTE is first in the sequence, due to commutativity (see page 27). Then, we have 2 disjoint zig-zags and can calculate the minimum number of triangles separately. ■

So, our new question is which one of these 3 cases is the minimum number of triangles for a given $Z_{a,b}$ zig-zag chain. While it is easy to see, by the induction hypothesis, the minimum number of triangles for Z_{a-1} (or $b-1$), what about the $Z_{a-2,2}$ chain. So, we can see that zig-zag chains of the form $Z_{n-2,2}$ appear frequently throughout DTE sequences that may be minimal. So, a natural place to continue is studying the minimum number of triangles on zig-zag chains of this type.

Theorem 4.7 *The minimum number of triangles for a $Z_{n-2,2}$ descendant is as follows:*

n	Minimum Number of triangles
$5m$	$2m+1$
$5m+1$	$2m+2$
$5m+2$	$2m+2$
$5m+3$	$2m+2$
$5m+4$	$2m+3$

Not surprisingly, theorem 4.7 has a similar theme behind it as theorem 4.1. Increasing the length of the first zig-zag peice in $Z_{n-2,2}$ by 5, increases the minimum number of triangles by 2. The base cases are slightly different though. A $Z_{n-2,2}$ zig-zag chain doesn't necessarily have the same minimum number of triangles as a Z_n zig-zag. This is due to the zig-zag containing less structure in the $Z_{n-2,2}$ graphs. The additional vertex that is only in one triangle potentially results in an additional triangle in a minimal descendant.

Before we prove this, there needs to be special attention payed to the $\leq N$ condition. The N is the N that is used in theorem 4.1. So, for this proof, we are using theorem 4.1 and the fact that zig-zags of length Z_n have a known minimum number of triangles for $n \leq N$. Eventually, we will use this theorem to show that this holds for $N+1$ as well, but let's not get ahead of ourselves.

Proof of 4.7 **Base Cases for Theorem 4.7** For all of our base cases, we will apply theorem 4.6. That is, we will simply take the minimum of the minimal number of triangles after performing a left DTE, right DTE and center DTE.

Since $n = 0, 1$ gives us a zig-zag of negative length and $n = 2$ gives us a single zig-zag piece, we will start with $n = 3$.

$Z_{1,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{1;2}$	3
Right DTE	$Z_{2;1}$	3
Left DTE	$Z_{0;2}$	2

From the table above, we can see that the minimum number of triangles of a $Z_{1;2}$ is indeed 2.

Similarly, for $n = 4, 5, 6, 7$, we have the following 3 cases to show that the minimum is indeed what the theorem states.

$Z_{2,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2;2}$	4
Right DTE	$Z_{2;1}$	3
Left DTE	$Z_{1;2}$	3

$Z_{3,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{3;2}$	4
Right DTE	$Z_{1,2;1}$	3
Left DTE	$Z_{2;2}$	4

$Z_{4,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{4;2}$	4
Right DTE	$Z_{2;2;1}$	4
Left DTE	$Z_{3;2}$	4

$Z_{5,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5;2}$	4
Right DTE	$Z_{2,3;1}$	4
Left DTE	$Z_{4;2}$	4

One can see from the above tables, that the base cases hold. That is, each chart shows that one of the splitting DTE leads to the desired minimum number of triangles.

Inductive Hypothesis Assume that theorem 4.7 holds true for all $n < N$.

The inductive step is proved using the same process as we used for the base cases. For each value mod 5, we check the minimum number of triangles using the three splitting DTE and conclude that the minimum among those 3 is the minimum number of triangles.

$Z_{N+1} = Z_{5m+1,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m+1;2}$	$2m+3$
Right DTE	$Z_{5m-1,2;1}$	$2m+3$
Left DTE	$Z_{5m;2}$	$2m+2$

$Z_{N+1} = Z_{5m+2,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m+2;2}$	$2m+4$
Right DTE	$Z_{5m,2;1}$	$2m+3$
Left DTE	$Z_{5m+1;2}$	$2m+3$

$Z_{N+1} = Z_{5m+3,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m+3;2}$	$2m+4$
Right DTE	$Z_{5m+1,2;1}$	$2m+3$
Left DTE	$Z_{5m+2;2}$	$2m+4$

$Z_{N+1} = Z_{5m+4,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m+4;2}$	$2m+4$
Right DTE	$Z_{5m+2,2;1}$	$2m+4$
Left DTE	$Z_{5m-3;2}$	$2m+4$

$Z_{N+1} = Z_{5m+5,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m+5;2}$	$2m+4$
Right DTE	$Z_{5m+3,2;1}$	$2m+4$
Left DTE	$Z_{5m+4;2}$	$2m+4$

Each of the charts has at least one of the splitting DTE result in the desired minimum number of triangles and none of the other splitting DTE resulting in a lower number. So our inductive hypothesis is true.

Using theorem 4.1, 4.6 and 4.7, we now know the minimum number of triangles of a zig-zag of the form $Z_{a,b}$, for $a, b \leq N$. Theorem 4.6 gives us the minimum in the form of the sum of the minimum number of triangles of two zig-zags, while theoremS 4.1 and 4.7 ensures we actually know the minimum number of triangles of both of those graphs.

We have shown what the minimum number of triangles for a zig-zag chain of length 2, with each piece being less than N, is. The next question is what is the minimum number of triangles for zig-zag pieces of size greater than two. We saw from theorem 4.6, that the number of splitting DTE one needs to consider is 3. However, we hope to use our understanding of the structure to ignore completely brute force computations.

There is another issue that arises that can be answered in a similar manner to what was done previously. A $Z_{a,b,c}$ zig-zag piece has a left DTE performed on the shared vertex of a, b and a right DTE performed on the shared vertex of b, c as a possible minimal DTE. However, we are then left with a $Z_{a-1;2,b-4,2;c-1}$ zig-zag. The Z_{a-1} and Z_{c-1} have known minumum number of triangles. However, the $Z_{2,b-4,2}$ is a graph with an unknown minimum number of triangles. These types of graph occur frequently in the same way as just described. So, we should figure out the minimum number of triangles of these graphs before going any further.

Theorem 4.8 *The minimum number of triangles for a $Z_{2,n-4,2}$ descendant is as follows:*

n	Minimum Number of triangles
$5m + 0$	$2m + 2$
$5m + 1$	$2m + 2$
$5m + 2$	$2m + 3$
$5m + 3$	$2m + 3$
$5m + 4$	$2m + 4$

Following a similar pattern as the previous theorems, we have that increasing n by 5 results in an increased minimum number of triangles by 2. Once again the base cases are slightly different, as a result of the extra loss of structure from the additional vertex in 2 triangles as opposed to 3. Regardless, the proof for theorem 4.8 is very similar to that of 4.7.

Proof of 4.8 **Base Cases for Theorem 4.8** For all of the base cases, we once again will apply theorem 4.6. So we are finding the minimum number of triangles after performing a left DTE, right DTE and center DTE. The minimum of those numbers is the minimum number of triangles for the general zig-zag. The following chart shows the 5 base cases.

$Z_{2,1,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,1;2}$	4
Right DTE	$Z_{2,2;1}$	4
Left DTE	$Z_{2;2}$	4

$Z_{2,2,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,2;2}$	5
Right DTE	$Z_{2;2;1}$	5
Left DTE	$Z_{2,1;2}$	4

$Z_{2,3,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,3;2}$	5
Right DTE	$Z_{2,1,2;1}$	5
Left DTE	$Z_{2,2;2}$	5

$Z_{2,4,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,4;2}$	6
Right DTE	$Z_{2,2,2;1}$	5
Left DTE	$Z_{2,3;2}$	5

$Z_{2,5,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5;2}$	6
Right DTE	$Z_{2,3,2;1}$	6
Left DTE	$Z_{2,4;2}$	6

One can see that the base cases hold from the tables above.

Inductive Step Assume that theorem holds true for all $n < N$. Once again, we are still bounding the theorem above by our original N.

Going through the cases to show the inductive step, one gets the following tables:

$Z_{N+1} = Z_{2,5m+1,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5m+1;2}$	$2m+4$
Right DTE	$Z_{2,5m-1,2;1}$	$2m+4$
Left DTE	$Z_{2,5m;2}$	$2m+4$

$Z_{N+1} = Z_{2,5m+2,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5m+2;2}$	$2m+5$
Right DTE	$Z_{2,5m,2;1}$	$2m+4$
Left DTE	$Z_{2,5m+1;2}$	$2m+4$

$Z_{N+1} = Z_{2,5m+3,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5m+3;2}$	$2m+5$
Right DTE	$Z_{2,5m+1,2;1}$	$2m+5$
Left DTE	$Z_{2,5m+2;2}$	$2m+5$

$Z_{N+1} = Z_{2,5m+4,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5m+4;2}$	$2m+6$
Right DTE	$Z_{2,5m+2,2;1}$	$2m+5$
Left DTE	$Z_{2,5m+3;2}$	$2m+5$

$Z_{N+1} = Z_{2,5m+5,2}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2,5m+5;2}$	$2m+6$
Right DTE	$Z_{2,5m+3,2;1}$	$2m+6$
Left DTE	$Z_{2,5m+4;2}$	$2m+6$

So, we know now the minimum number of triangles of zig-zags of the form Z_n, $Z_{n,2}$ and $Z_{2,n,2}$, if $n \leq N$. This means that when looking at some arbitrary zig-zag chain, one is able to determine the minimum number of triangles of the disjoint zig-zags after applying a left DTE, a right DTE or a center DTE. The next question, is how can we find the minimum number of triangles of a zig-zag chain using this new understanding of the structure of these commonly occuring zig-zags.

Chapter 5

Long Zig-Zags

Before continuing on to finding out the minimum number of triangles of any zig-zag chain, we will take a moment to reflect on the differences between the different zig-zag and zig-zag chains where we know the minimum number of triangles. The charts below show the differences in the minimum number of triangles for each of the different zig-zag chain types. These are found through theorems 4.1, 4.7 and 4.8. We haven't fully proved the theorems as we are still inside the triple induction, so they are still bounded from above by the inductive hypothesis of theorem 4.1. So, $n \leq N$.

n	Z_n	$Z_{n,2}$	$Z_{2,n,2}$
$5m+1$	$2m+1$	$2m+2$	$2m+4$
$5m+2$	$2m+2$	$2m+3$	$2m+4$
$5m+3$	$2m+2$	$2m+3$	$2m+5$
$5m+4$	$2m+2$	$2m+4$	$2m+5$
$5m+5$	$2m+2$	$2m+4$	$2m+6$

n	Z_n	$Z_{n-2,2}$	$Z_{2,n-4,2}$
$5m+1$	$2m+1$	$2m+2$	$2m+2$
$5m+2$	$2m+2$	$2m+2$	$2m+3$
$5m+3$	$2m+2$	$2m+2$	$2m+3$
$5m+4$	$2m+2$	$2m+3$	$2m+4$
$5m+5$	$2m+2$	$2m+3$	$2m+4$

The first chart is restating the information directly from the tables that appeared earlier. The second chart, perhaps the more useful one, shows what happens as a zig-zag loses some of its structure. That is, we can see that splitting a complete zig-zag into a zig-zag chain with a Z_2 on the end sometimes results in an additional triangle in a minimal descendant. However, this is not always the case. In some cases, losing a bit of structure from the zig-zag doesn't change the number of triangles in a minimal descendant. One can also see that a complete zig-zag having a Z_2 split off at both ends either results in an increase in the number of triangles in a minimal descendant by 1 or 2.

For example, we can see that Z_5 has 2 triangles in minimal descendants. Splitting off a 2 zig-zag, we get a $Z_{3,2}$ zig-zag chain, with minimum triangles 3. Similarly, splitting off a 2 zig-zag from the other end, we get a $Z_{2,1,2}$ zig-zag chain, with minimum triangles 4. So, in both steps, we increase the minimum number of triangles by 1. These observations will play an important role when dealing with longer zig-zag chains.

Definition 5.1 (Splitting Sequence) The splitting sequence of a zig-zag chain, G, is a sequence of left, right and center DTE (not the specific DTE, but rather just the name of the types) that when applied to the pairs of zig-zag pieces of G, result in a graph whose minimal descendants have the same number of triangles as the minimal descendants of G. There may be multiple different splitting sequences.

Definition 5.2 (G_L, G_R, G_C) Suppose $G = Z_{a,b,\ldots,s}$ for some length s and where each $a, b, ..., s \leq N$. Denote $G_L = Z_{a-1;2,b-2,\ldots s}$, $G_R = Z_{a-2,2;b-1,\ldots s}$ and $G_C = Z_{a;b,\ldots s}$. That is, G_L is the G after performing a left DTE on the first zig-zag pairs. Similarly, G_R and G_C are G after performing a right DTE and a center DTE on the first zig-zag pairs, respectively.

Theorem 5.3 *Suppose $G = Z_{a,b,\ldots,s}$ for some length s and where each $a, b, ..., s \leq N$. If there is a strict minimum amongst the splitting DTE applied to $Z_{a,b}$, then that same splitting DTE minimizes G_R, G_L, G_C.*

This theorem is stating that if one of the splitting DTE of the first pair of zig-zags results in a strict minimum compared to the other two, then there is a minimal DTE sequence for the entire chain that contains the splitting DTE. So, one can look at the first pair of zig-zags, determine if there is a strict minimum among the splitting DTE if so, can conclude what some of a splitting sequence looks like.

The reason this works for the first pair of zig-zags and not further ones is because the first zig-zag piece is not going to be changed by any other splitting DTE other than the first one. If we try applying this to a pair of zig-zags not at the beginning or end of the chain, then the splitting DTE applied to the other side of either zig-zag may cause problems. In other words, since the first zig-zag piece only has one splitting DTE applied to it, we can get it out of the way first with no worries that we will have to effect it later with a further splitting DTE.

Proof of 5.3 Suppose towards a contradiction that there exists some zig-zag chain G, such that there is a strict minimum amongst the splitting DTE for the first two zig-zag pieces, say the left DTE, for example, but $\triangle(G_L) > \triangle(G_C)$ or $\triangle(G_L) > \triangle(G_R)$. Without loss of generality, suppose that $\triangle(G_L) > \triangle(G_R)$. Suppose S_{G_R} is a minimal sequence for G_R. We can say without loss of generality here, since this proof doesn't depend on the structure that is left after the first splitting DTE on the first and second DTE.

Let S'_{G_R} be a minimal sequence for G_R with the splitting DTE performed first, in the order in which the pieces they effect appear in the zig-zag. That is, the splitting DTE that effects the first two zig-zags is first in the sequence, followed by the splitting DTE that effects the second and third zig-zag pieces, and so on. Then, we will show that there exists a splitting sequence from G_L to a graph with minimal triangles equal to the number of triangles in S'_{G_R}. Suppose $S'^{*}_{G_R} = R, s_2, s_3, ...s_z$ is the splitting sequence of S'_G. Then, we claim that the minimum number of triangles of G with the splitting sequence $S^*_{G_L} = L, s_2, s_3, ...s_z$ is at most the same as the number of triangles of G after applying S_{G_R}.

It is possible that some s_i is not a splitting DTE, but since the sequence is minimal, the zig-zag chain around the vertex that didn't have a splitting DTE applied to it is a $Z_{1,1}$ zig-zag. This is true, since any larger zig-zag chain would not be minimal. Then, we can apply a right DTE (or left DTE) and have a new minimal sequence with a splitting DTE at the previous graph where there was none. So, we will assume that our initial sequence already has all s_i as splitting DTE, since if any s_i wasn't splitting, we can replace it with a right DTE.

First, calculate the number of triangles after applying S'_{G_R} to G. After applying the splitting sequence $S^*_{G_R}$ to G, we get z disjoint zig-zags. Denote the sum of all zig-zag pieces excluding the first two as n. Denote the minimum number of triangles in the first two zig-zag pieces as a after applying R but before applying s_2 as a. Going through the 3 cases of what s_2 could be, we have:

1) If s_2 is a left DTE, then the minimum number of triangles between the first two zig-zag pieces is least $a-1$. So, the whole zig-zag has $n+a-1$ minimum triangles at least.

2) If s_2 is a center DTE, then the minimum number of triangles between the first two zig-zag pieces is at least a. So, the whole zig-zag has $n+a$ minimum triangles at least.

3) If s_2 is a right DTE, then the minimum number of triangles between the first two zig-zag pieces is at least a. So, the whole zig-zag has $n+a$ minimum triangles at least.

Next, calculating the number of triangles after applying $S^*_{G_L}$ to G. The sum of all the zig-zags excluding the first two is equal to n since the same splitting were applied as in

$S'^{*}_{G_R}$. So, all that's left is to figure out the minimum number of triangles of the first two zig-zag pieces. Since the minimum number of triangles of the first two zig-zag pieces after applying a left DTE is less than the minimum number of triangles after applying a right DTE to the first two zig-zag pieces, we have that the first two zig-zag pieces have at most $a-1$ triangles in a minimum descendant before applying s_2. Going through the three cases of what splitting DTE s_2 is, we have:

1) If s_2 is a left DTE, then the minimum number of triangles between the first zig-zag pieces is at most $a-1$. Recall that $a-1$ is the minimum number of triangles in the first two zig-zag pieces. After applying a left DTE, we either decrease the number of triangles, or it stays the same. Taking the worst case, we end up with the number of triangles staying the same, so we have $a-1$ triangles. So, the whole zig-zag has $n+a-1$ triangles in a minimum descendant at most.

2) If s_2 is a center DTE, then the minimum number of triangles between the first two zig-zag pieces is at most $a-1$ (since a center DTE will at worst not increase the minimum number of triangles). So, the whole zig-zag has at most $n+a-1$ triangles in a minimum descendant.

3) If s_2 is a right DTE, then the minimum number of triangles between the first two zig-zag pieces is at most a (since a right DTE will at worst increase the minimum number of triangles by 1). So, the whole zig-zag is has at most $n+a$ triangles in a minimum descendant.

In all three cases, we have the minimum number of triangles of G_R at least equal to the minimum number of triangles of G_L. So our theorem holds true. ■

So, with this theorem, we know that if there is a strict minimum between the first two zig-zag pieces using a specific splitting DTE, then that splitting DTE will result in a minimum number of triangles for the entire zig-zag chain. The reason this works for the first two zig-zag pieces is because the first zig-zag piece is only being acted upon by one splitting DTE. So, there is no further splitting DTE that could result in more triangles than we want. This is not true for any of the other zig-zag pieces (other than the last one in the chain) since there is a splitting DTE applied from both the zig-zag pieces to the left and right. However, if we start at the beginning, find a strict minimum splitting DTE between the first two pairs, then we can reapply the theorem again for the second and third zig-zags. That is, we will then have the first zig-zag piece disjoint from the rest of the chain and the second zig-zag piece will only have one more splitting DTE applied to it. So, once again, we can apply the theorem. In fact, we can keep applying the theorem as long as we have a strict minimum.

Keep in mind, what is important for finding the minimum number of triangle isn't the order of applying the splitting DTEs; it's just the types of splitting DTE themselves. The order is only important as that is what allows us to apply the theorem several times. If we were to know the splitting DTE that lead to a minimum number of triangles, we could apply them in any order. The splitting DTE effect different triangles, so they always commute.

Although this theorem is very helpful for finding the minimum of some zig-zag chains, what happens when there isn't a strict minimum amongst the splitting DTE. Suppose that both a left and right DTE on the first two zig-zag pieces of a zig-zag chain result in the minimum number of triangles of the first two zig-zag pieces. Which choice should we make for the first splitting DTE? Perhaps intuitively, one would expect that the splitting DTE that results in the smaller amount of triangles in the second zig-zag would be the better choice. After all, the second zig-zag piece is the piece that is going to have another splitting DTE applied to it. However, this is not the case. The splitting DTE that we would chose is the one that puts the second zig-zag piece in the best position for the next splitting DTE.

What do we mean by the best position. Well, we have 3 types of splitting DTE that are all being applied to the right end of the second zig-zag piece. What would we want for each splitting DTE? If the next splitting DTE is a left DTE, a triangle is getting destroyed from this second zig-zag piece. Ideally, if we are destroying a triangle, we would like the minimum number of triangles to decrease. If the next splitting DTE is a right DTE, the structure is getting broken up on the end of the zig-zag. So, we would like the minimum number of triangles to not increase. If the next splitting DTE is a center DTE, then the decision doesn't matter. More specifically, we have the following definition.

Definition 5.4 (Best Position, Better Position, Opposite Position) A zig-zag piece is in the best position if it would not gain a triangle in its minimal descendants from having Z_2 chipped off at an end and if it would lose a triangle in its minimal descendants from losing a triangle. A zig-zag piece is defined to be in a better position than another if it has more of these two properties than the other piece. Two zig-zag pieces are defined to be in opposite positions if each zig-zag meets exactly one of the previous conditions, but opposite ones of each other. .

Theorem 5.5 *Suppose $G = Z_{a,b,...,s}$ for some length s and where each $a, b, ..., s \leq N$. Further suppose that $G' = Z_{a,b}$. If there is are multiple minimums amongst $\triangle(G'_L), \triangle(G'_R), \triangle(G'_C)$, say $\triangle(G'_L)$ and $\triangle(G'_R)$ but b after applying a left DTE to its left end triangle is in a better position after b applying a right DTE to its left end triangle, then $\triangle(G_L) \leq \triangle(G_C), \triangle(G_R)$.*

Proof 5.5 Suppose towards a contradiction that there exists some zig-zag chain G, such that there is are multiple minimums amongst $\triangle(G'_L), \triangle(G'_R), \triangle(G'_C)$, say $\triangle(G'_L)$ and

$\triangle(G'_R)$ but b after applying a left DTE to its left end triangle is in a better position than b after applying a right DTE to its left end triangle and $\triangle(G_L) > \triangle(G_R)$. Suppose S_{G_R} is a minimum sequence for G_R. Let S'_{G_R} be a minimal sequence for G_R with the splitting DTE performed first, in the order in which the pieces they effect appear in the zig-zag. That is, the splitting DTE that effects the first two zig-zags is first in the sequence, followed by the splitting DTE that effects the second and third zig-zag pieces, and so on. Then we will show that there exists a splitting sequence from G_L to a graph with minimal triangles equal to the number of triangles in S'_{G_R}. Suppose $S'^{*}_{G_R} = R, s_2, s_3, ...s_z$ is the splitting sequence of S'_{G_R}. Then, we claim that the minimum number of triangles of G with the sequence $S'^{*}_{G_L} = L, s_2, s_3...s_z$ has at most the same minimum number of triangles as G with $S'^{*}_{G_R}$ applied to it as a splitting sequence.

First, calculating the number of triangles after applying S'_{G_R} to G. After applying the splitting sequence $S'^{*}_{G_R}$ to G, we get z disjoint zig-zags. Denote the sum of all zig-zag pieces excluding the first two as n. Denote the minimum number of triangles in the first two pieces as a after applying R but before applying s_2 as a. Similar for similar reason as the previous proof, we can assume that each s_i is a splitting DTE since if some aren't, we can make a new sequence where they are splitting DTE. Going through the 3 cases of what s_2 could be, we have:

1) If s_2 is a left DTE, then the minimum number of triangles between the first two zig-zag pieces is at least $a - 1 + c_1$, where $c_1 = 0$ or 1. We have c_1 here, as whatever value c_1 is here, will be used later. So, the whole zig-zag has $n + a - 1 + c_1$ triangles.

2) If s_2 is a center DTE, then the minimum number of triangles between the first two zig-zag pieces is at least a. So, the whole zig-zag has $n + a$ minimum triangles at least.

3) If s_2 is a right DTE, then the minimum number of triangles between the first two zig-zag pieces is at least $a + c_2$, where $c_2 = 0$ or 1. So, the whole zig-zag has $n + a + c_2$ triangles.

Next, calculating the number of triangles after applying $S'^{*}_{G_L}$ to G. The sum of all zig-zags excluding the first two is equal to n since the same splitting DTE were applied as in $S'^{*}_{G_R}$. The minimum number of triangles of the first two zig-zag pieces after applying a left DTE but before applying s_2 is a by hypothesis. Going through the three cases of what splitting DTE s_2 is, we have:

1) If s_2 is a left DTE, then the minimum number of triangles between the first two zig-zag pieces is at most $a - 1 + c_1$. Since b (this zig-zag where these DTE are being applied to) is in a better position after having a left DTE applied to it as opposed to a right DTE, we know that if b_R loses a triangle from a right DTE, then it also loses one from a right DTE. So, the whole zig-zag has at most $n + a - 1 + c_1$ triangles of a minimum descendant.

2) If s_2 is a center DTE, then the minimum number of triangles between the first zig-zag first two zig-zag pieces is at most a. So, the whole zig-zag has at most $n + a$ triangles in a minimum descendant.

3) If s_2 is a right DTE, then the minimum number of triangles between the first two zig-zag pieces is at most $a + c_2$. Once again, this b_L only gains a triangle from this DTE if b_R did as well. So, the whole zig-zag has at most $n + a + c_2$ triangles in a minimum descendant.

In all three cases, we have the minimum number of triangles of G_R is at least equal to the minimum number of triangles of G_L. This is a contradiction, so our theorem holds true. ■

If we have a splitting DTE on the first two pieces of a zig-zag chain that results in the strict minimum number of triangles with respect to the other splitting DTE, then theorem 5.4 tells us that this results in the minimum number of triangles for the entire chain. If we don't have a strict minimum amongst the splitting DTE, but have a splitting DTE that results in the best position of the second zig-zag piece amongst those splitting DTE, then theorem 5.5 tells us this results in the minimum number of triangles for the entire chain. So, the problem arises when there is no strict minimum amongst splitting DTEs, and the splitting DTEs that are minimal leave the second zig-zag piece in opposite positions of each other. Going through each of the values of mod 5, we will check if this problem arises.

The chart below shows each of the potential changes that each of the graphs could go through. That is, for each pair in the chart, the first value represents whether decreasing n by 1 results in the minimum number of triangles of the graph decreasing by 1 (represented by -) or whether it stays the same (represented by 0). The second value of each pair represents whether breaking off a 2 zig-zag results in an increase in the minimum number of triangles by 1 (represented by +) or whether it stays the same (represented by 0). The chart is created by using theorems 4.1, 4.7 and 4.8.

n	Z_n CDTE +/-	Z_{n-1} RDTE +/-	$Z_{n-2,2}$ LDTE +/-
$2m$	0/+	0/0	0/+
$2m+1$	-/0	0/+	-/0
$2m+2$	-/0	-/0	0/+
$2m+3$	0/0	-/0	0/+
$2m+4$	0/0	0/0	-/+

For example, consider that we have a $Z_{a,2m,\ldots n}$ zig-zag. The value of a is purposely ambiguous. Suppose that all splitting DTE applied to $Z_{a,2m}$ result in the same number of minimal triangles. Then, we want to set up the second zig-zag piece in the best position. Looking at the table, we see that both Z_{2m} and $Z_{2m-2,2}$ gain a triangle if we perform a right DTE on the second and third zig-zag pieces. However, Z_{2m-1} goes not gain a triangle. All 3 zig-zags, don't lose triangles if a left DTE is performed on them. So, we conclude that Z_{2m-1} is the best position, and so the splitting DTE that will be applied to $Z_{a,2m}$ will be a right DTE so that Z_{2m} becomes Z_{2m-1}.

For $n = 0 \bmod 5$, we can see that Z_{n-1} is the best position to leave the graph in, and there is no difference between Z_n and $Z_{n-2,2}$.

For $n = 1 \bmod 5$, we can see that Z_n and $Z_{n-2,2}$ are the best positions to the leave the graph in, and there is no difference between them.

For $n = 2 \bmod 5$, we can see that Z_n and Z_{n-1} are the best positions to leave the graph in, and there is no difference between them.

For $n = 3 \bmod 5$, we can see that Z_{n-1} is the best position, followed by Z_n.

For $n = 4 \bmod 5$, we have problems arise. Z_n and Z_{n-1} are the best choices if we want to split off a 2 zig-zag since we won't gain any triangles in minimum descendants, and that $Z_{n-2,2}$ is the best choice if we want to remove a triangle since we will lose a triangle in a minimum descendant. If both choices are equally appealing, we will use a similar strategy as before and try to set up the next zig-zag in the best position as possible. So now, looking at the value of the third zig-zag piece mod 5, denoted as Z_q, we get:

For $q = 0 \bmod 5$, we have the zig-zag both not losing a triangle in a minimal descendant upon having a triangle removed and gaining a triangle upon having a 2 zig-zag split off. So, we just want to set this zig-zag up for the next DTE. The best case is turning this zig-zag into its Z_{q-1} form, so we want to remove a triangle from this. So we want our initial zig-zag of length $n = 4 \bmod 5$ to have its structure destroyed. So, we chose to use Z_n or Z_{n-1} for our second zig-zag piece.

For $q = 1 \bmod 5$, we have the zig-zag will both lose a triangle in a minimal descendant upon having a triangle removed and not gain a triangle upon having a 2 zig-zag split off. Setting the zig-zag up for the next DTE, we see the best case is either Z_q or $Z_{q-2,2}$. So our $n = 4 \bmod 5$ zig-zag will lose a triangle to minimize.

For $q = 2 \bmod 5$, the same case as above but this time we want a Z_q or Z_{q-1} zig-zag. So, we will have our $n = 4 \bmod 5$ zig-zag have a 2 zig-zag break off so this zig-zag can lose a triangle.

For $q = 3$ and $q = 4 \bmod 5$, the cases are the same. The zig-zags will not lose a triangle in a minimal descendant if they lose a triangle nor will they lose a triangle upon a 2 zig-zag splitting away. So, our $n = 4 \bmod 5$ zig-zag will lose a zig-zag, and each of these zig-zags will have a 2 split away.

We will not go through the proofs of the cases of $q \bmod 5$, as it is the same proof as theorem 5.3, except instead of minimizing the minimum triangles of a zig-zag chain with 2 pieces, we are doing it over a zig-zag chain with 3 pieces.

Theorem 5.6 *Suppose we have a zig-zag chain. Then the splitting DTE that causes the first zig-zag piece to be disjoint and that results in the minimum number of triangles between the disjoint zig-zag and the remaining chain is determined as follows:*

1) If one of the splitting DTE results in a strict minimum number of triangles between the first two zig-zags, then this splitting DTE results in a minimum number of triangles for the entire chain.

2) If there are multiple splitting DTE that result in a minimum number of triangles between the first two zig-zags and there is a splitting DTE that results in a best position for the second zig-zag, then this splitting DTE results in a minimum number of triangles for the entire chain.

3) If there are multiple splitting DTE that result in a minimum number of triangles between the first two zig-zags and there are multiple splitting DTE among these splitting DTE with the same position for the second zig-zag, choose the position that results in the minimum number of triangles among the second and third zig-zags.

4) If there are multiple splitting DTE that result in a minimum number of triangles between the first two zig-zags, multiple splitting DTE among these splitting DTE with the same position for the second zig-zag and multiple splitting DTE the minimize the number of triangles between the second and third zig-zag pieces, choose the DTE that results in the best position for the third zig-zag.

Proof 5.6 1) is true by theorem 5.4 2) is true by theorem 5.5. 3) and 4) are true from the comments after theorem 5.5

So, given any zig-zag chain, we can use theorem 5.6 to find the splitting DTE that causes the first zig-zag piece to be disjoint and that results in the minimum number of triangles between the first zig-zag piece and the rest of the chain. We can keep applying this until all the initial zig-zag pieces are disjoint. Then, since all these zig-zags are in a form that has been studied in chapter 4, we can calculate the minimum number of triangles of each piece individually, and conclude that their sum is the minimum number of triangles of the whole chain. ■

We almost have enough machinery to prove the main theorem of the paper; the minimum number of triangles of an open zig-zag.

Theorem 5.7 $\triangle Z_{q,3,n-q-4} = \triangle Z_{q+5k,3,n-q-4-5k}$ for any k that allows all zig-zag pieces to be of non-negative size.

Proof of 5.7 Given any zig-zag chains, $G = Z_{q,3,n-q-4}$ and $G' = Z_{q+5k,3,n-q-4-5k}$, for some k that allows all zig-zag pieces to be of non-negative size, we know that since all the zig-zags are of sizes congruent mod 5, theorem 5.6 results in 3 disjoint zig-zags for each graph where the first and last zig-zags differ by size of $5k$. From theorems, 4.1, 4.6 and 4.7, we know that the minimum number of triangles for the first and last zig-zags differ by $2k$ (one being a $+2k$ and the other being $-2k$). So, the minimum number of triangles for both graphs are equal since the $2k$s cancel each other out.

As we are now ready to complete the proof of the induction hypothesis of theorem 4.1, let us recall that theorem:

Theorem 4.1 *The minimum number of triangles for a Z_n descendant is as follows:*

n	Minimum Number of triangles
$5m+0$	$2m$
$5m+1$	$2m+1$
$5m+2$	$2m+2$
$5m+3$	$2m+2$
$5m+4$	$2m+2$

The first DTE performed on Z_n results in a $Z_{q,3,n-q-4}$ for some $q \in [n-4]$. By theorem 5.7, we know that the minimum number of triangles depends only on the value of q mod 5 since increasing the length of the first zig-zag piece and decreasing the length of the last zig-zag piece by 5 results in no change in the minimum number of triangles (or vice-versa). So, the only cases that depend on q that we need to go through are each of the values of q where $q \in [5]$. However, we also need to go through each value of n mod 5. For each value of n, we need to go through the 5 values of q.

For each value of n mod 5, we have 5 cases to go through. That is, we need to go through the three splitting DTEs on the zig-zags $Z_{1,3,n-5}$, $Z_{2,3,n-6}$, $Z_{3,3,n-7}$, $Z_{4,3,n-8}$ and $Z_{5,3,n-9}$. The first splitting DTE doesn't depend on the value of n. So, before diving into the cases for each n, we will decide which splitting DTE is optimal for each value of q. By theorems 5.6, we need to find the DTE that results in the minimum number of triangles amongst the first zig-zag pair. If there isn't a strict minimum, we chose the DTE that also results in the second zig-zag piece being left in the best position for future DTE. Going through the cases we get:

5.1 First Splitting DTE for $Z_{q,3,n-q-4}$ Zig-Zags

$Z_{1,3}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{1;3}$	3
Right DTE	$Z_{2;2}$	4
Left DTE	$Z_{0;2,1}$	2

The zero in the zig-zag after the left DTE represents that a triangle was there but was destroyed. $Z_{0;2,1}$ is the same zig-zag as $Z_{2,1}$. Regardless, we conclude that a left DTE is used if $q = 1$ since the left DTE is a strict minimum amongst the splitting DTE.

$Z_{2,3}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{2;3}$	4
Right DTE	$Z_{2;2}$	4
Left DTE	$Z_{1;2,1}$	3

So, we conclude that a left DTE is used if $q = 2$ since the left DTE is a strict minimum amongst the splitting DTE.

$Z_{3,3}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{3;3}$	4
Right DTE	$Z_{1,2;2}$	4
Left DTE	$Z_{2;2,1}$	4

So, we conclude that a right or center DTE is used if $q = 3$ since they both leave the zig-zag in the best position for future DTE. Since both DTE are equivalent choices, we will use right DTE going forward for future charts. The charts later in this chapter could have been made with center DTE instead and have the same results.

$Z_{4,3}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{4;3}$	4
Right DTE	$Z_{2,2;2}$	5
Left DTE	$Z_{3;2,1}$	4

So, we conclude that a center DTE is used if $q = 4$ since both left and center DTE result in a minimum number of triangles, but a center zig-zag results in a better position for future DTE when compared to a left DTE.

$Z_{5,3}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5;3}$	4
Right DTE	$Z_{3,2;2}$	5
Left DTE	$Z_{4;2,1}$	4

So, we conclude that DTE is used if $q = 5$ for similar reasoning as for $q = 4$.

For each value of q mod 5, we know know which splitting DTE expansion needs to be done in order to find a minimal sequence. So there will be slightly different structures depending on the value of q. Keep in mind that the first splitting DTE only depends on the value of q, not n. So for each n, the first splitting DTE will be the one that is shown by the charts above. The charts below are a bit different from the ones in previous chapters as they are zig-zag chains of length 3 as opposed to 2. The charts already have the splitting DTE applied to them so the first zig-zag piece will already be disjoint. So, the third column of the chart will show the minimum number of triangles between the last two zig-zag pieces only. The fourth column will then show the minimum number of triangles for the entire zig-zag. So it will be the minimum number of triangles of the first piece (which will be $0, 1$ or 2 added to the minimum number of triangles on the last two zig-zag pieces). Going through the different values of n and q we get the following charts:

5.2 $n = 0$ mod 5

$N = 5m$

For $N = 5M$, we have 5 cases to go through. That is, we have to check each of the splitting DTEs for the zig-zags $Z_{1,3,m-5}$, $Z_{2,3,m-6}$, $Z_{3,3,m-7}$, $Z_{4,3,m-8}$ and $Z_{5,3,m-9}$. Since these are zig-zag chains of length 3, we require two splitting DTE to fully split the zig-zags up. We already showed what splitting DTE is optimal for the first pair of zig-zags in section 5.1. So, the zig-zags in the charts below already have the optimal splitting DTE applied to the first pair of zig-zag pieces; the charts are going through figuring out the optimal DTE applied to the second pair of zig-zag pieces.

$Z_{0;2,1,5m-5}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-5}$	$2m$	$2m$
Right DTE	$Z_{2,2;5m-6}$	$2m+1$	$2m+1$
Left DTE	$Z_{2;2,5m-7}$	$2m+1$	$2m+1$

Using the chart above as an example to the reader, we start with a $Z_{1,3,5n-5}$ zig-zag. From the charts in section 5.1, we can see that a left DTE minimizes a $Z_{1,3}$. Performing that splitting DTE on the $Z_{1,3}$ subgraph of $Z_{1,3,5n-5}$, we get a $Z_{0;2,1,5m-5}$ zig-zag, as stated in the chart. From this point, we are applying each of the possible splitting DTE on the zig-zag $Z_{2,1,5m-5}$, over the vertex shared by the zig-zag pieces of size 1 and $5m-5$. The third column above shows which splitting DTE minimizes this zig-zag, and the fourth column then shows the minimum number of triangles of the entire zig-zag chain. In this chart, the minimum number of triangles of the entire zig-zag chain is the same as the minimum number of triangles of the last three zig-zag pieces since the first zig-zag piece in the chain ends up being of size 0. In the next chart, the first zig-zag piece ends up being of size 1, so we add 1 to our third column. The other charts result in us adding 2. All charts remaining charts in chapter 5 follow this approach.

$Z_{1;2,1,5m-6}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-6}$	$2m$	$2m+1$
Right DTE	$Z_{2,2;5m-7}$	$2m+1$	$2m+2$
Left DTE	$Z_{2;2,5m-8}$	$2m$	$2m+1$

$Z_{1,2;2,5m-7}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2;5m-7}$	$2m$	$2m+2$
Right DTE	$Z_{2;5m-8}$	$2m$	$2m+2$
Left DTE	$Z_{1;2,5m-9}$	$2m-1$	$2m+1$

$Z_{3;3,5m-8}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-8}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-9}$	$2m-1$	$2m+1$
Left DTE	$Z_{2;2,5m-10}$	$2m$	$2m+2$

$Z_{5;3,5m-9}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-9}$	$2m-1$	$2m+1$
Right DTE	$Z_{1,2;5m-10}$	$2m-2$	$2m$
Left DTE	$Z_{2;2,5m-11}$	$2m$	$2m+2$

So, we can see that there are two ways choices in q and splitting DTE that result in a minimum of $2m$ triangles. There is no way to get lower than $2m$, so our theorem holds.

5.3 $n = 1$ mod 5

$N = 5m + 1$

$Z_{0;2,1,5m-4}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-4}$	$2m+1$	$2m+1$
Right DTE	$Z_{2,2;5m-5}$	$2m+1$	$2m+1$
Left DTE	$Z_{2;2,5m-6}$	$2m+2$	$2m+2$

$Z_{1;2,1,5m-5}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-5}$	$2m$	$2m+1$
Right DTE	$Z_{2,2;5m-6}$	$2m+1$	$2m+2$
Left DTE	$Z_{2;2,5m-7}$	$2m+1$	$2m+2$

$Z_{1,2;2,5m-6}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2;5m-6}$	$2m$	$2m+2$
Right DTE	$Z_{2;5m-7}$	$2m$	$2m+2$
Left DTE	$Z_{1;2,5m-8}$	$2m$	$2m+2$

$Z_{4;3,5m-7}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-7}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-8}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-9}$	$2m$	$2m+2$

$Z_{5;3,5m-8}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-8}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-9}$	$2m-1$	$2m+1$
Left DTE	$Z_{1;2,5m-10}$	$2m-1$	$2m+1$

There are several ways of getting $2m+1$ minimum triangles, but no way of getting less. So our theorem holds.

5.4 $n = 2$ mod 5

$N = 5m + 2$

$Z_{0;2,1,5m-3}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-3}$	$2m+2$	$2m+2$
Right DTE	$Z_{2,2;5m-4}$	$2m+2$	$2m+2$
Left DTE	$Z_{2;2,5m-5}$	$2m+2$	$2m+2$

$Z_{1;2,1,5m-4}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-4}$	$2m+1$	$2m+2$
Right DTE	$Z_{2,2;5m-5}$	$2m+1$	$2m+2$
Left DTE	$Z_{2;2,5m-6}$	$2m+2$	$2m+3$

$Z_{1,2;2,5m-5}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2;5m-5}$	$2m$	$2m+2$
Right DTE	$Z_{2;5m-6}$	$2m$	$2m+2$
Left DTE	$Z_{1;2,5m-7}$	$2m$	$2m+2$

$Z_{4;3,5m-6}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-6}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-7}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-8}$	$2m+1$	$2m+3$

$Z_{5;3,5m-7}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-7}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-8}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-9}$	$2m$	$2m+2$

There are several ways of getting $2m+2$ minimum triangles, but no way of getting less. So our theorem holds.

5.5 $n = 3$ mod 5

$N = 5m + 3$

$Z_{0;2,1,5m-2}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-2}$	$2m+2$	$2m+2$
Right DTE	$Z_{2,2;5m-3}$	$2m+3$	$2m+3$
Left DTE	$Z_{2;2,5m-4}$	$2m+2$	$2m+2$

$Z_{1;2,1,5m-3}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-3}$	$2m+2$	$2m+3$
Right DTE	$Z_{2,2;5m-4}$	$2m+3$	$2m+4$
Left DTE	$Z_{2;2,5m-5}$	$2m+2$	$2m+3$

$Z_{1,2;2,5m-4}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2;5m-4}$	$2m+1$	$2m+3$
Right DTE	$Z_{2;5m-5}$	$2m$	$2m+2$
Left DTE	$Z_{1;2,5m-6}$	$2m+1$	$2m+3$

$Z_{4;3,5m-5}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-5}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-6}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-7}$	$2m+1$	$2m+3$

$Z_{5;3,5m-6}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-6}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-7}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-8}$	$2m+1$	$2m+3$

There are several ways of getting $2m+2$ minimum triangles, but no way of getting less. So our theorem holds.

5.6 $n = 4$ mod 5

$N = 5m + 4$

$Z_{0;2,1,5m-1}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-1}$	$2m+2$	$2m+2$
Right DTE	$Z_{2,2;5m-2}$	$2m+3$	$2m+3$
Left DTE	$Z_{2;2,5m-3}$	$2m+3$	$2m+3$

$Z_{1;2,1,5m-2}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2,1;5m-2}$	$2m+2$	$2m+3$
Right DTE	$Z_{2,2;5m-3}$	$2m+3$	$2m+4$
Left DTE	$Z_{2;2,5m-4}$	$2m+2$	$2m+3$

$Z_{1,2;2,5m-3}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{2;5m-3}$	$2m+2$	$2m+4$
Right DTE	$Z_{2;5m-4}$	$2m+1$	$2m+3$
Left DTE	$Z_{1;2,5m-5}$	$2m+1$	$2m+3$

$Z_{4;3,5m-4}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-4}$	$2m+1$	$2m+3$
Right DTE	$Z_{1,2;5m-5}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-6}$	$2m+1$	$2m+3$

$Z_{5;3,5m-5}$			
DTE Type	Zig-zag Pieces	$\triangle$ of last zig-zag pair	$\triangle$ of zig-zag chain
Center DTE	$Z_{3;5m-5}$	$2m$	$2m+2$
Right DTE	$Z_{1,2;5m-6}$	$2m$	$2m+2$
Left DTE	$Z_{2;2,5m-7}$	$2m+1$	$2m+3$

There are several ways of getting $2m+2$ minimum triangles, but no way of getting less. So our theorem holds.

So, the triple induction is completed and theorems 4.1, 4.6, and 4.7 hold true for all values of N.

Chapter 6

Closed Zig-Zags

So far our all of our theorems and proofs have been about open zig-zags and open zig-zag chains. The final question we will be addressing in this thesis is what happens for closed zig-zag and closed zig-zag chains. Closed zig-zags are only a few well chosen DTE away from being an open zigzag. Similarly, closed zig-zag chains can be made into open zig-zag chains. Explicitly defining closed zig-zag chains, we have:

Definition 6.1 (Closed Zig-Zag Chain) An closed zig-zag chain, denoted $Z_{a,b...n:}$ is the zig-zag chain $Z_{a,b,...n}$ with the additional condition that the zig-zag of size a shares an end vertex with the vertex of size n. That is, the zig-zag loops back around and connects to itself.

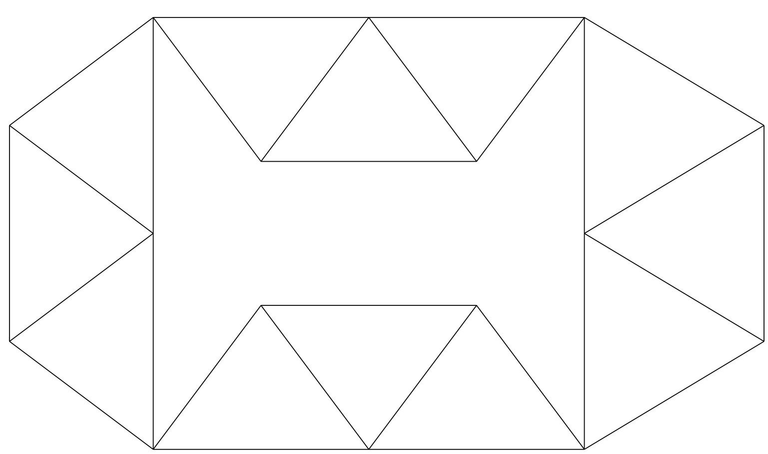

Figure 6.1: Closed Zig-Zag Chain; $Z_{3,3,3,3:}$

Given a closed zig-zag chain $Z_{a,b,\ldots n:}$, we have three splitting DTE able to be performed over the vertex shared by the zig-zags of length a and n. Each splitting DTE results in a (potentially different) open zig-zag chain.

Theorem 6.2 *Of these three open zig-zag chains, the one with the lowest minimum number of triangles, has the same minimum number of triangles as the initial closed zig-zag chain. More specifically:*

$\triangle(\mathrm{Z}_{a,b,\ldots,n:}) = \min(\triangle(\mathrm{Z}_{a,b\ldots n}),\ \triangle(\mathrm{Z}_{a-1,b\ldots n-2,2}),\ (Z_{2,a-2,b\ldots n-1}))$

Proof of 6.2 Suppose there exists some minimum sequence of $Z_{a,b,\ldots n}$ that doesn't have a splitting DTE performed on the zig-zag pieces of length a and n. Then, there exists a $Z_{1,1}$ zig-zag at their shared vertex in this minimum descendant since any larger graph would result in the descendant not being minimum. Then, we can create a new minimum sequence by performing a right (or left) DTE on this $Z_{1,1}$ subgraph. So, any minimum sequence of has a splitting DTE performed on the zig-zag pieces of size a and n or can have an additional splitting DTE included in the sequence that doesn't effect the number of triangles. One can create a new sequence where this splitting DTE is performed first in the sequence since the structure of the zig-zags being effect exists in the initial chain. One has three choices in splitting DTE, so the minimum of these choices results in the minimum of the original zig-zag. ■

This theorem results in turning any closed zig-zag chain into three open (potentially different) zig-zag chains. All our theorems from earlier chapters will now be able to be applied here. One will only ever need to do this once, so while closed zig-zags are slightly more complicated, they quickly get down to a very manageable position. Using this logic, we can calculate the minimum number of triangles of any closed zig-zag.

Theorem 6.3 *The minimum number of triangles for a $Z_{n:}$ descendant is as follows:*

n	Minimum Number of triangles
$5m+0$	$2m$
$5m+1$	$2m+1$
$5m+2$	$2m+2$
$5m+3$	$2m+2$
$5m+4$	$2m+2$

Proof of 6.3 Given any closed zig-zag, $Z_{n:}$, one knows that any minimal sequence contains no increasing or non-neutral DTE by theorem 3.8 and 3.10. So, every minimal sequence starts with the same DTE, that results in $Z_{n-4,3:}$. From here, we use theorems 4.1, 4.6, 4.7 and 5.6 to get the minimum of this zig-zag chains as the minimum of $\triangle(\mathrm{Z}_{n-4,3})$,

$\triangle(\mathrm{Z}_{n-5,1,2})$ and $\triangle(\mathrm{Z}_{2,n-6,2})$. We can now use the techniques used in previous chapters for each case of n mod 5.

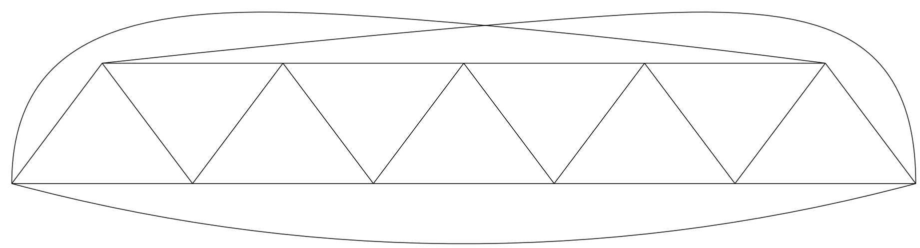

Figure 6.2: $Z_{9:}$

6.1 $Z_{5m:}$

Going through this section as an example to the reader, we start with our $Z_{5m:}$ zig-zag. Any decreasing DTE results in a $Z_{5m-4,3:}$ zig-zag chain. There are three choices in splitting DTE for the two pieces, which result in $Z_{5m-4,3:}$, $Z_{5m-5,1,2}$ and $Z_{2,5m-6,2}$ zig-zag chains. Two of these structures are already known from earlier chapters, so we don't need a new chart for them, as we can restate the results from earlier. The structure we don't know, the $Z_{5m-4,3:}$ chain, will have a full chart dedicated to it in order to see what its minimum number of triangles are. The minimum of all the values in the chart, as well as the two other zig-zag pieces, is the minimum for the entire chain.

$Z_{5m-4,3:}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m-4;3}$	$2m+1$
Right DTE	$Z_{5m-6,2;2}$	$2m+2$
Left DTE	$Z_{5m-5;2,1}$	$2m$

Minimum number of triangles for $Z_{5m-5,1,2} = 2m$.
Minimum number of triangles for $Z_{2,5m-6,2} = 2m+1$.
So, the minimum number of triangles is $2m$.

6.2 Z_{5m+1} :

$Z_{n:} = Z_{5m+1:} = Z_{5m-3,3:}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m-3;3}$	$2m+2$
Right DTE	$Z_{5m-5,2;2}$	$2m+2$
Left DTE	$Z_{5m-4;2,1}$	$2m+1$

Minimum number of triangles for $Z_{2,5m-5,2} = 2m+2$.
Minimum number of triangles for $Z_{5m-4,1,2} = 2m+1$.
So, the minimum number of triangles is $2m+1$.

6.3 Z_{5m+2}

$Z_{n:} = Z_{5m+2:} = Z_{5m-2,3:}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m-2;3}$	$2m+2$
Right DTE	$Z_{5m-4,2;2}$	$2m+2$
Left DTE	$Z_{5m-3;2,1}$	$2m+2$

Minimum number of triangles for $Z_{2,5m-2,2} = 2m+2$.
Minimum number of triangles for $Z_{5m-3,1,2} = 2m+2$.
So, the minimum number of triangles is $2m+2$.

6.4 $5m+3$

$Z_{n:} = Z_{5m+3:} = Z_{5m-1,3:}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m-1;3}$	$2m+2$
Right DTE	$Z_{5m-3,2;2}$	$2m+3$
Left DTE	$Z_{5m-2;2,1}$	$2m+2$

Minimum number of triangles for $Z_{2,5m-3,2} = 2m+2$.
Minimum number of triangles for $Z_{5m-2,1,2} = 2m+2$.
So, the minimum number of triangles is $2m+2$.

6.5 $Z_{5m+4:}$

$Z_{n:} = Z_{5m+4:} = Z_{5m,3:}$		
DTE Type	Zig-zag Pieces	Minimum Number of triangles
Center DTE	$Z_{5m;3}$	$2m+2$
Right DTE	$Z_{5m-2,2;2}$	$2m+3$
Left DTE	$Z_{5m-1;2,1}$	$2m+2$

Minimum number of triangles for $Z_{2,5m-2,2} = 2m+3$.
Minimum number of triangles for $Z_{5m-1,1,2} = 2m+2$.
So, the minimum number of triangles is $2m+2$.

We can see that the minimum number of triangles of $Z_{n:}$ is the equal to the minimum number of triangles as Z_n. So the extra structure of the closed zig-zag doesn't actually allow us to remove any additional triangles. Performing a center DTE then a left DTE on $Z_{n:}$ results in a $Z_{n-4,3}$ zig-zag chain then a $Z_{n-5;2,1}$ zig-zag chain. From these zig-zags, we can see that the minimum number of triangles is the same as the minimum number of triangles of Z_n since the minimum number of triangles of Z_{n-5} is 2 less than that of Z_n and then we have an additional 2 triangles from $Z_{2,1}$. It turns out that there is no case where we can save more triangles.

Chapter 7

Conclusion

We have successful answered the question we presented in chapter 2. Given any zig-zag chain, open or closed, we now have the tools to figure out the minimum number of triangles of this chain. Since K_5 descendants can be expressed as a edge disjoint union of zig-zag and zig-zag chains, by theorem 2.14, this means we now know the minimum number of triangles of any K_5 descendant.

The first step we took was showing that we only need decreasing and non-trivial DTEs. Being able to completely ignore increasing and trivial DTEs when looking at minimum descendants, allowed us to use the techniques in chapter 4, 5 and 6. That is, we can use that the fact that there is a finite number of decreasing and trivial DTEs that can be applied to any K_5 descendant, to systematically find out the minimum number of triangles of these graphs. The finite number of decreasing and trivial DTEs is due to both of these DTEs destroying the zig-zag structure of our graph.

Once we did, we were able to look solely at decreasing and non-trivial DTEs, we are able to focus on the splitting DTEs of minimal sequences for zig-zag chains. That is, we are able to take the DTEs that cause the zig-zag pieces of a zig-zag chain to split from each other, and perform these DTEs first in our minimal sequence. Once we do this, we can break down our zig-zag chain into more familar structures. These structures are zig-zags of the form $Z_n, Z_{n,2}$ and $Z_{n,4}$. Since these structures arise so commonly, we showed what the minimum number of triangles for each of these graphs are, as shown below:

n	Z_n	$Z_{n-2,2}$	$Z_{2,n-4,2}$
$5m+1$	$2m+1$	$2m+2$	$2m+2$
$5m+2$	$2m+2$	$2m+2$	$2m+3$
$5m+3$	$2m+2$	$2m+2$	$2m+3$
$5m+4$	$2m+2$	$2m+3$	$2m+4$
$5m+5$	$2m+2$	$2m+3$	$2m+4$

This direction of looking at K_5 descendants is basically complete from my eyes. Although there may be other zig-zag chains of interest in the future that we have not explicitly calculated the minimum number of triangles of, we now have the tools do so easily. Other problems that arise with K_5 descendants are largely different from what was talked about in this paper. Understanding the structure of K_5 descendants is another talked about problem but quite different from this. This paper mostly ignored edges that were not in triangles, but in practise, some K_5 descendants may consist almost entirely of edges not in triangles. Furthermore, the structure of the zig-zags in K_5 with a small number of triangles compared to the number of edges is also an interesting question. One can never turn an isolated zig-zag of length 2 into an isolated triangle. So, understanding when a zig-zag of length 2 occurs as opposed to an isolated triangle is another question yet to be answered. Still, we now have a much better understand of a specific type of K_5 descendant, those being minimal K_5.

The other problems mentioned in the introduction are the main directions going forward. Conjecture 1.8, regarding all graphs with $c_2 = -1$ being K_5 descendants would be a nice next conjecture to solve, but we are likely far away from that point.

Enumerating graphs with a specific level with respect to some number of vertices is a more approachable next problem. The level of a graph is equal to the number of vertices minus the number of triangles. Enumerating the number of K_5 descendants for a given number of vertices is known for levels 0 to 4. The case for the level equalling 5, although somewhat complicated, is definitely approachable. We are much further away from a general formula though.

This thesis has an interesting relationship to enumerating graphs with a specific level. While it is easier to enumerate low leveled graphs, this paper spent a great deal of focus on dealing with graphs with a low number of triangles, and hence a large level. The biggest issue is the fact that we didn't track edges not in triangles throughout this entire thesis, while these edges naturally lead to different graphs. Regardless, one can hope that this thesis reveals some structure of the large leveled graphs.

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